NELSON VCE

WHERE EVERY QUESTION IS EXAM PRACTICE

FURTHER MATHEMATICS

UNIT 4

DIRK STRASSER
GREG NEAL
KAREN MCMULLEN
NATALIE CARUSO
SERIES EDITOR: DIRK STRASSER

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*Please note that complimentary access to NelsonNet and the NelsonNetBook is only available to teachers who use the accompanying student textbook as a core educational resource in their classroom. Contact your sales representative for information about access codes and conditions.

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- written by Sue Thomson and Judy Binns, experts on teaching and writing for senior applied mathematics courses
- engaging students with interesting and relevant applications of mathematics, with an emphasis on numeracy and literacy
- featuring optional extension topics for students who wish to advance to the Further Mathematics pathway

Contents

About this series iii
About this book vi
About the authors viii

MODULE: MATRICES

8

Matrix arithmetic 1

8.1	Introduction to matrices	2
	Exam prep 8.1	8
	Exam practice 8.1	10
8.2	Addition, subtraction and scalar multiplication of matrices	14
	Exam prep 8.2	18
	Exam practice 8.2	19
8.3	Matrix multiplication	21
	Exam prep 8.3	30
	Exam practice 8.3	31
8.4	Inverse matrices	36
	Exam prep 8.4	42
	Exam practice 8.4	43
Summary		45

9

Matrix applications 47

9.1	Using matrices	48
	Exam prep 9.1	52
	Exam practice 9.1	54
9.2	Matrices and simultaneous equations	60
	Exam prep 9.2	65
	Exam practice 9.2	66
9.3	Communication and dominance matrices	71
	Exam prep 9.3	79
	Exam practice 9.3	81
9.4	Transition diagrams and matrices	86
	Exam prep 9.4	91
	Exam practice 9.4	93
9.5	Transition matrices and recurrence relations	97
	Exam prep 9.5	105
	Exam practice 9.5	106
9.6	Transitions with restocking and culling	113
	Exam prep 9.6	116
	Exam practice 9.6	117
Summary		123

Matrices – Examination 1 **125**
Matrices – Examination 2 **130**

MODULE: NETWORKS AND DECISION MATHEMATICS

10

Undirected graphs 133

10.1	Basic concepts of graphs and networks	134
	Exam prep 10.1	138
	Exam practice 10.1	140
10.2	Planar graphs and Euler's formula	146
	Exam prep 10.2	148
	Exam practice 10.2	149
10.3	Types of graphs	152
	Exam prep 10.3	156
	Exam practice 10.3	157
10.4	Exploring and travelling problems	159
	Exam prep 10.4	166
	Exam practice 10.4	169
10.5	Trees and minimum spanning trees	176
	Exam prep 10.5	180
	Exam practice 10.5	181
Summary		186

11

Directed graphs 189

11.1	The scheduling problem	190
	Exam prep 11.1	195
	Exam practice 11.1	196
11.2	Critical path analysis	199
	Exam prep 11.2	206
	Exam practice 11.2	207
11.3	The assignment problem and bipartite graphs	218
	Exam prep 11.3	224
	Exam practice 11.3	225
11.4	Network flow problems	230
	Exam prep 11.4	233
	Exam practice 11.4	234
11.5	Shortest path problems	239
	Exam prep 11.5	241
	Exam practice 11.5	241
Summary		243

Networks and decision mathematics – Examination 1 **247**
Networks and decision mathematics – Examination 2 **251**

MODULE: GEOMETRY AND MEASUREMENT

12 **Measurement and trigonometry** 255
12.1 Surface area and volume 256
Exam prep 12.1 263
Exam practice 12.1 264
12.2 Linear scale factors 272
Exam prep 12.2 278
Exam practice 12.2 279
12.3 Right-angled triangles 283
Exam prep 12.3 288
Exam practice 12.3 289
12.4 Non-right-angled triangles 294
Exam prep 12.4 299
Exam practice 12.4 300
12.5 Bearings 304
Exam prep 12.5 307
Exam practice 12.5 307
Summary 313

13 **Spherical geometry** 317
13.1 Circle measurement 318
Exam prep 13.1 322
Exam practice 13.1 323
13.2 Latitude and longitude 325
Exam prep 13.2 329
Exam practice 13.2 329
13.3 Shortest distance problems 331
Exam prep 13.3 333
Exam practice 13.3 334
13.4 Time zones 336
Exam prep 13.4 339
Exam practice 13.4 339
Summary 341

Geometry and measurement – Examination 1 343
Geometry and measurement – Examination 2 346

MODULE: GRAPHS AND RELATIONS

14 **Construction and interpretation of graphs** 349
14.1 Review of straight-line graphs 350
Exam prep 14.1 358
Exam practice 14.1 359
14.2 Modelling with straight-line graphs 362
Exam prep 14.2 367
Exam practice 14.2 368
14.3 Line segment and step graphs 372
Exam prep 14.3 378
Exam practice 14.3 380
14.4 Simultaneous equations 386
Exam prep 14.4 391
Exam practice 14.4 391
14.5 Break-even analysis 396
Exam prep 14.5 398
Exam practice 14.5 399
14.6 Interpreting non-linear graphs 403
Exam prep 14.6 404
Exam practice 14.6 405
14.7 Constructing non-linear graphs 409
Exam prep 14.7 416
Exam practice 14.7 417
Summary 423

15 **Linear programming** 427
15.1 Linear inequalities 428
Exam prep 15.1 438
Exam practice 15.1 438
15.2 Systems of linear inequalities and feasible regions 440
Exam prep 15.2 445
Exam practice 15.2 445
15.3 Objective functions 451
Exam prep 15.3 458
Exam practice 15.3 458
15.4 Integer solutions 466
Exam prep 15.4 468
Exam practice 15.4 468
Summary 472

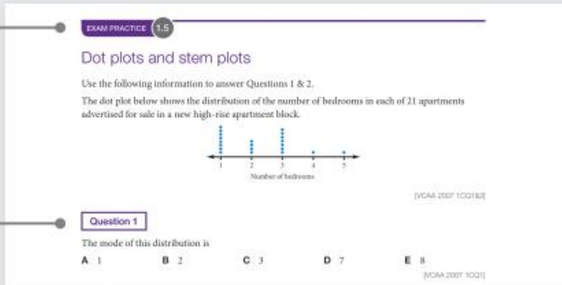
Graphs and relations – Examination 1 475
Graphs and relations – Examination 2 479

Solutions 482
Glossary and index 544

About this book

Exam Practice for every topic featuring carefully selected past exam questions

Questions graded in difficulty according to VCAA data on student exam performance


EXAM PRACTICE 1.5

Dot plots and stem plots

Use the following information to answer Questions 1 & 2.

The dot plot below shows the distribution of the number of bedrooms in each of 21 apartments advertised for sale in a new high-rise apartment block.

Number of bedrooms

VCAA 2007 1CQ1&2

Question 1

The mode of this distribution is

A 1 B 2 C 3 D 7 E 8

VCAA 2007 1CQ1

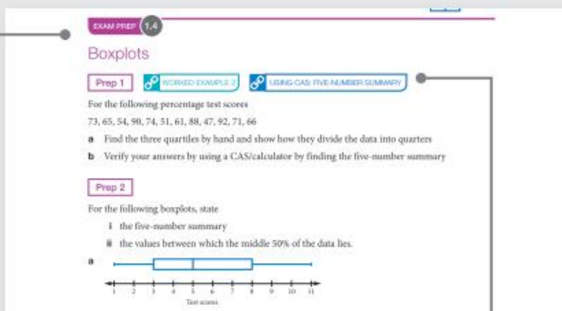
Questions requiring recall from previous textbook sections to support continuous revision

Exam practice

Are you under pressure to rush through course content to leave time to do sufficient exam practice at the end of the year?

Nelson VCE Maths eases the stress by integrating past exam and practice exam questions throughout the year.

Exam Prep focussing specifically on what is required to answer exam questions on a particular topic


EXAM PREP 1.4

Boxplots

Prep 1 WORKED EXAMPLE 2 USING CAS: FIVE-NUMBER SUMMARY

For the following percentage test scores

73, 65, 54, 90, 74, 51, 61, 88, 47, 92, 71, 66

a Find the three quartiles by hand and show how they divide the data into quarters

b Verify your answers by using a CAS/calculator by finding the five-number summary

Prep 2

For the following boxplots, state

i the five-number summary

ii the values between which the middle 50% of the data lies.

a

Test scores

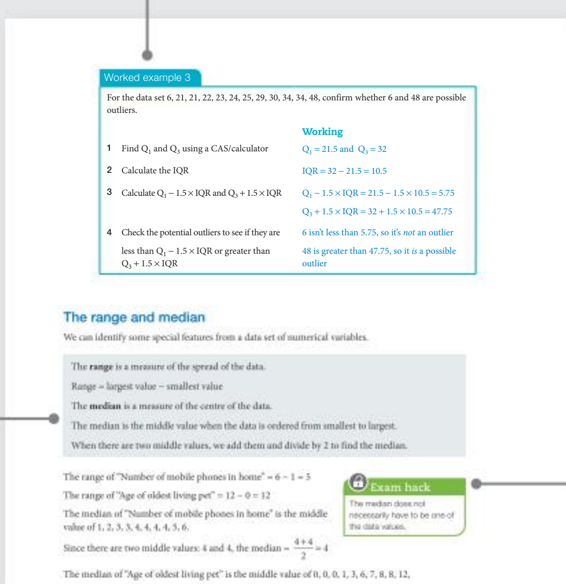
Links to matching worked examples and Using CAS

Exam preparation

Do your students struggle with the maths workload and complain you are 'going too fast'?

Nelson VCE Maths filters out the unnecessary busy work for students that usually fills VCE mathematics textbooks and targets the requirements of the exams, so students can concentrate on what matters.

Worked Examples with clear instructions and working.


Worked example 3

For the data set 6, 21, 21, 22, 23, 24, 25, 29, 30, 34, 34, 48, confirm whether 6 and 48 are possible outliers.

	Working
1 Find Q_1 and Q_3 using a CAS/calculator	$Q_1 = 21.5$ and $Q_3 = 32$
2 Calculate the IQR	$IQR = 32 - 21.5 = 10.5$
3 Calculate $Q_1 - 1.5 \times IQR$ and $Q_3 + 1.5 \times IQR$	$Q_1 - 1.5 \times IQR = 21.5 - 1.5 \times 10.5 = 5.75$ $Q_3 + 1.5 \times IQR = 32 + 1.5 \times 10.5 = 47.75$
4 Check the potential outliers to see if they are less than $Q_1 - 1.5 \times IQR$ or greater than $Q_3 + 1.5 \times IQR$	6 isn't less than 5.75, so it's *not* an outlier 48 is greater than 47.75, so it *is* a possible outlier

The range and median

We can identify some special features from a data set of numerical variables.

The **range** is a measure of the spread of the data.

Range = largest value − smallest value

The **median** is a measure of the centre of the data.

The median is the middle value when the data is ordered from smallest to largest.

When there are two middle values, we add them and divide by 2 to find the median.

The range of "Number of mobile phones in home" = 6 − 1 = 5

The range of "Age of oldest living pet" = 12 − 0 = 12

The median of "Number of mobile phones in home" is the middle value of 1, 2, 3, 3, 4, 4, 4, 4, 5, 6.

Since there are two middle values: 4 and 4, the median = $\frac{4+4}{2} = 4$

The median of "Age of oldest living pet" is the middle value of 0, 0, 0, 1, 3, 6, 7, 8, 8, 12,

Exam hack

The median does not necessarily have to be one of the data values.

Short, sharp summary boxes.

Targeted theory

Do your students find it difficult to make sense of overly detailed and convoluted theory in textbooks?

Nelson VCE Maths provides clear and concise theory, so your students don't get bogged down with wordy explanations.

Exam hacks with 'insider' information based on difficulties students have encountered in past exams.

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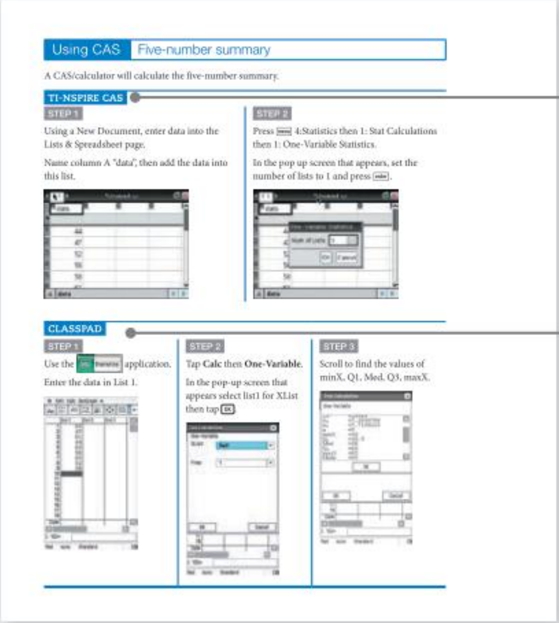
CAS focus

Do your students struggle to find relevant instructions on how to use their CAS at the point of need?

Nelson VCE Maths has CAS instructions clearly labelled and located where students need them.



Using CAS Five-number summary

A CAS/calculator will calculate the five-number summary.

TI-NSPIRE CAS

STEP 1

Using a New Document, enter data into the Lists & Spreadsheet page.

Name column A "data", then add the data into this list.

STEP 2

Press [menu] 4:Statistics then 1: Stat Calculations then 1: One-Variable Statistics.

In the pop up screen that appears, set the number of lists to 1 and press [enter].

CLASSPAD

STEP 1

Use the [Statistics] application.

Enter the data in List 1.

STEP 2

Tap **Calc** then **One-Variable**.

In the pop-up screen that appears select list1 for XList then tap [OK].

STEP 3

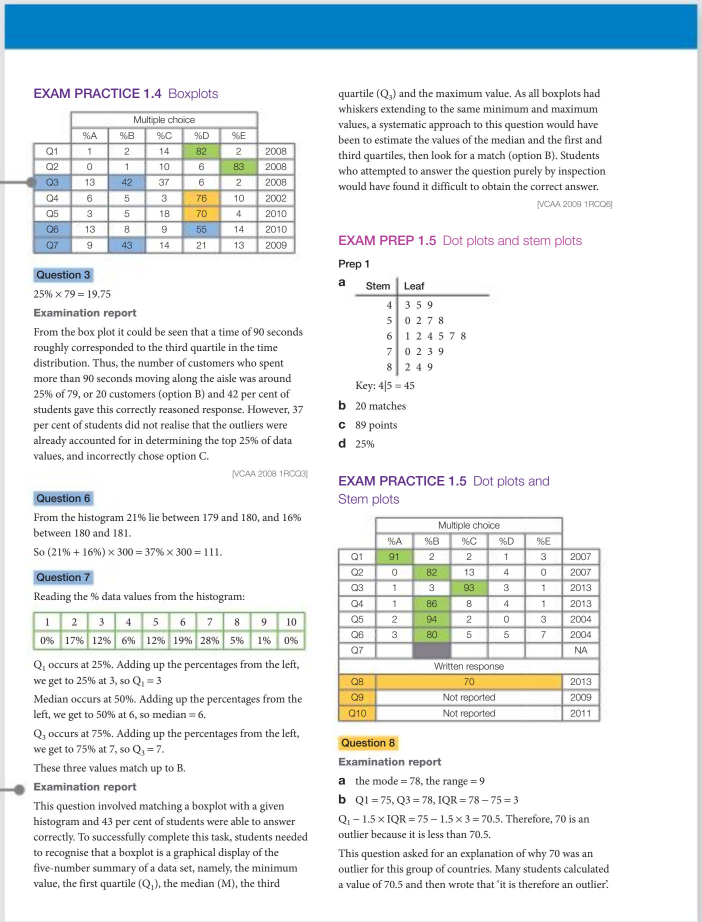
Scroll to find the values of minX, Q1, Med, Q3, maxX.

Both TI-Nspire CAS and ClassPad steps and screen shots.

Detailed solutions

Do your students sometimes need more than just answers at the back of the book?

For selected questions, ***Nelson VCE Maths*** also provides worked solutions and identification of common errors from the VCAA examination reports (worked solutions to *all* questions can be downloaded from the NelsonNet teacher website).



EXAM PRACTICE 1.4 Boxplots

	Multiple choice					
	%A	%B	%C	%D	%E	
Q1	1	2	14	82	2	2008
Q2	0	1	10	6	83	2008
Q3	13	42	37	6	2	2008
Q4	6	5	3	76	10	2002
Q5	3	5	18	70	4	2010
Q6	13	8	9	55	14	2010
Q7	9	43	14	21	13	2009

Question 3

$25\% \times 79 = 19.75$

Examination report

From the box plot it could be seen that a time of 90 seconds roughly corresponded to the third quartile in the time distribution. Thus, the number of customers who spent more than 90 seconds moving along the aisle was around 25% of 79, or 20 customers (option B) and 42 per cent of students gave this correctly reasoned response. However, 37 per cent of students did not realise that the outliers were already accounted for in determining the top 25% of data values, and incorrectly chose option C.

[VCAA 2008 1RCQ3]

Question 6

From the histogram 21% lie between 179 and 180, and 16% between 180 and 181.

So $(21\% + 16\%) \times 300 = 37\% \times 300 = 111$.

Question 7

Reading the % data values from the histogram:

1	2	3	4	5	6	7	8	9	10
0%	17%	12%	6%	12%	19%	28%	5%	1%	0%

Q_1 occurs at 25%. Adding up the percentages from the left, we get to 25% at 3, so $Q_1 = 3$

Median occurs at 50%. Adding up the percentages from the left, we get to 50% at 6, so median = 6.

Q_3 occurs at 75%. Adding up the percentages from the left, we get to 75% at 7, so $Q_3 = 7$.

These three values match up to B.

Examination report

This question involved matching a boxplot with a given histogram and 43 per cent of students were able to answer correctly. To successfully complete this task, students needed to recognise that a boxplot is a graphical display of the five-number summary of a data set, namely, the minimum value, the first quartile (Q_1), the median (M), the third quartile (Q_3) and the maximum value. As all boxplots had whiskers extending to the same minimum and maximum values, a systematic approach to this question would have been to estimate the values of the median and the first and third quartiles, then look for a match (option B). Students who attempted to answer the question purely by inspection would have found it difficult to obtain the correct answer.

[VCAA 2009 1RCQ6]

EXAM PREP 1.5 Dot plots and stem plots

Prep 1

a

Stem	Leaf
4	3 5 9
5	0 2 7 8
6	1 2 4 5 7 8
7	0 2 3 9
8	2 4 9

Key: 4|5 = 45

b 20 matches

c 89 points

d 25%

EXAM PRACTICE 1.5 Dot plots and Stem plots

	Multiple choice					
	%A	%B	%C	%D	%E	
Q1	91	2	2	1	3	2007
Q2	0	82	13	4	0	2007
Q3	1	3	93	3	1	2013
Q4	1	86	8	4	1	2013
Q5	2	94	2	0	3	2004
Q6	3	80	5	5	7	2004
Q7						NA
Written response						
Q8	70					2013
Q9	Not reported					2009
Q10	Not reported					2011

Question 8

Examination report

a the mode = 78, the range = 9

b $Q1 = 75, Q3 = 78, IQR = 78 - 75 = 3$

$Q_1 - 1.5 \times IQR = 75 - 1.5 \times 3 = 70.5$. Therefore, 70 is an outlier because it is less than 70.5.

This question asked for an explanation of why 70 was an outlier for this group of countries. Many students calculated a value of 70.5 and then wrote that 'it is therefore an outlier'.

Colour-coded percentages based on VCAA data indicating the difficulty level of exam questions

Question-specific extracts from the VCAA Examination report

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About the authors

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CHAPTER 8

MATRIX ARITHMETIC

8.1 Introduction to matrices
- What is a matrix?
- The order of a matrix
- Types of matrices
- Constructing matrices
- The transpose of a matrix

8.2 Addition, subtraction and scalar multiplication of matrices
- Addition and subtraction of matrices
- Using CAS: Adding and subtracting matrices
- Scalar multiplication
- Using CAS: Addition, subtraction and scalar multiplication of matrices

8.3 Matrix multiplication
- Multiplying matrices
- Using CAS: Multiplying matrices
- Matrix multiplication rules
- Summing matrices
- Powers of matrices
- Using CAS: Finding powers of matrices

8.4 Inverse matrices
- The identity matrix
- The inverse matrix
- Finding the determinant and the inverse of a matrix
- Using CAS: Finding the determinant and inverse of a matrix

Summary

Prior learning

8.1 Introduction to matrices

What is a matrix?

A **matrix** is a rectangular arrangement of numbers organised into rows and columns. **Matrices** are usually presented in square brackets, for example:

$$\begin{bmatrix} 3 & 2 & 4 \\ 1 & -2 & 4 \end{bmatrix} \qquad \begin{bmatrix} 0 & -1 \\ 5 & 13 \end{bmatrix}$$

$$\begin{bmatrix} 21 & 15 & -42 & 11 \end{bmatrix}$$

Consider the table showing how senior students get to their school each day. The data from the table can be presented in the matrix beside it, as shown.

Fairfax Photos/Justin McManus

Senior students' methods of travel to school

	Year 11	Year 12
Walk	32	18
Bus	78	38
Car (other driver)	27	12
Car (self-driven)	9	31

Matrix equivalent

$$A = \begin{bmatrix} 32 & 18 \\ 78 & 38 \\ 27 & 12 \\ 9 & 31 \end{bmatrix}$$

We usually name matrices using capital letters. Each value in a matrix is called an **element**. Matrix A has 8 elements. There are four rows in matrix A. The first row in matrix A contains the elements 32 and 18. There are two columns in matrix A. The second column contains the elements 18, 38, 12 and 31.

$$\begin{array}{l c} & \begin{array}{cc} \text{Column 1} & \text{Column 2} \\ \downarrow & \downarrow \end{array} \\ \begin{array}{l} \text{Row 1} \longrightarrow \\ \text{Row 2} \longrightarrow \\ \text{Row 3} \longrightarrow \\ \text{Row 4} \longrightarrow \end{array} & \begin{bmatrix} 32 & 18 \\ 78 & 38 \\ 27 & 12 \\ 9 & 31 \end{bmatrix} \end{array}$$

The order of a matrix

The **order** of a matrix tells us how many rows and columns it has. We always write the number of rows first, so Matrix A is a 'four by two' (or 4×2) matrix .

> The order of a matrix is written as $m \times n$, and we say 'm by n', where m is the number of rows and n is the number of columns.
>
> An $m \times n$ matrix has mn elements. For example, a 4×2 matrix has 8 elements.

Types of matrices

Type of matrix	Description	Example	Order of example
Row matrix	A matrix with just one row	$\begin{bmatrix} -2 & 3 & 11 \end{bmatrix}$	1×3
Column matrix	A matrix with just one column	$\begin{bmatrix} 12 \\ -1 \\ 0 \end{bmatrix}$	3×1
Zero matrix	A matrix where all the elements are 0	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	2×3
Binary matrix	A matrix where every element is 0 or 1	$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$	2×3
Square matrix	A matrix that has the same number of rows as columns	$\begin{bmatrix} 2 & 2 & 6 \\ 7 & 1 & 0 \\ 1 & 0 & 10 \end{bmatrix}$	3×3
Symmetric matrix	A square matrix where the elements are symmetric with respect to the **leading diagonal** (the diagonal running from the upper left to the lower right)	$\begin{bmatrix} \mathbf{4} & 6 & -8 \\ 6 & \mathbf{1} & 2 \\ -8 & 2 & \mathbf{10} \end{bmatrix}$	3×3
Upper triangular matrix	A square matrix where all the elements below the leading diagonal are 0	$\begin{bmatrix} -3 & 8 & 2 \\ 0 & -4 & 7 \\ 0 & 0 & 5 \end{bmatrix}$	3×3
Lower triangular matrix	A square matrix where all the elements above the leading diagonal are 0	$\begin{bmatrix} 2 & 0 & 0 \\ 7 & 6 & 0 \\ 0 & -1 & -3 \end{bmatrix}$	3×3
Diagonal matrix	A square matrix where the only non-zero elements are in the leading diagonal	$\begin{bmatrix} 4 & 0 & 0 \\ 0 & -8 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	3×3
Permutation matrix	A square matrix where every row and column has exactly one 1, with 0s everywhere else	$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	3×3

Worked example 1

	Sold by school office	Sold online	Sold at theatre
Student ticket	183	78	41
Adult ticket	4	140	5
Concession ticket	0	54	7

Different types of tickets to a school production of *Fame* have been sold in three different ways, as shown in the table. Write

a the matrix, T, that could be used to show this information and state the order of T

b the row matrix that could be used to show the numbers of concession tickets sold in each of the different ways and state the order of the matrix

c the 3×1 matrix that could be used to show the number of student tickets sold in each of the different ways

d the 1×3 matrix could be used to show the number of each type of ticket sold online

e the column matrix that could be used to show the total number of each type of ticket sold

f the column matrix that could be used to show the total number of each of the ways the tickets have been sold.

Working

a Rewrite the information in the table as a matrix.

$$T = \begin{bmatrix} 183 & 78 & 41 \\ 4 & 140 & 5 \\ 0 & 54 & 7 \end{bmatrix}$$

The order of T is 3×3.

b Find the information in the table and write it in correct matrix form.

$$\begin{bmatrix} 0 & 54 & 7 \end{bmatrix}$$

The order of the matrix is 1×3.

c Find the information in the table and write it in correct matrix form.

$$\begin{bmatrix} 183 \\ 78 \\ 41 \end{bmatrix}$$

d Find the information in the table and write it in correct matrix form.

$$\begin{bmatrix} 78 & 140 & 54 \end{bmatrix}$$

e Add the appropriate table entries and write in the correct matrix form.

$$\begin{bmatrix} 183+78+41 \\ 4+140+5 \\ 0+54+7 \end{bmatrix} = \begin{bmatrix} 302 \\ 149 \\ 61 \end{bmatrix}$$

f Add the appropriate table entries and write in the correct matrix form.

$$\begin{bmatrix} 183+4+0 \\ 78+140+54 \\ 41+5+7 \end{bmatrix} = \begin{bmatrix} 187 \\ 272 \\ 53 \end{bmatrix}$$

Worked example 2

Runners' Paradise has four stores: North, South, East and West, and they currently have only a limited number of sizes of the Run-fit running shoe brand in stock. The table lists the number of pairs of road-runners, trail-runners and cross-trainers that each of the four stores has in four popular sizes.

Store	Size 8	Size $8\frac{1}{2}$	Size 9	Size $9\frac{1}{2}$
North	3 cross-trainers	2 trail-runners	6 trail-runners	1 road-runner
South	2 road-runners	1 road-runner	3 cross-trainers	1 trail-runner
East	5 cross-trainers	3 cross-trainers	4 trail-runners	5 trail-runners
West	2 cross-trainers	1 cross-trainer	1 road-runner	3 cross-trainers

a Write a 3×4 matrix that shows the total number of Run-fit road-runners (R), trail-runners (T) and cross-trainers (C) in each size held at the four outlets.

b Write a 4×3 matrix that shows the same information as in part **a**.

Working

a Set up a matrix that has the type of running shoe as the rows and the sizes as the columns. Enter its elements by adding each relevant category across the four stores.

$$\begin{array}{c c} & \begin{array}{cccc} \text{Size 8} & \text{Size } 8\frac{1}{2} & \text{Size 9} & \text{Size } 9\frac{1}{2} \end{array} \\ \begin{array}{c} R \\ T \\ C \end{array} & \begin{bmatrix} 2 & 1 & 1 & 1 \\ 0 & 2 & 10 & 6 \\ 10 & 4 & 3 & 3 \end{bmatrix} \end{array}$$

b Rewrite the matrix so that the sizes are the rows and the types of running shoe are the columns.

$$\begin{array}{c c} & \begin{array}{ccc} R & T & C \end{array} \\ \begin{array}{c} \text{Size 8} \\ \text{Size } 8\frac{1}{2} \\ \text{Size 9} \\ \text{Size } 9\frac{1}{2} \end{array} & \begin{bmatrix} 2 & 0 & 10 \\ 1 & 2 & 4 \\ 1 & 10 & 3 \\ 1 & 6 & 3 \end{bmatrix} \end{array}$$

8.1

Constructing matrices

We indicate the position of each element in a matrix by referring to which row and column it is in.

For example, in the matrix $A = \begin{bmatrix} 5 & 2 \\ 8 & 9 \\ 3 & 7 \end{bmatrix}$:

a_{11} is the element in row 1 and column 1 $\quad a_{11} = 5$

The 11 in a_{11} is read as 1, 1 not 11.

a_{12} is the element in row 1 and column 2 $\quad a_{12} = 2$

a_{ij} is an element of matrix A where i is the row number and j is the column number.

For example, the elements of a 3×3 matrix can be written as $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Worked example 3

Construct the matrix, A, for each set of rules about the elements, a_{ij}.

a A is a 3×2 matrix, where $a_{12} = 6$, $a_{31} = 9$, $a_{32} = 7$ and all the other elements are 2.

b A is a 1×4 matrix, where $a_{ij} = i - j$.

c A is a 3×3 matrix, where $a_{ij} = 4$ when $i = j$ and $a_{ij} = 0$ when $i \neq j$.

Working

a Use the definition for a_{ij}, where i is the row number and j is the column number, to construct the matrix.

$$A = \begin{bmatrix} 2 & 6 \\ 2 & 2 \\ 9 & 7 \end{bmatrix}$$

b **1** List the elements of the matrix in a_{ij} form.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \end{bmatrix}$$

2 Use the rule to find all the elements of A.

For the rule $a_{ij} = i - j$,
$a_{11} = 1 - 1 = 0$, $a_{12} = 1 - 2 = -1$,
$a_{13} = 1 - 3 = -2$, $a_{14} = 1 - 4 = -3$

3 Write the matrix.

$$A = \begin{bmatrix} 0 & -1 & -2 & -3 \end{bmatrix}$$

c **1** List the elements of the matrix in a_{ij} form.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

2 Use the rule to find all the elements of A.

For $i = j$, $a_{ij} = 4$, so $a_{11} = a_{22} = a_{33} = 4$

For $i \neq j$, $a_{ij} = 0$, so every other element is 0.

3 Write the matrix.

$$A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

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The transpose of a matrix

A **transpose** of a matrix is a new matrix formed by turning all the rows of the original matrix into columns and vice versa. The transpose of matrix A is written A^T. For example:

$$\begin{bmatrix} 3 & 1 & 4 \\ 8 & 2 & 7 \end{bmatrix}^T = \begin{bmatrix} 3 & 8 \\ 1 & 2 \\ 4 & 7 \end{bmatrix}$$

The order of the transpose of a matrix is the reverse of the order of the original matrix. In this example, the original matrix was 2×3 and its transpose is 3×2.

For a square matrix, both the original matrix and its transpose have the same order. In the following example, they are both 2×2.

$$\begin{bmatrix} 5 & 9 \\ 4 & 1 \end{bmatrix}^T = \begin{bmatrix} 5 & 4 \\ 9 & 1 \end{bmatrix}$$

If a matrix, A, has elements a_{ij}, then the transpose of a matrix, A^T, has elements a_{ji}.

For example, the element a_{12} in the original matrix becomes a_{21} in the transpose matrix and the element a_{23} in the original matrix becomes a_{32} in the transpose matrix and so on.

8.1

Worked example 4

Find the transpose of the following matrices and state the orders of both A and A^T.

a $A = \begin{bmatrix} 0 & 3 \\ 6 & -3 \\ 2 & 9 \end{bmatrix}$ **b** $A = \begin{bmatrix} 1 & 9 & 3 & 8 \end{bmatrix}$ **c** $A = \begin{bmatrix} 7 & 1 & 0 \\ 12 & 6 & 2 \\ 3 & 0 & 10 \end{bmatrix}$

Working

a Turn all the rows of the original matrix into columns and vice versa.

$A^T = \begin{bmatrix} 0 & 6 & 2 \\ 3 & -3 & 9 \end{bmatrix}$

The order of A is 3×2.
The order of A^T is 2×3.

b Turn all the rows of the original matrix into columns and vice versa.

$A^T = \begin{bmatrix} 1 \\ 9 \\ 3 \\ 8 \end{bmatrix}$

The order of A is 1×4.
The order of A^T is 4×1.

c Turn all the rows of the original matrix into columns and vice versa.

$A^T = \begin{bmatrix} 7 & 12 & 3 \\ 1 & 6 & 0 \\ 0 & 2 & 10 \end{bmatrix}$

The order of A is 3×3.
The order of A^T is 3×3.

EXAM PREP 8.1

Introduction to matrices

Prep 1

State the order of each matrix, then answer the question.

a $\begin{bmatrix} 3 & 10 \\ 1 & -6 \end{bmatrix}$ Is this a column, square or zero matrix?

b $\begin{bmatrix} 2.1 \\ 3.7 \\ 1.5 \end{bmatrix}$ Is this a column, upper triangular or diagonal matrix?

c $\begin{bmatrix} -3 & -9 \end{bmatrix}$ Is this a square, binary or row matrix?

d $\begin{bmatrix} 8 & 0 \\ 1 & 6 \end{bmatrix}$ Is this a permutation, lower triangular or diagonal matrix?

e $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ Is this a permutation, binary or symmetric matrix?

f $\begin{bmatrix} 0 & 0 & 10 & 6 \\ 0 & 0 & 1 & 4 \\ 10 & 1 & 8 & -2 \\ 6 & 4 & -2 & 0 \end{bmatrix}$ Is this a symmetric, lower triangular or upper triangular matrix?

Prep 2

Answer true or false to each statement.

a A 4×3 matrix has 43 elements.

b A 2×10 matrix has 20 elements.

c The transpose of a 5×2 matrix has 10 elements.

d A permutation matrix with three 1s has 6 elements.

e If a matrix has 5 elements, it has to be either a row matrix or a column matrix.

Prep 3

WORKED EXAMPLE 1

	Type of cake				
Ingredient	Chocolate	Fruit	Tea	Banana	Butter
Sugar	100	80	80	75	100
Flour	225	125	150	175	150
Butter	125	100	150	150	175

The table shows the quantity (in grams) of the main ingredients used to bake various cakes. Write

a the matrix, T, that could be used to show this information and state the order of T

b the row matrix that could be used to show the amount of flour needed in each of the cakes and state the order of the matrix

c the 5×1 matrix that could be used to show the amount of sugar needed in each of the cakes

d the 1×3 matrix could be used to show the amount of each ingredient needed in a banana cake

e the column matrix that could be used to show the total amounts of each ingredient needed if one of every type of cake is made

f the column matrix that could be used to show the total grams of ingredients in each of the cakes.

8.1

Prep 4 WORKED EXAMPLE 2

The table shows the winning schools in four sports in a School Zone Sport Competition from 2014 to 2016.

	Football	Netball	Soccer	Basketball
2014	Valley College	Forest SC	Heights High	Forest SC
2015	Valley College	Heights High	Forest SC	Forest SC
2016	Forest SC	Valley College	Heights High	Valley College

a Write a 3×4 matrix that shows the number of times each sport was won by each of the three teams: Valley College (V), Forest SC (F) and Heights High (H).

b Write a 4×3 matrix that shows the same information as in part **a**.

Prep 5 WORKED EXAMPLE 3

Construct the matrix, A, for each set of rules about the elements, a_{ij}.

a A is a 2×3 matrix where $a_{13} = 8$, $a_{21} = 5$, $a_{23} = 7$ and all the other elements are 1.

b A is a 3×1 matrix where $a_{ij} = i + j$.

c A is a 3×3 matrix where $a_{ij} = 0$ when $i = j$ and $a_{ij} = 1$ when $i \neq j$.

Prep 6 WORKED EXAMPLE 4

Find the transpose of the following matrices and state the orders of both A and A^T.

a $A = \begin{bmatrix} 7 & 2 \\ 8 & -1 \\ 0 & 9 \\ 3 & 4 \end{bmatrix}$ **b** $A = \begin{bmatrix} 6 \\ 3 \\ 0 \end{bmatrix}$ **c** $A = \begin{bmatrix} 7 & 1 \\ 12 & 6 \end{bmatrix}$

Introduction to matrices

Senior students' favourite takeaway food

	Year 11	Year 12
Pizza	32	15
Chicken	38	27
Fish and chips	30	33
Hamburgers	9	32
Kebabs	5	2
Other	3	7

Question 1

List all the orders of the matrices that could be used to represent all the information in the table.

A 6×2 or 2×6

B 6×2 only

C 2×6 only

D 7×3 or 3×7

E 7×3 only

Question 2

For the matrix $A = \begin{bmatrix} 2 & 6 & 4 & 8 \\ 2 & 0 & 9 & 2 \\ 7 & 8 & 9 & 1 \\ 9 & 0 & 6 & 7 \end{bmatrix}$, which is the element a_{34}?

A 3 **B** 6 **C** 1 **D** 4 **E** 7

Question 3

If $K = \begin{bmatrix} 9 & 12 & 0 \\ 3 & 15 & 2 \end{bmatrix}$, what is the value of $k_{13} + k_{11}$?

A 12 **B** 9 **C** 3 **D** 5 **E** 18

Question 4

Which of the following matrices is the transpose of $\begin{bmatrix} -3 & 5 & 2 \\ -4 & 0 & 7 \end{bmatrix}$?

A $\begin{bmatrix} -4 & 0 & 7 \\ -3 & 5 & 2 \end{bmatrix}$

B $\begin{bmatrix} -3 & -4 \\ 5 & 0 \\ 2 & 7 \end{bmatrix}$

C $\begin{bmatrix} -3 & 2 & 5 \\ -4 & 7 & 0 \end{bmatrix}$

D $\begin{bmatrix} -4 & -3 \\ 0 & 5 \\ 7 & 2 \end{bmatrix}$

E $\begin{bmatrix} 5 & 0 \\ -3 & -4 \\ 2 & 7 \end{bmatrix}$

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Question 5

Which of these is *not* a correct description of the matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$?

A binary matrix **B** permutation matrix **C** square matrix
D diagonal matrix **E** 2×2 matrix

8.1

Question 6

Which one of the following matrices won't stay the same after it is transposed?

A $\begin{bmatrix} 5 & 3 \\ 3 & 7 \end{bmatrix}$ **B** $\begin{bmatrix} 5 & 0 & 3 \\ 0 & 5 & 0 \\ 3 & 0 & 5 \end{bmatrix}$ **C** $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}$

D $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ **E** $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Question 7

Which one of the following matrices is symmetric?

A $\begin{bmatrix} 3 & 4 & 1 & 5 \\ 4 & 6 & 2 & 5 \\ 1 & 2 & 6 & 10 \\ 5 & 4 & 10 & 3 \end{bmatrix}$ **B** $\begin{bmatrix} 0 & 4 & 0 & 4 \end{bmatrix}$ **C** $\begin{bmatrix} 21 & 12 \\ 12 & 17 \end{bmatrix}$

D $\begin{bmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \end{bmatrix}$ **E** $\begin{bmatrix} 5 & 0 \\ 0 & 5 \\ 5 & 0 \end{bmatrix}$

Question 8

Which one of the following rules could be used to construct the matrix $A = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$?

A $a_{12} = 2, a_{21} = -1, a_{11} = -3, a_{22} = 2$ **B** $a_{11} = 2, a_{12} = -3, a_{21} = -1, a_{22} = 2$
C $a_{12} = -3, a_{21} = -1$, and $a_{ij} = 2$ when $i = j$ **D** $a_{12} = -1, a_{21} = -3$, and $a_{ij} = 2$ when $i = j$
E $a_{11} = 2, a_{12} = -1, a_{13} = -1, a_{14} = 2$

Question 9

The matrix shows the airfares (in dollars) that are charged by Zeniff Airlines to fly between Adelaide (A), Melbourne (M) and Sydney (S).

$$\begin{array}{c} \text{From} \\ \begin{array}{ccc} A & M & S \end{array} \\ \begin{bmatrix} 0 & 85 & 89 \\ 85 & 0 & 99 \\ 97 & 101 & 0 \end{bmatrix} \begin{array}{c} A \\ M \\ S \end{array} \text{ To} \end{array}$$

The cost to fly from Melbourne to Sydney with Zeniff Airlines is

A \$85 **B** \$89 **C** \$97
D \$99 **E** \$101

[VCAA 2011 1MQ1]

Question 10

The number of tourists visiting three towns, Oldtown, Newtown and Twixtown, was recorded for three years. The data is summarised in the table.

	2004	2005	2006
Oldtown	975	1002	1390
Newtown	2105	1081	1228
Twixtown	610	1095	1380

The 3×1 matrix that could be used to show the number of tourists visiting the three towns in the year **2005** is

A $\begin{bmatrix} 975 & 1002 & 1390 \end{bmatrix}$

B $\begin{bmatrix} 1002 & 1081 & 1095 \end{bmatrix}$

C $\begin{bmatrix} 975 \\ 1002 \\ 1390 \end{bmatrix}$

D $\begin{bmatrix} 1002 \\ 1081 \\ 1095 \end{bmatrix}$

E $\begin{bmatrix} 975 & 1002 & 1390 \\ 2105 & 1081 & 1228 \\ 610 & 1095 & 1380 \end{bmatrix}$

[VCAA 2007 1MQ2]

Question 11

The order of the matrix $\begin{bmatrix} 2 & 2 \\ 2 & 2 \\ 2 & 2 \end{bmatrix}$ is

A 2×2 **B** 2×3 **C** 3×2 **D** 4 **E** 6

[VCAA 2010 1MQ1]

Question 12

Three teams, Blue (B), Green (G) and Red (R), compete for three different sporting competitions. The table shows the competition winners for the past three years.

	Athletics	Cross country	Swimming
2004	Green	Green	Blue
2005	Green	Red	Blue
2006	Blue	Green	Blue

A matrix that shows the **total number** of competitions won by each of the three teams in **each** of these three years could be

A $\begin{array}{c} \\ 2004 \\ 2005 \\ 2006 \end{array} \begin{array}{c} \begin{array}{ccc} B & G & R \end{array} \\ \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \end{array}$

B $\begin{array}{c} \\ 2004 \\ 2005 \\ 2006 \end{array} \begin{array}{c} \begin{array}{ccc} B & G & R \end{array} \\ \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 2 & 1 & 0 \end{bmatrix} \end{array}$

C $\begin{array}{c} \\ 2004 \\ 2005 \\ 2006 \end{array} \begin{array}{c} \begin{array}{ccc} B & G & R \end{array} \\ \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \end{array}$

D $\begin{array}{c} \\ 2004 \\ 2005 \\ 2006 \end{array} \begin{array}{c} \begin{array}{ccc} B & G & R \end{array} \\ \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$

E $\begin{array}{c} \\ 2004 \\ 2005 \\ 2006 \end{array} \begin{array}{c} \begin{array}{ccc} B & G & R \end{array} \\ \begin{bmatrix} 2 & 1 & 0 \\ 2 & 0 & 1 \\ 0 & 3 & 0 \end{bmatrix} \end{array}$

[VCAA 2006 1MQ4]

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Question 13

The number of people attending the morning, afternoon and evening sessions at a cinema is given in the table. The admission charges (in dollars) for each session are also shown in the table.

	Session		
	Morning	Afternoon	Evening
Number of people attending	25	56	124
Admission charge ($)	12	15	20

A column matrix that can be used to list the number of people attending each of the three sessions is

A $\begin{bmatrix} 25 & 56 & 124 \end{bmatrix}$

B $\begin{bmatrix} 25 \\ 56 \\ 124 \end{bmatrix}$

C $\begin{bmatrix} 12 & 15 & 20 \end{bmatrix}$

D $\begin{bmatrix} 12 \\ 15 \\ 20 \end{bmatrix}$

E $\begin{bmatrix} 25 & 56 & 124 \\ 12 & 15 & 20 \end{bmatrix}$

[VCAA 2009 1MQ3]

Question 14

A store has three outlets, *A*, *B* and *C*. These outlets sell dresses, jackets and skirts made by the fashion house Ocki.

The table lists the number of Ocki dresses, jackets and skirts that are currently held at each outlet.

	Size 10	Size 12	Size 14	Size 16
Outlet *A*	2 dresses	3 jackets	1 skirt	4 jackets
Outlet *B*	1 skirt	1 jacket	3 jackets	1 dress
Outlet *C*	2 skirts	2 dresses	2 dresses	1 jacket

A matrix that shows the total number of Ocki dresses (*D*), jackets (*J*) and skirts (*S*) in each size held at the three outlets is given by

A $\begin{array}{c c} & \begin{array}{ccc} D & J & S \end{array} \\ \begin{array}{c} \text{Size 10} \\ \text{Size 12} \\ \text{Size 14} \\ \text{Size 16} \end{array} & \begin{bmatrix} 2 & 1 & 2 \\ 3 & 1 & 2 \\ 1 & 3 & 2 \\ 4 & 1 & 1 \end{bmatrix} \end{array}$

B $\begin{array}{c c} & \begin{array}{ccc} D & J & S \end{array} \\ \begin{array}{c} \text{Size 10} \\ \text{Size 12} \\ \text{Size 14} \\ \text{Size 16} \end{array} & \begin{bmatrix} 2 & 0 & 3 \\ 2 & 4 & 0 \\ 2 & 3 & 1 \\ 1 & 5 & 0 \end{bmatrix} \end{array}$

C $\begin{array}{c c} & \begin{array}{ccc} D & J & S \end{array} \\ \begin{array}{c} \text{Size 10} \\ \text{Size 12} \\ \text{Size 14} \\ \text{Size 16} \end{array} & \begin{bmatrix} 2 & 3 & 1 \\ 4 & 1 & 1 \\ 3 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} \end{array}$

D $\begin{array}{c c} & \begin{array}{ccc} D & J & S \end{array} \\ \begin{array}{c} \text{Size 10} \\ \text{Size 12} \\ \text{Size 14} \\ \text{Size 16} \end{array} & \begin{bmatrix} 7 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$

E $\begin{array}{c c} & \begin{array}{ccc} D & J & S \end{array} \\ \begin{array}{c} \text{Size 10} \\ \text{Size 12} \\ \text{Size 14} \\ \text{Size 16} \end{array} & \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 0 \end{bmatrix} \end{array}$

[VCAA 2012 1MQ7]

8.2 Addition, subtraction and scalar multiplication of matrices

WS Addition and subtraction of matrices

Addition and subtraction of matrices

Addition and subtraction of matrices can only be performed with matrices of the same order. Addition of matrices involves adding the elements that are in corresponding positions in both matrices.

So, if we have matrix $A = \begin{bmatrix} 2 & 7 \\ 11 & 3 \end{bmatrix}$ and matrix $B = \begin{bmatrix} 1 & 3 \\ 5 & 12 \end{bmatrix}$, then

$$A + B = \begin{bmatrix} 2+1 & 7+3 \\ 11+5 & 3+12 \end{bmatrix} = \begin{bmatrix} 3 & 10 \\ 16 & 15 \end{bmatrix}$$

Similarly, to subtract matrix B from matrix A, we simply subtract each element in matrix B from its corresponding element in matrix A. So:

$$A - B = \begin{bmatrix} 2-1 & 7-3 \\ 11-5 & 3-12 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 6 & -9 \end{bmatrix}$$

Alamy/Tim Gainey

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Using CAS — Adding and subtracting matrices

8.2

Use a CAS/calculator to evaluate

$$\begin{bmatrix} 12 & 3 \\ 18 & 5 \end{bmatrix} + \begin{bmatrix} 13 & -9 \\ 6 & 22 \end{bmatrix}$$

TI-NSPIRE CAS

STEP 1

Use a New Document with a Calculator page.

Press .

Select the 2 × 2 matrix template $\begin{bmatrix} \square & \square \\ \square & \square \end{bmatrix}$ and press enter.

STEP 2

Enter the elements, pressing tab after each entry.

STEP 3

Press + then create a second 2 × 2 matrix using the method outlined above.

Press enter.

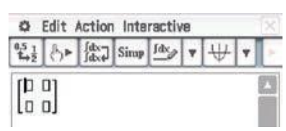
CLASSPAD

STEP 1

Use the Main $\sqrt{\alpha}$ application.

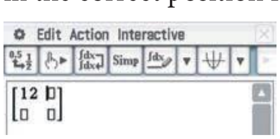
Press Keyboard.

Tap Math2 then tap the 2 × 2 matrix template .



STEP 2

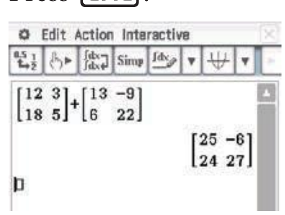
Enter the elements, using the arrow keys ▶ ▼ ◀ or the stylus to position the cursor in the correct position for each entry.



STEP 3

Press ▶ as necessary to move the cursor outside the first matrix. Press + then create a second 2 × 2 matrix using the method outlined in Step 2.

Press EXE.



Worked example 5

Find the values of x, y and z in the following

$$\begin{bmatrix} 6 & 12 \\ y & 7 \end{bmatrix} - \begin{bmatrix} 6 & -5 \\ 1 & 2z \end{bmatrix} = \begin{bmatrix} 0 & x \\ 4 & -3 \end{bmatrix}$$

		Working
1	Using the elements in row 1 and column 2 of each matrix, write down an equation.	$12 - (-5) = x$
2	Solve for x.	$x = 17$
3	Using the elements in row 2 and column 1 of each matrix, write down an equation.	$y - 1 = 4$
4	Solve for y.	$y = 5$
5	Using the elements in row 2 and column 2 of each matrix, write down an equation.	$7 - 2z = -3$
6	Solve for z.	$2z = 10$ $z = 5$

Scalar multiplication

Scalar multiplication involves multiplying a matrix by a number. When dealing with matrices, we use the word **scalar** to indicate a number that's not in a matrix. To multiply matrix A by the scalar 4, we simply multiply each of the elements in matrix A by 4.

If $A = \begin{bmatrix} 5 & 2 \\ 7 & -3 \\ 12 & 6 \end{bmatrix}$, then $4A = 4\begin{bmatrix} 5 & 2 \\ 7 & -3 \\ 12 & 6 \end{bmatrix} = \begin{bmatrix} 4\times 5 & 4\times 2 \\ 4\times 7 & 4\times(-3) \\ 4\times 12 & 4\times 6 \end{bmatrix} = \begin{bmatrix} 20 & 8 \\ 28 & -12 \\ 48 & 24 \end{bmatrix}$

Using CAS Addition, subtraction and scalar multiplication of matrices

Given that $Q = \begin{bmatrix} 3 & -5 & 10 \\ 11 & 6 & 17 \\ -2 & 15 & 0 \end{bmatrix}$ and $R = \begin{bmatrix} 12 & 2 & 1 \\ -3 & 4 & 7 \\ 22 & 14 & -7 \end{bmatrix}$, evaluate $4Q - 2R$ and $3Q + 7R$.

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TI-NSPIRE CAS

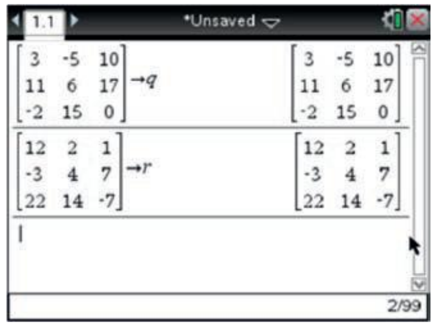
STEP 1

Use a new document with a calculator page.

Press [template], then select the 3 × 3 matrix template. Set the number of rows to 3 and the number of columns to 3.

Create the first 3 × 3 matrix, then press [ctrl] [var] [Q] [enter] to store this matrix as *q*.

Create the second 3 × 3 matrix, then press [ctrl] [var] [R] to store this matrix as *r*.

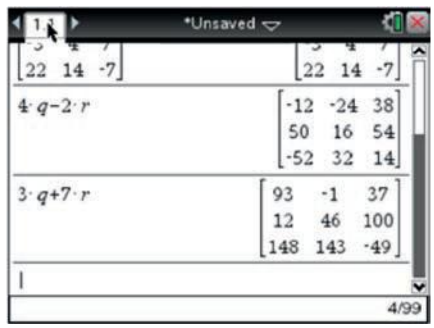


STEP 2

The necessary calculations can then be performed.

Open a new document to clear the stored matrices.

Press [menu]
1: Actions
4: Clear a-z
[enter] to clear all stored variables



CLASSPAD

STEP 1

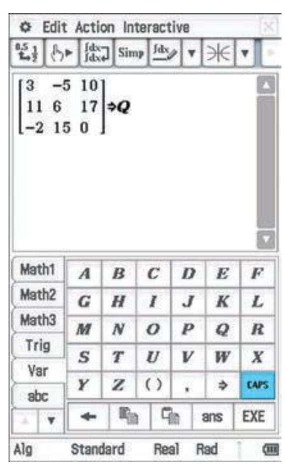
Use the Main $\sqrt{\alpha}$ application.

Press [Keyboard].

Tap [Math2] then tab the 2 × 2 matrix template [matrix] twice for a 3 × 3 matrix.

Create the first 3 × 3 matrix, then tap [⇒] [Var] [CAPS] *Q* and press [EXE] to store this matrix as *Q*.

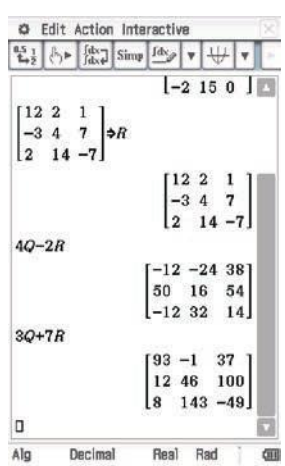
Create the second 3 × 3 matrix, then tap [⇒] [Var] *R* and press [EXE] to store this matrix as *R*.

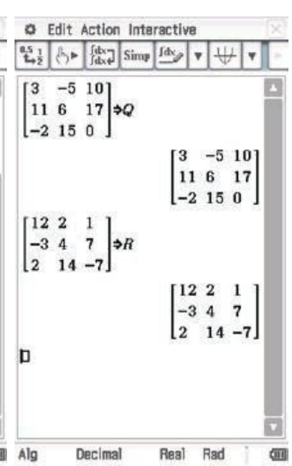


STEP 2

The necessary calculations can then be performed.

To clear stored matrices, tap **Edit**, then **Clear All Variables**, then **OK**.





Worked example 6

Find the values of the pronumerals in the following matrix equation.

$$3\begin{bmatrix} x & 2 & -1 \end{bmatrix} + 2\begin{bmatrix} 4 & 5 & y \end{bmatrix} = \begin{bmatrix} 14 & 16 & 11 \end{bmatrix}$$

Working

1 Perform the scalar multiplications on the left-hand side.

$$3\begin{bmatrix} x & 2 & -1 \end{bmatrix} + 2\begin{bmatrix} 4 & 5 & y \end{bmatrix}$$
$$= \begin{bmatrix} 3x & 6 & -3 \end{bmatrix} + \begin{bmatrix} 8 & 10 & 2y \end{bmatrix}$$

2 Add the corresponding elements.

$$= \begin{bmatrix} 3x+8 & 16 & -3+2y \end{bmatrix}$$

3 Now equate these to the right-hand side of the original equation.

$$\begin{bmatrix} 3x+8 & 16 & -3+2y \end{bmatrix} = \begin{bmatrix} 14 & 16 & 11 \end{bmatrix}$$

4 Equate corresponding elements and solve for the unknown pronumerals, using a CAS/calculator if necessary.

$3x + 8 = 14$ $\quad$ $-3 + 2y = 11$

$x = 2$ $\quad$ $y = 7$

EXAM PREP 8.2

Addition, subtraction and scalar multiplication of matrices

Prep 1

Using the matrices N, O, P and Q, find the following.

$$N = \begin{bmatrix} 16 & 5 \\ 2 & 12 \end{bmatrix}, O = \begin{bmatrix} 10 & 3 \\ 7 & 22 \end{bmatrix}, P = \begin{bmatrix} 15 & 1 \\ 8 & 34 \end{bmatrix}, Q = \begin{bmatrix} 11 & -5 \\ 29 & 36 \end{bmatrix}$$

a $N + O$ **b** $N + P$ **c** $Q - N$ **d** $N - Q$ **e** $P - P$

Prep 2

USING CAS: ADDING AND SUBTRACTING MATRICES

Use a CAS/calculator to evaluate the following.

a $\begin{bmatrix} 0 & 5 & -1.2 \\ 3.6 & -2 & 10 \\ 14 & 5.1 & -6 \end{bmatrix} + \begin{bmatrix} 3.1 & 8 & 6.2 \\ 5 & 12 & -16 \\ 3 & -3.2 & -5 \end{bmatrix}$

b $\begin{bmatrix} 8 & 5.2 & 17 \\ 5.8 & 12 & 16 \\ 18 & 9.2 & -11 \end{bmatrix} - \begin{bmatrix} 0 & 3.1 & 9 \\ 5 & 12 & -16 \\ 3 & -3.2 & -5 \end{bmatrix}$

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Prep 3 WORKED EXAMPLE 5

Find the value of the pronumeral in each of the following.

a $\begin{bmatrix} 21 \\ 35 \\ 16 \\ 15 \end{bmatrix} + \begin{bmatrix} 17 \\ 9 \\ 38 \\ 14 \end{bmatrix} = \begin{bmatrix} 38 \\ m \\ 54 \\ 29 \end{bmatrix}$

b $\begin{bmatrix} 27 & 13 \\ 9 & 44 \end{bmatrix} - \begin{bmatrix} 11 & 14 \\ 28 & 39 \end{bmatrix} = \begin{bmatrix} 16 & -1 \\ -19 & d \end{bmatrix}$

c $\begin{bmatrix} 31 & 19 \\ 28 & t \end{bmatrix} + \begin{bmatrix} 56 & 47 \\ 37 & 29 \end{bmatrix} = \begin{bmatrix} 87 & 66 \\ 65 & 77 \end{bmatrix}$

Prep 4 USING CAS: ADDITION, SUBTRACTION AND SCALAR MULTIPLICATION OF MATRICES

Given that $M = \begin{bmatrix} 5 & 9 \\ 3 & 8 \end{bmatrix}$ and $N = \begin{bmatrix} 3 & -2 \\ 5 & 4 \end{bmatrix}$, evaluate the following using a CAS/calculator.

a $2M + 2N$ **b** $3N - 2M$ **c** $5M + 4N$

d $6M - 4N$ **e** $12N - 3M$ **f** $M + 2M$

Prep 5 WORKED EXAMPLE 6

Find the values of the pronumerals in the following matrix equations.

a $5\begin{bmatrix} a & 3 \\ 7 & -2 \end{bmatrix} + 2\begin{bmatrix} 5 & b \\ 11 & 8 \end{bmatrix} = \begin{bmatrix} 25 & 37 \\ 57 & c \end{bmatrix}$

b $3\begin{bmatrix} 10 & 5 \\ 4 & x \\ 8 & 7 \end{bmatrix} + 4\begin{bmatrix} 1 & 9 \\ 8 & 6 \\ y & 2 \end{bmatrix} = \begin{bmatrix} 34 & 51 \\ z & 45 \\ 60 & 29 \end{bmatrix}$

c $4\begin{bmatrix} 5 & p \\ 7 & 9 \end{bmatrix} - 2\begin{bmatrix} 11 & 10 \\ q & 8 \end{bmatrix} = \begin{bmatrix} 2f & 32 \\ -10 & 20 \end{bmatrix}$

d $4\begin{bmatrix} a \\ -3 \\ b \\ 10 \end{bmatrix} - 7\begin{bmatrix} 12 \\ c \\ -5 \\ 6 \end{bmatrix} = \begin{bmatrix} -52 \\ 23 \\ 19 \\ d \end{bmatrix}$

EXAM PRACTICE 8.2

Addition, subtraction and scalar multiplication of matrices

Question 1

$2\begin{bmatrix} 3 & 2 \\ 0 & 4 \end{bmatrix} + 3\begin{bmatrix} -1 & 0 \\ 1 & 6 \end{bmatrix}$ is equal to

A $5\begin{bmatrix} 2 & 2 \\ 1 & 10 \end{bmatrix}$ **B** $\begin{bmatrix} 3 & 7 \\ 3 & 26 \end{bmatrix}$ **C** $6\begin{bmatrix} 2 & 2 \\ 1 & 10 \end{bmatrix}$ **D** $\begin{bmatrix} 3 & 4 \\ 3 & 26 \end{bmatrix}$ **E** $\begin{bmatrix} 5 & 4 \\ 3 & 8 \end{bmatrix}$

8.2

Question 2

The matrix sum $\begin{bmatrix} 0 & -4 \\ 2 & 5 \end{bmatrix} + \begin{bmatrix} 5 & 4 \\ -2 & 2 \end{bmatrix}$ is equal to

A $\begin{bmatrix} 5 & 0 \\ 0 & 7 \end{bmatrix}$ **B** $\begin{bmatrix} 0 & 0 \\ 0 & 7 \end{bmatrix}$ **C** $\begin{bmatrix} 5 & -4 \\ 0 & 7 \end{bmatrix}$

D $\begin{bmatrix} 0 & 5 & -4 & 4 \\ 2 & -2 & 5 & 2 \end{bmatrix}$ **E** $\begin{bmatrix} 0 & -4 & 5 & 4 \\ 2 & 5 & -2 & 2 \end{bmatrix}$

[VCAA 2007 1MQ1]

Question 3

$2 \times \begin{bmatrix} 2 & 8 \\ 4 & -1 \\ 3 & 5 \end{bmatrix} - \begin{bmatrix} 3 & 7 \\ 4 & 2 \\ 2 & 3 \end{bmatrix}$ equals

A $\begin{bmatrix} 1 & 1 \\ 0 & -3 \\ 4 & 2 \end{bmatrix}$ **B** $\begin{bmatrix} -2 & 2 \\ 0 & -6 \\ 2 & 4 \end{bmatrix}$ **C** $\begin{bmatrix} 1 & 9 \\ 12 & 0 \\ 8 & 13 \end{bmatrix}$ **D** $\begin{bmatrix} 1 & 9 \\ 4 & -4 \\ 4 & 7 \end{bmatrix}$ **E** $\begin{bmatrix} -1 & 1 \\ 0 & -3 \\ 1 & 2 \end{bmatrix}$

[VCAA 2012 1MQ1]

Question 4

If $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 7 \\ 8 & d \end{bmatrix} = \begin{bmatrix} 4 & 7 \\ 8 & 11 \end{bmatrix}$, then d is equal to

A −11 **B** −10 **C** 7 **D** 10 **E** 11

[VCAA 2008 1MQ1]

Question 5

$3\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} + 2\begin{bmatrix} -1 & 0 \\ 2 & -7 \end{bmatrix}$ equals

A $\begin{bmatrix} 4 & 3 \\ 4 & -5 \end{bmatrix}$ **B** $6\begin{bmatrix} 1 & 1 \\ 2 & -4 \end{bmatrix}$ **C** $\begin{bmatrix} 4 & 3 \\ 4 & 2 \end{bmatrix}$ **D** $5\begin{bmatrix} 1 & 1 \\ 2 & -4 \end{bmatrix}$ **E** $\begin{bmatrix} 3 & 6 \\ 7 & 4 \end{bmatrix}$

[VCAA 2009 1MQ1]

Question 6

The matrix $\begin{bmatrix} 12 & 36 \\ 0 & 24 \end{bmatrix}$ is equal to

A $12\begin{bmatrix} 0 & 3 \\ 0 & 2 \end{bmatrix}$ **B** $12\begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}$ **C** $12\begin{bmatrix} 0 & 24 \\ -12 & 12 \end{bmatrix}$ **D** $12\begin{bmatrix} 0 & 24 \\ 0 & 12 \end{bmatrix}$ **E** $12\begin{bmatrix} 1 & 3 \\ -12 & 2 \end{bmatrix}$

[VCAA 2006 1MQ1]

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Matrix multiplication

8.3

Multiplying matrices

Multiplying matrices

We have looked at the multiplication of a matrix by a scalar. It is also possible to multiply a matrix by a matrix. To multiply two matrices, we multiply pairs of elements, working *across the rows* in the first matrix and *down the columns* in the second matrix.

For example, if we have $A = \begin{bmatrix} 2 & 6 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 5 & 1 \\ 2 & 4 & 7 \end{bmatrix}$, and we need to find AB, the first element in AB is calculated by multiplying each element of A row 1 by its matching element in B column 1 and adding them together:

$AB = \begin{bmatrix} 2 & 6 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & 5 & 1 \\ 2 & 4 & 7 \end{bmatrix}$, so the first element of AB is $(2 \times 3 + 6 \times 2) = 18$.

Follow this pattern to find all the elements of AB:

$$\begin{bmatrix} (2\times3)+(6\times2)=18 & (2\times5)+(6\times4)=34 & (2\times1)+(6\times7)=44 \\ (0\times3)+(1\times2)=2 & (0\times5)+(1\times4)=4 & (0\times1)+(1\times7)=7 \end{bmatrix}$$

So $AB = \begin{bmatrix} 18 & 34 & 44 \\ 2 & 4 & 7 \end{bmatrix}$

Exam hack

Note that the row and column pairs you multiply together also tell you the position of the element in the *AB* matrix. For example, when you multiply '*A* row 1 × *B* column 3', the result is the *AB* element in row 1 column 3.

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Using CAS Multiplying matrices

Let $A = \begin{bmatrix} 5 & 3 \\ 2 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 5 \\ 3 & 7 \end{bmatrix}$.

a Calculate AB by hand and verify your answer using a CAS/calculator.

b Calculate BA by hand and verify your answer using a CAS/calculator.

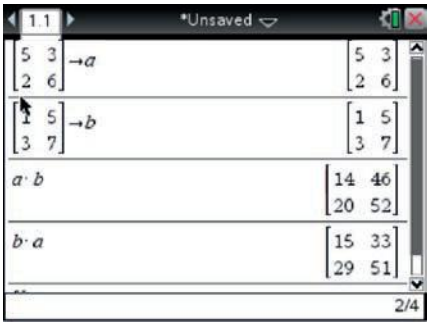
c What do you notice about the two products?

a $$AB = \begin{bmatrix} 5 & 3 \\ 2 & 6 \end{bmatrix}\begin{bmatrix} 1 & 5 \\ 3 & 7 \end{bmatrix}$$
$$= \begin{bmatrix} 5\times1+3\times3 & 5\times5+3\times7 \\ 2\times1+6\times3 & 2\times5+6\times7 \end{bmatrix}$$
$$= \begin{bmatrix} 14 & 46 \\ 20 & 52 \end{bmatrix}$$

b $$BA = \begin{bmatrix} 1 & 5 \\ 3 & 7 \end{bmatrix}\begin{bmatrix} 5 & 3 \\ 2 & 6 \end{bmatrix}$$
$$= \begin{bmatrix} 1\times5+5\times2 & 1\times3+5\times6 \\ 3\times5+7\times2 & 3\times3+7\times6 \end{bmatrix}$$
$$= \begin{bmatrix} 15 & 33 \\ 29 & 51 \end{bmatrix}$$

To calculate using a CAS/calculator, enter the matrices and assign them as in the previous Using CAS before doing the multiplication.

TI-NSPIRE CAS



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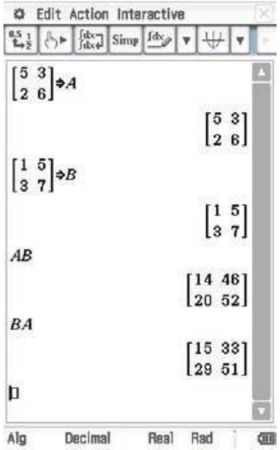

c AB and BA are not equal. Swapping the order when multiplying matrices has resulted in different answers.

Unlike when multiplying numbers, swapping the order when multiplying matrices usually gives a different answer. Generally for matrices, $AB \neq BA$.

Since the order in which you multiply matrices is important, we have two terms to describe different sorts of multiplication. When we are multiplying two matrices AB, we say we are pre-multiplying by A (i.e. A is *before* B) or post-multiplying by B (i.e. B is *after* A).

Matrix multiplication rules

Multiplication of two matrices is only possible if the number of columns in the first matrix is the same as the number of rows in the second matrix. If it's possible to do the multiplication, then the product matrix will have the same number of rows as the first matrix and the same number of columns as the second matrix.

Matrix product rule

If matrix A is of order $m \times n$ and matrix B is of order $n \times p$, then the product AB is defined, which means it's possible. Otherwise AB isn't undefined, which means it's not possible.

$$\begin{array}{ccc} \text{order of 1st matrix} & & \text{order of 2nd matrix} \\ (m \times n) & \times & (n \times p) \end{array}$$

must be the same
for AB to be defined

For example, $(3 \times 7) \times (7 \times 6)$ is a defined product, but $(3 \times 2) \times (3 \times 2)$ isn't.

Note the $\times$ inside the brackets in this context always indicates the order of a matrix, not multiplication.

Order of product matrix rule

If matrix A is of order $m \times n$ and matrix B is of order $n \times p$, then AB will be of order $m \times p$. This can be shown by a **matrix order equation**:

$$\begin{array}{ccccc} \text{order of 1st matrix} & & \text{order of 2nd matrix} & & \text{order of product matrix} \\ (m \times n) & \times & (n \times p) & = & (m \times p) \end{array}$$

These tell us the order of AB.

For example, the matrix order equation $(3 \times 4) \times (4 \times 6) = (3 \times 6)$ tells us that a matrix with 3 rows and 4 columns multiplied by a matrix with 4 rows and 6 columns will result in a matrix with 3 rows and 6 columns.

Exam hack

The matrix order equation can be used to work out the order of a product when more than two matrices are being multiplied together. For example, $(3 \times 7) \times (7 \times 2) \times (2 \times 5) = (3 \times 5)$.

This means, if the multiplications are defined,

- the number of rows in the product = the number rows in the *first* matrix
- the number of columns in the product = the number of columns in the *last* matrix

Summing matrices

Row and column matrices that consist entirely of 1s are called **summing matrices** because, when they are multiplied with other matrices, they sum elements. For example,

$$\begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} 10 & 2 & 1 \\ 3 & 7 & 5 \end{bmatrix} = \begin{bmatrix} 13 & 9 & 6 \end{bmatrix}$$ where the columns are summed.

$$\begin{bmatrix} 2 & 4 & 3 \\ 5 & 1 & 9 \\ 6 & 20 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 9 \\ 15 \\ 27 \end{bmatrix}$$ where the rows are summed.

It is also possible to use the summing matrices to calculate means of elements. For example,

$$\frac{1}{2}\begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} 10 & 2 & 1 \\ 3 & 7 & 5 \end{bmatrix} = \begin{bmatrix} 6.5 & 4.5 & 3 \end{bmatrix}$$ where the means of the columns are calculated.

$$\begin{bmatrix} 2 & 4 & 3 \\ 5 & 1 & 9 \\ 6 & 20 & 1 \end{bmatrix}\frac{1}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 9 \end{bmatrix}$$ where the means of the rows are calculated.

To sum the columns of a matrix, pre-multiply it by a suitable row matrix consisting of 1s, called a summing matrix.

To sum the rows of a matrix, post-multiply it by a suitable column matrix consisting of 1s, also called a summing matrix.

To find the mean of the columns of a matrix, pre-multiply it by the summing matrix and by $\frac{1}{n}$, where n is the number of columns in the summing matrix.

To find the mean of the rows of a matrix, post-multiply it by the summing matrix and by $\frac{1}{m}$, where m is the number of rows in the summing matrix.

Worked example 7

Let $A = \begin{bmatrix} 3 & 7 & 2 & 12 \end{bmatrix}$, $B = \begin{bmatrix} 5 \\ 9 \\ 7 \\ 2 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $X = \begin{bmatrix} 3 & 2 \\ 4 & 6 \\ 5 & 10 \end{bmatrix}$

and $Y = \begin{bmatrix} 18 & 2 \\ 12 & 5 \end{bmatrix}$.

a **i** Show how the orders of A and B will tell us what the order of AB will be.

ii Calculate AB.

b **i** Show how the orders of A and B will tell us what the order of BA will be.

ii Calculate BA.

c **i** Calculate CX and comment on what has been summed.

ii Find the means of the columns of C by multiplying CX by a scalar.

d **i** Calculate YD and comment on what has been summed.

ii Find the means of the rows of Y by multiplying YD by a scalar.

Working

a **i** Use the fact that $(m \times n) \times (n \times p)$ gives an $m \times p$ matrix.

A is of order 1×4

B is of order 4×1

AB is $(1 \times 4) \times (4 \times 1)$

So AB will be a 1×1 matrix.

ii Multiply the corresponding elements, then add them together.

$$AB = \begin{bmatrix} 3 & 7 & 2 & 12 \end{bmatrix} \begin{bmatrix} 5 \\ 9 \\ 7 \\ 2 \end{bmatrix}$$

$$= [3 \times 5 + 7 \times 9 + 2 \times 7 + 12 \times 2]$$

$$= [116]$$

b **i** Use the fact that $(m \times n) \times (n \times p)$ gives an $m \times p$ matrix.

BA is $(4 \times 1) \times (1 \times 4)$

So BA will be a 4×4 matrix.

ii Multiply the corresponding elements, then add them together.

$$BA = \begin{bmatrix} 5 \\ 9 \\ 7 \\ 2 \end{bmatrix} \begin{bmatrix} 3 & 7 & 2 & 12 \end{bmatrix}$$

$$= \begin{bmatrix} 15 & 35 & 10 & 60 \\ 27 & 63 & 18 & 108 \\ 21 & 49 & 14 & 84 \\ 6 & 14 & 4 & 24 \end{bmatrix}$$

c **i** Pre-multiply by the summing matrix.

$$CX = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 4 & 6 \\ 5 & 10 \end{bmatrix} = \begin{bmatrix} 12 & 18 \end{bmatrix}$$

The columns of C have been summed.

ii Multiply by $\frac{1}{n}$, where n is the number of columns in the summing matrix.

$$\frac{1}{3}CX = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 4 & 6 \\ 5 & 10 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 12 & 18 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 6 \end{bmatrix}$$

d **i** Post-multiply by the summing matrix.

$YD = \begin{bmatrix} 18 & 2 \\ 12 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 20 \\ 17 \end{bmatrix}$

The rows of Y have been summed.

ii Multiply by $\frac{1}{m}$, where m is the number of rows in the summing matrix. Since $\frac{1}{m}$ is a scalar, the order in which you multiply it doesn't matter.

$$Y\frac{1}{2}D = \frac{1}{2}YD$$
$$= \frac{1}{2}\begin{bmatrix} 18 & 2 \\ 12 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$= \frac{1}{2}\begin{bmatrix} 20 \\ 17 \end{bmatrix} = \begin{bmatrix} 10 \\ 8.5 \end{bmatrix}$$

Worked example 8

Let $X = \begin{bmatrix} 5 & 3 \\ 2 & 6 \\ 9 & 10 \end{bmatrix}$, $A = \begin{bmatrix} 1 & 5 \\ 9 & 11 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 7 \\ 1 & 12 \\ 5 & 11 \end{bmatrix}$.

For each matrix product **a** XA **b** XB **c** AX

i state whether the product is defined

ii for those products that are defined, state their order.

Working

X is of order 3×2, A is of order 2×2 and B is of order 3×2.

a **i** Does the number of columns in X equal the number of rows in A?

Is $(3 \times 2) \times (2 \times 2)$ defined?

Yes. The product XA exists.

ii Use a matrix order equation.

$(3 \times 2) \times (2 \times 2) = (3 \times 2)$

XA has order 3×2.

b **i**

Is $(3 \times 2) \times (3 \times 2)$ defined?

No. The product XB does not exist.

c **i**

Is $(2 \times 2) \times (3 \times 2)$ defined?

No. The product AX does not exist.

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Worked example 9

a A and B are both matrices with 2 rows and 4 columns. $AX = B$. What is the order of X?

b A is a matrix with 2 rows and 4 columns. X is another matrix. B is a matrix with 5 rows.

If $AXB = \begin{bmatrix} 6 & 1 & 7 \\ 2 & 8 & 5 \end{bmatrix}$, what are the orders of X and B?

c A is a matrix of order $m \times n$.

i What is the order of A^T, the transpose of A?

ii Show that the product AA^T is defined and state the order.

iii What type of matrix is AA^T?

	Working
a **1** Write down the orders for all the matrices, using pronumerals when they are unknown.	A is of order 2×4. B is of order 2×4. Let X be of order $m \times n$.
2 Write the matrix order equation.	$AX = B$ has the matrix order equation $(2 \times 4) \times (m \times n) = (2 \times 4)$
3 Use the matrix order equation to find m and n.	$m = 4$ (for the multiplication to be defined) $n = 4$ (since the number of columns in the second matrix = the number of columns in the product)
4 Write the answer.	The order of X is 4×4.
b **1** Write down the orders for all the matrices, using pronumerals when they are unknown.	A is of order 2×4. Let X be of order $m \times n$. B is of order $5 \times p$. AXB is of order 2×3.
2 Write the matrix order equation.	$AXB = \begin{bmatrix} 6 & 1 & 7 \\ 2 & 8 & 5 \end{bmatrix}$ has the matrix order equation $(2 \times 4) \times (m \times n) \times (5 \times p) = (2 \times 3)$
3 Use the matrix order equation to find m and n.	$m = 4$ (for the first multiplication to be defined) $n = 5$ (for the second multiplication to be defined) $p = 3$ (since the number of columns in the last matrix = the number of columns in the product)
4 Write the answer.	The order of X is 4×5. The order of B is 5×3.

8.3

c **i** In A^T, the rows and columns from A have swapped.

A has order $m \times n$, so A^T has order $n \times m$.

ii Write the matrix order equation for AA^T.

$(m \times n) \times (n \times m) = (m \times m)$

AA^T is defined because the number of columns in A equals the number of rows in A^T.

AA^T has order $(m \times m)$.

iii AA^T has the same number of rows as columns.

AA^T is a square matrix.

Worked example 10

Show that for the matrix equation

$$\begin{bmatrix} 2 & 1 \\ 5 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ a \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 6 & -1 \end{bmatrix}\begin{bmatrix} b \\ 2 \end{bmatrix},$$

the following equations could be used to find the values of a and b:

$$a - 3b = -12$$
$$a - 6b = -22$$

Working

1 Multiply the matrices.

$$\begin{bmatrix} 8+a \\ 20+a \end{bmatrix} = \begin{bmatrix} 3b-4 \\ 6b-2 \end{bmatrix}$$

2 Equate the matching elements and simplify, using a CAS/calculator if necessary.

$$8+a = 3b-4$$
$$a-3b = -12$$
$$20+a = 6b-2$$
$$a-6b = -22$$

So the equations $a - 3b = -12$, $a - 6b = -22$ could be used to find the values of a and b.

Powers of matrices

Just as 5^2 means 5×5, for a matrix named A, then A^2 means $A \times A$. Only square matrices can be raised to a power because otherwise the number of columns in the first matrix will be different to the number of rows in the second matrix and multiplication won't be possible. For example, a 2×3 matrix squared would give $(2 \times 3) \times (2 \times 3)$, which isn't defined.

Only square matrices can be raised to a power.

Any power of a matrix will always have the same order as the original matrix.

Using CAS Finding powers of matrices

If $X = \begin{bmatrix} 5 & 11 & 12 \\ 7 & 3 & 1 \\ -4 & 2 & 6 \end{bmatrix}$,

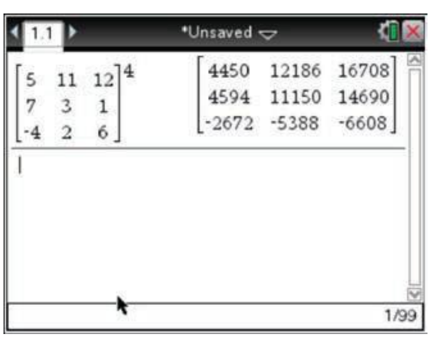
a find X^4.

b what order will X^{50} have?

a Create a 3×3 matrix and enter the elements as shown.

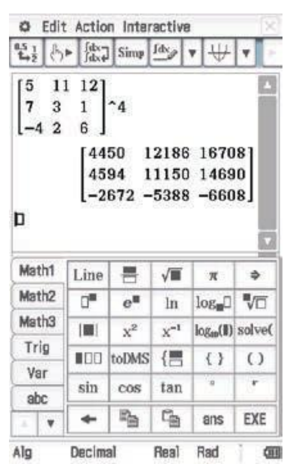
TI-NSPIRE CAS

Press [^] [4] then [enter] to raise the matrix to the power 4.



CLASSPAD

Press [^] 4 then [EXE] to raise the matrix to the power 4.



Write the answer.

$$X^4 = \begin{bmatrix} 4450 & 12\,186 & 16\,708 \\ 4594 & 11\,150 & 14\,690 \\ -2672 & -5\,388 & -6\,608 \end{bmatrix}$$

b The order of X^{50} will be 3×3, the same as the order of X.

Multiplying matrices

Prep 1

USING CAS: MULTIPLYING MATRICES

Let $X = \begin{bmatrix} 5 & 8 & 11 \\ 3 & 6 & 4 \end{bmatrix}$ and $Y = \begin{bmatrix} 1 & 5 \\ 6 & 4 \\ 3 & 7 \end{bmatrix}$.

a Calculate XY by hand and verify your answer using a CAS/calculator.

b Calculate YX by hand and verify your answer using a CAS/calculator.

c What do you notice about the two products?

Prep 2

WORKED EXAMPLE 7

Let $A = \begin{bmatrix} 2 & 1 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 3 \\ 4 \\ 10 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $X = \begin{bmatrix} 6 & 4 \\ 1 & 3 \\ 5 & 2 \end{bmatrix}$ and $Y = \begin{bmatrix} 9 & 20 \\ 3 & 9 \end{bmatrix}$.

a **i** Show how the orders of A and B will tell us what the order of AB will be.

ii Calculate AB.

b **i** Show how the orders of A and B will tell us what the order of BA will be.

ii Calculate BA.

c **i** Calculate CX and comment on what has been summed.

ii Find the means of the columns of C by multiplying CX by a scalar.

d **i** Calculate YD and comment on what has been summed.

ii Find the means of the rows of D by multiplying YD by a scalar.

Prep 3

WORKED EXAMPLE 8

Let $A = \begin{bmatrix} 2 & 4 & 1 \\ 5 & 6 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 4 \\ 6 & 8 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 3 \\ 5 & 2 \\ 1 & 6 \\ 4 & 7 \end{bmatrix}$ and $D = \begin{bmatrix} 4 & 3 \\ 5 & 1 \\ 2 & 6 \end{bmatrix}$.

For each matrix product below:

i state whether the product is defined

ii for those products that are defined, state their order.

a AB **b** BA **c** AD **d** AC **e** DA

f BC **g** CB **h** DB **i** CD **j** BB

Prep 4 WORKED EXAMPLE 9

a A and B are both matrices with 3 rows and 5 columns. $AX = B$. What is the order of X?

b A is a matrix with 2 rows and 5 columns. X is another matrix. B is a matrix with 6 rows.

If $AXB = \begin{bmatrix} 5 & 1 & 7 & 0 \\ 9 & 8 & 8 & 8 \end{bmatrix}$, what are the orders of X and B?

c A is a matrix of order $m \times n$.

i What is the order of A^T, the transpose of A?

ii Show that the product A^TA is defined and state the order.

iii What type of matrix is A^TA?

Prep 5 WORKED EXAMPLE 10

Show that for the matrix equation

$$\begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ a \end{bmatrix} = \begin{bmatrix} -2 & 4 \\ 6 & -1 \end{bmatrix}\begin{bmatrix} b \\ 1 \end{bmatrix},$$

the following equations could be used to find the values of a and b:

$$a + b = 3$$
$$a - 2b = 1$$

Prep 6 USING CAS: FINDING POWERS OF MATRICES

If $X = \begin{bmatrix} 3 & 7 \\ 9 & 4 \end{bmatrix}$ and $Y = \begin{bmatrix} 2.5 & 6.1 \\ 3.9 & 2.7 \end{bmatrix}$, find

a X^2 **b** X^3 **c** Y^2 **d** $X^2 + Y^2$

e X^5 **f** $X^5 - X^2$ **g** what order $X^{41} - Y^{26}$ has

EXAMPLE PRACTICE 8.3

Multiplying matrices

Question 1

If $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $C = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, then $AB + 2C$ equals

A $\begin{bmatrix} 0 \\ 3 \end{bmatrix}$ **B** $\begin{bmatrix} 3 \\ 0 \end{bmatrix}$ **C** $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ **D** $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$ **E** $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$

[VCAA 2011 1MQ2]

Question 2

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \\ 0 \\ 2 \end{bmatrix} - 2\times\begin{bmatrix} 0 \\ -1 \\ -1 \\ 0 \end{bmatrix}$$ equals

A $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ **B** $\begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix}$ **C** $\begin{bmatrix} 2 \\ -2 \\ -2 \\ 2 \end{bmatrix}$ **D** $\begin{bmatrix} 4 \\ 0 \\ 0 \\ 4 \end{bmatrix}$ **E** $\begin{bmatrix} 0 \\ 2 \\ 2 \\ 0 \end{bmatrix}$

[VCAA 2013 1MQ1]

Question 3

The matrix $\begin{bmatrix} 12 & 15 & 3 \\ -6 & 0 & 24 \end{bmatrix}$ can also be written as

A $\begin{bmatrix} 12 & 15 & 3 \end{bmatrix} + \begin{bmatrix} -6 & 0 & 24 \end{bmatrix}$

B $\begin{bmatrix} 12 \\ -6 \end{bmatrix} + \begin{bmatrix} 15 \\ 0 \end{bmatrix} + \begin{bmatrix} 3 \\ 24 \end{bmatrix}$

C $\begin{bmatrix} 3 \\ 6 \end{bmatrix}\begin{bmatrix} 4 & 5 & 1 \\ -1 & 0 & 4 \end{bmatrix}$

D $\frac{1}{3}\times\begin{bmatrix} 4 & 5 & 1 \\ -2 & 0 & 8 \end{bmatrix}$

E $3\times\begin{bmatrix} 4 & 5 & 1 \\ -2 & 0 & 8 \end{bmatrix}$

[VCAA 2009 1MQ2]

Question 4

Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$.

Then $A^3(B - C)$ equals

A $\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$ **B** $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ **C** $\begin{bmatrix} 3 & 6 \\ 6 & -3 \end{bmatrix}$ **D** $\begin{bmatrix} 3 & 0 \\ 0 & -3 \end{bmatrix}$ **E** $\begin{bmatrix} 5 & 10 \\ 10 & -5 \end{bmatrix}$

[VCAA 2006 1MQ3]

Question 5

Matrix A has three rows and two columns. Matrix B has four rows and three columns.

Matrix $C = B \times A$ has

A two rows and three columns.

B three rows and two columns.

C three rows and three columns.

D four rows and two columns.

E four rows and three columns.

[VCAA 2013 1MQ2]

8.3

Question 6

Let $A=\begin{bmatrix} -2 \\ 0 \end{bmatrix}$, $B=\begin{bmatrix} 0 & 9 \end{bmatrix}$ and $C=[2]$.

Using these matrices, the matrix product that is **not** defined is

A AB **B** AC **C** BA **D** BC **E** CB

[VCAA 2006 1MQ2]

Question 7

If $A=\begin{bmatrix} 8 & 1 \\ 4 & 2 \end{bmatrix}$ and $B=\begin{bmatrix} 3 & 11 \\ 6 & 0 \end{bmatrix}$, then the matrix $AB=\begin{bmatrix} 30 & 96 \\ 24 & 48 \end{bmatrix}$.

The element '24' in the matrix AB is correctly obtained by calculating

A $4 \times 6 + 2 \times 0$ **B** $4 \times 3 + 2 \times 6$ **C** $3 \times 4 + 12 \times 1$

D $4 \times 2 + 8 \times 2$ **E** $8 \times 3 + 1 \times 0$

[VCAA 2012 1MQ2]

Question 8

If $A=\begin{bmatrix} 8 & 4 \\ 5 & 3 \end{bmatrix}$ and the product $AX=\begin{bmatrix} 5 & 6 \\ 8 & 10 \end{bmatrix}$, then X is

A $\begin{bmatrix} 24 & -14 \\ 13 & -7.5 \end{bmatrix}$ **B** $\begin{bmatrix} -4.25 & -5.5 \\ 9.75 & 12.5 \end{bmatrix}$ **C** $\begin{bmatrix} -3.75 & 7 \\ -6.5 & 12 \end{bmatrix}$

D $\begin{bmatrix} 25 & 11 \\ -19.5 & -8.5 \end{bmatrix}$ **E** $\begin{bmatrix} 0.625 & 1.5 \\ 1.6 & 3.333 \end{bmatrix}$

[VCAA 2007 1MQ3]

Question 9

m and n are positive whole numbers. Matrix P is of order $m \times n$. Matrix Q is of order $n \times m$.

The matrix products PQ and QP are both defined

A for no values of m and n. **B** when m is equal to n only.

C when m is greater than n only. **D** when m is less than n only.

E for all values of m and n.

[VCAA 2010 1MQ8]

Question 10

Which matrix expression results in a matrix that contains the sum of the numbers 2, 5, 4, 1 and 8?

A $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \times \begin{bmatrix} 2 & 5 & 4 & 1 & 8 \end{bmatrix}$

B $\begin{bmatrix} 2 & 5 & 4 & 1 & 8 \end{bmatrix} \times \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

C $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 8 \end{bmatrix}$

D $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 2 \\ 5 \\ 4 \\ 1 \\ 8 \end{bmatrix}$

E $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 8 \end{bmatrix}$

[VCAA 2010 1MQ4]

Question 11

Matrix M is a 3×4 matrix. Matrix P has five rows. N is another matrix.
If the matrix product

$M(NP) = \begin{bmatrix} 4 & 1 & 7 & 2 \\ 0 & 9 & 7 & 4 \\ 4 & 3 & 3 & 1 \end{bmatrix}$, then the order of matrix N is

A 3×5 **B** 5×3 **C** 4×5 **D** 5×4 **E** 5×5

[VCAA 2007 1MQ9]

Question 12

$$\begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix} \times \begin{bmatrix} a \\ 3 \end{bmatrix} = \begin{bmatrix} 6 & 3 \\ 2 & -1 \end{bmatrix} \times \begin{bmatrix} 2 \\ b \end{bmatrix}$$

Which set of equations below could be used to determine the values of a and b that are shown in the matrix equation above?

A $a - b = 2$, $a + b = 0$

B $a + b = -2$, $a - b = 0$

C $a + b = 2$, $a - b = 0$

D $a - b = 8$, $a + b = 2$

E $a - b = 8$, $a + b = -2$

[VCAA 2012 1MQ9]

Question 13

If $A = \begin{bmatrix} 1 & 3 \\ 6 & 4 \\ 0 & 0 \end{bmatrix}$ and the matrix product $XA = \begin{bmatrix} 4 & 1 \\ 1 & 4 \\ 3 & 5 \end{bmatrix}$, then the order of matrix X is

A 2×2 **B** 2×3 **C** 3×1 **D** 3×2 **E** 3×3

[VCAA 2006 1MQ6]

8.3

Question 14

$A = \begin{bmatrix} 5 & 4 \\ 11 & 8 \\ 7 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 9 & 12 \\ 10 & 5 & 6 \end{bmatrix}$, $C = \begin{bmatrix} 3 & 8 \\ 1 & 0 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 7 \\ 3 & 8 \\ 9 & 2 \end{bmatrix}$

a Which pair of matrices can be added?

b Does the matrix product AC exist? Explain.

c List all of the possible matrix products formed by pairs from the given matrices.

d State the order of each matrix product that you listed for part **c**.

e Which matrix can be raised to a power? Why?

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8.4 Inverse matrices

The identity matrix

If we multiply a number by 1, it leaves the original number the same. The number 1 is called the **multiplicative identity** because the original number remains *identical* to what it was before multiplication.

The equivalent matrix to the number 1 is the **identity matrix**, I, the square matrix where all the elements in the leading diagonal are 1 and the other elements are 0. Multiplying a matrix by the identity matrix leaves the original matrix unchanged. Here are some identity matrices:

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Worked example 11

If $A = \begin{bmatrix} 3 & 10 \\ 2 & 5 \end{bmatrix}$ and I is the identity matrix, calculate AI and IA by hand.

What do you notice when you compare the two answers?

Working

1 Calculate AI.

$$AI = \begin{bmatrix} 3 & 10 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3\times1+10\times0 & 3\times0+10\times1 \\ 2\times1+5\times0 & 2\times0+5\times1 \end{bmatrix} = \begin{bmatrix} 3 & 10 \\ 2 & 5 \end{bmatrix}$$

2 Calculate IA.

$$IA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & 10 \\ 2 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 1\times3+0\times2 & 1\times10+0\times5 \\ 0\times3+1\times2 & 0\times10+1\times5 \end{bmatrix} = \begin{bmatrix} 3 & 10 \\ 2 & 5 \end{bmatrix}$$

3 Compare the two answers.

$AI = IA = A$

This is an example where we can change the order of the matrices and the product stays the same.

The identity matrix, I, is the square matrix with leading diagonal elements 1 and other elements 0, which when multiplied by a square matrix, A, of the same order, leaves A unchanged.

$AI = IA = A$

The inverse matrix

The **multiplicative inverse** of the number 5 is $\frac{1}{5}$. A multiplicative inverse is what you multiply the original number by to get 1. For example, $5 \times \frac{1}{5} = \frac{5}{5} = 1$. Similarly, the multiplicative inverse of a matrix is what you multiply the original matrix by to get the identity matrix, I.

The **inverse** of a square matrix A, written as A^{-1}, is the matrix where

$AA^{-1} = A^{-1}A = I$

Inverses are only defined for square matrices.

Worked example 12

Show that the matrices $\begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}$ and $\begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix}$ are inverses of each other.

Working

1 Show that the product of the two matrices equals the identity matrix.

$$\begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix}$$
$$= \begin{bmatrix} 3\times5+7\times(-2) & 3\times(-7)+7\times3 \\ 2\times5+5\times(-2) & 2\times(-7)+5\times3 \end{bmatrix}$$
$$= \begin{bmatrix} 15-14 & -21+21 \\ 10-10 & -14+15 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

2 Reverse the order of the two matrices and show that their product also equals the identity matrix.

$$\begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}$$
$$= \begin{bmatrix} 5\times3+(-7)\times2 & 5\times7+(-7)\times5 \\ -2\times3+3\times2 & -2\times7+3\times5 \end{bmatrix}$$
$$= \begin{bmatrix} 15-14 & 35-35 \\ -6+6 & -14+15 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Finding the determinant and the inverse of a matrix

To find the inverse of a matrix, we first need to find the **determinant**. We will start with 2×2 matrices. The calculations involved for finding determinants and inverses of higher order square matrices are significantly more complex and require a CAS/calculator.

For a matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the determinant is $\det(A) = ad - bc$ and

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{\det(A)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

The inverse doesn't exist when $\det(A) = 0$.

Worked example 13

For each of the following matrices find

i the determinant

ii the inverse (if it exists).

a $A = \begin{bmatrix} 6 & 2 \\ 8 & 3 \end{bmatrix}$ **b** $B = \begin{bmatrix} 2 & 5 \\ 3 & 7 \end{bmatrix}$ **c** $C = \begin{bmatrix} 8 & 2 \\ 8 & 2 \end{bmatrix}$

	Working
a **i** Calculate $\det(A)$.	$\det(A) = 6 \times 3 - 2 \times 8 = 18 - 16 = 2$
ii Find A^{-1} (if it exists).	$A^{-1} = \frac{1}{2}\begin{bmatrix} 3 & -2 \\ -8 & 6 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & -1 \\ -4 & 3 \end{bmatrix}$
b **i** Calculate $\det(B)$.	$\det(B) = 2 \times 7 - 5 \times 3 = 14 - 15 = -1$
ii Find B^{-1} (if it exists).	$B^{-1} = -1\begin{bmatrix} 7 & -5 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} -7 & 5 \\ 3 & -2 \end{bmatrix}$
c **i** Calculate $\det(C)$.	$\det(C) = 8 \times 2 - 2 \times 8 = 16 - 16 = 0$
ii Find C^{-1} (if it exists).	Since $\det(C) = 0$, C^{-1} doesn't exist.

Worked example 14

a If the determinant of $\begin{bmatrix} 5 & -3 \\ x & 3 \end{bmatrix}$ is equal to 21, what is the value of x?

b For the matrix $A = \begin{bmatrix} 2 & 3 \\ x & -4 \end{bmatrix}$, what is the value of x for which $A = A^{-1}$?

c Show that if A and B are both square matrices and $A + B$ is defined, then $A^{-1}B + A^4$ is defined.

8.4

Working

a **1** Find the determinant of the matrix.

$$\det = 5 \times 3 - (-3) \times x$$
$$= 15 + 3x$$

2 Let the determinant equal the number given and solve for x.

$$15 + 3x = 21$$
$$x = 2$$

b **1** Find the determinant of A.

$$\det(A) = 2 \times (-4) - 3 \times x$$
$$= -8 - 3x$$

2 Find A^{-1}.

$$A^{-1} = \frac{1}{-8-3x}\begin{bmatrix} -4 & -3 \\ -x & 2 \end{bmatrix} = \begin{bmatrix} \dfrac{-4}{-8-3x} & \dfrac{-3}{-8-3x} \\ \dfrac{-x}{-8-3x} & \dfrac{2}{-8-3x} \end{bmatrix}$$

3 Let $A = A^{-1}$ and, using the fact that for matrices to be equal every corresponding element has to be equal, set 2 corresponding elements equal to each other.

$$A = A^{-1}$$

$$\begin{bmatrix} 2 & 3 \\ x & -4 \end{bmatrix} = \begin{bmatrix} \dfrac{-4}{-8-3x} & \dfrac{-3}{-8-3x} \\ \dfrac{-x}{-8-3x} & \dfrac{2}{-8-3x} \end{bmatrix}$$

So, $2 = \dfrac{-4}{-8-3x}$

Solve for x, using a CAS/calculator if necessary.

$$x = -2$$

c **1** State the orders of A and B.

A and B are both square matrices, so let A be of order $m \times m$ and B be of order $n \times n$.

2 Use the information given about A and B.

Since $A + B$ is defined, A and B must be of the same order, so $m = n$.
This means both A and B are of order $m \times m$.

3 Use the orders of A and B to find the orders of the other matrices in the question.

Show that $A^{-1}B + A^4$ is defined.

If A is $m \times m$, then A^{-1} is $m \times m$ and A^4 is $m \times m$.

4 Write a matrix order equation and answer the question.

The matrix order equation for $A^{-1}B + A^4$ is

$$(m \times m) \times (m \times m) + (m \times m)$$
$$= (m \times m) + (m \times m)$$

Since the two matrices being added together are of the same order, $A^{-1}B + A^4$ is defined.

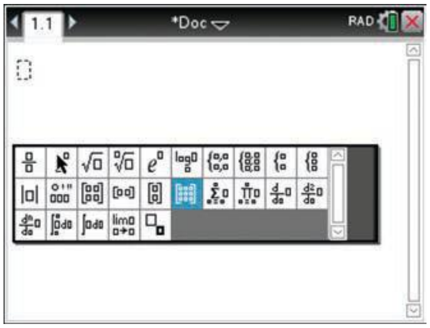
Using CAS Finding the determinant and inverse of a matrix

For $A = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}$, find $\det(A)$ and A^{-1}.

TI-NSPIRE CAS

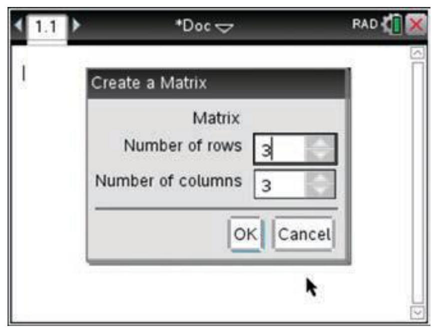
STEP 1

On a calculator page press templates



STEP 2

Create a 3×3 matrix and enter the elements for matrix A above. Press ctrl var A to store the matrix as a.



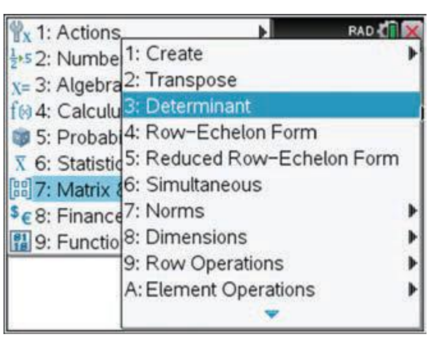
STEP 3

Press menu

7: Matrix & Vector

3: Determinant

Press A enter

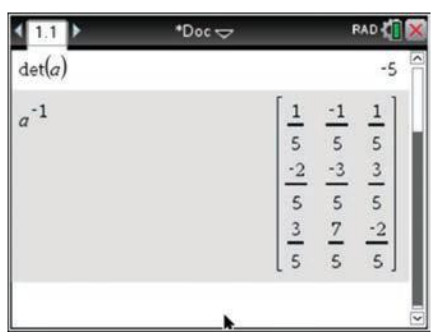


STEP 4

To calculate the inverse a^{-1}

type A ^ (–) 1 enter

To approximate, enter a^{-1} and press ctrl enter



STEP 5

Write the answer.

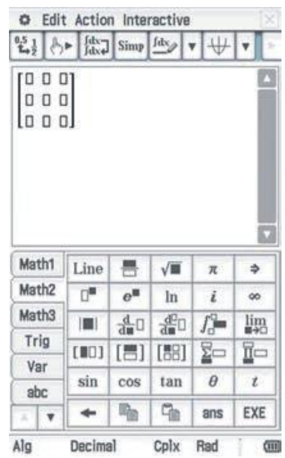
det $(A) = -5$

$$A^{-1} = \frac{1}{5}\begin{bmatrix} 1 & -1 & 1 \\ -2 & -3 & 3 \\ 3 & 7 & -2 \end{bmatrix} = \begin{bmatrix} 0.2 & -0.2 & 0.2 \\ -0.4 & -0.6 & 0.6 \\ 0.6 & 1.4 & -0.4 \end{bmatrix}$$

CLASSPAD

STEP 1

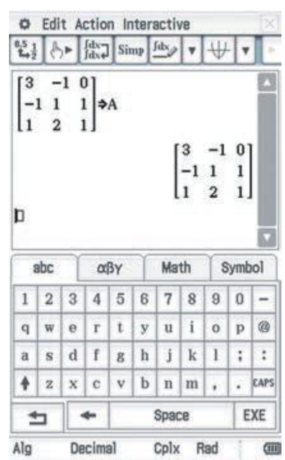
Use the Main $\sqrt[\text{}]{\alpha}$ application. Press Keyboard, then Math2 and tap on [▯▯] twice for a 3 × 3 matrix.



STEP 2

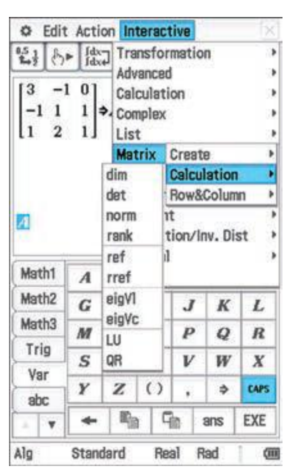
Enter the numbers in the matrix by tapping each square and typing the number. You can also use the arrows to move around the elements of the matrix.

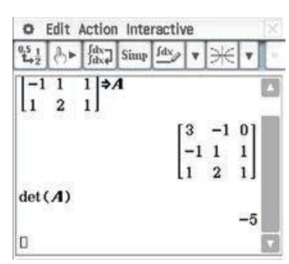
Assign a name to the matrix by tapping ⇒, then the name. Press EXE.



STEP 3

To find the determinant, highlight A that has been defined, then tap **Interactive**, **Matrix**, **Calculation**, **det** to get det (A) = –5.

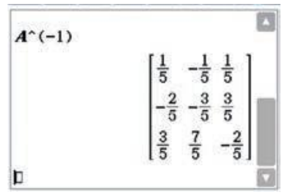




STEP 4

To calculate the inverse of A, A^{-1}, type A [^] (−1).

Press [EXE].



STEP 5

Write the answer.

$\det(A) = -5$

$$A^{-1} = \begin{bmatrix} 0.2 & -0.2 & 0.2 \\ -0.4 & -0.6 & 0.6 \\ 0.6 & 1.4 & -0.4 \end{bmatrix}$$

EXAM PREP 8.4

Inverse matrices

Prep 1 WORKED EXAMPLE 11

If $A = \begin{bmatrix} 5 & 2 \\ 1 & 11 \end{bmatrix}$ and I is the identity matrix, find AI and IA by hand.

What do you notice when you compare the two answers?

Prep 2 WORKED EXAMPLE 12

Show that the matrices $\begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix}$ and $\begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}$ are inverses of each other.

Prep 3 WORKED EXAMPLE 13

For each of the following matrices find

i the determinant **ii** the inverse (if it exists).

a $A = \begin{bmatrix} 2 & 8 \\ 2 & 7 \end{bmatrix}$ **b** $B = \begin{bmatrix} 5 & 10 \\ 1 & 1 \end{bmatrix}$ **c** $C = \begin{bmatrix} 5 & 2 \\ 10 & 4 \end{bmatrix}$

Prep 4 WORKED EXAMPLE 14

a If the determinant of $\begin{bmatrix} 6 & -12 \\ 3 & x \end{bmatrix}$ is equal to 12, what is the value of x?

b For the matrix $A = \begin{bmatrix} 5 & x \\ -4 & -5 \end{bmatrix}$, what is the value of x for which $A = A^{-1}$?

c Show that if A and B are both square matrices and $A + B$ is defined, then $A^2A^{-1}B^3 - B$ is defined.

Prep 5 | USING CAS: FINDING THE DETERMINANT AND INVERSE OF A MATRIX

For each of the following matrices, find

i $\det(A)$ **ii** A^{-1}

a $A = \begin{bmatrix} 4 & 3 & 6 \\ 2 & -10 & 7 \\ -1 & -8 & 1 \end{bmatrix}$ **b** $A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}$ **c** $A = \begin{bmatrix} 5 & 1 & 4 \\ 0 & -2 & 1 \\ -1 & -4 & 1 \end{bmatrix}$

8.4

EXAM PRACTICE 8.4

Inverse matrices

Question 1

Which of the following matrices has no inverse?

A $\begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$ **B** $\begin{bmatrix} 10 & 4 \\ 5 & 2 \end{bmatrix}$ **C** $\begin{bmatrix} 4 & 10 \\ 5 & 2 \end{bmatrix}$ **D** $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ **E** $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

Question 2

The inverse for the matrix $\begin{bmatrix} 1 & x \\ -1 & 2 \end{bmatrix}$ is

A $\begin{bmatrix} 2 & -x \\ 1 & 1 \end{bmatrix}$ **B** $\frac{1}{2+x}\begin{bmatrix} 2 & -x \\ 1 & 1 \end{bmatrix}$ **C** $\frac{1}{2-x}\begin{bmatrix} 2 & -x \\ 1 & 1 \end{bmatrix}$

D $\frac{1}{2+x}\begin{bmatrix} 2 & x \\ -1 & 1 \end{bmatrix}$ **E** $\frac{1}{2-x}\begin{bmatrix} 2 & x \\ 1 & 1 \end{bmatrix}$

Question 3

If A is a 2×3 matrix, B is a 3×2 matrix and C is a 3×3 matrix, which one of the following matrices can you be certain has no inverse?

A BA **B** C **C** C^2 **D** AB **E** CB

Question 4

If $A = \begin{bmatrix} 8 & 7 \\ 3 & 2 \end{bmatrix}$, which of the following is true?

A $\det(A) = \det(A^{-1})$ **B** $\det(A) = \det(A^2)$ **C** $\det(A) = \det(A^T)$

D $\det(A) = \det(A^3)$ **E** $\det(A) = \det(AA^{-1})$

Question 5

The determinant of $\begin{bmatrix} 3 & 2 \\ 6 & x \end{bmatrix}$ is equal to 9. The value of x is

A -7 **B** -4.5 **C** 1 **D** 4.5 **E** 7

[VCAA 2008 1MQ5]

Question 6

Matrix A is a 1×3 matrix. Matrix B is a 3×1 matrix.

Which one of the following matrix expressions involving A and B is defined?

A $A+\frac{1}{3}B$ **B** $2B \times 3A$ **C** A^2B **D** B^{-1} **E** $B - A$

[VCAA 2008 1MQ4]

Question 7

Matrix A is a 3×4 matrix. Matrix B is a 3×3 matrix.

Which one of the following matrix expressions is defined?

A BA^2 **B** $BA - 2A$ **C** $A + 2B$ **D** $B^2 - AB$ **E** A^{-1}

[VCAA 2011 1MQ4]

Question 8

Consider the following matrix A.

$$A = \begin{bmatrix} 3 & k \\ -4 & -3 \end{bmatrix}$$

A is equal to its inverse A^{-1} for a particular value of k. This value of k is

A -4 **B** -2 **C** 0 **D** 2 **E** 4

[VCAA 2011 1MQ8]

Question 9

P, Q, R and S are matrices such that the matrix product $P = QRS$ is defined.

Matrix Q and matrix S are square, non-zero matrices for which $Q + S$ is **not defined**. Which one of the following matrix expressions **is defined**?

A $R - S$ **B** $Q + R$ **C** P^2 **D** R^{-1} **E** $P \times S$

[VCAA 2013 1MQ9]

Matrix arithmetic

Practice quiz

SUMMARY

8

Matrix representation

- The order of a matrix is written as $m \times n$, and we say 'm by n', where m is the number of rows and n is the number of columns.
- Each value in a matrix is called an **element**. a_{ij} is an element of matrix A where i is the row number, j is the column number.
- An $m \times n$ matrix has mn elements.

Types of matrices

Row matrix	A matrix with just one row
Column matrix	A matrix with just one column
Zero matrix	A matrix where all the elements are 0
Binary matrix	A matrix where every element is 0 or 1
Square matrix	A matrix that has the same number of rows as columns
Symmetric matrix	A square matrix where the elements are symmetric with respect to the leading diagonal
Upper triangular matrix	A square matrix where all the elements below the leading diagonal are 0
Lower triangular matrix	A square matrix where all the elements above the leading diagonal are 0
Diagonal matrix	A square matrix where the only non-zero elements are in the leading diagonal
Permutation matrix	A square matrix where every row and every column has exactly one 1, with 0s everywhere else
Transpose matrix	A new matrix formed by turning all the rows of the original matrix into columns and vice-versa.

Addition and subtraction and scalar multiplication of matrices

- Addition/subtraction of matrices involves adding/subtracting the elements that are in corresponding positions in both matrices.
- When adding/subtracting two matrices, the result is a matrix of the same order as those being added/subtracted.
- **Scalar multiplication** involves multiplying every element in a matrix by the same number.

Matrix multiplication

- To multiply two matrices, AB, we multiply pairs of elements, working *across the rows* in A and *down the columns* in B.
- Generally for matrices $AB \neq BA$.
- If matrix A is of order $m \times n$ and matrix B is of order $n \times p$, then

order of 1st matrix $(m \times n)$ $\times$ order of 2nd matrix $(n \times p)$

must be the same
for AB to be defined

- If matrix A is of order $m \times n$ and matrix B is of order $n \times p$, then AB will be of order $m \times p$.
- The number of rows in a defined matrix product is the number rows in the *first* matrix and the number of columns is the number of columns in the *last* matrix.
- Only square matrices can be raised to a power.
- Any power of a matrix will always have the same order as the original matrix.

Summing matrices

- To sum the columns of a matrix, pre-multiply it by a suitable row matrix consisting of 1s.
- To sum the rows of a matrix, post-multiply it by a suitable column matrix consisting of 1s.
- To find the mean of the columns of a matrix, pre-multiply it by the summing matrix and by $\frac{1}{n}$, where n is the number of columns in the summing matrix.
- To find the mean of the rows of a matrix, post-multiply it by the summing matrix and by $\frac{1}{m}$, where m is the number of rows in the summing matrix.

Inverse matrices

- The identity matrix, I, is the square matrix with leading diagonal elements 1 and other elements 0, which when multiplied by a square matrix, A, of the same order, leaves A unchanged, i.e. $AI = IA = A$.
- The **inverse** of a square matrix A, written as A^{-1}, is the matrix where $AA^{-1} = A^{-1}A = I$
- For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the **determinant** is $\det(A) = ad - bc$
- The inverse of matrix A is $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{\det(A)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$
- Inverses are only defined for square matrices. The inverse doesn't exist when $\det(A) = 0$.

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MATRICES

Examination 1

Reading time: (5 minutes)
Writing time: (30 minutes)

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers, one bound reference, one approved graphics calculator or approved CAS calculator or CAS software and, if desired, one scientific calculator. Calculator memory DOES NOT need to be cleared.
- Students are NOT permitted to bring into the examination room: blank sheets of paper and/or white out liquid/tape.
- Students may refer to the sheet of miscellaneous formulas supplied.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

Instructions

Choose the response that is correct for the question.

A correct answer scores 1, an incorrect answer scores 0.

Marks will not be deducted for incorrect answers.

No marks will be given if more than one answer is completed for any question.

Unless otherwise indicated, the diagrams are not drawn to scale.

Question 1

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 2 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix} \text{ is equal to}$$

A. $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 15 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

B. $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

C. $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

D. $\begin{bmatrix} 0 & 0 & 4 & 0 \\ 4 & 1 & 1 & 9 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 9 & 0 \end{bmatrix}$

E. $\begin{bmatrix} 0 & 0 & 2 & 0 \\ 2 & 1 & 1 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix}$

[VCAA 2014 1MQ1]

EXAMINATION 1 – continued

Question 2

$$y - z = 8$$
$$5x - y = 0$$
$$x + z = 4$$

The system of three simultaneous linear equations above can be written in matrix form as

A. $\begin{bmatrix} 0 & 1 & -1 \\ 0 & 5 & -1 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 0 \\ 4 \end{bmatrix}$

B. $\begin{bmatrix} 0 & 1 & -1 \\ 5 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 0 \\ 4 \end{bmatrix}$

C. $\begin{bmatrix} 1 & -1 \\ 5 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 0 \\ 4 \end{bmatrix}$

D. $\begin{bmatrix} 0 & 5 & -1 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 0 \\ 4 \end{bmatrix}$

E. $\begin{bmatrix} 0 & 5 & 0 \\ -1 & -1 & 0 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 0 \\ 4 \end{bmatrix}$

[VCAA 2014 1MQ2]

Question 3

Regular customers at a hairdressing salon can choose to have their hair cut by Shirley, Jen or Narj. The salon has 600 regular customers who get their hair cut each month.

In June, 200 customers chose Shirley (*S*) to cut their hair, 200 chose Jen (*J*) to cut their hair and 200 chose Narj (*N*) to cut their hair.

The regular customers' choice of hairdresser is expected to change from month to month as shown in the transition matrix, *T*, below.

$$\begin{array}{c} \text{This month} \\ \begin{array}{cc} & \begin{array}{ccc} S & J & N \end{array} \\ T = & \begin{bmatrix} 0.75 & 0.10 & 0.10 \\ 0.10 & 0.75 & 0.15 \\ 0.15 & 0.15 & 0.75 \end{bmatrix} \begin{array}{c} S \\ J \\ N \end{array} \text{ Next month} \end{array} \end{array}$$

In the long term, the number of regular customers who are expected to choose Shirley is closest to

A. 150

B. 170

C. 185

D. 195

E. 200

[VCAA 2014 1MQ3]

EXAMINATION 1 – continued

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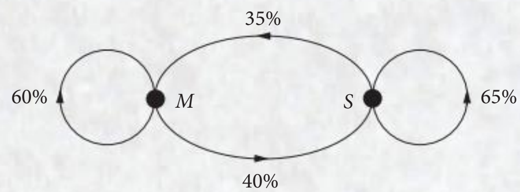
Question 4

Two hundred and fifty people buy bread each day from a corner store. They have a choice of two brands of bread: Megaslice (*M*) and Superloaf (*S*).

The customers' choice of brand changes daily according to the transition diagram below.



On a given day, 100 of these people bought Megaslice bread while the remaining 150 people bought Superloaf bread.

The number of people who are expected to buy each brand of bread the next day is found by evaluating the matrix product

A.
$$\begin{array}{c} \begin{matrix} M & \quad S \end{matrix} \\ \begin{bmatrix} 0.60 & 0.40 \\ 0.35 & 0.65 \end{bmatrix} \end{array} \begin{bmatrix} 100 \\ 150 \end{bmatrix} \begin{matrix} M \\ S \end{matrix}$$

B.
$$\begin{array}{c} \begin{matrix} M & \quad S \end{matrix} \\ \begin{bmatrix} 0.60 & 0.40 \\ 0.65 & 0.35 \end{bmatrix} \end{array} \begin{bmatrix} 100 \\ 150 \end{bmatrix} \begin{matrix} M \\ S \end{matrix}$$

C.
$$\begin{array}{c} \begin{matrix} M & \quad S \end{matrix} \\ \begin{bmatrix} 0.60 & 0.35 \\ 0.40 & 0.65 \end{bmatrix} \end{array} \begin{bmatrix} 100 \\ 150 \end{bmatrix} \begin{matrix} M \\ S \end{matrix}$$

D.
$$\begin{array}{c} \begin{matrix} M & \quad S \end{matrix} \\ \begin{bmatrix} 0.65 & 0.40 \\ 0.35 & 0.60 \end{bmatrix} \end{array} \begin{bmatrix} 100 \\ 150 \end{bmatrix} \begin{matrix} M \\ S \end{matrix}$$

E.
$$\begin{array}{c} \begin{matrix} M & \quad S \end{matrix} \\ \begin{bmatrix} 0.60 & 0.65 \\ 0.40 & 0.35 \end{bmatrix} \end{array} \begin{bmatrix} 100 \\ 150 \end{bmatrix} \begin{matrix} M \\ S \end{matrix}$$

[VCAA 2014 1MQ4]

Question 5

Students from Year 7 and Year 8 in a school sold trees to raise funds for a school trip.

The number of large, medium and small trees that were sold by each Year group is shown in the table below.

Year group	***Large***	***Medium***	***Small***
7	52	78	61
8	45	56	81

The large trees were sold for \$32 each, the medium trees were sold for \$26 each and the small trees were sold for \$18 each.

EXAMINATION 1 – continued

A matrix product that can be used to calculate the amount, in dollars, raised by each year group by selling trees is

A. $\begin{bmatrix} 52 & 78 & 61 \\ 32 & 26 & 18 \end{bmatrix}\begin{bmatrix} 45 \\ 56 \\ 81 \end{bmatrix}$

B. $\begin{bmatrix} 7 & 52 & 78 & 61 \\ 8 & 45 & 56 & 81 \end{bmatrix}\begin{bmatrix} 32 \\ 26 \\ 18 \\ 0 \end{bmatrix}$

C. $\begin{bmatrix} 32 & 26 & 18 \end{bmatrix}\begin{bmatrix} 52 & 45 \\ 78 & 56 \\ 61 & 81 \end{bmatrix}$

D. $\begin{bmatrix} 32 & 26 & 18 \end{bmatrix}\begin{bmatrix} 52 & 78 & 61 \\ 45 & 56 & 81 \\ 0 & 0 & 0 \end{bmatrix}$

E. $\left(\begin{bmatrix} 52 \\ 78 \\ 61 \end{bmatrix} + \begin{bmatrix} 45 \\ 56 \\ 81 \end{bmatrix}\right) \times \begin{bmatrix} 32 & 26 & 18 \end{bmatrix}$

[VCAA 2014 1MQ5]

Question 6

The order of matrix X is 3×2.

The element in row i and column j of matrix X is x_{ij} and it is determined by the rule

$$x_{ij} = i + j$$

The matrix X is

A. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$

B. $\begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 6 & 7 \end{bmatrix}$

C. $\begin{bmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$

D. $\begin{bmatrix} 1 & 2 \\ 3 & 3 \\ 4 & 4 \end{bmatrix}$

E. $\begin{bmatrix} 2 & 3 \\ 3 & 4 \\ 4 & 5 \end{bmatrix}$

[VCAA 2014 1MQ6]

Question 7

A transition matrix, T, and a state matrix, S_2, are defined as follows.

$$T = \begin{bmatrix} 0.5 & 0 & 0.5 \\ 0.5 & 0.5 & 0 \\ 0 & 0.5 & 0.5 \end{bmatrix} \qquad S_2 = \begin{bmatrix} 300 \\ 200 \\ 100 \end{bmatrix}$$

If $S_2 = TS_1$, the state matrix S_1 is

A. $\begin{bmatrix} 200 \\ 250 \\ 150 \end{bmatrix}$

B. $\begin{bmatrix} 300 \\ 200 \\ 100 \end{bmatrix}$

C. $\begin{bmatrix} 300 \\ 0 \\ 300 \end{bmatrix}$

D. $\begin{bmatrix} 400 \\ 0 \\ 200 \end{bmatrix}$

E. undefined

[VCAA 2014 1MQ7]

EXAMINATION 1 – continued

Question 8

Wendy will have lunch with one of her friends each day of this week. Her friends are Angela (A), Betty (B), Craig (C), Daniel (D) and Edgar (E). On Monday, Wendy will have lunch with Craig. Wendy will use the transition matrix to choose a friend to have lunch with for the next four days of the week.

$$\begin{array}{c} \text{Today} \\ \begin{array}{ccc} & \begin{array}{ccccc} A & B & C & D & E \end{array} & \\ T = & \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} & \begin{array}{c} A \\ B \\ C \\ D \\ E \end{array} \end{array} \end{array} \text{Tomorrow}$$

The order in which Wendy has lunch with her friends for the next four days is

A. Angela, Betty, Craig, Daniel

B. Daniel, Betty, Angela, Craig

C. Daniel, Betty, Angela, Edgar

D. Edgar, Angela, Daniel, Betty

E. Edgar, Daniel, Betty, Angela

[VCAA 2014 1MQ8]

Question 9

A and B are square matrices such that $AB = BA = I$, where I is an identity matrix. Which one of the following statements is **not** true?

A. $ABA = A$

B. $AB^2A = I$

C. B must equal A

D. B is the inverse of A

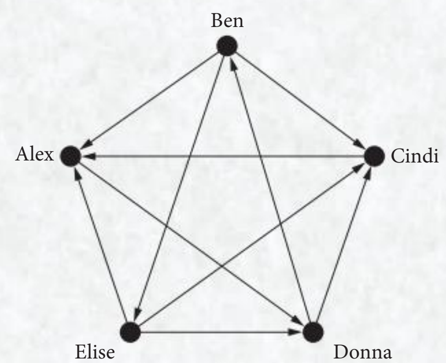
E. both A and B have inverses

[VCAA 2014 1MQ9]

Question 10

The graph shows the results of a chess competition between five players: Alex, Ben, Cindi, Donna and Elise.

Each arrow indicates the winner of individual games. For example, the arrow from Alex to Donna indicates that Alex beat Donna in their game. The sum of their one-step and two-step dominances is calculated to give each player a dominance score. The dominance scores are then used to rank the players. The ranking of the players in this competition, from highest to lowest dominance score, is



A. Ben, Elise, Donna, Alex, Cindi

B. Ben, Elise, Cindi, Donna, Alex

C. Ben, Elise, Donna, Cindi, Alex

D. Elise, Ben, Donna, Alex, Cindi

E. Elise, Ben, Donna, Cindi, Alex

[VCAA 2014 1NQ4]

EXAMINATION 1 – ended

MATRICES

Examination 2

Reading time: (5 minutes)
Writing time: (25 minutes)

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers, one bound reference, one approved graphics calculator or approved CAS calculator or CAS software and, if desired, one scientific calculator. Calculator memory DOES NOT need to be cleared.
- Students are NOT permitted to bring into the examination room: blank sheets of paper and/or white out liquid/tape.
- Students may refer to the sheet of miscellaneous formulas supplied.
- All written responses must be in English.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

Instructions

You need not give numerical answers as decimals unless instructed to do so. Alternative forms may involve, for example, π, surds or fractions.

Diagrams are not to scale unless specified otherwise.

Question 1 (6 marks)

A small city is divided into four regions: Northern (N), Eastern (E), Southern (S) and Western (W). The number of adult males (M) and the number of adult females (F) living in each of the regions in 2013 is shown in matrix V.

$$\begin{array}{cc} & \begin{array}{cc} M & F \end{array} \\ V = & \begin{bmatrix} 1360 & 1460 \\ 1680 & 1920 \\ 900 & 1060 \\ 1850 & 1700 \end{bmatrix} \begin{array}{c} N \\ E \\ S \\ W \end{array} \end{array}$$

a. Write down the order of matrix V. 1 mark

b. How many adult males lived in the Western region in 2013? 1 mark

c. In terms of the population of the city, what does the sum of the elements in the second column of matrix V represent? 1 mark

EXAMINATION 2 – continued

An election is to be held in the city. All of the adults in each of the regions of the city will vote in the election. One of the election candidates, Ms Aboud, estimates that she will receive 45% of the male votes and 55% of the female votes in the election. This information is shown in matrix P.

$$P = \begin{array}{c} \begin{bmatrix} 0.45 \\ 0.55 \end{bmatrix} \end{array} \begin{array}{c} M \\ F \end{array}$$

d. Explain, in terms of rows and columns, why the matrix product $V \times P$ is defined. 1 mark

The product of matrices V and P is shown.

$$V \times P = \begin{bmatrix} 1360 & 1460 \\ 1680 & 1920 \\ 900 & 1060 \\ 1740 & 1860 \end{bmatrix} \times \begin{bmatrix} 0.45 \\ 0.55 \end{bmatrix} = \begin{bmatrix} w \\ 1812 \\ 988 \\ 1806 \end{bmatrix}$$

e. Using appropriate elements from the matrix product $V \times P$, write a calculation to show that the value of w is 1415. 1 mark

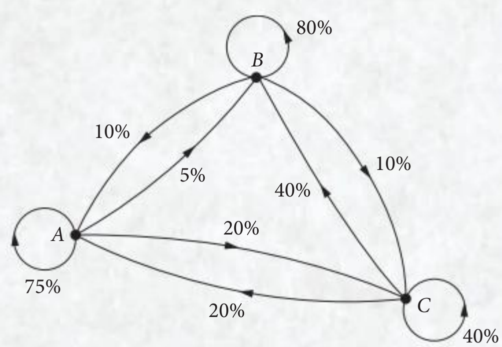
f. How many votes does Ms Aboud expect to receive in the election? 1 mark

[VCAA 2014 2MQ1]

Question 2 (6 marks)

There are three candidates in the election: Ms Aboud (A), Mr Broad (B) and Mr Choi (C). The election campaign will run for six months, from the start of January until the election at the end of June. A survey of voters found that voting preference can change from month to month leading up to the election.

The transition diagram shows the percentage of voters who are expected to change their preferred candidate from month to month.



a. **i.** Of the voters who prefer Mr Choi this month, what percentage are expected to prefer Ms Aboud next month? 1 mark

ii. Of the voters who prefer Ms Aboud this month, what percentage are expected to change their preferred candidate next month? 1 mark

In January, 12 000 voters are expected in the city. The number of voters in the city is expected to remain constant until the election is held in June. The state matrix that indicates the number of voters who are expected to have a preference for each candidate in January, S_1, is given.

$$S_1 = \begin{array}{c} \begin{bmatrix} 6000 \\ 3840 \\ 2160 \end{bmatrix} \end{array} \begin{array}{c} A \\ B \\ C \end{array}$$

b. How many voters are expected to change their preference to Mr Broad in February? 1 mark

The information in the transition diagram has been used to write the transition matrix, T.

$$\begin{array}{c} \\ \\ T = \end{array} \begin{array}{c} \text{This month} \\ \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0.75 & 0.10 & 0.20 \\ 0.05 & 0.80 & 0.40 \\ 0.20 & 0.10 & 0.40 \end{bmatrix} \end{array} \begin{array}{l} \\ \\ A \\ B \quad \text{Next month} \\ C \end{array}$$

c. **i.** Evaluate the matrix $S_3 = T^2 S_1$ and write down the elements, correct to the nearest whole number. 1 mark

ii. What information does matrix S_3 contain? 1 mark

d. Using matrix T, how many votes would the winner of the election in June be expected to receive? Write your answer, correct to the nearest whole number. 1 mark

[VCAA 2014 2MQ2]

Question 3 (3 marks)

Mr Choi may need to withdraw from the election at the end of May. Matrix T, shown on the right, shows the percentage of voters who change their preferred candidate from month to month, **before** Mr Choi would withdraw from the election.

$$\begin{array}{c} \\ \\ T = \end{array} \begin{array}{c} \text{This month} \\ \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0.75 & 0.10 & 0.20 \\ 0.05 & 0.80 & 0.40 \\ 0.20 & 0.10 & 0.40 \end{bmatrix} \end{array} \begin{array}{l} \\ \\ A \\ B \quad \text{Next month} \\ C \end{array}$$

Matrix T_1, shown on the right, shows the percentage of voters who change their preferred candidate from May to June, **after** Mr Choi would withdraw from the election.

Consider the voters who preferred Mr Broad in May and who were expected to prefer Mr Choi in June.

$$\begin{array}{c} \\ \\ T_1 = \end{array} \begin{array}{c} \text{May} \\ \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0.75 & 0.15 & 0.6 \\ 0.25 & 0.85 & 0.4 \\ 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{l} \\ \\ A \\ B \quad \text{June} \\ C \end{array}$$

a. What percentage of these voters are now expected to prefer Mr Broad in June? 1 mark

The state matrix that indicates the number of voters who are expected to have a preference for each candidate in January, S_1, is given on the right.

$$S_1 = \begin{bmatrix} 6000 \\ 3840 \\ 2160 \end{bmatrix} \begin{array}{l} A \\ B \\ C \end{array}$$

b. If Mr Choi withdraws, how many votes is Mr Broad expected to receive in the election in June? Write your answer, correct to the nearest vote. 2 marks

[VCAA 2014 2MQ3]

EXAMINATION 2 – ended

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CHAPTER 9

MATRIX APPLICATIONS

9.1 Using matrices

9.2 Matrices and simultaneous equations

Solving two simultaneous equations using matrices

Inconsistent and dependent systems of equations without solutions

Solving three or more simultaneous equations using matrices

Using CAS: Solving simultaneous equations using matrices

9.3 Communication and dominance matrices

Communication diagrams and matrices

Multi-step communication

Self-communication links

Dominance matrices

9.4 Transition diagrams and matrices

Constructing transition diagrams and matrices

Interpreting transition matrices

9.5 Transition matrices and recurrence relations

The state matrix recurrence relation

Using CAS: Generating matrix sequences through recursive computation

The state matrix rule

Long term trends

The equilibrium state matrix

9.6 Transitions with restocking and culling

Summary

9.1 Using matrices

Matrices are often a convenient way of representing real-life information, and matrix arithmetic can be used to do relevant calculations. The advantage of matrices when you use a CAS/calculator is that they allow you to do a large number of calculations quickly.

Worked example 1

Each week the coach of the Little Diggers basketball team awards the Best and Fairest Player Award to the player who scores the most game points. The results of their last game were as follows.

	3 pointers	2 pointers	1 pointer
Asma	0	5	0
Sarah	1	3	2
Lucy	0	7	0
Matt	2	5	0
Kelly	1	7	0
Min-Lee	0	2	2
Sophie	1	8	0

a Write a column matrix, P, to represent the 3 different scores possible.

b Write a matrix, G, to represent the data from the table.

c Use matrix multiplication to find a score matrix named S that represents the total scored by each of the players.

d Which player won the Best and Fairest Award?

e The opposition team scored a total of 87 points. Did the Little Diggers win the game?

Working

a When a player scores, they can either get 3 points, 2 points or 1 point.

$$P = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

b Matrix G will have 7 rows and 3 columns.

$$G = \begin{bmatrix} 0 & 5 & 0 \\ 1 & 3 & 2 \\ 0 & 7 & 0 \\ 2 & 5 & 0 \\ 1 & 7 & 0 \\ 0 & 2 & 2 \\ 1 & 8 & 0 \end{bmatrix}$$

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c To find S, we must find GP.

This will give us a $(7 \times 3) \times (3 \times 1)$, which will result in a 7×1 matrix that represents the personal total for each of the seven players.

$$S = \begin{bmatrix} 0 & 5 & 0 \\ 1 & 3 & 2 \\ 0 & 7 & 0 \\ 2 & 5 & 0 \\ 1 & 7 & 0 \\ 0 & 2 & 2 \\ 1 & 8 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 11 \\ 14 \\ 16 \\ 17 \\ 6 \\ 19 \end{bmatrix}$$

d The largest element in S is the highest personal score.

Sophie won with 19 points.

e The total score for Little Diggers can be found by adding all of the elements in S.

$$\text{Score} = 10 + 11 + 14 + 16 + 17 + 6 + 19$$
$$= 93$$

Write the answer.

93 is greater than 87, so Little Diggers won.

Worked example 2

The manager of a local hardware store purchases small metal fasteners for \$3 each and large metal fasteners for \$5. In the last two weeks, he purchased the number of fasteners shown.

a Find the two matrices that can be multiplied to give the total purchase cost of metal fasteners in each of the two weeks and complete the multiplication.

	Small metal fasteners	Large metal fasteners
Week 1	121	112
Week 2	95	84

The manager sells goods at a 55% mark-up. He recorded his purchase costs over the last 2 weeks for metal fasteners and three other items in the following table.

	Week 1	Week 2
Paint	\$1060	\$1555
Timber	\$3029	\$1124
Metal fasteners		
Gardening tools	\$896	\$1230

b **i** Represent these costs in a 4×2 cost matrix, C.

ii Using scalar multiplication, represent the selling prices of these goods in a 4×2 matrix, S.

c **i** Create a profit matrix.

ii Calculate the total profit to be made if all of the goods purchased over these 2 weeks are sold.

Working

a We require a matrix product that calculates No. of small × Cost of small + No. of large × Cost of large.

$$\begin{array}{l} \text{Week 1} \\ \text{Week 2} \end{array} \begin{bmatrix} 121 & 112 \\ 95 & 84 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 923 \\ 705 \end{bmatrix}$$

9.1

b i The table already has 4 rows and 2 columns. Fill in the missing information from part **a**.

$$C = \begin{bmatrix} 1060 & 1555 \\ 3029 & 1124 \\ 923 & 705 \\ 896 & 1230 \end{bmatrix}$$

ii Selling price = 1.55 × Cost price

Multiplying the cost matrix by the scalar 1.55 will multiply each element by 1.55, which represents a mark-up (increase) of 55% (0.55).

$$S = 1.55C = 1.55\begin{bmatrix} 1060 & 1555 \\ 3029 & 1124 \\ 923 & 705 \\ 896 & 1230 \end{bmatrix}$$

$$= \begin{bmatrix} 1643.00 & 2410.25 \\ 4694.95 & 1742.20 \\ 1430.65 & 1092.75 \\ 1388.80 & 1906.50 \end{bmatrix}$$

c i To create a profit matrix:

Profit = Selling price – Cost price

$$\text{Profit } = S - C$$

$$= \begin{bmatrix} 1643.00 & 2410.25 \\ 4694.95 & 1742.20 \\ 1430.65 & 1092.75 \\ 1388.80 & 1906.50 \end{bmatrix} - \begin{bmatrix} 1060 & 1555 \\ 3029 & 1124 \\ 923 & 705 \\ 896 & 1230 \end{bmatrix}$$

$$= \begin{bmatrix} 583.00 & 855.25 \\ 1665.95 & 618.20 \\ 507.65 & 387.75 \\ 492.80 & 676.50 \end{bmatrix}$$

ii The total profit can be found by adding all of the elements in the profit matrix.

$$\begin{aligned}\text{Total profit} &= 583.00 + 855.25 + 1665.95 + 618.20 \\ &\quad + 507.65 + 387.75 + 492.80 + 676.50 \\ &= \$5787.10\end{aligned}$$

Worked example 3

Matrix B shows the number of sightings of rare birds made by members of a birdwatching club.

$$B = \begin{array}{l} \\ \text{Harry} \\ \text{Esther} \\ \text{Aniela} \\ \text{Gordon} \end{array} \begin{array}{c} \begin{array}{ccc} \text{Flitter} & \text{Bitwing} & \text{Redtwit} \end{array} \\ \begin{bmatrix} 13 & 3 & 4 \\ 2 & 5 & 2 \\ 1 & 1 & 3 \\ 5 & 7 & 10 \end{bmatrix} \end{array} \qquad P = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \qquad Q = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

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a **i** Calculate the product $A = PB$.

ii What information does the element a_{13} give?

b **i** Calculate $C = BQ$ and describe the information given by C.

ii What information does the element c_{21} give?

c Calculate $\frac{1}{4}PBQ$ and describe the information given by this matrix.

Working

a **i** Do the multiplication. The matrix order equation tells us the order of the product:

$(1 \times 4) \times (4 \times 3) = (1 \times 3)$

$$A = PB = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 13 & 3 & 4 \\ 2 & 5 & 2 \\ 1 & 1 & 3 \\ 5 & 7 & 10 \end{bmatrix}$$

$$= \begin{bmatrix} 21 & 16 & 19 \end{bmatrix}$$

The product involves summing each type of bird.

A gives the totals for each type of bird sighted by the club members.

ii $(PB)_{ij}$ is the element in the ith row and jth column in the product PB.

a_{13} tells us that a total of 19 redtwits were sighted by the club members.

b **i** Do the multiplication. The matrix order equation tells us the order of the product:

$(4 \times 3) \times (3 \times 1) = (4 \times 1)$

$$C = BQ = \begin{bmatrix} 13 & 3 & 4 \\ 2 & 5 & 2 \\ 1 & 1 & 3 \\ 5 & 7 & 10 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 20 \\ 9 \\ 5 \\ 22 \end{bmatrix}$$

The product involves summing the birds that each birdwatcher sighted.

C gives the total number of birds sighted by each birdwatcher.

ii $(BQ)_{ij}$ is the element in the ith row and jth column in the product BQ.

c_{21} tells us that Esther sighted a total of 9 birds.

c Do the multiplication. The matrix order equation tells us the order of the product:

$(1 \times 4) \times (4 \times 1) = (1 \times 1)$

$$\frac{1}{4}PBQ = \frac{1}{4}PC = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 20 \\ 9 \\ 5 \\ 22 \end{bmatrix}$$

$$= \frac{1}{4}[56] = [14]$$

The product involves adding elements and dividing by the number of elements.

$\frac{1}{4}PBQ$ tells us that the mean number of birds sighted by the birdwatchers is 14.

Exam hack

Even if your answer for a matrix multiplication is a single number, you must give your answer with matrix brackets around it, or it won't be marked as correct.

Alamy/Chris Jobs

EXAM PREP 9.1

Using matrices

Prep 1 WORKED EXAMPLE 1

Each week the coach of the Little Clunkers cricket team awards the Best Batter to the player who makes the most runs. The results of their last game are shown in the table.

	Single runs	Fours	Sixes
Heshan	26	5	2
Sam	12	2	0
Ahmat	18	7	2
Toni	9	2	0
Vishna	30	3	0
Jordan	5	0	0
Yasara	3	0	0

a Write a column matrix, W, to represent the 3 different ways of scoring runs.

b Write a matrix, R, to represent the data from the table.

c Use matrix multiplication to find a score matrix named S that represents the total runs made by each of the players.

d Which player won the Best Batter award?

e The opposition team made 207 runs. Did the Little Clunkers win the game?

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Prep 2 WORKED EXAMPLE 2

A chemist purchases small bottles of vitamin C tablets for \$2 each and large bottles of vitamin C tablets for \$3. In the last two weeks, he has purchased the number of vitamin C bottles shown.

	Small bottles of vitamin C	Large bottles of vitamin C
Week 1	75	60
Week 2	47	82

a Find the two matrices that can be multiplied to give the total purchase cost of vitamin C bottles in each of the two weeks and complete the multiplication.

The chemist sells goods at a 75% mark-up. He recorded his purchase costs over the last 2 weeks for vitamin C tablets and three other items in the following table.

	Week 1	Week 2
Vitamin C bottles		
Vitamin D bottles	\$473	\$542
Multivitamin bottles	\$628	\$745
Calcium bottles	\$263	\$220

b **i** Represent these costs in a 4×2 cost matrix, C.

ii The selling price is calculated by finding 175% of the cost price. Convert 175% to a decimal.

iii Using scalar multiplication, represent the selling prices of these goods in a 4×2 matrix, S.

c **i** Create a profit matrix.

ii Calculate the total profit to be made if all goods purchased over these 2 weeks are sold.

Prep 3 WORKED EXAMPLE 3

Members of a trainspotting club recorded the number of different trains they saw over a weekend.

$$B = \begin{array}{r} \\ \text{Kristin} \\ \text{Amy} \\ \text{Seth} \\ \text{Steve} \\ \text{Judd} \end{array} \begin{array}{c} \begin{array}{ccc} \text{Redrattlerr} & \text{Blueghost} & \text{Silverstreak} \end{array} \\ \begin{bmatrix} 6 & 7 & 0 \\ 2 & 6 & 3 \\ 7 & 10 & 1 \\ 6 & 9 & 2 \\ 2 & 5 & 4 \end{bmatrix} \end{array}$$

Let $P = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix}$ and $Q = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

a **i** Calculate the product $A = PB$ and describe what information is given by this matrix.

ii What information does the element a_{12} give?

b **i** Calculate $C = BQ$ and describe what information is given by C.

ii What information does the element c_{41} give?

c Calculate $\frac{1}{5}PBQ$ and describe what information is given by this matrix.

Using matrices

Question 1

Five students, Richard (R), Brendon (B), Lee (L), Arif (A) and Karl (K), were asked whether they played each of the following sports: football (F), golf (G), soccer (S) or tennis (T). Their responses are displayed in the table.

Student	Sport played			
	Football (F)	Golf (G)	Soccer (S)	Tennis (T)
R	yes	no	no	yes
B	yes	yes	yes	no
L	no	no	no	yes
A	no	yes	no	yes
K	yes	no	no	yes

If 1 is used to indicate that the student plays a particular sport and 0 is used to indicate that the student does not play a particular sport, which one of the following matrices could be used to represent the information in the table?

A $\begin{array}{cc} \begin{array}{ccccc} R & B & L & A & K \end{array} & \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} & \begin{array}{c} F \\ G \\ S \\ T \end{array} \end{array}$

B $\begin{array}{cc} \begin{array}{cccc} F & G & S & T \end{array} & \\ \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} & \begin{array}{c} R \\ B \\ L \\ A \\ K \end{array} \end{array}$

C $\begin{array}{cc} \begin{array}{ccccc} R & B & L & A & K \end{array} & \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 \end{bmatrix} & \begin{array}{c} F \\ G \\ S \\ T \end{array} \end{array}$

D $\begin{array}{cc} \begin{array}{cccc} F & G & S & T \end{array} & \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} & \begin{array}{c} R \\ B \\ L \\ A \\ K \end{array} \end{array}$

E $\begin{array}{cc} \begin{array}{cccc} F & G & S & T \end{array} & \\ \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} & \begin{array}{c} R \\ B \\ L \\ A \\ K \end{array} \end{array}$

[VCAA 2013 1MQ5]

Question 2

A company makes regular (R), queen (Q) and king (K) size beds. Each bed comes in either the classic style or the more expensive deluxe style. The price of each style of bed, in dollars, is listed in a price matrix P, where

$$P = \begin{array}{cc} \begin{array}{ccc} R & Q & K \end{array} & \\ \begin{bmatrix} 145 & 210 & 350 \\ 185 & 270 & 410 \end{bmatrix} & \begin{array}{l} \text{Classic} \\ \text{Deluxe} \end{array} \end{array}$$

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The company wants to increase the price of all beds. A new price matrix, listing the increased prices of the beds, can be generated from P by forming a matrix product with the matrix, M, where

$$M = \begin{bmatrix} 1.2 & 0 \\ 0 & 1.35 \end{bmatrix}$$

This new price matrix is

A $\begin{bmatrix} 145 & 210 & 350 \\ 185 & 270 & 410 \end{bmatrix}$ **B** $\begin{bmatrix} 234.90 & 340.20 & 567 \\ 299.70 & 437.40 & 664.20 \end{bmatrix}$ **C** $\begin{bmatrix} 174 & 252 & 420 \\ 222 & 324 & 492 \end{bmatrix}$

D $\begin{bmatrix} 174 & 252 & 420 \\ 249.75 & 364.50 & 553.50 \end{bmatrix}$ **E** $\begin{bmatrix} 195.75 & 283.50 & 472.50 \\ 249.75 & 364.50 & 553.50 \end{bmatrix}$

[VCAA 2006 1MQ5]

Question 3

The cost prices of three different electrical items in a store are \$230, \$290 and \$310 respectively.

The selling price of each of these three electrical items is 1.3 times the cost price plus a commission of \$20 for the salesman.

A matrix that lists the selling price of each of these three electrical items is determined by evaluating

A $1.3 \times \begin{bmatrix} 230 \\ 290 \\ 310 \end{bmatrix} + [20]$ **B** $1.3 \times \begin{bmatrix} 230 \\ 290 \\ 310 \end{bmatrix} + 1.3 \times 20$ **C** $1.3 \times \begin{bmatrix} 230 \\ 290 \\ 310 \end{bmatrix} + \begin{bmatrix} 20 \\ 20 \\ 20 \end{bmatrix}$

D $1.3 \times \begin{bmatrix} 230 \\ 290 \\ 310 \end{bmatrix} + 1.3 \times \begin{bmatrix} 20 \\ 20 \\ 20 \end{bmatrix}$ **E** $1.3 \times \begin{bmatrix} 230+20 \\ 290+20 \\ 310+20 \end{bmatrix}$

[VCAA 2008 1MQ3]

Question 4

Peter bought only apples and bananas from his local fruit shop.

The matrix $N = \begin{array}{c} \begin{array}{cc} A & B \end{array} \\ \begin{bmatrix} 3 & 4 \end{bmatrix} \end{array}$ lists the number of apples (A) and bananas (B) that Peter bought.

The matrix $C = \begin{bmatrix} 0.37 \\ 0.43 \end{bmatrix} \begin{array}{c} A \\ B \end{array}$ lists the cost (in dollars) of one apple and one banana respectively.

The matrix product, NC, gives

A the total amount spent by Peter on the fruit that he bought.

B the total number of pieces of fruit that Peter bought.

C the individual amounts that Peter spent on apples and bananas respectively.

D the total number of pieces of fruit that Peter bought and the total amount that he spent.

E the individual number of apples and bananas that Peter bought and the individual amounts that Peter spent on these apples and bananas respectively.

[VCAA 2010 1MQ2]

Question 5

Apples cost \$3.50 per kg, bananas cost \$4.20 per kg and carrots cost \$1.89 per kg. Ashley buys 3 kg of apples, 2 kg of bananas and 1 kg of carrots. A matrix product to calculate the total cost of these items is

A $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}\begin{bmatrix} 3.50 \\ 4.20 \\ 1.89 \end{bmatrix}$

B $\begin{bmatrix} 3 & 2 & 1 \end{bmatrix}\begin{bmatrix} 3.50 & 4.20 & 1.89 \end{bmatrix}$

C $\begin{bmatrix} 3.50 \times 2 & 4.20 \times 3 & 1.89 \times 1 \end{bmatrix}$

D $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}\begin{bmatrix} 3.50 & 4.20 & 1.89 \end{bmatrix}$

E $\begin{bmatrix} 3.50 & 4.20 & 1.89 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$

[VCAA 2008 1MQ2]

Question 6

The table shows the number of classes and the number of students in each class at each year level in a secondary school.

	Year level			
	9	10	11	12
Number of classes	7	5	6	4
Students per class	22	20	18	24

Let

$F = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$, $G = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $M = \begin{bmatrix} 7 & 5 & 6 & 4 \end{bmatrix}$, $N = \begin{bmatrix} 7 \\ 5 \\ 6 \\ 4 \end{bmatrix}$, $P = \begin{bmatrix} 22 & 0 & 0 & 0 \\ 0 & 20 & 0 & 0 \\ 0 & 0 & 18 & 0 \\ 0 & 0 & 0 & 24 \end{bmatrix}$

A matrix product that displays the total number of students in Years 9–12 at this school is

A $M \times P \times F$

B $P \times G \times M$

C $F \times P \times N$

D $P \times N \times F$

E $F \times N \times P$

[VCAA 2012 1MQ6]

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Question 7

A school has three computer classes, A, B and C. There are 15 students in each class. Each student is given a mark out of 100 based on their performance in a test. Matrix M below displays the marks obtained by these 45 students, listed by class.

$$M = \begin{array}{c} \\ \left[\begin{array}{ccccccccccccccc} 56 & 78 & 79 & 43 & 67 & 56 & 80 & 85 & 75 & 89 & 55 & 64 & 95 & 34 & 63 \\ 90 & 45 & 56 & 65 & 76 & 79 & 27 & 45 & 69 & 73 & 70 & 63 & 65 & 34 & 59 \\ 76 & 76 & 89 & 47 & 50 & 66 & 68 & 89 & 88 & 90 & 45 & 67 & 78 & 45 & 87 \end{array}\right] \end{array} \begin{array}{l} A \\ B \\ C \end{array} \quad \text{Class}$$

Two other matrices, S and R, are defined as follows.

$$S = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad R = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$$

Which one of the following matrix expressions can be used to generate a matrix that displays the mean mark obtained for each class?

A $\frac{1}{45}M$

B $\frac{1}{3}R \times M$

C $\frac{1}{3}R \times M \times S$

D $\frac{1}{15}M \times S$

E $\frac{1}{15}S \times R \times M$

[VCAA 2013 1MQ7]

Question 8

Vince, Nev and Rani all service office equipment. The matrix T shows the time that it takes (in minutes) for each of Vince (V), Nev (N) and Rani (R) to service a photocopier (P), a fax machine (F) and a scanner (S).

$$T = \begin{array}{c} \begin{array}{ccc} V & N & R \end{array} \\ \begin{bmatrix} 12 & 15 & 14 \\ 8 & 7 & 8 \\ 20 & 19 & 17 \end{bmatrix} \end{array} \begin{array}{l} \\ P \\ F \\ S \end{array}$$

The matrix U displays the number of photocopiers, fax machines and scanners to be serviced in three schools, Alton (A), Borton (B) and Carlon (C).

$$U = \begin{array}{c} \begin{array}{ccc} P & F & S \end{array} \\ \begin{bmatrix} 5 & 3 & 2 \\ 4 & 4 & 3 \\ 6 & 1 & 2 \end{bmatrix} \end{array} \begin{array}{l} \\ A \\ B \\ C \end{array}$$

A matrix that displays the time that it would take each of Vince, Nev and Rani, working alone, to service the photocopiers, fax machines and scanners in each of the three schools is

A $\begin{bmatrix} 17 & 18 & 16 \\ 12 & 11 & 11 \\ 26 & 20 & 19 \end{bmatrix}$

B $\begin{bmatrix} 204 & 110 & 97 \\ 116 & 60 & 53 \\ 278 & 153 & 131 \end{bmatrix}$

C $\begin{bmatrix} 124 & 134 & 128 \\ 140 & 145 & 139 \\ 120 & 135 & 126 \end{bmatrix}$

D $\begin{bmatrix} 7 & 12 & 12 \\ 4 & 3 & 5 \\ 14 & 18 & 15 \end{bmatrix}$

E $\begin{bmatrix} 60 & 15 & 28 \\ 32 & 35 & 24 \\ 120 & 19 & 34 \end{bmatrix}$

[VCAA 2010 1MQ6]

Question 9

Three types of cheese, cheddar (C), gouda (G) and blue (B), will be bought for a school function. The cost matrix P lists the prices of these cheeses, in dollars, at two stores, Foodway and Safeworth.

$$P = \begin{array}{c} \begin{bmatrix} 6.80 & 5.30 & 6.20 \\ 7.30 & 4.90 & 6.15 \end{bmatrix} \end{array} \begin{array}{l} \text{Foodway} \\ \text{Safeworth} \end{array}$$

a What is the order of matrix P? 1 mark

The number of packets of each type of cheese needed is listed in the quantity matrix Q.

$$Q = \begin{bmatrix} 8 \\ 11 \\ 3 \end{bmatrix} \begin{array}{c} C \\ G \\ B \end{array}$$

b **i** Evaluate the matrix $W = PQ$. 1 mark

ii At which store will the total cost of the cheese be lower? 1 mark

[VCAA 2009 2MQ1]

Question 10

Two subjects, Biology and Chemistry, are offered in the first year of a university science course. The matrix N lists the number of students enrolled in each subject.

$$N = \begin{bmatrix} 460 \\ 360 \end{bmatrix} \begin{array}{l} \text{Biology} \\ \text{Chemistry} \end{array} \qquad P = \begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 0.05 & 0.125 & 0.175 & 0.45 & 0.20 \end{bmatrix} \end{array}$$

The matrix P lists the proportion of these students expected to be awarded an A, B, C, D or E grade in each subject.

a Write down the order of matrix P. 1 mark

b Let the matrix $R = NP$.

i Evaluate the matrix R. 1 mark

ii Explain what the matrix *element* R_{24} represents. 1 mark

c Students enrolled in Biology have to pay a laboratory fee of \$110, while students enrolled in Chemistry pay a laboratory fee of \$95.

i Write down a clearly labelled row matrix, called F, that lists these fees. 1 mark

ii Show a matrix calculation that will give the total laboratory fees, L, paid in dollars by the students enrolled in Biology and Chemistry. Find this amount. 1 mark

[VCAA 2008 2MQ1]

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Question 11

A manufacturer sells three products, A, B and C, through outlets at two shopping centres, Eastown (E) and Noxland (N). The number of units of each product sold per month through each shop is given by the matrix Q.

$$Q = \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 2500 & 3400 & 1890 \\ 1765 & 4588 & 2456 \end{bmatrix} \end{array} \begin{array}{c} \\ E \\ N \end{array}$$

a Write down the order of matrix Q. 1 mark

The matrix P gives the selling price, in dollars, of products A, B, C.

$$P = \begin{bmatrix} 14.50 \\ 21.60 \\ 19.20 \end{bmatrix} \begin{array}{c} A \\ B \\ C \end{array}$$

b **i** Evaluate the matrix M, where $M = QP$. 1 mark

ii What information do the elements of matrix M provide? 1 mark

c Explain why the matrix PQ is not defined. 1 mark

[VCAA 2006 2MQ1]

Question 12

Rosa uses the following six-digit pin for her bank account: 216342. With her knowledge of matrices, she decides to use matrix multiplication to disguise this pin. First, she writes the six digits in the 2×3 matrix A.

$$A = \begin{bmatrix} 2 & 6 & 4 \\ 1 & 3 & 2 \end{bmatrix}$$

Next, she creates a new matrix by forming the matrix product, $C = BA$, where $B = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$.

a **i** Determine the matrix $C = BA$. 1 mark

ii From the matrix C, Rosa is able to write down a six-digit number that disguises her original pin. She uses the same pattern that she used to create matrix A from the digits 216342.

Write down the new six-digit number that Rosa uses to disguise her pin. 1 mark

b Show how the original matrix A can be regenerated from matrix C. 1 mark

[VCAA 2012 2MQ2]

Question 13

In a game of basketball, a successful shot for goal scores one point, two points, or three points, depending on the position from which the shot is thrown. G is a column matrix that lists the number of points scored for each type of successful shot.

$$G = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

In one game, Oscar was successful with 4 one-point shots for goal, 8 two-point shots for goal, and 2 three-point shots for goal.

a Write a row matrix, N, that shows the number of each type of successful shot for goal that Oscar had in that game. 1 mark

b Evaluate matrix P found by multiplying matrix N with matrix G so that $P = N \times G$.

c In this context, what does the information in matrix P provide? 1 mark

[VCAA 2010 2MQ1]

9.1

9.2 Matrices and simultaneous equations

Solving two simultaneous equations using matrices

Matrices can be used to solve simultaneous equations.

Worked example 4

a Show that the simultaneous equations $5x + 3y = 13$

$6x + 4y = 16$

are generated by the matrix equation $\begin{bmatrix} 5 & 3 \\ 6 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 13 \\ 16 \end{bmatrix}$.

b Solve the simultaneous equations using the matrix equation.

Working

a **1** Multiply the matrices on the left of the equation.

$$\begin{bmatrix} 5 & 3 \\ 6 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 13 \\ 16 \end{bmatrix}$$

$$\begin{bmatrix} 5x+3y \\ 6x+4y \end{bmatrix} = \begin{bmatrix} 13 \\ 16 \end{bmatrix}$$

2 Equate the elements.

$5x + 3y = 13$
$6x + 4y = 16$

b **1** Find the inverse of the square matrix in the matrix equation.

The inverse of $\begin{bmatrix} 5 & 3 \\ 6 & 4 \end{bmatrix}$ is

$$\frac{1}{20-18}\begin{bmatrix} 4 & -3 \\ -6 & 5 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 4 & -3 \\ -6 & 5 \end{bmatrix} = \begin{bmatrix} 2 & -1.5 \\ -3 & 2.5 \end{bmatrix}$$

2 Pre-multiply both sides of the matrix equation by the inverse.

$$\begin{bmatrix} 2 & -1.5 \\ -3 & 2.5 \end{bmatrix}\begin{bmatrix} 5 & 3 \\ 6 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & -1.5 \\ -3 & 2.5 \end{bmatrix}\begin{bmatrix} 13 \\ 16 \end{bmatrix}$$

3 Use the fact that a matrix multiplied by its inverse is the identity matrix.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & -1.5 \\ -3 & 2.5 \end{bmatrix}\begin{bmatrix} 13 \\ 16 \end{bmatrix}$$

4 A matrix multiplied by the identity matrix leaves the matrix unchanged.

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & -1.5 \\ -3 & 2.5 \end{bmatrix}\begin{bmatrix} 13 \\ 16 \end{bmatrix}$$

5 Find the product on the right of the equation.

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 26-24 \\ -39+40 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

6 Equate the elements to solve the simultaneous equations.

$x = 2, y = 1.$

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> **Exam hack**
>
> When solving simultaneous equations using matrices, always pre-multiply by the inverse.

Inconsistent and dependent systems of equations without solutions

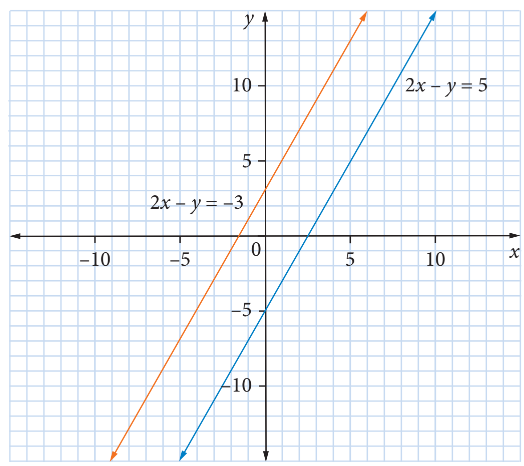
9.2

Not every set of simultaneous equations has a unique solution.

The lines represented by the equations may be parallel. For example:

$2x - y = 5$ (or $y = 2x - 5$)
$2x - y = -3$ (or $y = 2x + 3$)

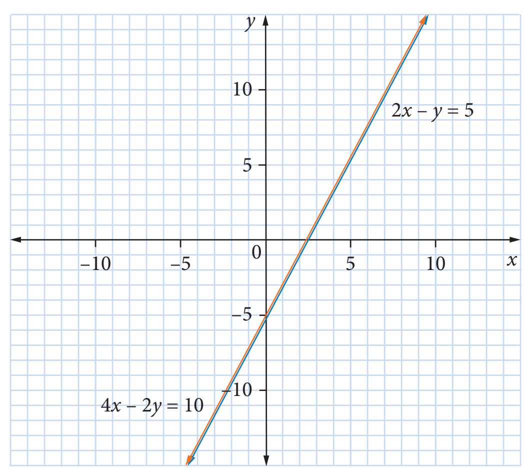
We can see they both have a slope of 2 and hence no intersection. This means the simultaneous equations have no solution. They are called an **inconsistent system of equations**.



The lines represented by the equations may actually be the same line. For example:

$2x - y = 5$
$4x - 2y = 10$

If we divide both sides of the second equation by 2, then we get $2x - y = 5$, which is the same as the first equation. This means that there is an infinite number of points of intersection between the two lines, so there is no unique solution. They are called a **dependent system of equations**.



When solving simultaneous equations using matrices, if there is no unique solution you will get a determinant of zero for the square matrix, which means no inverse exists. If you are using a CAS/calculator, it will give you an error message: either ‘Singular matrix’ or ‘Undefined’.

Worked example 5

For each pair of simultaneous equations, use matrices to find the value of e for which the equations do not have a unique solution.

a $7x + 5y = 12$
$ex + 5y = 11$

b $11x - 3y = -9$
$22x + ey = 2$

Working

a **1** Write the simultaneous equations in matrix form.

$$\begin{bmatrix} 7 & 5 \\ e & 5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 12 \\ 11 \end{bmatrix}$$

2 Find the determinant of the square matrix.

$$\begin{aligned} \det &= 7 \times 5 - 5 \times e \\ &= 35 - 5e \end{aligned}$$

3 Let the determinant equal zero and solve using a CAS/calculator if necessary.

$$\begin{aligned} 35 - 5e &= 0 \\ 5e &= 35 \\ e &= 7 \end{aligned}$$

b **1** Write the simultaneous equations in matrix form.

$$\begin{bmatrix} 11 & -3 \\ 22 & e \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -9 \\ 2 \end{bmatrix}$$

2 Find the determinant.

$$\begin{aligned} \det &= 11 \times e - (-3) \times 22 \\ &= 11e + 66 \end{aligned}$$

3 Let the determinant equal zero and solve for e.

$$\begin{aligned} 11e + 66 &= 0 \\ 11e &= -66 \\ e &= -6 \end{aligned}$$

Solving three or more simultaneous equations using matrices

The matrix method is extremely effective when solving three or more simultaneous equations.

Using CAS Solving simultaneous equations using matrices

Solve these simultaneous equations using matrices.

$$\begin{aligned} 2x + 3y - z &= 7 \\ 3x + + 2z &= 2 \\ x + y + z &= 7 \end{aligned}$$

STEP 1

Write the simultaneous equations as a matrix equation. Insert zeros where a pronumeral is missing.

$$\begin{bmatrix} 2 & 3 & -1 \\ 3 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \\ 7 \end{bmatrix}$$

STEP 2

Identify the matrices and write them in the form $AX = B$.

$$A = \begin{bmatrix} 2 & 3 & -1 \\ 3 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 7 \\ 2 \\ 7 \end{bmatrix}$$

$AX = B$

STEP 3

As long as $\det(A) \neq 0$, the solution is found by pre-multiplying both sides by A^{-1}.

$X = A^{-1}B$

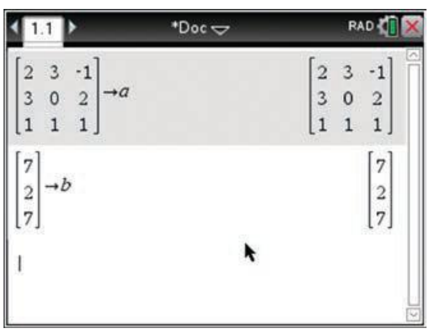
STEP 4

Enter A and B into your CAS/calculator.

TI-NSPIRE CAS

Create a 3×3 matrix, enter the elements shown and store as a.

Create a 3×1 matrix, enter the elements shown and store as b.



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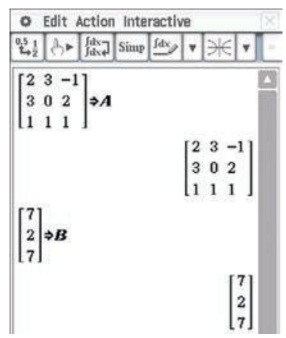
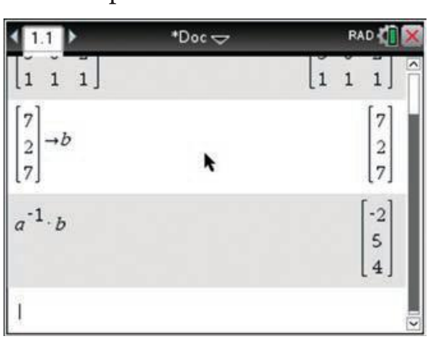
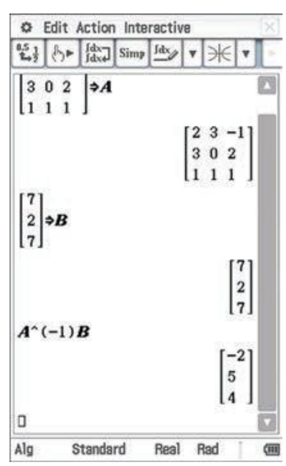


STEP 5

Find the product $a^{-1}b$.





STEP 6

Write your answer.

$$X = \begin{bmatrix} -2 \\ 5 \\ 4 \end{bmatrix}$$

$x = -2$, $y = 5$ and $z = 4$

Worked example 6

A factory has three robots that assemble three different models of computers (*A*, *B* and *C*).

Robot 1 assembles three model *A*s, two model *B*s and five model *C*s in 113 minutes.
Robot 2 assembles one model *A* and three model *B*s in 56 minutes.
Robot 3 assembles two model *B*s and one model *C* in 40 minutes.

Let a = the amount of time in minutes it takes for a robot to assemble one model *A*.
Let b = the amount of time in minutes it takes a robot to assemble one model *B*.
Let c = the amount of time in minutes it takes a robot to assemble one model *C*.

a Write three simultaneous equations in terms of a, b and c.

b Write the simultaneous equations in matrix form.

c Solve the matrix equation and hence find how long it takes a robot to assemble each of the three computer models.

Working

a Use the information in the question to write three simultaneous equations.

$$\begin{aligned} 3a + 2b + 5c &= 113 \\ a + 3b &= 56 \\ 2b + c &= 40 \end{aligned}$$

b Rewrite in matrix form, adding zeros where necessary.

$$\begin{bmatrix} 3 & 2 & 5 \\ 1 & 3 & 0 \\ 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 113 \\ 56 \\ 40 \end{bmatrix}$$

c **1** Solve with a CAS/calculator by finding the inverse of the 3×3 matrix and pre-multiplying the inverse on both sides of the equation.

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} \frac{3}{17} & \frac{8}{17} & -\frac{15}{17} \\ -\frac{1}{17} & \frac{3}{17} & \frac{5}{17} \\ \frac{2}{17} & -\frac{6}{17} & \frac{7}{17} \end{bmatrix}\begin{bmatrix} 113 \\ 56 \\ 40 \end{bmatrix} = \begin{bmatrix} 11 \\ 15 \\ 10 \end{bmatrix}$$

2 Write the answer.

It takes a robot 11 minutes to assemble model *A*, 15 minutes to assemble model *B*, and 10 minutes to assemble model *C*.

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Matrices and simultaneous equations

Prep 1

WORKED EXAMPLE 4

a Show that the matrix equation $\begin{bmatrix} 4 & -1 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -5 \\ 7 \end{bmatrix}$ generates the simultaneous equations

$$\begin{aligned} 4x - y &= -5 \\ x - 3y &= 7 \end{aligned}$$

b Solve the simultaneous equations using the matrix equation, showing all the steps.

Prep 2

WORKED EXAMPLE 5

For each pair of simultaneous equations, use matrices to find the value of e for which the equations do not have a unique solution.

a $\begin{aligned} 6x - 4y &= 16 \\ ex - 4y &= 13 \end{aligned}$

b $\begin{aligned} 12x + 3y &= -3 \\ 4x + ey &= -1 \end{aligned}$

Prep 3

USING CAS: SOLVING THREE OR MORE SIMULTANEOUS EQUATIONS USING MATRICES

Solve each set of simultaneous equations using matrices.

a $\begin{aligned} 5x - y + 2z &= 1 \\ -x - 3y &= -5 \\ x + 5z &= 19 \end{aligned}$

b $\begin{aligned} x + y + z &= 12 \\ 2x + 2y - 3z &= 9 \\ -x + y - z &= -4 \end{aligned}$

c $\begin{aligned} 10x - 5y &= -40 \\ 3x + 2z &= 7 \\ 4y - z &= 38 \end{aligned}$

Prep 4

WORKED EXAMPLE 6

A worker assembles a particular model of laptop, printer, modem and router.

She assembles

- two laptops, five printers, three modems and one router in 167 minutes
- four laptops and ten printers in 240 minutes
- six printers, twelve modems and three routers in 273 minutes
- five modems and two routers in 82 minutes

Let a = the amount of time in minutes it takes her to assemble one laptop.

Let b = the amount of time in minutes it takes her to assemble one printer.

Let c = the amount of time in minutes it takes her to assemble one modem.

Let d = the amount of time in minutes it takes her to assemble one router.

a Write four simultaneous equations in terms of a, b, c and d.

b Write the simultaneous equations in matrix form.

c Solve the matrix equation and hence find how long it takes her to assemble each of the four items.

EXAM PRACTICE 9.2

Matrices and simultaneous equations

Question 1

The matrix equation $\begin{bmatrix} 4 & 2 & 8 \\ 2 & 0 & 3 \\ 0 & 3 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \\ 6 \end{bmatrix}$ can be used to solve the system of simultaneous linear equations

A $4x + 2y + 8z = 7$
$2x + 3y = 2$
$3x - y = 6$

B $4x + 2y + 8z = 7$
$2x + 3y = 2$
$3y - z = 6$

C $4x + 2y + 8z = 7$
$2y + 3z = 2$
$3x - z = 6$

D $4x + 2y + 8z = 7$
$2x + 3z = 2$
$3y - z = 6$

E $4x + 2y + 8z = 7$
$2x + 3z = 2$
$3x - z = 6$

[VCAA 2009 1MQ4]

Question 2

The solution of the matrix equation $\begin{bmatrix} 0 & -3 & 2 \\ 1 & 1 & 1 \\ -2 & 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 11 \\ 5 \\ 8 \end{bmatrix}$ is

A $\begin{bmatrix} 1 \\ 24 \\ 2 \end{bmatrix}$ **B** $\begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$ **C** $\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$ **D** $\begin{bmatrix} -11 \\ \frac{4}{3} \\ 8 \end{bmatrix}$ **E** $\begin{bmatrix} 11 \\ 5 \\ 8 \end{bmatrix}$

[VCAA 2008 1MQ6]

Question 3

A system of three simultaneous linear equations is written in matrix form as follows.

$$\begin{bmatrix} 1 & -2 & 0 \\ 1 & 0 & 3 \\ 0 & 2 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 11 \\ -5 \end{bmatrix}$$

One of the three linear equations is

A $x - 2y + z = 4$
B $x + y + 3z = 11$
C $2x - y = -5$
D $x + 3z = 11$
E $3y - z = -5$

[VCAA 2010 1MQ5]

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Question 4

Each of the following four matrix equations represents a system of simultaneous linear equations.

$$\begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 4 \\ 8 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 4 \\ 8 \end{bmatrix} \qquad \begin{bmatrix} 0 & 3 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 6 \\ 12 \end{bmatrix}$$

How many of these systems of simultaneous linear equations have a unique solution?

A 0 **B** 1 **C** 2 **D** 3 **E** 4

[VCAA 2011 1MQ3]

Question 5

Consider the following system of three simultaneous linear equations.

$2x + z = 5$
$x - 2y = 0$
$y - z = -1$

This system of equations can be written in matrix form as

A $\begin{bmatrix} 2 & 1 \\ 1 & -1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 5 \\ 0 \\ -1 \end{bmatrix}$

B $\begin{bmatrix} 2 & 0 & 1 \\ 1 & -2 & 0 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 5 \\ 0 \\ -1 \end{bmatrix}$

C $\begin{bmatrix} 2 & 1 & 5 \\ 1 & -2 & 0 \\ 1 & -1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 5 \\ 0 \\ -1 \end{bmatrix}$

D $\begin{bmatrix} 2 & 1 & 0 \\ 1 & -2 & 0 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 5 \\ 0 \\ -1 \end{bmatrix}$

E $\begin{bmatrix} 2 & 1 \\ 1 & -2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 5 \\ 0 \\ -1 \end{bmatrix}=\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

[VCAA 2007 1MQ4]

Question 6

$2.8x + 0.7y = 10$
$1.4x + ky = 6$

The set of simultaneous linear equations does **not** have a solution if k equals

A –0.35 **B** –0.250 **C** 0 **D** 0.25 **E** 0.35

[VCAA 2013 1MQ4]

Question 7

How many of the following five sets of simultaneous linear equations have a unique solution?

$4x + 2y = 10$	$x = 0$	$x - y = 3$	$2x + y = 5$	$x = 8$
$2x + \ \ y = 5$	$x + y = 6$	$x + y = 3$	$2x + y = 10$	$y = 2$

A 1 **B** 2 **C** 3 **D** 4 **E** 5

[VCAA 2006 1MQ7]

Question 8

A worker can assemble 10 bookcases and four desks in 360 minutes, and eight bookcases and three desks in 280 minutes. If each bookcase takes b minutes to assemble and each desk takes d minutes to assemble, the matrix $\begin{bmatrix} b \\ d \end{bmatrix}$ will be given by

A $\begin{bmatrix} -1.5 & 2 \\ 4 & -5 \end{bmatrix}\begin{bmatrix} 360 \\ 280 \end{bmatrix}$ **B** $\begin{bmatrix} 10 & 4 \\ 8 & 3 \end{bmatrix}\begin{bmatrix} 360 \\ 280 \end{bmatrix}$ **C** $\begin{bmatrix} 3 & -4 \\ -8 & 10 \end{bmatrix}\begin{bmatrix} 360 \\ 280 \end{bmatrix}$

D $\begin{bmatrix} 5 & -2 \\ -4 & 1.5 \end{bmatrix}\begin{bmatrix} 360 \\ 280 \end{bmatrix}$ **E** $\begin{bmatrix} 10 \\ 4 \end{bmatrix}[360]+\begin{bmatrix} 8 \\ 3 \end{bmatrix}[280]$

[VCAA 2013 1MQ6]

Question 9

$x + z = 6$
$2y + z = 8$
$2x + y + 2z = 15$

The **solution** of the simultaneous equations is given by

A $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 & -1 & 2 \\ -2 & 0 & 1 \\ 4 & 1 & -2 \end{bmatrix}\begin{bmatrix} 6 \\ 8 \\ 15 \end{bmatrix}$ **B** $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}\begin{bmatrix} 6 \\ 8 \\ 15 \end{bmatrix}$

C $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\frac{3}{2} & \frac{1}{2} & \frac{1}{2} \\ -1 & 1 & 0 \\ 2 & -1 & 0 \end{bmatrix}\begin{bmatrix} 6 \\ 8 \\ 15 \end{bmatrix}$ **D** $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ -2 & 0 & 1 \\ 2 & -1 & 0 \end{bmatrix}\begin{bmatrix} 6 \\ 8 \\ 15 \end{bmatrix}$

E $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 & -1 & 1 \\ -2 & 0 & 1 \\ 2 & 1 & -1 \end{bmatrix}\begin{bmatrix} 6 \\ 8 \\ 15 \end{bmatrix}$

[VCAA 2012 1MQ3]

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Question 10

The total cost of one ice-cream and three soft drinks at Catherine's shop is \$9. The total cost of two ice-creams and five soft drinks is \$16. Let x be the cost of an ice-cream and y be the cost of a soft drink.

The matrix $\begin{bmatrix} x \\ y \end{bmatrix}$ is equal to

A $\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

B $\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} 9 \\ 16 \end{bmatrix}$

C $\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} 9 \\ 16 \end{bmatrix}$

D $\begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 9 \\ 16 \end{bmatrix}$

E $\begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 9 \\ 16 \end{bmatrix}$

[VCAA 2010 1MQ3]

Question 11

Tickets for a function are sold at the school office, the function hall and online. Different prices are charged for students, teachers and parents. The table shows the number of tickets sold at each place and the total value of sales.

	School office	Function hall	Online
Student tickets	283	35	84
Teacher tickets	28	4	3
Parent tickets	5	2	7
Total sales	\$8712	\$1143	\$2609

For this function student tickets cost \$$x$, teacher tickets cost \$$y$, and parent tickets cost \$$z$.

a Use the information in the table to find the two missing elements in the following matrix equation.

$$\begin{bmatrix} 283 & 28 & 5 \\ & 4 & \\ 84 & 3 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8712 \\ 1143 \\ 2609 \end{bmatrix}$$

1 mark

b Use the matrix equation to find the cost of a teacher ticket to the school function. 1 mark

[VCAA 2009 2MQ2]

Question 12

Oscar's basketball coach has written three linear equations that can be used to predict the number of points, p, rebounds, r, and assists, a, that Oscar will have in his next game. The equations are

$$\begin{aligned} p + r + a &= 33 \\ 2p - r + 3a &= 40 \\ p + 2r + a &= 43 \end{aligned}$$

a Write these equations in matrix form. 1 mark

This matrix equation can be solved in the following way.

$$\begin{bmatrix} p \\ r \\ a \end{bmatrix} = \begin{bmatrix} 7 & -1 & -4 \\ -1 & 0 & 1 \\ x & 1 & 3 \end{bmatrix} \begin{bmatrix} 33 \\ 40 \\ 43 \end{bmatrix}$$

b Determine the value of x shown in the matrix equation. 1 mark

c How many rebounds is Oscar predicted to have in his next game? 1 mark

[VCAA 2010 2MQ3]

Question 13

Market researchers claim that the ideal number of bookshops (x), sports shoe shops (y) and music stores (z) for a shopping centre can be determined by solving the equations

$$\begin{aligned} 2x + y + z &= 12 \\ x - y + z &= 1 \\ 2y - z &= 6 \end{aligned}$$

a Write the equations in matrix form. 1 mark

b Do the equations have a unique solution? Provide an explanation to justify your response. 1 mark

c Write down an inverse matrix that can be used to solve these equations. 1 mark

d Solve the equations and hence write down the estimated ideal number of bookshops, sports shoe shops and music stores for a shopping centre. 1 mark

[VCAA 2006 2MQ3]

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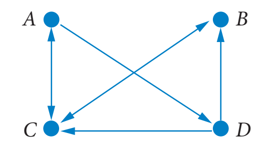
Communication and dominance matrices

9.3

Communication diagrams and matrices



Matrices are useful when we are investigating communications involving computer systems, friendship groups, social media, the military and travel. This **communication diagram** shows how four computers communicate with each other. The arrows indicate which way the communication goes. The communication between computer A and computer C is **two-way**: A can communicate with C, and C can communicate with A. However, the communication between A and D is **one-way**. A can communicate with D, but D can't communicate with A.

The **communication matrix** M, which represents this communications diagram, is:

$$M = \begin{array}{c} \text{From} \\ \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ \\ A \\ B \\ C \\ D \end{array} \ \text{To}$$

where an element is 0 if there is no communication and 1 if there is.

A communication matrix is a square binary matrix where communication is indicated by a 1 and non-communication is indicated by a 0.

The leading diagonal of a communication matrix always consists of 0s since self-communication isn't considered communication in this context.

A communication matrix where every communication is two-way is symmetric about the leading diagonal.

Exam hack

Communication matrices are sometimes written with the 'from' as the rows and the 'to' as the columns. The two ways of constructing these matrices are transposes of each other (i.e. the rows of one matrix are the columns of the other matrix and vice versa).

$$M = \text{From} \begin{array}{c} \\ \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \text{To} \\ \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$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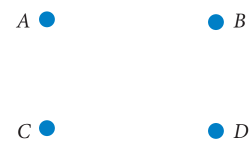
Worked example 7

a The communication matrix M shows the direct flights for an airline that flies between four cities Arkton, Brenton, Corkton and Dunton, indicated by their first letters. Complete the communication diagram by drawing arrows to indicate direct flights.

$$\begin{array}{c} \\ M = \end{array} \begin{array}{c} \text{From} \\ \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ \\ A \\ B \\ C \\ D \end{array} \quad \text{To}$$



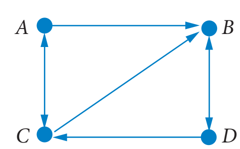
b Evaluate the matrix product $N = KM$, where $K = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$

c What effect has pre-multiplying by K had on M, and what information does the matrix N contain?

d Which of the cities has the fewest direct flights to the other three cities?

Working

a Draw single arrows to indicate one-way communication/connection and double-ended arrows for two-way communication/connection.



b Multiply the matrices.

$$N = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 & 2 & 2 \end{bmatrix}$$

c Use the fact that K is a summing matrix.

Pre-multiplying by K has summed the elements in the columns in M. Matrix N shows the total number of direct flights from each city to the other three cities.

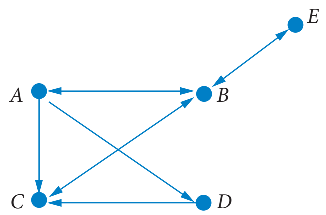
d Look at the elements of N.

Matrix N shows that there is only one direct flight from Brenton to the other cities, whereas the other cities each have two direct flights. So Brenton has the fewest direct flights to the other three cities.

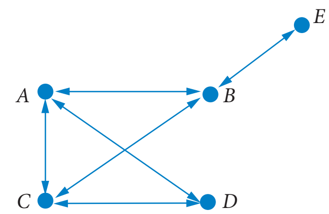
Worked example 8

a Find the communication matrix M that matches each communication diagram.

i

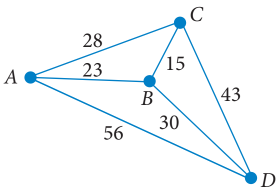


ii



b Explain why the second communication matrix is symmetric about the leading diagonal.

c This diagram shows the driving times in minutes between four towns. Write a matrix that represents this information.



Working

a i Indicate communication by a 1 and non-communication by a 0.

$$M = \begin{array}{c} \text{From} \\ \begin{matrix} A & B & C & D & E \end{matrix} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{matrix} A \\ B \\ C \\ D \\ E \end{matrix} \quad \text{To}$$

ii Indicate communication by a 1 and non-communication by a 0.

$$M = \begin{array}{c} \text{From} \\ \begin{matrix} A & B & C & D & E \end{matrix} \\ \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{matrix} A \\ B \\ C \\ D \\ E \end{matrix} \quad \text{To}$$

b Comment on the two-way communication involved.

All the arrows in the second communication diagram go both ways. Communication matrices where every communication is two-way are always symmetric about the leading diagonal.

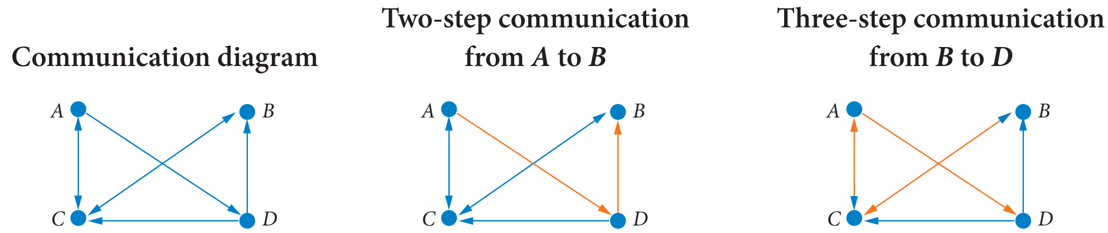
c The leading diagonal elements are 0 since the driving time from a town to the same town is zero minutes. The matrix is symmetric since none of the roads is one-way.

$$M = \begin{array}{c} \text{From} \\ \begin{matrix} A & B & C & D \end{matrix} \\ \begin{bmatrix} 0 & 23 & 28 & 56 \\ 23 & 0 & 15 & 30 \\ 28 & 15 & 0 & 43 \\ 56 & 30 & 43 & 0 \end{bmatrix} \end{array} \begin{matrix} A \\ B \\ C \\ D \end{matrix} \quad \text{To}$$

Multi-step communication

Communication involves more than direct **one-step communication**. For example, in the communication diagram below, computers *A* and *B* don't communicate directly, but *A* can communicate with *D*, which can then communicate with *B*. This is called a **two-step communication** between *A* and *B*.

Similarly, *B* can't communicate directly with *D*, but *B* can communicate with *C*, which can communicate with *A*, which can communicate with *D*. This is called a **three-step communication** between *B* and *D*.



For a communications matrix M:

M^2 gives the number of two-step communications.

M^3 gives the number of three-step communications.

M^4 gives the number of four-step communications, and so on.

$M + M^2$ gives us the number of one- or two-step communications.

$M + M^2 + M^3$ gives us the number of one-, two- or three-step communications, and so on.

Exam hack

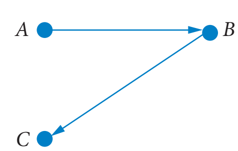
Note the difference between two-way communication and two-step communication.

In this example:

- *A* and *B* have two-way communication.
- $A \rightarrow B \rightarrow A$ is a two-step communication from *A* to *A*.

In this example,

- there is no two-way communication
- $A \rightarrow B \rightarrow C$ is a two-step communication from *A* to *C*.



Self-communication links

There is not much point in most communication situations where the sender and receiver are the same. These are called **self-communication links**. We included zeros in the leading diagonal of communication matrices so that we don't record one-step self-communication. However, once we start taking powers of communication matrices to find multi-step communication, non-zero elements appear in the leading diagonal.

For example, for

$$M = \begin{array}{c} \text{From} \\ \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} A \\ B \\ C \\ D \end{array} \text{To} \qquad \text{we get} \qquad M^2 = \begin{array}{c} \text{From} \\ \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 2 \end{bmatrix} \end{array} \begin{array}{c} A \\ B \\ C \\ D \end{array} \text{To}$$

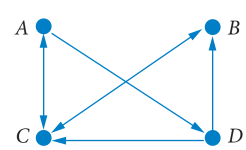
The leading diagonal in M^2 shows the following self-communication links:

- one two-step communication from A to A
- one two-step communication from B to B
- two two-step communications from D to D.

In most practical situations, self-communication should be ignored.

Worked example 9

For the communication diagram representing four computers and matching matrix shown, find the following.



$$M = \begin{array}{c} \text{From} \\ \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} A \\ B \\ C \\ D \end{array} \text{To}$$

a **i** The two-step communication matrix.

ii The number of two-step self-communications from A to A, listing all the paths.

iii The number of two-step communications from A to B, listing all the paths.

b **i** The three-step communication matrix.

ii The number of three-step self-communications from A to A, listing all the paths.

iii The number of three-step communications from C to B, listing all the paths.

c **i** The matrix that shows the number of one- or two-step communications.

ii The number of one- or two-step communications from D to C, listing all the paths.

iii The communications that can't be made with either one or two steps.

d **i** The matrix that shows the number of communications of no more than three steps.

ii Evidence that every one of the four computers can communicate with every other computer using one-, two- or three-step communications.

Working

a **i** Find M^2 using a CAS/calculator.

$$M^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{array}{c} \text{From} \\ \begin{array}{cccc} A & B & C & D \end{array} \\ = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 2 & 1 & 0 & 1 \\ 1 & 0 & 2 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{array}{c} A \\ B \\ C \\ D \end{array} \text{ To} \end{array}$$

ii Find the number of two-step communications from M^2, and find the paths from the communication diagram.

There is one two-step self-communication from A to A.

$A \to C \to A$

iii

There are two two-step communications from A to B.

$A \to D \to B$
$A \to C \to B$

b **i** Find M^3 using a CAS/calculator.

$$M^3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{array}{c} \text{From} \\ \begin{array}{cccc} A & B & C & D \end{array} \\ = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 1 & 0 & 3 & 1 \\ 3 & 2 & 1 & 2 \\ 1 & 1 & 0 & 1 \end{bmatrix} \begin{array}{c} A \\ B \\ C \\ D \end{array} \text{ To} \end{array}$$

ii Find the number of three-step communications from M^3.

There is one three-step self-communication from A to A.

$A \to D \to C \to A$

iii

There are three three-step communications from C to B.

$C \to A \to D \to B$
$C \to A \to C \to B$
$C \to B \to C \to B$

c **i** Find $M + M^2$.

$$M + M^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 0 & 1 \\ 2 & 1 & 0 & 1 \\ 1 & 0 & 2 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\begin{array}{c} \text{From} \\ \begin{array}{cccc} A & B & C & D \end{array} \\ = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 2 \\ 2 & 1 & 2 & 2 \\ 1 & 0 & 1 & 0 \end{bmatrix} \begin{array}{c} A \\ B \\ C \\ D \end{array} \text{To} \end{array}$$

ii Find the number of one- or two-step communications from $M + M^2$.

There are two one- or two-step communications from D to C.

$D \to C$
$D \to B \to C$

iii Find the zeros in $M + M^2$.

B can't communicate with D in one or two steps.
D can't communicate with D in one or two steps.

d **i** 'No more than three steps' means 'one-, two- or three-steps', so find $M + M^2 + M^3$. Add the answer to part **c i** to the answer to part **b i**.

$$M + M^2 + M^3 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 2 \\ 2 & 1 & 2 & 2 \\ 1 & 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 2 & 1 \\ 1 & 0 & 3 & 1 \\ 3 & 2 & 1 & 2 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

$$\begin{array}{c} \text{From} \\ \begin{array}{cccc} A & B & C & D \end{array} \\ = \begin{bmatrix} 2 & 1 & 3 & 2 \\ 3 & 1 & 4 & 3 \\ 5 & 3 & 3 & 4 \\ 2 & 1 & 1 & 1 \end{bmatrix} \begin{array}{c} A \\ B \\ C \\ D \end{array} \text{To} \end{array}$$

ii Look for zeros in $M + M^2 + M^3$.

There are no zeros in $M + M^2 + M^3$, so every computer can communicate with every other computer using one-, two- or three-step communications.

Dominance matrices

Dominance situations are similar to communication situations, except *all* the arrows in a **dominance diagram** are *one-way*. Examples of dominance situations involve competitions where there are winners and losers, and food chains where there are animals that eat other animals and those that are eaten. An arrow from A to B in a dominance diagram shows A has dominated B (e.g. A has defeated B in a competition, or A eats B in a food chain).

Direct dominance is a one-step dominance where A dominates B. **Indirect dominance** of A over B is a two-step dominance where A dominates C, which dominates B. A **dominance matrix** is a square binary matrix where direct dominance is indicated by a 1 and other elements are 0.

9.3

For a square dominance matrix M in the form

$$M = \begin{array}{c} \begin{matrix} A & B & C & \cdots \end{matrix} \\ \left[\begin{matrix} \quad & \quad & \quad & \quad \\ & & & \\ & & & \\ & & & \end{matrix}\right] \end{array} \begin{matrix} A \\ B \\ C \\ \vdots \end{matrix}$$

- the sum of the elements in each column of M gives us the number of **direct dominances** for each of $A, B, C, \ldots$
- the sum of the elements in each column of M^2 gives us the number of **indirect dominances** for each of $A, B, C, \ldots$
- the sum of the elements in each column of $M + M^2$ gives us the number of direct and indirect dominances for each of $A, B, C, \ldots$, which can be used to establish overall dominance when direct dominances are equal.

Worked example 10

Abbie, Bess, Cath and Denise play off against each other in a series of one-on-one basketball games. In these games:

- Abbie beat Bess
- Bess had no wins
- Cath beat Bess and Denise
- Denise beat Abbie and Bess

a Complete a dominance diagram to indicate who has defeated whom.

b Find M, the dominance matrix.

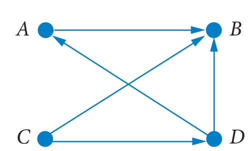
c Find how many wins each person had.

d Find M^2 and explain what information this matrix gives us.

e Find $M + M^2$ and explain how this could be used to work out who won the play-off.

Working

a Draw arrows between letters representing the players. All arrows are one-way only.



b Indicate a win (dominance) by a 1 and a loss by a 0.

$$M = \begin{array}{c} \begin{matrix} A & B & C & D \end{matrix} \\ \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{array} \begin{matrix} A \\ B \\ C \\ D \end{matrix}$$

c Summing each column tells us how many direct wins each person had.

- Abbie had one win
- Bess had no wins
- Cath had two wins
- Denise had two wins

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d Use a CAS/calculator to find M^2. Interpret what two-step dominance means in this case.

$$M^2 = \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ A \\ B \\ C \\ D \end{array}$$

M^2 tells us how often a player defeated a player who had defeated another player. It gives us the number of indirect wins.

So, although Abbie and Cath didn't play each other, we can see that Cath defeated a player who had defeated Abbie.

We can also see that Cath defeated a player who had defeated Bess, and that Denise defeated a player who had defeated Bess.

e Find $M + M^2$. The sum of the elements in each column tells us how many direct and indirect wins each person had.

$$M + M^2 = \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ A \\ B \\ C \\ D \end{array}$$

Although Cath and Denise both had two direct wins, summing the columns of $M + M^2$ tells us that Cath had a total of 4 direct and indirect wins, and Denise had 3 direct and indirect wins. Based on this, Cath has dominance and should be declared the winner.

EXAM PREP 9.3

Communication and dominance matrices

Prep 1 WORKED EXAMPLE 7

The following communication matrix M shows the direct shuttles for a bus company that shuttles between four locations: the Art Gallery, the Beach, Central Station and Docklands, indicated by their first letters.

a Copy and complete the communication diagram shown by drawing arrows between the letters to indicate direct shuttles.

$$M = \begin{array}{c} \text{From} \\ \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ \\ A \\ B \\ C \\ D \end{array} \text{ To}$$

b Evaluate the matrix product $N = KM$, where $K = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$

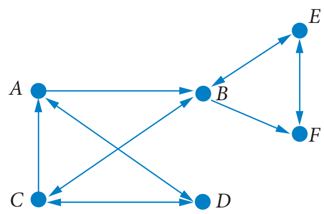
c What effect has pre-multiplying by K had on M, and what information does the matrix N contain?

d Which of the locations has the most direct shuttles to the other three locations?

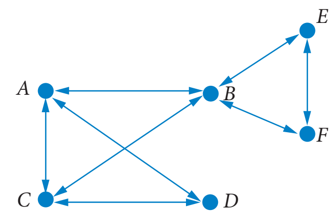
Prep 2 WORKED EXAMPLE 8

a Find the communication matrix M that matches each of these communication diagrams.

i

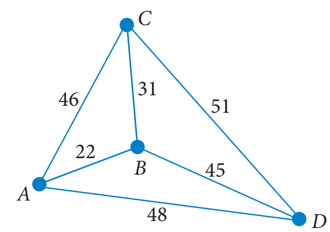


ii



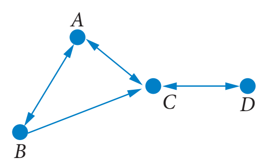
b Explain why the second communication matrix is symmetric about the leading diagonal.

c This diagram shows the driving times in minutes between four towns. None of the roads is one way. Write a matrix that could be used to show this information.



Prep 3 WORKED EXAMPLE 9

For the communication diagram representing four ships in communication range and matching matrix shown, find the following.



$$\begin{array}{cc} & \text{From} \\ & \begin{array}{cccc} A & B & C & D \end{array} \\ M = & \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ \\ A \\ B \\ C \\ D \end{array} \quad \text{To}$$

a **i** The two-step communication matrix.

ii The number of two-step self-communications from A to A, listing all the paths.

iii The number of two-step communications from A to B, listing all the paths.

b **i** The three-step communication matrix.

ii The number of three-step self-communications from A to A, listing all the paths.

iii The number of three-step communications from C to A, listing all the paths.

c **i** The matrix that shows the number of one- or two-step communications.

ii The number of one- or two-step communications from A to C, listing all the paths.

iii The communications that can't be made with either one or two steps.

d **i** The matrix that shows the number of communications of no more than three steps.

ii Evidence that every one of the four ships can communicate with every other ship using one-, two- or three-step communications.

Prep 4 WORKED EXAMPLE 10

Four players, Albie, Bertie, Collie and Dougie, played in a round-robin darts competition. The results were:

- Albie beat Bertie and Collie
- Bertie beat Dougie
- Collie beat Bertie and Dougie
- Dougie had no wins

a Copy and complete the dominance diagram shown by drawing arrows between the letters of the players' first names to indicate who has defeated whom.

b Find M, the dominance matrix that matches this dominance diagram.

c Find how many wins each person had from the matrix M.

d Find M^2 and explain what information this matrix gives us.

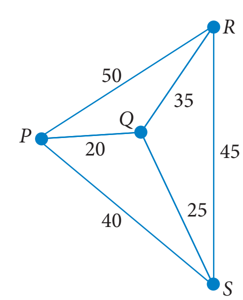
e Find $M + M^2$ and explain how this could be used to work out who won the competition.

EXAM PRACTICE 9.3

Communication and dominance matrices

Question 1

The diagram shows the tracks directly linking four camping sites P, Q, R and S in a national park. The shortest time that it takes to walk between the camping sites (in minutes) along each of these tracks is also shown. A matrix that could be used to present the same information is



A $\begin{array}{cc} \begin{matrix} P & Q & R & S \end{matrix} & \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} & \begin{matrix} P \\ Q \\ R \\ S \end{matrix} \end{array}$

B $\begin{array}{cc} \begin{matrix} P & Q & R & S \end{matrix} & \\ \begin{bmatrix} 0 & 20 & 40 & 50 \\ 20 & 0 & 45 & 25 \\ 40 & 45 & 0 & 35 \\ 50 & 25 & 35 & 0 \end{bmatrix} & \begin{matrix} P \\ Q \\ R \\ S \end{matrix} \end{array}$

C $\begin{array}{cc} \begin{matrix} P & Q & R & S \end{matrix} & \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 20 & 1 & 0 & 0 \\ 50 & 35 & 1 & 0 \\ 40 & 25 & 45 & 1 \end{bmatrix} & \begin{matrix} P \\ Q \\ R \\ S \end{matrix} \end{array}$

D $\begin{array}{cc} \begin{matrix} P & Q & R & S \end{matrix} & \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} & \begin{matrix} P \\ Q \\ R \\ S \end{matrix} \end{array}$

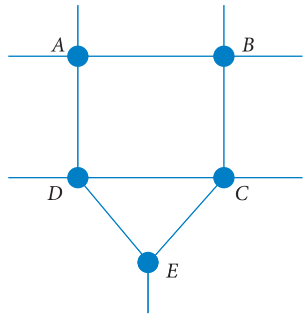
E $\begin{array}{cc} \begin{matrix} P & Q & R & S \end{matrix} & \\ \begin{bmatrix} 0 & 20 & 50 & 40 \\ 20 & 0 & 35 & 25 \\ 50 & 35 & 0 & 45 \\ 40 & 25 & 45 & 0 \end{bmatrix} & \begin{matrix} P \\ Q \\ R \\ S \end{matrix} \end{array}$

[VCAA 2012 1MQ4]

Question 2

A, B, C, D and E are five intersections joined by roads, as shown in the diagram.

Some of these roads are one-way only.



$$\begin{array}{c c l} & \text{From intersection} & \\ & \begin{array}{ccccc} A & B & C & D & E \end{array} & \\ & \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix} & \begin{array}{c} A \\ B \\ C \\ D \\ E \end{array} \quad \text{To intersection} \end{array}$$

The matrix indicates the direction that cars can travel along each of these roads. In this matrix

- 1 in column A and row B indicates that cars can travel directly from A to B
- 0 in column B and row A indicates that cars cannot travel directly from B to A (either it is a one-way road or no road exists).

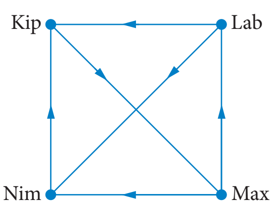
Cars can travel in both directions between intersections

A A and D **B** B and C **C** C and D **D** D and E **E** C and E

[VCAA 2009 1MQ5]

Question 3

The graph shows the one-step dominances between four farm dogs, Kip, Lab, Max and Nim. In this graph, an arrow from Lab to Kip indicates that Lab has a one-step dominance over Kip. From this graph, it can be concluded that Kip has a two-step dominance over



A Max only. **B** Nim only. **C** Lab and Nim only.

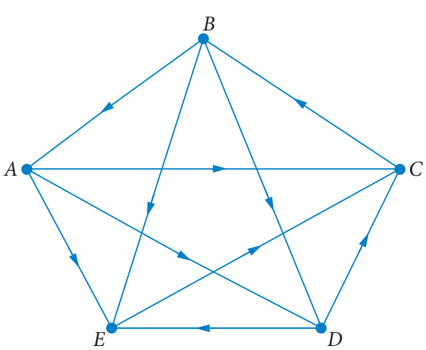
D all of the other three dogs. **E** none of the other three dogs.

[VCAA 2011 1NQ2]

Question 4

There are five teams, A, B, C, D and E, in a volleyball competition. Each team played each other team once in 2007. The results are summarised in the directed graph. An arrow from A to E signifies that A defeated E.

In 2007, the team that had the highest number of two-step dominances was



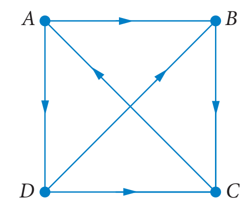
A team A **B** team B **C** team C

D team D **E** team E

[VCAA 2007 1NQ8]

Question 5

Four people, Ash (A), Binh (B), Con (C) and Dan (D), competed in a table tennis tournament. In this tournament, each competitor played each of the other competitors once. The results of the tournament are summarised in the directed graph. Each arrow shows the winner of a game played in the tournament. For example, the arrow from C to A shows that Con defeated Ash.



In the tournament, each competitor was given a ranking that was determined by calculating the sum of their one-step and two-step dominances. The competitor with the highest sum is ranked number one (1). The competitor with the second-highest sum was ranked number two (2), and so on.

Using this method, the rankings of the competitors in this tournament were

A Dan (1), Ash (2), Con (3), Binh (4)

B Dan (1), Ash (2), Binh (3), Con (4)

C Con (1), Dan (2), Ash (3), Binh (4)

D Ash (1), Dan (2), Binh (3), Con (4)

E Ash (1), Dan (2), Con (3), Binh (4)

[VCAA 2013 1NQ5]

Question 6

Matrix F shows the flight connections for an airline that serves four cities, Anvil (A), Berga (B), Cantor (C) and Dantel (D).

In this matrix, the '1' in column C row B, for example, indicates that, using this airline, you can fly directly from Cantor to Berga. The '0' in column C row D, for example, indicates that you cannot fly directly from Cantor to Dantel.

$$\begin{array}{cccc} & \text{From} & & \\ & \begin{array}{cccc} A & B & C & D \end{array} & & \\ F = & \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} & \begin{array}{c} A \\ B \\ C \\ D \end{array} & \text{To} \end{array}$$

a Copy and complete the following sentence.

On this airline, you can fly directly from Berga to $\boxed{}$ and $\boxed{}$. 1 mark

b List the route that you must follow to fly from Anvil to Cantor. 1 mark

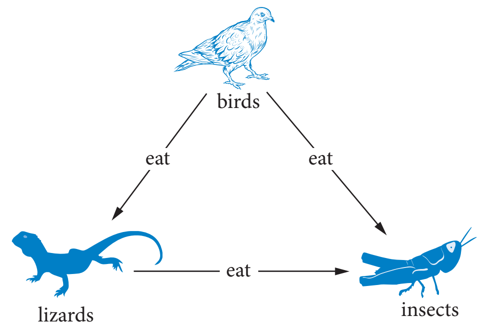
c Evaluate the matrix product $G = KF$, where $K = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$. 1 mark

d In the context of the problem, what information does matrix G contain? 1 mark

[VCAA 2012 2MQ1]

Question 7

The diagram shows the feeding paths for insects (I), birds (B) and lizards (L). The matrix E has been constructed to represent the information in this diagram. In matrix E, a '1' is read as 'eat' and a '0' is read as 'do not eat'.



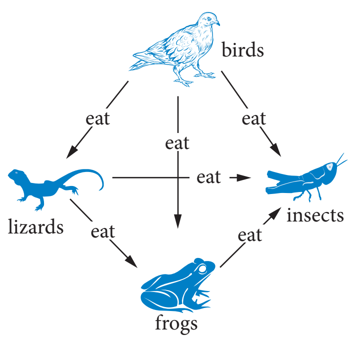
$$E = \begin{array}{c} \begin{array}{ccc} I & B & L \end{array} \\ \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ I \\ B \\ L \end{array}$$

a Referring to insects, birds or lizards

i what does the '1' in column B, row L, of matrix E indicate? 1 mark

ii what does the row of zeros in matrix E indicate? 1 mark

The diagram shows the feeding paths for insects (I), birds (B), lizards (L) and frogs (F). The matrix Z has been set up to represent the information in this diagram. Matrix Z has not been completed.



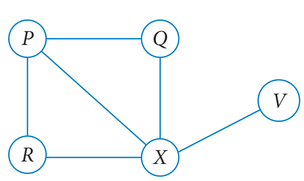
$$Z = \begin{array}{c} \begin{array}{cccc} I & B & L & F \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & - \\ 0 & 0 & 0 & - \\ 0 & 1 & 0 & - \\ - & - & - & - \end{bmatrix} \end{array} \begin{array}{c} \\ I \\ B \\ L \\ F \end{array}$$

b Copy and complete the matrix Z by writing in the seven missing elements. 1 mark

[VCAA 2011 2MQ1]

Question 8

Five trout-breeding ponds, P, Q, R, X and V, are connected by pipes, as shown in the diagram. The matrix W is used to represent the information in this diagram.



$$W = \begin{array}{c} \begin{array}{ccccc} P & Q & R & X & V \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ P \\ Q \\ R \\ X \\ V \end{array}$$

In matrix W the 1 in column 1, row 2, for example, indicates that a pipe directly connects pond P and pond Q. The 0 in column 1, row 5, for example, indicates that pond P and pond V are not directly connected by a pipe.

a Find the sum of the elements in row 3 of matrix W. 1 mark

b In terms of the breeding ponds described, what does the sum of the elements in row 3 of matrix W represent? 1 mark

The pipes connecting pond P to pond R and pond P to pond X are removed. Matrix N will be used to show this situation. However, it has missing elements.

c Copy and complete matrix N by filling in the missing elements in row 1 and column 1. 1 mark

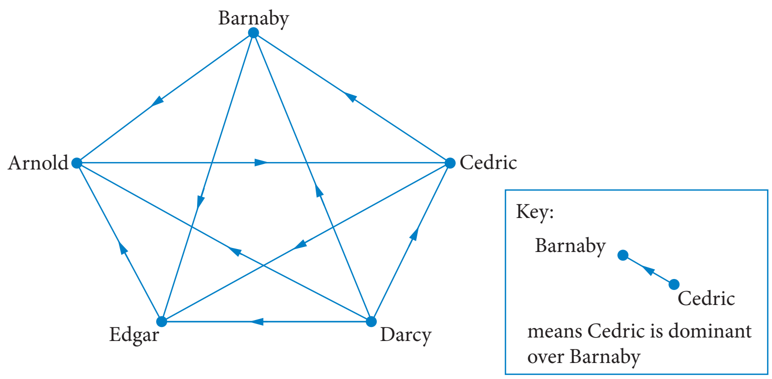
$$N = \begin{array}{c} \begin{array}{ccccc} P & Q & R & X & V \end{array} \\ \begin{bmatrix} 0 & - & - & - & - \\ - & 0 & 0 & 1 & 0 \\ - & 0 & 0 & 1 & 0 \\ - & 1 & 1 & 0 & 1 \\ - & 0 & 0 & 1 & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ P \\ Q \\ R \\ X \\ V \end{array}$$

[VCAA 2013 2MQ1]

9.3

Question 9

The children are taken to the zoo where they observe the behaviour of five young male lion cubs. The lion cubs are named Arnold, Barnaby, Cedric, Darcy and Edgar. A dominance hierarchy has emerged within this group of lion cubs. In the dominance diagram, the directions of the arrows show which lions are dominant over others.



a Name the two pairs of lion cubs who have equal totals of one-step dominances. 2 marks

b Over which lion does Cedric have both a one-step dominance and a two-step dominance? 1 mark

In determining the final order of dominance, the number of one-step dominances and two-step dominances are added together.

c Copy and complete the table for the final order of dominance. 1 mark

Final order of dominance	Lion
1st	Darcy
2nd	
3rd	
4th	
5th	

Over time, the pattern of dominance changes until each lion cub has a one-step dominance over two other lion cubs.

d Determine the total number of two-step dominances for this group of five lion cubs. 1 mark

[VCAA 2008 2NQ4]

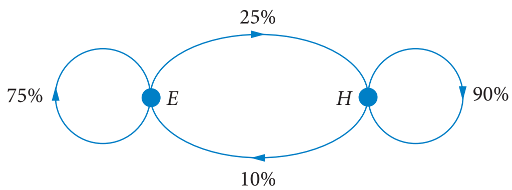
9.4 Transition diagrams and matrices

Constructing transition diagrams and matrices

A **transition matrix** is a square matrix that shows a change from one **state** (a condition or a location at a point in time) to another, where the change follows the same rules each time.

This **transition diagram** and a matching transition matrix show how shoppers at two shopping centres, Eastworld and Highton, change the centre they visit from one week to the next.

Transition diagram



Transition matrix

This week

$$\begin{array}{cc} & \begin{array}{cc} E & H \end{array} & \\ T = & \begin{bmatrix} 0.75 & 0.1 \\ 0.25 & 0.9 \end{bmatrix} & \begin{array}{c} E \\ H \end{array} \end{array}$$

Next week

- 75% of shoppers who shop at Eastworld one week will shop at Eastworld the next week
- 25% of shoppers who shop at Eastworld one week will shop at Highton the next week
- 90% of shoppers who shop at Highton one week will shop at Highton the next week
- 10% shoppers who shop at Highton one week will shop at Eastworld the next week

> Transition diagrams show transitions using percentages.
>
> In a transition diagram, all the arrow percentages *from* a single point add up to 100%.
>
> Transition matrices give the percentages from the transition diagram as decimals.
>
> Each value in a transition matrix is from 0 to 1, and each column adds up to 1.

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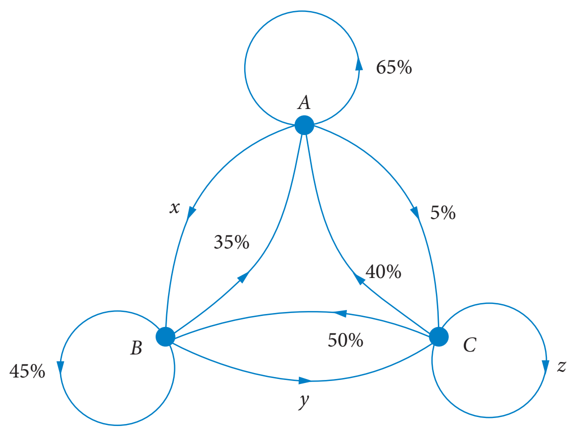
Exam hack

There are different ways of labelling transition matrices, but the current state is always the column entry and the next state is always the row entry.

Worked example 11

a For this transition diagram, find x, y, z and find the matching matrix.



b Find x, y and z for this transition matrix and draw the matching transition diagram.

$$\begin{array}{c c}
\text{Current state} & \\
\begin{array}{ccc} A & B & C \end{array} & \\
\begin{bmatrix} 0.2 & y & 0 \\ x & 0.1 & 0 \\ 0.4 & 0.6 & z \end{bmatrix} & \begin{array}{c} A \\ B \\ C \end{array} \quad \text{Next state}
\end{array}$$

Working

a **i** All the arrow percentages *from* a single point add up to 100%.

For arrows from A: $x + 65 + 5 = 100$

$x = 30\%$

For arrows from B: $45 + y + 35 = 100$

$y = 20\%$

For arrows from C: $50 + 40 + z = 100$

$z = 10\%$

ii Convert the percentages to decimals.

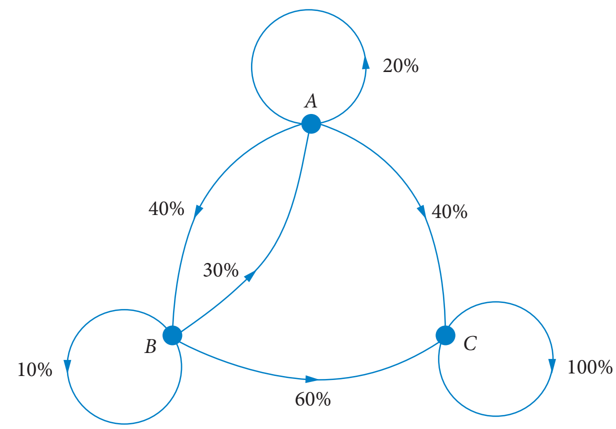
$$\begin{array}{c c}
\text{Current state} & \\
\begin{array}{ccc} A & B & C \end{array} & \\
\begin{bmatrix} 0.65 & 0.35 & 0.4 \\ 0.3 & 0.45 & 0.5 \\ 0.05 & 0.2 & 0.1 \end{bmatrix} & \begin{array}{c} A \\ B \\ C \end{array} \quad \text{Next state}
\end{array}$$

b **i** Use the fact that the columns of a transition matrix must add up to 1.

$0.2 + x + 0.4 = 1$ $\quad x = 0.4$ $\qquad y + 0.1 + 0.6 = 1$ $\quad y = 0.3$ $\qquad 0 + 0 + z = 1$ $\quad z = 1$

ii Convert the decimals to percentages.

Don't show arrow lines for 0%.



Interpreting transition matrices

Worked example 12

A certain species of bird nests in three different locations A, B and C on a peninsula. The birds change their nesting location each year according to the transition matrix T, where

$$\begin{array}{cc} & \text{This year} \\ & \begin{array}{ccc} A & B & C \end{array} \\ T = & \begin{bmatrix} 0.4 & 0.2 & 0.3 \\ 0.25 & 0.55 & 0 \\ 0.35 & 0.25 & 0.7 \end{bmatrix} \begin{array}{l} A \\ B \\ C \end{array} \text{Next year} \end{array}$$

This year there are 400 birds in location A, 660 birds nested in location B and 210 birds nested in location C. How many birds:

a nesting in location B this year would be nesting in location C next year?

b nesting in location C this year would be nesting in location C next year?

c will be in location A next year?

d will be in the same location next year as they were this year?

Working

a Locate the relevant element in the transition matrix and multiply by the number of birds.

0.25 of birds that nest in location B one year will nest in location C the following year.

$0.25 \times 660 = 165$ birds

b Locate the relevant element in the transition matrix and multiply by the number of birds.

0.7 of birds that nest in location C one year will nest in location C the following year.

$0.7 \times 210 = 147$ birds

c Locate the relevant elements in the transition matrix and multiply by the number of birds in each case and add.

$0.4 \times 400 + 0.2 \times 660 + 0.3 \times 210 = 355$

355 birds will be in location A next year.

d Locate the relevant elements in the transition matrix and multiply by the number of birds in each case and add.

$0.4 \times 400 + 0.55 \times 660 + 0.7 \times 210 = 670$

670 birds will be in the same location next year as they were this year.

Worked example 13

Find the transition matrix T for each of the following situations.

a If a train arrives at a certain station on time, the next train has an 85% chance of being on time.

If a train arrives at a certain station late, the next train has a 30% chance of being late.

b Lazlo has the following system for answering multiple-choice questions when there are five options A, B, C, D and E involved.

- He never picks the same option twice in a row.
- If he has picked D, he always chooses C next.
- If he has picked E, he always chooses A next.
- If he has picked A, B or C, he tosses a coin to decide which of the next two in alphabetical order he will choose next (e.g. if he has chosen B, then he tosses a coin to decide whether he should choose C or D next).

Working

a **1** Set up a 2 × 2 matrix using O for 'on time' and L for 'late' with a transition from 'This train' to 'Next train'. Enter the percentages from the question as decimals.

This train

$$T = \begin{array}{c} \begin{array}{cc} O & L \end{array} \\ \begin{bmatrix} 0.85 & \\ & 0.3 \end{bmatrix} \end{array} \begin{array}{c} O \\ L \end{array} \text{ Next train}$$

2 Enter the remaining elements of the matrix by using the fact that the columns of a transition matrix must add up to 1.

$$T = \begin{array}{c} \text{This train} \\ \begin{array}{cc} O & \quad L \end{array} \\ \begin{bmatrix} 0.85 & 0.7 \\ 0.15 & 0.3 \end{bmatrix} \end{array} \begin{array}{c} \\ \\ O \\ L \end{array} \ \text{Next train}$$

b 1 Set up a 5 × 5 matrix for options A, B, C, D and E with a transition from 'This question' to 'Next question'.

$$T = \begin{array}{c} \text{This question} \\ \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} \quad & \quad & \quad & \quad & \quad \\ & & & & \\ & & & & \\ & & & & \\ & & & & \end{bmatrix} \end{array} \begin{array}{c} \\ \\ A \\ B \\ C \\ D \\ E \end{array} \ \text{Next question}$$

2 'Never picking the same option twice in a row' means the leading diagonal consists of all 0s.

Always picking C after a D means it's a 100% certainty, so this element is a 1.

Always picking A after an E means it's a 100% certainty, so this element is a 1.

B and C have a 50% chance of being selected after A, so these elements are 0.5 each.

C and D have a 50% chance of being selected after B, so these elements are 0.5 each.

D and E have a 50% chance of being selected after C, so these elements are 0.5 each.

$$T = \begin{array}{c} \text{This question} \\ \begin{array}{ccccc} A & \ \ B & \ \ C & \ \ D & \ \ E \end{array} \\ \begin{bmatrix} 0 & & & & 1 \\ 0.5 & 0 & & & \\ 0.5 & 0.5 & 0 & 1 & \\ & 0.5 & 0.5 & 0 & \\ & & 0.5 & & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ \\ A \\ B \\ C \\ D \\ E \end{array} \ \text{Next question}$$

3 Use the fact that the columns of a transition matrix must add up to 1 and enter 0s as the remaining elements.

$$T = \begin{array}{c} \text{This question} \\ \begin{array}{ccccc} A & \ \ B & \ \ C & \ \ D & \ \ E \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0.5 & 0 & 0 & 0 & 0 \\ 0.5 & 0.5 & 0 & 1 & 0 \\ 0 & 0.5 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ \\ A \\ B \\ C \\ D \\ E \end{array} \ \text{Next question}$$

Transition diagrams and matrices

Prep 1

State whether each matrix could be a transition matrix.

a $\begin{bmatrix} 0.6 & 0.7 \\ 0.4 & 0.3 \end{bmatrix}$

b $\begin{bmatrix} 0.6 & 0.7 \\ 0.3 & 0.1 \\ 0.1 & 0.2 \end{bmatrix}$

c $\begin{bmatrix} 0.9 & 0.1 \\ 0.4 & 0.9 \end{bmatrix}$

d $\begin{bmatrix} 0.25 & 0.85 & 0.7 \\ 0.45 & 0.1 & 0.15 \\ 0.3 & 0.05 & 0.15 \end{bmatrix}$

e $\begin{bmatrix} 1 & 0.5 & 0 & 0 \\ 0 & 0.5 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

f $\begin{bmatrix} 0.4 & 0 & 0.5 & 1 \\ 0.6 & 1 & 0.5 & 0 \end{bmatrix}$

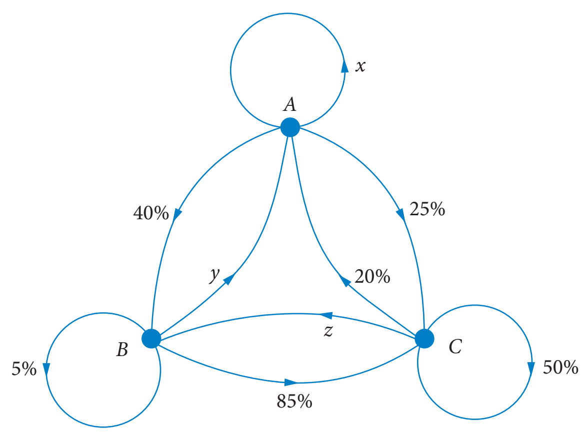
g The 6×6 identity matrix

h A 5×5 permutation matrix

Prep 2

WORKED EXAMPLE 11

a **i** Find x, y and z in the following transition diagram.



ii Find the matching transition matrix.

b Find the missing values x, y and z and draw the matching transition diagram.

$$\begin{array}{c c} \text{Current state} & \\ \begin{array}{ccc} A & B & C \end{array} & \\ \begin{bmatrix} 0.7 & 0.35 & z \\ 0.2 & y & 1 \\ x & 0.15 & 0 \end{bmatrix} & \begin{array}{l} A \\ B \\ C \end{array} \text{ Next state} \end{array}$$

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Prep 3 WORKED EXAMPLE 12

A polling company has established a pattern of voter behaviour in a particular electorate regarding the three major political parties, Labor (L), Coalition (C) and Greens (G). The pattern from one election to the next has been incorporated into the transition matrix T where

$$\begin{array}{c} \\ \\ T = \end{array}\begin{array}{c} \text{This election} \\ \begin{array}{ccc} L & C & G \end{array} \\ \begin{bmatrix} 0.8 & 0.1 & 0.5 \\ 0 & 0.8 & 0 \\ 0.2 & 0.1 & 0.5 \end{bmatrix} \end{array}\begin{array}{l} \\ \\ \begin{array}{l} L \\ C \\ G \end{array} \end{array} \text{Next election}$$

In the most recent election, 32 425 people voted Labor, 26 650 voted Coalition and 10 044 voted Green.

a How many people who voted Labor in this election would vote Greens in the next election?

b How many people who voted Coalition in this election would vote Coalition in the next election?

c How many people will vote Labor in the next election?

d How many people will vote in the next election for the same party they voted for in this election?

Prep 4 WORKED EXAMPLE 13

Find the transition matrix T for each of the following situations.

a The Bureau of Meteorology has established that in a particular town, if it rains on any day, there is a 65% chance of it raining the next day, and if it is dry on any day, there is a 90% chance it will be dry the next day.

b Valeria has the following system for answering multiple-choice questions when there are five options A, B, C, D and E involved.

- She never picks the same option twice in a row.
- If she has picked A, she always chooses E next.
- If she has picked B, she always chooses D next.
- If she has picked C, she tosses a coin to decide whether to pick D or E next.
- If she has picked D, she tosses a coin to decide whether to pick A or E next.
- If she has picked E, she tosses a coin to decide whether to pick A or B next.

EXAM PRACTICE 9.4

Transition diagrams and matrices

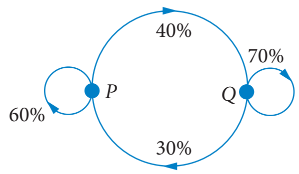
Question 1

T is a transition matrix, where

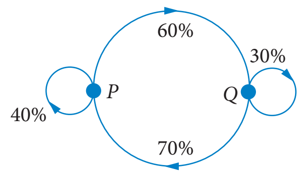
$$\begin{array}{cc} & \text{From} \\ & \begin{array}{cc} P & Q \end{array} \\ T = & \begin{bmatrix} 0.6 & 0.7 \\ 0.4 & 0.3 \end{bmatrix} \begin{array}{c} P \\ Q \end{array} \ \text{To} \end{array}$$

An equivalent transition diagram, with proportions expressed as percentages, is

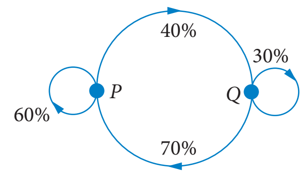
A



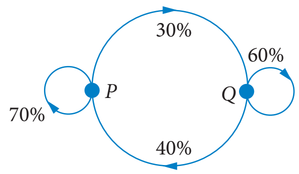
B



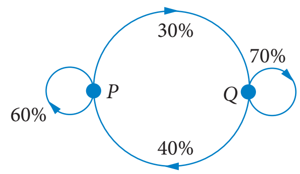
C



D



E



[VCAA 2009 1MQ6]

Question 2

In a country town, people only have the choice of doing their food shopping at a store called Marks (M) or at a newly opened store called Foodies (F). A market researcher predicts that

- of those who do their food shopping at Marks this week, 70% will shop at Marks next week and 30% will shop at Foodies
- of those who do their food shopping at Foodies this week, 90% will shop at Foodies next week and 10% will shop at Marks.

A transition matrix that can be used to represent this situation is

A

$$\begin{array}{cc} & \text{This week} \\ & \begin{array}{cc} M & F \end{array} \\ T = & \begin{bmatrix} 0.7 & 0.9 \\ 0.3 & 0.1 \end{bmatrix} \begin{array}{c} M \\ F \end{array} \ \text{Next week} \end{array}$$

B

$$\begin{array}{cc} & \text{This week} \\ & \begin{array}{cc} M & F \end{array} \\ T = & \begin{bmatrix} 0.7 & 0.1 \\ 0.3 & 0.9 \end{bmatrix} \begin{array}{c} M \\ F \end{array} \ \text{Next week} \end{array}$$

C

$$\begin{array}{cc} & \text{This week} \\ & \begin{array}{cc} M & F \end{array} \\ T = & \begin{bmatrix} 0.3 & 0.7 \\ 0.9 & 0.1 \end{bmatrix} \begin{array}{c} M \\ F \end{array} \ \text{Next week} \end{array}$$

D

$$\begin{array}{cc} & \text{This week} \\ & \begin{array}{cc} M & F \end{array} \\ T = & \begin{bmatrix} 0.7 & 0.3 \\ 0.1 & 0.9 \end{bmatrix} \begin{array}{c} M \\ F \end{array} \ \text{Next week} \end{array}$$

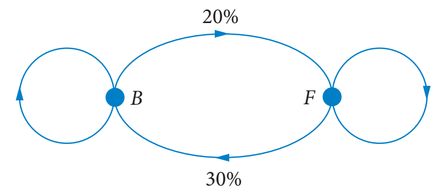
E

$$\begin{array}{cc} & \text{This week} \\ & \begin{array}{cc} M & F \end{array} \\ T = & \begin{bmatrix} 0.3 & 0.9 \\ 0.7 & 0.1 \end{bmatrix} \begin{array}{c} M \\ F \end{array} \ \text{Next week} \end{array}$$

[VCAA 2009 1MQ8]

Question 3

There are two fast-food shops in a country town: Big Burgers (B) and Fast Fries (F). Every week, each family in the town will purchase takeaway food from one of these shops. The transition diagram shows the way families in the town change their preferences for fast food from one week to the next.



A transition matrix that provides the same information as the transition diagram is

A $$\begin{array}{c} \text{From} \\ \begin{array}{cc} B & F \end{array} \\ \begin{bmatrix} 0 & 0.3 \\ 0.2 & 0 \end{bmatrix} \end{array} \begin{array}{c} B \\ F \end{array} \ \text{To}$$

B $$\begin{array}{c} \text{From} \\ \begin{array}{cc} B & F \end{array} \\ \begin{bmatrix} 1.2 & -0.3 \\ -0.2 & 1.3 \end{bmatrix} \end{array} \begin{array}{c} B \\ F \end{array} \ \text{To}$$

C $$\begin{array}{c} \text{From} \\ \begin{array}{cc} B & F \end{array} \\ \begin{bmatrix} 0.7 & 0.3 \\ 0.2 & 0.8 \end{bmatrix} \end{array} \begin{array}{c} B \\ F \end{array} \ \text{To}$$

D $$\begin{array}{c} \text{From} \\ \begin{array}{cc} B & F \end{array} \\ \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} \end{array} \begin{array}{c} B \\ F \end{array} \ \text{To}$$

E $$\begin{array}{c} \text{From} \\ \begin{array}{cc} B & F \end{array} \\ \begin{bmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{bmatrix} \end{array} \begin{array}{c} B \\ F \end{array} \ \text{To}$$

[VCAA 2012 1MQ5]

Question 4

Australians go on holidays either within Australia or overseas. Market research shows that

- 95% of those who had their last holiday in Australia said that their next holiday would be in Australia
- 20% of those who had their last holiday overseas said that their next holiday would also be overseas.

A transition matrix that could be used to describe this situation is

A $\begin{bmatrix} 0.95 \\ 0.20 \end{bmatrix}$

B $\begin{bmatrix} 0.95 \\ 0.05 \end{bmatrix} + \begin{bmatrix} 0.20 \\ 0.80 \end{bmatrix}$

C $\begin{bmatrix} 0.95 & 0.95 \\ 0.20 & 0.20 \end{bmatrix}$

D $\begin{bmatrix} 0.95 & 0.20 \\ 0.05 & 0.80 \end{bmatrix}$

E $\begin{bmatrix} 0.95 & 0.80 \\ 0.05 & 0.20 \end{bmatrix}$

[VCAA 2006 1MQ8]

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Question 5

Each year, a family always goes on its holiday to one of three places: Portland (P), Quambatook (Q) or Rochester (R). They never go to the same place two years in a row. For example, if they went to Portland one year, they would not go to Portland the next year; they would go to Quambatook or Rochester instead.

A transition matrix that can be used to model this situation is

A $$\text{Next year}\quad \begin{array}{c c} & \begin{array}{ccc} & \text{This year} & \\ P & Q & R \end{array} \\ \begin{array}{c} P \\ Q \\ R \end{array} & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

B $$\text{Next year}\quad \begin{array}{c c} & \begin{array}{ccc} & \text{This year} & \\ P & Q & R \end{array} \\ \begin{array}{c} P \\ Q \\ R \end{array} & \begin{bmatrix} 0 & 1 & 0 \\ 0.9 & 0 & 0 \\ 0.1 & 0 & 1 \end{bmatrix} \end{array}$$

C $$\text{Next year}\quad \begin{array}{c c} & \begin{array}{ccc} & \text{This year} & \\ P & Q & R \end{array} \\ \begin{array}{c} P \\ Q \\ R \end{array} & \begin{bmatrix} 0 & 0.3 & 0.8 \\ 0.3 & 0.4 & 0.2 \\ 0.7 & 0.3 & 0 \end{bmatrix} \end{array}$$

D $$\text{Next year}\quad \begin{array}{c c} & \begin{array}{ccc} & \text{This year} & \\ P & Q & R \end{array} \\ \begin{array}{c} P \\ Q \\ R \end{array} & \begin{bmatrix} 0 & 0.2 & 0 \\ 0.3 & 0 & 0.8 \\ 0.5 & 0.6 & 0 \end{bmatrix} \end{array}$$

E $$\text{Next year}\quad \begin{array}{c c} & \begin{array}{ccc} & \text{This year} & \\ P & Q & R \end{array} \\ \begin{array}{c} P \\ Q \\ R \end{array} & \begin{bmatrix} 0 & 0 & 0.1 \\ 0.5 & 0 & 0.9 \\ 0.5 & 1 & 0 \end{bmatrix} \end{array}$$

[VCAA 2007 1MQ7]

9.4

Question 6

Kerry sat for a multiple-choice test consisting of six questions. Each question had four alternative answers, A, B, C or D. He selected D for his answer to the first question. He then determined the answers to the remaining questions by following the transition matrix.

$$\text{Next question}\quad \begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \text{This question} \\ \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

The answers that he gave to the six test questions, starting with D, were

A

Question	1	2	3	4	5	6
Answer	D	B	C	A	D	B

B

Question	1	2	3	4	5	6
Answer	D	B	C	A	A	A

C

Question	1	2	3	4	5	6
Answer	D	B	C	A	C	A

D

Question	1	2	3	4	5	6
Answer	D	A	C	B	D	D

E

Question	1	2	3	4	5	6
Answer	D	C	B	A	B	C

[VCAA 2007 1MQ8]

Question 7

Robbie completed a test of four multiple-choice questions. Each question had four alternatives, A, B, C or D. Robbie randomly guessed the answer to the first question. He then determined his answers to the remaining three questions by following the transition matrix on the right.

$$\begin{array}{c} \text{This question} \\ \begin{array}{cccc} A & B & C & D \end{array} \end{array}$$
$$T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{array}{c} A \\ B \\ C \\ D \end{array} \quad \text{Next question}$$

Which of the following statements is **true**?

A It is impossible for Robbie to give the same answer to all four questions.

B Robbie would always give the same answer to the first and fourth questions.

C Robbie would always give the same answer to the second and third questions.

D If Robbie answered A for question one, he would have answered B for question two.

E It is possible that Robbie gave the same answer to exactly three of the four questions.

[VCAA 2010 1MQ9]

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Transition matrices and recurrence relations

9.5

The state matrix recurrence relation

The power of transition matrices is that they can be used to make predictions. They allow you to deal with situations where multiple probabilities are involved.

Since problems associated with transition matrices involve regular set changes occurring cumulatively over time, we can use recurrence relations to solve them. The principle is the same as the recurrence relations covered in chapters 6 and 7.

For example, in the situation where

- 75% of shoppers who shop at Eastworld one week will shop at Eastworld the next week
- 90% of shoppers who shop at Highton one week will shop at Highton the next week,

we had the following transition matrix:

$$\begin{array}{cccc} & \text{This week} & & \\ & \begin{array}{cc} E & H \end{array} & & \\ T = & \begin{bmatrix} 0.75 & 0.1 \\ 0.25 & 0.9 \end{bmatrix} & \begin{array}{c} E \\ H \end{array} & \text{Next week} \end{array}$$

Suppose that we know that this week 23 000 shoppers go to Eastworld and 12 000 go to Highton. We can multiply the column matrix containing this information by the transition matrix to predict how many shoppers will go to each shopping centre next week.

$\begin{bmatrix} 0.75 & 0.1 \\ 0.25 & 0.9 \end{bmatrix}\begin{bmatrix} 23\,000 \\ 12\,000 \end{bmatrix} = \begin{bmatrix} 18\,450 \\ 16\,550 \end{bmatrix}$, so next week 18 450 shoppers are predicted to go to Eastworld and 16 550 shoppers are predicted to go to Highton.

To predict how many shoppers will go to each centre the week after that, we would multiply the *new* shopper numbers by the transition matrix.

$\begin{bmatrix} 0.75 & 0.1 \\ 0.25 & 0.9 \end{bmatrix}\begin{bmatrix} 18\,450 \\ 16\,550 \end{bmatrix} = \begin{bmatrix} 15\,493 \\ 19\,508 \end{bmatrix}$ rounding to the nearest person, so in the next week 15 493 shoppers are predicted to go to Eastworld and 19 508 shoppers are predicted to go to Highton.

To find the number of shoppers at the two centres each week, we use the following recurrence relation:

$$S_0 = \begin{bmatrix} 23\,000 \\ 12\,000 \end{bmatrix}, S_{n+1} = \begin{bmatrix} 0.75 & 0.1 \\ 0.25 & 0.9 \end{bmatrix} S_n$$

S_0 is called the **initial state matrix** and S_n is a **state matrix**.

The recurrence relation that generates a sequence of state matrices from a transition matrix is given by

S_0 = initial state matrix, $S_{n+1} = TS_n$

where T is a transition matrix and S_n is a column state matrix.

Using CAS — Generating matrix sequences through recursive computation

A market research company has analysed the movement of shoppers from Eastworld (E) and Highton (H) shopping centres and have established that the movement occurs according to this transition matrix.

$$\begin{array}{cc} & \text{This week} \\ & \begin{array}{cc} E & H \end{array} \\ T = & \begin{bmatrix} 0.75 & 0.1 \\ 0.25 & 0.9 \end{bmatrix} \begin{array}{c} E \\ H \end{array} \text{ Next week} \end{array}$$

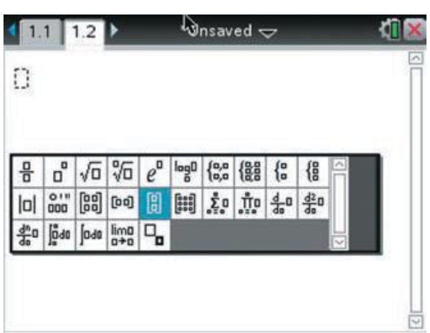
This week, 23 000 shoppers went to Eastworld and 12 000 went to Highton. Use the recursive computation function on your CAS/calculator to find how many shoppers go to each shopping centre after 4 weeks.

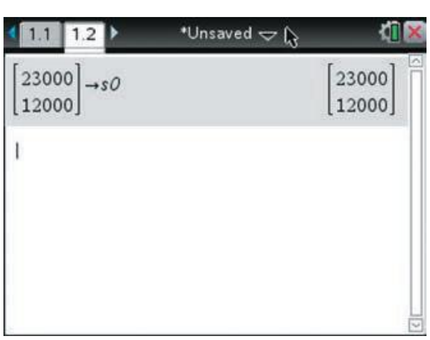
TI-NSPIRE CAS

STEP 1

Use the math expression template key ( ) to call up the displays and choose a 2-row matrix, press enter.

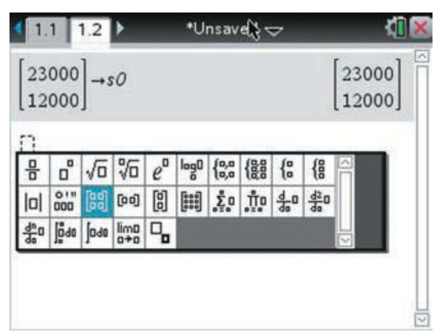
Enter the initial values 23 000 and 12 000 and use ctrl sto→ var to store this as $s0$.



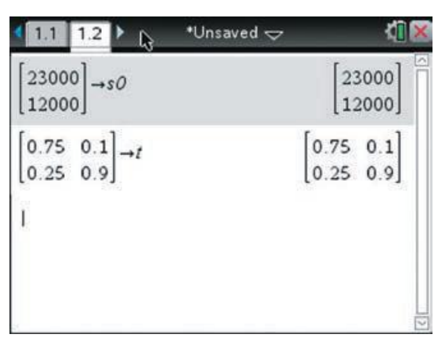


STEP 2

Use the button to call up the displays and choose a 2 × 2 matrix, press enter.

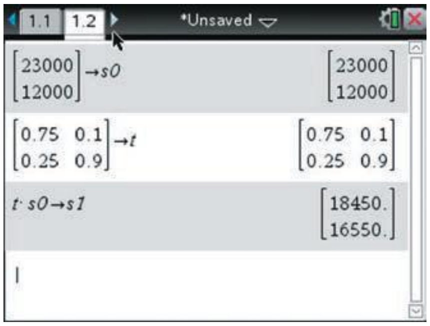


Enter the initial values transitional matrix values 0.75, 0.1, 0.25 and 0.9 and use ctrl sto→ var to store this as t.



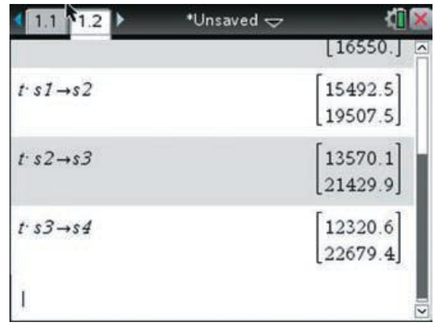
STEP 3

Calculate the number of shoppers going to Eastworld and Highton the next week by multiplying *T* and *s*0. Define this as *s*1.



STEP 4

Keep finding the number of shoppers per week until you reach *s*4.



STEP 5

Write the answer.

12 321 shoppers go to Eastworld and 22 679 shoppers go to Highton after 4 weeks.

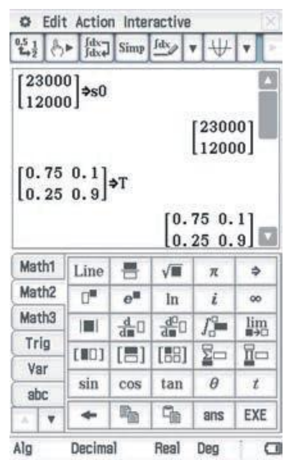
CLASSPAD

STEP 1

In the main menu Main $\sqrt{\alpha}$, enter the shopper matrix by tapping Keyboard, Math2 and selecting [▭]. Type the values into the 2 × 1 column matrix that appears.

Define this column matrix as s0 using ⇒ and abc.

Do the same for the transition matrix, but use [▭▭] for a 2 × 2 matrix.

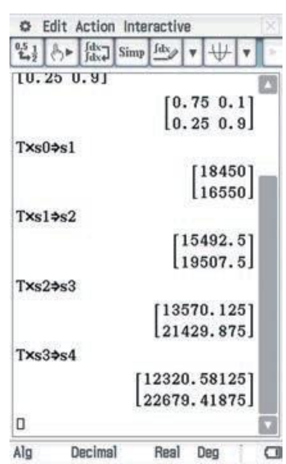


STEP 2

Calculate the number of shoppers going to Eastworld and Highton the next week by multiplying T and s0. Define this as s1.

Keep finding the number of shoppers per week until you reach s4.

Because the abc instead of Var is being used here to get s0, s1 and so on, the times sign × is needed for multiplying. When using Var it is not required.



STEP 3

Round the numbers up and write the answer.

12 321 shoppers go to Eastworld and 22 679 shoppers go to Highton after 4 weeks.

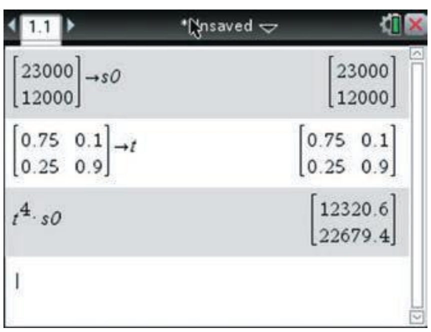
The state matrix rule

We can use the state matrix recurrence relation to find a state matrix rule that makes calculations easier.

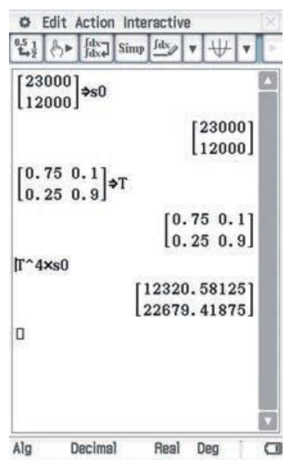
	Using the state matrix recurrence relation
Initial state:	S_0
After one transition:	$S_1 = T \times S_0$
After two transitions:	$S_2 = T \times S_1 = T \times (T \times S_0) = T^2 \times S_0$
After three transitions:	$S_3 = T \times S_2 = T \times (T^2 \times S_0) = T^3 \times S_0$
After four transitions:	$S_4 = T \times S_3 = T \times (T^3 \times S_0) = T^4 \times S_0$
…	…
After n transitions:	$S_n = T^n \times S_0$

Using this state matrix rule, we can reduce and simplify CAS calculations.

TI-NSPIRE CAS



CLASSPAD



The rule for finding S_n, the state matrix after n transitions, is

$S_n = T^n \times S_0$

where T is a transition matrix, and S_0 is the initial state matrix.

Worked example 14

A fleet of trucks starts each day at one of two depots, A or B. By the end of the day, the trucks end up at either of the two depots according to the following transition matrix.

$$\begin{array}{ccc} & \text{Start of this day} & \\ & \begin{array}{cc} A & B \end{array} & \\ T = & \begin{bmatrix} 0.95 & 0.15 \\ 0.05 & 0.85 \end{bmatrix} & \begin{array}{c} A \\ B \end{array} \quad \text{Start of next day} \end{array}$$

At the start of a particular day, there are 40 trucks at depot A and 100 trucks at depot B.

a How many trucks are at each depot after 6 days?

b How many trucks are at each depot after 10 days?

Working

a **1** Use your CAS/calculator and the rule $S_n = T^n \times S_0$.

$S_0 = \begin{bmatrix} 40 \\ 100 \end{bmatrix}, n = 6$

$S_6 = T^6 \times S_0$

$= \begin{bmatrix} 0.95 & 0.15 \\ 0.05 & 0.85 \end{bmatrix}^6 \times \begin{bmatrix} 40 \\ 100 \end{bmatrix} \approx \begin{bmatrix} 88 \\ 52 \end{bmatrix}$

2 Write the answer.

After 6 days, 88 trucks will be at depot A and 52 trucks will be at depot B.

b **1** Use your CAS/calculator and the rule $S_n = T^n \times S_0$.

$S_0 = \begin{bmatrix} 40 \\ 100 \end{bmatrix}, n = 10$

$S_{10} = T^{10} \times S_0$

$= \begin{bmatrix} 0.95 & 0.15 \\ 0.05 & 0.85 \end{bmatrix}^{10} \times \begin{bmatrix} 40 \\ 100 \end{bmatrix} \approx \begin{bmatrix} 98 \\ 42 \end{bmatrix}$

2 Write the answer.

After 10 days, 98 trucks will be at depot A and 42 trucks will be at depot B.

Long term trends

It's often possible to make statements about long term trends directly from transition matrices. For example, if the transition matrix for the movement of trucks from one location to another is

$$\begin{array}{cc} & \text{This day} \\ & \begin{array}{cc} A & B \end{array} \\ T = & \begin{bmatrix} 0.6 & 0 \\ 0.4 & 1 \end{bmatrix} \begin{array}{c} A \\ B \end{array} \text{ Next day} \end{array}$$

then we can see that 0% are moving from B to A, and that 100% are staying at B each time. Since 40% are moving from A to B each time, then we can see that in the long term there will be no trucks at A and all of them will be at B.

If the transition matrix is

$$\begin{array}{cc} & \text{This day} \\ & \begin{array}{cc} A & B \end{array} \\ T = & \begin{bmatrix} 0.6 & 0.7 \\ 0.4 & 0.3 \end{bmatrix} \begin{array}{c} A \\ B \end{array} \text{ Next day} \end{array}$$

then we can see that the percentage of those moving from B to A (70%) is greater than the percentage of those moving from A to B (40%), which means we can say that in the long term, more will be moving to A than to B.

For a 2×2 transition matrix T where

$$\begin{array}{cc} & \text{Current state} \\ & \begin{array}{cc} A & B \end{array} \\ T = & \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{array}{c} A \\ B \end{array} \text{ Next state} \end{array}$$

- if $a_{21} > a_{12}$, in the long term more will transition to B than to A.
- if $a_{12} > a_{21}$, in the long term more will transition to A than to B.

The equilibrium state matrix

Although transition matrices involve regular change and movement, often in the long term the state matrix gets to a point where it isn't changing from one transition to the next. So in the truck depot example, although some trucks continue to swap between depots each day, the number of trucks in each depot at the end of each day settles on a fixed number.

By taking higher and higher powers of the transition matrix T, we can get to a point where there is no longer any noticeable change from one state matrix to the next. This is called the **equilibrium state matrix** or **steady-state matrix**.

In practice, if we get the same state matrix for two consecutive powers of T in a row, we can say we have found the equilibrium state matrix. Usually testing T^n for $n = 29$ and 30 is enough to find a close approximation to the equilibrium state matrix (although with a CAS/calculator it's just as easy to calculate even higher powers of T^n).

A transition matrix T will give an equilibrium state solution if T has no zero elements or if T^2 (or *any* higher powers of T) has no zero elements.

So, for example

$T = \begin{bmatrix} 0.4 & 0.3 \\ 0.6 & 0.7 \end{bmatrix}$ will give an equilibrium state solution because it has no zero elements.

$T = \begin{bmatrix} 0 & 1 \\ 0.6 & 0.4 \end{bmatrix}$ will give an equilibrium state solution because $T^2 = \begin{bmatrix} 0.6 & 0.4 \\ 0.24 & 0.76 \end{bmatrix}$ has no zero elements.

9.5

Exam hack

- If there is an equilibrium state, and the initial state matrix doesn't contain any zeros, then none of the state matrices will contain zeros.
- Rows of zeros in consecutive high powers of the transition matrix, T, that give an equilibrium state solution tell us important information about long term trends. For example, if

$$T^{29} = T^{30} = \begin{array}{c} \\ \\ \\ \\ \end{array}\begin{array}{cc} \begin{array}{ccc} A & B & C \end{array} & \\ \begin{bmatrix} 0.3 & 0.6 & 0.1 \\ 0 & 0 & 0 \\ 0.7 & 0.4 & 0.9 \end{bmatrix} & \begin{array}{c} A \\ B \\ C \end{array} \end{array}$$

, then we know that B doesn't occur in the long term.

iStockPhoto/KingWu

Worked example 15

This is the transition matrix from Worked example 14.

At the start of 'this' day there were 40 trucks at depot A and 100 trucks at depot B.

$$\begin{array}{cc} & \text{Start of this day} \\ & \begin{array}{cc} A & B \end{array} \\ T = & \begin{bmatrix} 0.95 & 0.15 \\ 0.05 & 0.85 \end{bmatrix} \begin{array}{c} A \\ B \end{array} \text{ Start of next day} \end{array}$$

a In the long term which depot will have the more trucks?

b How do we know there will be an equilibrium state solution?

c How may trucks are at each depot after 29 days?

d How may trucks are at each depot after 30 days?

e Use your answers to estimate the equilibrium state matrix. What is the long term trend?

Working

a Check whether $a_{21} > a_{12}$ or $a_{12} > a_{21}$.

0.15 > 0.05, so in the long term more trucks will be at depot A.

b Check to see if there are any zero elements in the transition matrix.

Since T has no zero elements, there will be an equilibrium state solution.

c **1** Use your CAS/calculator and the rule $S_n = T^n \times S_0$.

$S_0 = \begin{bmatrix} 40 \\ 100 \end{bmatrix}$, $n = 29$

$$S_{29} = T^{29} \times S_0$$
$$= \begin{bmatrix} 0.95 & 0.15 \\ 0.05 & 0.85 \end{bmatrix}^{29} \times \begin{bmatrix} 40 \\ 100 \end{bmatrix} \approx \begin{bmatrix} 105 \\ 35 \end{bmatrix}$$

2 Write the answer.

After 29 days, 105 trucks will be at depot A and 35 trucks will be at depot B.

d **1** Use your CAS/calculator and the rule $S_n = T^n \times S_0$.

$S_0 = \begin{bmatrix} 40 \\ 100 \end{bmatrix}$, $n = 30$

$$S_{30} = T^{30} \times S_0$$
$$= \begin{bmatrix} 0.95 & 0.15 \\ 0.05 & 0.85 \end{bmatrix}^{30} \times \begin{bmatrix} 40 \\ 100 \end{bmatrix} \approx \begin{bmatrix} 105 \\ 35 \end{bmatrix}$$

2 Write the answer.

After 30 days, 105 trucks will be at depot A and 35 trucks will be at depot B.

e Use the fact that we have found two consecutive transitions where the state matrix hasn't changed.

An estimate of the equilibrium state matrix is $\begin{bmatrix} 105 \\ 35 \end{bmatrix}$.

In the long term, 105 trucks will be at depot A and 35 trucks will be at depot B.

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Transition matrices and recurrence relations

Prep 1

USING CAS: FIVE-NUMBER SUMMARY

Two fast food outlets, FushNChups (F) and BurgerHQ (B) are situated in a small town. A market research company has analysed the fast food buying patterns of people in the town and have established that they behave according to this transition matrix.

$$\begin{array}{c} \text{This week} \\ \begin{array}{cc} F & B \end{array} \end{array}$$

$$T = \begin{array}{c} \begin{bmatrix} 0.6 & 0.25 \\ 0.4 & 0.75 \end{bmatrix} \end{array} \begin{array}{c} F \\ B \end{array} \quad \text{Next week}$$

In the current week, 64 people bought from FushNChups and 82 bought from BurgerHQ. Use the recursive computation function on your CAS/calculator to find how many people go to each fast food outlet after 5 weeks.

Prep 2

WORKED EXAMPLE 14

A number of train carriages start each day at one of two depots, A or B. By the end of the day, the train carriages end up at either of the two depots according to this transition matrix.

$$\begin{array}{c} \text{Start of this day} \\ \begin{array}{cc} A & B \end{array} \end{array}$$

$$T = \begin{bmatrix} 0.8 & 0.15 \\ 0.2 & 0.85 \end{bmatrix} \begin{array}{c} A \\ B \end{array} \quad \text{Start of next day}$$

At the start of today, there are 81 train carriages at depot A and 49 train carriages at depot B.

a How many train carriages are at each depot after 5 days?

b How many train carriages are at each depot after 12 days?

Prep 3

WORKED EXAMPLE 15

Answer these questions for the situation described in Prep 2.

a In the long term, which depot will have the more train carriages?

b How do we know there will be an equilibrium state solution?

c How may train carriages are at each depot after 29 days?

d How may train carriages are at each depot after 30 days?

e Use your answers to estimate the equilibrium state matrix. What is the long term trend?

EXAM PRACTICE 9.5

Transition matrices and recurrence relations

Question 1

In a country town, people only have the choice of doing their food shopping at a store called Marks (M) or at a newly opened store called Foodies (F). In the first week that Foodies opened, only 300 of the town's 800 shoppers did their food shopping at Marks. The remainder did their food shopping at Foodies.

A state matrix S_0 that can be used to represent this situation is

A $S_0 = \begin{array}{c} \begin{bmatrix} 300 \\ 800 \end{bmatrix} \end{array} \begin{array}{c} M \\ F \end{array}$ **B** $S_0 = \begin{bmatrix} 500 \\ 300 \end{bmatrix} \begin{array}{c} M \\ F \end{array}$ **C** $S_0 = \begin{bmatrix} 800 \\ 300 \end{bmatrix} \begin{array}{c} M \\ F \end{array}$

D $S_0 = \begin{bmatrix} 300 \\ 500 \end{bmatrix} \begin{array}{c} M \\ F \end{array}$ **E** $S_0 = \begin{bmatrix} 800 \\ 500 \end{bmatrix} \begin{array}{c} M \\ F \end{array}$

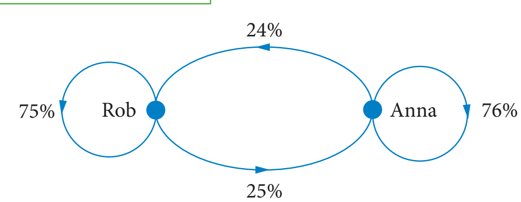
[VCAA 2009 1MQ7]

Use the following information to answer Questions 2 & 3.

Two politicians, Rob and Anna, are the only candidates for a forthcoming election. At the beginning of the election campaign, people were asked for whom they planned to vote. The numbers were as follows.

Candidate	Number of people who plan to vote for the candidate
Rob	5692
Anna	3450

During the election campaign, it is expected that people may change the candidate that they plan to vote for each week according to the transition diagram.



Question 2

The total number of people who are expected to change the candidate that they plan to vote for one week after the election campaign begins is

A 828 **B** 1423 **C** 2251 **D** 4269 **E** 6891

[VCAA 2011 1MQ5]

Question 3

The election campaign will run for ten weeks. If people continue to follow this pattern of changing the candidate they plan to vote for, the expected winner after ten weeks will be

A Rob by about 50 votes. **B** Rob by about 100 votes. **C** Rob by fewer than 10 votes.

D Anna by about 100 votes. **E** Anna by about 200 votes.

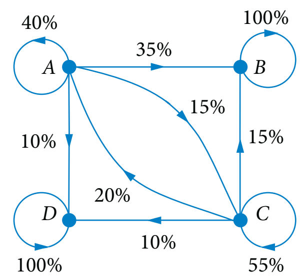
[VCAA 2011 1MQ6]

Use the following information to answer Questions 4–6.

A large population of mutton birds migrates each year to a remote island to nest and breed. There are four nesting sites on the island, A, B, C and D.

Researchers suggest that the following transition matrix can be used to predict the number of mutton birds nesting at each of the four sites in subsequent years. An equivalent transition diagram is also given.

$$T = \begin{array}{c} \text{This year} \\ \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0.4 & 0 & 0.2 & 0 \\ 0.35 & 1 & 0.15 & 0 \\ 0.15 & 0 & 0.55 & 0 \\ 0.1 & 0 & 0.1 & 1 \end{bmatrix} \end{array} \begin{array}{c} A \\ B \\ C \\ D \end{array} \quad \text{Next year}$$



Question 4

Two thousand eight hundred mutton birds nest at site C in 2008.

Of these 2800 mutton birds, the number that nest at site A in 2009 is predicted to be

A 560 **B** 980 **C** 1680

D 2800 **E** 3360

[VCAA 2008 1MQ7]

Question 5

This transition matrix predicts that, in the long term, the mutton birds will

A nest only at site A.

B nest only at site B.

C nest only at sites A and C.

D nest only at sites B and D.

E continue to nest at all four sites.

[VCAA 2008 1MQ8]

Question 6

Six thousand mutton birds nest at site B in 2008. Assume that an equal number of mutton birds nested at each of the four sites in 2007. The same transition matrix applies.

The total number of mutton birds that nested on the island in 2007 was

A 6000 **B** 8000 **C** 12 000

D 16 000 **E** 24 000

[VCAA 2008 1MQ9]

Question 7

$T = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}$ is a transition matrix and $S_3 = \begin{bmatrix} 1150 \\ 850 \end{bmatrix}$ is a state matrix.

If $S_3 = TS_2$, then S_2 equals

A $\begin{bmatrix} 1000 \\ 1000 \end{bmatrix}$ **B** $\begin{bmatrix} 1090 \\ 940 \end{bmatrix}$ **C** $\begin{bmatrix} 1100 \\ 900 \end{bmatrix}$

D $\begin{bmatrix} 1150 \\ 850 \end{bmatrix}$ **E** $\begin{bmatrix} 1175 \\ 825 \end{bmatrix}$

[VCAA 2009 1MQ9]

Question 8

A coffee shop sells three types of coffee, Brazilian (B), Italian (I) and Kenyan (K). The regular customers buy one cup of coffee each per day and choose the type of coffee they buy according to the following transition matrix, T.

$$\begin{array}{cc} & \text{Choose today} \\ & \begin{array}{ccc} B & I & K \end{array} \\ T = & \begin{bmatrix} 0.8 & 0.1 & 0.1 \\ 0 & 0.8 & 0.1 \\ 0.2 & 0.1 & 0.8 \end{bmatrix} \begin{array}{c} B \\ I \\ K \end{array} \text{ Choose tomorrow} \end{array}$$

On a particular day, 84 customers bought Brazilian coffee, 96 bought Italian coffee and 81 bought Kenyan coffee. If these same customers continue to buy one cup of coffee each per day, the number of these customers who are expected to buy each of the three types of coffee in the long term is

A Brazilian 85
Italian 85
Kenyan 91

B Brazilian 87
Italian 58
Kenyan 116

C Brazilian 88
Italian 86
Kenyan 87

D Brazilian 89
Italian 89
Kenyan 83

E Brazilian 116
Italian 89
Kenyan 58

[VCAA 2013 1MQ3]

Question 9

A new colony of several hundred birds is established on a remote island. The birds can feed at two locations, A and B. The birds are expected to change feeding locations each day according to the transition matrix

$$\begin{array}{cc} & \text{This day} \\ & \begin{array}{cc} A & B \end{array} \\ T = & \begin{bmatrix} 0.4 & 0.3 \\ 0.6 & 0.7 \end{bmatrix} \begin{array}{c} A \\ B \end{array} \text{ Next day} \end{array}$$

In the beginning, approximately equal numbers of birds feed at each site each day. Which of the following statements is **not** true?

A 70% of the birds that feed at B on a given day will feed at B the next day.

B 60% of the birds that feed at A on a given day will feed at B the next day.

C In the long term, more birds will feed at B than at A.

D The number of birds that change feeding locations each day will decrease over time to zero.

E In the long term, some birds will always be found feeding at each location.

[VCAA 2010 1MQ7]

Question 10

Each night, a large group of mountain goats sleep at one of two locations, A or B. On the first night, equal numbers of goats are observed to be sleeping at each location. From night to night, goats change their sleeping locations according to a transition matrix T. It is expected that, in the long term, more goats will sleep at location A than at location B. Assuming that the total number of goats remains constant, a transition matrix T that would predict this outcome is

A $$\begin{array}{cc} & \text{This night} \\ & \begin{array}{cc} A & B \end{array} \\ T = & \begin{bmatrix} 0.8 & 0.4 \\ 0.2 & 0.6 \end{bmatrix} \begin{array}{c} A \\ B \end{array} \text{ Next night} \end{array}$$

B $$\begin{array}{cc} & \text{This night} \\ & \begin{array}{cc} A & B \end{array} \\ T = & \begin{bmatrix} 0.7 & 0.1 \\ 0.3 & 0.9 \end{bmatrix} \begin{array}{c} A \\ B \end{array} \text{ Next night} \end{array}$$

C $$\begin{array}{cc} & \text{This night} \\ & \begin{array}{cc} A & B \end{array} \\ T = & \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix} \begin{array}{c} A \\ B \end{array} \text{ Next night} \end{array}$$

D $$\begin{array}{cc} & \text{This night} \\ & \begin{array}{cc} A & B \end{array} \\ T = & \begin{bmatrix} 0.6 & 0.2 \\ 0.4 & 0.8 \end{bmatrix} \begin{array}{c} A \\ B \end{array} \text{ Next night} \end{array}$$

E $$\begin{array}{cc} & \text{This night} \\ & \begin{array}{cc} A & B \end{array} \\ T = & \begin{bmatrix} 0.1 & 0.8 \\ 0.9 & 0.2 \end{bmatrix} \begin{array}{c} A \\ B \end{array} \text{ Next night} \end{array}$$

[VCAA 2011 1MQ7]

Question 11

There are 30 children in a Year 6 class. Each week, every child participates in one of three activities: cycling (C), orienteering (O) or swimming (S). The activities that the children select each week change according to the transition matrix.

$$\begin{array}{cc} & \text{This week} \\ & \begin{array}{ccc} C & O & S \end{array} \\ T = & \begin{bmatrix} 0.5 & 0.3 & 0.3 \\ 0.1 & 0.6 & 0.2 \\ 0.4 & 0.1 & 0.5 \end{bmatrix} \end{array} \begin{array}{c} \\ \\ C \\ O \\ S \end{array} \quad \text{Next week}$$

From the transition matrix it can be concluded that

A in the first week of the program, ten children do cycling, ten children do orienteering and ten children do swimming.

B at least 50% of the children do not change their activities from the first week to the second week.

C in the long term, all of the children will choose the same activity.

D orienteering is the most popular activity in the first week.

E 50% of the children will do swimming each week.

[VCAA 2012 1MQ8]

Question 12

A colony of fruit bats feeds nightly at three different locations, A, B and C. Initially, the number of bats from the colony feeding at each of the locations was as follows.

Location	Number of bats
A	1568
B	1105
C	894

$$\begin{array}{cc} & \text{This night} \\ & \begin{array}{ccc} A & B & C \end{array} \\ T = & \begin{bmatrix} 0.8 & 0.1 & 0.2 \\ 0.1 & 0.6 & 0.1 \\ 0.1 & 0.3 & 0.7 \end{bmatrix} \end{array} \begin{array}{c} \\ \\ A \\ B \\ C \end{array} \quad \text{Next night}$$

The bats change feeding locations according to the transition matrix T. If this pattern of feeding continues, the number of bats feeding at location A in the long term will be closest to

A 1254 **B** 1543 **C** 1568 **D** 1605 **E** 1725

[VCAA 2007 1MQ6]

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Question 13

A large population of birds lives on a remote island. Every night, each bird settles at either location A or location B. It was found on the first night that the number of birds at each location was the same. On each subsequent night, a percentage of birds changed the location at which they settled. This movement of birds is described by the transition matrix

$$\begin{array}{cc} & \begin{array}{cc} A & B \end{array} \\ \begin{array}{c} A \\ B \end{array} & \begin{bmatrix} 0.8 & 0 \\ 0.2 & 1 \end{bmatrix} \end{array}$$

Assume this pattern of movement continues. In the long term, the number of birds that settle at location A will

A not change.

B gradually decrease to zero.

C eventually settle at around 20% of the island's bird population.

D eventually settle at around 80% of the island's bird population.

E gradually increase.

[VCAA 2006 1MQ9]

Question 14

By the end of each academic year, students at the university will have either passed, failed or deferred the year. Experience has shown that

- 88% of students who pass this year will also pass next year
- 10% of students who pass this year will fail next year
- 2% of students who pass this year will defer next year
- 52% of students who fail this year will pass next year
- 44% of students who fail this year will fail next year
- 4% of students who fail this year will defer next year
- 65% of students who defer this year will pass next year
- 10% of students who defer this year will fail next year
- 25% of students who defer this year will defer next year.

Twelve hundred and thirty students began a business degree in 2007. By the end of the 2007 academic year, 880 students had passed, 230 had failed, while 120 had deferred the year. No students have dropped out of the business degree permanently. Use this information to predict the number of business students who will **defer** the 2009 academic year.

[VCAA 2008 2MQ4]

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Question 15

A new shopping centre called Shopper Heaven (S) is about to open. It will compete for customers with Eastown (E) and Noxland (N). Market research suggests that each shopping centre will have a regular customer base, but attract and lose customers on a weekly basis as follows.

- 80% of Shopper Heaven customers will return to Shopper Heaven next week
- 12% of Shopper Heaven customers will shop at Eastown next week
- 8% of Shopper Heaven customers will shop at Noxland next week
- 76% of Eastown customers will return to Eastown next week
- 9% of Eastown customers will shop at Shopper Heaven next week
- 15% of Eastown customers will shop at Noxland next week
- 85% of Noxland customers will return to Noxland next week
- 10% of Noxland customers will shop at Shopper Heaven next week
- 5% of Noxland customers will shop at Eastown next week

a Use this information to copy and complete this transition matrix T (express percentages as proportions, for example, write 76% as 0.76).

This week

$$T = \begin{array}{c} S \quad E \quad N \\ \left[\begin{array}{ccc} \quad & \quad & \quad \\ & & \\ & & \end{array}\right] \end{array} \begin{array}{l} S \\ E \\ N \end{array} \quad \text{Next week}$$

2 marks

During the week that Shopper Heaven opened, it had 300 000 customers.

In the same week, Eastown had 120 000 customers and Noxland had 180 000 customers.

b Use this information to copy and complete this column matrix, K_0.

$$K_0 = \left[\begin{array}{c} \quad \\ \quad \\ \quad \end{array}\right] \begin{array}{l} S \\ E \\ N \end{array}$$

c Use T and K_0 to write and evaluate a matrix product that determines the number of customers expected at each of the shopping centres during the following week.

d Show, by calculating at least two appropriate state matrices, that, in the long term, the number of customers expected at each centre each week is given by the matrix

$$K = \left[\begin{array}{c} 194\,983 \\ 150\,513 \\ 254\,504 \end{array}\right]$$

[VCAA 2006 2MQ2]

Transitions with restocking and culling

9.6

Alamy/Kevin Wheal

In all the transition situations we've dealt with so far, the *total* number of things involved (shoppers, birds, trucks, etc.) has remained the same throughout each transition. We are now going to look at transitions where things are either added or taken away. Adding something to a transition situation is called **restocking** and taking something away is called **culling**.

These sorts of transitions are based on the matrix recurrence relation

S_0 = initial state matrix, $S_{n+1} = TS_n + B$

where B is a column matrix.

When using the restocking and culling recurrence relation, there is no simple rule that involves powers of the transition matrix to solve problems. You need to solve the problem the long way using the recurrence relation.

Worked example 16

A company wishes to start a commercial whiting breeding farm and it starts with 20 000 whiting eggs, 4000 baby whiting, and 500 adult whiting.

The following could happen in a year:

- eggs (E) could die (D) or they could live and become baby whiting (B)
- baby whiting (B) could die (D) or they could live and become adult whiting (A)
- adult whiting (A) could die (D) or they could live for a while but will eventually die.

From one year to the next, this situation can be represented by the transition matrix T, where

$$\begin{array}{c} \text{This year} \\ \begin{array}{cc} T = \begin{array}{c} \begin{array}{cccc} E & B & A & D \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0.3 & 0 & 0 & 0 \\ 0 & 0.35 & 0.6 & 0 \\ 0.7 & 0.65 & 0.4 & 1 \end{bmatrix} \end{array} & \begin{array}{c} \\ E \\ B \\ A \\ D \end{array} \text{Next year} \end{array} \end{array}$$

The company removes 500 adult whiting each year to sell to restaurants, adds an extra 200 baby whiting each year, and estimates that 8000 extra eggs will be laid by the adult whiting during each year. It establishes that the population of whiting can be modelled by the recurrence relation

$$S_0 = \begin{bmatrix} 20\,000 \\ 4000 \\ 500 \\ 0 \end{bmatrix}, S_{n+1} = TS_n + B$$

a Find B.

b Find the number of adult whiting after 1 year.

c Find the number of adult whiting after 2 years.

d Instead of removing just 500 adult whiting after the second year, the company decides to remove enough so that the number of adult whiting remains the same as it was at the very start. How many do they remove?

e The company decides to start another farm under exactly the same circumstances, but with one change. The breeders realise that the number of eggs laid by the adult whiting during each year depends on how many female adult whiting they had each year. They come up with the following modified recurrence relation where they estimate 50% of the adults (i.e. the females) each lay 200 eggs a year.

$$S_0 = \begin{bmatrix} 20\,000 \\ 4000 \\ 500 \\ 0 \end{bmatrix}, S_{n+1} = TS_n + \begin{bmatrix} 0 \\ 200 \\ -500 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 100 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} S_n$$

How many eggs will there be after the first year using this model?

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Working

a Use the information about what is added and taken away.

$$B = \begin{bmatrix} 8000 \\ 200 \\ -500 \\ 0 \end{bmatrix}$$

b Use the recurrence relation to find S_1.

$$S_0 = \begin{bmatrix} 20\,000 \\ 4000 \\ 500 \\ 0 \end{bmatrix}, S_{n+1} = TS_n + \begin{bmatrix} 8000 \\ 200 \\ -500 \\ 0 \end{bmatrix}$$

$$S_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0.3 & 0 & 0 & 0 \\ 0 & 0.35 & 0.6 & 0 \\ 0.7 & 0.65 & 0.4 & 1 \end{bmatrix} \begin{bmatrix} 20\,000 \\ 4000 \\ 500 \\ 0 \end{bmatrix} + \begin{bmatrix} 8000 \\ 200 \\ -500 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 6000 \\ 1700 \\ 16\,800 \end{bmatrix} + \begin{bmatrix} 8000 \\ 200 \\ -500 \\ 0 \end{bmatrix} = \begin{bmatrix} 8000 \\ 6200 \\ 1200 \\ 16\,800 \end{bmatrix}$$

There are 1200 adult whiting after the first year.

c Use the recurrence relation to find S_2.

$$S_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0.3 & 0 & 0 & 0 \\ 0 & 0.35 & 0.6 & 0 \\ 0.7 & 0.65 & 0.4 & 1 \end{bmatrix} \begin{bmatrix} 8000 \\ 6200 \\ 1200 \\ 16\,800 \end{bmatrix} + \begin{bmatrix} 8000 \\ 200 \\ -500 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 2400 \\ 2890 \\ 26\,910 \end{bmatrix} + \begin{bmatrix} 8000 \\ 200 \\ -500 \\ 0 \end{bmatrix} = \begin{bmatrix} 8000 \\ 2600 \\ 2390 \\ 26\,910 \end{bmatrix}$$

There are 2390 adult whiting after the second year.

d Subtract the number of adult whiting at the very start to find how many are removed.

The company removes 2390 – 500 = 1890 adult whiting after the second year.

e Use the recurrence relation to find S_1.

$$S_0 = \begin{bmatrix} 20\,000 \\ 4000 \\ 500 \\ 0 \end{bmatrix}$$

$$S_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0.3 & 0 & 0 & 0 \\ 0 & 0.35 & 0.6 & 0 \\ 0.7 & 0.65 & 0.4 & 1 \end{bmatrix} \begin{bmatrix} 20\,000 \\ 4000 \\ 500 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 200 \\ -500 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 100 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 20\,000 \\ 4000 \\ 500 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 6000 \\ 1700 \\ 16\,800 \end{bmatrix} + \begin{bmatrix} 0 \\ 200 \\ -500 \\ 0 \end{bmatrix} + \begin{bmatrix} 50\,000 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 50\,000 \\ 6200 \\ 1200 \\ 16\,800 \end{bmatrix}$$

There are 50 000 eggs after the first year.

EXAM PREP 9.6

Transitions with restocking and culling

Prep 1 WORKED EXAMPLE 16

A cooperative wishes to start a commercial salmon breeding farm and it starts with 15 000 salmon eggs, 5000 baby salmon, and 600 adult salmon.

The following could happen in a year

- eggs (E) could die (D) or they could live and become baby salmon (B)
- baby salmon (B) could die (D) or they could live and become adult salmon (A)
- adult salmon (A) could die (D) or they could live for a while but will eventually die.

From one year to the next, this situation can be represented by the transition matrix T, where

$$T = \begin{array}{c} \text{This year} \\ \begin{array}{cccc} E & B & A & D \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0.25 & 0 & 0 & 0 \\ 0 & 0.4 & 0.7 & 0 \\ 0.75 & 0.6 & 0.3 & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ \\ E \\ B \\ A \\ D \end{array} \text{ Next year}$$

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The cooperative removes 1000 adults each year to sell to restaurants, adds an extra 400 baby salmon each year, and estimates that 9000 extra eggs will be laid by the adult salmon during each year. It establishes that the population of salmon can be modelled by the recurrence relation

$$S_0 = \begin{bmatrix} 15\,000 \\ 5000 \\ 600 \\ 0 \end{bmatrix},\ S_{n+1} = TS_n + B$$

a Find B.

b Find the number of adult salmon after 1 year.

c Find the number of adult salmon after 2 years.

d Instead of removing just 1000 adult salmon after the second year, the cooperative decides to remove enough so that the number of adult salmon remains the same as it was at the very start. How many do they remove?

e The cooperative decides to start another farm under exactly the same circumstances but with one change. The breeders realise that the number of eggs laid by the adult salmon during each year depends on how many female adult salmon they had each year. They come up with the following modified recurrence relation where they estimate that 50% of the adults (i.e. the females) each lay 120 eggs a year.

$$S_0 = \begin{bmatrix} 15\,000 \\ 5000 \\ 600 \\ 0 \end{bmatrix},\ S_{n+1} = TS_n + \begin{bmatrix} 0 \\ 400 \\ -1000 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 60 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} S_n$$

How many eggs will there be after the first year using this model?

EXAM PRACTICE 9.6

Transitions with restocking and culling

Question 1

The matrix S_{n+1} is determined from the matrix S_n using the rule $S_{n+1} = TS_n - C$, where T, S_0 and C are defined as follows.

$$T = \begin{bmatrix} 0.5 & 0.6 \\ 0.5 & 0.4 \end{bmatrix},\quad S_0 = \begin{bmatrix} 100 \\ 250 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 20 \\ 20 \end{bmatrix}$$

Given this information, the matrix S_2 equals

A $\begin{bmatrix} 100 \\ 250 \end{bmatrix}$ **B** $\begin{bmatrix} 148 \\ 122 \end{bmatrix}$ **C** $\begin{bmatrix} 170 \\ 140 \end{bmatrix}$ **D** $\begin{bmatrix} 180 \\ 130 \end{bmatrix}$ **E** $\begin{bmatrix} 190 \\ 160 \end{bmatrix}$

[VCAA 2013 1MQ9]

Question 2

The bookshop manager at the university has developed a matrix formula for determining the number of Mathematics and Physics textbooks he should order each year.

For 2009, the starting point for the formula is the column matrix S_{2008}. This lists the number of Mathematics and Physics textbooks sold in 2008.

$$S_{2008} = \begin{array}{c} \\ \end{array}\begin{bmatrix} 456 \\ 350 \end{bmatrix} \begin{array}{l} \text{Mathematics} \\ \text{Physics} \end{array}$$

O_{2009} is a column matrix listing the number of Mathematics and Physics textbooks to be ordered for 2009.

O_{2009} is given by the matrix formula

$$O_{2009} = AS_{2008} + B, \text{ where } A = \begin{bmatrix} 0.75 & 0 \\ 0 & 0.68 \end{bmatrix} \text{ and } B = \begin{bmatrix} 18 \\ 12 \end{bmatrix}.$$

a Determine O_{2009}. 1 mark

The matrix formula above only allows the manager to predict the number of books he should order one year ahead. A new matrix formula enables him to determine the number of books to be ordered two or more years ahead.

The new matrix formula is $O_{n+1} = CO_n - D$ where O_n is a column matrix listing the number of Mathematics and Physics textbooks to be ordered for year n.

$$\text{Here, } C = \begin{bmatrix} 0.8 & 0 \\ 0 & 0.8 \end{bmatrix} \text{ and } D = \begin{bmatrix} 40 \\ 38 \end{bmatrix}.$$

The number of books ordered in 2008 was given by $O_{2008} = \begin{bmatrix} 500 \\ 360 \end{bmatrix} \begin{array}{l} \text{Mathematics} \\ \text{Physics} \end{array}$

b Use the new matrix formula to determine the number of Mathematics textbooks the bookshop manager should order in 2010. 2 marks

[VCAA 2008 2MQ3]

Question 3

A school entered a Rock Eisteddfod competition and a series of extra rehearsals commenced in April. Each week participants could choose extra dancing rehearsals or extra singing rehearsals.

A matrix equation used to determine the number of students expected to attend these extra rehearsals is given by $L_{n+1} = \begin{bmatrix} 0.85 & 0.25 \\ 0.15 & 0.75 \end{bmatrix} \times L_n - \begin{bmatrix} 5 \\ 7 \end{bmatrix}$ where L_n is the column matrix that lists the number of students attending in week n.

The attendance matrix for the first week of extra rehearsals is given by $L_1 = \begin{bmatrix} 95 \\ 97 \end{bmatrix} \begin{array}{l} \text{dancing} \\ \text{singing} \end{array}$

a Calculate the number of students who are expected to attend the extra singing rehearsals in week 3. 1 mark

b Of the students who attended extra rehearsals in week 3, how many are not expected to return for any extra rehearsals in week 4? 1 mark

[VCAA 2009 2MQ4]

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Question 4

10 000 trout eggs, 1000 baby trout and 800 adult trout are placed in a pond to establish a trout population. In establishing this population,

- eggs (E) may die (D) or they may live and eventually become baby trout (B)
- baby trout (B) may die (D) or they may live and eventually become adult trout (A)
- adult trout (A) may die (D) or they may live for a period of time but will eventually die.

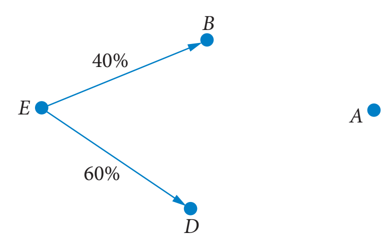
From year to year, this situation can be represented by the transition matrix T, where

$$\begin{array}{ccl} & \text{This year} & \\ & \begin{array}{cccc} E & A & B & D \end{array} & \\ T = & \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0.4 & 0 & 0 & 0 \\ 0 & 0.25 & 0.5 & 0 \\ 0.6 & 0.75 & 0.5 & 1 \end{bmatrix} & \begin{array}{c} E \\ A \\ B \\ D \end{array} \ \text{Next year} \end{array}$$

a Use the information in the transition matrix T to

i determine the number of eggs in the population that die in the first year — 1 mark

ii copy and complete the transition diagram, showing the relevant percentages. — 2 marks



The initial state matrix for this trout population, S_0, can be written as $S_0 = \begin{bmatrix} 10\,000 \\ 1\,000 \\ 800 \\ 0 \end{bmatrix} \begin{array}{c} E \\ B \\ A \\ D \end{array}$

Let S_n represent the state matrix describing the trout population after n years.

b Using the rule $S_n = TS_{n-1}$, determine each of the following.

i S_1 — 1 mark

ii The number of adult trout predicted to be in the population after four years. — 1 mark

c The transition matrix T predicts that, in the long term, all of the eggs, baby trout and adult trout will die.

i How many years will it take for all of the adult trout to die (that is, when the number of adult trout in the population is first predicted to be less than one)? — 1 mark

ii What is the largest number of adult trout that is predicted to be in the pond in any one year? — 1 mark

d Determine the number of eggs, baby trout and adult trout that, if added to or removed from the pond at the end of each year, will ensure that the number of eggs, baby trout and adult trout in the population remains constant from year to year. — 2 marks

The rule $S_n = TS_{n-1}$ that was used to describe the development of the trout in this pond does not take into account new eggs added to the population when the adult trout begin to breed.

e To take breeding into account, assume that 50% of the adult trout lay 500 eggs each year. The matrix describing the population after one year, S_1, is now given by the new rule

$S_1 = TS_0 + 500MS_0$

$$\text{where } T = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0.40 & 0 & 0 & 0 \\ 0 & 0.25 & 0.50 & 0 \\ 0.60 & 0.75 & 0.50 & 1.0 \end{bmatrix}, M = \begin{bmatrix} 0 & 0 & 0.50 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } S_0 = \begin{bmatrix} 10\,000 \\ 1\,000 \\ 800 \\ 0 \end{bmatrix}$$

i Use this new rule to determine S_1. 1 mark

ii This pattern continues so that the matrix describing the population after n years, S_n, is given by the rule $S_n = TS_{n-1} + 500MS_{n-1}$

Use this rule to determine the number of eggs in the population after two years. 2 marks

[VCAA 2013 2MQ2]

Question 5

To study the life-and-death cycle of an insect population, a number of insect eggs (E), juvenile insects (J) and adult insects (A) are placed in a closed environment.

The initial state of this population can be described by the column matrix $S_0 = \begin{array}{c} \begin{bmatrix} 400 \\ 200 \\ 100 \\ 0 \end{bmatrix} \end{array} \begin{array}{c} E \\ J \\ A \\ D \end{array}$

A row has been included in the state matrix to allow for insects and eggs that die (D).

a What is the total number of insects in the population (including eggs) at the beginning of the study? 1 mark

In this population

- eggs may die, or they may live and grow into juveniles
- juveniles may die, or they may live and grow into adults
- adults will live a period of time, but they will eventually die.

In this population, the adult insects have been sterilised so that no new eggs are produced. In these circumstances, the life-and-death cycle of the insects can be modelled by the transition matrix

$$T = \begin{array}{c} \text{This week} \\ \begin{array}{cccc} E & J & A & D \end{array} \\ \begin{bmatrix} 0.4 & 0 & 0 & 0 \\ 0.5 & 0.4 & 0 & 0 \\ 0 & 0.5 & 0.8 & 0 \\ 0.1 & 0.1 & 0.2 & 1 \end{bmatrix} \end{array} \begin{array}{c} E \\ J \\ A \\ D \end{array} \text{ Next week}$$

9.6

b What proportion of eggs turn into juveniles each week? 1 mark

c **i** Evaluate the matrix product $S_1 = TS_0$ 1 mark

ii Write down the number of live **juveniles** in the population after one week. 1 mark

iii Determine the number of live **juveniles** in the population after four weeks. Write your answer correct to the nearest whole number. 1 mark

iv After a number of weeks, there will be no live eggs (less than one) left in the population. When does this first occur? 1 mark

v Write down the exact steady-state matrix for this population. 1 mark

d If the study is repeated with unsterilised adult insects, eggs will be laid and potentially grow into adults. Assuming that 30% of adults lay eggs each week, the population matrix after one week, S_1, is now given by $S_1 = TS_0 + BS_0$

$$\text{where } B = \begin{bmatrix} 0 & 0 & 0.3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } S_0 = \begin{bmatrix} 400 \\ 200 \\ 100 \\ 0 \end{bmatrix} \begin{array}{c} E \\ J \\ A \\ D \end{array}$$

i Copy and complete S_1. 1 mark

$$S_1 = \begin{bmatrix} \quad \\ \quad \\ \quad \\ \quad \end{bmatrix} \begin{array}{c} E \\ J \\ A \\ D \end{array}$$

This pattern continues. The population matrix after n weeks, S_n, is given by $S_n = TS_{n-1} + BS_{n-1}$

ii Determine the number of live eggs in this insect population after two weeks. 1 mark

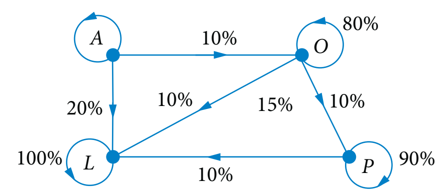
[VCAA 2007 2MQ2]

Question 6

When a new industrial site was established at the beginning of 2011, there were 350 staff at the site. The staff comprised 100 apprentices (A), 200 operators (O) and 50 professionals (P). At the beginning of each year, staff can choose to stay in the same job, move to a different job at the site or leave the site (L). The number of staff in each category at the beginning of 2011 is given in the matrix

$$S_{2011} = \begin{bmatrix} 100 \\ 200 \\ 50 \\ 0 \end{bmatrix} \begin{array}{c} A \\ O \\ P \\ L \end{array}$$

The transition diagram shows the way in which staff are expected to change their jobs at the site each year.



a How many staff at the site are expected to be working in their same jobs after one year? 1 mark

The information in the transition diagram has been used to write the transition matrix T.

$$T = \begin{array}{c} \text{This year} \\ \begin{array}{cccc} A & O & P & L \end{array} \\ \begin{bmatrix} 0.70 & 0 & 0 & 0 \\ 0.10 & 0.80 & 0 & 0 \\ 0 & 0.10 & 0.90 & 0 \\ 0.20 & 0.10 & 0.10 & 1.00 \end{bmatrix} \end{array} \begin{array}{c} A \\ O \\ P \\ L \end{array} \text{ Next year}$$

b Explain the meaning of the entry in the fourth row and fourth column of transition matrix T. 1 mark

If staff at the site continue to change their jobs in this way, the matrix S_n will contain the number of apprentices (A), operators (O), professionals (P) and staff who leave the site (L) at the beginning of the nth year.

c Use the rule $S_{n+1} = TS_n$ to find

- **i** S_{2012} 1 mark
- **ii** the expected number of operators at the site at the beginning of 2013 1 mark
- **iii** the beginning of which year the number of operators at the site first drops below 30 1 mark
- **iv** the total number of staff at the site in the longer term. 1 mark

Suppose the manager decides to bring 30 new apprentices, 20 new operators and 10 new professionals to the site at the beginning of each year.

The matrix S_{n+1} will then be given by $S_{n+1} = TS_n + A$, where

$$S_{2011} = \begin{bmatrix} 100 \\ 200 \\ 50 \\ 0 \end{bmatrix} \begin{array}{c} A \\ O \\ P \\ L \end{array} \quad \text{and} \quad A = \begin{bmatrix} 30 \\ 20 \\ 10 \\ 0 \end{bmatrix} \begin{array}{c} A \\ O \\ P \\ L \end{array}$$

d Find the expected number of operators at the site at the beginning of 2013. 1 mark

[VCAA 2012 2MQ3]

Matrix applications

Practice quiz

SUMMARY

9

Solving two simultaneous equations using matrices

- Matrices can be used to solve simultaneous equations.
- When solving simultaneous equations using matrices, always *pre-multiply* both sides by the inverse.
- When solving simultaneous equations using matrices, if there is no unique solution, a CAS/calculator will give you an error message.

Communication matrices

- A **communication diagram** shows one-way or two-way arrows between points indicating when communication occurs.
- A **communication matrix** is a square binary matrix where communication is indicated by a 1 and non-communication or self-communication is indicated by a 0.
- **One-step communication** is direct communication between A and B.
- **One-way communication** is when A can communicate with B, but B can't communicate with A.
- **Two-way communication** is when A can communicate with B, and B can communicate with A.
- An example of a **two-step communication** between A and B is $A \rightarrow D \rightarrow B$, where A can communicate with D, which can then communicate with B.
- A communication matrix where every communication is two-way is symmetric about the leading diagonal.
- For a communication matrix M:
  - M^2 gives the number of two-step communications
  - M^3 gives the number of three-step communications.
  - M^4 gives the number of four-step communications, and so on.
  - $M + M^2$ gives us the number of one- or two-step communications.
  - $M + M^2 + M^3$ gives us the number of one-, two- or three-step communications, and so on.
- The leading diagonals in M^2, M^3, etc. show self-communication links.

Dominance matrices

- **Dominance diagrams** are communication diagrams where *all* the arrows are *one-way*.

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- A **dominance matrix** is a square binary matrix where dominance is indicated by a 1 and other elements are 0.
- **Direct dominance** occurs when A dominates B.
- **Indirect dominance** of A over B occurs when A dominates C, which dominates B.
- For a square dominance matrix M
  - the sum of the elements in each column of M gives us the number of direct dominances
  - the sum of the elements in each column of M^2 gives us the number of indirect dominances
  - the sum of the elements in each column of $M + M^2$ gives us the number of direct and indirect dominances.

Transition matrices

- **Transition diagrams** show transitions from one **state** to another using arrows with corresponding percentages.
- In a transition diagram all the arrow percentages *from* a single point add up to 100%.
- A **transition matrix** is a square matrix that shows a change from one state to another, where the change follows the same rules each time.
- Transition matrices give the percentages from the transition diagram as decimals.
- Each value in a transition matrix is from 0 to 1, and each column adds up to 1.

Transition matrices and recurrence relations

- A **state matrix** is a column matrix representing a state at a point in time of a transition.
- The recurrence relation that generates a sequence of **state matrices** from a transition matrix is given by S_0 = **initial state matrix**, $S_{n+1} = TS_n$ where T is a transition matrix and S_n is a state matrix.
- The rule for finding S_n, the state matrix after n transitions, is $S_n = T^n \times S_0$ where T is a transition matrix and S_0 is the initial state matrix.
- For a 2×2 transition matrix T where

Current state

$$T = \begin{array}{c} \begin{array}{cc} A & B \end{array} \\ \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \end{array} \begin{array}{c} A \\ B \end{array} \text{ Next state,}$$

  - if $a_{21} > a_{12}$, in the long term more will transition to B
  - if $a_{12} > a_{21}$, in the long term more will transition to A.

- The **equilibrium state matrix** is a column matrix that occurs when we take a higher power of the transition matrix T and there is no longer any noticeable change to the state matrix.
- Transitions involving **restocking** (adding something) and **culling** (taking something away) are based on the matrix recurrence relation S_0 = initial state matrix , $S_{n+1} = TS_n + B$ where B is a column matrix.
- When using the restocking and culling recurrence relation, problems need to be solved using the recurrence relation, not using powers of the transition matrix.

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CHAPTER 10

UNDIRECTED GRAPHS

10.1 Basic concepts of graphs and networks

Components of a graph

Adjacency matrix

10.2 Planar graphs and Euler's formula

Connected graphs

Planar graphs

Euler's formula

10.3 Types of graphs

10.4 Exploring and travelling problems

Walks, trails, paths, circuits, cycles and bridges

Eulerian trails and circuits

Hamiltonian paths and cycles

Weighted graphs and Dijkstra's algorithm

10.5 Trees and minimum spanning trees

Trees and spanning trees

Minimum spanning trees

Prim's algorithm

Summary

Prior learning

10.1 Basic concepts of graphs and networks

Components of a graph

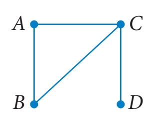
A **network** is a group of interconnected people or things. Graph theory is used to solve network problems in biochemistry, electrical engineering, scheduling, communications and computer science.

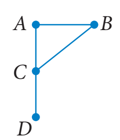
A **graph** is a network diagram that consists of points called **vertices** that are connected by lines called **edges**.

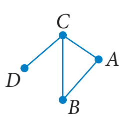
Two vertices that are connected by one or more edges are referred to as **adjacent vertices**.

An **undirected graph** is one where there are no directions associated with the edges.

Vertices are labelled with single letters only. A graph with vertices A, B, C, D and edges AB, AC, BC, DC is shown below drawn in three different ways. This graph can be drawn in different ways as long as the edges are connecting the same vertices. Different-looking graphs that have exactly the same information, with the same edges connecting the same **vertices**, are actually the same graph.





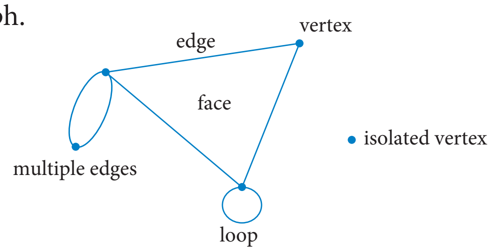


Graphs can also contain regions called **faces**. A graph will contain a face when an area is enclosed by three or more edges. The region outside the graph is also considered a face.

Exam hack

Since the same graph can be drawn in more than one way, some questions will require you to look at a graph in a different way by redrawing connecting edges in order to determine how many faces it has. Always think of edges as movable pieces of string; they can be rearranged, but still must be connected to the same vertices.

The diagram illustrates the different components of a graph.



In a graph, the **degree** of a vertex (also called the **order** of the vertex) is the number of edges linked to that vertex, with loops counted twice.

The degree sum of a graph is the sum of the degrees of all the vertices. For any graph,

Degree sum = $2 \times$ number of edges

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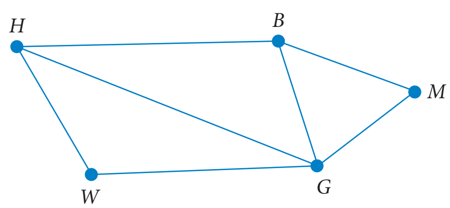
Worked example 1

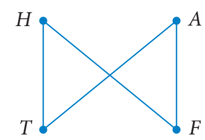
For each graph, identify the number of

i vertices **ii** edges **iii** faces

a



b



Working

a A vertex (point) occurs at the end of any edge (line).

i There are 5 vertices: *H, B, M, G, W*

ii There are 7 edges: *HB, BM, MG, GW, HW, HG, BG*

A face is the region enclosed by three or more edges, with the space outside the graph also included as a face.

iii There are 4 faces: 3 enclosed and 1 outside the graph

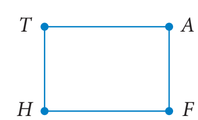
b **1** This graph has edges that intersect but there is no vertex at the point of intersection.

i There are 4 vertices: *H, A, T, F*

ii There are 4 edges: *HT, HF, TA, FA*

2 Redrawing the graph as follows will make it easier to see the number of faces.

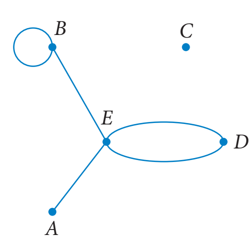
iii



There are 2 faces: 1 enclosed and 1 outside the graph.

Worked example 2

Determine the degree sum for the following graph.



Working

The graph has 5 edges: *BB, BE, ED, ED, EA*

Substitute the number of edges into the degree sum equation.

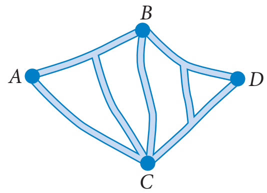
$$\begin{aligned}\text{Degree sum} &= 2 \times \text{number of edges}\\ &= 2 \times 5\\ &= 10\end{aligned}$$

10.1

Worked example 3

The map shows the roads connecting four towns, *A*, *B*, *C* and *D*.

Sketch a network diagram showing the roads connecting the four towns.



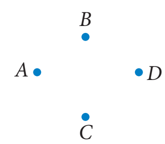
Working

1 Consider all the possible direct routes. These direct routes will become the edges.

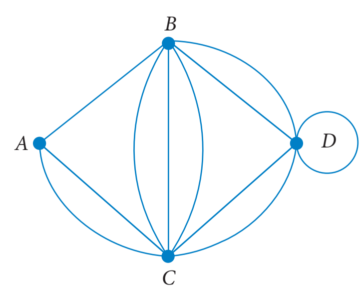
Note that it is possible to travel from *D* to *D* without passing another town; this is a loop.

AC: 2	*AB*: 1	*BC*: 3
BD: 2	*CD*: 2	*DD*: 1

2 Draw the vertices first.



3 Connect the vertices with the correct number of edges determined previously.



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Exam hack

When matching a road network to its graph, think about how many different ways you can get directly from one destination to the next. Pay special attention to loops – these are present if you can leave one destination (vertex) and return without passing through any other destination.

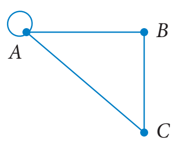
Adjacency matrix

Adjacency matrices

10.1

Graphs can be represented using an **adjacency matrix**. A graph containing n vertices can be represented using an $n \times n$ matrix. To represent the graph, the names of the vertices are generally above the columns and to the left of the rows. The number of edges connecting the vertices is placed at the intersection of the corresponding row and column.

For example, in the adjacency matrix below, the number in the cell for row A column B indicates the number of edges connecting vertex A to vertex B. In an undirected graph this number is also the same for row B column A.



$$\begin{array}{cc} & \begin{matrix} A & B & C \end{matrix} \\ \begin{matrix} A \\ B \\ C \end{matrix} & \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \end{array}$$

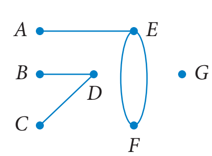
Remember that a loop is an edge that has both ends connected to the same vertex. Even though a loop counts for 2 when calculating the degree of a vertex, when you complete an adjacency matrix for any graph the loop only counts as 1 edge. In the graph above, there is a loop at vertex A.

When representing a graph using an adjacency matrix:

- a '0' represents no edge connecting the vertices
- a '1' represents 1 edge connecting the vertices
- a number greater than 1 is used when there are multiple edges connecting the same vertices
- a '1' along the leading diagonal indicates a loop at that vertex.

Worked example 4

Represent the following graph using an adjacency matrix



Working

1 Set up a blank matrix, placing the vertex names across the top and to the left.

Look at each vertex individually, starting with vertex A. Vertex A is connected to vertex E only. Therefore, we place a 1 at the intersections of vertex A and vertex E.

$$\begin{array}{cc} & \begin{matrix} A & B & C & D & E & F & G \end{matrix} \\ \begin{matrix} A \\ B \\ C \\ D \\ E \\ F \\ G \end{matrix} & \begin{bmatrix} \; & \; & \; & \; & 1 & \; & \; \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ 1 & & & & & & \\ & & & & & & \\ & & & & & & \end{bmatrix} \end{array}$$

2 Vertex B is connected to vertex D only. Therefore we place a 1 at the intersections of vertex B and vertex D.

Vertex C is connected to vertex D only. Therefore we place a 1 at the intersection of vertex C and vertex D.

$$\begin{array}{cc} & \begin{array}{ccccccc} A & B & C & D & E & F & G \end{array} \\ \begin{array}{c} A\\B\\C\\D\\E\\F\\G \end{array} & \left[\begin{array}{ccccccc} & & & & 1 & & \\ & & & 1 & & & \\ & & & 1 & & & \\ & 1 & 1 & & & & \\ 1 & & & & & & \\ & & & & & & \\ & & & & & & \end{array}\right] \end{array}$$

3 Vertex E is connected to vertex F with two edges, therefore we place a 2 at the intersections of vertex E and vertex F.

$$\begin{array}{cc} & \begin{array}{ccccccc} A & B & C & D & E & F & G \end{array} \\ \begin{array}{c} A\\B\\C\\D\\E\\F\\G \end{array} & \left[\begin{array}{ccccccc} & & & & 1 & & \\ & & & 1 & & & \\ & & & 1 & & & \\ & 1 & 1 & & & & \\ 1 & & & & & 2 & \\ & & & & 2 & & \\ & & & & & & \end{array}\right] \end{array}$$

4 Each edge has now been represented. Complete the matrix by placing a 0 in all unoccupied places.

$$\begin{array}{cc} & \begin{array}{ccccccc} A & B & C & D & E & F & G \end{array} \\ \begin{array}{c} A\\B\\C\\D\\E\\F\\G \end{array} & \left[\begin{array}{ccccccc} 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

Exam hack

The number of rows (or columns) of an adjacency matrix tells us the number of vertices the graph has. Halving the total of all the elements in an adjacency matrix tells us how many edges the graph has.

EXAM PREP 10.1

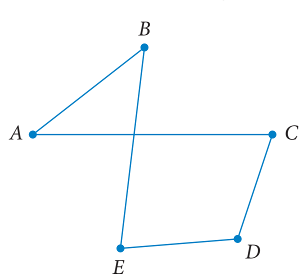
Basic concepts of graphs and networks

Prep 1 WORKED EXAMPLE 1

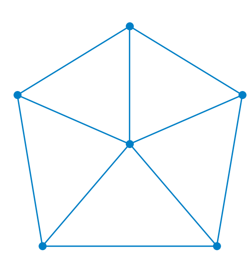
For the following graphs, determine the number of

i vertices **ii** edges **iii** faces

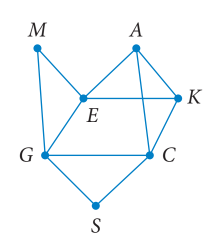
a



b



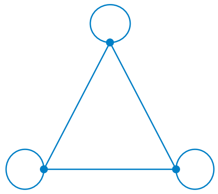
c



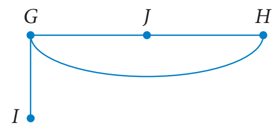
Prep 2 WORKED EXAMPLE 2

Determine the degree sum for the following graphs.

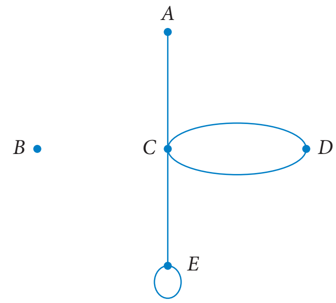
a



b



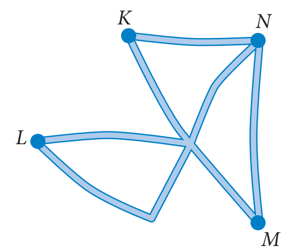
c



Prep 3 WORKED EXAMPLE 3

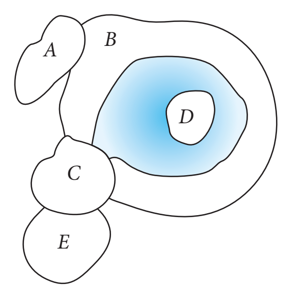
a The following map shows the roads connecting four towns, *K*, *L*, *M* and *N*.

Sketch a network diagram showing the roads connecting the four towns.



b The city of Nerak is divided into five suburbs labelled as *A* to *E* on the map. A lake, which is in the middle of the city, is shaded on the map.

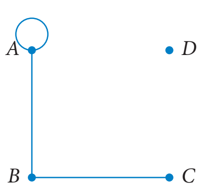
Sketch a network diagram connecting the five suburbs.



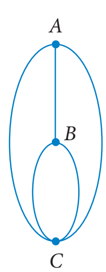
Prep 4 WORKED EXAMPLE 4

Represent each graph using an adjacency matrix.

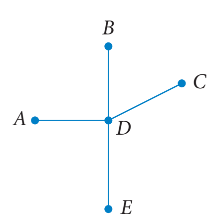
a



b



c



EXAM PRACTICE 10.1

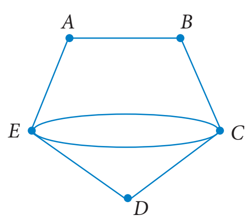
Basic concepts of graphs and networks

Question 1

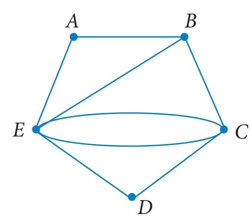
$$\begin{array}{c c}
\begin{array}{ccccc} A & B & C & D & E \end{array} & \\
\begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 2 & 1 & 0 \end{bmatrix} & \begin{array}{c} A \\ B \\ C \\ D \\ E \end{array}
\end{array}$$

A graph that can be drawn from the adjacency matrix is

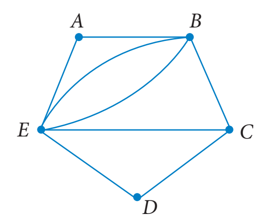
A



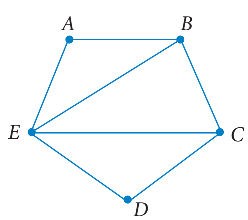
B



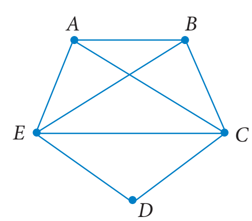
C



D



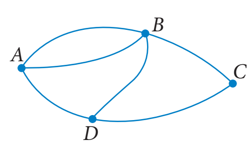
E



[VCAA 2010 1NQ3]

Question 2

Consider the following graph.



An adjacency matrix that could be used to represent this graph is

A $\begin{bmatrix} 0 & 2 & 0 & 1 \\ 2 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$

B $\begin{bmatrix} 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

C $\begin{bmatrix} 0 & 1 & 0 & 1 \\ 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$

D $\begin{bmatrix} 0 & 2 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$

E $\begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

[VCAA 2007 1NQ3]

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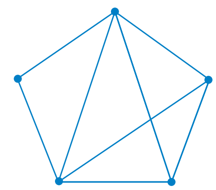
Question 3



The number of edges in the graph is

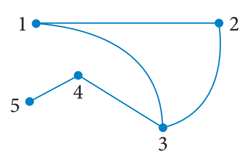
A 5 **B** 7 **C** 8

D 10 **E** 11

[VCAA 2010 1NQ2]

Question 4

Which of the following adjacency matrices could be used to represent the graph?



A $\begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$

B $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}$

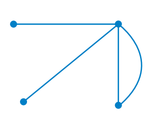
C $\begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

D $\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$

E $\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$

[VCAA 2002 1NQ3]

Question 5



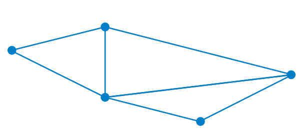
The sum of the degrees of all the vertices in this network diagram is

A 6 **B** 7 **C** 8

D 9 **E** 10

[VCAA 2005 1NQ1]

Question 6



The number of vertices with an odd degree in the network diagram is

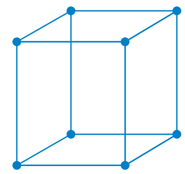
A 1 **B** 2 **C** 3

D 4 **E** 5

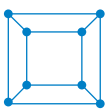
[VCAA 2006 1NQ1]

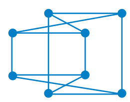
Question 7

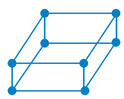
A network is represented by the following graph.

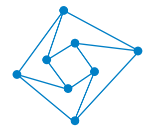


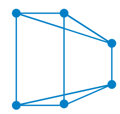
Which one of the following graphs could **not** be used to represent the same network?

A 

B 

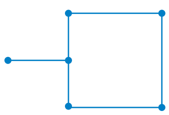
C 

D 

E 

[VCAA 2011 1NQ5]

Question 8



The sum of the degrees of all vertices in the graph is

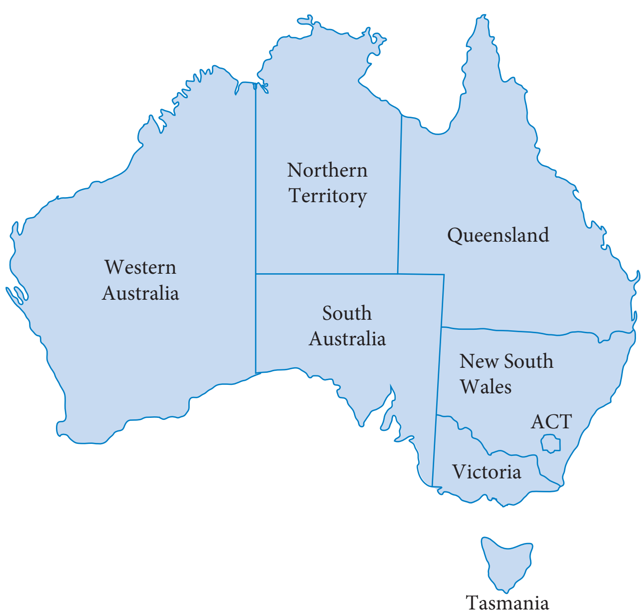
A 6 **B** 8 **C** 9

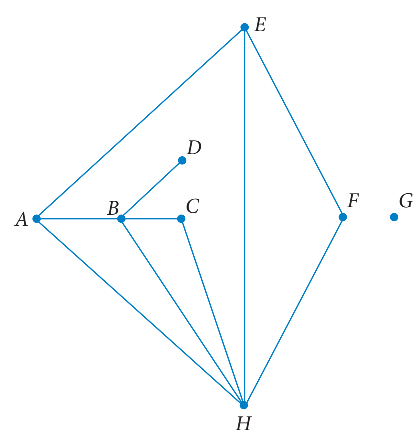
D 11 **E** 12

[VCAA 2012 1NQ1]

Use the following information to answer Questions 9 & 10.

The map of Australia shows the six states, the Northern Territory and the Australian Capital Territory (ACT). In the network diagram, each of the vertices A to H represents one of the states or territories shown on the map of Australia. The edges represent a border shared between two states or between a state and territory.





Question 9

In the network diagram, the order of the vertex that represents the Australian Capital Territory (ACT) is

A 0 **B** 1 **C** 2 **D** 3 **E** 4

[VCAA 2011 1NQ3]

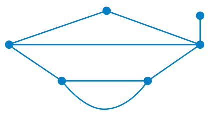
Question 10

In the network diagram, Queensland is represented by

A vertex A **B** vertex B **C** vertex C **D** vertex D **E** vertex E

[VCAA 2011 1NQ4]

Question 11



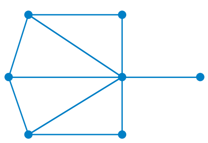
The sum of the degrees of all the vertices in the network diagram is

A 6 **B** 7 **C** 8

D 15 **E** 16

[VCAA 2003 1NQ4]

Question 12



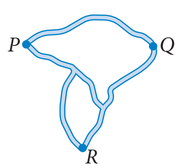
In the network diagram shown, the number of vertices of even degree is

A 2 **B** 3 **C** 4

D 5 **E** 6

[VCAA 2011 1NQ1]

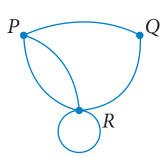
Question 13



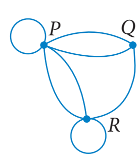
The map shows the road connections between three towns, P, Q and R.

The graph that could be used to model these road connections is

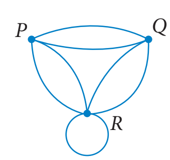
A



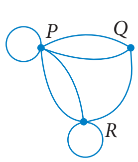
B



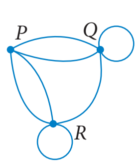
C



D



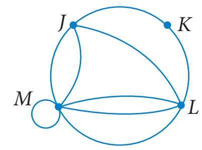
E



[VCAA 2013 1NQ6]

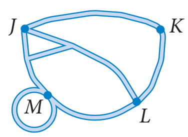
Question 14

The diagram represents a network of road connections between the four towns J, K, L and M.

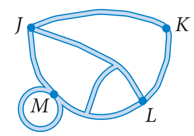


The map that corresponds to the roads that are represented by the network diagram could be

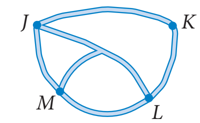
A



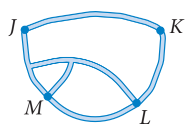
B



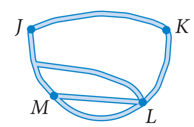
C



D



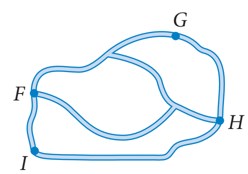
E



[VCAA 2005 1NQ7]

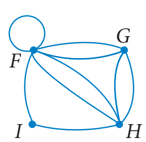
Question 15

The diagram on the right shows a map of the roads between four towns, F, G, H and I.

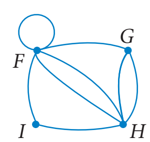


A network diagram that represents all the connections between the four towns on the map is

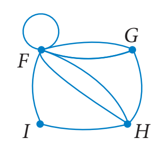
A



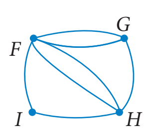
B



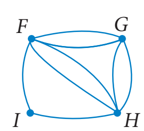
C



D

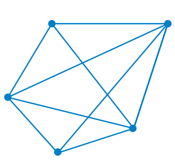


E



[VCAA 2004 1NQ4]

Question 16



This graph has

A 4 faces **B** 5 faces **C** 6 faces

D 8 faces **E** 9 faces

[VCAA 2008 1NQ7]

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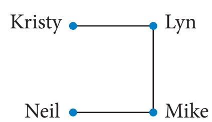
Question 17

In a competition, members of a team work together to complete a series of challenges. The members of one team are Kristy (K), Lyn (L), Mike (M) and Neil (N).

In one of the challenges, these four team members are only allowed to communicate directly with each other as indicated by the edges of the following network diagram.

The adjacency matrix also shows the allowed lines of communication.



$$\begin{array}{cc} \begin{array}{cccc} K & L & M & N \end{array} & \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & f & 0 & 1 \\ 0 & g & 1 & 0 \end{bmatrix} & \begin{array}{c} K \\ L \\ M \\ N \end{array} \end{array}$$

a Explain the meaning of a **zero** in the adjacency matrix. 1 mark

b Write down the values of f and g in the adjacency matrix. 1 mark

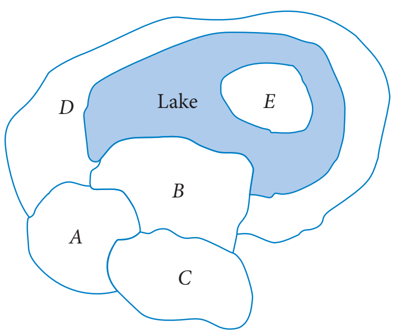
[VCAA 2010 2NQ1]

Question 18

The city of Robville is divided into five suburbs labelled as A to E on the map.

A lake that is situated in the city is shaded on the map.

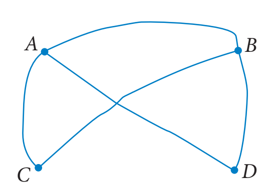
An adjacency matrix is constructed to represent the number of land borders between the suburbs.



$$\begin{array}{cc} & \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \\ E \end{array} & \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 2 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

a Explain why all values in the final row and final column are zero. 1 mark

In the network diagram, vertices represent suburbs and edges represent land borders between suburbs. The diagram has been started but is not finished.



b The network diagram is missing one edge and one vertex. Copy the diagram, and on it

i draw the missing edge 1 mark

ii draw and label the missing vertex. 1 mark

[VCAA 2009 2NQ1]

10.2 Planar graphs and Euler's formula

Connected graphs

A **connected graph** contains no isolated vertices and therefore every vertex is **reachable** from every other vertex; that is, there is a path between any two vertices of the graph. If a graph is connected, then you will be able to connect each vertex without taking your pen off the paper.

WS
Planar graphs

Planar graphs

In order for a graph to be considered a **planar graph** it must be connected. The second requirement (and the most important) is that the graph must not contain any edges that cross over.

Although a graph may appear non-planar (as in Diagram A), if it can be redrawn without any of the edges intersecting (as shown in Diagram B), then it's planar.

Diagram A

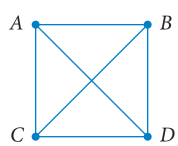
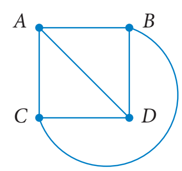


Diagram B

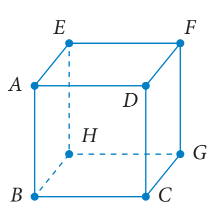


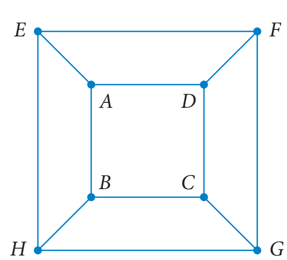
Exam hack

To determine whether a graph is planar or not, it is a good idea to imagine the edges as pieces of string that have been tied to nails on a board. If two (or more) pieces of string are crossed, then think about whether you can move a piece of string so that they are no longer crossed.

Euler's formula

Euler's formula shows the relationship between vertices, edges and faces. Remember that a face is the enclosed region in a graph. This cube has six faces – just like a die has six sides. This graph is planar because it can be redrawn without any two edges crossing. In fact, any prism will be planar.





Euler's formula can be used to determine whether a graph is planar.

vertices + faces – edges = 2

$v + f - e = 2$

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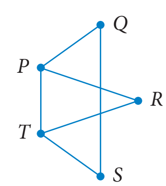
Shutterstock.com/Thinair

Worked example 5

Show that the following graph is planar by:

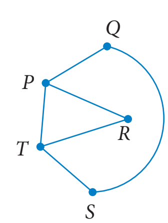
a redrawing the graph so that no edges intersect

b using Euler's formula.



Working

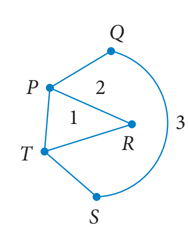
a Edge QS is crossing edge PQ and edge RT, therefore move edge QS to the outside of vertex R. The graph is clearly planar.



b This graph has 5 vertices, 6 edges and 3 faces.

$v + f - e = 5 + 3 - 6 = 2$

Euler's formula is true for this graph, so the graph is planar.



Worked example 6

The adjacency matrix represents a planar graph with 4 vertices.

Determine the number of faces.

$$\begin{array}{c c} \begin{array}{cccc} A & B & C & D \end{array} & \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} & \begin{array}{c} A \\ B \\ C \\ D \end{array} \end{array}$$

Working

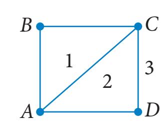
1 Use the adjacency matrix to determine the number of edges (AB, AC, AD, BC, CD). Alternatively, count the 1s below the zero diagonal.

There are 5 edges.

2 Use Euler's formula with $v = 4$ and $e = 5$ to determine the number of faces.

Alternatively, you could sketch the graph and count the number of faces.

$$\begin{aligned} v + f - e &= 2 \\ 4 + f - 5 &= 2 \\ f &= 3 \end{aligned}$$



There are 3 faces.

Worked example 7

A connected planar graph of 12 edges divides the plane into 5 distinct regions.

Determine the number of vertices for this graph.

Working

Substitute $e = 12$ and $f = 5$ into Euler's formula to find v.

$$\begin{aligned} v + f - e &= 2 \\ v + 5 - 12 &= 2 \\ v &= 9 \end{aligned}$$

The number of vertices is 9.

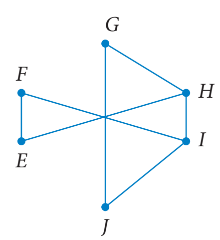
EXAM PREP 10.2

Planar graphs and Euler's formula

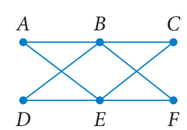
Prep 1 WORKED EXAMPLE 5

Redraw each graph to show that it is planar.

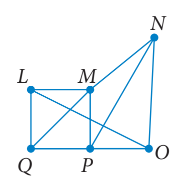
a



b



c



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Prep 2 WORKED EXAMPLE 6

a The adjacency matrix represents a planar graph with 4 vertices. Determine the number of faces.

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

b The adjacency matrix represents a planar graph with 5 vertices. Determine the number of faces.

$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

Prep 3 WORKED EXAMPLE 7

a A connected planar graph has 10 vertices and 6 faces.

Determine the number of edges for this graph.

b A connected planar graph has 4 vertices and 6 edges.

Determine the number of faces for this graph.

c A connected planar graph has 8 edges and 5 faces.

Determine the number of vertices for this graph.

EXAM PRACTICE 10.2

Planar graphs and Euler's formula

Question 1

A connected planar graph of 15 edges divides the plane into eight distinct regions. The number of vertices in this graph is

A 3 **B** 7 **C** 9 **D** 23 **E** 25

[VCAA 2002 1NQ1]

Question 2

A connected planar graph has 12 edges. This graph could have

A 5 vertices and 6 faces
B 5 vertices and 8 faces
C 6 vertices and 8 faces
D 6 vertices and 9 faces
E 7 vertices and 9 faces

[VCAA 2007 1NQ2]

Question 3

A connected planar graph has 10 edges and 10 faces. The number of vertices for this graph is

A 2 **B** 5 **C** 8 **D** 12 **E** 20

[VCAA 2009 1NQ4]

Question 4

A connected planar graph has 10 vertices and 15 edges. A number of edges are removed to leave a connected graph with 10 vertices and 3 faces. The number of edges that were removed is

A 4 **B** 5 **C** 6 **D** 7 **E** 8

[VCAA 2005 1NQ5]

Question 5

$$\begin{array}{c} \\ P \\ Q \\ R \\ S \end{array} \begin{array}{c} \begin{array}{cccc} P & Q & R & S \end{array} \\ \begin{bmatrix} 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 1 \\ 2 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

The adjacency matrix represents a planar graph with four vertices. The number of faces (regions) on the planar graph is

A 1 **B** 2 **C** 3

D 4 **E** 5

[VCAA 2012 1NQ4]

Question 6

Five graphs are each represented by an adjacency matrix as shown below.

$$M = \begin{bmatrix} 0 & 2 & 1 & 0 \\ 2 & 0 & 2 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \quad N = \begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 2 \\ 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{bmatrix} \quad O = \begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \\ 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 2 \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \quad Q = \begin{bmatrix} 0 & 2 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Which adjacency matrix represents a **connected** graph?

A M **B** N **C** O **D** P **E** Q

[VCAA 2003 1NQ9]

Question 7

A connected planar graph has an **equal** number of vertices and faces.

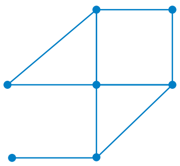
If there are 20 edges in this graph, the number of vertices must be

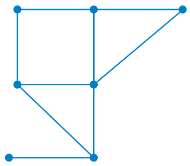
A 9 **B** 10 **C** 11 **D** 20 **E** 22

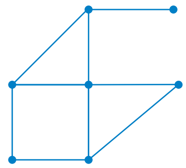
[VCAA 2003 1NQ5]

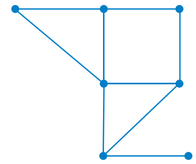
Question 8

The diagrams show four connected planar graphs.


Graph J


Graph K


Graph L


Graph M

Equivalent graphs are

A J and L only
B J and K and L only
C J and K and M only
D J and L and M only
E J and K and L and M

[VCAA 2004 1NQ2]

Question 9

A connected graph consists of five vertices and four edges. Consider the following five statements.

- The graph is planar.
- The graph has more than one face.
- All vertices are of even degree.
- The sum of the degrees of the vertices is eight.
- The graph cannot have a loop.

How many of these statements are always true for such a graph?

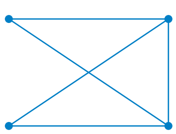
A 1
B 2
C 3
D 4
E 5

[VCAA 2013 1NQ7]

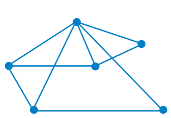
Question 10

Euler's formula, relating vertices, faces and edges, does **not** apply to which one of the following graphs?

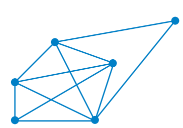
A



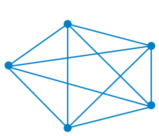
B



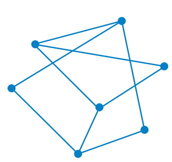
C



D



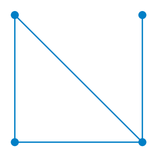
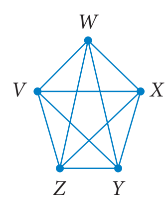
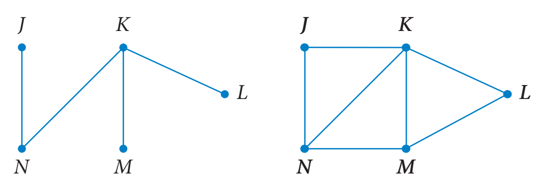
E



[VCAA 2006 1NQ8]

10.3 Types of graphs

Three types of graphs and their real-life applications are outlined below.

Definition	Diagram	Application
A **simple graph** contains no loops or multiple edges.		Roads connecting towns
A **complete graph** has every vertex connected to every other vertex. A complete graph with n vertices has $\frac{n(n-1)}{2}$ edges.		Sporting team playing a round-robin competition
A **subgraph** is a graph that exists within another graph. It contains only vertices and edges from the original graph, although it may be drawn differently to the original graph.		Electricity cables connecting houses

Exam hack

Exams often ask questions that require you to understand how a certain situation can be represented using a network diagram. Make sure you are familiar with the above graphs and their different types of applications.

Getty Images/Dan Mullan/Stringer

A complete graph with n vertices has $\frac{n(n-1)}{2}$ edges.

Worked example 8

A local soccer club is holding a round robin tournament where every team plays every other team once. There are five teams in the competition.

a What type of graph is this?

b Without constructing a graph, how many games will be played?

c Check your answer to part **b** by constructing the graph representing this tournament.

Working

a Vertices represent the teams and edges represent the games played. Every vertex will be connected to every other vertex as each team plays each other once.

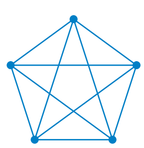
It is a complete graph.

b A complete graph with n vertices has $\frac{n(n-1)}{2}$ edges. There are 5 vertices in this graph, therefore $n = 5$.

$$\text{Edges} = \frac{n(n-1)}{2} = \frac{5(5-1)}{2} = 10$$

There are 10 edges, therefore 10 games will be played.

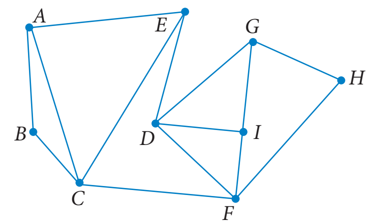
c Draw the 5 vertices and connect each vertex with an edge. Count the number of edges.



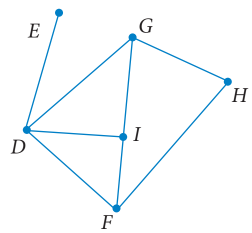
There are 10 edges, therefore part **b** is correct.

Worked example 9

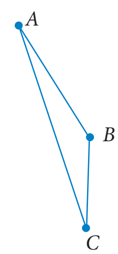
Determine which of the following are subgraphs of this graph.



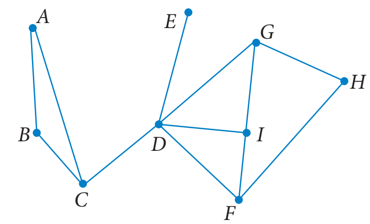
a



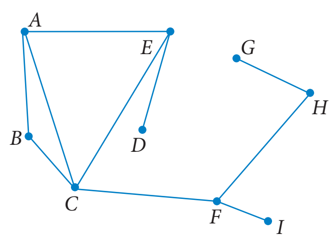
b



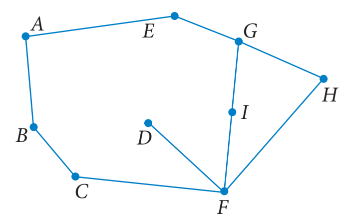
c



d



e



Working

a This graph contains only vertices and edges from the original graph.

This is a subgraph.

b This graph contains only vertices and edges from the original graph.

This is a subgraph (even though vertex B is in a different position).

c This graph contains an extra edge between C and D that wasn't in the original graph.

This is not a subgraph.

d This graph contains only vertices and edges from the original graph.

This is a subgraph (even though vertex I is in a different position).

e This graph contains an extra edge between E and G that wasn't in the original graph.

This is not a subgraph.

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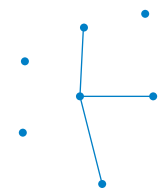
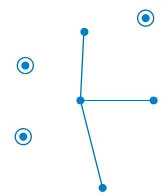
Worked example 10

Determine the smallest number of edges that need to be added to make this a connected graph.

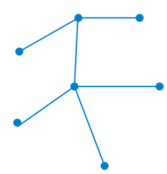


Working

1 Every vertex must have a path to every other vertex. This graph has three isolated vertices.



2 Draw edges so that these vertices are no longer isolated.



3 Three edges were drawn.

To make this graph connected, a minimum of three vertices must be added (these edges could be placed between other vertices to make another connected graph).

Worked example 11

Determine the number of edges needed to make a complete graph with 10 vertices.

Working

A complete graph with n vertices has $\frac{n(n-1)}{2}$ edges. There are 10 vertices in this graph, therefore $n = 10$.

$$\text{Edges} = \frac{n(n-1)}{2} = \frac{10(10-1)}{2} = 45$$

A complete graph with 10 vertices will have 45 edges.

EXAM PREP 10.3

Types of graphs

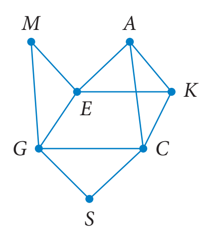
Prep 1

WORKED EXAMPLE 8

a A local netball club is holding a round-robin tournament where every team plays every other team once. There are six teams in the competition.

- **i** What type of graph is this?
- **ii** Without constructing a graph, how many games will be played?
- **iii** Check your answer to part **b** by constructing the graph representing this tournament.

b Emily has received a friend request on Facebook from Sally. To find out how she knows Sally, she has drawn the following graph with each edge representing a friendship between the two people.



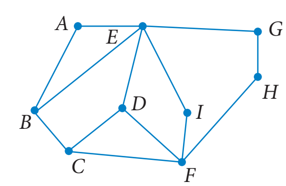
KEY:
E = Emily
G = George
K = Kath
S = Sally
A = Alison
C = Celeste
M = Mark

- **i** Who is friends with both Emily and Sally?
- **ii** Who is Emily not friends with?
- **iii** If Emily wanted to be introduced to Sally, which friend should she ask to make this introduction?

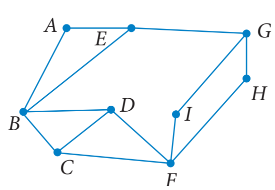
Prep 2

WORKED EXAMPLE 9

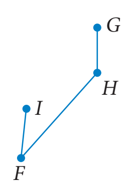
Determine which of the following are subgraphs of this graph.



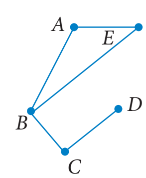
a



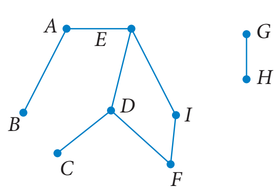
b



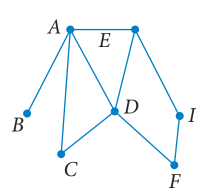
c



d



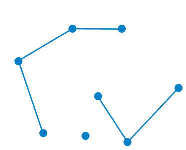
e



Prep 3

WORKED EXAMPLE 10

Determine the smallest number of edges that need to be added to make this a connected graph.



Prep 4   WORKED EXAMPLE 11

a Determine the number of edges needed to make a complete graph with 15 vertices.

b A complete graph has 66 edges. Determine how many vertices it has.

c At a maths conference, a group of 10 professors shook hands with one another. What was the number of handshakes?

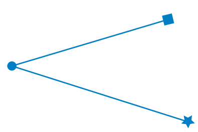
d In 2013, four teams played in a round-robin competition where each team played every other team once. In the previous year, there were six teams in the round-robin competition. Knowing that a complete graph with n vertices has $\frac{n(n-1)}{2}$ edges, calculate the number of matches played

i in 2012   **ii** in 2013.

EXAM PRACTICE 10.3

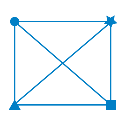
Types of graphs

Question 1

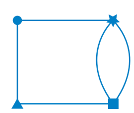


The graph is a subgraph of which one of the following graphs?

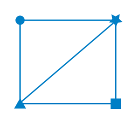
A



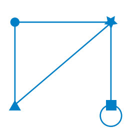
B



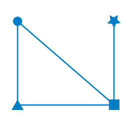
C



D

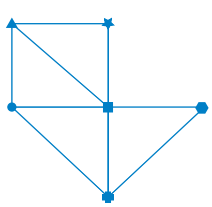


E



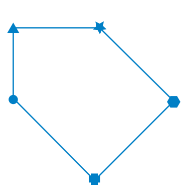
[VCAA 2010 1NQ1]

Question 2

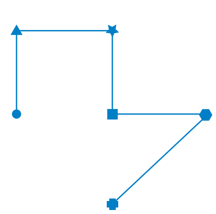


A subgraph of the network graph is

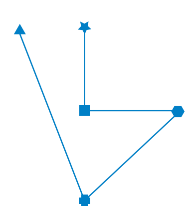
A



B

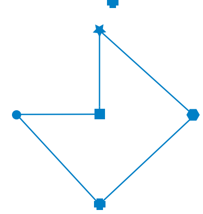


C



D

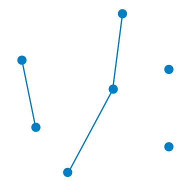
E



[VCAA 2004 1NQ1]

Question 3

Consider the following graph. The smallest number of edges that need to be added to make this a connected graph is

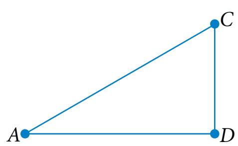


A 1 **B** 2 **C** 3

D 4 **E** 5

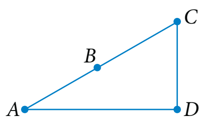
[VCAA 2009 1NQ1]

Question 4

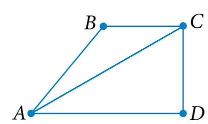


The graph is a subgraph of which of the following graphs?

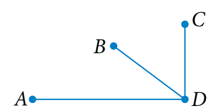
A



B

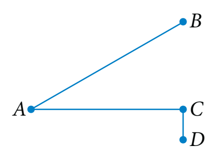


C



D

E



[VCAA 2008 1NQ2]

Question 5

A complete graph with six vertices is drawn. This network diagram would best represent

A the journey of a paper boy who delivers to six homes, covering the minimum distance.

B the cables required to connect six houses to pay television that minimises the length of cables needed.

C a six-team basketball competition where all teams play each other once.

D a project where six tasks must be performed between the start and finish.

E the allocation of different assignments to a group of six students.

[VCAA 2006 1NQ7]

Question 6

The number of edges needed to make a complete graph with four vertices is

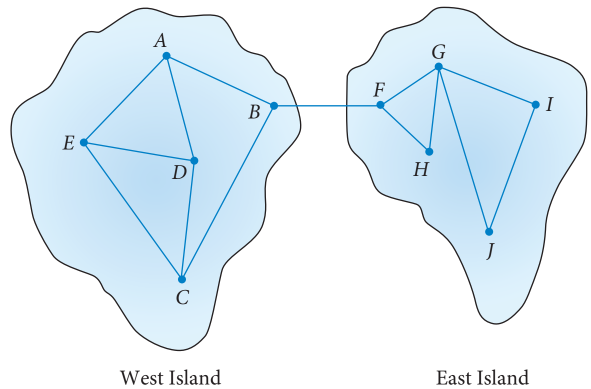
A 2 **B** 3 **C** 4 **D** 5 **E** 6

[VCAA 2013 1NQ2]

Exploring and travelling problems 10.4

Walks, trails, paths, circuits, cycles and bridges

Travelling and exploring a graph involves moving from one vertex to another via an edge and is described by listing the vertices that are visited in order. A number of terms used when discussing travelling and exploring are listed below with examples from the following graph.



Term	Description	Example
walk	Any sequence of vertices of a graph along edges which may include repeated vertices or repeated edges	*A-B-F-H-G-F-B*
trail	A walk with no repeated edges	*G-F-H-G-I-J*
path	A walk with no repeated vertices	*B-A-D-E-C*
circuit	A walk with no repeated edges that starts and finishes at the same vertex	*G-F-H-G-I-J-G*
cycle	A walk with no repeated vertices that starts and finishes at the same vertex	*A-B-C-D-E-A*
bridge	An edge that keeps a graph connected	*BF*

Exam hack

If there are no repeated vertices in a walk, there are also no repeated edges.

This means paths and cycles have no repeated edges as well as no repeated vertices.

Exam hack

A graph can have more than one bridge. Any edge that, if removed, creates an isolated vertex or that disconnects two parts of a graph is a bridge.

WS

Eulerian circuits

Eulerian trails and Eulerian circuits

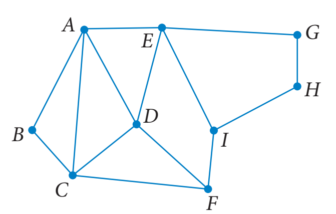
Eulerian trail

- An Eulerian trail is a walk that includes *every* edge once only, and starts and finishes at *different* vertices.
- For an Eulerian trail to exist, the connected graph must have *exactly two* vertices of odd degree.
- To find the Eulerian trail, you must start at one vertex of odd degree and finish at the other vertex of odd degree.

Eulerian circuit

- An Eulerian circuit is a walk which includes *every* edge once only and starts and finishes at the *same* vertex.
- An Eulerian circuit will only exist if *all* vertices are of even degree.

For example, this graph has exactly two vertices with odd degrees (F and I), so an Eulerian trail exists and it could start at F and finish at I, or start at I and finish at F. An Eulerian circuit doesn't exist.



Exam hack

When a question refers to an Eulerian trail or Eulerian circuit, always determine the degree of every vertex to quickly identify whether the correct conditions apply.

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Hamiltonian paths and cycles

WS
Hamiltonian paths and cycles

- A **Hamiltonian path** is a walk that includes *every vertex* in a graph once only, and starts and finishes at *different* vertices.
- A **Hamiltonian cycle** is a walk that includes *every vertex* in a graph once only, and starts and finishes at the *same* vertex.
- Unlike Eulerian trails and circuits, to determine whether a Hamiltonian path or cycle exists, you must use trial and error.

Worked example 12

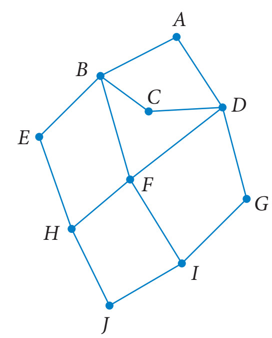
For the following graph, determine whether any of the following exist. For any that exist, describe the walk.

a Eulerian trail

b Eulerian circuit

c Hamiltonian path

d Hamiltonian cycle



Working

a **1** Determine how many vertices have an odd degree.

Vertex *I* and vertex *H* are of odd degree. Since only two vertices are of odd degree, an Eulerian trail exists.

2 The Eulerian trail must start and finish at vertices of odd degree. It must also include each edge once only.

H-E-B-A-D-C-B-F-D-G-I-F-H-J-I

b Determine if all vertices are of even degree.

From part **a**, we know that two vertices are of odd degree, therefore an Eulerian circuit does not exist.

c A Hamiltonian path must cover each vertex once only. Trial and error is the only method that can be used to decide whether such a path exists.

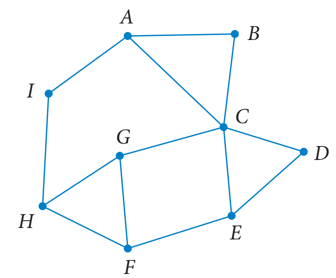
A Hamiltonian path does not exist.

d A Hamiltonian cycle must cover each vertex once only and return to the starting vertex.

A Hamiltonian cycle does not exist.

Worked example 13

A salesman is travelling to seven country towns to sell flour to the local bakeries. A map of the towns is shown, with the vertices representing the towns and the edges representing the roads connecting the towns.



The salesman wants to travel from town A to visit every town once on his journey and then return back to town A.

a What is this walk an example of?

b Describe a route the salesman could take.

c List the edges that the salesman does not travel on.

Working

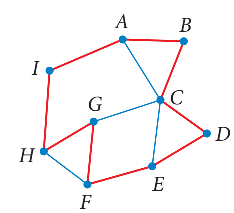
a The salesman must go to every vertex and return to the vertex he started at.

Hamiltonian cycle.

b The salesman must start at A and travel to every other town once only before returning to A.

A-I-H-G-F-E-D-C-B-A

c **1** Highlight the path the salesman took.



2 The unhighlighted edges are the edges the salesman did not travel on.

HF, GC, CE and AC.

Exam hack

To recall the difference between Eulerian and Hamiltonian, remember **E**ulerian trails and circuits must travel on every **E**dge, and **H**amiltonian paths and cycles are like visiting every **H**ouse (vertex).

Weighted graphs and Dijkstra's algorithm

A **weighted graph** is a graph where extra information such as distances, times or costs is labelled on the edges. When the extra information is a real-life quantity, the weighted graph is also called a network.

To find the shortest path on a network, do it by inspection or, if the network is complicated, use **Dijkstra's algorithm**. Dijkstra's algorithm finds the shortest path between any two vertices, or between a given vertex and every other vertex in a weighted graph.

10.4

Dijkstra's algorithm for finding the shortest path

1 Assign a value of 0 to the Start vertex, and draw a box around the vertex name and value.

2 Assign the value of each vertex connected to the Start by adding the value on the edge to the Start value of 0. Write each of these values next to their vertex names.

3 Find the vertex with the lowest value, and draw a box around the vertex name and value. All boxed values are locked in and won't change. Any unboxed value may change.

4 Consider all the unboxed vertices connected to the vertex you have just boxed. Add each edge value to the value you have just boxed, and write the answer next to each of the connecting vertices. If a connecting vertex already has a value and the answer is less than it, replace it with the new value. Otherwise leave the existing value unchanged.

5 Find the unboxed vertex with the lowest value, and draw a box around the vertex name and value. If there are two or more equal values at any stage, choose any one.

6 Repeat steps 4 and 5 until the Destination vertex and value is boxed. The value in the Destination box is the value of the shortest path.

7 To find the sequence of the shortest path, backtrack through the boxed vertices:

- Start at the Destination box and move to the connected boxed vertex whose value is the Destination value – the edge value.
- Move from this boxed vertex to the next connected boxed vertex using the same procedure.
- Continue until you reach the Start vertex.

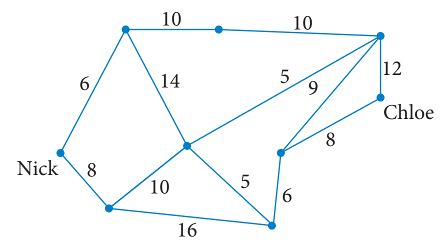
Exam hack

- The lowest value can be anywhere on the diagram. Don't just look at the most recent values that you've entered.
- Remember that the algorithm is only complete when you have boxed the destination value, not just when you give the destination vertex a value.
- Sometimes there are two or more equal shortest paths. The backtracking process gives you all of these.

Worked example 14

The network shows the travel times, in minutes, along a series of roads that connect Chloe's home to Nick's home.

Determine the shortest time, in minutes, that it takes Nick to travel to Chloe's house.



Working

As this network is not too complex, it is quicker to determine the shortest path by inspection rather than by Dijkstra's algorithm.

$8 + 10 + 5 + 12 = 35$

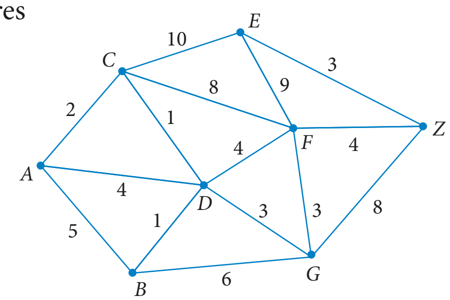
The shortest time needed to get from Nick's to Chloe's house is 35 minutes.

Worked example 15

The map shows the paths that connect different enclosures in a zoo. The zookeeper wants to lock the front gate at *A* and travel to the exit gate *Z* in the shortest time.

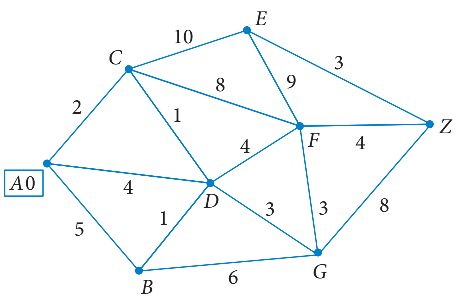
The travel times (in minutes) are allocated on each edge.

Determine the length of the shortest path from *A* to *Z* and draw this path on the map.



Working

1 Use Dijkstra's algorithm to find the shortest path. Draw a box around the starting vertex *A* and enter a value of zero.



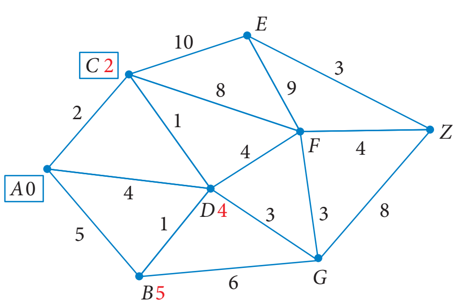
2 The boxed vertex *A* is connected to vertices *B*, *C* and *D*. Add the times on the connecting edges to the boxed value of 0 and write the sums next to the vertices, shown in red on the diagram.

Vertex $B = 0 + 5 = 5$

Vertex $C = 0 + 2 = 2$

Vertex $D = 0 + 4 = 4$

In this case the smallest value is 2, so box *C* 2. Once a value is boxed, it doesn't change. Unboxed values may change as we continue.



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3 Consider all the unboxed vertices connected to the vertex C, which are E, F and D.

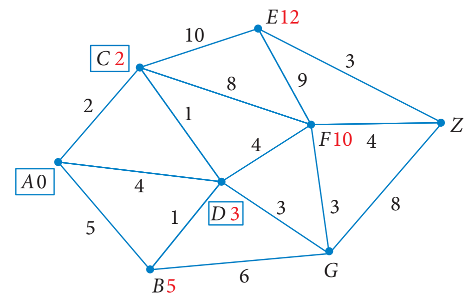
Vertex $E = 2 + 10 = 12$

Vertex $F = 2 + 8 = 10$

Vertex $D = 2 + 1 = 3$

A D value already exists. The new D value of 3 is smaller than the old value of 4, so replace D 4 with D 3.

Box the smallest value, which is D 3.



4 Consider all the unboxed vertices connected to the vertex D:

Vertex $B = 3 + 1 = 4$

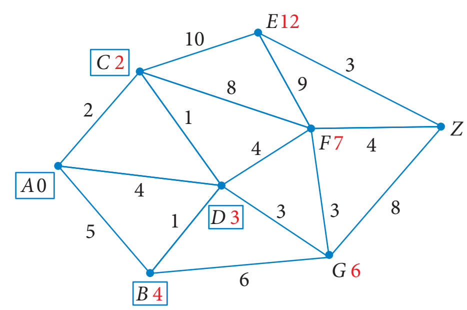
Vertex $F = 3 + 4 = 7$

Vertex $G = 3 + 3 = 6$

The new value of B is smaller than the old value of B so replace B 5 with B 4.

The new value of F is smaller than the old value of F so replace F 10 with F 7.

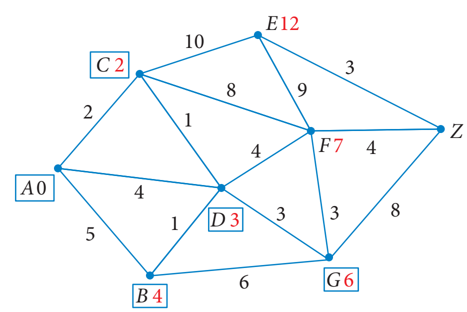
Box the smallest value, which is B 4.



5 The only unboxed vertex from B is G.

Vertex $G = 4 + 6 = 10$

The value already at G is smaller than this, so it should remain G 6. Box G 6.



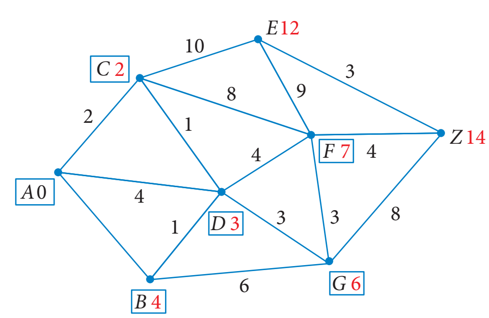
6 From G, the unboxed vertices are F and Z.

Vertex $F = 6 + 3 = 9$

The value already at F is smaller than this, so it should remain F 7. Box F 7.

Vertex $Z = 6 + 8 = 14$

Box the smaller value, F 7.



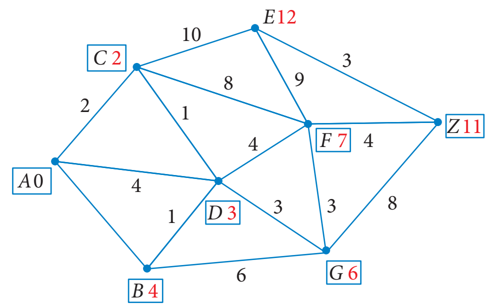
7 From F 7:

Vertex $E = 7 + 9 = 16$

Vertex $Z = 7 + 4 = 11$

The value already at E is smaller than 16, so it should remain E 12.

The value at Z is currently 14, so change this to 11. Z 11 is smaller so box it.



8 Since the destination vertex Z is now boxed, we have found the shortest path. The value of Z is the number of minutes. To find the path, backtrack from Z to A.

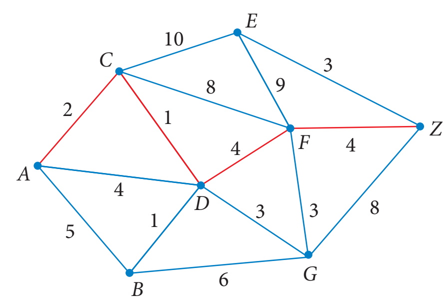
Start from Z and move to F, since $11 - 4 = 7$.

From F move to D, since $7 - 4 = 3$.

From D move to C, since $3 - 1 = 2$.

From C move to A (the start vertex), since $2 - 2 = 0$.

The shortest time path of 11 minutes is A-C-D-F-Z.

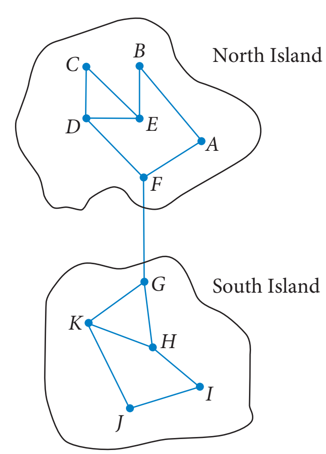


EXAM PREP 10.4

Exploring and travelling problems

Prep 1

Zlatko is touring two islands off the coast of Australia. Define the following as a walk, trail, path, circuit, cycle or bridge.



a	G-K-H-I-J-K-H-G	**b**	FG
c	C-E-B-A-F-D-C	**d**	D-E-C-D-F-G
e	G-K-J-I-H-G	**f**	A-F-G-K-H

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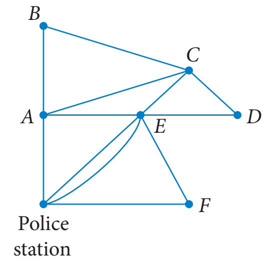
Prep 2 WORKED EXAMPLE 12

Police are patrolling inner-city streets to prevent crime. Police officials must determine the best route for officers so that they do not walk along a street more than once. Officers must start and finish at the police station.



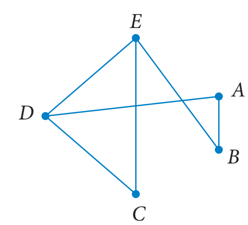
a Describe a route that the officers can take.

b Explain why this graph has an Eulerian circuit and not an Eulerian trail.

c If one of the roads connecting the police station with E was closed, would officers still be able to patrol every road without going over the same road twice? (They don't have to return to the police station.)

Prep 3 WORKED EXAMPLE 13

The Nixon family is travelling to five country towns to visit relatives. A map of the towns is shown on the right, with the vertices representing the towns and the edges representing the roads connecting the towns.



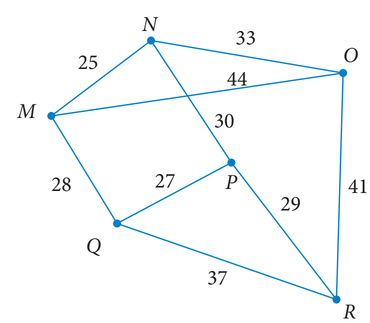
The family wants to travel from town A, visit every town once and then return to town A.

a Describe a route that the family could take.

b List the edge(s) that the family does not travel on.

Prep 4 WORKED EXAMPLE 14

Merrick, Nottinham, Opple, Printone, Quetec and Rusher are six towns shown on the following graph. The numbered edges show how the towns are connected by roads and the distances between these towns in kilometres.



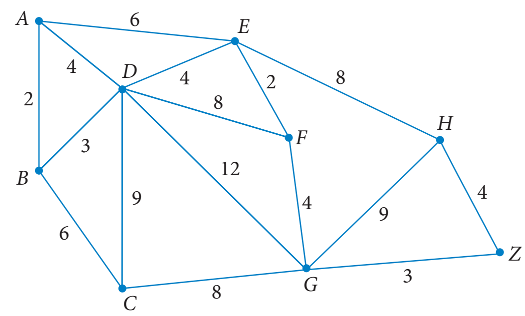
a In kilometres, what is the shortest distance between Nottinham and Rusher?

b How many different ways are there to travel between Nottinham and Rusher without passing through any town more than once? List the paths.

c A visiting politician wants to visit each town. What is the shortest distance that the politician has to travel if she starts at Merrick and finishes at Rusher?

10.4

Prep 5 | WORKED EXAMPLE 15



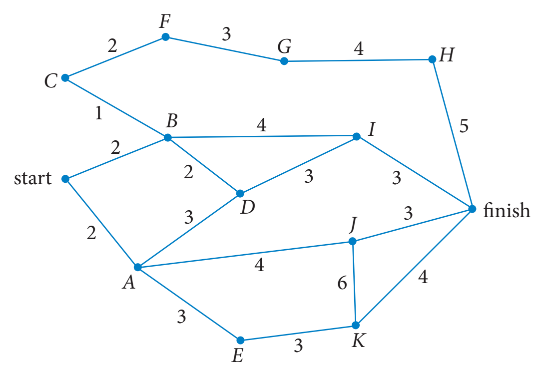
The following network represents the paths between buildings at a local secondary school. The vertices represent the buildings and the weights on the edges represent the time, in minutes, it takes to walk along the path between adjacent buildings.

Connor has to travel from his locker located at building A to his maths class in building Z.

a Determine the shortest time it will take Connor to get from his locker to his maths class.

b State the path that Connor took.

Prep 6 | WORKED EXAMPLE 15



Hayley, Danielle and Karen participated in a race through a local national park. Each participant took a different route from start to finish. The graph shows the distances in kilometres between adjacent points.

a Hayley went from the start and followed the path B-C-F, believing it to be the start of the shortest path. If she continued onto GH to the finish, how far did she travel?

b When Hayley arrived she realised that Danielle and Karen had already arrived. Despite both taking a different route, Danielle and Karen each travelled the same minimum distance. What was that distance?

c Describe the two routes Danielle and Karen travelled.

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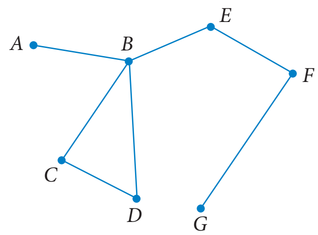
Exploring and travelling problems

Use the following information to answer Questions 1 & 2.



Question 1

How many bridges are there in the graph?

A 0 **B** 1 **C** 2 **D** 3 **E** 4

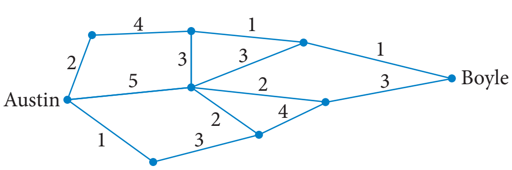
Question 2

Which of the follow statements about the graph is *not* true?

A *G-F-E-B-A* is a path
B *A-B-C-D-B-E* is a trail
C *B-C-D-B* is a cycle
D *E-B-C-D-B-E* is a circuit
E *E-B-C-D-B-E-F* is a walk

Question 3

The network shows the distances, in kilometres, along roads that connect the cities of Austin and Boyle.



The shortest distance, in kilometres, from Austin to Boyle is

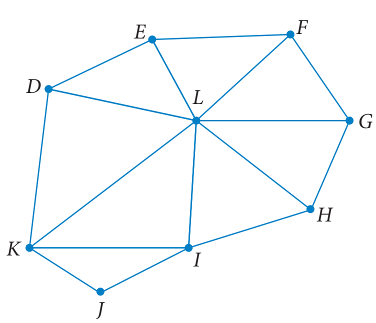
A 7 **B** 8 **C** 9 **D** 10 **E** 11

[VCAA 2009 1NQ2]

Question 4

A Hamiltonian cycle for the graph is

A *KJIHGLFEDK*
B *DKLIJHGFED*
C *DEFGHIJKD*
D *JIKDLHGFE*
E *GHILKJILDEFG*

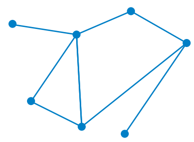


[VCAA 2008 1NQ3]

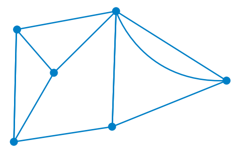
Question 5

Which one of the following graphs contains an Eulerian circuit?

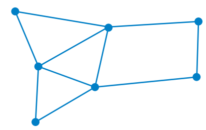
A



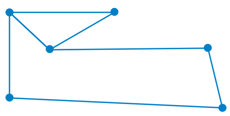
B



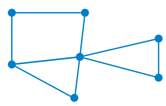
C



D

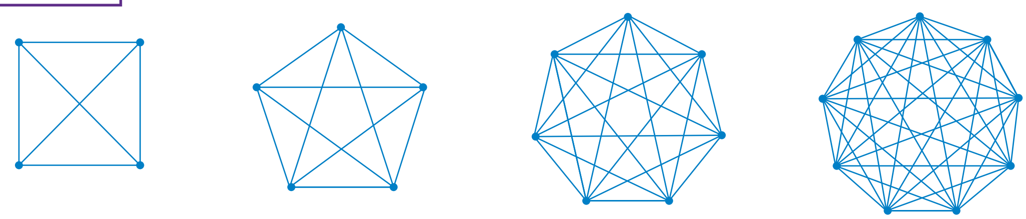


E



[VCAA 2002 1NQ4]

Question 6



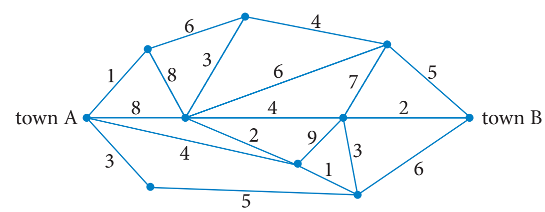
How many of the four complete graphs above will have an Eulerian circuit?

A 0 **B** 1 **C** 2 **D** 3 **E** 4

[VCAA 2012 1NQ5]

Question 7

The following network shows the distances, in kilometres, along a series of roads that connect town A to town B.



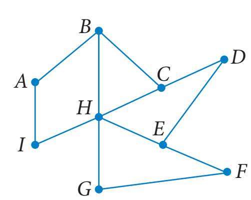
The shortest distance, in kilometres, to travel from town A to town B is

A 9 **B** 10 **C** 11 **D** 12 **E** 13

[VCAA 2007 1NQ4]

Question 8

For the network diagram, an Eulerian **trail** can be found



A without altering the network diagram.

B by adding an edge that joins A to H.

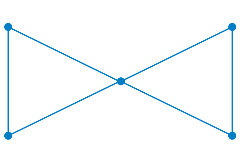
C by adding an edge that joins C to F.

D by removing the edge that joins B to C.

E by removing the edge that joins D to E.

[VCAA 2005 1NQ2]

Question 9



The number of Hamiltonian cycles involving all five vertices in the graph is

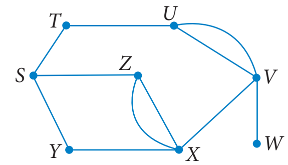
A 0 **B** 1 **C** 2

D 3 **E** 4

[VCAA 2012 1NQ2]

Question 10

In the network diagram, an Eulerian trail can be created by adding **one** new edge. Adding which one of the following edges creates an Eulerian trail?

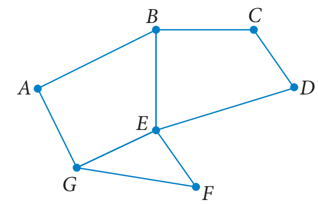


A *ST* **B** *SU* **C** *SX*
D *XW* **E** *ZY*

[VCAA 2003 1NQ3]

Question 11

Which one of the following statements is **true** regarding the network diagram?



A *ABCDEFG* is a Hamiltonian cycle.
B Only one Hamiltonian path exists.
C *CBAGFEDC* is an Eulerian circuit.
D At least two Eulerian trails exist.
E There are no circuits.

[VCAA 2006 1NQ3]

Question 12

A connected planar graph has five vertices, *A*, *B*, *C*, *D* and *E*. The degree of each vertex is given in the table.

Vertex	Degree
A	3
B	4
C	3
D	5
E	3

Which of the following statements regarding this planar graph is true?

A The sum of the vertices equals 15.
B It contains more than one Eulerian trail.
C It contains an Eulerian circuit.
D Euler's formula $v + f = e + 2$ could not be used.
E The addition of one further edge could create an Eulerian trail.

[VCAA 2008 1NQ5]

Question 13

An Eulerian trail through a network diagram commences at vertex *P* and ends at vertex *Q*. Consider the following five statements about this Eulerian trail and network.

- In the network diagram, there could be three vertices with degree equal to one.
- The trail could have passed through an isolated vertex.
- The trail could have included vertex *Q* more than once.
- The sum of the degrees of vertices *P* and *Q* could equal seven.
- The sum of the degrees of all vertices in the network diagram could equal seven.

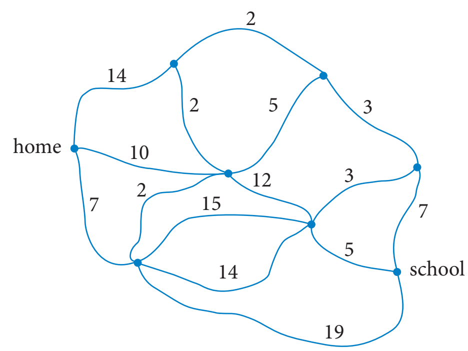
How many of these statements are true?

A 0 **B** 1 **C** 2 **D** 3 **E** 4

[VCAA 2011 1NQ9]

Question 14

The network shows the travel times, in minutes, along a series of roads that connect a student's home to school.



The shortest time, in minutes, for this student to travel from home to school is

A 22 **B** 23 **C** 24

D 25 **E** 26

[VCAA 2003 1NQ8]

Question 15

An undirected connected graph has five vertices.

Three of these vertices are of even degree and two of the existing vertices are of odd degree.

One extra edge is added. It joins two of the existing vertices.

In the resulting graph, it is **not** possible to have five vertices that are

A all of even degree.

B all of equal degree.

C one of even degree and four of odd degree.

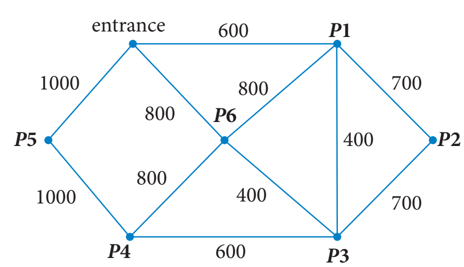
D three of even degree and two of odd degree.

E four of even degree and one of odd degree.

[VCAA 2009 1NQ8]

Question 16

The vertices in the network diagram show the entrance to a wildlife park and six picnic areas in the park: *P*1, *P*2, *P*3, *P*4, *P*5 and *P*6. The numbers on the edges represent the lengths, in metres, of the roads joining these locations.



a In this graph, what is the degree of the vertex at the entrance to the wildlife park? 1 mark

b What is the shortest distance, in metres, from the entrance to picnic area *P*3? 1 mark

c A park ranger starts at the entrance and drives along every road in the park once.

  i At which picnic area will the park ranger finish? 1 mark

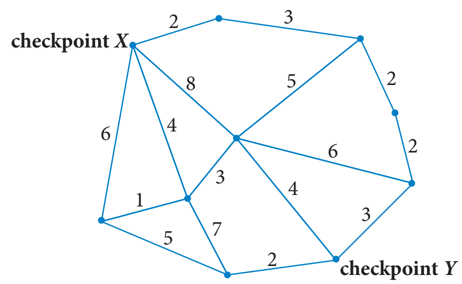
  ii What mathematical term is used to describe the route the park ranger takes? 1 mark

d A park cleaner follows a route that starts at the entrance and passes through each picnic area once, ending at picnic area *P*1. Write down the order in which the park cleaner will visit the six picnic areas. 1 mark

[VCAA 2013 2NQ1]

Question 17

The diagram shows a network of tracks (represented by edges) between checkpoints (represented by vertices) in a short-distance running course. The numbers on the edges indicate the time, in minutes, a team would take to run along each track.



A challenge requires teams to run from checkpoint X to checkpoint Y using these tracks.

a What would be the shortest possible time taken for a team to run from checkpoint X to checkpoint Y? 1 mark

b Teams are required to follow a route from checkpoint Y that passes through every checkpoint once only.

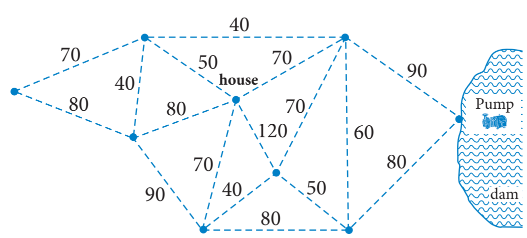
i What mathematical term is used to describe such a route? 1 mark

ii Copy the network diagram and draw in the route from checkpoint X to checkpoint Y that passes through every checkpoint once only. 1 mark

[VCAA 2010 2NQ2]

Question 18

Water will be pumped from a dam to eight locations on a farm. The pump and the eight locations (including the house) are shown as vertices in the network diagram. The numbers on the edges joining the vertices give the shortest distances, in metres, between locations.



a Determine the shortest distance between the house and the pump. 1 mark

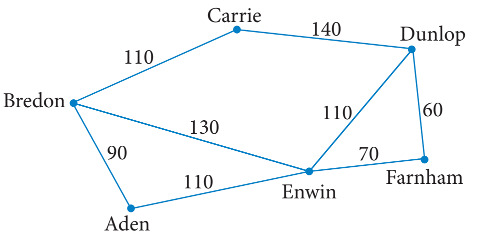
b How many vertices on the network diagram have an odd degree? 1 mark

c The total length of all edges in the network is 1180 metres. A journey starts and finishes at the house and travels along every edge in the network. Determine the shortest distance travelled. 1 mark

[VCAA 2012 2NQ1a]

Question 19

Aden, Bredon, Carrie, Dunlop, Enwin and Farnham are six towns. The network shows the road connections and distances between these towns in kilometres.



a In kilometres, what is the shortest distance between Farnham and Carrie? 1 mark

b How many different ways are there to travel from Farnham to Carrie without passing through any town more than once? 1 mark

An engineer plans to inspect all of the roads in this network. He will start at Dunlop and inspect each road only once.

c At which town will the inspection finish? 1 mark

Another engineer decides to start and finish her road inspection at Dunlop.

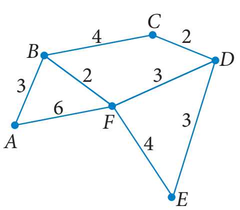
If an assistant inspects **two** of the roads, this engineer can inspect the remaining six roads and visit each of the other five towns only once.

d How many kilometres of road will the assistant need to inspect? 1 mark

[VCAA 2011 2NQ1]

Question 20

The following network diagram shows the distances, in kilometres, along the roads that connect six intersections A, B, C, D, E and F.



a If a cyclist started at intersection B and cycled along every road in this network once only, at which intersection would she finish? 1 mark

b The next challenge involves cycling along every road in this network at least once. Teams have to start and finish at intersection A.

The blue team does this and cycles the shortest possible total distance.

i Apart from intersection A, through which intersections does the blue team pass more than once? 1 mark

ii How many kilometres does the blue team cycle? 1 mark

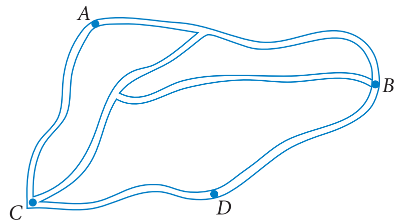
c The red team does not follow the rules and cycles along a bush path that connects two of the intersections. This route allows the red team to ride along every road only once. Which two intersections does the bush path connect? 1 mark

[VCAA 2010 2NQ3]

Question 21

Four children each live in a different town.

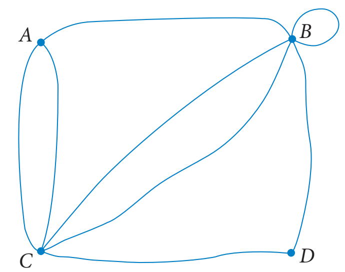
The following is a map of the roads that link the four towns, A, B, C and D.



a In how many different ways may a vehicle travel from town A to town D without travelling along any road more than once? 1 mark

James' father has begun to draw a network diagram that represents all the routes between the four towns on the map.

In this network diagram, vertices represent towns and edges represent routes between towns that do not pass through any other town.



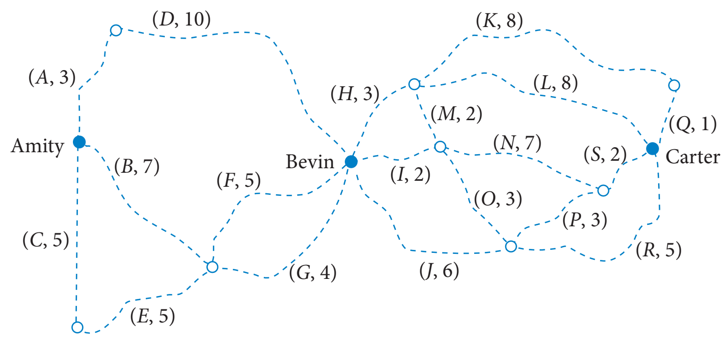
b **i** One more edge needs to be added to complete this network diagram. Copy the diagram and draw in this edge clearly. 1 mark

ii With reference to the network diagram, explain why a motorist at A could not drive each of these routes once only and arrive back at A. 1 mark

[VCAA 2008 2NQ2]

Question 22

The map below shows the roads that connect the towns of Amity, Bevin and Carter. The towns and major intersections (indicated by open circles) form the nodes of this network of roads. The labels on the roads indicate their names and lengths in kilometres. For instance, (E, 5) indicates that Road E is 5 km long.



a What is the length (in kilometres) of the shortest path from Amity to Bevin? 1 mark

b Copy the map and draw this path on it. 1 mark

[VCAA 2002 2NQ1]

10.5 Trees and minimum spanning trees

Trees and spanning trees

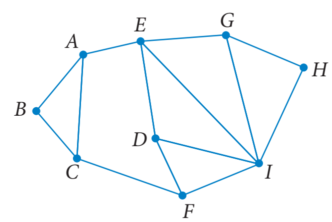
A **tree** is a connected graph with no loops, multiple edges, or cycles.

A **spanning tree** is a tree subgraph that includes all the vertices of the original graph.

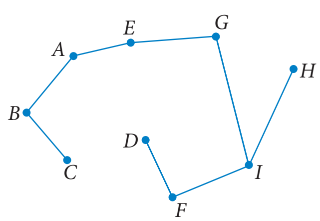
> Every connected graph contains one or more trees.
>
> A tree with n vertices has $n - 1$ edges.

Worked example 16

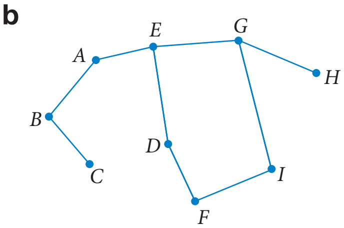
Determine which of the following graphs are a spanning tree of the graph on the right, and for those that are, verify that they each have $n - 1$ edges, where n is the number of vertices.



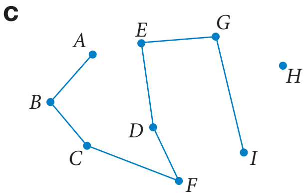
a



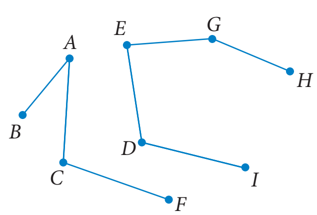
b



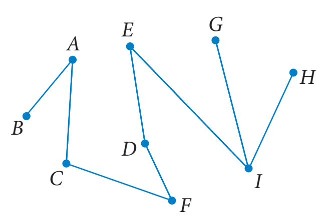
c



d



e



Working

a This graph is connected, has all the vertices of the original graph, and has no loops, multiple edges, or cycles.

This graph is a spanning tree.

It has $n = 9$ vertices and $n - 1 = 8$ edges.

b This graph has a cycle.

This graph is not a spanning tree.

c This graph has an isolated vertex, so it's not connected.

This is not a spanning tree.

d This graph is not connected.

This is not a spanning tree.

e This graph is connected, has all the vertices of the original graph, and has no loops, multiple edges, or cycles.

This is a spanning tree.

It has $n = 9$ vertices and $n - 1 = 8$ edges.

Shutterstock.com/nostal6ie

Minimum spanning trees

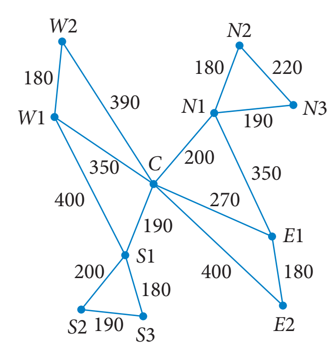
WS
Minimum spanning trees

A **minimum spanning tree** is a spanning tree for weighted graphs that represents the minimum distance (or minimum time or minimum cost) between all the vertices. Minimum spanning trees have practical applications such as finding the minimum length of pipes required in a water system, or the minimum length of pathways required to connect buildings.

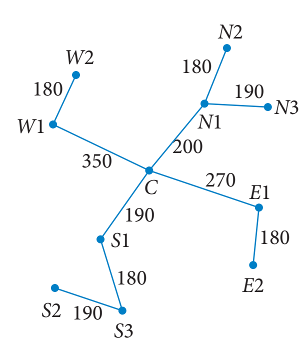
Prim's algorithm

Prim's algorithm is a series of steps to find a minimum spanning tree for a graph.

The map on the right outlines where certain buildings are within a university. The vertices represent the buildings and the edges represent existing paths (and their distances in metres).



The maintenance team at the university want to cover some of the paths so that, no matter which building the students need to go to, they will be able to stay dry when it is raining. The team want to achieve this by covering the shortest distance possible. We use Prim's algorithm to find the mimimum spanning tree to solve this problem.



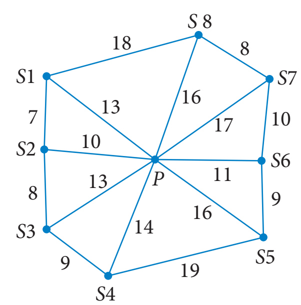
Prim's algorithm to find a minimum spanning tree

1. Choose any vertex within the network.
2. Find the edge with the lowest value connecting this vertex to another vertex.
3. Look at *all* the edges connecting these two vertices (not just the last one connected) and choose the edge that has the lowest value.
4. Look at *all* the vertices covered so far (not just the last one connected) and select the edge with the lowest value.
5. Repeat step 4 until all the vertices in the graph are included in the tree.

Note that if there is more than one edge with the lowest value, then just choose any one of them.

Worked example 17

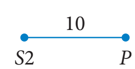
Edgar wishes to connect electricity to 8 sites in a caravan park. This weighted graph shows the direct distance between each site and the next (in metres) and the location of the electricity source (P).



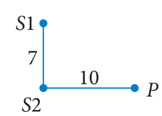
Use Prim's algorithm to find the smallest amount of cable needed to connect electricity to all 8 sites.

Working

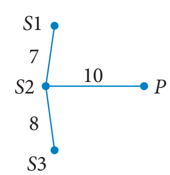
1 Start at P and select the edge with the shortest length.



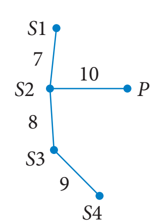
2 Then choose the shortest edge connecting either P or $S2$.



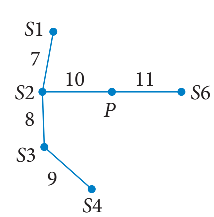
3 Choose the shortest edge connecting P, $S2$ or $S1$.



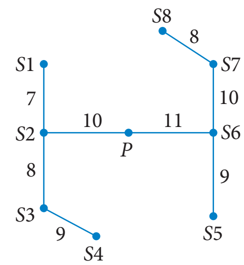
4 Choose the shortest edge connecting P, $S2$, $S1$ or $S3$.



5 Choose the shortest edge connecting P, $S2$, $S1$, $S3$ or $S4$.



6 Continue this process until all vertices are connected.

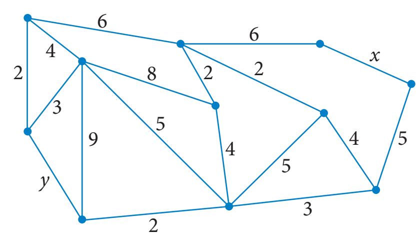


7 Add all the edges together in the minimum spanning tree to get the minimum length of cable required.

$7 + 8 + 9 + 10 + 11 + 8 + 10 + 9 = 72$ m

Worked example 18

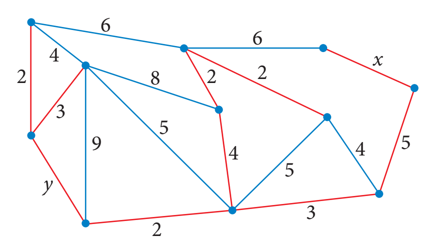
The minimal spanning tree for the network includes edges with weightings x and y.



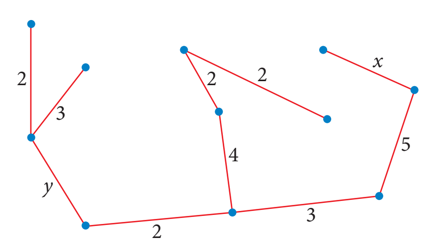
The length of the minimal spanning tree is 25. Determine the values of x and y.

Working

1 Use Prim's algorithm to determine the minimum spanning tree.



2 The spanning tree can easily be seen when the non-required edges are removed.



3 The minimum spanning tree must add to 25.

$$2+3+2+4+2+2+3+5+x+y=25$$
$$23+x+y=25$$
$$x+y=2$$

Since each edge must have a weight, x and y are both equal to 1.

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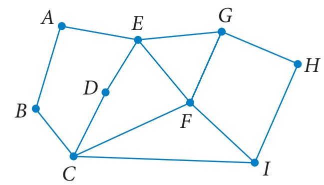
EXAM PREP 10.5

Trees and minimum spanning trees

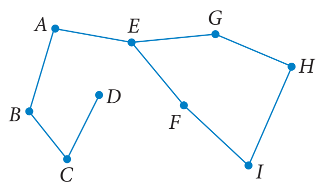
Prep 1

WORKED EXAMPLE 16

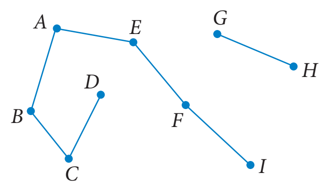
Determine which of the following graphs are a spanning tree of this graph shown, and for those that are, verify that they each have $n - 1$ edges, where n is the number of vertices.



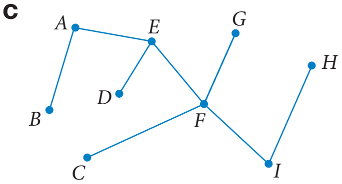
a



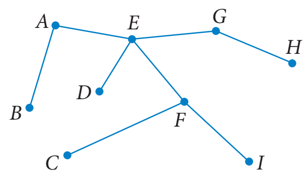
b



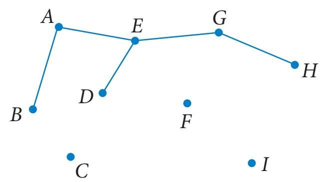
c



d



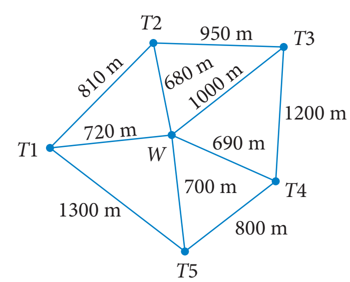
e



Prep 2

WORKED EXAMPLE 17

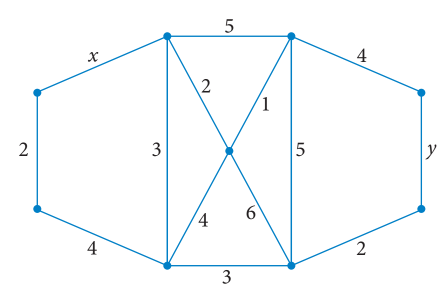
Elijah wishes to connect plumbing to 5 toilet blocks in a national park. Use Prim's algorithm to determine the smallest length of piping required to connect water from the water source (W) to the 5 toilet blocks.



Prep 3

WORKED EXAMPLE 18

The minimal spanning tree for the network includes edges with weightings x and y. The length of the minimal spanning tree is 15. Determine the values of x and y.



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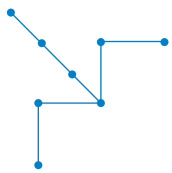
EXAM PRACTICE 10.5

Trees and minimum spanning trees

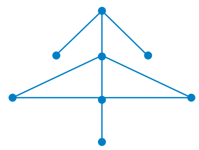
Question 1

Which of the following graphs is a tree?

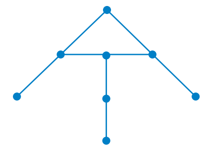
A



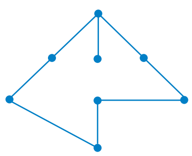
B



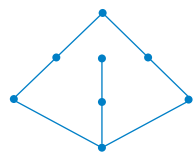
C



D



E



[VCAA 2013 1NQ1]

Question 2

A store manager is directly in charge of five department managers. Each department manager is directly in charge of six sales people in their department. This staffing structure could be represented graphically by

A a tree

B a circuit

C an Eulerian path

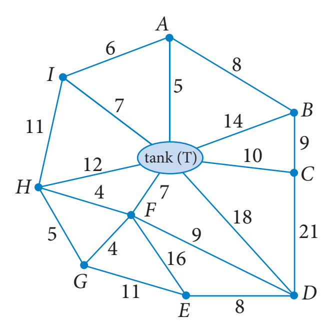
D a Hamiltonian path

E a complete graph

[VCAA 2011 1NQ6]

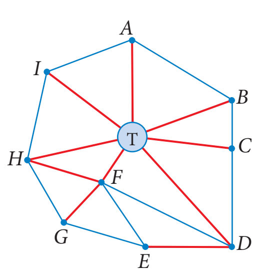
Question 3

The network diagram shows distances (in metres) between different points in a garden, *A*, *B*, *C*, *D*, *E*, *F*, *G*, *H* and *I*. Also shown are the distances from a tank (*T*) to these points. Pipes will carry water to each of these nine points. In the network diagrams, the water pipes are indicated in red.

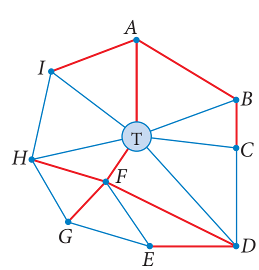


Which of these network diagrams (including **D** and **E** next page) shows the minimum length of water pipe required to connect the tank (*T*) to each of the nine points?

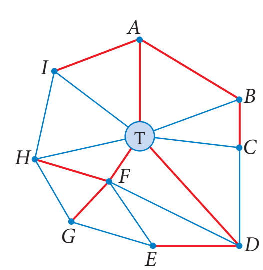
A



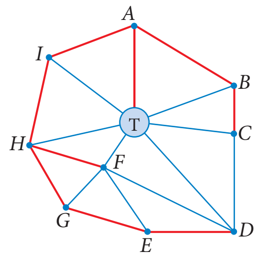
B



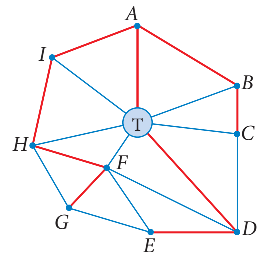
C



D



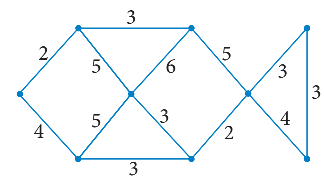
E



[VCAA 2005 1NQ4]

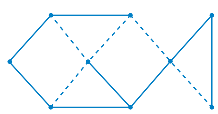
Question 4

The vertices of the graph represent nine computers in a building. The computers are to be connected with optical fibre cables, which are represented by edges. The numbers on the edges show the costs, in hundreds of dollars, of linking these computers with optical fibre cables.

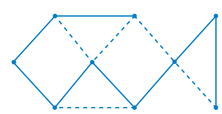


Based on the same set of vertices and edges, which one of the following graphs shows the cable layout (in bold) that would link all the computers with optical fibre cables for the minimum cost?

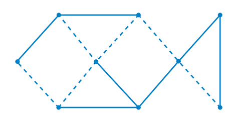
A



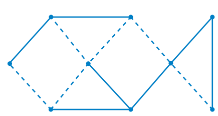
B



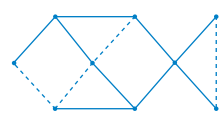
C



D



E



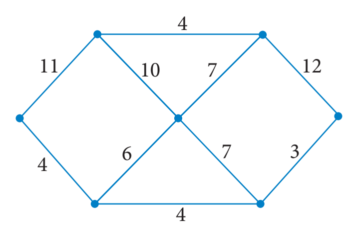
[VCAA 2013 1NQ3]

Question 5

The length of the minimal spanning tree for this network is

A 15 **B** 22 **C** 28

D 34 **E** 35



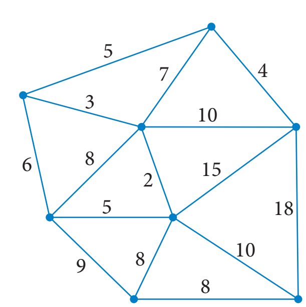
[VCAA 2004 1NQ3]

Question 6

For the network, the length of the minimal spanning tree is

A 30 **B** 31 **C** 35

D 39 **E** 45



[VCAA 2010 1NQ5]

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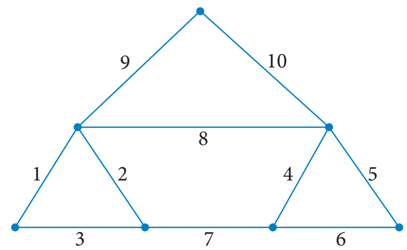
Question 7

The minimal spanning tree for the network will include the edge that has a weight of

A 3 **B** 6 **C** 8

D 9 **E** 10



[VCAA 2006 1NQ4]

Question 8

A simple connected graph with 3 edges has 4 vertices. This graph must be

A a complete graph **B** a tree

C a non-planar graph **D** a graph that contains a loop

E a graph that contains a circuit

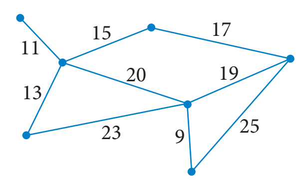
[VCAA 2008 1NQ4]

Question 9

For the graph, the total weight on the minimum weight spanning tree is

A 9 **B** 25 **C** 42

D 84 **E** 85



[VCAA 2002 1NQ6]

Question 10

Underground water pipes are needed to water a new golf course. Water will be pumped from the dam in the back corner of the course. To find the smallest total length of water pipe needed, we must find

A a critical path **B** a minimal spanning tree

C the shortest Eulerian circuit **D** the shortest Hamiltonian cycle

E the perimeter of the golf course

[VCAA 2004 1NQ5]

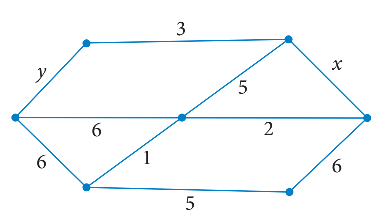
Question 11

The minimal spanning tree for the network includes two edges with weightings x and y. The length of the minimal spanning tree is 19. The values of x and y could be

A $x = 1$ and $y = 7$ **B** $x = 2$ and $y = 5$

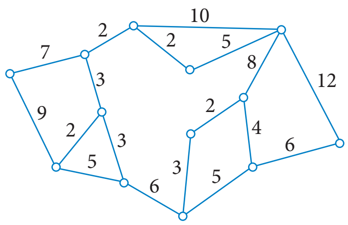
C $x = 3$ and $y = 5$ **D** $x = 4$ and $y = 5$

E $x = 5$ and $y = 6$



[VCAA 2007 1NQ7]

Question 12



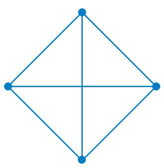
The length of the minimal spanning tree for this network is

A 37 **B** 38 **C** 45

D 47 **E** 51

[VCAA 2003 1NQ6]

Question 13



A mathematical term that could **not** be used to describe the graph shown is

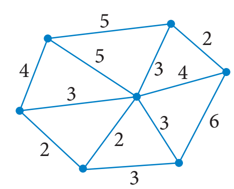
A complete **B** planar **C** simple

D undirected **E** tree

[VCAA 2007 1NQ1]

Question 14

James, Dante, Tahlia and Chanel are four children playing a game.
In this children's game, seven posts are placed in the ground. The network shows the distances, in metres, between the seven posts. The aim of the game is to connect the posts with ribbon using the shortest length of ribbon. This will be a minimal spanning tree.



a Copy the diagram and draw in a minimal spanning tree for this network. 1 mark

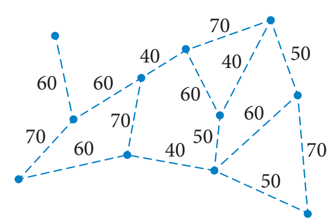
b Determine the length, in metres, of this minimal spanning tree. 1 mark

c How many different minimal spanning trees can be drawn for this network? 1 mark

[VCAA 2008 2NQ1]

Question 15

At the Farnham showgrounds, eleven locations require access to water. These locations are represented by vertices on the network diagram shown. The dashed lines on the network diagram represent possible water pipe connections between adjacent locations. The numbers on the dashed lines show the minimum length of pipe required to connect these locations in metres.



All locations are to be connected using the smallest total length of water pipe possible.

a Copy the diagram and show where these water pipes will be placed. 1 mark

b Calculate the total length, in metres, of water pipe that is required. 1 mark

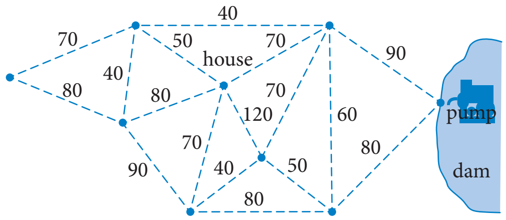
[VCAA 2011 2NQ2]

Question 16

The total length of pipe that supplies water from the pump to the eight locations on the farm is a minimum. This minimum length of pipe is laid along some of the edges in the network.

a Copy the diagram and **draw** in the minimum length of pipe that is needed to supply water to all locations on the farm. 1 mark

b What is the mathematical term that is used to describe this minimum length of pipe in part **a**? 1 mark



[VCAA 2012 2NQ1b]

Question 17

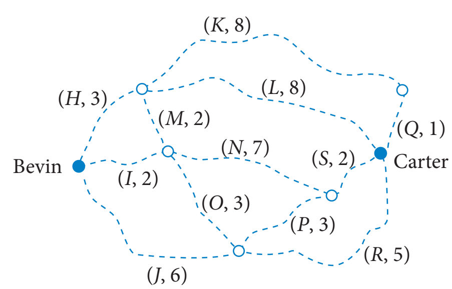
The Water Authority wants to lay water mains along the roads in order to put a fire hydrant at every node on the network shown on this map section. It decides that a minimal spanning tree for this network is suitable.

a Copy the map section and draw in a minimal spanning tree for this network.

b Each week, Andrew, who lives in Bevin, must travel through this network to inspect each of the fire hydrants and then return to Bevin.

i Write down, in order, the road sections that Andrew must travel to complete a circuit of shortest length, beginning at Bevin. He does this by travelling along a circuit that prevents him from travelling along any road more than once. 1 mark

ii How many kilometres is the total length of this shortest circuit? 2 marks



[VCAA 2002 2NQ3]

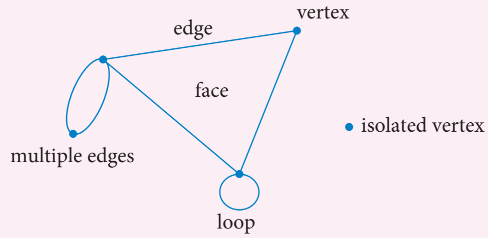
SUMMARY

10

Undirected graphs

Practice quiz

- A **graph** is a network diagram that consists of points, called **vertices**, which are connected by lines, called **edges**.



- An **adjacency matrix** shows the number of edges connecting the vertices in a graph.

$$\begin{array}{c c} & \begin{array}{ccc} A & B & C \end{array} \\ \begin{array}{c} A \\ B \\ C \end{array} & \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \end{array}$$

- A **planar graph** is a connected graph that has no edges that cross. **Euler's formula** can be used to determine whether a graph is planar: $v + f - e = 2$
- A **simple graph** contains no loops or multiple edges.
- A **complete graph** has every vertex connected to every other vertex. A complete graph with n vertices has $\frac{n(n-1)}{2}$ edges.
- A **subgraph** is a graph that exists within another graph. It contains only vertices and edges from the original graph, although it may be drawn differently to the original graph.

Walks, trails, paths, circuits, cycles and bridges

- A **walk** is any sequence of vertices of a graph along edges.
- A **trail** is a walk with no repeated edges.
- A **path** is a walk with no repeated vertices.
- A **circuit** is a walk with no repeated edges that starts and finishes at the same vertex.
- A **cycle** is a walk with no repeated vertices that starts and finishes at the same vertex.
- A **bridge** is an edge that keeps a graph connected.
- An **Eulerian trail** is a walk which includes *every edge* once only and starts and finishes at *different* vertices. For an Eulerian trail to exist, the connected graph must have *exactly two* vertices of odd degree. The trail must start at one vertex of odd degree and finish at the other vertex of odd degree.

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- An **Eulerian circuit** is a walk which includes *every edge* once only and starts and finishes at the *same* vertex. An Eulerian circuit will only exist if all vertices are of even degree.
- A **Hamiltonian path** is a walk that includes *every vertex* in a graph once only, and starts and finishes at *different* vertices.
- A **Hamiltonian cycle** is a walk that includes *every vertex* in a graph once only, and starts and finishes at the *same* vertex.
- Trial and error needs to be used to determine whether a Hamiltonian path or cycle exists.

Weighted graphs and Dijkstra's algorithm

A **weighted graph** has extra information such as distances, times or costs labelled on the edges. **Dijkstra's algorithm** is used to find the shortest path between two vertices:

1. Assign a value of 0 to the Start vertex, and draw a box around the vertex name and value.
2. Assign the value of each vertex connected to the Start by adding the value on the edge to the Start value of 0. Write each of these values next to their vertex names.
3. Find the unboxed vertex with the lowest value, and draw a box around the vertex name and value. All boxed values are locked in and won't change. Any unboxed value may change.
4. Consider all the unboxed vertices connected to the vertex you have just boxed. Add each edge value to the value you have just boxed, and write the answer next to each of the connecting vertices. If a connecting vertex already has a value and the answer is less than it, replace it with the new value. Otherwise leave the existing value unchanged.
5. Find the unboxed vertex with the lowest value, and draw a box around the vertex name and value. If there are two or more equal values at any stage, choose any one.
6. Repeat steps 4 and 5 until the Destination vertex and value is boxed. The value in the Destination box is the value of the shortest path.
7. To find the sequence of the shortest path, backtrack through the boxed vertices:
   - Start at the Destination box and move to the connected boxed vertex whose value is the Destination value – the edge value.
   - Move from this boxed vertex to the next connected boxed vertex using the same procedure.
   - Continue until you reach the Start vertex.

Trees and spanning trees

- A **tree** is a connected graph with no loops, multiple edges, or cycles.
- Every connected graph contains one or more trees.
- A tree with n vertices has $n - 1$ edges.
- A **spanning tree** is a tree subgraph that includes all the vertices of the original graph.
- A **minimum spanning tree** is a spanning tree for weighted graphs that represents the minimum distance (or minimum time or minimum cost) between all the vertices.

Prim's algorithm

Prim's algorithm is used to determine the minimum spanning tree.

1 Choose any vertex within the network.

2 Find the edge with the lowest value connecting this vertex to another vertex.

3 Look at all the edges connecting these two vertices and choose the edge that has the lowest value.

4 Look at all the vertices covered so far and select the edge with the lowest value.

5 Repeat step 4 until all the vertices in the graph are included in the tree.

Note that if there is more than one edge with the lowest value, then just choose any one of them.

NETWORKS AND DECISION MATHEMATICS

Examination 1

Reading time: (5 minutes)
Writing time: (30 minutes)

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers, one bound reference, one approved graphics calculator or approved CAS calculator or CAS software and, if desired, one scientific calculator. Calculator memory DOES NOT need to be cleared.
- Students are NOT permitted to bring into the examination room: blank sheets of paper and/or white out liquid/tape.
- Students may refer to the sheet of miscellaneous formulas supplied.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

Instructions

Choose the response that is correct for the question.

A correct answer scores 1, an incorrect answer scores 0.

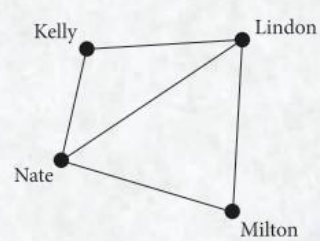
Marks will not be deducted for incorrect answers.

No marks will be given if more than one answer is completed for any question.

Unless otherwise indicated, the diagrams are **not** drawn to scale.

Question 1

The graph shows the roads connecting four towns: Kelly, Lindon, Milton and Nate. A bus starts at Kelly, travels through Nate and Lindon, then stops when it reaches Milton. The mathematical term for this route is



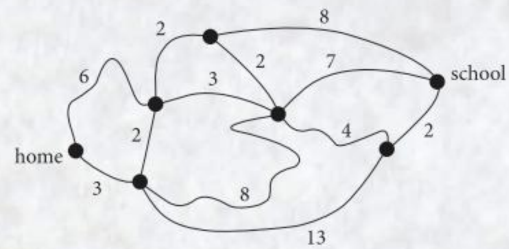
A. a loop
B. an Eulerian path
C. an Eulerian circuit
D. a Hamiltonian path
E. a Hamiltonian circuit

[VCAA 2014 1NQ1]

EXAMINATION 1 – continued

Question 2

The diagram shows the network of roads that Stephanie can use to travel between home and school. The numbers on the roads show the time, in minutes, that it takes her to ride a bicycle along each road. Using this network of roads, the shortest time that it will take Stephanie to ride her bicycle from home to school is



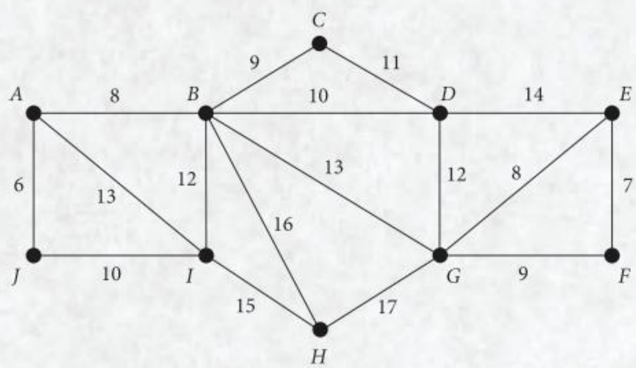
A. 12 minutes **B.** 13 minutes

C. 14 minutes **D.** 15 minutes

E. 16 minutes

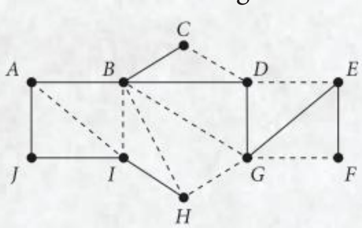
[VCAA 2014 1NQ3]

Question 3

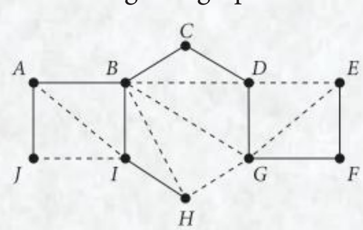


Which one of the following is the minimal spanning tree for the weighted graph shown?

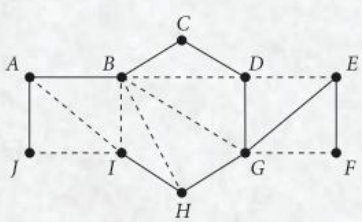
A.



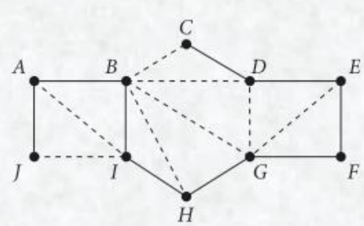
B.



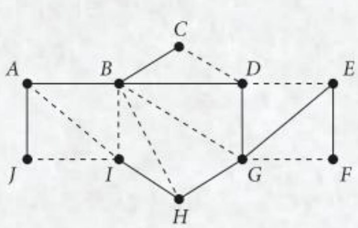
C.



D.



E.

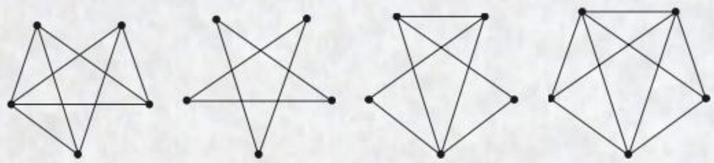


[VCAA 2014 1NQ5]

EXAMINATION 1 – continued

Use the following information to answer Questions 4 and 5.

Consider the following four graphs.



Question 4

How many of these four graphs have an Eulerian circuit?

A. 0 **B.** 1 **C.** 2 **D.** 3 **E.** 4

[VCAA 2014 1NQ6]

Question 5

How many of these four graphs are planar?

A. 0 **B.** 1 **C.** 2 **D.** 3 **E.** 4

[VCAA 2014 1NQ7]

Question 6

Which one of the following statements about critical paths is true?

A. There can be only one critical path in a project.

B. A critical path always includes at least two activities.

C. A critical path will always include the activity that takes the longest time to complete.

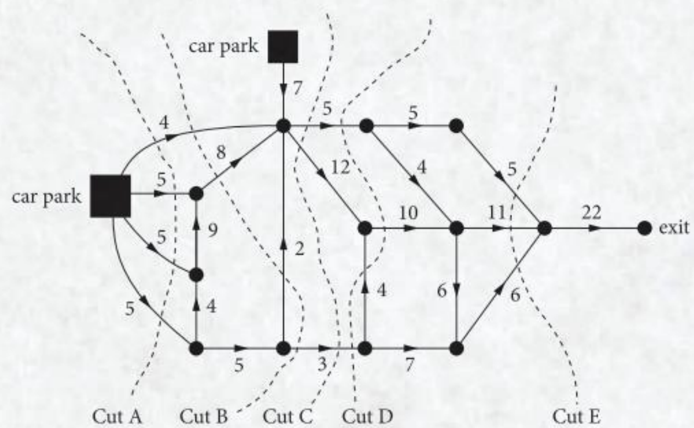
D. Reducing the time of any activity on a critical path for a project will always reduce the minimum completion time for the project.

E. If there are no other changes, increasing the time of any activity on a critical path will always increase the completion time of a project.

[VCAA 2014 1NQ8]

Question 7

A network of tracks connects two car parks in a festival venue to the exit, as shown in the directed graph. The arrows show the direction that cars can travel along each of the tracks and the numbers show each track's capacity in cars per minute. Five cuts are drawn on the diagram. The maximum number of cars per minute that will reach the exit is given by the capacity of



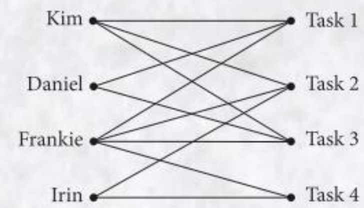
A. Cut A **B.** Cut B **C.** Cut C **D.** Cut D **E.** Cut E

[VCAA 2014 1NQ9]

EXAMINATION 1 – continued

Question 8

The bipartite graph shows the tasks that each of four people is able to undertake. All tasks must be allocated and each person can only be allocated one task. A possible allocation is



A.

Kim	Task 3
Daniel	Task 2
Frankie	Task 4
Irin	Task 1

B.

Kim	Task 3
Daniel	Task 1
Frankie	Task 4
Irin	Task 2

C.

Kim	Task 4
Daniel	Task 1
Frankie	Task 3
Irin	Task 2

D.

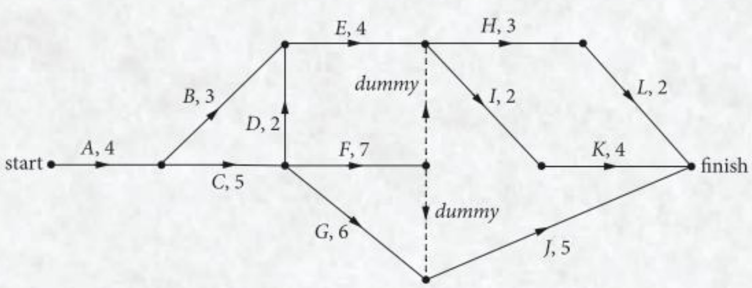
Kim	Task 2
Daniel	Task 1
Frankie	Task 4
Irin	Task 3

E.

Kim	Task 4
Daniel	Task 2
Frankie	Task 1
Irin	Task 3

Question 9

The diagram shows the tasks that must be completed in a project. Also shown are the completion times, in minutes, for each task.



The minimum project completion time is

A. 20 minutes

B. 21 minutes

C. 16 minutes

D. 22 minutes

E. 17 minutes

[VCAA 2011 1NQ8]

EXAMINATION 1 – ended

NETWORKS AND DECISION MATHEMATICS

Examination 2

Reading time: (5 minutes)
Writing time: (25 minutes)

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers, one bound reference, one approved graphics calculator or approved CAS calculator or CAS software and, if desired, one scientific calculator. Calculator memory DOES NOT need to be cleared.
- Students are NOT permitted to bring into the examination room: blank sheets of paper and/or white out liquid/tape.
- Students may refer to the sheet of miscellaneous formulas supplied.
- All written responses must be in English.

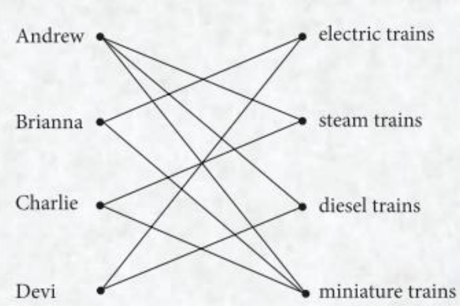
Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

Instructions

You need not give numerical answers as decimals unless instructed to do so. Alternative forms may involve, for example, π, surds or fractions.
Diagrams are not to scale unless specified otherwise.

Question 1 (2 marks)

Four members of a train club, Andrew, Brianna, Charlie and Devi, have joined one or more interest groups for electric, steam, diesel or miniature trains. The edges of the bipartite graph on the right show the interest groups that these four train club members have joined.



a. How many of these four members have joined the steam trains interest group? 1 mark

b. Which interest group have both Brianna and Charlie joined? 1 mark

[VCAA 2014 2NQ1]

EXAMINATION 2 – continued

Question 2 (4 marks)

Planning a train club open day involves four tasks. Table 1 shows the number of hours that each club member would take to complete these tasks.

Table 1

Task	*Andrew*	*Brianna*	*Charlie*	*Devi*
publicity	13	12	10	10
finances	9	10	11	11
equipment	8	12	11	10
catering	9	10	11	8

The Hungarian algorithm will be used to allocate the tasks to club members so that the total time taken to complete the tasks is minimised. The first step of the Hungarian algorithm is to subtract the smallest element in each row of Table 1 from each of the elements in that row. The result of this step is shown in Table 2 below.

a. Copy and complete Table 2 by filling in the missing numbers for Andrew. 1 mark

Table 2

Task	*Andrew*	*Brianna*	*Charlie*	*Devi*
publicity	3	2	0	0
finances		1	2	2
equipment		4	3	2
catering		2	3	0

After completing Table 2, Andrew decided that an allocation of tasks to minimise the total time taken was not yet possible using the Hungarian algorithm.

b. Explain why Andrew made this decision. 1 mark

Table 3 shows the final result of all steps of the Hungarian algorithm.

Table 3

Task	*Andrew*	*Brianna*	*Charlie*	*Devi*
publicity	4	2	0	1
finances	0	0	1	2
equipment	0	3	2	2
catering	1	1	2	0

c. **i.** Which task should be allocated to Andrew? 1 mark

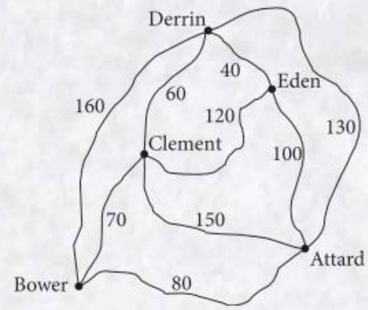
ii. How many hours in total are used to plan for the open day? 1 mark

[VCAA 2014 2NQ2]

EXAMINATION 2 – continued

Question 3 (4 marks)

The diagram on the right shows a network of train lines between five towns: Attard, Bower, Clement, Derrin and Eden. The numbers indicate the distances, in kilometres, that are travelled by train between connected towns. Charlie followed an Eulerian path through this network of train lines.



a. **i.** Write down the names of the towns at the start and at the end of Charlie's path. 1 mark

ii. What distance did he travel? 1 mark

Brianna will follow a Hamiltonian path from Bower to Attard.

b. What is the shortest distance that she can travel? 1 mark

The train line between Derrin and Eden will be removed. If one other train line is removed from the network, Andrew would be able to follow an Eulerian circuit through the network of train lines.

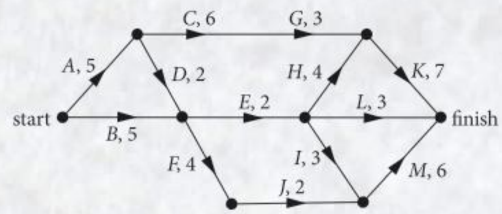
c. Which other train line should be removed? Write down the pair of towns that this train line connects. 1 mark

[VCAA 2014 2NQ3]

EXAMINATION 2 – continued

Question 4 (5 marks)

To restore a vintage train, 13 activities need to be completed. The network on the right shows these 13 activities and their completion times in hours.



a. Determine the earliest starting time of activity *F*. 1 mark

The minimum time in which all 13 activities can be completed is 21 hours.

b. What is the latest starting time of activity *L*? 1 mark

c. What is the float time of activity *J*? 1 mark

Just before they started restoring the train, the members of the club needed to add another activity, *X*, to the project. Activity *X* will take seven hours to complete. Activity *X* has no predecessors, but must be completed before activity *G* starts.

d. What is the latest starting time of activity *X* if it is not to increase the minimum completion time of the project? 1 mark

Activity *A* can be crashed by up to four hours at an additional cost of $90 per hour. This may reduce the minimum completion time for the project, including activity *X*.

e. Determine the least cost of crashing activity *A* to give the greatest reduction in the minimum completion time of the project. 1 mark

[VCAA 2014 2NQ4]

EXAMINATION 2 – ended

CHAPTER 11

DIRECTED GRAPHS

11.1 The scheduling problem
- Directed graphs
- Drawing a directed graph from an activity table
- Reachability
- Dummy activities

11.2 Critical path analysis
- Forward scanning to determine EST
- Backward scanning to determine LST
- Identifying critical paths
- Activity float time and project crashing

11.3 The assignment problem and bipartite graphs
- Bipartite graphs
- Matrix form
- Optimum assignments
- The Hungarian algorithm

11.4 Network flow problems
- Flow capacity and maximum flow
- The capacity of a cut
- Maximum flow – minimum cut

11.5 Dijkstra's algorithm and directed graphs
- Dijkstra's shortest path algorithm

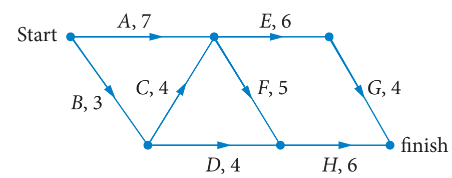
Summary

Prior learning

11.1 The scheduling problem

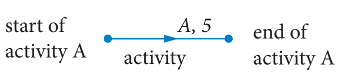
Directed graphs

A **directed graph**, or digraph, is a set of vertices connected by edges, where the edges have a direction. In a **weighted digraph** there is a number associated with each edge.



Drawing a directed graph from an activity table

When a large number of activities are involved in the completion of a task or project, it is often useful to draw a directed graph to show the sequence of activities and the time required for each activity.



Each **activity** is represented by an edge, shown by a line with an arrow. The activity is labelled using a letter and the letter plus the activity time is marked on the line. It is only possible to move along the network from the start to the finish in the direction of the arrows.

Each vertex or node is shown by a circle, representing the start or finish of a particular activity.

An **activity table** shows the order and estimated time for each activity. The table below shows 7 activities that must be completed. An **immediate predecessor** is any activity that must be completed before the current activity can commence.

Activity	Activity time (hours)	Immediate predecessor
A	3	–
B	4	*A*
C	3	*A*
D	8	*A*
E	3	*B*
F	5	*C*
G	7	*E, D, F*

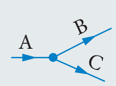
There are three different types of connections.

1 One preceding event

Event *B* is preceded by *A*.

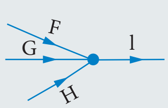
2 More than one activity with the same preceding event

Events *B* and *C* are preceded by *A*.



3 An activity with more than one preceding event

Event *I* is preceded by *F*, *G* and *H*.



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Guidelines to follow when drawing a network diagram

- Use a vertex to represent the start of the network.
- Look for any activities that do not have any predecessors. These will be your starting activities.
- Multiple predecessors to an activity will all end at the same vertex.
- An activity should not be represented by more than one edge in the network.
- Two vertices can be connected by one edge only.
- A vertex indicating the completion of the project needs to be included in the network.

Worked example 1

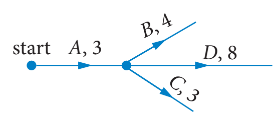
Draw the directed graph for the activity table shown on the previous page.

Working

1 Activity *A* has no predecessors.

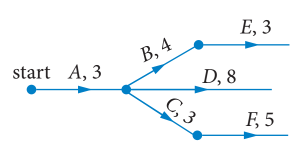
Label the first vertex 'start' and from this draw a directed edge labelled *A*, 3.

2 Activities *B*, *C* and *D* are all preceded by *A*. From the vertex at the end of edge *A* draw 3 directed edges labelled *B*, 4, *C*, 3 and *D*, 8.

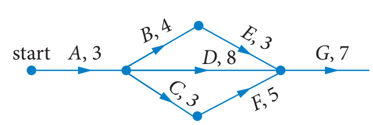


3 Activity *E* is preceded by *B*. From the vertex at the end of *B* draw an edge labelled *E*, 3.

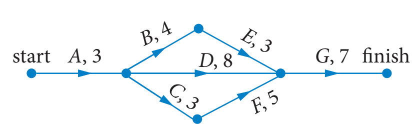
Activity *F* is preceded by *C*. From the vertex at the end of *C* draw an edge labelled *F*, 5.



4 Activity *G* is preceded by activities *E*, *D* and *F*. Draw a vertex and connect the edges *E*, *F* and *D* to it. From this vertex draw an edge labelled *G*, 7.



5 The final vertex is drawn at the end of edge *G* and labelled 'finish'.



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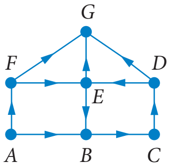
Reachability

Reachability is the ability to get from one vertex to another vertex in a directed graph.

Worked example 2

For this directed graph, determine which vertices are **not** reachable from vertex *C*.

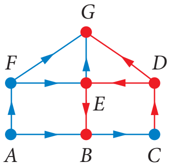


Working

1 Trace the paths from vertex *C* to every possible vertex.

The vertices that can be reached are coloured red and the paths have red edges.

Vertices *B*, *D*, *E* and *G* are reachable from vertex *C*.



2 If a path does not exist from vertex *C* to a vertex, it is **not** reachable.

Vertices *A* and *F* are not reachable from vertex *C*.

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Dummy activities

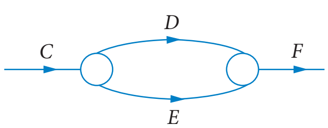
A **dummy activity** needs to be added to a network to ensure that no two vertices are connected by multiple edges or to maintain precedence structure.

A dummy activity has zero time and is shown as a directed edge with a broken line.

1. Multiple edges

A network cannot be drawn with multiple edges connecting the same vertices. In the example below, the directed graph cannot be drawn with two vertices both connected by activities *D* and *E*. A dummy activity needs to be included. It can occur before or after activities *D* or *E* to overcome this problem.

Activity	Immediate predecessor
D	*C*
E	*C*
F	*D*, *E*



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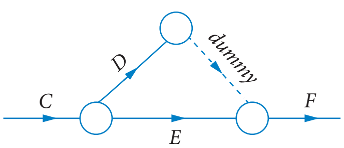
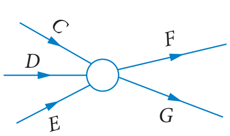


correct

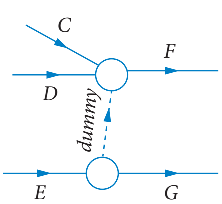
2. Logic difficulties in the precedence structure

The second use of dummy activities is to ensure the logic of the network is maintained. The first network has activity *G* preceded by activities *C*, *D* and *E*, which does not follow the logic of the activity table. A second vertex and a dummy activity need to be included so activities *F* and *G* do not both start from the same vertex and activities *C*, *D* and *E* do not finish at the same vertex.

Activity	Immediate predecessor
F	*C*, *D*, *E*
G	*E*



incorrect



correct

Worked example 3

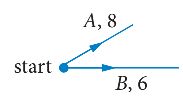
Draw the directed graph for the project shown, including dummy activities where required.

Activity	Activity time (hours)	Immediate predecessor
A	8	–
B	6	–
C	4	*A*, *B*
D	5	*A*, *B*
E	7	*C*, *D*

Working

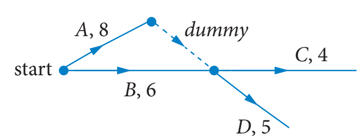
1 Activities *A* and *B* have no predecessors.

Label the first vertex ‘start’ and from this draw directed edges labelled *A*, 8 and *B*, 6.

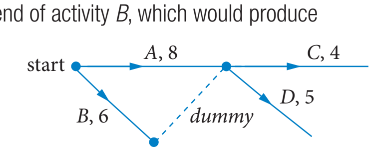


2 Activities *C* and *D* are both preceded by *A* and *B*; however two vertices cannot be connected by multiple edges. A dummy activity must be included after activity *A* or activity *B*.

Draw two directed edges labelled *C*, 4 and *D*, 5 from the vertex at the end of activity *B*.

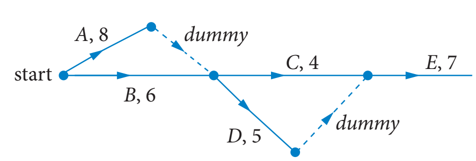


The dummy activity could also be placed at the end of activity *B*, which would produce

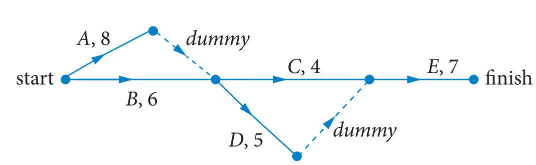


3 Activity *E* is preceded by activities *C* and *D*. A dummy activity is required so that two vertices are not connected by multiple edges.

Draw a directed edge labelled *E*, 7 from the vertex at the end of activity *C*.



4 Draw a vertex labelled ‘finish’ at the end of activity *E*.



Exam hack

Always check your directed graph to ensure that

- it contains all the connections and times specified in the activity table
- it does not have two vertices connected with multiple edges
- it has a start vertex and a finish vertex.

The scheduling problem

Prep 1

WORKED EXAMPLE 1

For a particular project there are nine activities that must be completed.

These activities, their times and their immediate predecessors are given in the following table.

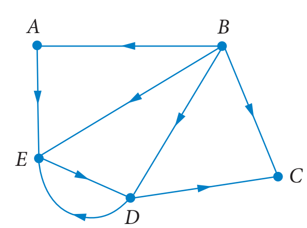
Activity	Activity time (days)	Immediate predecessor
A	1	–
B	4	–
C	6	A
D	4	A
E	7	B, C
F	5	B, C
G	7	F
H	6	D
I	2	G

Draw the directed graph for this project.

Prep 2

WORKED EXAMPLE 2

The directed graph shows a series of rivers that flow between logging towns A, B, C, D and E. Logs from the towns' lumber yards are transported via the rivers. The direction of flow is indicated on the edge. Which towns can the lumber yard at town E transport to?



Prep 3

WORKED EXAMPLE 3

Draw the directed graph for the project shown, including dummy activities where required.

Activity	Activity time (hours)	Immediate predecessor
A	8	–
B	4	–
C	7	–
D	5	A, B, C
E	7	A, B, C

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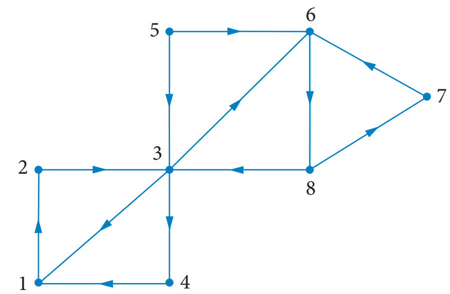
The scheduling problem

Question 1

The following directed graph represents a series of one-way streets with intersections numbered as nodes 1 to 8.



All intersections can be reached from

A intersection 4

B intersection 5

C intersection 6

D intersection 7

E intersection 8

[VCAA 2006 1NQ2]

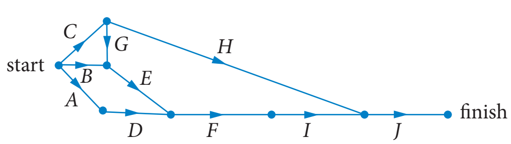
Question 2

For a particular project there are ten activities that must be completed. These activities and their immediate predecessors are given in the following table.

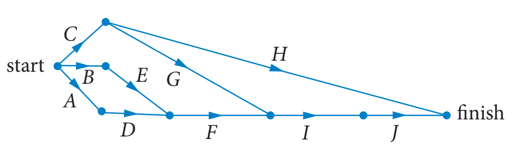
Activity	Immediate predecessors
A	–
B	–
C	–
D	*A*
E	*B*
F	*D*, *E*
G	*C*
H	*C*
I	*F*, *G*
J	*H*, *I*

A directed graph that could represent this project is

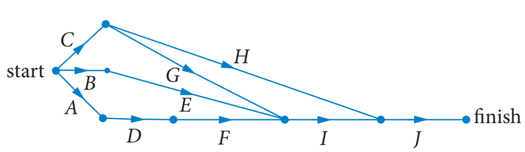
A



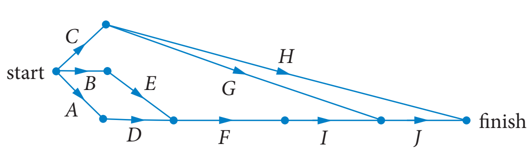
B



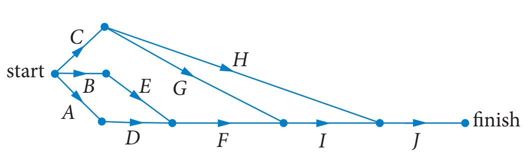
C



D



E



[VCAA 2006 1NQ5]

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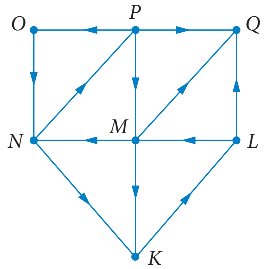
Question 3



For the directed graph shown, vertex O **cannot** be reached from vertex

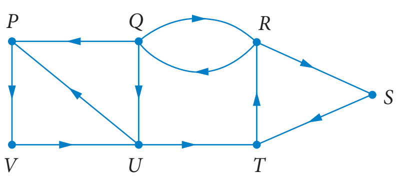
A L **B** M **C** N

D P **E** Q

[VCAA 2003 1NQ2]

11.1

Question 4



In the digraph, all vertices are reachable from every other vertex. All vertices would still be reachable from every other vertex if we remove the edge in the direction from

A Q to U **B** R to S **C** S to T

D T to R **E** V to U

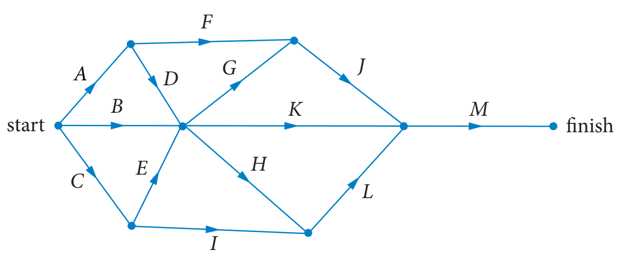
[VCAA 2012 1NQ6]

Question 5

The network shows the activities that are needed to complete a particular project. The total number of activities that need to be completed before activity L may begin is

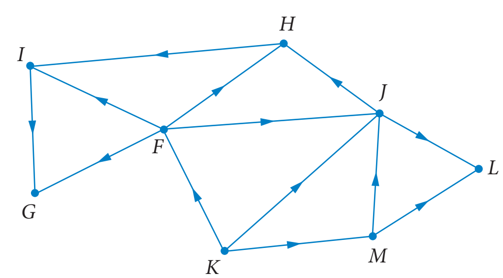
A 2 **B** 4 **C** 6

D 7 **E** 8



[VCAA 2009 1NQ5]

Question 6



One of the landmarks in the city is a hedge maze. The maze contains eight statues. The statues are labelled F to M on the following directed graph. Walkers within the maze are only allowed to move in the directions of the arrows.

a Write down the two statues that a walker could not reach from statue M. 1 mark

b One way that statue H can be reached from statue K is along path K-F-H. List the three other ways that statue H can be reached from statue K. 1 mark

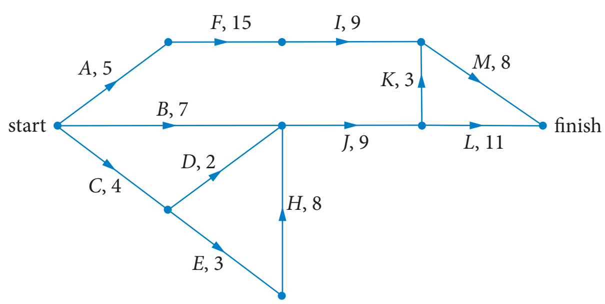
[VCAA 2009 2NQ2]

Question 7

A project will be undertaken in the wildlife park. This project involves the 13 activities shown in the table. The duration, in hours, and predecessor(s) of each activity are also included in the table.

Activity	Duration (hours)	Predecessor(s)
A	5	–
B	7	–
C	4	–
D	2	C
E	3	C
F	15	A
G	4	B, D, H
H	8	E
I	9	F, G
J	9	B, D, H
K	3	J
L	11	J
M	8	I, K

Activity G is missing from the network diagram for this project, which is shown.



Copy and complete the network diagram by inserting activity G. 1 mark

[VCAA 2013 2NQ2]

Critical path analysis

11.2

Critical path analysis is a step-by-step project management technique that is used to examine every activity in a project and how each affects the project completion time.

Forward scanning to determine EST

Forward scanning through a network enables us to determine the earliest start time (EST) for every activity in the network.

The **earliest start time** (EST) for an activity is the earliest time it is possible to start the activity.

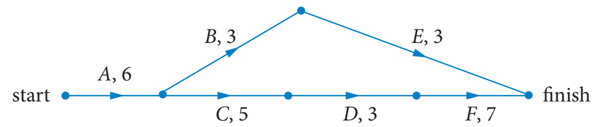
To determine the earliest start times

1. Begin with the first vertex, which has an earliest start time of 0. We will use the convention of writing the earliest starting times in the top box next to each vertex.
2. ESTs are calculated from left to right.
3. Add the activity time to the EST of the previous vertex. If more than one activity leads to the vertex, the highest figure obtained becomes the new EST.
4. Continue until the finish is reached.

Shutterstock.com/Stephanie Periquet

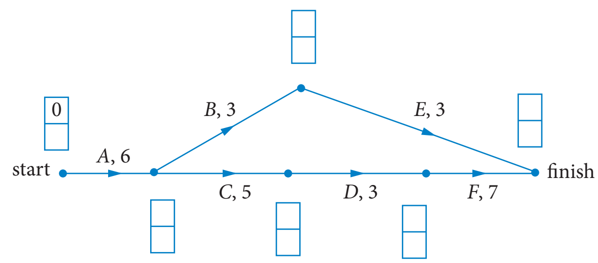
Worked example 4

Determine the earliest start times (EST) for each activity in the network. Activity times shown are in hours.



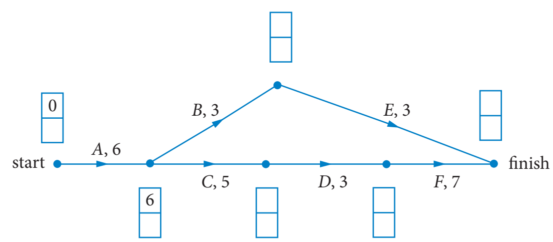
Working

1 Draw boxes above each vertex and enter 0 as the EST of the 'start' vertex.



2 To find the EST for activities B and C, add the activity time of 6 to the previous EST.

This gives an earliest start time for activities B and C as 6 hours.

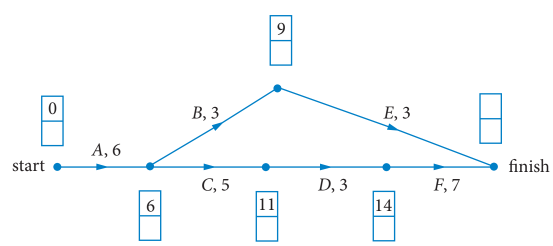


3 The EST for activity E is $6 + 3 = 9$. This is written in the top box at the start of activity E.

This process is repeated:

EST for activity $D = 6 + 5 = 11$

EST for activity $F = 11 + 3 = 14$

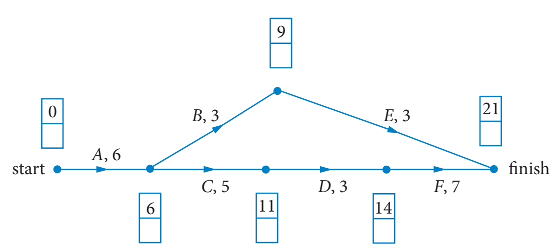


4 To calculate the EST for the finish, find the totals for the path containing activity E and the path containing activity F. The highest figure obtained is the final EST.

Activity E: $9 + 3 = 12$

Activity F: $14 + 7 = 21$

Therefore, 21 hours is the final EST.



Backward scanning to determine LST

Calculating EST and LST

Backward scanning is the process used to find the **latest start time** (LST) for an activity. This is the latest time you can start the activity without affecting the project completion time.

The LST at the left vertex = the LST at the right vertex − the activity time

To determine latest start times (LST)

1 Commence LST calculations at the 'finish' vertex. At the 'finish' vertex, LST = EST.

2 Work backwards from left to right. LSTs are written in the bottom box next to each vertex.

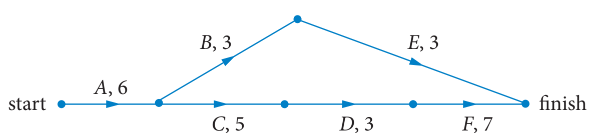
3 To find the LST at the left vertex, work backwards, subtracting the activity time from the LST at the right vertex.

4 Where there is more than one path to the previous vertex, take the smallest result as the LST.

5 The LST at the first vertex must be zero.

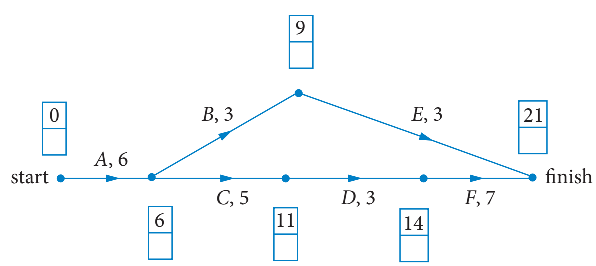
Worked example 5

Determine the latest start times (LST) for each activity in the network. Activity times shown are in hours.

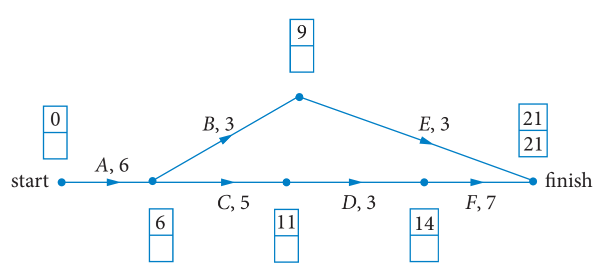


Working

1 First calculate ESTs for each activity. These were completed in worked example 4.



2 Work backwards from right to left. The LST for the finish is equal to the EST.



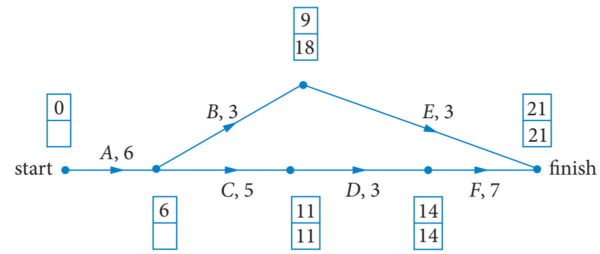
3 The LST for activity E is

$21 - 3 = 18$. This is written in the bottom box at the start of activity E.

The LST for:

Activity $F = 21 - 7 = 14$

Activity $D = 14 - 3 = 11$



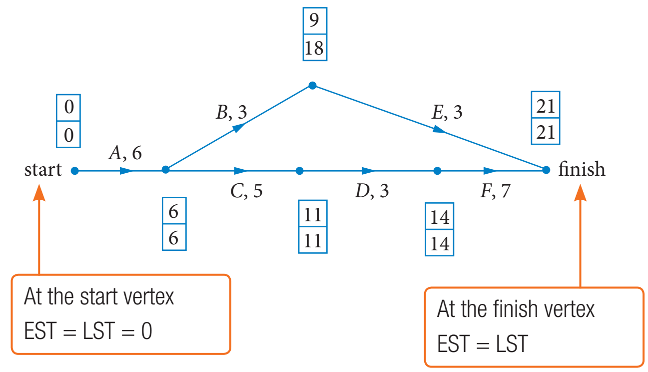
4 To calculate the LST for activities B and C, take the smallest figure obtained from:

Activity B: $18 - 3 = 15$

Activity C: $11 - 5 = 6$

Therefore 6 hours is the LST for activity A.

The start LST $= 6 - 6 = 0$



Identifying critical paths

There are often multiple paths between the start and finish vertices of a network. The **critical path** is the longest time path between the start and finish and determines the project completion time.

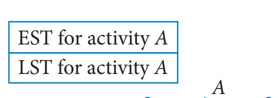
- On the critical path, activities have equal EST and LST values.
- Activities on the critical path are called critical activities.
- Activities not on the critical path are called non-critical activities.

To determine the critical path,

1. Use forward scanning to determine the earliest start times for each activity.
2. Identify the path or paths that produce the final EST.
3. Use backward scanning to determine the latest start times for each activity.
4. Activities for which EST = LST are on the critical path.

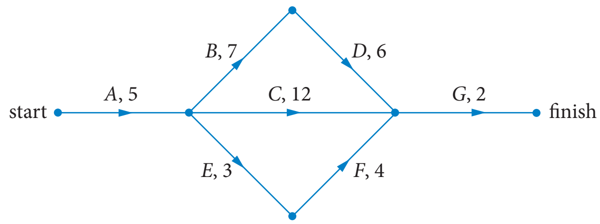
Exam hack

The earliest start time (EST) and latest start time (LST) that are associated with an activity are the values written in the boxes at the start of the activity.



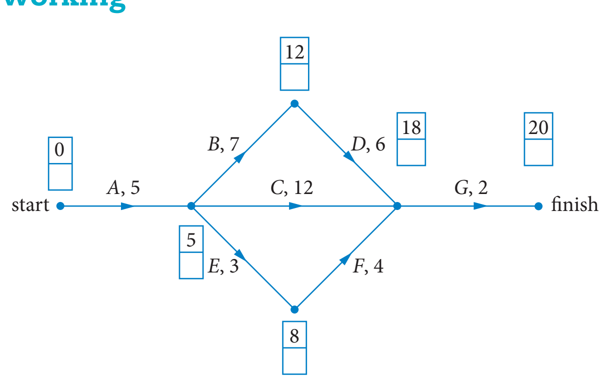
Worked example 6

Determine the earliest start times (EST) and latest start times (LST) for each activity in the network and hence determine the critical path. Activity times shown are in hours.

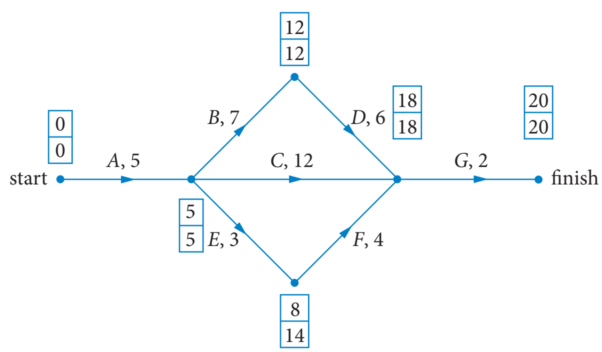


Working

1 First calculate the ESTs for each activity.



2 Calculate the LSTs for each activity.



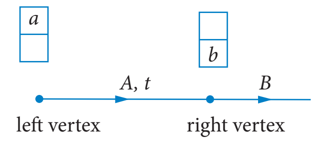
3 Activities where EST = LST are critical activities and are on the critical path.

The critical path is *A-B-D-G*.

Critical paths and activity float times

Activity float time and project crashing

The **float time** for an activity is the maximum time an activity can be extended or postponed without affecting the project completion time. Activities on the critical path all have float times of zero.



In the diagram shown,

a = the EST of activity A

b = the LST for activity B, and

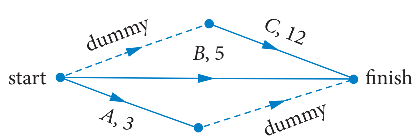
t = the activity time for activity A

Float for activity $A = b - a - t$

> For a non-critical activity, Float = $\text{LST}_{\text{right}} - \text{EST}_{\text{left}}$ – activity time
>
> For critical activities, Float = 0

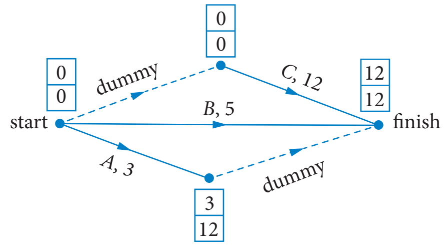
Worked example 7

Determine critical activities and the float times for the non-critical activities for the project shown. Times shown are in days.



Working

1 First calculate ESTs and LSTs for each activity.



2 Identify the critical path.

This is the longest time path for the project and contains activity C.

The critical activity is C.

3 Calculate float times for activities A and B using the formula

Float = $\text{LST}_{\text{end}} - \text{EST}_{\text{start}}$ – activity time

Float for activity $B = 12 - 0 - 5 = 7$ days

Float for activity $A = 12 - 0 - 3 = 9$ days

It may be possible to speed up a project by employing more resources so some activities can be completed more quickly. When this occurs with critical activities, the process is called **crashing**.

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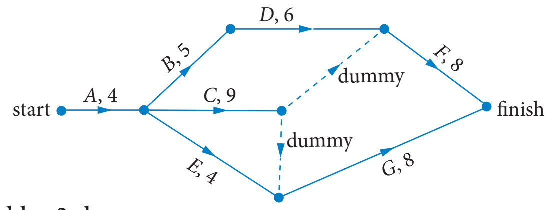
Worked example 8

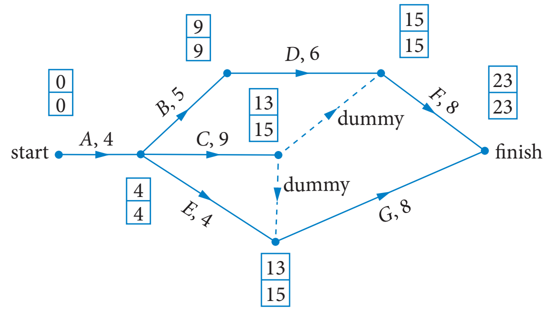
a Determine the critical path and the minimum project completion time for the project. Times shown are in days.



b Extra resources are used to speed up the original project, resulting in activities *B*, *D* and *F* each being reduced by 3 days. Find the new critical path and the minimum project completion time.

Working

a **1** First calculate the ESTs and LSTs for each activity.



2 On the critical path EST = LST.

Critical path = *A*-*B*-*D*-*F*

The minimum project completion time is EST or LST for the finish vertex.

Project completion time = 23 days

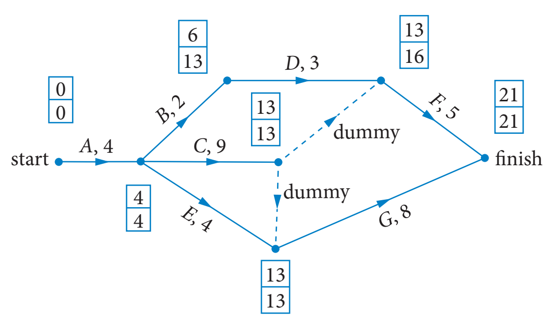
b **1** Reduce the times for activities *B*, *D* and *F* by 3 days.

Activity time for *B* = 5 – 3 = 2 days

Activity time for *D* = 6 – 3 = 3 days

Activity time for *F* = 8 – 3 = 5 days

2 Calculate the ESTs and LSTs for each activity with the new network.



3 Identify the activities where EST = LST to find the critical path.

The critical path is *A*-*C*-*G* and the project completion time is 21 days.

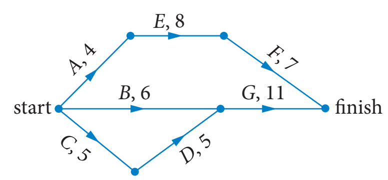
The dummy activity is not included in the critical path.

EXAM PREP 11.2

Critical path analysis

Prep 1 WORKED EXAMPLE 4

Determine the earliest start times (EST) for each activity in the following directed graph.

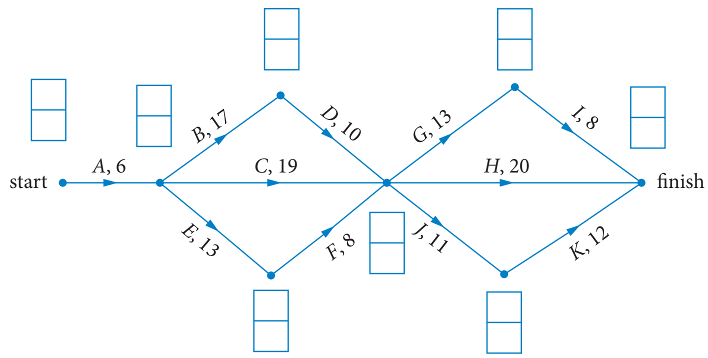


Prep 2 WORKED EXAMPLE 5

Determine the latest start times (LST) for the directed graph in Prep 1.

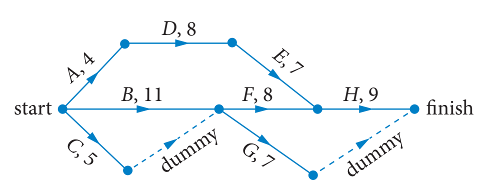
Prep 3 WORKED EXAMPLE 6

Determine the earliest starting times (EST) and latest starting times (LST) for each activity in the network and hence determine the critical path. Activity times shown are in hours.



Prep 4 WORKED EXAMPLE 7

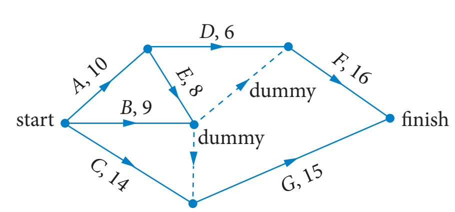
Determine critical activities and the float times for the non-critical activities for the project shown. Times shown are in days.



Prep 5 WORKED EXAMPLE 8

a Determine the critical path and the minimum project completion time for the project shown. Times shown are in days.

b Extra resources are used to speed up the original project, resulting in activities C, E and F each being reduced by 5 days. Find the new critical path and the minimum project completion time.



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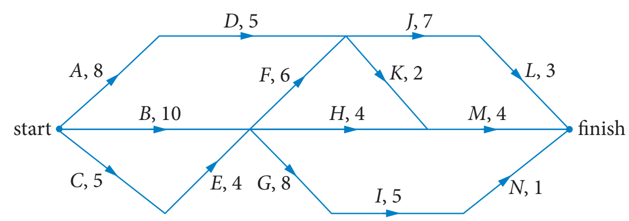
EXAM PRACTICE 11.2

Critical path analysis

Question 1

The activities and their completion times (days) needed to complete a project are shown in the digraph. For the network shown, the length of the critical path is



A 22 days **B** 23 days

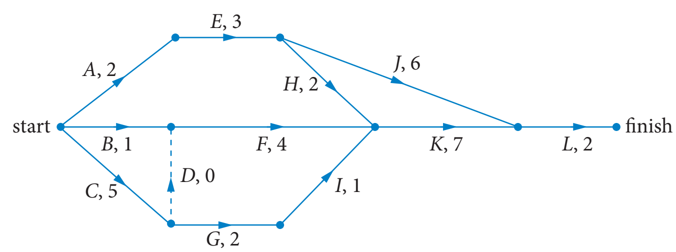
C 25 days **D** 26 days

E 28 days

[VCAA 2004 1NQ8]

Use the following information to answer Questions 2 & 3.

The following network shows the activities that are needed to complete a project and their completion times (in hours).



Question 2

Which one of the following statements regarding this project is **false**?

A Activities *A*, *B* and *C* all have the same earliest start time.

B There is only one critical path for this project.

C Activity *J* may start later than activity *H*.

D The shortest path gives the minimum time for project completion.

E Activity *L* must be on the critical path.

[VCAA 2007 1NQ5]

Question 3

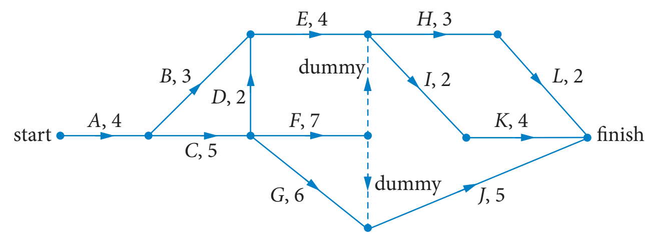
The earliest start time for activity *L*, in hours, is

A 11 **B** 12 **C** 14 **D** 15 **E** 16

[VCAA 2007 1NQ7]

Question 4

The diagram shows the tasks that must be completed in a project. Also shown are the completion times, in minutes, for each task. The critical path for this project includes activities



A B and I **B** C and H **C** D and E **D** F and K **E** G and J

[VCAA 2011 1NQ8]

Question 5

Which one of the following is a **true** statement about a **critical path** in a project?

A Knowledge of the critical path can be used to decide if any tasks in a project can be delayed without extending the length of time of the project.

B All tasks on the critical path must be completed before any other task in the same project can be started.

C Decreasing the times of tasks not on the critical path will decrease the length of time of the project.

D The critical path must always include at least two tasks in a project.

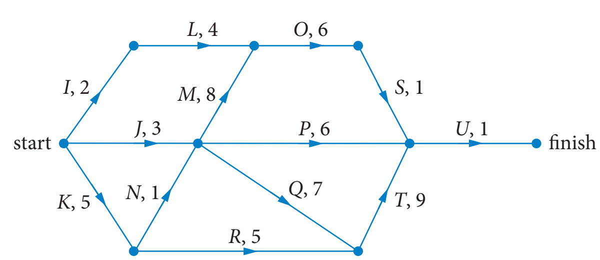
E There is only one critical path in any project.

[VCAA 2004 1NQ6]

Question 6

The following network gives the times, in hours, taken to complete the 12 tasks required to finish a project.

The critical path for this project is



A J-P-U **B** K-R-T-U

C J-M-O-S-U **D** K-N-Q-T-U

E K-N-M-O-S-U

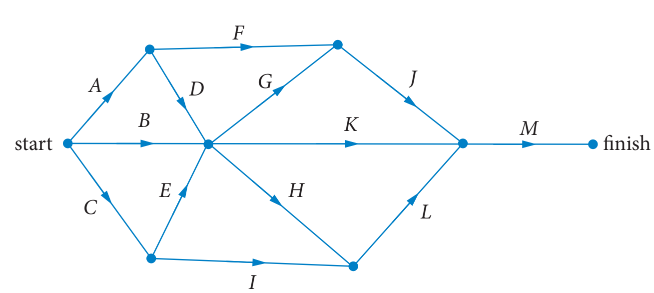
[VCAA 2003 1NQ6]

Question 7

The network shows the activities that are needed to complete a particular project.

The duration of every activity is initially 5 hours. For an extra cost, the completion times of both activity F and activity K can be reduced to 3 hours each.

If this is done, the completion time for the project will be



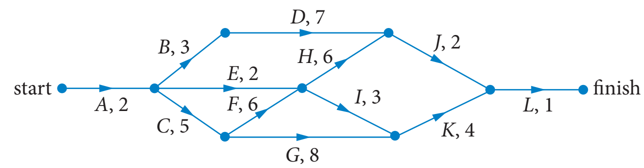
A decreased by 2 hours **B** decreased by 3 hours **C** decreased by 4 hours

D decreased by 6 hours **E** unchanged

[VCAA 2009 1NQ6]

Use the following information to answer Questions 8 & 9.

The network shows the activities that are needed to finish a particular project and their completion times (in days).



Question 8

The earliest start time for Activity K, in days, is

A 7 **B** 15 **C** 16 **D** 19 **E** 20

[VCAA 2008 1NQ8]

Question 9

This project currently has one critical path.

A second critical path, in addition to the first, would be created by

A increasing the completion time of D by 7 days.

B increasing the completion time of G by 1 day.

C increasing the completion time of I by 2 days.

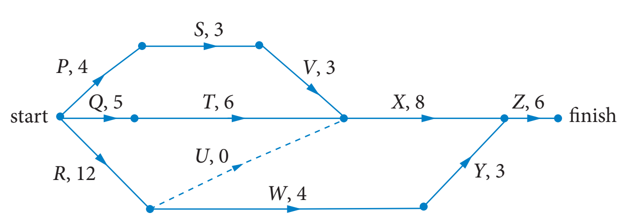
D decreasing the completion time of C by 1 day.

E decreasing the completion time of H by 2 days.

[VCAA 2008 1NQ9]

Question 10

The activities and their completion times (in hours) that are needed to complete a project are shown in the network. For this project, the minimum time taken to complete the whole project would be increased if one of the activities was delayed. This activity is

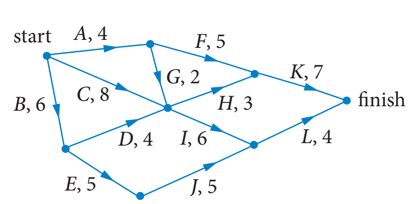


A P **B** Q **C** R **D** T **E** W

[VCAA 2005 1NQ8]

Question 11

The following network gives the time, in days, taken to complete the 12 activities for a building project. The minimum time in which this building project can be completed is



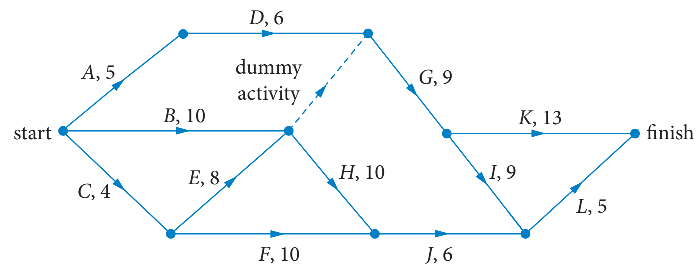
A 8 days **B** 16 days **C** 18 days

D 20 days **E** 59 days

[VCAA 2002 1NQ7]

Question 12

A project has 12 activities. The network gives the time (in hours) that it takes to complete each activity. The critical path for this project is



A *ADGK* **B** *ADGIL*

C *BHJL* **D** *CEGIL*

E *CEHJL*

[VCAA 2010 1NQ8]

Question 13

The table lists the six activities in a project and the earliest start time, in hours, and the predecessor(s) of each task.

The time taken for activity E is two hours. Without affecting the time taken for the entire project, the time taken for activity C could be increased by

A 0 hours **B** 8 hours

C 9 hours **D** 11 hours

E 27 hours

Task	Predecessor	Earliest start time
A	–	0
B	–	0
C	*A*	8
D	*B*	15
E	*C*	22
F	*D*, *E*	35

[VCAA 2004 1NQ9]

Question 14

Andy, Brian and Caleb must complete three activities in total (K, L and M).

The table shows the person selected to complete each activity, the time it will take to complete the activity (in minutes), and the immediate predecessor for each activity.

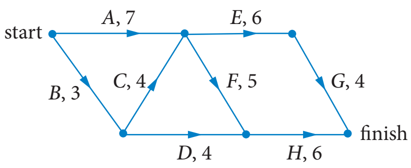
Person	Activity	Duration	Immediate predecessor
Andy	*K*	13	–
Brian	*L*	5	*K*
Caleb	*M*	16	*L*

All three activities must be completed in a total of 40 minutes. The instant that Andy starts his activity, Caleb gets a telephone call. The maximum time, in minutes, that Caleb can speak on the telephone before he must start his allocated activity is

A 5 **B** 13 **C** 18 **D** 24 **E** 34

[VCAA 2011 1NQ7]

Question 15



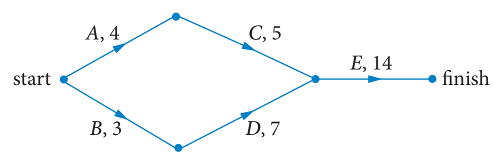
Eight activities, A, B, C, D, E, F, G and H, must be completed for a project. The graph shows these activities and their usual duration in hours. The duration of each activity can be reduced by one hour. To complete this project in 16 hours, the minimum number of activities that must be reduced by one hour each is

A 1 **B** 2 **C** 3 **D** 4 **E** 5

[VCAA 2012 1NQ8]

Question 16

The graph shows five activities, A, B, C, D and E, that must be completed to finish a project. The number next to each letter shows the completion time, in hours, for the activity.



Each of the five activities can have its completion time reduced by a maximum of one hour at a cost of \$100 per hour. The lowest cost to achieve the greatest reduction in the time taken to finish the project is

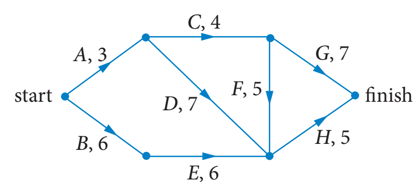
A \$100 **B** \$200 **C** \$300 **D** \$400 **E** \$500

[VCAA 2013 1NQ8]

Question 17

John, Ken and Lisa must work together to complete eight activities, A, B, C, D, E, F, G and H, in minimum time. The directed network shows the activities, their completion times in days, and the order in which they must be completed.

Several activities need special skills. Each of these activities may be completed only by a specified person.



- Activities A and F may only be completed by John.
- Activities B and C may only be completed by Ken.
- Activities D and E may only be completed by Lisa.
- Activities G and H may be completed by any one of John, Ken or Lisa.

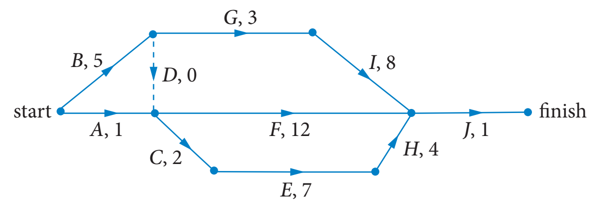
With these conditions, the minimum number of days required to complete these eight activities is

A 14 **B** 17 **C** 20 **D** 21 **E** 24

[VCAA 2012 1NQ9]

Question 18

The network shows the activities and their completion times (in hours) that are needed to complete a project. The project is to be crashed by reducing the completion time of **one** activity only. This will reduce the completion time of the project by a maximum of



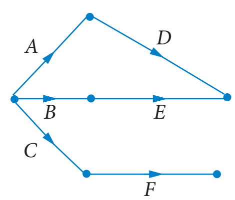
A 1 hour **B** 2 hours **C** 3 hours **D** 4 hours **E** 5 hours

[VCAA 2006 1NQ9]

Question 19

Five musicians are to record an album. This will involve nine activities. The activities and their immediate predecessors are shown in the following table. The duration of each activity is not yet known.

Activity	Immediate predecessors
A	–
B	–
C	–
D	*A*
E	*B*
F	*C*
G	*D*, *E*
H	*F*
I	*G*, *H*



a Use the information in the table to copy and complete the network by including activities *G*, *H* and *I*. 1 mark

There is only one critical path for this project.

b How many **non-critical** activities are there? 1 mark

The following table gives the earliest start times (EST) and latest start times (LST) for **three of the activities only**. All times are in hours.

Activity	EST	LST
A	0	2
C	0	1
I	12	12

c Write down the critical path for this project. 1 mark

The minimum time required for this project to be completed is 19 hours.

d What is the duration of activity *I*? 1 mark

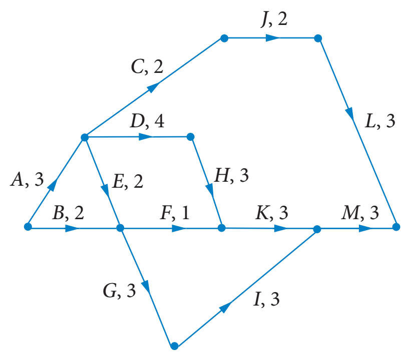
The duration of activity *C* is 3 hours.

e Determine the maximum combined duration of activities *F* and *H*. 1 mark

[VCAA 2006 2NQ3]

Question 20

The Bowen Yard Buster team specialises in backyard improvement projects. The team has identified the activities required for a backyard improvement. The network diagram shows the activities identified and the **actual** times, in hours, needed to complete each activity, that is, the duration of each activity. The table lists the activities, their immediate predecessor(s) and the **earliest starting times** (EST), in hours, of each of the activities. Activity X is not yet drawn on the network diagram



	Immediate predecessor(s)	EST
A	–	0
B	–	0
C	A	3
D	A	3
E		3
F	B, E	5
G	B, E	5
H	D	7
I	G	
J	C, X	8
K	F, H	10
L	J	10
M	I, K	
X	D	7

a Use the information in the network diagram to find the missing values in the shaded cells of the table. 3 marks

b Copy the network diagram, then draw and label activity X on it, including its direction and duration. 1 mark

c The path A–D–H–K–M is the only critical path in this project.

Write down (in hours) the duration of path A–D–H–K–M. 1 mark

d Explain the importance of the critical path in completing the project. 1 mark

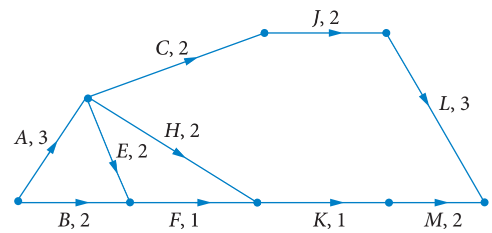
e When the weather is poor, activity B takes more than two hours to complete. What is the maximum time (in hours) that could be allowed to complete activity B without delaying the completion of the entire project? 1 mark

[VCAA 2003 2NQ2]

11.2

Question 21

To save money, Bowen Yard Busters decide to revise the project and leave out activities *D*, *G*, *I* and *X*. This results in a reduction in the time needed to complete activities *H*, *K* and *M*, as shown.



a For this revised project network, what is the **earliest starting time** for activity *K*? 1 mark

b Write down the critical path for this revised project network. 1 mark

c Without affecting the earliest completion time for this entire **revised** project, what is the **latest starting time** for activity *M*? 1 mark

[VCAA 2003 2NQ3]

Question 22

In the final challenge, each team has to complete a construction project that involves activities *A* to *I*.

Activity	EST (minutes)	LST (minutes)	Duration (minutes)	Immediate predecessor
A	0	0	5	–
B	5	5	6	*A*
C	5	6	4	*A*
D	11	11	2	*B*
E	5	9	7	*A*
F		10	6	*C*
G	9	13	1	*C*
H	13	13	3	*D*
I	10	14	2	*G*

The table shows the earliest start time (EST), latest start time (LST) and duration, in minutes, for each activity. The immediate predecessor is also shown. The earliest start time for activity *F* is missing.

a What is the smallest number of activities that must be completed before activity *F* can commence? 1 mark

b What is the earliest start time for activity *F*? 1 mark

c Write down all the activities that must be completed before activity *G* can commence. 1 mark

d What is the float time, in minutes, for activity *G*? 1 mark

e What is the shortest time, in minutes, in which this construction project can be completed? 1 mark

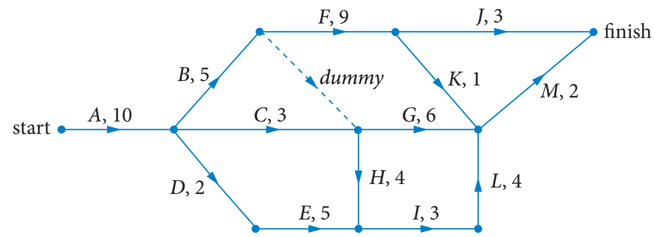
f Write down the critical path for this network. 1 mark

[VCAA 2010 2NQ4]

9780170371186

Question 23

Thirteen activities must be completed before the produce grown on a farm can be harvested. The directed network shows these activities and their completion times in days.



a Determine the earliest starting time, in days, for activity *E*. 1 mark

b A *dummy* activity starts at the end of activity *B*.
Explain why this *dummy* activity is used on the network diagram. 1 mark

c Determine the earliest starting time, in days, for activity *H*. 1 mark

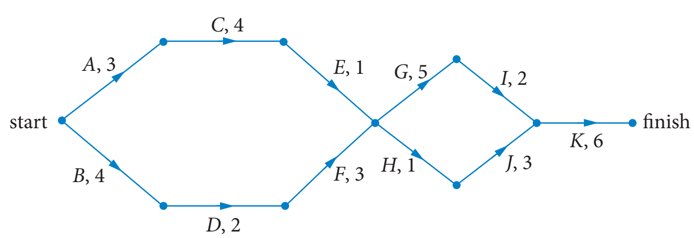
d In order, list the activities on the critical path. 1 mark

e Determine the latest starting time, in days, for activity *J*. 1 mark

[VCAA 2012 2NQ2]

Question 24

A walkway is to be built across the lake. Eleven activities must be completed for this building project. The directed network shows the activities and their completion times in weeks.



a What is the earliest start time for activity *E*? 1 mark

b Write down the critical path for this project. 1 mark

c The project supervisor correctly writes down the float time for each activity that can be delayed and makes a list of these times. Determine the longest float time, in weeks, on the supervisor's list. 1 mark

A twelfth activity, *L*, with duration three weeks, is to be added without altering the critical path. Activity *L* has an **earliest** start time of four weeks and a **latest** start time of five weeks.

d Copy the network diagram and draw in activity *L* on it. 1 mark

e Activity *L* starts, but then takes four weeks longer than originally planned.

Determine the total overall time, in weeks, for the completion of this building project. 1 mark

[VCAA 2009 2NQ4]

11.2

Question 25

A section of the Farnham showgrounds has flooded due to a broken water pipe. The public will be stopped from entering the flooded area until repairs are made and the area has been cleaned up.

The table shows the nine activities that need to be completed in order to repair the water pipe. Also shown are some of the durations, earliest start times (EST) and the immediate predecessors for the activities.

Activity	Activity description	Duration (hours)	EST	Immediate predecessor(s)
A	Erect barriers to isolate the flooded area	1	0	–
B	Turn off the water to the showgrounds		0	–
C	Pump water from the flooded area	1	2	*A*, *B*
D	Dig a hole to find the broken water pipe	1		*C*
E	Replace the broken water pipe	2	4	*D*
F	Fill in the hole	1	6	*E*
G	Clean up the entire affected area	4	6	*E*
H	Turn on the water to the showgrounds	1	6	*E*
I	Take down the barriers	1	10	*F*, *G*, *H*

a What is the duration of activity *B*? 1 mark

b What is the earliest start time (EST) for activity *D*? 1 mark

c Once the water has been turned off (Activity *B*), which of the activities *C* to *I* could be delayed without affecting the shortest time taken to complete all activities? 1 mark

It is more complicated to replace the broken water pipe (Activity *E*) than expected. It will now take four hours to complete instead of two hours.

d Determine the shortest time in which activities *A* to *I* can now be completed. 1 mark

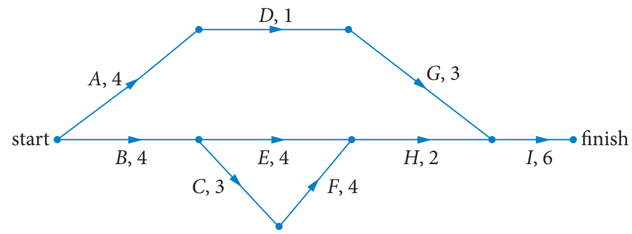
Turning on the water to the showgrounds (Activity *H*) will also take more time than originally expected. It will now take five hours to complete instead of one hour.

e With the increased durations for Activity *H* and Activity *E*, determine the shortest time in which activities *A* to *I* can be completed. 1 mark

[VCAA 2011 2NQ3]

Question 26

A community centre is to be built on the new housing estate. Nine activities have been identified for this building project. The directed network shows the activities and their completion times in weeks.



11.2

a Determine the minimum time, in weeks, taken to complete this project. 1 mark

b Determine the slack time, in weeks, for activity *D*. 2 marks

The builders of the community centre are able to speed up the project. Some of the activities can be reduced in time at an additional cost. The activities that can be reduced in time are *A*, *C*, *E*, *F* and *G*.

c Which of these activities, if reduced in time individually, would **not** result in an earlier completion of the project? 1 mark

The owner of the estate is prepared to pay the additional cost to achieve early completion. The cost of reducing the time of each activity is $5000 per week. The maximum reduction in time for each one of the five activities, *A*, *C*, *E*, *F*, *G*, is 2 weeks.

d Determine the minimum time taken, in weeks, for the project to be completed now that certain activities can be reduced in time. 1 mark

e Determine the minimum additional cost of completing the project in this reduced time. 1 mark

[VCAA 2007 2NQ4]

11.3 The assignment problem and bipartite graphs

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The **assignment problem** involves finding the best way of matching the elements in the two groups, like a group of workers to a set of tasks, that optimises a stated objective such as minimising cost, distance or time.

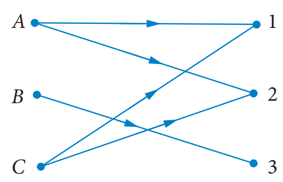
Bipartite graphs

A **bipartite graph** has its vertices in two distinct sets and the edges join elements in the first set to elements in the second set. Bipartite graphs can be undirected or directed. Since allocating a task to a person could also be viewed as allocating a person to a task, arrows are not always included in assignment problems.

In the bipartite graph below, there are three tasks {1, 2, 3} that can be performed by three people {A, B, C}. The edges show which people are qualified to perform which tasks.

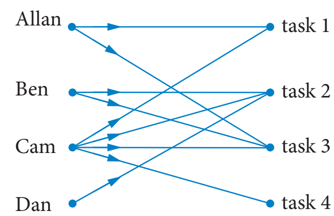
The bipartite graph shows that

- A can perform tasks 1 and 2
- B can perform task 3
- C can perform tasks 1 and 2



Worked example 9

The bipartite graph shows the tasks that each of four people is able to undertake. If each person must complete one task, find a valid allocation.

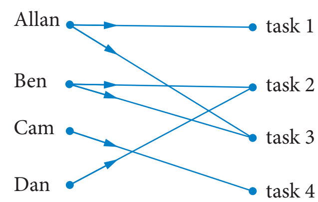


Working

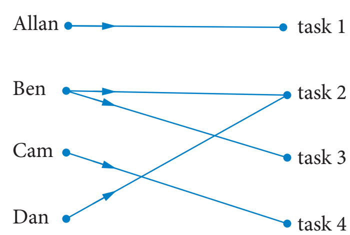
1 Identify tasks that have the smallest number of links.

Cam is the only person who can do task 4.

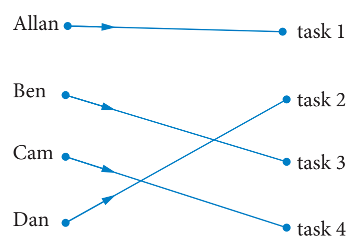
2 Allocate Cam to task 4, then eliminate links from Cam to tasks 1, 2 and 3.



3 Allocate Allan to task 1, then eliminate links from Allan to task 3.



4 Allocate Ben to task 3, which leaves Dan allocated task 2.



Task allocation:

Allan – task 1 Ben – task 3

Cam – task 4 Dan – task 2

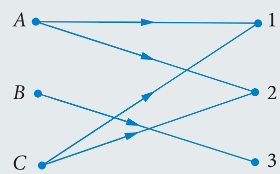
11.3

Matrix form

A matrix can be used to represent a bipartite graph.

In the matrix, 1 represents a connection and 0 indicates that there is no connection.

The bipartite graph



is represented by the matrix

$$\begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \end{array}$$

Worked example 10

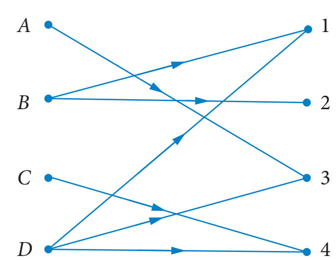
Four workers *A*, *B*, *C* and *D* need to be allocated one task each from task 1, task 2, task 3 and task 4. The workers are only qualified to perform certain tasks and this is indicated in the following matrix. Draw a bipartite graph from the matrix and determine a valid allocation.

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

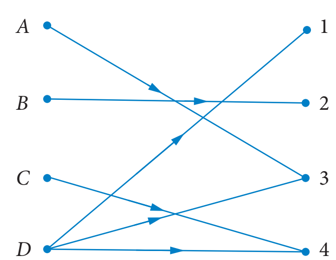
Working

1 Draw connections for every 1 in the matrix.

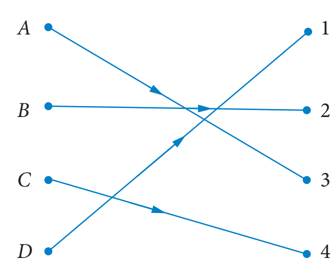
*A*3, *B*1, *B*2, *C*4, *D*1, *D*3 and *D*4



2 Start with the tasks with the smallest number of links. Allocate *B* to task 2 and remove the other link from *B* to 1.



3 Task 1 can only be done by *D* so allocate *D* to task 1 and remove any other links from *D* to tasks 3 and 4.



The allocations are *A*3, *B*2, *C*4 and *D*1.

Optimum assignments

The aim of the allocation problem is to find a way of assigning the elements in one set to the elements in a second set to meet an objective like minimising cost or time.

Row and column reduction method

1. Choose the smallest number in each row and subtract it from every element in the same row.
2. For every column that does not have a zero value, choose the smallest number in the column and subtract it from every element in the same column.
3. Cover all the zeros with the smallest number of lines (horizontal or vertical). The process is complete if the number of lines is equal to the number of rows.

11.3

Worked example 11

Three workers A, B, and C need to be allocated one task each from task 1, task 2 and task 3. The time in hours that each worker takes to complete the tasks is shown in the table.

If the tasks must be completed in the minimum time, find how the tasks are assigned.

	1	2	3
A	8	10	9
B	9	11	7
C	7	9	8

Working

1 Identify the smallest number in each row and subtract it from the other numbers in the same row.

Row A: subtract 8
Row B: subtract 7
Row C: subtract 7

	1	2	3
A	0	2	1
B	2	4	0
C	0	2	1

2 For every column that does not have a zero value, choose the smallest number in the column and subtract it from every element in the same column.

Column 1 has a zero
Column 2: subtract 2
Column 3 has a zero

	1	2	3
A	0	0	1
B	2	2	0
C	0	0	1

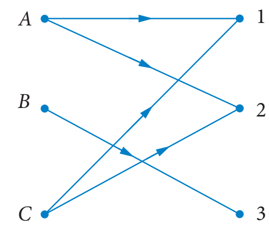
3 Cover all the zeros with the smallest number of lines (horizontal or vertical).

	1	2	3
A	0	0	1
B	2	2	0
C	0	0	1

There are three lines and three rows, so a solution has been found.

4 The zeros indicate the allocations. Draw a bipartite graph connecting $A1$, $A2$, $B3$, $C1$ and $C2$.

B needs to complete task 3, but A or C can complete either task 1 or task 2. So the two possible allocations are $A2$, $B3$, $C1$ and $A1$, $B3$, $C2$.



The Hungarian algorithm

WS The Hungarian algorithm

The **Hungarian algorithm** is an extension to row and column reduction when this does **not** produce an optimum solution.

The Hungarian algorithm

1 Perform a row and column reduction and cover all the zeros with the smallest number of horizontal or vertical lines possible.

If the number of lines is equal to the number of rows, a solution has been found.

2 If the number of lines does not equal the number of rows, find the smallest uncovered number and add it to every covered number. Numbers that are covered twice have this number added twice.

3 Subtract the smallest number in the matrix or table from all of the numbers in the table.

4 Cover all the zeros with the smallest number of horizontal or vertical lines and if the number of lines is equal to the number of rows, the process is complete. If this is not the case, repeat steps 2, 3 and 4 until a solution is found.

5 Draw a bipartite graph and use it to determine the optimum allocation.

Worked example 12

Four workers A, B, C and D all provide quotes for jobs 1, 2, 3 and 4. Their quotes are shown in the matrix, where the rows represent the workers A, B, C and D and the columns represent the jobs 1, 2, 3 and 4.

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 15 & 24 & 40 & 21 \\ 24 & 18 & 19 & 20 \\ 28 & 13 & 30 & 22 \\ 10 & 18 & 17 & 13 \end{bmatrix} \end{array}$$

a Determine the best allocation of workers that minimises the total quote.

b Find the cost of the four jobs.

Working

a **1** Perform a row reduction.

Row A: subtract 15
Row B: subtract 18
Row C: subtract 13
Row D: subtract 10

	1	2	3	4
A	0	9	25	6
B	6	0	1	2
C	15	0	17	9
D	0	8	7	3

2 Perform a column reduction.

Column 3: subtract 1
Column 4: subtract 2

	1	2	3	4
A	0	9	24	4
B	6	0	0	0
C	15	0	16	7
D	0	8	6	1

3 Cover the zeros.

There are 3 lines and 4 rows.

As the number of lines does not equal the number of rows, the Hungarian algorithm must be used.

	1	2	3	4
A	0	9	24	4
B	6	0	0	0
C	15	0	16	7
D	0	8	6	1

4 The smallest uncovered number is 1. Add this to every covered number but add 2 to the numbers covered twice.

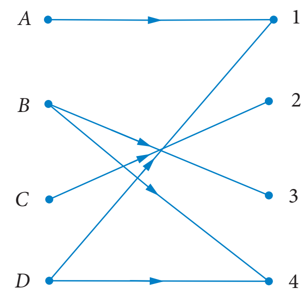
	1	2	3	4
A	1	10	24	4
B	8	2	1	1
C	16	1	16	7
D	1	9	6	1

5 The smallest value is 1, so subtract this from every element in the matrix.

Now cover the zeros. There are now 4 lines and 4 rows, so the allocation is complete.

	1	2	3	4
A	0	9	23	3
B	7	1	0	0
C	15	0	15	6
D	0	8	5	0

6 Draw the bipartite graph.



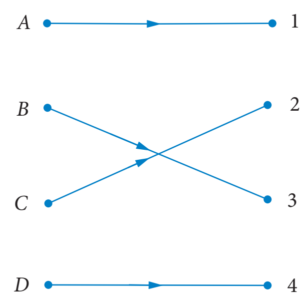
Determine the allocation.

Two jobs have a single worker connected.

- Worker C for job 2
- Worker B for job 3

This leaves

- Worker D for job 4
- Worker A for job 1



Allocation $A1$, $B3$, $C2$, $D4$

b Look at the original matrix to determine the cost.

$A1 = 15$, $B3 = 19$, $C2 = 13$, $D4 = 13$

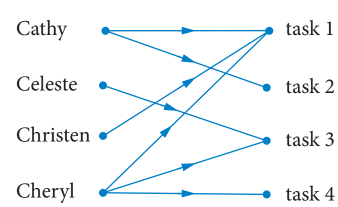
$$\begin{aligned}\text{Cost} &= \$15 + \$19 + \$13 + \$13\\ &= \$60\end{aligned}$$

EXAM PREP 11.3

The assignment problem and bipartite graphs

Prep 1 WORKED EXAMPLE 9

Determine the allocation of tasks from the following bipartite graph.



Prep 2 WORKED EXAMPLE 10

Workers A, B, C and D are each to be assigned one task from tasks 1, 2, 3 and 4. The matrix representing the workers that are qualified to perform the tasks is shown.

Draw a bipartite graph and hence determine a valid allocation.

$$\begin{array}{c} \\ A\\ B\\ C\\ D \end{array}\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array}\\ \begin{bmatrix} 0 & 0 & 1 & 0\\ 1 & 0 & 1 & 1\\ 0 & 0 & 1 & 1\\ 0 & 1 & 0 & 1 \end{bmatrix} \end{array}$$

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Prep 3 WORKED EXAMPLE 11

Perform row and column reductions on the matrix that represents the time, in hours, for workers A, B, C and D to complete tasks 1, 2, 3 and 4, and hence determine the optimum allocation and the minimum time to complete all four tasks.

$$\begin{array}{cc} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} A \\ B \\ C \\ D \end{matrix} & \begin{bmatrix} 25 & 14 & 7 & 15 \\ 6 & 10 & 8 & 9 \\ 13 & 5 & 9 & 5 \\ 8 & 5 & 10 & 11 \end{bmatrix} \end{array}$$

Prep 4 WORKED EXAMPLE 12

A taxi company has 3 taxis A, B and C available and there are three customers 1, 2 and 3 who require a taxi. The distance (km) that each taxi must travel to get to the customers is shown in the table.

Use the Hungarian algorithm to find the optimum allocation that minimises the distance travelled.

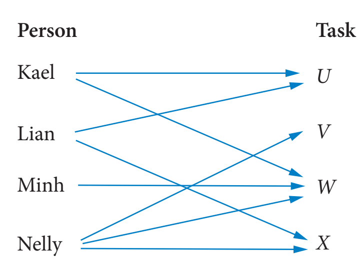
	1	2	3
A	11	19	17
B	21	15	13
C	15	18	21

EXAM PRACTICE 11.3

The assignment problem and bipartite graphs

Question 1

The bipartite graph represents the tasks that four people are able to undertake.



The matrix representation for this task allocation is

A
$$\begin{array}{cc} & \begin{matrix} U & V & W & X \end{matrix} \\ \begin{matrix} \text{Kael} \\ \text{Lian} \\ \text{Minh} \\ \text{Nelly} \end{matrix} & \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

B
$$\begin{array}{cc} & \begin{matrix} U & V & W & X \end{matrix} \\ \begin{matrix} \text{Kael} \\ \text{Lian} \\ \text{Minh} \\ \text{Nelly} \end{matrix} & \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

C
$$\begin{array}{cc} & \begin{matrix} U & V & W & X \end{matrix} \\ \begin{matrix} \text{Kael} \\ \text{Lian} \\ \text{Minh} \\ \text{Nelly} \end{matrix} & \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

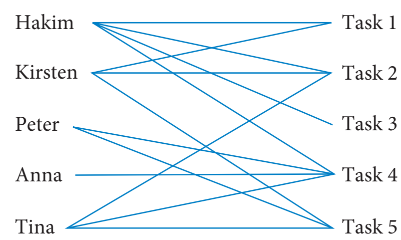
D
$$\begin{array}{cc} & \begin{matrix} U & V & W & X \end{matrix} \\ \begin{matrix} \text{Kael} \\ \text{Lian} \\ \text{Minh} \\ \text{Nelly} \end{matrix} & \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

E
$$\begin{array}{cc} & \begin{matrix} U & V & W & X \end{matrix} \\ \begin{matrix} \text{Kael} \\ \text{Lian} \\ \text{Minh} \\ \text{Nelly} \end{matrix} & \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

[VCAA 2005 1NQ3]

Question 2

The bipartite graph shows the tasks that each of five people are able to undertake.



If each person is to be allocated one task only, then a feasible task allocation is

A

Hakim	3
Kirsten	1
Peter	5
Anna	4
Tina	2

B

Hakim	3
Kirsten	2
Peter	5
Anna	4
Tina	1

C

Hakim	3
Kirsten	1
Peter	2
Anna	4
Tina	5

D

Hakim	3
Kirsten	5
Peter	1
Anna	4
Tina	2

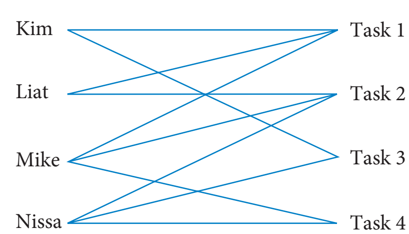
E

Hakim	3
Kirsten	5
Peter	1
Anna	2
Tina	4

[VCAA 2003 1NQ1]

Question 3

The bipartite graph shows the tasks that each of four people is able to undertake.



All tasks must be allocated and each person can be allocated one task only. A valid task allocation is

A

Kim	task 1
Liat	task 2
Mike	task 3
Nissa	task 4

B

Kim	task 3
Liat	task 1
Mike	task 2
Nissa	task 3

C

Kim	task 3
Liat	task 2
Mike	task 1
Nissa	task 4

D

Kim	task 1
Liat	task 3
Mike	task 4
Nissa	task 2

E

Kim	task 2
Liat	task 1
Mike	task 4
Nissa	task 3

[VCAA 2012 1NQ3]

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Question 4

Kate, Lexie, Mei and Nasim enter a competition as a team. In this competition, the team must complete four tasks, *W*, *X*, *Y* and *Z*, as quickly as possible. The table shows the time, in minutes, that each person would take to complete each of the four tasks.

	Kate	Lexie	Mei	Nasim
W	6	3	4	6
X	4	3	5	5
Y	5	7	9	6
Z	3	2	3	2

If each team member is allocated one task only, the minimum time in which this team would complete the four tasks is

A 10 minutes
B 12 minutes
C 13 minutes
D 14 minutes
E 15 minutes

[VCAA 2013 1NQ4]

Question 5

Four workers, Anna, Bill, Caitlin and David, are each to be assigned a different task.

The table gives the time, in minutes, that each worker takes to complete each of the four tasks.

	Task 1	Task 2	Task 3	Task 4
Anna	7	5	15	9
Bill	8	5	18	10
Caitlin	4	6	22	4
David	7	11	16	10

The tasks are allocated so as to minimise the total time taken to complete the four tasks. This total time, in minutes, is

A 21
B 28
C 31
D 34
E 38

[VCAA 2009 1NQ7]

Question 6

The table shows the time (in minutes) that each of four people, Aiden, Bing, Callum and Dee, would take to complete each of the tasks *U*, *V*, *W* and *X*. If each person is allocated one task only, the minimum total time for this group of people to complete all four tasks is

A 22 minutes
B 28 minutes
C 29 minutes
D 30 minutes
E 32 minutes

	Task			
	U	*V*	*W*	*X*
Aiden	3	2	9	9
Bing	5	6	12	12
Callum	9	6	12	14
Dee	8	3	8	12

[VCAA 2010 1NQ9]

11.3

Question 7

Five people are to be each allocated one of five tasks (A, B, C, D, E). The table shows the time, in hours, that each person takes to complete the tasks. The tasks must be completed in the least possible total amount of time. If no person can help another, Francis should be allocated task

A A **B** B **C** C

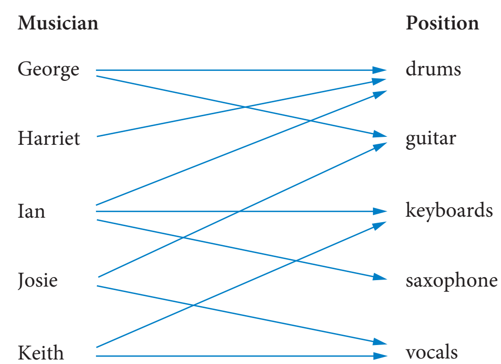
D D **E** E

$$\begin{array}{l} \\ \\ \text{Francis} \\ \text{David} \\ \text{Herman} \\ \text{Indira} \\ \text{Natalie} \end{array} \begin{array}{c} \text{Task} \\ \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 12 & 15 & 99 & 10 & 14 \\ 10 & 9 & 10 & 7 & 12 \\ 99 & 10 & 11 & 6 & 12 \\ 8 & 8 & 12 & 9 & 99 \\ 8 & 99 & 9 & 8 & 11 \end{bmatrix} \end{array}$$

[VCAA 2004 1NQ7]

Question 8

George, Harriet, Ian, Josie and Keith are a group of five musicians. They are forming a band where each musician will fill one position only. The following bipartite graph illustrates the positions that each is able to fill.



a Which musician **must** play the guitar? 1 mark

b Copy and complete the table showing the positions that the following musicians **must** fill in the band. 2 marks

Person	Position
Harriet	
Ian	
Keith	

[VCAA 2006 2NQ1]

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Question 9

Four tasks, W, X, Y and Z, must be completed. Four workers, Julia, Ken, Lana and Max, will each do one task. Table 1 shows the time, in minutes, that each person would take to complete each of the four tasks.

The tasks will be allocated so that the total time of completing the four tasks is a minimum. The Hungarian method will be used to find the optimal allocation of tasks.

Table 1

		Worker			
		Julia	Ken	Lana	Max
Task	W	26	21	22	25
	X	31	26	21	38
	Y	29	26	20	27
	Z	38	26	26	35

Step 1 of the Hungarian method is to subtract the minimum entry in each row from each element in the row.

a Complete step 1 for task X by writing down the number missing from the shaded cell in Table 2. 1 mark

Table 2

		Worker			
		Julia	Ken	Lana	Max
Task	W	5	0	1	4
	X	10	5	0	
	Y	9	6	0	7
	Z	12	0	0	9

The second step of the Hungarian method ensures that all columns have at least one zero. The numbers that result from this step are shown in Table 3.

b Following the Hungarian method, the smallest number of lines that can be drawn to cover the zeros is shown dashed in Table 3. These dashed lines indicate that an optimal allocation cannot be made yet. Give a reason why. 1 mark

Table 3

		Worker			
		Julia	Ken	Lana	Max
Task	W	0	0	1	0
	X	5	5	0	13
	Y	4	6	0	3
	Z	7	0	0	5

c Complete the steps of the Hungarian method to produce a table from which the optimal allocation of tasks can be made. 1 mark

d Write the name of the task that each person should do for the optimal allocation of tasks. 2 marks

[VCAA 2012 2NQ3]

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11.4 Network flow problems

The applications of flow networks range from transporting water through a network of water pipes to moving people or product by road or a rail network.

Flow capacity and maximum flow

WS

Network flow capacity

The first vertex of the network is called the **source** and the final vertex of the network is called the **sink**.

The **inflow capacity** is the total flow capacity entering a vertex and the **outflow capacity** is the total flow capacity leaving the vertex.

The **maximum flow through a vertex** is the smaller of the total inflow capacity of the vertex and the total outflow capacity of the vertex.

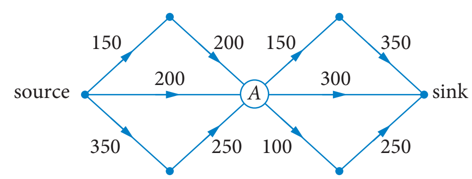
Worked example 13

The directed graph shows the flow capacities in litres per minute. Determine

a the inflow capacity of vertex *A*

b the outflow capacity of vertex *A*

c the maximum flow out of vertex *A*.



Working

a The total inflow capacity of vertex *A* can be found by adding the values on the edges entering vertex *A*.

The total inflow capacity of vertex *A*
= 200 + 200 + 250
= 650 litres per minute

b The total outflow capacity of vertex *A* can be found by adding the values on the edges leaving vertex *A*.

The total outflow capacity of vertex *A*
= 150 + 300 + 100
= 550 litres per minute

c The maximum flow out of vertex *A* is the smaller of inflow capacity and outflow capacity.

The flow out of vertex *A* = 550 litres per minute.

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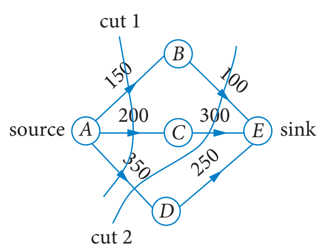
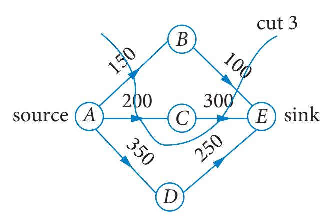
The capacity of a cut

A **cut** through a network must stop all flow from the start (*source*) to the end (*sink*).

One method of finding the maximum flow through a network is to use the minimum capacity of the cuts. In the network shown, cut 1 and cut 2 are both valid. The cuts sever the pipes in the network in such a way that there is no flow path from the source (A) to the sink (E). Cut 3 is **not** a valid cut because it does not stop all the flow between the source and the sink. It is still possible for flow to occur from A to D to E.



valid cuts

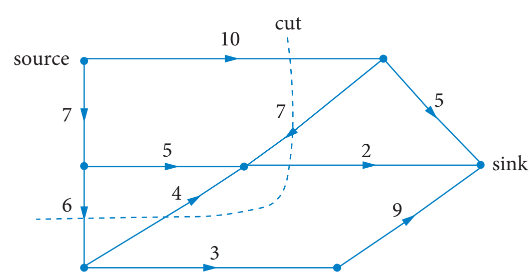


invalid cuts

The **capacity of a cut** can be determined by adding all the flow capacities that pass across the cut in the direction from source to sink.

Worked example 14

In the network, the numbers on the edges show the maximum possible flow between the vertices. The direction of the arrow indicates the direction of the flow. A cut separating the sink from the source is also shown. Determine the capacity of the cut.



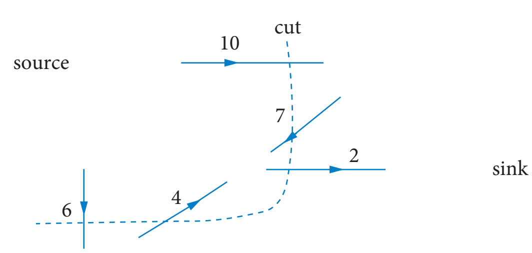
Working

Add the flows where the arrow crosses the cut in the direction from the source side of the cut to the sink side of the cut.

Flow of cut $= 6 + 10 + 2$

$= 18$

4 and 7 are not included as they flow in the direction sink to source across the cut.



Exam hack

The source is the first vertex (on the left) and the sink is the last vertex (on the right). To determine the capacity of a cut, only add flow capacities that pass across the cut in the direction left (source) to right (sink).

Maximum flow – minimum cut

The **maximum flow** from source to sink can be determined by finding the cut that produces the minimum value.

To determine the maximum flow for a network,

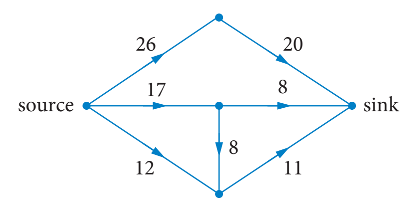
1 identify cuts through the network

2 find the capacity of each cut

3 maximum flow = capacity of the minimum cut

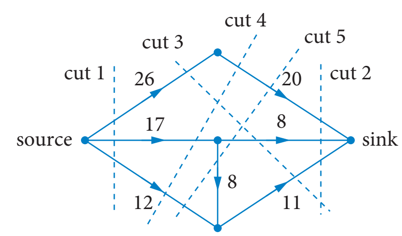
Worked example 15

Find the maximum flow for the network.



Working

1 Identify cuts that stop the flow from source to sink.



2 Find the capacity of each cut.

Cut 1 capacity = 26 + 17 + 12 = 55
Cut 2 capacity = 20 + 8 + 11 = 39
Cut 3 capacity = 26 + 8 + 11 = 45
Cut 4 capacity = 12 + 17 + 20 = 49
Cut 5 capacity = 12 + 8 + 20 = 40

3 Maximum flow = minimum cut

The maximum flow from source to sink = 39.

Shutterstock.com/Tooykrub

11.4

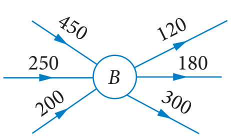
EXAM PREP 11.4

Network flow problems

Prep 1

WORKED EXAMPLE 13

The traffic capacity, in vehicles per hour, of roads in and out of an intersection, labelled vertex *B*, is shown. Determine



a the inflow capacity of vertex *B*

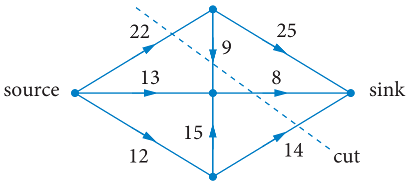
b the outflow capacity of vertex *B*

c the maximum flow out of vertex *B*.

Prep 2

WORKED EXAMPLE 14

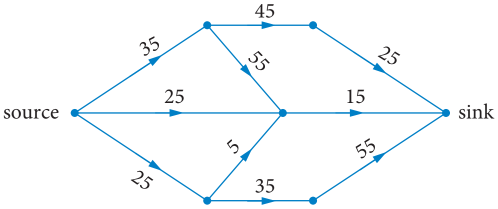
Determine the capacity of the cut.



Prep 3

WORKED EXAMPLE 15

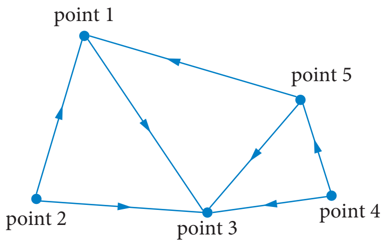
Find the maximum flow for the network.



Network flow problems

Question 1

Steel water pipes connect five points underground. The directed graph shows the directions of the flow of water through these pipes between these points. The directed graph shows that water can flow from



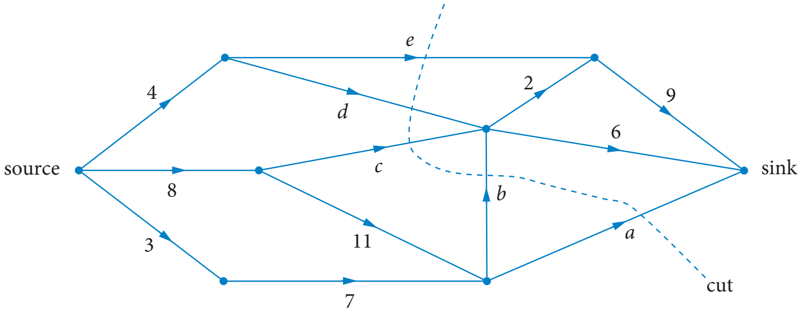
A point 1 to point 2 **B** point 1 to point 4

C point 4 to point 1 **D** point 4 to point 2a

E point 5 to point 2

[VCAA 2008 1NQ1]

Question 2



In the directed graph, the weight of each edge is non-zero. The capacity of the cut shown is

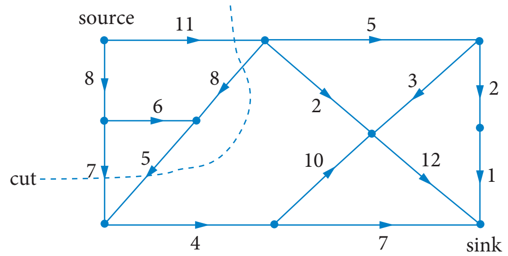
A $a + b + c + d + e$ **B** $a + c + d + e$ **C** $a + b + c + e$

D $a + b + c - d + e$ **E** $a - b + c - d + e$

[VCAA 2006 1NQ6]

Question 3

In the network, the values on the edges give the maximum flow possible between each pair of vertices. The arrows show the direction of flow. A cut that separates the source from the sink in the network is also shown. The capacity of this cut is



A 14 **B** 18 **C** 23 **D** 31 **E** 40

[VCAA 2010 1NQ6]

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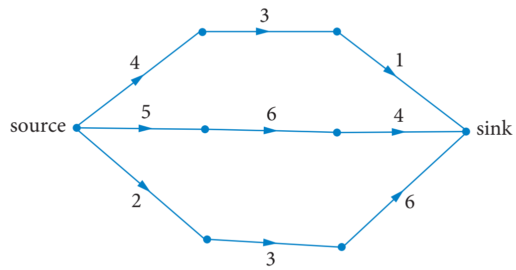
Question 4



The maximum flow from source to sink through the network shown is

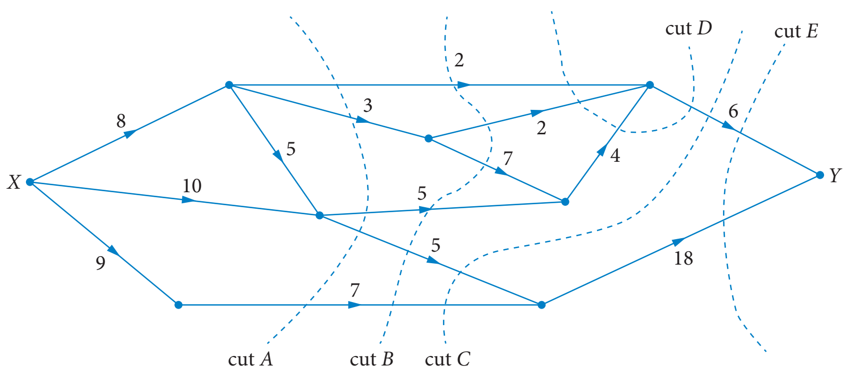
A 6 **B** 7 **C** 8

D 11 **E** 16

[VCAA 2009 1NQ3]

11.4

Question 5

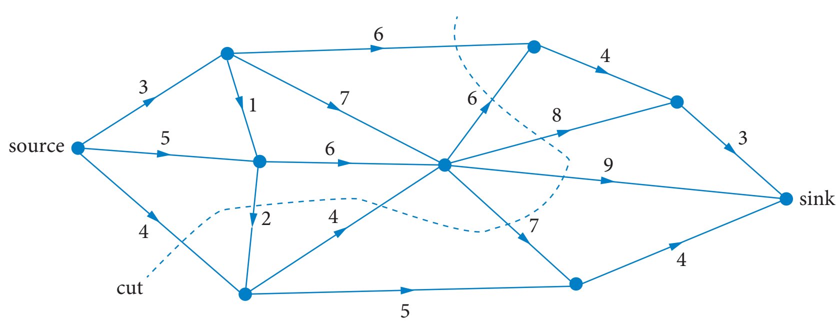


On the directed graph, the values on the edges give the maximum flow between nodes in the direction of the arrows. Five cuts have been made on the diagram. Which cut allows you to find the maximum flow from point X to point Y?

A cut A **B** cut B **C** cut C **D** cut D **E** cut E

[VCAA 2005 1NQ6]

Question 6



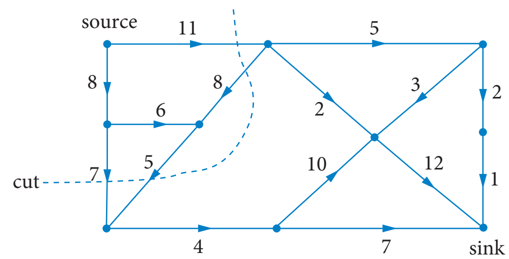
For the graph, the capacity of the cut shown is

A 33 **B** 36 **C** 40 **D** 42 **E** 46

[VCAA 2008 1NQ6]

Question 7

In the network, the values on the edges give the maximum flow possible between each pair of vertices. The arrows show the direction of flow. A cut that separates the source from the sink in the network is also shown. The maximum flow between source and sink through the network is

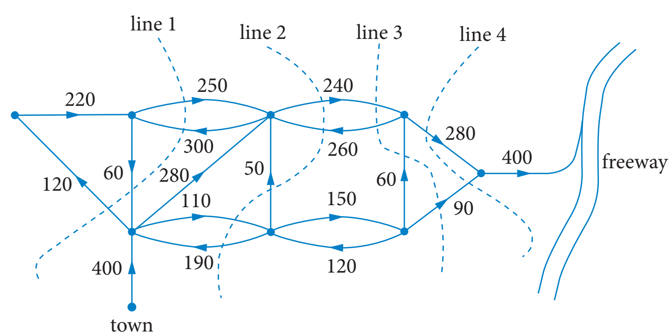


A 7 **B** 10 **C** 11 **D** 12 **E** 20

[VCAA 2010 1NQ7]

Question 8

Vehicles from a town can drive onto a freeway along a network of one-way and two-way roads, as shown in the network diagram. The numbers indicate the maximum number of vehicles per hour that can travel along each road in this network. The arrows represent the permitted direction of travel. One of the four dotted lines shown on the diagram is the minimum cut for this network.



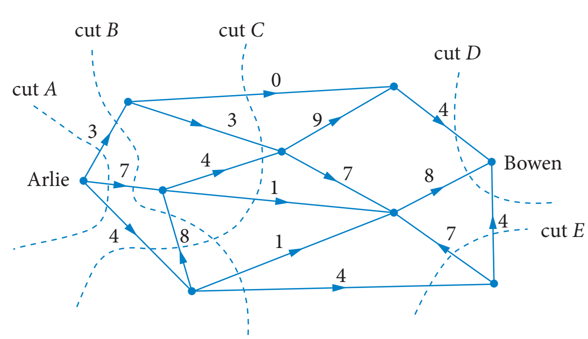
The maximum number of vehicles per hour that can travel through this network from the town onto the freeway is

A 310 **B** 330 **C** 350 **D** 370 **E** 390

[VCAA 2012 1NQ7]

Question 9

A train journey consists of a connected sequence of stages formed by edges on the following directed network from Arlie to Bowen. The number of available seats for each stage is indicated beside the corresponding edge, as shown on the diagram. The five cuts, *A*, *B*, *C*, *D* and *E*, shown on the network, are attempts to find the maximum number of available seats that can be booked for a journey from Arlie to Bowen.



a Write down the capacities of cut *A*, cut *B* and cut *C*. 3 marks

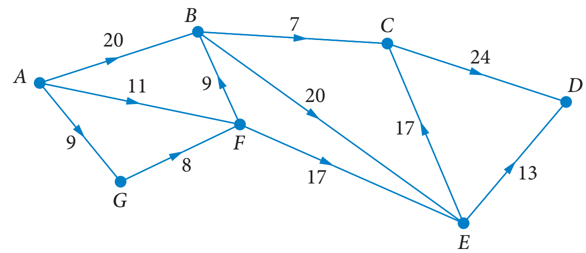
b Cut *E* is not a valid cut when trying to find the minimum cut between Arlie and Bowen. Why? 1 mark

c Determine the maximum number of available seats for a train journey from Arlie to Bowen. 1 mark

[VCAA 2003 2NQ1]

Question 10

The rangers at a wildlife park restrict access to the walking tracks through areas where the animals breed. The edges on the directed network diagram represent one-way tracks through the breeding areas. The direction of travel on each track is shown by an arrow. The numbers on the edges indicate the maximum number of people who are permitted to walk along each track each day.



a Starting at *A*, how many people, in total, are permitted to walk to *D* each day? 1 mark

One day, all the available walking tracks will be used by students on a school excursion. The students will start at *A* and walk in four separate groups to *D*.

Students must remain in the same groups throughout the walk.

b Group 1 will have 17 students. This is the maximum group size that can walk together from *A* to *D*. Write down the path that group 1 will take. 1 mark

c Groups 2, 3 and 4 will each take different paths from *A* to *D*.

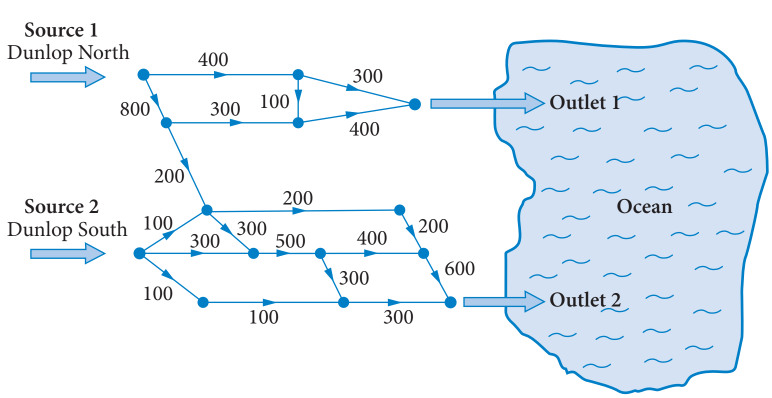
Copy the table and write the six missing entries. 2 marks

Group	Maximum group size	Path taken from *A* to *D*
1	17	answered in **part b**
2		
3		
4		

[VCAA 2013 2NQ3]

Question 11

Stormwater enters a network of pipes at either Dunlop North (Source 1) or Dunlop South (Source 2) and flows into the ocean at either Outlet 1 or Outlet 2. On the network diagram, the pipes are represented by straight lines with arrows that indicate the direction of the flow of water. Water cannot flow through a pipe in the opposite direction. The numbers next to the arrows represent the maximum rate, in kilolitres per minute, at which stormwater can flow through each pipe.



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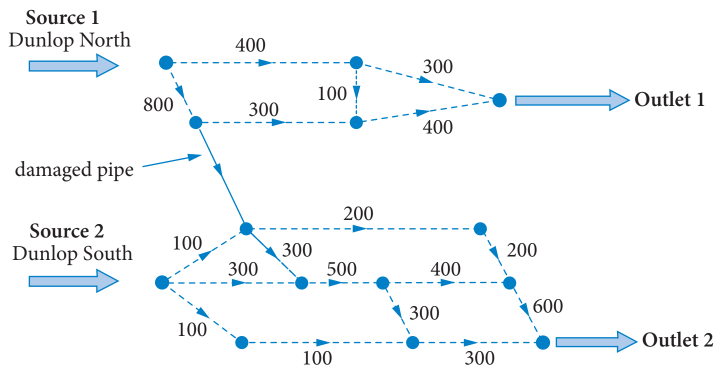
a Copy and complete the following sentence for this network of pipes by writing either the number 1 or 2 in each box.

Stormwater from Source $\boxed{}$ **cannot** reach Outlet $\boxed{}$ 1 mark

b Determine the maximum rate, in kilolitres per minute, that water can flow from these pipes into the ocean at **i** Outlet 1 and **ii** Outlet 2. 2 marks

A length of pipe, shown by a solid line on the network diagram, has been damaged and will be replaced with a larger pipe.

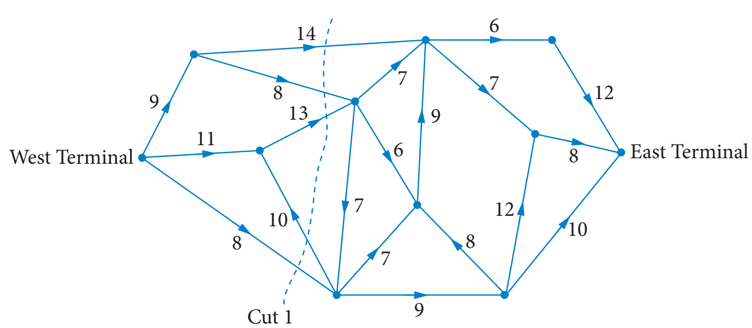


c The new pipe must enable the greatest possible rate of flow of stormwater into the ocean from Outlet 2. What minimum rate of flow through the pipe, in kilolitres per minute, will achieve this? 1 mark

[VCAA 2011 2NQ4]

Question 12

As an attraction for young children, a miniature railway runs throughout the new housing estate. The trains travel through stations that are represented by nodes on the directed network diagram.
The number of seats available for children, between each pair of stations, is indicated beside the corresponding edge.



Cut 1, through the network, is shown in the diagram.

a Determine the capacity of Cut 1. 1 mark

b Determine the maximum number of seats available for children for a journey that begins at the West Terminal and ends at the East Terminal. 1 mark

On one particular train, 10 children set out from the West Terminal. No new passengers board the train on the journey to the East Terminal.

c Determine the maximum number of children who can arrive at the East Terminal on this train. 1 mark

[VCAA 2007 2NQ3]

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Dijkstra's algorithm and directed graphs

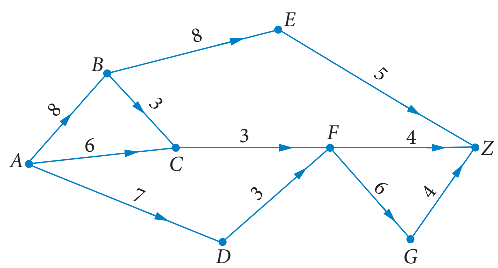
11.5

Dijkstra's shortest path algorithm

Dijkstra's algorithm can also be used for finding the shortest path of directed graphs. In these cases the possible shortest paths are affected by the direction indicated on the edge.

Worked example 16

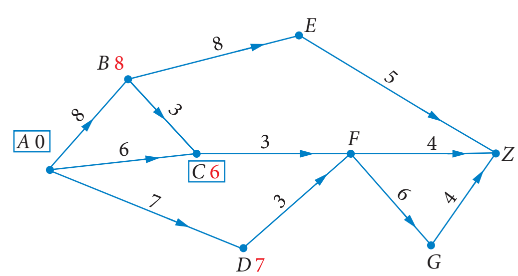
The times, in minutes, to various tracks in an orienteering event are shown in the network. Find the quickest path from A to Z using Dijkstra's algorithm.



Working

1 Draw a box around the starting vertex A and enter a value of zero.

2 The boxed vertex A is connected to B, C and D. Add the times on the connecting edges to the boxed value of 0 and write these totals next to the vertices, shown in red on the diagram.



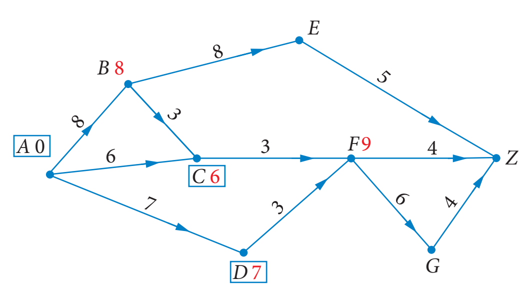
Vertex $B = 0 + 8 = 8$
Vertex $C = 0 + 6 = 6$
Vertex $D = 0 + 7 = 7$

Box the smallest value, that is, C 6.

3 Consider all the unboxed vertices connected to the vertex C, which is F only.

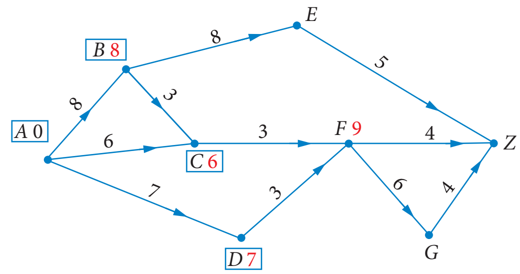
Vertex $F = 6 + 3 = 9$

Box the smallest value, that is, D 7.



4 D is connected to F only: $7 + 3 = 10$, which is larger than the current vertex value, so leave 9 as the value for F.

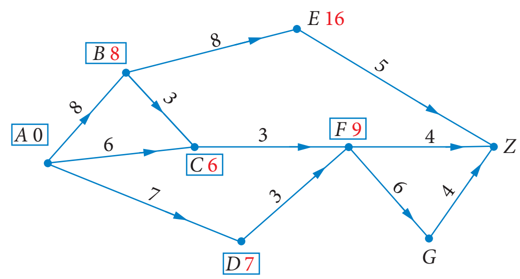
Box the smallest value, that is, B 8.



5 Consider all the unboxed vertices connected to the vertex B, which is E only.

Vertex $E = 8 + 8 = 16$.

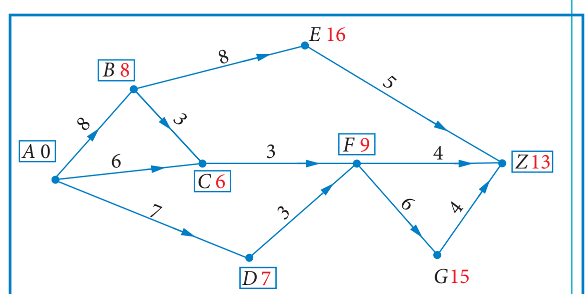
Box the smallest value, that is, F 9.



6 Consider all the unboxed vertices connected to the vertex F:

Vertex $G = 9 + 6 = 15$
Vertex $Z = 9 + 4 = 13$

Box the smallest value, that is, Z 13.



7 Backtrack from vertex Z to vertex A.

The shortest time path of 13 minutes is A-C-F-Z.

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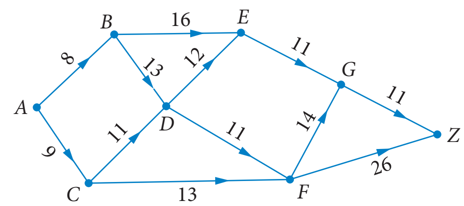
EXAM PREP 11.5

Dijkstra's algorithm and directed graphs

Prep 1 WORKED EXAMPLE 16

This network shows distances in kilometres. Find the shortest path and the shortest distance from *A* to *Z* using Dijkstra's algorithm.

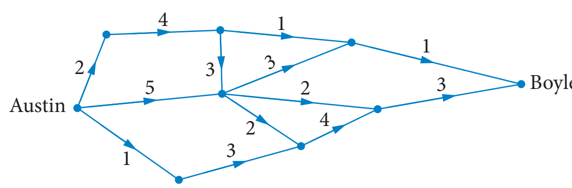


EXAM PRACTICE 11.5

Dijkstra's algorithm and directed graphs

Question 1

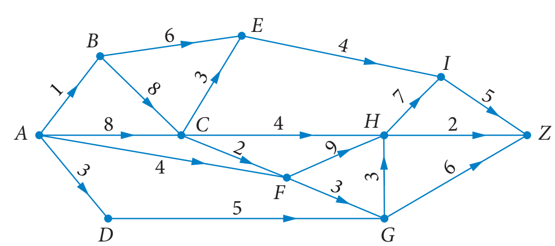
The network shows the distances, in kilometres, along a series of one-way roads that connect the cities of Austin and Boyle. The shortest distance between Austin and Boyle is



A 11 **B** 8 **C** 9 **D** 10 **E** 14

Use the following information to answer Questions 2 – 4.

The time in minutes taken to traverse a series of one-way tracks is shown in the directed graph.



Question 2

The shortest time taken to travel from vertex *A* to vertex *H* is

A 13 **B** 11 **C** 4 **D** 10 **E** 12

Question 3

The shortest path form vertex *A* to vertex *Z* is

A *A-F-G-H-Z* **B** *A-F-G-Z* **C** *A-D-G-Z* **D** *A-B-E-I-Z* **E** *A-B-C-H-Z*

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Question 4

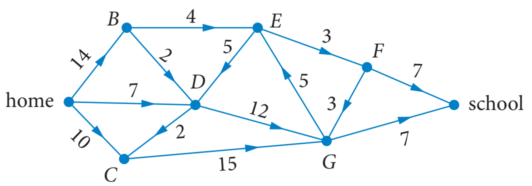
The shortest time to travel from vertex A to vertex Z is

A 13 minutes **B** 14 minutes **C** 12 minutes

D 15 minutes **E** 11 minutes

Use the following information to answer Questions 5 & 6.

The network shows the travel times, in minutes, along a series of roads that connect a student's home to school. The student may only travel in the direction indicated by the arrows.



Question 5

The shortest time, in minutes, for this student to travel from home to school is

A 28 **B** 22 **C** 26 **D** 29 **E** 23

Question 6

The shortest path from home to school is

A home-D-G-school **B** home-D-E-F-school **C** home-B-E-F-school

D home-C-G-school **E** home-B-D-G-school

Directed graphs

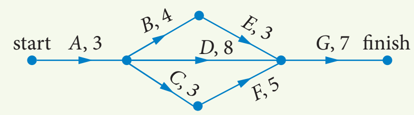
Practice quiz

SUMMARY

11

The scheduling problem

- The edges on a **directed graph**, or digraph, have a direction. In a **weighted digraph** there is a number associated with each edge.

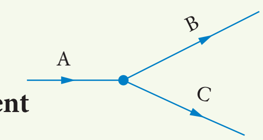


- An activity table shows the order and estimated time for each activity.
- An **immediate predecessor** is any activity that must be completed before this activity can commence.

There are three different types of connections.

1. **One preceding event**

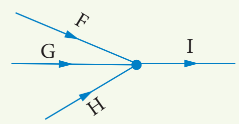
   Event *B* is preceded by *A*.



2. **More than one activity with the same preceding event**

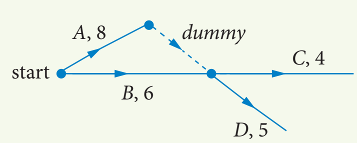
   Events *B* and *C* are preceded by *A*.

3. **An activity with more than one preceding event**



   Event *I* is preceded by *F*, *G* and *H*.

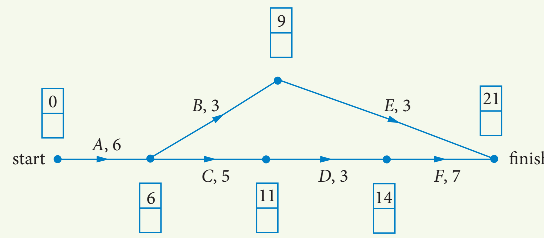
- When drawing a network diagram
  - Use a vertex to represent the start of the network.
  - Look for any activities that do not have any predecessors. These will be your starting activities.
  - Multiple predecessors to an activity will all end at the same vertex.
  - An activity should not be represented by more than one edge in the network.
  - Two vertices can be connected by one edge only.
  - A vertex indicating the completion of the project needs to be included in the network.
- **Reachability** is the ability to get from one vertex to another vertex in a directed graph.
- A **dummy activity** needs to be added to a network to ensure that no two vertices are connected by multiple edges or to maintain precedence structure.



- A dummy activity has zero time and is shown as a directed edge with a broken line.

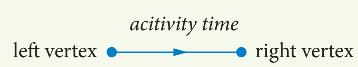
Forward scanning to determine EST

- The earliest start time (EST) for an activity is the earliest time it is possible to start the activity.
- Forward scanning through a network enables us to determine the earliest start time (EST) for every activity in the network.
  1. Begin with the first vertex, which has an earliest start time of 0. We will use the convention of writing the earliest starting times in the top box next to each vertex.
  2. ESTs are calculated from left to right.
  3. Add the activity time to the EST of the previous vertex. If more than one activity leads to the vertex, then the highest figure obtained becomes the new EST.
  4. Continue until the finish is reached.



Backward scanning to determine LST

- The latest start time (LST) is the latest time you can start an activity without affecting the project completion time.
- The LST at the left vertex = the LST at the right vertex – the activity time



- Backward scanning is the process used to find the latest start times (LST) for activities.
  1. Commence LST calculations at the finish vertex. At the finish vertex, LST = EST.
  2. Work backwards from left to right. LSTs are written in the bottom box next to each vertex.
  3. To find the LST at the left vertex, work backwards, subtracting the activity time from the LST at the right vertex.
  4. Where there is more than one path to the previous vertex, take the smallest result as the LST.
  5. The LST at the first vertex must be zero.

The critical path

- The critical path is the longest time path in the network.
- To determine the critical path
  1. Use forward scanning to determine the earliest start times for each activity.
  2. Identify the path or paths that produce the final EST.
  3. Use backward scanning to determine the latest start times for each activity.
  4. Activities for which EST = LST are on the critical path.

Activity float time and project crashing

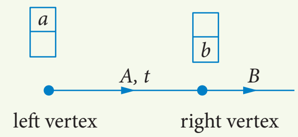
- The **float time** for an activity is the maximum time an activity can be extended or postponed without affecting the project completion time.

  In the diagram shown

  a = the EST of activity A

  b = the LST for activity B

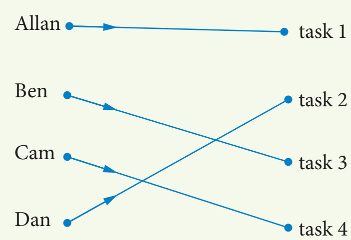
  t = the activity time for activity A



- Float for non-critical activity $A = b - a - t$
- For critical activities, Float = 0

Bipartite graphs

- A bipartite graph has its vertices in two distinct sets and the edges join elements in the first set to elements in the second set.



Row and column reduction method

1. Choose the smallest number in each row and subtract it from every element in the same row.
2. For every column that does not have a zero value, choose the smallest number in the column and subtract it from every element in the same column.
3. Cover all the zeros with the smallest number of lines (horizontal or vertical).

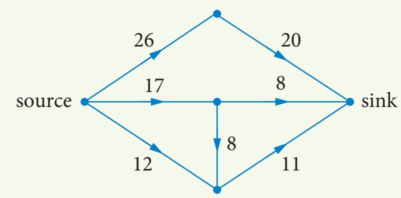
   The process is complete if the number of lines is equal to the number of rows.

The Hungarian algorithm

1. Perform a row and column reduction and cover all the zeros with the smallest number of horizontal or vertical lines possible. If the number of lines is equal to the number of rows, a solution has been found.
2. If the number of lines does not equal the number of rows, find the smallest uncovered number and add it to every covered number. Numbers that are covered twice have this number added twice.
3. Subtract the smallest number in the matrix or table from all of the numbers in the table.
4. Cover all the zeros with the smallest number of horizontal or vertical lines and if the number of lines is equal to the number of rows, the process is complete. If this is not the case, then repeat steps 2, 3 and 4 until a solution is found.
5. Draw a bipartite graph and use it to determine the optimum allocation.

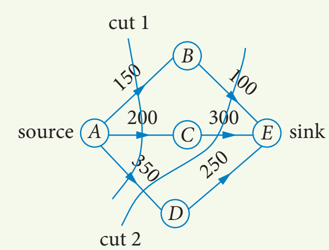
Flow capacity and maximum flow

- The first vertex of the network is called the source and the final vertex of the network is called the sink.
- The **inflow capacity** is the total flow capacity entering a vertex and the **outflow capacity** is the total flow capacity leaving the vertex.
- The **maximum flow through a vertex** is the smaller of the total inflow capacity of the vertex and the total outflow capacity of the vertex.



Maximum flow – minimum cut

- A **cut** through a network must stop all flow from the start (*source*) to the end (*sink*).
- The maximum flow from source to sink can be determined by finding the cut that produces the minimum value.
- To determine the maximum flow for a network
  1. Identify cuts through the network.
  2. Find the capacity of each cut.
  3. maximum flow = capacity of the minimum cut



valid cuts

Dijkstra's shortest path algorithm

- Dijkstra's algorithm can be used for finding the shortest path for directed graphs.

CHAPTER 12

MEASUREMENT AND TRIGONOMETRY

12.1 Surface area and volume
- Pythagoras' theorem
- Perimeter and circumference
- Areas of shapes
- Total surface area
- Volume

12.2 Scale factors
- Scales and similar figures
- Polygons and similarity
- Triangles and similarity
- Length, area and volume scale factors

12.3 Right-angled triangles
- Finding side lengths
- Finding angles
- Angles of elevation and depression

12.4 Non-right-angled triangles
- The sine rule
- Using CAS: Solving sine rule equations to find an angle
- The ambiguous case for the sine rule
- The cosine rule

12.5 Bearings

Summary

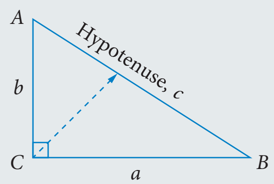
Prior learning 1

Prior learning 2

12.1 Surface area and volume

Pythagoras' theorem

Pythagoras' theorem is used to find the length of a side in a right-angled triangle.



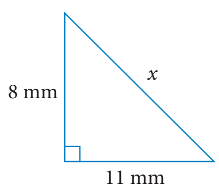
In $\triangle ABC$: $c^2 = a^2 + b^2$

The **hypotenuse** is the longest side of a right-angled triangle and is always opposite the right angle.

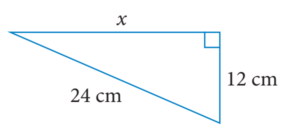
Worked example 1

Find the value of x correct to two decimal places in each right-angled triangle.

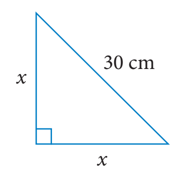
a



b



c



	Working
a Substitute $a = 8$, $b = 11$ and $c = x$ into the formula $c^2 = a^2 + b^2$ and solve for x, using a CAS/calculator if necessary.	$c^2 = a^2 + b^2$ $x^2 = 8^2 + 11^2$ $x^2 = 185$ $x = \sqrt{185} \approx 13.60$ mm
b **1** Substitute $a = 12$, $b = x$ and $c = 24$.	$24^2 = 12^2 + x^2$
2 Solve the equation for x.	$x^2 = 24^2 - 12^2$ $x = \sqrt{432} \approx 20.78$ cm
c **1** Substitute $a = x$, $b = x$ and $c = 30$.	$30^2 = x^2 + x^2$
2 Solve the equation for x, using a CAS/calculator if necessary.	$2x^2 = 30^2$ $2x^2 = 900$ $x^2 = 450$ $x = \sqrt{450} \approx 21.21$ cm

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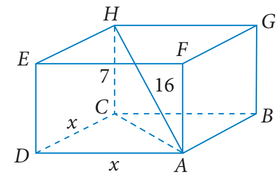
Worked example 2

This rectangular box has a square base. The lengths of AH and CH are 16 cm and 7 cm respectively. Find x, correct to three decimal places.



Working

1 Use Pythagoras' theorem to find AC in triangle ACH.

$a^2 + b^2 = c^2$

$(AC)^2 + 7^2 = 16^2$

$(AC)^2 = 16^2 - 7^2$

$(AC)^2 = 207$

$AC \approx 14.387$

2 Use Pythagoras' theorem to find x in triangle ADC.

$c^2 = a^2 + b^2$

$(AC)^2 = x^2 + x^2$

$207 = 2x^2$ ← $(AC)^2 = 207$ from above.

$x^2 = 103.5$

$x = \sqrt{103.5} \approx 10.173$ cm

12.1

Perimeter and circumference

The **perimeter** is the distance around a two-dimensional shape.
The perimeter of a circle is called the **circumference**.

WS Units of length and perimeter

Circumference of a circle

The circumference of a circle can be calculated by using one of the following formulas.

Circumference = $\pi \times$ diameter (or $C = \pi d$)

Circumference = $2 \times \pi \times$ radius (or $C = 2\pi r$)

Worked example 3

Find the radius of a circle, correct to one decimal place, given that its circumference is 100 cm.

Working

1 Substitute $C = 100$ into the formula $C = 2\pi r$.

$C = 2\pi r$

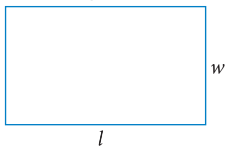
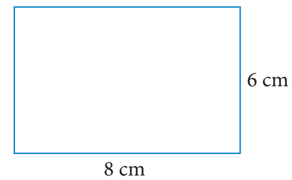
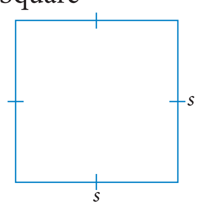
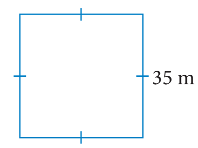
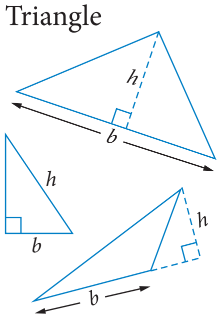
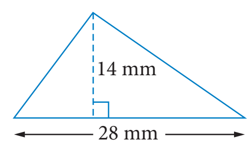
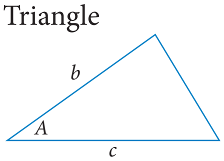
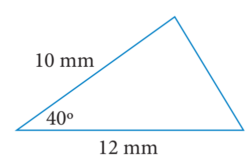
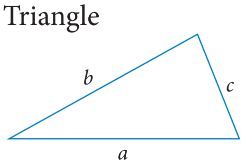
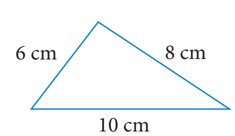
$100 = 2\pi r$

2 Solve the equation for r, using a CAS/calculator if necessary.

$r = \dfrac{100}{2\pi} \approx 15.9$ cm

WS
Areas of composite shapes

Areas of shapes

Shape	Area formula	Example
Rectangle 	$A = \text{length} \times \text{width}$ $A = lw$	 $A = 8 \times 6$ $= 48\text{ cm}^2$
Square 	$A = \text{side} \times \text{side}$ $A = (\text{side})^2$ $A = s^2$	 $A = 35^2$ $= 1225\text{ m}^2$
Triangle 	$A = \frac{1}{2} \times \text{base} \times \text{perpendicular height}$ $A = \frac{1}{2}\,bh$	 $A = \frac{1}{2} \times 28 \times 14$ $= 196\text{ mm}^2$
Triangle 	Area $= \frac{1}{2}$ product of two sides $\times$ sin(**included angle**) $A = \frac{1}{2}bc\sin(A)$	 $A = \frac{1}{2} \times 12 \times 10 \times \sin(40°)$ $\approx 38.57\text{ mm}^2$
Triangle 	**Heron's formula** $A = \sqrt{s(s-a)(s-b)(s-c)}$ The semi-perimeter (s) is $s = \frac{1}{2}(a+b+c)$	 $s = \frac{1}{2}(6+8+10)$ $= 12\text{ cm}$ $A = \sqrt{12(12-8)(12-10)(12-6)}$ $= \sqrt{12 \times 4 \times 2 \times 6}$ $= 24\text{ cm}^2$

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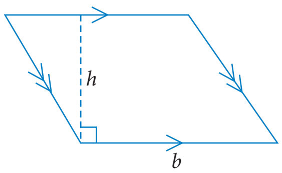
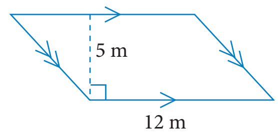
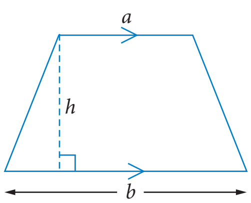
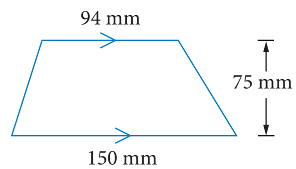
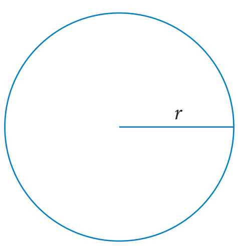
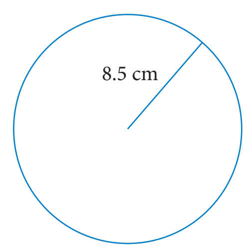
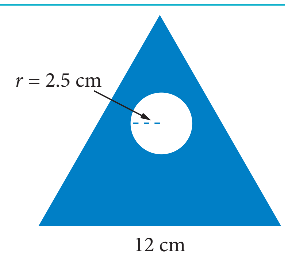
Shape	Area formula	Example
Parallelogram 	$A = \text{base} \times \text{perpendicular height}$ $A = bh$	 $A = 12 \times 5$ $= 60 \text{ m}^2$
Trapezium 	$A = \frac{1}{2} \times \text{sum of parallel sides} \times$ perpendicular distance between sides $A = \frac{1}{2}(a+b)h$	 $A = \frac{1}{2} \times (94 + 150) \times 75$ $= 9150 \text{ mm}^2$
Circle 	$A = \pi \times (\text{radius})^2$ $A = \pi r^2$	 $A = \pi \times 8.5^2$ $\approx 227 \text{ cm}^2$

Worked example 4

Find the shaded area for an equilateral triangle with a circular hole of radius 2.5 cm cut out of it. Express your answer in cm^2 correct to two decimal places.



Working

1 Find the areas of the triangle and circle.

The equilateral triangle has equal angles of 60°. Use the formulas

Triangle area: $A = \frac{1}{2}bc\sin(A)$

Circle area: $A = \pi r^2$

Triangle area $= \frac{1}{2} \times 12 \times 12 \times \sin(60°)$

$\approx 62.354 \text{ cm}^2$

Heron's formula can also be used for the area of the triangle.

Circle area $= \pi \times 2.5^2$

$\approx 19.635 \text{ cm}^2$

2 Find the difference between the two areas.

Shaded area $\approx 62.354 - 19.635$

$\approx 42.72 \text{ cm}^2$

Surface area of solids

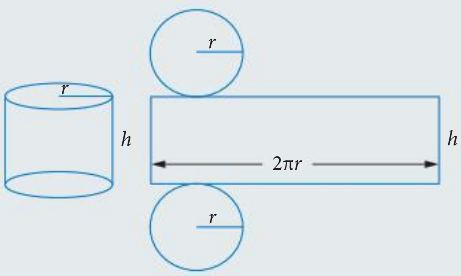
Surface area

Total surface area

The **total surface area** of a solid object is the sum of all the areas on the surface. Many total surface areas can be determined using the area rules for shapes.

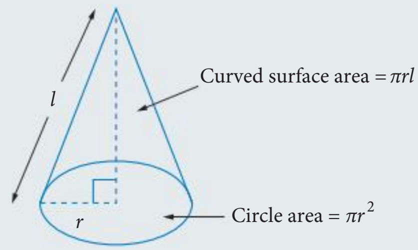
Total surface area formulas for common solids

Cylinder



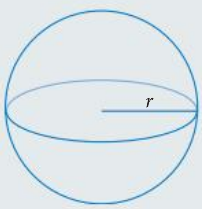
Surface area = $\pi r^2 + \pi r^2 + 2\pi rh$
$= 2\pi r^2 + 2\pi rh$

Cone



Surface area = $\pi r^2 + \pi rl$

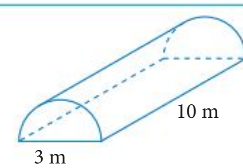
Sphere



Surface area = $4\pi r^2$

Worked example 5

A greenhouse made of polythene has semicircular ends with a diameter of 3 metres. If the length of the greenhouse is 10 metres, find the area that is covered by polythene.



Working

The shapes on the surface of the greenhouse are two semicircles and a curved surface area. The total surface area is half the surface area of a cylinder.

Total surface area

$= 2\left(\frac{1}{2}\pi r^2\right) + \frac{1}{2}(2\pi rh)$

$= 2\left(\frac{1}{2}\pi \times 1.5^2\right) + \frac{1}{2}(2\pi \times 1.5 \times 10)$

$\approx 54.19 \text{ m}^2$

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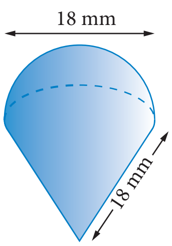
12.1

Worked example 6

Calculate the total surface area of this composite solid made with a cone and a hemisphere. Express your answer in mm^2 correct to two decimal places.



Working

1 The cone and the hemisphere each have a radius of 9 mm.

Calculate the curved surface area of the hemisphere.

Curved surface area of the hemisphere

$= \frac{1}{2}\left(4\pi r^2\right)$

$= \frac{1}{2}\left(4\pi \times 9^2\right)$

$\approx 508.938\ \text{mm}^2$

2 Find the curved surface area of the cone.

Curved surface area of the cone

$= \pi r l$

$= \pi \times 9 \times 18$

$\approx 508.938\ \text{mm}^2$

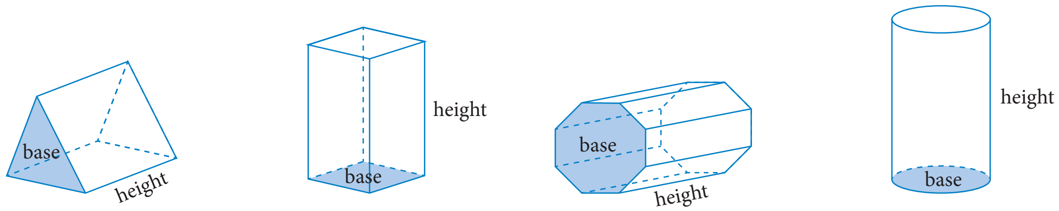
3 Find the sum of the two surface areas.

Total surface area $\approx 508.938 + 508.938$

$\approx 1017.88\ \text{mm}^2$

Volume

A **prism** is a solid with two identical ends and flat sides. The shape of the **base** gives each prism its name. The base of a **cylinder** is a circle. Unlike a prism, cylinders have curved sides.



Triangular prism | Square prism | Octagonal prism | Cylinder

Volume of a prism = area of base × perpendicular height

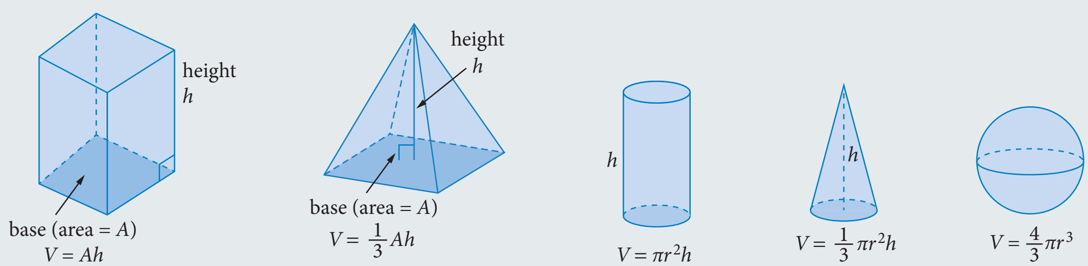
$V = A \times h$

Cones and pyramids are tapered solids, narrowing to a point called an apex.

The volume of a tapered solid = $\frac{1}{3}$ the area of the base × the perpendicular height

$V = \frac{1}{3} A \times h$

Volume formulas



Prism | Pyramid | Cylinder | Cone | Sphere

Worked example 7

A hollow cone with a radius of 10 centimetres and a height of 15 centimetres is filled with water. The water in the cone is poured into an empty cylinder with a radius of 8 centimetres and a height of 20 centimetres. Find the height of the water in the cylinder in centimetres, correct to one decimal place.

Working

1 Find the volume of the cone.

$$V = \frac{1}{3}\pi r^2 h$$
$$= \frac{1}{3}\pi \times 10^2 \times 15$$
$$\approx 1570.80 \text{ cm}^3$$

2	Substitute $V = 1570.80$, $r = 8$ and $h = x$ into the formula for the volume of a cylinder.	$V = \pi r^2 h$ $1570.80 = \pi \times 8^2 \times x$
3	Solve the equation for x, using a CAS/calculator if necessary.	$x = \dfrac{1570.80}{\pi \times 8^2}$ $= \dfrac{1570.80}{64\pi}$ ≈ 7.8 cm The height of water in the cylinder is 7.8 cm.

12.1

EXAM PREP 12.1

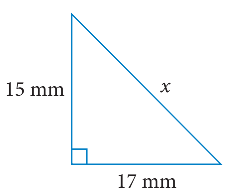
Surface area and volume

Prep 1

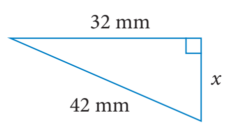
WORKED EXAMPLE 1

Find the value of x correct to two decimal places, in each right-angled triangle.

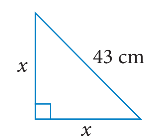
a



b



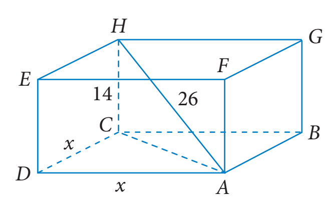
c



Prep 2

WORKED EXAMPLE 2

The rectangular prism *ABCDEFGH* has a square base. The lengths of *AH* and *CH* are 26 cm and 14 cm respectively. Find the length of *AB* correct to three decimal places.



Prep 3

WORKED EXAMPLE 3

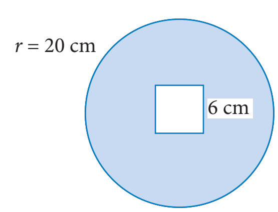
Find the radius of a circle, correct to one decimal place, given that its circumference is 5500 cm.

Prep 4

WORKED EXAMPLE 4

Find the shaded area for a circle of radius 20 cm with a square hole with a side length of 6 cm.

Answer in cm^2 correct to two decimal places.

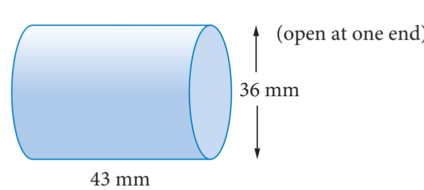


Prep 5

WORKED EXAMPLE 5

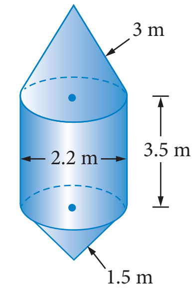
Find the total surface area for an open cylinder with a diameter of 36 mm and a height of 43 mm.

Express your answer in square millimetres, correct to one decimal place.



Prep 6 WORKED EXAMPLE 6

Find the total surface area of the solid in square metres, correct to two decimal places.



Prep 7 WORKED EXAMPLE 7

A hemisphere with a radius of 12 centimetres is filled with water. The contents of the hemisphere are poured into an empty cylinder with a radius of 10 centimetres and a height of 25 centimetres. Find the height of the water in the cylinder in centimetres, correct to one decimal place.

EXAM PRACTICE 12.1

Surface area and volume

Question 1

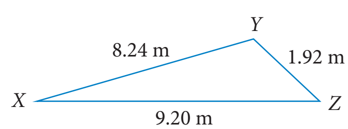
The perimeter of a regular pentagon is 100 cm. The side length of this pentagon, in cm, is

A 5 **B** 10 **C** 20 **D** 25 **E** 50

[VCAA 2013 1GTQ1]

Question 2

The area (in m^2) of triangle XYZ can be found using Heron's formula



$A = \sqrt{s(s-a)(s-b)(s-c)}$

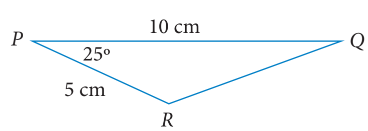
with $a = 1.92$, $b = 8.24$, $c = 9.20$ and $s =$

A 4.40 **B** 6.45 **C** 9.20 **D** 9.68 **E** 19.36

[VCAA 2009 1GTQ2]

Question 3

The area of the triangle PQR is closest to



A 3.3 cm^2 **B** 6.3 cm^2 **C** 10.6 cm^2

D 12.5 cm^2 **E** 22.7 cm^2

[VCAA 2005 1GTQ1]

Question 4

A block of land is triangular in shape. The three sides measure 36 m, 58 m and 42 m. To calculate the area, Heron's formula is used. The correct application of Heron's formula for this triangle is

A Area $= \sqrt{136(136-36)(136-58)(136-42)}$

B Area $= \sqrt{136(136-18)(136-29)(136-21)}$

C Area $= \sqrt{68(68-36)(68-58)(68-42)}$

D Area $= \sqrt{68(68-18)(68-29)(68-21)}$

E Area $= \sqrt{68(136-36)(136-58)(136-42)}$

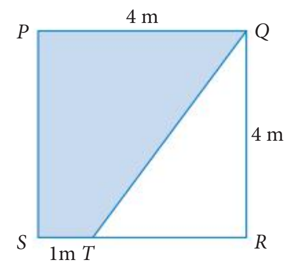
[VCAA 2006 1GTQ5]

Question 5

PQRS is a square of side length 4 m, as shown in the diagram.
The distance *ST* is 1 m.

The shaded area *PQTS* shown in the diagram, in m^2, is closest to

A 6 **B** 8 **C** 9
D 10 **E** 12



[VCAA 2012 1GTQ2]

Question 6

The radius of a circle is 6.5 centimetres. A square has the same area as this circle.

The length of each side of the square, in centimetres, is closest to

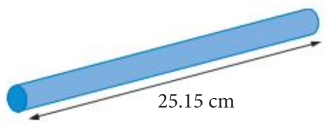
A 6.4 **B** 10.2 **C** 11.5 **D** 23.0 **E** 33.2

[VCAA 2011 1GTQ3]

Question 7

The solid cylindrical rod shown has a volume of 490.87 cm^3.
The length is 25.15 cm. The radius (in cm) of the cross-section of the rod, correct to one decimal place, is

A 2.5 **B** 5.0 **C** 6.3
D 12.5 **E** 19.6



[VCAA 2008 1GTQ4]

Question 8

Sand is poured out of a truck and forms a pile in the shape of a right circular cone. The diameter of the base of the pile of sand is 2.6 m. The height is 1.2 m.

The volume (in m^3) of sand in the pile is closest to

A 2.1 **B** 3.1 **C** 6.4 **D** 8.5 **E** 25.5

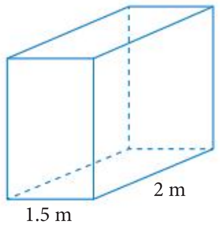
[VCAA 2008 1GTQ7]

Question 9

The rectangular box shown in this diagram is closed at the top and at the bottom. It has a volume of 6 m^3. The base dimensions are 1.5 m × 2 m.

The total surface area of this box is

A 10 m^2 **B** 13 m^2 **C** 13.5 m^2
D 20 m^2 **E** 27 m^2



[VCAA 2006 1GTQ6]

Question 10

A rectangle is 3.79 m wide and has a perimeter of 24.50 m.

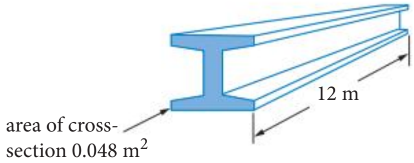
Correct to one decimal place, the length of the diagonal of this rectangle is

A 9.2 m **B** 9.3 m **C** 12.2 m **D** 12.3 m **E** 12.5 m

[VCAA 2007 1GTQ3]

Question 11

A steel beam used for constructing a building has a cross-sectional area of 0.048 m^2 as shown.



The beam is 12 m long.

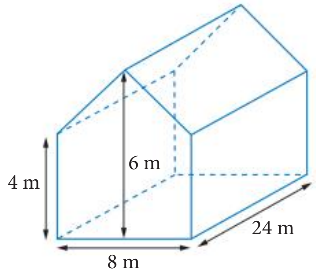
In cubic metres, the volume of this steel beam is closest to

A 0.576 **B** 2.5 **C** 2.63 **D** 57.6 **E** 2500

[VCAA 2007 1GTQ4]

Question 12

The building shown in the diagram is 8 m wide and 24 m long. The side walls are 4 m high.



The peak of the roof is 6 m vertically above the ground.

In cubic metres, the volume of this building is

A 384 **B** 576 **C** 960

D 1152 **E** 4608

[VCAA 2006 1GTQ4]

Question 13

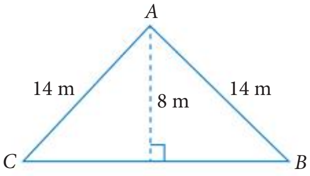
A circle has a circumference of 10 cm. The radius of this circle is closest to

A 1.3 cm **B** 1.6 cm **C** 1.8 cm **D** 3.2 cm **E** 5.0 cm

[VCAA 2010 1GTQ2]

Question 14

In triangle ABC, the length CB is closest to



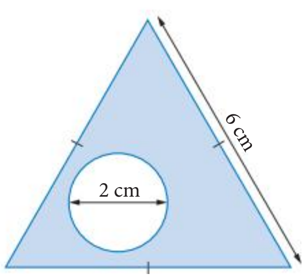
A 11.5 m **B** 12.0 m **C** 16.1 m

D 19.8 m **E** 23.0 m

[VCAA 2005 1GTQ3]

Question 15

An equilateral triangle of side length 6 cm is cut from a sheet of cardboard. A circle is then cut out of the triangle, leaving a hole of diameter 2 cm as shown.



The area of cardboard remaining, as shown by the shaded region in the diagram, is closest to

A 3 cm^2 **B** 9 cm^2 **C** 12 cm^2

D 15 cm^2 **E** 16 cm^2

[VCAA 2010 1GTQ3]

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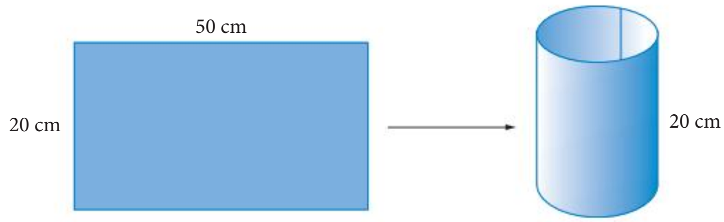
Question 16

A rectangular sheet of cardboard has length 50 cm and width 20 cm. This sheet of cardboard is made into an open-ended cylinder by joining the two shorter sides, with no overlap. This is shown in the diagram.



The radius of this cylinder, in cm, is closest to

A 6.4 **B** 8.0 **C** 15.6 **D** 15.9 **E** 17.8

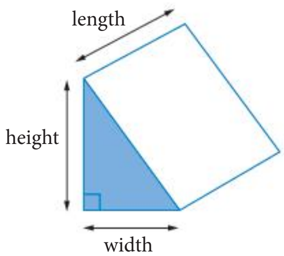
[VCAA 2012 1GTQ3]

Question 17

A right triangular prism has a volume of 160 cm^3. A second right triangular prism is made with the same width, twice the height and three times the length of the prism shown. The volume of the second prism (in cm^3) is

A 320 **B** 640 **C** 960

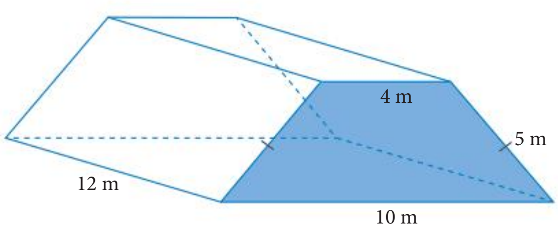
D 1280 **E** 1920



[VCAA 2009 1GTQ5]

Question 18

A greenhouse is built in the shape of a trapezoidal prism, as shown in the diagram. The cross-section of the greenhouse (shaded) is an isosceles trapezium. The parallel sides of this trapezium are 4 m and 10 m respectively. The two equal sides are each 5 m. The length of the greenhouse is 12 m.



The five exterior surfaces of the greenhouse, not including the base, are made of glass. The total area, in m^2, of the glass surfaces of the greenhouse is

A 196 **B** 212 **C** 224 **D** 344 **E** 672

[VCAA 2013 1GTQ7]

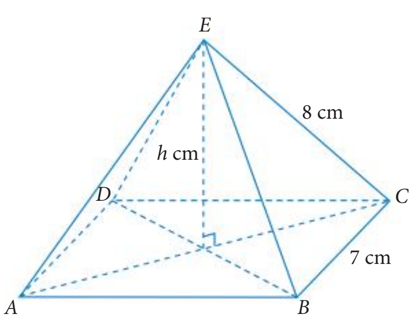
Question 19

For the square-based right pyramid *ABCDE* shown, the sides of the base are 7 cm and the slant edges are 8 cm in length.

The vertical height, *h* cm, of this pyramid is closest to

A 3.9 cm **B** 6.3 cm **C** 7.2 cm

D 10.6 cm **E** 12.7 cm

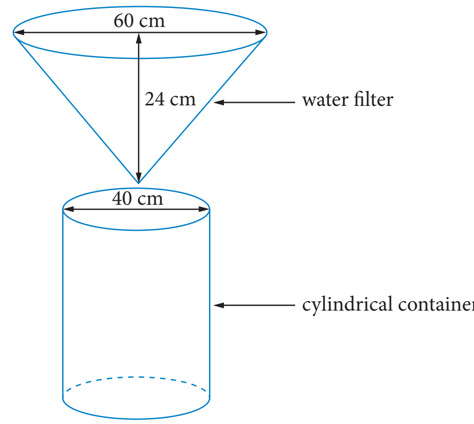


[VCAA 2002 1GTQ7]

12.1

Question 20

A conical water filter has a diameter of 60 cm and a depth of 24 cm. It is filled to the top with water. The water filter sits above an empty cylindrical container that has a diameter of 40 cm. The water is allowed to flow from the water filter into the cylindrical container. When the water filter is empty, the depth of water in the cylindrical container will be

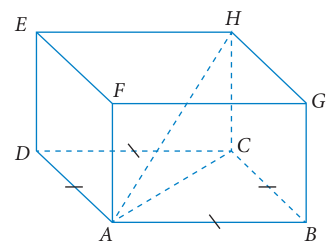


A 8 cm **B** 18 cm **C** 24 cm
D 32 cm **E** 96 cm

[VCAA 2010 1GTQ9]

Question 21

A rectangular prism with a square base, *ABCD*, is shown. The diagonal of the prism, *AH*, is 8 cm. The height of the prism, *HC*, is 4 cm. The volume of this rectangular prism is



A 64 cm^3 **B** 96 cm^3 **C** 128 cm^3
D 192 cm^3 **E** 256 cm^3

[VCAA 2013 1GTQ9]

Question 22

A cylinder of radius R and height H has volume V. The volume of a cylinder with radius $3R$ and height $3H$ is

A $3V$ **B** $6V$ **C** $9V$ **D** $27V$ **E** $81V$

[VCAA 2012 1GTQ6]

Question 23

A closed cubic box of side length 36 cm is to contain a thin straight metal rod. The maximum possible length of the rod is closest to

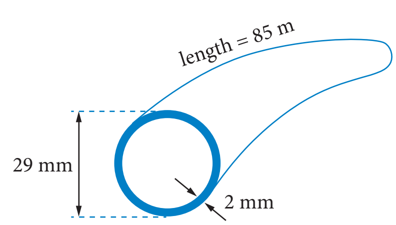
A 36 cm **B** 51 cm **C** 62 cm **D** 108 cm **E** 216 cm

[VCAA 2007 1GTQ7]

Question 24

A hose with a circular cross-section is 85 metres long.

The outside diameter of the hose is 29 millimetres. Its walls are 2 millimetres thick. One litre of water occupies a volume of 1000 cm^3. When the hose is full with water, the volume it holds (in litres) is closest to

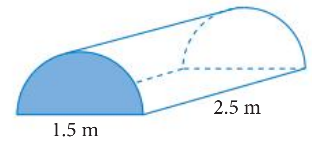


A 4 **B** 42 **C** 49 **D** 56 **E** 167

[VCAA 2009 1GTQ8]

Question 25

A tent with semicircular ends is in the shape of a prism. The diameter of the ends is 1.5 m. The tent is 2.5 m long. The total surface area (in m^2) of the tent, including the base, is closest to



A 5.5 **B** 7.7 **C** 8.8 **D** 11.4 **E** 15.3

[VCAA 2008 1GTQ6]

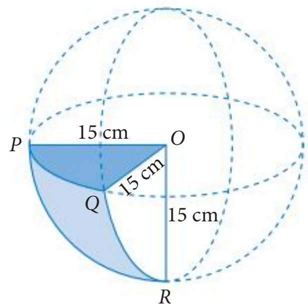
Question 26

The solid *OPQR*, as shown, is one-eighth of a sphere of radius 15 cm. The point *O* is the centre of the sphere and the points *P*, *Q* and *R* are on the surface of the sphere.

The total surface area of the solid *OPQR*, in cm^2, is closest to

A 619 **B** 648 **C** 706

D 884 **E** 1767

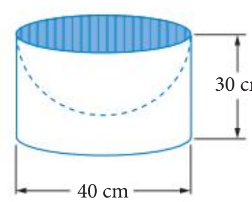


[VCAA 2012 1GTQ9]

Question 27

A solid cylinder has a height of 30 cm and a diameter of 40 cm. A hemisphere is cut out of the top of the cylinder as shown.

In square centimetres, the total surface area of the remaining solid (including its base) is closest to



A 1260 **B** 2510 **C** 6280 **D** 7540 **E** 10 050

[VCAA 2007 1GTQ6]

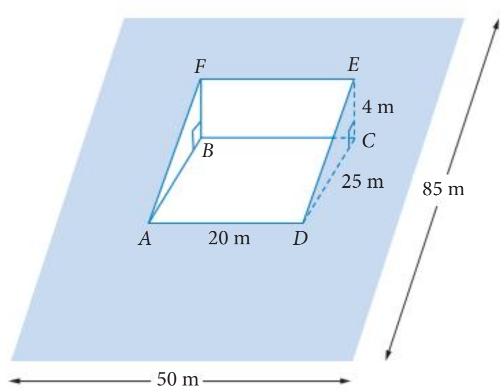
Question 28

A rectangular block of land has width 50 metres and length 85 metres.

a Calculate the area of this block of land. Write your answer in m^2. 1 mark

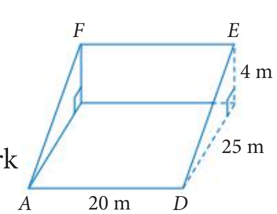
To build a house, the builders dig a hole in the block of land. The hole has the shape of a right triangular prism, *ABCDEF*. The width *AD* = 20 m, length *DC* = 25 m and height *EC* = 4 m are shown in the diagram.

b Calculate the volume of the right triangular prism, *ABCDEF*. Write your answer in m^3. 1 mark



Once the right triangular prism shape has been dug, a fence will be placed along the two sloping edges, *AF* and *DE*, and along the edges *AD* and *FE*.

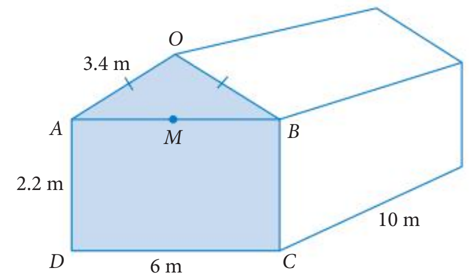
c Calculate the total length of fencing that will be required. Write your answer, in metres, correct to one decimal place. 1 mark



[VCAA 2012 2GTQ1]

Question 29

A shed has the shape of a prism. Its front face, *AOBCD*, is shaded in the diagram. *ABCD* is a rectangle and *M* is the midpoint of *AB*.



a Show that the length of *OM* is 1.6 m. 1 mark

b Show that the area of the front face of the shed, *AOBCD*, is 18 m^2. 1 mark

c Find the volume of the shed in m^3. 1 mark

d All inside surfaces of the shed, including the floor, will be painted.

i Find the total area that will be painted in m^2. 1 mark

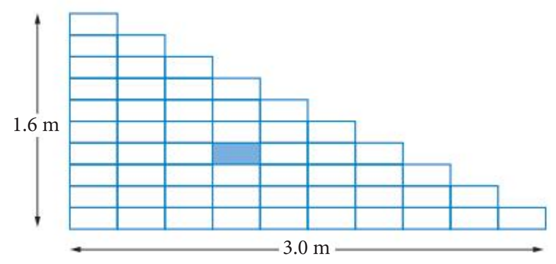
One litre of paint will cover an area of 16 m^2.

ii Determine the number of litres of paint that is required. 1 mark

[VCAA 2008 2GTQ2]

Question 30

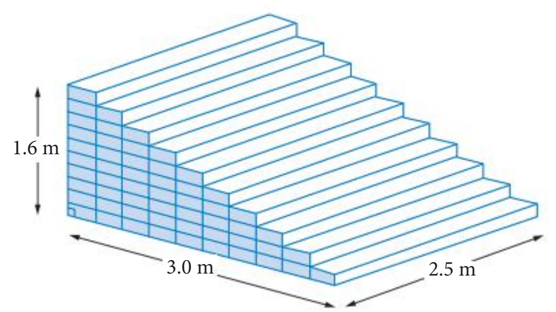
A concrete staircase leading up to the grandstand has 10 steps. The staircase is 1.6 m high and 3.0 m deep. Its cross-section comprises identical rectangles. One of these rectangles is shaded in the diagram.



a Find the area of the shaded rectangle in square metres. 1 mark

The concrete staircase is 2.5 m wide.

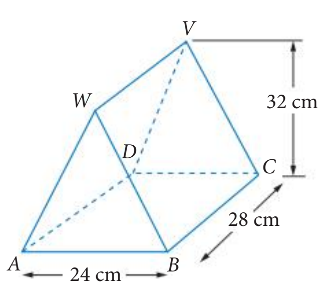
b Find the volume of the solid concrete staircase in cubic metres. 2 marks



[VCAA 2013 2GTQ2]

Question 31

Tessa carves a triangular prism from a block of wood the shape of a rectangular prism. Using point *V*, halfway between the top corners on the back face, she constructs the triangular prism shown.



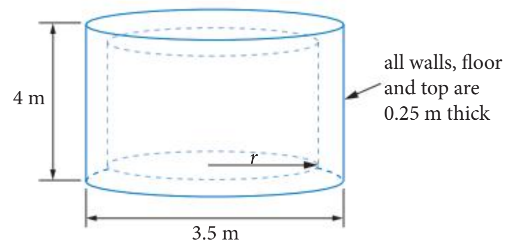
a Show that, correct to the nearest centimetre, length *AW* is 34 cm. 1 mark

b Using length *AW* as 34 cm, find the total surface area, in cm^2, of the triangular prism. 2 marks

[VCAA 2007 2GTQ2]

Question 32

A *closed* cylindrical water tank has external diameter 3.5 metres. The external height of the tank is 2.4 metres. The walls, floor and top of the tank are made of concrete 0.25 m thick.



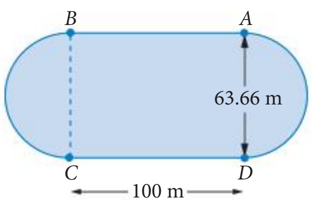
a What is the internal radius, r, of the tank? 1 mark

b Determine the maximum amount of water this tank can hold. Write your answer correct to the nearest cubic metre. 2 marks

[VCAA 2006 2GTQ3]

Question 33

A grassed region in the athletics ground is shown shaded in the diagram.



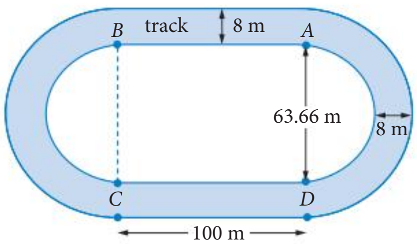
The perimeter of the grassed region comprises two parallel lines, *BA* and *CD*, each 100 m in length, and two semicircles, *BC* and *AD*.

In total, the perimeter of the grassed region is 400 m.

a The diameter of the semicircle *AD* is 63.66 m, correct to two decimal places. Show how this value could be obtained. 1 mark

b Determine the area of the grassed region, correct to the nearest square metre. 1 mark

A running track, shown shaded in the diagram, surrounds the grassed region. This running track is 8 m wide at all points.



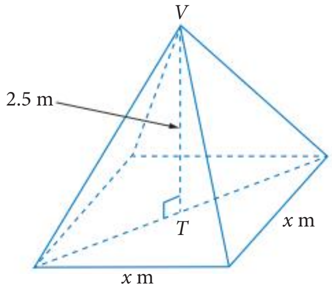
c The running track is to be resurfaced with special rubber material that is 0.1 m deep. Find the volume of rubber material that is needed to resurface the running track. Write your answer, correct to the nearest cubic metre. 2 marks

[VCAA 2013 2GTQ3]

Question 34

A concrete, square pyramid with volume 1.8 m^3 sits on the flat top of the hill. The length of the square base of the pyramid is x metres. The height of the pyramid, *VT*, is 2.5 metres.

Find the value of x, correct to two decimal places. 2 marks



[VCAA 2010 2GTQ3]

12.2 Scale factors

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Scales and similar figures

WS Interpreting an office plan

A **scale** is a **ratio** used to compare two similar objects. Two objects are **similar** if they have the same shape, but one is an enlargement of the other. A scale of 1 : 100, for example, could be used to indicate that the real length of something is one hundred times the length on a drawing.

A **scale factor** measures how many times something needs to be enlarged or reduced to produce the similar figure. A scale factor can be a whole number, a proper fraction or an improper fraction.

For a scale $a : b$, the scale factor $= \frac{b}{a}$.

$$\text{Scale factor} = \frac{\text{length of second figure}}{\text{corresponding length of first figure}}$$

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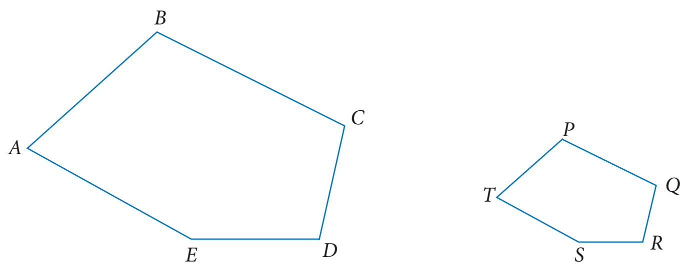
Polygons and similarity

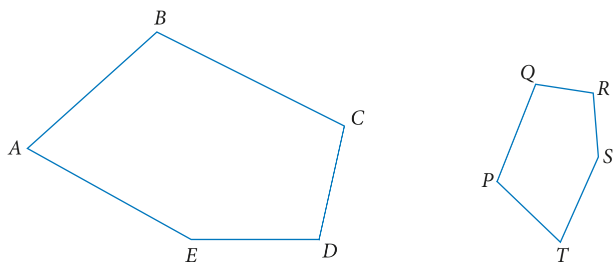
Polygons are similar if they have

- corresponding angles equal, and
- corresponding sides in the same ratio.

Orientation doesn't affect similarity. For example, these two polygons are similar.



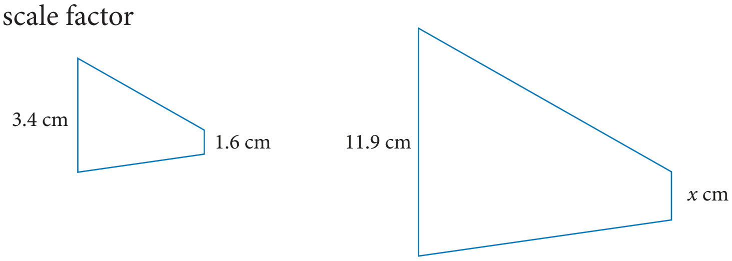
The following two polygons are also similar.



Worked example 8

For these similar figures, find the

a scale factor **b** value of x



Working

a Write the scale factor as a whole number, a proper fraction or an improper fraction, using a CAS/calculator if necessary.

$$\text{Scale factor} = \frac{\text{length of second figure}}{\text{corresponding length of first figure}}$$

$$= \frac{11.9}{3.4}$$

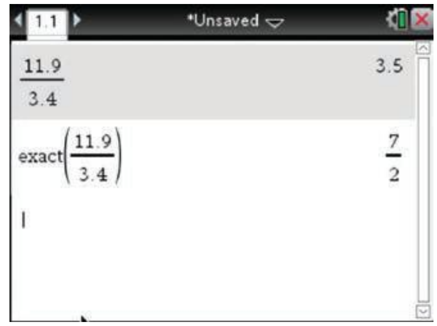
$$= 3.5$$

$$= \frac{7}{2}$$

TI-NSPIRE CAS

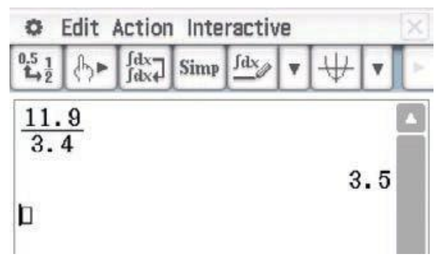
In AUTO mode, if decimals are used in the input, then the CAS will automatically return a decimal answer.

To obtain a fractional answer, type **exact** $\left(\frac{11.9}{3.4}\right)$, then press enter.

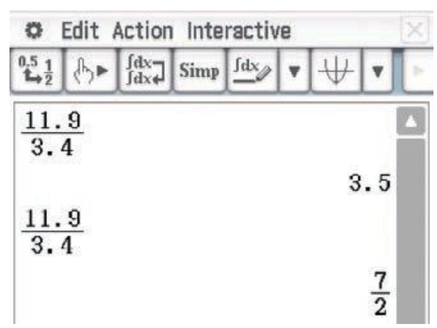


CLASSPAD

In **Decimal** mode, the CAS will automatically return a decimal answer.



To obtain a fractional answer, tap **Decimal** on the bottom toolbar to change the mode to **Standard**.



b The scale factor compares corresponding sides. Identify the corresponding side to the unknown. Use ↔ to indicate corresponding sides.

Write and solve an equation using the ratio of corresponding sides.

$7 \leftrightarrow 2, \quad x \text{ cm} \leftrightarrow 1.6 \text{ cm}$

$$\frac{7}{2} = \frac{x}{1.6}$$

$$x = \frac{7}{2} \times 1.6$$

$$= 5.6$$

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Triangles and similarity

Finding sides in similar triangles

12.2

Method for solving similar triangles

1 Identify the corresponding sides by noting that corresponding sides are opposite equal angles.

2 Write a ratio equation for the corresponding sides in the form

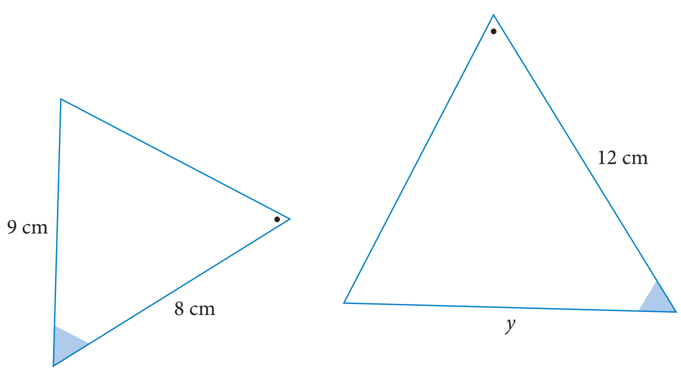
$$\frac{\text{side 1 of triangle 1}}{\text{side 1 of triangle 2}} = \frac{\text{side 2 of triangle 1}}{\text{side 2 of triangle 2}}$$

3 Solve the equation for the unknown.

Note: It doesn't matter whether triangle 1 or triangle 2 lengths appear as the numerator, as long as it's the same on both sides of the equation.

Worked example 9

Find the value of y in the pair of similar triangles.



Working

1 Identify the corresponding sides. $9 \leftrightarrow y,\ 8 \leftrightarrow 12$

These are opposite equal angles.

2 Write a ratio equation.

$$\frac{9}{y} = \frac{8}{12}$$

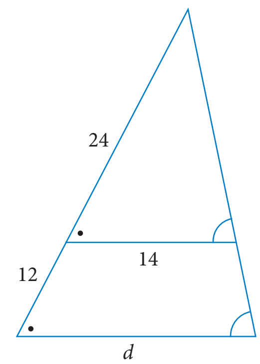
The sides for the small triangle are in the numerator of each fraction and the corresponding sides for the large triangle are in the denominator.

3 Solve the equation, using a CAS/calculator if necessary.

$$9 \times 12 = 8 \times y$$
$$8y = 108$$
$$y = 13.5 \text{ cm}$$

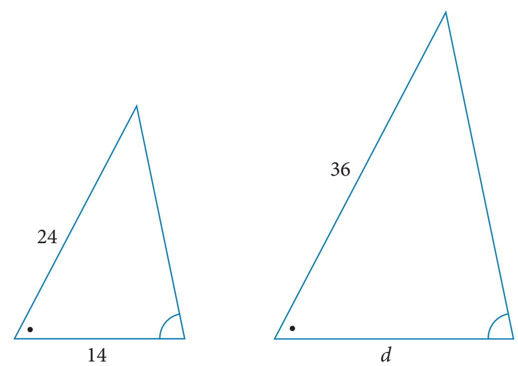
Worked example 10

Find the value of d in the pair of similar triangles.



Working

1 Redraw as two separate triangles.



2 Identify the corresponding sides. $14 \leftrightarrow d, \quad 24 \leftrightarrow 36$

3 Write a ratio equation. $\frac{14}{d} = \frac{24}{36}$

4 Solve the equation, using a CAS/calculator if necessary.

$$14 \times 36 = 24 \times d$$
$$24d = 504$$
$$d = 21$$

Length, area and volume scale factors

The ratios of the area and volume can be determined from the ratio of the side lengths in two similar figures.

For two similar figures with length ratio $a : b$, and length scale factor k

- Area ratio $= a^2 : b^2$ and Area scale factor $= k^2$
- Volume ratio $= a^3 : b^3$ and Volume scale factor $= k^3$

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Worked example 11

A photograph is 8 cm long. Its enlargement is 12 cm long.
If the area of the small photo is 48 cm^2, what is the area of the enlargement?

Shutterstock.com/Bildagentur Zoonar GmbHJ

Working

1. Write the length ratio $a : b$ — Length ratio $8 : 12 = 2 : 3$

   and find the area ratio $a^2 : b^2$ — Area ratio $2^2 : 3^2 = 4 : 9$

2. Write a ratio equation for the two areas. The ratio is written in the order small photo : large photo — Let x = the area of the large photo

   Areas $4 : 9 = 48 : x$

3. Write as a fraction equation and solve the equation, using a CAS/calculator if necessary.

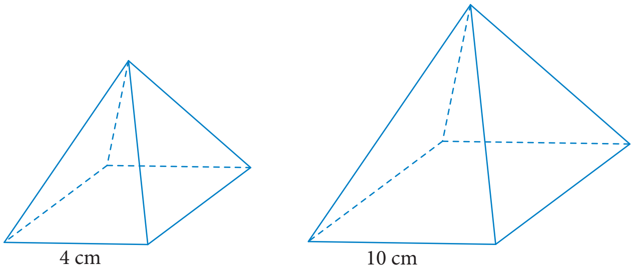
$$\frac{4}{9} = \frac{48}{x}$$

$$4x = 48 \times 9$$
$$= 432$$
$$x = 108$$

The area of the large photo is 108 cm^2.

Worked example 12

A square-based pyramid with base of length 4 cm is enlarged to produce a similar pyramid with base length 10 cm.



a Find the length ratio.

b Find the volume ratio.

c If the volume of the smaller pyramid is 32 cm^3, what is the volume of the larger pyramid?

Working

a Write the length ratio in the order small pyramid : big pyramid — Length ratio = $4 : 10 = 2 : 5$

b Write the volume ratio using the rule Length ratio $a : b$ Volume ratio $a^3 : b^3$	Volume ratio $= 2^3 : 5^3 = 8 : 125$
c **1** Write a ratio equation for the volume.	Let x = the volume of the large pyramid $8 : 125 = 32 : x$
2 Write as a fraction equation and solve for x, using a CAS/calculator if necessary.	$\frac{8}{125} = \frac{32}{x}$ $8x = 4000$ $x = 500$ The volume of the larger pyramid is 500 cm^3.

EXAM PREP 12.2

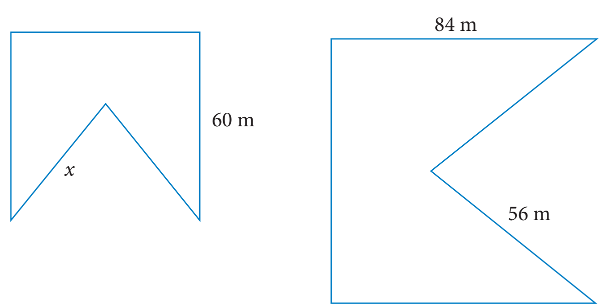
Scale factors

Prep 1 WORKED EXAMPLE 8

For these similar figures, find the

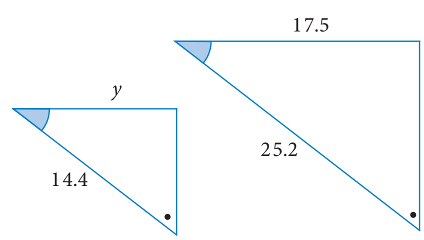
a scale factor

b value of x.



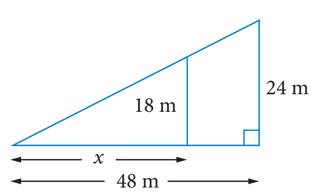
Prep 2 WORKED EXAMPLE 9

Find the value of y in the similar triangles.



Prep 3 WORKED EXAMPLE 10

Find the value of x using similar triangles.



Prep 4 WORKED EXAMPLE 11

The corresponding sides of two similar triangles are 9 cm and 15 cm. If the area of the smaller triangle is 63 cm^2, find the area of the larger triangle.

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Prep 5 WORKED EXAMPLE 12

Two similar cones have bases with radius 20 cm and 24 cm respectively.

a What is the scale factor of the heights of the cones?

b What is the volume scale factor of the cones?

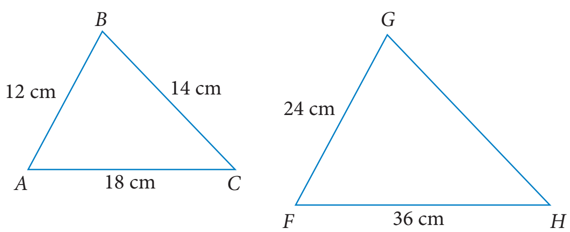
c If the volume of the smaller cone is 937.5 cm^3, what is the volume of the larger cone?

12.2

EXAM PRACTICE 12.2

Scale factors

Question 1



The two triangles, *ABC* and *FGH*, are similar.
The length of *GH* is

A 14 cm **B** 24 cm

C 26 cm **D** 28 cm

E 32 cm

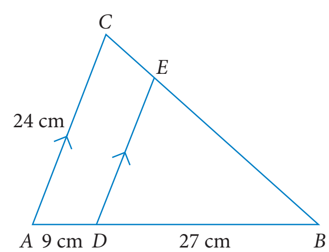
[VCAA 2009 1GTQ1]

Question 2

In the diagram, $AD = 9$ cm, $AC = 24$ cm and $DB = 27$ cm.
Line segments *AC* and *DE* are parallel. The length of *DE* is

A 6 cm **B** 8 cm **C** 12 cm

D 16 cm **E** 18 cm



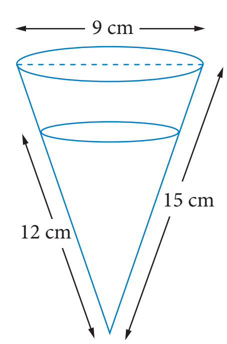
[VCAA 2006 1GTQ7]

Question 3

Two right cones, as shown, have the same angle at the base. The larger cone has a slant height of 15 cm and the smaller cone has a slant height of 12 cm. The diameter of the larger cone is 9 cm. The diameter of the smaller cone is

A 2.0 cm **B** 3.6 cm **C** 4.5 cm

D 6.0 cm **E** 7.2 cm



[VCAA 2003 1GTQ9]

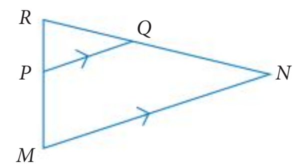
Question 4

A triangle has sides of length 20 cm, 48 cm and 52 cm. A second triangle, which is similar to the first triangle, has a longest side of 65 cm. The perimeter of the second triangle is

A 96 cm **B** 120 cm **C** 125 cm **D** 133 cm **E** 150 cm

[VCAA 2004 1GTQ4]

Question 5



In triangle MNR, point P lies on side MR and point Q lies on side NR. The lines PQ and MN are parallel. The length of RQ is 4 cm, the length of QN is 6 cm and the length of PQ is 5 cm. The length of MN, in cm, is equal to

A 7.5 **B** 8.3 **C** 12.0 **D** 12.5 **E** 15.0

[VCAA 2013 1GTQ6]

Question 6

The scale used on a map is 1 : 50 000. On this map, a distance of 4 km would be represented by

A 2.0 cm **B** 5.0 cm **C** 8.0 cm **D** 12.5 cm **E** 20.0 cm

[VCAA 2013 1GTQ5]

Question 7

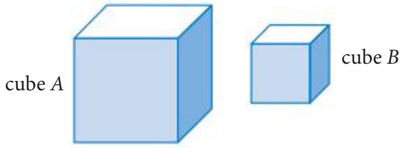
The scale on a particular map is 1 : 10 000.

A distance of 5 cm on this map would correspond to an actual distance of

A 0.5 km **B** 2 km **C** 5 km **D** 20 km **E** 50 km

[VCAA 2003 1GTQ5]

Question 8

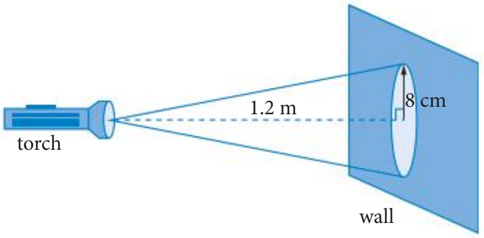


Cube A and cube B are shown. The side length of cube A is 1.5 times the side length of cube B. The surface area of cube B is 256 cm^2. The surface area of cube A is

A 114 cm^2 **B** 256 cm^2 **C** 384 cm^2 **D** 576 cm^2 **E** 864 cm^2

[VCAA 2010 1GTQ9]

Question 9

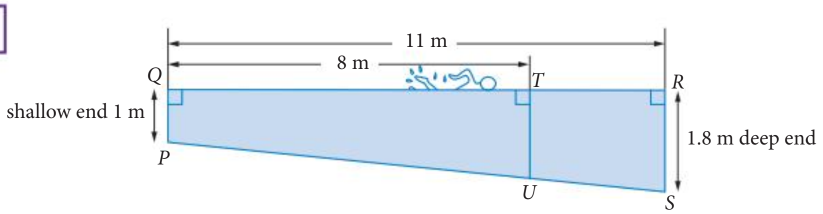


A torch, which is held horizontally, is shone on to a wall from a distance of 1.2 metres, as shown. The circular area of light it creates on the wall has a radius of 8 centimetres. The torch is now moved an additional 2 metres away from the wall. The radius of the circular area of light on the wall is now closest to

A 3 cm **B** 10 cm **C** 13 cm **D** 16 cm **E** 21 cm

[VCAA 2002 1GTQ4]

Question 10



The diagram shows a cross-section, $PQRS$, of a swimming pool. The swimming pool is 11 metres long and the depth increases uniformly from 1 metre at the shallow end to 1.8 metres at the deep end. The depth of the water at a point 8 metres from the shallow end, represented by TU on the diagram, is closest to

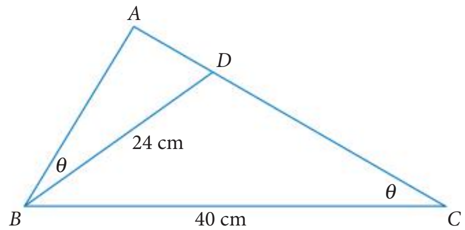
A 1.25 metres **B** 1.31 metres **C** 1.34 metres **D** 1.58 metres **E** 1.62 metres

[VCAA 2011 1GTQ8]

12.2

Question 11

In the diagram, $\angle ABD = \angle ACE = \theta$. $BD = 24$ cm and $BC = 40$ cm. The area of triangle ABD is 100 cm^2. The area of triangle ABC, in cm^2, is closest to



A 167 **B** 178 **C** 267
D 278 **E** 378

[VCAA 2011 1GTQ9]

Question 12

A juice container in the shape of a rectangular prism has a total surface area of 220 cm^2. An enlarged scale model is made so that each side is five times longer than the corresponding side of the actual container. The total surface area of the scale model is

A 220 cm^2 **B** 1100 cm^2 **C** 5500 cm^2 **D** 22 000 cm^2 **E** 27 500 cm^2

[VCAA 2003 1GTQ7]

Question 13

A plan for a mouse maze is drawn to a scale of 1 : 6. On the plan, the mouse maze covers an area of 720 cm^2. The area of the actual mouse maze is

A 1200 cm^2 **B** 4320 cm^2 **C** 8640 cm^2 **D** 25 920 cm^2 **E** 129 600 cm^2

[VCAA 2004 1GTQ5]

Question 14

A factory floor is rectangular in shape with an area of 1440 m^2. One of the linear dimensions of the original floor is 36 m. It is to be enlarged to a similar shape with an area of 2250 m^2. When the area of the floor is enlarged, the corresponding linear dimension will be

A 40 m **B** 45 m **C** 50 m **D** 56.25 m **E** 62.5 m

[VCAA 2005 1GTQ8]

Question 15

A cafe sells two sizes of cupcakes with a similar shape. The large cupcake is 6 cm wide at the base and the small cupcake is 4 cm wide at the base. The price of a cupcake is proportional to its volume. If the large cupcake costs \$5.40, then the small cupcake will cost

A \$1.60 **B** \$2.32 **C** \$2.40 **D** \$3.40 **E** \$3.60

[VCAA 2013 1GTQ4]

Question 16

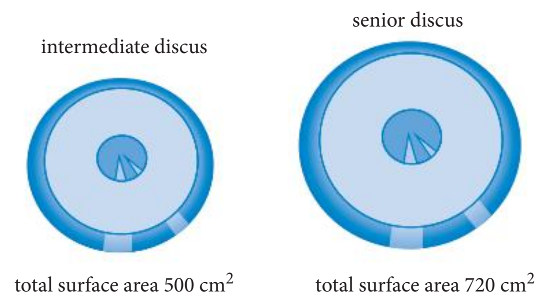
A block of land has an area of 4000 m^2. When represented on a map, this block of land has an area of 10 cm^2. On the map, 1 cm would represent an actual distance of

A 10 m **B** 20 m **C** 40 m **D** 400 m **E** 4000 m

[VCAA 2007 1GTQ5]

Question 17

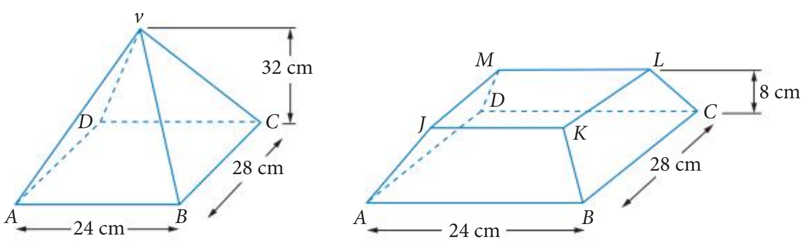
Competitors in the intermediate division of the discus use a smaller discus than the one used in the senior division, but of a similar shape. The total surface area of each discus is given. By what value can the volume of the intermediate discus be multiplied to give the volume of the senior discus? 2 marks



[VCAA 2013 2GTQ4]

Question 18

Tessa removes the top 24 cm of the height of a pyramid. The pyramid and the shape remaining are shown below. The top surface, *JKLM*, is parallel to the base, *ABCD*.

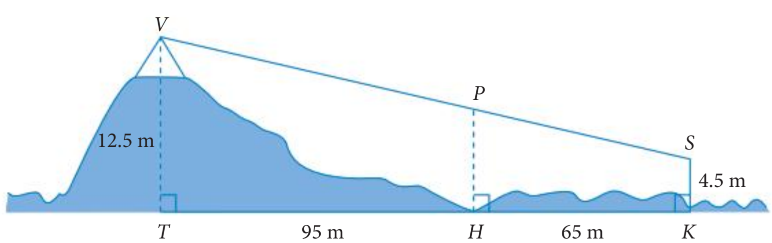


a What fraction of the height of the pyramid has Tessa removed? 1 mark

b What fraction of the volume of the pyramid remains? 2 marks

[VCAA 2007 2GTQ4]

Question 19



A flying fox suspension wire begins at *V*, 12.5 metres above *T*, as shown in the diagram. It ends at *S*, 4.5 metres above *K*. At *P*, the flying fox wire passes over *H*. The horizontal distances *TH* and *HK* are 95 metres and 65 metres respectively. Calculate the vertical distance, *PH*, in metres. 2 marks

[VCAA 2010 2GTQ4]

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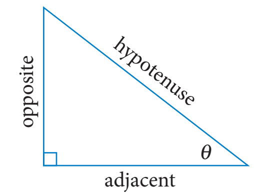
Right-angled triangles

12.3

Trigonometry is used to find unknown sides and angles in right-angled triangles. The opposite and adjacent sides of the right-angled triangle are named in relation to a given angle θ (theta).

Each trigonometric function is defined as the ratio between a different pair of sides in the triangle.



WS

Identifying the correct trigonometric ratio

Function	Abbreviation	Ratio	Initials
sine	sin	$\sin(\theta)=\dfrac{\text{opposite}}{\text{hypotenuse}}$	SOH
cosine	cos	$\cos(\theta)=\dfrac{\text{adjacent}}{\text{hypotenuse}}$	CAH
tangent	tan	$\tan(\theta)=\dfrac{\text{opposite}}{\text{adjacent}}$	TOA

Alamy/Cols Travel

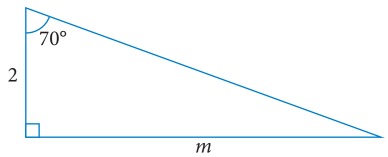
Finding side lengths

Make sure your CAS/calculator mode is set in degrees for all trigonometry calculations.

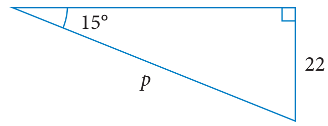
Worked example 13

Find the value of each pronumeral, correct to two decimal places.

a

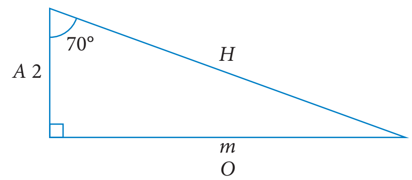


b



Working

a **1** Label the sides of the right-angled triangle with the letters O, A or H.



2 The labelled sides are 2, which is adjacent, and m, which is opposite. The trigonometric ratio is tan (θ).

Write the equation in terms of m.

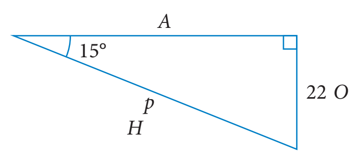
$$\tan(\theta) = \frac{\text{opposite}}{\text{adjacent}}$$

$$\tan(70°) = \frac{m}{2}$$

3 Solve for m, using a CAS/calculator if necessary.

$$m = 2 \times \tan(70°)$$
$$\approx 5.49$$

b **1** Label the sides of the right-angled triangle with the letters O, A or H.



2 The labelled sides are 22, which is opposite, and p, which is the hypotenuse. The trigonometric ratio is sin (θ).

$$\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\sin(15°) = \frac{22}{p}$$

3 Write the equation in terms of p.

Solve for p, using a CAS/calculator if necessary.

$$p = \frac{22}{\sin(15°)}$$
$$\approx 85.00$$

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Finding angles

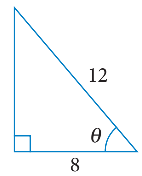
Trigonometric ratios can be used to find angles in right-angled triangles when two sides are known. The equations can be solved using the inverse trigonometric functions $\sin^{-1}$, $\cos^{-1}$ and $\tan^{-1}$.

WS
Calculating lengths and angles

12.3

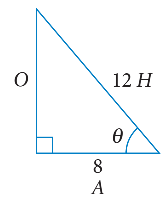
Worked example 14

Find the value of θ correct to the nearest degree.



Working

1 Label the sides in the triangle.



2 The side 8 is adjacent and the side 12 is the hypotenuse. The trigonometric ratio is cos (θ).

$$\cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{8}{12} = \frac{2}{3}$$

3 Solve the equation using $\cos^{-1}$.

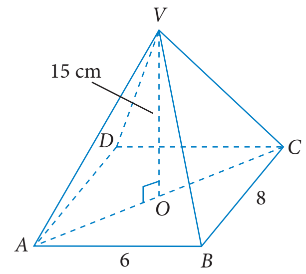
$$\theta = \cos^{-1}\left(\frac{2}{3}\right) \approx 48°$$

Shutterstock.com/LittleStocker

Worked example 15

In the right rectangular pyramid, the height VO is 15 cm.

Find the angle VCO correct to the nearest degree.



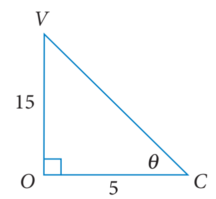
Working

1 Find the length of AC using Pythagoras' theorem.

$$(AC)^2 = 6^2 + 8^2$$
$$(AC)^2 = 100$$
$$AC = 10$$

2 Redraw triangle VCO and label the angle VCO.

$$OC = \frac{1}{2}AC$$
$$OC = 5$$



3 In the right-angled triangle, 15 is the opposite side and 5 is the adjacent side. The trigonometric ratio is tan (θ).

$$\tan(\theta) = \frac{\text{opposite}}{\text{adjacent}}$$
$$= \frac{15}{5}$$
$$= 3$$

4 Solve the equation using $\tan^{-1}$.

$$\theta = \tan^{-1}(3)$$
$$\approx 72°$$

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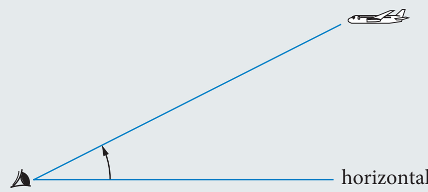
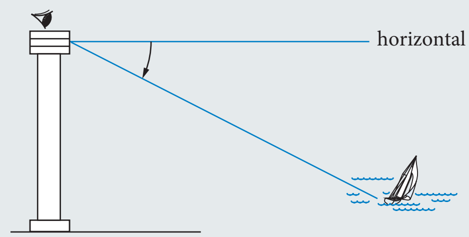
Angles of elevation and depression

Angles of elevation and depression are measured from the horizontal.

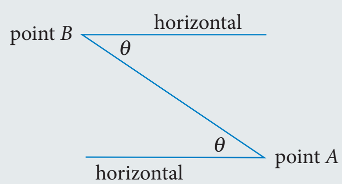
The **angle of elevation** involves looking up.



The **angle of depression** involves looking down.



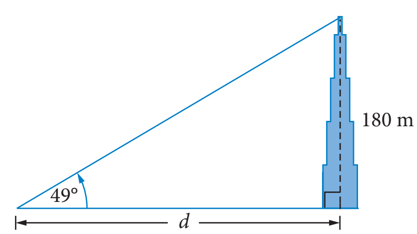
The angle of elevation from point A up to point B is equal to the angle of depression from point B down to point A.



Worked example 16

From street level, Sonja sees the top of a 180 m tower at an angle of elevation of 49°.

How far, to the nearest metre, is she from the tower?



Working

1 Determine the trigonometric ratio.

180 is opposite and d is adjacent, so the ratio is tangent.

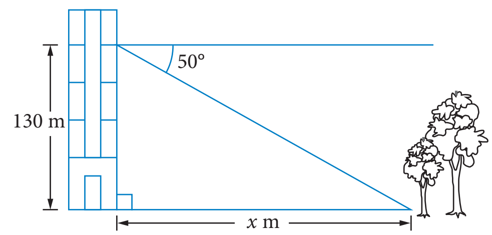
$\tan(49°) = \dfrac{180}{d}$

2 Solve for d.

$d = \dfrac{180}{\tan(49°)} \approx 156 \text{ m}$

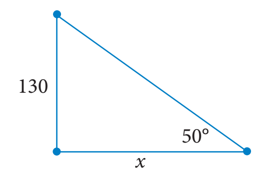
Worked example 17

From her apartment, 130 m above ground level, Maddy sights the park at an angle of depression of 50°. How far, to the nearest metre, is the park from the base of Maddy's building?



Working

1 In the triangle, the angle of elevation from the ground up to Maddy is also 50°. Draw this angle in the triangle.



2 Determine the trigonometric ratio and write an equation in terms of x.

$$\tan(50°) = \frac{130}{x}$$

3 Solve for x.

$$x = \frac{130}{\tan(50°)} \approx 109 \text{ m}$$

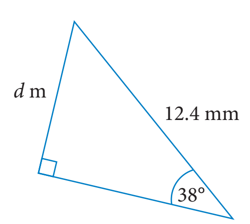
EXAM PREP 12.3

Right-angled triangles

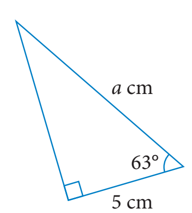
Prep 1 WORKED EXAMPLE 13

Find the value of each pronumeral. Express your answer correct to one decimal place.

a

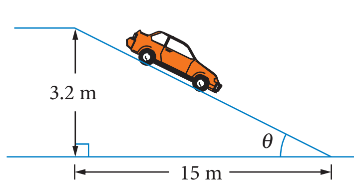


b



Prep 2 WORKED EXAMPLE 14

Find the angle of inclination, θ, of this ramp in a multistorey car park, correct to the nearest degree.



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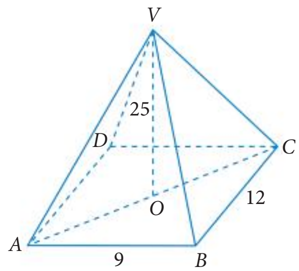
Prep 3 WORKED EXAMPLE 15

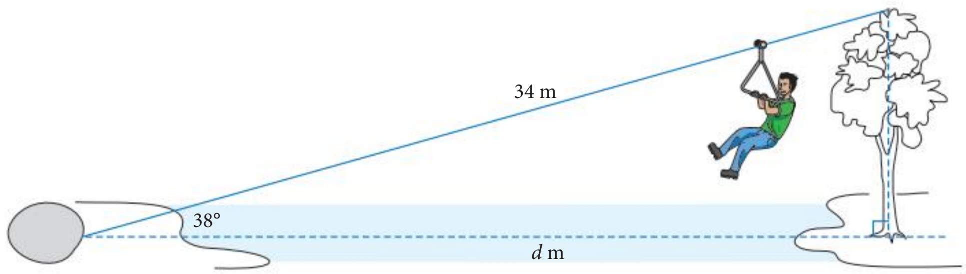
In the right pyramid. the height *VO* is 25 cm.

Find the angle *VCO*, correct to the nearest degree.



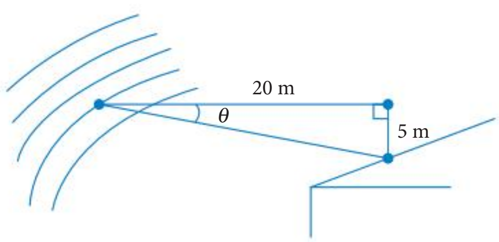
Prep 4 WORKED EXAMPLE 16

A flying fox is constructed between the top of a tree and a large rock. It has a length of 34 m and an angle of elevation of 38°. Which of the following is the (horizontal) distance between the tree and the rock, correct to one decimal place?



Prep 5 WORKED EXAMPLE 17

In a concert hall, Liam's seat is 20 m from the stage and 5 m above it. What is his angle of depression to the stage, correct to the nearest degree?

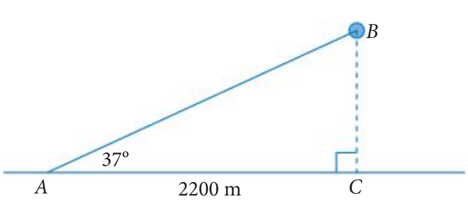


EXAMPLE PRACTICE 12.3

Right-angled triangles

Question 1

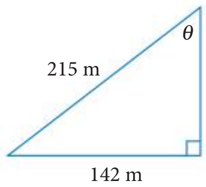
For an observer on the ground at *A*, the angle of elevation of a weather balloon at *B* is 37°. *C* is a point on the ground directly under the balloon. The distance *AC* is 2200 m. To the nearest metre, the height of the weather balloon above the ground is



A 1324 m **B** 1658 m **C** 1757 m **D** 2919 m **E** 3655 m

[VCAA 2007 1GTQ2]

Question 2



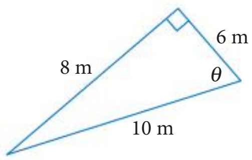
For the triangle shown, the size of angle θ is closest to

A 33° **B** 41° **C** 45°

D 49° **E** 57°

[VCAA 2006 1GTQ1]

Question 3



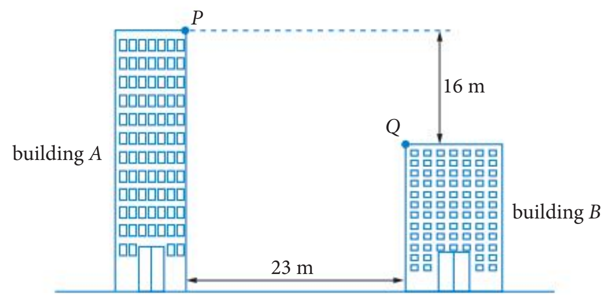
For the triangle shown, the value of cos (θ) is equal to

A $\frac{6}{10}$ **B** $\frac{6}{8}$ **C** $\frac{8}{10}$ **D** $\frac{10}{8}$ **E** $\frac{8}{6}$

[VCAA 2007 1GTQ1]

Question 4

The point Q on building B is visible from point P on building A, as shown in the diagram. Building A is 16 metres taller than building B. The horizontal distance between point P and point Q is 23 metres. The angle of depression of point Q from point P is closest to

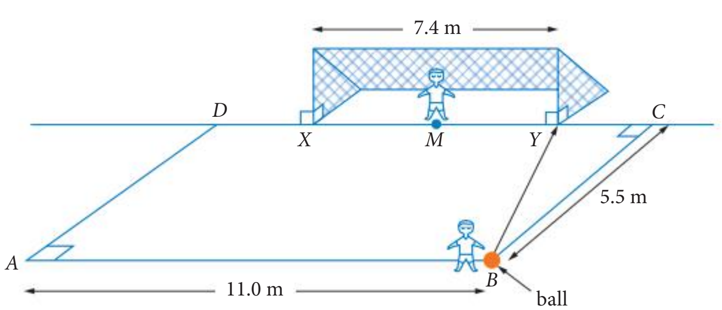


A 35° **B** 41° **C** 44° **D** 46° **E** 55°

[VCAA 2011 1GTQ2]

Use the following information to answer Questions 5 & 6.

A soccer goal is 7.4 metres wide. A rectangular region $ABCD$ is marked out directly in front of the goal. In this rectangular region, $AB = DC = 11.0$ metres and $AD = BC = 5.5$ metres. The goal line XY lies on DC and M is the midpoint of both DC and XY.



Question 5

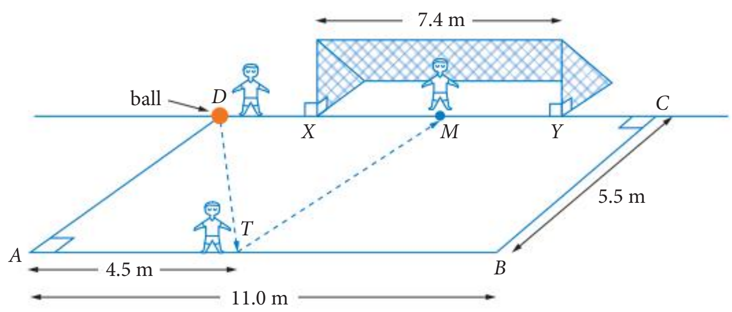
Ben kicks the ball from point B. It travels in a straight line to the base of the goal post at point Y on the goal line. Angle CBY, the angle that the path of the ball makes with the line BC, is closest to

A 18° **B** 33° **C** 45° **D** 67° **E** 72°

[VCAA 2010 1GTQ5]

Question 6

David kicks the ball from point D in a straight line to Tara. Tara is standing near point T on the line AB, a distance of 4.5 metres from point A. Tara then kicks the ball from point T in a straight line to the midpoint of the goal line at M.



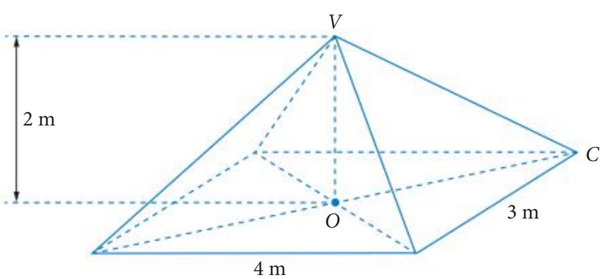
The total distance that the ball will travel in moving from point D to T to M is closest to

A 5.5 m **B** 12.1 m **C** 12.5 m **D** 12.7 m **E** 12.9 m

[VCAA 2010 1GTQ6]

Question 7

A right pyramid, shown, has a rectangular base with length 4 m and width 3 m. The height of the pyramid is 2 m. The angle VCO that the sloping edge VC makes with the base of the pyramid, to the nearest degree, is



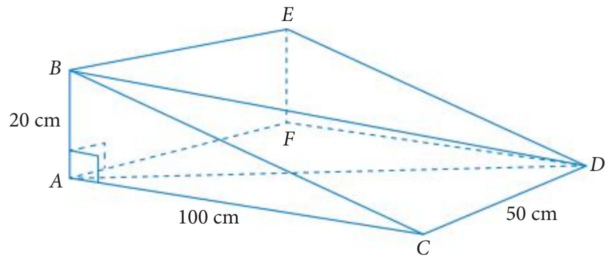
A 22° **B** 27° **C** 34°

D 39° **E** 45°

[VCAA 2011 1GTQ5]

Question 8

A right-triangular prism $ABCDEF$ is as shown, with lengths $AB = 20$ cm, $AC = 100$ cm and $CD = 50$ cm. The size of angle ADB is



A 10.1° **B** 11.3°

C 30.5° **D** 59.0°

E 59.5°

[VCAA 2004 1GTQ9]

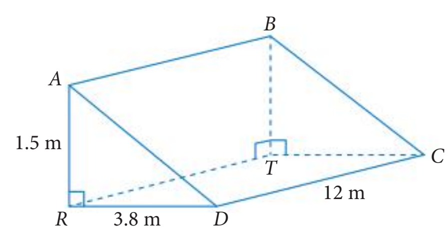
Use the following information to answer Questions 9 & 10.

$ABCD$ is a sloping rectangular roof above a horizontal rectangular ceiling, $TCDR$.

$AB = DC = 12$ metres

$RD = TC = 3.8$ metres

$AR = BT = 1.5$ metres



Question 9

The angle of depression of D from A is closest to

A 21.5° **B** 23.3° **C** 66.7° **D** 68.5° **E** 111.5°

[VCAA 2009 1GTQ6]

Question 10

The angle ACR is closest to

A 6.80° **B** 6.84° **C** 7.13° **D** 18.80° **E** 21.54°

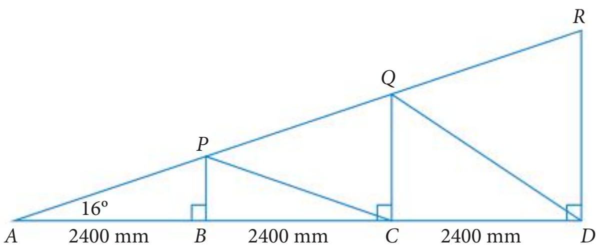
[VCAA 2009 1GTQ7]

Question 11

The structure of a roof frame is shown in the diagram. In this diagram, $AB = BC = CD = 2400$ mm and $\angle PAB = 16°$. The length of QD, in mm, is closest to

A 2741 **B** 2767 **C** 2830

D 3394 **E** 5201



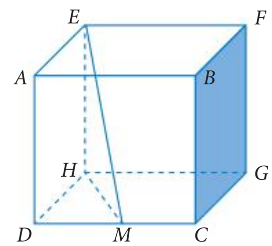
[VCAA 2011 1GTQ7]

Question 12

The cube has sides four metres long. M is the midpoint of DC. The angle EMH is closest to

A 41.8° **B** 48.2° **C** 49.1°

D 54.7° **E** 70.5°



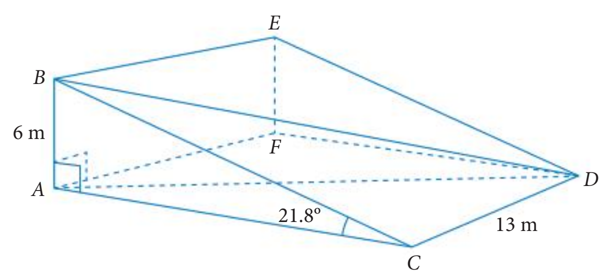
[VCAA 2005 1GTQ9]

Question 13

The diagram shows a right-triangular prism $ABCDEF$. In this prism, $AB = 6$ m, angle $ACB = 21.8°$ and $CD = 13$ m. The size of the angle CBD is closest to

A 21.6° **B** 26.7° **C** 38.8°

D 40.9° **E** 51.2°



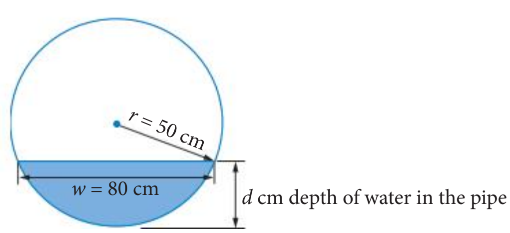
[VCAA 2010 1GTQ7]

Question 14

The cross-section of a water pipe is circular with a radius, r, of 50 cm, as shown. The surface of the water has a width, w, of 80 cm. The depth of water in the pipe, d, could be

A 20 cm **B** 25 cm **C** 30 cm

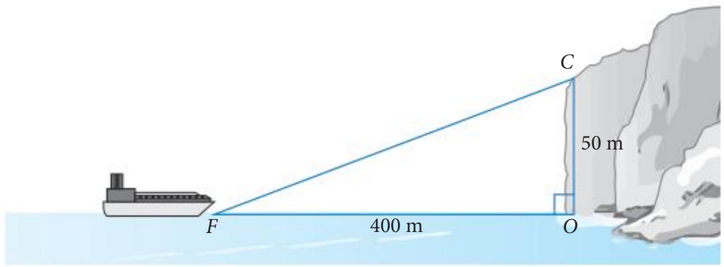
D 40 cm **E** 50 cm



[VCAA 2006 1GTQ8]

Question 15

A ferry, F, is 400 metres from point O at the base of a 50 metre high cliff, OC.



a Show that the gradient of the line FC in the diagram is 0.125. 1 mark

b Calculate the angle of elevation of point C from F.
Write your answer in degrees, correct to one decimal place. 1 mark

c Calculate the distance FC, in metres, correct to one decimal place. 1 mark

[VCAA 2009 2GTQ1]

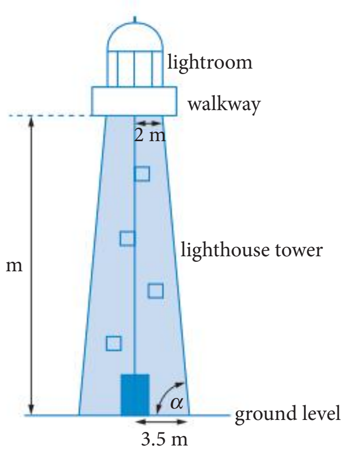
Question 16

The lighthouse tower, shaded on the figure to the right, is in the shape of a truncated cone.

It has circular cross-sections that decrease uniformly from a radius of 3.5 metres at ground level to a radius of 2 metres at the walkway.

The height of the lighthouse tower is 18 metres.

The angle marked α is the angle that the outer wall of the lighthouse tower makes with the horizontal at ground level.

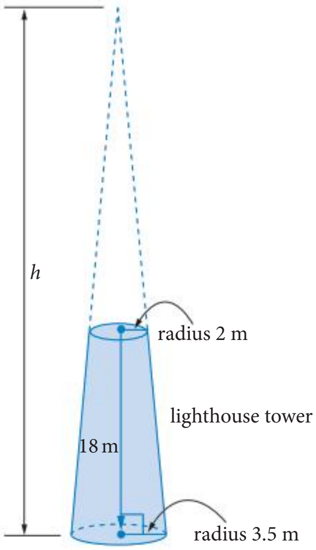


a Determine the size of angle α. Write your answer in degrees, correct to one decimal place. 1 mark

The lighthouse tower is part of a cone. The height of this cone is h metres and its base radius is 3.5 metres, as shown.

b **i** Determine h, the height of this cone, in metres. 2 marks

ii Determine the volume of the lighthouse tower. Write your answer correct to the nearest cubic metre. 1 mark



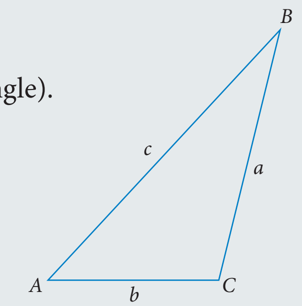
[VCAA 2011 2GTQ4]

12.4 Non-right-angled triangles

The convention when labelling a triangle ABC is

- A, B and C represent the angles (as well as the vertices of the triangle).
- a, b and c represent the sides.

Each angle is opposite the side length with the same letter.



The sine rule and the cosine rule are used to find sides or angles in non-right-angled triangles.

The sine rule

WS The sine rule—Finding lengths of sides

WS The sine rule—Finding angles

The **sine rule** can be used to solve a non-right-angled triangle if we have been given either

- one side and two angles or
- two sides and the angle that is **not** included between the two sides.

The sine rule can be written in two forms

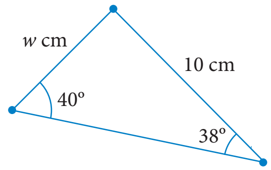
When finding sides: $\dfrac{a}{\sin(A)} = \dfrac{b}{\sin(B)} = \dfrac{c}{\sin(C)}$

When finding angles: $\dfrac{\sin(A)}{a} = \dfrac{\sin(B)}{b} = \dfrac{\sin(C)}{c}$

amanaimages/Alastair Philip Wiper/View

Worked example 18

Find the value of w in centimetres, correct to one decimal place.

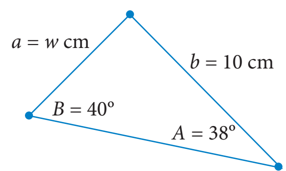


Working

1 The triangle can be solved with the sine rule as the given information is two angles and a side.

$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)}$$

2 Identify side a, angle A (side w is opposite angle 38°), side length b and angle B (side length 10 cm is opposite the angle 40°) on the triangle.



Substitute into the formula.

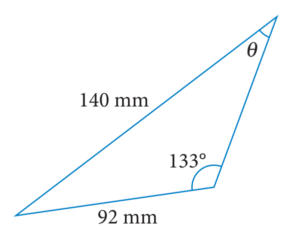
$$\frac{w}{\sin(38°)} = \frac{10}{\sin(40°)}$$

3 Solve the equation for w, using a CAS/calculator if necessary.

$$w = \frac{10}{\sin(40°)} \times \sin(38°)$$
$$\approx 9.6 \text{ cm}$$

Using CAS Solving sine rule equations to find an angle

Find the value of θ in degrees, correct to one decimal place.



The triangle can be solved with the sine rule as the given information is two sides and the non-included angle.

To find the angle θ, use the formula

$$\frac{\sin(A)}{a} = \frac{\sin(B)}{b} = \frac{\sin(C)}{c}$$

TI-NSPIRE CAS

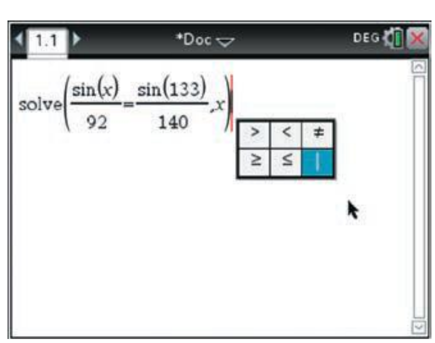
STEP 1

On a calculator page, press menu, 3: Algebra, 1: Solve.

Enter the equation

$$\frac{\sin(x)}{92} = \frac{\sin(133°)}{140}$$

,x) enter



STEP 2

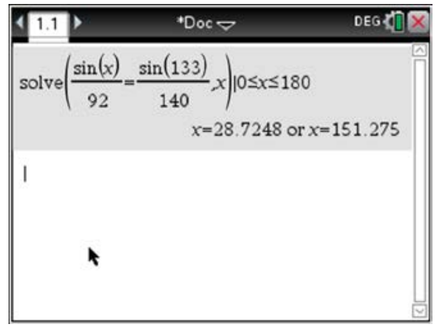
The domain must also be added.

Press ctrl =, select |

Enter the domain

$0 \le x \le 180$

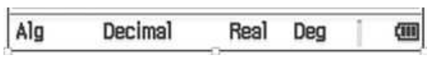
As the triangle has one angle of 133°, x cannot be 151° because this would produce an angle sum greater than 180°. The only possible solution is 28.7°.



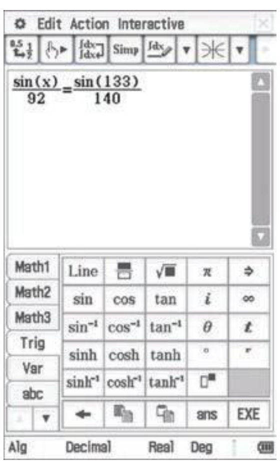
CLASSPAD

STEP 1

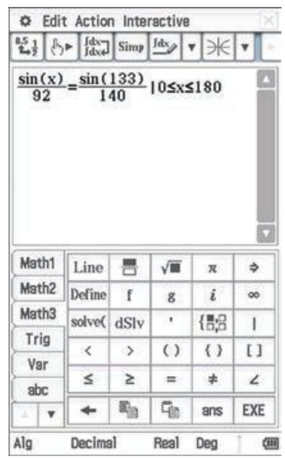
Make sure your calculator is set up as



Using Trig and the Fraction button type in the required equation.

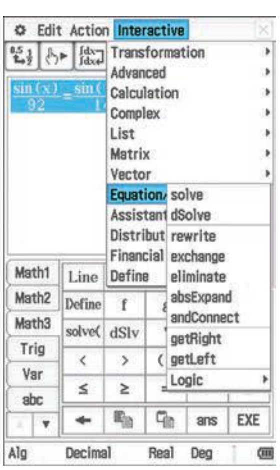


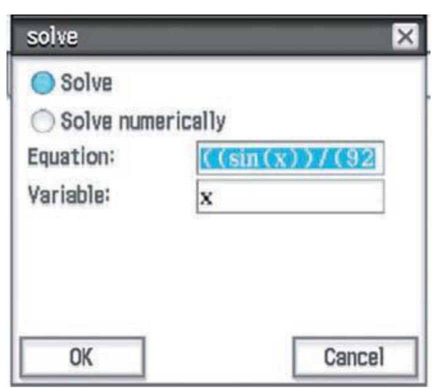
Now add a restriction because your answer will be between 0 and 180 degrees for a triangle. Use | from Math3



STEP 2

Highlight and tap **Interactive**, **Equation/ Inequality**, **solve** and tap **OK**.

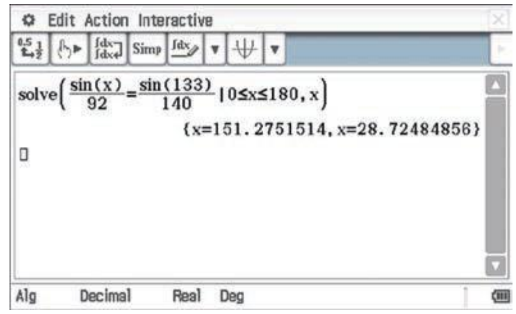




STEP 3

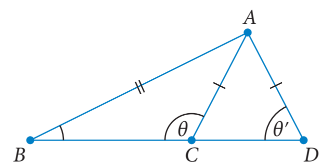
Rotate screen using Rotate to see the set of full solutions.

As the triangle has one angle of 133°, x cannot be 151° because this would produce an angle sum greater than 180°. The only possible solution is 28.7°.



The ambiguous case for the sine rule

When the sine rule is used to find an angle, there are sometimes two possible solutions. This can happen when you are given two sides and an angle which isn't between the two given sides (that is, a **non-included angle**). As shown in the diagram, triangles BAC and BAD both have the same two sides and angle but there are two possible values for θ, one greater than 90° and one less than 90°.

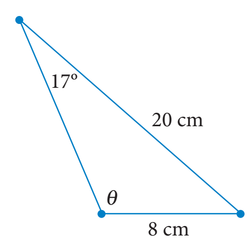


The two possible solutions to this using the sine rule are $\theta°$ and $180° - \theta°$.

Always check the angle sum of the triangle to verify if both solutions are valid.

Worked example 19

Find the two possible values for θ, correct to the nearest degree.

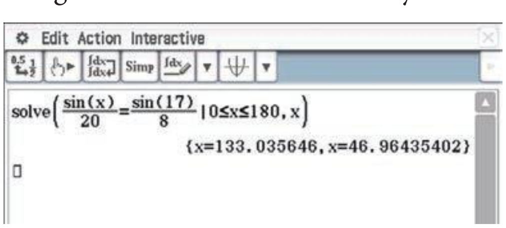


Working

1. The problem can be solved with the sine rule as the given information is two sides and the non-included angle.

$$\frac{\sin(A)}{a} = \frac{\sin(B)}{b}$$

2. Substitute into the sine rule and solve for θ, using a CAS/calculator if necessary.



$$\frac{\sin(\theta)}{20} = \frac{\sin(17)}{8}$$

$$\sin(\theta) = \frac{\sin(17)}{8} \times 20$$

$$\sin(\theta) = 0.7309\ldots$$

$$\theta = \sin^{-1}(0.7309\ldots)$$

3. Find two possible values for θ.

$$\theta = 47° \text{ and } \theta = 180° - 47°$$

$$\theta = 47° \text{ and } \theta = 133°$$

4 Verify that the angles do not produce an angle sum more than 180°.

Both solutions are valid as

Triangle 1 would have an angle sum of

$17° + 47° + C = 64° + C$

Triangle 2 would have an angle sum of

$17° + 133° + C = 150° + C$

The cosine rule—Angles and sides

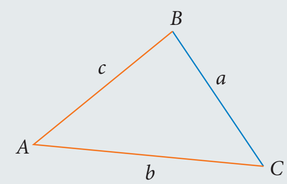
The cosine rule

The **cosine rule** is used to solve non-right-angled triangles if we have been given either

- two sides and the included angle, or
- three sides.

The cosine rule can be used in two forms.

- When finding a side $a^2 = b^2 + c^2 - 2bc\cos(A)$
- When finding an angle $\cos(A) = \dfrac{b^2 + c^2 - a^2}{2bc}$

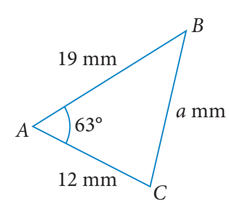


In the cosine rule, angle A is the included angle between sides b and c.

Worked example 20

Find the value of a in the triangle.

Express your answer in millimetres, correct to one decimal place.



Working

1 Use the cosine rule because the given information is two sides and an included angle.

$a^2 = b^2 + c^2 - 2bc\cos(A)$

2 Identify the values of b, c and angle A.

Side b is opposite angle B and side c is opposite angle C.

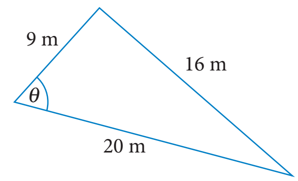
$b = 12, c = 19, A = 63°$

3 Substitute into the formula and solve, using a CAS/calculator if necessary.

$a^2 = 12^2 + 19^2 - 2 \times 12 \times 19 \times \cos(63°)$
$= 297.98\ldots$
$a \approx 17.3$

Worked example 21

Find the value of θ, correct to the nearest degree.



Working

1. Identify the values of a, b and c in relation to the angle θ.

   θ is the included angle between sides b and c.

   $a = 16$, $b = 20$, $c = 9$

2. Substitute into the cosine rule for finding angles.

   $$\cos(A) = \frac{b^2 + c^2 - a^2}{2bc}$$

   $$\cos(\theta) = \frac{20^2 + 9^2 - 16^2}{2 \times 20 \times 9} = \frac{5}{8}$$

3. Solve for θ using the inverse cosine function and round the answer to the nearest whole degree.

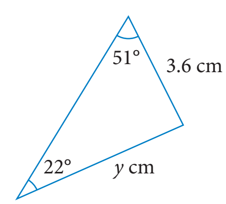
   $$\theta = \cos^{-1}\left(\frac{5}{8}\right) \approx 51°$$

EXAM PREP 12.4

Non-right-angled triangles

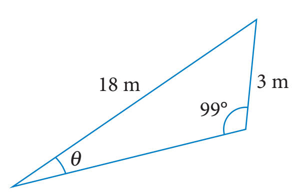
Prep 1 WORKED EXAMPLE 18

Find the value of y in centimetres, correct to two decimal places.



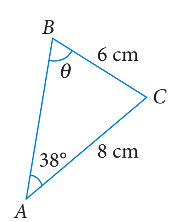
Prep 2 USING CAS: SOLVING SINE RULE EQUATIONS TO FIND AN ANGLE

Find the value of θ, correct to the nearest degree.



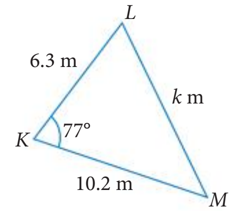
Prep 3 WORKED EXAMPLE 19

Find two possible values for the angle θ, correct to the nearest degree.



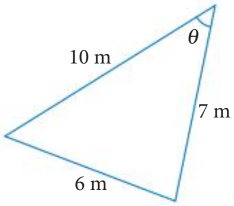
Prep 4 WORKED EXAMPLE 20

Find the value of k in metres, correct to one decimal place.



Prep 5 WORKED EXAMPLE 21

Find the value of θ, correct to the nearest tenth of a degree.

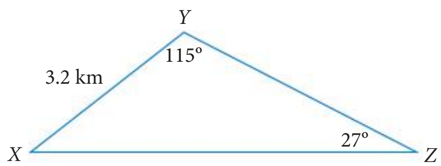


EXAM PRACTICE 12.4

Non-right-angled triangles

Question 1

The diagram shows the route of a cross-country race. Point X lies due west of point Z. Given that the length XY is 3.2 km, the length XZ is closest to



A 1.5 km **B** 1.6 km **C** 6.4 km **D** 7.0 km **E** 7.6 km

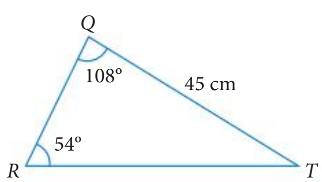
[VCAA 2004 1GTQ3]

Question 2

The length of RT in the triangle shown is closest to

A 17 cm **B** 33 cm **C** 45 cm

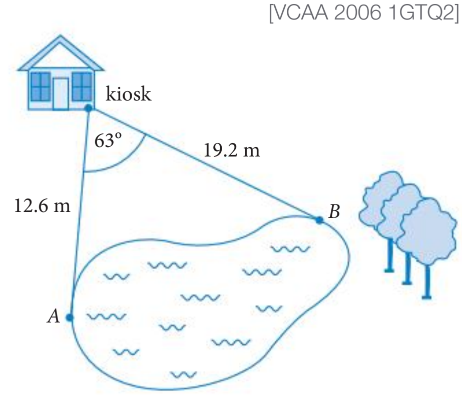
D 53 cm **E** 57 cm



[VCAA 2006 1GTQ2]

Question 3

The distances from a kiosk to points A and B on opposite sides of a pond are found to be 12.6 m and 19.2 m respectively. The angle between the lines joining these points to the kiosk is 63°. The distance, in m, across the pond between points A and B can be found by evaluating



A $\frac{1}{2}\times 12.6\times 19.2\times\sin(63°)$ **B** $\frac{19.2\times\sin(63°)}{12.6}$

C $\sqrt{12.6^2+19.2^2}$ **D** $\sqrt{12.6^2+19.2^2-2\times 12.6\times 19.2\times\cos(63°)}$

E $\sqrt{s(s-12.6)(s-19.2)(s-63)}$, where $s=\frac{1}{2}(12.6+10.2+63)$

[VCAA 2013 1GTQ2]

9780170371186

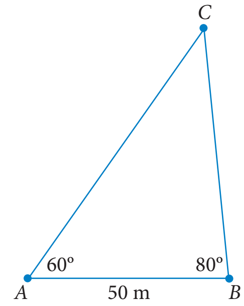
Question 4

In triangle ACB, $\angle CAB = 60°$ and $\angle ABC = 80°$. The length of side $AB = 50$ m.

The length of side AC is closest to

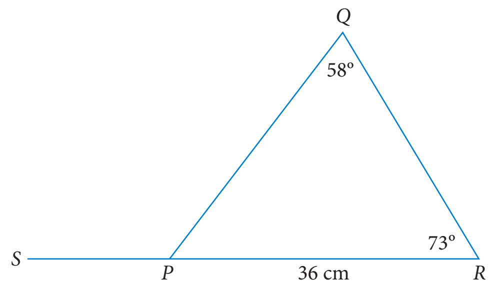
A 57 m **B** 67 m **C** 77 m

D 81 m **E** 100 m



[VCAA 2011 1GTQ4]

Use the following information to answer Questions 5 & 6.



Question 5

The size of $\angle SPQ$ is exactly

A 41° **B** 49° **C** 107° **D** 122° **E** 131°

[VCAA 2003 1GTQ1]

Question 6

Given that the length of PR is 36 cm, the length of PQ is

A 31.9 cm **B** 34.4 cm **C** 40.6 cm **D** 42.5 cm **E** 43.7 cm

[VCAA 2003 1GTQ2]

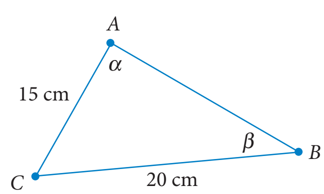
Question 7

In triangle ABC, $\sin(\alpha) = 0.8$.

$\sin(\beta)$ is equal to

A 0.5 **B** 0.6 **C** 0.75

D 0.8 **E** 0.9375



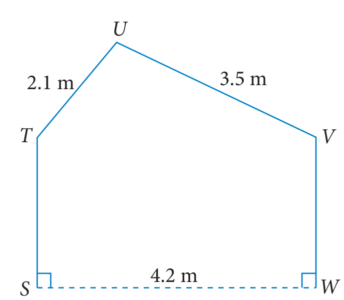
[VCAA 2005 1GTQ7]

Question 8

A cross-section of a glass greenhouse is shown in the diagram. The sides of the glass panels TU and UV are 2.1 metres and 3.5 metres long respectively. The greenhouse is 4.2 metres wide. The walls ST and WV are vertical and equal in height. The size of $\angle TUV$ is

A 44.4° **B** 45.6° **C** 86.2°

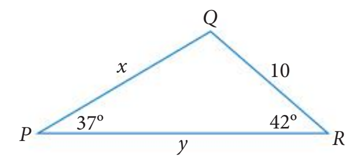
D 93.8° **E** 109.6



[VCAA 2003 1GTQ8]

Question 9

PQR is a triangle with side lengths x, 10 and y, as shown.
In this triangle, angle $RPQ = 37°$ and angle $QRP = 42°$.
Which one of the following expressions is correct for triangle *PQR*?



A $x = \dfrac{10}{\sin(37°)}$

B $y = \dfrac{10}{\tan(37°)}$

C $x = 10 \times \dfrac{\sin(42°)}{\sin(37°)}$

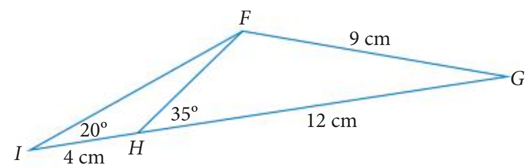
D $y = 10 \times \dfrac{\sin(37°)}{\sin(101°)}$

E $10^2 = x^2 + y^2 - 2xy\cos(42°)$

[VCAA 2012 1GTQ7]

Question 10

In the diagram, the length of *FH* is equal to



A $4\tan(55°)$

B $\sqrt{12^2 - 9^2}$

C $\sqrt{12^2 + 9^2 - 216\cos(35°)}$

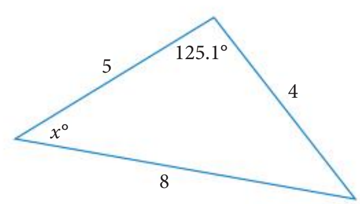
D $\dfrac{4\sin(20°)}{\sin(15°)}$

E $12\sin(55°)$

[VCAA 2004 1GTQ8]

Question 11

For the triangle shown, the value of $\sin(x°)$ is given by



A $\dfrac{\sin(125.1°)}{2}$

B $\dfrac{5^2 + 4^2 - 8^2}{2 \times 5 \times 4}$

C $2 \times \sin(125.1°)$

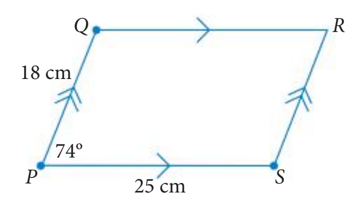
D $\dfrac{5^2 + 8^2 - 4^2}{2 \times 5 \times 8}$

E $\dfrac{5 \times \sin(125.1°)}{8}$

[VCAA 2008 1GTQ5]

Question 12

In parallelogram *PQRS*, $\angle QPS = 74°$. In this parallelogram, $PQ = 18$ cm and $PS = 25$ cm. The length of the longer diagonal of this parallelogram is closest to



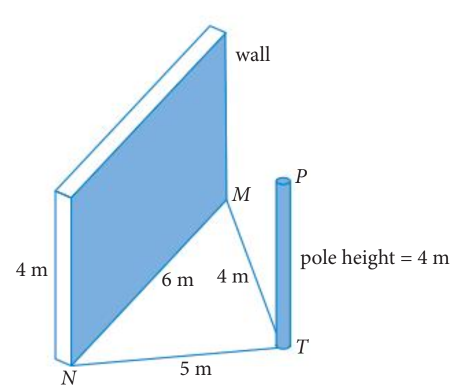
A 26.5 cm **B** 30.1 cm **C** 30.8 cm

D 34.6 cm **E** 39.9 cm

[VCAA 2011 1GTQ6]

Question 13

A vertical pole, *TP*, is 4 metres tall and stands on level ground near a vertical wall. The wall is 6 metres long and 4 metres high. The base of the pole, *T*, is 5 metres from one end of the wall at *N* and 4 metres from the other end of the wall at *M*. The pole falls and hits the wall. The maximum height above ground level at which the pole could hit the wall is closest to



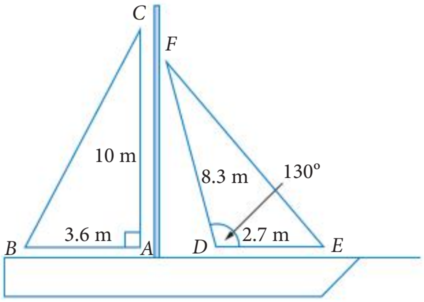
A 0 m **B** 1.5 m **C** 2.3 m

D 2.7 m **E** 3.3 m

[VCAA 2009 1GTQ9]

Question 14

A yacht has two flat triangular sails as shown in the diagram.



The sail ABC is in the shape of a right-angled triangle.
The height AC is 10 metres and the length AB is 3.6 metres.

a Calculate angle ABC. Write your answer correct to the nearest degree. 1 mark

b Calculate the length BC. Write your answer in metres, correct to one decimal place. 1 mark

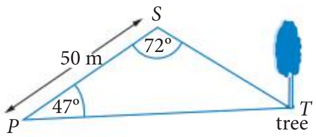
The sail DEF has side lengths $DE = 2.7$ metres and $DF = 8.3$ metres. The angle EDF is 130°.

c Calculate the length EF. Write your answer in metres, correct to one decimal place. 1 mark

d Calculate the area of the sail DEF. Write your answer in square metres, correct to one decimal place. 1 mark

[VCAA 2004 2GTQ1]

Question 15



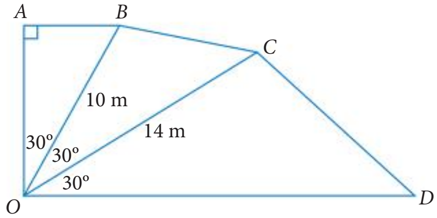
A tree is growing near the block of land. The base of the tree, T, is at the same level as the corners, P and S, of the block of land.

a Show that, correct to two decimal places, distance ST is 41.81 metres. 1 mark

b From point S, the angle of elevation to the top of the tree is 22°. Calculate the height of the tree. Write your answer, in metres, correct to one decimal place. 1 mark

[VCAA 2012 2GTQ3]

Question 16



$OABCD$ has three triangular sections, as shown in the diagram. Triangle OAB is a right-angled triangle. Length OB is 10 m and length OC is 14 m.
Angle AOB = angle BOC = angle COD = 30°

a Calculate the length of OA. Write your answer in metres, correct to two decimal places. 1 mark

b Determine the area of triangle OAB. Write your answer in m^2, correct to one decimal place. 1 mark

c Triangles OBC and OCD are similar. The area of triangle OBC is 35 m^2. Find the area of triangle OCD, in m^2. 1 mark

d Determine angle CDO. Write your answer correct to the nearest degree. 2 marks

[VCAA 2012 2GTQ4]

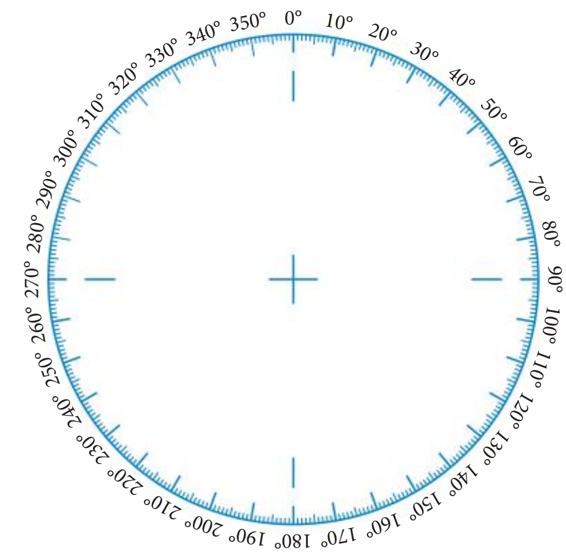
12.5 Bearings

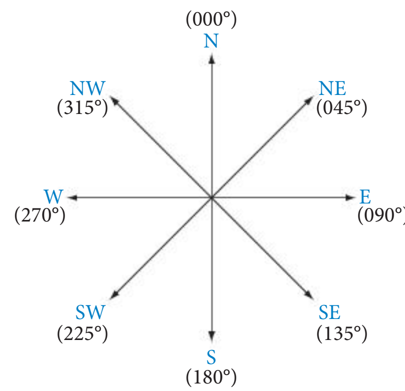
Identifying bearings

Bearings

Bearings use angles to show the direction of a location from a given point and are found by connecting the two points with a straight line.

Three-figure bearings (also known as **true bearings**) are angles measured in a clockwise direction from due north and are written as three digits. The diagram below shows how the angles in one revolution of 360° match with a **compass rose**.





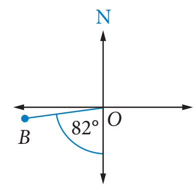
Exam hack

Three-figure bearings must be written with three digits. For example, 5° clockwise from due north must be written as 005°.

Alamy/robertharding

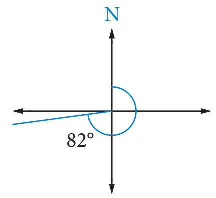
Worked example 22

Write the three-figure bearing of B from O.



Working

1 The three-figure bearing is the angle clockwise from due north. Mark this angle on the diagram.



2 Calculate the bearing by adding the straight angle of 180°.

$$\text{Three-figure bearing} = 180° + 82° = 262°$$

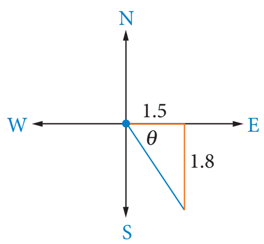
Worked example 23

A hot air balloon travelled 1.5 km due east but drifted 1.8 km south. What is its three-figure bearing from its starting point, to the nearest degree?

Working

1 Draw a compass rose and include the information on this diagram.

The start is the centre of the compass rose.



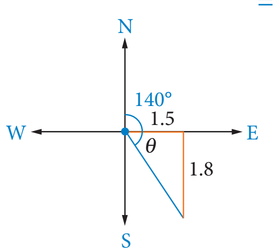
2 Find θ using the tangent ratio.

$$\tan(\theta) = \frac{1.8}{1.5}$$

$$\theta = \tan^{-1}\left(\frac{1.8}{1.5}\right) \approx 50°$$

3 Mark the three-figure bearing angle on the diagram and calculate it by adding a right angle of 90° to θ.

$$\text{Three-figure bearing} = 90° + 50° = 140°$$



Worked example 24

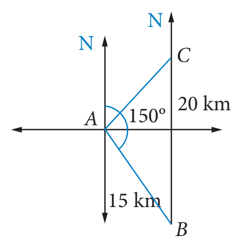
Emily is going on a three-day bushwalking adventure. She starts at camp site A and walks for 15 km on a bearing of 150° to camp site B. On the second day, she walks 20 km due north to campsite C. On the third day, Emily plans to return to campsite A.

a Find the distance, to the nearest kilometre, that Emily needs to travel to return to camp site A.

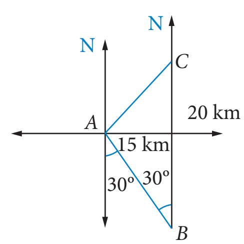
b What bearing does Emily need to travel on in order to return to camp site A? Express your answer correct to the nearest degree.

Working

a **1** Draw a compass rose and include the information on this diagram.



2 The angle between the 15 km line and due south is 30°. The angle at B is alternate to 30°. Draw this information on the diagram.



3 Find the distance AC using the cosine rule.

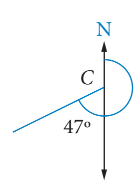
$$(AC)^2 = 15^2 + 20^2 - 2 \times 15 \times 20 \times \cos(30°)$$
$$AC = 10.265\ldots \approx 10 \text{ km}$$

b **1** The bearing is taken from camp site C, so find the angle in the triangle at C.

Angle C can be found using the sine rule.

$$\frac{\sin(C)}{15} = \frac{\sin(30°)}{10.265\ldots}$$
$$\sin(C) = \frac{15\sin(30°)}{10.265\ldots}$$
$$C \approx 47°$$

2 Write the angle as a three-figure bearing.



Bearing from camp site C = 180° + 47° = 227°

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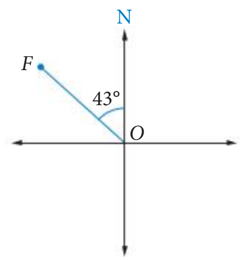
EXAM PREP 12.5

Bearings

Prep 1 WORKED EXAMPLE 22

Write the three-figure bearing of point F from O.



Prep 2 WORKED EXAMPLE 23

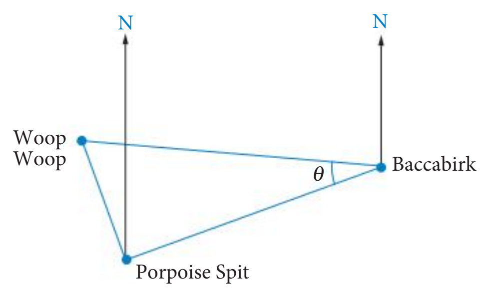
A ship sails due south for 20 km and due east for a further 15 km. Find the three-figure bearing, to the nearest degree, of the ship from its starting point.

Prep 3 WORKED EXAMPLE 24

The town of Woop Woop is 22 km from the town of Porpoise Spit, on a three-figure bearing of 340°. The town of Baccabirk is 48 km from Porpoise Spit, on a three-figure bearing of 070°.

a Find θ to the nearest degree.

b Find the three-figure bearing of Woop Woop from Baccabirk. (Hint: Use alternate angles.)



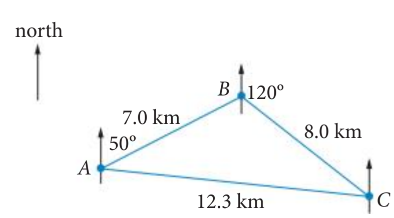
EXAM PRACTICE 12.5

Bearings

Question 1

An orienteering course is triangular in shape and is marked by three points, A, B and C, as shown in the diagram.

In this course, B is 7.0 km from A, C is 8.0 km from B and A is 12.3 km from C. The area (in km^2) enclosed by this course is closest to

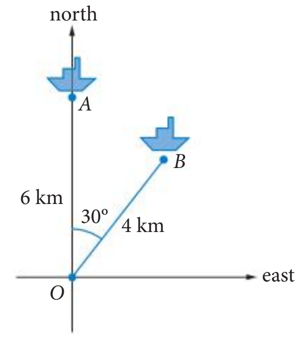


A 21 **B** 24 **C** 25 **D** 26 **E** 28

[VCAA 2008 1GTQ3]

Question 2

Two ships are observed from point O. At a particular time, their positions A and B are as shown. The distance between the ships at this time is



A 3.0 km **B** 3.2 km **C** 4.5 km
D 9.7 km **E** 10.4 km

[VCAA 2005 1GTQ4]

Question 3

A triangular course for a yacht race has three stages.

Stage 1 is from the Start to Marker 1; a distance of 3.5 km on a bearing of 055°.

Stage 2 is from Marker 1 to Marker 2; a distance of 4.6 km on a bearing of 145°.

Stage 3 is from Marker 2 back to the Start.

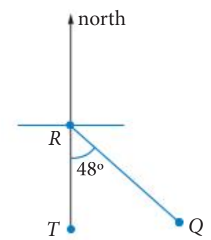
The distance travelled on Stage 3, in km, is closest to

A 4.9 **B** 5.3 **C** 5.8 **D** 6.0 **E** 7.7

[VCAA 2012 1GTQ8]

Question 4

The locations of three towns, Q, R and T, are shown in the diagram. Town T is due south of town R. The angle TRQ is 48°. The bearing of town R from town Q is



A 048° **B** 132° **C** 138°
D 228° **E** 312°

[VCAA 2009 1MMTQ3]

Question 5

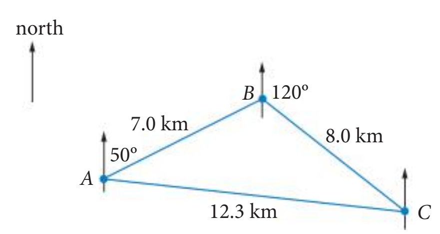
Dan takes his new aircraft on a test flight. He starts from his local airport and flies 10 km on a bearing of 045° until he reaches his brother's farm. From here, he flies 18 km on a bearing of 300° until he reaches his parents' farm. Finally, he flies back directly from his parents' farm to his local airport. The total distance (in km) that he flies is closest to

A 37 **B** 42 **C** 46 **D** 59 **E** 61

[VCAA 2010 1GTQ8]

Question 6

An orienteering course is triangular in shape and is marked by three points, A, B and C, as shown in the diagram. In this course, the bearing of B from A is 050° and the bearing of C from B is 120°. The bearing of B from C is



A 060° **B** 120° **C** 240°
D 300° **E** 310°

[VCAA 2008 1GTQ2]

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Question 7

There are four telecommunications towers in a city. The towers are called Grey Tower, Black Tower, Silver Tower and White Tower. Grey Tower is 10 km due west of Black Tower. Silver Tower is 10 km from Grey Tower on a bearing of 300°. White Tower is 10 km due north of Silver Tower. Correct to the nearest degree, the bearing of Black Tower from White Tower is

A 051° **B** 129° **C** 141° **D** 309° **E** 321°

[VCAA 2013 1GTQ8]

12.5

Question 8

Two hikers, Anton and Beth, walk in different directions from the same camp. Beth walks for 12 km on a bearing of 135° to a picnic ground. Anton walks for 6 km on a bearing of 045° to a lookout tower. On what bearing (to the nearest degree) should Anton walk from the lookout tower to meet Beth at the picnic ground?

A 063° **B** 108° **C** 153° **D** 162° **E** 180°

[VCAA 2008 1GTQ9]

Question 9

The points M, N and P form the vertices of a triangular course for a yacht race. $MN = MP = 4$ km. The bearing of N from M is 070°. The bearing of P from M is 180°. Three people perform different calculations to determine the length of NP in kilometres.

Graeme $NP = \sqrt{16+16-2\times4\times4\times\cos(110°)}$

Shelley $NP = 2 \times 4 \times \cos(35°)$

Tran $NP = \dfrac{4\times\sin(110°)}{\sin(35°)}$

The correct length of NP would be found by

A Graeme only

B Tran only

C Graeme and Shelley only

D Graeme and Tran only

E Graeme, Shelley and Tran.

[VCAA 2007 1GTQ9]

Question 10

Points M and P are the same distance from a third point O.

The bearing of M from O is 038° and the bearing of P from O is 152°. The bearing of P from M is

A between 000° and 090°

B between 090° and 180°

C exactly 180°

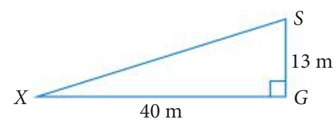
D between 180° and 270°

E between 270° and 360

[VCAA 2006 1GTQ9]

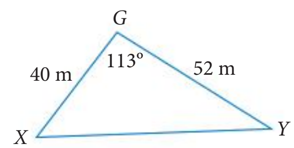
Question 11

A spectator, S, in the grandstand of an athletics ground is 13 m vertically above point G. Competitor X, on the athletics ground, is at a horizontal distance of 40 m from G.



a Find the distance, SX, correct to the nearest metre. 1 mark

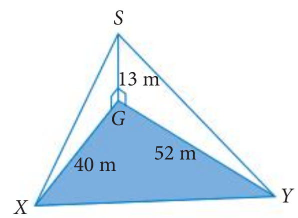
Competitor X is 40 m from G and competitor Y is 52 m from G. The angle XGY is 113°.



b **i** Calculate the distance, XY, correct to the nearest metre. 1 mark

ii Find the area of triangle XGY, correct to the nearest square metre. 1 mark

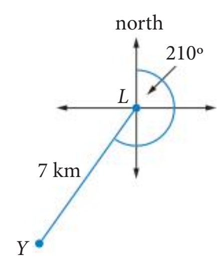
c Determine the angle of elevation of spectator S from competitor Y, correct to the nearest degree. Note that X, G and Y are on the same horizontal level. 1 mark



[VCAA 2013 2GTQ1]

Question 12

A yacht, Y, is 7 km from a lighthouse, L, on a bearing of 210°, as shown in the diagram.



a A ferry can also be seen from the lighthouse. The ferry is 3 km from L on a bearing of 135°. Copy the diagram and label the position of the ferry, F, and show an angle to indicate its bearing. 1 mark

b Determine the angle between LY and LF. 1 mark

c Calculate the distance, in km, between the ferry and the yacht, correct to two decimal places. 1 mark

d Determine the bearing of the lighthouse from the ferry. 1 mark

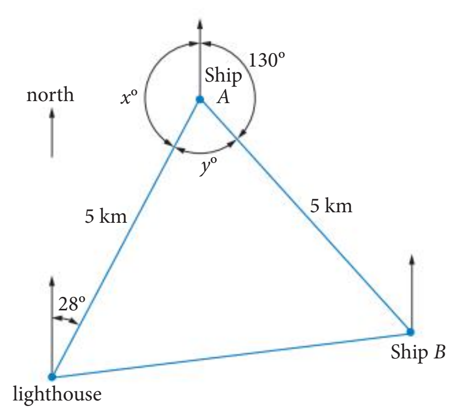
[VCAA 2009 2GTQ2]

Question 13

Ship A and Ship B can both be seen from the lighthouse.

Ship A is 5 kilometres from the lighthouse, on a bearing of 028°.

Ship B is 5 kilometres from Ship A, on a bearing of 130°.



a Two angles, x and y, are shown.

i Determine the size of the angle x in degrees. 1 mark

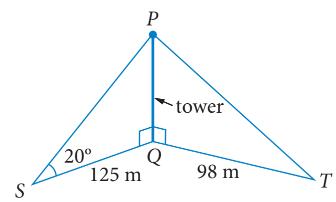
ii Determine the size of the angle y in degrees. 1 mark

b Determine the bearing of the lighthouse from Ship A. 1 mark

c Determine the bearing of Ship B from the lighthouse. 1 mark

[VCAA 2011 2GTQ2]

Question 14



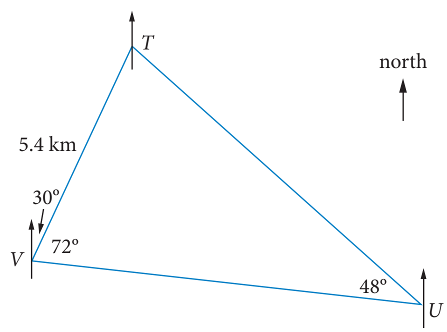
The allotment of land contains a communications tower, *PQ*. Points *S*, *Q* and *T* are situated on level ground. From *S*, the angle of elevation of *P* is 20°. Distance *SQ* is 125 metres. Distance *TQ* is 98 metres.

a Determine the height, *PQ*, of the communications tower. Write your answer, in metres, correct to one decimal place. 1 mark

b Determine the angle of depression of *T* from *P*. Write your answer, in degrees, correct to one decimal place. 1 mark

[VCAA 2006 2GTQ2]

Question 15



A course for a yacht race is triangular in shape and is marked by three buoys *T*, *U* and *V*.

Starting from buoy *V*, the yachts sail 5.4 kilometres on a bearing of 030° to buoy *T*. They then sail to buoy *U* and back to buoy *V*. The angle *TVU* is 72° and the angle *TUV* is 48°.

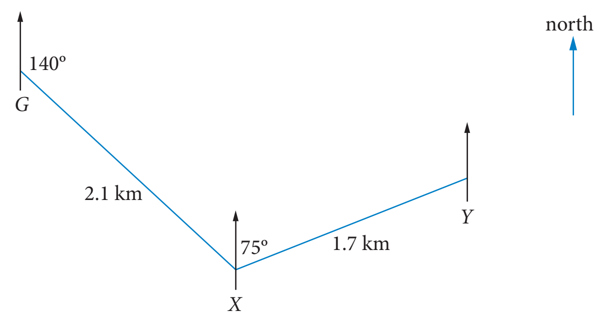
a Determine the bearing of *V* from *U*. 2 marks

b Determine the distance *TU*. Write your answer in kilometres, correct to one decimal place. 1 mark

c Determine the shortest distance needed to complete the race. Write your answer in kilometres, correct to one decimal place. 2 marks

[VCAA 2004 2GTQ2]

Question 16



The land near the camping ground is flat and suitable for orienteering. Checkpoint *X* is situated 2.1 km from camping ground *G* on a bearing of 140°. Checkpoint *Y* is situated 1.7 km from checkpoint *X* on a bearing of 075°.

a **i** How far south of *G* is checkpoint *X*? Write your answer in kilometres correct to one decimal place. 1 mark

ii How far south of *G* is checkpoint *Y*? Write your answer in kilometres correct to one decimal place. 2 marks

b Determine the size of angle *GXY*. 1 mark

c Calculate the distance *GY*. Write your answer in kilometres correct to one decimal place. 1 mark

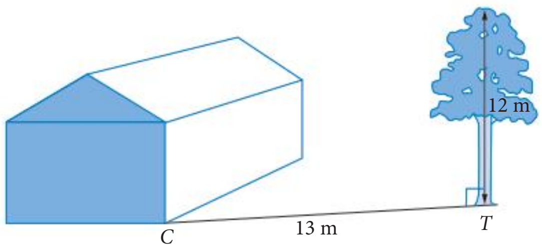
d Determine the bearing of checkpoint *Y* from camping ground *G*. Write your answer correct to the nearest degree. 2 marks

[VCAA 2005 2GTQ2]

12.5

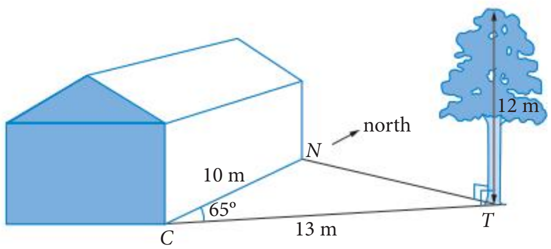
Question 17

A tree, 12 m tall, is growing at point T near the shed. The distance, CT, from corner C of the shed to the centre base of the tree is 13 m.



a Calculate the angle of elevation of the top of the tree from point C. Write your answer, in degrees, correct to one decimal place. 1 mark

N and C are two corners at the base of the shed. N is due north of C. The angle TCN is 65°.



b Show that, correct to one decimal place, the distance NT is 12.6 m. 1 mark

c Calculate the angle CNT correct to the nearest degree. 1 mark

d Determine the bearing of T from N. Write your answer correct to the nearest degree. 1 mark

e Is it possible for the tree to hit the shed if it falls? Explain your answer, showing appropriate calculations. 2 marks

[VCAA 2008 2GTQ3]

Question 18

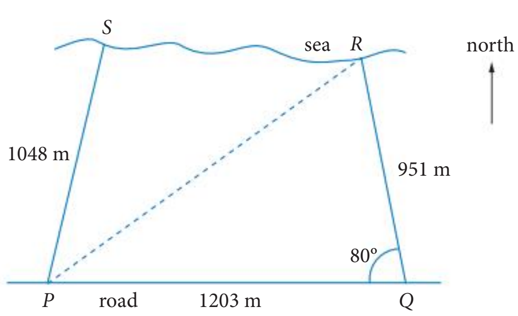
Daniel threw a javelin a distance of 68.32 m on a bearing of 057° on his first throw. On his second throw from the same point, he threw the javelin a distance of 72.51 m. The second throw landed at a point on a bearing of 125°, measured from the point where the first throw landed.

Determine the distance between the point where Daniel's first throw landed and the point where his second throw landed. Write your answer in metres, correct to one decimal place. 2 marks

[VCAA 2013 2GTQ5]

Question 19

The diagram shows a camping ground by the sea. The boundary PQ is 1203 metres long and runs beside an east–west road. The boundary PS is 1048 metres long and the bearing of S from P is 015° true. The boundary QR is 951 metres long and the angle PQR is 80°. The fourth boundary of the camping ground is along a cliff edge by the sea.



a Find the area of triangle PQR in square metres, correct to the nearest square metre. 2 marks

b Find the distance from P to R, correct to the nearest metre. 2 marks

c Find the bearing of R from P, correct to the nearest degree. 2 marks

d Find the area of triangle PSR in square metres, correct to the nearest 10 square metres. 2 marks

e Hence find the approximate area of the camping ground. Give your answer correct to the nearest 10 square metres. 1 mark

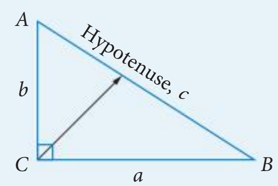
[VCAA 2002 2GTQ1]

Measurement and trigonometry

Practice quiz

Pythagoras' theorem

- In $\triangle ABC$: $c^2 = a^2 + b^2$
- The **hypotenuse** is the longest side of a right-angled triangle and is always opposite the right angle.



Circumference of a circle

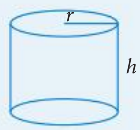
$C = \pi d$ or $C = 2\pi r$

Area formulas

Shape	Area formula
Rectangle	$A = \text{length} \times \text{width}$ $A = lw$
Square	$A = \text{side} \times \text{side}$ $A = (\text{side})^2$ $A = s^2$
Triangle	$A = \frac{1}{2} \times \text{base} \times \text{perpendicular height}$ $A = \frac{1}{2} bh$ $A = \frac{1}{2} \text{ product of two sides} \times \sin(\text{included angle})$ $A = \frac{1}{2} bc \sin(A)$ **Heron's formula** $A = \sqrt{s(s-a)(s-b)(s-c)}$ The semi perimeter (s) is $s = \frac{1}{2}(a+b+c)$
Parallelogram	$A = \text{base} \times \text{perpendicular height}$ $A = bh$
Trapezium	$A = \frac{1}{2} \times \text{sum of parallel sides} \times \text{perpendicular distance between sides}$ $A = \frac{1}{2}(a+b)h$

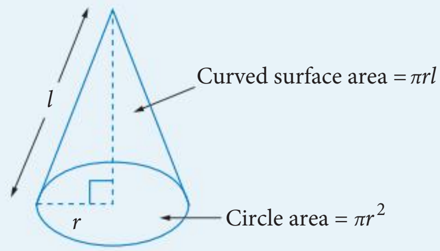
Total surface area formulas

Cylinder



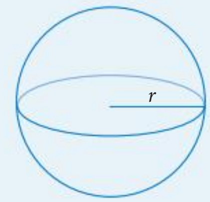
Surface area
$= \pi r^2 + \pi r^2 + 2\pi rh$
$= 2\pi r^2 + 2\pi rh$

Cone



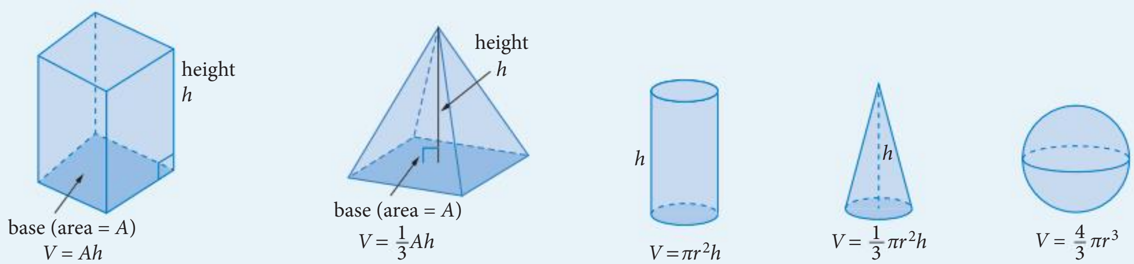
Surface area = $\pi r^2 + \pi rl$

Sphere



Surface area = $4\pi r^2$

Volume formulas



Prism Pyramid Cylinder Cone Sphere

Similarity

- **Scale factor** = $\dfrac{\text{length of second figure}}{\text{corresponding length of first figure}}$
- Two objects are **similar** if they have the same shape, but one is an enlargement of the other.
- To solve similar triangles, write a ratio equation for the corresponding sides in the form

  $$\frac{\text{side 1 of triangle 1}}{\text{side 1 of triangle 2}} = \frac{\text{side 2 of triangle 1}}{\text{side 2 of triangle 2}}$$

  and solve the equation for the unknown.

Length, area and volume ratios

For two similar figures with length ratio $a : b$,

area ratio = $a^2 : b^2$

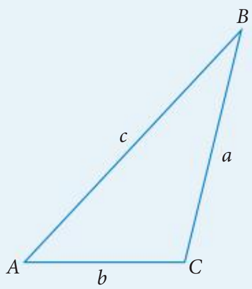
volume ratio = $a^3 : b^3$

Right-angled triangles

Ratio	Initials
$\sin(\theta)=\frac{\text{opposite}}{\text{hypotenuse}}$	SOH
$\cos(\theta)=\frac{\text{adjacent}}{\text{hypotenuse}}$	CAH
$\tan(\theta)=\frac{\text{opposite}}{\text{adjacent}}$	TOA

The sine rule

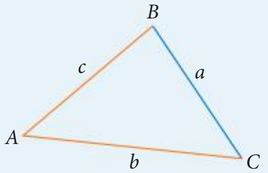
- The sine rule can be used to solve a non-right-angled triangle given
  - one side and two angles, or
  - two sides and the angle that is **not** included between the two sides.
- When finding sides use $\frac{a}{\sin(A)}=\frac{b}{\sin(B)}=\frac{c}{\sin(C)}$
- When finding angles use $\frac{\sin(A)}{a}=\frac{\sin(B)}{b}=\frac{\sin(C)}{c}$



The cosine rule

- The cosine rule is used to solve a non-right-angled triangle given
  - two sides and the included angle, or
  - three sides.
- When finding a side use $a^2 = b^2 + c^2 - 2bc\cos(A)$
- When finding an angle use $\cos(A)=\frac{b^2+c^2-a^2}{2bc}$

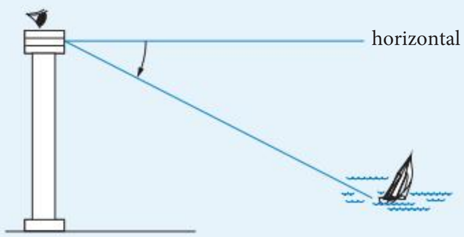
In the cosine rule, angle A is the included angle between sides b and c.



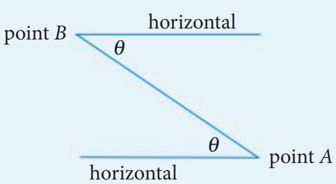
Angles of elevation and depression

- The angle of elevation involves looking up.

- The angle of depression involves looking down.

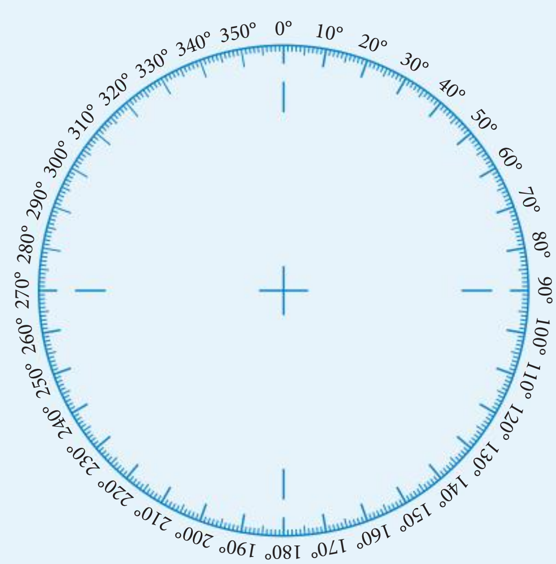


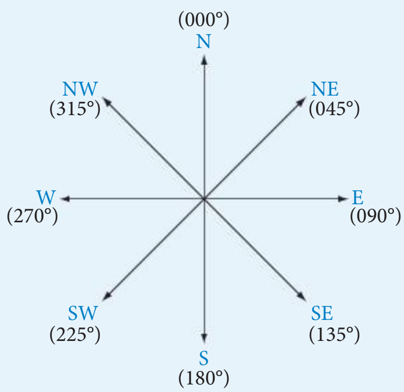
- The angle of elevation from point A up to point B is equal to the angle of depression from point B down to point A.



Bearings

Three-figure bearings are angles measured in a clockwise direction from due north and are written as three digits.





GEOMETRY AND MEASUREMENT

Examination 1

Reading time: (5 minutes)
Writing time: (30 minutes)

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers, one bound reference, one approved graphics calculator or approved CAS calculator or CAS software and, if desired, one scientific calculator. Calculator memory DOES NOT need to be cleared.
- Students are NOT permitted to bring into the examination room: blank sheets of paper and/or white out liquid/tape.
- Students may refer to the sheet of miscellaneous formulas supplied.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

Instructions

Choose the response that is correct for the question.

A correct answer scores 1, an incorrect answer scores 0.

Marks will not be deducted for incorrect answers.

No marks will be given if more than one answer is completed for any question.

Unless otherwise indicated, the diagrams are **not** drawn to scale.

EXAMINATION 1 – continued

Question 1

An arc subtends an angle of 72° at the centre of a circle of radius 14.5 cm.

The length of the arc correct to the nearest centimetre is

A. 36 cm **B.** 29 cm **C.** 264 cm **D.** 18 cm **E.** 9 cm

Question 2

A plane must fly between two places on the Earth (of radius 6400 km) with the same meridian of longitude. The shortest distance between A with coordinates 35° N 110° W and B with coordinates 15°S 110° W to the nearest kilometre is

A. 5585 km **B.** 12 287 km **C.** 2234 km

D. 889 km **E.** 622 km

Question 3

The time in Port Moresby (9° S 147° E) when it is 9 a.m. on Saturday in New York City (41° N 75° W) is

A. 7 p.m. Friday **B.** 12 a.m. Sunday **C.** 5 p.m. Friday

D. 11 p.m. Saturday **E.** 6 p.m. Friday

Question 4

Paddy lives in Dublin (53° N 6° W) and he calls a friend in Melbourne (38° S 145° E). He tells his friend to expect a call at 7:00 p.m. on Saturday night (Melbourne time) and he talks on the phone for 1 hour. The time in Dublin when Paddy finishes the call is

A. 6 a.m. Sunday **B.** 10 a.m. Saturday **C.** 5 a.m. Saturday

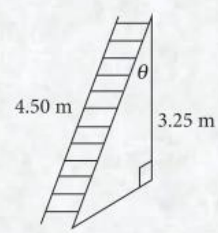
D. 11 a.m. Saturday **E.** 9 a.m. Saturday

Question 5

The top of a ladder that is 4.50 m long rests 3.25 m up a wall, as shown in the diagram. The angle, θ, that the ladder makes with the wall is closest to

A. 36° **B.** 44° **C.** 46°

D. 50° **E.** 54°

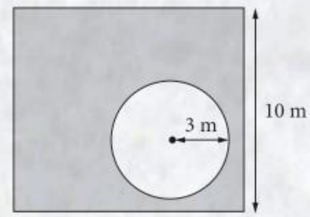


[VCAA 2014 1GTQ1]

EXAMINATION 1 – continued

Question 6

A circular pool is located in a square lawn, as shown. The sides of the square lawn are 10 m in length. The pool has a radius of 3 m. The area of the lawn surrounding the pool, in square metres, is closest to

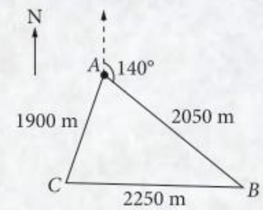


A. 21 **B.** 59 **C.** 72
D. 81 **E.** 128

[VCAA 2014 1GTQ2]

Use the following information to answer Questions 7 & 8.

A cross-country race is run on a triangular course.
The points A, B and C mark the corners of the course, as shown.
The distance from A to B is 2050 m.
The distance from B to C is 2250 m.
The distance from A to C is 1900 m.
The bearing of B from A is 140°.



Question 7

The bearing of C from A is closest to

A. 032° **B.** 069° **C.** 192° **D.** 198° **E.** 209°

[VCAA 2014 1GTQ6]

Question 8

The area within the triangular course ABC, in square metres, can be calculated by evaluating

A. $\sqrt{3100\times1200\times1050\times850}$

B. $\sqrt{3100\times2250\times2050\times1900}$

C. $\sqrt{6200\times4300\times4150\times3950}$

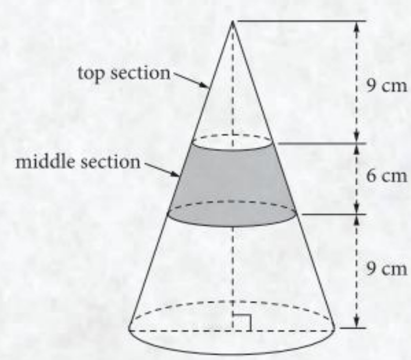
D. $\frac{1}{2}\times2050\times2250\times\sin(140°)$

E. $\frac{1}{2}\times2050\times2250\times\sin(40°)$

[VCAA 2014 1GTQ7]

Question 9

The middle section of a cone is shaded, as shown in the diagram. The surface area of the unshaded top section of the cone is 180 cm^2. The surface area of the middle section of the cone, in square centimetres, is



A. 80 **B.** 120 **C.** 300
D. 320 **E.** 500

[VCAA 2014 1GTQ9]

EXAMINATION 1 – ended

GEOMETRY AND MEASUREMENT

Examination 2

Reading time: (5 minutes)
Writing time: (25 minutes)

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers, one bound reference, one approved graphics calculator or approved CAS calculator or CAS software and, if desired, one scientific calculator. Calculator memory DOES NOT need to be cleared.
- Students are NOT permitted to bring into the examination room: blank sheets of paper and/or white out liquid/tape.
- Students may refer to the sheet of miscellaneous formulas supplied.
- All written responses must be in English.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

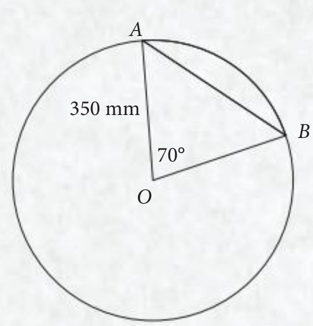
Instructions

You need not give numerical answers as decimals unless instructed to do so. Alternative forms may involve, for example, π, surds or fractions.

Diagrams are not to scale unless specified otherwise.

Question 1

The arc of a circle with a radius of 350 mm subtends an angle of 70° at the centre.



a. Find the length of the arc in millimetres correct to two decimal places. 1 mark

The chord AB divides the circle into two segments.

b. Find the area of the major segment correct to the nearest square millimetre. 1 mark

EXAMINATION 2 – continued

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Question 2

The coordinates of the city of Ballarat, Victoria, are 37.55° S 143.85° E.

a. What is the latitude of Ballarat? 1 mark

b. Find the shortest distance of Ballarat to the equator, to the nearest kilometre, using the radius of the Earth as 6400 km. 1 mark

c. Find the radius of the parallel of latitude for Ballarat. 1 mark

Question 3

Andrew catches a plane from Tokyo with coordinates 35.7° N 139.7° E at 3:30 p.m. on Monday afternoon bound for Beijing with coordinates 39.9° N 116.4° E.

a. Find the number of hours ahead or behind Greenwich Mean Time (GMT) for the city of Tokyo. 1 mark

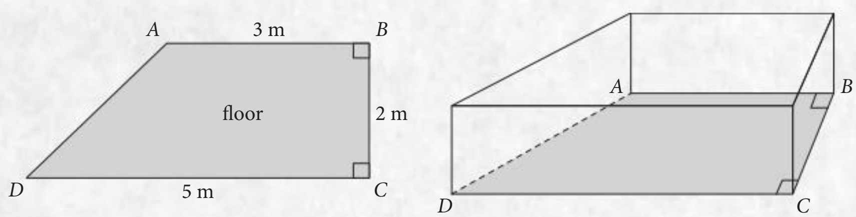
b. Find the time in Beijing when Andrew catches the plane. 1 mark

c. If Andrew takes a direct flight that takes 3 hours, at what time and day will Andrew arrive in Beijing? 1 mark

Question 4

The floor of a chicken coop is in the shape of a trapezium.

The floor, *ABCD*, and the chicken coop are shown.



$AB = 3$ m, $BC = 2$ m and $CD = 5$ m.

a. What is the area of the floor of the chicken coop?
Write your answer in square metres. 1 mark

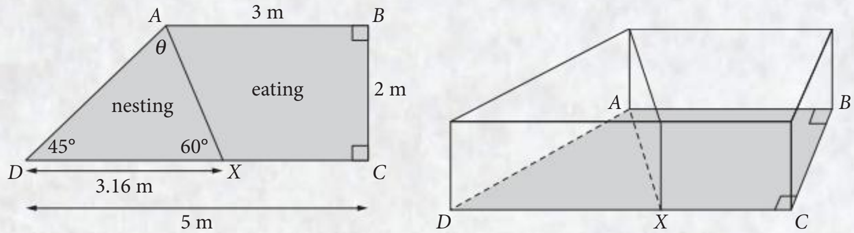
b. What is the perimeter of the floor of the chicken coop?
Write your answer in metres, correct to one decimal place. 1 mark

[VCAA 2014 2GTQ1]

EXAMINATION 2 – continued

Question 5

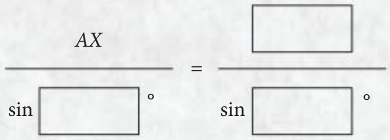
The chicken coop has two spaces, one for nesting and one for eating. The nesting and eating spaces are separated by a wall along the line AX, as shown in the diagrams.



$DX = 3.16$ m, $\angle ADX = 45°$ and $\angle AXD = 60°$.

a. Write down a calculation to show that the value of θ is 75°. 1 mark

b. The sine rule can be used to calculate the length of the wall AX.
Copy and complete.



1 mark

c. What is the length of AX?
Write your answer in metres, correct to two decimal places. 1 mark

d. Calculate the area of the floor of the nesting space, ADX.
Write your answer in square metres, correct to one decimal place. 1 mark

[VCAA 2014 2GTQ2]

Question 6

One of the chickens escapes into a neighbouring field through an open gate. The chicken's owner is 50 m due north of the gate, searching for the chicken. The chicken is 40 m from the gate on a bearing of 295°. What is the bearing of the chicken from its owner?

Write your answer correct to the nearest degree. 2 marks

[VCAA 2014 2GTQ4]

EXAMINATION 2 – ended

CHAPTER 13

SPHERICAL GEOMETRY

13.1 Circle measurement
Arc length
Area of a sector
Area of a segment
Arc length on a sphere
13.2 Latitude and longitude
Great circles and meridians of longitude
Small circles and parallels of latitude
Locating places on the Earth using latitude and longitude
13.3 Shortest distance problems
Distances between points with the same longitude
Distances between points with the same latitude
13.4 Time zones
Time zones and longitude
Summary

13.1 Circle measurement

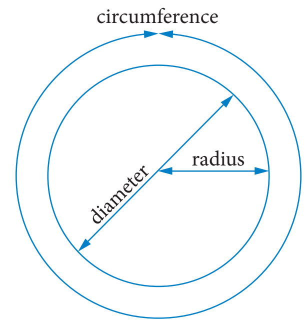
Arc length

The circumference is the distance around a circle.

$C = \pi d$ or $C = 2\pi r$

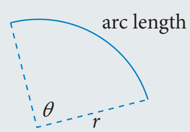
where C is the circumference, d is the diameter and r is the radius.

The **arc length** is a fraction of the circumference. In an arc that subtends an angle of θ at the centre, the fraction is $\frac{\theta}{360}$.



The **arc length** l can be determined by the formula

$$l = \frac{\theta}{360} \times 2\pi r$$



where r is the radius and θ is the angle in degrees subtended by the arc at the centre.

Worked example 1

Find the length of arc that subtends an angle of 65° at the centre of a circle with a radius of 15 cm. Express the answer in centimetres, correct to three decimal places.

	Working
1 Write the formula.	$l = \frac{\theta}{360} \times 2\pi r$
2 Substitute $r = 15$ and $\theta = 65$.	$= \frac{65}{360} \times 2\pi \times 15$
Calculate the answer.	≈ 17.017 cm

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Worked example 2

An arc of length 28.5 cm subtends an angle of θ at the centre of a circle with a diameter of 42 cm. Find the value of θ correct to the nearest degree.

Working

1 Substitute $l = 28.5$ and $r = 21$ into the formula.

Diameter = 42 cm
Radius = 42 ÷ 2 = 21

$$28.5 = \frac{\theta}{360} \times 2\pi \times 21$$

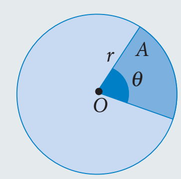
2 Solve, using a CAS/calculator if necessary.

$\theta \approx 78°$

Area of a sector

The area of a circle with radius r is $A = \pi r^2$

The area of a **sector** is the fraction $\frac{\theta}{360}$ of the circle area and is given by the formula $A = \frac{\theta}{360}\pi r^2$, where θ is the angle subtended by the arc at the centre.



Worked example 3

Find the area of a sector with a radius of 35 cm that subtends an angle of 105° at the centre. Express the answer in square centimetres, correct to two decimal places.

Working

1 Write the formula.

$$A = \frac{\theta}{360}\pi r^2$$

2 Substitute $r = 35$ and $\theta = 105$.

$$= \frac{105}{360} \times \pi \times 35^2$$

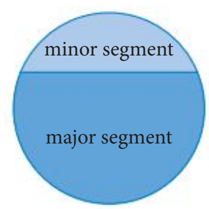
Calculate the answer.

$$\approx 1122.46 \text{ cm}^2$$

Alamy/Aerial Archives

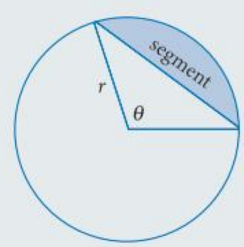
Area of a segment

A **segment** of a circle is the area bound by a chord and the arc subtended by the chord. A chord will produce two segments in a circle: a **minor segment** is the smaller segment and a **major segment**, which is the larger segment.



The area of a segment is given by the formula $A = \frac{\theta}{360}\pi r^2 - \frac{1}{2}r^2 \sin(\theta)$

where θ is the angle in degrees subtended by the chord at the centre.



Exam hack

Make sure you set the calculator mode to degrees when calculating the area of a segment.

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Worked example 4

Find the area of a segment of a circle with a radius of 50 cm if the chord subtends an angle of 54° at the centre. Express the answer in square centimetres correct to two decimal places.

Working

1 Write the formula.

$$A = \frac{\theta}{360}\pi r^2 - \frac{1}{2}r^2\sin(\theta)$$

2 Substitute $r = 50$ and $\theta = 54$ into the formula.

$$= \frac{54}{360}\pi \times 50^2 - \frac{1}{2}\times 50^2\sin(54°)$$

$$\approx 166.83 \text{ cm}^2$$

Worked example 5

The segment of a circle with a radius of r cm has an area of 353.77 cm^2. Find the length of the arc in the segment if the angle subtended at the centre by the arc is 120°. Express the answer correct to three decimal places.

Working

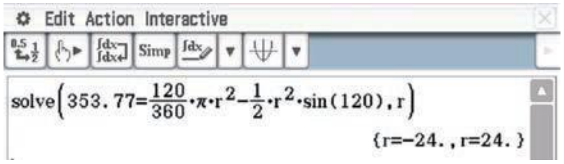
1 Write the formula for the segment area.

$$A = \frac{\theta}{360}\pi r^2 - \frac{1}{2}r^2\sin(\theta)$$

2 Substitute $A = 353.77$ and $\theta = 120°$ into the segment area formula to find r.

$$353.77 = \frac{120}{360}\pi \times r^2 - \frac{1}{2}\times r^2\sin(120°)$$

3 Use a CAS/calculator and solve for r.



$r = 24$ (r must be positive)

4 Substitute into the arc length formula.

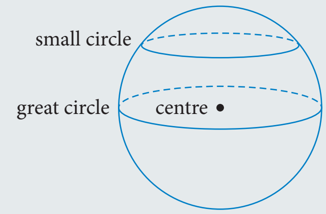
$$l = \frac{\theta}{360}\times 2\pi r$$

$$= \frac{120}{360}\times 2\pi \times 24$$

$$\approx 50.265 \text{ cm}$$

Arc length on a sphere

A **great circle** is the largest circle that can be drawn on a sphere. The centre of a great circle is the centre of the sphere. The centre of a **small circle** isn't the same as the centre of the sphere. The shortest distance between two points on the surface of a sphere occurs on a great circle.



Worked example 6

Find the shortest distance between two points on a sphere of radius 50 cm that subtend an angle of 100° at the centre. Express the answer correct to two decimal places.

	Working
1 Write the arc length formula.	$l = \frac{\theta}{360} \times 2\pi r$
2 The shortest distance is the great circle distance. This is on an arc with a radius of 50 cm subtending an angle $\theta = 100°$. Substitute into the formula.	$= \frac{100}{360} \times 2\pi \times 50$ ≈ 87.27 cm

EXAM PREP 13.1

Circle measurement

Prep 1 WORKED EXAMPLE 1

Find the arc length of a circle with a radius of 60 cm that subtends an angle of 40° at the centre. Express the answer in centimetres, correct to three decimal places.

Prep 2 WORKED EXAMPLE 2

An arc of length 145 cm subtends an angle of θ° at the centre of a circle with a diameter of 200 cm. Find the value of θ correct to the nearest tenth of a degree.

Prep 3 WORKED EXAMPLE 3

Find the area of a sector with a radius of 6.5 cm that subtends an angle of 75° at the centre. Express the answer in square centimetres correct to two decimal places.

Prep 4 WORKED EXAMPLE 4

Find the area of a segment of a circle with a radius of 220 cm if the chord subtends an angle of 150° at the centre. Express the answer in square centimetres correct to one decimal place.

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Prep 5

WORKED EXAMPLE 5

The arc length of a circle of radius 12.5 cm is 9.5 cm. Find the area of the corresponding sector correct to two decimal places.

Prep 6

WORKED EXAMPLE 6

Find the shortest distance between two points on a sphere of radius 300 cm that subtend an angle of 80° at the centre. Express the answer correct to two decimal places.

13.1

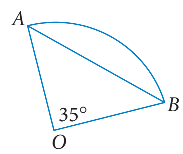
EXAM PRACTICE 13.1

Circle measurement

Question 1

The area of triangle OAB is 964.756 cm^2.

The radius of the sector is closest to



A 56 cm **B** 50 cm **C** 63 cm

D 54 cm **E** 58 cm

Question 2

The minor segment within a circle of radius 36 m has an area of 50.733 m^2. If the segment subtends an angle of $\theta°$ at the centre, the value of $\theta°$ is closest to

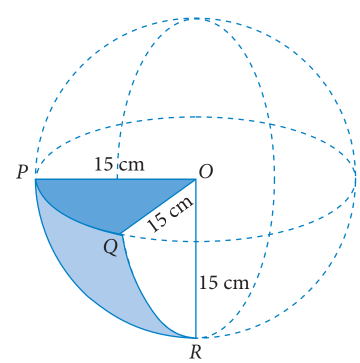
A 100° **B** 50° **C** 45° **D** 32° **E** 115°

Question 3

The solid $OPQR$, as shown, is one-eighth of a sphere of radius 15 cm.

The point O is the centre of the sphere and the points P, Q and R are on the surface of the sphere. $\angle POQ = \angle QOR = \angle ROP = 90°$

The arc length PQ, in cm, is closest to



A 48 **B** 24 **C** 15

D 54 **E** 177

Question 4

A circle has a circumference of 10 cm. The radius of this circle is closest to

A 1.3 cm **B** 1.6 cm **C** 1.8 cm **D** 3.2 cm **E** 5.0 cm

[VCAA 2010 1GTQ2]

Question 5

Find the arc length of a circle with a radius of 11 cm that subtends an angle of 64° at the centre. Express the answer in centimetres correct to three decimal places.

Question 6

The arc length of a circle is 105 cm. Find the circumference of the circle if the angle subtended by the arc is 60°.

Question 7

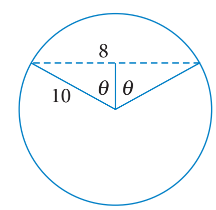
Calculate the shortest distance between two points on the Earth's surface that subtend an angle of 3° at the centre. Use 6400 km as the radius of the Earth and round your answer to the nearest metre.

Question 8

The minute hand of a clock has a length of 12 cm. Find the area the hand sweeps through in 40 minutes. Express your answer in cm^2 correct to one decimal place.

Question 9

A chord with a length of 8 cm subtends an angle of 2θ at the centre of a circle with a radius of 10 cm.



a Show that $\sin(\theta) = \dfrac{2}{5}$

b Use the answer in part **a** to find the area of the minor segment correct to two decimal places.

Latitude and longitude

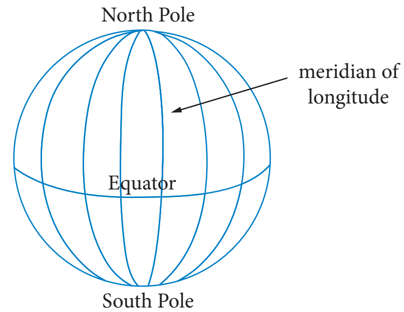
13.2

Great circles and meridians of longitude

The Earth is spherical with an approximate radius of 6400 km.

All circles that run down the Earth's surface and pass through both the North and South Poles are great circles. Each half of a great circle joining the North and South Poles is a **meridian of longitude**.

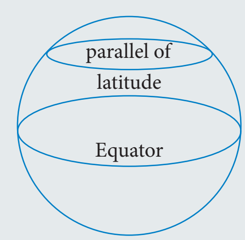
The **Greenwich meridian** (or prime meridian) has a longitude of 0° and passes through the Greenwich Observatory in London. All other meridians of longitude are measured in degrees east or west of the Greenwich meridian.

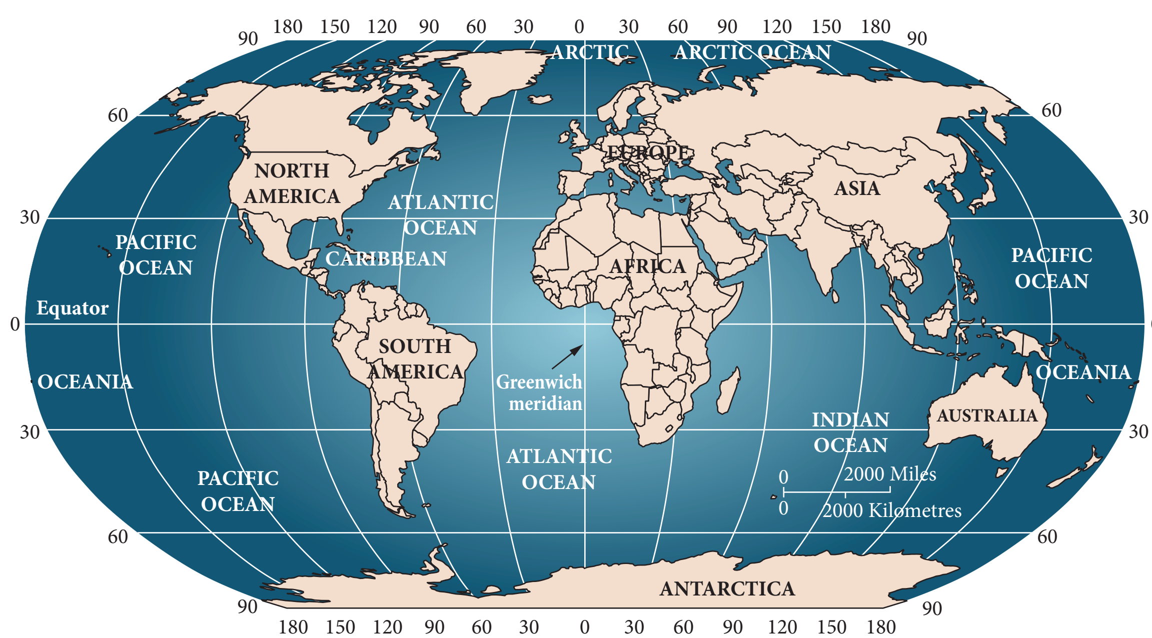


Small circles and parallels of latitude

The **Equator** is a great circle that runs across the Earth's surface.

Small circles parallel to the Equator are called **parallels of latitude**. The Equator has a latitude of 0°. All other parallels of latitude are measured in degrees north or south of the Equator.





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WS
Map of the world

Locating places on the Earth using latitude and longitude

Latitude and longitude is a coordinate system that is used to locate any position on the globe.

The **coordinates** for Perth are 32° S 116° E, which is on a parallel of latitude 34° south of the Equator and a meridian of longitude 116° east of the Greenwich meridian.

Coordinates are written in the order: angle of latitude N or S, angle of longitude E or W.

Shutterstock.com/Artist1704

Worked example 7

WS Australian coordinates

13.2

Identify the places on the map of Australia below that have the following coordinates

a 26° S 133° E **b** 17° S 145° E

Working

a **1** Use the parallels of latitude to estimate the 26° S parallel.

The 26° S parallel of latitude passes close to places including Carnarvon, Uluru (Ayers Rock), Kulgera and Bundaberg.

2 Use the meridians of longitude to estimate the 133° E meridian.

Kulgera (NT) has the coordinates 26° S 133° E.

b **1** Use the parallels of latitude to estimate the 17° S parallel.

The 17° S parallel passes close to places including Derby and Cairns.

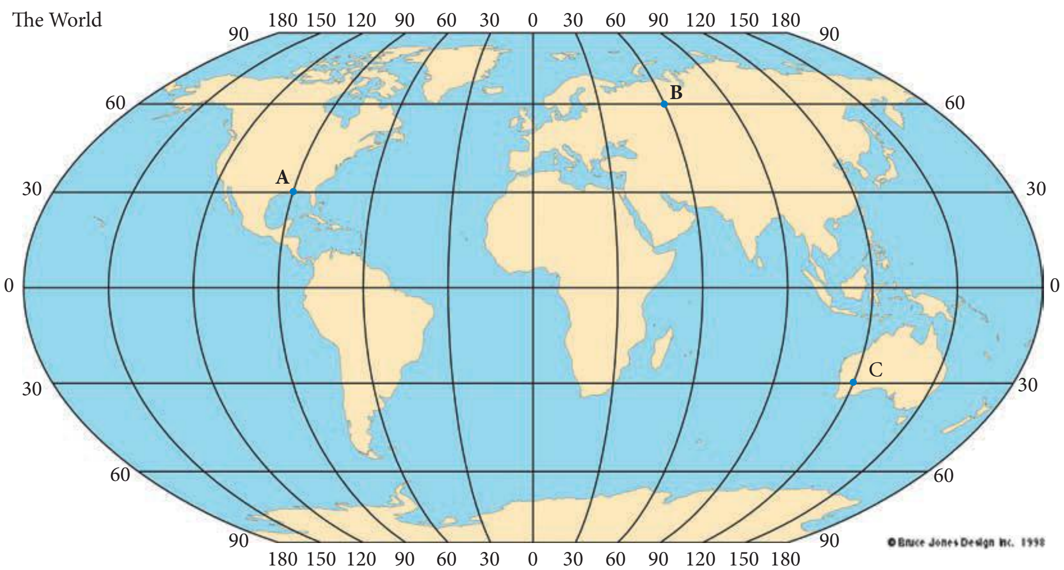
2 Use the meridians of longitude to estimate the 145° E meridian.

Cairns (Qld) has the coordinates 17° S 145° E.

WS
Latitude and longitude

Worked example 8

Write the coordinates of points A, B and C shown on the world map below.



Alamy/Simon Brandt

			Working
A	**1**	A is on the 30° parallel of latitude north of the Equator.	Latitude = 30° N
	2	A is on the 90° meridian of longitude west of the Greenwich meridian.	Longitude = 90° W
	3	Write the coordinates for point A in the order latitude then longitude.	Coordinates of A = 30° N 90° W
B	**1**	B is on the 60° parallel of latitude north of the Equator.	Latitude = 60° N
	2	B is on the 60° meridian of longitude east of the Greenwich meridian.	Longitude = 60° E
	3	Write the coordinates for point B.	Coordinates of B = 60° N 60° E
C	**1**	C is on the 30° parallel of latitude south of the Equator.	Latitude = 30° S
	2	C is on the 120° meridian of longitude east of the Greenwich meridian.	Longitude = 120° E
	3	Write the coordinates for point C.	Coordinates of C = 30° S 120° E

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EXAM PREP 13.2

Latitude and longitude

Prep 1 WORKED EXAMPLE 7

Identify places on the map of Australia on page 327 that have the coordinates

a 25° S 114° E **b** 33° S 152° E **c** 18° S 122° E

Prep 2 WORKED EXAMPLE 8

Use the map of Australia on page 327 to find the approximate coordinates of:

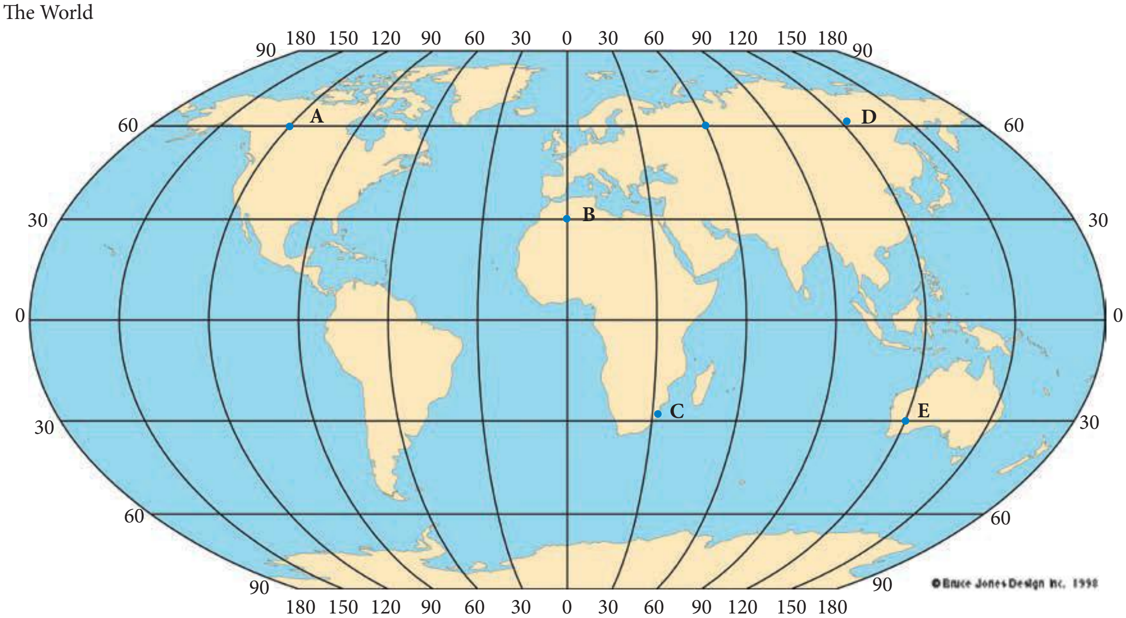
a Port Augusta **b** Cunnamulla **c** Port Hedland

EXAM PRACTICE 13.2

Latitude and longitude

Use the following information to answer Questions 1 – 4.

The World



Alamy/Simon Brandt

Question 1

The coordinates of point A are

A 60° N 120° W **B** 60° N 120° E **C** 60° S 120° E

D 60° S 120° W **E** 120° N 60° E

Question 2

The coordinates of point D are

A 60° N 120° E
B 60° S 120° W
C 60° N 120° W
D 60° S 120° E
E 120° N 60° W

Question 3

Which of the following statements is not true of point E?

A Point E is 30° south of the Equator.

B Point E is 120° east of the Greenwich meridian.

C The coordinates of point E are 30° S 120° E.

D Point E has the same meridian of longitude as point D.

E Point E has the same parallel of latitude as point D.

Question 4

The point that is on the Greenwich meridian is

A point B
B point C
C point E
D point D
E point A

Question 5

An aeroplane starts its flight at a position 19° S 125° W and flies 40° due north along the meridian of longitude. The location of the plane's final position is

A 59° N 125° W
B 59° S 125° W
C 19° S 85° W
D 21° N 125° W
E 19° S 165° W

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Shortest distance problems

13.3

Distances between points with the same longitude

Points with the same longitude are on the same great circle, so the shortest distance between two points is the great circle distance calculated on the meridian of longitude.

To calculate the distance between two points with the same longitude:

1 Substitute into the formula $l = \frac{\theta}{360} \times 2\pi r$ where $r = 6400$ km.

2 If the points are on the same side of the Equator, θ is the difference between the angles of latitude.

3 If the points are on different sides of the Equator, θ is the sum of the angles of latitude.

When calculating the shortest distance to the Equator or the Poles, use:

- the Equator has a parallel of latitude of 0°
- the North Pole has a parallel of latitude of 90° N
- the South Pole has a parallel of latitude of 90° S.

Worked example 9

Find the shortest distance, to the nearest kilometre, between point A with coordinates 32°S 120° W and point B with coordinates 15°N 120° W.

	Working
1 Write the formula for a great circle distance where $r = 6400$ km, as the points have the same longitude.	$l = \frac{\theta}{360} \times 2\pi \times 6400$
2 The points are on different sides of the Equator (south and north), so add the angles to find the angle between the points.	$\theta = 32 + 15 = 47$
3 Substitute into the formula and evaluate.	$l = \frac{47}{360} \times 2\pi \times 6400$ ≈ 5250 km

Worked example 10

Find the shortest distance, to the nearest kilometre, between a point with coordinates 55° S 110° E and the Equator.

Working

1 Write the formula for a great circle distance. $l = \frac{\theta}{360} \times 2\pi \times 6400$

2 Find the angle between the point and the Equator. $\theta = 55$

3 Substitute into the formula and evaluate. $l = \frac{55}{360} \times 2\pi \times 6400 \approx 6144$ km

Worked example 11

Find the shortest distance, to the nearest kilometre, between a point with coordinates 55°S 150° W and the North Pole.

Working

1 Write the formula for a great circle distance. $l = \frac{\theta}{360} \times 2\pi \times 6400$

2 The points are on different sides of the Equator (south and north), so add the angles to find the angle between the points. $\theta = 55 + 90 = 145$

3 Substitute into the formula and evaluate. $l = \frac{145}{360} \times 2\pi \times 6400 \approx 16\,197$ km

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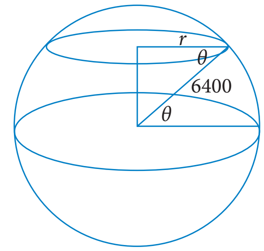
Distance between points with the same latitude

To determine the distance between points along a parallel of latitude, calculate the radius of the small circle first and then find the arc length for the small circle. From the diagram



$$\cos(\theta) = \frac{r}{6400}$$

$$r = 6400 \times \cos(\theta)$$

The radius, r, of a parallel of latitude is given by $r = 6400 \times \cos(\theta)$, where θ is the angle of latitude.

The distance, l, along the parallel is $l = \frac{\alpha}{360} \times 2\pi r$, where α is the angle between the two points along the parallel of latitude.

Worked example 12

Find the distance, to the nearest kilometre, along the parallel of latitude between point A with coordinates 35° S 30° E and point B with coordinates 35° S 150° E.

Working

1. Write the formula for the radius of the parallel of latitude. $r = 6400 \times \cos(\theta)$
2. Substitute $\theta = 35°$. $= 6400 \times \cos(35°)$ ≈ 5242.57
3. Calculate the angle between 30° E and 150° E. $\alpha = 150 - 30$ $= 120$
4. Write the formula for the distance along the parallel. $l = \frac{\alpha}{360} \times 2\pi r$
5. Calculate the distance along the 35° S parallel. $= \frac{120}{360} \times 2\pi \times 5242.57$ $\approx 10\,980$ km

EXAM PREP 13.3

Shortest distance problems

Prep 1 WORKED EXAMPLE 9

Find the shortest distance, to the nearest kilometre, between Point A, with approximate coordinates 32° N 30° E, and Point B, with approximate coordinates 21° N 30° E.

Prep 2 WORKED EXAMPLE 9

Find the shortest distance, to the nearest kilometre, between Point A, with approximate coordinates 18° S 15° E, and Point B, with approximate coordinates 56° N 15° E.

Prep 3 WORKED EXAMPLE 10

Find the shortest distance, to the nearest kilometre, between Virginia Beach, USA, with approximate coordinates 37° N 76° W, and the Equator.

Prep 4 WORKED EXAMPLE 11

Find the shortest distance, to the nearest kilometre, between Virginia Beach, USA, with approximate coordinates 37° N 76° W, and the North Pole.

Prep 5 WORKED EXAMPLE 12

Find the distance, to the nearest kilometre, along the parallel of latitude between Vienna, Austria, with approximate coordinates 48° N 16° E, and Paris, France, with approximate coordinates 48° N 2° E.

EXAMPLE PRACTICE 13.3

Shortest distance problems

Question 1

The distance along the meridian of longitude between Kiev, Ukraine, with approximate coordinates 50° N 30° E, and Kigali, Rwanda, with approximate coordinates 2° S 30° E is given by

A $l = \frac{48}{360} \times 2\pi \times 6400$ **B** $l = \frac{50}{360} \times 2\pi \times 6400$ **C** $l = \frac{52}{360} \times 2\pi \times 6400$

D $l = \frac{30}{360} \times 2\pi \times 6400$ **E** $l = \frac{20}{360} \times 2\pi \times 6400$

Question 2

The distance along the meridian, to the nearest kilometre, between Kiev, Ukraine, with approximate coordinates 50° N 30° E, and Alexandria, Egypt, with approximate coordinates 32° N 30° E is

A 2011 km **B** 5808 km **C** 3351 km **D** 5585 km **E** 40 212 km

Question 3

The distance along the meridian of longitude, to the nearest kilometre, between Vienna, Austria, with approximate coordinates 48° N 16° E, and the Equator is

A 1787 km **B** 15 415 km **C** 4691 km **D** 5362 km **E** 8266 km

Question 4

The shortest distance between point A, with coordinates 45° N 120° W, and point B, with coordinates 65° N 60° E, is given by

A $l = \frac{180}{360} \times 2\pi \times 6400$ **B** $l = \frac{70}{360} \times 2\pi \times 6400$ **C** $l = \frac{45}{360} \times 2\pi \times 6400$

D $l = \frac{65}{360} \times 2\pi \times 6400$ **E** $l = \frac{60}{360} \times 2\pi \times 6400$

13.3

Question 5

The shortest distance, to the nearest tenth of a kilometre, between San Diego, California, with approximate coordinates 33° N 117° E, and the North Pole is

A 3686.1 km **B** 13 739.2 km **C** 13 069.0 km **D** 3015.9 km **E** 6367.0 km

Question 6

The shortest distance between San Diego, California, with approximate coordinates 33° N 117° E, and the South Pole is given by

A $l = \frac{57}{360} \times 2\pi \times 6400$ **B** $l = \frac{33}{360} \times 2\pi \times 6400$ **C** $l = \frac{27}{360} \times 2\pi \times 6400$

D $l = \frac{117}{360} \times 2\pi \times 6400$ **E** $l = \frac{123}{360} \times 2\pi \times 6400$

Question 7

Find the shortest distance along the meridian of longitude, to the nearest kilometre, between Kiev, Ukraine, with approximate coordinates 50° N 30° E and Zurich, Switzerland, with approximate coordinates 21° N 30° E.

Question 8

A plane flies on a path along the parallel of latitude between point A, with coordinates 64° S 31° E, and point B, with coordinates 64° S 150° E. Find

a the radius of the parallel of latitude, to the nearest metre

b the distance travelled by the plane, to the nearest kilometre.

Question 9

Find the shortest distance, to the nearest kilometre, between Moscow, Russia with coordinates 55.75° N 37.61° E and

a the Equator **b** the North Pole **c** the South Pole

13.4 Time zones

Time zones and longitude

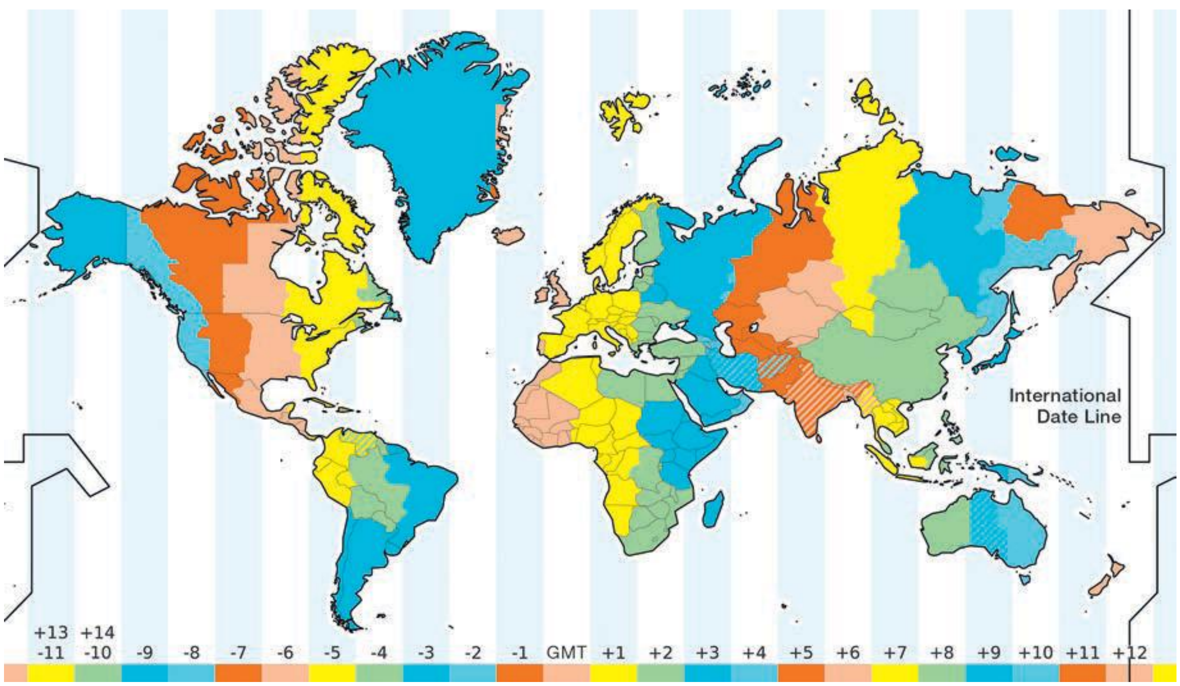
WS Time differences

WS Table of time zones

WS Map of the world

Time zones are regions that observe the same time of day. The reference time zone is **Greenwich Mean Time** (**GMT**), which has the Greenwich meridian as its centre. Because the Earth rotates from west to east (the Sun rises from the east), anywhere east of the Greenwich meridian is ahead of GMT and anywhere west is behind GMT. While the abbreviation GMT is commonly used to identify different time zones, the official abbreviation for the time standard is UTC, which stands for **Coordinated Universal Time**.

Although time zones are based on longitude, as can be seen on the world time zone map below, they are affected by political and geographical factors. We will also be ignoring daylight saving-type time changes made by many countries.



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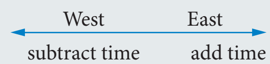
Greenwich Mean Time (GMT) is the time zone used as a reference for other time zones.

Since the Earth rotates 360° from west to east in 24 hours (and $360 \div 24 = 15$), we can think of times zones as:

15° difference in longitude = 1 hour difference in time zones.

- To find the time east of the current location, add the time difference.
- To find the time west of the current location, subtract the time difference.



The **International Date Line** is 180° east or 180° west of the Greenwich meridian. If you cross this line in the easterly direction, you will need to subtract a day, while crossing in the westerly direction you will need to add a day.

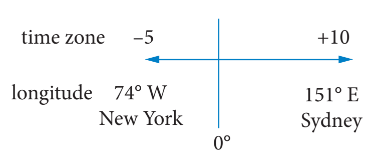
If you are travelling east, the addition of an hour for every 15 degrees of longitude adds up to 24 hours by the time you get back to where you started. This compensates for the day that you subtracted when you crossed the International Date Line. The reverse of this happens when you travel west. The 24 hours you subtract compensates for the day that you added when you crossed the International Date Line.

Exam hack

In practice, flying any significant distance east from Australia (e.g. to the USA) involves crossing the International Date Line, while flying west from Australia (e.g. to Europe) doesn't.

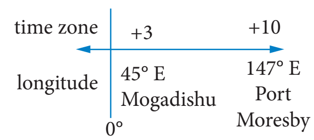
Worked example 13

Find the time in Sydney, 34° S 151° E, when it is 10 p.m. on Tuesday in New York City, 41° N 74° W.

	Working	
1 Divide the longitudes by 15.	Sydney	$151 \div 15 \approx 10.1$ east
	New York City	$74 \div 15 \approx 4.9$ west
2 Allocate a time zone value by rounding.	Sydney	+10
East is positive. West is negative.	New York City	–5
3 Find the time difference. Draw an east–west line.		
	Time difference = $10 - (-5) = 15$ hours	
4 As Sydney is east of New York City, add 15 hours to the New York City time.	Time in Sydney = 10 p.m. Tuesday + 15 h = 1 p.m. Wednesday	

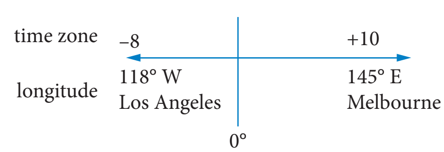
Worked example 14

Find the time in Port Moresby, 9° S 147° E, when it is 7 a.m. on Saturday in Mogadishu, 2° N 45° E.

	Working
1 Divide the longitudes by 15.	Port Moresby $147 \div 15 \approx 9.8$ east Mogadishu $45 \div 15 \approx 3$ east
2 Allocate a time zone value. East is positive. West is negative.	Port Moresby $+10$ Mogadishu $+3$
3 Find the time difference. Draw an east–west line.	 Time difference is $10 - 3 = 7$ hours
4 As Port Moresby is east of Mogadishu, add 7 hours to the Mogadishu time.	Time in Port Moresby = 7 a.m. Saturday + 7 h = 2 p.m. Saturday

Worked example 15

A plane leaves Los Angeles, 34° N 118° W, at 11:00 p.m. on Sunday and flies west across the International Date Line to Melbourne, 38° S 145° E. If the flight takes 14 hours, find the arrival time in Melbourne.

	Working
1 Divide the longitudes by 15.	Los Angeles $118 \div 15 \approx 7.9$ west Melbourne $145 \div 15 \approx 9.7$ east
2 Allocate a time zone value. East is positive. West is negative.	Los Angeles -8 Melbourne $+10$
3 Find the time difference. Draw an east–west line.	 Time difference $= 10 - (-8) = 18$ hours
4 As Melbourne is west of Los Angeles, subtract 18 hours from the Los Angeles time. Since the plane is travelling west across the International Date Line, we need to add a day.	Time in Melbourne = 11 p.m. Sunday – 18 h + 1 day = 5 a.m. Sunday + 1 day = 5 a.m. Monday
5 Add the 14 hour flight time onto the converted Melbourne time.	Flight arrival time = 5 a.m. Monday + 14 h = 7 p.m. Monday

EXAM PREP 13.4

Time zones

Prep 1 WORKED EXAMPLE 13

Calculate the time zone for

a Tweed Heads, 28° S 154° E **b** Shanghai, 31° N 121° E **c** Tampa, 28° N 82° W

Prep 2 WORKED EXAMPLE 13

Find the time in Perth, 32° S 116° E, when it is 3 p.m. on Thursday in Dallas, 33° N 97° W.

Prep 3 WORKED EXAMPLE 14

Find the time in Vienna, 48° N 16° E, when it is 5 p.m. on Saturday in New Delhi, 29° N 77° E.

Prep 4 WORKED EXAMPLE 15

A plane leaves Tokyo, 36° N 140° E, at 6:00 p.m. on Friday, and flies east across the International Date Line to Montreal, 46° N 74° W. If the journey takes 20 hours, find the arrival time in Montreal.

EXAM PRACTICE 13.4

Time zones

Question 1

The time in Perth, 32° S 116° E, when it is 9 a.m. on Thursday in New York City, 41° N 74° W, can be calculated using the equation

A New York City time = Perth time – 14
B New York City time = Perth time + 15
C New York City time = Perth time – 13
D New York City time = Perth time – 15
E New York City time = Perth time + 14

Question 2

The time in Tokyo, 36° N 140° E, when it is 7 a.m. on Wednesday in Florence, 44° N 11° E, is

A 10 p.m. Tuesday **B** 11 p.m. Tuesday **C** 4 p.m. Wednesday
D 3 p.m. Wednesday **E** 2 p.m. Wednesday

Question 3

The time in Montreal, 46° N 74° W, when it is 2 a.m. on Saturday in Glasgow, 56° N 4° W, is

A 9 p.m. Friday **B** 7 a.m. Saturday **C** 2 a.m. Saturday
D 10 p.m. Friday **E** 6 a.m. Saturday

Use the following information to answer Questions 4 & 5.

A plane leaves Melbourne, 38° S 145° E, at 10:00 a.m. on Monday, and flies to San Juan, Puerto Rico, 18° N 66° W.

Question 4

The time in San Juan when the flight leaves is

A 10 a.m. Monday **B** 9 p.m. Sunday **C** 12 a.m. Tuesday

D 11 p.m. Monday **E** 8 p.m. Sunday

Question 5

If the flight takes 19 hours, the arrival time in San Juan is

A 4 p.m. Monday **B** 3 p.m. Monday **C** 5 a.m. Tuesday

D 6 a.m. Tuesday **E** 7 p.m. Tuesday

Question 6

Adam catches a plane that leaves Perth, 32° S 116° E, bound for Paris where the time zone is 1 hour ahead of GMT. The flight will take 22 hours, which includes a stopover. If Adam wants to arrive in Paris at 8:00 a.m. on Monday morning, the time he will need to leave Perth is

A 4 p.m. Sunday **B** 12 a.m. Sunday **C** 8 p.m. Sunday

D 10 a.m. Sunday **E** 5 p.m. Sunday

Question 7

Stavros lives in Athens, 38° N 24° E, and he calls a friend in Dublin, 53° N 6° W, at 8:30 p.m. on Sunday night. If the call lasts for 1 hour, at what time in Dublin does the call end?

Question 8

Cindy catches a plane from Hobart, 43° S 147° E, at 5:30 p.m. on Wednesday night bound for Beijing, 40° N 116° E. If she is on a direct flight that takes 7 hours, at what time will she arrive?

Question 9

A plane leaves Sydney, 34° S 151° E, at 7:00 a.m. on Thursday and flies to Tampa, 28° N 82° W. Find the time in Tampa when the flight arrives if the flight takes 18 hours.

Spherical geometry

Crossword of the Earth

Practice quiz

SUMMARY

13

Circle measurement

- Circumference of a circle $C = \pi d$ or $2\pi r$.
- The arc length l is a fraction of the circumference and is given by the formula $l = \frac{\theta}{360} \times 2\pi r$, where r is the radius and θ is the angle in degrees subtended by the arc at the centre.
- Area of a circle $A = \pi r^2$.
- The area A of a sector is a fraction of the circle area and is given by the formula $A = \frac{\theta}{360}\pi r^2$, where θ is the angle subtended by the arc at the centre.
- The area of a segment is given by the formula $A = \frac{\theta}{360}\pi r^2 - \frac{1}{2}r^2\sin(\theta)$, where θ is the angle in degrees subtended by the chord at the centre.

Great circles and meridians of longitude

- A **great circle** on a sphere has a centre that is the centre of the sphere. The shortest distance between two points on the surface of a sphere occurs on a great circle.
- The Earth is spherical with an approximate radius of 6400 km.
- All circles that run down the Earth's surface and pass through both the North and South Poles are great circles. Each half of a great circle joining the North and South Poles is a **meridian of longitude**.
- The **Greenwich meridian** (or prime meridian) has a longitude of 0° and is the reference line for all other meridians of longitude, which are measured in degrees east or west of the Greenwich meridian.

Small circles or parallels of latitude

- The **Equator** is a great circle that has a latitude of 0° and runs across the Earth's surface.
- Small circles parallel to the Equator are called **parallels of latitude** and are measured in degrees north or south of the Equator.
- Coordinates are written in the order: angle of latitude N or S, angle of longitude E or W.

Distances between points with the same longitude

- Points with the same longitude are on the same great circle. The shortest distance between two points is the great circle distance calculated on the meridian of longitude.
- To calculate the distance between two points with the same longitude:

  1 Substitute into the formula $l = \frac{\theta}{360} \times 2\pi r$, where $r = 6400$ km.

  2 If the points are on the same side of the Equator, θ is the difference between the angles of latitude.

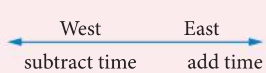
  3 If the points are on different sides of the Equator, θ is the sum of the angles of latitude.
- When calculating the shortest distance to the Equator or the Poles, use:
  - the Equator has a parallel of latitude of 0°
  - the North Pole has a parallel of latitude of 90° N
  - the South Pole has a parallel of latitude of 90° S

Distance between points with the same latitude

- The radius, r, of a parallel of latitude is given by $r = 6400 \times \cos(\theta)$, where θ is the angle of latitude.
- The distance, l, along the parallel is $l = \frac{\alpha}{360} \times 2\pi r$, where α is the angle between the two points along the parallel of latitude.

Time zones

- Greenwich Mean Time (GMT) is the time of day in the United Kingdom.
- 15° difference in longitude = 1 hour difference in time zones.
- To find the time east of the current location, add the time difference.
- To find the time west of the current location, subtract the time difference.



- If you cross the International Date Line in the easterly direction, you will need to subtract a day, while crossing in the westerly direction you will need to add a day.
- When calculating time zones, divide the longitudes by 15 and allocate a time zone value to the position. East is positive, west is negative.

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CHAPTER 14

CONSTRUCTION AND INTERPRETATION OF GRAPHS

14.1 Review of straight-line graphs

Linear equations and straight-line graphs

The slope of a straight line

The intercept–slope form of a straight line

Finding the equation of a line from any two points

Using CAS Finding the equation of a line from two points

14.2 Modelling with straight-line graphs

Linear modelling

Predictions, interpolation and extrapolation

Modelling rates

Modelling profit and loss

14.3 Line segment and step graphs

Drawing line segment linear graphs

Modelling with line segment linear graphs

Step graphs

14.4 Simultaneous equations

Graphing and solving simultaneous equations

Simultaneous equations without unique solutions

Using CAS: Solving simultaneous equations

Modelling with simultaneous equations

14.5 Break-even analysis

The break-even point

14.6 Interpreting non-linear graphs

14.7 Constructing non-linear graphs

Graphs of relations of the form $y = kx^n$

Linearisation

Summary

Prior learning

14.1 Review of straight-line graphs

Linear equations and straight-line graphs

WS
Linear graphs

A **linear equation** is one in which the **variables** involved are raised to the power 1.
The graph of a linear equation is a straight line.

Worked example 1

Sketch $4x + 3y = 8$ by hand by first finding the x- and y-intercepts.

Working

1 Write the equation.

$4x + 3y = 8$

Find the ***x*-intercept** by letting $y = 0$.

When $y = 0$, $4x + 0 = 8$

Solve the equation.

$x = 2$

The coordinates of the x-intercept are $(2, 0)$.

2 Find the ***y*-intercept** by letting $x = 0$.

When $x = 0$, $0 + 3y = 8$

Solve the equation.

$y = \frac{8}{3}$

$= 2\frac{2}{3}$

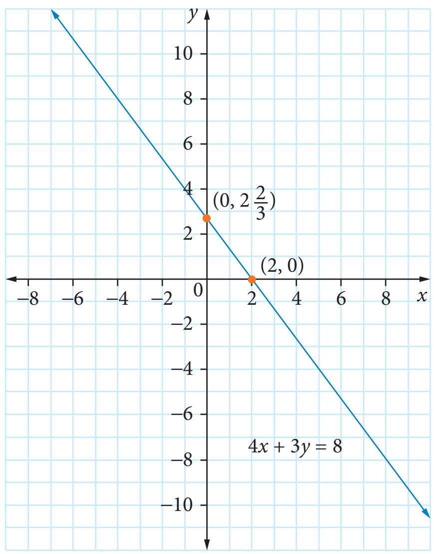
The coordinates of the y-intercept are $(0, 2\frac{2}{3})$.

3 Plot the intercepts.

Join them to draw the line.

Extend the line and place arrows on both ends of the line.

Label the graph with its equation.



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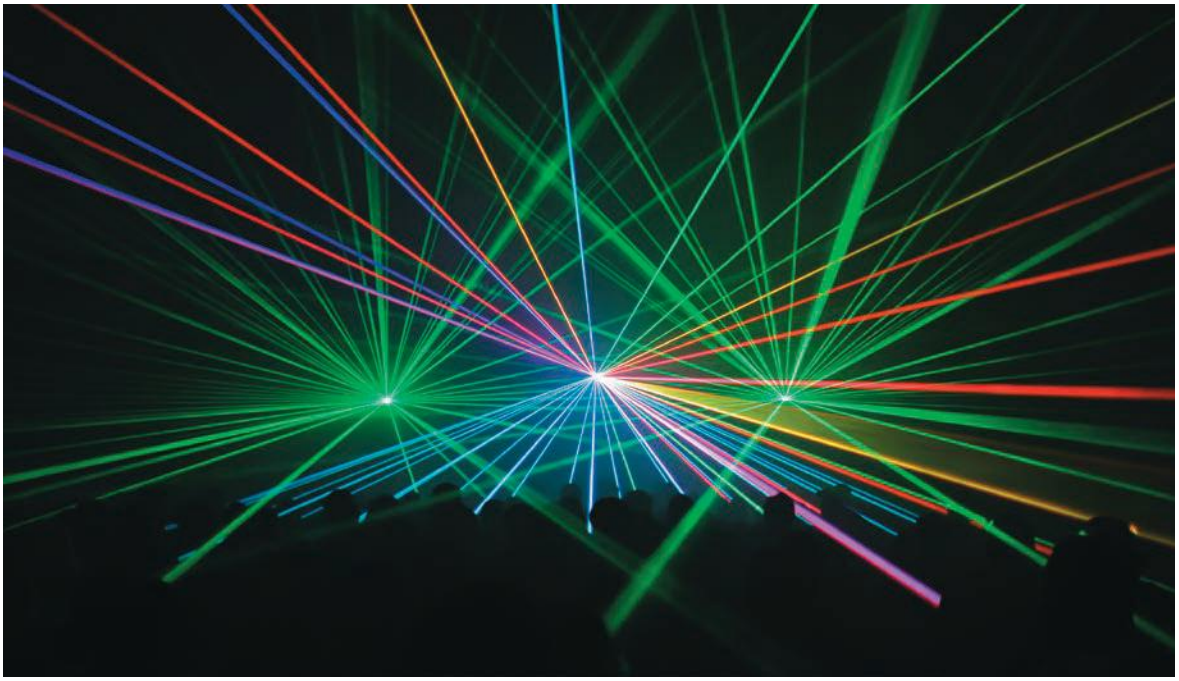


iStockPhoto/schaffert

14.1

Worked example 2

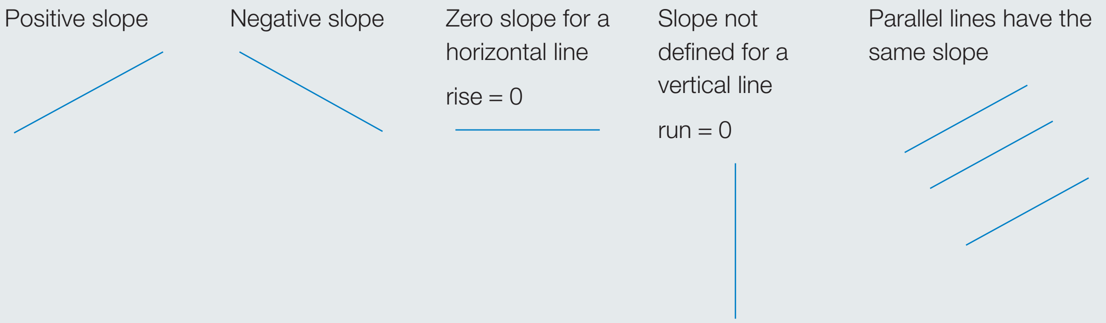
Determine if the point (–2, 3) lies on each line.

a $y = 2x + 7$ **b** $3x - y = 8$

	Working
a **1** Identify the x-coordinate and the y-coordinate of the point (–2, 3).	$x = -2$ and $y = 3$.
2 Substitute the values into the equation of the line and evaluate.	$y = 2x + 7$ $3 = 2 \times (-2) + 7$ $3 = -4 + 7$
3 State whether the equation is true or false and write the answer.	True. (–2, 3) lies on the line $y = 2x + 7$.
b **1** Identify the x-coordinate and the y-coordinate of the point (–2, 3).	$x = -2$ and $y = 3$.
2 Substitute the values into the equation of the line and evaluate.	$3x - y = 8$ $3 \times (-2) - 3 = 8$ $-6 - 3 = 8$
3 State whether the equation is true or false and write the answer.	False. (–2, 3) does not lie on the line $3x - y = 8$.

The slope of a straight line

- The **slope** of a line, also known as the **gradient**, is a measure of the steepness of a line.
- The slope of a line can be calculated using the formula Slope $= \dfrac{\text{rise}\uparrow}{\text{run}\rightarrow}$, where the rise is the vertical distance between two points and the run is the horizontal distance between the same two points.
- The slope measures the rate of change of y as x changes.



The intercept–slope form of a straight line

WS
Gradient and y-intercept

WS
Finding the gradient between two points on a line

- Slope $= \dfrac{y_2 - y_1}{x_2 - x_1}$, where (x_1, y_1) and (x_2, y_2) are any two points on the line.
- $y = a + bx$ is called the **intercept–slope form** of the equation of the straight line, where a = the y-intercept of the line and b = the slope of the line.

Exam hack

The formula Slope $= \dfrac{y_2 - y_1}{x_2 - x_1}$ can be used for *any* two points on the line. Choose the ones where the numbers are the easiest to calculate. It doesn't matter which point you call (x_1, y_1) and which one you call (x_2, y_2), but make sure that, once you decide, you get the values in the formula in the right order.

Worked example 3

Calculate the slope of the straight line drawn from the following table of values.

x	–2	0	2
y	16	6	–4

Working

1 Select two points from the table, say (0, 6) and (2, –4).
Let $(x_1, y_1) = (0, 6)$ and $(x_2, y_2) = (2, -4)$.

2 Write the formula.
$$\text{Slope} = \frac{y_2 - y_1}{x_2 - x_1}$$

3 Substitute these points into the formula and simplify.
$$\text{Slope} = \frac{-4-6}{2-0} = \frac{-10}{2} = -5$$

4 Write the answer.
The slope is –5.

Worked example 4

Find the y-intercept and slope of the graph of each equation.

a $2x - 3y = 8$ **b** $y = 6$

Working

a 1 Rewrite the equation in the intercept–slope form of $y = a + bx$, that is, rearrange the equation to make y the subject. Use a CAS/calculator if necessary.

$$2x - 3y = 8$$
$$2x = 8 + 3y$$
$$2x - 8 = 3y$$
$$y = -\frac{8}{3} + \frac{2}{3}x$$

2 Identify a = y-intercept and b = slope.

State the values of the y-intercept and the slope.

$a = -\frac{8}{3}$, $b = \frac{2}{3}$

The y-intercept is $-\frac{8}{3}$ (or $-2\frac{2}{3}$) and the slope is $\frac{2}{3}$.

b 1 Rewrite the equation in the intercept–slope form of $y = a + bx$.

$y = 6$
$y = 6 + 0x$

2 Identify a = y-intercept and b = slope.

State the values of the y-intercept and the slope.

$a = 6$, $b = 0$

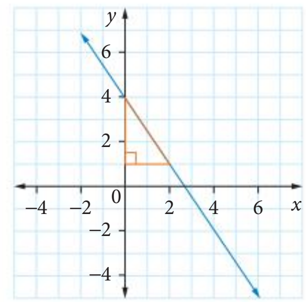
Note that this is a horizontal line.

The y-intercept is 6 and the slope is 0.

Worked example 5

a Find the y-intercept and slope of the graph and hence find the equation of the line.

b Is this line parallel to $y+\frac{3}{2}x=-25$?



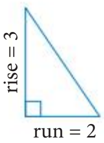
Working

a Identify the y-intercept.

$a = 4$

Calculate the slope using rise over run for two suitable points using the formula or as shown by the triangle marked on the graph.

$b = \frac{\text{rise}}{\text{run}} = -\frac{3}{2}$



Substitute into the intercept–slope equation: $y = a + bx$.

The equation is $y = 4-\frac{3}{2}x$.

b Parallel lines have the same slope. Compare the slopes of the two lines to see if they are the same.

$y = 4-\frac{3}{2}x$ has a slope of $-\frac{3}{2}$.

$y+\frac{3}{2}x=-25$ in intercept–slope form is

Sometimes the equation has a negative term first. It can be rearranged so that the positive term is written first. For example, $y = -8 + 4x$ can also be written as $y = 4x - 8$. The equations are the same.

$y=-25-\frac{3}{2}x$, so it has a slope of $-\frac{3}{2}$.

The two lines are different but have the same slopes, so they are parallel.

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Worked example 6

What is the value of k if

a the point $(k, 2)$ lies on the line $10y = 2 - 3x$?

b the line $y = kx + 5$ goes through the point $(1, 3)$?

c the line $5x + 3y = 2k$ has an x-intercept of 6?

	Working
a Substitute the x and y values of the point into the equation and solve for k, using a CAS/calculator if necessary.	For $(k, 2)$, $x = k$ and $y = 2$. Substituting into $10y = 2 - 3x$ and solving we get $20 = 2 - 3k$ $k = -6$
b Substitute the x and y values of the point into the equation and solve for k, using a CAS/calculator if necessary.	For $(1, 3)$, $x = 1$ and $y = 3$. Substituting into $y = kx + 5$ and solving we get $3 = k + 5$ $k = -2$
c Substitute the x and y values of the point into the equation and solve for k, using a CAS/calculator if necessary.	For x-intercept at $(6, 0)$, $x = 6$ and $y = 0$. Substituting into $5x + 3y = 2k$ and solving we get $30 + 0 = 2k$ $k = 15$

Finding the equation of a line from any two points

To find the equation of a straight line given any two points,

- use the Regression function in the Statistics section of your CAS/calculator
- OR use the formula $y - y_1 = b(x - x_1)$ where b = the slope of the line and (x_1, y_1) is any point on the line.

Using CAS Finding the equation of a line from two points

Find the equation of the straight line that goes through the points (3, 2) and (5, 10).

Solve by hand if it is quicker than using a CAS/calculator.

Let $(x_1, y_1) = (3, 2)$ and $(x_2, y_2) = (5, 10)$.

$$\text{Slope} = b = \frac{y_2 - y_1}{x_2 - x_1} = \frac{10-2}{5-3} = \frac{8}{2} = 4$$

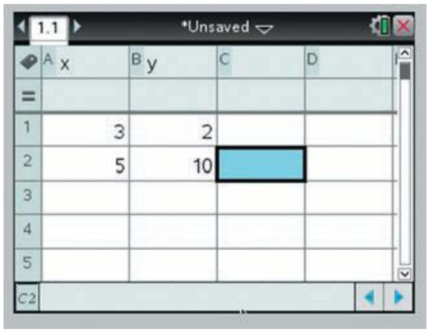
Use $y - y_1 = b(x - x_1)$

$$y - 2 = 4(x - 3)$$
$$y - 2 = 4x - 12$$
$$y = -10 + 4x$$

TI-NSPIRE CAS

STEP 1

Open a new document and select **Add Lists & Spreadsheet**. Name lists x and y. Enter the values into the lists.

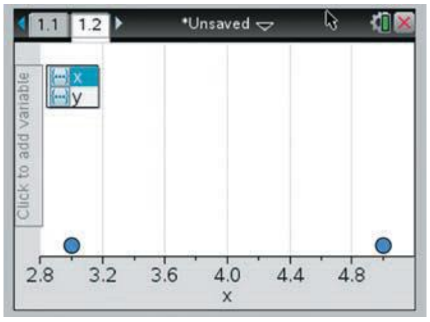


STEP 2

Press the home button. Select **Add Data & Statistics**.

STEP 3

Click on the bottom axis label box and select 'x'.

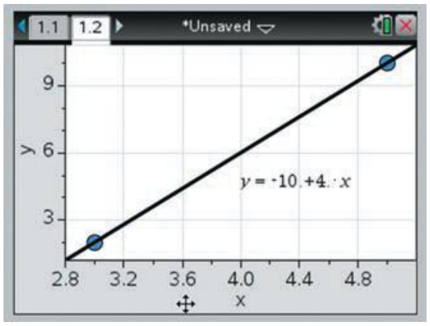


STEP 4

Click on the side axis label box and select 'y'.

STEP 5

Press menu, 4: Analyze, 6: Regression, 2: Show Linear (a+bx).



STEP 6

Write the answer.

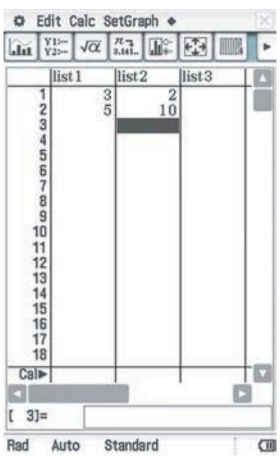
$y = -10 + 4x$

CLASSPAD

STEP 1

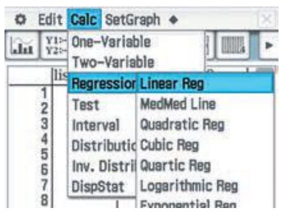
From Menu, select the **Statistics** application Statistics.

Enter the x values in **list 1** and the corresponding y values in **list 2**.



STEP 2

Tap **Calc**, **Regression** and **Linear Reg**.



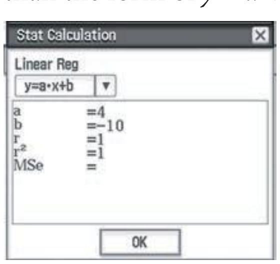
STEP 3

On the Set Calculation screen that pops up, check that the details are correct before hitting OK.

STEP 4

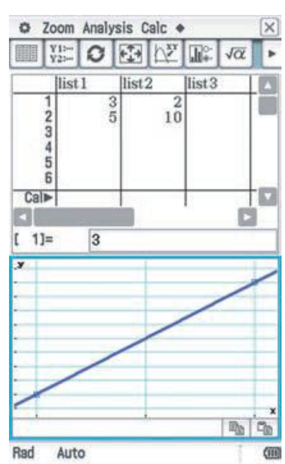
The screen below gives the equation of the line as $y = 4x - 10$.

Note: The ClassPad gives the equation as $y = ax + b$ rather than the form of $y = a + bx$.



STEP 5

Hitting OK will plot the line passing through the two points.



EXAM PREP 14.1

Review of straight-line graphs

Prep 1

WORKED EXAMPLE 1

Sketch the lines by hand by first finding the x- and y-intercepts.

a $y = 3x - 4$ **b** $y = 2x + 1$ **c** $y = 4 - 3x$

d $y = -x - 3$ **e** $2x + y = 5$ **f** $3x - y = 4$

Prep 2

WORKED EXAMPLE 2

a Does the point (5, 20) lie on the line $y = 15x - 55$?

b Does the point (3, −4) lie on the line $2x + 3y + 6 = 0$?

c Does the point (−10, 15) lie on the line $5x - 3y = -5$?

Prep 3

WORKED EXAMPLE 3

Calculate the slope of the straight line drawn from the following table of values.

x	0	4	7	10
y	−1	11	20	29

Prep 4

WORKED EXAMPLE 4

State the y-intercept and slope of the graph of each equation.

a $8x - 2y = 6$ **b** $4x + 2y - 16 = 0$ **c** $y = x$ **d** $y = -4$

Prep 5

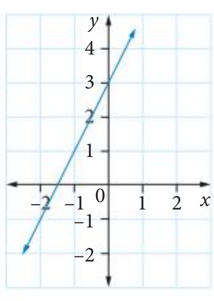
WORKED EXAMPLE 5

For each of the following,

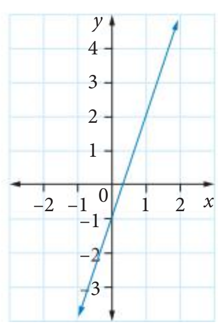
i find the y-intercept and slope, and hence find the equation

ii state whether the line is parallel to $y - 3x = 15$.

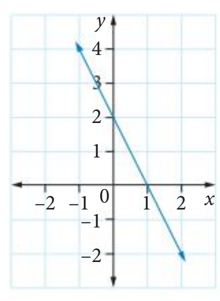
a



b



c



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Prep 6 — WORKED EXAMPLE 6

What is the value of k if

a the point $(4, k)$ lies on the line $2y - 5x = -10$?

b the line $y = kx - 7$ goes through the point $(1, 4)$?

c the line $2x + 7y = 3k$ has an x-intercept of 6?

14.1

Prep 7 — USING CAS: FINDING THE EQUATION OF A LINE FROM TWO POINTS

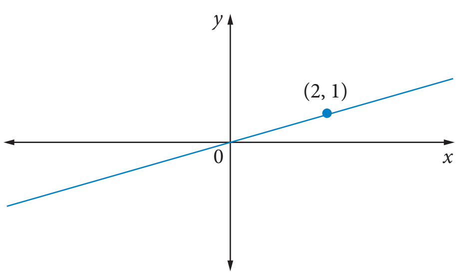
Find the equation of the straight line that goes through each pair of points.

a $(4, 3)$ and $(6, 9)$ **b** $(1, 3)$ and $(4, 6)$ **c** $(5, -3)$ and $(-1, 3)$ **d** $(2, -2)$ and $(1, 4)$

EXAM PRACTICE 14.1

Review of straight-line graphs

Question 1



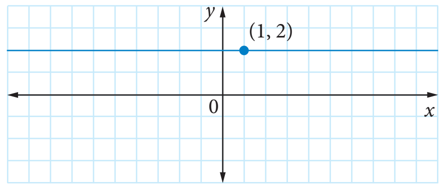
The line passes through the origin and the point $(2, 1)$. The slope of this line is

A -2 **B** -1 **C** $-\frac{1}{2}$

D $\frac{1}{2}$ **E** 2

[VCAA 2007 1GRQ1]

Question 2



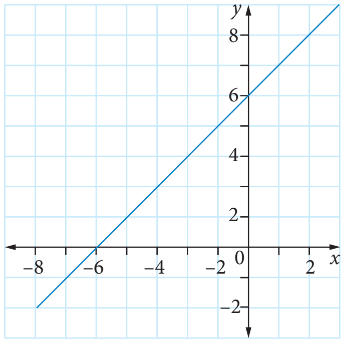
On the graph, the equation of the line passing through the point $(1, 2)$ is

A $x = 1$ **B** $y = 1$ **C** $x = 2$

D $y = 2$ **E** $y = x + 1$

[VCAA 2006 1GRQ1]

Question 3



The equation of the line shown on the graph is

A $y = x - 6$

B $y = x + 6$

C $y = 6 - x$

D $y = -6$

E $y = 6$

[VCAA 2013 1GRQ1]

Question 4

A point that lies on the graph of $3x - 2y = -5$ is

A $(3, -2)$ **B** $(1, 1)$ **C** $(1, -1)$ **D** $(2, -3)$ **E** $(-1, 1)$

[VCAA 2013 1GRQ2]

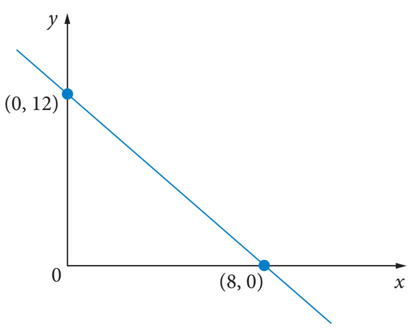
Question 5

The point $(2, 1)$ lies on the line $y = 3x + c$. The value of c is

A -7 **B** -5 **C** -1 **D** 5 **E** 7

[VCAA 2004 1GRQ2]

Question 6



If the line has equation $3x + 2y = 4k$, then the value of k must be

A 2 **B** 3 **C** 6

D 8 **E** 12

[VCAA 2005 1GRQ4]

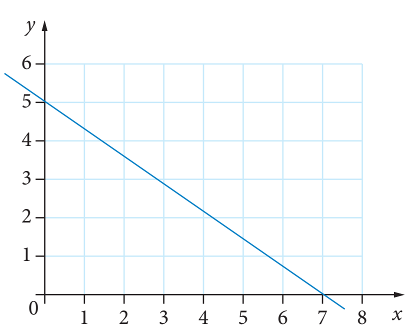
Question 7

An equation for the straight line that passes through the points $(10, 1)$ and $(4, -2)$ is

A $x + 2y = 12$ **B** $2x + y = 6$ **C** $4x + y = 14$ **D** $x - 4y = 14$ **E** $x - 2y = 8$

[VCAA 2010 1GRQ3]

Question 8



The equation of this straight line is

A $x = 7$

B $y = 5$

C $7x + 5y = 35$

D $5x + 7y = 35$

E $7x - 5y = 35$

[VCAA 2002 1GRQ1]

Question 9

The lines $y + 8 = 0$ and $x - 12 = 0$ intersect at the point

A $(-12, 8)$ **B** $(-8, 12)$ **C** $(0, 0)$ **D** $(8, -12)$ **E** $(12, -8)$

[VCAA 2004 1GRQ3]

Question 10

The point of intersection of two lines is $(2, -2)$. One of these two lines could be

A $x - y = 0$ **B** $2x + 2y = 8$ **C** $2x + 2y = 0$ **D** $2x - 2y = 4$ **E** $2x - 2y = 0$

[VCAA 2006 1GRQ6]

Question 11

Which one of the following statements about the line with equation $12x - 4y = 0$ is not true?

A The line passes through the origin.

B The line has a slope of 12.

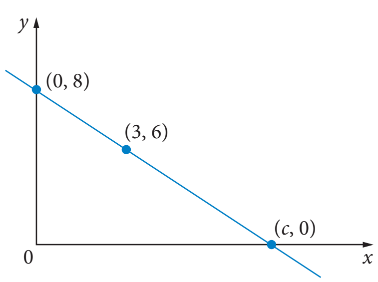
C The line has the same slope as the line with the equation $12x - 4y = 12$.

D The point $(1, 3)$ lies on the line.

E For this line, as x increases, y increases.

[VCAA 2006 1GRQ5]

Question 12



For the straight-line graph, the value of c is

A 8 **B** 11 **C** 12

D 14 **E** 16

[VCAA 2003 1GRQ3]

Question 13

Which of the following statements is not true?

A The line with equation $7x - 4y = 0$ passes through the point $(4, 7)$.

B The point $(3, 5)$ lies in the region defined by $7x - 4y \geq 0$.

C The line with equation $3x + 5y = 0$ has a positive gradient.

D The lines $7x - 4y = 0$ and $3x + 5y = 0$ meet at the origin.

E For the line with the equation $7x - 4y = 0$, y increases as x increases.

[VCAA 2005 1GRQ3]

14.2 Modelling with straight-line graphs

Linear modelling

WS Practical applications

A linear graph can often be used to model real-life situations. As was discussed in Chapter 3, the x variable equivalent is called the **explanatory variable** (or independent variable) and the y variable equivalent is called the **response variable** (or dependent variable). The explanatory variable is what we would expect to affect the response variable.

Worked example 7

Jules is organising the annual Year 12 dance. The total cost for the event will include \$500 venue hire, \$280 for the DJ and \$25 per head for food. The maximum capacity of the venue is 200. The table shows the total cost, \$$C$, for the numbers of students, n, attending the dance.

No. of students (n)	80	100	120	140	160
Total cost (\$$C$)	2780	3280	3780	4280	4780

A linear model of the form $C = a + bn$ can be used to find the total cost of the event, C, for n students.

a Find the values of a and b and hence write down the linear model for C in terms of n.

b What does the a in $C = a + bn$ represent? Why doesn't this value make sense as a minimum cost?

c Sketch a graph of C against n, for values of n from 0 to 50.

d What would be the total cost of the dance if 105 students attended?

e According to the linear model, what would be the total cost of the dance if 1000 students attended? Why isn't this figure a reliable prediction of the cost?

	Working
a **1** Choose 2 points on the line from the table.	(80, 2780) and (100, 3280)
2 Use your CAS/calculator to find the equation of the straight line from the two points.	$C = 780 + 25n$
b a is the vertical axis (i.e. the C-axis) intercept and is therefore the cost when $n = 0$.	a represents the cost of the dance if no students attend. It doesn't make sense as a minimum cost because in real life, if the numbers were looking low, Jules would cancel the booking long before the date. The money he would lose would be any deposits he paid rather than the \$780 venue hire and DJ costs.

c To sketch the graph for values of n from 0 to 50, find the coordinates of two points in that range. We already know the C-axis intercept, so we only need one more point (e.g. find the C value when $n = 20$).

The C-axis intercept has coordinates (0, 780).

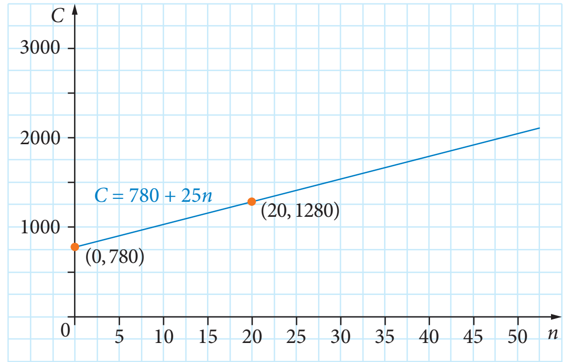
$n = 20$ gives us $C = 780 + 25 \times 20 = 1280$

Another point on the line is (20, 1280).

Sketch the graph.

Label the graph with its equation.

Note that a negative number of students or negative cost makes no sense in this situation, so only the positive axes should be included.



d Write the equation for the total cost.

Substitute $n = 105$ into the equation, find C and write the answer in words.

$$C = 780 + 25n$$
$$= 780 + 25 \times 105$$
$$= 3405$$

The cost of the dance will be \$3405 if 105 students attend.

e Substitute $n = 1000$, find C and write the answer in words.

$$C = 780 + 25n$$
$$= 780 + 25 \times 1000$$
$$= 25\,780$$

The cost of the dance, according to the model, will be \$25 780 if 1000 students attend.

This figure isn't a reliable prediction of the cost because the venue has a maximum capacity of 200, so Jules would need to find another venue and a different model would apply.

Alamy/imageBROKER

Predictions, interpolation and extrapolation

We can easily use linear models to make predictions by simply substituting values into the equation. However, real-life issues mean that the predictions aren't always reliable. In the previous example, the predictions of the cost were only reliable for the range of values given (minimum of 80 to a maximum of 200).

Predicting within the known data range is called **interpolation** and predicting outside the known data range is called **extrapolation**. Predictions that involve interpolation are always more reliable than predictions that involve extrapolation.

Modelling rates

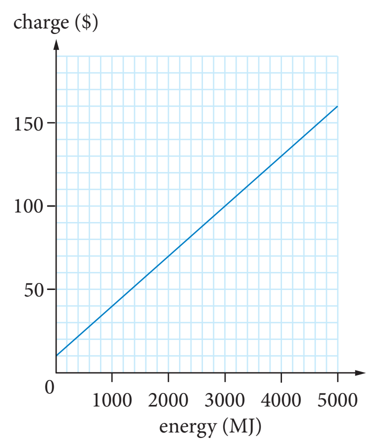
Linear graphs model rate problems well because, as we've seen, the slope of the graph measures the rate of change of y as x changes.

Worked example 8

The TruBlu Energy Company charges its customers a combination of an upfront cost (in dollars) and a rate per megajoule (MJ) of electricity used according to the graph.

a What is the upfront cost?

b What is the rate of charge per megajoule?



Working

a The upfront cost is the charge before any energy is used (when energy = 0).

The vertical intercept is 10.

The upfront cost is $10.

b The rate of charge per megajoule is the slope of the line.

Use (0, 10) and (5000, 160).

$$\text{Slope} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{160-10}{5000-0} = \frac{150}{5000} = 0.03$$

The charge is $0.03 per megajoule.

Modelling profit and loss

Linear graphs can often be used to model both cost (expenses) and revenue (income) for a business. We can then use Profit = revenue – cost to model profit. If profit according to this formula is negative, then it's a loss.

Worked example 9

Snow domes are sold for $6.50 each, and the cost of making n snow domes is given by the equation Cost = 130 + 5n.

a Find the revenue equation in terms of n.

b Find the profit equation in terms of n.

c How much profit would be made if 100 snow domes were sold?

d How much profit would be made if 80 snow domes were sold?

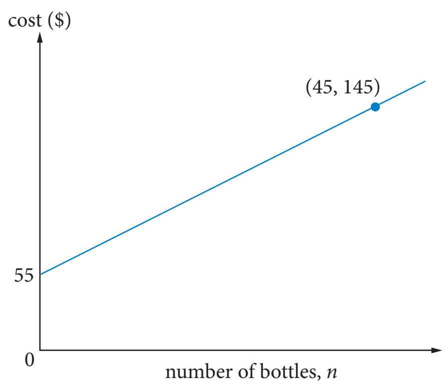
e How many snow domes need to be sold to make at least $1000 profit?

	Working
a Use the price of one item to calculate the revenue from selling n items.	Revenue = 6.50 × number of snow domes sold = 6.5n
b Use the profit formula and simplify.	Profit = revenue – cost = 6.5n – (130 + 5n) = 1.5n – 130
c Substitute n = 100 into the profit equation and write the answer.	Profit = 1.5 × 100 – 130 = 20 100 snow domes would make a profit of $20.
d Substitute n = 80 into the profit equation and write the answer.	Profit = 1.5 × 80 – 130 = –10 80 snow domes would make a loss of $10.
e **1** Let profit equal 1000 and solve for n, using a CAS/calculator if necessary.	1.5n – 130 = 1000 n ≈ 753.33
2 Round the answer according to the question. The answer needs to be a whole number since n represents the number of snow domes.	The profit needs to be *at least* $1000, and selling 753 snow domes won't quite make that profit. So the number of snow domes that need to be sold is 754.

Exam hack

Always look at the context of the question when rounding, particularly if no specific instructions are given. If the answer needs to be a whole number, decide whether it makes sense to break the normal rounding rules of '0 – 4 round down' and '5 – 9 round up'.

Worked example 10



The graph shows the cost (in dollars) of producing bottles of OJJ orange juice.

a What is the cost equation in terms of n, the number of bottles sold?

b What is the cost of 200 bottles?

c If p is the selling price of one bottle, what would be the revenue made on 200 bottles?

d What is the selling price of one bottle if the profit from the sale of 200 bottles is $295?

Working

a Find the cost equation from the graph.

The vertical intercept is 55.

The two points shown are (0, 55) and (45, 145).

The slope $= \frac{145-55}{45-0} = \frac{90}{45} = 2$

So cost $= 55 + 2n$

b Substitute $n = 200$ into the cost equation, and write the answer.

Cost $= 55 + 2 \times 200 = 455$.

The cost of 200 bottles is $455.

c Write an equation for the revenue in terms of p.

The revenue from selling 200 bottles is given by

Revenue $= 200 \times p = 200p$

d **1** Use the profit formula, previous answers, and the information given in the question.

The profit from selling 200 bottles is $295.

Profit = revenue − cost

$295 = 200p - 455$

2 Solve for p, using a CAS/calculator if necessary, and write the answer.

$p = 3.75$

The selling price of one bottle is $3.75.

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EXAM PREP 14.2

Modelling with straight-line graphs

Prep 1

WORKED EXAMPLE 7

During Australian summers, crickets chirp faster at night if the temperature is higher. There is a linear relationship between the temperature and a cricket's chirping rate, as shown in the table.

Temperature, T (°C)	12	15	19	22	28
Chirp rate, n (chirps/min)	72	96	128	152	200

A linear model of the form $n = a + bT$ can be used to find the chirp rate, n, for a given temperature (°C).

a Find the values of a and b and hence write down the linear model for n in terms of T.

b What does the a in $n = a + bT$ represent? Why doesn't this value make sense in this context?

c Sketch a graph of n against T for values of T between 3 and 30.

d What will be the chirp rate of a cricket when the temperature is 26°C? Does this involve interpolation or extrapolation?

e According to the linear model, what will be the chirp rate of a cricket if the temperature is 4°C? Why isn't this figure a reliable prediction of the chirp rate? Does this involve interpolation or extrapolation?

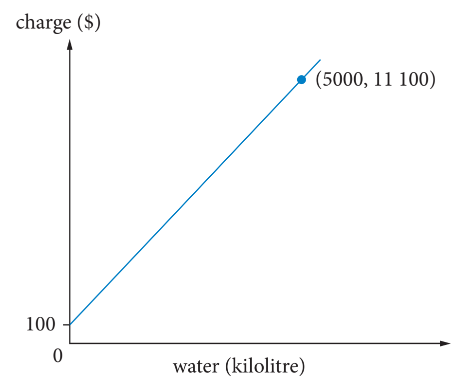
Prep 2

WORKED EXAMPLE 8

The Wally Water Company charges its customers a combination of an upfront cost (in dollars) and a rate per kilolitre used, according to the graph.

a What is the upfront cost?

b What is the rate of charge per kilolitre?



Prep 3

WORKED EXAMPLE 9

Key rings are sold for \$5.50 each, and the cost of making n key rings is given by the equation Cost = 110 + 4n.

a Find the revenue equation in terms of n.

b Find the profit equation in terms of n.

c How much profit would be made if 200 key rings were sold?

d How much profit would be made if 50 key rings were sold?

e How many key rings need to be sold to make at least \$600 profit?

Prep 4 WORKED EXAMPLE 10

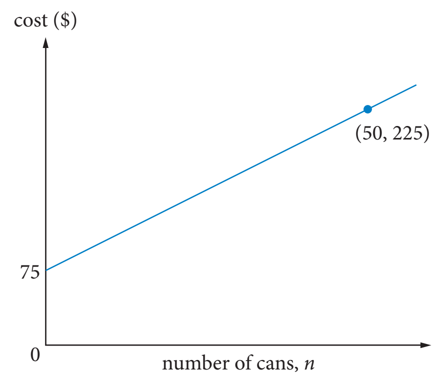
The graph shows the cost (in dollars) of producing cans of BB baked beans.

a What is the cost equation in terms of the n, number of cans sold?

b What is the cost of 100 cans?

c If p is the selling price of one can, what would be the revenue made on 100 cans?

d What is the selling price of one can if the profit from the sale of 100 cans is $265?



EXAM PRACTICE 14.2

Modelling with straight-line graphs

Question 1

At a convenience store, one doughnut costs $2.40 and one drink costs $3.00. A customer purchased five doughnuts and a number of drinks at a total cost of $24.00. The number of drinks purchased was

A 4 **B** 5 **C** 6 **D** 9 **E** 10

[VCAA 2012 1GRQ2]

Question 2

Initially there are 5000 litres of water in a tank. Water starts to flow out of the tank at the constant rate of 2 litres per minute until the tank is empty. After t minutes, the number of litres of water in the tank, V, will be

A $V = 5000 - 2t$ **B** $V = 2t - 5000$ **C** $V = 5000 + 2t$

D $V = 2 - 5000t$ **E** $V = 5000t - 2$

[VCAA 2008 1GRQ2]

Question 3

An electrician charges a fixed call-out fee of $50 and then charges $65 per hour for each hour worked. For n hours worked, the total charge in dollars is

A 115 **B** $n + 115$ **C** $50n + 65$ **D** $65n + 50$ **E** $115n$

[VCAA 2005 1GRQ5]

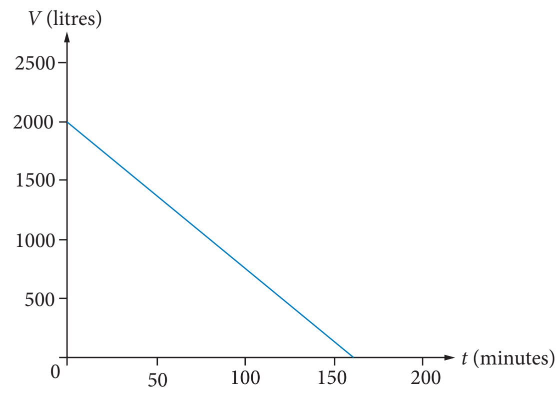
Question 4

A full tank holds 2000 litres of water. Water is pumped out of the tank at a constant rate. The graph shows how the volume of water in the tank, V, changes with time, t.

The constant rate, in litres per minute, at which the water is being pumped out of the tank is

A 0.8 **B** 2.0 **C** 12.5

D 80.0 **E** 160.0



[VCAA 2013 1GRQ3]

Question 5

A builder's fee, C dollars, can be determined from the rule $C = 60 + 55n$, where n represents the number of hours worked. According to this rule, the builder's fee will be

A \$60 for 1 hour of work **B** \$110 for 2 hours of work **C** \$500 for 8 hours of work

D \$550 for 10 hours of work **E** \$1150 for 10 hours of work

[VCAA 2007 1GRQ2]

Question 6

The cost in dollars, C, of making n pottery mugs is given by the equation $C = 150 + 6n$. A loss will result from selling

A 60 mugs at \$9.00 each **B** 70 mugs at \$8.50 each **C** 80 mugs at \$7.50 each

D 90 mugs at \$8.00 each **E** 100 mugs at \$9.50 each

[VCAA 2010 1GRQ5]

Question 7

The cost, \$$C$, of making x kilograms of chocolate fudge is given by $C = 60 + 5x$. The revenue, \$$R$, from selling x kilograms of chocolate fudge is given by $R = 15x$. A particular quantity of chocolate fudge was made and sold. It resulted in a loss of \$20. The number of kilograms of chocolate fudge made and sold was

A 2 **B** 4 **C** 8 **D** 12 **E** 16

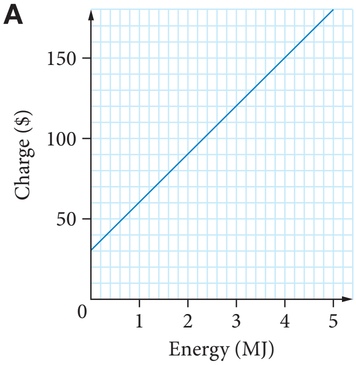
[VCAA 2011 1GRQ5]

14.2

Question 8

In one month, an energy company charges a $30 service fee plus a supply charge of two cents per megajoule (MJ) of energy used. The graph that best models this situation is

A



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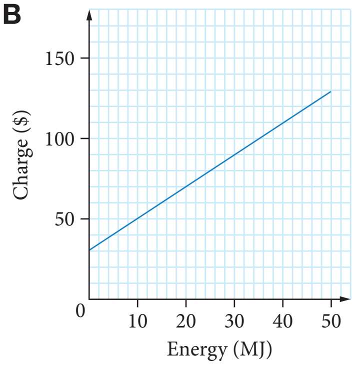

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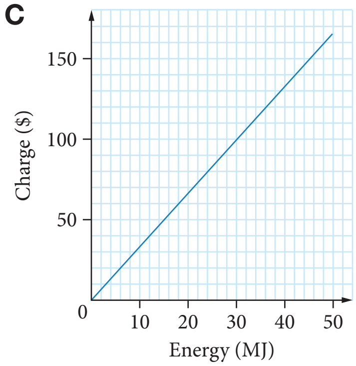
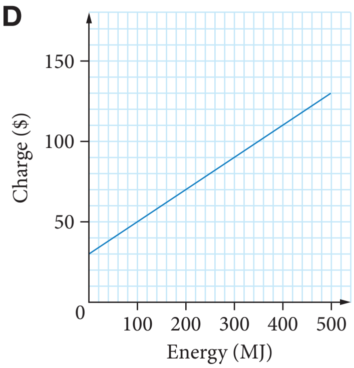
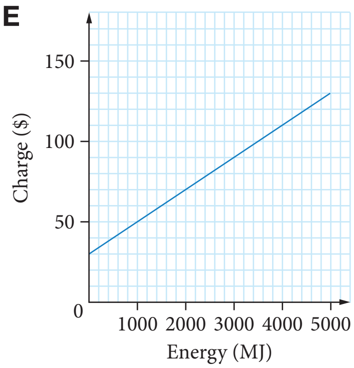


D



E

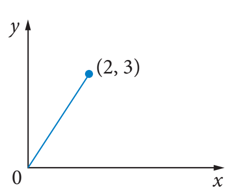


[VCAA 2013 1GRQ6]

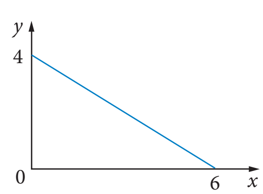
Question 9

A mixture contains two liquids, A and B. Liquid A costs $2 per litre and liquid B costs $3 per litre. Let x be the volume (in litres) of liquid A purchased. Let y be the volume (in litres) of liquid B purchased. Which graph shows all possible volumes of liquid A and liquid B that can be purchased for exactly $12?

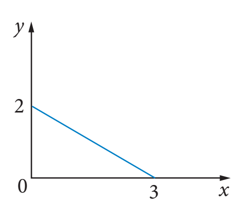
A



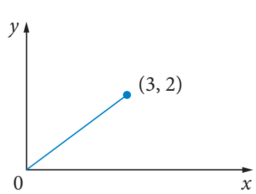
B



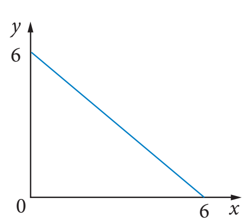
C



D



E



[VCAA 2008 1GRQ5]

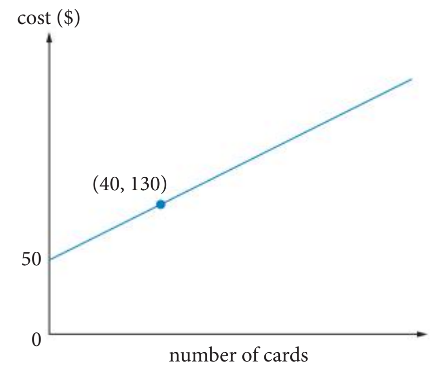
Question 10

The graph shows the cost (in dollars) of producing birthday cards.

If the profit from the sale of 150 birthday cards is \$175, the selling price of one card is

A \$0.30 **B** \$1.60 **C** \$3.10

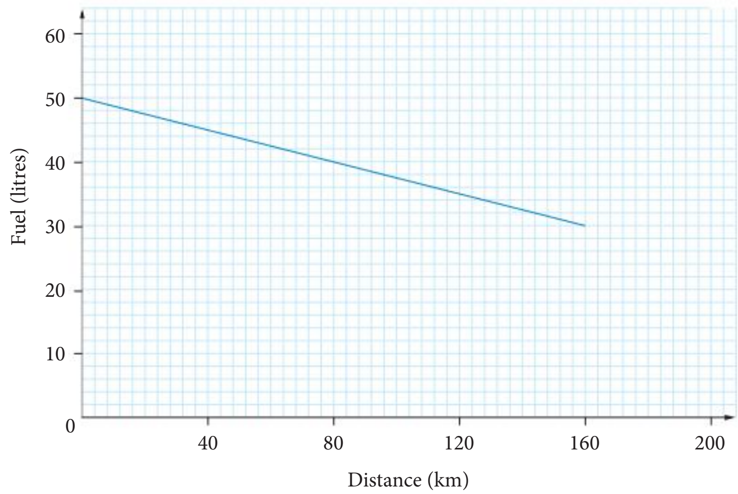
D \$3.50 **E** \$4.40



[VCAA 2009 1GRQ9]

Question 11

In one particular week, Harry began with 50 litres of fuel in the tank of his van. After he had travelled 160 km, there were 30 litres of fuel left in the tank of his van. The amount of fuel remaining in the tank of Harry's van followed a linear trend as shown in the graph.



a Determine the equation of the line shown in the graph. 2 marks

b Assume that this linear trend continues and that Harry does not add fuel to the tank of his van. How much **further** will he be able to travel before the tank is empty? 1 mark

c Harry stopped to refuel his van when there were 12 litres of fuel left in the tank. He completely filled the tank in $3\frac{1}{2}$ minutes when fuel was flowing from the pump at a rate of 18 litres per minute. How much fuel does the tank hold when it is completely full? Write your answer in litres. 1 mark

[VCAA 2006 2GRQ2]

Question 12

A rock-climbing activity will be offered to students on a camp. Each student who participates will pay \$24. The organisers have to pay the rock-climbing instructor \$260 for the afternoon. They also have to pay an insurance cost of \$6 per student. Let n be the total number of students who participate in rock climbing.

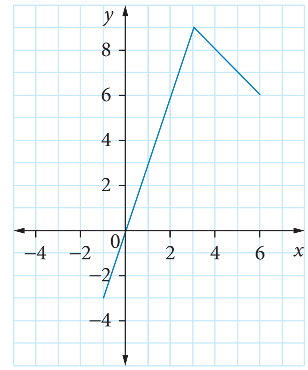
a Write an expression for the profit that the organisers will make in terms of n. 1 mark

b The organisers want to make a profit of at least \$500. Determine the minimum number of students who will need to participate in rock climbing. 1 mark

[VCAA 2013 2GRQ3]

14.3 Line segment and step graphs

Drawing line segment linear graphs



A **line segment linear graph**, also known as a **piecewise linear graph**, is a combination of two or more straight line segments.

An example of a piecewise linear function is

$$y=\begin{cases} 3x & -1 \le x < 3 \\ 12-x & 3 \le x \le 6 \end{cases}$$

which means

- $y = 3x$ for all the x values between -1 and 3, where -1 is included but 3 is not included
- $y = 12 - x$ for all the x values between 3 and 6, where both 3 and the 6 are included.

The graph for this a piecewise linear function is shown above.

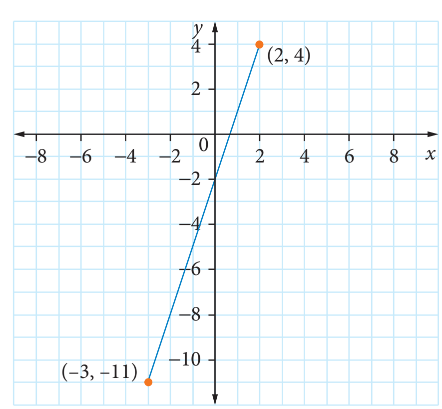
Exam hack

The important points when sketching a piecewise graph are the end points of each line segment. To sketch the graph, we find the coordinates of these end points and then join them in order.

Worked example 11

a Sketch the graph of $y = 3x - 2$ for $-3 \le x < 2$.

b On the same set of axes, sketch the graph of $y = 4$ for $2 \le x \le 5$.

		Working
a	**1** Write the equation.	$y = 3x - 2$
	2 The graph of $y = 3x - 2$ is only required for x values between -3 and 2. Find the value of y at each of the end points, that is, where $x = -3$ and $x = 2$.	When $x = -3$, $y = -9 - 2 = -11$ When $x = 2$, $y = 6 - 2 = 4$ The end points are $(-3, -11)$ and $(2, 4)$.
	3 Plot the coordinates of the end points: $(-3, -11)$ and $(2, 4)$. Join these points to sketch the line.	

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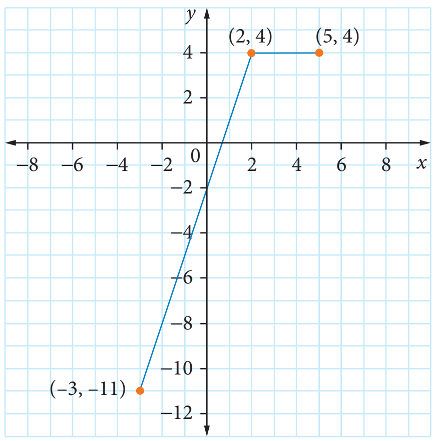
b **1** Write the equation. $y = 4$

2 The graph of $y = 4$ is only required for x values between 2 and 5. The end points are (2, 4) and (5, 4).

3 The graph of $y = 4$ starts where the graph of $y = 3x - 2$ finishes.

Draw a horizontal line from (2, 4) to (5, 4).

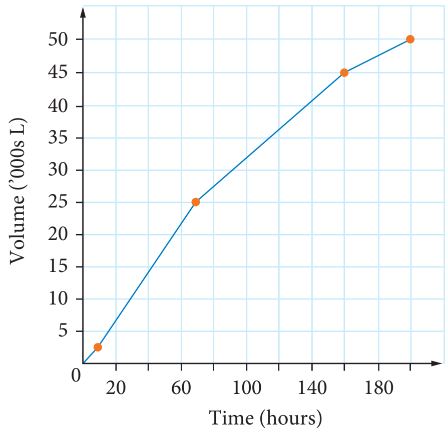


Modelling with line segment linear graphs

Line segment linear graphs are useful for displaying information that involves different rates of change.

Worked example 12

A rainwater tank holds 50 000 litres. The line segment linear graph shows the rate at which the rainwater tank fills with water from the start of winter.



a Does the rainwater tank fill at the same rate until it reaches capacity? Give reasons for your answer.

b Approximately how many litres does the rainwater tank hold after 160 hours?

c After how many hours from the start of winter is the rainwater tank filled to capacity?

d Approximately during which times does the tank fill the fastest?

e What is the rate, in litres per hour, that the tank is filling in the last 20 hours?

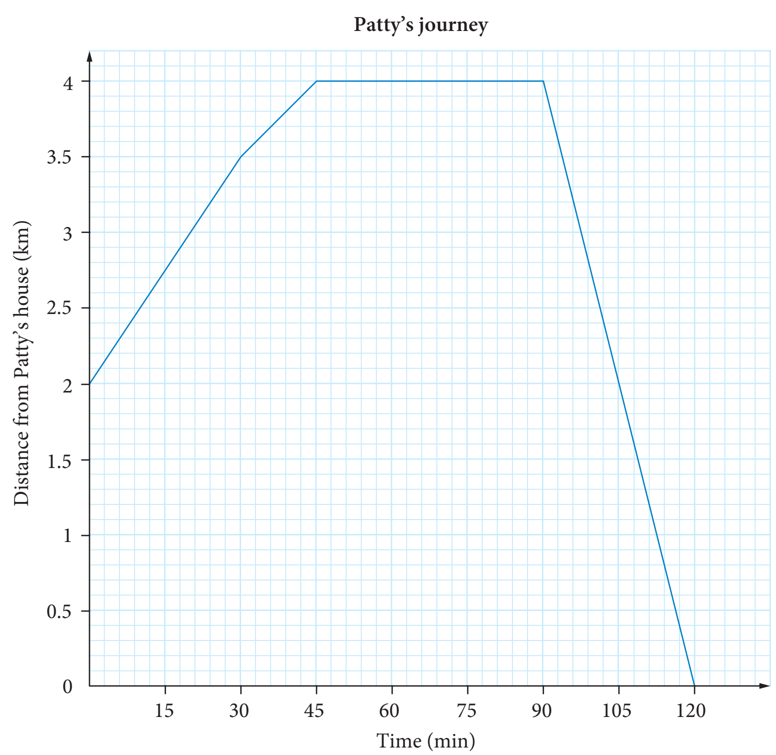
Working

a Refer to the line segments and their slopes. No. The graph is made up of four different line segments and each line segment has a different slope. Hence, the rainwater tank is filling at four different rates.

b	Note that the vertical scale is in thousands of litres.	After 160 hours, the rainwater tank holds 45 000 litres.
c	Read from the graph.	The rainwater tank filled to capacity after 200 hours.
d	Identify which of the line segments has the greatest slope.	The second time period has the greatest slope, so the tank fills the fastest from approximately 10 hours to 70 hours after the start of winter.
e	Find the slope of the last line segment.	Slope of last line segment $= \frac{\text{rise}\uparrow}{\text{run}\rightarrow} = \frac{5}{40} = 0.125$ thousand litres per hour.
	Convert the rate to litres per hour.	The tank is filling at a rate of $0.125 \times 1000 = 125$ litres per hour in the last 20 hours.

Worked example 13

Patty walked to Susan's house, stayed there for a while, then borrowed Susan's bike and cycled home. This travel graph illustrating Patty's movements has four distinct sections.



a The vertical intercept of this graph is 2 km. What does this mean?

b Did Patty start walking faster or slower after 30 minutes?

c How long did Patty stay at Susan's house?

d What is the distance between Patty's house and Susan's house?

e Calculate Patty's cycling speed in kilometres/hour during the last section of her journey.

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Working

a The vertical scale measures Patty's distance from her own house.

Patty started her journey 2 km from her own house.

b The slope of each line segment measures the speed (since speed = distance divided by time).

The slope of the line segment after 30 minutes is less than the slope of the line segment before 30 minutes, so Patty starts walking slower after 30 minutes.

c The horizontal line indicates that Patty is the same distance from her house during this time (that is, she is at Susan's house).

Patty stayed at Susan's house from the 45 minute mark to the 90 minute mark, so Patty stayed at Susan's house for 45 minutes.

d Read the distance using the vertical scale of the graph.

The distance between Patty's house and Susan's house is 4 km.

e Calculate the slope from the graph in kilometres/minute and convert to kilometres/hour.

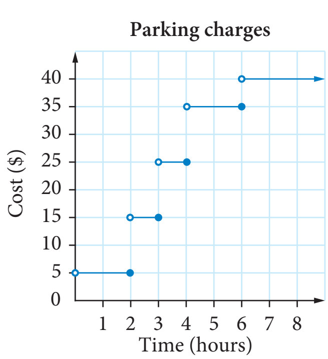
$$\text{Cycling speed} = \frac{\text{rise}\uparrow}{\text{run}\rightarrow} = \frac{4\text{ km}}{30\text{ min}} = \frac{4\text{ km}}{\frac{1}{2}\text{ h}} = 8\text{ km/h}$$

Step graphs

Step graphs

A **step graph** is made up of horizontal line segments that look like steps. The ends of each line segment in a step graph are labelled with circles. These can be open or closed. An open circle means that the point is *not* included as part of the graph, while a closed circle means that the point *is* included as part of the graph.

Worked example 14



This step graph shows the parking charges at a car park.

a Find the charge for parking for:

i 3 hours

ii 55 minutes

iii $3\frac{1}{2}$ hours

iv 4 hours 20 minutes.

b For what range of times can a driver park for $25?

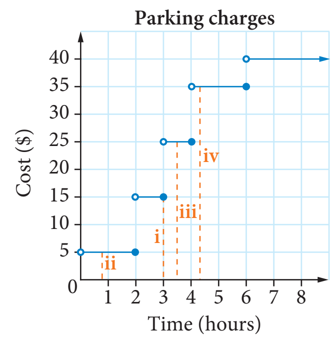
c What does the arrow on the $40 step mean?

d If Sarah parked in the car park for two days in a row and paid a total of $40, what would be the maximum number of total hours that she could have parked?

Working

a **i** Find 3 hours on the horizontal axis and draw a vertical line from this point until you reach one of the steps. The vertical line meets the second step at the end of the interval. The circle at the end of the step is closed, so 3 hours is included in this interval.

According to the step graph, if a driver parks for less than or equal to 2 hours, they will pay $5. If a driver parks for more than 2 hours but less than or equal to 3 hours, they will pay $15, and so on.



3 hours of parking will cost $15.

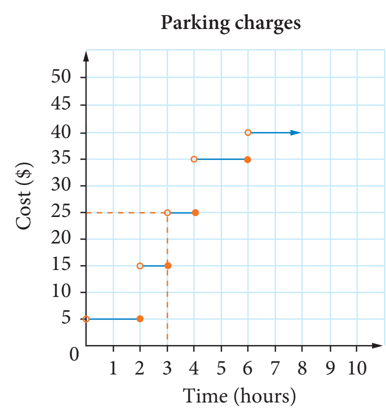
ii 55 minutes of parking will cost $5.

iii $3\frac{1}{2}$ hours of parking will cost $25.

iv 4 hours 20 minutes of parking will cost $35.

b **1** Using the graph, find $25 dollars on the vertical axis.

Move across horizontally until you reach the step that is in line with 25. The step ranges from 3 to 4. The open circle at the start means that 3 is not included. The closed circle at the end means that 4 is included.



2 Write the answer.

The driver can park for a time that is more than 3 hours but less than or equal to 4 hours for $25.

c The arrow indicates the line continues.

After 6 hours, the parking charge remains constant at $40. The maximum daily charge is $40.

d **1** List all the possible charges and find the combinations of two of these charges that add to $40.

The possible charges are $5, $15, $25, $35 and $40. The two day charges that add to $40 are: $5 + $35 and $15 + $25.

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2 Use the graph to find the maximum number of parking hours in each case, and sum them to find the total maximum number of hours.

For \$5, the maximum number of parking hours is 2. For \$35, the maximum number is 6. So, for this combination, the total maximum number of hours is 8.

For \$15, the maximum number of parking hours is 3. For \$25, the maximum number is 4. So, for this combination, the total maximum number of hours is 7.

3 Write the answer.

The maximum number of total hours she could have parked is 8.

14.3

Worked example 15

Sketch the following step graph.

$$y = \begin{cases} 1 & -2 < x \le 1 \\ 3 & 1 < x \le 4 \\ 5 & 4 < x \le 7 \\ 7 & 7 < x \le 10 \end{cases}$$

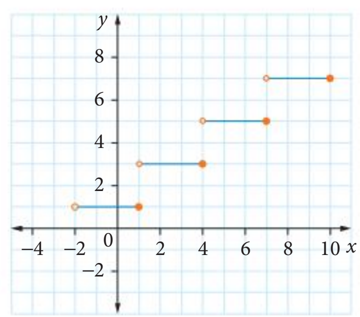
Working

1 There are 4 steps in this graph. The first step or interval has the equation $y = 1$ when x is greater than -2 but less than or equal to 1.

Draw a horizontal line from -2 to 1 at $y = 1$.

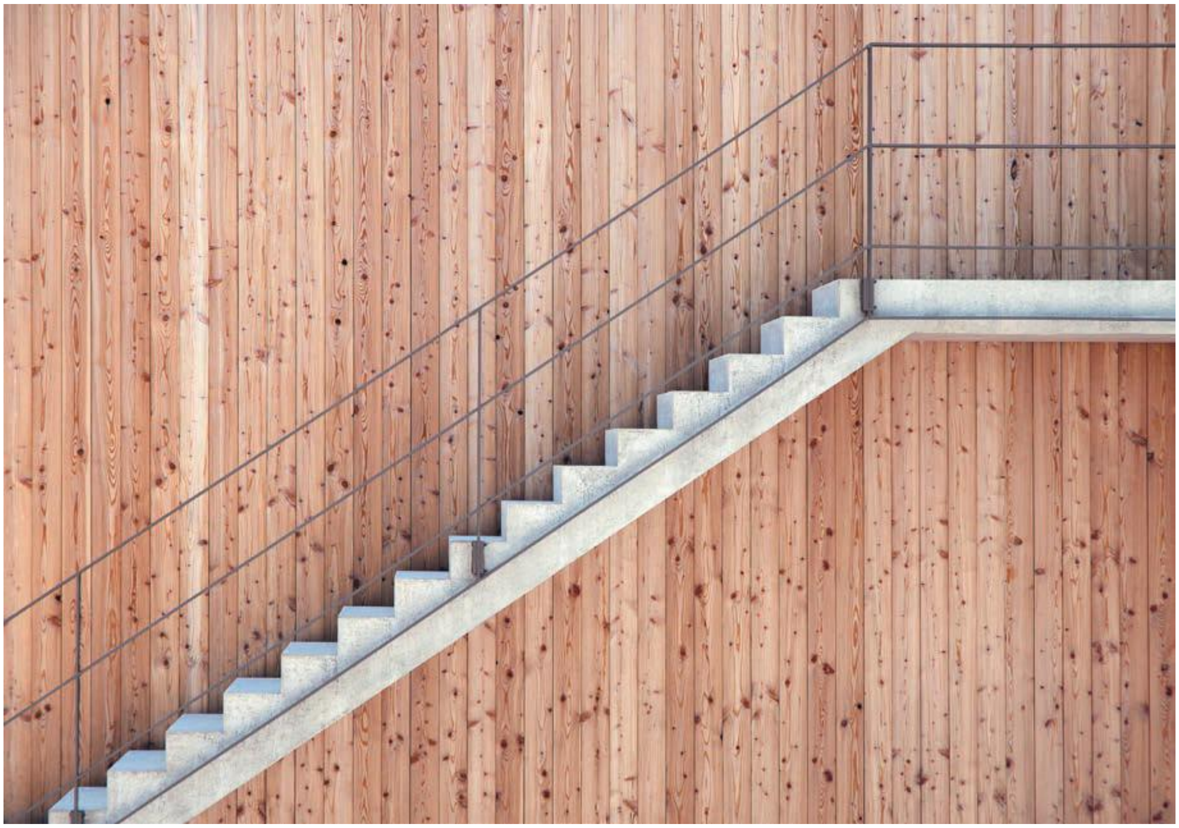
-2 is not included, so place an open circle at this end.

1 is included, so place a closed circle at this end.



2 Repeat this process for the remaining 3 steps to complete the graph.

Each step should have the open circle at the start and the closed circle at the end.



amanaimages/RG Images/STOCK48

EXAM PREP 14.3

Line segment and step graphs

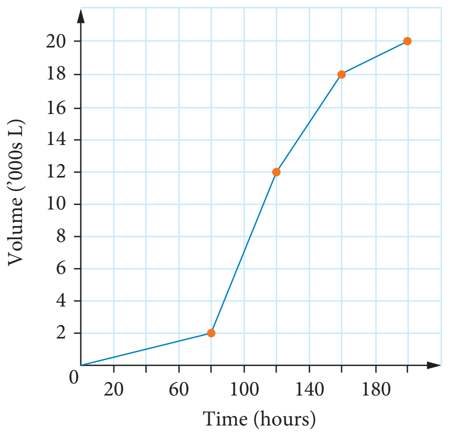
Prep 1 WORKED EXAMPLE 11

a Sketch the graph of $y = \frac{1}{2}x$ for $-8 \le x < 2$.

b On the same set of axes, sketch the graph of $y = 1$ for $2 \le x \le 8$.

Prep 2 WORKED EXAMPLE 12

A rainwater tank holds 20 000 litres. The line segment linear graph shows the rate at which the rainwater tank fills with water from the start of the rainy season.



a Does the rainwater tank fill at the same rate until it reaches capacity? Give reasons for your answer.

b Approximately how many litres does the rainwater tank hold after 160 hours?

c After how many hours from the start of the rainy season is the rainwater tank filled to capacity?

d Approximately during which times does the tank fill the fastest?

e What is the rate, in litres per hour, at which the tank is filling in the first 80 hours?

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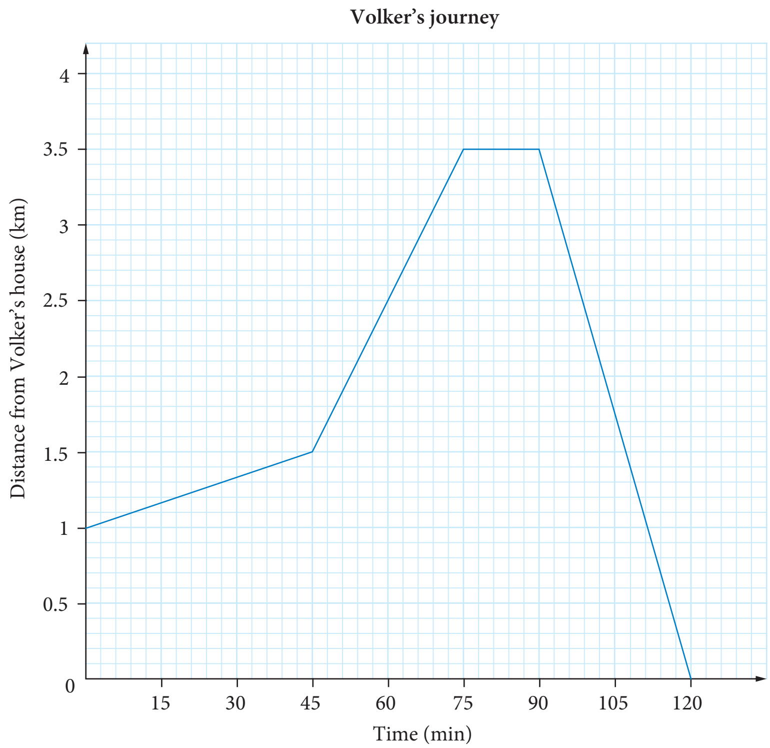
Prep 3 WORKED EXAMPLE 13

Volker and Herb walked from the park to Herb's house. Volker stayed there a while and then jogged home. This travel graph illustrating Volker's movements has four distinct sections.



a The vertical intercept of this graph is 1 km. What does this mean?

b Did Volker and Herb start walking faster or slower after 45 minutes?

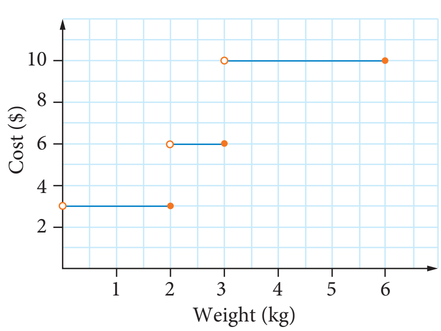
c How long did Volker stay at Herb's house?

d What is the distance between Volker's house and Herb's house?

e Calculate Volker's jogging speed in kilometres/hour during the last section of his journey.

Prep 4 WORKED EXAMPLE 14

This step graph shows the cost of sending parcels of different weights by air freight.



a Find the cost of sending:

i one parcel weighing 3.4 kg

ii one parcel weighing 1.7 kg

iii two parcels, weighing 2 kg and 3 kg

b What range of weights for a single parcel can be sent for $6?

c Why do you think there is no arrow on the $10 step?

d If Craig has a large number of small items which can be split up into more than one parcel, what is the maximum total weight he could send for $12?

Prep 5

WORKED EXAMPLE 15

Sketch the following step graphs.

a $y = \begin{cases} -3 & -5 \le x < -1 \\ 1.5 & -1 \le x < 1 \\ 4 & 1 \le x < 6 \end{cases}$

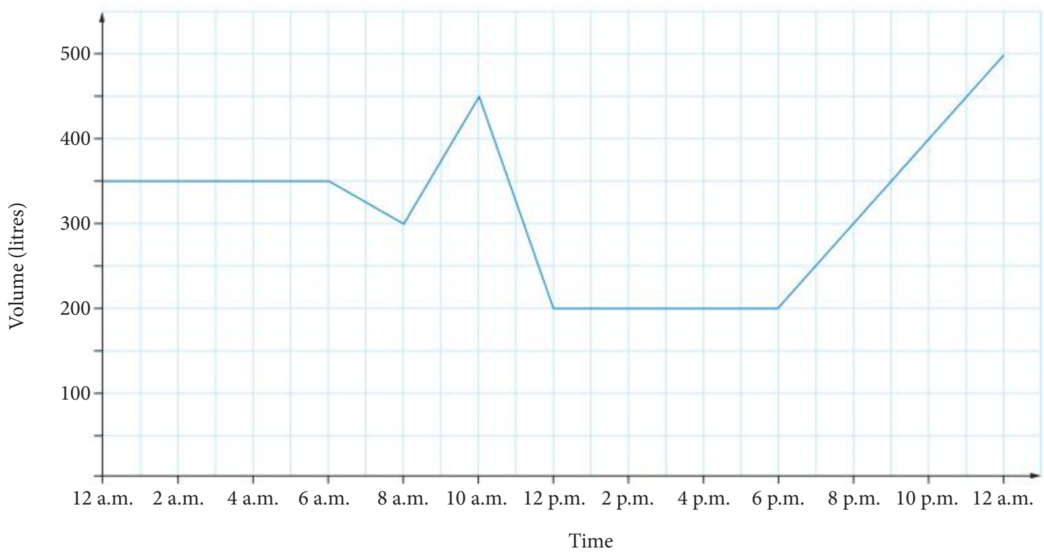
b $y = \begin{cases} -3.5 & -3 < x \le -1 \\ -1 & -1 < x \le 2.5 \\ 3.5 & 2.5 < x \le 5.5 \\ 6.5 & 5.5 < x \le 8 \end{cases}$

EXAM PRACTICE 14.3

Line segment and step graphs

Use the following information to answer Questions 1 & 2.

The volume of water that is stored in a tank over a 24-hour period is shown in the graph.



Question 1

What is the difference in the volume of water (in litres) in the tank between 8 a.m. and 6 p.m.?

A 50 **B** 100 **C** 120 **D** 200 **E** 400

[VCAA 2010 1GRQ1]

Question 2

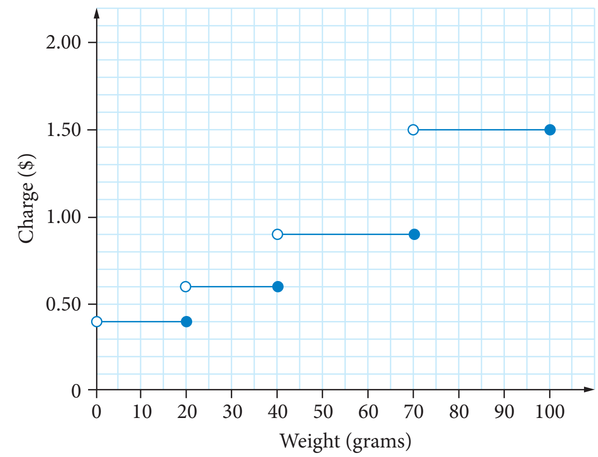
The rate of increase in the volume of water in the tank (in litres/hour) between 8 a.m. and 10 a.m. is

A 37.5 **B** 50 **C** 75 **D** 125 **E** 150

[VCAA 2010 1GRQ2]

Use the following information to answer Questions 3 & 4.

The charges for posting letters that weigh 100 g or less are shown in the graph.



Question 3

The charge for posting a 35 g letter is

A $0.40 **B** $0.60

C $0.90 **D** $1.50

E $2.00

[VCAA 2011 1GRQ1]

Question 4

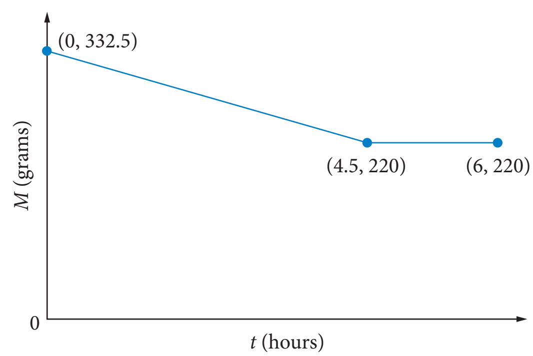
Two letters are posted. The total postage charge **cannot** be

A $0.80 **B** $1.20 **C** $1.40 **D** $2.10 **E** $3.00

[VCAA 2011 1GRQ2]

Use the following information to answer Questions 5 & 6.

A gas-powered camping lamp is lit and the gas is left on for six hours. During this time, the lamp runs out of gas. The graph shows how the mass, M, of the gas container (in grams) changes with time, t (in hours), over this period.



Question 5

Assume that the loss in weight of the gas container is due only to the gas being burnt.
From the graph it can be seen that the lamp runs out of gas after

A 1.5 hours **B** 3 hours **C** 4.5 hours **D** 6 hours **E** 220 hours

[VCAA 2006 1GRQ3]

Question 6

Which one of the following rules could be used to describe the graph?

A $M = \begin{cases} 332.5 - 25t & 0 \le t \le 4.5 \\ 220 & 4.5 < t \le 6 \end{cases}$

B $M = \begin{cases} 332.5 - 25t & 0 \le t \le 4.5 \\ 220t & 4.5 < t \le 6 \end{cases}$

C $M = \begin{cases} 332.5 + 25t & 0 \le t \le 4.5 \\ 220t & 4.5 < t \le 6 \end{cases}$

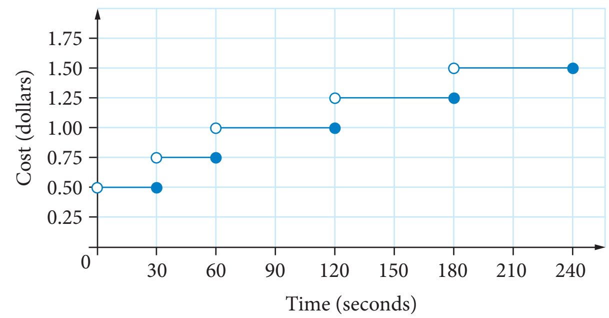
D $M = \begin{cases} 332.5 - 12.5t & 0 \le t \le 4.5 \\ 220t & 4.5 < t \le 6 \end{cases}$

E $M = \begin{cases} 332.5 - 12.5t & 0 \le t \le 4.5 \\ 220 & 4.5 < t \le 6 \end{cases}$

[VCAA 2006 1GRQ4]

Question 7

The graph shows the cost (dollars) of mobile telephone calls up to 240 seconds long.



The cost of making a 90-second call followed by a 30-second call is

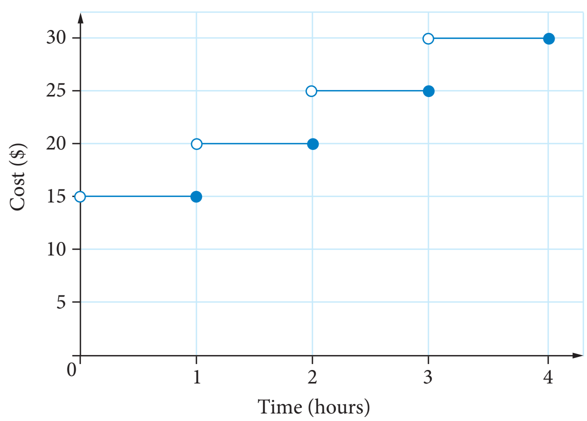
A \$1.00 **B** \$1.20

C \$1.25 **D** \$1.50

E \$1.75

[VCAA 2004 1GRQ1]

Question 8



The cost of hiring one motorbike for up to 4 hours is shown in the graph. Two motorbikes were hired. The **total** charge for hiring the two motorbikes was \$45.

The time for which each motorbike was hired could have been

A 1 hour and 2 hours

B 1 hour and 3 hours

C 1.5 hours and 2 hours

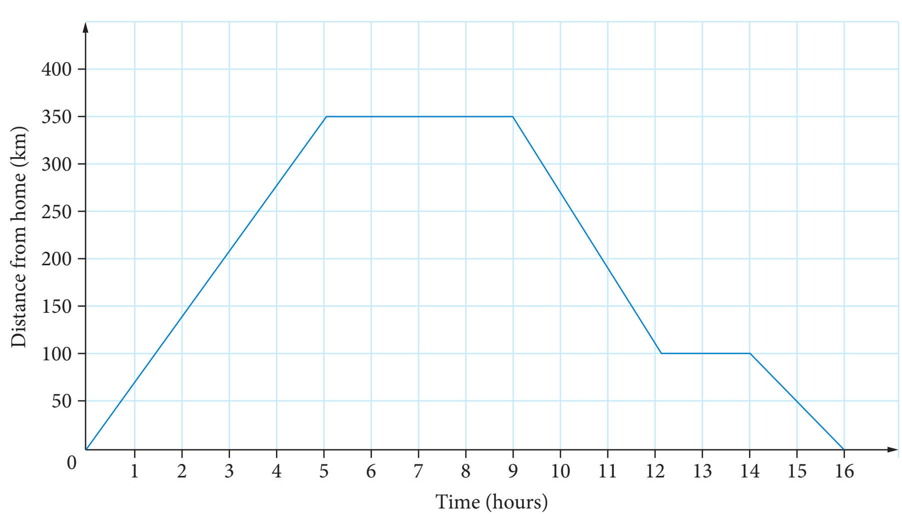
D 1.5 hours and 3 hours

E 2 hours and 3.5 hours

[VCAA 2007 1GRQ5]

Use the following information to answer Questions 9 & 10.

The graph shows a distance–time graph for a car travelling home along a long straight road over a 16-hour period.



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Question 9

In which of the time intervals is the speed of the car the greatest?

A 0 to 5 hours **B** 5 to 9 hours **C** 9 to 12 hours

D 12 to 14 hours **E** 14 to 16 hours

[VCAA 2004 1GRQ4]

Question 10

After twelve hours the car has travelled a total distance of

A 100 km **B** 350 km **C** 450 km **D** 600 km **E** 700 km

[VCAA 2004 1GRQ5]

Use the following information to answer Questions 11 & 12.

Question 11

The graph shows the volume of water, V litres, in a tank at time t minutes. The equation of this line between $t = 50$ and $t = 85$ minutes is

A $V = 1700 - 20t$

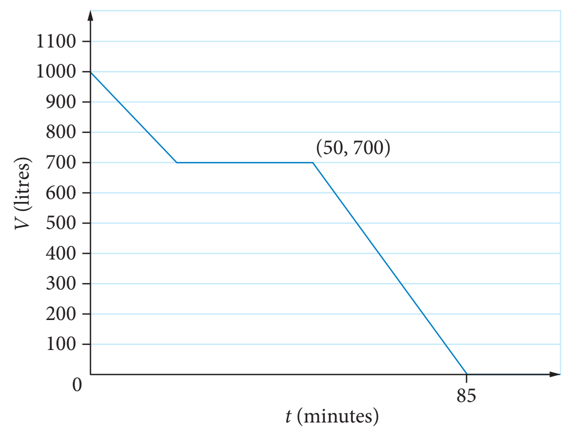
B $V = 700 - 20t$

C $V = 20t + 1700$

D $V = 20t + 700$

E $V = 35t - 700$

[VCAA 2012 1GRQ4]



Question 12

During the 85 minutes that it took to empty the tank, the volume of water in the tank first decreased at the rate of 15 litres per minute and then did not change for a period of time. The period of time, in minutes, for which the volume of water in the tank did not change is

A 15 **B** 20 **C** 30 **D** 50 **E** 85

[VCAA 2012 1GRQ5]

Question 13

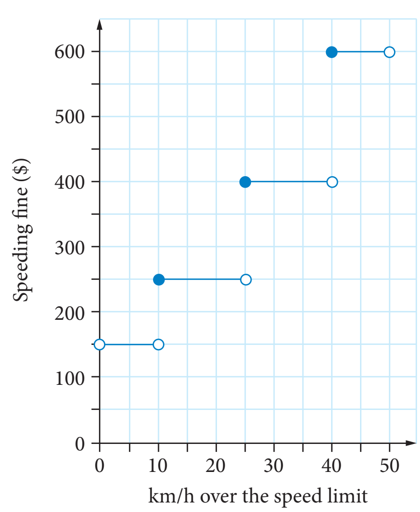
The step graph shows the speeding fines that are given for exceeding the speed limit by different amounts.

A driver was fined for driving at a speed of 65 km/h in a zone with a speed limit of 40 km/h. The fine given was

A \$65 **B** \$150 **C** \$250

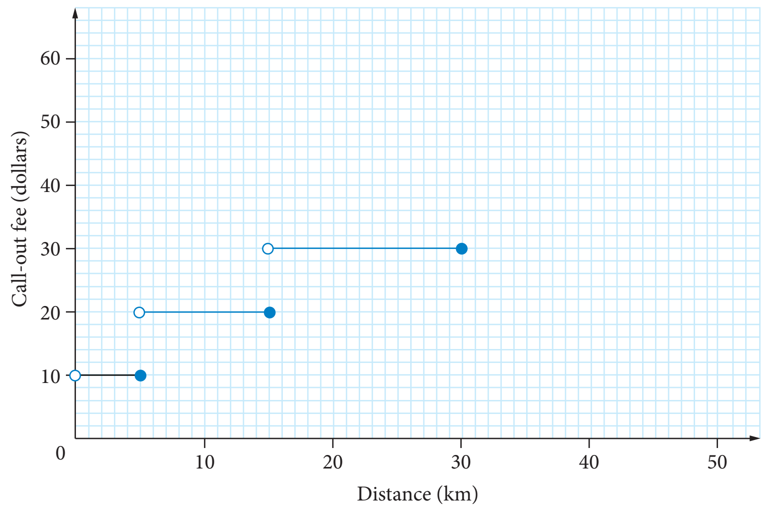
D \$400 **E** \$600

[VCAA 2013 1GRQ5]



Question 14

Harry operates a mobile pet care service. The call-out fee charged depends on the distance he has to travel to tend to a pet. The call-out fees for distances up to 30 km are shown on the graph.



a According to this graph

i what is the call-out fee to travel a distance of 20 km? 1 mark

ii what is the maximum distance travelled for a call-out fee of $10? 1 mark

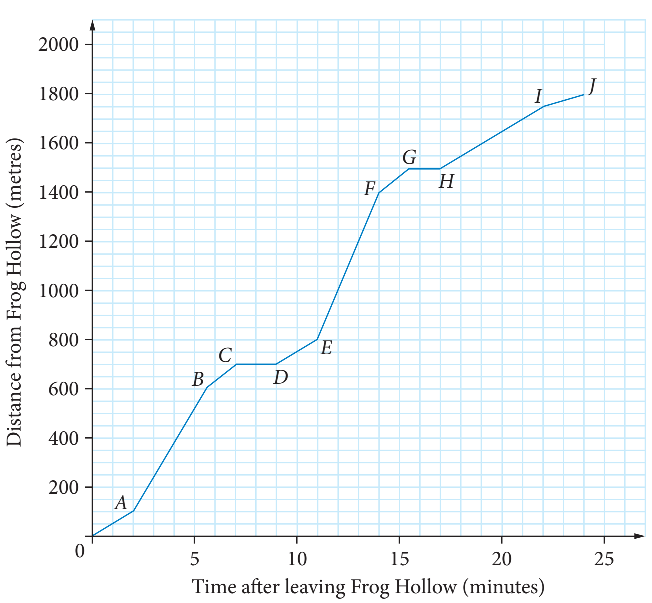
A call-out fee of $50 is charged to travel distances of more than 30 km but less than or equal to 40 km.

b Copy the graph and draw this information on it. 1 mark

[VCAA 2006 2GRQ1]

Question 15

At the Gum Flat Fun Park there are many attractions. One that appeals especially to the younger visitors is the train Puffing Polly. The distance–time graph represents a train trip for Puffing Polly from Fog Hollow to Eagle Hill, stopping at two stations on the way.



a What is the total time for which Puffing Polly is stopped at the two stations on the way? 1 mark

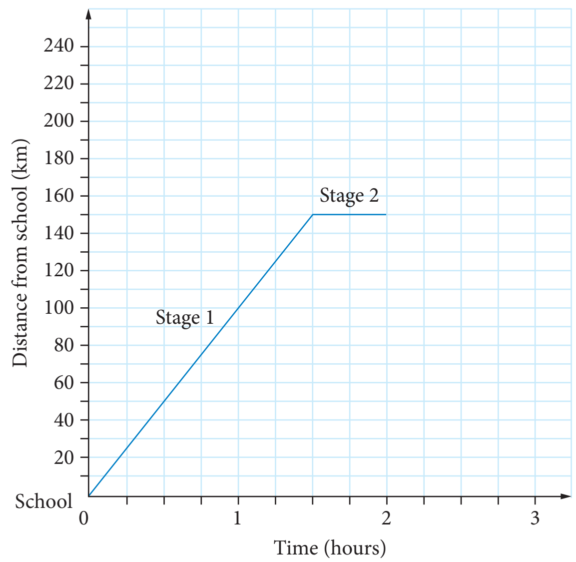
b **i** Which line segment of the graph represents the section of the trip when Puffing Polly is travelling the fastest? 1 mark

ii Find its speed for this section of the trip, stating clearly the units used in your answer. 2 marks

[VCAA 2002 2GRQ1]

Question 16

The distance–time graph shows the first two stages of a bus journey from a school to a camp.



a At what constant speed, in kilometres per hour, did the bus travel during stage 1 of the journey? 1 mark

b For how many minutes did the bus stop during stage 2 of the journey? 1 mark

The third stage of the journey is missing from the graph. During stage 3, the bus continued its journey to the camp and travelled at a constant speed of 60 km/h for one hour.

c Copy the graph and draw a line segment to represent stage 3 of the journey. 1 mark

d Find the average speed of the bus over the three hours. Write your answer in kilometres per hour. 1 mark

The distance, D km, of the bus from the school t hours after departure is given by

$$D = \begin{cases} 100t & 0 \le t \le 1.5 \\ 150 & 1.5 \le t \le 2 \\ 60t + k & 2 \le t \le 3 \end{cases}$$

e Determine the value of k. 1 mark

[VCAA 2013 2GRQ1]

Question 17

Singlets are produced and sold in large quantities. The revenue, R_S dollars, generated from the sale of x singlets is given by the equation

$$R_S = \begin{cases} 10x & x \le 500 \\ 6x + 2000 & x > 500 \end{cases}$$

a Calculate the revenue, R_S, generated by the sale of 620 singlets. 1 mark

b Sketch a graph of the revenue, R_S, for $0 \le x \le 1000$.

c If the cost, C_S dollars, of producing x singlets is $C_S = 4x + 1500$, determine the number of singlets that would need to be produced and sold to obtain a profit of \$2000. 2 marks

[VCAA 2004 2GRQ3]

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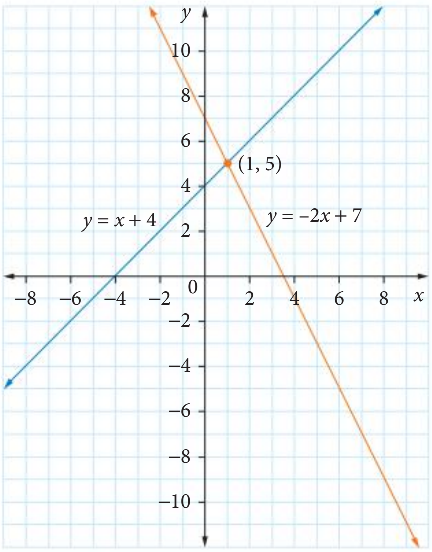
14.4 Simultaneous equations

Graphing and solving simultaneous equations

Sketching simultaneous equations

Solving simultaneous equations

A pair of equations such as $y = x + 2$ and $y = 2x - 5$ that can be solved together to find the values of x and y that satisfy *both* equations are called **simultaneous equations**. If simultaneous linear equations are graphed on the same Cartesian plane, their solution is the coordinates of the point where their graphs intersect.



In the graph shown, the point of intersection of $y = x + 4$ and $y = -2x + 7$ is the point (1, 5), so the solution to the simultaneous equations is $x = 1$ and $y = 5$.

You can check that your solution to simultaneous equations is correct by substituting back into the original equations. So, in the above example, if you substitute $x = 1$ into $y = x + 4$ and $y = -2x + 7$ and you get $y = 5$, verifying that your solution is correct.

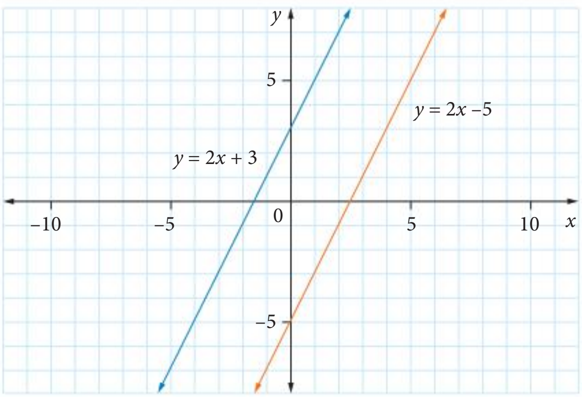
Simultaneous equations without unique solutions

Not every pair of linear simultaneous equations has a unique solution because lines don't always have a point of intersection. There are two ways this can occur for straight lines.

1. *The lines are parallel*. For example:

   $2x - y = 5$ or $y = 2x - 5$

   $2x - y = -3$ or $y = 2x + 3$

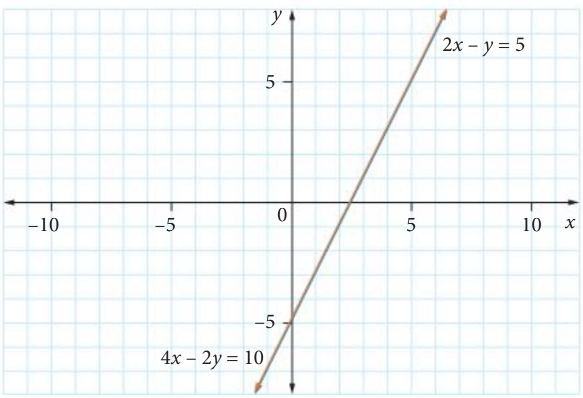


Both lines have a slope of 2 and hence no intersection, which means that the simultaneous equations have no solution.

2. *The lines involved are actually the same line*. For example:

   $2x - y = 5$

   $4x - 2y = 10$



If we divide both sides of the second equation by 2, then we get $2x - y = 5$, which is the same as the first equation. This means that there is an infinite number of points of intersection between the two lines, so there is no unique solution.

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14.4

Using CAS Solving simultaneous equations

Solve the following simultaneous equations and check that your solution is correct.

$y = 2x + 1$

$4x + y + 5 = 0$

Solve by hand, if it is quicker.

1	Make y the subject of both equations.	$y = 2x + 1$ $y = -4x - 5$
2	Equate the two y terms and solve for x.	$2x + 1 = -4x - 5$ $6x = -6$ $x = -1$
3	Substitute the x value into either one of the original two equations and solve for y.	$y = 2 \times (-1) + 1 = -1$
4	Write the solution.	$x = -1, y = -1$

TI-NSPIRE CAS

STEP 1

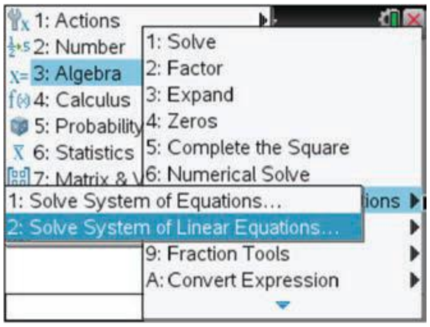
Open a New Document with a Calculator page.

Press menu.

Select 3: Algebra.

Select 7: Solve System of Equations …

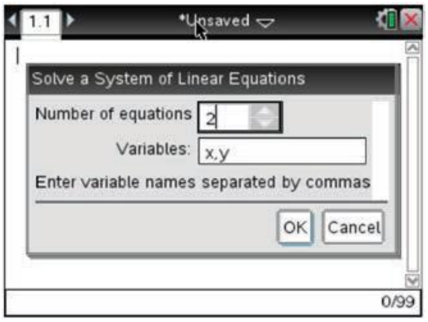
Select 2: Solve System of Linear Equations …



STEP 2

Complete the windows shown on the screen.

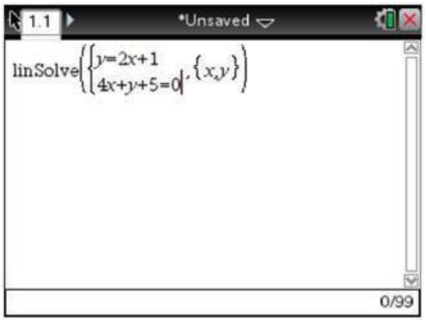
When OK is highlighted, press enter.



STEP 3

Enter the equations into the boxes as shown.

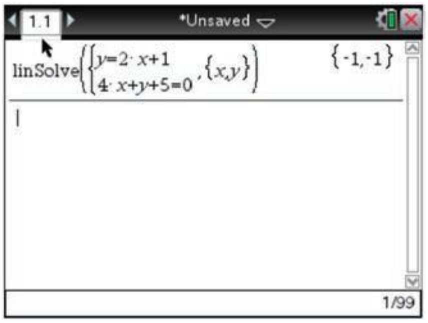
You can move between boxes by using the tab key.



STEP 4

Press enter to show the solution:

$x = -1, y = -1$.



STEP 5

Check if the solution is correct by substituting back into the original equations.

Substitute $x = -1$, $y = -1$ into equations [1] and [2].

Write the answer.

[1] $\quad -1 = 2 \times (-1) + 1$ ✓

[2] $\quad 4 \times (-1) - 1 + 5 = 0$

$\quad -4 + 4 = 0$ ✓

The solution to the simultaneous equations is $x = -1, y = -1$.

CLASSPAD

STEP 1

Use the $\sqrt{\alpha}$ Main application.

Press Keyboard then tap using Math1.

STEP 2

In the top line type $y = 2x + 1$.

In the bottom line type $4x + y + 5 = 0$.

After the vertical line, provided by the template, type x,y.

Press EXE.

STEP 3

Write the answer.

The solution to the simultaneous equations is $x = -1, y = -1$.

Modelling with simultaneous equations

Steps for solving worded problems for simultaneous equations.

1. Read the question carefully.
2. Identify the unknowns and assign a pronumeral for each of them.
3. Set up the equations by converting the given information into mathematical sentences.
4. Solve the simultaneous equations using the most appropriate method.
5. Check the solution in *both* of the original equations.
6. Answer the original question in sentence form.

Exam hack

Always check to see if both variables are given in the same units. Convert units where necessary.

Worked example 16

The length of a rectangle is 5 cm greater than its width. If the perimeter of the rectangle is 38 cm, find the dimensions of the rectangle.

	Working
1 Define the variables.	Let l = length and w = width.
2 You can draw a diagram to help you. Use the given information to write two linear equations in terms of the two variables.	 Length is 5 cm greater than its width. $l = w + 5$ Perimeter of a rectangle = $2l + 2w$, so $38 = 2l + 2w$
3 Solve using a CAS/calculator.	$l = 12$, $w = 7$.
4 Remember to check that the solution is correct by substituting back into the original equations.	Substitute $l = 12$ and $w = 7$ into the equations. $12 = 7 + 5$ ✓ $38 = 2 \times 12 + 2 \times 7$ $38 = 24 + 14$ ✓
5 Write the answer.	The length of the rectangle is 12 cm and the width of the rectangle is 7 cm.

Worked example 17

Toni spent \$16.40 on pens and pencils. She purchased a total of 8 items. If the pens cost \$2.80 each and the pencils cost 80 cents each, how many pens and pencils did Toni purchase?

	Working
1 Define the variables.	Let p = number of pens, q = number of pencils.
2 Use the given information to write two linear equations. Remember to convert the 80 cents to \$0.80, so that all the units are in dollars.	There are 8 items, so $p + q = 8$ Pens cost \$2.80 each, pencils cost \$0.80 each, and the total cost is \$16.40, so $2.8p + 0.8q = 16.4$
3 Solve using a CAS/calculator.	$p = 5$, $q = 3$.
4 Remember to check that the solution is correct and write the answers.	Toni purchased 5 pens and 3 pencils.

EXAM PREP 14.4

Simultaneous equations

Prep 1 USING CAS: SOLVING SIMULTANEOUS EQUATIONS

Solve the following pairs of simultaneous equations using a CAS/calculator.

a $y = x + 3$
$2x - y = -4$

b $x + y = 2$
$x - y = -6$

c $2x + y = 3$
$4x - y = 3$

d $3x - y = 4$
$2x - y = 2$

Prep 2 WORKED EXAMPLE 16

The length of a rectangle is three times its width. If the perimeter of the rectangle is 52 cm, find the dimensions of the rectangle.

Prep 3 WORKED EXAMPLE 17

Beachside Ice-creams sells ice-cream in single cones and double cones. The price of a single cone is \$3.80, whereas the price of a double cone is \$5.20. On a particular day, they sold a total of 450 cones and their total takings for the day were \$1990. How many of each type of cone did Beachside Ice-creams sell on the day?

EXAM PRACTICE 14.4

Simultaneous equations

Question 1

The total playing time of three CDs and four DVDs is 690 minutes. The total playing time of five CDs and seven DVDs is 1192 minutes. All of the CDs have the same playing time as each other and all of the DVDs have the same playing time as each other. Let x be the playing time of a CD. Let y be the playing time of a DVD. The set of simultaneous linear equations that can be solved to find the playing time of a CD and the playing time of a DVD is

A $4x + 3y = 690$
$7x + 5y = 1192$

B $3x + 4y = 690$
$5x + 7y = 1192$

C $3x + 5y = 690$
$4x + 7y = 1192$

D $3x + 4y = 1192$
$5x + 7y = 690$

E $4x + 3y = 1192$
$7x + 5y = 690$

[VCAA 2009 1GRQ4]

Question 2

Bruce and John both work in a factory. They assemble bicycles and scooters. It takes 45 minutes to assemble a bicycle and 15 minutes to assemble a scooter. Bruce assembled 7 bicycles and 8 scooters. In the same time, John assembled 6 bicycles and a number of scooters. The number of scooters that John assembled is

A 3 **B** 6 **C** 11 **D** 13 **E** 31

[VCAA 2010 1GRQ7]

Question 3

The Blue Caps cricket club has different prices for its junior and senior subscriptions. The total cost for two junior subscriptions and one senior subscription is \$225. The cost of a senior subscription is three times the cost of a junior subscription. The cost of a senior subscription is

A \$45 **B** \$75 **C** \$90 **D** \$135 **E** \$180

[VCAA 2013 1GRQ4]

Question 4

Two lines intersect at point A on a graph. The equation of one of the lines is $3x + 4y = 26$.
The coordinates of point A could be

A (2, 5) **B** (3, 4) **C** (4, 3) **D** (5, 2) **E** (7, 22)

[VCAA 2011 1GRQ3]

Use the following information to answer Questions 5 & 6.

A publisher produces a restaurant guide each year. To produce x copies the cost is C dollars, where $C = 15\,000 + 15x$. If all of the x copies produced are sold, then the revenue gained is R dollars, where $R = 25x$.

Question 5

Which one of the following statements is not true?

A The cost and revenue equations are linear.

B The selling price for each copy of the guide is \$25.

C It will cost \$30 000 to produce 1000 copies of the guide.

D The revenue raised from selling 1000 copies of the guide is \$15 000.

E The revenue is more than the cost if 1600 copies of the guide are sold.

[VCAA 2003 1GRQ4]

Question 6

If x copies of the guide are produced and sold, then the profit made is P dollars, where P is given by

A $P = 15\,000 - 10x$ **B** $P = 10x - 15\,000$ **C** $P = 15x - 15\,000$

D $P = 40x - 15\,000$ **E** $P = 15\,000 - 40x$

[VCAA 2003 1GRQ5]

Question 7

At the local bakery, the cost of four doughnuts and six buns is \$14.70. The cost of three doughnuts and five buns is \$11.90. At this bakery, the cost of one doughnut and two buns will be

A \$2.80 **B** \$3.80 **C** \$3.85 **D** \$4.55 **E** \$4.85

[VCAA 2008 1GRQ6]

Question 8

The fare, \$$F$, to travel a distance of n kilometres in a taxi is given by the rule $F = a + bn$. To travel a distance of 20 kilometres, the taxi fare is \$18.20. To travel a distance of 30 kilometres, the taxi fare is \$25.70. The charge per kilometre, b, is

A \$0.75 **B** \$0.88 **C** \$0.91 **D** \$1.33 **E** \$3.20

[VCAA 2011 1GRQ4]

Question 9

For the pair of simultaneous equations $2x - 3y = 7$ and $3x = 5 - y$, the solution is

A $x = -2, y = -1$ **B** $x = -1, y = -3$ **C** $x = -1, y = 2$
D $x = 2, y = -3$ **E** $x = 2, y = -1$

[VCAA 2003 1GRQ6]

Question 10

For the pair of simultaneous equations $4x = 7 - y$, $5x + 7y = 3$ the solution is

A $x = -2, y = -1$ **B** $x = -2, y = 1$ **C** $x = 1, y = 3$
D $x = -1, y = 2$ **E** $x = 2, y = -1$

[VCAA 2002 1GRQ4]

Question 11

Which one of the following pairs of simultaneous linear equations has no solution?

A $3x - y = 5$
$4x + y = 9$

B $2x - y = 1$
$4x - 2y = 3$

C $x + 3y = 0$
$2x - y = 7$

D $x - 3y = 10$
$3x + 2y = 8$

E $4x + y = -6$
$2x - y = 0$

[VCAA 2007 1GRQ8]

Question 12

One afternoon at the beach, Mr Smith bought four ice-creams and three drinks for his family at a cost of \$21.40. Mrs Brown bought five of the same ice-creams and two of the same drinks for \$20.80. Based on these prices, the cost of one drink is

A \$2.80 **B** \$2.90 **C** \$3.00 **D** \$3.30 **E** \$3.40

[VCAA 2005 1GRQ6]

Question 13

Two lines have equations $y = -5$ and $y = -x + 5$ respectively. The point that lies on both of these lines is

A $(-10, 5)$ **B** $(-5, 5)$ **C** $(0, -5)$ **D** $(5, -5)$ **E** $(10, -5)$

[VCAA 2005 1GRQ2]

Question 14

The cost, \$$C$, of hiring a boat for x hours is given by the equation $C = ax + b$, where a is the hourly rate and b is a fixed booking fee. When the boat is hired for 4 hours, the cost is \$320. When the boat is hired for 6 hours, the cost is \$450. When the boat is hired for one hour, the cost is

A \$65 **B** \$75 **C** \$77 **D** \$80 **E** \$125

[VCAA 2004 1GRQ6]

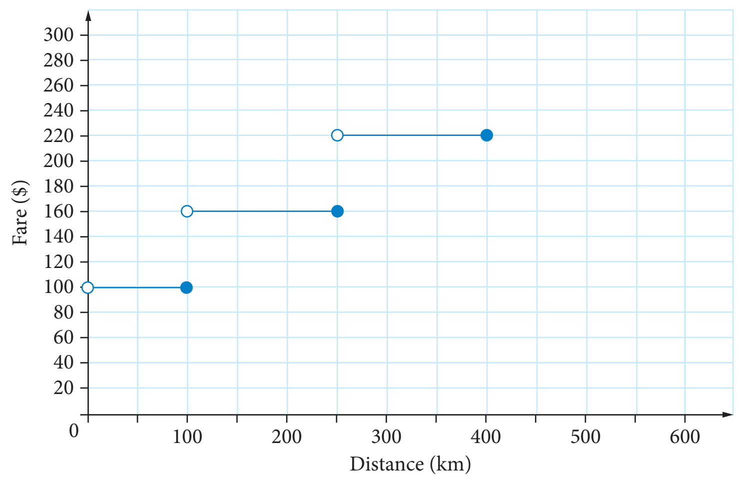
Question 15

Fair Go Airlines offers air travel between destinations in regional Victoria. The table shows the fares for some distances travelled.

Distance (km)	Fare
0 < distance ≤ 100	$100
100 < distance ≤ 250	$160
250 < distance ≤ 400	$220

a What is the maximum distance a passenger could travel for $160? 1 mark

The fares for the distances travelled in the table are graphed below.



b The fare for a distance longer than 400 km, but not longer than 550 km, is $280. Copy the graph and draw this information on it.

Fair Go Airlines is planning to change its fares. A new fare will include a service fee of $40, plus 50 cents per kilometre travelled. An equation used to determine this new fare is given by

Fare = 40 + 0.5 × distance.

c A passenger travels 300 km. How much will this passenger save on the fare calculated using the equation above compared to the fare shown in the table? 1 mark

d At a certain distance between 250 km and 400 km, the fare, when calculated using either the new equation or the table, is the same. What is the distance? 2 marks

e An equation connecting the maximum distance that may be travelled for each fare in the table can be written as Fare = $a + b \times$ maximum distance. Determine a and b. 2 marks

[VCAA 2009 2GRQ1]

Question 16

Two bushwalkers, Malinda and Christos, set out to walk from Fishbone Creek to Snake Gully, a distance of 20 km. They start at the same time and follow the same route.

a Malinda walks at a constant speed of 4 km/h for the entire journey and takes no rest periods. How far does she travel in 1.5 hours? 1 mark

b The distance walked by Malinda from Fishbone Creek, in kilometres, is given by the equation $D_m = 4t$ for $0 \le t \le 5$, where t is the time in hours since she began walking. Draw and label the graph of D_m against t. 2 marks

Christos started walking at the same time as Malinda and followed the same route. At the start, he walked at a constant speed of 6 km/h. However, after walking at this speed for two hours he developed sore feet. Rather than stopping, he slowed down to a constant speed of 2 km/h for the remainder of the trip.

c Let D_c represent the distance walked by Christos. Draw and label the graph of D_c against t on the set of axes you used for part **b**. 2 marks

d Malinda eventually catches up to Christos. How many hours after they start walking does this happen? 1 mark

e The equations give the distance, D_c, in kilometres, walked by Christos at any time t hours.

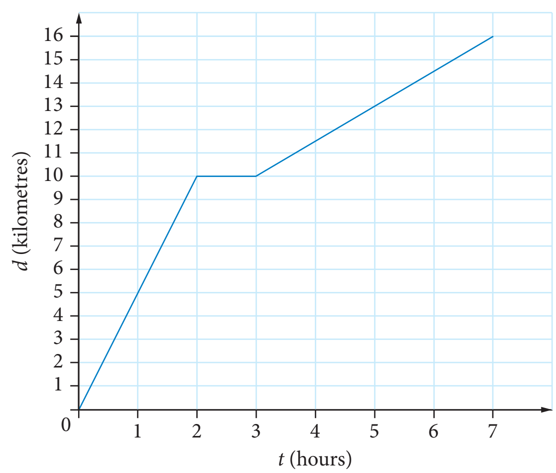
$$D_c = \begin{cases} at, & 0 \le t \le 2 \quad \text{hours} \\ bt + h, & 2 < t \le d \quad \text{hours} \end{cases}$$

Determine the values for a, b, h and d. 3 marks

[VCAA 2003 2GRQ1]

Question 17

Michael began his hike at the national park office and followed a track towards a camping ground, 16 km away. The distance–time graph shows Michael's distance from the national park office, d kilometres, after t hours of hiking.



It took Michael seven hours to complete this hike.

a What was Michael's average speed, in kilometres per hour, during this hike? Write your answer correct to one decimal place. 1 mark

The equation of Michael's distance–time graph from $t = 3$ to $t = 7$ is $d = at + b$.

b Determine the value of both a and b. 2 marks

Katie hiked along the same track as Michael, but hiked in the opposite direction. She began at the camping ground and hiked towards the national park office. Katie's distance from the national park office, d kilometres, after t hours of hiking, can be determined from the equation $D = -3t + 16$.

Katie and Michael both started hiking at the same time.

c After how many hours did Katie pass Michael? 1 mark

Katie and Michael both carry radio transmitters that allow them to talk to each other while hiking. The transmitters will not work if Katie and Michael are more than three kilometres apart.

d For how many hours during the hike were Michael and Katie able to use the radio transmitters to talk to each other? Write your answer in hours, correct to two decimal places. 2 marks

[VCAA 2011 2GRQ2]

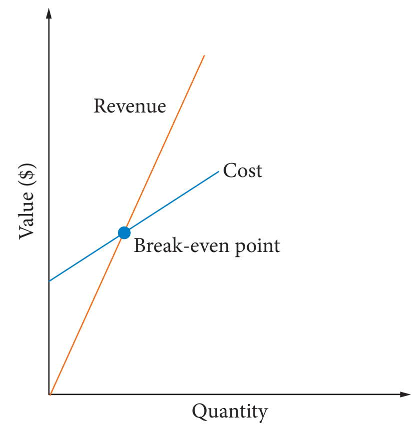
14.5 Break-even analysis

The break-even point

We saw earlier in the chapter that a cost equation and a revenue equation can be combined to find the profit equation:

Profit = revenue – cost

An important figure in business is the **break-even point**, the point where the cost equals the revenue (i.e. when the profit is exactly zero). It's the point of intersection between the cost equation and the revenue equation and can be found by solving the equations simultaneously.



Worked example 18

The cost and revenue equations for the Super Products Company are given below.

Cost: $V = 2000 + \dfrac{5x}{3}$ Revenue: $V = 15x$

where the value, V, is measured in thousands of dollars and x is the number of units sold.

a Graph both equations on the same set of axes for $0 \leq x \leq 300$.

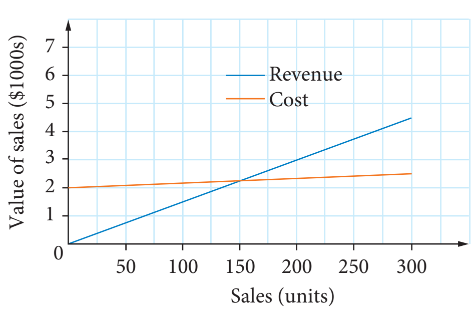
b Find the break-even point for the company.

c How many units need to be sold for the cost to be equal to the revenue?

d What are the costs when the break-even point has been reached?

Working

a Graph both lines on the same set of axes.



b	Use a CAS/calculator (unless solving by hand is quicker) to solve the simultaneous equations and find the point of intersection. Note the vertical scale is in \$1000s.	$x = 150$, $V = 2250$ The break-even point for the company = point of intersection = (150, 2250).
c	Use the point to write the answer.	150 units need to be sold for the cost to be equal to the revenue.
d	Use the point to write the answer.	The costs when the break-even point has been reached are \$2250.

14.5

Worked example 19

A bakery sells the first 50 Nutella doughnuts each day at a discount price of \$2.00 per doughnut. After that, a doughnut is sold for \$4.00.

The revenue, R, in dollars, made from selling n Nutella doughnuts each day is given by the rule

$$R = \begin{cases} 2n, & 0 \le n \le 50 \\ 4n - 100, & n > 50 \end{cases}$$

The cost, C, in dollars, of making n nutella doughnuts each day is $C = 250 + 0.5n$.

How many doughnuts need to be sold to break even each day?

		Working
1	Find the break-even number of doughnuts for both parts of the revenue equation. Find the break-even number of doughnuts for the first part of the revenue equation using a CAS/calculator (unless solving by hand is quicker).	For $0 \le n \le 50$, find the break-even number of doughnuts for the value, V. Revenue: $V = 2n$ [1] Cost: $V = 250 + 0.5n$ [2] Solving these equations gives $n \approx 167$. Since $0 \le n \le 50$, this answer is outside the range of values for n and can't be correct.
2	Find the break-even number of doughnuts for the second part of the revenue equation using a CAS/calculator (unless solving by hand is quicker).	For $n > 50$, find the break-even number of doughnuts for the value, V. Revenue: $V = 4n - 100$ [1] Cost: $V = 250 + 0.5n$ [2] Solving these equations gives $n = 100$ Since $n > 50$, this answer is correct.
3	Write the answer.	100 doughnuts need to be sold to break even each day.

Newsphoto/Newspix/Adam Elwood

EXAM PREP 14.5

Break-even analysis

Prep 1 WORKED EXAMPLE 18

The cost and revenue equations for Chimaera Enterprises are given below.

Cost: $V = 2000 + \dfrac{20x}{3}$ Revenue: $V = 20x$

where the value, V, is measured in thousands of dollars and x is the number of units sold.

a Graph both equations on the same set of axes for $0 \le x \le 300$.

b Find the break-even point for the company.

c How many units need to be sold for the cost to be equal to the revenue?

d What are the costs when the break-even point has been reached?

Prep 2 WORKED EXAMPLE 19

A street vendor sells the first 100 churros each day at a discount price of \$2.00 per churro. After that, a churro is sold for \$5.00.

The revenue, R, in dollars, made from selling n churros each day is given by the rule

$$R = \begin{cases} 2n, & 0 \le n \le 100 \\ 5n - 300, & n > 100 \end{cases}$$

The cost, C, in dollars, of making n churros each day is $C = 660 + 0.2n$. How many churros need to be sold to break even each day?

9780170371186

Break-even analysis

Question 1

Paul makes rulers. There is a fixed cost of $60 plus a manufacturing cost of $0.20 per ruler. Last week, Paul was able to break even by selling his rulers for $1 each. The number of rulers Paul sold last week was

A 50 **B** 75 **C** 90 **D** 120 **E** 150

[VCAA 2007 1GRQ4]

Question 2

Brian, a landscaping contractor, charges by the hour for his company's services. To complete a particular job, he will have to use three workers and pay each of them $20 per hour. The fixed costs for the job are $150 and it will take four hours to complete the job. To break even on this job, his hourly charge to the client should be

A $38.25 **B** $57.50 **C** $97.50 **D** $127.50 **E** $132.50

[VCAA 2009 1GRQ8]

Question 3

The cost of manufacturing a number of frying pans consists of a fixed cost of $400 plus a cost of $50 per frying pan. The manufacturer could break even by selling

A 10 frying pans at $90 each
B 10 frying pans at $45 each
C 15 frying pans at $60 each
D 15 frying pans at $30 each
E 20 frying pans at $50 each

[VCAA 2006 1GRQ8]

Question 4

A cafe sells the first 200 cups of hot chocolate each day at a special price of $3.00 per cup. After that, a cup of hot chocolate will be sold for $4.50. The revenue, R, in dollars, made from selling n cups of hot chocolate each day is given by the rule

$$R = \begin{cases} 3n, & 0 \le n \le 200 \\ 4.50n - 300, & n > 200 \end{cases}$$

The cost, C, in dollars, of making n cups of hot chocolate each day is $C = 500 + 1.30n$.

To break even, the number of cups of hot chocolate that must be sold each day is

A 63 **B** 200 **C** 250 **D** 295 **E** 300

[VCAA 2013 1GRQ9]

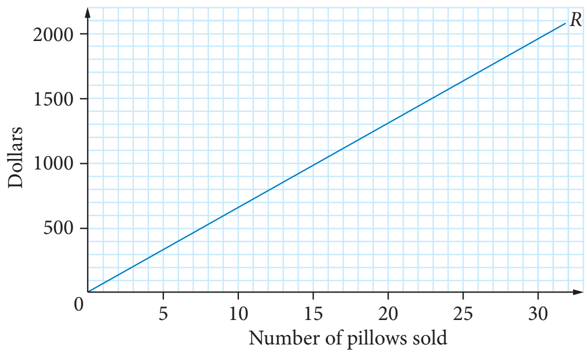
Question 5

Anne sells Softsleep pillows for $65 each.

a Write an equation for the revenue, R dollars, that Anne receives from the sale of x Softsleep pillows. 1 mark

b The cost, C dollars, of making x Softsleep pillows is given by $C = 500 + 40x$. Find the cost of making 30 Softsleep pillows. 1 mark

The revenue, R, from the sale of x Softsleep pillows is graphed below.



c Copy the graph and on it draw the graph of $C = 500 + 40x$. 1 mark

d How many Softsleep pillows will Anne need to sell in order to break even? 1 mark

e Anne also sells Resteasy pillows. Last week she sold 35 Softsleep and m Resteasy pillows. The selling price per pillow is shown in the table.

The total revenue from pillow sales last week was $4275. Find m, the number of Resteasy pillows sold.

Type	Selling price per pillow	Number sold
Softsleep	$65	35
Resteasy	$50	m

1 mark

[VCAA 2010 2GRQ1&2]

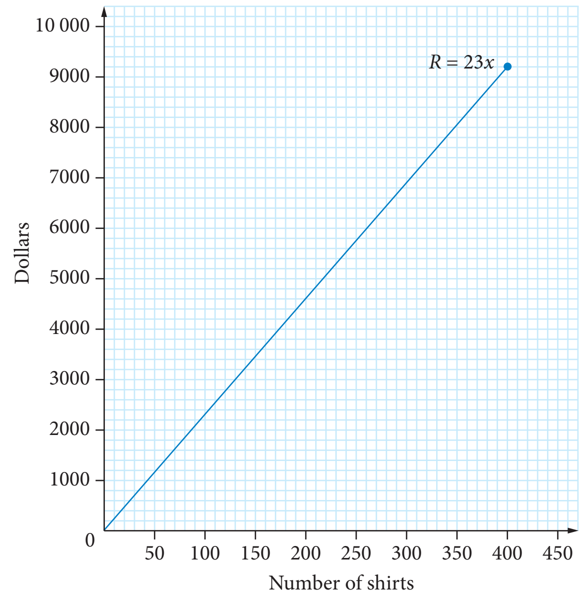
Question 6

A clothing manufacturer finds that the cost, C dollars, of producing x shirts is given by the equation $C = 8x + 2400$.

a Determine the cost of producing 400 shirts. 1 mark

b Determine the maximum number of shirts that can be produced for $3000. 1 mark

c Assuming all the shirts are sold, the revenue, R dollars, from the sale of x shirts produced is given by an equation $R = 23x$. A graph of the revenue equation $R = 23x$ for $0 \le x \le 400$ is drawn on the axes shown. Copy this graph and on these same axes draw a graph of the cost equation $C = 8x + 2400$ for $0 \le x \le 400$. 2 marks



d Determine the number of shirts that need to be produced and sold for the manufacturer to break even. 1 mark

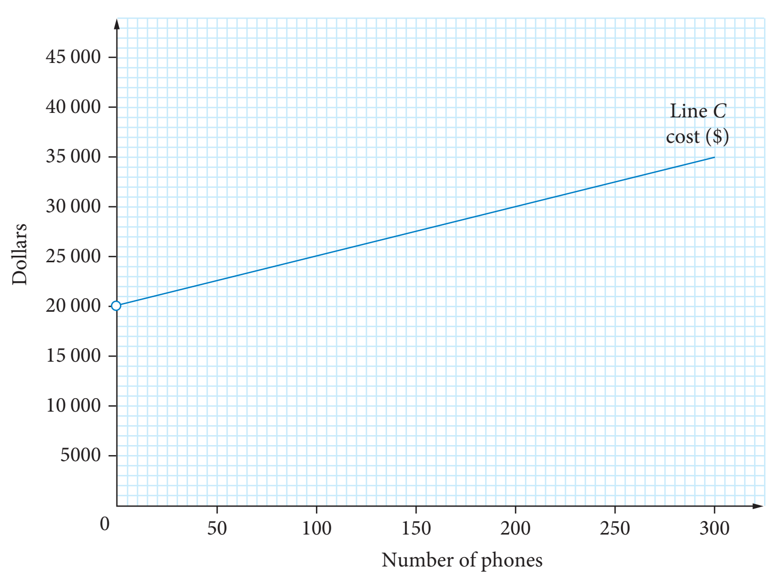
e Given the cost equation is $C = 8x + 2400$ and the revenue equation is $R = 23x$, write an equation for the profit, P dollars, from the production and sale of x shirts. 2 marks

f Calculate the profit from the production and sale of 345 shirts. 1 mark

g The manufacturer also produces jackets. They receive an order for 250 jackets. The cost of producing the 250 jackets is \$4800. Determine the selling price per jacket to achieve an overall profit of \$3000. 2 marks

[VCAA 2004 2GRQ1&2]

Question 7



The cost, C, in dollars, of making n phones is shown by the line in the graph.

a **i** Calculate the gradient of the line, C. 1 mark

ii Write an equation for the cost, C, in dollars, of making n phones. 1 mark

b The revenue, R, in dollars, obtained from selling n phones is given by $R = 150n$.

  i Copy the graph shown and draw this line on it. 1 mark

  ii How many phones would need to be sold to obtain $54 000 in revenue? 1 mark

c Determine the number of phones that would need to be made and sold to break even. 1 mark

[VCAA 2012 2GRQ1]

Question 8

Tiffany decides to enter a charity event involving running and cycling. There is a $35 fee to enter.

a Write an equation that gives the total amount, R dollars, collected from entry fees where there are x competitors in the event. 1 mark

The event costs the organisers $50 625 plus $12.50 per competitor.

b Write an equation that gives the total cost, C, in dollars, of the event when there are x competitors. 1 mark

c **i** Determine the number of competitors required for the organisers to break even. 1 mark

The number of competitors who entered the event was 8670.

  ii Determine the profit made by the organisers. 1 mark

[VCAA 2008 2GRQ2]

Question 9

The cost, C, and revenue, R, in dollars, for making and selling n laptops respectively is given by

Cost: $C = 320n + 125\,000$ Revenue: $R = 600n$

a What is the minimum number of laptops that should be made and sold in order to obtain a profit? 1 mark

The cost of making each laptop increases by $50.

b The selling price of each laptop will need to increase to offset this cost increase.
Find the new selling price of each laptop so that the break-even point occurs when 400 laptops are made and sold. 1 mark

[VCAA 2012 2GRQ2]

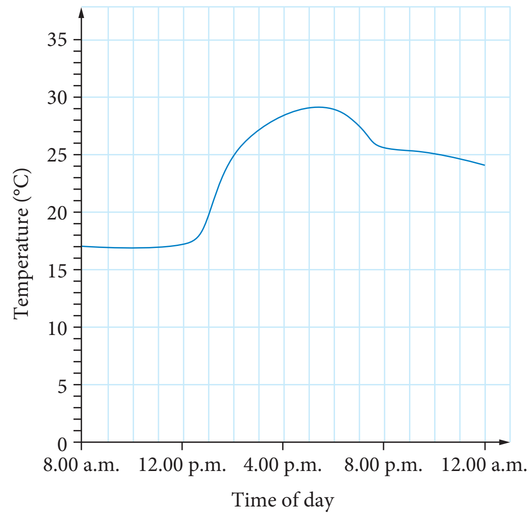
Interpreting non-linear graphs

14.6

We have so far only explored graphs involving straight lines. Most graphs of real-life phenomena aren't linear.

Worked example 20

The graph represents the temperature, in degrees Celsius, over four 4-hour time intervals.



a What is the approximate maximum temperature?

b What was the approximate temperature at 2.00 p.m.?

c During which of the four time intervals was the temperature the most stable?

d During which of the four time intervals did the greatest increase in temperature occur?

e Approximately during which times was the temperature between 20°C and 25°C?

f What is the average rate of change in temperature per 4-hour time interval over the period shown?

g What do you think would be the temperature at 6 a.m. earlier on this day? How reliable do you think your prediction is? Give reasons.

	Working
a Find the high point of the graph.	29°C
b Read from the graph.	25°C
c Find the part of the graph that is closest to being horizontal.	8.00 a.m. to 12.00 p.m.
d Find the 4-hour time interval where the graph is the steepest.	12.00 p.m. to 4.00 p.m.
e Read from the graph.	Between 1.00 p.m. and 2.00 p.m., and between around 11.00 p.m. and 12.00 a.m.
f Find the difference between the temperature at the start and end of the period and divide by the number of intervals.	$\frac{24-17}{4}=\frac{7}{4}=1.75$ The average rate of change in temperature per 4-hour time interval is 1.75°C.

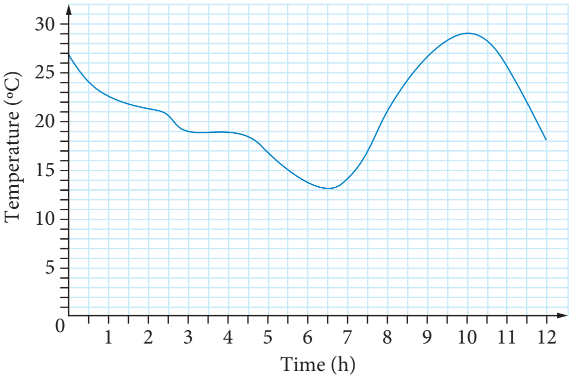
g Extrapolate two hours earlier from 8 a.m.	It looks like the temperature would be around 17°C, but this may not be a reliable prediction because it involves extrapolation. For example, a cool change may have occurred between 6.00 a.m. and 8.00 a.m. with a sudden significant drop in temperature.

EXAM PREP 14.6

Interpreting non-linear graphs

Prep 1 WORKED EXAMPLE 20

The graph represents the temperature, in degrees Celsius, over a 12-hour period.



a What is the approximate maximum temperature?

b What was the approximate temperature after $2\frac{1}{2}$ hours?

c During which hour was the temperature the most stable?

d During which half-hour period did the greatest increase in temperature occur?

e Approximately during which times was the temperature between 14°C and 17°C?

f What is the average rate of change in temperature per hour over the period shown?

g What do you think would be the temperature after 14 hours? How reliable do you think your prediction is? Give reasons.

Shutterstock.com/Andrey Armyagov

Interpreting non-linear graphs

Question 1

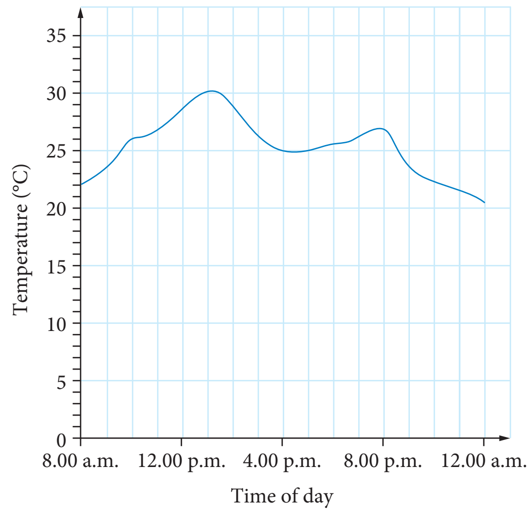
The graph represents the temperature, in degrees Celsius, over a 16-hour period. During this period, the minimum temperature occurred at

A 8.00 a.m. **B** 1.20 p.m.

C 4.15 p.m. **D** 7.45 p.m.

E 12.00 a.m.

[VCAA 2007 1GRQ3]



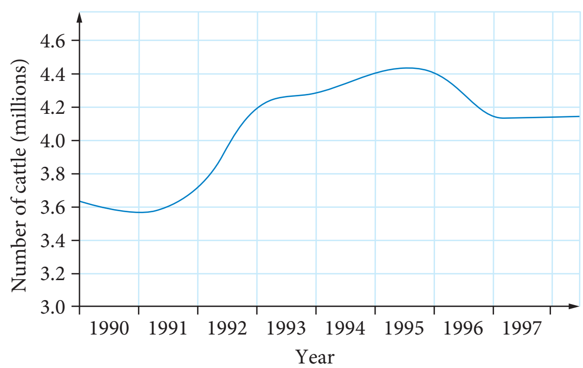
Question 2

The following graph shows the number of cattle in a particular state during the 1990s. Of the years listed, in which year did the number of cattle in the state remain relatively unchanged?

A 1991 **B** 1992

C 1994 **D** 1996

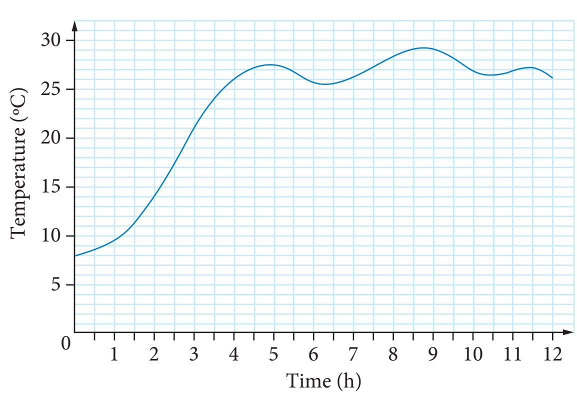
E 1997



[VCAA 2002 1GRQ3]

Use the following information to answer Questions 3 & 4.

The graph shows the water temperature in a fish tank over a 12-hour period.



Question 3

Over the 12-hour period, the temperature of the tank is increasing most rapidly

A during the first 2 hours

B from 2 to 4 hours

C from 4 to 6 hours

D from 6 to 8 hours

E from 8 to 10 hours

[VCAA 2009 1GRQ1]

Question 4

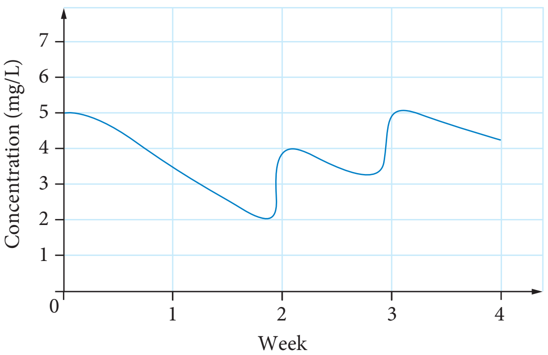
The fish tank is considered to be a safe environment for a type of fish if the water temperature is maintained between 24°C and 28°C. Over the 12-hour period, the length of time (in hours) that the environment was safe for this type of fish was closest to

A 1.5 **B** 5.0 **C** 7.0 **D** 8.5 **E** 10.5

[VCAA 2009 1GRQ2]

Question 5

The concentration (in mg/L) of a particular chemical in a swimming pool is graphed over a four-week period. For this four-week period, the concentration of the chemical was greater than 3 mg/L for



A exactly four weeks

B between three and four weeks

C exactly two weeks

D exactly one week

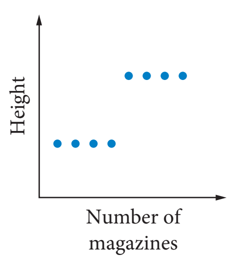
E less than one week

[VCAA 2008 1GRQ1]

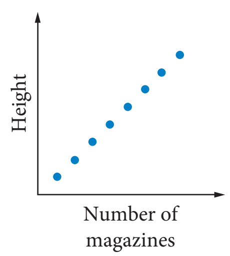
Question 6

A newsagent has four thick magazines and four thin magazines. The magazines are stacked one by one into a pile. The thick magazines are all placed in the bottom part of the pile, and the thin magazines are all placed in the top part of the pile. A graph that indicates the changing height of the pile of magazines as each magazine is added to the pile could be

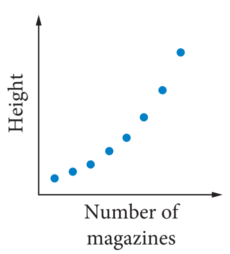
A



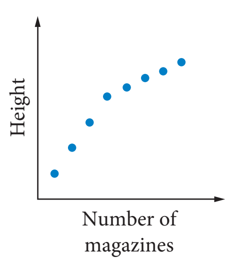
B



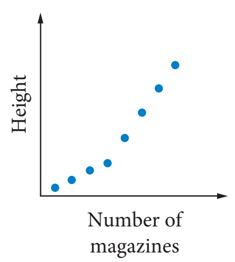
C



D



E



[VCAA 2011 1GRQ6]

Question 7

The graph shows the temperature (in degrees Celsius) over a 24-hour period. The period of greatest temperature increase was

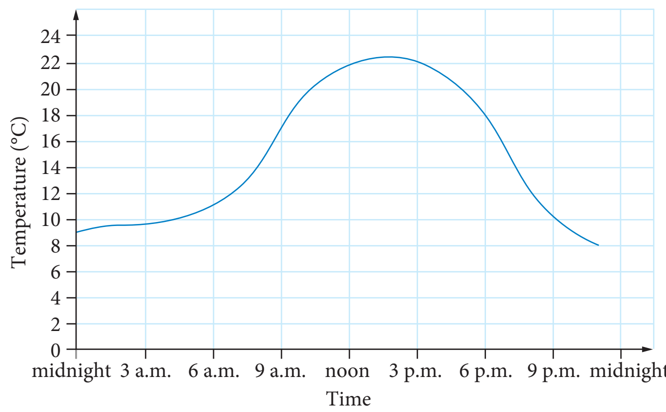
A 6 a.m. – 9 a.m.

B 9 a.m. – noon

C noon – 3 p.m.

D 3 p.m. – 6 p.m.

E 6 p.m. – 9 p.m.



[VCAA 2005 1GRQ1]

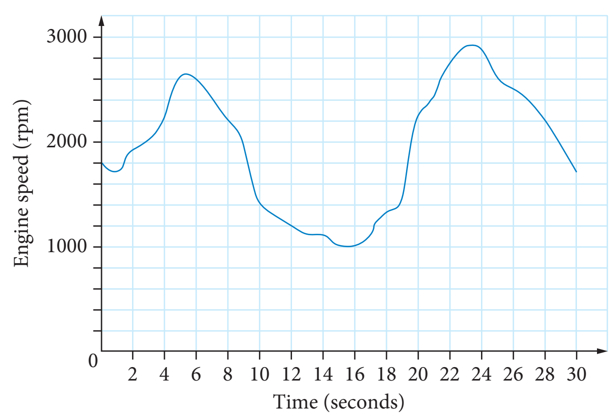
Question 8

The graph shows the engine speed of a car measured in revolutions per minute (rpm) over a period of 30 seconds. The total time that the engine speed was above 2200 rpm is

A 4 seconds **B** 6 seconds

C 12 seconds **D** 20 seconds

E 24 seconds



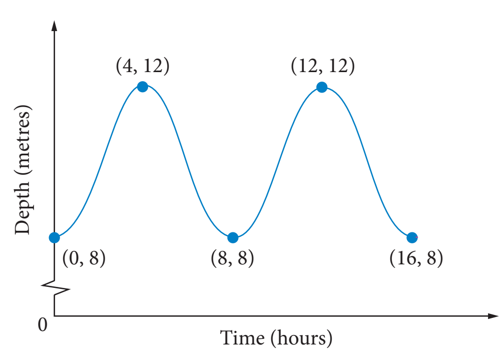
[VCAA 2003 1GRQ1]

Question 9

The graph represents the depth of water in a channel (in metres) as it changes over time (in hours). During the time interval shown, the number of times the depth of the water in the channel is 10 metres is

A 0 **B** 1 **C** 2

D 3 **E** 4

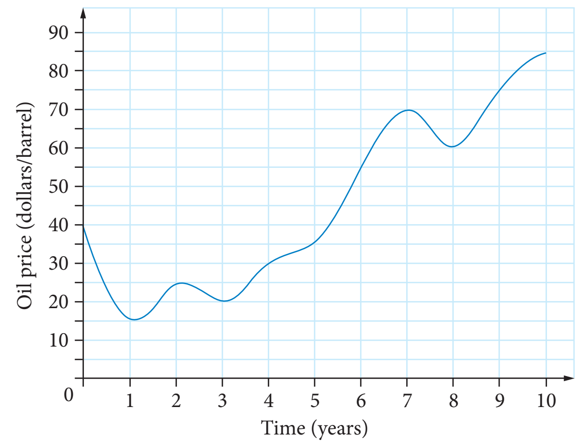


[VCAA 2006 1GRQ2]

14.6

Question 10

The oil price (dollars/barrel) over a ten-year period is shown in the graph. Which one of the following statements is **true**?



A The highest oil price over the ten-year period is \$70.

B The oil price decreased for exactly two of the ten years.

C The oil price changed most rapidly during the sixth year.

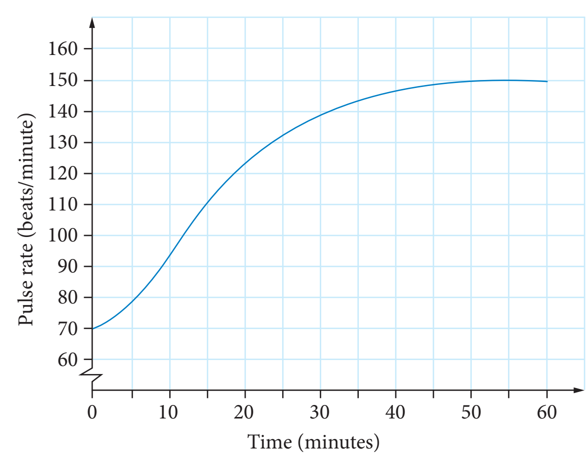
D On average, the oil price changed by \$4.50 per year over the ten years.

E The difference between the highest and lowest oil price during this ten-year period is \$45.

[VCAA 2012 1GRQ6]

Question 11

Tiffany's pulse rate (in beats/minute) during the first 60 minutes of a long-distance run is shown in the graph.



a What was Tiffany's pulse rate (in beats/minute) 15 minutes after she started her run? 1 mark

b By how much did Tiffany's pulse rate increase over the first 60 minutes of her run? Write your answer in beats/minute. 1 mark

c The recommended maximum pulse rate for adults during exercise is determined by subtracting the person's age in years from 220.

i Write an equation in terms of the variables *maximum pulse rate* and *age* that can be used to determine a person's recommended maximum pulse rate from his or her age. 1 mark

The target zone for aerobic exercise is between 60% and 75% of a person's maximum pulse rate. Tiffany is 20 years of age.

ii Determine the values between which Tiffany's pulse rate should remain so that she exercises within her target zone. Write your answers correct to the nearest whole number. 1 mark

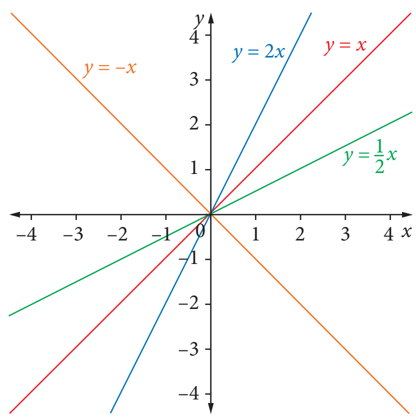
[VCAA 2008 2GRQ1]

Constructing non-linear graphs

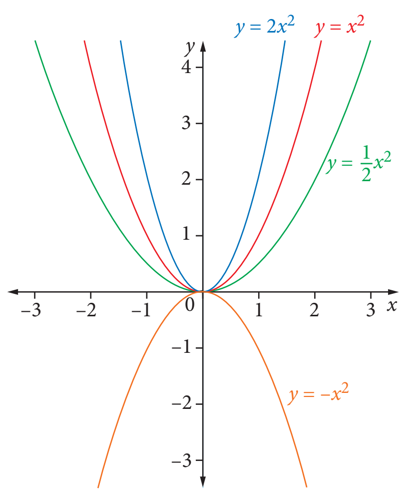
14.7

Graphs of relations of the form $y = kx^n$

Graphs in the form $y = kx^n$, where $n = 1, 2, 3, -1, -2$ and k is called the **constant of proportionality**, are shown.

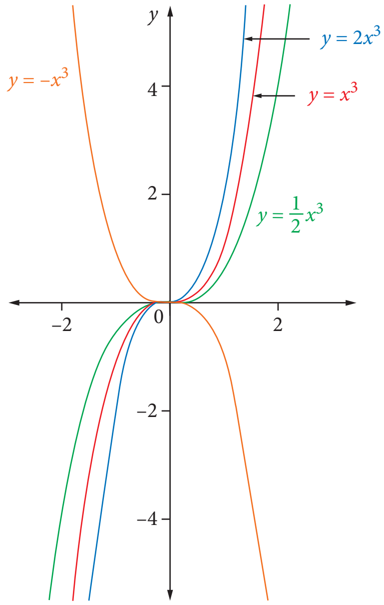
<table>
<tr><th>Power of x</th><th>Examples</th><th>Description</th></tr>
<tr><td>$n = 1$</td><td>$y = x$, $y = 2x$, $y = \frac{1}{2}x$, $y = -x$
</td><td>

- straight lines going through the origin
- positive k means positive slope
- negative k means negative slope
- $k > 1$ means steeper slope than $y = x$
- $0 < k < 1$ means less steep slope than $y = x$

</td></tr>
<tr><td>$n = 2$</td><td>$y = x^2$, $y = 2x^2$, $y = \frac{1}{2}x^2$, $y = -x^2$
</td><td>

- parabolas with turning points at the origin and symmetric about the y-axis
- positive k means the parabola is above the x-axis.
- negative k means the parabola is below the x-axis.
- $k > 1$ means the parabola is narrower than $y = x^2$
- $0 < k < 1$ means the parabola is wider than $y = x^2$

</td></tr>
</table>

<table>
<tr>
<td>$n = 3$</td>
<td>$y = x^3$, $y = 2x^3$, $y = \frac{1}{2}x^3$, $y = -x^3$

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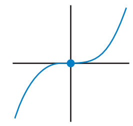
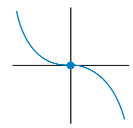
<td>

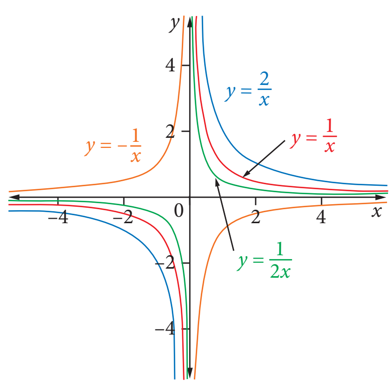
- cubics with points of inflection at the origin
- positive k means the basic shape is



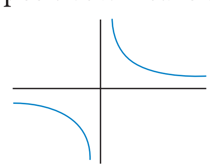
- negative k means the basic shape is



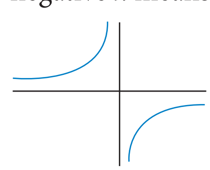
- $k > 1$ means the cubic is narrower than $y = x^3$
- $0 < k < 1$ means the cubic is wider than $y = x^3$

</td>
</tr>
<tr>
<td>$n = -1$</td>
<td>$y = \frac{1}{x}$ (or $y = x^{-1}$)
$y = \frac{2}{x}$ (or $y = 2x^{-1}$)
$y = \frac{1}{2x}$ (or $y = \frac{1}{2}x^{-1}$)
$y = -\frac{1}{x}$ (or $y = -x^{-1}$)

</td>
<td>

- hyperbolas that approach but never reach the x- and y-axes (where these lines they approach are called **asymptotes**).
- positive k means the basic shape is



- negative k means the basic shape is



- $k > 1$ means the two parts of the hyperbola are further apart than for $y = \frac{1}{x}$
- $0 < k < 1$ means the two parts of the hyperbola are closer together than for $y = \frac{1}{x}$

</td>
</tr>
</table>

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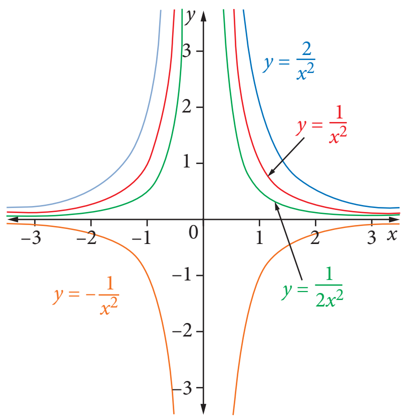
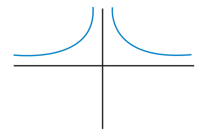
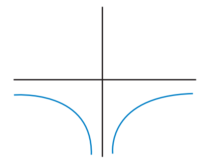
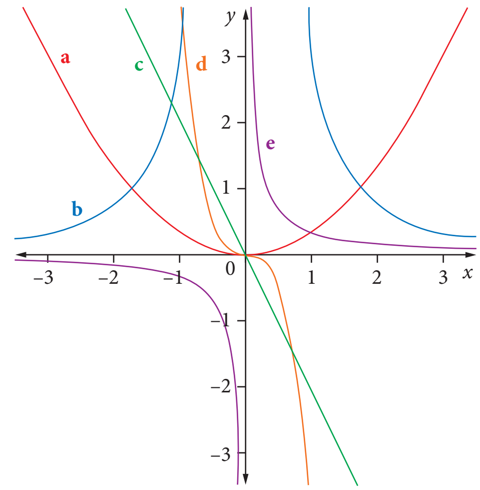
$n = -2$	$y = \frac{1}{x^2}$ (or $y = x^{-2}$) $y = \frac{2}{x^2}$ (or $y = 2x^{-2}$) $y = \frac{1}{2x^2}$ (or $y = \frac{1}{2}x^{-2}$) $y = -\frac{1}{x^2}$ (or $y = -x^{-2}$) 	• truncuses that, like hyperbolas, approach but never reach the x and y axes • positive k means the basic shape is  • negative k means the basic shape is  • $k > 1$ means the two parts are further apart than for $y = \frac{1}{x^2}$ • $0 < k < 1$ means the two parts are closer together than for $y = \frac{1}{x^2}$

iStockPhoto/Akabei

Worked example 21

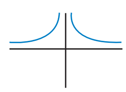
Match the graphs with their equation.

$y = \dfrac{1}{3x}$, $y = \dfrac{1}{3}x^2$, $y = -4x^3$, $y = -2x$, $y = \dfrac{3}{x^2}$.

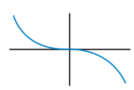


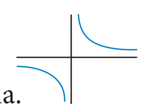
Working

a This is a parabola above the x-axis. $\qquad y = \dfrac{1}{3}x^2$

b The basic shape of this is  . $\qquad y = \dfrac{3}{x^2}$

c This is a straight line with a negative slope. $\qquad y = -2x$

d The basic shape of this is  , so it's a negative cubic. $\qquad y = -4x^3$

e The basic shape of this is  , so it's a positive hyperbola. $\qquad y = \dfrac{1}{3x}$

Worked example 22

Find the value of k in each of the following.

a The graph of $y = kx^2$ that goes through the point (3, 18)

b The graph of $y = kx^3$ that goes through the point (2, –80)

c The graph of $y = kx^{-1}$ that goes through the point (4, 5)

Working

a Substitute the x and y values into the equation and solve for k, using a CAS/calculator if necessary.

$x = 3, y = 18$

$18 = k \times 3^2$

$9k = 18$

$k = 2$

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b Substitute the x and y values into the equation and solve for k, using a CAS/calculator if necessary.

$x = 2, y = -80$

$-80 = k \times 2^3$

$8k = -80$

$k = -10$

c Substitute the x and y values into the equation and solve for k, using a CAS/calculator if necessary.

$x = 4, y = 5$

$5 = k \times 4^{-1}$

$5 = \frac{k}{4}$

$k = 20$

Linearisation

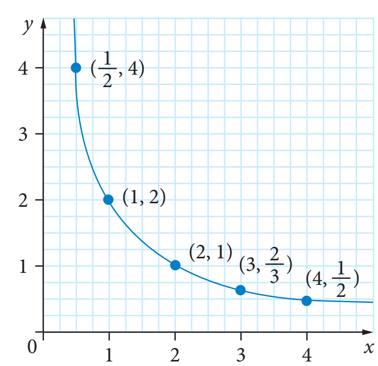
Linearisation involves putting something into straight-line form. It's possible to linearise non-linear graphs to make them simpler to work with. Let's look at $y = \frac{2}{x}$ for $x > 0$.

The table of values and graphs show that $y = \frac{2}{x}$ can be represented by a straight line if instead of graphing y against x, we graph y against $\frac{1}{x}$:

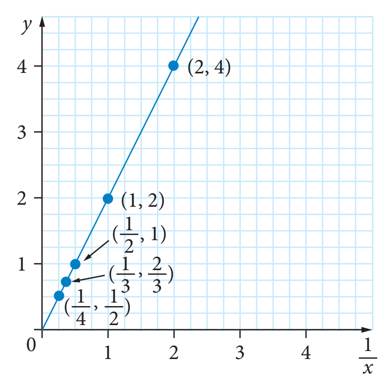
Table of values for $y = \frac{2}{x}, x > 0$

x	$\frac{1}{2}$	1	2	3	4
$\frac{1}{x}$	2	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$
y	4	2	1	$\frac{2}{3}$	$\frac{1}{2}$

Graph of y against x for $y = \frac{2}{x}, x > 0$



Graph of y against $\frac{1}{x}$ for $y = \frac{2}{x}, x > 0$



The slope of the straight line that we get is 2, which was the value of k, the constant of proportionality, in the non-linear equation.

If we graph $y = kx^n$ against x^n, where $n = 1, 2, 3, -1, -2$, then the graph will always be a straight line going through (0, 0) with slope = k. (Note the point (0 , 0) itself may not be included.)

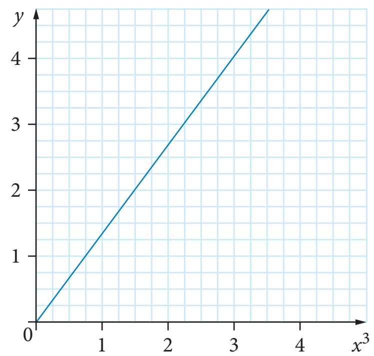
Exam hack

Always check the label on the horizontal axis to see if it is a non-linear axis.

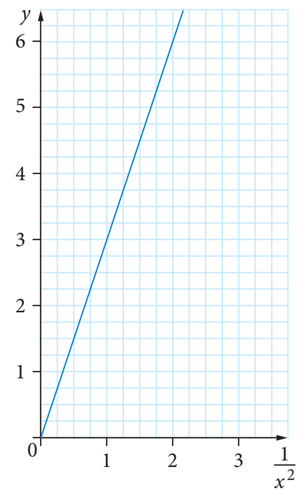
Worked example 23

Find the equation that matches the graph for each of the following.

a



b



Working

a **1** Identify what is being graphed.

y is being graphed against x^3.

2 Find the value of k in $y = kx^3$ by finding the slope of the line.

Two points on the line are (0, 0) and (3, 4), so the slope is $\frac{4-0}{3-0} = \frac{4}{3}$.

$k = \frac{4}{3}$

3 Write the equation that matches the graph.

$y = \frac{4}{3}x^3$

b **1** Identify what is being graphed.

y is being graphed against $\frac{1}{x^2}$.

2 Find the value of k in $y = \frac{k}{x^2}$ by finding the slope of the line.

Two points on the line are (0, 0) and (2, 6), so the slope is $\frac{6-0}{2-0} = \frac{6}{2} = 3$

$k = 3$

3 Write the equation that matches the graph.

$y = \frac{3}{x^2}$

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Worked example 24

A ball is dropped from a skyscraper and the relationship between the distance it has fallen, d metres, and the time taken, t seconds, is $d = kt^2$. The ball fell 20 m in the first 2 seconds.

a How far has it fallen after 4 seconds?

b Sketch the graph of d against t for $0 \leq t \leq 3$, showing the coordinates of the point at $t = 3$.

c Sketch the graph of d against t^2 for $0 \leq t^2 \leq 9$, showing the coordinates of the point at $t^2 = 9$.

d Verify that the slope of the line in part **c** is equal to the value of k.

e Verify that the point (100, 500) lies on the graph of d against t^2, and explain what the point tells us about the distance the ball has fallen and the time taken.

Working

a 1 Use the information given in the question and solve to find k.

When $t = 2$, $d = 20$.

Substituting into the equation:

$$20 = k \times 2^2$$
$$4k = 20$$
$$k = 5$$

2 Replace k in the equation and substitute the given value of t into the equation.

The equation is $d = 5t^2$.

Substituting $t = 4$ gives us $d = 5 \times 4^2 = 5 \times 16 = 80$

3 Write the answer.

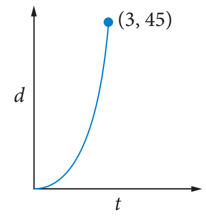
The ball has fallen 80 m after 4 seconds.

b 1 Find the point (t, d).

At $t = 3$, $d = 5 \times 3^2 = 45$

So the point is (3, 45).

2 Sketch the graph, using the fact that the equation represents a parabola.

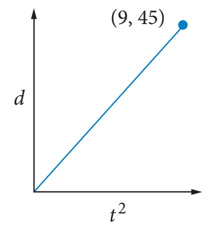


c 1 Find the point (t^2, d).

At $t^2 = 9$, $d = 5 \times 9 = 45$

So the point is (9, 45).

2 Sketch the graph, using the fact that it is a straight line going through (0, 0) and (9, 45).



d		Calculate the slope of the line.	Verify that the slope = $k = 5$. Slope $= \frac{45-0}{9-0} = \frac{45}{9} = 5$
e	**1**	Verify that the point (100, 500) lies on a straight line going through (0, 0), with a slope of 5.	Slope $= \frac{500-0}{100-0} = \frac{500}{100} = 5$, so (100, 500) lies on the line.
	2	Use the fact that the point is in the form (t^2, d).	(100, 500) tells us that when $t^2 = 100$, $d = 500$. So when $t = 10$, $d = 500$. This means that the ball has fallen 500 metres after 10 seconds.

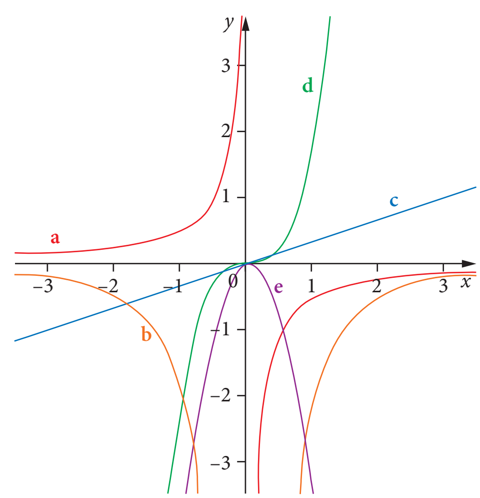
EXAM PREP 14.7

Constructing non-linear graphs

Prep 1 WORKED EXAMPLE 21

Match the graphs with their equation.

$$y = \frac{1}{3}x,\ y = -4x^2,\ y = -\frac{2}{x^2},\ y = 2x^3,\ y = -\frac{1}{2x}$$



Prep 2 WORKED EXAMPLE 22

Find the value of k in each of the following.

a The graph of $y = kx^2$ that goes through the point (2, –44)

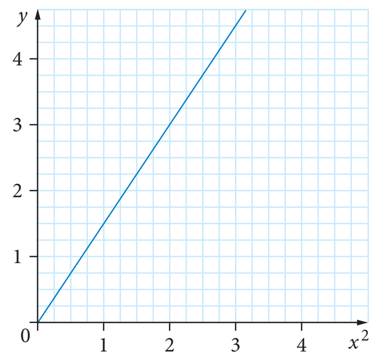
b The graph of $y = kx^3$ that goes through the point (10, 6000)

c The graph of $y = kx^{-2}$ that goes through the point (–3, 5)

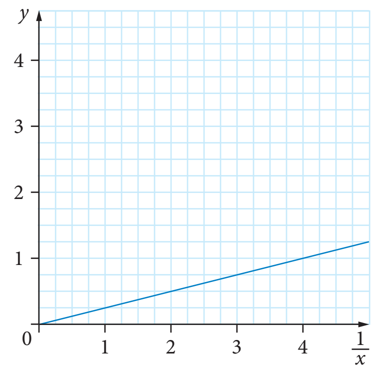
Prep 3 — WORKED EXAMPLE 23

Find the equation that matches the graph for each of the following.

a



b



Prep 4 — WORKED EXAMPLE 24

The equation that describes the volume of water (in litres) being poured into a container is $V = kd^3$, where d is the depth (in cm) of the water. When the depth of the water is 10 cm, the volume is one litre.

a What is the volume when the depth is 30 cm?

b Sketch the graph of V against d for $0 \le d \le 20$, showing the coordinates of the point at $d = 20$.

c Sketch the graph of V against d^3 for $0 \le d^3 \le 8000$, showing the coordinates of the point at $d^3 = 8000$.

d Verify that the slope of the line in part **c** is equal to the value of k.

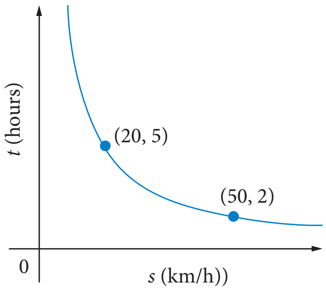
e Verify that the point (1 000 000, 1000) lies on the graph of V against d^3, and explain what the point tells us about the depth of the water and the volume.

EXAM PRACTICE 14.7

Constructing non-linear graphs

Question 1

The graph shows the time, t, in hours, taken to travel 100 km at an average speed of s km/h.



Which statement is false?

A As the average speed increases, the time taken to travel 100 km decreases.

B It will take 2 hours to travel 100 km at an average speed of 50 km/h.

C The relationship between time and average speed is linear.

D When travelling at an average speed of 20 km/h, the 100 km journey takes 5 hours to complete.

E A formula that relates s and t is $t = \dfrac{100}{s}$, $s > 0$.

[VCAA 2008 1GRQ3]

Question 2

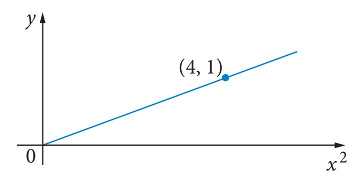
If the point (3, –2) lies on the curve with equation $y = \frac{k}{x^2}$, then the value of k is

A –18 **B** –12 **C** –6 **D** 12 **E** 18

[VCAA 2003 1GRQ2]

Question 3

The graph shows the relationship between y and x^2.
The relationship between y and x is



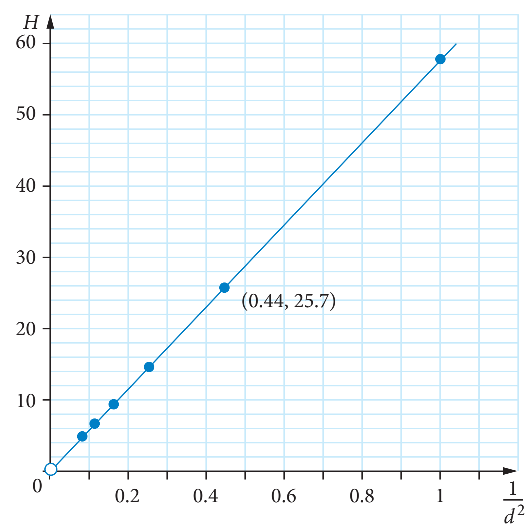
A $y = 4x$ **B** $y = \frac{1}{4}x$ **C** $y = \frac{1}{4}x^2$

D $y = 16x^2$ **E** $y = \frac{1}{16}x^2$

[VCAA 2008 1GRQ7]

Question 4

The heat intensity of a fire, H, is recorded at different distances, d, from the fire. When H is plotted against $\frac{1}{d^2}$, the data points lie on a straight line, as shown. The point (0.44, 25.7) lies on the line. Given this information, the rule that relates the intensity of the fire, H, to the distance, d, from the fire is closest to



A $H = \frac{58.4}{d^2}$ **B** $H = \frac{38.7}{d^2}$

C $H = \frac{4.98}{d^2}$ **D** $H = 38.7d$

E $H = 58.4d$

[VCAA 2013 1GRQ7]

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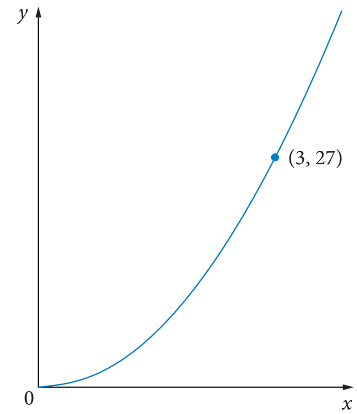
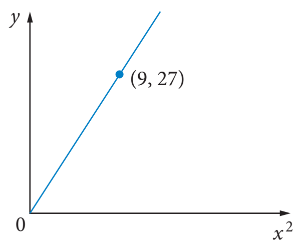
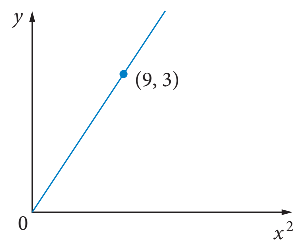
Question 5

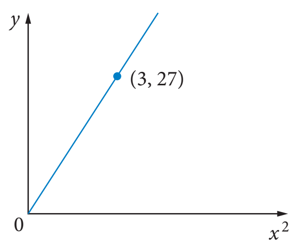
The graph of $y = 3x^2$ is shown on the right.

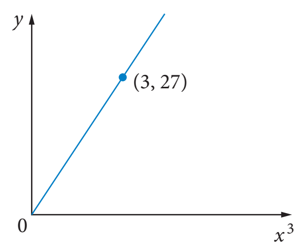


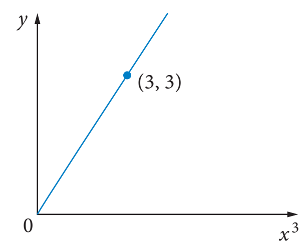
Another graph that represents this relationship between x and y is

A


B


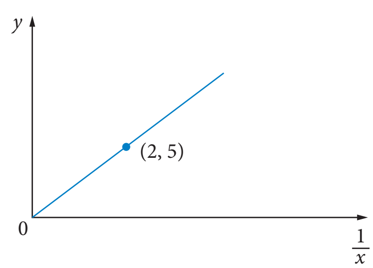
C


D


E


[VCAA 2010 1GRQ8]

Question 6



A graph of y versus $\frac{1}{x}$ is shown. The rule connecting x and y is

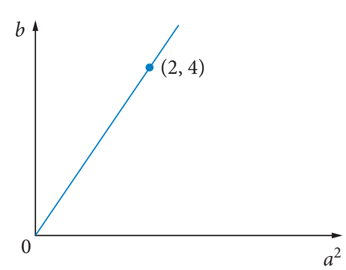
A $y = \frac{5}{2x}$ **B** $y = \frac{5}{2}x$ **C** $y = \frac{2}{5x}$

D $y = \frac{2}{5}x$ **E** $y = \frac{10}{x}$

[VCAA 20 1GRQ7]

Question 7

The relationship between the variables a and b, as shown in the graph, is



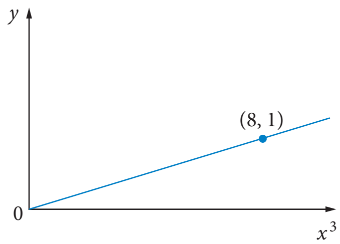
A $b = a^2$ **B** $b = 2a^2$

C $b = 2a$ **D** $a = 2b$

E $a^2 = 2b$

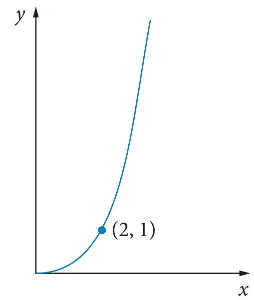
[VCAA 2005 1GRQ7]

Question 8

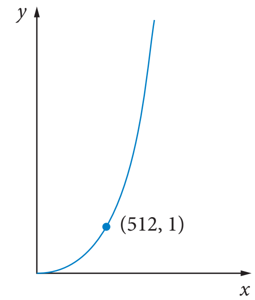


The graph represents a relationship $y = kx^3$ for $x \geq 0$. A graph that shows this relationship when plotted against x is

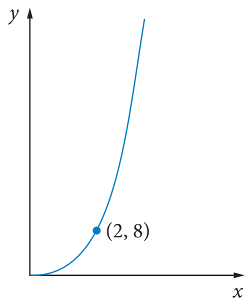
A



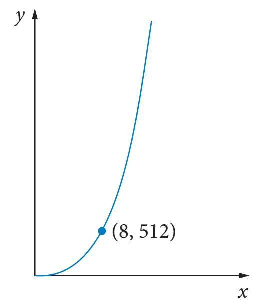
B



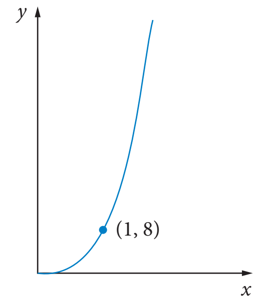
C



D



E

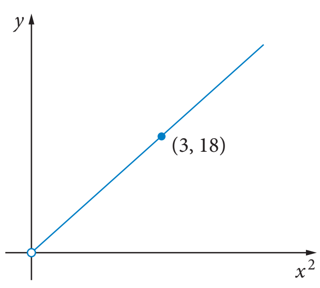


[VCAA 2004 1GRQ9]

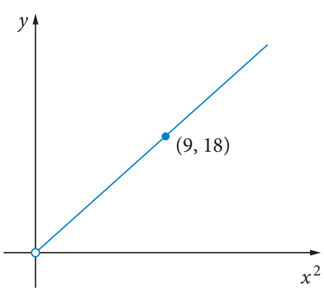
Question 9

The relation $y = 2x^2$, for $x > 0$, is represented by

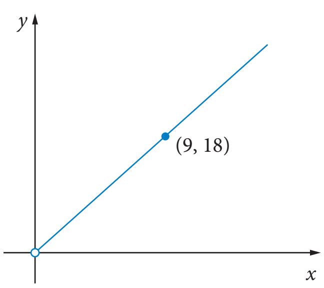
A



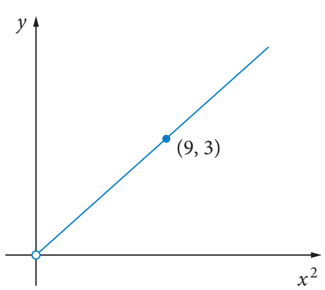
B



C



D



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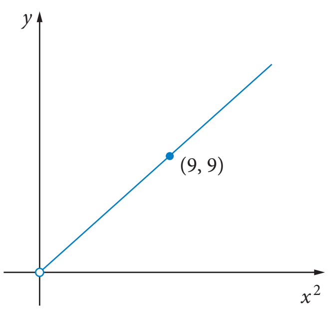
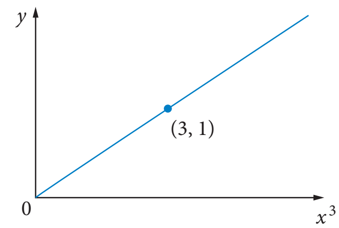


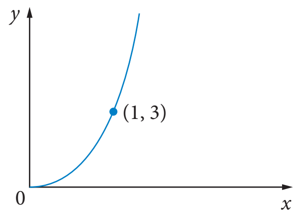
[VCAA 2002 1GRQ7]

Question 10

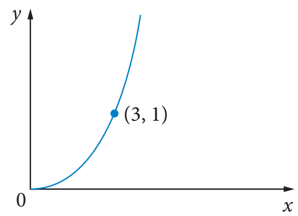
The graph shows a relationship between y and x^3. The graph that shows the same relationship between y and x is



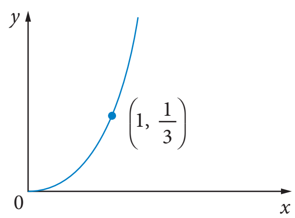
A



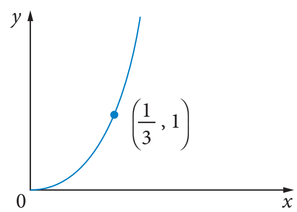
B



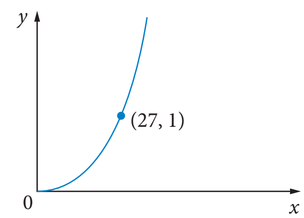
C



D



E

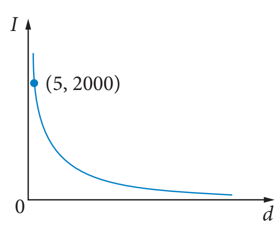


[VCAA 2007 1GRQ7]

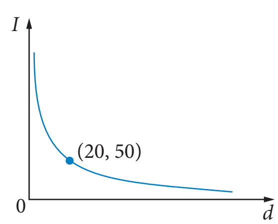
Question 11

The relationship between the intensity of sound, and the distance from the source of the sound, is $I = \frac{k}{d^2}$, where I is the intensity of sound measured in watts per square metre (W/m^2), d is the distance in metres from the source, and k is a constant. The intensity of sound is 20 W/m^2 at a distance of 50 m from the source. A graph showing the relationship between intensity of sound and distance from the source is

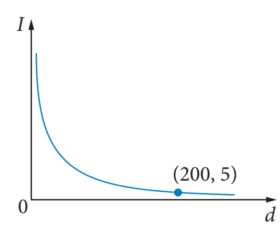
A



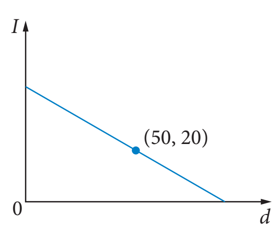
B



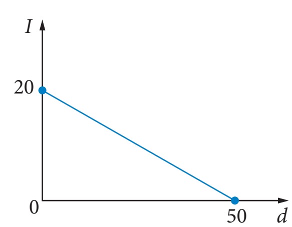
C



D



E



[VCAA 2005 1GRQ9]

14.7

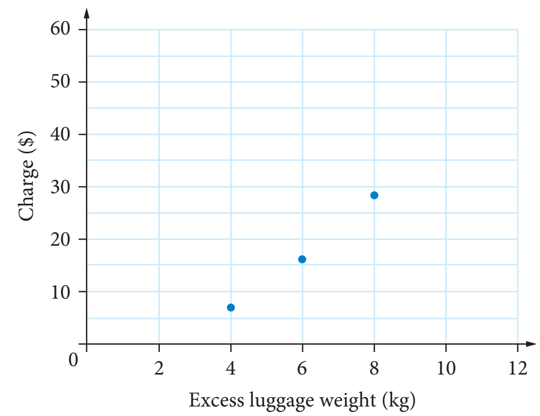
Question 12

Luggage over 20 kg in weight is called excess luggage. Fair Go Airlines charges for transporting excess luggage. The charges for some excess luggage weights are shown in the table.

Excess luggage weight (kg)	4	6	8	10
Charge ($)	$7.20	$16.20	$28.80	$45.00

a Copy and complete this graph by plotting the charge for excess luggage weight of 10 kg. Mark this point with a cross (×).

1 mark



b A graph of the charge against (*excess luggage weight*)^2 is to be constructed. Find the missing (*excess luggage weight*)^2 value in this table, then copy the graph and plot this point with a cross (×).

1 mark

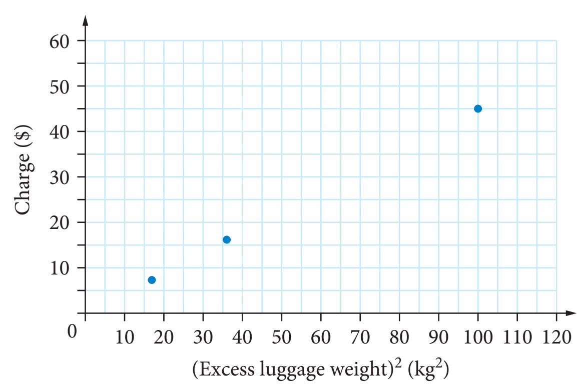
Excess luggage weight (kg)	4	6	8	10
(excess luggage weight)^2 (kg^2)	16	36		100
Charge ($)	$7.20	$16.20	$28.80	$45.00

c The graph can be used to find the value of k in the equation:

Charge = $k \times$ (*excess luggage weight*)^2

Find k. 1 mark

d Calculate the charge for transporting 12 kg of excess luggage. Write your answer in dollars correct to the nearest cent. 1 mark



e Another company, Cheapstar Airlines, uses the two equations to calculate the total cost of a flight. The passenger fare, in dollars, for a given distance, in km, is calculated using the equation Fare = 20 + 0.47 × distance.

The charge, in dollars, for a particular excess luggage weight, in kg, is calculated using the equation Charge = m × (excess luggage weight)^2.

Suzie will fly 450 km with 15 kg of excess luggage on Cheapstar Airlines. She will pay $299 for this flight. Determine the value of m.

2 marks

[VCAA 2009 2GRQ2&3]

Construction and interpretation of graphs

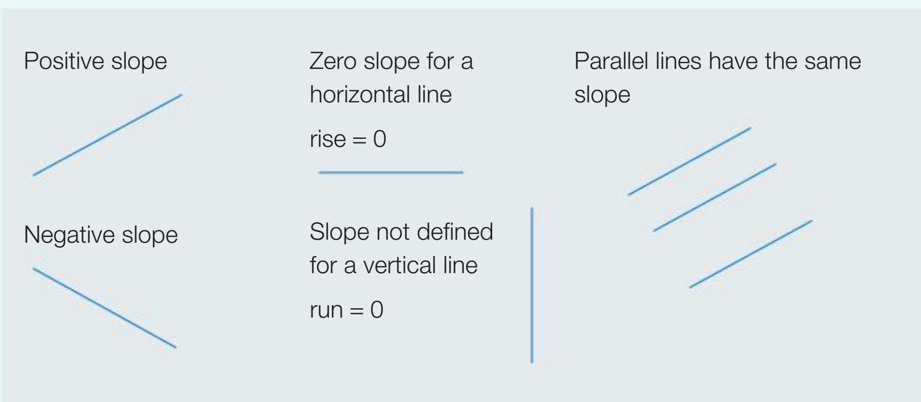
Practice quiz

SUMMARY

14

Linear equations and straight-line graphs

- To determine whether a point lies on a line, substitute the x-coordinate and the y-coordinate into the equation of the line. If the equation is true, then the point lies on the line, and if it is false, then the point does not lie on the line.
- Slope $= \dfrac{\text{rise}\uparrow}{\text{run}\rightarrow}$, where the rise is the vertical distance between two points and the run is the horizontal distance between the same two points.
- The slope measures the rate of change of y as x changes.



The intercept–slope form of a straight line

- Slope $= \dfrac{y_2 - y_1}{x_2 - x_1}$, where (x_1, y_1) and (x_2, y_2) are any two points on the line.
- $y = a + bx$, is called the **intercept–slope form** of the equation of the straight line, where a = the y-intercept of the line, b = the slope of the line
- To find the equation for a straight line given any two points
  - use the Regression function in the Statistics section of your CAS/calculator, OR
  - use the formula $y - y_1 = b(x - x_1)$, where b = the slope of the line and (x_1, y_1) is any point on the line.

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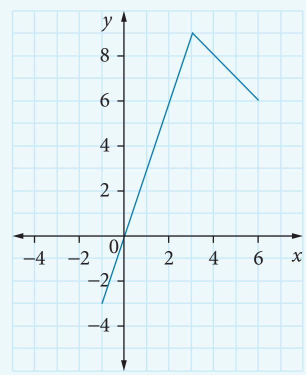
Linear modelling

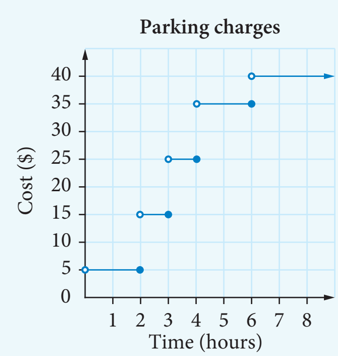
- We can use linear models to make predictions by substituting values into the equation.
- Predicting within the known data range is called **interpolation** and predicting outside the known data range is called **extrapolation**. Predictions that involve interpolation are always more reliable than predictions that involve extrapolation.
- Linear graphs model rate problems, where the slope of the graph measures the rate of change of y as x changes.
- Linear graphs can often be used to model both cost and revenue for a business.
- The formula Profit = revenue – cost can be used to model profit. If profit according to this formula is negative, then it's a loss.

Line segment and step graphs

- A **line segment linear graph** is a combination of two or more line segments.
- Line segment linear graphs are useful for displaying information that involves different rates of change.

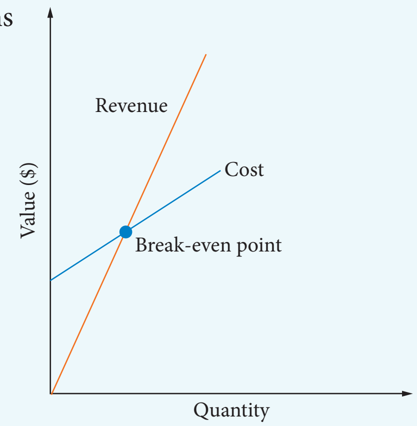


- A **step graph** is made up of horizontal line segments that look like steps, where the ends of each line segment can be open or closed. An open circle means that the point is *not* included as part of the graph, whereas a closed circle means that the point *is* included as part of the graph.



Simultaneous equations and break-even analysis

- If simultaneous linear equations are graphed on the same Cartesian plane, their solution is the coordinates of the point where their graphs intersect.
- Not every pair of linear simultaneous equations has a unique solution because lines don't always have a point of intersection. This can occur when the lines are parallel or the lines involved are actually the same line.
- Steps for solving worded problems for simultaneous equations
  1. Read the question carefully.
  2. Identify the unknowns and assign a pronumeral for each of them.
  3. Set up the equations by converting the given information into mathematical sentences.
  4. Solve the simultaneous equations using the most appropriate method.
  5. Check the solution in *both* of the original equations.
  6. Answer the original question in sentence form.

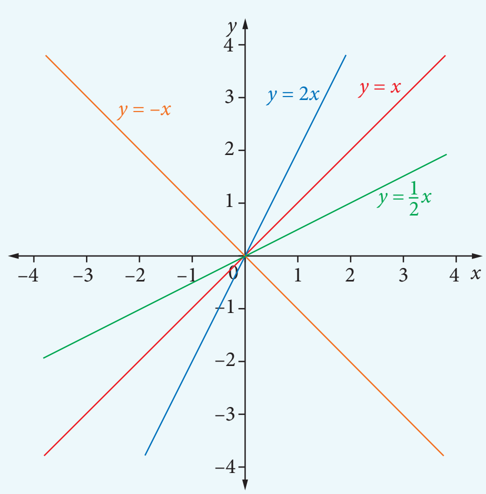


- The **break-even point** is the point where cost equals revenue.

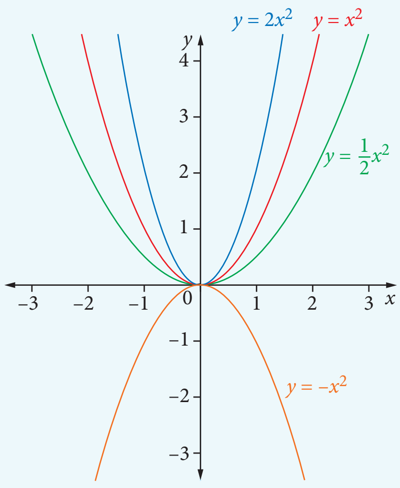
Non-linear graphs

- Graphs in the form $y = kx^n$, where $n = 1, 2, 3, -1, -2$ and k is called the **constant of proportionality**, are shown.

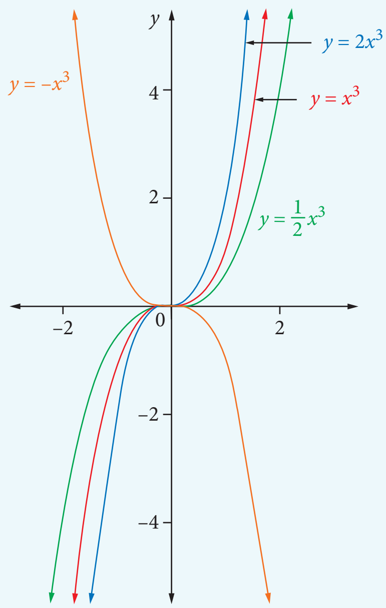
Straight lines



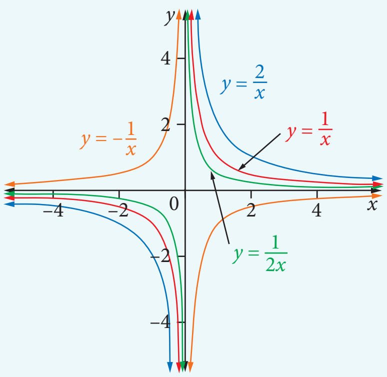
Parabolas



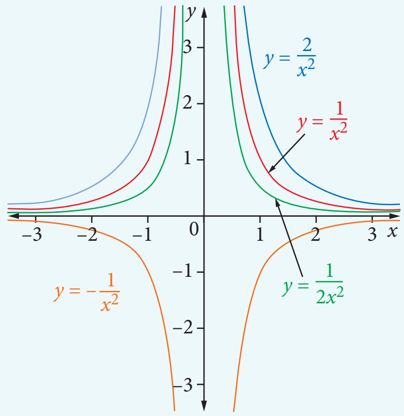
Cubics



Hyperbolas



Truncuses



- Non-linear graphs can be put into straight-line form to make them simpler to work with.
- If we graph $y = kx^n$ against x^n, where $n = 1, 2, 3, -1, -2$, then the graph will always be a straight line going through (0, 0) with slope = k. (Note the point (0 , 0) itself may not be included.)

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GRAPHS AND RELATIONS

Examination 1

Reading time: (5 minutes)
Writing time: (25 minutes)

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers, one bound reference, one approved graphics calculator or approved CAS calculator or CAS software and, if desired, one scientific calculator. Calculator memory DOES NOT need to be cleared.
- Students are NOT permitted to bring into the examination room: blank sheets of paper and/or white out liquid/tape.
- Students may refer to the sheet of miscellaneous formulas supplied

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

Instructions

Choose the response that is correct for the question.

A correct answer scores 1, an incorrect answer scores 0.

Marks will not be deducted for incorrect answers.

No marks will be given if more than one answer is completed for any question.

Unless otherwise indicated, the diagrams are **not** drawn to scale.

Question 1

The graph shows the altitude, in metres, of a balloon over a six-hour flight. Over the six-hour period, the length of time, in hours, where the altitude of the balloon was at least 1500 m is

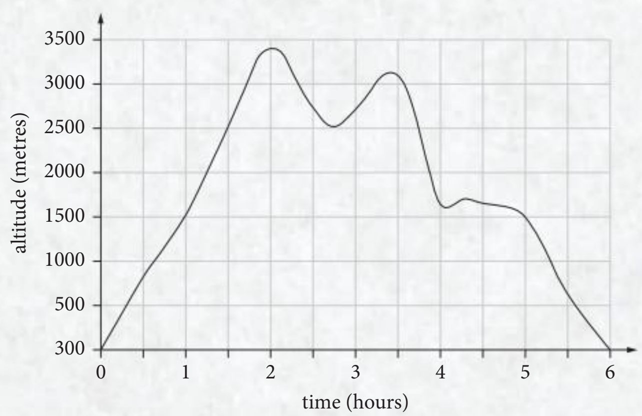
A. 3

B. 4

C. 5

D. 6

E. 7



[VCAA 2014 1GRQ1]

EXAMINATION 1 – continued

Question 2

The vertical line that passes through the point (3, 2) has the equation

A. $x + y = 5$ **B.** $xy = 6$ **C.** $3y = 2x$

D. $y = 2$ **E.** $x = 3$

[VCAA 2014 1GRQ2]

Question 3

The point (2, 20) lies on the graph of $y = \frac{k}{x}$, as shown.

The value of k is

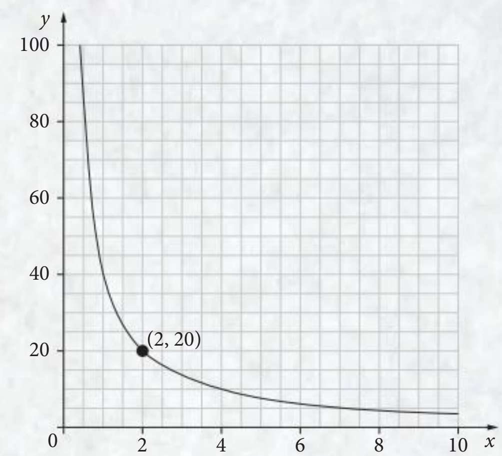
A. 5

B. 10

C. 20

D. 40

E. 80



[VCAA 2014 1GRQ3]

Question 4

A line passes through the points (–1, 1) and (3, 5). Another point that lies on this line is

A. (0, 1) **B.** (1, 3) **C.** (2, 6) **D.** (3, 4) **E.** (4, 7)

[VCAA 2014 1GRQ4]

Question 5

The distance–time graph shows a train's journey between two towns. During the journey, the train stopped for 30 minutes.

The average speed of the train, in kilometres per hour, for the journey is closest to

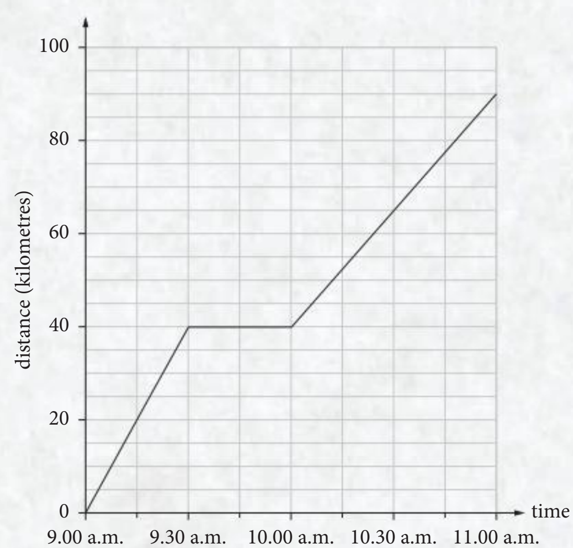
A. 45

B. 50

C. 60

D. 65

E. 80



[VCAA 2014 1GRQ5]

EXAMINATION 1 – continued

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Question 6

The Domestics Cleaning Company provides household cleaning services. For two hours of cleaning, the cost is \$55. For four hours of cleaning, the cost is \$94. The rule for the cost of cleaning services is Cost = $a + b \times$ hours where a is a fixed charge, in dollars, and b is the charge per hour of cleaning, in dollars per hour. Using this rule, the cost for five hours of cleaning is

A. \$19.50

B. \$97.50

C. \$99.50

D. \$113.50

E. \$121.50

[VCAA 2014 1GRQ6]

Question 7

Consider the following statements that relate to the solution of linear programming problems.
Which one of the following statements is **true**?

A. Only one point can be a solution.

B. No point outside the feasible region can be a solution.

C. To have a solution, the feasible region must be bounded.

D. Only the corner points of a feasible region can be a solution.

E. Only the corner points with integer coordinates can be a solution.

[VCAA 2014 1GRQ7]

Question 8

The constraints of a linear programming problem are given by the following set of inequalities.

$$x + y \le 8 \qquad 3x + 5y \le 30 \qquad x \ge 0 \qquad y \ge 0$$

The coordinates of the points that define the boundaries of the feasible region for this linear programming problem are

A. (0, 0), (0, 6), (3, 5), (8, 0)

B. (0, 0), (0, 6), (5, 3), (8, 0)

C. (0, 0), (0, 6), (5, 3), (10, 0)

D. (0, 0), (0, 8), (5, 3), (8, 0)

E. (0, 0), (0, 8), (5, 3), (10, 0)

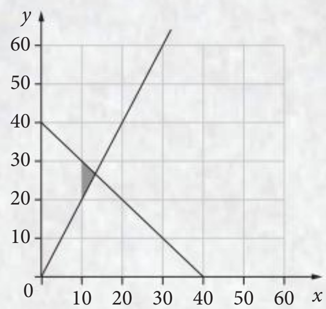
[VCAA 2014 1GRQ8]

EXAMINATION 1 – continued

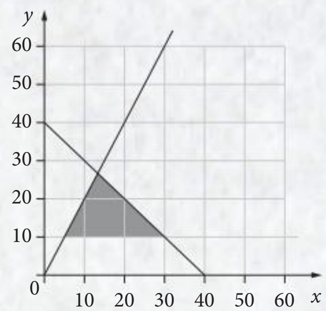
Question 9

Xavier and Yvette share a job. Yvette must work at least twice as many hours as Xavier. They must work at least 40 hours each week, in total. Xavier must work at least 10 hours each week. Yvette can only work for a maximum of 30 hours each week. Let x represent the number of hours that Xavier works each week. Let y represent the number of hours that Yvette works each week. In which one of the following graphs does the shaded area show the feasible region defined by these conditions?

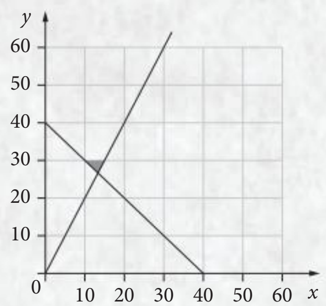
A.



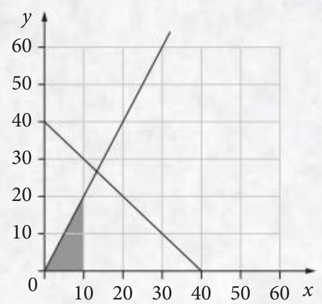
B.



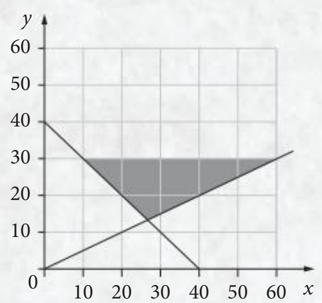
C.



D.



E.



[VCAA 2014 1GRQ9]

EXAMINATION 1 – ended

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GRAPHS AND RELATIONS

Examination 2

Reading time: (5 minutes)
Writing time: (25 minutes)

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers, one bound reference, one approved graphics calculator or approved CAS calculator or CAS software and, if desired, one scientific calculator. Calculator memory DOES NOT need to be cleared.
- Students are NOT permitted to bring into the examination room: blank sheets of paper and/or white out liquid/tape.
- Students may refer to the sheet of miscellaneous formulas supplied.
- All written responses must be in English.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

Instructions

You need not give numerical answers as decimals unless instructed to do so. Alternative forms may involve, for example, π, surds or fractions.

Diagrams are not to scale unless specified otherwise.

EXAMINATION 2 – continued

Question 1 (8 marks)

Fastgrow and Booster are two tomato fertilisers that contain the nutrients nitrogen and phosphorus.

The amount of nitrogen and phosphorus in each kilogram of Fastgrow and Booster is shown in the table.

	1 kg of Booster	**1 kg of Fastgrow**
Nitrogen	0.05 kg	0.05 kg
Phosphorus	0.02 kg	0.06 kg

a. How many kilograms of phosphorus are in 2 kg of Booster? 1 mark

b. If 100 kg of Booster and 400 kg of Fastgrow are mixed, how many kilograms of nitrogen would be in the mixture? 1 mark

Arthur is a farmer who grows tomatoes. He mixes quantities of Booster and Fastgrow to make his own fertiliser. Let x be the number of kilograms of Booster in Arthur's fertiliser. Let y be the number of kilograms of Fastgrow in Arthur's fertiliser. Inequalities 1 to 4 represent the nitrogen and phosphorus requirements of Arthur's tomato field.

Inequality 1	$x \geq 0$	Inequality 2	$y \geq 0$
Inequality 3 (nitrogen)	$0.05x + 0.05y \geq 200$	Inequality 4 (phosphorus)	$0.02x + 0.06y \geq 120$

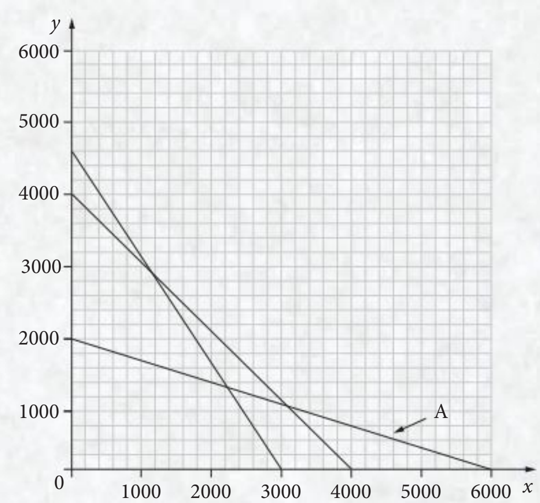
Arthur's tomato field also requires at least 180 kg of the nutrient potassium. Each kilogram of Booster contains 0.06 kg of potassium. Each kilogram of Fastgrow contains 0.04 kg of potassium.

c. Inequality 5 represents the potassium requirements of Arthur's tomato field. Write down Inequality 5 in terms of x and y. 1 mark

The lines that represent the boundaries of Inequalities 3, 4 and 5 are shown in the graph.

d. **i.** Using the graph, write down the equation of line A. 1 mark

ii. Copy the graph and shade the region that satisfies Inequalities 1 to 5. 1 mark



Arthur would like to use the least amount of his own fertiliser to meet the nutrient requirements of his tomato field and still satisfy Inequalities 1 to 5.

e. **i.** What weight of his own fertiliser will Arthur need to make? 1 mark

ii. On the graph you drew, show the point(s) where this solution occurs. 2 marks

[VCAA 2014 2GRQ1]

EXAMINATION 2 – continued

Question 2 (3 marks)

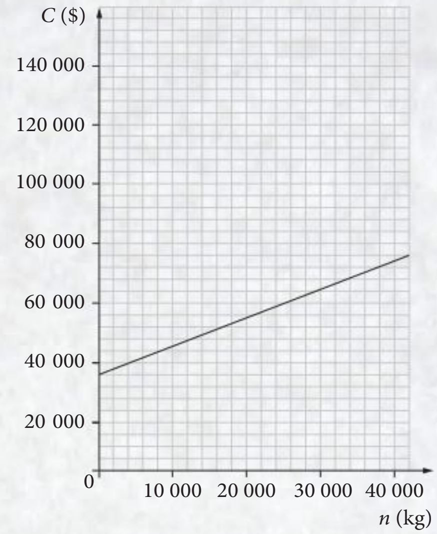
The cost, C, in dollars, of producing n kilograms of tomatoes is given by

$$C = 1.25n + 36\,000 \qquad 0 \le n \le 40\,000$$

The revenue, R, in dollars, from selling n kilograms of tomatoes is given by

$$R = 3.5n \qquad 0 \le n \le 40\,000$$

The cost, C, for the production of n kilograms of tomatoes has been graphed.



a. Copy the graph and draw the revenue equation line, $R = 3.5n$. 1 mark

b. What profit will Arthur make if he sells a total of 20 000 kg of tomatoes? 2 marks

[VCAA 2014 2GRQ2]

Question 3 (4 marks)

A shop owner bought 100 kg of Arthur's tomatoes to sell in her shop. She bought the tomatoes for \$3.50 per kilogram. The shop owner will offer a discount to her customers based on the number of kilograms of tomatoes they buy in one bag. The revenue, in dollars, that the shop owner receives from selling the tomatoes is given by

$$\text{Revenue} = \begin{cases} 5.4n, & 0 < n \le 2 \\ 10.8 + 4(n-2), & 2 < n \le 10 \\ a + 2(n-10), & 10 < n < 100 \end{cases}$$

where n is the number of kilograms of tomatoes that a customer buys in one bag.

a. What is the revenue that the shop owner receives from selling 8 kg of tomatoes in one bag? 1 mark

b. Show that a has the value 42.8 in the revenue equation. 1 mark

c. Find the maximum number of kilograms of tomatoes that a customer can buy in one bag, so that the shop owner never makes a loss. 2 marks

[VCAA 2014 2GRQ3]

EXAMINATION 2 – ended

CHAPTER 15

LINEAR PROGRAMMING

15.1 Linear inequalities

Linear programming and inequalities

Linear inequalities with one variable

Using CAS: Graphing inequalities with one variable

Linear inequalities with two variables

Using CAS: Graphing inequalities with two variables

Writing linear inequalities

15.2 Systems of linear inequalities and feasible regions

Simultaneous linear inequalities

Using CAS: Graphing a system of linear inequalities

Feasible regions

15.3 Objective functions

The corner point method

Using CAS: Finding the corner points of a feasible region

Using CAS: Substituting corner points into the objective function

The sliding line method

15.4 Integer solutions

Summary

15.1 Linear inequalities

Linear programming and inequalities

Linear programming is a technique used to maximise profit or minimise cost given limited resources. It involves finding the maximum and minimum values of a quantity for a region defined by linear **inequalities**.

An equation is a mathematical statement showing that two expressions are equal using the symbol =. $3x + 2y = 6$ is an equation.

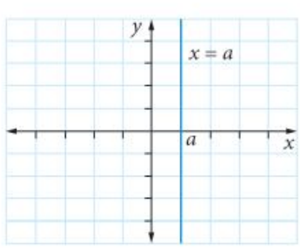
An inequality is a mathematical statement showing that two expressions are not equal using the symbol $<$, $\leq$, $>$, $\geq$ or $\neq$. $3x + 2y \leq 6$ is an inequality.

Inequality	meaning
$<$	Less than
$\leq$	Less than or equal to
$>$	Greater than
$\geq$	Greater than or equal to
$\neq$	Not equal to

Linear inequalities with one variable

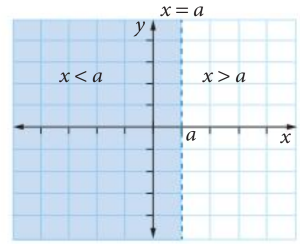
A linear equation produces a straight line while a linear inequality is an area or region which shows all the ordered pairs that satisfy the inequality. The convention of identifying the required region as the shaded area will be used in this chapter.

The equation $x = a$ is a vertical line.

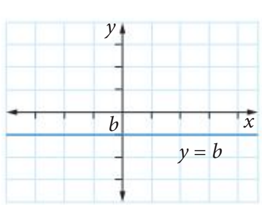


The region $x < a$ is on the left side of the line.

The region $x > a$ is on the right side of the line.

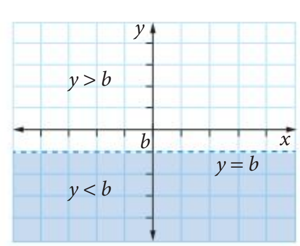


The line $y = b$ is a horizontal line.



The region $y < b$ is below the line.

The region $y > b$ is above the line.



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Alamy/Jim West

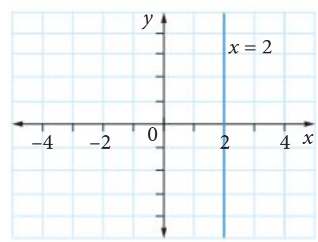
15.1

Worked example 1

Graph the inequality $x \geq 2$.

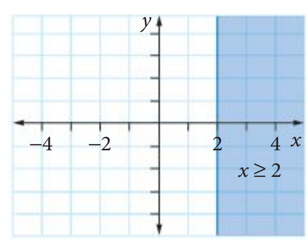
Working

1 Draw the vertical line $x = 2$.



2 Shade the right-hand side of the line where x is greater than 2.

Indicate the required region is the shaded part of the graph by including a legend.



Include a legend indicating the shaded part of the graph is the required region.

required region

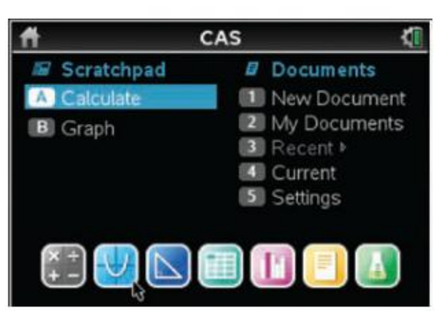
Using CAS Graphing inequalities with one variable

Graph $x \geq 2$ using a CAS/calculator.

TI-NSPIRE CAS

STEP 1

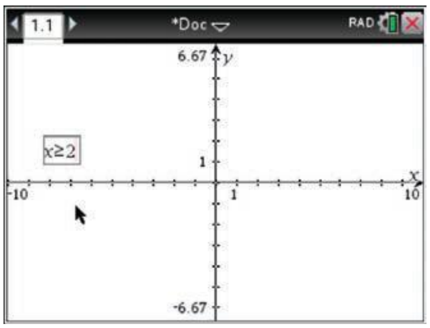
From the main menu select the graph icon .



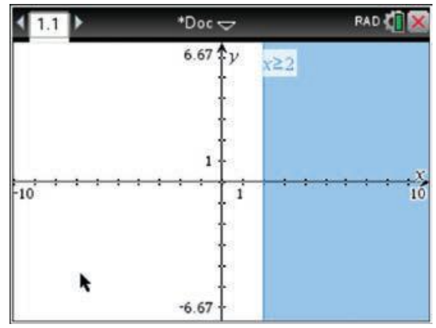
STEP 2

Press tab tab.
This will remove the function entry f1(x).
Press ctrl menu and select 5: Text.
Press X ctrl =.
Select ≥.
Press enter 2 enter.

Use the touch pad to drag the equation onto the X axis near $x = 2$.



The required region is shaded.



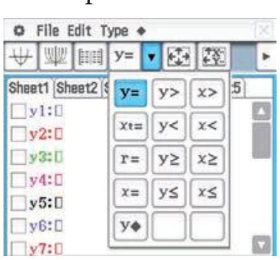
CLASSPAD

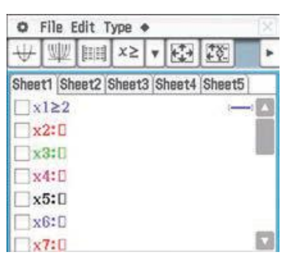
STEP 1

Using the main menu  tap  to enter Graph and Table.

STEP 2

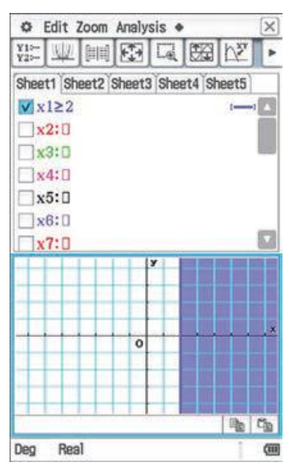
Tap the downward arrow next to the y= icon, then tap **x ≥** and **2**. Tick the box next to x1.





STEP 3

Tap the graph  icon and the graph and shading will appear. Set **View Window** if necessary.



Inequalities with ≤ or ≥ have the linear equation drawn with a solid line.

Inequalities with < or > have the linear equation drawn with a broken line.

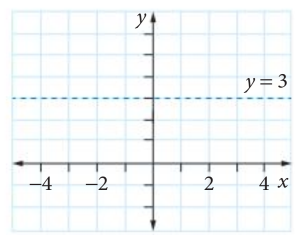
Worked example 2

Graph the inequality $y < 3$.

Working

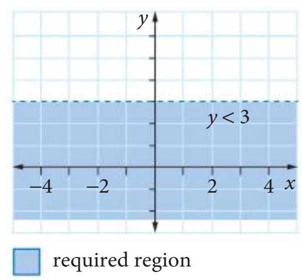
1 Draw the horizontal line $y = 3$ as a broken line.

The inequality $y < 3$ does not include the line $y = 3$.



2 Shade the region below the line where y is less than 3.

Indicate that the shaded part of the graph is the required region with a legend.



Linear inequalities with two variables

A linear inequality written in terms of x and y contains two variables.

Method for sketching inequalities

1 Replace the inequality sign <, ≤, > or ≥ with the equality sign = to make the equation of a line.

2 Find two points that the line passes through and draw a line that passes through these points. These points could be the x-intercept where $y = 0$ and the y-intercept where $x = 0$.

3 Draw a solid line if the inequality has a ≤ or ≥.

Draw a broken line if the inequality has a < or >.

4 Select a point on one side of the line and test it to determine if it satisfies the inequality.

5 If the point satisfies the inequality, then shade the same side of the line as the point. If the point does not satisfy the inequality, then shade the other side of the line to the point.

Inequalities can also be graphed using CAS.

Worked example 3

Graph the inequality $3x + 2y \le 12$.

Working

1 Write the inequality as the equation $3x + 2y = 12$ and determine the x- and y-intercepts.

x-intercept ($y = 0$)

$3x = 12$

$x = 4$

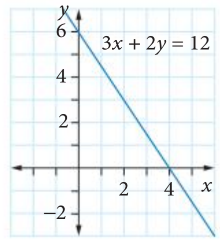
y-intercept ($x = 0$)

$2y = 12$

$y = 6$

2 Draw a straight line with a y-intercept of 6 and an x-intercept of 4.

Draw a solid line as the inequality contains the symbol $\le$.



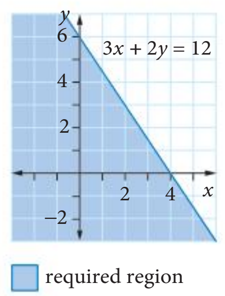
3 Test the point (0, 0) on the inequality $3x + 2y \le 12$.

Substitute $x = 0$ and $y = 0$ into $3x + 2y$.

$3 \times 0 + 2 \times 0 = 0 \le 12$ True

4 The point (0, 0) is below the line.

As 0 is less than 12, this point satisfies the inequality. This makes the area below the line the required region so we shade this side of the line.



Worked example 4

Determine if the point (2, 4) satisfies the inequality $5x + y > 15$.

Working

1 Substitute $x = 2$ and $y = 4$ into the left-hand side of the inequality.

$$\text{Left-hand side} = 5x + y = 5 \times 2 + 4 = 14$$

2 The inequality states that $5x + y$ is greater than 15. But 14 is not greater than 15.

The point (2, 4) does not satisfy the inequality.

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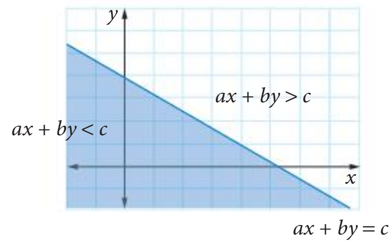
Exam hack

The following are short cuts for deciding which region to shade without the need to test a point.

If the line is in the form $ax + by = c$ and a, b and c are positive:

‘Less than’ is the region below the line. $ax + by < c$

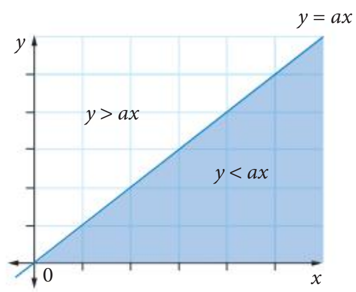
‘Greater than’ is the region above the line. $ax + by > c$



If the line is in the form $y = ax$ (i.e. it passes through (0, 0) and has a gradient of a):

‘Less than’ is the region below the line. $y < ax$

‘Greater than’ is the region above the line. $y > ax$



Using CAS Graphing inequalities with two variables

Graph the inequality $3x + 2y \leq 12$ using a CAS/calculator

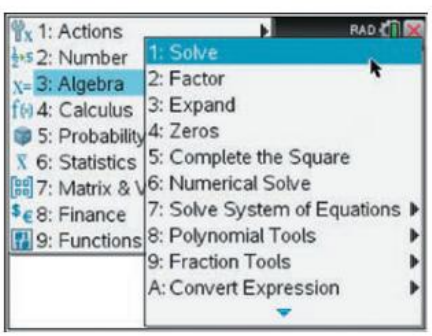
TI-NSPIRE CAS

STEP 1

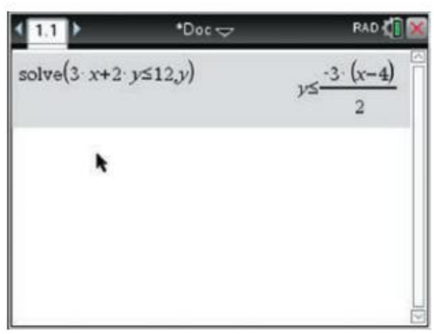
Add a calculator page.

Press [menu], select 3: Algebra, 1: Solve.

Enter the inequality $3x+2y\leq12$.



Press [,] [Y] [)] and [enter].



STEP 2

Press ▲[enter], select the right-hand side of the inequality $\dfrac{-3(x-4)}{2}$ and press [ctrl] [C].

This will copy the expression.

STEP 3

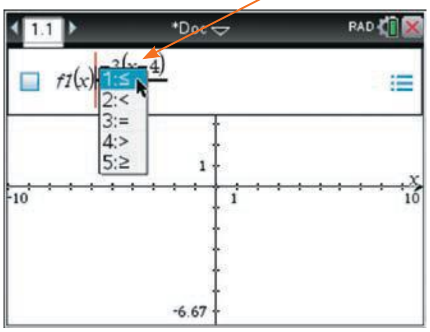
Press [ctrl] [doc▾], add a graphing page and press [ctrl] [V].

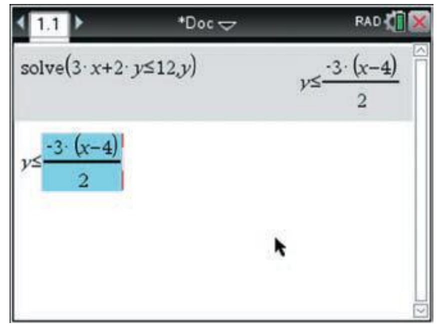
This will paste the expression.

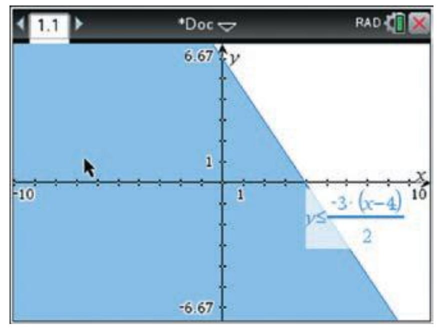
Delete the = sign and replace with ≤.

This will ensure the required region is shaded.

The inequality sign in the graph (≤) must be the same as the one in the solved equation (≤).







CLASSPAD

STEP 1

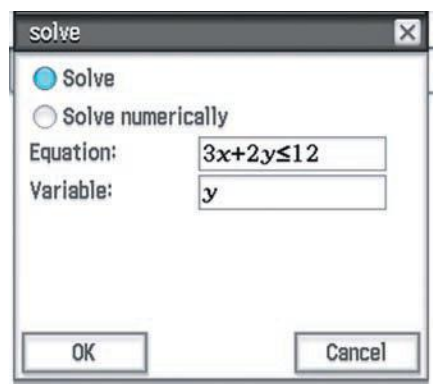
In the main menu, have your mode in Standard and enter the inequality

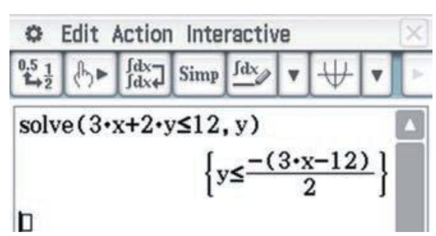
$3x+2y\le12$

You will find ≤ by pressing [Keyboard] and tapping [Math3].

Highlight the inequality and tap **Interactive**, **Equation/Inequality**, **Solve**.

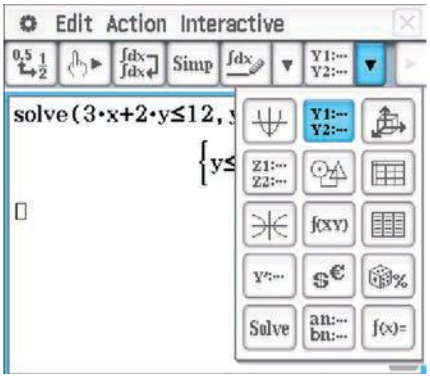
Select y as the variable and tap **OK**.





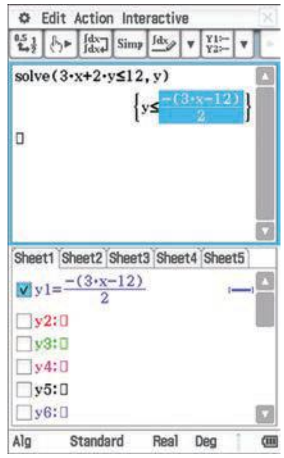
STEP 2

Tap the graph icon and then tap Y1: Y2:.



STEP 3

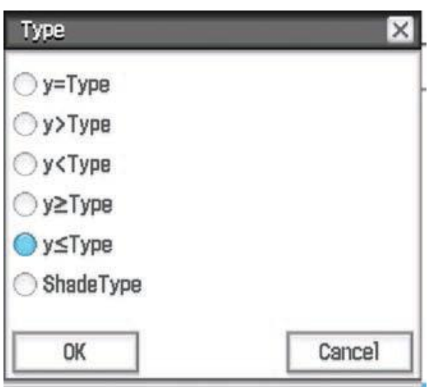
Tap the box next to y1 so a tick appears. Highlight the RHS of the inequality and drag to table section.



STEP 4

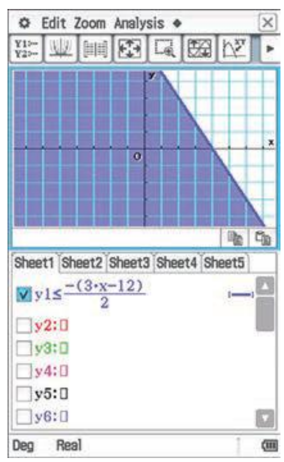
Tap the equals sign next to y1 and then tap **y≤type**.

Tap **OK**.



STEP 5

Tap graph icon to create the inequality graph. Set **View Window** if necessary.



15.1

Writing linear inequalities

A linear programming problem will normally involve two variables x and y. Such problems involve real-life **constraints** which are restrictions placed on the possible values of x and y, written as inequalities.

Worked example 5

A furniture maker produces two types of chairs in her workshop. The standard chair requires 4 hours of machining and 2 hours for assembly, while the deluxe chair requires 6 hours of machining and 3 hours for assembly. There are at most 30 hours of machine time and 20 hours of assembly time available each week. If x is the number of standard chairs the workshop produces each week, and y is the number of deluxe chairs the workshop produces each week, find the inequality representing

a the constraint on machining time

b the constraint on assembly time.

Working

a **1** Write the information referring to machining time in a table.

Machining time

Standard x	Deluxe y	Available time
4 h	6 h	30 h

2 Write an expression for the number of hours of machining time used in terms of x and y. For example, x standard chairs @ 4 h per chair = $4x$ h

Machine time used = $4x + 6y$

3 Write an inequality for the machining constraint. The maximum available time for machining is 30 h.

Machining constraint.

$4x + 6y \leq 30$

b **1** Write the information referring to assembly time in a table.

Assembly time

Standard x	Deluxe y	Available time
2 h	3 h	20 h

2 Write an expression for the number of hours of assembly time used in terms of x and y.

Assembly time = $2x + 3y$

3 Write an inequality for the assembly constraint. The maximum available time for assembly is 20 h.

Assembly constraint

$2x + 3y \leq 20$

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Worked example 6

Ashleigh is designing a necklace to be made from orange beads and black beads. She has to use at most 50 beads and must use at least twice as many orange beads as black beads. If x is the number of orange beads used and y is the number of black beads used in the necklace, write the four constraints.

	Working
1 The number of beads cannot be negative.	$x \geq 0, y \geq 0$
2 The total number of beads, $x + y$, is less than or equal to 50.	Total constraint $x + y \leq 50$
3 At least twice as many orange beads as black beads. Write this as a ratio inequality and transpose to make y the subject.	Ratio of black to orange constraint. At least twice as many orange (x) as black (y) $x : y \geq 2 : 1$ $\frac{x}{y} \geq \frac{2}{1}$ $x \geq 2y$ $\frac{1}{2}x \geq y$ $y \leq \frac{1}{2}x$
4 Test the inequality by substituting in possible values for x and y.	If y (black) = 2, x (orange) could be 4, 5, 6, … Substitute $y = 2$ and $x = 6$ into the inequality $y \leq \frac{1}{2}x$ $2 \leq \frac{1}{2} \times 6$ $2 \leq 3$ This shows that the inequality is correct.
5 Write the four constraints.	Constraints $x \geq 0, y \geq 0, x + y \leq 50, \; y \leq \frac{1}{2}x$

Exam hack

Always substitute possible values for x and y into inequalities of the form $y \leq \frac{a}{b}x$ to test whether a valid inequality has been written.

15.1

EXAM PREP 15.1

Linear inequalities

Prep 1 WORKED EXAMPLE 1 USING CAS: GRAPHING INEQUALITIES WITH ONE VARIABLE

Graph the inequality $x > -2$ by hand and verify using a CAS/calculator.

Prep 2 WORKED EXAMPLE 2 USING CAS: GRAPHING INEQUALITIES WITH ONE VARIABLE

Graph the inequality $y \leq 4$ by hand and verify using a CAS/calculator.

Prep 3 WORKED EXAMPLE 3 USING CAS: GRAPHING INEQUALITIES WITH TWO VARIABLES

Graph the inequality $5x + 2y \geq 10$ by hand and verify using a CAS/calculator.

Prep 4 WORKED EXAMPLE 4

Find if the point (2, 5) satisfies the inequality $3x + 4y > 2$.

Prep 5 WORKED EXAMPLE 5

Chris enjoys rock climbing and surfing. He allocates at most $200 on these recreational activities. Rock climbing costs $15 per hour and surfing costs $10 per hour. If x is the number of hours that Chris spends rock climbing each week and y is the number of hours that Chris spends surfing each week, write a linear inequality representing the activities cost constraint.

Prep 6 WORKED EXAMPLE 6

A firm employs skilled workers and apprentices. Their workforce cannot exceed 20 workers and they must employ at least twice as many skilled workers as apprentices. If x is the number of skilled workers employed and y is the number of apprentices employed, write the four constraints.

EXAM PRACTICE 15.1

Linear inequalities

Question 1

An English tutorial session takes 1.5 hours. A History tutorial session takes 30 minutes.

Kathy has no more than 15 hours available in a week for tutorial sessions.

Let x represent the number of English tutorial sessions Kathy has each week.

Let y represent the number of History tutorial sessions Kathy has each week.

An inequality representing the constraint on Kathy's tutorial time each week (in hours) is

A $1.5x + 30y = 15$ **B** $1.5x + 30y \geq 15$ **C** $1.5x + 30y \leq 15$

D $1.5x + 0.5y \geq 15$ **E** $1.5x + 0.5y \leq 15$

[VCAA 2009 1GRQ5]

Question 2

The manager of an office is ordering finger food for an office party. Hot items cost \$2.15 each and cold items cost \$1.50 each.

Let x be the number of hot items ordered. Let y be the number of cold items ordered.

The manager can spend no more than \$5 for each of the 200 employees. An inequality that can be used to represent this constraint is

A $1.5x + 2.15y \le 5$ **B** $1.5x + 2.15y \le 200$ **C** $1.5x + 2.15y \le 1000$

D $2.15x + 1.5y \le 200$ **E** $2.15x + 1.5y \le 1000$

[VCAA 2010 1GRQ4]

Question 3

Russell is a wine producer. He makes both red and white wine. Let x represent the number of bottles of red wine he makes and y represent the number of bottles of white wine he makes.

This year he plans to make at least twice as many bottles of red wine as white wine.

An inequality representing this situation is

A $y \le x + 2$ **B** $y \le 2x$ **C** $y \ge 2x$ **D** $x \le 2y$ **E** $x \ge 2y$

[VCAA 2007 1GRQ6]

Question 4

A company uses both trainee and qualified inspectors for quality control on an assembly line. The trainee inspectors can examine 15 items per hour, while the qualified inspectors can examine 25 items per hour. The company needs at least 1800 items inspected in each eight-hour day. Let x be the number of trainee inspectors employed by the company, and y be the number of qualified inspectors employed. The constraint imposed by the number of items that need to be inspected in an eight-hour day is expressed by the inequality

A $8x + 8y \ge 1800$ **B** $15x + 25y \le 1800$ **C** $15x + 25y \ge 1800$

D $120x + 200y \ge 1800$ **E** $120x + 200y \le 1800$

[VCAA 2002 1GRQ9]

Question 5

Students at a camp can participate in two different water sport activities: canoeing and surfing. The cost of canoeing is \$30 per hour and the cost of surfing is \$20 per hour. The budget allows each student to spend up to \$200, in total, on water sport activities. The way in which a student decides to spend the \$200 is described by the following inequality.

$30 \times \textit{hours canoeing} + 20 \times \textit{hours surfing} \le 200$

Hillary wants to spend exactly two hours canoeing during the camp.

Calculate the maximum number of hours she could spend surfing. 1 mark

[VCAA 2013 2GRQ2]

15.2 Systems of linear inequalities and feasible regions

Every linear inequality will graph as a line with a region or area that defines all the ordered pairs that satisfy the inequality. A series of linear inequalities will produce lines and regions that intersect or overlap.

Niels Quist

Simultaneous linear inequalities

The solution to a set of linear inequalities can be shown by graphing the equations and shading to show the region required. The points where the lines intersect and the regions that are shaded show every possible solution graphically.

Method for graphing systems of simultaneous inequalities

1 Sketch each equation on a set of axes, calculating x- and y-intercepts where necessary.

2 Indicate with an arrow the side of the line that is to be shaded.

3 Shade the area that is common for each inequality.

4 Use a test point where necessary to determine which side of the line requires shading.

Worked example 7

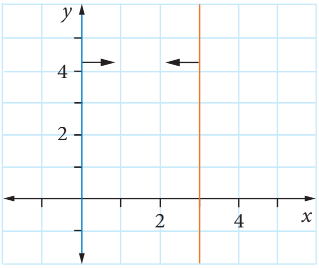
Graph the following system of inequalities.

$x \geq 0, y \geq 0, x \leq 3, y \leq 2, x + y \leq 4$

Working

1 Sketch the vertical lines $x = 0$ and $x = 3$.

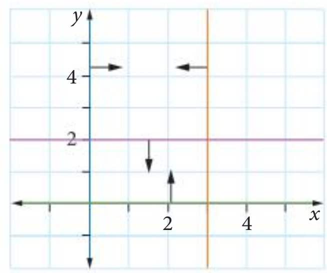
Indicate with an arrow to shade on the right of $x = 0$ and the left of $x = 3$.



2 Sketch the horizontal lines

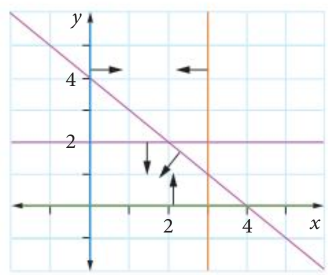
$y = 0$ and $y = 2$.

Indicate with an arrow to shade above $y = 0$ and below $y = 2$.



3 Sketch the final line $x + y = 4$ after finding x- and y-intercepts.

Indicate with an arrow to shade the area below the line $x + y = 4$.



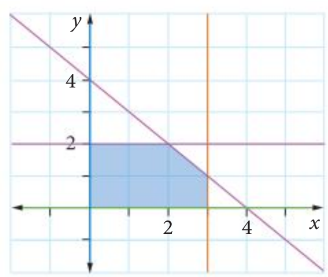
$x + y = 4$

x-intercept ($y = 0$)
$x = 4$

y-intercept ($x = 0$)
$y = 4$

4 Shade the section where the areas overlap.

Include a legend stating that the shaded area is the required region.



required region

Using CAS — Graphing a system of linear inequalities

Graph the following system of linear inequalities using a CAS/calculator.

$x \geq 0, y \geq 0, x \leq 3, y \leq 2, x + y \leq 4$

TI-NSPIRE CAS

STEP 1

Add a graphing page.

Graph the vertical lines.

On a graphing page, press [tab] [tab].

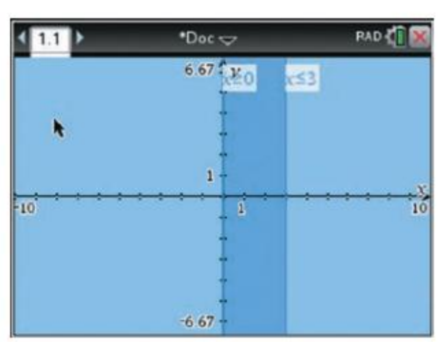
This will remove the function entry.

Press [ctrl] [menu], 5: Text.

Press [X] [ctrl] [=], select ≥ and press [0].

Drag the text to the x-axis near the origin.

Repeat for the constraint $x \leq 3$.



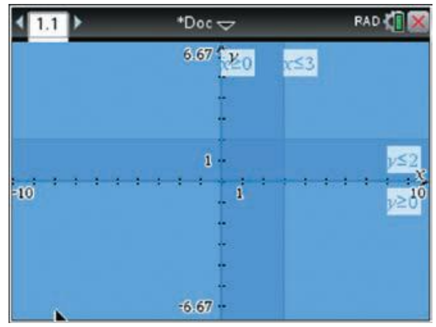
STEP 2

Graph the horizontal lines.

Press [tab] and select ≥ and press [0].

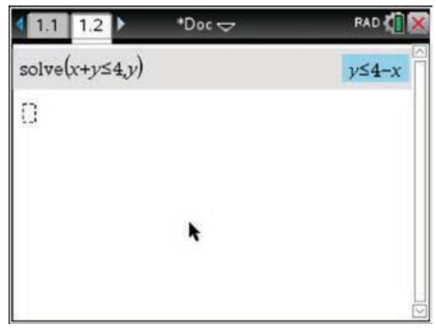
Press [tab].

Select ≤ and press [2] [enter].



STEP 3

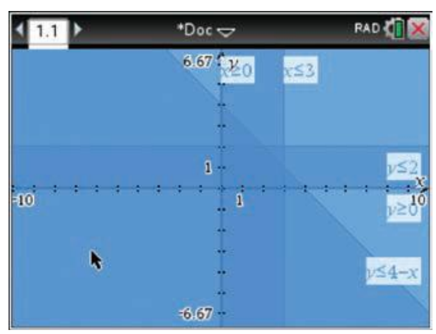
Press [ctrl] [doc▾].
Add a calculator page.
Press [menu].
3: Algebra
1: Solve
Enter $x + y \leq 4, x$).
Press [enter].
Press ▲
and press [ctrl] [C].
This will copy the inequality.



STEP 4

Press [ctrl] ◀.
This will return you to the graphing page.
Press [ctrl] [menu].
Select 5: Text.

Press [enter] [ctrl] [V] [enter].
To graph the inequality, use the touch pad to drag the inequality towards the Y axis.



CLASSPAD

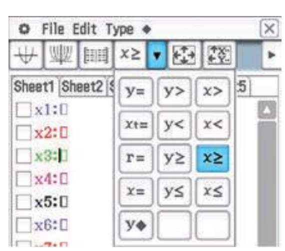
STEP 1

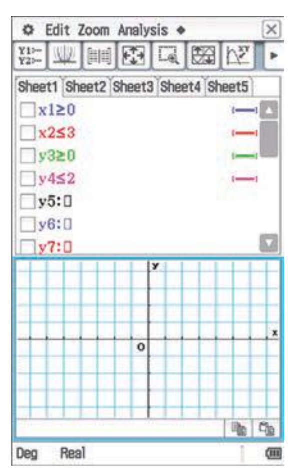
Using the main menu Menu, tap  to enter Graph and Table.

Tap $x \geq$ and [EXE].

Repeat for

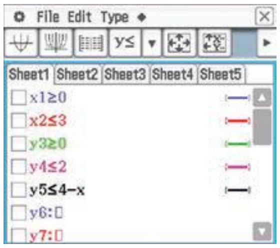
$x \leq 3, y \geq 0, y \leq 2$.





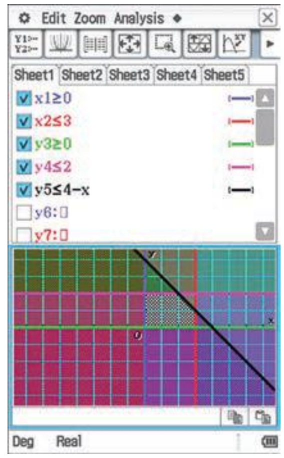
STEP 2

Enter the inequality as y5.



STEP 3

Tap to select all 5 graphs and tap graph icon.



Worked example 8

Which of these points (if any) satisfy all the following constraints?

(0, 4), (2, 2), (5, 1), (3, 2), (2, 4)

$4x + 5y \leq 20$, $x \geq 1$, $2x + y \leq 6$

Working

1 Identify the points that satisfy the constraint $x \geq 1$.

(2, 2), (5, 1), (3, 2) and (2, 4) satisfy the constraint $x \geq 1$.

2 Of these, identify the points that satisfy the constraint $4x + 5y \leq 20$.

The point satisfies the constraint if $4x + 5y$ is less than or equal to 20.

Point	(2, 2)	(5, 1)	(3, 2)	(2, 4)
$4x + 5y$	8 + 10 = 18	20 + 5 = 25	12 + 10 = 22	8 + 20 = 28

Only (2, 2) satisfies the constraint $4x + 5y \leq 20$.

3 Check to see if the remaining point satisfies $2x + y \leq 6$.

Substitute (2, 2) to check if it is less than or equal to 6.

Point	(2, 2)
$2x + y$	4 + 2 = 6

So (2, 2) satisfies the constraint $2x + y \leq 6$.

4 Write the answer.

The point (2 , 2) satisfies all three constraints.

Feasible regions

The required region from a system of simultaneous linear inequalities is also called a **feasible region**.

Worked example 9

Determine the feasible region by graphing the following constraints, and hence determine if the point (2, 1) satisfies the constraints.

$x \geq 0, y \geq 0, 3x + 4y \leq 12, y \leq 2x$

Working

1 Graph the inequalities, using a CAS/calculator if necessary. Identify points that the lines $y = 2x$ and $3x + 4y = 12$ pass through.

$3x + 4y = 12$

x-intercept ($y = 0$) | y-intercept ($x = 0$)

$3x = 12$ | $4y = 12$

$x = 4$ | $y = 3$

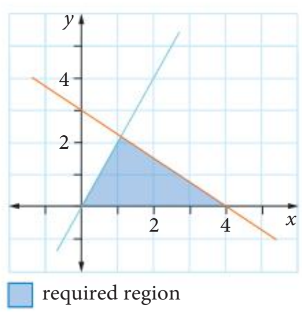
$3x + 4y = 12$ goes through the points (4, 0) and (0, 3).

$y = 2x$

$x = 0, y = 0$

$x = 1, y = 2 \times 1 = 1$

$y = 2x$ goes through the points (0, 0) and (1, 2).



2 Determine if the point (2, 1) lies in the feasible region.

The point (2, 1) satisfies the constraints.

Exam hack

When drawing the straight lines it is important to identify two points that the line passes through and not just approximate the graph from the calculator display.

EXAM PREP 15.2

Systems of linear inequalities and feasible regions

Prep 1

WORKED EXAMPLE 7 USING CAS: GRAPHING A SYSTEM OF LINEAR INEQUALITIES

Graph the following system of inequalities by hand and verify using a CAS/calculator.

$x \geq 0, y \geq 0, y \leq 3, x + y \leq 4, 2x + y \geq 4$

Prep 2

WORKED EXAMPLE 8

Which of these points (if any) satisfy all the following constraints?

(2, 3), (3, 2), (2, 2), (4, 1), (1, 4)

$x \geq 0, y \geq 0, y \leq 3, x + y \leq 4, 2x + y \geq 4$

Prep 3

WORKED EXAMPLE 9

Determine the feasible region by graphing the following constraints, and hence determine if the point (1, 2) satisfies the constraints.

$x \geq 0, 2x + y \leq 6, y \geq \frac{1}{3}x$

EXAM PRACTICE 15.2

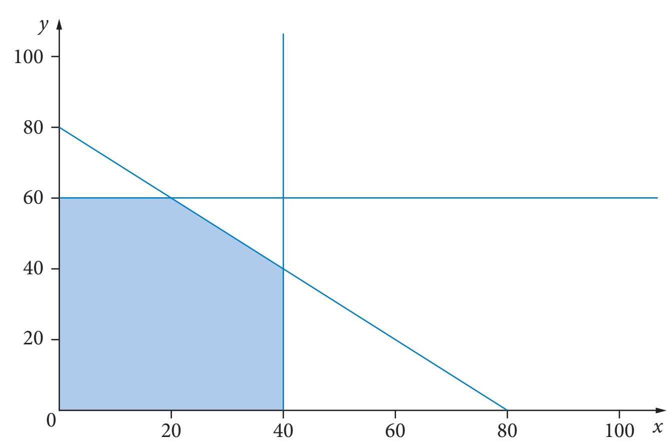
Systems of linear inequalities and feasible regions

Question 1

In a linear programming problem involving animal management on a farm

- x represents the number of cows on the farm
- y represents the number of sheep on the farm.

The feasible region (with boundaries included) for the problem is indicated by the shaded region on the diagram.



One of the constraints defining the feasible region indicates that

A there must be 20 cows and 60 sheep.

B there must be 40 cows and 40 sheep.

C the number of sheep cannot exceed 40.

D the number of cows must be at least 60.

E the total number of cows and sheep cannot exceed 80.

[VCAA 2006 1GRQ7]

Question 2

Daisey's bread shop makes white and brown bread subject to the following constraints.

- No more than 240 loaves of bread can be made each day.
- At least five loaves of white bread will be made for every loaf of brown bread that is made.

Let w be the number of loaves of white bread that are made each day.

Let b be the number of loaves of brown bread that are made each day.

A pair of inequalities that could be written to represent these constraints is

A $w + b \leq 240$ and $w \geq 5b$

B $w + b \leq 240$ and $w \leq 5b$

C $w + b < 240$ and $w > \frac{b}{5}$

D $w + b < 240$ and $w < 5b$

E $w + b \leq 240$ and $w \leq \frac{b}{5}$

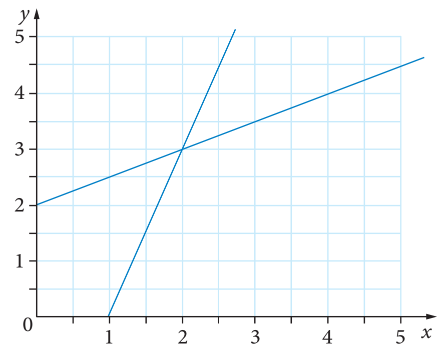
[VCAA 2012 1GRQ8]

Question 3

The graphs of the linear relations $x - 2y = -4$ and $3x - y = 3$ are shown. A point that satisfies both the inequalities $x - 2y \geq -4$ and $3x - y \geq 3$ is

A (1, 2) **B** (1, 2.5) **C** (2, 4)

D (3, 2) **E** (3, 4)



[VCAA 2010 1GRQ6]

Question 4

These four inequalities were used to construct the feasible region for a linear programming problem: $x \geq 0, y \geq 0, x + y \leq 9, y \leq \frac{1}{2}x$.

A point that lies within this feasible region is

A (4, 4) **B** (5, 3) **C** (6, 2) **D** (6, 4) **E** (7, 3)

[VCAA 2006 1GRQ9]

Question 5

Kathy prefers to have no more than 18 tutorial sessions in total each week.

She prefers to have at least 4 English tutorial sessions.

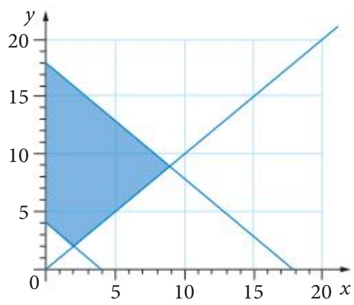
She also prefers to have at least as many History tutorial sessions as English tutorial sessions.

Let x represent the number of English tutorial sessions Kathy has each week.

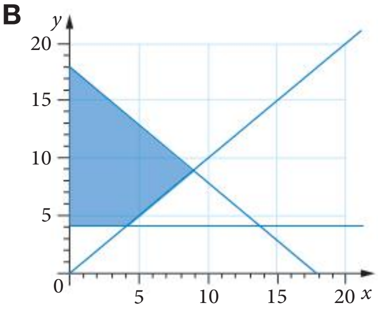
Let y represent the number of History tutorial sessions Kathy has each week.

The shaded region that satisfies all of these constraints is

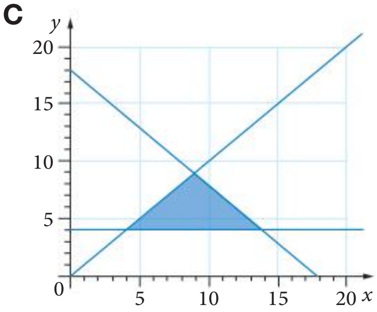
A



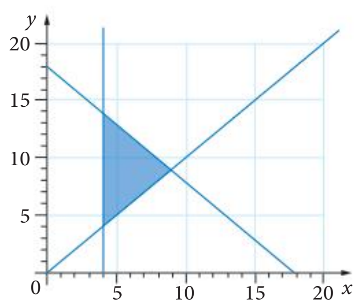
B



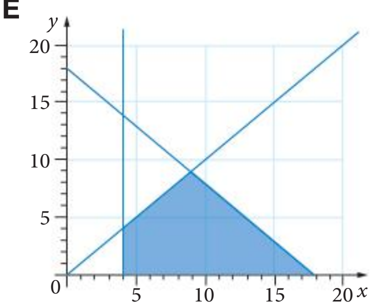
C



D



E



[VCAA 2009 1GRQ6]

Question 6

A region is defined by the following inequalities: $y \geq -4x + 10$, $y - x \geq 1$

A point that lies within this region is

A (1, 3) **B** (2, 1) **C** (3, 2) **D** (4, 6) **E** (5, 1)

[VCAA 2008 1GRQ8]

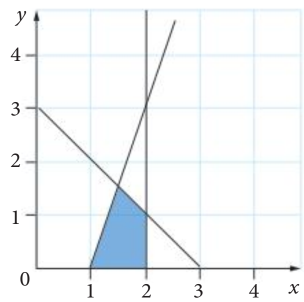
Question 7

The following inequalities define a region in the x-y plane.

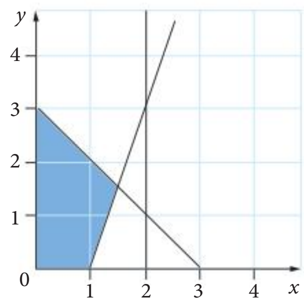
$x \geq 0$, $x \leq 2$, $y \geq 0$, $3x - y \geq 3$, $x + y \leq 3$

Which one of the following diagrams represents this region?

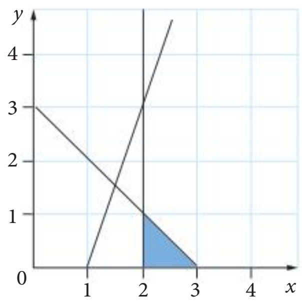
A



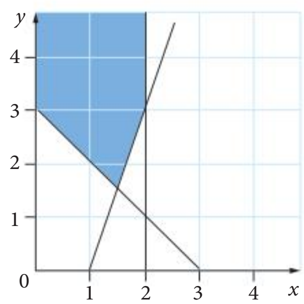
B



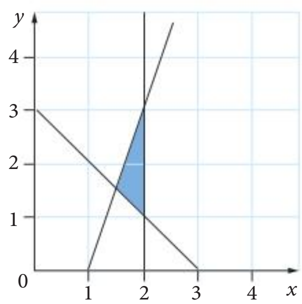
C



D



E



[VCAA 2003 1GRQ7]

Question 8

The shaded region shown in the graph (with boundaries included) is described by

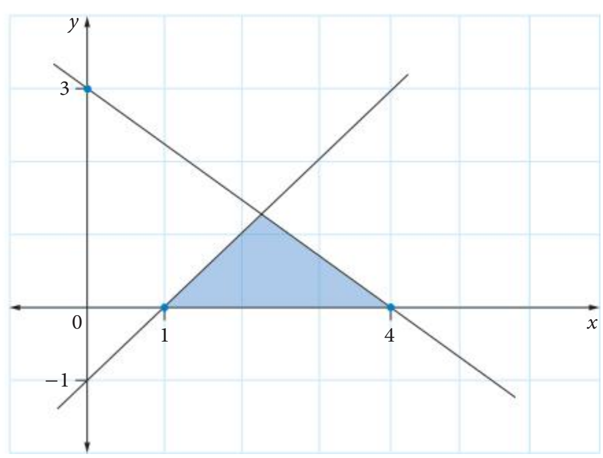
A $3x + 4y \leq 12$, $x - y \leq 1$, $x \geq 0$, $y \geq 0$

B $3x + 4y \leq 12$, $x - y \geq 1$, $x \geq 0$, $y \geq 0$

C $3x + 4y \geq 12$, $x - y \geq 1$, $x \geq 0$, $y \geq 0$

D $4x + 3y \leq 12$, $x - y \leq 1$, $x \geq 0$, $y \geq 0$

E $4x + 3y \leq 12$, $x - y \geq 1$, $x \geq 0$, $y \geq 0$



[VCAA 2004 1GRQ8]

9780170371186

Question 9

Jensen has two jobs, one at a nursery and the other in a restaurant. Each week he works for at least 18 hours; he works at least 4 hours at the nursery and at most 16 hours in the restaurant. Also, each week, Jensen works at least twice as many hours in the restaurant than he does at the nursery. Let x be the number of hours per week that Jensen works at the nursery and y be the number of hours per week that Jensen works in the restaurant.

The set of constraints that apply to Jensen's working hours is

A $x \leq 4, y \geq 16, x + y \geq 18, y \geq 2x$

B $x \leq 4, y \geq 16, x + y \geq 18, 2y \geq x$

C $x \geq 4, y \leq 16, x + y \leq 18, 2y \geq x$

D $x \geq 4, y \leq 16, x + y \geq 18, y \geq 2x$

E $x \geq 4, y \leq 16, x + y \geq 18, 2y \geq x$

[VCAA 2003 1GRQ9]

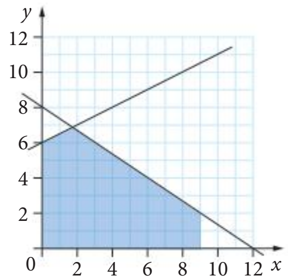
Question 10

The following inequalities define a region in the x-y plane.

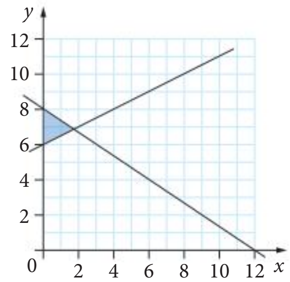
$x \geq 0, x \leq 9, 2x + 3y \leq 24, 2y - x \leq 12$

Which one of the following graphs represents this region?

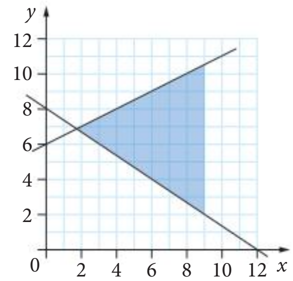
A



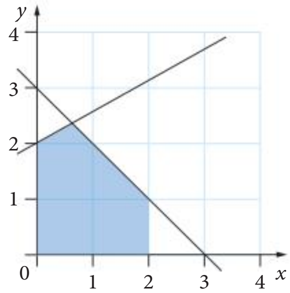
B



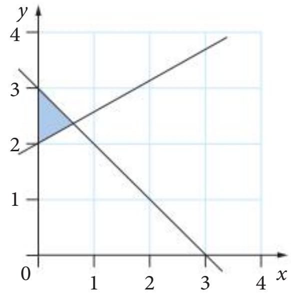
C



D



E



[VCAA 2002 1GRQ8]

Question 11

The Goldsmith family are going on a driving holiday in Western Australia. Gas is generally cheaper than petrol. Their car must run on petrol for some of the driving time. Let x be the number of hours driving using gas and y be the number of hours driving using petrol.

Inequalities 1 to 5 below represent the constraints on driving a car over a 24-hour period. Explanations are given for Inequalities 3 and 4.

Inequality 1: $x \geq 0$

Inequality 2: $y \geq 0$

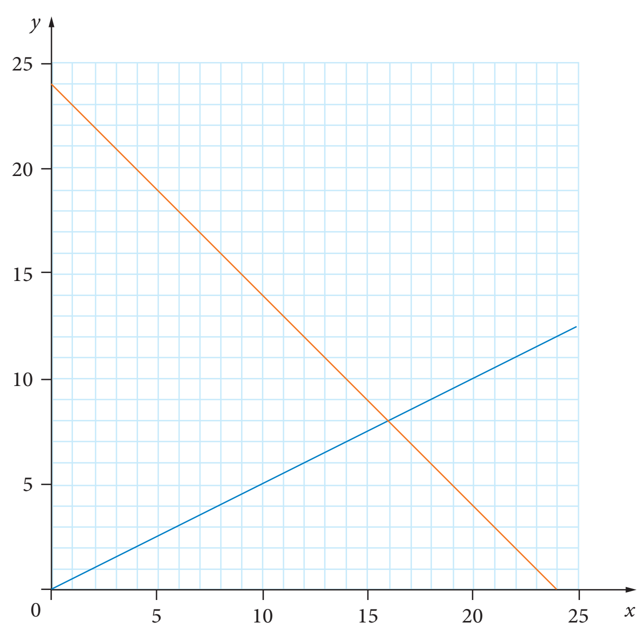
Inequality 3: $y \leq \frac{1}{2}x$. The number of hours driving using petrol must not exceed half the number of hours driving using gas.

Inequality 4: $y \geq \frac{1}{3}x$. The number of hours driving using petrol must be at least one third the number of hours driving using gas.

Inequality 5: $x + y \leq 24$

a Explain the meaning of Inequality 5 in terms of the context of this problem. 1 mark

The lines $x + y = 24$ and $y = \frac{1}{2}x$ are drawn on the graph.



b Copy the graph and on it

i draw the line $y = \frac{1}{3}x$ 1 mark

ii clearly shade the feasible region represented by Inequalities 1 to 5. 1 mark

On a particular day, the Goldsmiths plan to drive for 15 hours. They will use gas for 10 of these hours.

c Will the Goldsmiths comply with all constraints? Justify your answer. 1 mark

[VCAA 2007 2GRQ3]

Objective functions

15.3

amanaimages/Chris Howes/incamerastock

The final stage in a linear programming problem is to maximise or minimise an **objective function**. The objective function is written in terms of the variables x and y and may refer to cost, profit or some other quantity that is to be maximised or minimised.

The two common methods to maximise or minimise the objective function are

1 the corner point method

2 the sliding line method.

The corner point method

The optimum value of the objective function will occur at one or more corners of the feasible region or on a line between two corners.

The **corner points** are the intersection points of the lines that are the boundary of the feasible region.

Corner point method

- Find the coordinates of the corner points of the feasible region.
- Substitute the values of x and y for each corner into the objective function.
- The solution for maximising are the coordinates that give the largest objective function value.
- The solution for minimising are the coordinates that give the smallest objective function value.

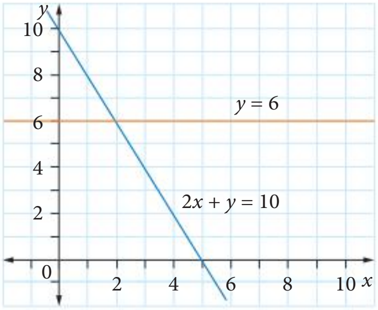
Worked example 10

a Graph the constraints $x \geq 0$, $y \geq 0$, $2x + y \leq 10$, $y \leq 6$, labelling the coordinates of the corner points on your graph.

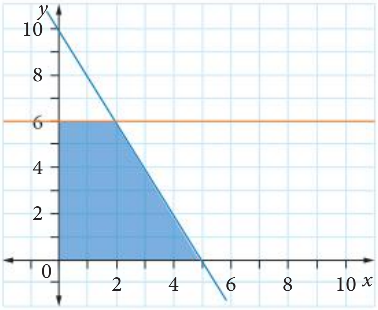
b Find the maximum value of $Z = x + 2y$ subject to these constraints.

Working

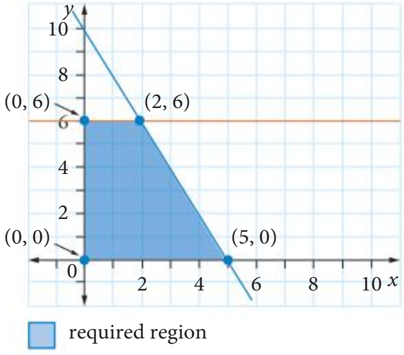
a **1** Graph the equations.



2 Shade the required region.



3 Find the intersection points and label these on the graph, using a CAS/calculator if necessary.



b Substitute the x and y coordinates of the corner points into the objective function.

Corner points	$Z = x + 2y$
(0, 0)	$0 + 2 \times 0 = 0$
(0, 6)	$0 + 2 \times 6 = 12$
(5, 0)	$5 + 2 \times 0 = 5$
(2, 6)	$2 + 2 \times 6 = 14$

The largest value obtained is the maximum value of Z.

The maximum value of Z is 14.

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Using CAS | Finding the corner points of a feasible region

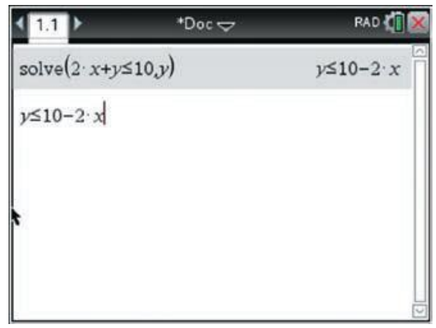
Find the corner points of the feasible region defined by the constraints $x \geq 0$, $y \geq 0$, $2x + y \leq 10$, $y \leq 6$, using a CAS/calculator.

TI-NSPIRE CAS

STEP 1

On a calculator page, solve the inequality $2x + y \leq 10$ for y.

Select $10 - 2x$ and copy by pressing ctrl C.



STEP 2

Press ctrl doc▾ and add a graphing page.

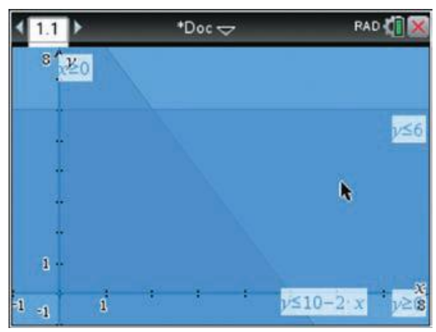
Enter each inequality so that the required region is shaded.

Press del and select ≤, then press ctrl V to paste the first equation.

To graph $x \geq 0$, press ctrl menu, 5: Text.

Enter $x \geq 0$, enter then drag to the origin.

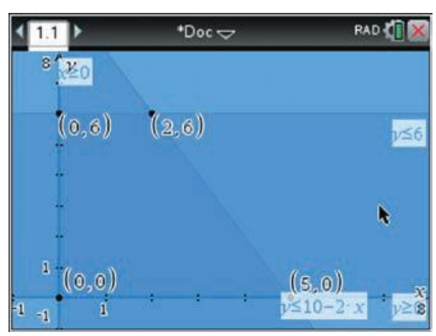
Graph the other two inequalities.



STEP 3

Press menu, 8: Geometry, 1: Points & lines, 3: Intersection points.

Then use the nav pad to select the intersecting lines and the coordinates will appear.



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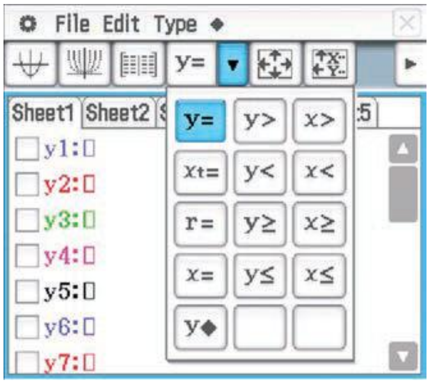
CLASSPAD

STEP 1

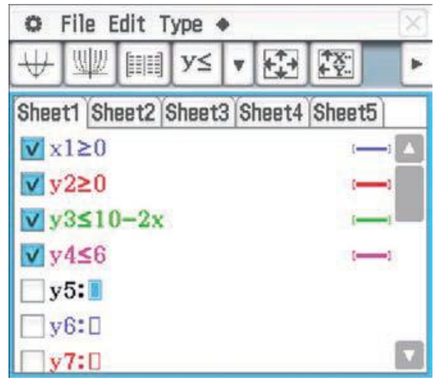
As before, enter in 

the inequalities $x \geq 0$, $y \geq 0$, $2x + y \leq 10$, $y \leq 6$, using the drop down arrow next to y=.



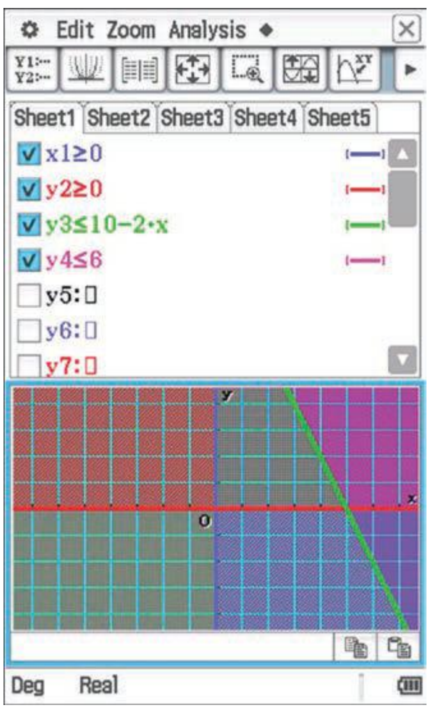
STEP 2

Tap all inequalities.



STEP 3

Tap the graph icon.



STEP 4

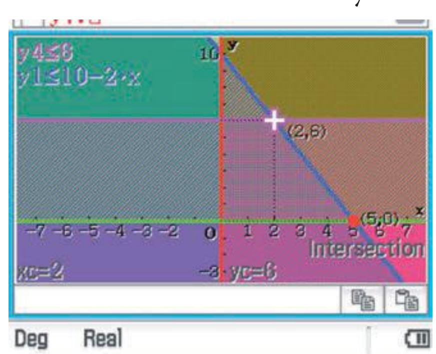
To find the coordinates of the corner points, tap **Analysis**, **G-Solve**, **Intersection**.

Use the up, down curser to select the y1 inequality and press EXE, then select the y4 inequality and press EXE.

This process can be repeated to give the other intersection points shown.

The other two corner points (0, 0) and (5, 0) can be seen by inspection.

Set **View Window** if necessary.



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Using CAS Substituting corner points into the objective function

Substitute the corner points (0, 0), (0, 6), (5, 0), (2, 6) into the objective function $Z = x + 2y$ using a CAS/calculator.

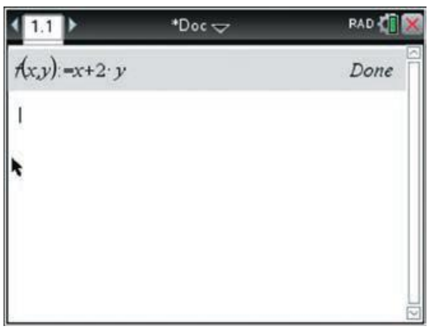
TI-NSPIRE CAS

STEP 1

Define the objective function $Z = x + 2y$ as f(x,y)=x+2y.

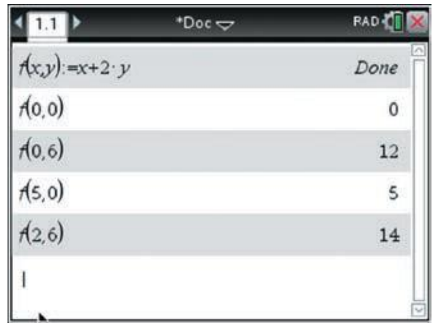
Enter f(x, y), press [:=] and enter the expression x+2y.

Press [enter].



STEP 2

Substitute each coordinate into f(x,y) as shown.



CLASSPAD

STEP 1

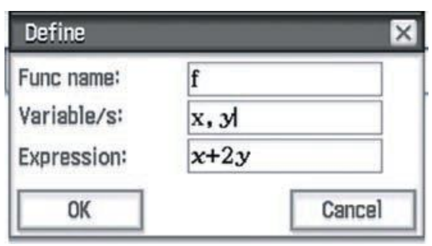
Define the objective function $Z = x + 2y$ as f(x,y)=x+2y.

Enter x+2y and select the expression.

Tap **Interactive**, **Define**.

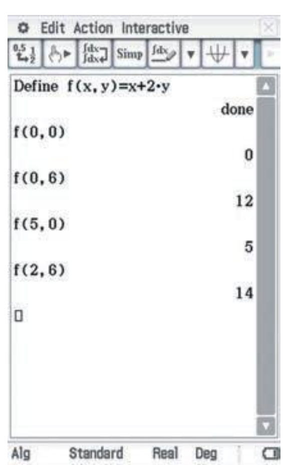
Enter x, y as the variables.

Tap **OK**.



STEP 2

Substitute each coordinate into f(x, y) as shown.



The sliding line method

The second method for maximising or minimising the objective function is the **sliding line** method. This method identifies the solution graphically.

Sliding line method

1 Choose a value for Z in the objective function $Z = ax + by$ that will give **integer** values for the x- and y-intercepts. $Z = ab$ is a good starting point.

2 Plot the objective function as a broken line on the graph of the constraints. Ideally, this broken line should be inside the feasible region.

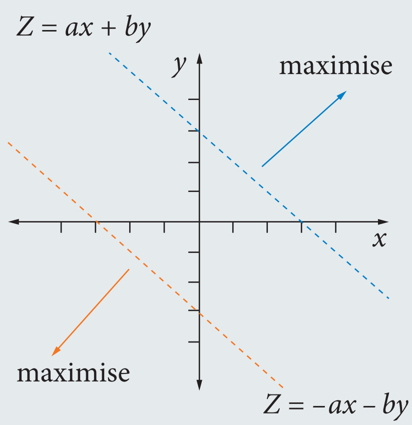
3 Slide a ruler along the page, keeping the ruler parallel to the original broken line at all times. Slide the line away from the origin to maximise and towards the origin to minimise.

4 Identify the last corner point that the sliding line contacts inside the feasible region. This point will provide the optimum solution.

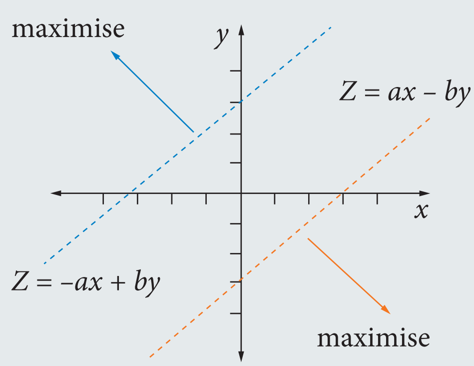
For an objective function $Z = ax + by$, where $Z = ab$, the x-intercept is $(b, 0)$ and the y-intercept is $(0, a)$.

If Z is a multiple of ab, then the x- and y-intercepts are the same multiple of $(b, 0)$ and $(0, a)$.

Negative gradient

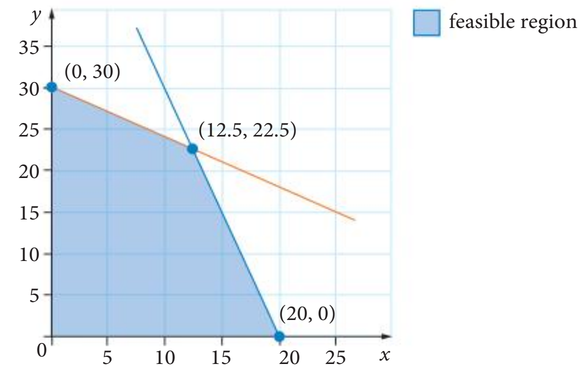


Positive gradient



Worked example 11

Find the maximum value of the objective function $Z = 2x + 3y$ for the constraints graphed below.



Working

1 Choose a value for the objective function $Z = 2x + 3y$.

Start with $Z = 2 \times 3$.

For the objective function $2x + 3y = 6$:

x-intercept ($y = 0$): $x = 3$

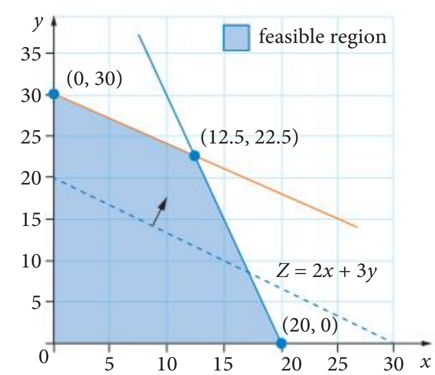
y-intercept ($x = 0$): $y = 2$

As this is too small for the graph, let $2x + 3y = 60$

x-intercept ($y = 0$): $x = 30$

y-intercept ($x = 0$): $y = 20$

2 Sketch the objective function on the graph as a broken line and label it $Z = 2x + 3y$.



3 Identify the last point the sliding line will contact as the broken line slides away from the origin.

The last point the sliding line contacts in the feasible region is (12.5, 22.5).

The maximum value occurs at the point $x = 12.5$, $y = 22.5$.

4 Find the maximum value by substituting the point (12.5, 22.5) into the objective function Z.

$$\begin{aligned}\text{Maximum value of } Z &= 2x + 3y \\ &= 2 \times 12.5 + 3 \times 22.5 \\ &= 92.5\end{aligned}$$

EXAM PREP 15.3

Objective functions

Prep 1

WORKED EXAMPLE 10

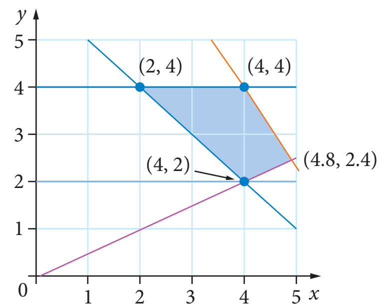
USING CAS: FINDING THE CORNER POINTS OF A FEASIBLE REGION

USING CAS: SUBSTITUTING CORNER POINTS INTO THE OBJECTIVE FUNCTION

The graph shows the feasible region for a linear programming problem. Use the corner point method to find

a the maximum value of the objective function $Z = x + 3y$

b the minimum value of the objective function $Z = 2x - y$.



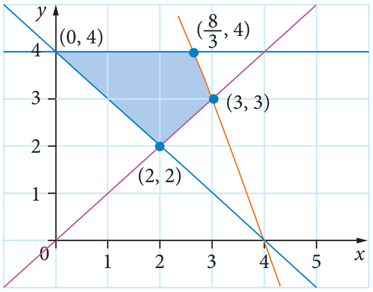
Prep 2

WORKED EXAMPLE 11

The graph shows the feasible region for a linear programming problem. Use the sliding line method to find

a the maximum value of the objective function $Z = x + 2y$

b the minimum value of $Z = -x + y$.



EXAM PRACTICE 15.3

Objective functions

Question 1

The shaded area in the graph represents the feasible region for a linear programming problem. The minimum value of the objective $P = 2x - y$ for this feasible region is

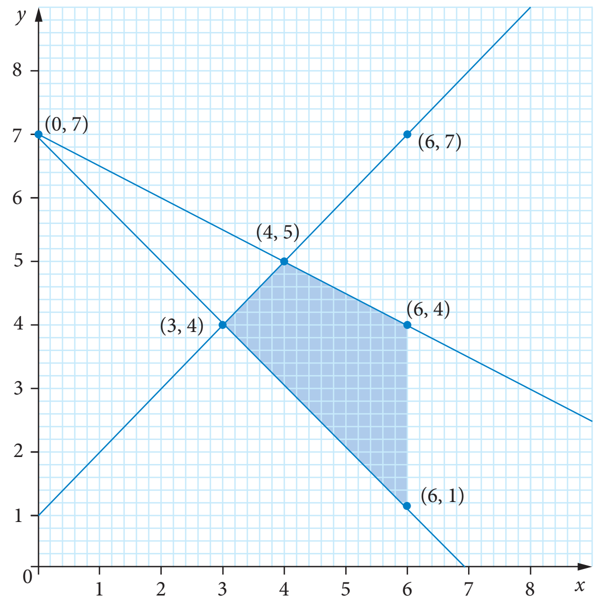
A −7

B 0

C 1

D 2

E 3



[VCAA 2012 1GRQ3]

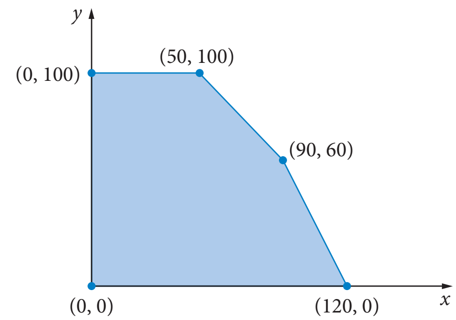
Question 2

For the shaded region (with boundaries included), the value of the objective function

$P = 4x - 3y$ is a maximum at the point

A (0, 0) **B** (0, 100) **C** (50, 100)

D (90, 60) **E** (120, 0)



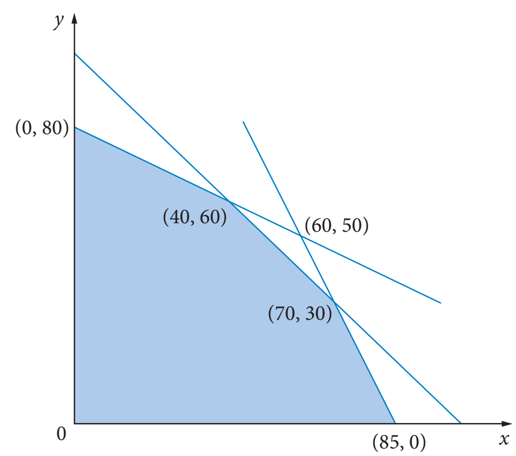
[VCAA 2005 1GRQ8]

Question 3

In the diagram, the shaded region (with boundaries included) represents the feasible region for a linear programming problem with the objective function $Z = 5x + 3y$. The **maximum** value of Z for this feasible region occurs at the point with coordinates

A (0, 80) **B** (40, 60) **C** (60, 50)

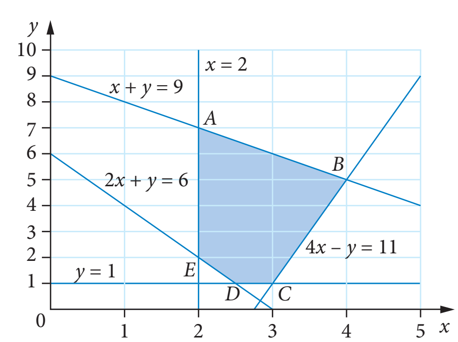
D (70, 30) **E** (85, 0)



[VCAA 2003 1GRQ8]

Use the following information to answer Questions 4 and 5.

Craig plays sport and computer games every Saturday. Let x be the number of hours that he spends playing sport. Let y be the number of hours that he spends playing computer games. Craig has placed some constraints on the amount of time that he spends playing sport and computer games. These constraints define the feasible region shown shaded in the graph. The equations of the lines that define the boundaries of the feasible region are also shown.



Question 4

One of the constraints that defines the feasible region is

A $y \leq 1$ **B** $x \leq 2$ **C** $x + y \geq 9$ **D** $2x + y \leq 6$ **E** $4x - y \leq 11$

[VCAA 2011 1GRQ7]

Question 5

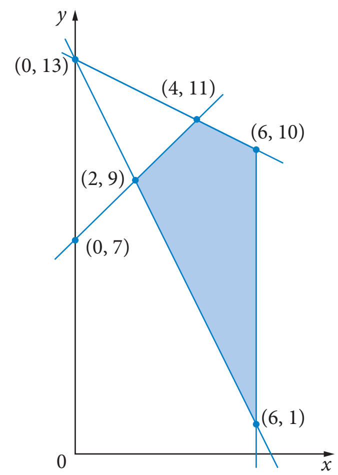
By spending Saturday playing sport and computer games, Craig believes he can improve his health. Let W be the health rating Craig achieves by spending a day playing sport and computer games. The value of W is determined by using the rule $W = 5x - 2y$. For the feasible region shown in the graph, the maximum value of W occurs at

A point A **B** point B **C** point C **D** point D **E** point E

[VCAA 2011 1GRQ8]

Question 6

The shaded region shown in the graph (with boundaries included) represents the feasible region for a linear programming problem. The maximum value of the objective function $y - 2x + 20$, for this feasible region, is



A 18

B 23

C 25

D 27

E 33

[VCAA 2004 1GRQ7]

Question 7

For linear programming problems in general, which of the following statements is **false**?

A The objective function is a linear expression.

B The expressions defining the constraints are linear inequalities.

C There may be more than one optimal solution.

D The value of the objective function must be positive.

E The constraints are used to define the feasible region.

[VCAA 2002 1GRQ6]

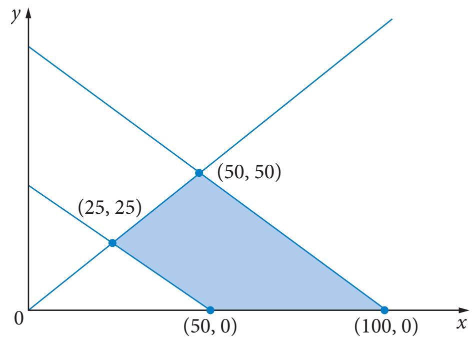
Question 8

The following five constraints apply to a linear programming problem.

$x \geq 0, y \geq 0, x + y \geq 50, x + y \leq 100, y \leq x$

In the diagram, the shaded region (with boundaries included) represents the feasible region for this linear programming problem.

The aim is to maximise the objective function $Z = 2x + ky$. If the maximum value of Z occurs **only** at the point (100, 0), then a possible value for k is



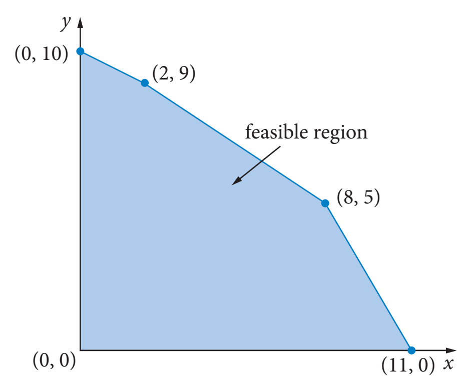
A 1 **B** 2 **C** 3 **D** 4 **E** 5

[VCAA 2007 1GRQ9]

Question 9

The feasible region for a particular linear programming problem is shown shaded on the following graph.

All relevant vertices are labelled. The **minimum** value of the expression $4x - 2y$ for this feasible region is



A −20 **B** −10 **C** 0

D 22 **E** 44

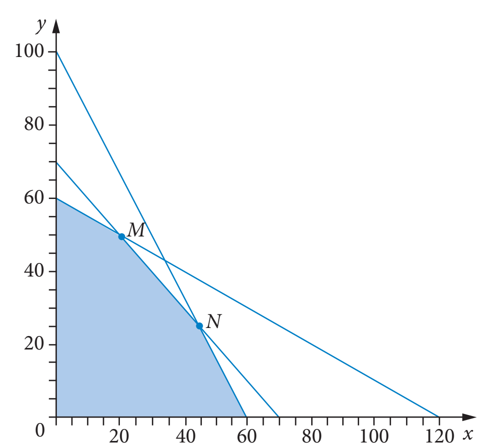
[VCAA 2002 1GRQ5]

Question 10

The shaded region in the graph represents the feasible region for a linear programming problem.

An objective function $Z = ax + by$ has its value maximised at both vertex M and vertex N.

The values of a and b could be



A $a = 15$ and $b = -15$

B $a = 15$ and $b = 15$

C $a = 15$ and $b = 25$

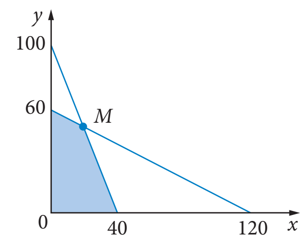
D $a = 25$ and $b = 50$

E $a = 50$ and $b = -25$

[VCAA 2013 1GRQ8]

Question 11

The shaded region in the graph represents the feasible region for a linear programming problem. Which objective function, Z, has its maximum value at the point M?



A $Z = x + y$ **B** $Z = x - y$ **C** $Z = 3x + y$

D $Z = 3x - 2y$ **E** $Z = x + 4y$

[VCAA 2008 1GRQ9]

Question 12

Students at the camp can participate in two different water sport activities: canoeing and surfing. The cost of canoeing is \$30 per hour and the cost of surfing is \$20 per hour. The budget allows each student to spend up to \$200, in total, on water sport activities. The way in which a student decides to spend the \$200 is described by the following inequality.

$30 \times$ hours canoeing $+ 20 \times$ hours surfing ≤ 200

Dennis would like to spend an equal amount of time canoeing and surfing. If he spent a total of \$200 on these activities, determine the maximum number of hours he could spend on each activity. 1 mark

[VCAA 2013 2GRQ2]

Question 13

Michael is preparing to hike through a national park. He decides to make some trail mix to eat on the hike. The trail mix consists of almonds and raisins. The table shows some information about the amount of carbohydrate and protein contained in each gram of almonds and raisins.

	1 g of almonds	1 g of raisins
Carbohydrate	0.2 g	0.8 g
Protein	0.2 g	0.04 g

a If Michael mixed 180 g of almonds and 250 g of raisins to make some trail mix, calculate the weight, in grams, of carbohydrate in the trail mix. 1 mark

Michael wants to make some trail mix that contains 72 g of protein. He already has 320 g of almonds.

b How many grams of raisins does he need to add? 2 marks

The trail mix Michael takes on his hike must satisfy his dietary requirements.
Let x be the weight, in grams, of almonds Michael puts into the trail mix.

Let y be the weight, in grams, of raisins Michael puts into the trail mix.

Inequalities 1 to 4 represent Michael's dietary requirements for the weight of carbohydrate and protein in the trail mix.

Inequality 1: $x \geq 0$ Inequality 3 (carbohydrate): $0.2x + 0.8y \geq 192$

Inequality 2: $y \geq 0$ Inequality 4 (protein): $0.2x + 0.04y \leq 40$

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Michael also requires a minimum of 16 g of **fibre** in the trail mix. Each gram of almonds contains 0.1 g of fibre. Each gram of raisins contains 0.04 g of fibre.

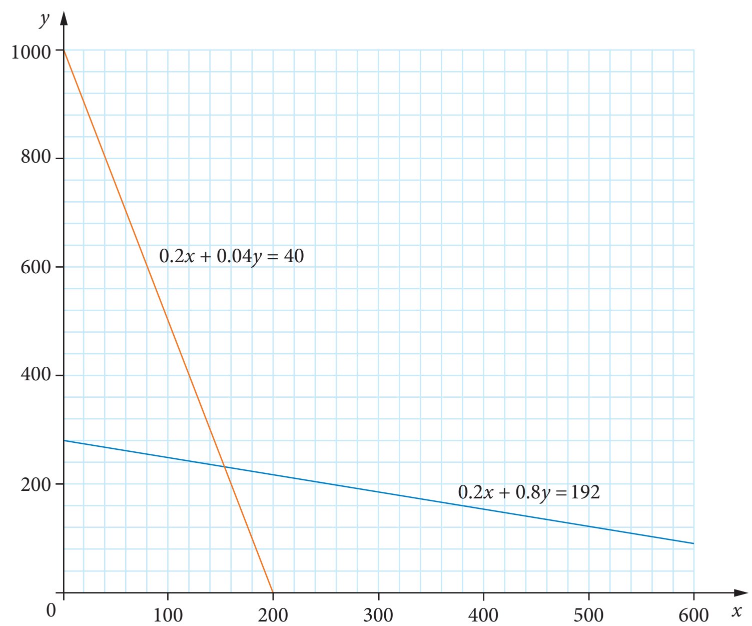
c Write down an inequality, in terms of x and y, that represents this dietary requirement. Call this Inequality 5 (fibre). 1 mark

The graphs of $0.2x + 0.8y = 192$ and $0.2x + 0.04y = 40$ are shown.

d Copy the graph and

i draw the straight line that relates to Inequality 5. 1 mark

ii shade the region that satisfies Inequalities 1 to 5. 1 mark



e What is the maximum weight in grams of trail mix that satisfies Michael's dietary requirements? 1 mark

Michael plans to carry at least 500 g of trail mix on his hike. He would also like this trail mix to contain the greatest possible weight of almonds. The trail mix must satisfy all of Michael's dietary requirements.

f What is the weight of the almonds, in grams, in this trail mix? 2 marks

[VCAA 2011 2GRQ1]

15.3

Question 14

Let x be the number of Softsleep pillows that are sold each week and y be the number of Resteasy pillows that are sold each week.

A constraint on the number of pillows that can be sold each week is given by

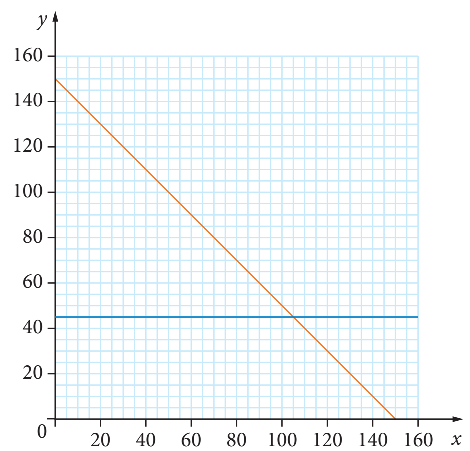
Inequality 1: $x + y \leq 150$

a Explain the meaning of Inequality 1 in terms of the context of this problem. 1 mark

Each week, Anne sells at least 30 Softsleep pillows and at least k Resteasy pillows. These constraints may be written as

Inequality 2: $x \geq 30$ Inequality 3: $y \geq k$

The graphs of $x + y = 150$ and $y = k$ are shown.



b State the value of k. 1 mark

c Copy the graph and

i draw the graph of $x = 30$

ii shade the region that satisfies Inequalities 1, 2 and 3. 2 marks

d Softsleep pillows sell for \$65 each and Resteasy pillows sell for \$50 each. What is the maximum possible weekly revenue that Anne can obtain? 2 marks

Anne decides to sell a third type of pillow, the Snorestop. She sells two Snorestop pillows for each Softsleep pillow sold. She cannot sell more than 150 pillows in total each week.

e Show that a new inequality for the number of pillows sold each week is given by

Inequality 4: $3x + y \leq 150$

where x is the number of Softsleep pillows that are sold each week and y is the number of Resteasy pillows that are sold each week. 1 mark

Softsleep pillows sell for \$65 each. Resteasy pillows sell for \$50 each. Snorestop pillows sell for \$55 each.

f Write an equation for the revenue, R dollars, from the sale of all three types of pillow, in terms of the variables x and y. 1 mark

g Use Inequalities 2, 3 and 4 to calculate the maximum possible weekly revenue from the sale of all three types of pillow. 2 marks

[VCAA 2010 2GRQ3]

Question 15

An event involves running for 10 km and cycling for 30 km. Let x be the time taken (in minutes) to run 10 km and y be the time taken (in minutes) to cycle 30 km.

Event organisers set constraints on the time taken, in minutes, to run and cycle during the event. Inequalities 1 to 6 represent all time constraints on the event.

Inequality 1: $x \geq 0$

Inequality 2: $y \geq 0$

Inequality 3: $x \leq 120$

Inequality 4: $y \leq 150$

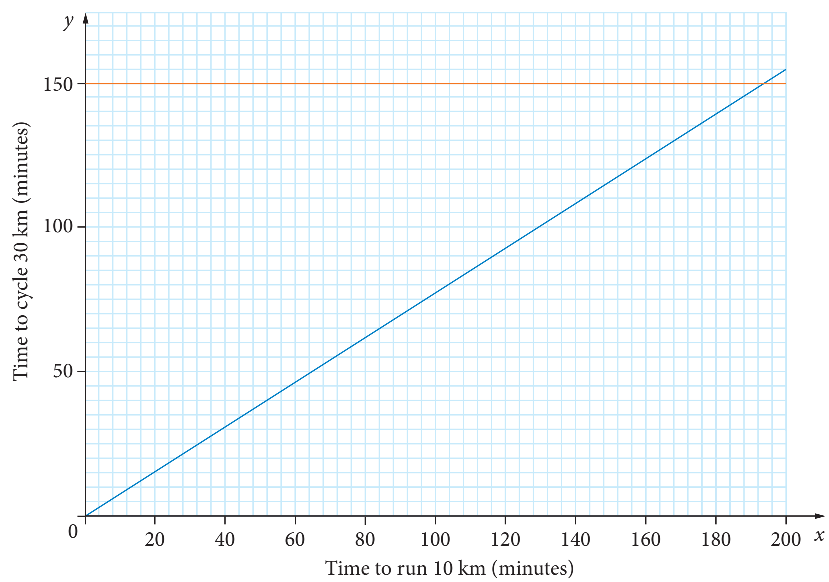
Inequality 5: $y \leq 1.5x$

Inequality 6: $y \geq 0.8x$

15.3

a Explain the meaning of Inequality 3 in terms of the context of this problem. 1 mark

The lines $y = 150$ and $y = 0.8x$ are drawn on the graph.



b Copy the graph and

i **draw** and **label** the lines $x = 120$ and $y = 1.5x$ 2 marks

ii clearly **shade** the feasible region represented by Inequalities 1 to 6. 1 mark

One competitor, Jenny, took 100 minutes to complete the run.

c Between what times, in minutes, can she complete the cycling and remain within the constraints set for the event? 1 mark

d Competitors who complete the event in 90 minutes or less qualify for a prize.

Tiffany qualified for a prize.

i Determine the maximum number of minutes for which Tiffany could have cycled. 1 mark

ii Determine the maximum number of minutes for which Tiffany could have run. 1 mark

[VCAA 2008 2GRQ3]

15.4 Integer solutions

In some linear programming problems the decision variables x and y must be whole numbers. **Integer solutions** exist when the variables x and y are defined as the *number* of X and Y items. The only ordered pairs that can be considered must be inside the feasible region with whole number values for x and y. Decimal or fraction values of corner points cannot be solutions for this type of linear programming problem.

Worked example 12

A carpenter makes two types of chairs, armchairs and rocking chairs. The armchairs take 8 hours to make and sell at a profit of \$40. The rocking chairs take 10 hours to make and sell at a profit of \$30. The carpenter makes at least twice as many armchairs as rocking chairs and has 40 hours production time available each week. Let x be the number of armchairs made each week and y the number of rocking chairs made each week.

a Find the four inequalities in terms of x and y.

b Find the objective function for the profit in the form $P = ax + by$.

c Show all the points that satisfy the four inequalities.

d Find the maximum profit the carpenter can make each week.

Working

a **1** Write the production time information in a table and write the first inequality.

Inequality 1

Production time

Armchairs x	Rocking chairs y	Available time
8 h	10 h	40 h

$8x + 10y \le 40$

Divide both sides by 2.

$4x + 5y \le 20$

2 Write the inequality for at least twice as many armchairs (x) as rocking chairs (y).

Inequality 2

$x : y \ge 2 : 1$

$$\frac{x}{y} \ge \frac{2}{1}$$

$$x \ge 2y$$

$$y \le \frac{1}{2}x$$

3 There cannot be a negative number of chairs.

Inequalities 3 and 4

$x \ge 0,\ y \ge 0$

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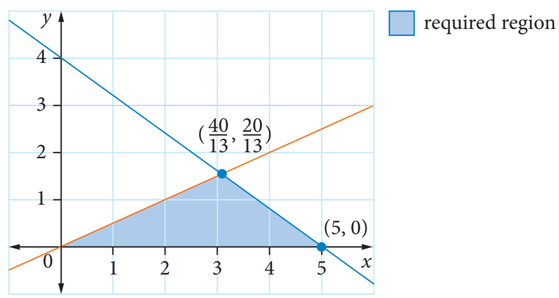
b **1** Write the objective function.

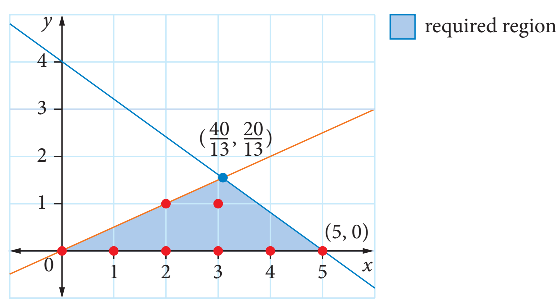
Armchairs x	Rocking chairs y	Total profit
\$40	\$30	P

$P = 40x + 30y$

2 Graph the inequalities using a CAS/calculator.



c Since both x and y must be whole numbers, the points that satisfy the inequalities lie on the intersections of the grid lines within the feasible region.



The red dots show the points that satisfy the four inequalities.

d **1** Set a value for the objective function so it can be graphed near the feasible region.

If $40x + 30y = 40 \times 30$, the x-intercept is (30, 0) and the y-intercept is (0, 40).

This is too big for the graph so let $40x + 30y = 120$

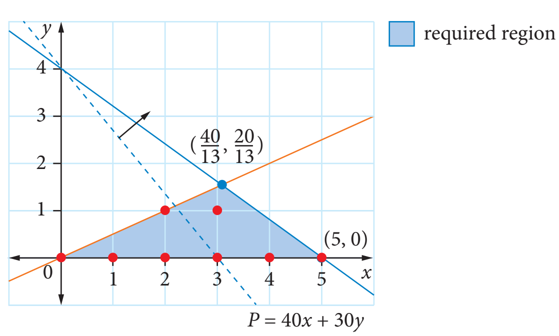
Let the objective function P be

$40x + 30y = 120$

Plot as a broken line on the graph.

x-intercept ($y = 0$) $x = 3$

y-intercept ($x = 0$) $y = 4$



15.4

2	Identify the possible points in the feasible region. Slide the line in the direction of the arrow until the final point in the feasible region is reached.	As x and y must both be whole numbers, (5, 0) will produce the maximum profit. Maximum profit = $40 \times 5 + 30 \times 0 = \$200$ This is achieved when 5 armchairs and no rocking chairs are produced each week.

Exam hack

When dealing with variables that can only take whole number values in a linear programming problem, the solution may not be one of the corner points. This would have occurred in the previous problem if the solid blue line had crossed the x-axis at, for example, (5.5, 0) rather than (5, 0).

EXAM PREP 15.4

Integer solutions

Prep 1 WORKED EXAMPLE 12

A small bicycle workshop makes two styles of bicycles – sports bikes and commuter bikes. Let x be the number of sports bicycles the workshop makes each week and y be the number of commuter bikes the workshop makes each week.

Inequalities 1 and 2 give some of the restrictions on x and y.

Inequality 1: $x \geq 2$ Inequality 2: $x + y \leq 10$

a Explain the meaning of Inequality 2.

For every 3 sports bikes, the workshop must make at least 2 commuter bikes each week.

b Write this inequality in the form $y \leq \frac{a}{b}x$.

c Graph the inequalities, and show all the points that satisfy the inequalities.

d The profit equation is $P = 2x + y$. Find the maximum weekly profit and the number of each type of bike that is made each week.

EXAM PRACTICE 15.4

Integer solutions

Question 1

A company repairs phones and laptops. Let x be the number of phones repaired each day and y be the number of laptops repaired each day. It takes 35 minutes to repair a phone and 50 minutes to repair a laptop. The constraints on the company are as follows.

Constraint 1: $x \geq 0$ Constraint 3: $35x + 50y \leq 1750$

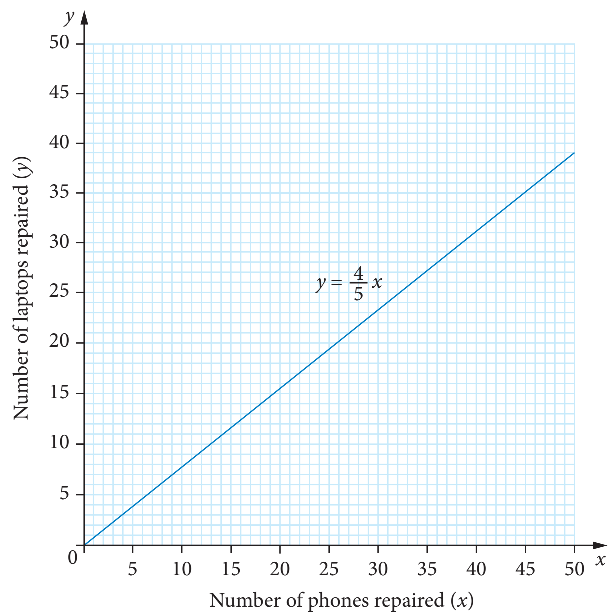
Constraint 2: $y \geq 0$ Constraint 4: $y \leq \frac{4x}{5}$

a Explain the meaning of Constraint 3 in terms of the time available to repair phones and laptops. 1 mark

b Constraint 4 describes the maximum number of phones that may be repaired relative to the number of laptops repaired. Use this constraint to copy and complete the following sentence.

For every ten phones repaired, at most $\boxed{}$ laptops may be repaired. 1 mark

The line $y = \frac{4}{5}x$ is drawn on the graph.



c Copy the graph and draw the line $35x + 50y = 1750$. 1 mark

d Within Constraints 1 to 4, what is the maximum number of laptops that can be repaired each day? 1 mark

e On a day in which exactly nine laptops are repaired, what is the maximum number of phones that can be repaired? 1 mark

The profit from repairing one phone is \$60 and the profit from repairing one laptop is \$100.

f **i** Determine the number of phones and the number of laptops that should be repaired each day in order to maximise the total profit. 2 marks

ii What is the maximum total profit per day that the company can obtain from repairing phones and laptops? 1 mark

[VCAA 2012 2GRQ3]

iStockPhoto/Jmichl

Question 2

Harriet offers dog washing and dog clipping services. Let x be the number of dogs washed in one day and y be the number of dogs clipped in one day.

It takes 20 minutes to wash a dog and 25 minutes to clip a dog. There are 200 minutes available each day to wash and clip dogs. This information can be written as inequalities 1 to 3.

Inequality 1: $x \geq 0$ Inequality 2: $y \geq 0$ Inequality 3: $20x + 25y \leq 200$

a Draw a diagram with appropriate axes and draw the line that represents $20x + 25y = 200$ on it. 1 mark

In any one day the number of dogs clipped is **at least** twice the number of dogs washed.

b Write an inequality to describe this information in terms of x and y. Call it Inequality 4. 1 mark

c **i** On your graph draw and clearly indicate the boundaries of the region represented by Inequalities 1 to 4. 2 marks

ii On a day when exactly five dogs are clipped, what is the maximum number of dogs that could be washed? 1 mark

The profit from washing one dog is \$40 and the profit from clipping one dog is \$30. Let P be the total profit obtained in one day from washing and clipping dogs.

d Write an equation for the total profit, P, in terms of x and y. 1 mark

e **i** Determine the number of dogs that should be washed and the number of dogs that should be clipped in one day in order to maximise the total profit. 1 mark

ii What is the maximum total profit that can be obtained from washing and clipping dogs in one day? 1 mark

[VCAA 2006 2GRQ3]

Question 3

The school group may hire two types of camp sites: powered sites and unpowered sites. Let x be the number of powered camp sites hired and y be the number of unpowered camp sites hired.

Inequality 1 and Inequality 2 give some restrictions on x and y.

Inequality 1: $x \leq 5$ Inequality 2: $y \leq 10$

There are 48 students to accommodate in total. A powered camp site can accommodate up to six students and an unpowered camp site can accommodate up to four students. Inequality 3 gives the restrictions on x and y based on the maximum number of students who can be accommodated at each type of camp site.

Inequality 3: $ax + by \geq 48$

a Write down the values of a and b in Inequality 3. 1 mark

b School groups must hire at least two unpowered camp sites for every powered camp site they hire. Write this restriction in terms of x and y as Inequality 4. 1 mark

The graph shows the three lines that represent the boundaries of Inequalities 1, 3 and 4.

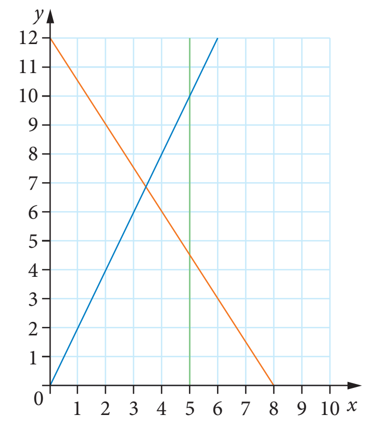
c Copy the graph, then show the points that satisfy Inequalities 1, 2, 3 and 4. 1 mark

d Determine the minimum number of camp sites that the school would need to hire. 1 mark

e The cost of each powered camp site is \$60 per day and the cost of each unpowered camp site is \$30 per day.

i Find the minimum cost per day, in total, of accommodating 48 students. 1 mark

School regulations require boys and girls to be accommodated separately. The girls must all use one type of camp site and the boys must all use the other type of camp site.



ii Determine the minimum cost per day, in total, of accommodating the 48 students if there is an equal number of boys and girls. 1 mark

[VCAA 2013 2GRQ4]

Question 4

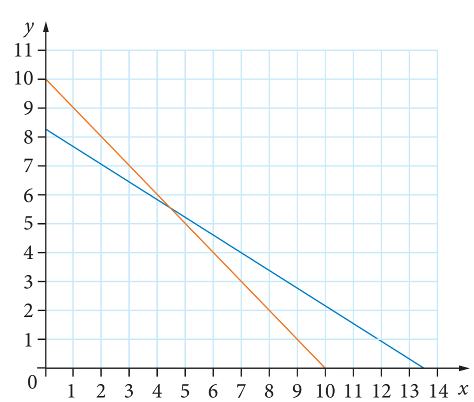
Cheapstar Airlines wishes to find the optimum number of flights per day on two of its most popular routes: Alberton to Bisley and Alberton to Crofton. Let x be the number of flights per day from Alberton to Bisley and y be the number of flights per day from Alberton to Crofton.

The table shows the constraints on the number of flights per day and the number of crew per flight.

	Alberton to Bisley	Alberton to Crofton	Maximum per day	Constraint
Number of flights per day	x	y	10	$x + y \leq 10$
Number of crew per flight	3	5	41	$3x + 5y \leq 41$

The lines $x + y = 10$ and $3x + 5y = 41$ have been graphed.

A profit of \$1300 is made on each flight from Alberton to Bisley and a profit of \$2100 is made on each flight from Alberton to Crofton. Determine the maximum total profit that Cheapstar Airlines can make per day from these flights. 2 marks



[VCAA 2009 2GRQ4]

SUMMARY

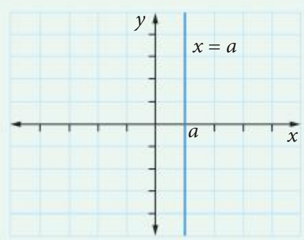
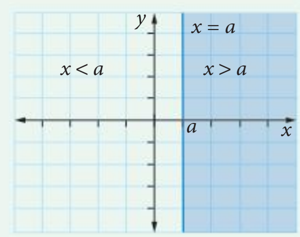
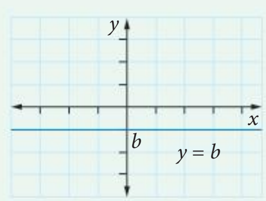
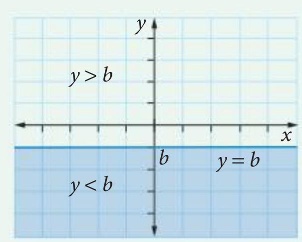
15

Linear programming

Practice quiz

Linear inequalities

- An inequality is a mathematical statement showing that two expressions are not equal using the symbol <, ≤, >, ≥ or ≠.
- Inequalities with ≤ or ≥ have the linear equation drawn with a solid line.
- Inequalities with < or > have the linear equation drawn with a broken line.

<table>
<tr>
<td>The equation $x = a$ is a vertical line.
</td>
<td>The region $x < a$ is on the left side of the line.
The region $x > a$ is on the right side of the line.
</td>
</tr>
<tr>
<td>The line $y = b$ is a horizontal line.
</td>
<td>The region $y < b$ is below the line.
The region $y > b$ is above the line.
</td>
</tr>
</table>

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Method for graphing inequalities

1. Replace the inequality sign <, ≤, > or ≥ with the equality sign = to make the equation of a line.
2. Find two points that the line passes through and draw a line that passes through these points. These points could be the x-intercept, where $y = 0$, and the y-intercept, where $x = 0$.
3. Draw a solid line if the inequality has a ≤ or ≥. Draw a broken line if the inequality has a < or > .
4. Select a point on one side of the line and test to determine if it satisfies the inequality.
5. If the point satisfies the inequality, shade the same side of the line as the point. If the point does not satisfy the inequality, shade the other side of the line.

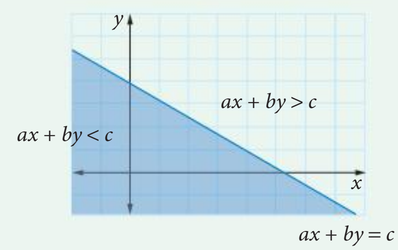
Inequalities can also be graphed using CAS.

Graphing linear inequalities

- If the line is in the form $ax + by = c$ and a, b and c are positive:

  'Less than' is the region below the line. $ax + by < c$

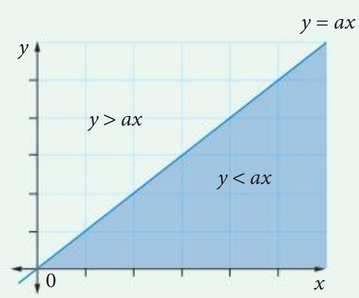
  'Greater than' is the region above the line. $ax + by > c$



- If the line is in the form $y = ax$ (i.e. it passes through (0, 0) and has a gradient of a):

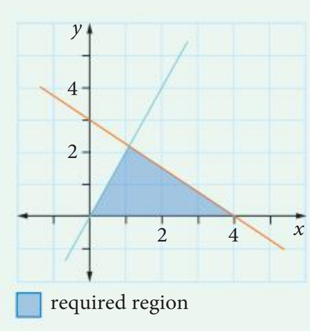
  'Less than' is the region below the line. $y < ax$

  'Greater than' is the region above the line. $y > ax$



Method for graphing systems of simultaneous inequalities

1. Sketch each equation on a set of axes, calculating x- and y-intercepts where necessary.
2. Indicate with a pointing arrow the side of the line that is to be shaded.
3. Shade the area that is common for each inequality.
4. Use a test point where necessary to determine which side of the line requires shading.



Corner point method

1 Find the coordinates of the corner points of the feasible region.

2 Substitute the values of x and y for each corner into the objective function.

3 The solution for maximising are the coordinates that give the largest objective function value.

4 The solution for minimising are the coordinates that give the smallest objective function value.

Sliding line method

1 Choose a value for Z in the objective function $Z = ax + by$ that will give integer values for the x- and y-intercepts. $Z = ab$ is a good starting point.

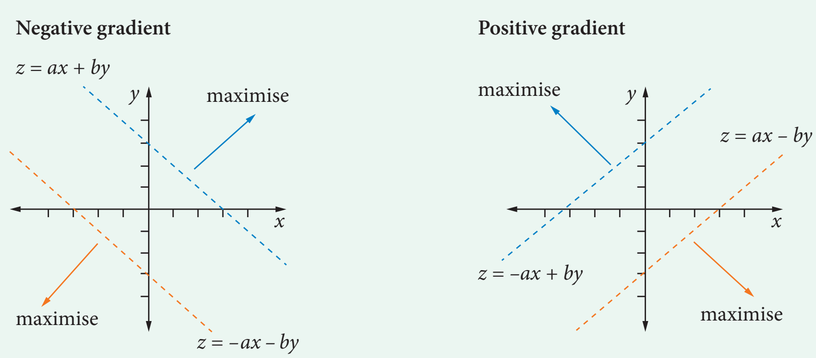
2 Plot the objective function as a broken line on the graph of the constraints. Ideally, this broken line should be inside the feasible region.

3 Slide a ruler along the page, keeping the ruler parallel to the original broken line at all times. Slide the line away from the origin to maximise and towards the origin to minimise.

4 Identify the last corner point that the sliding line contacts inside the feasible region. This point will provide the optimum solution.

For an objective function $Z = ax + by$ where $Z = ab$

- The x-intercept is $(b, 0)$ and the y-intercept is $(0, a)$.
- If Z is a multiple of ab, the x- and y-intercepts are the same multiple of $(b, 0)$ and $(0, a)$.
- Slide the objective function away from the origin to maximise and towards the origin to minimise.



- Some linear programming problems have only whole number solutions.

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SOLUTIONS

- For multiple-choice questions, the tables show the percentage of students who chose each option in the VCAA examination. The correct answer is indicated by highlighting.
- The written response question percentages represent the average percentage score achieved by students in the VCAA examination for that question.
- Colour coding has been used to show question grading: ■ = 80–100%, ■ = 60–79%, ■ = 0–59%, ■ = percentage unknown or not applicable.
- Highlighting of a question number indicates that working and/or the relevant section from the VCAA examination report has been included.

CHAPTER 8
Matrix arithmetic

EXAM PREP 8.1 Introduction to matrices

Prep 1

a 2×2; square matrix **b** 3×1; column matrix

c 1×2; row matrix **d** 2×2; lower triangular

e 3×3; binary matrix **f** 4×4; symmetric matrix

Prep 2

a False **b** True **c** True **d** False **e** True

Prep 3

a $T = \begin{bmatrix} 100 & 80 & 80 & 75 & 100 \\ 225 & 125 & 150 & 175 & 150 \\ 125 & 100 & 150 & 150 & 175 \end{bmatrix}$

The order of T is 5×3.

b $\begin{bmatrix} 225 & 125 & 150 & 175 & 150 \end{bmatrix}$

The order of the matrix is 1×5.

c $\begin{bmatrix} 100 \\ 80 \\ 80 \\ 75 \\ 100 \end{bmatrix}$ **d** $\begin{bmatrix} 75 & 175 & 150 \end{bmatrix}$

e $\begin{bmatrix} 435 \\ 825 \\ 700 \end{bmatrix}$ **f** $\begin{bmatrix} 450 \\ 305 \\ 380 \\ 400 \\ 425 \end{bmatrix}$

Prep 4

a $\begin{array}{c c} & \begin{array}{cccc} \text{Football} & \text{Netball} & \text{Soccer} & \text{Basketball} \end{array} \\ \begin{array}{c} V \\ F \\ H \end{array} & \begin{bmatrix} 2 & 1 & 0 & 1 \\ 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & 0 \end{bmatrix} \end{array}$

b $\begin{array}{c c} & \begin{array}{ccc} V & F & H \end{array} \\ \begin{array}{r} \text{Football} \\ \text{Netball} \\ \text{Soccer} \\ \text{Basketball} \end{array} & \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix} \end{array}$

Prep 5

a $\begin{bmatrix} 1 & 1 & 8 \\ 5 & 1 & 7 \end{bmatrix}$ **b** $\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$ **c** $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

Prep 6

a $A^T = \begin{bmatrix} 7 & 8 & 0 & 3 \\ 2 & -1 & 9 & 4 \end{bmatrix}$

The order of A is 4×2. The order of A^T is 2×4.

b $A^T = \begin{bmatrix} 6 & 3 & 0 \end{bmatrix}$

The order of A is 3×1. The order of A^T is 1×3.

c $A^T = \begin{bmatrix} 7 & 12 \\ 1 & 6 \end{bmatrix}$

The order of A is 2×2. The order of A^T is 2×2.

EXAM PRACTICE 8.1 Introduction to matrices

	Multiple choice					
	%A	%B	%C	%D	%E	
Q1						
Q2						
Q3						
Q4						
Q5						
Q6						
Q7						
Q8						
Q9	2	1	0	2	95	2011
Q10	0	3	1	93	2	2007
Q11	1	6	92	0	0	2010
Q12	3	88	6	2	1	2006
Q13	16	75	1	1	7	2009
Q14	22	72	3	2	1	2012

EXAM PREP 8.2 Addition, subtraction and scalar multiplication of matrices

Prep 1

a $\begin{bmatrix} 26 & 8 \\ 9 & 34 \end{bmatrix}$ **b** $\begin{bmatrix} 31 & 6 \\ 10 & 46 \end{bmatrix}$ **c** $\begin{bmatrix} -5 & -10 \\ 27 & 24 \end{bmatrix}$

d $\begin{bmatrix} 5 & 10 \\ -27 & -24 \end{bmatrix}$ **e** $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Prep 2

a $\begin{bmatrix} 3.1 & 13 & 5 \\ 8.6 & 10 & -6 \\ 17 & 1.9 & -11 \end{bmatrix}$ **b** $\begin{bmatrix} 8 & 2.1 & 8 \\ 0.8 & 0 & 32 \\ 15 & 12.4 & -6 \end{bmatrix}$

Prep 3

a $m = 44$ **b** $d = 5$ **c** $t = 48$

Prep 4

a $\begin{bmatrix} 16 & 14 \\ 16 & 24 \end{bmatrix}$ **b** $\begin{bmatrix} -1 & -24 \\ 9 & -4 \end{bmatrix}$ **c** $\begin{bmatrix} 37 & 37 \\ 35 & 56 \end{bmatrix}$

d $\begin{bmatrix} 18 & 62 \\ -2 & 32 \end{bmatrix}$ **e** $\begin{bmatrix} 21 & -51 \\ 51 & 24 \end{bmatrix}$ **f** $\begin{bmatrix} 15 & 27 \\ 9 & 24 \end{bmatrix}$

Prep 5

a $a = 3, b = 11, c = 6$ **b** $x = 7, y = 9, z = 44$

c $p = 13, q = 19, f = -1$ **d** $a = 8, b = -4, c = -5, d = -2$

EXAM PRACTICE 8.2 Addition, subtraction and scalar multiplication of matrices

	Multiple choice					
	%A	%B	%C	%D	%E	
Q1						
Q2	97	1	1	0	0	2007
Q3	1	2	1	95	1	2012
Q4	1	0	1	94	4	2008
Q5	93	2	1	4	0	2009
Q6	6	84	4	3	2	2006

EXAM PREP 8.3 Matrix multiplication

Prep 1

a $\begin{bmatrix} 86 & 134 \\ 51 & 67 \end{bmatrix}$ **b** $\begin{bmatrix} 20 & 38 & 31 \\ 42 & 72 & 82 \\ 36 & 66 & 61 \end{bmatrix}$

c XY and YX are not equal. Swapping the order when multiplying matrices has resulted in different answers.

Prep 2

a **i** A is of order 1×3. B is of order 3×1. AB is $(1 \times 3) \times (3 \times 1)$. So AB will be a 1×1 matrix.

ii $AB = \begin{bmatrix} 60 \end{bmatrix}$

b **i** B is of order 3×1. A is of order 1×3. BA is $(3 \times 1) \times (1 \times 3)$. So BA will be a 3×3 matrix.

ii $BA = \begin{bmatrix} 6 & 3 & 15 \\ 8 & 4 & 20 \\ 20 & 10 & 50 \end{bmatrix}$

c **i** $CX = \begin{bmatrix} 12 & 9 \end{bmatrix}$

The columns of X have been summed.

ii $\frac{1}{3}CX = \begin{bmatrix} 4 & 3 \end{bmatrix}$

d **i** $YD = \begin{bmatrix} 9 & 20 \\ 3 & 9 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 29 \\ 12 \end{bmatrix}$

The rows of D have been summed.

ii $\frac{1}{2}YD = \begin{bmatrix} 14.5 \\ 6 \end{bmatrix}$

Prep 3

a **i** No

b **i** Yes **ii** 2×3

c **i** Yes **ii** 2×2

d **i** No

e **i** Yes **ii** 3×3

f **i** No

g **i** Yes **ii** 4×2

h **i** Yes **ii** 3×2

i **i** No

j **i** Yes **ii** 2×2

Prep 4

a The order of X is 5×5.

b The order of X is 5×6.

The order of B is 6×4.

c **i** A has order $m \times n$, so A^T has order $n \times m$.

ii A^TA is defined because the number of columns in A^T equals the number of rows in A.

A^TA has order $(n \times n)$.

iii A^TA is a square matrix.

Prep 5

$$\begin{bmatrix} -2+2a \\ -4+3a \end{bmatrix} = \begin{bmatrix} -2b+4 \\ 6b-1 \end{bmatrix}$$

Prep 6

a $\begin{bmatrix} 72 & 49 \\ 63 & 79 \end{bmatrix}$ **b** $\begin{bmatrix} 657 & 700 \\ 900 & 757 \end{bmatrix}$

c $\begin{bmatrix} 30.04 & 31.72 \\ 20.28 & 31.08 \end{bmatrix}$ **d** $\begin{bmatrix} 102.04 & 80.72 \\ 83.28 & 110.08 \end{bmatrix}$

e $\begin{bmatrix} 91\,404 & 87\,493 \\ 112\,491 & 103\,903 \end{bmatrix}$ **f** $\begin{bmatrix} 91\,332 & 87\,444 \\ 112\,428 & 103\,824 \end{bmatrix}$

g 2×2

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EXAM PRACTICE 8.3 Matrix multiplication

	Multiple choice					
	%A	%B	%C	%D	%E	
Q1	92	1	4	1	2	2011
Q2	2	91	3	1	2	2013
Q3	2	4	6	2	85	2009
Q4	84	5	4	4	1	2006
Q5	1	3	11	80	4	2013
Q6	5	9	5	75	5	2006
Q7	20	72	4	1	2	2012
Q8	2	65	8	2	22	2007
Q9	7	20	9	4	59	2010
Q10	8	57	9	19	6	2010
Q11	23	9	50	12	4	2007
Q12	13	39	25	13	9	2012
Q13	26	17	5	29	23	2006
	Written response					
Q14						

Question 9

If P is of order $m \times n$ and Q is of order $n \times m$, then the matrix order equation for PQ is $(m \times n) \times (n \times m) = (m \times m)$, which will be defined for all values of m and n.

The matrix order equation for QP is $(n \times m) \times (m \times n) = (n \times n)$, which will also be defined for all values of m and n. So the matrix products PQ and QP are both defined for all values of m and n.

Question 10

This question can be done by calculating all the matrices, but a quicker way could be to look at the orders of the matrices. The 'sum of the numbers 2, 5, 4, 1, 8' indicates that the answer is a 1×1 matrix. Option B, which has the matrix order equation $(1 \times 5) \times (5 \times 1) = (1 \times 1)$, is clearly the only option that will give a 1×1 matrix.

Question 11

M is of order 3×4. P is of order $5 \times p$.

Let N be of order $m \times n$. $M(NP)$ is of order 3×4.

$M(NP)$ has the matrix order equation

$(3 \times 4) \times (m \times n) \times (5 \times p) = (3 \times 4)$

$m = 4$ (for the first multiplication to be defined)

$n = 5$ (for the second multiplication to be defined)

$p = 4$ (since the number of columns in the last matrix = the number of columns in the product)

The order of N is 4×5.

Question 12

Examination report

One possible solution strategy is as follows.

Simplify the equation by completing the matrix products on each side of the equation.

$$\begin{bmatrix} 3a+12 \\ a+6 \end{bmatrix} = \begin{bmatrix} 12+3b \\ 4-b \end{bmatrix}$$

Thus $3a + 12 = 12 + 3b$ or $a - b = 0$ and $a + 6 = 4 - b$ or $a + b = -2$

[VCAA 2012 1RMQ9]

Question 13

A is of order 3×2. XA is of order 3×2.

Let X be of order $m \times n$.

XA has the matrix order equation

$(m \times n) \times (3 \times 2) = (3 \times 2)$

$m = 3$ (since the number of rows in the first matrix = the number of rows in the product)

$n = 3$ (for the multiplication to be defined)

The order of X is 3×3.

Question 14

a A and D

b Yes. C has the same number of rows as A has columns.

c AC, AB, BA, BD, CB, DC, DB

d AC is of order 3×2, AB is of order 3×3, BA is of order 2×2, BD is of order 2×2, CB is of order 2×3, DC is of order 3×2, DB is of order 3×3.

e C – only square matrices can be raised to a power.

EXAM PREP 8.4 Inverse matrices

Prep 1

a $AI = \begin{bmatrix} 5 & 2 \\ 1 & 11 \end{bmatrix}$ **b** $IA = \begin{bmatrix} 5 & 2 \\ 1 & 11 \end{bmatrix}$

c $AI = IA = A$

This is an example where we can change the order of the matrices and the product stays the same.

Prep 2

$$\begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Prep 3

a **i** $\det(A) = -2$

ii $A^{-1} = -\frac{1}{2}\begin{bmatrix} 7 & -8 \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} -\frac{7}{2} & 4 \\ 1 & -1 \end{bmatrix}$

b **i** $\det(B) = -5$

ii $B^{-1} = -\frac{1}{5}\begin{bmatrix} 1 & -10 \\ -1 & 5 \end{bmatrix} = \begin{bmatrix} -\frac{1}{5} & 2 \\ \frac{1}{5} & -1 \end{bmatrix}$

c **i** $\det(C) = 0$ **ii** C^{-1} doesn't exist.

Prep 4

a $x = -4$ **b** $x = 6$

c Since $A + B$ is defined, A and B must be of the same order, so both A and B are of order $m \times m$.

If A is $m \times m$, then A^{-1} is $m \times m$ and A^2 is $m \times m$.

If B is $m \times m$, then B^3 is $m \times m$.

The matrix order equation for $A^2A^{-1}B^3 - B$ is

$(m \times m) \times (m \times m) \times (m \times m) - (m \times m) =$

$(m \times m) - (m \times m)$

Since the two matrices being subtracted are of the same order, $A^2A^{-1}B^3 - B$ is defined.

Prep 5

a **i** $\det(A) = 1$

ii $A^{-1} = \begin{bmatrix} 46 & -51 & 81 \\ -9 & 10 & -16 \\ -26 & 29 & -46 \end{bmatrix}$

b **i** $\det(A) = -3$

ii $A^{-1} = \begin{bmatrix} \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ -\frac{2}{3} & -\frac{1}{3} & \frac{1}{3} \\ 1 & 1 & 0 \end{bmatrix}$

c **i** $\det(A) = 1$

ii $A^{-1} = \begin{bmatrix} 2 & -17 & 9 \\ -1 & 9 & -5 \\ -2 & 19 & -10 \end{bmatrix}$

EXAM PRACTICE 8.4 Inverse matrices

	Multiple choice					
	%A	%B	%C	%D	%E	
Q1						
Q2						
Q3						
Q4						
Q5	2	3	10	5	80	2008
Q6	6	67	11	3	12	2008
Q7	17	57	9	8	8	2011
Q8	10	8	9	49	24	2011
Q9	12	17	19	12	39	2013

Question 7

Option A isn't defined because only powers of square matrices are defined, so BA^2 isn't defined.

Option B has the matrix order equation $(3 \times 3) \times (3 \times 4) + (3 \times 4) = (3 \times 4) + (3 \times 4)$. This sum is defined because the two matrices being added are of the same order. So option B *is* defined.

Option C isn't defined because it involves adding two matrices of different orders.

Option D isn't defined because AB isn't defined since the orders are $(3 \times 4) \times (3 \times 3)$.

Option E isn't defined because only square matrices have inverses.

Question 8

$\det(A) = -9 - [k \times (-4)] = -9 + 4k$

$$A^{-1} = \frac{1}{-9+4k}\begin{bmatrix} -3 & -k \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} \frac{-3}{-9+4k} & \frac{-k}{-9+4k} \\ \frac{4}{-9+4k} & \frac{3}{-9+4k} \end{bmatrix}$$

$A = A^{-1}$

$$\begin{bmatrix} 3 & k \\ -4 & -3 \end{bmatrix} = \begin{bmatrix} \frac{-3}{-9+4k} & \frac{-k}{-9+4k} \\ \frac{4}{-9+4k} & \frac{3}{-9+4k} \end{bmatrix}$$

So, $3 = \frac{-3}{-9+4k}$

Solving for k using a CAS/calculator gives $k = 2$.

Question 9

Examination report

One solution strategy is as follows.

Because the product $P = QRS$ is defined, and Q and S are square, we can deduce (in general terms) the order of these matrices as follows.

Let the order of $Q = m \times m$ and the order of $S = n \times n$, where $m \neq n$ because $Q + S$ is not defined.

Since QRS is defined, the order of R must be $m \times n$.

Thus, considering each option

- A: $R - S$ is not defined as R and S have different orders
- B: $Q + R$ is not defined as Q and R have different orders
- C: $P^2 = P \times P$ is not defined because $m \neq n$
- D: R^{-1} is not defined because R is not square
- E: $P \times S$ is defined because the number of columns in P, n, equals the number of rows in S, n.

[VCAA 2013 1RMQ9]

CHAPTER 9
Matrix applications

EXAM PREP 9.1 Using matrices

Prep 1

a $W = \begin{bmatrix} 1 \\ 4 \\ 6 \end{bmatrix}$

b $R = \begin{bmatrix} 26 & 5 & 2 \\ 12 & 2 & 0 \\ 18 & 7 & 2 \\ 9 & 2 & 0 \\ 30 & 3 & 0 \\ 5 & 0 & 0 \\ 3 & 0 & 0 \end{bmatrix}$

c $S = \begin{bmatrix} 26 & 5 & 2 \\ 12 & 2 & 0 \\ 18 & 7 & 2 \\ 9 & 2 & 0 \\ 30 & 3 & 0 \\ 5 & 0 & 0 \\ 3 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 58 \\ 20 \\ 58 \\ 17 \\ 42 \\ 5 \\ 3 \end{bmatrix}$

d The award was a tie between Heshan and Ahmat.

e The Little Clunkers made 203, so they lost.

Prep 2

a $\begin{array}{l} \text{Week 1} \\ \text{Week 2} \end{array}\begin{bmatrix} 75 & 60 \\ 47 & 82 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 330 \\ 340 \end{bmatrix}$

b **i** $\begin{bmatrix} 330 & 340 \\ 473 & 542 \\ 628 & 745 \\ 263 & 220 \end{bmatrix}$

ii $175\% = 175 \div 100 = 1.75$

iii $S = 1.75C$

$= \begin{bmatrix} 577.50 & 595.00 \\ 827.75 & 948.50 \\ 1099.00 & 1303.75 \\ 460.25 & 385.00 \end{bmatrix}$

c **i** Profit $= S - C$

$\begin{bmatrix} 247.50 & 255.00 \\ 354.75 & 406.50 \\ 471.00 & 558.75 \\ 197.25 & 165.00 \end{bmatrix}$

ii $247.50 + 255.00 + \ldots + 165.00 = \$2655.75$

Prep 3

a **i** $A = PB = \begin{bmatrix} 23 & 37 & 10 \end{bmatrix}$

A gives the totals for each type of train spotted by the club on the weekend.

ii a_{12} tells us that there were a total of 37 Blueghosts sighted on the weekend.

b **i** $C = \begin{bmatrix} 13 \\ 11 \\ 18 \\ 17 \\ 11 \end{bmatrix}$

C gives the total number of trains spotted by each trainspotter on the weekend.

ii c_{41} tells us that Steve spotted a total of 17 trains on the weekend.

c $\frac{1}{5}PBQ = \frac{1}{5}\begin{bmatrix} 70 \end{bmatrix} = \begin{bmatrix} 14 \end{bmatrix}$

$\frac{1}{5}PBQ$ tells us that the mean number of trains sighted by club members over the weekend is 14.

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EXAM PRACTICE 9.1 Using matrices

	Multiple choice					
	%A	%B	%C	%D	%E	
Q1	1	3	86	7	2	2013
Q2	3	2	11	78	4	2006
Q3	17	1	72	2	8	2008
Q4	70	3	15	4	9	2010
Q5	20	2	2	15	61	2008
Q6	10	13	60	10	6	2012
Q7	11	10	19	43	16	2013
Q8	4	66	19	1	9	2010
	Written response					
Q9	90					2009
Q10a–bii	70					2008
Q10ci–cii	Not reported					2008
Q11	68					2006
Q12	67					2012
Q13	Not reported					2010

Question 7

The mean of a class involves adding 15 results and dividing the sum by 15. This means only D and E are options. There are 3 classes, so the matrix will be either 1×3 or 3×1.

E isn't an option since $S \times R \times M$ is $(15 \times 1) \times (1 \times 3) \times (3 \times 15)$, giving a product matrix of 15×15.

$M \times S$ is $(3 \times 15) \times (15 \times 1)$, giving a product matrix of 3×1, so this is the correct option.

Question 8

The product of the matrices needs to calculate the numbers of each of the photocopiers, fax machines and scanners multiplied by the number of minutes it takes for each to be serviced. The product UT does this.

$$UT = \begin{bmatrix} 5 & 3 & 2 \\ 4 & 4 & 3 \\ 6 & 1 & 2 \end{bmatrix}\begin{bmatrix} 12 & 13 & 14 \\ 8 & 7 & 8 \\ 20 & 19 & 17 \end{bmatrix} = \begin{bmatrix} 124 & 134 & 128 \\ 140 & 145 & 139 \\ 120 & 135 & 126 \end{bmatrix}$$

Question 9

a 2×3

b **i** $\begin{bmatrix} 131.30 \\ 130.75 \end{bmatrix}$ **ii** Safeworth

Question 10

Examination report

a 1×5

b **i** $\begin{bmatrix} 23 & 57.5 & 80.5 & 207 & 92 \\ 18 & 45 & 63 & 162 & 72 \end{bmatrix}$

ii The number of students who are expected to get a D in Chemistry.

c **i** $\begin{array}{cc} B & C \\ \end{array}$
$\begin{bmatrix} 110 & 95 \end{bmatrix}$

Despite a row matrix being required, many students wrote a column matrix instead. The question expected students to label the matrix as shown.

ii $\begin{bmatrix} 110 & 95 \end{bmatrix}\begin{bmatrix} 460 \\ 360 \end{bmatrix} = \begin{bmatrix} 84\,800 \end{bmatrix}$

[VCAA 2008 2RMQ1]

Question 11

Examination report

a 2×3

b **i** $M = \begin{bmatrix} 145\,978.00 \\ 171\,848.50 \end{bmatrix}$

ii Total revenue from selling products A, B and C at Eastown and Noxland.

c Number of columns in $P \neq$ numbers of rows in Q.

Many answers illustrated a poor understanding of the need for the number of columns in matrix P to be the same as the number of rows in matrix Q.

[VCAA 2006 2RMQ1]

Question 12

Examination report

a **i** $C = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 2 & 6 & 4 \\ 1 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 2 \\ 3 & 9 & 6 \end{bmatrix}$

ii 133926

b $A = B^{-1}C$

Other equivalent forms of this answer were accepted.

Question 13

Examination report

a $\begin{bmatrix} 4 & 8 & 2 \end{bmatrix}$

b $\begin{bmatrix} 26 \end{bmatrix}$ The required answer needed to be a matrix.

c The total of all the points scored by Oscar in this game.

Explanations should have been applicable to the context of the question. Since there were different points available for different types of shots at goal, an answer of 'The total of all goals shot by Oscar' was not accepted.

[VCAA 2010 2RMQ1]

EXAM PREP 9.2 Matrices and simultaneous equations

Prep 1

a $\begin{bmatrix} 4x-y \\ x-3y \end{bmatrix} = \begin{bmatrix} -5 \\ 7 \end{bmatrix}$ **b** $x=-2, y=-3$

Prep 2

a Solving $-24 + 4e = 0$ gives $e = 6$.

b Solving $12e - 12 = 0$ gives $e = 1$.

Prep 3

a $x=-1, y=2, z=4$ **b** $x=5, y=4, z=3$

c $x=1, y=10, z=2$

Prep 4

a
$$\begin{aligned} 2a + 5b + 3c + d &= 167 \\ 4a + 10b &= 240 \\ 6b + 12c + 3d &= 273 \\ 5c + 2d &= 82 \end{aligned}$$

b
$$\begin{bmatrix} 2 & 5 & 3 & 1 \\ 4 & 10 & 0 & 0 \\ 0 & 6 & 12 & 3 \\ 0 & 0 & 5 & 2 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 167 \\ 240 \\ 273 \\ 82 \end{bmatrix}$$

c
$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 2 & 5 & 3 & 1 \\ 4 & 10 & 0 & 0 \\ 0 & 6 & 12 & 3 \\ 0 & 0 & 5 & 2 \end{bmatrix}^{-1}\begin{bmatrix} 167 \\ 240 \\ 273 \\ 82 \end{bmatrix} = \begin{bmatrix} 20 \\ 16 \\ 12 \\ 11 \end{bmatrix}$$

20 minutes to assemble a laptop, 16 minutes for a printer, 12 minutes for a modem, and 11 minutes for a router.

EXAM PRACTICE 9.2 Matrices and simultaneous equations

	Multiple choice					
	%A	%B	%C	%D	%E	
Q1	4	6	5	82	3	2009
Q2	11	75	4	4	6	2008
Q3	11	4	5	73	5	2010
Q4	4	13	67	10	5	2011
Q5	26	64	3	4	3	2007
Q6	7	7	23	7	55	2013
Q7	14	27	42	9	7	2006
Q8	41	42	7	3	6	2013
Q9	41	49	4	3	2	2012
Q10	7	43	9	4	35	2010
	Written response					
Q11	70					2009
Q12	Not reported					2010
Q13	35					2006

Question 6

The equation in matrix form is $\begin{bmatrix} 2.8 & 0.7 \\ 1.4 & k \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 10 \\ 6 \end{bmatrix}$.

The set of simultaneous equations doesn't have a solution when the determinant of the square matrix is zero.

$\det = 2.8k - 0.7 \times 1.4 = 2.8k - 0.98 = 0$

Solving for k gives $k = 0.35$.

Question 7

Rewrite these in matrix form and find the number of square matrices that have a determinant which isn't zero.

$\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \det = 0$, $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \det = 1$, $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \det = 2$,

$\begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} \det = 0$, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \det = 1$

Three matrices have non-zero determinants, so three have a unique solution.

Question 8

The two simultaneous equations involved are

$$\begin{bmatrix} 10 & 4 \\ 8 & 3 \end{bmatrix}\begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} 360 \\ 280 \end{bmatrix}$$

To solve for b and d, find the inverse of the square matrix and pre-multiply on both sides.

$$\begin{bmatrix} b \\ d \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} 3 & -4 \\ -8 & 10 \end{bmatrix}\begin{bmatrix} 360 \\ 280 \end{bmatrix} = \begin{bmatrix} -1.5 & 2 \\ 4 & -5 \end{bmatrix}\begin{bmatrix} 360 \\ 280 \end{bmatrix}$$

Question 9

The matrix form of the simultaneous equations is

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 6 \\ 8 \\ 15 \end{bmatrix}$$

The inverse of the square matrix is $\begin{bmatrix} -3 & -1 & 2 \\ -2 & 0 & 1 \\ 4 & 1 & -2 \end{bmatrix}$.

Solving, we get $\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} -3 & -1 & 2 \\ -2 & 0 & 1 \\ 4 & 1 & -2 \end{bmatrix}\begin{bmatrix} 6 \\ 8 \\ 15 \end{bmatrix}$

Question 10

The simultaneous equations are

$$\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 9 \\ 16 \end{bmatrix}$$

The inverse of the square matrix is

$$-1\begin{bmatrix} 5 & -3 \\ -2 & 1 \end{bmatrix}=\begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix}$$

Solving, we get $\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 9 \\ 16 \end{bmatrix}$

Question 11

a 35 and 2

b $32

Question 12

a $\begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 3 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} p \\ r \\ a \end{bmatrix}=\begin{bmatrix} 33 \\ 40 \\ 43 \end{bmatrix}$

b $x = -5$

c 10

Question 13

Examination report

a $\begin{bmatrix} 2 & 1 & 1 \\ 1 & -1 & 1 \\ 0 & 2 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 12 \\ 1 \\ 6 \end{bmatrix}$

Many students did not deal well with the missing x term in the third equation. A common error was to put zero in the wrong position on the third line or to simply leave a gap. Neither of these two options was awarded a mark.

b There is a unique solution since $\det\begin{bmatrix} 2 & 1 & 1 \\ 1 & -1 & 1 \\ 0 & 2 & -1 \end{bmatrix} = 1 \neq 0$

c $\begin{bmatrix} -1 & 3 & 2 \\ 1 & -2 & -1 \\ 2 & -4 & -3 \end{bmatrix}$

A common incorrect answer was $\begin{bmatrix} 2 & 1 & 1 \\ 1 & -1 & 1 \\ 0 & 2 & -1 \end{bmatrix}^{-1}$.

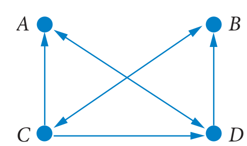
d 3 bookshops, 4 sports shoe shops and 2 music stores

[VCAA 2006 2RMQ3]

EXAM PREP 9.3 Communication and dominance matrices

Prep 1

a



b $N = \begin{bmatrix} 1 & 1 & 3 & 2 \end{bmatrix}$

c Pre-multiplying by K has summed the elements in the columns in M. Matrix N shows the total number of direct shuttles from each of the four locations to the other three locations.

d Matrix N shows that there are three direct shuttles from Central Station to the other locations, whereas the other locations have fewer direct shuttles leaving from them. So Central Station has the most direct shuttles to the other three locations.

Prep 2

a **i**
$$M = \begin{array}{c} \text{From} \\ \begin{array}{cccccc} A & B & C & D & E & F \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ \\ A \\ B \\ C \\ D \\ E \\ F \end{array} \ \text{To}$$

ii
$$M = \begin{array}{c} \text{From} \\ \begin{array}{cccccc} A & B & C & D & E & F \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ \\ A \\ B \\ C \\ D \\ E \\ F \end{array} \ \text{To}$$

b All the arrows in the second communication diagram go both ways. This means every communication is two-way. Communication matrices where every communication is two-way are always symmetric about the leading diagonal.

c
$$M = \begin{array}{c} \text{From} \\ \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 22 & 46 & 68 \\ 22 & 0 & 31 & 45 \\ 46 & 31 & 0 & 51 \\ 48 & 45 & 51 & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ \\ A \\ B \\ C \\ D \end{array} \ \text{To}$$

Prep 3

a **i**
$$M^2 = \begin{array}{c} \text{From} \\ \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 2 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 2 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ \\ A \\ B \\ C \\ D \end{array} \ \text{To}$$

ii 2; $A \to B \to A$ and $A \to C \to A$ **iii** 0

b **i**
$$M^3 = \begin{array}{c} \text{From} \\ \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 1 & 0 & 1 \\ 3 & 3 & 1 & 2 \\ 1 & 1 & 2 & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ \\ A \\ B \\ C \\ D \end{array} \ \text{To}$$

ii 1; $A \to B \to C \to A$

iii 3; $C \to D \to C \to A$ $\quad C \to A \to C \to A$
$C \to A \to B \to A$

c **i**
$$M + M^2 = \begin{array}{c} \text{From} \\ \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 2 & 2 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ \\ A \\ B \\ C \\ D \end{array} \ \text{To}$$

ii 2; $A \to C \quad A \to B \to C$

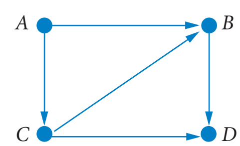
iii D can't communicate with B in one or two steps.

d **i**
$$M + M^2 + M^3 = \begin{array}{c} \text{From} \\ \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 3 & 4 & 4 & 1 \\ 3 & 2 & 1 & 1 \\ 5 & 5 & 3 & 3 \\ 2 & 2 & 3 & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ \\ A \\ B \\ C \\ D \end{array} \ \text{To}$$

ii There are no zeros in $M + M^2 + M^3$, so every one of the four ships can communicate with every other ship using one-, two- or three-step communications.

Prep 4

a



b
$$M = \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ A \\ B \\ C \\ D \end{array}$$

c Albie 2, Bertie 1, Collie 2, Dougie 0.

d
$$M^2 = \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ A \\ B \\ C \\ D \end{array}$$

M^2 tells us how often a player defeated a player who had defeated another player. It gives us the number of indirect wins. So, we can see that Albie defeated a player who had defeated Bertie, and Albie also defeated two players who had defeated Dougie. We can also see that Collie defeated a player who had defeated Dougie.

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e $$M + M^2 = \begin{array}{cc} \begin{array}{cccc} A & B & C & D \end{array} & \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 2 & 1 & 2 & 0 \end{bmatrix} & \begin{array}{c} A \\ B \\ C \\ D \end{array} \end{array}$$

Although Albie and Collie both had two direct wins, summing the columns of $M + M^2$ tells us that Albie had a total of 5 direct and indirect wins, and Collie had 3 direct and indirect wins. Based on this, Albie has dominance and should be declared the winner.

EXAM PRACTICE 9.3 Communication and dominance matrices

	Multiple choice					
	%A	%B	%C	%D	%E	
Q1	1	1	2	12	84	2012
Q2	5	5	10	4	75	2009
Q3	9	5	74	3	9	2011
Q4	22	65	5	5	2	2007
Q5	8	7	6	17	62	2013
	Written response					
Q6	80					2012
Q7	Not reported					2011
Q8	73					2013
Q9	50					2008

Question 6

Examination report

a Anvil and Dantel.

b Anvil – Berga – Dantel – Cantor

c $G = KF = \begin{bmatrix} 1 & 2 & 1 & 1 \end{bmatrix}$

d The matrix G lists, for each city, the total number of direct flight connections from that city to another city in the network.

[VCAA 2012 2RMQ1]

Question 7

Examination report

a **i** Birds eat lizards. A common inaccurate interpretation was 'one bird eats one lizard'.

ii No birds, lizards or insects eat birds.

Many students had difficulty with this question. Common incorrect answers included:

- no lizards or insects eat birds (this omitted one of the zeros)
- insects do not eat insects, birds or lizards (confused a column with a row)
- none of these animals eat their own kind (confused a diagonal with a row).

b $$Z = \begin{array}{cc} \begin{array}{cccc} I & B & L & F \end{array} & \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} & \begin{array}{c} I \\ B \\ L \\ F \end{array} \end{array}$$

[VCAA 2011 2RMQ1]

Question 8

a 2

b The number of ponds connected directly by pipe to pond R.

c $$N = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Question 9

Examination report

a Arnold and Edgar; Barnaby and Cedric

Both pairs are expected. One mark was awarded for each correct pair.

b Edgar

c Cedric, Barnaby, Arnold, Edgar

d 20

[VCAA 2008 2RNQ4]

EXAM PREP 9.4 Transition diagrams and matrices

Prep 1

a yes **b** no **c** no **d** yes

e no **f** no **g** yes **h** yes

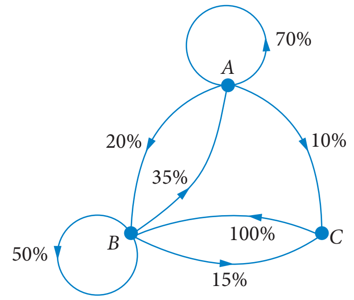
Prep 2

a **i** $x = 35\%, y = 10\%, z = 30\%$

ii $$\begin{array}{cc} \text{Current state} & \\ \begin{array}{ccc} A & B & C \end{array} & \\ \begin{bmatrix} 0.35 & 0.1 & 0.2 \\ 0.4 & 0.05 & 0.3 \\ 0.25 & 0.85 & 0.5 \end{bmatrix} & \begin{array}{c} A \\ B \\ C \end{array} \text{ Next state} \end{array}$$

b **i** $x = 0.1, y = 0.5, z = 0$

ii



Prep 3

a 6485 **b** 21 320 **c** 33 627 **d** 52 282

Prep 4

$$\textbf{a}\quad \begin{array}{cc} & \text{This day} \\ & \begin{array}{cc} R & D \end{array} \\ T = & \begin{bmatrix} 0.65 & 0.1 \\ 0.35 & 0.9 \end{bmatrix} \end{array} \begin{array}{c} \\ \\ R \\ D \end{array} \text{Next day}$$

$$\textbf{b}\quad \begin{array}{cc} & \text{This question} \\ & \begin{array}{ccccc} A & B & C & D & E \end{array} \\ T = & \begin{bmatrix} 0 & 0 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0.5 & 0 & 0 \\ 1 & 0 & 0.5 & 0.5 & 0 \end{bmatrix} \end{array} \begin{array}{c} A \\ B \\ C \\ D \\ E \end{array} \text{Next question}$$

EXAM PRACTICE 9.4 Transition diagrams and matrices

	Multiple choice					
	%A	%B	%C	%D	%E	
Q1	6	3	87	1	2	2009
Q2	12	76	2	8	1	2009
Q3	13	1	6	74	5	2012
Q4	12	5	6	22	56	2006
Q5	33	4	7	8	48	2007
Q6	8	48	19	14	11	2007
Q7	35	29	12	8	15	2010

Question 4

Let A indicate a holiday in Australia, O overseas.

$$\begin{array}{cc} & \text{This holiday} \\ & \begin{array}{cc} A & O \end{array} \\ T = & \begin{bmatrix} 0.95 & \\ & 0.20 \end{bmatrix} \end{array} \begin{array}{c} A \\ O \end{array} \text{Next holiday}$$

Completing this so that the columns add to 1 gives us E.

Question 5

'Never going to the same place twice' means that there are zeros in the leading diagonal. This means D and E are the only two options. Of those two, only E has the elements in each column adding to 1, so E is the correct option.

Question 6

This can be solved by starting with D for 'this question' in the transition matrix and following the subsequent choices which are all 100% (i.e. certain) choices.

If Kerry selects D, the matrix says he will select B next. If he selects B, then the matrix says he will select C next, and so on. This process gives D, B, C, A, A, A (i.e. alternative B) as the correct option.

Question 7

The test consists of only four multiple choice questions, so we can list the results of each selection according to the transition matrix. The 1s in the matrix indicate 100% (i.e. certain) choices.

If A is selected first: A, A, A, A

If B selected first: B, D, C, B

If C selected first: C, B, D, C

If D selected first: D, C, B, D

Looking at the four sets of results, the only true statement is that Robbie always gives the same answer to the first and fourth question, so B is the correct option.

EXAM PREP 9.5 Transition matrices and recurrence relations

Prep 1

56 at FushNChups and 90 at BurgerHQ.

Prep 2

a 59 train carriages at depot A and 71 at depot B.

b 56 train carriages at depot A and 74 at depot B.

Prep 3

a 0.2 > 0.15, so more train carriages will be at depot *B*.

b Since *T* has no zero elements, there will be an equilibrium state solution.

c, d 56 train carriages at depot *A* and 74 at depot *B*.

e $\begin{bmatrix} 56 \\ 74 \end{bmatrix}$ In the long term, 56 train carriages will be at depot *A* and 74 train carriages will be at depot *B*.

EXAM PRACTICE 9.5 Transition matrices and recurrence relations

	Multiple choice					
	%A	%B	%C	%D	%E	
Q1	3	8	3	84	1	2009
Q2	5	9	68	16	2	2011
Q3	7	15	10	18	48	2011
Q4	68	8	16	4	4	2008
Q5	2	4	13	57	24	2008
Q6	17	8	15	25	34	2008
Q7	6	8	65	6	15	2009
Q8	18	63	9	5	4	2013
Q9	5	6	14	60	15	2010
Q10	58	6	13	8	15	2011
Q11	8	55	11	11	14	2012
Q12	17	15	10	51	6	2007
Q13	11	40	12	14	22	2006
	Written response					
Q14	Not reported					2008
Q15 a–b	80					2006
Q15 c–d	45					2006

Question 3

$$T = \begin{array}{c} \begin{array}{cc} R & A \end{array} \\ \begin{bmatrix} 0.6 & 0.25 \\ 0.4 & 0.75 \end{bmatrix} \end{array} \begin{array}{c} R \\ A \end{array}$$

$$\begin{bmatrix} 0.6 & 0.25 \\ 0.4 & 0.75 \end{bmatrix}^{10} \begin{bmatrix} 5629 \\ 3450 \end{bmatrix} \approx \begin{bmatrix} 4479 \\ 4663 \end{bmatrix}$$

Anna has more votes by 4663 – 4479 = 184 (about 200).

Question 5

$$T^{29} = T^{30} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \frac{3}{4} & 1 & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{4} & 0 & \frac{1}{3} & 1 \end{bmatrix}$$, so in the long term the mutton birds will only nest on sites *B* and *D*.

Question 6

The same number of birds are on each site in 2007.

Let x = the number of birds on each site. So the relevant part of the matrix multiplication is

$$\begin{bmatrix} 0.35 & 1 & 0.15 & 0 \\ & & & \end{bmatrix} \begin{bmatrix} x \\ x \\ x \\ x \end{bmatrix} = \begin{bmatrix} 6000 \\ \\ \end{bmatrix}$$, which gives

$0.35x + x + 0.15x = 6000$

Solving this, we get $x = 4000$. This is the number on each site, so the total number is $4x = 16\,000$.

Question 9

A and B are clearly true from the transition matrix.

Since 0.6 > 0.3, C is true.

There is an equilibrium state, since the initial state matrix doesn't contain any zeros, and therefore the steady state matrix will never contain any zeros. This means that some birds will always be found at each location, so E is true.

If the equilibrium state solution has no zeros, then there will always be some changes to the feeding location, that is, although the totals will not change, an (equal) number of birds will swap locations each year. So D is not true.

Question 10

$$T = \begin{array}{c} \begin{array}{cc} A & B \end{array} \\ \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix} \end{array} \begin{array}{c} A \\ B \end{array}$$

If $t_{12} > t_{21}$, in the long term more will transition to *A*.
So look for the option where $t_{12} > t_{21}$.

Question 11

Reject A, D and E because the initial state matrix isn't given.

We can conclude from *T* that 50% of children doing cycling this week do cycling next week, 60% doing orienteering this week do orienteering next week, and 50% of children doing swimming this week do swimming next week. So at least 50% do not change their activity from the first week to the second week.

Reject C because there are no zeros in the transition matrix, hence there will be no zeros in the steady state matrix.

Question 12

$$\begin{bmatrix} 0.8 & 0.1 & 0.2 \\ 0.1 & 0.6 & 0.1 \\ 0.1 & 0.3 & 0.7 \end{bmatrix}^{29} \begin{bmatrix} 1568 \\ 1105 \\ 894 \end{bmatrix} = \begin{bmatrix} 1605 \\ 713 \\ 1248 \end{bmatrix}$$

$$\begin{bmatrix} 0.8 & 0.1 & 0.2 \\ 0.1 & 0.6 & 0.1 \\ 0.1 & 0.3 & 0.7 \end{bmatrix}^{30} \begin{bmatrix} 1568 \\ 1105 \\ 894 \end{bmatrix} = \begin{bmatrix} 1605 \\ 713 \\ 1248 \end{bmatrix}$$

So location A is closest to 1605.

Question 13

According to the transition matrix, 100% of birds at location B stay at location B and 0% move to location A. With 20% of birds at location A moving to location B each night, then the movement of birds is all one way and the birds at location A will gradually decrease to zero.

Question 14

Examination report

$$42;\ S_2 = \begin{bmatrix} 0.88 & 0.52 & 0.65 \\ 0.10 & 0.44 & 0.10 \\ 0.02 & 0.04 & 0.25 \end{bmatrix}^{2} \begin{bmatrix} 880 \\ 230 \\ 120 \end{bmatrix} = \begin{bmatrix} 996.9 \\ 191.4 \\ 41.7 \end{bmatrix}$$

Round 41.7 to conclude that 42 students will defer the 2009 academic year. Few students wrote down the relevant transition matrix, which would have earned at least one mark if correct. Of those who did write it down, 2% and 4% were sometimes incorrectly written as 0.2 and 0.4.

[VCAA 2008 2RMQ4]

Question 15

Examination report

a $T = \begin{bmatrix} 0.80 & 0.09 & 0.10 \\ 0.12 & 0.76 & 0.05 \\ 0.08 & 0.15 & 0.85 \end{bmatrix}$

Some students wrote percentages despite being asked for proportions. Some simply copied the figures in order, as printed into the three columns. The percentage, 8%, was sometimes written as a proportion of 0.8.

b $K_0 = \begin{bmatrix} 300\,000 \\ 120\,000 \\ 180\,000 \end{bmatrix}$

c $K_1 = T\,K_0 = \begin{bmatrix} 0.80 & 0.09 & 0.10 \\ 0.12 & 0.76 & 0.05 \\ 0.08 & 0.15 & 0.85 \end{bmatrix} \begin{bmatrix} 300\,000 \\ 120\,000 \\ 180\,000 \end{bmatrix} = \begin{bmatrix} 268\,800 \\ 136\,200 \\ 195\,000 \end{bmatrix}$

There are 268 800 at Shopper Heaven, 136 200 at Eastown and 195 000 at Noxland. The expression and the result were both required for full marks.

d Any two products $T^n K_0$ where $n \geq 38$. For example:

$$K_{38} = T^{38} K_0 = \begin{bmatrix} 0.80 & 0.09 & 0.10 \\ 0.12 & 0.76 & 0.05 \\ 0.08 & 0.15 & 0.85 \end{bmatrix}^{38} \begin{bmatrix} 300\,000 \\ 120\,000 \\ 180\,000 \end{bmatrix} = \begin{bmatrix} 194\,983 \\ 150\,513 \\ 254\,504 \end{bmatrix}$$

and

$$K_{39} = T^{39} K_0 = \begin{bmatrix} 0.80 & 0.09 & 0.10 \\ 0.12 & 0.76 & 0.05 \\ 0.08 & 0.15 & 0.85 \end{bmatrix}^{39} \begin{bmatrix} 300\,000 \\ 120\,000 \\ 180\,000 \end{bmatrix} = \begin{bmatrix} 194\,983 \\ 150\,513 \\ 254\,504 \end{bmatrix}$$

and this is the same as K_{38}.

[VCAA 2006 2RMQ2]

EXAM PREP 9.6 Transitions with restocking and culling

Prep 1

a $B = \begin{bmatrix} 9000 \\ 400 \\ -1000 \\ 0 \end{bmatrix}$ **b** 1420 **c** 1654

d The company removes 1054 adult salmon.

e 36 000

EXAM PRACTICE 9.6 Transitions with restocking and culling

	Multiple choice					
	%A	%B	%C	%D	%E	
Q1	5	41	19	21	13	2013
	Written response					
Q2	Not reported					2008
Q3	Not reported					2009
Q4 ai–aii	57					2013
Q4 bi–d	40					2013
Q4 ei–eii	43					2013
Q5 a–b	Not reported					2007
Q5c	50					2007
Q5d	30					2007
Q6	38					2012

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Question 1

$$S_1 = \begin{bmatrix} 0.5 & 0.6 \\ 0.5 & 0.4 \end{bmatrix}\begin{bmatrix} 100 \\ 250 \end{bmatrix} - \begin{bmatrix} 20 \\ 20 \end{bmatrix} = \begin{bmatrix} 180 \\ 130 \end{bmatrix}$$

$$S_2 = \begin{bmatrix} 0.5 & 0.6 \\ 0.5 & 0.4 \end{bmatrix}\begin{bmatrix} 180 \\ 130 \end{bmatrix} - \begin{bmatrix} 20 \\ 20 \end{bmatrix} = \begin{bmatrix} 148 \\ 122 \end{bmatrix}$$

Question 2

Examination report

a $O_{2009} = \begin{bmatrix} 0.75 & 0 \\ 0 & 0.68 \end{bmatrix}\begin{bmatrix} 456 \\ 350 \end{bmatrix} + \begin{bmatrix} 18 \\ 12 \end{bmatrix} = \begin{bmatrix} 360 \\ 250 \end{bmatrix}$

b 248

$$O_{2010} = \begin{bmatrix} 0.8 & 0 \\ 0 & 0.8 \end{bmatrix}\begin{bmatrix} 360 \\ 250 \end{bmatrix} - \begin{bmatrix} 40 \\ 38 \end{bmatrix} = \begin{bmatrix} 248 \\ 162 \end{bmatrix}$$

The added column matrix at each year is applied after each transition matrix is applied. Consequently, simply squaring the transition matrix is not appropriate in this question. [VCAA 2008 2RMQ3]

Question 3

Examination report

a 68

$$L_2 = T \times L_1 - \begin{bmatrix} 5 \\ 7 \end{bmatrix}$$

$$\Rightarrow L_2 = \begin{bmatrix} 0.85 & 0.25 \\ 0.15 & 0.75 \end{bmatrix}\begin{bmatrix} 95 \\ 97 \end{bmatrix} - \begin{bmatrix} 5 \\ 7 \end{bmatrix} = \begin{bmatrix} 100 \\ 80 \end{bmatrix}$$

$$\therefore L_3 = T \times L_2 - \begin{bmatrix} 5 \\ 7 \end{bmatrix}$$

$$\Rightarrow L_3 = \begin{bmatrix} 0.85 & 0.25 \\ 0.15 & 0.75 \end{bmatrix}\begin{bmatrix} 100 \\ 80 \end{bmatrix} - \begin{bmatrix} 5 \\ 7 \end{bmatrix} = \begin{bmatrix} 100 \\ 68 \end{bmatrix}$$

A common error was for students to square the transition matrix first and then use the initial state matrix.

Students should note that:

$$L_3 \neq T^2 \times L_1 - \begin{bmatrix} 5 \\ 7 \end{bmatrix} = \begin{bmatrix} 0.85 & 0.25 \\ 0.15 & 0.75 \end{bmatrix}^2 \times \begin{bmatrix} 95 \\ 97 \end{bmatrix} - \begin{bmatrix} 5 \\ 7 \end{bmatrix}$$

b 12

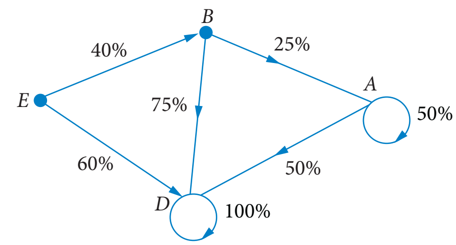
The subtracted column matrix $\begin{bmatrix} 5 \\ 7 \end{bmatrix}$ indicates that, each week, another 5 + 7 = 12 students will no longer turn up to any rehearsal. [VCAA 2009 2RMQ4]

Question 4

Examination report

a **i** 60% × 10 000 = 6000

ii



The 100% cycle drawn at D (all that die this year will be dead next year) was a common omission.

Students should not add edges marked 0% that are in the opposite direction to the given edges. Similarly, loops with 0% at E and B should not be included.

b **i** $S_1 = \begin{bmatrix} 0 \\ 4000 \\ 650 \\ 7150 \end{bmatrix}$

$$S_1 = T\,S_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0.4 & 0 & 0 & 0 \\ 0 & 0.25 & 0.5 & 0 \\ 0.6 & 0.75 & 0.5 & 1 \end{bmatrix}\begin{bmatrix} 10\,000 \\ 1000 \\ 800 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 4000 \\ 650 \\ 7150 \end{bmatrix}$$

ii 331.25 adult trout

$$S_4 = T^4 S_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0.4 & 0 & 0 & 0 \\ 0 & 0.25 & 0.5 & 0 \\ 0.6 & 0.75 & 0.5 & 1 \end{bmatrix}^4 \times \begin{bmatrix} 10\,000 \\ 1000 \\ 800 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 331.25 \\ 11\,468.75 \end{bmatrix}$$

On average, the state matrix predicts that there will be 331.25 adult trout after four years. Since this average is a predicted value, a decimal number of adult trout is valid. An answer of 331 adult trout was common and also accepted.

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c **i** 13 years

$$S_{12}=T^{12}S_0=\begin{bmatrix}0\\0\\1.2939\ldots\\11\ 790.706\ldots\end{bmatrix}\quad S_{13}=T^{13}S_0=\begin{bmatrix}0\\0\\0.646\ 97\ldots\\11\ 799.353\ldots\end{bmatrix}$$

S_{13} is the first state matrix where the number of adult trout is less than one, as asked for in the question.

ii 1325 adult trout

$$S_1=TS_0=\begin{bmatrix}0\\4000\\650\\7150\end{bmatrix}\ S_2=T^2S_0=\begin{bmatrix}0\\0\\1325\\10\ 475\end{bmatrix}\ S_3=T^3S_0=\begin{bmatrix}0\\0\\662.5\\11\ 137.5\end{bmatrix}\ S_4=T^3S_0=\begin{bmatrix}0\\0\\331.25\\11\ 468.75\end{bmatrix}$$

S_2 gives the greatest number of adult trout.

It was not sufficient to assume that the number of adult trout was decreasing without checking the values given by S_2 and S_3.

A common incorrect answer was 800, taken directly from S_0. This may have been based on observing that S_1 had fewer (650) adult trout and S_4 had even fewer (331.25) adult trout. The assumption that this meant there was a decrease from S_0 through to S_4 should have been checked.

d Add 10 000 eggs, remove 3000 baby trout and add 150 adult trout.

$$S_0-S_1=\begin{bmatrix}10\ 000\\1000\\800\\0\end{bmatrix}-\begin{bmatrix}0\\4000\\650\\7150\end{bmatrix}=\begin{bmatrix}10\ 000\\-3000\\150\\-7150\end{bmatrix}$$

Many students did not answer this question.

e **i** $S_1=\begin{bmatrix}200\ 000\\4000\\650\\7150\end{bmatrix}$

$$S_1=\begin{bmatrix}0&0&0&0\\0.40&0&0&0\\0&0.25&0.50&0\\0.60&0.75&0.50&1.0\end{bmatrix}\begin{bmatrix}10\ 000\\1000\\800\\0\end{bmatrix}+500\times\begin{bmatrix}0&0&0.50&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}\begin{bmatrix}10\ 000\\1000\\800\\0\end{bmatrix}=\begin{bmatrix}200\ 000\\4000\\650\\7150\end{bmatrix}$$

ii 162 500

$$S_2=\begin{bmatrix}0&0&0&0\\0.40&0&0&0\\0&0.25&0.50&0\\0.60&0.75&0.50&1.0\end{bmatrix}\times S_1+500\times\begin{bmatrix}0&0&0.50&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}\times S_1$$

$$=\begin{bmatrix}0&0&0&0\\0.40&0&0&0\\0&0.25&0.50&0\\0.60&0.75&0.50&1.0\end{bmatrix}\begin{bmatrix}200\ 000\\4000\\650\\7150\end{bmatrix}\times\begin{bmatrix}0&0&0.50&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}\begin{bmatrix}200\ 000\\4\ 000\\650\\7150\end{bmatrix}=\begin{bmatrix}162\ 500\\80\ 000\\1325\\130\ 475\end{bmatrix}$$

Using the equation given, the state matrix for each year is calculated by using the state matrix from the year before.

Many students instead found $S_2 = T^2S_0 + 500MS_0$ rather than using S_1 to determine S_2.

Of those who did a correct calculation, some left the matrix S_2 as their answer. The number of eggs had to be extracted from matrix S_2 for full marks.

[VCAA 2013 2RMQ2]

Question 5

Examination report

a 700 **b** 0.5; 50% and $\frac{1}{2}$ were also accepted.

c i $\begin{bmatrix} 0.4 & 0 & 0 & 0 \\ 0.5 & 0.4 & 0 & 0 \\ 0 & 0.5 & 0.8 & 0 \\ 0.1 & 0.1 & 0.2 & 1 \end{bmatrix}\begin{bmatrix} 400 \\ 200 \\ 100 \\ 0 \end{bmatrix} = \begin{bmatrix} 160 \\ 280 \\ 180 \\ 80 \end{bmatrix}$

ii 280

iii 56

$$\begin{bmatrix} 0.4 & 0 & 0 & 0 \\ 0.5 & 0.4 & 0 & 0 \\ 0 & 0.5 & 0.8 & 0 \\ 0.1 & 0.1 & 0.2 & 1 \end{bmatrix}^4\begin{bmatrix} 400 \\ 200 \\ 100 \\ 0 \end{bmatrix} = \begin{bmatrix} 10.24 \\ 56.32 \\ 312.96 \\ 320.48 \end{bmatrix}$$

iv 7 weeks

v $\begin{bmatrix} 0.4 & 0 & 0 & 0 \\ 0.5 & 0.4 & 0 & 0 \\ 0 & 0.5 & 0.8 & 0 \\ 0.1 & 0.1 & 0.2 & 1 \end{bmatrix}^{80} \approx \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

Some students apparently did not raise their transition matrix to a high enough power to find this steady-state matrix. Others did not interpret a calculator answer such as 5.8460065E–30 as representing zero, or 699.999828 as representing 700 in this practical context. Such unrounded answers were not accepted.

d i $\begin{bmatrix} 0.4 & 0 & 0 & 0 \\ 0.5 & 0.4 & 0 & 0 \\ 0 & 0.5 & 0.8 & 0 \\ 0.1 & 0.1 & 0.2 & 1 \end{bmatrix}\begin{bmatrix} 400 \\ 200 \\ 100 \\ 0 \end{bmatrix}$

$$+\begin{bmatrix} 0 & 0 & 0.3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 400 \\ 200 \\ 100 \\ 0 \end{bmatrix} = \begin{bmatrix} 190 \\ 280 \\ 180 \\ 80 \end{bmatrix}$$

ii 130

$$\begin{bmatrix} 0.4 & 0 & 0 & 0 \\ 0.5 & 0.4 & 0 & 0 \\ 0 & 0.5 & 0.8 & 0 \\ 0.1 & 0.1 & 0.2 & 1 \end{bmatrix}\begin{bmatrix} 190 \\ 280 \\ 180 \\ 80 \end{bmatrix}$$

$$+\begin{bmatrix} 0 & 0 & 0.3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 190 \\ 280 \\ 180 \\ 80 \end{bmatrix} = \begin{bmatrix} 130 \\ 207 \\ 284 \\ 163 \end{bmatrix}$$

Simply stating the resulting 4×1 matrix did not answer this question.

[VCAA 2007 2RMQ2]

Question 6

Examination report

a $70\% \times 100 + 80\% \times 200 + 90\% \times 50 = 275$

b Once a worker leaves the site, they never return (or equivalent statement).

Most students were unable to explain the meaning of the 1.00 figure in the matrix within the context of the question. Students were expected to respond with a statement such as 'Once a worker leaves the site, they never return'.

The most common unacceptable answer was '100% of those who leave this year will leave the next year', while others said that '100% of the people who do job *L* this year will again do job *L* next year'.

c i $S_{2012} = \begin{bmatrix} 0.70 & 0 & 0 & 0 \\ 0.10 & 0.80 & 0 & 0 \\ 0 & 0.10 & 0.90 & 0 \\ 0.20 & 0.10 & 0.10 & 1.00 \end{bmatrix}\begin{bmatrix} 100 \\ 200 \\ 50 \\ 0 \end{bmatrix} = \begin{bmatrix} 70 \\ 170 \\ 65 \\ 45 \end{bmatrix}$

ii 143

$$S_{2013} = \begin{bmatrix} 0.70 & 0 & 0 & 0 \\ 0.10 & 0.80 & 0 & 0 \\ 0 & 0.10 & 0.90 & 0 \\ 0.20 & 0.10 & 0.10 & 1.00 \end{bmatrix}^2\begin{bmatrix} 100 \\ 200 \\ 50 \\ 0 \end{bmatrix} = \begin{bmatrix} 49 \\ 143 \\ 75.5 \\ 82.5 \end{bmatrix}$$

The most common incorrect answer was 170, found from S_{2012} rather than from S_{2013} as required.

iii 2021

$$S_{2019} = \begin{bmatrix} 5.8 \\ 44.6 \\ 81.7 \\ 218 \end{bmatrix} \quad S_{2020} = \begin{bmatrix} 4 \\ 36.2 \\ 78 \\ 231.8 \end{bmatrix} \quad S_{2021} = \begin{bmatrix} 2.8 \\ 29.4 \\ 73.8 \\ 244 \end{bmatrix}$$

An answer of 'the 10th year' was not accepted, since actual years were used and required.

iv No staff will remain at the site in the long term.

$$\begin{bmatrix} 0.70 & 0 & 0 & 0 \\ 0.10 & 0.80 & 0 & 0 \\ 0 & 0.10 & 0.90 & 0 \\ 0.20 & 0.10 & 0.10 & 1.00 \end{bmatrix}^{100} \approx \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

The most common incorrect answer was 350.

d 182

$$S_{2012} = \begin{bmatrix} 0.70 & 0 & 0 & 0 \\ 0.10 & 0.80 & 0 & 0 \\ 0 & 0.10 & 0.90 & 0 \\ 0.20 & 0.10 & 0.10 & 1.00 \end{bmatrix} \begin{bmatrix} 100 \\ 200 \\ 50 \\ 0 \end{bmatrix} + \begin{bmatrix} 30 \\ 20 \\ 10 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 100 \\ 190 \\ 75 \\ 45 \end{bmatrix}$$

$$\therefore S_{2013} = \begin{bmatrix} 0.70 & 0 & 0 & 0 \\ 0.10 & 0.80 & 0 & 0 \\ 0 & 0.10 & 0.90 & 0 \\ 0.20 & 0.10 & 0.10 & 1.00 \end{bmatrix} \begin{bmatrix} 100 \\ 190 \\ 75 \\ 45 \end{bmatrix} + \begin{bmatrix} 30 \\ 20 \\ 10 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 100 \\ 182 \\ 96.5 \\ 91.5 \end{bmatrix}$$

From the equation given, the state matrix for each year is calculated by using the state matrix from the year before.

Many students instead found $S_{2013} = T^2S_{2011} + A$ rather than $S_{2012} = TS_{2011} + A$ and therefore $S_{2013} = S_{2012} + A$.

Other students inappropriately used the matrix

$S_{2012} = \begin{bmatrix} 70 \\ 170 \\ 65 \\ 45 \end{bmatrix}$ from part **c i** to find the matrix here for S_{2013}.

The matrix for S_{2012} cannot be applied here since it was determined from a different relation that did not involve matrix A. The most common incorrect answer was 163.

[VCAA 2012 2RMQ3]

EXAMINATION Solutions

MATRICES – EXAMINATION 1

1	2	3	4	5	6	7	8	9	10
A	B	B	C	C	E	D	E	C	A

MATRICES – EXAMINATION 2

Question 1

a 4×2 **b** 1850 adults

c Total number of adult females living in this small city

d Number of columns in V = number of rows in P

e $1360 \times 0.45 + 1460 \times 0.55 = 1415$

f $1415 + 1812 + 988 + 1806 = 6021$ votes

Question 2

a **i** 20% **ii** 25%

b $5\% \times 6000 + 40\% \times 2160 = 1164$

c **i** $S_3 = \begin{bmatrix} 4900 \\ 4634 \\ 2466 \end{bmatrix}$

ii The number of preferences for each candidate (predicted) for March.

d 5303

Question 3

a 50% **b** 6451

CHAPTER 10 Undirected graphs

EXAM PREP 10.1 Basic concepts of graphs and networks

Prep 1

a **i** 5 **ii** 5 **iii** 2

b **i** 6 **ii** 10 **iii** 6

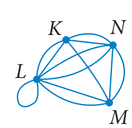
c **i** 7 **ii** 11 **iii** 6

Prep 2

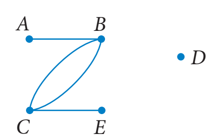
a 12 **b** 8 **c** 10

Prep 3

a



b



Prep 4

a
$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

b
$$\begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \\ 2 & 2 & 0 \end{bmatrix} \end{array}$$

c
$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

EXAM PRACTICE 10.1 Basic concepts of graphs and networks

	Multiple choice					
	%A	%B	%C	%D	%E	
Q1	2	95	1	1	1	2010
Q2	93	2	2	1	2	2007
Q3	3	2	89	5	0	2010
Q4	88	4	4	1	3	2002
Q5	6	3	87	2	1	2005
Q6	1	85	6	1	7	2006
Q7	2	12	2	3	81	2011
Q8	15	1	1	8	76	2012
Q9	9	63	7	9	11	2011
Q10	69	3	4	3	20	2011
Q11	9	2	8	19	62	2003
Q12	30	61	3	1	5	2011
Q13	28	30	40	2	1	2013
Q14	15	31	13	33	8	2005
Q15	26	24	6	23	21	2004
Q16	1	3	11	13	72	2008
Written response						
Q17	80					2010
Q18	77					2009

Question 13

Always consider how many routes you can take directly (not go through any other town) between two towns. Each town has two possible routes (edges) to each other town (vertices). It is also possible to go from R to R without passing through another town, so there is a loop at R.

Question 14

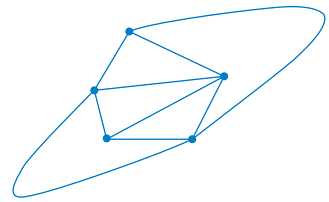
This is the reverse of Question 13. Remember that the edges represent the number of possible routes between towns. Although both Option A and Option B clearly show a loop, it is important to remember that in this situation the loop just indicates travelling along a road and returning to the town without passing through another town. The difference between Option B and Option D is the three possible routes between towns M and L.

Question 15

This is similar to Question 13. The first thing to look for are any loops. There is a loop at F (can travel from F to F without passing other towns), so eliminate D and E. Then look at the other towns and how many direct routes there are, matched by the edges in the network diagram.

Question 16

The graph needs to be redrawn so that none of the edges cross each other. One way of doing this is shown below. Once this has been done, we can see the graph clearly has six regions. (option C).



Question 17

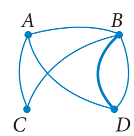
a A zero means the two people are not allowed to communicate directly with each other.

b $f = 1, g = 0$

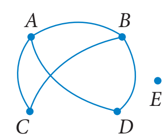
Question 18

a There is no land border between E and any other suburb.

b **i**



ii

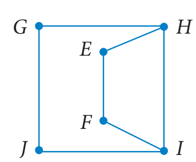


The vertex E had to be clearly isolated from all edges and other vertices.

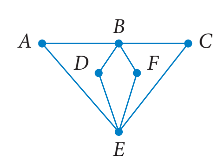
EXAM PREP 10.2 Planar graphs and Euler's formula

Prep 1

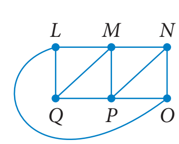
a



b



c



Note that other answers are possible.

Prep 2

a 2 **b** 2

Prep 3

a 14 **b** 4 **c** 5

EXAM PRACTICE 10.2 Planar graphs and Euler's formula

	Multiple choice					
	%A	%B	%C	%D	%E	
Q1	2	10	76	7	5	2002
Q2	5	7	76	7	5	2007
Q3	68	6	10	9	6	2009
Q4	66	10	9	9	6	2005
Q5	1	6	10	63	20	2012
Q6	13	60	13	6	8	2003
Q7	5	14	59	13	9	2003
Q8	40	4	7	4	44	2004
Q9	14	35	39	9	2	2013
Q10	37	13	11	28	11	2006

Question 7

Since the graph is planar, we can use Euler's formula.

Let the number of vertices = x and the number of faces = x (since they are equal).

$$v + f - e = 2$$
$$x + x - 20 = 2$$
$$2x = 22$$
$$x = 11$$ There are 11 vertices.

Question 8

Write the degree of each vertex as this will help you see clearly how many edges connect each vertex. Remember that edges can be moved as long as they are still connected to the same vertices; this condition will enable you to realise that all four graphs are the same.

Question 9

Examination report

One way of answering this question was to construct all possible different connected graphs with five vertices and four edges. There are three.

Inspection of the graphs then shows that, of the five statements, only three are true for all graphs (option C)

- the graph is planar
- the sum of the degrees of the vertices is eight
- the graph cannot have a loop.

[VCAA 2013 1RNQ7]

Question 10

Examination report

The key to answering this question correctly was that only planar graphs satisfied Euler's formula. Of the five graphs presented, only graph D could not be redrawn as a planar graph. Interestingly, 37 per cent of students incorrectly chose option A, perhaps unaware of the fact that any graph with four or fewer vertices is planar.

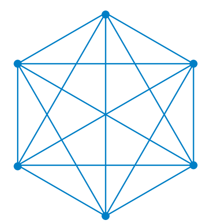
[VCAA 2006 1RNQ8]

EXAM PREP 10.3 Types of graphs

Prep 1

a **i** Complete graph **ii** 15 games

iii



b **i** George **ii** Sally and Celeste **iii** George

Prep 2

a No **b** Yes **c** Yes

d Yes **e** No

Prep 3

2 edges

Prep 4

a 105 edges **b** 12 **c** 45

d **i** 15 **ii** 6

EXAM PRACTICE 10.3 Types of graphs

	Multiple choice					
	%A	%B	%C	%D	%E	
Q1	88	1	2	2	7	2010
Q2	8	83	2	1	5	2004
Q3	2	3	74	4	16	2009
Q4	7	67	3	20	3	2008
Q5	14	12	59	11	3	2006
Q6	1	11	27	2	58	2013

9780170371186

Question 5

A. This is an example of a shortest path where not every edge is covered.

B. This is a minimum spanning tree where not every edge is covered.

C. The vertices represent the teams and the edges represent the games. Since every team plays each other, this is a complete graph.

D. This is a digraph where edges are given a direction.

E. This is a bipartite graph; two separate groups.

Question 6

$$\text{Edges} = \frac{4(4-1)}{2} = \frac{12}{2} = 6$$

EXAM PREP 10.4 Exploring and travelling problems

Prep 1

a walk **b** bridge **c** circuit

d trail **e** cycle **f** path

Prep 2

a Station, A, B, C, A, E, C, D, E, F, Station, E, Station

b This graph contains no vertices of odd degree.

c Removing this edge will give two vertices an odd degree, allowing an Eulerian trail to exist and therefore the officers will be able to patrol every road.

Prep 3

a A-B-E-C-D-A **b** DE

Prep 4

a 59 km **b** 9 ways

N-P-R, N-P-Q-R, N-O-R, N-P-Q-M-O-R, N-O-M-Q-R, N-O-M-Q-P-R, N-M-O-R, N-M-Q-R, N-M-Q-P-R

c 159 km

Prep 5

a 15 minutes **b** A-E-F-G-Z

Prep 6

a 17 km **b** 9 km

c start-B-I-finish, start-A-J-finish

EXAM PRACTICE 10.4 Exploring and travelling problems

	Multiple choice					
	%A	%B	%C	%D	%E	
Q1						
Q2						
Q3	1	93	3	2	1	2009
Q4	89	1	2	5	2	2008
Q5	2	5	79	10	4	2002
Q6	9	7	11	67	6	2012
Q7	1	66	28	2	4	2007
Q8	13	7	7	65	7	2005
Q9	60	5	8	3	25	2012
Q10	3	59	6	27	5	2003
Q11	18	13	16	43	10	2006
Q12	1	14	13	30	41	2008
Q13	8	30	37	20	4	2011
Q14	2	28	36	14	20	2003
Q15	15	28	15	16	25	2009
	Written response					
Q16	90					2013
Q17	80					2010
Q18	Not reported					2012
Q19	Not reported					2011
Q20	53					2010
Q21	Not reported					2008
Q22	Not reported					2002

Question 10

For an Eulerian trail to exist, there must be two vertices of odd degree only, with the rest being of even degree. Vertices S, U, W and Z are all of odd degree, so an edge must be placed between any two of these to make only two vertices of odd degree. The only option that fits the criteria is SX.

Question 11

A. This is not a cycle as the path does not return to the same vertex.

B. Hamiltonian paths must visit every vertex only once. This is quite a simple graph and therefore more than one Hamiltonian path exists.

C. An Eulerian circuit could not exist because not all vertices are of even degree.

D. There are only two vertices of odd degree and therefore an Eulerian trail exists. One is B-A-G-E-B-C-D-E-F-G. There are others.

E. There are multiple circuits in this network.

Question 12

A. The degree sum is 18.

B. An Eulerian trail can't exist because there are more than two vertices of odd degree.

C. An Eulerian circuit can't exist because not every vertex is even.

D. Euler's formula can be used as the question states that the graph is planar.

E. If one edge was added between two vertices of odd degree, this would create an Eulerian trail.

Question 13

- If an Eulerian trail exists, then there must only be two vertices of odd degree, not three as stated.
- An Eulerian trail cannot contain an isolated vertex.
- It is possible to pass vertex *Q* more than once as an Eulerian trail is only allowed to cover edges once, but can pass vertices multiple times.
- Since it's an Eulerian trail, the trail must start and end at vertices of odd degree. This means it is not possible for the sum of degrees of vertices *P* and *Q* to equal 7 as no two odd numbers add to 7.
- It is not possible for the degree sum to equal 7 as the vertices *P* and *Q* must be of odd degree. This would mean that all the other vertices would still add to an odd number, which is not possible for an Eulerian trail.

Therefore, 'the path could have included vertex *Q* more than once' is the only true statement.

Question 14

Examination report

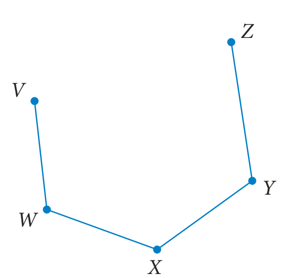
The fact that many chose the option that corresponded to a travelling time of 24 minutes suggested that these students likely knew what to do, but were perhaps not careful enough in eliminating other feasible solutions before deciding on their answer.

[VCAA 2003 1RNQ8]

Question 15

Examination report

A strategy to answer this question was to construct a simple graph that satisfied the given conditions, and to use this graph to test each of the options in a systematic manner.



The graph shown is one such graph; it has five vertices, three of even degree (*X*, *Y* and *W*) and two of odd degree (*V* and *Z*). From this graph it can be seen that:

- adding an edge between *V* and *Z* gives a graph in which all vertices are even and all vertices are of equal degree, so both option A and option B are possible
- adding an edge between *W* and *Y* gives a graph with one vertex of even degree and four of odd degree, so option C is possible
- adding an edge between *V* and *X* gives a graph with three vertices of even degree and two of odd degree, so option D is possible
- there is no way that a single edge can be added to the graph so that it has four vertices of even degree and one of odd degree, so option E was the correct answer.

[VCAA 2009 1RNQ8]

Question 16

a 3 **b** 1000 m

c **i** *P*4 **ii** Eulerian trail.

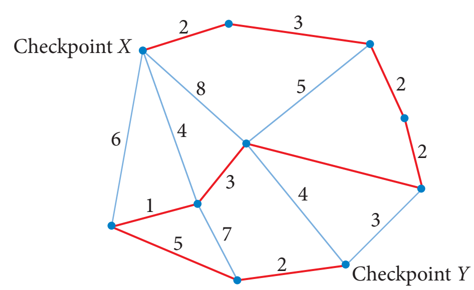
d *E*–*P*5–*P*4–*P*6–*P*3–*P*2–*P*1

Question 17

a 11 minutes

b **i** Hamiltonian path

ii



Question 18

Examination report

a 160 m **b** 2 **c** 1180 + 70 = 1250 m

An Eulerian circuit would be an ideal solution, but this is not possible due to the presence of two odd vertices: one at the house and one at the end of the edge marked 70, leading from the house. However, an 1180-metre long Eulerian path commencing at the house is possible, provided it ended at the other odd vertex. To return to the house, we must then add 70 metres for the length of the shortest path between these two odd vertices. This question was very poorly answered, with a common incorrect answer of 1180.

[VCAA 2012 2RNQ1a]

Question 19

a 200 km

b 6 *F-D-C, F-E-B-C, F-D-E-B-C, F-E-D-C, F-E-A-B-C, F-D-E-A-B-C*

c Bredon **d** 240 km

Question 20

a *D* **b** **i** *B, D* **ii** 32 km

c *B* and *D*

Take note of the wording in part b. It says 'at least once' so this is not requiring an Eulerian circuit; edges can be travelled on more than once. The walk is *A, F, E, D, B, F, D, F, B, A*.

This is an example of an Eulerian circuit as the red team is travelling on every edge only once with the addition of an extra edge. For an Eulerian circuit to exist, every vertex must be of even degree. Vertices *D* and *B* are the only vertices of odd degree so linking them with an edge will make them of even degree, hence an Eulerian circuit will exist.

Question 21

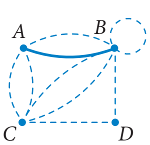
Examination report

a 7

The unnamed intersections are labelled as T_1 and T_2 from top to bottom. The seven possible paths are: A-T_1-B-D, A-T_1-T_2-B-D, A-T_1-T_2-C-D, A-T_1-B-T_2-C-D, A-C-D, A-C-T_2-B-D, A-C-T_2-T_1-B-D

Few students managed to find that there were seven paths.

b **i**



ii To meet the requirements, there should be an Eulerian circuit, which can only exist if all vertices have an even degree. In this network, vertices *C* and *B* have odd degrees.

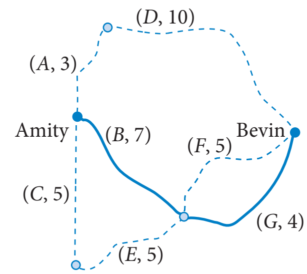
[VCAA 2008 2RNQ2]

Question 22

Examination report

a 11 km. Many chose the roads *B* and *F* as the shortest path and wrote 12 km. One consequential mark was awarded if path *B–F* was indicated on the graph for part **b**.

b



B-F was the most common incorrect shortest path from Amity to Bevin.

[VCAA 2002 2RNQ1]

EXAM PREP 10.5 Trees and minimum spanning trees

Prep 1

a No **b** No **c** Yes **d** Yes **e** No

Prep 2

3740 m

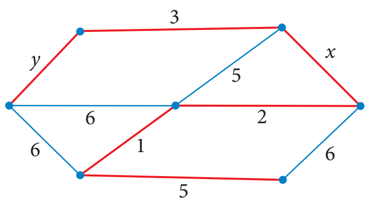
Prep 3

$x = 1, y = 1$

EXAM PRACTICE 10.5 Trees and minimum spanning trees

	Multiple choice					
	%A	%B	%C	%D	%E	
Q1	91	2	2	2	2	2013
Q2	82	3	4	5	5	2011
Q3	2	80	6	8	3	2005
Q4	78	6	12	3	1	2013
Q5	3	5	75	9	7	2004
Q6	3	6	76	9	5	2010
Q7	8	6	9	70	7	2006
Q8	7	69	12	6	5	2008
Q9	5	4	9	67	15	2002
Q10	17	64	9	8	2	2004
Q11	13	18	54	8	6	2007
Q12	5	11	48	28	8	2003
Q13	7	37	7	5	44	2007
	Written response					
Q14	Not reported					2008
Q15	Not reported					2011
Q16	Not reported					2012
Q17	Not reported					2002

Question 11

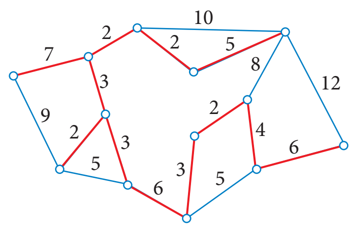


Use Prim's algorithm and include x and y.

$$3+2+1+5+x+y=19$$
$$x+y=8$$

There are two options: A and C. If we chose A, this would mean that the edge y would not be included in the spanning tree as other edges could be used as they have a lower weight. Therefore, option C is the correct answer.

Question 12



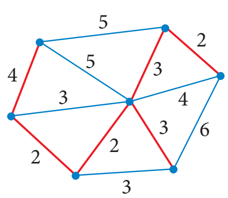
Question 13

The graph is not a tree because it has cycles.

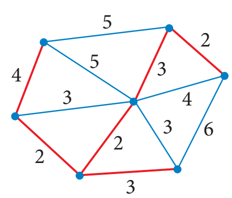
Question 14

Examination report

a



or



Either of these two trees was accepted. A number of students drew circuits and these were not accepted.

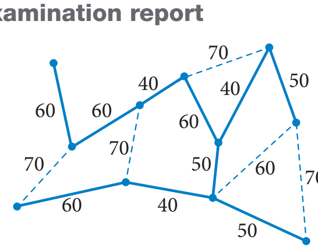
b 16 **c** Two, as shown in part **a** above.

[VCAA 2008 2RNQ1]

Question 15

Examination report

a



b 510 m

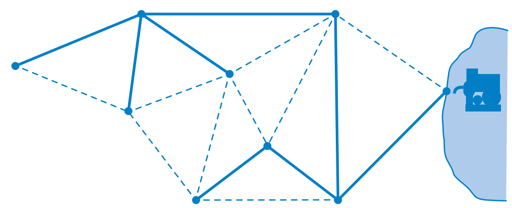
A consequential mark was available for the correct total length of any spanning tree drawn in part **2a**. A common error was for students to omit the length of one of the edges from their diagram.

[VCAA 2011 2RNQ2]

Question 16

Examination report

a



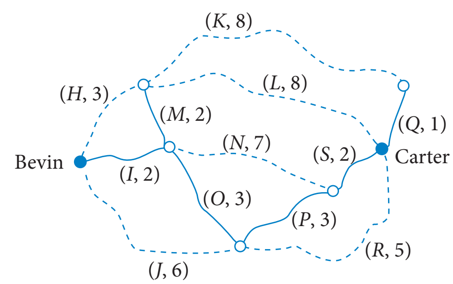
b Minimal spanning tree

Many students provided incorrect answers such as maximum flow, Hamiltonian path, minimum cut, shortest path and others.

[VCAA 2012 2RNQ1b]

Question 17

a

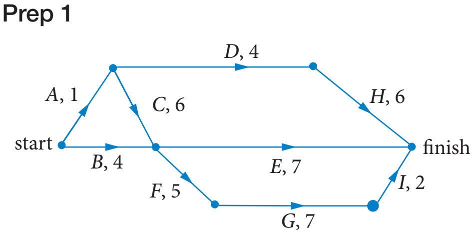


b **i** *I-O-P-S-Q-K-H* **ii** 22 km

CHAPTER 11
Directed graphs

EXAM PREP 11.1 The scheduling problem

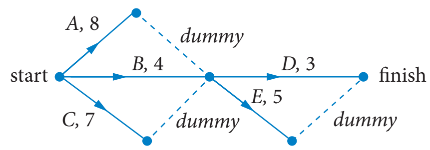
Prep 1



Prep 2

Town *D* and Town *C*.

Prep 3

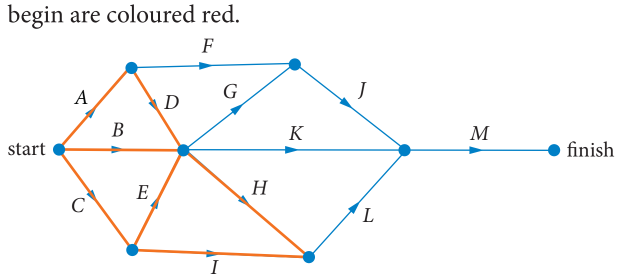


EXAM PRACTICE 11.1 The scheduling problem

	Multiple choice					
	%A	%B	%C	%D	%E	
Q1	3	87	4	3	3	2006
Q2	3	8	6	3	80	2006
Q3	3	3	4	1	89	2003
Q4	74	7	9	6	4	2012
Q5	30	15	7	45	3	2009
	Written response					
Q6a	90					2009
Q6b	90					2009
Q7	80					2013

Question 5

The activities that must be completed before activity *L* can begin are coloured red.

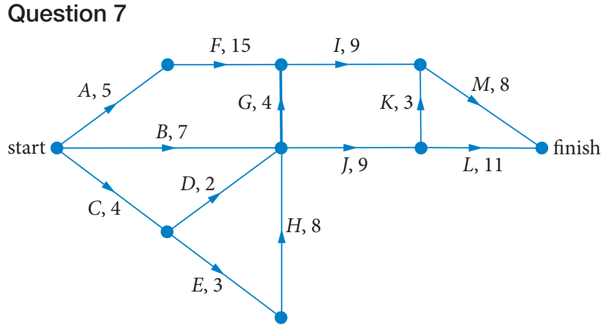


The seven activities *A*, *B*, *C*, *D*, *E*, *H* and *I* must all be completed before activity *L* can commence.

Question 6

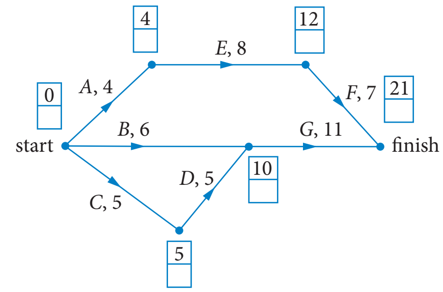
a *K* and *F* **b** (*K*)-*J*-*H*, (*K*)-*F*-*J*-*H* and (*K*)-*M*-*J*-*H*

Question 7

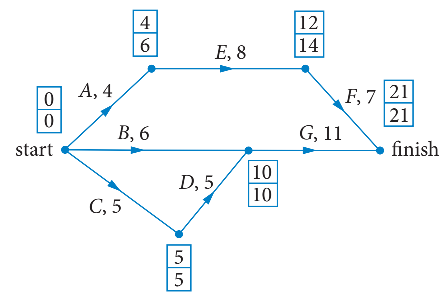


EXAM PREP 11.2 Critical path analysis

Prep 1

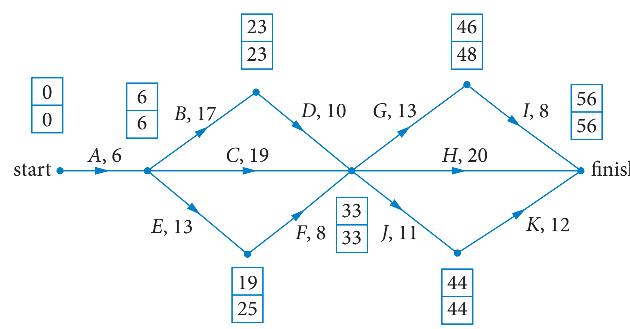


Prep 2



Prep 3

Critical path is *A*-*B*-*D*-*J*-*K*.



Prep 4

Critical activities *A*, *D*, *E* and *H*.

Float for non-critical activities: *C* 8, *B* 2, *F* 2, *G* 10.

Prep 5

a *A*-*E*-*F*, 34 days **b** *A*-*E*-*G*, 28 days

EXAM PRACTICE 11.2 Critical path analysis

	Multiple choice					
	%A	%B	%C	%D	%E	
Q1	6	19	9	63	2	2004
Q2	14	11	29	43	2	2007
Q3	21	7	10	42	19	2007
Q4	12	8	11	56	13	2011
Q5	54	14	4	6	17	2004
Q6	20	10	7	54	9	2003
Q7	14	9	15	10	52	2009
Q8	22	17	53	5	4	2008
Q9	35	21	19	11	13	2008
Q10	16	7	51	14	11	2005
Q11	2	35	9	51	3	2002
Q12	13	24	12	41	9	2010
Q13	26	19	15	34	5	2004
Q14	17	8	38	33	4	2011
Q15	12	43	30	8	6	2012
Q16	21	20	25	23	9	2013
Q17	11	45	14	18	12	2012
Q18	28	26	19	17	9	2006
	Written response					
Q19a–b	63					2006
Q19c–e	34					2006
Q20a–d	50					2003
Q21a–c	50					2003
Q22a–f	51					2010
Q23a–e	48					2012
Q24a–e	44					2009
Q25a–e	42					2011
Q26a–e	40					2007

Question 3

The reason for the low percentage of correct answers in the exam was because the dummy activity arrow was not shown in the question.

Question 9

Examination report

The original critical path is A-C-F-H-J-L. A second critical path, A-B-D-J-L, emerges when the completion time of D is increased by 7 days (option A).

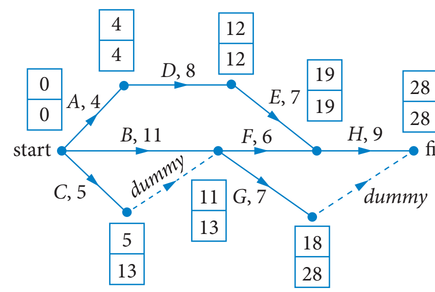
[VCAA 2008 1RNQ9]

Question 13

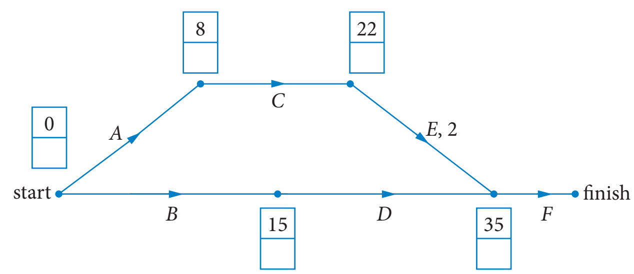
Examination report

One reason for the difficulty with this question may have been the need for students to construct an appropriate network diagram to help them organise and make sense of the information presented.

[VCAA 2004 1RNQ9]



The information provided produces the directed graph above. From this information activity times can be added.



A-C-E is currently 24 hours and can be 35 hours without increasing the project time, therefore activity C can be increased by 11 hours. (option D)

Question 14

Examination report

To answer this question correctly, students needed to realise that it was the latest starting time for activity M, 24 minutes (option D), and not the earliest starting time of 18 minutes (option C) that determined the time that Caleb could speak on the phone before commencing activity M.

[VCAA 2011 1RNQ7]

Question 15

Examination report

In answering this question, it was necessary to realise that there are two paths, A-F-H and B-C-F-H, of length 18 hours (the critical paths) and two paths, A-E-G and B-C-E-G, of length 17 hours. Reducing activities A and B by one hour each reduces the lengths of these two paths to 17 and 16 hours respectively. Reducing either F or H by one hour will then further reduce the critical paths A-F-H and B-C-F-H to 16 hours as required. Thus, a minimum of three activities must be reduced by one hour each to reduce the project completion time to 16 hours. It was not sufficient to just reduce the length of paths *A-F-H* and *B-C-F-H* to 16 hours, for example by reducing the durations of *F* and *H* by one hour each because it would still take 17 hours to complete the activities on the paths *A-E-G* and *B-C-E-G*.

[VCAA 2012 1RNQ8]

Question 16

Examination report

The critical path is *B-D-E* and has a completion time of 24 hours. The other path is *A-C-E* that has a completion time of 23 hours. As activity *E* is common to paths *B-D-E* and *A-C-E*, reducing the completion time of activity *E* reduces the completion times to these paths to 23 and 22 hours respectively.

Further reductions can now be considered for path *A-C* (9 hours) and path *B-D* (10 hours), which have a common start and end point. If the completion time of activities *A*, *B*, *C* and *D* is reduced by one hour each, then the time to complete path *A-C* becomes 7 hours and the time to complete path *B-D* becomes 8 hours.

Since path *B-D* takes at least 8 hours to complete, there is no point in paying for the completion time for path *A-C* to be reduced to less than 8 hours. Therefore, reduce only **one of** activity *A* or activity *C*. This means that the completion times of the three activities *E*, *B*, *D*, plus either *A* or *C*, should be reduced by one hour each at a cost of $400 (option D).

[VCAA 2013 1RNQ8]

Question 17

Examination report

Without restrictions, the critical paths are *A-C-F-H* and *B-E-H*, giving a minimum completion time of 17 days.

With restrictions

- the length of time required to complete path *A-C-F-H* is increased by three days because Ken must complete activity *B* before starting activity *C*, giving a revised completion time of $17 + 3 = 20$ hours. Requiring John to complete task *A* does not change this.
- the length of time required to complete path *B-E-H* is increased by four days because Lisa must complete activity *D* before starting activity *E*, giving a revised completion time of $17 + 4 = 21$ hours. Requiring Ken to complete task *B* does not change this.

Thus the minimum completion time required for the project with the restrictions given is 21 hours.

[VCAA 2012 1RNQ9]

Question 18

Examination report

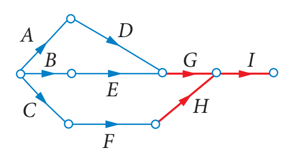
To correctly answer this question, students had first to identify *B-D-C-E-H-J* as the critical path. When choosing an activity to be crashed on the critical path, care needed to be taken to ensure that crashing this activity did not create a new critical path. This restricted the amounts by which critical path activities *C*, *E*, *H* or *J* could be crashed to one hour. However, inspection shows that the remaining activity, *B*, could be crashed by a maximum of four hours without creating a new critical path. Note that *D* is a dummy activity, so it could not be crashed.

[VCAA 2006 1RNQ9]

Question 19

Examination report

a Arrows were required for full marks.



b 5

There are only three complete paths from start to finish. Without even considering the duration of any activities, any one of these three paths leaves five activities unaccounted for in the other two paths.

c *B-E-G-I*

Every activity on a critical path has zero slack time (float). From the given information, this means that *A* and *C* cannot be on a critical path, which leaves only one complete path as an option.

d 7 hours **e** 8 hours

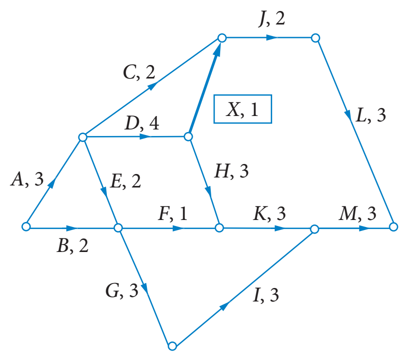
[VCAA 2006 2RNQ3]

Question 20

Examination report

a Immediate predecessor for E is A. EST for I is 8 and EST for M is 13 (one mark for each answer). The predecessor was generally found by most students. The EST figures were generally poorly found; many did not fully understand the table or the application.

b



D is a predecessor of X and so X must start at the end of D.

Also, X (EST of 7), is a predecessor for J (EST of 8). Therefore, the duration of X must be 1 and must terminate at the start of J.

A common incorrect diagram for X began at the end of D but then did not connect with any other point on the network but simply finished at a disconnected point. Others were able to find the correct location for X but assigned it the value of 7.

c 16 hours

Most got this as a simple addition of durations.

d The critical path gives the minimum time for completing the whole project.

Most were unable to clearly explain the significance of the critical path in terms of the overall project completion time.

Common unacceptable errors were: 'It gives the maximum time for the completion of the project' and 'Nothing else can be done until this path is finished'.

e 7 hours

The entire project can be completed in 16 hours as found in part **c**. Therefore, any path involving B can be up to 16 hours long.

Since B is not on the original critical path, we can increase its duration until it is on a new, and second, critical path. This new path becomes B-G-I-M, which was originally 11 hours long. Therefore B 2 can be increased by 5 hours to a total of 7 hours as a maximum value without increasing the whole project beyond the established 16 hours.

Common and incorrect answers included 3 and 5.

[VCAA 2003 2RNQ2]

Question 21

Examination report

a 6 hours

Many simply looked at B-F as providing the EST for K and gave an answer of 3. This indicates confusion with a 'shortest path' problem.

b A-C-J-L

Generally not well done, as many students did not explore all possible paths. Several students included M in their critical path by writing A-C-J-L-M. Other common incorrect answers were A-E-F-K-M and B-F-K-M.

c 8 hours

Many did not get this as they had not found the correct critical path in part **b**.

[VCAA 2003 2RNQ3]

Question 22

Examination report

a 2	**b** 9	**c** A and C
d 4	**e** 16	**f** A-B-D-H

The project time and the critical path could be found without a drawn network.

Activities that are not predecessors for any other activity will not have any successor before completion of the project.

These were activities I, H, F, E in this problem.

The minimum time to complete the project is the maximum of earliest start time + activity time for all these activities.

The critical path for the project will then end at the activity that has this maximum value. That is, the maximum of

E: $5 + 7 = 12$ F: $9 + 6 = 15$

H: $13 + 3 = 16$ I: $10 + 2 = 10$

Therefore the critical path ends with H where H depends upon D; D depends upon B; B depends upon A.

[VCAA 2010 2RNQ4]

Question 23

Examination report

a 12 days

b Activity F has only activity B as a predecessor, while activities G and H have both B and C as predecessors. As there cannot be two activities called B, a dummy activity (with zero time) is drawn as a form of extension of B to the start of G and H to indicate that B is a predecessor for these two activities as well.

c 15 days

Many students ignored the dummy activity and obtained the incorrect answer of 13.

d *A-B-H-I-L-M*

Common incorrect answers: *A-B-F-J* and *A-C-G-M*.

e 25 days

A consequential mark was available for a correct calculation that showed the addition of the times, in days, for each of the activities in an incorrect critical path given for part **e**. Instead, most students wrote a single number here without showing the calculation and were ineligible for the mark.

[VCAA 2012 2RNQ2]

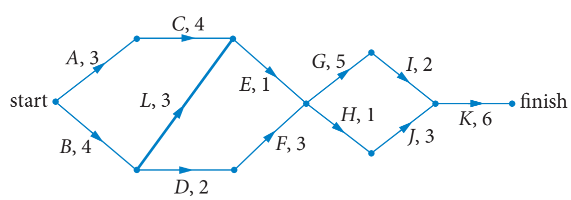
Question 24

Examination report

a 7 weeks **b** *B-D-F-G-I-K* **c** 3 weeks

H or *J* can be delayed for a maximum of three weeks.

d



The answer required the correct edge with an arrow marked in the correct direction.

e 25 weeks

A new critical path is created through *B-L-E-G-I-K* with a duration of $4 + 7 + 1 + 5 + 2 + 6 = 25$ weeks.

[VCAA 2009 2RNQ4]

Question 25

Examination report

a 2 hours

A common incorrect answer was one hour.

b 3 hours

c *F* and *H*

A common incorrect answer included activity *G*, either by itself or with other activities. This activity was not on the critical path (it was not a predecessor for activity *I*) and could be delayed by one hour.

d 13 hours

While not required by the question, an activity diagram would have been helpful when answering this question.

e 14 hours

[VCAA 2011 2RNQ3]

Question 26

Examination report

a 19 weeks

b EST of $D = 4$, $G = 5$ LST of $D = 9$, $G = 10$

EFT of $G = 8$, $H = 13$

Therefore $9 - 4 = 5$ or $10 - 5 = 5$ or $13 - 8 = 5$ weeks

c *A*, *E* and *G*

Since these three activities are not on a critical path, crashing any of these will not affect the completion time of the project.

d 15 weeks

e \$25 000

Reduce *C* by two weeks, *F* by two weeks and *E* by one week = 5 weeks at \$5000 per week.

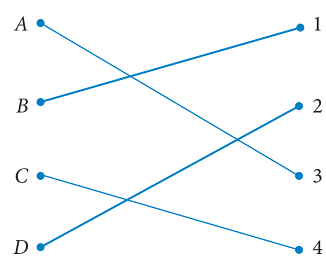
[VCAA 2007 2RNQ4]

EXAM PREP 11.3 The assignment problem and bipartite graphs

Prep 1

1 – Christen, 2 – Cathy, 3 – Celeste, 4 – Cheryl

Prep 2



Allocation: *A*3, *B*1, *C*4, *D*2

Prep 3

	1	2	3	4
A	18	7	0	8
B	0	4	2	3
C	8	0	4	0
D	3	0	5	6

Allocation: *A*3, *B*1, *C*4, *D*2 Time = 23 hours

Prep 4

*A*1, *B*3, *C*2 for a minimum distance of 42 km.

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EXAM PRACTICE 11.3 The assignment problem and bipartite graphs

	Multiple choice					
	%A	%B	%C	%D	%E	
Q1	2	94	1	2	1	2005
Q2	92	3	3	1	1	2003
Q3	2	8	86	2	3	2012
Q4	5	5	8	73	9	2013
Q5	5	14	61	14	5	2009
Q6	7	60	20	8	4	2010
Q7	19	3	5	29	43	2004
	Written response					
Q8	90					2006
Q9a–b	70					2012
Q9c–d	67					2012

Question 8

a George

b

Person	Position
Harriet	Drums
Ian	Saxophone
Keith	Keyboard

Question 9

a 17

b Four lines are needed before an allocation of 4 tasks to 4 people may be attempted and there are only 3.

c

	Worker			
Task	Julia	Ken	Lana	Max
W	0	0	4	0
X	2	2	0	10
Y	1	3	0	0
Z	7	0	3	5

d W Julia, X Lana, Y Max, Z Ken

EXAM PREP 11.4 Network flow problems

Prep 1

a 900 vehicles per hour **b** 600 vehicles per hour

c 600 vehicles per hour

Prep 2

capacity = 22 + 8 + 14 = 44

Prep 3

65

EXAM PRACTICE 11.4 Network flow problems

	Multiple choice					
	%A	%B	%C	%D	%E	
Q1	1	1	95	3	0	2008
Q2	13	3	63	16	4	2006
Q3	2	21	50	24	2	2010
Q4	4	44	17	28	6	2009
Q5	6	12	39	27	14	2005
Q6	4	10	12	35	39	2008
Q7	12	24	36	13	14	2010
Q8	28	10	23	29	9	2012
	Written response					
Q9	51					2003
Q10	43					2013
Q11	43					2011
Q12	33					2007

Question 5

Cut $A = 2 + 3 + 5 + 5 + 7 = 22$

Cut $B = 2 + 2 + 5 + 5 + 7 = 21$

Cut $C = 7 + 5 + 6 = 18$ Cut D not valid

Cut $E = 6 + 18 = 24$ Maximum flow = minimum cut

Question 6

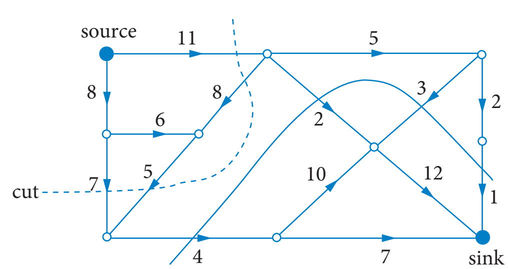
Examination report

For an individual flow to contribute to the capacity of the cut, the direction of the flow must be from the source region to the sink region. For the cut shown, one of the edges indicates a flow of 4 units in the reverse direction. This precludes this particular flow from contributing to the capacity of the cut. It was surprising that 39 per cent of students apparently did not understand this key idea and incorrectly chose option E.

[VCAA 2008 1RNQ6]

Question 7

cut = 1 + 3 + 2+ 4 = 10



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Question 8

Examination report

This question required students to determine the maximum possible flow in a network by identifying the minimum cut. The key to answering this question was to recognise that to apply the 'minimum cut–maximum flow' theorem, the cut must separate the 'source' of traffic (the town) from the 'sink' (the freeway). Of the cuts shown on the network, only those represented by lines 2, 3 and 4 separated the source from the sink and could potentially be used to determine the maximum flow. The solution then proceeds as follows.

Line 1 does not separate the source from the sink.

Flow across the cut represented by line 2 = 240 + 110 = 350

Flow across the cut represented by line 3 = 240 + 60 + 90 = 390

Flow across the cut represented by line 4 = 280 + 90 = 370

Thus, line 2 is the minimum cut, implying that the maximum flow of vehicles through the network from the town to the freeway is 350 per hour.

[VCAA 2012 1RNQ7]

Question 9

Examination report

a Cut A = 14, Cut B = 23, Cut C = 12

Cut A – generally OK.

Cut B – poorly done with a common incorrect answer of 15. This ignored the value of the edge marked with an 8 and which is directed toward Bowen despite the direction apparently to the left as drawn on the diagram.

Cut C – poorly done with a common incorrect figure of 20 or 4. In this cut, the edge marked with an 8 is directed toward Arlie and must be ignored, not subtracted nor added.

b Cut E does not entirely isolate Bowen from Arlie.

A cut that is to be relevant in a maximum flow problem must totally cut all access between the *source* (start, or Arlie in this case) and the *sink* (end, or Bowen in this case). In this example, there is still access from Arlie to Bowen if we were to exclude all the edges cut by E. Many incorrect answers referred to the direction of the edge marked with a 7 and suggested that this was heading in the wrong direction; or certainly not toward Bowen.

c 12 seats

One mark was available for correctly using the smallest, incorrect answer from either part **a** or from another labelled cut on the diagram. The most common error was to add up all the seats available along what may have been regarded as the 'shortest path'. Another common error was *maximum flow = maximum cut*.

[VCAA 2003 2RNQ1]

Question 10

Examination report

a 37

The numbers on the edges of the directed network gave the maximum number of people who are permitted to walk along any one of the tracks. The question required the maximum number of people permitted to walk from A to D each day. Many students did not see this question as a minimum cut/maximum flow problem.

Maximum flow = minimum cut of 37 through CD and ED or through AB, FB and FE or through BC, EC and ED.

b A-B-E-C-D

c

Group	Maximum group size	Path taken from A to D
1	17	Answered in part **b**
2	11	A-F-E-D
3	7	A-G-F-B-C-D
4	2	A-B-E-D

Once 17 students in group 1 had plotted their route A-B-E-C-D, there was a maximum of another 37 – 17 = 20 students who would be permitted to walk from A to D.

The table required the maximum group size for the remaining 20 students, from which the second-largest possible group was 11, leaving 9 students remaining to form another one group or more.

Often, group sizes were unreasonable since they exceeded 24, which was the number of students permitted initially on the most vacant track C to D.

[VCAA 2013 2RNQ3]

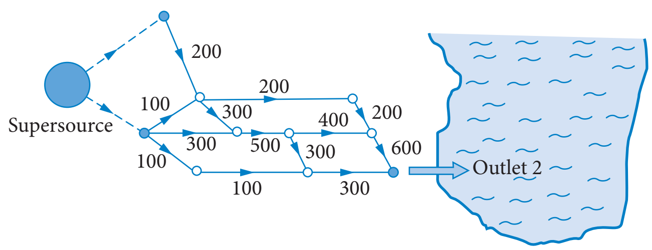
Question 11

Examination report

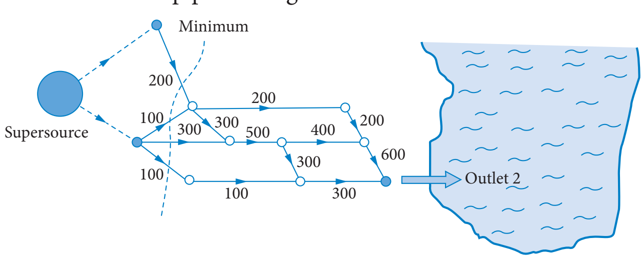
a 2, 1

b **i** Outlet 1: 700 kL/min **ii** Outlet 2: 700 kL/min

Water that eventually flowed through Outlet 2 came from Source 2 and also along the pipe labelled 200 from Source 1. To address this, a single Supersource can be considered, as shown in the diagram below.



It is now clearer that the required minimum cut must separate the Supersource from Outlet 2 and, in this case, includes the 200 pipe coming down from Source 1.



c 300

Question **b ii** gave a minimum cut of 700 that included the damaged pipe. The next smallest cut in the lower pipe system is 800. Therefore, the original 200 damaged pipe can be lengthened by 100 so that the cut from part **b ii** is now also 800 and so the replacement pipe should allow 300 kilolitres per minute.

[VCAA 2011 2RNQ4]

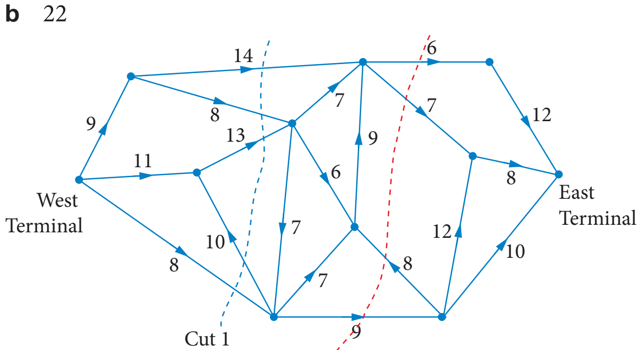
Question 12

Examination report

a 43

The edge with the 10 should not have been counted as its flow was in the reverse direction.

b 22



Minimum cut ⇒ maximum flow. Despite this, some answers given here were greater than the answer for part **a**.

The minimum cut is shown here. The edge marked 8 is not counted as its flow is in the reverse direction.

c 7

A common incorrect answer was 8.

[VCAA 2007 2RNQ3]

EXAM PREP 11.5 Dijkstra's algorithm and directed graphs

Prep 1

A-B-E-G-Z 46 km

EXAM PRACTICE 11.5 Dijkstra's algorithm and directed graphs

Q1	Q2	Q3	Q4	Q5	Q6
B	D	A	C	C	A

EXAMINATION Solutions

NETWORKS AND DECISION MATHEMATICS – EXAMINATION 1

1	2	3	4	5	6	7	8	9
D	C	A	B	E	E	D	B	D

NETWORKS AND DECISION MATHEMATICS – EXAMINATION 2

Question 1

a 2 **b** Miniature trains

Question 2

a 0, 0, 1

b The minimum number of lines to cover all zeros is less than four.

c **i** Equipment **ii** 36 hours

Question 3

a **i** Bower, Eden **ii** 910 km

b 270 km Bower–Clement–Derrin–Eden

c Between Bower and Derrin.

Question 4

a 7 hours **b** 18 hours **c** 2 hours

d 4 hours **e** $270

CHAPTER 12
Measurement and trigonometry

EXAM PREP 12.1 Surface area and volume

Prep 1 **a** 22.67 mm **b** 27.20 mm **c** 30.41 cm

Prep 2 15.492 cm

Prep 3 875.4 cm

Prep 4 1220.64 cm^2

Prep 5 5881.1 mm^2

Prep 6 39.74 m^2

Prep 7 11.5 cm

EXAM PRACTICE 12.1 Surface area and volume

	Multiple choice					
	%A	%B	%C	%D	%E	
Q1	2	5	87	5	1	2013
Q2	2	5	3	86	4	2009
Q3	4	4	81	5	7	2005
Q4	13	2	80	2	2	2006
Q5	4	5	8	73	10	2012
Q6	5	6	72	3	14	2011
Q7	71	5	10	6	7	2008
Q8	70	9	11	8	2	2008
Q9	4	9	8	68	11	2006
Q10	13	66	9	8	3	2007
Q11	66	4	12	15	2	2007
Q12	3	5	65	21	5	2006
Q13	2	64	8	7	18	2010
Q14	10	2	11	13	63	2005
Q15	5	10	61	17	7	2010
Q16	10	60	14	10	5	2012
Q17	9	15	59	11	4	2009
Q18	8	12	58	18	3	2013
Q19	10	48	23	16	3	2002
Q20	6	48	14	23	8	2010
Q21	9	47	17	18	7	2013
Q22	8	3	23	47	19	2012
Q23	17	29	46	4	3	2007
Q24	12	43	21	14	9	2009
Q25	6	30	11	39	14	2008
Q26	13	12	33	31	10	2012
Q27	12	25	27	28	7	2007

	Written response	
Q28	70	2012
Q29	60	2008
Q30	53	2013
Q31	50	2007
Q32	50	2006
Q33	43	2013
Q34	40	2010

Question 25

Examination report

It was explicitly stated in the question that the base of the tent was to be included in the calculation. Despite this, 30 per cent of students apparently ignored this instruction and incorrectly chose the surface area of the tent excluding the base (option D). This highlights the need for students to read a question carefully before proceeding with their calculations.

[VCAA 2008 1RGTQ6]

Question 26

Examination report

This question was challenging for many students.

One possible approach is as follows.

The solid is bounded by the curved surface PQR and three sectors of a circle of equal area QOR, POQ and POR (hidden).

Thus, the total surface area required

= area of curved surface $PQR + 3 \times$ area of sector QOR

$= \frac{1}{8} \times$ the surface area of the sphere of radius

15 cm $+ 3 \times \frac{1}{4} \times$ the area of the circle of radius 15 cm

$$= \frac{1}{8} \times 4\pi \times 15^2 + 3 \times \frac{1}{4} \times \pi \times 15^2$$

$= 883.57\ldots$ cm^2

[VCAA 2012 1RGTQ9]

Question 27

Examination report

Most students had started correctly but either failed to include the base of the solid (the 27 per cent who chose option C) or focused solely on the hemispherical surface (the 25 per cent who chose option B). To ensure that they include all of the surfaces involved, students might find it helpful to begin the solution to questions such as this by writing down a statement like: Total surface area = surface area of the hemispherical bowl + surface area of the side of the cylinder + surface area of the base of the cylinder.

[VCAA 2007 1RGTQ6]

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Question 28

a 4250 m^2 **b** 1000 m^3 **c** 90.6 m

Question 29

a $OM = \sqrt{3.4^2 - 3^2}$ **b** $\text{Area} = 2.2 \times 6 + \frac{1}{2} \times 6 \times 1.6$

c 180 m^3 **d** **i** 208 m^2 **ii** 13 litres

Question 30

Examination report

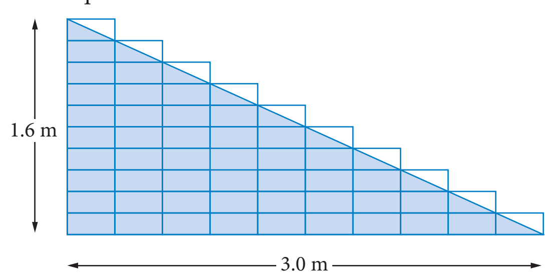
a 0.048 m^2

Converting units of area was an issue for some students. Many students correctly found the shaded area to be 480 cm^2 but then incorrectly converted this to 4.8 m^2.

b 6.6 m^3

Area of the cross-section = 55 × area of one of the rectangles

Many students inappropriately used $A = \frac{1}{2} \times 3.0 \times 1.6$. This would give the area shaded in the diagram below and is the equivalent of five rectangles fewer than the cross-section area required.



[VCAA 2013 2RGTQ2]

Question 31

Examination report

a $AW = \sqrt{32^2 + 12^2}$

$AW = \sqrt{1168} = 34.18\ldots \approx 34$

This calculation was often accompanied by a suitable diagram.

b SA = area of base + area of two sides + area of two ends

$$= 24 \times 28 + 2 \times (28 \times 34) + 2 \times \left(\frac{1}{2} \times 24 \times 32\right)$$

$$= 672 + 1904 + 768$$

$$= 3344 \text{ cm}^2$$

The solid in this question is a triangular prism where the triangle does not have a right angle. A significant number of students used an incorrect formula that had obviously been copied from a textbook and failed to score any marks.

[VCAA 2007 2RGTQ2]

Question 32

Examination report

a 1.5 m

An incorrect value of 1.75 was a common response.

b 13 m^3

Many students did not find the correct internal height (1.9 m) of the tank, which was necessary for a method mark.

[VCAA 2006 2RGTQ3]

Question 33

Examination report

a $200 + \pi d = 400$

The given diameter of 63.66 m needed to be the end result of a calculation in this 'show that' question. A common unacceptable answer included an appropriate equation involving the radius, r, but did not include an explanation of how this gave the diameter, d, as required. While this last step might appear to be unnecessary, it is essential that connections between variables are explained.

b 9549 m^2

$$A = 100 \times 63.66 + \pi\left(\frac{63.66}{2}\right)^2 = 9548.9\ldots$$

c 340 m^3

Area within outer boundary

$$= 100 \times 79.66 + \pi\left(\frac{79.66}{2}\right)^2 = 12949.9\ldots$$

Volume = (12 950 – 9549) × 0.1 = 340.1

Common errors included using a diameter of 63.66 + 8 = 71.66 m for the outside track and using the inappropriate formula for the volume of a sphere. Some students treated the running track as a prism, 0.1 m deep, 8 m wide and 400 m long.

[VCAA 2013 2RGTQ3]

Question 34

Examination report

$$1.8 = \frac{1}{3} \times x^2 \times 2.5$$

$x \approx 1.4697 \approx 1.47 \text{ m}$

It was concerning that a number of students seemed confused by the volume being given as 1.8 m^3 and wrote an equation involving 1.8^3.

[VCAA 2010 1RGTQ3]

EXAM PREP 12.2 Scale factors

Prep 1

a $\frac{7}{5}$ **b** 40

Prep 2

$y = 10$

Prep 3

$x = 36$ m

Prep 4

175 cm^2

Prep 5

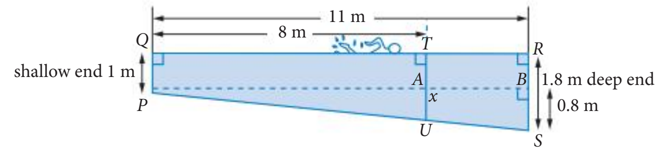
a 5 : 6 **b** 125 : 216 **c** 1620 cm^3

EXAM PRACTICE 12.2 Scale factors

	Multiple choice					
	%A	%B	%C	%D	%E	
Q1	1	2	1	95	1	2009
Q2	2	10	5	13	69	2006
Q3	2	4	6	23	65	2003
Q4	2	5	3	25	64	2004
Q5	23	8	11	55	3	2013
Q6	15	6	52	16	11	2013
Q7	47	3	16	3	31	2003
Q8	5	2	42	46	5	2010
Q9	5	10	24	15	46	2002
Q10	4	31	11	40	13	2011
Q11	11	8	18	37	25	2011
Q12	2	50	37	6	5	2003
Q13	5	57	6	31	1	2004
Q14	3	27	7	52	10	2005
Q15	22	2	6	11	59	2013
Q16	4	18	14	58	7	2007
	Written response					
Q17	18					2013
Q18	17					2007
Q19	10					2010

Question 10

Examination report



The most common mistake in answering this question was to assume that the relationship between similar triangles also applies to trapeziums. Making this assumption leads to 1.31 metres (option B), which was incorrect.

Similar triangles could be used to answer the question in part by drawing in the dotted line to create the similar triangles *PBS* and *PAU* as shown above, where $AU = x$ and $BS = 0.8$ m. We can then write:

$\frac{x}{0.8} = \frac{8}{11}$ or $x = 0.8 \times \frac{8}{11} = 0.58$, correct to two decimal places

Thus, the depth of the swimming pool at a distance of 8 metres from the shallow end of the pool is depth = TU = $TA + AU = 1.00 + 0.58 = 1.58$ metres, correct to two decimal places.

[VCAA 2011 1RGTQ8]

Question 11

Examination report

The most efficient way to solve this problem was to use scaling. The problem was to determine the linear scale factor *k*. Because the triangles *ABC* and *ABD* are similar, it is clear that $k = \frac{40}{24}$.

Thus, the area of triangle $ABC = k^2 \times$ area of triangle $ABD = \left(\frac{40}{24}\right)^2 \times 100 = 277.777\ldots\text{cm}^2 \approx 278\ \text{cm}^2$.

Some students obtained the answer 278 cm^2, but then added in an area of 100 cm^2 to obtain the answer 378 cm^2 (option E). This could occur if they thought that the similar triangles were *ABD* and *DCB* rather than *ABD* and *ABC*.

[VCAA 2011 1RGTQ9]

Question 12

Examination report

This question involved the scaling up of an area. While 37 per cent correctly used a scaling factor of 25 to give the answer 5500 cm^2 (option C), 50 per cent incorrectly used a linear scaling factor of five to arrive at an answer of 1100 cm^2 (option B). Students failed to recognise that the scaling factor required depends on the dimension of the quantity being scaled.

[VCAA 2003 1RGTQ7]

Question 13

Examination report

While 31 per cent of students correctly used a scaling factor of 36 to give the answer 25 920 cm^2 (option D), 57 per cent incorrectly used a linear scaling factor of 6 to arrive at an answer of 4320 cm^2 (option B). Students failed to recognise that the required scaling factor depended on the dimension of the quantity being scaled.

[VCAA 2004 1RGTQ5]

Question 14

Examination report

Although 27 per cent of students used the correct scale factor of 1.25 to give the correct answer 45 m (option B), 52 per cent incorrectly used a scale factor of 1.5625 to arrive at an answer of 56.25 (option D). These students failed to take into account the dimensions of the quantities being scaled.

[VCAA 2005 1RGTQ8]

Question 15

Examination report

One solution strategy to this question is: the cost of the cupcakes is proportional to their volume, so that, if k is the linear scale factor,

$$\frac{\text{cost}_{\text{large}}}{\text{cost}_{\text{small}}} = \frac{\text{v}_{\text{large}}}{\text{v}_{\text{small}}} = k^3$$

The linear scale factor, k, can be determined by comparing the widths of the two cakes, to give $k = \frac{6}{4} = \frac{3}{2}$ so that

$$\text{cost}_{\text{small}} = \frac{\$5.40}{\left(\frac{3}{2}\right)^3} = \$1.60 \text{ (option A)}$$

[VCAA 2013 1RGTQ4]

Question 16

Examination report

From this information a scale factor for area ($k^2 = 400$) can be determined. The majority of students (58 per cent) apparently obtained this area scale factor but then incorrectly applied it directly to scaling the given length rather than first converting it into the corresponding linear scale factor, $k = \sqrt{400} = 20$.

[VCAA 2007 1RGTQ5]

Question 17

Examination report

$$k^2 = \frac{720}{500} = \frac{36}{25}$$

$$\therefore\ k = \frac{6}{5} = 1.2$$

$$\therefore\ k^3 = 1.2^3 = \frac{216}{125} = 1.728$$

The ratio of dimensions of the larger discus to the smaller discus was required.

The value of $k > 1$ since a larger volume is scaled up from the smaller volume. A common error was to work with the reciprocal of the area ratio. This would give a value less than one for the linear and the volume ratios. If this is multiplied with the smaller discus dimensions, the result would be a smaller discus, not a larger one. This question was very poorly answered. Many students did not attempt this question.

[VCAA 2013 2RGTQ4]

Question 18

Examination report

a $\frac{24}{32} = \frac{3}{4}$

b Volume removed $= \left(\frac{3}{4}\right)^3 = \left(\frac{27}{64}\right)$ of the original volume

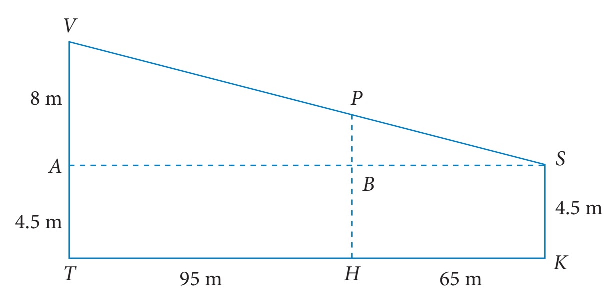
Hence volume remaining $= 1 - \frac{27}{64} = \frac{37}{64}$

[VCAA 2007 2RGTQ4]

Question 19

Examination report

a 7.75 m



Similarity could be applied to two triangles found by first drawing a horizontal line through point S.

$$\frac{PB}{8} = \frac{65}{160}$$

$$PB = \frac{8 \times 65}{160} = 3.25 \text{ metres}$$

and $PH = PB + BH = 3.25 + 4.5 = 7.75$ m

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Another correct method considered *VS* as a straight line on a pair of *XY* axes with the origin at point *T*, finding its equation and then substituting $x = 95$.

A third correct method was to find the area of the trapezium *VSKT*. Equating this to the sum of the areas of two trapezia, *VPHT* and *PSKH*, with a common unknown length *PH*, resulted in an equation with one unknown to be solved.

The most common error was to inappropriately apply similarity to 'corresponding' lengths of two trapezia to give an incorrect answer of 5.08 m.

[VCAA 2010 2RGTQ4]

EXAM PREP 12.3 Right-angled triangles

Prep 1

a 7.6 mm **b** 11.0 cm

Prep 2

12°

Prep 3

73°

Prep 4

26.8 m

Prep 5

14°

EXAM PRACTICE 12.3 Right-angled triangles

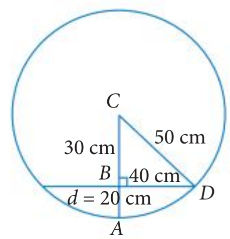
	Multiple choice					
	%A	%B	%C	%D	%E	
Q1	6	79	6	8	1	2007
Q2	5	77	7	8	3	2006
Q3	75	5	14	2	4	2007
Q4	71	6	8	3	11	2011
Q5	66	19	7	3	4	2010
Q6	4	13	14	55	13	2010
Q7	6	11	14	58	10	2011
Q8	58	17	13	7	5	2004
Q9	53	9	8	28	3	2009
Q10	55	11	10	11	12	2009
Q11	11	54	16	14	4	2011
Q12	51	17	13	11	7	2005
Q13	16	15	49	12	7	2010
Q14	22	11	39	22	4	2006
	Written response					
Q15	77					2009
Q16	18					2011

Question 14

Examination report

Option C, which corresponded to the distance of the surface from the centre of the pipe, was given by 39 per cent of students. A further 22 per cent of students wrongly assumed that the cross-section of the surface was semi-circular and chose option D.

- Draw in a radius line that is a perpendicular bisector of the water surface line.
- Label points as shown on the diagram below.
- From right-angled triangle *BCD*, $CB = 30$ cm $\left(= \sqrt{50^2 - 40^2} \text{ cm}\right)$
- Then $d = CA - CB = 50 - 30 = 20$ cm [option A]



[VCAA 2006 1RGTQ8]

Question 15

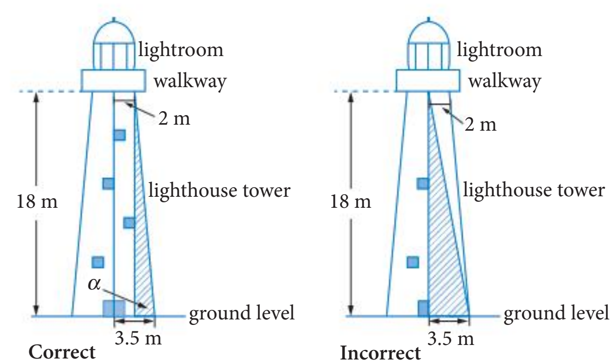
a $\frac{50}{400} = 0.125$ **b** 7.1° **c** 403.1 m

Question 16

Examination report

a 85.2°

$\tan^{-1}\left(\frac{18}{1.5}\right) = 85.236\ldots$

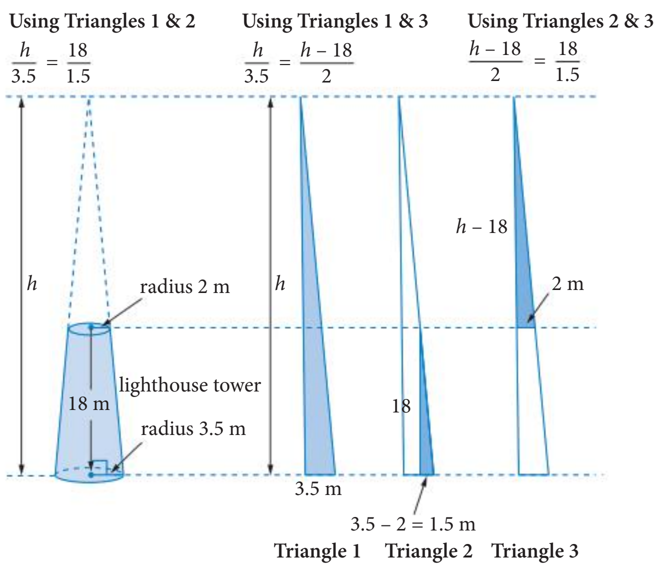


Many students drew an inappropriate right triangle with base = 3.5 m and perpendicular height = 18 m and then found the angle formed at the corresponding corner by using $\tan^{-1}\left(\frac{18}{3.5}\right) = 78.996\ldots$

[VCAA 2011 1RGTQ4]

b **i** 42 m

Various similar triangles (such as those labelled 1, 2 and 3 in the diagrams below) could be used to form the relevant equation.



Most students were unable to identify the required similar triangles. A common error was to equate the ratios $\frac{h}{18} = \frac{3.5}{2}$. This attempts to equate a triangle with a trapezium.

ii 438 m^3

Large cone – small cone

$$= \left(\frac{1}{3} \times \pi \times 3.5^2 \times 42\right) - \left(\frac{1}{3} \times \pi \times 2^2 \times 24\right) = 438.469\ldots$$

Of those who did attempt this question, many students gave incomplete/incorrect answers consisting of only the volume of a large cone with height = 42 m or height = 18 m.

[VCAA 2011 2RGTQ4]

EXAM PREP 12.4 Non-right-angled triangles

Prep 1

$y = 7.47$ cm

Prep 2

$\theta = 9°$

Prep 3

$\theta = 55°$ or $\theta = 125°$

Prep 4

$k = 10.7$ m

Prep 5

$\theta = 36.2°$

EXAM PRACTICE 12.4 Non-right-angled triangles

	Multiple choice					
	%A	%B	%C	%D	%E	
Q1	1	4	83	8	4	2004
Q2	3	5	4	82	5	2006
Q3	5	8	4	81	2	2013
Q4	6	8	75	6	4	2011
Q5	3	20	4	3	70	2003
Q6	7	10	75	6	2	2003
Q7	12	68	10	5	5	2005
Q8	6	9	13	64	8	2003
Q9	11	7	61	10	10	2012
Q10	7	9	26	52	5	2004
Q11	41	7	7	18	27	2008
Q12	25	12	20	34	9	2011
Q13	19	19	27	19	14	2009
	Written response					
Q14	70					2004
Q15	50					2012
Q16	45					2012

Question 11

Examination report

27 per cent of students chose option E, suggesting that many students were able to recognise that an application of the sine rule was required but misapplied the rule or merely chose the response that appeared correct rather than working it out for themselves.

[VCAA 2008 1RGTQ5]

Question 12

Examination report

This involved a routine application of the cosine rule. The most common error was to choose the answer 26.5 cm (option A). This is the correct length of the shortest diagonal in the parallelogram.

[VCAA 2011 1RGTQ6]

Question 13

Examination report

The key to answering this question was to recognise that the maximum height occurs when the pole falls at right angles to the wall.

[VCAA 2009 1RGTQ9]

Question 14

a 70° **b** 10.6 m **c** 10.2 m **d** 8.6 m^2

Question 15

Examination report

a $\dfrac{ST}{\sin(47°)} = \dfrac{50}{\sin(61°)}$

$ST = \dfrac{50\sin(47°)}{\sin(61°)} \approx 41.81\ldots$

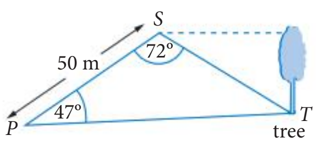
This 'show that' question required the end result of an appropriate calculation to be 41.81.

b 16.9 m

Height = 41.81 × tan (22°) = 16.892…

The most common incorrect answer came from students who connected point S to the top of the tree, as shown in the diagram below, then incorrectly assumed a right triangle had been formed with ST as the hypotenuse and then used

$\sin(22°) = \dfrac{x}{41.81} = 15.7$



[VCAA 2012 2RGTQ3]

Question 16

Examination report

a 8.66 m $\quad 10 \times \cos(30°) = 8.660\ldots$

b 21.7 m^3 $\quad \dfrac{1}{2} \times 10 \times 8.66 \times \sin(30°) = 21.65\ldots$

Some applied Heron's rule instead of $A = \dfrac{1}{2}bc\sin(A°)$

c 68.6 m^2 $\quad \left(\dfrac{14}{10}\right)^2 \times 35 = 68.6$

While most students correctly found the scale factor $\dfrac{14}{10}$, many did not apply it properly by failing to square it when calculating the scaled-up area.

d 43°

$BC = \sqrt{10^2 + 14^2 - 2(10)(14)\cos(30°)} = 7.315\ldots$

$\therefore \dfrac{\sin(BCO)}{10} = \dfrac{\sin(30°)}{BC = 7.315\ldots}$

$\therefore \angle BCO = 43.118\ldots = \angle CDO$

Many students did not attempt this question. The key to the quickest solution starts with realising that angle CDO is the same as angle BCO.

[VCAA 2012 2RGTQ4]

EXAM PREP 12.5 Bearings

Prep 1

317°

Prep 2

143°

Prep 3

a $\theta = 25°$ **b** 275°

EXAM PRACTICE 12.5 Bearings

	Multiple choice					
	%A	%B	%C	%D	%E	
Q1	5	5	8	68	14	2008
Q2	11	60	9	2	18	2005
Q3	10	11	58	10	10	2012
Q4	8	27	7	3	54	2009
Q5	14	18	49	14	4	2010
Q6	30	8	10	43	10	2008
Q7	12	39	25	13	10	2013
Q8	11	12	20	38	19	2008
Q9	18	16	11	17	37	2007
Q10	12	31	19	31	7	2006
	Written response					
Q11	78					2013
Q12	65					2009
Q13	50					2011
Q14	50					2006
Q15	50					2004
Q16	42					2005
Q17	35					2008
Q18	18					2013
Q19	Not reported					2002

Question 6

Examination report

This question was a routine bearings question, yet only 43 per cent of students correctly chose 300° (option D).

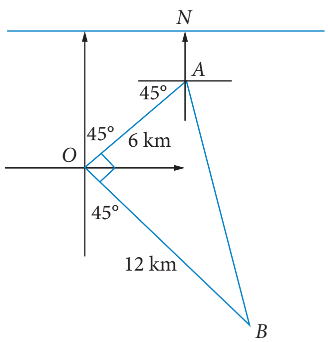
[VCAA 2008 1RGTQ2]

Question 8

Examination report

The key to answering this question was for students to draw a clear diagram and include all of the relevant information.

Note that $\triangle OAB$ is a right-angled triangle.



To find the bearing of B from A, $\angle NAB$, first find $\angle OAB$.

$\tan(\angle OAB) = \frac{12}{6} = 2$ so $\angle OAB = 63.43\ldots°$

Then the bearing of B from A,

$\angle NAB = 270° - (45° + \angle OAB)$

$= 270° - (45° + 63.43\ldots°)$

$= 161.56\ldots°$ or 162° to the nearest degree (option D)

[VCAA 2008 1RGTQ9]

Question 9

Examination report

To answer this question, students were required to draw a diagram and then carefully assess all of the alternatives offered before arriving at a solution. As it turned out, all three alternatives given were valid methods (option E), but this option was chosen by only 37 per cent of students. The relatively even distribution of student responses over the other incorrect options suggests that most students either failed to systematically test all of the alternative methods, or failed to get started and just guessed.

[VCAA 2007 1RGTQ9]

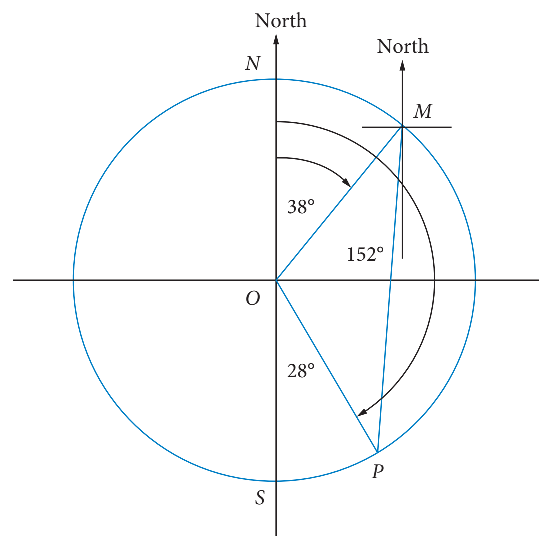
Question 10

Examination report

Correctly answering this question, a bearings problem, depended critically on being able to construct an appropriate diagram from the information given. This was clearly beyond most students, with only 31 per cent giving the correct response, option D. A possible solution strategy is as follows.

- As M and P are the same distance from O, they must lie on the circumference of the same circle.
- Use this information to construct a diagram as shown below.
- A carefully drawn diagram that takes into account that the angle MON must be greater than SOP (from the bearings), shows that the bearing of P from M is between 180° and 270°.

Note: For the bearing to be exactly 180°, the bearing of P would have to be 142°. For the bearing to be between 090° and 180°, the bearing of P would have to be less than 142°.



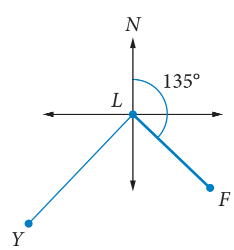
[VCAA 2006 1RGTQ9]

Question 11

a 42 m **b** **i** 77 m **ii** 957 m^2 **c** 14°

Question 12

a



b 75°

c 6.87 km

d 315°

Question 13

Examination report

a **i** 152° **ii** 78° **b** 208° **c** 079°

ΔABL is an isosceles triangle with base angles $= t$

$t = \frac{180 - (28 + 50)}{2} = \frac{102}{2} = 51°$

Therefore angle from North to LB is $28° + 51° = 79°$.

True (three-figure) bearings were required so the bearing was 079°, rather than just 79°.

[VCAA 2011 2RGTQ2]

Question 14

a 45.5 m **b** 24.9°

Question 15

Examination report

a 282° T

A significant number of students were unable to find an appropriate angle that could usefully be applied to find the bearing.

b 6.9 km Generally well done.

c 18.6 km

One of the two marks was awarded for a correct calculation for the length of *UV*. Many incorrectly quoted the length of the shortest of the three sides of the triangular course.

[VCAA 2004 2RGTQ2]

Question 16

Examination report

a i 1.6 km

A common incorrect answer was 2.1 km (the length of *GX*).

ii 1.2 km

The rounded answer to part **a i** needed to be used in the calculation here. If that answer was wrong, marks were still available here if the correct calculation was applied.

The question involved two steps. Many students rounded off their answer to the first step and used this in the second step. Exact numbers should be retained in the calculator and used in the second step unless the first step was the answer to a specific question.

Several students applied Pythagoras' theorem to the non-right-angled triangle *GXY* to obtain 2.7 km. Others assumed that *X* was twice as far south of *G* as *Y*.

b 115°

Common errors included answers of 75°, 150° and 215°.

c 3.2 km

The answer to part **b** was required here. If that answer was wrong, a consequential mark was still available here for correct use of the cosine rule.

Many students who used the cosine rule forgot to take the square root at the end. Pythagoras' theorem was also incorrectly used here by some students.

d 111°

[VCAA 2005 2RGTQ2]

Question 17

Examination report

a 42.7°

b $NT^2 = 10^2 + 13^2 - 2 \times 10 \times 13 \times \cos(65°)$

$NT^2 = 159.119$

$NT = \sqrt{159.119}$

$NT = 12.614\ldots \approx 12.6$

Some students did not show that a square root had to be taken at the last step and simply went from

$\therefore NT^2 \approx 159.119$ to

$\therefore NT = 12.6$, which was given in the question.

c 69°

d $180° - 69° = 111°$

e The nearest distance to the shed = 13 sin (65°) = 11.8 metres

∴ It is possible for the tree to hit the shed as its height of 12 metres > 11.8 m.

There is a section of the shed between *C* and *N* that is within 12 metres of the tree. Full marks were awarded for any correct calculation that identified any point within this section and then used this calculation to justify a written conclusion that the tree would hit.

Some students calculated the shortest distance correctly and said that the tree would hit the shed but gave no mathematical comparison between this distance and the tree height to justify their statement.

An answer of 'yes' without justification by calculation and comparison gained no marks. Remarkably, most students seemed to have difficulty understanding the three-dimensional diagram. Only a minority understood that the tree did not have to fall along the line *CT* or the line *NT*. Many students incorrectly concluded that, as the tree was shorter than the distance *CT* or *NT*, it would not hit the shed.

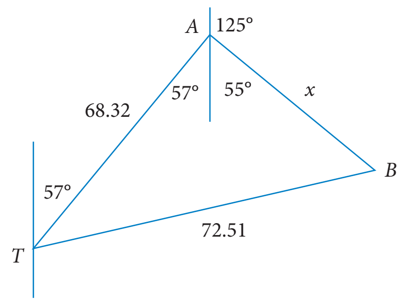
Others used Pythagoras' theorem to incorrectly calculate the distance from the tree to a point halfway between *C* and *N* on the shed. The triangle involved in these calculations was not a right triangle and Pythagoras' theorem could not be used. A small number of students calculated the distance from the top of the tree to point *C* and used this 17.4 metre distance to justify their assertion that the tree would not hit.

[VCAA 2008 2RMTQ3]

Question 18

Examination report

9.7 m



One method to determine the length AB uses the sine rule to first determine $\angle ABT$. Then, since $\angle TAB = 112°$, $\angle ATB$ can be calculated and used in the cosine rule to find the length AB. Many students drew a diagram that showed all of the bearings. However, many misread the question and incorrectly used 125° as the bearing of point B from T rather than of point B from A. Others incorrectly used 72.51 m as the length of AB.

[VCAA 2013 2RGTQ5]

Question 19

Examination report

a 563 336 m^2

An answer in the range [563 330, 563 340] was accepted in recognition of different ways this question could have been tackled. One mark for substituting relevant dimensions into

$$\text{Area} = \frac{1}{2}bc\sin(A)$$

Generally well done although there continue to be many who insist on using $A = \frac{1}{2}bh$ for all triangles.

b 1398 m

One mark for an appropriate use of the sine or cosine rules toward finding the length PR. Many did not consider, or understand, that the length of any one side of a triangle cannot exceed the sum of the other two lengths. Consequently, some impossible lengths for PR were offered. Several students tried to apply Pythagoras' theorem to triangle QPR.

c 048° T, 48° was also accepted

One mark for calculating angle $QPR = 33°$. Many did not seem to fully understand bearings and did not know what to do with the 015° T given for the bearing from P to S. A number calculated angle PRQ and gave this answer as the required bearing. Others assumed angle SPQ was a right angle. A common incorrect answer was 42°.

d 398 980 m^2

An answer within the range of [398 200, 399 100] was accepted. This allowed for students who used rounded off figures through their calculations or other extended methods that might have included breaking up into two right triangles. There was one consequential mark for finding angle SPR by 90 – 15 (their angle QPR from within part **c**) and another for using their angle SPR in

$\text{Area} = \frac{1}{2}bc\sin(A)$ to get a final answer.

Rounding off to 'the nearest 10 m^2, caused some difficulties with many simply dropping off the units column, for example 563 336 became 56 333 or 56 334.

e 962 320 m^2

Most students got this consequential mark as the sum of their answers for parts **a** and **d**.

[VCAA 2002 2RGTQ1]

CHAPTER 13
Spherical geometry

EXAM PREP 13.1 Circle measurement

Prep 1

41.888 cm

Prep 2

83.1°

Prep 3

27.65 cm^2

Prep 4

51 255.5 cm^2

Prep 5

59.38 cm^2

Prep 6

418.88 cm

EXAM PRACTICE 13.1 Circle measurement

	Multiple choice					
	%A	%B	%C	%D	%E	
Q1						
Q2						
Q3						
Q4	2	64	8	7	18	2010

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Question 5

12.287 cm

Question 6

630 cm

Question 7

$$l = \frac{\theta}{360} \times 2\pi r = \frac{3}{360} \times 2\pi \times 6400 = 335.103 \text{ km}$$

Question 8

301.6 cm^2

Question 9

a $\sin(\theta) = \frac{4}{10} = \frac{2}{5}$

b $\theta = \sin^{-1}\left(\frac{2}{5}\right) \approx 23.578°$

Angle at the centre = 47.156°

$$A = \frac{47.156}{360}\pi \times 10^2 - \frac{1}{2} \times 10^2 \sin(47.156°) \approx 4.49 \text{ cm}^2$$

EXAM PREP 13.2 Latitude and longitude

Prep 1

a Carnarvon **b** Newcastle **c** Broome

Prep 2

a 32° or 33° S 138° E **b** 28° S 146° E

c 20° S 119° E

EXAM PRACTICE 13.2 Latitude and longitude

Q1	Q2	Q3	Q4	Q5
A	A	E	A	D

EXAM PREP 13.3 Shortest distance problems

Prep 1

$$l = \frac{11}{360} \times 2\pi \times 6400 \approx 1229 \text{ km}$$

Prep 2

8266 km

Prep 3

4133 km

Prep 4

5920 km

Prep 5

1046 km

EXAM PRACTICE 13.3 Shortest distance problems

Q1	Q2	Q3	Q4	Q5	Q6
C	A	D	B	E	E

Question 7

$$l = \frac{29}{360} \times 2\pi \times 6400 \approx 3239 \text{ km}$$

Question 8

a 2 805 575 m **b** 5827 km

Question 9

a 6227 km **b** 3826 km **c** 16 280 km

EXAM PREP 13.4 Time zones

Prep 1

a +10 **b** +8 **c** −5

Prep 2

5 a.m. Friday

Prep 3

1 p.m. Saturday

Prep 4

4 a.m. Sunday

EXAM PRACTICE 13.4 Time zones

Q1	Q2	Q3	Q4	Q5	Q6
C	D	A	E	B	E

Question 7

7:30 p.m. Sunday night

Question 8

10:30 p.m. Wednesday

Question 9

Tampa time zone = –5   Sydney time zone = +10

At 7:00 a.m. Thursday in Sydney the time in Tampa = 7:00 a.m. – 15 h = 4 p.m. Wednesday.

Add flight time of 18 h.

Arrival time in Tampa = 4 p.m. Wednesday + 18 h
= 10 a.m. Thursday

EXAMINATION Solutions

GEOMETRY AND MEASUREMENT – EXAMINATION 1

1	2	3	4	5	6	7	8	9
D	A	B	B	B	C	E	A	D

GEOMETRY AND MEASUREMENT – EXAMINATION 2

Question 1

a 427.61 mm   **b** 367 570 mm^2

Question 2

a 37.55° S   **b** 4194 km   **c** 5074 km

Question 3

a +9 or 9 hours ahead of GMT

b 2:30 p.m. Monday   **c** 5:30 p.m. Monday

Question 4

a 8 m^2 **b** 12.8 m

Question 5

a 180 – (45 + 60) = 75

b $\frac{AX}{\sin(45°)} = \frac{3.16}{\sin(75°)}$ or $\frac{AX}{\sin(45°)} = \frac{\sqrt{8}}{\sin(60°)}$

c 2.31 m   **d** 3.2 m^2

Question 6

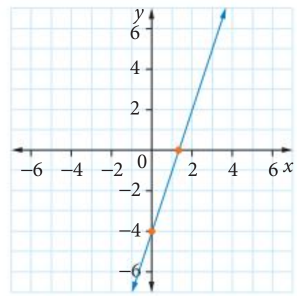
228°

CHAPTER 14
Construction and interpretation of graphs

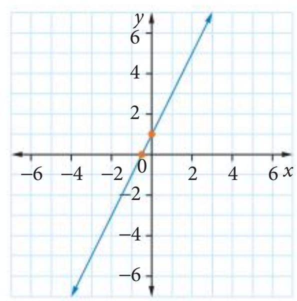
EXAM PREP 14.1 Review of straight-line graphs

Prep 1

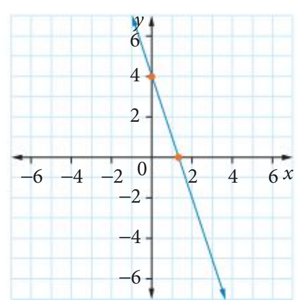
a $(1\frac{1}{3}, 0)$ $(0, –4)$



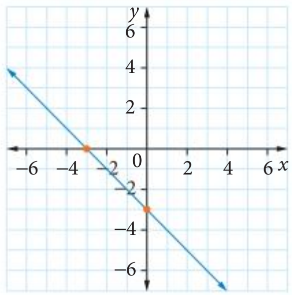
b $(–\frac{1}{2}, 0)$ $(0, 1)$



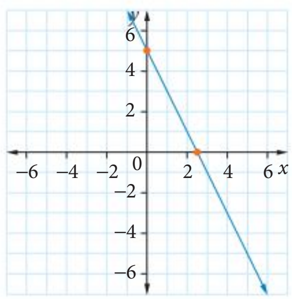
c $(1\frac{1}{3}, 0)$ $(0, 4)$



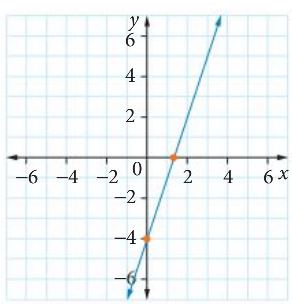
d (−3, 0) (0, −3)



e $(2\frac{1}{2}, 0)$ (0, 5)



f $(1\frac{1}{3}, 0)$ (0, −4)



Prep 2

a Yes **b** Yes **c** No

Prep 3

3

Prep 4

a The y-intercept is −3 and the slope is 4.

b The y-intercept is 8 and the slope is −2.

c The y-intercept is 0 and the slope is 1.

d The y-intercept is −4 and the slope is 0.

Prep 5

a **i** $y = 3 + 2x$

ii Slope of 2, so it is not parallel to $y - 3x = 15$ which has a slope of 3.

b **i** $y = -1 + 3x$

ii Both have a slope of 3, so they are parallel.

c **i** $y = 2 - 2x$

ii Slope of −2, so it is not parallel to $y - 3x = 15$ which has a slope of 3.

Prep 6

a $k = 5$ **b** $k = 11$ **c** $k = 4$

Prep 7

a $y = -9 + 3x$ **b** $y = 2 + x$

c $y = 2 - x$ **d** $y = 10 - 6x$

EXAM PRACTICE 14.1 Review of straight-line graphs

	Multiple choice					
	%A	%B	%C	%D	%E	
Q1	4	3	4	71	18	2007
Q2	6	6	4	70	14	2006
Q3	12	69	13	2	4	2013
Q4	12	3	8	8	68	2013
Q5	3	63	14	11	9	2004
Q6	8	8	56	6	22	2005
Q7	11	12	10	9	56	2010
Q8	1	2	25	53	19	2002
Q9	14	12	11	12	50	2004
Q10	6	8	46	24	15	2006
Q11	10	44	17	14	15	2006
Q12	42	8	42	5	3	2003
Q13	19	14	38	12	16	2005

Question 6

Substitute either (0, 12) or (8, 0) into $3x + 2y = 4k$ and solve for k. Substituting $x = 0$ and $y = 12$ we get

$$0 + 24 = 4k$$
$$k = 6$$

Question 7

Substitute the values of the two points into the equations to see if they both lie on the line, or use a CAS/calculator or the formula $y - y_1 = a(x - x_1)$ to find the equation of the line.

Let $(x_1, y_1) = (10, 1)$ and $(x_2, y_2) = (4, -2)$.

$$\text{Slope} = a = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-2-1}{4-10} = \frac{-3}{-6} = \frac{1}{2}$$

Using $y - y_1 = a(x - x_1)$

$$y - 1 = \frac{1}{2}(x - 10)$$
$$y = \frac{1}{2}x - 4$$
$$x - 2y = 8$$

Question 8

The line is neither vertical nor horizontal, so options A and B aren't correct. Substitute the values of the two known intercept points (0, 5) and (8, 0) into the equations to see which line both points lie on.

Question 9

Substitute the values of each point into both equations to see which point lies on both lines. Alternatively sketch $y = -8$ and $x = 12$ on the same axes to see where they intersect.

Question 10

Substitute the values of the point into the lines to see which line the point lies on.

Question 11

A The line passes through the origin means (0, 0) is on the line. Substituting the values of this point into $12x - 4y = 0$ shows this is true.

B Writing the equation in intercept–slope form gives $y = 3x$, which has a slope of 3, so it is *not* true that the line has a slope of 12.

C The line with the equation $12x - 4y = 12$ in intercept–slope form gives $y = -3 + 3x$, which has a slope of 3, so this is true.

D Substituting the values of (1, 3) into $12x - 4y = 0$ shows this is true.

E Since the slope of the line is positive, as x increases y increases, so this is true.

Question 12

Find the equation of the line that goes through the point (0, 8) and (3, 6).

a = the y-intercept = 8 $\qquad b = \frac{6-8}{3-0} = -\frac{2}{3}$

So the equation is $y = 8 - \frac{2}{3}x$.

Substitute the values of $(c, 0)$ into this equation to find c.

$$0 = 8 - \frac{2}{3}c$$

$$c = \frac{3 \times 8}{2} = 12$$

Question 13

A Substituting the values of (4, 7) into $7x - 4y = 0$ shows this is true.

B Substituting the values of (3, 5) into $7x - 4y \geq 0$ shows this is true. $21 - 20 = 1 \geq 0$

C Rewriting $3x + 5y = 0$ in intercept–slope form gives $y = -\frac{3}{5}x$, which has a negative slope. So this is *not* true.

D Substituting the values of (0, 0) into both equations shows this is true.

E Rewriting $7x - 4y = 0$ in intercept–slope form gives $y = \frac{7}{4}x$, which has a positive slope and, therefore, y increases as x increases. So this is true.

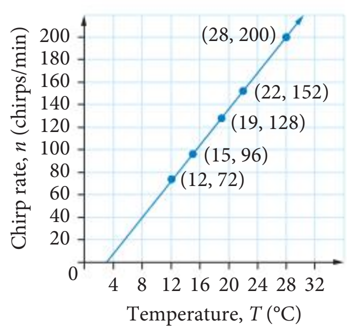
EXAM PREP 14.2 Modelling with straight-line graphs

Prep 1

a $n = -24 + 8T$

b The a represents the number of cricket chirps when the temperature is 0°C. The equation gives –24 chirps, which is impossible.

c



Note that since it makes no sense to talk about negative chirps, the graph shouldn't extend below the horizontal axis.

d 184 chirps/min, interpolation.

e 8 chirps/min. This figure isn't a reliable prediction of the chirp rate because 4°C is an unrealistic temperature for an Australian summer. This involves extrapolation.

Prep 2

a \$100 **b** \$2.20 per kilolitre

Prep 3

a Revenue = $5.5n$ **b** Profit = $1.5n - 110$

c \$190 **d** Loss of \$35 **e** 474

Prep 4

a Cost = $75 + 3n$ **b** \$375

c Revenue = $100p$ **d** \$6.40

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EXAM PRACTICE 14.2 Modelling with straight-line graphs

	Multiple choice					
	%A	%B	%C	%D	%E	
Q1	96	1	1	1	0	2012
Q2	93	3	2	1	1	2008
Q3	1	2	3	91	2	2005
Q4	3	3	87	4	3	2013
Q5	4	4	86	2	4	2007
Q6	12	8	68	7	3	2010
Q7	6	65	18	7	3	2011
Q8	10	17	3	17	52	2013
Q9	14	48	12	18	7	2008
Q10	7	30	12	37	14	2009
	Written response					
Q11	55					2006
Q12	50					2013

Question 8

The \$30 service fee means that for 0 MJ there is a charge of \$30, so 30 is the y-intercept. This eliminates option C. For each megajoule of energy, there is an increase of 2 cents in cost. This means the slope of the line needs to be 0.02.

Using the two endpoints to calculate the slope in each case.

The two endpoints for option E are (0, 30) and (5000, 130).

Slope $= \frac{130-30}{5000-0} = \frac{100}{5000} = 0.02$, so E best models the situation.

Question 9

$2x + 3y = 12$

Find the x- and y-intercepts: $y = 4$ when $x = 0$ and $x = 6$ when $y = 0$. So option B is correct.

Question 10

Examination report

Profit = Revenue – Costs = $R - C$

Let S be the selling price of one card. Then, the revenue from selling 150 cards is $R = 150 \times S$

From the graph, the cost C of producing n cards is given by $C = 50 + 2n$, so the cost of producing 150 cards is

$C = 50 + 2 \times 150 = \$350$

The profit from selling 150 cards is \$175, so

$175 = 150 \times S - 350$ or $S = \$3.50$

[VCAA 2009 1RGRQ9]

Question 11

Examination report

a $f = -\frac{1}{8}d + 50$

The negative sign on the gradient was missed by a number of students.

b 240 km

An incorrect answer of 400 km was common, due to incomplete reading of the question.

c 75 litres

An incorrect answer of 63 litres was common, due to incomplete reading of the question.

[VCAA 2006 2RGRQ2]

Question 12

Examination report

a $18n - 260$

Profit $= 24n - (6n + 260)$

Many students forgot to include the bracket needed for the expression for cost.

b 43 students

$18n - 260 \geq 500$

$\therefore n \geq 42.2$

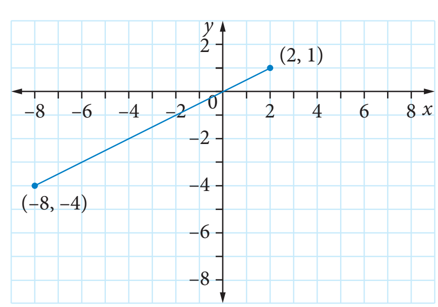
Students must take the context of a question into account before rounding numbers. Many students rounded 42.2 down to an answer of 42. The question asked for the minimum number of students who will need to participate in order to make a profit of at least \$500, which will not quite be achieved with only 42 students.

[VCAA 2013 2RGRQ3]

EXAM PREP 14.3 Line segment and step graphs

Prep 1

a



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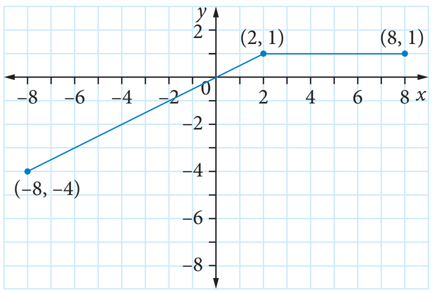
b



Prep 2

a No, each line segment has a different slope. Hence, the rainwater tank is filling at four different rates.

b 18 000 litres **c** 200 hours

d approximately 80 hours to 120 hours

e 25 litres per hour

Prep 3

a They started their journey at the park 1 km from Volker's house.

b The slope of the line segment after 45 minutes is greater, so Volker and Herb start walking faster.

c 15 minutes **d** 3.5 km **e** 7 km/h

Prep 4

a **i** \$10 **ii** \$3 **iii** \$9

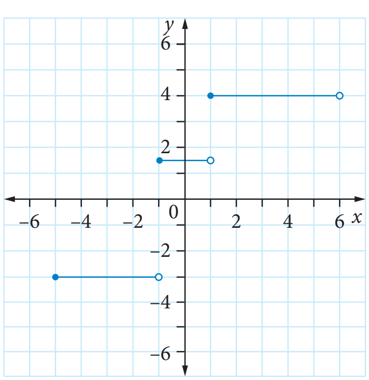
b More than 2 kg, but less than or equal to 3 kg.

c 6 kg is the heaviest item that can be sent in this type of parcel delivery.

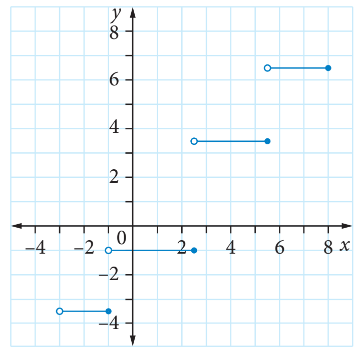
d 8 kg

Prep 5

a



b



EXAM PRACTICE 14.3 Line segment and step graphs

	Multiple choice					
	%A	%B	%C	%D	%E	
Q1	6	88	1	4	2	2010
Q2	2	7	67	2	21	2010
Q3	2	97	1	0	0	2011
Q4	13	5	56	14	11	2011
Q5	2	2	81	11	4	2006
Q6	50	23	8	9	9	2006
Q7	17	2	5	68	8	2004
Q8	10	13	12	60	5	2007
Q9	24	15	59	1	2	2004
Q10	8	4	11	61	16	2004
Q11	49	26	4	6	14	2012
Q12	6	20	64	7	2	2012
Q13	0	1	34	57	7	2013
	Written response					
Q14	87					2006
Q15	Not reported					2002
Q16a–c	87					2013
Q16d–e	45					2013
Q17a–b	37					2004
Q17c	20					2004

Question 4

The four charges are \$0.40, \$0.60, \$0.90 and \$1.50. It's impossible for any two of these to give a total of \$1.40.

Question 6

The second segment of the graph has the equation $M = 220$, so only A and E are options. The slope of the first segment is $\frac{332.5 - 220}{4.5 - 0} = -25$, so A is correct.

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Question 9

The speed is the slope of the line segments. The horizontal lines have slope zero so speed is 0 for these. Find which of the other three line segments has the largest slope.

For 0 to 5 hours, speed $= \dfrac{350}{5} = 70$ km/h

For 9 to 12 hours, speed $= \dfrac{250}{3} \approx 83.3$ km/h

For 14 to 16 hours, speed $= \dfrac{100}{2} = 50$ km/h

So the speed of the car is the greatest from 9 to 12 hours.

Question 11

The two points of the line segment are (85, 0) and (50, 700).

Use a CAS/calculator to find the equation of the line going through these two points or solve by hand.

Slope $= \dfrac{700-0}{50-85} = -\dfrac{700}{35} = -20$

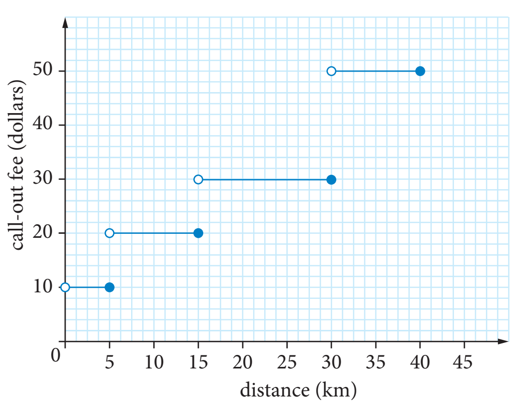
Using the formula $y - y_1 = a(x - x_1)$,

$V = -20(t - 85)$ or $V = 1700 - 20t$

Question 14

a **i** $30 **ii** 5 km

b



Question 15

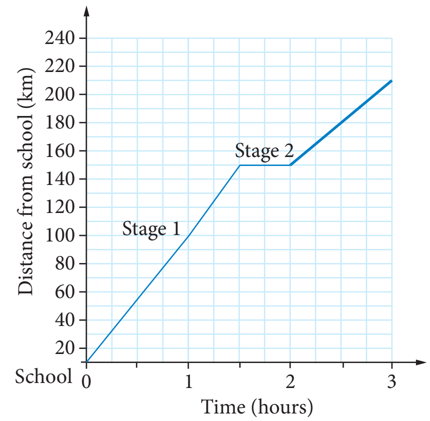
a 3.5 minutes

b **i** Section E–F **ii** 200 m/min, or 12 km/h.

Question 16

a 100 km/h **b** 30 minutes

c



d 70 km/h **e** $k = 30$

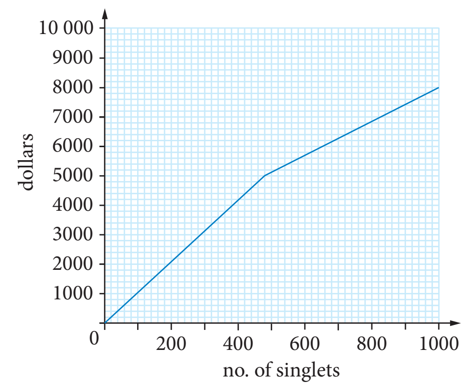
Question 17

Examination report

a $5720

Many students did not understand how to interpret this formula. Many used both formulas and quoted both answers, while others decided that the way to get only one answer was to subtract or add the two results.

b



Many students sketched only one line. Others typically sketched two complete, intersecting lines and gained one mark if these were correct. The second mark came from correctly terminating each section of the graph at (500, 5000).

c 750

Poorly answered. The solution to the equation $2000 = R_s - C_s$ was needed, with the appropriate revenue formula applied. If $R_s = 10x$ was used, $x = 583$ should have been discarded as inappropriate to $R_s = 10x$. The most common error was to write $2000 = 4x + 1500$, giving an answer of 125.

[VCAA 2004 2RGRQ3]

EXAM PREP 14.4 Simultaneous equations

Prep 1

a $x = -1, y = 2$ **b** $x = -2, y = 4$

c $x = 1, y = 1$ **d** $x = 2, y = 2$

Prep 2

$l = 19.5$ cm, $w = 6.5$ cm

Prep 3

250 single cones and 200 double cones

EXAM PRACTICE 14.4 Simultaneous equations

	Multiple choice					
	%A	%B	%C	%D	%E	
Q1	2	93	3	2	1	2009
Q2	3	6	83	5	1	2010
Q3	8	7	4	78	2	2013
Q4	68	12	9	6	4	2011
Q5	5	9	11	68	7	2003
Q6	14	38	28	14	6	2003
Q7	7	10	10	65	7	2008
Q8	64	9	20	5	2	2011
Q9	6	10	11	11	62	2003
Q10	6	10	16	8	60	2002
Q11	8	59	15	9	9	2007
Q12	15	13	9	6	57	2005
Q13	3	12	14	14	56	2005
Q14	22	8	9	15	45	2004
	Written response					
Q15a–b	90					2009
Q15c–e	60					2009
Q16a–c	61					2003
Q16d–e	34					2003
Q17	30					2011

Question 6

$C = 15\,000 + 15x$ and $R = 25x$, so $P = 25x - (15\,000 + 15x)$
$= 25x - 15\,000 - 15x = 10x - 15\,000$

Question 11

Writing the 2 equations in intercept–slope form gives us $y = -1 + 2x$ and $y = -\frac{3}{2} + 2x$, so we can see they are two lines with the same slope, but different y-intercepts. This means they are parallel and have no intersection, so the equations have no solution.

Question 12

Let x = the cost of one ice-cream and y = the cost of one drink. $4x + 3y = 21.4$ and $5x + 2y = 20.8$. Using a CAS/calculator to solve simultaneously, we get $y = 3.4$, so the cost of one drink is \$3.40.

Question 13

Since the y value of the point needs to be -5, only C, D and E are options. Substitute the options into the other equation to see which one of the points lies on the line.

Question 14

Using the information, we get the two equations $4a + b = 320$ and $6a + b = 450$. Solving using a CAS/calculator, we get $a = 125$, so the boat hire for one hour costs \$125.

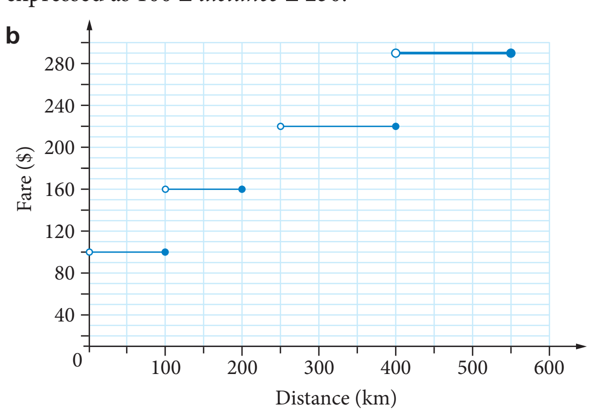
Question 15

Examination report

a 250 km

The most common incorrect answer was 249 km, suggesting confusion with a range for the *distance* expressed as $100 \le distance \le 250$.

b



Open and closed terminals were required as shown. The most common error showed the line ending at a distance of 525 km.

c \$30

d 360 km

$220 = 40 + 0.5d$

$\therefore d = 360$

Students must clearly identify their solution to a question. For this question, several answers consisted merely of the line $220 = 40 + 0.5 \times 360$.

e $a = 60$, $b = \frac{2}{5} = 0.4$

This involved solving two simultaneous equations selected from:

- $100 = a + 100b$
- $160 = a + 250b$
- $220 = a + 400b$.

Some students found only the equation $160 = a + 250b$ and then made up a value for a (usually $a = 0$) to find a value of b.

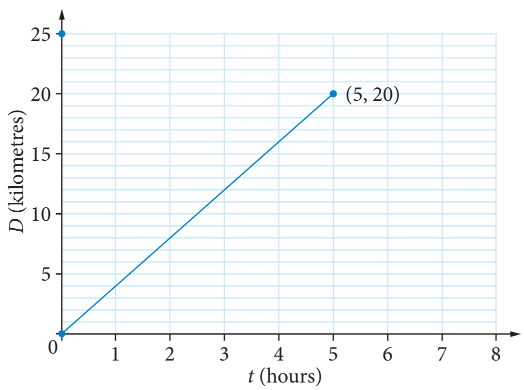
[VCAA 2009 2RGRQ1]

Question 16

Examination report

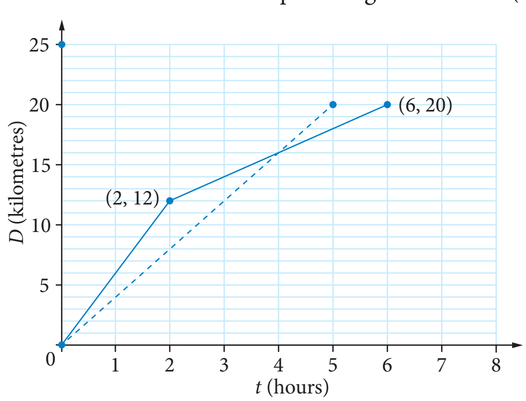
a 6 km

b



Most students got both marks, although many did not terminate their line at (5, 20). In this question, the terminal was disregarded. Some did not join points with a line (necessary for a time series) and lost 1 mark. One mark was given for a reasonable line with positive gradient from (0, 0).

c



Most students were able to draw a two-segment graph which had to end at (6, 20). Many stopped at $t = 5$, below the point for Malinda. Others went beyond (6, 20). The correct terminal point was required for full marks and the points had to be joined. A method mark was available for a reasonable two-segment graph from (0, 0).

d 4 hours

e $a = 6$, $b = 2$, $h = 8$, $d = 6$

One mark was awarded for any two correct values, a second mark for the third and then a third mark for the final correct value. Most got the value for a but fewer got the value for b, which had a variety of answers offered. The value of h caused most problems with $h = 12$ a common error. Similarly, $d = 5$ was a common error and usually reflected terminating their second portion of the graph for D_c at (5, 18).

[VCAA 2003 2RGRQ1]

Question 17

Examination report

a 2.3 km/h $\quad \frac{16}{7} = 2.285\ldots$

b $a = 1.5$, $b = 5.5$

Solve $10 = 3a + b \quad [1]$

$16 = 7a + b \quad [2]$

The correct value of a was common, but the value of b evaded many.

c 2 hours

Solve $d = -3t + 16 \quad [1]$

$d = 5t \quad [2]$

While not required by the question, very few students chose to add the line for Katie's hike on the graph as part of their working. Consequently, most equated Katie's equation with the equation $d = 1.5t + 5.5$ (from part **b**) for the wrong section of Michael's graph. This usually led to an incorrect answer of 2.3 hours.

d 3 km apart at $(-3t + 16) - 5t = 3$

$\therefore t = \frac{13}{8} = 1.625$

Also 3 km apart at $t = 3$

$\therefore$ can talk for $3 - 1.625 = 1.375 \text{ h} \approx 1.38 \text{ h}$

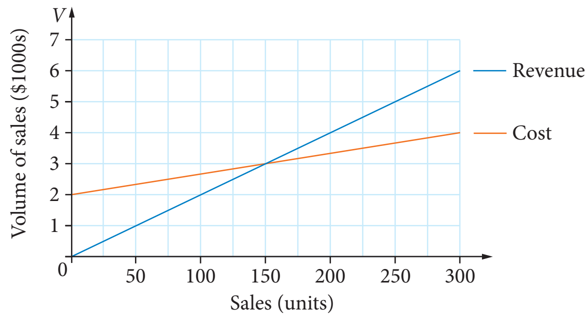
As most students had not drawn in Katie's graph, few were able to complete this question correctly.

[VCAA 2011 2RGRQ2]

EXAM PREP 14.5 Break-even analysis

Prep 1

a



b (150, 3000) **c** 150 **d** $3000

Prep 2

200 churros need to be sold to break even each day.

EXAM PRACTICE 14.5 Break-even analysis

	%A	%B	%C	%D	%E	
	Multiple choice					
Q1	8	75	7	7	2	2007
Q2	8	15	65	8	3	2009
Q3	64	14	7	7	7	2006
Q4	7	12	52	20	8	2013
	Written response					
Q5a–d	80					2010
Q5e	Not reported					2010
Q6a–c	78					2004
Q6d–g	55					2004
Q7	72					2012
Q8	70					2008
Q9	40					2012

Question 4

Find the break-even value for $0 \le n \le 200$:

Revenue: $V = 3n$ and Cost: $V = 500 + 1.30n$.

Using a CAS/calculator, $n = 294.12$, which is outside the range of values so it can't be the correct answer.

Find the break-even value for $n > 200$:

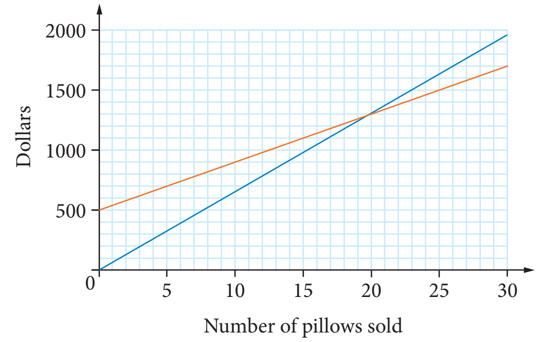
Revenue: $V = 4.5n - 300$ and Cost: $V = 500 + 1.30n$.

Using a CAS/calculator, $n = 250$. Since $n > 200$, this answer is correct. So 250 is the break-even number of cups that need to be sold each day.

Question 5

a $R = 65x$ **b** $1700

c



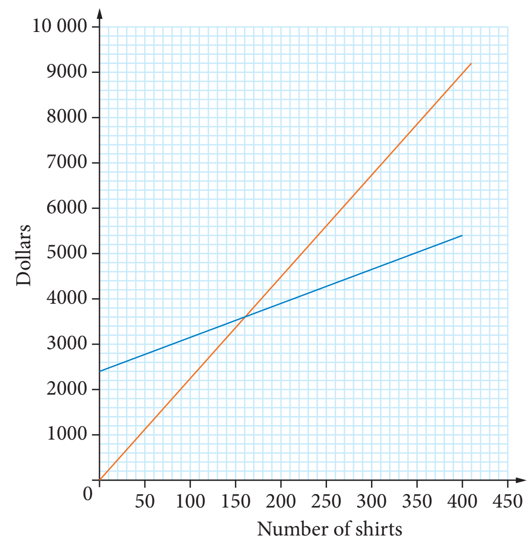
d 20 **e** $m = 40$

Question 6

Examination report

a $5600 **b** 75 shirts

c



d 160

This could have been calculated or read off the graph.

e $P = 15x - 2400$

The most common error was in simplifying $P = R - C$ where brackets were not applied to the cost function. This gave the incorrect answer of $P = 15x + 2400$. This was awarded one of the two available marks. The relationship of $P = R - C$ does not seem to be well understood by a number of students.

f $2775

This mark was awarded for a correct substitution in the student's equation for part **e**.

g $31.20

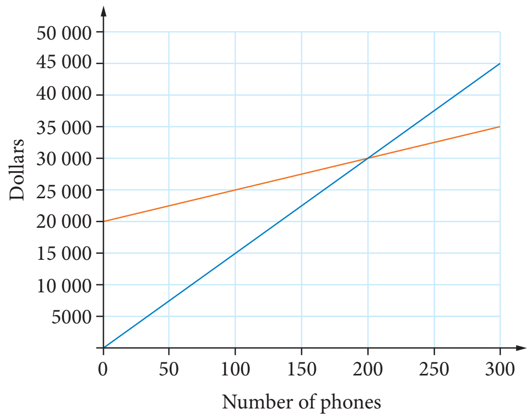
Several students wrote this as $31.2 which was awarded the mark, although teachers should address such notation.

[VCAA 2004 2RGRQ1&2]

Question 7

a **i** 50 **ii** $C = 50n + 20\,000$

b **i**



ii 360

c 200 phones

Question 8

a $R = 35x$ **b** $C = 50\,625 + 12.5x$

c **i** 2250 **ii** \$144 450

Question 9

Examination report

a 447 laptops

$320n + 125\,000 = 600n$

$\therefore$ break even at 446.428 laptops and so must sell more than 446 for a profit.

A common error involved not relating the answer to the context and rounding 446.428… down to 446, which is then on the loss side of the break-even point, rather than the profit side as required.

b \$682.50

New cost equation is $C = 370n + 125\,000$

For 400 laptops, cost = \$273 000

$\therefore$ Per laptop $= \dfrac{273000}{400} = 682.5$

[VCAA 2012 2RGRQ2]

EXAM PREP 14.6 Interpreting non-linear graphs

Prep 1

a 29°C **b** 21°C

c During the fourth hour.

d Between $7\frac{1}{2}$ and 8 hours.

e Between 5 and 6 hours, and between 7 and $7\frac{1}{2}$ hours.

f –0.75°C/h

g If we continue the curve from 12 to 14 hours, it gives a temperature under 5°C. Given the range of temperatures shown during the 12-hour period, this is highly unlikely to be reliable. This is an example of where extrapolation can be extremely misleading.

EXAM PRACTICE 14.6 Interpreting non-linear graphs

	Multiple choice					
	%A	%B	%C	%D	%E	
Q1	1	3	1	0	94	2007
Q2	3	1	2	1	93	2002
Q3	4	92	2	1	1	2009
Q4	2	12	65	19	2	2009
Q5	2	90	5	1	2	2008
Q6	5	7	6	75	7	2011
Q7	73	6	18	0	3	2005
Q8	8	16	72	2	2	2003
Q9	10	2	18	5	65	2006
Q10	1	7	29	49	14	2012
	Written response					
Q11	80					20

Question 10

A is false because the highest oil price over the ten-year period is \$85 not \$70.

B is false because oil price decreased for three, not two, of the ten years.

C is false. The first year is from 0–1, the second year is from 1–2, the sixth year is from 5-6 and has the greatest increase.

D is true because on average over the ten years the oil price change was $\dfrac{85-40}{10} = \$4.50$ per year.

E is false because the difference between the highest and lowest oil price during this ten-year period is \$85 – \$15 = \$70.

Question 11

a 110 beats/minute **b** 80 beats/minute

c **i** *maximum pulse rate* = 220 – *age*

ii Between 120 and 150 beats/minute.

EXAM PREP 14.7 Constructing non-linear graphs

Prep 1

a $y = -\dfrac{1}{2x}$ **b** $y = -\dfrac{2}{x^2}$ **c** $y = \dfrac{1}{3}x$

d $y = 2x^3$ **e** $y = -4x^2$

Prep 2

a $k = -11$ **b** $k = 6$ **c** $k = 45$

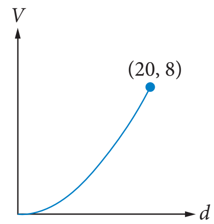
Prep 3

a $y = \frac{3}{2}x^2$ **b** $y = \frac{1}{4x}$

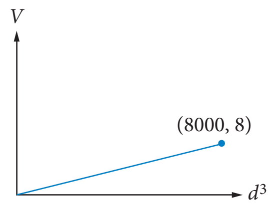
Prep 4

a 27 litres

b



c



d Slope $= \frac{8-0}{8000-0} = \frac{8}{8000} = \frac{1}{1000} = k$

e Slope $= \frac{1000-0}{1000000-0} = \frac{1000}{1000000} = \frac{1}{1000}$, so

(1 000 000, 1000) lies on the line.

(1 000 000, 1000) tells us that when $d^3 = 1\,000\,000$, $V = 1000$. So when $d = 100$, $V = 1000$.

This means that when the depth of the container is 100 cm, the volume of water in the container is 1000 litres.

EXAM PRACTICE 14.7 Constructing non-linear graphs

	Multiple choice					
	%A	%B	%C	%D	%E	
Q1	6	5	70	7	11	2008
Q2	63	10	15	8	4	2003
Q3	11	23	50	5	10	2008
Q4	46	14	20	6	12	2013
Q5	45	13	23	9	10	2010
Q6	40	25	9	10	15	2012
Q7	33	38	13	4	12	2005
Q8	34	33	10	10	12	2004
Q9	43	29	12	11	5	2002
Q10	8	18	26	21	27	2007
Q11	23	38	12	17	9	2005
	Written response					
Q12a	96					2009
Q12b–d	70					2009
Q12e	55					2009

Question 3

Slope $= \frac{1}{4}$, so $k = \frac{1}{4}$. Since the horizontal axis is x^2, the equation is $y = \frac{1}{4}x^2$.

Question 4

Slope $= \frac{25.7}{0.44} \approx 58.4$, so $k = 58.4$. Since the horizontal axis is $\frac{1}{d^2}$, the equation is $H = \frac{58.4}{d^2}$.

Question 5

Since $k = 3$, the slope of the linearised graph needs to be 3. Only option A has a slope of 3.

Question 6

Examination report

In this question, students were asked to find the relationship between two variables, y and x, given that a graph of y plotted against $\frac{1}{x}$ was a straight line passing through the point (2, 5). The correct answer was $y = \frac{5}{2x}$. A significant number of students chose the incorrect answer $y = \frac{5x}{2}$ by ignoring the fact that the graph was a plot of y against $\frac{1}{x}$, not x.

[VCAA 2012 1RGRQ7]

Question 7

Examination report

The key to answering this question correctly was to recognise that in the linear plot the coordinates (2, 4) represent the values of a^2 and b respectively, not a and b. This is known to be conceptually difficult for students, so the high percentage of incorrect responses was not unexpected.

[VCAA 2005 1RGRQ7]

Question 8

Examination report

The key to answering this question was to recognise that, in the linear plot, the coordinates (8, 1) represented the values of x^3 and y respectively, not x and y.

[VCAA 2004 1RGRQ9]

9780170371186

Question 9

Examination report

Most students failed to recognise that the coordinates on the correct line would be (x^2, y) and not (x, y).

[VCAA 2002 1RGRQ7]

Question 10

Examination report

The key to answering this question was to recognise that, in the linear plot, the coordinates (3, 1) represented the values of x^3 and y respectively, not of x and y.

- The equation of a linear graph of y plotted against x^3 as shown is $y = kx^3$.
- The coordinates on the graph tell us that when $x^3 = 3$, $y = 1$ so that $1 = k \times 3$ or $k = \frac{1}{3}$, so $y = \frac{1}{3}x^3$.
- When y is plotted against x (in the first quadrant), $y = \frac{1}{3}x^3$ is a cubic curve with the shape shown in all options A to E.
- The next step is to determine which of the coordinates shown lies on the curve $y = \frac{1}{3}x^3$. Inspection, or systematic testing, shows that when $x = 1$, $y = \frac{1}{3} \times 1^3 = \frac{1}{3}$.

Thus the graph shown in option C is correct.

[VCAA 2007 1RGRQ7]

Question 11

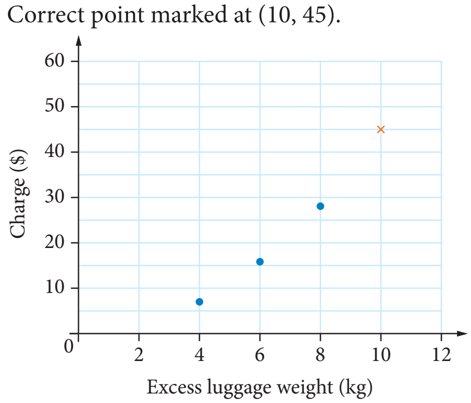
Examination report

The key to answering this question correctly was to recognise that the constant k first needed to be determined using the information given in the stem of the question.

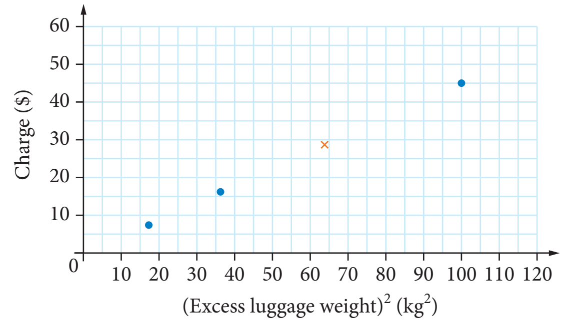
[VCAA 2005 1RGRQ9]

Question 12

a Correct point marked at (10, 45).



b 64 (on the table) and (64, 28.8) on the graph

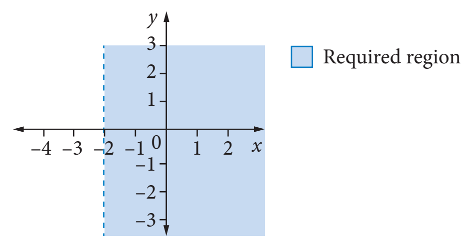


c $k = 0.45$ **d** \$64.80 **e** $m = 0.3$

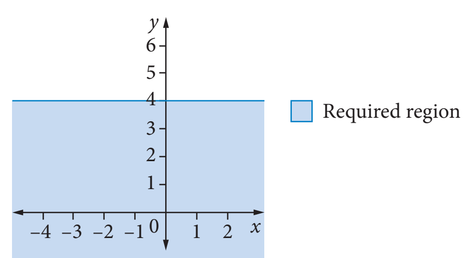
CHAPTER 15
Linear programming

EXAM PREP 15.1 Linear inequalities

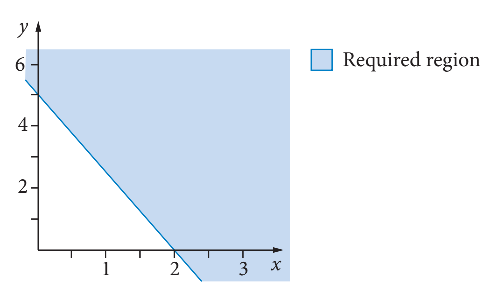
Prep 1



Prep 2



Prep 3



Prep 4

3(2) + 4(5) = 26, so (2, 5) satisfies the inequality.

Prep 5

$15x + 10y \le 200$

Prep 6

$x \ge 0, y \ge 0, x + y \le 20, y \le \frac{1}{2}x$

EXAM PRACTICE 15.1 Linear inequalities

	Multiple choice					
	%A	%B	%C	%D	%E	
Q1	2	3	14	7	73	2009
Q2	7	2	4	17	70	2010
Q3	2	43	15	6	34	2007
Q4	5	17	45	26	7	2002
	Written response					
Q5	85					2013

Question 3

Examination report

The most popular, but incorrect, choice made by 43 per cent of students was $y \le 2x$ (option B). In terms of the problem at hand, this inequality directly translates to 'the number of bottles of white wine is less than or equal to twice the number of bottles of red wine'. This is a true statement but is not the constraint specified.

[VCAA 2007 1RGRQ6]

Question 4

Examination report

The challenge for students in answering this question was that students had to realise that, while the inspection rates were given in items per hour, the quota for the minimum items to be inspected was given for an eight-hour day.

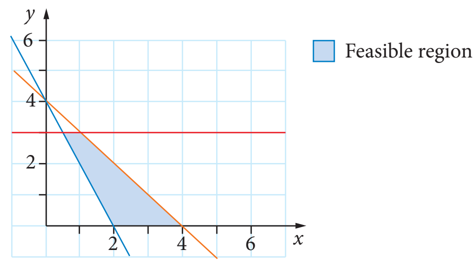
[VCAA 2002 1RGRQ9]

Question 5

7 hours

EXAM PREP 15.2 Systems of linear inequalities and feasible regions

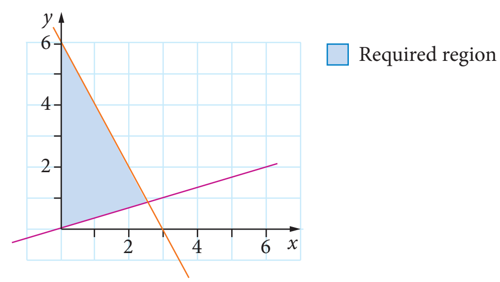
Prep 1



Prep 2

Only the point (2, 2).

Prep 3



Since the point (1, 2) lies within the feasible region, it satisfies the constraints.

EXAM PRACTICE 15.2 Systems of linear inequalities and feasible regions

	Multiple choice					
	%A	%B	%C	%D	%E	
Q1	9	13	10	5	64	2006
Q2	55	14	6	3	21	2012
Q3	11	10	11	54	13	2010
Q4	15	14	54	10	6	2006
Q5	10	10	12	51	16	2009
Q6	14	17	15	46	7	2008
Q7	38	27	9	12	14	2003
Q8	21	35	12	16	16	2004
Q9	8	10	11	33	38	2003
Q10	40	32	20	5	3	2002
	Written response					
Q11a–d	34					2007

9780170371186

Question 8

Examination report

It would appear that a lack of care in checking all of the inequalities that defined the shaded region was the significant contributor to students responding incorrectly.

[VCAA 2004 1RGRQ8]

Question 9

Examination report

The significant proportion of students who chose the incorrect response E, was indicative of an inability to translate correctly the statement 'Jensen works at least twice as many hours in the restaurant than he does at the nursery' by writing $2y \geq x$ rather than $y \geq 2x$.

[VCAA 2003 1RGRQ9]

Question 10

Examination report

The low success rate in this question seemed to be a product of students not taking sufficient care in noting the directions of the final inequality.

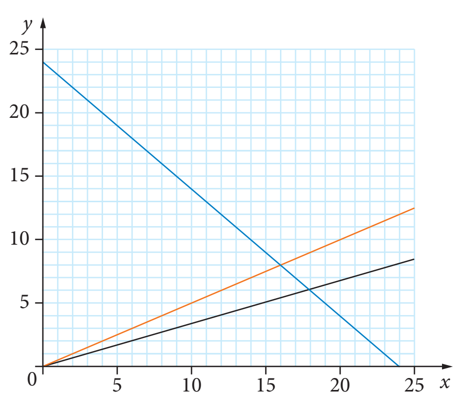
[VCAA 2002 1RGRQ8]

Question 11

Examination report

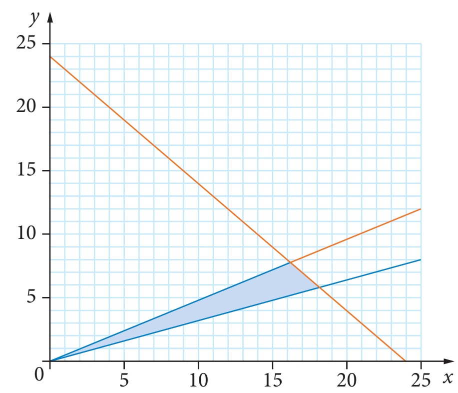
a The total number of hours of driving with petrol plus the hours driving with gas must be no more than 24.

b **i**



It was easy to see if this line had been drawn accurately or not since the correct line $y = \frac{1}{3}x$ went through the point (15, 5). Some students tried to plot the point for $x = 25$ rather than one that can be more accurately located, such as x equals any multiple of three. Accuracy of graph drawing is very important. A straight edge is expected for lines drawn in this module.

ii



It was difficult for the correct feasible region to be shaded if the line $y = \frac{1}{3}x$ had been drawn in the wrong location. If it had been drawn near the correct location and the feasible region had been shaded in the relevant area, the mark was awarded.

c Yes. Ten hours on gas and five hours on petrol lies on the line $y = \frac{1}{2}x$ and is within the feasible region. A reference had to be made to the need for five hours using petrol.

[VCAA 2007 2RGRQ3]

EXAM PREP 15.3 Objective functions

Prep 1

a Maximum = 16 **b** Minimum = 0

Prep 2

a $Z = \frac{8}{3} + 2 \times 4 = \frac{32}{3}$ **b** $Z = -2 + 2 = -3 + 3 = 0$

EXAM PRACTICE 15.3 Objective functions

	Multiple choice					
	%A	%B	%C	%D	%E	
Q1	6	3	11	73	7	2012
Q2	2	4	15	13	65	2005
Q3	6	9	13	64	8	2003
Q4	5	8	16	13	59	2011
Q5	10	19	59	6	6	2011
Q6	9	14	58	8	10	2004
Q7	8	9	19	55	9	2002
Q8	53	15	12	10	8	2007
Q9	44	12	32	10	2	2002
Q10	7	35	19	32	5	2013
Q11	17	6	17	19	41	2008
	Written response					
Q12	85					2013
Q13	52					2011
Q14	48					2010
Q15	40					2008

Question 10

Examination report

The key to answering this question was to recognise that, for the objective function to have its minimum value at both vertices M and N, all points on the line between M and N must also minimise the objective function.

Thus, the family of lines representing the objective function $Z = ax + by$ must be parallel to the constraint passing through both M and N. This constraint has a slope of -1. For the objective function to have a slope of -1, a must equal b, leading to option B.

[VCAA 2013 1RGRQ8]

Question 11

Examination report

The key to answering this question was to recognise that not all objective functions have their maximum value at the same vertex. It is true that the objective function, $Z = x + 4y$ (option D), chosen by 47 per cent of students, had a larger value at the vertex M (20, 50) than the objective function $Z = x + y$ (option A). However, the maximum value of the objective function $Z = x + 4y$ occurs at the vertex (0, 60), not at M(20, 50). A check of the values of each objective function at each of the vertices will show that, of the objective functions given, only $Z = x + y$ (option A) has its maximum value at the vertex M.

[VCAA 2008 1RGRQ9]

Question 12

4 hours each

Question 13

Examination report

a 236 g

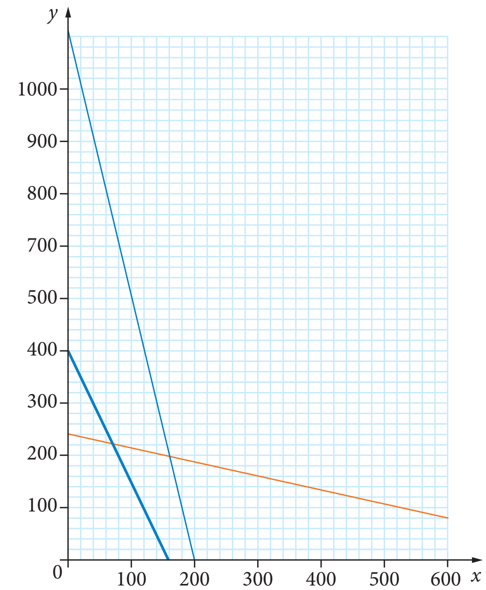
b $320 \times 0.2 + 0.04 \times r = 72$

$\therefore r = 200$ g

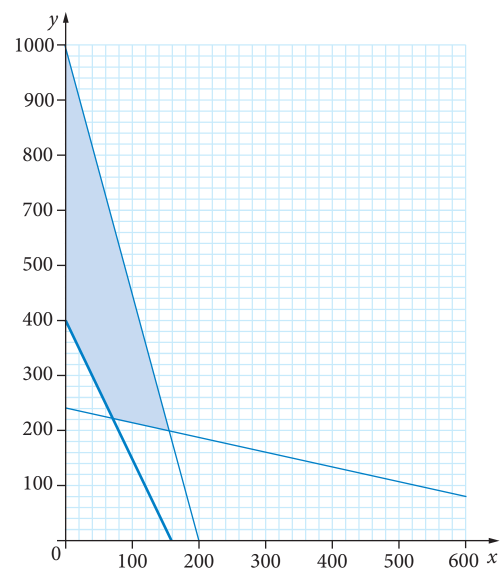
A common incorrect answer was 8 g, which is the number of grams of protein needed from the raisins. If the calculation method could be followed, this answer could have gained a method mark. However, this was not often the case as many students chose to write only their final answers.

c $0.1x + 0.04y \geq 16$

d **i**

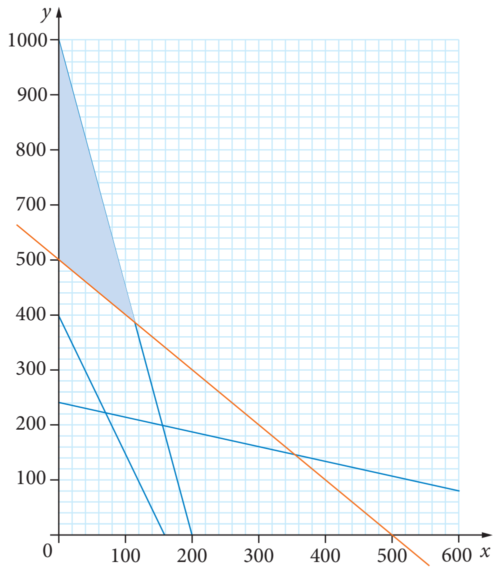


ii



e 1000 g

f 125 g



Draw in the new constraint $x + y \geq 500$.

This gives a new feasible region shaded above.

The line $x + y = 500$ crosses $0.2x + 0.04y = 40$ at (125, 375).

This gives the maximum $x = 125$.

This question was generally answered poorly, with the majority of students ignoring the new constraint $x + y \leq 500$. This usually produced an incorrect answer of 160 g.

[VCAA 2011 2RGRQ1]

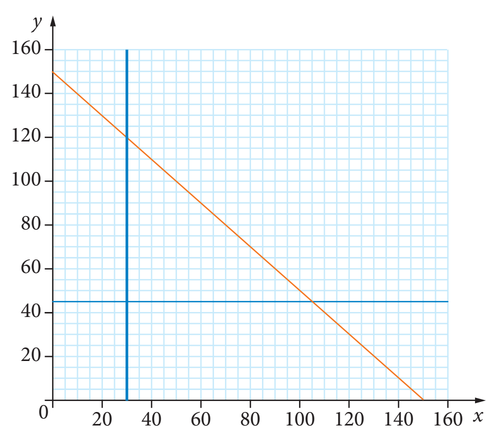
Question 14

Examination report

a The total number of pillows sold cannot exceed 150.

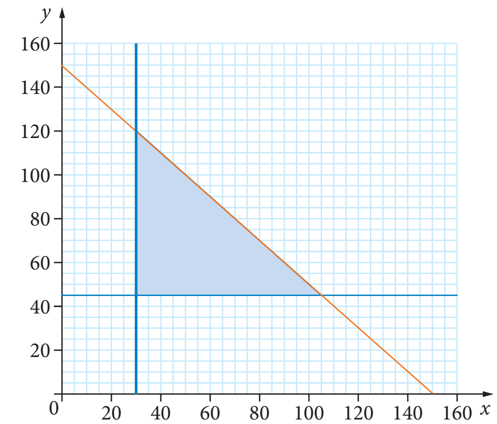
b $k = 45$

c i



If this was the only line drawn on the graph, then labelling was not necessary. However, some students also drew another line as working for a later part of the question. In this case, the line $x = 30$ had to be clearly identified.

ii



The instruction to shade the feasible region was included in the question; however, many students found the feasible region by shading the excluded region. Such answers could only be accepted if there is notation or a legend that clearly identifies the feasible region as blank or shaded in a different style.

d \$9075

$R = 65x + 50y$

$(105, 45) \Rightarrow \$9075$

$(30, 120) \Rightarrow \$7950$

e Let z = number of Snorestop pillows.

$\therefore x + y + z \leq 150$ and $z = 2x$

$\therefore x + y + 2x \leq 150$

$\therefore 3x + y \leq 150$

Many students attempted only written explanations and were not successful in presenting the mathematics that resulted in the equation $3x + y \leq 150$.

f $R = 65x + 50y + 55z$

But $z = 2x$

$\therefore R = 65x + 50y + 55 \times 2x$

$\therefore R = 175x + 50y$

g New intersection at (35, 45).

$R = 175 \times 35 + 50 \times 45 = \$8375$

A method mark was available for finding the new intersection point due to Inequality 4.

[VCAA 2010 2RGRQ3]

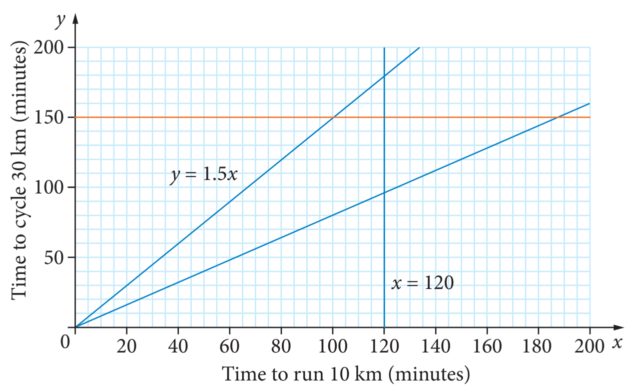
Question 15

Examination report

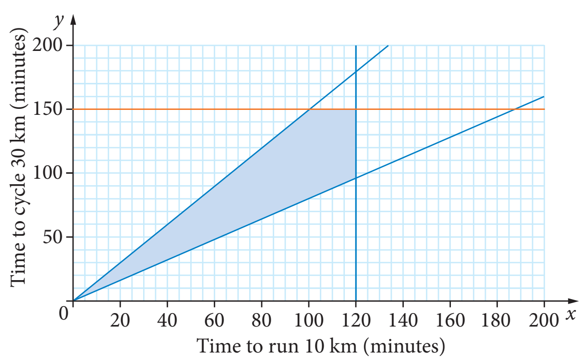
a Must run 10 km in no more than 120 minutes.

The correct answer included 120 minutes as an acceptable time to run 10 km. Therefore, an answer of 'run 10 km in less than 120 minutes' was not accepted.

b i



ii



Many students shaded the excluded section. This was acceptable as long as the correct feasible region was then identified, perhaps by a legend. This was essential as the question asked for the feasible region to be shaded, which can allowably be interpreted as 'identified.' Several students drew the line $x = 120$ as a horizontal line and a few shaded in above the line $y = 150$.

c Between 80 and 150 minutes.

d i 54 minutes

Maximum cycle time (on $y = 1.5x$ line)

$x + y \leq 90$ and $y = 1.5x$

$\therefore 2.5x \leq 90$

$\therefore x \leq 36$

$\therefore y \leq 54$

ii 50 minutes

Maximum run time (on $y = 0.8x$ line)

$x + y \leq 90$ and $y = 0.8x$

$\therefore 1.8x \leq 90$

$\therefore x \leq 50$

[VCAA 2008 2RGRQ3]

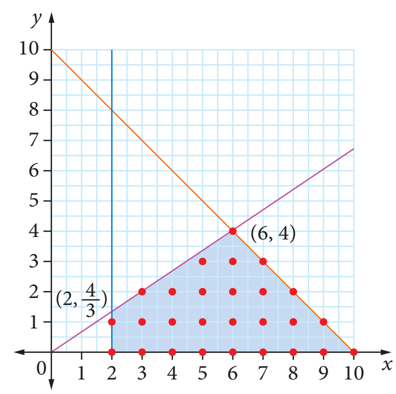
EXAM PREP 15.4 Integer solutions

Prep 1

a The total number of sports and commuter bikes made each week must be less than or equal to 10.

b $y \geq \frac{2}{3}x$

c



d $P = 20$, 10 sports bikes and no commuter bikes.

EXAM PRACTICE 15.4 Integer solutions

	Written response	
Q1	41	2012
Q2	34	2006
Q3	32	2013
Q4	10	2009

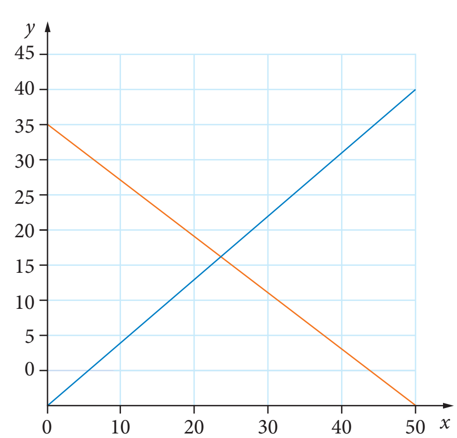
Question 1

Examination report

a Given the time it takes to repair each phone and laptop, the time available for repairing phones and laptops is not more than 1750 minutes each day.

b 8

c



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d 18

Intersection of lines at (23.3, 18.7). The maximum number of laptops must be fewer than 18.7. The most common incorrect answer was 35, which resulted from not addressing Constraint 4. Many students found $x \approx 18.7$ and rounded up to give 19 laptops. However, it can be seen that this value is above the intersection point and thus outside the feasible region.

e 37 phones

$35x + 50 \times (y = 9) = 1750$

$\therefore\ x = 37.14\ldots$

From the graph, the line $y = 9$ crosses both $y = \frac{4}{5}x$ and $35x + 50y \le 1750$, with the greatest value for x occurring at the second line crossing through $35x + 50y \le 1750$.

A common incorrect answer was 11 phones, found by using only the rule or graph of $y = \frac{4}{5}x$.

f **i** 24 phones and 18 laptops

One mark was awarded for each of the two numbers correctly labelled. By substitution, we can determine that the maximum profit occurs at the intersection of the two lines or at a point below this intersection.

Alternatively, the 'sliding line' method could be used.

$P = 60x + 100y$ (gradient = –0.6)

$35x + 50y = 1750$ (gradient = –0.7), which is steeper than –0.6.

Therefore, slide the profit line up to get the maximum value of y at 18 (see answer to part **d**).

Then substitute $y = 18$ into Constraint 3.

$\therefore\ 35x + 50 \times (y = 18) = 1750$

$\therefore\ x = 24.28\ldots$

When $y = 18$, $x = 23$ or $x = 24$, as both fit inside the feasible region. Take the larger value for x. Then, maximum profit is at (24, 18). The most common incorrect answer was 23 laptops and 19 phones.

ii $P = 60 \times 24 + 100 \times 18 = \$3240$

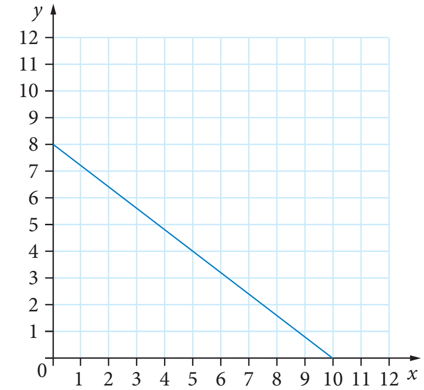
Very few students obtained this answer.

[VCAA 2012 2RGRQ3]

Question 2

Examination report

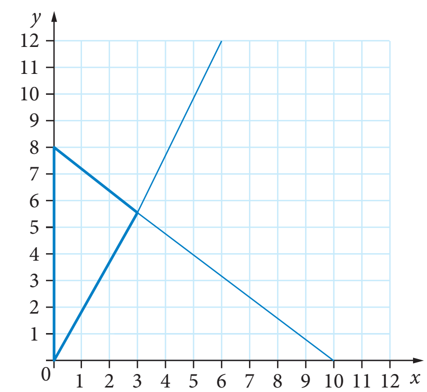
a



b $y \ge 2x$

The equation $2y \ge x$ was a common incorrect response.

c **i**



A method mark was available for correctly plotting an incorrect equation from part **b**. Although the question asked for the boundaries of the region to be highlighted, most students shaded the feasible region, which was accepted.

ii 2 dogs

A consequential mark was available here but many students were unable to interpret the x value from their graph when $y = 5$.

d $P = 40x + 30y$

e **i** 2 washes and 6 clips

The maximum value was not at a vertex of the feasible region. Incorrect answers often included decimal values.

ii \$260

[VCAA 2006 2RGRQ3]

Question 3

Examination report

a $a = 6$, $b = 4$

Unacceptable answers included $a \le 6$ and $b \le 4$. The values of a and b are constants within an inequality and only equals signs can apply to the constants in this question.

b $y \ge 2x$

At least two unpowered sites (y) were needed for every powered site (x). To write this as an inequality, it may have been useful to first explore the minimal relationship between x and y; that is, to look first at exactly two unpowered sites for every one powered site to build a table.

x	y
1	2
2	4
3	6

This leads to the equation $y = 2x$

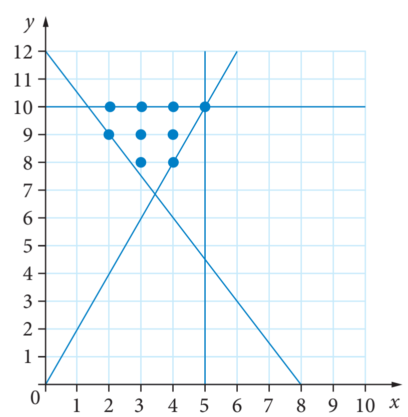
However, if at least two unpowered sites for every one powered site were needed, then the following table would also be suitable.

x	y
1	2
2	5
3	10

Here, the value of y is greater than, or equal to, twice the value of x. Hence, the inequality is $y \ge 2x$.

A common incorrect answer was $2y \ge x$ and, if tested against the second table, would have been found to be inappropriate.

c



Only whole numbers apply to the number of powered and unpowered sites. Therefore, it was expected that nine points within the feasible region, as above, were identified or listed. However, in this question for this year, a correctly shaded region was accepted.

d Either of the points (2, 9) or (3, 8) give the minimum number of powered and unpowered sites, 11.

e **i** \$390

The two points (2, 9) or (3, 8) can now be tested to find the lowest cost.

(3, 8) gives Cost $= 60 \times 3 + 30 \times 8 =$ \$420

(2, 9) gives Cost $= 60 \times 2 + 30 \times 9 =$ \$390

ii \$480

There are 24 boys and 24 girls. It doesn't matter which type each sex uses.

This requires at least $x = 4$ powered sites and $y = 6$ unpowered sites. However, the point (4, 6) is not inside the correct feasible region. It does not satisfy the correct inequality 4 from part **b** as at least two unpowered sites (y) are needed for every powered camp site (x).

$y \ge 2x$ is needed.

[VCAA 2013 2RGRQ4]

Question 4

Examination report

\$17 300

The objective function was $P = 1300x + 2100y$. Substituting (2, 7) gave the maximum profit of \$17 300. Once the feasible region was identified, most students found the point of intersection at (4.5, 5.5) and substituted these values into the objective function without considering the context of the question. The two variables, number of crew and number of flights, cannot have fractional values and so (4.5, 5.5) did not answer this question. Students had to find integer values within the correct feasible region. A method mark was available for applying any two applicable integer pairs other than (0, 0). Some students applied the point (0, 10), but this was not within the feasible region.

[VCAA 2009 2RGRQ4]

EXAMINATION Solutions

GRAPHS AND RELATIONS – EXAMINATION 1

1	2	3	4	5	6	7	8	9
B	E	D	B	A	D	B	B	C

GRAPHS AND RELATIONS – EXAMINATION 2

Question 1

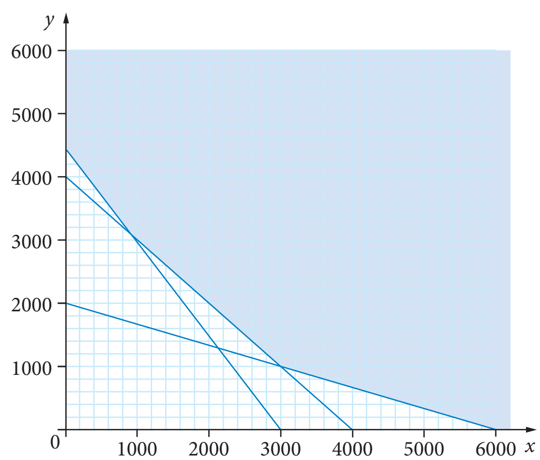
a 0.04 kg

b 25 kg

c $0.06x + 0.04y \geq 180$

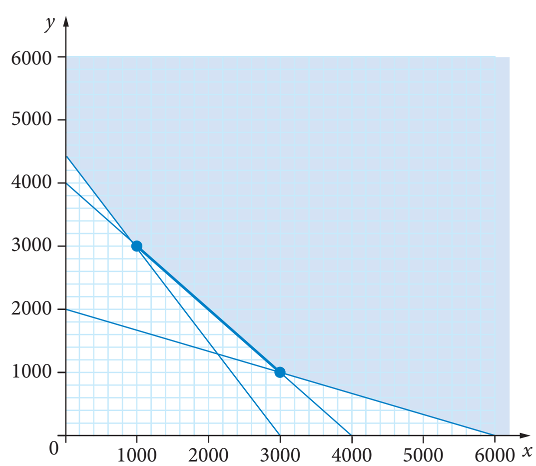
d **i** $0.02x + 0.06y = 120$ or $y = -\frac{1}{3}x + 2000$

ii



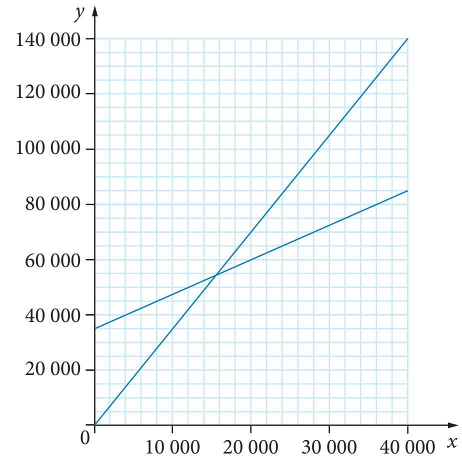
e **i** 4000 kg

ii



Question 2

a



b $9000

Question 3

a $34.80

b $10.8 + 4(10 - 2) = 42.8$

c 15.2 kg

GLOSSARY AND INDEX

activity (directed graphs) Interconnected steps represented by an edge in a directed graph and shown by a line with an arrow. (p. 190)
activity table A table showing the order and estimated time for a series of activities. (p. 190)
adjacency matrix A representation of how the vertices are connected by edges. (p. 137)
adjacent vertex Two vertices that are connected by one edge. (p. 134)
angle of depression The angle between the horizontal and a line looking down to an object. (p. 287)
angle of elevation The angle between the horizontal and a line looking up to an object. (p. 287)
arc length A fraction of the circumference of a circle. (p. 318)
assignment problem The process of finding the best way to match the elements in two groups, such as a group of workers to a set of tasks, to optimise a stated objective such as minimising cost, distance or time. (p. 218)
asymptote A line that a graph approaches but never touches. (p. 410)

base The surface a solid stands on. (p. 262)
bearings Angles that show the location of one point from another point. (p. 304)
binary matrix A matrix where every element is 0 or 1. (p. 3)
bipartite graph A graph whose set of vertices can be split into two distinct groups, where vertices cannot connect other vertices from the same group. (p. 218)
break-even point The point where the cost equals the revenue, which can be found by solving the cost and revenue equations simultaneously. (p. 396)
bridge An edge that keeps a graph connected. (p. 186)

capacity of a cut The total of all the flow capacities passing across a cut in the direction from source to sink. (p. 231)
circuit A walk with no repeated edges that starts and finishes at the same vertex. (p. 186)
circumference The perimeter of a circle. (p. 257)
column matrix A matrix that has just one column. (p. 3)
communication diagram A diagram showing one-way or two-way arrows between points indicating when communication occurs. (p. 71)
communication matrix A square binary matrix where communication is indicated by a 1 and non-communication is indicated by a 0. (p. 71)
complete graph A graph where each vertex is connected to every other vertex in the graph. (p. 152)
compass rose A diagram indicating compass points. (p. 304)
connected graph A graph where there is a path between each pair of vertices. (p. 146)
constant of proportionality The constant value of the ratio between two proportional quantities, such as k in $y = kx^n$, where $n = 1, 2, 3, -1, -2$. (p. 409)
constraints Restrictions placed on the possible values of x and y that are written as inequalities. (p. 436)
Coordinated Universal Time (UTC) The official name for the time standard measured from the Greenwich meridian. (p. 336)
coordinates A set of values showing an exact position. (p. 326)
corner points The intersection points of the lines that are the boundaries of the feasible region. (p. 451)
cosine rule A formula that is used to find sides and angles in non-right-angled triangles. (p. 298)
crashing The process of speeding up a project by employing more resources for critical activities, enabling them to be completed more quickly. (p. 204)
critical path The longest time path between the start and finish, which determines the project completion time. (p. 202)
critical path analysis A step-by-step project management technique that is used to examine every activity in a project and how each affects the project completion time. (p. 199)

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culling The process of deleting something from a transition. (p. 113)
cut A line in a network that stops all flow from source to sink. (p. 231)
cycle A walk with no repeated vertices that starts and finishes at the same vertex. (p. 186)

degree (**order of a vertex**) The number of edges connected to a vertex. (p. 134)
dependent system of equations Simultaneous equations with an infinite number of solutions. (p. 61)
determinant A number that plays an important role in finding the inverse of a matrix. (p. 38)
diagonal matrix A square matrix where the only non-zero elements are in the leading diagonal. (p. 3)
digraph *See* directed graph.
Dijkstra's algorithm A method used to find the shortest path between two vertices. (p. 163)
direct dominance A one-step dominance where *A* dominates *B*. (p. 77)
directed graph (**digraph**) A set of vertices connected by edges, where the edges have a direction. (p. 190)
dominance diagram A communication diagram where all the arrows are one-way. (p. 77)
dominance matrix A square binary matrix where dominance is indicated by a 1 and other elements are 0. (p. 77)
dummy activity An activity of zero time, shown as a directed edge with a broken line added to a network to ensure that no two vertices are connected by multiple edges or to maintain precedence structure. (p. 193)

earliest start time (EST) The earliest time it is possible to start the activity. (p. 199)
edge The lines connecting the vertices of a graph. (p. 134)
element A value in a matrix. (p. 2)
Equator The great circle that is perpendicular to the axis of the Earth, has a latitude of 0°, and is the reference for all parallels of latitude. (p. 325)
equilibrium state matrix (**steady state matrix**) A column matrix representing the final state of a transition. (p. 102)
Euler's formula A formula used to test whether a graph is planar:

$$\text{vertices} + \text{faces} - \text{edges} = 2$$

(p. 146)
Eulerian circuit A walk which includes *every edge* once only and starts and finishes at the *same* vertex. (p. 160)
Eulerian trail A walk which includes *every edge* once only, and starts and finishes at *different* vertices. (p. 160)
explanatory variable A variable that we expect to affect another variable. (p. 362)
extrapolation A prediction made outside the original data range. (p. 364)

face A region bound by edges. The region surrounding a planar graph is also considered a face. (p. 134)
feasible region A region or area obtained by graphing a set of inequalities. (p. 444)
float time The maximum time an activity can be extended or postponed without affecting the project completion time. Activities on the critical path all have float times of zero. (p. 204)

GMT The abbreviation for Greenwich Mean Time. (p. 336)
gradient *See* slope.
graph (**network diagram**) A diagram that consists of a set of points, called vertices, that are joined by a set of lines, called edges. (p. 134)
great circle The circle on a sphere that has a centre coinciding with the centre of a sphere. The shortest distance between two points on the surface of a sphere occurs on a great circle. (p. 322)
Greenwich Mean Time (GMT) The time zone used as a reference for other time zones. (p. 336)
Greenwich meridian The prime meridian of longitude, which has a longitude of 0° and is the reference point for all other meridians of longitude. (p. 325)

Hamiltonian cycle A walk that includes *every vertex* in a graph once only, and starts and finishes at the *same* vertex. (p. 161)

Hamiltonian path A walk that includes *every vertex* in a graph once only, and starts and finishes at *different* vertices. (p. 161)

Heron's formula A rule for calculating the area of a triangle from the lengths of its three sides. (p. 258)

Hungarian algorithm An extension to row and column reduction when this does *not* produce an optimum solution. (p. 222)

hypotenuse The longest side of a right-angled triangle. (p. 256)

identity matrix A square matrix where all the elements in the leading diagonal are 1 and the other elements are 0. (p. 36)

immediate predecessor An activity that must be completed before another activity can commence. (p. 190)

included angle The angle between two sides of a triangle. (p. 258)

inconsistent system of equations Simultaneous equations with no solution. (p. 61)

indirect dominance A two-step dominance of A over B, where A dominates C, which dominates B. (p. 77)

inequality A mathematical statement showing that two expressions are not equal using the symbol $<, \leq, >, \geq$ or $\neq$. (p. 428)

inflow capacity The total flow capacity entering a vertex. (p. 230)

initial state matrix A column matrix representing the starting state of a transition. (p. 97)

integers The set of positive whole numbers, negative whole numbers and zero. (p. 456)

integer solution A solution to a linear programming problem where both numbers in the ordered pairs are whole numbers. (p. 466)

intercept–slope form A linear equation written in the form $y = a + bx$, where a is the y-intercept of the line and b is the slope of the line. (p. 352)

International Date Line The line 180° east or 180° west of the Greenwich meridian, where crossing in the easterly direction means adding a day and crossing in the westerly direction means subtracting a day. (p. 337)

interpolation A prediction made within the original data range. (p. 364)

inverse (of a matrix) The matrix you multiply another matrix by to get the identity matrix. (p. 37)

isolated vertex A vertex that has no edges connected to it. (p. 134)

latest start time (LST) The latest time it is possible to start an activity without affecting the project completion time. (p. 201)

leading diagonal The diagonal in a matrix running from the upper left to the lower right. (p. 3)

line segment linear graph (**piecewise linear graph**) A combination of two or more straight line segments with different slopes, where each line segment starts from where the previous line segment finishes. (p. 372)

linear equation An equation where the variables involved are raised to the power 1. (p. 350)

linear modelling Using a linear equation or rule to represent patterns in nature and real-life situations. (p. 362)

linear programming The process of finding the maximum and minimum values of a quantity for a region defined by linear inequalities. (p. 428)

linearisation The process of putting something into a straight-line form. (p. 413)

loop A vertex that has an edge connected to itself. (p. 134)

lower triangular matrix A square matrix where all the elements above the leading diagonal are 0. (p. 3)

major segment The larger segment of a circle. (p. 320)

matrices Plural of matrix.

matrix A rectangular arrangement of numbers organised into rows and columns, usually presented in square brackets. (p. 2)

matrix multiplication The multiplication of a matrix by another matrix. (p. 21)

matrix order equation An equation involving the orders of matrices. (p. 23)

maximum flow The capacity of the minimum cut. (p. 230)

meridian of longitude Any great circle line on the Earth's surface that passes through the North Pole and the South Pole. (p. 325)
minimum spanning tree A spanning tree where the total weight of all the edges is a minimum. (p. 177)
minor segment The smaller segment of a circle. (p. 320)
multiple edges Two vertices that are connected by more than one edge. (p. 193)
multiplicative identity The number which when multiplied by any number leaves the original unchanged. (p. 36)
multiplicative inverse (of a matrix) *See* inverse (of a matrix).
multiplicative inverse (of a number) The number you multiply another number by to get 1. (p. 37)

network A group of interconnected people or things. (p. 134)
network diagram *See* graph.
non-included angle An angle that isn't between two given sides of a triangle. (p. 297)

objective function A function written in terms of the variables x and y, which may refer to cost, profit or some other quantity that is to be maximised or minimised. (p. 451)
one-step communication A direct communication between A and B. (p. 74)
one-way communication A communication where A can communicate with B, but B can't communicate with A. (p. 71)
order (of a matrix) The number of rows and columns in a matrix. (p. 2)
order of a vertex *See* degree.
outflow capacity The total flow capacity leaving the vertex. (p. 230)

parallels of latitude Small circles parallel to the Equator that are measured in degrees north or south of the Equator. (p. 325)
path A walk where there are no repeated vertices. (p. 186)
perimeter The distance around a two-dimensional shape. (p. 257)
permutation matrix A square matrix where every row and every column has exactly one 1, with 0s everywhere else (p. 3)
piecewise linear graph *See* line segment linear graph.
planar graph A connected graph which can be drawn without any edges crossing over. (p. 146)
Prim's algorithm A method used to determine the minimum spanning tree. (p. 177)
prism A solid with with two identical ends and flat sides. (p. 262)
Pythagoras' theorem A rule for calculating the third side of a right-angled triangle given the length of the other two sides. (p. 256)

ratio The relationship between two numbers indicating how much of one quantity there is compared to the second quantity. (p. 272)
reachability The ability to get from one vertex to another vertex in a directed graph. (p. 192)
reachable A vertex is reachable if it is connected to another vertex by one or more edges. (p. 146)
response variable A variable that we expect to be affected or changed by another variable. (p. 362)
restocking The process of adding something to a transition. (p. 113)
row matrix A matrix that has just one row. (p. 3)

scalar A number that is not in a matrix. (p. 16)
scalar multiplication The multiplication of every element in a matrix by the same number. (p. 16)
scale A ratio used to compare two similar objects. (p. 272)
scale factor The ratio of any two matching lengths in two similar figures. (p. 272)
sector area The fraction $\frac{\theta}{360}$ of the circle area. (p. 319)
segment The area bound by a chord and the arc subtended by the chord. (p. 320)
self-communication links Communication links where the sender and receiver are the same. (p. 75)
similar figures Figures that are exactly the same shape but not necessarily the same size. (p. 272)
simple graph A graph that contains no loops or multiple edges. (p. 152)
simultaneous equations Two or more linear equations, each with two or more variables, which are being solved to find values that are common solutions to all the equations. (p. 386)

sine rule A formula that is used to find sides and angles in non-right-angled triangles. (p. 294)
sink The final vertex of a flow network. (p. 230)
sliding line method A method used to find the maximum or minimum value of an objective function in a linear programming problem, where the initial line has the same gradient as the objective function and is moved parallel to the initial line until a maximum or minimum value is obtained inside the feasible region. (p. 456)
slope (gradient) The measure of the steepness of a line. (p. 352)
small circle A circle on a sphere whose centre *doesn't* coincide with the centre of a sphere. (p. 322)
source The first vertex of a flow network. (p. 230)
spanning tree A tree subgraph that includes all the vertices of the original graph. (p. 176)
square matrix A matrix that has the same number of rows as columns. (p. 3)
state A condition or a location at a point in time. (p. 86)
state matrix A column matrix representing a state at a point in time of a transition. (p. 97)
steady state matrix *See* equilibrium state matrix.
step graph A graph made up of horizontal line segments that look like steps, where the ends of each line segment can be open or closed. (p. 375)
subgraph A graph that exists within another graph and contains only vertices and edges from the original graph. (p. 152)
summing matrices Row and column matrices consisting entirely of 1s which sum elements when multiplied with other matrices. (p. 24)
symmetric matrix A square matrix where the elements are symmetric with respect to the leading diagonal. (p. 3)

three-figure bearings (**true bearings**) Three-digit angular directions measured in degrees in a clockwise direction from north. (p. 304)
three-step communication A communication between *A* and *B*, where *A* can communicate with *D*, which can then communicate with *C*, which can then communicate with *B*. (p. 74)
time zones Regions that observe the same time in reference to Greenwich mean time (GMT), the time observed in the zone with the Greenwich meridian as its centre. (p. 336)
total surface area The sum of all the areas on the surface of a solid object. (p. 260)
trail A walk where there are no repeated edges. (p. 186)
transition A change from one state to another. (p. 86)
transition diagram A diagram that shows transitions from one state to another using arrows with corresponding percentages. (p. 86)
transition matrix A square matrix that shows a change from one state to another, where the change follows the same rules each time. (p. 86)
transpose (of a matrix) A new matrix formed by turning all the rows of the original matrix into columns and vice-versa. (p. 7)
tree A connected graph with no loops, multiple edges, or cycles. (p. 176)
trigonometry A branch of mathematics that studies the relationship between the lengths of sides and the sizes of angles in a triangle. (p. 283)
true bearings *See* three-figure bearings.
two-step communication A communication between *A* and *B*, where *A* can communicate with *C*, which can then communicate with *B*. (p. 74)
two-way communication A communication where *A* can communicate with *B*, and *B* can communicate with *A*. (p. 71)

undirected graph A graph where there are no directions associated with the edges. (p. 134)
upper triangular matrix A square matrix where all the elements below the leading diagonal are 0. (p. 3)
UTC The abbreviation for Coordinated Universal Time. (p. 336)

variable A letter or symbol used to represent a quantity that can have many different values in a particular situation. (p. 350)
vertex The points on a graph. (p. 134)
vertices Plural of vertex.

walk Any sequence of vertices of a graph along edges which may include repeated vertices or repeated edges. (p. 186)

weighted graph A graph with quantities on its edges representing distance, time, etc. (p. 163)

***x*-intercept** The x value of a point where a line crosses the x-axis. (p. 350)

***y*-intercept** The y value of a point where the line crosses the y-axis. (p. 350)

zero matrix A matrix where all the elements are 0. (p. 3)

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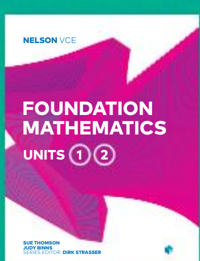

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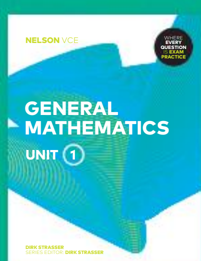
Foundation Mathematics Units 1 & 2



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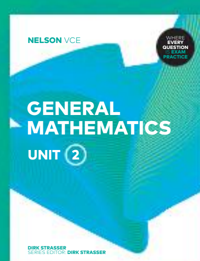

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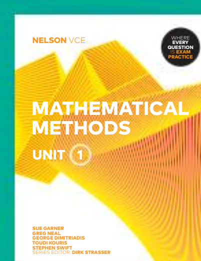

9780170370943
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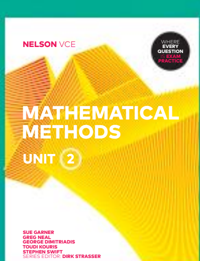
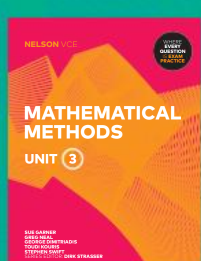


9780170371025
Mathematical Methods Unit 2



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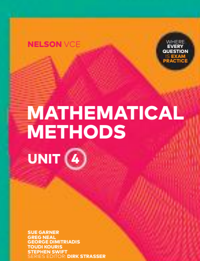
Mathematical Methods Unit 3



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