CAMBRIDGE

SPECIALIST MATHEMATICS

UNITS 1 & 2

SECOND EDITION

CAMBRIDGE SENIOR MATHEMATICS FOR QUEENSLAND

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INCLUDES INTERACTIVE TEXTBOOK POWERED BY CAMBRIDGE HOTMATHS

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Online appendices accessed through the Interactive Textbook or PDF Textbook

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About the lead author and consultants

About the lead author

Michael Evans was a consultant to ACARA on the writing of the Australian Curriculum from which the Queensland syllabus has evolved. He is a consultant with the Australian Mathematical Sciences Institute, and is coordinating author of the ICE-EM 7–10 series also published by Cambridge.

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Introduction and overview

Cambridge Senior Mathematics for Queensland Specialist Mathematics Units 1 & 2 provides complete and aligned coverage of the QCAA syllabus to be implemented in Year 11 from 2025. Its four components – the print book, downloadable PDF textbook, online Interactive Textbook (ITB) and Online Teaching Resource (OTS) – contain a huge range of resources, including worked solutions and revision of Year 10 material, available to schools in a single package at one convenient price (the OTS is included with class adoptions, conditions apply). There are no extra subscriptions or per-student charges to pay.

New features in the second edition:

- **Learning intentions** complemented by a Skills Checklist at the end of each chapter that allows you to check your understanding and tick off your achievements,
- **Technology-free and technology-active** short-response and multiple-choice questions are clearly labelled in Chapter review sections and Unit Revision chapters.
- **The problem-solving and modelling task (Appendix C):** There are three appendices at the end of the book. Appendix C is written by consultant Joel Speranza, providing advice on how to complete problem-solving and modelling tasks (PSMTs). This is supported by video resources accessed through QR codes and in the Interactive Textbook.
  Trigonometric functions are an important component of many topics in both Specialist Mathematics Units 1 & 2 and Specialist Mathematics Units 3 & 4. Appendix A provides material which enables the reader the opportunity to review trigonometric functions.
  Appendix B provides a preparation for several topics in Specialist Mathematics Units 3 & 4.

The Second Edition also features significantly revised and updated material from the first edition, including:

Additional material: Material that is additional to the syllabus is provided in Chapters 1 to 5, Appendix A and Appendix B and several other sections of the book. This has been done to prepare the students for this course, giving sufficient background, and to provide the opportunity for students to have a thorough preparation for Specialist Mathematics Units 3 & 4.

Degree of difficulty classification of questions: in the exercises, questions are classified as simple familiar SF, complex familiar CF, or complex unfamiliar CU questions and are indicated by a strip along the margin. The revision chapters described below also contain model questions for each of these categories, and tests are also provided in the teacher resources, made up of such categorised model questions.

Three revision chapters of material covered in the course: The first two of these chapters each cover an entire unit, and the last revision chapter contains questions revising the whole book. Each is divided into technology-free and technology-active short-response and multiple-choice questions, followed by problem-solving and modelling questions and investigations.

The problem-solving and modelling questions are multi-part questions where the students undertake modelling or problem solving with an indication of the path to be taken. These can be used in introducing the techniques of problem-solving and modelling. The investigation section is less structured and the student will have to make decisions about the pathway they will follow. Answers for them are not given in the textbook, but model solutions are in the Online Teaching Suite.

Calculator guidance: Throughout the book there is guidance for the use of the TI-Nspire CX non-CAS and the Casio fx-CG20AU and fx-CG50AU graphics calculators for the solution of problems. Guidance on the TI-84Plus CE is included in the Interactive Textbook, accessed via icons next to the TI-Nspire boxes. There are also online guides for the general use of each of these calculators.

Assessment: Examination practice questions and assessment tasks are provided in the revision chapters and the Online Teaching Suite. Check the updates there for developments as more guidance is published closer to implementation of the new assessment.

Interactive Textbook (ITB)

The Interactive Textbook (ITB) is an online HTML version of the print textbook powered by the HOTmaths platform, included with the print book or available as a separate purchase.

Updated and revised for the new syllabus, the Interactive Textbook includes:

- **Video demonstrations** of all worked examples
- **Quick quizzes** containing auto-marked multiple-choice questions have been thoroughly updated and revised, enabling students to check their understanding.
- A **success criteria** checklist at the end of each chapter with linked questions and examples available for download
- Comprehensive **worked solutions** for all questions are provided in the Interactive Textbook as an option that teacher can choose to enable for their students.
- **Downloadable skillsheets** can be used for homework or in class to focus on a single skill or small set of related skills.
- **Definitions** pop up for key terms in the text, and are also provided in a dictionary.

The Online Teaching Suite (OTS)

The Online Teaching Suite is automatically enabled with a teacher account and is integrated with the teacher's copy of the Interactive Textbook. All the teacher resources are in one place for easy access. The features include:

- A teacher's view of a student's working and self-assessment which enables them to modify the student's self-assessed marks, and respond where students flag that they had difficulty.

- The **task manager** allowing to direct students on a custom activity sequence based on their scores in measurable activities
- Quickly create customised tests from a bank of multiple-choice questions using the **test generator**. Tests are auto-marked in the Interactive Textbook or can be printed and used for homework or assessment practice.
- An expanded and revised suite of **chapter tests** and **assignments**
- Editable **curriculum grids** and **teaching programs**.
- A brand-new **Exam Generator**, allowing the creation of customised printable and online trial exams (see below for more).

More about the Exam Generator

The Online Teaching Suite, available from 2025, will include a comprehensive bank of QCAA exam questions, augmented by exam-style questions written by experts, to allow teachers to create custom trial exams.

Custom exams can model end-of-year exams, or target specific topics or types of questions that students may be having difficulty with.

Features include:

- Filtering by question-type, topic and degree of difficulty
- Searchable by key words
- Answers provided to teachers
- Worked solutions for all questions
- QCAA marking scheme
- Multiple-choice exams can be auto-marked if completed online, with filterable reports
- All custom exams can be printed and completed under exam-like conditions or used as revision.

Acknowledgements

The authors and publishers wish to thank **Peter Flynn** and **Jacinta Foley** for their expertise and timeliness in preparing the calculator screenshots, instructions and how-to guides for this edition.

Thanks to **Joel Speranza** for the creation of the PSMT appendix and supplementary supporting videos.

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1 Algebra

Chapter contents

This chapter is provided for review of previous years.

Algebra is the language of mathematics. Algebra helps us to state ideas more simply. It also enables us to make general statements about mathematics, and to solve problems that would be difficult to solve otherwise.

We know by basic arithmetic that $9 \times 7 + 2 \times 7 = 11 \times 7$. We could replace the number 7 in this statement by any other number we like, and so we could write down infinitely many such statements. These can all be captured by the algebraic statement $9x + 2x = 11x$, for any number x. Thus algebra enables us to write down general statements.

Formulas enable mathematical ideas to be stated clearly and concisely. An example is the well-known formula for compound interest. Suppose that an initial amount P is invested at an interest rate R, with interest compounded annually. Then the amount, A_n, that the investment is worth after n years is given by $A_n = P(1 + R)^n$.

In this chapter we review some of the techniques which you have met in previous years. Algebra plays a central role in Specialist Mathematics at Years 11 and 12. It is important that you become fluent with the techniques introduced in this chapter and in Chapter 4.

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1A Indices

Learning intentions

- To be able to use the index laws to simplify expressions.

Review of index laws

For all non-zero real numbers a and b and all integers m and n:

- $a^m \times a^n = a^{m+n}$
- $a^m \div a^n = a^{m-n}$
- $(a^m)^n = a^{mn}$
- $(ab)^n = a^n b^n$
- $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$
- $a^{-n} = \frac{1}{a^n}$
- $\frac{1}{a^{-n}} = a^n$
- $a^0 = 1$

Rational indices

If a is a positive real number and n is a natural number, then $a^{\frac{1}{n}}$ is defined to be the nth root of a. That is, $a^{\frac{1}{n}}$ is the positive number whose nth power is a. For example: $9^{\frac{1}{2}} = 3$.

If n is odd, then we can define $a^{\frac{1}{n}}$ when a is negative. If a is negative and n is odd, define $a^{\frac{1}{n}}$ to be the number whose nth power is a. For example: $(-8)^{\frac{1}{3}} = -2$.

In both cases we can write:

$$a^{\frac{1}{n}} = \sqrt[n]{a} \quad \text{with} \quad \left(a^{\frac{1}{n}}\right)^n = a$$

In general, the expression a^x can be defined for rational indices, i.e. when $x = \frac{m}{n}$, where m and n are integers, by defining

$$a^{\frac{m}{n}} = \left(a^{\frac{1}{n}}\right)^m$$

To employ this definition, we will always first write the fractional power in simplest form.

Note: The index laws hold for rational indices m and n whenever both sides of the equation are defined (for example, if a and b are positive real numbers).

Example 1

Simplify each of the following:

a $x^2 \times x^3$ **b** $\frac{x^4}{x^2}$ **c** $x^{\frac{1}{2}} \div x^{\frac{4}{5}}$ **d** $(x^3)^{\frac{1}{2}}$

Solution	Explanation
a $x^2 \times x^3 = x^{2+3} = x^5$	$a^m \times a^n = a^{m+n}$
b $\frac{x^4}{x^2} = x^{4-2} = x^2$	$\frac{a^m}{a^n} = a^{m-n}$
c $x^{\frac{1}{2}} \div x^{\frac{4}{5}} = x^{\frac{1}{2}-\frac{4}{5}} = x^{-\frac{3}{10}}$	$\frac{a^m}{a^n} = a^{m-n}$
d $(x^3)^{\frac{1}{2}} = x^{\frac{3}{2}}$	$(a^m)^n = a^{mn}$

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Example 2

Evaluate:

a $125^{\frac{2}{3}}$ **b** $\left(\frac{1000}{27}\right)^{\frac{2}{3}}$

Solution

a $125^{\frac{2}{3}} = \left(125^{\frac{1}{3}}\right)^2 = 5^2 = 25$

b $\left(\frac{1000}{27}\right)^{\frac{2}{3}} = \left(\left(\frac{1000}{27}\right)^{\frac{1}{3}}\right)^2 = \left(\frac{10}{3}\right)^2 = \frac{100}{9}$

Explanation

$125^{\frac{1}{3}} = \sqrt[3]{125} = 5$

$\left(\frac{1000}{27}\right)^{\frac{1}{3}} = \sqrt[3]{\frac{1000}{27}} = \frac{10}{3}$

Example 3

Simplify $\dfrac{\sqrt[4]{x^2y^3}}{x^{\frac{1}{2}}y^{\frac{2}{3}}}$.

Solution

$$\frac{\sqrt[4]{x^2y^3}}{x^{\frac{1}{2}}y^{\frac{2}{3}}} = \frac{(x^2y^3)^{\frac{1}{4}}}{x^{\frac{1}{2}}y^{\frac{2}{3}}} = \frac{x^{\frac{2}{4}}y^{\frac{3}{4}}}{x^{\frac{1}{2}}y^{\frac{2}{3}}}$$

$$= x^{\frac{2}{4}-\frac{1}{2}}y^{\frac{3}{4}-\frac{2}{3}}$$

$$= x^0y^{\frac{1}{12}}$$

$$= y^{\frac{1}{12}}$$

Explanation

$(ab)^n = a^nb^n$

$\frac{a^m}{a^n} = a^{m-n}$

$a^0 = 1$

Summary 1A

- **Index laws**
  - $a^m \times a^n = a^{m+n}$
  - $a^m \div a^n = a^{m-n}$
  - $(a^m)^n = a^{mn}$
  - $(ab)^n = a^nb^n$
  - $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$
  - $a^{-n} = \frac{1}{a^n}$
  - $\frac{1}{a^{-n}} = a^n$
  - $a^0 = 1$
- **Rational indices**
  - $a^{\frac{1}{n}} = \sqrt[n]{a}$
  - $a^{\frac{m}{n}} = \left(a^{\frac{1}{n}}\right)^m$

Exercise 1A

Example 1

SF

1 Simplify each of the following using the appropriate index laws:

a $x^3 \times x^4$ **b** $a^5 \times a^{-3}$ **c** $x^2 \times x^{-1} \times x^2$ **d** $\frac{y^3}{y^7}$

e $\frac{x^8}{x^{-4}}$ **f** $\frac{p^{-5}}{p^2}$ **g** $a^{\frac{1}{2}} \div a^{\frac{2}{3}}$ **h** $(a^{-2})^4$

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SF

i $(y^{-2})^{-7}$ **j** $(x^5)^3$ **k** $(a^{-20})^{\frac{3}{5}}$ **l** $\left(x^{-\frac{1}{2}}\right)^{-4}$

m $(n^{10})^{\frac{1}{5}}$ **n** $2x^{\frac{1}{2}} \times 4x^3$ **o** $(a^2)^{\frac{5}{2}} \times a^{-4}$ **p** $\dfrac{1}{x^{-4}}$

q $\left(2n^{-\frac{2}{5}}\right)^5 \div (4^3n^4)$ **r** $x^3 \times 2x^{\frac{1}{2}} \times -4x^{-\frac{3}{2}}$

s $(ab^3)^2 \times a^{-2}b^{-4} \times \dfrac{1}{a^2b^{-3}}$ **t** $(2^2p^{-3} \times 4^3p^5 \div (6p^{-3}))^0$

Example 2

2 Evaluate each of the following:

a $25^{\frac{1}{2}}$ **b** $64^{\frac{1}{3}}$ **c** $\left(\dfrac{16}{9}\right)^{\frac{1}{2}}$ **d** $16^{-\frac{1}{2}}$

e $\left(\dfrac{49}{36}\right)^{-\frac{1}{2}}$ **f** $27^{\frac{1}{3}}$ **g** $144^{\frac{1}{2}}$ **h** $64^{\frac{2}{3}}$

i $9^{\frac{3}{2}}$ **j** $\left(\dfrac{81}{16}\right)^{\frac{1}{4}}$ **k** $\left(\dfrac{23}{5}\right)^0$ **l** $128^{\frac{3}{7}}$

3 Use your calculator to evaluate each of the following, correct to two decimal places:

a 4.35^2 **b** 2.4^5 **c** $\sqrt{34.6921}$ **d** $(0.02)^{-3}$ **e** $\sqrt[3]{0.729}$

f $\sqrt[4]{2.3045}$ **g** $(345.64)^{-\frac{1}{3}}$ **h** $(4.568)^{\frac{2}{5}}$ **i** $\dfrac{1}{(0.064)^{-\frac{1}{3}}}$

CF

4 Simplify each of the following, giving your answer with positive index:

a $\dfrac{a^2b^3}{a^{-2}b^{-4}}$ **b** $\dfrac{2a^2(2b)^3}{(2a)^{-2}b^{-4}}$ **c** $\dfrac{a^{-2}b^{-3}}{a^{-2}b^{-4}}$

d $\dfrac{a^2b^3}{a^{-2}b^{-4}} \times \dfrac{ab}{a^{-1}b^{-1}}$ **e** $\dfrac{(2a)^2 \times 8b^3}{16a^{-2}b^{-4}}$ **f** $\dfrac{2a^2b^3}{8a^{-2}b^{-4}} \div \dfrac{16ab}{(2a)^{-1}b^{-1}}$

5 Write $\dfrac{2^n \times 8^n}{2^{2n} \times 16}$ in the form 2^{an+b}.

6 Write $2^{-x} \times 3^{-x} \times 6^{2x} \times 3^{2x} \times 2^{2x}$ as a power of 6.

7 Simplify each of the following:

a $2^{\frac{1}{3}} \times 2^{\frac{1}{6}} \times 2^{-\frac{2}{3}}$ **b** $a^{\frac{1}{4}} \times a^{\frac{2}{5}} \times a^{-\frac{1}{10}}$ **c** $2^{\frac{2}{3}} \times 2^{\frac{5}{6}} \times 2^{-\frac{2}{3}}$

d $\left(2^{\frac{1}{3}}\right)^2 \times \left(2^{\frac{1}{2}}\right)^5$ **e** $\left(2^{\frac{1}{3}}\right)^2 \times 2^{\frac{1}{3}} \times 2^{-\frac{2}{5}}$

Example 3

8 Simplify each of the following:

a $\sqrt[3]{a^3b^2} \div \sqrt[3]{a^2b^{-1}}$ **b** $\sqrt{a^3b^2} \times \sqrt{a^2b^{-1}}$ **c** $\sqrt[5]{a^3b^2} \times \sqrt[5]{a^2b^{-1}}$

d $\sqrt{a^{-4}b^2} \times \sqrt{a^3b^{-1}}$ **e** $\sqrt{a^3b^2c^{-3}} \times \sqrt{a^2b^{-1}c^{-5}}$ **f** $\sqrt[5]{a^3b^2} \div \sqrt[5]{a^2b^{-1}}$

g $\dfrac{\sqrt{a^3b^2}}{a^2b^{-1}c^{-5}} \times \dfrac{\sqrt{a^{-4}b^2}}{a^3b^{-1}} \times \sqrt{a^3b^{-1}}$

1B Scientific notation

Learning intentions

- To be able to write natural numbers with scientific notation.
- To be able to write a natural number to a given number of significant figures.

Often when dealing with real-world problems, the numbers involved may be very small or very large. For example:

- The distance from Earth to the Sun is approximately 150 000 000 kilometres.
- The mass of an oxygen atom is approximately 0.000 000 000 000 000 000 000 026 grams.

To help deal with such numbers, we can use a more convenient way to express them. This involves expressing the number as a product of a number between 1 and 10 and a power of 10 and is called **standard form** or **scientific notation**.

These examples written in standard form are:

- 1.5×10^8 kilometres
- 2.6×10^{-23} grams

Multiplication and division with very small or very large numbers can often be simplified by first converting the numbers into standard form. When simplifying algebraic expressions or manipulating numbers in standard form, a sound knowledge of the index laws is essential.

Example 4

Write each of the following in standard form:

a 3 453 000 **b** 0.00675

Solution

a $3\,453\,000 = 3.453 \times 10^6$ **b** $0.00675 = 6.75 \times 10^{-3}$

Example 5

Determine the value of $\dfrac{32\,000\,000 \times 0.000\,004}{16\,000}$.

Solution

$$\begin{aligned}\frac{32\,000\,000 \times 0.000\,004}{16\,000} &= \frac{3.2 \times 10^7 \times 4 \times 10^{-6}}{1.6 \times 10^4}\\ &= \frac{12.8 \times 10^1}{1.6 \times 10^4}\\ &= 8 \times 10^{-3}\\ &= 0.008\end{aligned}$$

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Significant figures

When measurements are made, the result is recorded to a certain number of significant figures. For example, if we say that the length of a piece of ribbon is 156 cm to the nearest centimetre, this means that the length is between 155.5 cm and 156.5 cm. The number 156 is said to be correct to three significant figures. Similarly, we may record π as being 3.1416, correct to five significant figures.

When rounding off to a given number of significant figures, first identify the last significant digit and then:

- if the next digit is 0, 1, 2, 3 or 4, round down
- if the next digit is 5, 6, 7, 8 or 9, round up.

It can help with rounding off if the original number is first written in scientific notation.

So $\pi = 3.141\,592\,653\ldots$ is rounded off to

$$3, \quad 3.1, \quad 3.14, \quad 3.142, \quad 3.1416, \quad 3.14159, \quad \text{etc.}$$

depending on the number of significant figures required.

Writing a number in scientific notation makes it clear how many significant figures have been recorded. For example, it is unclear whether 600 is recorded to one, two or three significant figures. However, when written in scientific notation as

$$6.00 \times 10^2, \quad 6.0 \times 10^2 \quad \text{or} \quad 6 \times 10^2$$

it is clear how many significant figures are recorded.

Example 6

Evaluate $\dfrac{\sqrt[5]{a}}{b^2}$ if $a = 1.34 \times 10^{-10}$ and $b = 2.7 \times 10^{-8}$.

Solution

$$\begin{aligned}
\frac{\sqrt[5]{a}}{b^2} &= \frac{\sqrt[5]{1.34 \times 10^{-10}}}{(2.7 \times 10^{-8})^2} \\
&= \frac{(1.34 \times 10^{-10})^{\frac{1}{5}}}{2.7^2 \times (10^{-8})^2} \\
&= 1.45443\ldots \times 10^{13} \\
&= 1.45 \times 10^{13} \quad \text{to three significant figures}
\end{aligned}$$

Note: Many calculators can display numbers in scientific notation. The format will vary from calculator to calculator. For example, the number $3\,245\,000 = 3.245 \times 10^6$ may appear as 3.245E6 or 3.245^{06}.

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Summary 1B

- A number is said to be in **scientific notation** (or **standard form**) when it is written as a product of a number between 1 and 10 and an integer power of 10.
  For example: $6547 = 6.547 \times 10^3$ and $0.789 = 7.89 \times 10^{-1}$
- Writing a number in scientific notation makes it clear how many **significant figures** have been recorded.
- When rounding off to a given number of significant figures, first identify the last significant digit and then:
  - if the next digit is 0, 1, 2, 3 or 4, round down
  - if the next digit is 5, 6, 7, 8 or 9, round up.

Exercise 1B

SF

Example 4

1 Express each of the following numbers in standard form:

a 47.8 **b** 6728 **c** 79.23 **d** 43 580
e 0.0023 **f** 0.000 000 56 **g** 12.000 34 **h** 50 million
i 23 000 000 000 **j** 0.000 000 0013 **k** 165 thousand **l** 0.000 014 567

2 Express each of the following in scientific notation:

a The mass of a water molecule is 0.000 000 000 000 000 000 000 0299 g.
b There are 31 536 000 seconds in one year.
c There are 3 057 000 000 000 000 000 000 atoms in one gram of gold.

3 Express each of the following as an ordinary number:

a The diameter of the Sun is 1.39×10^9 m.
b The diameter of a red blood cell is 7.5×10^{-6} m.
c The diameter of an electron is 5.6×10^{-15} m.

4 Write each of the following in scientific notation, correct to the number of significant figures indicated in the brackets:

a 456.89 (4) **b** 34 567.23 (2) **c** 5679.087 (5)
d 0.045 36 (2) **e** 0.090 45 (2) **f** 4568.234 (5)

Example 5

5 Without using a calculator, determine the value of:

a $\dfrac{5\,000\,000 \times 0.08}{400\,000}$ **b** $\dfrac{320\,000 \times 0.0006}{4000}$ **c** $\dfrac{5\,400\,000 \times 0.8}{0.000\,02}$

d $\dfrac{1200 \times 0.000\,03}{14.4}$ **e** $\dfrac{\sqrt{3.6 \times 10^7}}{3 \times 10^{-2}}$ **f** $\dfrac{2.2 \times 10^9}{\sqrt{1.21 \times 10^4}}$

Example 6

6 Evaluate the following correct to three significant figures:

a $\dfrac{\sqrt[3]{a}}{b^4}$ if $a = 2 \times 10^9$ and $b = 3.215$ **b** $\dfrac{\sqrt[4]{a}}{4b^4}$ if $a = 2 \times 10^{12}$ and $b = 0.05$

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1C Solving linear equations and simultaneous linear equations

Learning intentions

- To be able to solve linear equations and simple simultaneous linear equations.

Many problems may be solved by first translating them into mathematical equations and then solving the equations using algebraic techniques. An equation is solved by determining the value or values of the variables that would make the statement true.

Linear equations are simple equations that can be written in the form $ax + b = 0$. There are a number of standard techniques that can be used for solving linear equations.

Example 7

a Solve $\frac{x}{5} - 2 = \frac{x}{3}$.

b Solve $\frac{x-3}{2} - \frac{2x-4}{3} = 5$.

Solution

a Multiply both sides of the equation by the lowest common multiple of 3 and 5:

$$\frac{x}{5} - 2 = \frac{x}{3}$$

$$\frac{x}{5} \times 15 - 2 \times 15 = \frac{x}{3} \times 15$$

$$3x - 30 = 5x$$

$$3x - 5x = 30$$

$$-2x = 30$$

$$x = \frac{30}{-2}$$

$$\therefore \quad x = -15$$

b Multiply both sides of the equation by the lowest common multiple of 2 and 3:

$$\frac{x-3}{2} - \frac{2x-4}{3} = 5$$

$$\frac{x-3}{2} \times 6 - \frac{2x-4}{3} \times 6 = 5 \times 6$$

$$3(x-3) - 2(2x-4) = 30$$

$$3x - 9 - 4x + 8 = 30$$

$$3x - 4x = 30 + 9 - 8$$

$$-x = 31$$

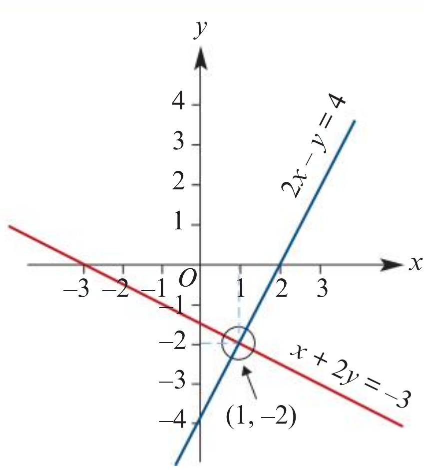
$$x = \frac{31}{-1}$$

$$\therefore \quad x = -31$$

Simultaneous linear equations

The intersection point of two straight lines can be found graphically; however, the accuracy of the solution will depend on the accuracy of the graphs.

Alternatively, the intersection point may be found algebraically by solving the pair of simultaneous equations. We shall consider two techniques for solving simultaneous equations.



Example 8

Solve the equations $2x - y = 4$ and $x + 2y = -3$.

Solution

Method 1: Substitution

$$2x - y = 4 \qquad (1)$$
$$x + 2y = -3 \qquad (2)$$

Using one of the two equations, express one variable in terms of the other variable.

From equation (2), we get $x = -3 - 2y$.

Substitute in equation (1):

$$2(-3 - 2y) - y = 4$$
$$-6 - 4y - y = 4$$
$$-5y = 10$$
$$y = -2$$

Then substitute this expression into the other equation (reducing it to an equation in one variable, y). Solve the equation for y.

Substitute the value of y into (2):

$$x + 2(-2) = -3$$
$$x = 1$$

Substitute this value for y in one of the equations to determine the other variable, x.

Check in (1): LHS $= 2(1) - (-2) = 4$

RHS $= 4$

A check can be carried out with the other equation.

Method 2: Elimination

$$2x - y = 4 \qquad (1)$$
$$x + 2y = -3 \qquad (2)$$

If one of the variables has the same coefficient in the two equations, we can eliminate that variable by subtracting one equation from the other.

To eliminate x, multiply equation (2) by 2 and subtract the result from equation (1).

It may be necessary to multiply one of the equations by a constant to make the coefficients of x or y the same in the two equations.

When we multiply equation (2) by 2, the pair of equations becomes:

$$2x - y = 4 \qquad (1)$$
$$2x + 4y = -6 \qquad (2')$$

Subtract (2′) from (1):

$$-5y = 10$$
$$y = -2$$

Now substitute for y in equation (2) to determine x, and check as in the substitution method.

Note: This example shows that the point $(1, -2)$ is the point of intersection of the graphs of the two linear relations.

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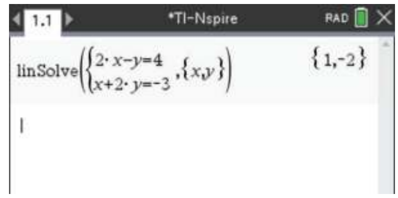
Using the TI-Nspire CX non-CAS

Method 1: Using a Calculator application

Simultaneous linear equations can be solved in a **Calculator** application.

- Use (menu) > **Algebra** > **Solve System of Linear Equations**.
- Complete the pop-up screen.
- Enter the equations as shown to give the solution to the simultaneous equations $2x - y = 4$ and $x + 2y = -3$.
- Hence the solution is $x = 1$ and $y = -2$.

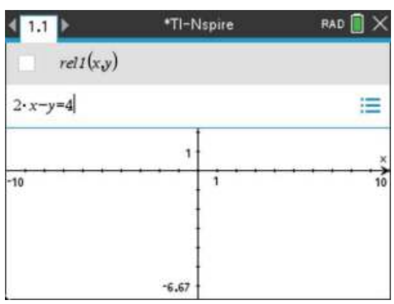


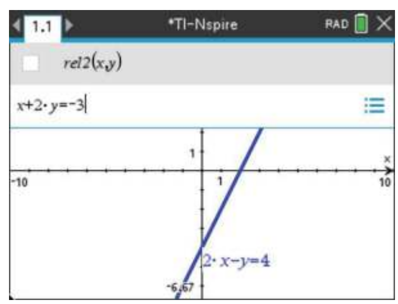
Method 2: Using a Graphs application

The simultaneous equations can also be solved graphically in a **Graphs** application.

- Equations of the form $a \cdot x + b \cdot y = c$ can be entered directly using (menu) > **Graph Entry/Edit** > **Relation**

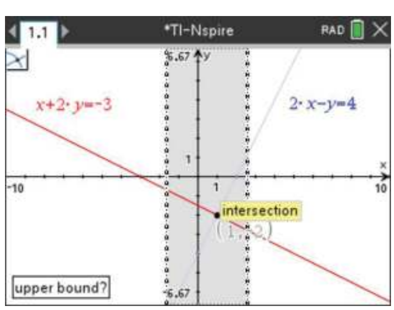
  Note: Equations of the form $a \cdot x + b \cdot y = c$ can also be entered directly using (menu) > **Graph Entry/Edit** > **Equation Templates** > **Line** > **Line Standard** $a \cdot x + b \cdot y = c$.
- Alternatively, rearrange each equation to make y the subject, and enter as a standard function (e.g. $f1(x) = 2x - 4$).

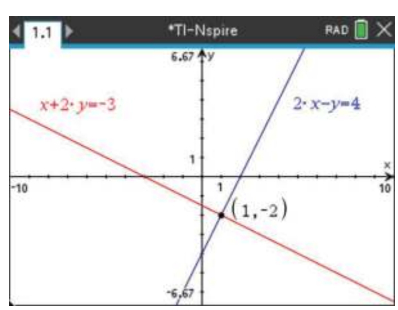




Note: If the entry line is not visible, press (tab). Pressing (enter) will hide the entry line. If you want to add more equations, use ▼ to add the next equation.

- The intersection point can be found using (menu) > **Analyze Graph** > **Intersection**.
- Move the cursor to the left of the intersection point (lower bound), click, move to the right of the intersection point (upper bound) and click again. The intersection point's coordinates will appear on the screen.



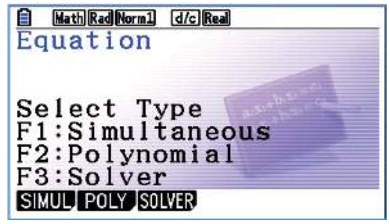


Using the Casio

Simultaneous linear equations can be solved in **Equation** mode or in **Graph** mode.

Method 1: Using Equation mode

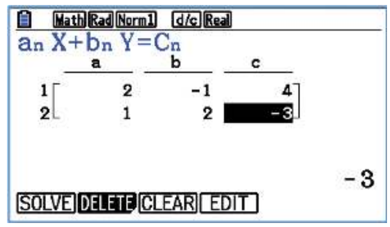
- Press (MENU) and then select **Equation** mode by pressing (ALPHA) (X,θ,T).
- Select **Simultaneous** (F1).



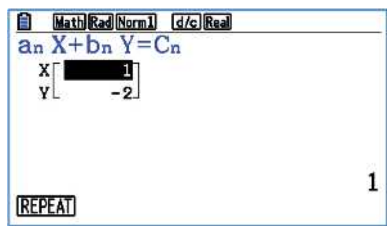
- Enter the coefficients of the two equations $2x - y = 4$ and $x + 2y = -3$ as shown:

  (2) (EXE) ((–)) (1) (EXE) (4) (EXE)

  (1) (EXE) (2) (EXE) ((–)) (3) (EXE)



- Select **Solve** (F1).
- Hence the solution is $x = 1$ and $y = -2$.



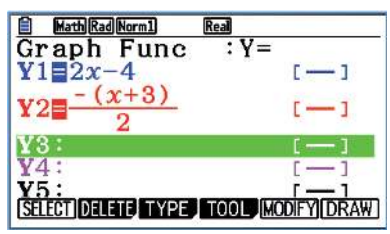
Method 2: Using Graph mode

- Press (MENU) (5) to select **Graph** mode.
- Transpose $2x - y = 4$ to make y the subject, and enter the equation in $Y1$:

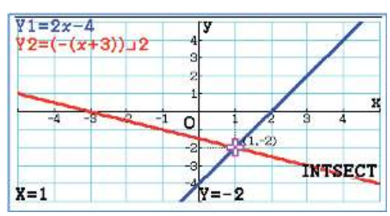
  (2) (X,θ,T) (–) (4) (EXE)

- Transpose $x + 2y = -3$ to make y the subject, and enter the equation in $Y2$:

  (▭/▭) ((–)) (() (X,θ,T) (+) (3) ()) ▼ (2) (EXE)



- Select **Draw** (F6).
- Go to the **G-Solve** menu (SHIFT) (F5) and select **Intersection** (F5).



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Summary 1C

- An equation is solved by determining the value or values of the variables that would make the statement true.
- A linear equation is one in which the ‘unknown’ is to the first power.
- There are often several different ways to solve a linear equation. The following steps provide some suggestions:
  1. Expand brackets and, if the equation involves fractions, multiply through by the lowest common denominator of the terms.
  2. Group all of the terms containing a variable on one side of the equation and the terms without the variable on the other side.
- Methods for solving simultaneous linear equations in two variables by hand:

  Substitution
  - Make one of the variables the subject in one of the equations.
  - Substitute for that variable in the other equation.

  Elimination
  - Choose one of the two variables to eliminate.
  - Obtain the same or opposite coefficients for this variable in the two equations. To do this, multiply both sides of one or both equations by a number.
  - Add or subtract the two equations to eliminate the chosen variable.

Exercise 1C

SF

Example 7a

1 Solve the following linear equations:

a $3x + 7 = 15$ **b** $8 - \frac{x}{2} = -16$ **c** $42 + 3x = 22$

d $\frac{2x}{3} - 15 = 27$ **e** $5(2x + 4) = 13$ **f** $-3(4 - 5x) = 24$

g $3x + 5 = 8 - 7x$ **h** $2 + 3(x - 4) = 4(2x + 5)$ **i** $\frac{2x}{5} - \frac{3}{4} = 5x$

j $6x + 4 = \frac{x}{3} - 3$

Example 7b

2 Solve the following linear equations:

a $\frac{x}{2} + \frac{2x}{5} = 16$ **b** $\frac{3x}{4} - \frac{x}{3} = 8$

c $\frac{3x - 2}{2} + \frac{x}{4} = -8$ **d** $\frac{5x}{4} - \frac{4}{3} = \frac{2x}{5}$

e $\frac{x - 4}{2} + \frac{2x + 5}{4} = 6$ **f** $\frac{3 - 3x}{10} - \frac{2(x + 5)}{6} = \frac{1}{20}$

g $\frac{3 - x}{4} - \frac{2(x + 1)}{5} = -24$ **h** $\frac{-2(5 - x)}{8} + \frac{6}{7} = \frac{4(x - 2)}{3}$

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Example 8

3 Solve each of the following pairs of simultaneous equations:

a $3x + 2y = 2$
$2x - 3y = 6$

b $5x + 2y = 4$
$3x - y = 6$

c $2x - y = 7$
$3x - 2y = 2$

d $x + 2y = 12$
$x - 3y = 2$

e $7x - 3y = -6$
$x + 5y = 10$

f $15x + 2y = 27$
$3x + 7y = 45$

SF

1D Solving problems with linear equations

Learning intentions

- To be able to solve problems with linear equations.

Many problems can be solved by translating them into mathematical language and using an appropriate mathematical technique to determine the solution. By representing the unknown quantity in a problem with a symbol (called a pronumeral or a variable) and constructing an equation from the information, the value of the unknown can be found by solving the equation.

Before constructing the equation, each variable and what it stands for (including the units) should be stated. All the elements of the equation must be in units of the same system.

Example 9

For each of the following, form the relevant linear equation and solve it for x:

a The length of the side of a square is $(x - 6)$ cm. Its perimeter is 52 cm.

b The perimeter of a square is $(2x + 8)$ cm. Its area is 100 cm^2.

Solution

a Perimeter = 4 × side length

Therefore $4(x - 6) = 52$

$$x - 6 = 13$$
$$x = 19$$

b The perimeter of the square is $2x + 8$.

The length of one side is $\frac{2x + 8}{4} = \frac{x + 4}{2}$.

Thus the area is

$$\left(\frac{x+4}{2}\right)^2 = 100$$

As the side length must be positive, this gives the linear equation

$$\frac{x+4}{2} = 10$$

Therefore $x = 16$.

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Example 10

An athlete trains for an event by gradually increasing the distance she runs each week over a five-week period. If she runs an extra 5 km each successive week and over the five weeks runs a total of 175 km, how far did she run in the first week?

Solution

Let the distance run in the first week be x km.

Then the distance run in the second week is $x + 5$ km, and the distance run in the third week is $x + 10$ km, and so on.

The total distance run is $x + (x + 5) + (x + 10) + (x + 15) + (x + 20)$ km.

$$\therefore \quad 5x + 50 = 175$$
$$5x = 125$$
$$x = 25$$

The distance she ran in the first week was 25 km.

Example 11

A man bought 14 CDs at a sale. Some cost \$15 each and the remainder cost \$12.50 each. In total he spent \$190. How many \$15 CDs and how many \$12.50 CDs did he buy?

Solution

Let n be the number of CDs costing \$15.
Then $14 - n$ is the number of CDs costing \$12.50.

$$\therefore \quad 15n + 12.5(14 - n) = 190$$
$$15n + 175 - 12.5n = 190$$
$$2.5n + 175 = 190$$
$$2.5n = 15$$
$$n = 6$$

He bought 6 CDs costing \$15 and 8 CDs costing \$12.50.

Summary 1D

Steps for solving a word problem with a linear equation

- Read the question carefully and write down the known information clearly.
- Identify the unknown quantity that is to be found.
- Assign a variable to this quantity.
- Form an expression in terms of x (or the variable being used) and use the other relevant information to form the equation.
- Solve the equation.
- Write a sentence answering the initial question.

Skill-sheet

Exercise 1D

SF

Example 9

1 For each of the cases below, write down a relevant equation involving the variables defined, and solve the equation for parts **a**, **b** and **c**.

a The length of the side of a square is $(x - 2)$ cm. Its perimeter is 60 cm.

b The perimeter of a square is $(2x + 7)$ cm. Its area is 49 cm^2.

c The length of a rectangle is $(x - 5)$ cm. Its width is $(12 - x)$ cm. The rectangle is twice as long as it is wide.

d The length of a rectangle is $(2x + 1)$ cm. Its width is $(x - 3)$ cm. The perimeter of the rectangle is y cm.

e n people each have a meal costing \$$p$. The total cost of the meal is \$$Q$.

f S people each have a meal costing \$$p$. A 10% service charge is added to the cost. The total cost of the meal is \$$R$.

g A machine working at a constant rate produces n bolts in 5 minutes. It produces 2400 bolts in 1 hour.

h The radius of a circle is $(x + 3)$ cm. A sector subtending an angle of 60° at the centre is cut off. The arc length of the minor sector is a cm.

CF

Example 10

2 Bronwyn and Noel have a women's clothing shop in Summerland. Bronwyn manages the shop and her sales are going up steadily over a particular period of time. They are going up by \$500 per week. If over a five-week period her sales total \$17 500, how much did she earn in the first week?

Example 11

3 Bronwyn and Noel have a women's clothing shop in Summerland. Sally, Adam and baby Lana came into the shop and Sally bought dresses and handbags. The dresses cost \$65 each and the handbags cost \$26 each. Sally bought 11 items and in total she spent \$598. How many dresses and how many handbags did she buy?

4 A rectangular courtyard is three times as long as it is wide. If the perimeter of the courtyard is 67 m, determine the dimensions of the courtyard.

5 A wine merchant buys 50 cases of wine. He pays full price for half of them, but gets a 40% discount on the remainder. If he paid a total of \$2260, how much was the full price of a single case?

6 A real-estate agent sells 22 houses in six months. He makes a commission of \$11 500 per house on some and \$13 000 per house on the remainder. If his total commission over the six months was \$272 500, on how many houses did he make a commission of \$11 500?

7 Three boys compare their marble collections. The first boy has 14 fewer than the second boy, who has twice as many as the third. If between them they have 71 marbles, how many does each boy have?

CF

8 Three girls are playing Scrabble. At the end of the game, their three scores add up to 504. Annie scored 10% more than Belinda, while Cassie scored 60% of the combined scores of the other two. What did each player score?

9 A biathlon event involves running and cycling. Kim can cycle 30 km/h faster than she can run. If Kim spends 48 minutes running and a third as much time again cycling in an event that covers a total distance of 60 km, how fast can she run?

10 The mass of a molecule of a certain chemical compound is 2.45×10^{-22} g. If each molecule is made up of two chlorine atoms and six carbon atoms and the mass of a carbon atom is one-third that of a chlorine atom, determine the mass of a carbon atom.

11 Mother's pearl necklace fell to the floor. One-sixth of the pearls rolled under the fridge, one-third rolled under the couch, one-fifth of them behind the bookshelf, and nine were found at her feet. How many pearls are there?

12 Parents say they don't have favourites, but everyone knows that's a lie. A father distributes \$96 to his three children according to the following instructions: The middle child receives \$12 less than the oldest, and the youngest receives one-third as much as the middle child. How much does each receive?

13 Kavindi has achieved an average mark of 88% on her first four maths tests. What mark would she need on her fifth test to increase her average to 90%?

14 In a particular class, 72% of the students have black hair. Five black-haired students leave the class, so that now 65% of the students have black hair. How many students were originally in the class?

15 Two tanks are being emptied. Tank A contains 100 litres of water and tank B contains 120 litres of water. Water runs from Tank A at 2 litres per minute, and water runs from tank B at 3 litres per minute. After how many minutes will the amount of water in the two tanks be the same?

16 Suppose that candle A is initially 10 cm tall and burns out after 2 hours. Candle B is initially 8 cm tall and burns out after 4 hours. Both candles are lit at the same time. Assuming 'constant burning rates':

a When is the height of candle A the same as the height of candle B?
b When is the height of candle A half the height of candle B?
c When is candle A 1 cm taller than candle B?

17 A motorist drove 320 km in $\frac{10}{3}$ hours. He drove part of the way at an average speed of 100 km/h and the rest of the way at an average speed of 90 km/h. What is the distance he travelled at 100 km/h?

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18 Jarmila travels regularly between two cities. She takes $\frac{14}{3}$ hours if she travels at her usual speed. If she increases her speed by 3 km/h, she can reduce her time taken by 20 minutes. What is her usual speed?

CF

1E Solving problems with simultaneous linear equations

Learning intentions

- To be able to solve problems with simultaneous linear equations.

When the relationship between two quantities is linear, we can determine the constants which determine this linear relationship if we are given two sets of information satisfying the relationship. Simultaneous linear equations enable this to be done.

Another situation in which simultaneous linear equations may be used is where it is required to determine the point of the Cartesian plane which satisfies two linear relations.

Example 12

There are two possible methods for paying gas bills:

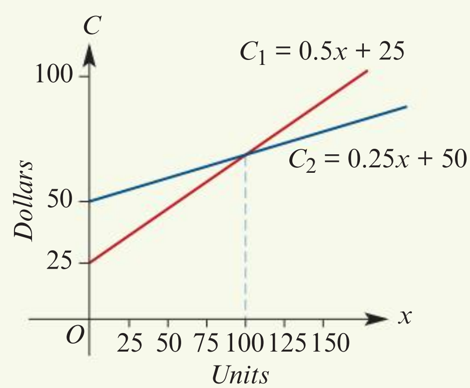
Method A A fixed charge of \$25 per quarter + 50c per unit of gas used
Method B A fixed charge of \$50 per quarter + 25c per unit of gas used

Determine the number of units which must be used before method B becomes cheaper than method A.

Solution

Let C_1 = charge (\$) using method A
C_2 = charge (\$) using method B
x = number of units of gas used

Then $C_1 = 25 + 0.5x$
$C_2 = 50 + 0.25x$



From the graph we see that method B is cheaper if the number of units exceeds 100.

The solution can be obtained by solving simultaneous linear equations:

$$C_1 = C_2$$
$$25 + 0.5x = 50 + 0.25x$$
$$0.25x = 25$$
$$x = 100$$

Example 13

If 3 kg of jam and 2 kg of butter cost \$29, and 6 kg of jam and 3 kg of butter cost \$54, determine the cost per kilogram of jam and of butter.

Solution

Let the cost of 1 kg of jam be x dollars and the cost of 1 kg of butter be y dollars.

Then $3x + 2y = 29$ (1)

and $6x + 3y = 54$ (2)

Multiply (1) by 2: $6x + 4y = 58$ (1′)

Subtract (1′) from (2): $-y = -4$

$y = 4$

Substitute in (2): $6x + 3(4) = 54$

$6x = 42$

$x = 7$

Jam costs \$7 per kilogram and butter costs \$4 per kilogram.

Summary 1E

Steps for solving a word problem with simultaneous linear equations

- Read the question carefully and write down the known information clearly.
- Identify the two unknown quantities that are to be found.
- Assign variables to these two quantities.
- Form expressions in terms of x and y (or other suitable variables) and use the other relevant information to form the two equations.
- Solve the system of equations.
- Write a sentence answering the initial question.

Exercise 1E

Example 12

1 A car hire firm offers the option of paying \$108 per day with unlimited kilometres, or \$63 per day plus 32 cents per kilometre travelled. How many kilometres would you have to travel in a given day to make the unlimited-kilometres option more attractive?

2 Company A will cater for your party at a cost of \$450 plus \$40 per guest. Company B offers the same service for \$300 plus \$43 per guest. How many guests are needed before Company A's charge is less than Company B's?

Example 13

3 A basketball final is held in a stadium which can seat 15 000 people. All the tickets have been sold, some to adults at \$45 and the rest for children at \$15. If the revenue from the tickets was \$525 000, determine the number of adults who bought tickets.

CF

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CF

4 A contractor employed eight men and three boys for one day and paid them a total of \$2240. Another day he employed six men and eighteen boys for \$4200. What was the daily rate he paid each man and each boy?

5 The sum of two numbers is 212 and their difference is 42. Determine the two numbers.

6 Each adult paid \$30 to attend a concert and each student paid \$20. A total of 1600 people attended. The total paid was \$37 000. How many adults and how many students attended the concert?

7 A chemical manufacturer wishes to obtain 700 litres of a 24% acid solution by mixing a 40% solution with a 15% solution. How many litres of each solution should be used?

8 Two children had 220 marbles between them. After one child had lost half her marbles and the other had lost 40 marbles, they had an equal number of marbles. How many did each child start with and how many did each child finish with?

9 An investor received \$31 000 interest per annum from a sum of money, with part of it invested at 10% and the remainder at 7% simple interest. She found that if she interchanged the amounts she had invested she could increase her return by \$1000 per annum. Calculate the total amount she had invested.

1F Substitution and transposition of formulas

Learning intentions

- To be able to substitute in formulas and transpose formulas.

An equation that states a relationship between two or more quantities is called a **formula**; e.g. the area of a circle is given by $A = \pi r^2$. The value of A, the subject of the formula, may be found by substituting a given value of r and the value of π.

Example 14

Using the formula $A = \pi r^2$, determine the value of A correct to two decimal places if $r = 2.3$ and $\pi = 3.142$ (correct to three decimal places).

Solution

$$\begin{aligned} A &= \pi r^2 \\ &= 3.142(2.3)^2 \\ &= 16.62118 \end{aligned}$$

$\therefore \quad A = 16.62$ correct to two decimal places

The formula $A = \pi r^2$ can also be transposed to make r the subject.

When transposing a formula, follow a similar procedure to solving a linear equation. Whatever has been done to the variable required is 'undone'.

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Example 15

a Transpose the formula $A = \pi r^2$ to make r the subject.

b Hence determine the value of r correct to three decimal places if $A = 24.58$ and $\pi = 3.142$ (correct to three decimal places).

Solution

a
$$A = \pi r^2$$
$$\frac{A}{\pi} = r^2$$
$$r = \sqrt{\frac{A}{\pi}}$$

b
$$r = \sqrt{\frac{A}{\pi}} = \sqrt{\frac{24.58}{3.142}}$$
$$= 2.79697\ldots$$

$r = 2.797$ correct to three decimal places

Summary 1F

- A formula relates different quantities: for example, the formula $A = \pi r^2$ relates the radius r with the area A of the circle.
- The variable on the left is called the subject of the formula: for example, in the formula $A = \pi r^2$, the subject is A.
- To calculate the value of a variable which is not the subject of a formula:
  Method 1 Substitute the values for the known variables, then solve the resulting equation for the unknown variable.
  Method 2 Rearrange to make the required variable the subject, then substitute values.

Exercise 1F

Example 14

1 Substitute the specified values to evaluate each of the following, giving the answers correct to two decimal places:

a v if $v = u + at$ and $u = 15$, $a = 2$, $t = 5$

b I if $I = \dfrac{PrT}{100}$ and $P = 600$, $r = 5.5$, $T = 10$

c V if $V = \pi r^2 h$ and $r = 4.25$, $h = 6$

d S if $S = 2\pi r(r + h)$ and $r = 10.2$, $h = 15.6$

e V if $V = \dfrac{4}{3}\pi r^2 h$ and $r = 3.58$, $h = 11.4$

f s if $s = ut + \dfrac{1}{2}at^2$ and $u = 25.6$, $t = 3.3$, $a = -1.2$

g T if $T = 2\pi\sqrt{\dfrac{\ell}{g}}$ and $\ell = 1.45$, $g = 9.8$

h f if $\dfrac{1}{f} = \dfrac{1}{v} + \dfrac{1}{u}$ and $v = 3$, $u = 7$

i c if $c^2 = a^2 + b^2$ and $a = 8.8$, $b = 3.4$

j v if $v^2 = u^2 + 2as$ and $u = 4.8$, $a = 2.5$, $s = 13.6$

SF

Example 15

2 Transpose each of the following to make the symbol in brackets the subject:

a $v = u + at$ (a)

b $S = \frac{n}{2}(a + \ell)$ (ℓ)

c $A = \frac{1}{2}bh$ (b)

d $P = I^2R$ (I)

e $s = ut + \frac{1}{2}at^2$ (a)

f $E = \frac{1}{2}mv^2$ (v)

g $Q = \sqrt{2gh}$ (h)

h $-xy - z = xy + z$ (x)

i $\frac{ax + by}{c} = x - b$ (x)

j $\frac{mx + b}{x - b} = c$ (x)

3 The formula $F = \frac{9C}{5} + 32$ is used to convert temperatures given in degrees Celsius (C) to degrees Fahrenheit (F).

a Convert 28 degrees Celsius to degrees Fahrenheit.

b Transpose the formula to make C the subject and determine C if $F = 135°$.

4 The sum, S of the interior angles of a polygon with n sides is given by the formula $S = 180(n - 2)$.

a Determine the sum of the interior angles of an octagon.

b Transpose the formula to make n the subject and hence determine the number of sides of a polygon whose interior angles add up to 1260°.

5 The volume, V, of a right cone is given by the formula $V = \frac{1}{3}\pi r^2 h$, where r is the radius of the base and h is the height of the cone.

a Determine the volume of a cone with radius 3.5 cm and height 9 cm.

b Transpose the formula to make h the subject and hence determine the height of a cone with base radius 4 cm and volume 210 cm^3.

c Transpose the formula to make r the subject and hence determine the radius of a cone with height 10 cm and volume 262 cm^3.

6 For a particular type of sequence of numbers, the sum S of the terms in the sequence is given by the formula

$$S = \frac{n}{2}(a + \ell)$$

where n is the number of terms in the sequence, a is the first term and ℓ is the last term.

a Determine the sum of such a sequence of seven numbers whose first term is -3 and whose last term is 22.

b What is the first term of such a sequence of 13 numbers whose last term is 156 and whose sum is 1040?

c How many terms are there in the sequence $25 + 22 + 19 + \cdots + (-5) = 110$?

SF

CF

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1G Algebraic fractions

Learning intentions

- To be able to add, subtract, multiply and divide algebraic fractions

The principles involved in addition, subtraction, multiplication and division of algebraic fractions are the same as for simple numerical fractions.

Addition and subtraction

To add or subtract, all fractions must be written with a common denominator.

Example 16

Simplify:

a $\dfrac{x}{3}+\dfrac{x}{4}$

b $\dfrac{2}{x}+\dfrac{3a}{4}$

c $\dfrac{5}{x+2}-\dfrac{4}{x-1}$

d $\dfrac{4}{x+2}-\dfrac{7}{(x+2)^2}$

Solution

a $\dfrac{x}{3}+\dfrac{x}{4}=\dfrac{4x+3x}{12}$

$=\dfrac{7x}{12}$

b $\dfrac{2}{x}+\dfrac{3a}{4}=\dfrac{8+3ax}{4x}$

c $\dfrac{5}{x+2}-\dfrac{4}{x-1}=\dfrac{5(x-1)-4(x+2)}{(x+2)(x-1)}$

$=\dfrac{5x-5-4x-8}{(x+2)(x-1)}$

$=\dfrac{x-13}{(x+2)(x-1)}$

d $\dfrac{4}{x+2}-\dfrac{7}{(x+2)^2}=\dfrac{4(x+2)-7}{(x+2)^2}$

$=\dfrac{4x+1}{(x+2)^2}$

Multiplication and division

Before multiplying and dividing algebraic fractions, it is best to factorise numerators and denominators where possible so that common factors can be readily identified.

Example 17

Simplify:

a $\dfrac{3x^2}{10y^2}\times\dfrac{5y}{12x}$

b $\dfrac{2x-4}{x-1}\times\dfrac{x^2-1}{x-2}$

c $\dfrac{x^2-1}{2x-2}\times\dfrac{4x}{x^2+4x+3}$

d $\dfrac{x^2+3x-10}{x^2-x-2}\div\dfrac{x^2+6x+5}{3x+3}$

Solution

a $\dfrac{3x^2}{10y^2}\times\dfrac{5y}{12x}=\dfrac{x}{8y}$

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b $\dfrac{2x-4}{x-1} \times \dfrac{x^2-1}{x-2} = \dfrac{2(x-2)}{x-1} \times \dfrac{(x-1)(x+1)}{x-2}$

$= 2(x+1)$

c $\dfrac{x^2-1}{2x-2} \times \dfrac{4x}{x^2+4x+3} = \dfrac{(x-1)(x+1)}{2(x-1)} \times \dfrac{4x}{(x+1)(x+3)}$

$= \dfrac{2x}{x+3}$

d $\dfrac{x^2+3x-10}{x^2-x-2} \div \dfrac{x^2+6x+5}{3x+3} = \dfrac{(x+5)(x-2)}{(x-2)(x+1)} \times \dfrac{3(x+1)}{(x+1)(x+5)}$

$= \dfrac{3}{x+1}$

More examples

The following two examples involve algebraic fractions and rational indices.

Example 18

Express $\dfrac{3x^3}{\sqrt{4-x}} + 3x^2\sqrt{4-x}$ as a single fraction.

Solution

$$\frac{3x^3}{\sqrt{4-x}} + 3x^2\sqrt{4-x} = \frac{3x^3 + 3x^2\sqrt{4-x}\sqrt{4-x}}{\sqrt{4-x}}$$

$$= \frac{3x^3 + 3x^2(4-x)}{\sqrt{4-x}}$$

$$= \frac{12x^2}{\sqrt{4-x}}$$

Example 19

Express $(x-4)^{\frac{1}{5}} - (x-4)^{-\frac{4}{5}}$ as a single fraction.

Solution

$$(x-4)^{\frac{1}{5}} - (x-4)^{-\frac{4}{5}} = (x-4)^{\frac{1}{5}} - \frac{1}{(x-4)^{\frac{4}{5}}}$$

$$= \frac{(x-4)^{\frac{1}{5}}(x-4)^{\frac{4}{5}} - 1}{(x-4)^{\frac{4}{5}}}$$

$$= \frac{x-5}{(x-4)^{\frac{4}{5}}}$$

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Summary 1G

- **Simplifying algebraic fractions**
  - First factorise the numerator and denominator.
  - Then cancel any factors common to the numerator and denominator.
- **Adding and subtracting algebraic fractions**
  - First obtain a common denominator and then add or subtract.
- **Multiplying and dividing algebraic fractions**
  - First factorise each numerator and denominator completely.
  - Then complete the calculation by cancelling common factors.

Skill-sheet

Exercise 1G

SF

Example 16

1 Simplify each of the following:

a $\frac{2x}{3}+\frac{3x}{2}$ **b** $\frac{3a}{2}-\frac{a}{4}$ **c** $\frac{3h}{4}+\frac{5h}{8}-\frac{3h}{2}$

d $\frac{3x}{4}-\frac{y}{6}-\frac{x}{3}$ **e** $\frac{3}{x}+\frac{2}{y}$ **f** $\frac{5}{x-1}+\frac{2}{x}$

g $\frac{3}{x-2}+\frac{2}{x+1}$ **h** $\frac{2x}{x+3}-\frac{4x}{x-3}-\frac{3}{2}$ **i** $\frac{4}{x+1}+\frac{3}{(x+1)^2}$

j $\frac{a-2}{a}+\frac{a}{4}+\frac{3a}{8}$ **k** $2x-\frac{6x^2-4}{5x}$ **l** $\frac{2}{x+4}-\frac{3}{x^2+8x+16}$

m $\frac{3}{x-1}+\frac{2}{(x-1)(x+4)}$ **n** $\frac{3}{x-2}-\frac{2}{x+2}+\frac{4}{x^2-4}$

o $\frac{5}{x-2}+\frac{3}{x^2+5x+6}+\frac{2}{x+3}$ **p** $x-y-\frac{1}{x-y}$

q $\frac{3}{x-1}-\frac{4x}{1-x}$ **r** $\frac{3}{x-2}+\frac{2x}{2-x}$

Example 17

2 Simplify each of the following:

a $\frac{x^2}{2y}\times\frac{4y^3}{x}$ **b** $\frac{3x^2}{4y}\times\frac{y^2}{6x}$ **c** $\frac{4x^3}{3}\times\frac{12}{8x^4}$

d $\frac{x^2}{2y}\div\frac{3xy}{6}$ **e** $\frac{4-x}{3a}\times\frac{a^2}{4-x}$ **f** $\frac{2x+5}{4x^2+10x}$

g $\frac{(x-1)^2}{x^2+3x-4}$ **h** $\frac{x^2-x-6}{x-3}$ **i** $\frac{x^2-5x+4}{x^2-4x}$

j $\frac{5a^2}{12b^2}\div\frac{10a}{6b}$ **k** $\frac{x-2}{x}\div\frac{x^2-4}{2x^2}$ **l** $\frac{x+2}{x(x-3)}\div\frac{4x+8}{x^2-4x+3}$

m $\frac{2x}{x-1}\div\frac{4x^2}{x^2-1}$ **n** $\frac{x^2-9}{x+2}\times\frac{3x+6}{x-3}\div\frac{9}{x}$ **o** $\frac{3x}{9x-6}\div\frac{6x^2}{x-2}\times\frac{2}{x+5}$

SF

3 Express each of the following as a single fraction:

a $\frac{1}{x-3}+\frac{2}{x-3}$ **b** $\frac{2}{x-4}+\frac{2}{x-3}$ **c** $\frac{3}{x+4}+\frac{2}{x-3}$

d $\frac{2x}{x-3}+\frac{2}{x+4}$ **e** $\frac{1}{(x-5)^2}+\frac{2}{x-5}$ **f** $\frac{3x}{(x-4)^2}+\frac{2}{x-4}$

g $\frac{1}{x-3}-\frac{2}{x-3}$ **h** $\frac{2}{x-3}-\frac{5}{x+4}$ **i** $\frac{2x}{x-3}+\frac{3x}{x+3}$

j $\frac{1}{(x-5)^2}-\frac{2}{x-5}$ **k** $\frac{2x}{(x-6)^3}-\frac{2}{(x-6)^2}$ **l** $\frac{2x+3}{x-4}-\frac{2x-4}{x-3}$

Example 18

4 Express each of the following as a single fraction:

a $\sqrt{1-x}+\frac{2}{\sqrt{1-x}}$ **b** $\frac{2}{\sqrt{x-4}}+\frac{2}{3}$ **c** $\frac{3}{\sqrt{x+4}}+\frac{2}{\sqrt{x+4}}$

d $\frac{3}{\sqrt{x+4}}+\sqrt{x+4}$ **e** $\frac{3x^3}{\sqrt{x+4}}-3x^2\sqrt{x+4}$ **f** $\frac{3x^3}{2\sqrt{x+3}}+3x^2\sqrt{x+3}$

Example 19

CF

5 Simplify each of the following:

a $(6x-3)^{\frac{1}{3}}-(6x-3)^{-\frac{2}{3}}$ **b** $(2x+3)^{\frac{1}{3}}-2x(2x+3)^{-\frac{2}{3}}$ **c** $(3-x)^{\frac{1}{3}}-2x(3-x)^{-\frac{2}{3}}$

1H Literal equations

Learning intentions

- To be able to solve literal linear equations and simultaneous linear literal equations in two variables

A literal equation in x is an equation whose solution will be expressed in terms of pronumerals rather than numbers. For the equation $2x + 5 = 7$, the solution is the number 1.

For the literal equation $ax + b = c$, the solution is $x = \frac{c-b}{a}$.

Literal equations are solved in the same way as numerical equations. Essentially, the literal equation is transposed to make x the subject.

Example 20

Solve the following for x:

a $px - q = r$ **b** $ax + b = cx + d$ **c** $\frac{a}{x} = \frac{b}{2x} + c$

Solution

a
$$px - q = r$$
$$px = r + q$$
$$\therefore \quad x = \frac{r+q}{p}$$

b
$$ax + b = cx + d$$
$$ax - cx = d - b$$
$$x(a-c) = d - b$$
$$\therefore \quad x = \frac{d-b}{a-c}$$

c Multiply both sides by $2x$:
$$\frac{a}{x} = \frac{b}{2x} + c$$
$$2a = b + 2xc$$
$$2a - b = 2xc$$
$$\therefore \quad x = \frac{2a-b}{2c}$$

Simultaneous literal equations

Simultaneous literal equations are solved by the same methods that are used for solving simultaneous equations, i.e. substitution and elimination.

Example 21

Solve each of the following pairs of simultaneous equations for x and y:

a $y = ax + c$
$y = bx + d$

b $ax - y = c$
$x + by = d$

Solution

a Equate the two expressions for y:

$$ax + c = bx + d$$
$$ax - bx = d - c$$
$$x(a - b) = d - c$$

Thus $x = \dfrac{d - c}{a - b}$

and
$$y = a\left(\frac{d - c}{a - b}\right) + c = \frac{ad - ac + ac - bc}{a - b} = \frac{ad - bc}{a - b}$$

b We will use the method of elimination, and eliminate y.

First number the two equations:

$$ax - y = c \quad (1)$$
$$x + by = d \quad (2)$$

Multiply (1) by b:

$$abx - by = bc \quad (1')$$

Add (1′) and (2):

$$abx + x = bc + d$$
$$x(ab + 1) = bc + d$$
$$\therefore \quad x = \frac{bc + d}{ab + 1}$$

Substitute in (1):

$$y = ax - c = a\left(\frac{bc + d}{ab + 1}\right) - c = \frac{ad - c}{ab + 1}$$

Summary 1H

- An equation for the variable x in which all the coefficients of x, including the constants, are pronumerals is known as a **literal equation**.
- The methods for solving linear literal equations or simultaneous linear literal equations are exactly the same as when the coefficients are given numbers.

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Exercise 1H

Example 20

1 Solve each of the following for x:

a $ax + n = m$

b $ax + b = bx$

c $\frac{ax}{b} + c = 0$

d $px = qx + 5$

e $mx + n = nx - m$

f $\frac{1}{x + a} = \frac{b}{x}$

g $\frac{b}{x - a} = \frac{2b}{x + a}$

h $\frac{x}{m} + n = \frac{x}{n} + m$

i $-b(ax + b) = a(bx - a)$

j $p^2(1 - x) - 2pqx = q^2(1 + x)$

k $\frac{x}{a} - 1 = \frac{x}{b} + 2$

l $\frac{x}{a - b} + \frac{2x}{a + b} = \frac{1}{a^2 - b^2}$

m $\frac{p - qx}{t} + p = \frac{qx - t}{p}$

n $\frac{1}{x + a} + \frac{1}{x + 2a} = \frac{2}{x + 3a}$

SF

2 For the simultaneous equations $ax + by = p$ and $bx - ay = q$, show that

$x = \frac{ap + bq}{a^2 + b^2}$ and $y = \frac{bp - aq}{a^2 + b^2}$.

3 For the simultaneous equations $\frac{x}{a} + \frac{y}{b} = 1$ and $\frac{x}{b} + \frac{y}{a} = 1$, show that $x = y = \frac{ab}{a + b}$.

Example 21

4 Solve each of the following pairs of simultaneous equations for x and y:

a $ax + y = c$
$x + by = d$

b $ax - by = a^2$
$bx - ay = b^2$

c $ax + by = t$
$ax - by = s$

d $ax + by = a^2 + 2ab - b^2$
$bx + ay = a^2 + b^2$

e $(a + b)x + cy = bc$
$(b + c)y + ax = -ab$

f $3(x - a) - 2(y + a) = 5 - 4a$
$2(x + a) + 3(y - a) = 4a - 1$

CF

5 Write s in terms of a only in the following pairs of equations:

a $s = ah$
$h = 2a + 1$

b $s = ah$
$h = a(2 + h)$

c $as = a + h$
$h + ah = 1$

d $as = s + h$
$ah = a + h$

e $s = h^2 + ah$
$h = 3a^2$

f $as = a + 2h$
$h = a - s$

g $s = 2 + ah + h^2$
$h = a - \frac{1}{a}$

h $3s - ah = a^2$
$as + 2h = 3a$

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Review

Chapter summary

- **Indices**
  - $a^m \times a^n = a^{m+n}$
  - $a^m \div a^n = a^{m-n}$
  - $(a^m)^n = a^{mn}$
  - $(ab)^n = a^n b^n$
  - $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$
  - $a^{-n} = \frac{1}{a^n}$
  - $a^0 = 1$
  - $a^{\frac{1}{n}} = \sqrt[n]{a}$
- A number is expressed in **standard form** or **scientific notation** when written as a product of a number between 1 and 10 and an integer power of 10; e.g. 1.5×10^8.
- **Linear equations**

  First identify the steps done to construct an equation; the equation is then solved by 'undoing' these steps. This is achieved by doing 'the opposite' in 'reverse order'.

  e.g.: Solve $3x + 4 = 16$ for x.

  Note that x has been multiplied by 3 and then 4 has been added.

  Subtract 4 from both sides: $3x = 12$

  Divide both sides by 3: $x = 4$
- An equation that states a relationship between two or more quantities is called a **formula**; e.g. the area of a circle is given by $A = \pi r^2$. The value of A, the subject of the formula, may be found by substituting a given value of r and the value of π.

  A formula can be transposed to make a different variable the subject by using a similar procedure to solving linear equations, i.e. whatever has been done to the variable required is 'undone'.
- A **literal equation** is solved using the same techniques as for a numerical equation: transpose the literal equation to make the required variable the subject.

Skills checklist

Check-list

Download this checklist from the Interactive Textbook, then print it and fill it out to check your skills.

1A **1** I can use the index laws to simplify expressions. ☐

See Example 1, Example 2, Example 3 and Questions 1, 2 and 8

1B **2** I can write numbers in standard form (scientific notation). ☐

See Example 4, Example 5 and Questions 1 and 5

1B **3** I can write numbers to a given number of significant figures. ☐

See Example 6 and Question 6

1C **4** I can solve linear simultaneous equations by using substitution and elimination and by using a calculator. ☐

See Example 7, Example 8 and Questions 1, 2 and 3

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Review

1D **5** I can solve suitable problems posed in a context by using linear equations. ☐

See Example 9, Example 10, Example 11 and Questions 1, 2 and 3

1E **6** I can solve suitable problems posed in a context by using simultaneous linear equations. ☐

See Example 12, Example 13 and Questions 1, 2 and 3

1F **7** I can substitute in formulas and transpose formulas. ☐

See Example 14, Example 15 and Questions 1 and 2

1G **8** I can perform operations on algebraic fractions. ☐

See Example 16, Example 17 and Questions 1 and 2

1G **9** I can perform operations on algebraic fractions involving rational indices. ☐

See Example 18, Example 19 and Questions 4 and 5

1H **10** I can solve linear literal equations. ☐

See Example 20 and Question 1

1H **11** I can solve simultaneous linear literal equations. ☐

See Example 21 and Question 4

Short-response questions

Technology-free short-response questions

SF

1 Simplify the following:

a $(x^3)^4$ **b** $(y^{-12})^{\frac{3}{4}}$ **c** $3x^{\frac{3}{2}} \times -5x^4$ **d** $(x^3)^{\frac{4}{3}} \times x^{-5}$

2 Express the product $23 \times 10^{-6} \times 14 \times 10^{15}$ in standard form.

3 Simplify the following:

a $\dfrac{3x}{5} + \dfrac{y}{10} - \dfrac{2x}{5}$ **b** $\dfrac{4}{x} - \dfrac{7}{y}$ **c** $\dfrac{5}{x+2} + \dfrac{2}{x-1}$

d $\dfrac{3}{x+2} + \dfrac{4}{x+4}$ **e** $\dfrac{5x}{x+4} + \dfrac{4x}{x-2} - \dfrac{5}{2}$ **f** $\dfrac{3}{x-2} - \dfrac{6}{(x-2)^2}$

4 Simplify the following:

a $\dfrac{x+5}{2x-6} \div \dfrac{x^2+5x}{4x-12}$ **b** $\dfrac{3x}{x+4} \div \dfrac{12x^2}{x^2-16}$

c $\dfrac{x^2-4}{x-3} \times \dfrac{3x-9}{x+2} \div \dfrac{9}{x+2}$ **d** $\dfrac{4x+20}{9x-6} \times \dfrac{6x^2}{x+5} \div \dfrac{2}{3x-2}$

Review

CF

5 Phoebe has bought a 3 terabyte external hard drive (3×10^{12} bytes).

a How many photos could be stored on this hard drive if the average size of a photo is 1.5 megabytes (1.5×10^6 bytes)?

b How long would it take to copy 3 terabytes of data onto the hard drive if the data transfer rate is 120 Mbps (120×10^6 bits per second, where 1 byte equals 8 bits)?

6 Swifts Creek Soccer Team has played 54 matches over the past three seasons. They have drawn one-third of their games and won twice as many games as they have lost. How many games have they lost?

7 A music store specialises in three types of CDs: classical, blues and heavy metal. In one week they sold a total of 420 CDs. They sold 10% more classical than blues, while sales of heavy metal CDs constituted 50% more than the combined sales of classical and blues CDs. How many of each type of CD did they sell?

8 The volume, V, of a cylinder is given by the formula $V = \pi r^2 h$, where r is the radius of the base and h is the height of the cylinder.

a Determine the volume of a cylinder with base radius 5 cm and height 12 cm.

b Transpose the formula to make h the subject and hence determine the height of a cylinder with a base radius of 5 cm and a volume of 585 cm^3.

c Transpose the formula to make r the subject and hence determine the radius of a cylinder with a height of 6 cm and a volume of 768 cm^3.

SF

9 Solve for x:

a $xy + ax = b$

b $\dfrac{a}{x} + \dfrac{b}{x} = c$

c $\dfrac{x}{a} = \dfrac{x}{b} + 2$

d $\dfrac{a - dx}{d} + b = \dfrac{ax + d}{b}$

10 Simplify:

a $\dfrac{p}{p+q} + \dfrac{q}{p-q}$

b $\dfrac{1}{x} - \dfrac{2y}{xy - y^2}$

c $\dfrac{x^2 + x - 6}{x + 1} \times \dfrac{2x^2 + x - 1}{x + 3}$

d $\dfrac{2a}{2a + b} \times \dfrac{2ab + b^2}{ba^2}$

11 A is three times as old as B. In three years' time, B will be three times as old as C. In fifteen years' time, A will be three times as old as C. What are their present ages?

CF

12 **a** Solve the following simultaneous equations for a and b:

$$a - 5 = \frac{1}{7}(b + 3) \qquad b - 12 = \frac{1}{5}(4a - 2)$$

b Solve the following simultaneous equations for x and y:

$$(p - q)x + (p + q)y = (p + q)^2$$
$$qx - py = q^2 - pq$$

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13 A man has to travel 50 km in 4 hours. He does it by walking the first 7 km at x km/h, cycling the next 7 km at $4x$ km/h and motoring the remainder at $(6x + 3)$ km/h. Determine x.

14 Simplify each of the following:

a $2n^2 \times 6nk^2 \div (3n)$

b $\dfrac{8c^2x^3y}{6a^2b^3c^3} \div \dfrac{\frac{1}{2}xy}{15abc^2}$

15 Solve the equation $\dfrac{x+5}{15} - \dfrac{x-5}{10} = 1 + \dfrac{2x}{15}$.

16 Simplify each of the following:

a $\left(1 + \dfrac{b}{a}\right) \div \left(1 - \dfrac{a}{b}\right)$

b $\left(1 - \dfrac{c}{d}\right) \div \left(1 - \dfrac{d}{c}\right)$

c $\left(x + \dfrac{x}{y}\right) \div \left(x - \dfrac{x}{y}\right)$

d $\left(\dfrac{1}{p} - \dfrac{1}{q}\right) \div \left(\dfrac{p}{q} - \dfrac{q}{p}\right)$

17 Determine p, in terms of a and b, if $\dfrac{a+p}{b-p} = \dfrac{4a}{3b}$.

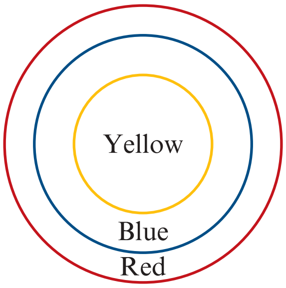
18 Henry and Thomas Wong collect basketball cards. Henry has five-sixths the number of cards that Thomas has. The Wright family also collect cards. George Wright has half as many cards again as Thomas, Sally Wright has 18 fewer than Thomas, and Zeb Wright has one-third the number Thomas has.

a Write an expression for each child's number of cards in terms of the number Thomas has.

b The Wright family owns six more cards than the Wong family. Write an equation representing this information.

c Solve the equation from part **b** and use the result to determine the number of cards each child has collected

19 A new advertising symbol is to consist of three concentric circles as shown, with the outer circle having a radius of 10 cm. It is desired that the three coloured regions cover the same area. Determine the radius of the innermost circle in the figure shown.



20 Temperatures in Fahrenheit (F) can be converted to Celsius (C) by the formula

$$C = \frac{5}{9}(F - 32)$$

Determine the temperature which has the same numerical value in both scales.

21 A cyclist goes up a long slope at a constant speed of 15 km/h. He turns around and comes down the slope at a constant speed of 40 km/h. Determine his average speed over a full circuit.

Review

Technology-active short-response questions

SF

22 The volume, V cm^3, of a cylinder with radius r cm and height h cm is given by the formula $V = \pi r^2 h$. Determine the value of h correct to 2 decimal places if $V = 8.6$ and $r = 2.7$.

23 A sum of \$50 000 is invested for 6 years compounded annually at 6.24%. How much is the investment worth at the end of this period?

24 When an object is shot up into the air, from ground level, with a speed of u m/s, its height above the ground h m and time of flight t seconds are related by the equation $h = ut - 4.9t^2$. Determine the speed that the object was fired if it reached a height of 28.2 m after 5.2 seconds. Give your answer in m/s correct to one decimal place.

25 The volume V cm^3 of metal in a tube is given by the formula $V = \pi\ell(r^2 - (r - m)^2)$ where ℓ cm is the length of the tube, r cm is the radius of the outside surface and m cm is the thickness of the material. Determine r, correct to two decimal places, if $V = 80$, $\ell = 10$ and $m = 0.18$.

26 Evaluate each of the following, giving answers in scientific notation correct to 3 significant figures:

a 3.56×0.078

b $0.0275^2 \times \sqrt{0.678}$

c $(5.64 \times 10^{-8}) \div (3.425 \times 10^{-6})$

d $(6.657 \times 10^{-8})^2$

27 How long does it take light to travel to the Earth from the Sun in seconds, assuming that the Earth is 1.5208×10^8 km from the Sun and the speed of light is approximately 2.9972×10^5 km/s? Give your answer in scientific notation correct to 2 significant figures.

CF

28 The area of a rectangle is reduced by 9 cm^2 if its length is reduced by 5 cm and the width is increased by 3 cm. If we increase the length by 3 cm and the width by 2 cm, then the area is increased by 69 cm^2. Determine the dimensions of the original rectangle. Give your answers correct to one decimal place.

CU

29 Jack cycles home from work, a distance of $10x$ km. Benny leaves at the same time and drives the $40x$ km to his home.

a Write an expression in terms of x for the time taken for Jack to reach home if he cycles at an average speed of 8 km/h.

b Write an expression in terms of x for the time taken for Benny to reach home if he drives at an average speed of 70 km/h.

c In terms of x, determine the difference in times of the two journeys.

d If Jack and Benny arrive at their homes 30 minutes apart:

  i Determine x, correct to three decimal places

  ii Determine the distance from work of each home, correct to the nearest kilometre.

CU

30 Sam's plastic dinghy has sprung a leak and water is pouring in the hole at a rate of 27 000 cm^3 per minute. He grabs a cup and frantically starts bailing the water out at a rate of 9000 cm^3 per minute. The dinghy is shaped like a circular prism (cylinder) with a base radius of 40 cm and a height of 30 cm.

a How fast is the dinghy filling with water?

b Write an equation showing the volume of water, V cm^3, in the dinghy after t minutes.

c Determine an expression for the depth of water, h cm, in the dinghy after t minutes.

d If Sam is rescued after 9 minutes, is this before or after the dinghy has completely filled with water?

31 The gravitational force between two objects, F N, is given by the formula

$$F = \frac{6.67 \times 10^{-11} m_1 m_2}{r^2}$$

where m_1 and m_2 are the masses (in kilograms) of the two objects and r is the distance (in metres) between them.

a What is the gravitational force between two objects each weighing 200 kg if they are 12 m apart? Express the answer in standard form (to two significant figures).

b Transpose the above formula to make m_1 the subject.

c The gravitational force between a planet and an object 6.4×10^6 m away from the centre of the planet is found to be 2.4×10^4 N. If the object has a mass of 1500 kg, calculate the approximate mass of the planet, giving the answer in standard form (to two significant figures).

32 A water storage reservoir is 3 km wide, 6 km long and 30 m deep. (The water storage reservoir is assumed to be a cuboid.)

a Write an equation to show the volume of water, V m^3, in the reservoir when it is d metres full.

b Calculate the volume of water, V_F m^3, in the reservoir when it is completely filled.

The water flows from the reservoir down a long pipe to a hydro-electric power station in a valley below. The amount of energy, E J, that can be obtained from a full reservoir is given by the formula

$$E = kV_F h$$

where k is a constant and h m is the length of the pipe.

c Determine k, given that $E = 1.06 \times 10^{15}$ when $h = 200$, expressing the answer in standard form correct to three significant figures.

d How much energy could be obtained from a full reservoir if the pipe was 250 m long?

e If the rate of water falling through the pipe is 5.2 m^3/s, how many days without rain could the station operate before emptying an initially full reservoir?

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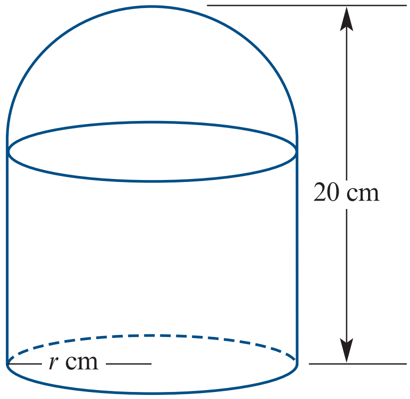
Review

CU

33 A container has a cylindrical base and a hemispherical top, as shown in the figure. The height of the container is 20 cm and its capacity is to be exactly 2 litres. Let r cm be the radius of the base.

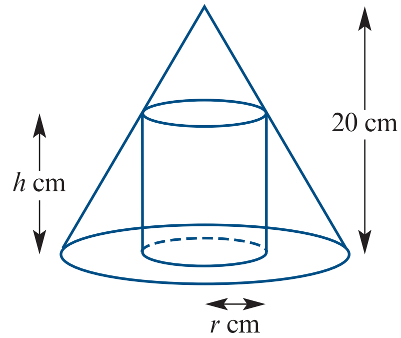


- **a** Express the height of the cylinder, h cm, in terms of r.
- **b** **i** Express the volume of the container in terms of r.
  - **ii** Determine r and h if the volume is 2 litres.

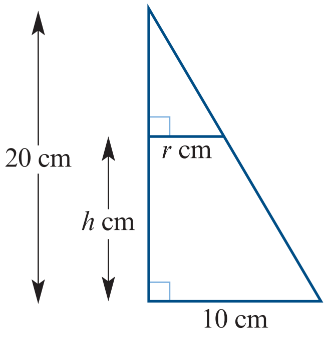
34 **a** Two bottles contain mixtures of wine and water. In bottle A there is two times as much wine as water. In bottle B there is five times as much water as wine. Bottle A and bottle B are used to fill a third bottle, which has a capacity of 1 litre. How much liquid must be taken from each of bottle A and bottle B if the third bottle is to contain equal amounts of wine and water?

- **b** Repeat for the situation where the ratio of wine to water in bottle A is $1 : 2$ and the ratio of wine to water in bottle B is $3 : 1$.
- **c** Generalise the result for the ratio $m : n$ in bottle A and $p : q$ in bottle B.

35 A cylinder is placed so as to fit into a cone as shown in the diagram.
The cone has a height of 20 cm and a base radius of 10 cm. The cylinder has a height of h cm and a base radius of r cm.



- **a** Use similar triangles to determine h in terms of r.
- **b** The volume of the cylinder is given by the formula $V = \pi r^2 h$. Determine the volume of the cylinder in terms of r.
- **c** Use a calculator to determine the values of r and h for which the volume of the cylinder is 500 cm^3.



Multiple-choice questions

Technology-free multiple-choice questions

1 For non-zero values of x and y, if $5x + 2y = 0$, then the ratio $\frac{y}{x}$ is equal to

A $-\frac{5}{2}$ **B** $-\frac{2}{5}$ **C** $\frac{2}{5}$ **D** 1

2 The solution of the simultaneous equations $3x + 2y = 36$ and $3x - y = 12$ is

A $x = \frac{20}{3},\ y = 8$ **B** $x = 2,\ y = 0$

C $x = 1,\ y = -3$ **D** $x = \frac{20}{3},\ y = 6$

3 The solution of the equation $t - 9 = 3t - 17$ is

A $t = -4$ **B** $t = \frac{11}{2}$ **C** $t = 4$ **D** $t = 2$

4 If $m = \frac{n-p}{n+p}$, then $p =$

A $\frac{n(1-m)}{1+m}$ **B** $\frac{n(m-1)}{1+m}$ **C** $\frac{n(1+m)}{1-m}$ **D** $\frac{n(1+m)}{m-1}$

5 $\frac{3}{x-3} - \frac{2}{x+3} =$

A 1 **B** $\frac{x+15}{x^2-9}$ **C** $\frac{15}{x-9}$ **D** $\frac{x+3}{x^2-9}$

6 $9x^2y^3 \div (15(xy)^3)$ is equal to

A $\frac{18xy}{5}$ **B** $\frac{3y}{5x}$ **C** $\frac{3x}{5}$ **D** $\frac{3}{5x}$

7 Transposing the formula $V = \frac{1}{3}h(\ell + w)$ gives $\ell =$

A $\frac{hw}{3V}$ **B** $\frac{3V}{h} - w$ **C** $\frac{3V-2w}{h}$ **D** $\frac{3Vh}{2} - w$

8 $\frac{(3x^2y^3)^2}{2x^2y} =$

A $\frac{9}{2}x^2y^7$ **B** $\frac{9}{2}x^2y^5$ **C** $\frac{9}{2}x^6y^7$ **D** $\frac{9}{2}x^6y^6$

9 If X is 50% greater than Y and Y is 20% less than Z, then

A X is 30% greater than Z **B** X is 20% greater than Z

C X is 20% less than Z **D** X is 10% less than Z

10 The average of two numbers is $5x + 4$. One of the numbers is x. The other number is

A $4x + 4$ **B** $9x + 8$ **C** $9x + 4$ **D** $10x + 8$

Technology-active multiple-choice questions

11 The variables x and y satisfy the simultaneous equations

$$0.5x - 0.7y = 2.2$$
$$1.8x - 2.2y = 8.8$$

The values of $x + y$ is

A 11 **B** 44 **C** 22 **D** 33

12 $\dfrac{\sqrt{1.21 \times 10^{-6}}}{2 \times 10^{-2}}$ is equal to

A 1.1×10^{-5} **B** 1.74×10^{-4} **C** 5.5×10^{-2} **D** 1.74

13 If $a = 2$ and $x = -2.1$, the value of $ax^2 + a^2x^3$ is

A 28.224 **B** 45.864 **C** 32.674 **D** -28.224

14 When 0.0017449 is expressed to 3 significant figures it becomes

A 0.00170 **B** 0.00174 **C** 0.00175 **D** 0.00180

15 What is the maximum number of solid iron cubes of side length 2 cm that could be cast from an ingot of iron in the shape of a cylinder of radius 2 cm and length 13.9 cm.

A 11 **B** 21 **C** 44 **D** 66

16 The velocity, V m/s, of sound in air at $t°$ C is obtainable from the formula

$$V = 332\sqrt{1 + 0.00367 \times t}$$

If $V = 352$ the value of t correct to three decimal places is

A 33.817 **B** 32.754 **C** 32.755 **D** 33.818

17 Places A and B are 100 km apart on a straight road. One car starts from A and another from B at the same time. If the cars travel in the same direction at speeds x km/h and y km/h ($x > y$) they meet in 5 hours. If the cars travel towards each other at speeds x km/h and y km/h ($x > y$) they meet in one hour. The values of x and y are

A $x = 60, y = 40$ **B** $x = 55, y = 45$
C $x = 58, y = 48$ **D** $x = 62, y = 48$

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2

Number systems and sets

Chapter contents

This chapter is provided as preparation for Chapter 7 Number and proof. It also contains material on how to change a recurring decimal into fraction form and vice versa.

This chapter introduces set notation and discusses sets of numbers and their properties. Set notation is used widely in mathematics and in this book it is employed where appropriate. In this chapter we discuss natural numbers, integers and rational numbers, and then continue on to consider irrational numbers.

Irrational numbers such as $\sqrt{2}$ naturally arise when applying Pythagoras' theorem. When solving a quadratic equation, using the method of completing the square or the quadratic formula, we obtain answers such as $x = \frac{1}{2}(1 \pm \sqrt{5})$. These numbers involve surds.

Since these numbers are irrational, we cannot express them in exact form using decimals or fractions. Sometimes we may wish to approximate them using decimals, but mostly we prefer to leave them in exact form. Thus we need to be able to manipulate these types of numbers and to simplify combinations of them which arise when solving a problem.

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2A Set notation

Learning intentions

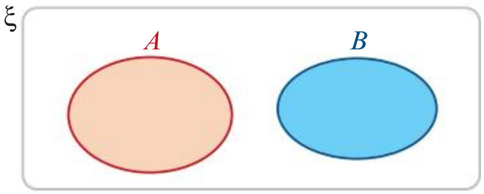
- To be able to use set notation.

A **set** is a general name for any collection of things or numbers. There must be a way of deciding whether any particular object is a member of the set or not. This may be done by referring to a list of the members of the set or a statement describing them. For example:
$A = \{-3, 3\} = \{ x : x^2 = 9 \}$

Note: $\{ x : \dots \}$ is read as 'the set of all x such that ...'.

- The symbol $\in$ means 'is a member of' or 'is an element of'.
  For example: $3 \in$ {prime numbers} is read '3 is a member of the set of prime numbers'.
- The symbol $\notin$ means 'is not a member of' or 'is not an element of'.
  For example: $4 \notin$ {prime numbers} is read '4 is not a member of the set of prime numbers'.
- Two sets are **equal** if they contain exactly the same elements, not necessarily in the same order. For example: if A = {prime numbers less than 10} and $B = \{2, 3, 5, 7\}$, then $A = B$.
- The set with no elements is called the **empty set** and is denoted by $\varnothing$.
- The **universal set** will be denoted by ξ. The universal set is the set of all elements which are being considered.
- If all the elements of a set B are also elements of a set A, then the set B is called a **subset** of A. This is written $B \subseteq A$. For example: $\{a, b, c\} \subseteq \{a, b, c, d, e, f, g\}$ and $\{3, 9, 27\} \subseteq$ {multiples of 3}. We note also that $A \subseteq A$ and $\varnothing \subseteq A$.

Venn diagrams are used to illustrate sets. For example, the diagram on the right shows two subsets A and B of a universal set ξ such that A and B have no elements in common. Two such sets are said to be **disjoint**.



The union of two sets

The set of all the elements that are members of set A or set B (or both) is called the **union** of A and B. The union of A and B is written $A \cup B$.

Example 1

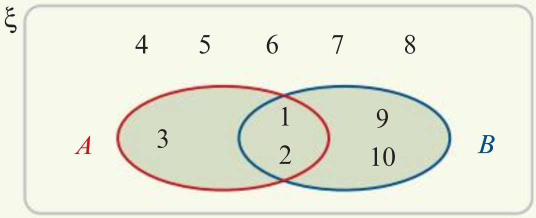
Let $\xi = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, $A = \{1, 2, 3\}$ and $B = \{1, 2, 9, 10\}$.

Determine $A \cup B$ and illustrate on a Venn diagram.

Solution

$A \cup B = \{1, 2, 3, 9, 10\}$

The shaded area illustrates $A \cup B$.



The intersection of two sets

The set of all the elements that are members of both set A and set B is called the **intersection** of A and B. The intersection of A and B is written $A \cap B$.

Example 2

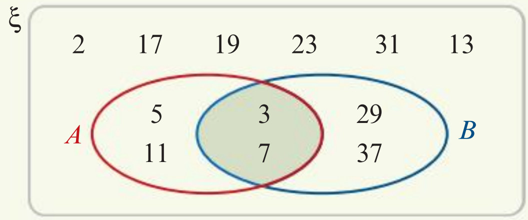
Let $\xi = \{\text{prime numbers less than } 40\}$, $A = \{3, 5, 7, 11\}$ and $B = \{3, 7, 29, 37\}$.

Determine $A \cap B$ and illustrate on a Venn diagram.

Solution

$A \cap B = \{3, 7\}$

The shaded area illustrates $A \cap B$.



The complement of a set

The **complement** of a set A is the set of all elements of ξ that are not members of A. The complement of A is denoted by A'.

If $\xi = \{\text{students at Highland Secondary College}\}$ and $A = \{\text{students with blue eyes}\}$, then A' is the set of all students at Highland Secondary College who do not have blue eyes.

Similarly, if $\xi = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and $A = \{1, 3, 5, 7, 9\}$, then $A' = \{2, 4, 6, 8, 10\}$.

Example 3

Let $\xi = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

$A = \{\text{odd numbers}\} = \{1, 3, 5, 7, 9\}$

$B = \{\text{multiples of } 3\} = \{3, 6, 9\}$

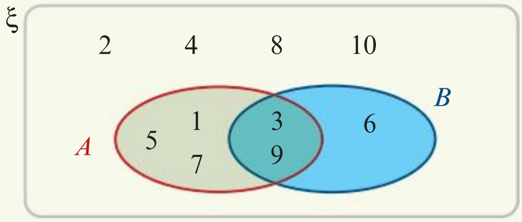
a Show these sets on a Venn diagram.

b Use the diagram to list the following sets:

i A' **ii** B' **iii** $A \cup B$ **iv** the complement of $A \cup B$, i.e. $(A \cup B)'$ **v** $A' \cap B'$

Solution

a



b From the diagram:

i $A' = \{2, 4, 6, 8, 10\}$

ii $B' = \{1, 2, 4, 5, 7, 8, 10\}$

iii $A \cup B = \{1, 3, 5, 6, 7, 9\}$

iv $(A \cup B)' = \{2, 4, 8, 10\}$

v $A' \cap B' = \{2, 4, 8, 10\}$

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Finite and infinite sets

When all the elements of a set may be counted, the set is called a **finite** set. For example, the set $A = \{\text{months of the year}\}$ is finite. The number of elements of a set A will be denoted $|A|$. In this example, $|A| = 12$. If $C = \{\text{letters of the alphabet}\}$, then $|C| = 26$.

Sets which are not finite are called **infinite** sets. For example, the set of real numbers, $\mathbb{R}$, and the set of integers, $\mathbb{Z}$, are infinite sets.

Summary 2A

- If x is an element of a set A, we write $x \in A$.
- If x is not an element of a set A, we write $x \notin A$.
- The **empty set** is denoted by $\varnothing$ and the **universal set** by ξ.
- If every element of B is an element of A, we say B is a **subset** of A and write $B \subseteq A$.
- The set $A \cup B$ is the **union** of A and B, where $x \in A \cup B$ if and only if $x \in A$ or $x \in B$.
- The set $A \cap B$ is the **intersection** of A and B, where $x \in A \cap B$ if and only if $x \in A$ and $x \in B$.
- The **complement** of A, denoted by A', is the set of all elements of ξ that are not in A.
- If two sets A and B have no elements in common, we say that they are **disjoint** and write $A \cap B = \varnothing$.

Exercise 2A

Example 1–3

SF

1 Let $\xi = \{1, 2, 3, 4, 5\}$, $A = \{1, 2, 3, 5\}$ and $B = \{2, 4\}$.
Show these sets on a Venn diagram and use the diagram to determine:

a A' **b** B' **c** $A \cup B$ **d** $(A \cup B)'$ **e** $A' \cap B'$

2 Let $\xi = \{\text{natural numbers less than } 17\}$, $P = \{\text{multiples of } 3\}$ and $Q = \{\text{even numbers}\}$.
Show these sets on a Venn diagram and use it to determine:

a P' **b** Q' **c** $P \cup Q$ **d** $(P \cup Q)'$ **e** $P' \cap Q'$

3 Let $\xi = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$, $A = \{\text{multiples of } 4\}$ and $B = \{\text{even numbers}\}$.
Show these sets on a Venn diagram and use this diagram to list the sets:

a A' **b** B' **c** $A \cup B$ **d** $(A \cup B)'$ **e** $A' \cap B'$

4 Let $\xi = \{\text{natural numbers from } 10 \text{ to } 25\}$, $P = \{\text{multiples of } 4\}$ and $Q = \{\text{multiples of } 5\}$.
Show these sets on a Venn diagram and use this diagram to list the sets:

a P' **b** Q' **c** $P \cup Q$ **d** $(P \cup Q)'$ **e** $P' \cap Q'$

5 Let $\xi = \{p, q, r, s, t, u, v, w\}$, $X = \{r, s, t, w\}$ and $Y = \{q, s, t, u, v\}$.
Show ξ, X and Y on a Venn diagram, entering all members. Hence list the sets:

a X' **b** Y' **c** $X' \cap Y'$ **d** $X' \cup Y'$ **e** $X \cup Y$ **f** $(X \cup Y)'$

Which two sets are equal?

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SF

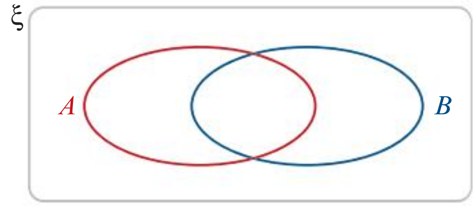
6 Let $\xi = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$, $X = \{\text{factors of } 12\}$ and $Y = \{\text{even numbers}\}$. Show ξ, X and Y on a Venn diagram, entering all members. Hence list the sets:

a X' **b** Y' **c** $X' \cup Y'$ **d** $(X \cap Y)'$ **e** $X \cup Y$ **f** $(X \cup Y)'$

Which two sets are equal?

7 Draw this diagram six times.
Use shading to illustrate each of the following sets:

a A' **b** B' **c** $A' \cap B'$
d $A' \cup B'$ **e** $A \cup B$ **f** $(A \cup B)'$



8 Let ξ = {different letters in the word *GENERAL*},
A = {different letters in the word *ANGEL*},
B = {different letters in the word *LEAN*}

Show these sets on a Venn diagram and use this diagram to list the sets:

a A' **b** B' **c** $A \cap B$ **d** $A \cup B$ **e** $(A \cup B)'$ **f** $A' \cup B'$

9 Let ξ = {different letters in the word *MATHEMATICS*}
A = {different letters in the word *ATTIC*}
B = {different letters in the word *TASTE*}

Show ξ, A and B on a Venn diagram, entering all the elements. Hence list the sets:

a A' **b** B' **c** $A \cap B$ **d** $(A \cup B)'$ **e** $A' \cup B'$ **f** $A' \cap B'$

2B Sets of numbers

Learning intentions

- To be familiar with the notation for sets of numbers, $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ and $\mathbb{R}$.
- To be able to write a rational number as a recurring decimal and vice versa.
- To be able to use interval notation.

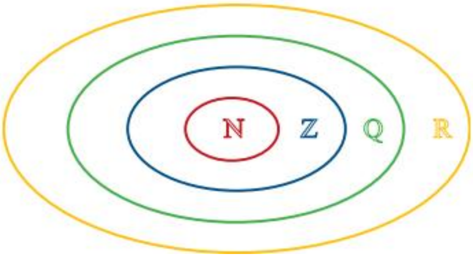
Recall that the elements of $\{1, 2, 3, 4, \dots\}$ are called **natural numbers**, and the elements of $\{\dots, -2, -1, 0, 1, 2, \dots\}$ are called **integers**.

The numbers of the form $\frac{p}{q}$, with p and q integers, $q \neq 0$, are called **rational numbers**.

The real numbers which are not rational are called **irrational**. Some examples of irrational numbers are $\sqrt{2}$, $\sqrt{3}$, π, $\pi + 2$ and $\sqrt{6} + \sqrt{7}$. These numbers cannot be written in the form $\frac{p}{q}$, for integers p, q; the decimal representations of these numbers do not terminate or repeat.

- The set of real numbers is denoted by $\mathbb{R}$.
- The set of rational numbers is denoted by $\mathbb{Q}$.
- The set of integers is denoted by $\mathbb{Z}$.
- The set of natural numbers is denoted by $\mathbb{N}$.

It is clear that $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$, and this may be represented by the diagram on the right.



We can use set notation to describe subsets of the real numbers.

For example:

- $\{ x : 0 < x < 1 \}$ is the set of all real numbers strictly between 0 and 1
- $\{ x : x \geq 3 \}$ is the set of all real numbers greater than or equal to 3
- $\{ 2n : n \in \mathbb{Z} \}$ is the set of all even integers.

The set of all ordered pairs of real numbers is denoted by $\mathbb{R}^2$. That is,

$$\mathbb{R}^2 = \{ (x, y) : x, y \in \mathbb{R} \}$$

This set is known as the **Cartesian product** of $\mathbb{R}$ with itself.

Rational numbers

Every rational number can be expressed as a terminating or recurring decimal.

To determine the decimal representation of a rational number $\frac{m}{n}$, perform the division $m \div n$.

For example, to determine the decimal representation of $\frac{3}{7}$, divide $3.0000000\ldots$ by 7.

$$\begin{array}{r|l} & 0.\;\;4\;\;2\;\;8\;\;5\;\;7\;\;1\;\;4\;\ldots \\ \hline 7 & 3.\;{}^{3}0\;{}^{2}0\;{}^{6}0\;{}^{4}0\;{}^{5}0\;{}^{1}0\;{}^{3}0\;\ldots \end{array}$$

Therefore $\frac{3}{7} = 0.\dot{4}2857\dot{1}$.

Theorem

Every rational number can be written as a terminating or recurring decimal.

Proof Consider any two natural numbers m and n. At each step in the division of m by n, there is a remainder. If the remainder is 0, then the division algorithm stops and the decimal is a terminating decimal.

If the remainder is never 0, then it must be one of the numbers $1, 2, 3, \ldots, n - 1$. (In the above example, $n = 7$ and the remainders can only be 1, 2, 3, 4, 5 and 6.) Hence the remainder must repeat after at most $n - 1$ steps.

Further examples:

$$\frac{1}{2} = 0.5, \quad \frac{1}{5} = 0.2, \quad \frac{1}{10} = 0.1, \quad \frac{1}{3} = 0.\dot{3}, \quad \frac{1}{7} = 0.\dot{1}4285\dot{7}$$

Theorem

A real number has a terminating decimal representation if and only if it can be written as

$$\frac{m}{2^{\alpha} \times 5^{\beta}}$$

for some $m \in \mathbb{Z}$ and some $\alpha, \beta \in \mathbb{N} \cup \{0\}$.

Proof Assume that $x = \dfrac{m}{2^\alpha \times 5^\beta}$ with $\alpha \geq \beta$. Multiply the numerator and denominator by $5^{\alpha-\beta}$. Then

$$x = \frac{m \times 5^{\alpha-\beta}}{2^\alpha \times 5^\alpha} = \frac{m \times 5^{\alpha-\beta}}{10^\alpha}$$

and so x can be written as a terminating decimal. The case $\alpha < \beta$ is similar.

Conversely, if x can be written as a terminating decimal, then there is $m \in \mathbb{Z}$ and $\alpha \in \mathbb{N} \cup \{0\}$ such that $x = \dfrac{m}{10^\alpha} = \dfrac{m}{2^\alpha \times 5^\alpha}$.

The method for determining a rational number $\dfrac{m}{n}$ from its decimal representation is demonstrated in the following example.

Example 4

Write each of the following in the form $\dfrac{m}{n}$, where m and n are integers:

a 0.05 **b** $0.\dot{4}2857\dot{1}$

Solution

a $0.05 = \dfrac{5}{100} = \dfrac{1}{20}$

b We can write

$$0.\dot{4}2857\dot{1} = 0.428571428571\ldots \qquad (1)$$

Multiply both sides by 10^6:

$$0.\dot{4}2857\dot{1} \times 10^6 = 428571.428571428571\ldots \qquad (2)$$

Subtract (1) from (2):

$$0.\dot{4}2857\dot{1} \times (10^6 - 1) = 428571$$

$$\therefore \quad 0.\dot{4}2857\dot{1} = \frac{428571}{10^6 - 1}$$

$$= \frac{3}{7}$$

Real numbers

The set of real numbers is made up of two important subsets: the **algebraic numbers** and the **transcendental numbers**.

An algebraic number is a solution to a polynomial equation of the form

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0, \quad \text{where } a_0, a_1, \ldots, a_n \text{ are integers}$$

Every rational number is algebraic. The irrational number $\sqrt{2}$ is algebraic, as it is a solution of the equation

$$x^2 - 2 = 0$$

It can be shown that π is not an algebraic number; it is a transcendental number. The proof is too difficult to be given here.

The proof that $\sqrt{2}$ is irrational is presented in Chapter 8.

Interval notation

Among the most important subsets of $\mathbb{R}$ are the **intervals**. The following is an exhaustive list of the various types of intervals and the standard notation for them. We suppose that a and b are real numbers with $a < b$.

$$(a, b) = \{ x : a < x < b \} \qquad [a, b] = \{ x : a \le x \le b \}$$
$$(a, b] = \{ x : a < x \le b \} \qquad [a, b) = \{ x : a \le x < b \}$$
$$(a, \infty) = \{ x : a < x \} \qquad [a, \infty) = \{ x : a \le x \}$$
$$(-\infty, b) = \{ x : x < b \} \qquad (-\infty, b] = \{ x : x \le b \}$$

Intervals may be represented by diagrams as shown in Example 5.

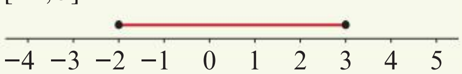
Example 5

Illustrate each of the following intervals of real numbers:

a $[-2, 3]$ **b** $(-3, 4]$ **c** $(-\infty, 2]$ **d** $(-2, 4)$ **e** $(-3, \infty)$

Solution

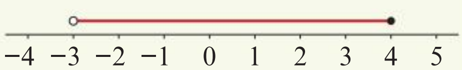
a $[-2, 3]$



Explanation

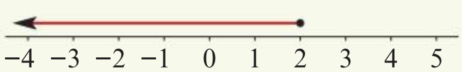
The square brackets indicate that the endpoints are included; this is shown with closed circles.

b $(-3, 4]$



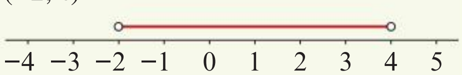
The round bracket indicates that the left endpoint is not included; this is shown with an open circle. The right endpoint is included.

c $(-\infty, 2]$



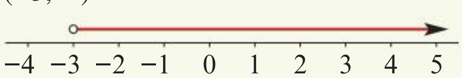
The symbol $-\infty$ indicates that the interval continues indefinitely (i.e. forever) to the left; it is read as ‘negative infinity’. The right endpoint is included.

d $(-2, 4)$



Both brackets are round; the endpoints are not included.

e $(-3, \infty)$



The symbol ∞ indicates that the interval continues indefinitely (i.e. forever) to the right; it is read as ‘infinity’. The left endpoint is not included.

Note: The ‘closed’ circle (●) indicates that the number is included.
The ‘open’ circle (○) indicates that the number is not included.

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The following are subsets of the real numbers for which we have special notation:

- Positive real numbers: $\mathbb{R}^+ = \{ x : x > 0 \}$
- Negative real numbers: $\mathbb{R}^- = \{ x : x < 0 \}$
- Real numbers excluding zero: $\mathbb{R} \setminus \{0\}$

Summary 2B

- Sets of numbers
  - Real numbers: $\mathbb{R}$
  - Rational numbers: $\mathbb{Q}$
  - Integers: $\mathbb{Z}$
  - Natural numbers: $\mathbb{N}$
- For real numbers a and b with $a < b$, we can consider the following **intervals**:

$$(a, b) = \{ x : a < x < b \} \qquad [a, b] = \{ x : a \le x \le b \}$$

$$(a, b] = \{ x : a < x \le b \} \qquad [a, b) = \{ x : a \le x < b \}$$

$$(a, \infty) = \{ x : a < x \} \qquad [a, \infty) = \{ x : a \le x \}$$

$$(-\infty, b) = \{ x : x < b \} \qquad (-\infty, b] = \{ x : x \le b \}$$

Exercise 2B

SF

1 **a** Is the sum of two rational numbers also rational?
b Is the product of two rational numbers also rational?
c Is the quotient of two rational numbers also rational (if defined)?

2 **a** Is the sum of two irrational numbers always irrational?
b Is the product of two irrational numbers always irrational?
c Is the quotient of two irrational numbers always irrational?

Example 4

3 Write each of the following in the form $\frac{m}{n}$, where m and n are integers:

a 0.45 **b** $0.\dot{2}\dot{7}$ **c** 0.12
d $0.\dot{2}8571\dot{4}$ **e** $0.\dot{3}\dot{6}$ **f** $0.\dot{2}$

4 Give the decimal representation of each of the following rational numbers:

a $\frac{2}{7}$ **b** $\frac{5}{11}$ **c** $\frac{7}{20}$ **d** $\frac{4}{13}$ **e** $\frac{1}{17}$

Example 5

5 Illustrate each of the following intervals of real numbers:

a $[-1, 4]$ **b** $(-2, 2]$ **c** $(-\infty, 3]$ **d** $(-1, 5)$ **e** $(-2, \infty)$

6 Write each of the following sets using interval notation:

a $\{ x : x < 3 \}$ **b** $\{ x : x \ge -3 \}$ **c** $\{ x : x \le -3 \}$
d $\{ x : x > 5 \}$ **e** $\{ x : -2 \le x < 3 \}$ **f** $\{ x : -2 \le x \le 3 \}$
g $\{ x : -2 < x \le 3 \}$ **h** $\{ x : -5 < x < 3 \}$

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2C Surds

Learning intentions

- To be able to use the arithmetic of surds including the rationalisation of denominators.

A **quadratic surd** is a number of the form $\sqrt{a}$, where a is a rational number which is not the square of another rational number.

Note: $\sqrt{a}$ is taken to mean the positive square root.

In general, a **surd of order *n*** is a number of the form $\sqrt[n]{a}$, where a is a rational number which is not a perfect nth power.

For example:

- $\sqrt{7}, \sqrt{24}, \sqrt{\frac{9}{7}}, \sqrt{\frac{1}{2}}$ are quadratic surds
- $\sqrt{9}, \sqrt{16}, \sqrt{\frac{9}{4}}$ are *not* surds
- $\sqrt[3]{7}, \sqrt[3]{15}$ are surds of order 3
- $\sqrt[4]{100}, \sqrt[4]{26}$ are surds of order 4

Quadratic surds hold a prominent position in school mathematics. For example, the solutions of quadratic equations often involve surds:

$x = \dfrac{1+\sqrt{5}}{2}$ is a solution of the quadratic equation $x^2 - x - 1 = 0$.

Some well-known values of trigonometric functions involve surds. For example:

$$\sin 60^\circ = \frac{\sqrt{3}}{2}, \qquad \sin 15^\circ = \frac{\sqrt{6}-\sqrt{2}}{4}$$

Exact solutions are often required in Mathematical Methods Units 3 & 4 and Specialist Mathematics Units 3 & 4.

Properties of square roots

The following properties of square roots are often used.

For positive numbers a and b:

- $\sqrt{ab} = \sqrt{a}\sqrt{b}$ e.g. $\sqrt{50} = \sqrt{25} \times \sqrt{2} = 5\sqrt{2}$
- $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$ e.g. $\sqrt{\frac{7}{9}} = \frac{\sqrt{7}}{\sqrt{9}} = \frac{\sqrt{7}}{3}$

Properties of surds

Simplest form

If possible, a factor which is the square of a rational number is 'taken out' of a square root. When the number under the square root has no factors which are squares of a rational number, the surd is said to be in **simplest form**.

Example 6

Write each of the following in simplest form:

a $\sqrt{72}$ **b** $\sqrt{28}$ **c** $\sqrt{\dfrac{700}{117}}$ **d** $\sqrt{\dfrac{99}{64}}$

Solution

a $\sqrt{72} = \sqrt{36 \times 2} = 6\sqrt{2}$

b $\sqrt{28} = \sqrt{4 \times 7} = 2\sqrt{7}$

c $\sqrt{\dfrac{700}{117}} = \dfrac{\sqrt{700}}{\sqrt{117}} = \dfrac{\sqrt{7 \times 100}}{\sqrt{9 \times 13}}$

$= \dfrac{10}{3}\sqrt{\dfrac{7}{13}}$

d $\sqrt{\dfrac{99}{64}} = \dfrac{\sqrt{99}}{\sqrt{64}} = \dfrac{\sqrt{9 \times 11}}{8}$

$= \dfrac{3\sqrt{11}}{8}$

Like surds

Surds which have the same 'irrational factor' are called **like surds**.

For example: $3\sqrt{7}$, $2\sqrt{7}$ and $\sqrt{7}$ are like surds.

The sum or difference of two like surds can be simplified:

- $m\sqrt{p} + n\sqrt{p} = (m + n)\sqrt{p}$
- $m\sqrt{p} - n\sqrt{p} = (m - n)\sqrt{p}$

Example 7

Express each of the following as a single surd in simplest form:

a $\sqrt{147} + \sqrt{108} - \sqrt{363}$

b $\sqrt{3} + \sqrt{5} + \sqrt{20} + \sqrt{27} - \sqrt{45} - \sqrt{48}$

c $\sqrt{50} + \sqrt{2} - 2\sqrt{18} + \sqrt{8}$

Solution

a $\sqrt{147} + \sqrt{108} - \sqrt{363}$

$= \sqrt{7^2 \times 3} + \sqrt{6^2 \times 3} - \sqrt{11^2 \times 3}$

$= 7\sqrt{3} + 6\sqrt{3} - 11\sqrt{3}$

$= 2\sqrt{3}$

b $\sqrt{3} + \sqrt{5} + \sqrt{20} + \sqrt{27} - \sqrt{45} - \sqrt{48}$

$= \sqrt{3} + \sqrt{5} + 2\sqrt{5} + 3\sqrt{3} - 3\sqrt{5} - 4\sqrt{3}$

$= 0\sqrt{3} + 0\sqrt{5}$

$= 0$

c $\sqrt{50} + \sqrt{2} - 2\sqrt{18} + \sqrt{8}$

$= 5\sqrt{2} + \sqrt{2} - 2 \times 3\sqrt{2} + 2\sqrt{2}$

$= 8\sqrt{2} - 6\sqrt{2}$

$= 2\sqrt{2}$

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Rationalising the denominator

In the past, a labour-saving procedure with surds was to **rationalise** any surds in the denominator of an expression. This is still considered to be a neat way of expressing final answers.

For $\sqrt{5}$, a rationalising factor is $\sqrt{5}$, as $\sqrt{5} \times \sqrt{5} = 5$.

For $1 + \sqrt{2}$, a rationalising factor is $1 - \sqrt{2}$, as $(1 + \sqrt{2})(1 - \sqrt{2}) = 1 - 2 = -1$.

For $\sqrt{3} + \sqrt{6}$, a rationalising factor is $\sqrt{3} - \sqrt{6}$, as $(\sqrt{3} + \sqrt{6})(\sqrt{3} - \sqrt{6}) = 3 - 6 = -3$.

Example 8

Rationalise the denominator of each of the following:

a $\dfrac{1}{2\sqrt{7}}$ **b** $\dfrac{1}{2 - \sqrt{3}}$ **c** $\dfrac{1}{\sqrt{3} - \sqrt{6}}$ **d** $\dfrac{3 + \sqrt{8}}{3 - \sqrt{8}}$

Solution

a $\dfrac{1}{2\sqrt{7}} \times \dfrac{\sqrt{7}}{\sqrt{7}} = \dfrac{\sqrt{7}}{14}$

b $\dfrac{1}{2 - \sqrt{3}} \times \dfrac{2 + \sqrt{3}}{2 + \sqrt{3}} = \dfrac{2 + \sqrt{3}}{4 - 3}$

$= 2 + \sqrt{3}$

c $\dfrac{1}{\sqrt{3} - \sqrt{6}} \times \dfrac{\sqrt{3} + \sqrt{6}}{\sqrt{3} + \sqrt{6}} = \dfrac{\sqrt{3} + \sqrt{6}}{3 - 6}$

$= \dfrac{-1}{3}(\sqrt{3} + \sqrt{6})$

d $\dfrac{3 + \sqrt{8}}{3 - \sqrt{8}} = \dfrac{3 + 2\sqrt{2}}{3 - 2\sqrt{2}} \times \dfrac{3 + 2\sqrt{2}}{3 + 2\sqrt{2}}$

$= \dfrac{9 + 12\sqrt{2} + 8}{9 - 8}$

$= 17 + 12\sqrt{2}$

Example 9

Expand the brackets in each of the following and collect like terms, expressing surds in simplest form:

a $(3 - \sqrt{2})^2$ **b** $(3 - \sqrt{2})(1 + \sqrt{2})$

Solution

a $(3 - \sqrt{2})^2$

$= (3 - \sqrt{2})(3 - \sqrt{2})$

$= 3(3 - \sqrt{2}) - \sqrt{2}(3 - \sqrt{2})$

$= 9 - 3\sqrt{2} - 3\sqrt{2} + 2$

$= 11 - 6\sqrt{2}$

b $(3 - \sqrt{2})(1 + \sqrt{2})$

$= 3(1 + \sqrt{2}) - \sqrt{2}(1 + \sqrt{2})$

$= 3 + 3\sqrt{2} - \sqrt{2} - 2$

$= 1 + 2\sqrt{2}$

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Skill-sheet

Exercise 2C

Example 6

1 Express each of the following in terms of the simplest possible surds:

a $\sqrt{8}$ **b** $\sqrt{12}$ **c** $\sqrt{27}$ **d** $\sqrt{50}$
e $\sqrt{45}$ **f** $\sqrt{1210}$ **g** $\sqrt{98}$ **h** $\sqrt{108}$
i $\sqrt{25}$ **j** $\sqrt{75}$ **k** $\sqrt{512}$

SF

Example 7

2 Simplify each of the following:

a $\sqrt{8} + \sqrt{18} - 2\sqrt{2}$ **b** $\sqrt{75} + 2\sqrt{12} - \sqrt{27}$
c $\sqrt{28} + \sqrt{175} - \sqrt{63}$ **d** $\sqrt{1000} - \sqrt{40} - \sqrt{90}$
e $\sqrt{512} + \sqrt{128} + \sqrt{32}$ **f** $\sqrt{24} - 3\sqrt{6} - \sqrt{216} + \sqrt{294}$

3 Simplify each of the following:

a $\sqrt{75} + \sqrt{108} + \sqrt{14}$ **b** $\sqrt{847} - \sqrt{567} + \sqrt{63}$
c $\sqrt{720} - \sqrt{245} - \sqrt{125}$ **d** $\sqrt{338} - \sqrt{288} + \sqrt{363} - \sqrt{300}$
e $\sqrt{12} + \sqrt{8} + \sqrt{18} + \sqrt{27} + \sqrt{300}$ **f** $2\sqrt{18} + 3\sqrt{5} - \sqrt{50} + \sqrt{20} - \sqrt{80}$

Example 8

4 Express each of the following with rational denominators:

a $\dfrac{1}{\sqrt{5}}$ **b** $\dfrac{1}{\sqrt{7}}$ **c** $-\dfrac{1}{\sqrt{2}}$ **d** $\dfrac{2}{\sqrt{3}}$ **e** $\dfrac{3}{\sqrt{6}}$

f $\dfrac{1}{2\sqrt{2}}$ **g** $\dfrac{1}{\sqrt{2}+1}$ **h** $\dfrac{1}{2-\sqrt{3}}$ **i** $\dfrac{1}{4-\sqrt{10}}$ **j** $\dfrac{2}{\sqrt{6}+2}$

k $\dfrac{1}{\sqrt{5}-\sqrt{3}}$ **l** $\dfrac{3}{\sqrt{6}-\sqrt{5}}$ **m** $\dfrac{1}{3-2\sqrt{2}}$

Example 9

5 Express each of the following in the form $a + b\sqrt{c}$:

a $\dfrac{2}{3-2\sqrt{2}}$ **b** $(\sqrt{5}+2)^2$ **c** $(1+\sqrt{2})(3-2\sqrt{2})$ **d** $(\sqrt{3}-1)^2$

e $\sqrt{\dfrac{1}{3}} - \dfrac{1}{\sqrt{27}}$ **f** $\dfrac{\sqrt{3}+2}{2\sqrt{3}-1}$ **g** $\dfrac{\sqrt{5}+1}{\sqrt{5}-1}$ **h** $\dfrac{\sqrt{8}+3}{\sqrt{18}+2}$

6 Expand and simplify each of the following:

a $(2\sqrt{a}-1)^2$ **b** $(\sqrt{x+1}+\sqrt{x+2})^2$

7 Since $8 = \sqrt{64}$ and $3\sqrt{7} = \sqrt{63}$, it is easy to see that $8 > 3\sqrt{7}$. Using the same idea, order these numbers from smallest to largest: $7,\ 3\sqrt{5},\ 5\sqrt{2},\ 4\sqrt{3}$.

CF

8 For real numbers a and b, we have $a > b$ if and only if $a - b > 0$. Use this to state the larger of:

a $5 - 3\sqrt{2}$ and $6\sqrt{2} - 8$ **b** $2\sqrt{6} - 3$ and $7 - 2\sqrt{6}$

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CF

9 For positive real numbers a and b, we have $a > b$ if and only if $a^2 - b^2 > 0$. Use this to state the larger of:

a $\frac{2}{\sqrt{3}}$ and $\frac{3}{\sqrt{2}}$ **b** $\frac{\sqrt{7}}{3}$ and $\frac{\sqrt{5}}{2}$ **c** $\frac{\sqrt{3}}{7}$ and $\frac{\sqrt{5}}{5}$ **d** $\frac{\sqrt{10}}{2}$ and $\frac{8}{\sqrt{3}}$

10 For each of the following, determine the values of b and c for which the quadratic equation $x^2 + bx + c = 0$ has the two given solutions:

a $\sqrt{3},\ -\sqrt{3}$ **b** $2\sqrt{3},\ -2\sqrt{3}$ **c** $1 - \sqrt{2},\ 1 + \sqrt{2}$

d $2 - \sqrt{3},\ 2 + \sqrt{3}$ **e** $3 - 2\sqrt{2},\ 3 + 2\sqrt{2}$ **f** $4 - 7\sqrt{5},\ 3 + 2\sqrt{5}$

11 Express $\frac{1}{\sqrt{2} + \sqrt{3} + \sqrt{5}}$ with a rational denominator.

12 **a** Show that $a - b = \left(a^{\frac{1}{3}} - b^{\frac{1}{3}}\right)\left(a^{\frac{2}{3}} + a^{\frac{1}{3}}b^{\frac{1}{3}} + b^{\frac{2}{3}}\right)$.

b Express $\frac{1}{1 - 2^{\frac{1}{3}}}$ with a rational denominator.

13 Evaluate $\frac{1}{\sqrt{4} + \sqrt{5}} + \frac{1}{\sqrt{5} + \sqrt{6}} + \cdots + \frac{1}{\sqrt{24} + \sqrt{25}}$.

2D Natural numbers

Learning intentions

- To be able to determine the prime decomposition of a natural number.
- To be able to determine the highest common factor and lowest common multiple of two natural numbers.

Factors and composites

The factors of 8 are 1, 2, 4 and 8.

The factors of 24 are 1, 2, 3, 4, 6, 8, 12 and 24.

The factors of 5 are 1 and 5.

A natural number a is a **factor** of a natural number b if there exists a natural number k such that $b = ak$.

A natural number greater than 1 is called a **prime number** if its only factors are itself and 1.

The prime numbers less than 100 are

2	3	5	7	11	13	17	19	23	29	31	37	41	43	47
53	59	61	67	71	73	79	83	89	97					

A natural number m is called a **composite number** if it can be written as a product $m = a \times b$, where a and b are natural numbers greater than 1 and less than m.

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Prime decomposition

Expressing a composite number as a product of powers of prime numbers is called **prime decomposition**. For example:

$$3000 = 3 \times 2^3 \times 5^3$$

$$2294 = 2 \times 31 \times 37$$

This is useful for determining the factors of a number. For example, the prime decomposition of 12 is given by $12 = 2^2 \times 3$. The factors of 12 are

$$1, \quad 2, \quad 2^2 = 4, \quad 3, \quad 2 \times 3 = 6 \quad \text{and} \quad 2^2 \times 3 = 12$$

This property of natural numbers is described formally by the following theorem.

Fundamental theorem of arithmetic

Every natural number greater than 1 either is a prime number or can be represented as a product of prime numbers. Furthermore, this representation is unique apart from rearrangement of the order of the prime factors.

Example 10

Give the prime decomposition of 17 248 and hence list the factors of this number.

Solution

The prime decomposition can be found by repeated division, as shown on the right.

2	17 248
2	8624
2	4312
2	2156
2	1078
7	539
7	77
11	11
	1

The prime decomposition of 17 248 is

$$17\,248 = 2^5 \times 7^2 \times 11$$

Therefore each factor must be of the form

$$2^\alpha \times 7^\beta \times 11^\gamma$$

where $\alpha = 0, 1, 2, 3, 4, 5$, $\beta = 0, 1, 2$ and $\gamma = 0, 1$.

The factors of 17 248 can be systematically listed as follows:

1	2	2^2	2^3	2^4	2^5
7	2×7	$2^2 \times 7$	$2^3 \times 7$	$2^4 \times 7$	$2^5 \times 7$
7^2	2×7^2	$2^2 \times 7^2$	$2^3 \times 7^2$	$2^4 \times 7^2$	$2^5 \times 7^2$
11	2×11	$2^2 \times 11$	$2^3 \times 11$	$2^4 \times 11$	$2^5 \times 11$
7×11	$2 \times 7 \times 11$	$2^2 \times 7 \times 11$	$2^3 \times 7 \times 11$	$2^4 \times 7 \times 11$	$2^5 \times 7 \times 11$
$7^2 \times 11$	$2 \times 7^2 \times 11$	$2^2 \times 7^2 \times 11$	$2^3 \times 7^2 \times 11$	$2^4 \times 7^2 \times 11$	$2^5 \times 7^2 \times 11$

Highest common factor

The **highest common factor** of two natural numbers a and b is the largest natural number that is a factor of both a and b. It is denoted by HCF(a, b).

For example, the highest common factor of 15 and 24 is 3. We write $\text{HCF}(15, 24) = 3$.

Note: The highest common factor is also called the **greatest common divisor**.

Using prime decomposition to determine HCF

Prime decomposition can be used to determine the highest common factor of two numbers.

For example, consider the numbers 140 and 110. Their prime factorisations are

$$140 = 2^2 \times 5 \times 7 \quad \text{and} \quad 110 = 2 \times 5 \times 11$$

A number which is a factor of both 140 and 110 must have prime factors which occur in both these factorisations. The highest common factor of 140 and 110 is $2 \times 5 = 10$.

Next consider the numbers

$$396\,000 = 2^5 \cdot 3^2 \cdot 5^3 \cdot 11 \quad \text{and} \quad 1\,960\,200 = 2^3 \cdot 3^4 \cdot 5^2 \cdot 11^2$$

To obtain the highest common factor, we take the lower power of each prime factor:

$$\text{HCF}(396\,000, 1\,960\,200) = 2^3 \cdot 3^2 \cdot 5^2 \cdot 11$$

Example 11

a Determine the highest common factor of 528 and 3168.

b Determine the highest common factor of 3696 and 3744.

Solution

a $528 = 2^4 \times 3 \times 11$

$3168 = 2^5 \times 3^2 \times 11$

$\therefore\ \text{HCF}(528, 3168) = 2^4 \times 3 \times 11 = 528$

b $3696 = 2^4 \times 3 \times 7 \times 11$

$3744 = 2^5 \times 3^2 \times 13$

$\therefore\ \text{HCF}(3696, 3744) = 2^4 \times 3 = 48$

Using the TI-Nspire CX non-CAS

- The prime decomposition of a natural number can be obtained using menu > **Number** > **Factor** as shown.

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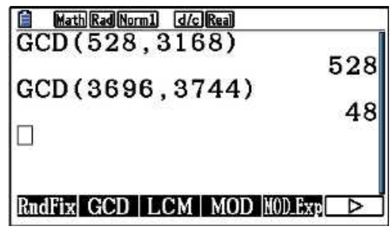
- The highest common factor of two numbers (also called their *greatest common divisor*) can be found by using the command **gcd( )** from menu > **Number** > **Greatest Common Divisor**, or by just typing it in, as shown.

Note: Nested **gcd( )** commands may be used to determine the greatest common divisor of several numbers.

Using the Casio

To determine the highest common factor of two natural numbers:

- Press MENU 1 to select **Run-Matrix** mode.
- Go to the **Numeric** menu OPTN F6 F4 and select **GCD** F6 F2.
- Enter the two numbers as shown and press EXE.



Note: The highest common factor is also called the *greatest common divisor* (GCD).

Lowest common multiple

A natural number a is a **multiple** of a natural number b if there exists a natural number k such that $a = kb$.

The **lowest common multiple** of two natural numbers a and b is the smallest natural number that is a multiple of both a and b. It is denoted by $\mathrm{LCM}(a, b)$.

For example: $\mathrm{LCM}(24, 36) = 72$ and $\mathrm{LCM}(256, 100) = 6400$

Using prime decomposition to determine LCM

Consider again the numbers

$$396\,000 = 2^5 \cdot 3^2 \cdot 5^3 \cdot 11 \quad \text{and} \quad 1\,960\,200 = 2^3 \cdot 3^4 \cdot 5^2 \cdot 11^2$$

To obtain the lowest common multiple, we take the higher power of each prime factor:

$$\mathrm{LCM}(396\,000, 1\,960\,200) = 2^5 \cdot 3^4 \cdot 5^3 \cdot 11^2$$

TI-84

Using the TI-Nspire CX non-CAS

The lowest common multiple of two numbers (also called their *least common multiple*) can be found by using the command **lcm( )** from menu > **Number** > **Least Common Multiple**, or by just typing it in, as shown.

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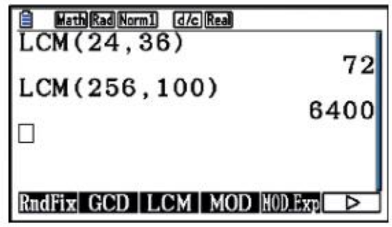
Using the Casio

To determine the lowest common multiple of two natural numbers:

- Press MENU 1 to select **Run-Matrix** mode.
- Go to the **Numeric** menu OPTN F6 F4 and select **LCM** F6 F3.
- Enter the two numbers as shown and press EXE.



Summary 2D

- A natural number a is a **factor** of a natural number b if there exists a natural number k such that $b = ak$.
- A natural number greater than 1 is a **prime number** if its only factors are itself and 1.
- A natural number m is a **composite number** if it can be written as a product $m = a \times b$, where a and b are natural numbers greater than 1 and less than m.
- Every composite number can be expressed as a product of powers of prime numbers; this is called **prime decomposition**. For example: $1300 = 2^2 \times 5^2 \times 13$
- The **highest common factor** of two natural numbers a and b, denoted by HCF(a, b), is the largest natural number that is a factor of both a and b.
- The **lowest common multiple** of two natural numbers a and b, denoted by LCM(a, b), is the smallest natural number that is a multiple of both a and b.

Exercise 2D

SF

Example 10 **1** Give the prime decomposition of each of the following numbers:

a 60 **b** 676 **c** 228 **d** 900 **e** 252
f 6300 **g** 68 640 **h** 96 096 **i** 32 032 **j** 544 544

Example 11 **2** Determine the highest common factor of each of the following pairs of numbers:

a 4361, 9281 **b** 999, 2160 **c** 5255, 716 845 **d** 1271, 3875 **e** 804, 2358

CF

3 **a** List all the factors of 18 and all the factors of 36.
b Why does 18 have an even number of factors and 36 an odd number of factors?
c Determine the smallest number greater than 100 with exactly three factors.

4 A woman has three children and two of them are teenagers, aged between 13 and 19. The product of their three ages is 1050. How old is each child?

5 By using prime decomposition, determine a natural number n such that $22^2 \times 55^2 = 10^2 \times n^2$.

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CF

6 Determine the smallest natural number n such that $60n$ is a square number.
Hint: First determine the prime decomposition of 60.

7 The natural number n has exactly eight different factors. Two of these factors are 15 and 21. What is the value of n?

8 Let n be the smallest of three natural numbers whose product is 720. What is the largest possible value of n?

9 When all eight factors of 30 are multiplied together, the product is 30^k. What is the value of k?

10 A bell rings every 36 minutes and a buzzer rings every 42 minutes. If they sound together at 9 a.m., when will they next sound together?

11 The LCM of two numbers is $2^5 \times 3^3 \times 5^3$ and the HCF is $2^3 \times 3 \times 5^2$. Determine all the possible numbers.

2E Problems involving sets

Learning intentions

- To be able to solve problems using sets.

Sets can be used to help sort information, as each of the following examples demonstrates. Recall that, if A is a finite set, then the number of elements in A is denoted by $|A|$.

Example 12

Two hundred and eighty students each travel to school by either train or bus or both. Of these students, 150 travel by train, and 60 travel by both train and bus.

a Show this information on a Venn diagram.

b Hence determine the number of students who travel by:

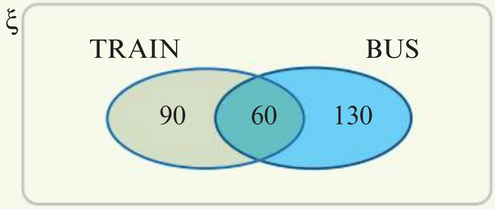
i bus

ii train but not bus

iii just one of these modes of transport.

Solution

a



$|\xi| = 280$

b **i** $|\text{BUS}| = 130 + 60 = 190$

ii $|\text{TRAIN} \cap (\text{BUS})'| = 90$

iii $|\text{TRAIN} \cap (\text{BUS})'| + |(\text{TRAIN})' \cap \text{BUS}| = 90 + 130 = 220$

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Example 13

An athletics team has 18 members. Each member competes in at least one of three events: sprints (S), jumps (J) and hurdles (H). Every hurdler also jumps or sprints. The following additional information is available:

$$|S| = 11, \quad |J| = 10, \quad |J \cap H' \cap S'| = 5, \quad |J' \cap H' \cap S| = 5 \quad \text{and} \quad |J \cap H'| = 7$$

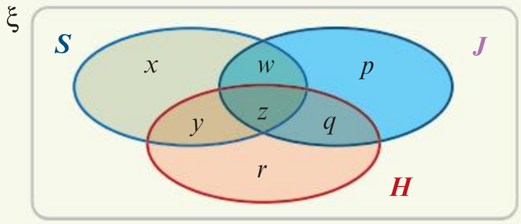
a Draw a Venn diagram.

b Determine:

i $|H|$ **ii** $|S \cap H \cap J|$ **iii** $|S \cup J|$ **iv** $|S \cap J \cap H'|$

Solution

a Assign a variable to the number of members in each region of the Venn diagram.



The information in the question can be summarised in terms of these variables:

$$x + y + z + w = 11 \quad \text{as } |S| = 11 \qquad (1)$$
$$p + q + z + w = 10 \quad \text{as } |J| = 10 \qquad (2)$$
$$x + y + z + w + p + q + r = 18 \quad \text{as all members compete} \qquad (3)$$
$$p = 5 \quad \text{as } |J \cap H' \cap S'| = 5 \qquad (4)$$
$$x = 5 \quad \text{as } |J' \cap H' \cap S| = 5 \qquad (5)$$
$$r = 0 \quad \text{as every hurdler also jumps or sprints} \qquad (6)$$
$$w + p = 7 \quad \text{as } |J \cap H'| = 7 \qquad (7)$$

From (4) and (7): $w = 2$.
Equation (3) now becomes

$$5 + y + z + 2 + 5 + q = 18$$
$$\therefore \quad y + z + q = 6 \qquad (8)$$

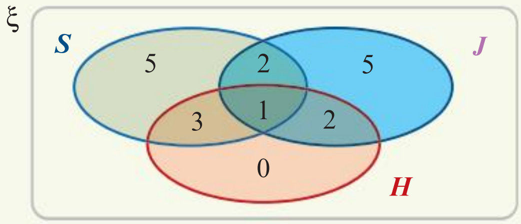
Equation (1) becomes

$$y + z = 4$$

Therefore from (8): $q = 2$.
Equation (2) becomes

$$5 + 2 + z + 2 = 10$$
$$\therefore \quad z = 1$$
$$\therefore \quad y = 3$$

The Venn diagram can now be completed as shown.



b **i** $|H| = 6$ **ii** $|S \cap H \cap J| = 1$ **iii** $|S \cup J| = 18$ **iv** $|S \cap J \cap H'| = 2$

Skill-sheet

Exercise 2E

Example 12

1 There are 28 students in a class, all of whom take either History or Economics or both. Of the 14 students who take History, five also take Economics.

a Show this information on a Venn diagram.

b Hence determine the number of students who take:

i Economics **ii** History but not Economics **iii** just one of these subjects.

2 **a** Draw a Venn diagram to show three sets A, B and C in a universal set ξ. Enter numbers in the correct parts of the diagram using the following information:

$$|A \cap B \cap C| = 2, \quad |A \cap B| = 7, \quad |B \cap C| = 6,$$
$$|A \cap C| = 8, \quad |A| = 16, \quad |B| = 20, \quad |C| = 19, \quad |\xi| = 50$$

b Use the diagram to determine:

i $|A' \cap C'|$ **ii** $|A \cup B'|$ **iii** $|A' \cap B \cap C'|$

3 In a border town in the Balkans, 60% of people speak Bulgarian, 40% speak Greek and 20% speak neither. What percentage of the town speak both Bulgarian and Greek?

4 At an international conference there were 105 delegates. Seventy spoke English, 50 spoke French and 50 spoke Japanese. Twenty-five spoke English and French, 15 spoke French and Japanese and 30 spoke Japanese and English. Everybody spoke at least one of these three languages.

a How many delegates spoke all three languages?

b How many spoke Japanese only?

5 A restaurant serves lunch to 350 people. It offers three desserts: profiteroles, gelati and fruit. Forty people have all three desserts, 70 have gelati only, 50 have profiteroles only and 60 have fruit only. Forty-five people have fruit and gelati only, 30 people have gelati and profiteroles only and 10 people have fruit and profiteroles only. How many people do not have a dessert?

Example 13

6 Forty travellers were questioned about the various methods of transport they had used the previous day. Every traveller used at least one of the following methods: car (C), bus (B), train (T). Of these travellers:

- eight had used all three methods of transport
- four had travelled by bus and car only
- two had travelled by car and train only
- the number (x) who had travelled by train only was equal to the number who had travelled by bus and train only.

If 20 travellers had used a train and 33 had used a bus, determine:

a the value of x

b the number who travelled by bus only

c the number who travelled by car only.

CF

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CF

7 Let ξ be the set of all integers and let

$$X = \{ x : 21 < x < 37 \}, \quad Y = \{ 3y : 0 < y \le 13 \}, \quad Z = \{ z^2 : 0 < z < 8 \}$$

a Draw a Venn diagram representing these sets.

b **i** Determine $X \cap Y \cap Z$. **ii** Determine $|X \cap Y|$.

CU

8 A number of students bought red, green and black pens. Three bought one of each colour. Of the students who bought two colours, three did not buy red, five not green and two not black. The same number of students bought red only as bought red with other colours. The same number bought black only as bought green only. More students bought red and black but not green than bought black only. More bought only green than bought green and black but not red. How many students were there and how many pens of each colour were sold?

9 For three subsets B, M and F of a universal set ξ,

$$|B \cap M| = 12, \quad |M \cap F \cap B| = |F'|, \quad |F \cap B| > |M \cap F|,$$

$$|B \cap F' \cap M'| = 5, \quad |M \cap B' \cap F'| = 5, \quad |F \cap M' \cap B'| = 5, \quad |\xi| = 28$$

Determine $|M \cap F|$.

10 A group of 80 students were interviewed about which sports they play. It was found that 23 do athletics, 22 swim and 18 play football. If 10 students do athletics and swim only, 11 students do athletics and play football only, six students swim and play football only and 46 students do none of these activities on a regular basis, how many students do all three?

11 At a certain secondary college, students have to be proficient in at least one of the languages Italian, French and German. In a particular group of 33 students, two are proficient in all three languages, three in Italian and French only, four in French and German only and five in German and Italian only. The number of students proficient in Italian only is x, in French only is x and in German only is $x + 1$. Determine x and then determine the total number of students proficient in Italian.

12 At a certain school, 201 students study one or more of Mathematics, Physics and Chemistry. Of these students: 35 take Chemistry only, 50% more students study Mathematics only than study Physics only, four study all three subjects, 25 study both Mathematics and Physics but not Chemistry, seven study both Mathematics and Chemistry but not Physics, and 20 study both Physics and Chemistry but not Mathematics. Determine the number of students studying Mathematics.

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Chapter summary

Sets

- **Set notation**

$x \in A$	x is an element of A
$x \notin A$	x is not an element of A
ξ	the universal set
$\varnothing$	the empty set
$A \subseteq B$	A is a subset of B
$A \cup B$	the union of A and B consists of all elements that are in either A or B or both
$A \cap B$	the intersection of A and B consists of all elements that are in both A and B
A'	the complement of A consists of all elements of ξ that are not in A

- **Sets of numbers**

$\mathbb{N}$	Natural numbers	$\mathbb{Z}$	Integers
$\mathbb{Q}$	Rational numbers	$\mathbb{R}$	Real numbers

Surds

- A **quadratic surd** is a number of the form $\sqrt{a}$, where a is a rational number which is not the square of another rational number.
- A **surd of order n** is a number of the form $\sqrt[n]{a}$, where a is a rational number which is not a perfect nth power.
- When the number under the square root has no factors which are squares of a rational number, the surd is said to be in **simplest form**.
- Surds which have the same 'irrational factor' are called **like surds**. The sum or difference of two like surds can be simplified:

$$m\sqrt{p} + n\sqrt{p} = (m+n)\sqrt{p} \quad \text{and} \quad m\sqrt{p} - n\sqrt{p} = (m-n)\sqrt{p}$$

Natural numbers

- A natural number a is a **factor** of a natural number b if there exists a natural number k such that $b = ak$.
- A natural number greater than 1 is a **prime number** if its only factors are itself and 1.
- A natural number m is a **composite number** if it can be written as a product $m = a \times b$, where a and b are natural numbers greater than 1 and less than m.
- Every composite number can be expressed as a product of powers of prime numbers; this is called **prime decomposition**. For example: $1300 = 2^2 \times 5^2 \times 13$
- The **highest common factor** of two natural numbers a and b, denoted by $\text{HCF}(a, b)$, is the largest natural number that is a factor of both a and b.
- The **lowest common multiple** of two natural numbers a and b, denoted by $\text{LCM}(a, b)$, is the smallest natural number that is a multiple of both a and b.

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Review

Skills checklist

Check-list

Download this checklist from the Interactive Textbook, then print it and fill it out to check your skills. ✓

2A **1** I can work with set notation including, intersection, union and complements . ☐

See Example 1, Example 2, Example 3 and Question 1

2B **2** I can write a rational number as a recurring decimal and vice versa. ☐

See Example 4 and Question 4

2B **3** I can use and interpret interval notation. ☐

See Example 5 and Question 5

2C **4** I can write a surd in simplest form. ☐

See Example 6 and Question 1

2C **5** I can add and subtract like surds. ☐

See Example 7 and Question 2

2C **6** I can rationalise the denominator of a quadratic surd. ☐

See Example 8 and Question 4

2C **7** I can expand brackets and collect like terms in expressions involving surds. ☐

See Example 9 and Question 5

2D **8** I can determine the prime decomposition of a natural number. ☐

See Example 10 and Question 1

2D **9** I can determine the highest common factor and lowest common denominator of two natural numbers. ☐

See Example 11 and Question 2

2E **10** I can solve problems using set notation. ☐

See Example 12, Example 13 and Questions 1 and 6

Short-response questions

Technology-free short-response questions

1 Express the following as fractions in their simplest form: SF

a $0.0\dot{7}$ **b** $0.\dot{4}\dot{5}$ **c** 0.005 **d** 0.405 **e** $0.2\dot{6}$ **f** $0.1\dot{7}1428\dot{5}$

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2 Express 504 as a product of powers of prime numbers.

3 Express each of the following with a rational denominator:

a $\frac{2\sqrt{3}-1}{\sqrt{2}}$ **b** $\frac{\sqrt{5}+2}{\sqrt{5}-2}$ **c** $\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}$

4 Express $\frac{3+2\sqrt{75}}{3-\sqrt{12}}$ in the form $a+b\sqrt{3}$, where $a, b \in \mathbb{Q} \setminus \{0\}$.

5 Express each of the following with a rational denominator:

a $\frac{6\sqrt{2}}{3\sqrt{2}-2\sqrt{3}}$ **b** $\frac{\sqrt{a+b}-\sqrt{a-b}}{\sqrt{a+b}+\sqrt{a-b}}$

6 In a class of 100 students, 55 are girls, 45 have blue eyes, 40 are blond, 25 are blond girls, 15 are blue-eyed blonds, 20 are blue-eyed girls, and five are blue-eyed blond girls. Determine:

a the number of blond boys

b the number of boys who are neither blond nor blue-eyed.

7 A group of 30 students received prizes in at least one of the subjects of English, Mathematics and French. Two students received prizes in all three subjects. Fourteen received prizes in English and Mathematics but not French. Two received prizes in English alone, two in French alone and five in Mathematics alone. Four received prizes in English and French but not Mathematics.

a How many received prizes in Mathematics and French but not English?

b How many received prizes in Mathematics?

c How many received prizes in English?

8 Fifty people are interviewed. Twenty-three people like Brand X, 25 like Brand Y and 19 like Brand Z. Eleven like X and Z. Eight like Y and Z. Five like X and Y. Two like all three. How many like none of them?

9 Three rectangles A, B and C overlap (intersect). Their areas are 20 cm^2, 10 cm^2 and 16 cm^2 respectively. The area common to A and B is 3 cm^2, that common to A and C is 6 cm^2 and that common to B and C is 4 cm^2. How much of the area is common to all three if the total area covered is 35 cm^2?

10 Express $\sqrt{112}-\sqrt{63}-\frac{224}{\sqrt{28}}$ in simplest form.

11 If $\frac{\sqrt{7}-\sqrt{3}}{x}=\frac{x}{\sqrt{7}+\sqrt{3}}$, determine the values of x.

12 Express $\frac{1+\sqrt{2}}{\sqrt{5}+\sqrt{3}}+\frac{1-\sqrt{2}}{\sqrt{5}-\sqrt{3}}$ in the form $a\sqrt{5}+b\sqrt{6}$.

13 Simplify $\sqrt{27}-\sqrt{12}+2\sqrt{75}-\sqrt{\frac{48}{25}}$.

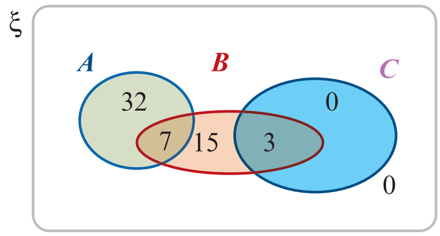
Review

SF

CF

Review

14 A, B and C are three sets and $\xi = A \cup B \cup C$. The number of elements in the regions of the Venn diagram are as shown. Determine:



a the number of elements in $A \cup B$
b the number of elements in C
c the number of elements in $B' \cap A$.

15 Using the result that $(\sqrt{a} + \sqrt{b})^2 = a + b + 2\sqrt{ab}$, determine the square root of $17 + 6\sqrt{8}$.

16 Prove that the product of two odd integers is odd. (You may assume that the sum and product of any two integers is an integer.)

17 The sum of the ages of Tom and Fred is 63. Tom is twice as old as Fred was when Tom was as old as Fred is now. What are the ages of Tom and Fred?

18 In this question, we consider the set $\{ a + b\sqrt{3} : a, b \in \mathbb{Q} \}$.
In Chapter 15, the set $\mathbb{C}$ of complex numbers is introduced, where $\mathbb{C} = \{ a + b\sqrt{-1} : a, b \in \mathbb{R} \}$.

a If $(2 + 3\sqrt{3}) + (4 + 2\sqrt{3}) = a + b\sqrt{3}$, determine a and b.
b If $(2 + 3\sqrt{3})(4 + 2\sqrt{3}) = p + q\sqrt{3}$, determine p and q.
c If $\dfrac{1}{3 + 2\sqrt{3}} = a + b\sqrt{3}$, determine a and b.
d Solve each of the following equations for x:
  i $(2 + 5\sqrt{3})x = 2 - \sqrt{3}$ **ii** $(x - 3)^2 - 3 = 0$ **iii** $(2x - 1)^2 - 3 = 0$
e Explain why every rational number is a member of $\{ a + b\sqrt{3} : a, b \in \mathbb{Q} \}$.

19 **a** Show that $\dfrac{1}{2 + \sqrt{3}} = 2 - \sqrt{3}$.
b Use the substitution $t = \left(\sqrt{2 + \sqrt{3}}\right)^x$ and part **a** to show that the equation

$$\left(\sqrt{2 + \sqrt{3}}\right)^x + \left(\sqrt{2 - \sqrt{3}}\right)^x = 4$$

can be written as $t + \dfrac{1}{t} = 4$.
c Show that the solutions of the equation are $t = 2 - \sqrt{3}$ and $t = 2 + \sqrt{3}$.
d Use this result to solve the equation $\left(\sqrt{2 + \sqrt{3}}\right)^x + \left(\sqrt{2 - \sqrt{3}}\right)^x = 4$.

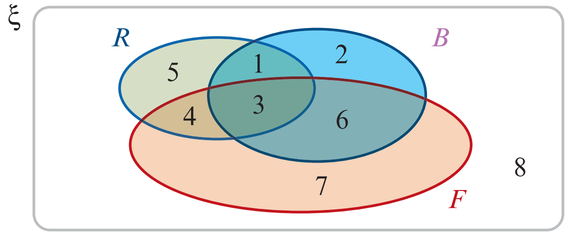
20 Use Venn diagrams to illustrate:
a $|A \cup B| = |A| + |B| - |A \cap B|$
b $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$

21 A quadratic equation with integer coefficients $x^2 + bx + c = 0$ has a solution $x = 2 - \sqrt{3}$.
a Determine the values of b and c.
Hint: Use the result that, for m, n rational, if $m + n\sqrt{3} = 0$, then $m = 0$ and $n = 0$.
b Determine the other solution to this quadratic equation.
c Now consider a quadratic equation with integer coefficients $x^2 + bx + c = 0$ that has a solution $x = m - n\sqrt{q}$, where q is not a perfect square. Show that:
  i $b = -2m$ **ii** $c = m^2 - n^2 q$

Hence show that:

iii $x^2 + bx + c = (x - (m - n\sqrt{q}))(x - (m + n\sqrt{q}))$

22 **a** The Venn diagram shows the set ξ of all students enrolled at Argos Secondary College. Set R is the set of all students with red hair. Set B is the set of all students with blue eyes. Set F is the set of all female students.



The numbers on the diagram are to label the eight different regions.

i Identify the region in the Venn diagram which represents male students who have neither red hair nor blue eyes.

ii Describe the gender, hair colour and eye colour of students represented in region 1 of the diagram.

iii Describe the gender, hair colour and eye colour of students represented in region 2 of the diagram.

b It is known that, at Argos Secondary College, 250 students study French (F), Greek (G) or Japanese (J). Forty-one students do not study French. Twelve students study French and Japanese but not Greek. Thirteen students study Japanese and Greek but not French. Thirteen students study only Greek. Twice as many students study French and Greek but not Japanese as study all three. The number studying only Japanese is the same as the number studying both French and Greek.

i How many students study all three languages?

ii How many students study only French?

23 In a certain city, three Sunday newspapers (A, B and C) are available. In a sample of 500 people from this city, it was found that:

- nobody regularly reads both A and C
- a total of 100 people regularly read A
- 205 people regularly read only B
- of those who regularly read C, exactly half of them also regularly read B
- 35 people regularly read A and B but not C
- 35 people don't read any of the papers at all.

a Draw a Venn diagram showing the number of regular readers for each possible combination of A, B and C.

b How many people in the sample were regular readers of C?

c How many people in the sample regularly read A only?

d How many people are regular readers of A, B and C?

Review

SF

CU

Technology-active short-response questions

24 Use your calculator to determine the highest common factor and least common multiple of each of the following:

a 25 272 000 and 61 158 240 **b** 50 544 000 and 3 538 080

25 Determine the highest common factor of 25 272 000, 61 158 240 and 3 538 080.

26 Determine the greatest natural number which on dividing 1401 and 984 by this number leaves remainders 1 and 4 respectively.

27 Determine the greatest natural number n which on dividing each of 441, 363 and 1095 by n gives the same remainder.

28 The prime factors of 323 400 are all less than or equal to 11. Write the prime decomposition of 323 400 and hence write the surd $\sqrt{323400}$ in simplest form.

29 **a** Show that $(\sqrt{x} + \sqrt{y})^2 = x + y + 2\sqrt{xy}$.

b Substitute $x = 3$ and $y = 5$ in the identity from part **a** to show that

$$\sqrt{3} + \sqrt{5} = \sqrt{8 + 2\sqrt{15}}$$

c Use this technique to determine the square root of:

i $14 + 2\sqrt{33}$ (**Hint:** Use $x = 11$ and $y = 3$.) **ii** $15 - 2\sqrt{56}$ **iii** $51 - 36\sqrt{2}$

30 A **Pythagorean triple** (x, y, z) consists of three natural numbers x, y, z such that $x^2 + y^2 = z^2$. For example: $(3, 4, 5)$ and $(5, 12, 13)$ are Pythagorean triples.
A Pythagorean triple is in simplest form if x, y, z have no common factor. Up to swapping x and y, all Pythagorean triples in simplest form may be generated by:

$$x = 2mn, \quad y = m^2 - n^2, \quad z = m^2 + n^2 \qquad \text{where } m, n \in \mathbb{N} \text{ and HCF } (m, n) = 1.$$

For example, if $m = 2$ and $n = 1$, then $x = 4$, $y = 3$ and $z = 5$.

a Determine the Pythagorean triple for

i $m = 5$ and $n = 2$. **ii** $m = 7$ and $n = 5$

b Determine the values of m and n to produce the Pythagorean triples

i $(48, 55, 73)$ **ii** $(72, 65, 97)$

c Verify that, if $x = 2mn$, $y = m^2 - n^2$ and $z = m^2 + n^2$, where $m, n \in \mathbb{N}$ then $x^2 + y^2 = z^2$.

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Multiple-choice questions

Technology-free multiple-choice questions

1 $\dfrac{4}{3+2\sqrt{2}}$ expressed in the form $a + b\sqrt{2}$ is

A $12 - 8\sqrt{2}$ **B** $3 + 2\sqrt{2}$

C $\dfrac{3}{17} - \dfrac{8}{17}\sqrt{2}$ **D** $\dfrac{3}{17} + \dfrac{8}{17}\sqrt{2}$

2 The prime decomposition of 86 400 is

A $2^5 \times 3^2 \times 5$ **B** $2^6 \times 3^3 \times 5^2$

C $2^7 \times 3^3 \times 5^2$ **D** $2^7 \times 3^3 \times 5$

3 $(\sqrt{6} + 3)(\sqrt{6} - 3)$ is equal to

A $3 - 12\sqrt{6}$ **B** $-3 - 6\sqrt{6}$ **C** $-3 + 6\sqrt{6}$ **D** -3

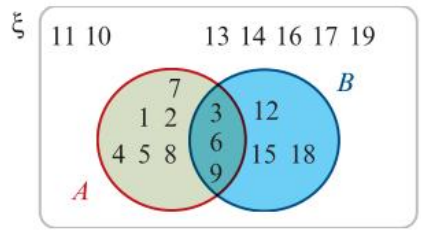
4 For the Venn diagram shown, ξ is the set of natural numbers less than 20, A is the set of natural numbers less than 10, and B is the set of natural numbers less than 20 that are divisible by 3. The set $B' \cap A$ is

A $\{3, 6, 9\}$

B $\{12, 15, 18\}$

C $\{1, 2, 4, 5, 7, 8\}$

D $\{10, 11, 13, 14, 16, 17, 19\}$



5 $(3, \infty) \cap (-\infty, 5] =$

A $(-\infty, 3)$ **B** $(-\infty, 5]$ **C** $(3, 5]$ **D** $\mathbb{R}$

6 A bell is rung every 6 minutes and a gong is sounded every 14 minutes. If these occur together at a particular time, then the smallest number of minutes until the bell and the gong are again heard simultaneously is

A 10 **B** 20 **C** 72 **D** 42

7 If X is the set of multiples of 2, Y the set of multiples of 7, and Z the set of multiples of 5, then $X \cap Y \cap Z$ can be described as

A the set of multiples of 2 **B** the set of multiples of 70

C the set of multiples of 35 **D** the set of multiples of 14

8 In a class of students, 50% play football, 40% play tennis and 30% play neither. The percentage that plays both is

A 10 **B** 20 **C** 30 **D** 50

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9 $\dfrac{\sqrt{7}-\sqrt{6}}{\sqrt{7}+\sqrt{6}} =$

A $5+2\sqrt{7}$ **B** $13+2\sqrt{6}$ **C** $13-2\sqrt{42}$ **D** $1+2\sqrt{42}$

10 There are 40 students in a class, all of whom take either Literature or Economics or both. Twenty take Literature and five of these also take Economics. The number of students who take only Economics is

A 20 **B** 5 **C** 10 **D** 15

11 The number of factors that the integer $2^p3^q5^r$ has is

A $\dfrac{(p+q+r)!}{p!\,q!\,r!}$ **B** pqr

C $p+q+r$ **D** $(p+1)(q+1)(r+1)$

12 The number of pairs of integers (m, n) which satisfy the equation $m+n=mn$ is

A 1 **B** 2 **C** 3 **D** 4

Technology-active multiple-choice questions

13 The prime decomposition of 125 897 412 is

A $587 \times 61 \times 3 \times 2^3$ **B** $587 \times 293 \times 61 \times 3 \times 2^2$

C $587 \times 293 \times 7 \times 3^3 \times 2^2$ **D** $2^2 \times 3 \times 293$

14 Let $N = 24 \times 35 \times 60 \times 77 \times 110 \times 230$. The smallest prime number which is not a factor of N is

A 7 **B** 11 **C** 23 **D** 13

15 If the least common multiple of m and n is 24, then the smallest integer greater than 3050 that is divisible by both m and n is

A 3052 **B** 3072 **C** 3060 **D** 3062

16 If the least common multiple of the first 900 positive integers is n, then what is the least common multiple of the first 907 positive integers in terms of n ?

A $907! \times n$ **B** $11 \times 41 \times 907 \times n$

C $907n$ **D** $907 \times 901 \times n$

17 Consider the expression $4n^3+6n^2+4n+1$. It can be shown that this is a composite number for all natural number n. The smallest natural number n such that $4n^3+6n^2+4n+1$ has 3 distinct prime factors is

A 2 **B** 4 **C** 5 **D** 6

18 We know that $a \times 13^5 + b \times 13^4 + c \times 13^3 + d \times 13^2 + e \times 13 + f = 1\,000\,000$. The smallest value of $a+b+c+d+e+f$ is

A 12 **B** 14 **C** 16 **D** 17

3

Trigonometric ratios and applications

Chapter contents

This chapter is provided to review previous years, in preparation for Chapter 8*, Vectors in the plane and as a foundation for Units* 3 *and* 4*.*

Trigonometry deals with the side lengths and angles of a triangle: the word *trigonometry* comes from the Greek words for triangle and measurement.

In this chapter, we review and extend your study of trigonometry from previous years. An understanding of trigonometry will be important for our study of vectors in Chapter 8.

We start this chapter by revising the four standard congruence tests for triangles. If you have the information about a triangle given in one of the congruence tests, then the triangle is uniquely determined (up to congruence). You can determine the unknown side lengths and angles of the triangle using the **sine rule** or the **cosine rule**.

In this chapter, we establish these two rules, and apply them in practical problems involving angles of elevation, angles of depression and compass bearings. We will explore similar problems using vectors in Chapter 8.

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3A Reviewing trigonometry

Learning intentions

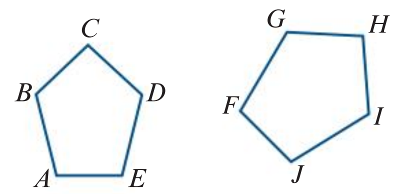
- To be able to use the trigonometric ratios to solve right-angled triangles.

Congruent triangles

Two plane figures are called **congruent** if one figure can be moved on top of the other figure, by a sequence of translations, rotations and reflections, so that they coincide exactly.

Congruent figures have exactly the same shape and size. For example, the two figures shown are congruent. We can write:

$$\text{pentagon } ABCDE \equiv \text{pentagon } FGHIJ$$

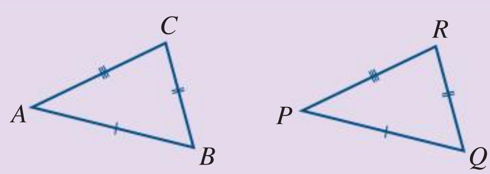


When two figures are congruent, we can determine a transformation that pairs up every part of one figure with the corresponding part of the other, so that:

- paired angles have the same size
- paired line segments have the same length
- paired regions have the same area.

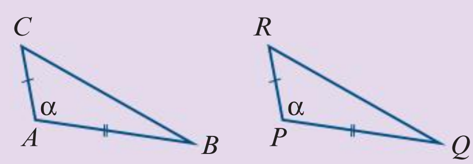
There are four standard tests for two triangles to be congruent.

- **The SSS congruence test**
  If the three sides of one triangle are respectively equal to the three sides of another triangle, then the two triangles are congruent.



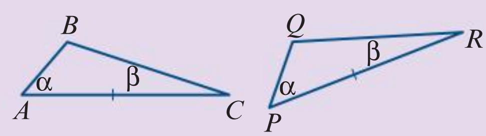
$\triangle ABC \equiv \triangle PQR$

- **The SAS congruence test**
  If two sides and the included angle of one triangle are respectively equal to two sides and the included angle of another triangle, then the two triangles are congruent.



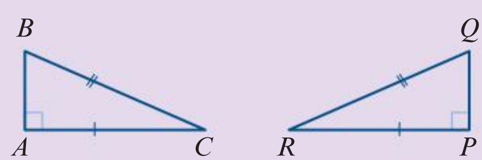
$\triangle ABC \equiv \triangle PQR$

- **The AAS congruence test**
  If two angles and one side of one triangle are respectively equal to two angles and the matching side of another triangle, then the two triangles are congruent.



$\triangle ABC \equiv \triangle PQR$

- **The RHS congruence test**
  If the hypotenuse and one side of one right-angled triangle are respectively equal to the hypotenuse and one side of another right-angled triangle, then the two triangles are congruent.



$\triangle ABC \equiv \triangle PQR$

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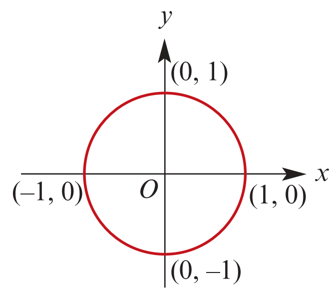
Defining sine and cosine

In this section we review sine, cosine and tangent for angles between 0° and 180°.

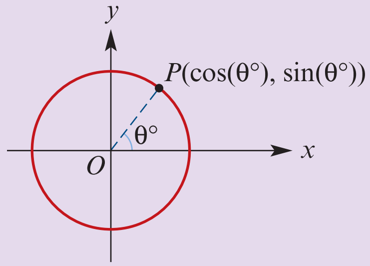
We can define the sine and cosine of any angle by using the **unit circle**. The unit circle is a circle of radius 1 with centre at the origin.



Unit-circle definition of sine and cosine

For each angle θ°, there is a point P on the unit circle as shown. The angle is measured anticlockwise from the positive direction of the x-axis.

- $\cos(\theta°)$ is defined as the x-coordinate of the point P
- $\sin(\theta°)$ is defined as the y-coordinate of the point P



The trigonometric ratios

For acute angles, the unit-circle definition of sine and cosine given above is equivalent to the ratio definition.

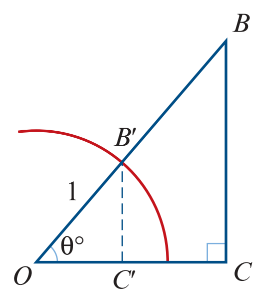
For a right-angled triangle OBC, we can construct a similar triangle $OB'C'$ that lies in the unit circle. From the diagram:

$$B'C' = \sin(\theta°) \quad \text{and} \quad OC' = \cos(\theta°)$$

As triangles OBC and $OB'C'$ are similar, we have

$$\frac{BC}{OB} = \frac{B'C'}{1} \quad \text{and} \quad \frac{OC}{OB} = \frac{OC'}{1}$$

$$\therefore \quad \frac{BC}{OB} = \sin(\theta°) \quad \text{and} \quad \frac{OC}{OB} = \cos(\theta°)$$

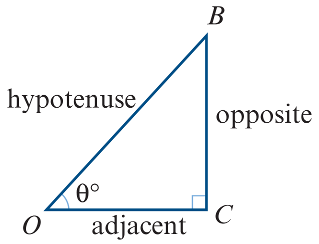


This gives the ratio definition of sine and cosine for a right-angled triangle. The naming of sides with respect to an angle θ° is as shown.

$$\sin(\theta°) = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\cos(\theta°) = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\tan(\theta°) = \frac{\text{opposite}}{\text{adjacent}}$$



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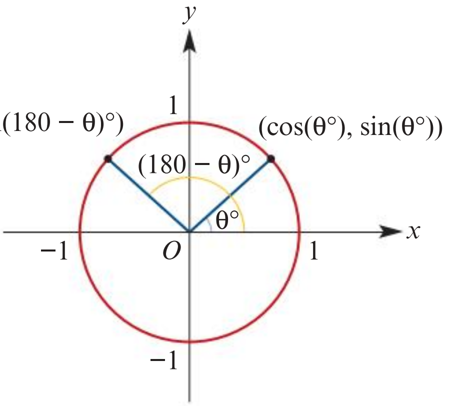
Obtuse angles

From the unit circle, we see that

$$\sin(180 - \theta)^\circ = \sin(\theta^\circ)$$
$$\cos(180 - \theta)^\circ = -\cos(\theta^\circ)$$



For example:

$$\sin 135^\circ = \sin 45^\circ$$
$$\cos 135^\circ = -\cos 45^\circ$$

In this chapter, we will generally use the ratio definition of tangent for acute angles. But we can also determine the tangent of an obtuse angle by defining

$$\tan(\theta^\circ) = \frac{\sin(\theta^\circ)}{\cos(\theta^\circ)}$$

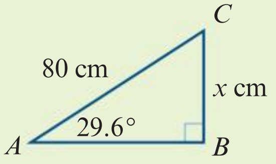
For working with triangles, we only need to consider angles between 0° and 180°. We will consider angles less than 0° and greater than 180° in Section 3F.

Solving right-angled triangles

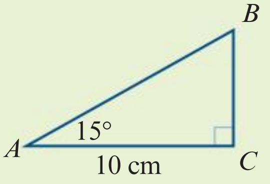
Here we provide some examples of using the trigonometric ratios.

Example 1

a Determine the value of x correct to two decimal places.



b Determine the length of the hypotenuse correct to two decimal places.



Solution

a $\dfrac{x}{80} = \sin 29.6^\circ$

$\therefore\ x = 80 \sin 29.6^\circ$

$= 39.5153\ldots$

Hence $x = 39.52$, correct to two decimal places.

b $\dfrac{10}{AB} = \cos 15^\circ$

$10 = AB \cos 15^\circ$

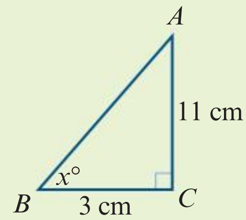
$\therefore\ AB = \dfrac{10}{\cos 15^\circ}$

$= 10.3527\ldots$

The length of the hypotenuse is 10.35 cm, correct to two decimal places.

Example 2

Determine the magnitude of $\angle ABC$.



Solution

$$\tan x = \frac{11}{3}$$

$$\therefore \quad x = \tan^{-1}\left(\frac{11}{3}\right)$$

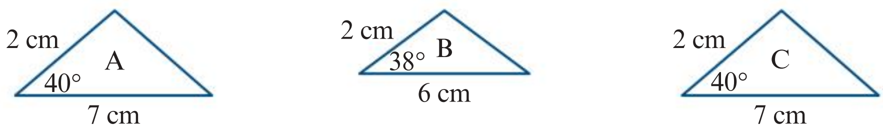
$$= (74.7448\ldots)^\circ$$

Hence $x = 74.74^\circ$, correct to two decimal places.

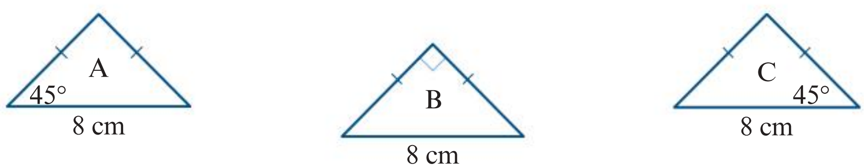
Exercise 3A

1 In each part, determine pairs of congruent triangles. State the congruence tests used.

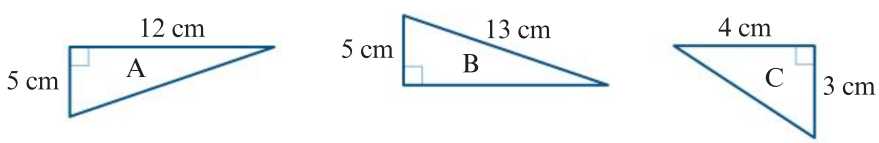
a



b

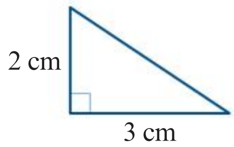


c



2 For each of the following, state whether or not the given labelling uniquely determines a triangle (up to congruence):

a



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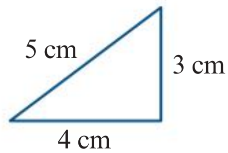

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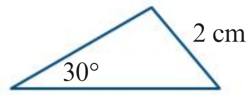
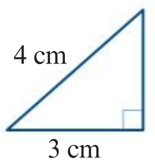
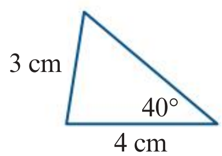


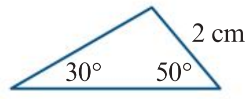
d



e



f



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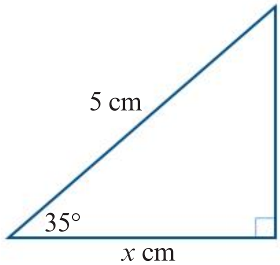
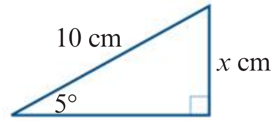
Example 1, 2

3 Determine the value of x in each of the following:

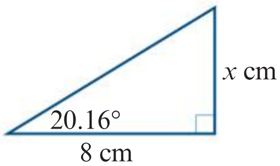
a



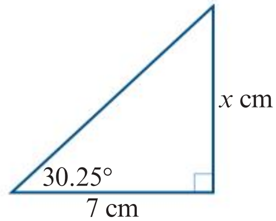
b



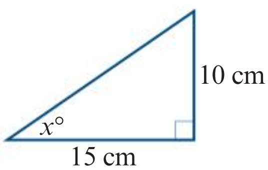
c



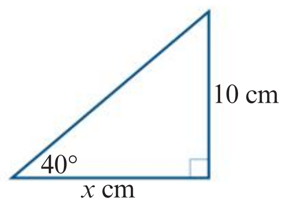
d



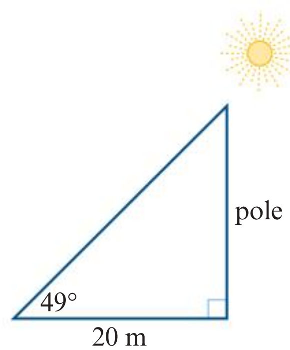
e



f



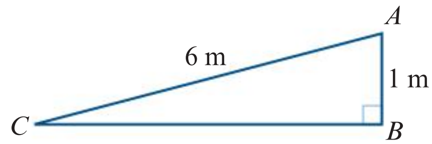
4 A pole casts a shadow 20 m long when the altitude of the sun is 49°. Calculate the height of the pole.



5 This figure represents a ramp.

a Determine the magnitude of angle ACB.

b Determine the distance BC.



6 A ladder leaning against a vertical wall makes an angle of 26° with the wall. If the foot of the ladder is 3 m from the wall, calculate:

a the length of the ladder

b the height it reaches above the ground.

7 An engineer is designing a straight concrete entry ramp, 60 m long measured along the slope, for a car park that is 13 m above street level. Calculate the angle of the ramp to the horizontal.

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8 A vertical mast is secured from its top by straight cables 200 m long fixed at the ground. The cables make angles of 66° with the ground. What is the height of the mast?

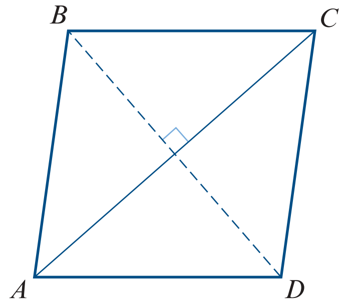
9 A mountain railway rises 400 m at a uniform slope of 16° with the horizontal. What is the distance travelled by a train for this rise?

10 The base of an isosceles triangle is 12 cm long and the equal sides are 15 cm long. Determine the magnitude of each of the three angles of the triangle.

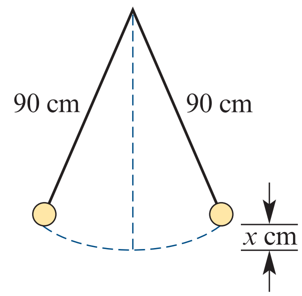
11 The diagonals of a rhombus bisect each other at right angles. If $BD = AC = 10$ cm, determine:

a the length of the sides of the rhombus

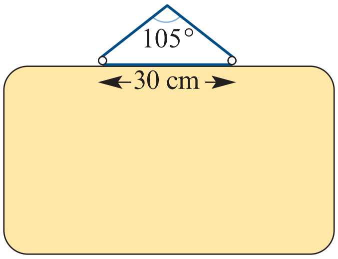
b the magnitude of angle ABC.



12 A pendulum swings from the vertical through an angle of 15° on each side of the vertical. If the pendulum is 90 cm long, what is the distance, x cm, between its highest and lowest points?

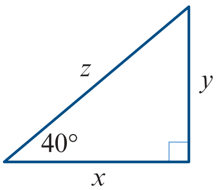


13 A picture is hung symmetrically by means of a string passing over a nail, with the ends of the string attached to two rings on the upper edge of the picture. The distance between the rings is 30 cm, and the string makes an angle of 105° at the nail. Determine the length of the string.



14 A ladder 4.7 m long is placed against a wall. The foot of the ladder must not be placed in a flower bed, which extends a distance of 1.7 m out from the base of the wall. How high up the wall can the ladder reach?

15 The area of the triangle shown is 7 square units. Determine the lengths of the three sides of the triangle, correct to two decimal places.



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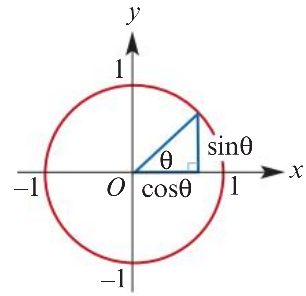
3B Exact values of sine, cosine and tangent

Learning intentions

- To be able to use the exact values of the trigonometric functions for $0^\circ, 90^\circ, 30^\circ, 60^\circ$ and 45° and the obtuse angles corresponding to $30^\circ, 60^\circ$ and 45°.

A calculator can be used to determine the values of sine, cosine and tangent for different angles. For many angles, the calculator gives an approximation. We now consider some angles for which the exact values can be found.

Exact values for 0° and 90°



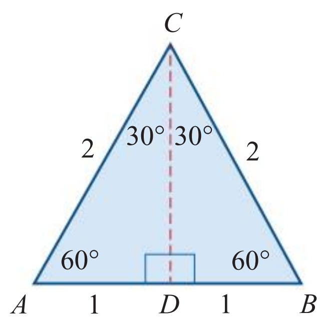
From the unit circle:

$$\sin 0^\circ = 0 \qquad \sin 90^\circ = 1$$

$$\cos 0^\circ = 1 \qquad \cos 90^\circ = 0$$

$$\tan 0^\circ = 0 \qquad \tan 90^\circ \text{ is undefined}$$

Exact values for 30° and 60°



Consider an equilateral triangle ABC of side length 2 units. In $\triangle ACD$, by Pythagoras' theorem, $CD = \sqrt{AC^2 - AD^2} = \sqrt{3}$.

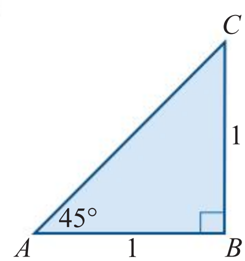
$$\sin 30^\circ = \frac{AD}{AC} = \frac{1}{2} \qquad \sin 60^\circ = \frac{CD}{AC} = \frac{\sqrt{3}}{2}$$

$$\cos 30^\circ = \frac{CD}{AC} = \frac{\sqrt{3}}{2} \qquad \cos 60^\circ = \frac{AD}{AC} = \frac{1}{2}$$

$$\tan 30^\circ = \frac{AD}{CD} = \frac{1}{\sqrt{3}} \qquad \tan 60^\circ = \frac{CD}{AD} = \frac{\sqrt{3}}{1} = \sqrt{3}$$

Exact values for 45°

For the triangle ABC shown on the right, we have $AC = \sqrt{1^2 + 1^2} = \sqrt{2}$.



$$\sin 45^\circ = \frac{BC}{AC} = \frac{1}{\sqrt{2}}$$

$$\cos 45^\circ = \frac{AB}{AC} = \frac{1}{\sqrt{2}}$$

$$\tan 45^\circ = \frac{BC}{AB} = 1$$

Example 3

Evaluate:

a $\cos 150^\circ$

b $\sin 120^\circ$

Solution

a $\cos 150^\circ = \cos(180 - 30)^\circ$

$= -\cos 30^\circ = -\dfrac{\sqrt{3}}{2}$

b $\sin 120^\circ = \sin(180 - 60)^\circ$

$= \sin 60^\circ = \dfrac{\sqrt{3}}{2}$

Note: Remember that $\sin(180 - \theta)^\circ = \sin(\theta^\circ)$ and $\cos(180 - \theta)^\circ = -\cos(\theta^\circ)$.

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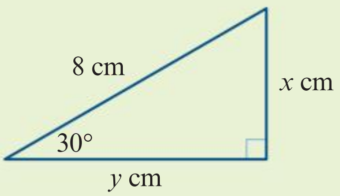
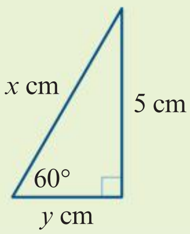
Example 4

Determine the exact values of x and y in each of the following:

a



b



Solution

a $x = 8\sin 30^\circ$

$= 8 \times \frac{1}{2}$

$= 4$

$y = 8\cos 30^\circ$

$= 8 \times \frac{\sqrt{3}}{2}$

$= 4\sqrt{3}$

b $\frac{5}{x} = \sin 60^\circ$

$\therefore\ x = \frac{5}{\sin 60^\circ} = \frac{10}{\sqrt{3}} = \frac{10\sqrt{3}}{3}$

$\frac{5}{y} = \tan 60^\circ$

$\therefore\ y = \frac{5}{\tan 60^\circ} = \frac{5}{\sqrt{3}} = \frac{5\sqrt{3}}{3}$

Summary 3B

As an aid to memory, the exact values of the trigonometric functions can be tabulated.

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
0°	0	1	0
30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
45°	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
90°	1	0	undefined

Exercise 3B

SF

Example 3

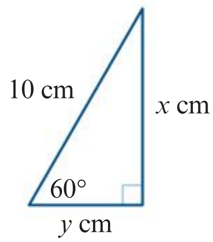
1 Determine the exact value of each of the following:

a $\cos 135°$ **b** $\sin 135°$ **c** $\cos 120°$ **d** $\sin 150°$

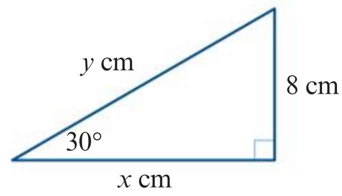
Example 4

2 Determine the exact values of x and y in each of the following:

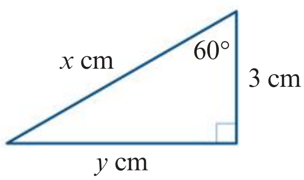
a



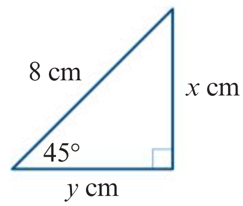
b



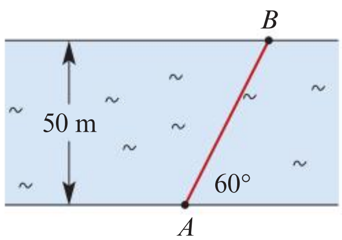
c



d



3 A river is known to be 50 m wide. A swimmer sets off from A to cross the river, and the path of the swimmer AB is as shown. How far does the person swim?



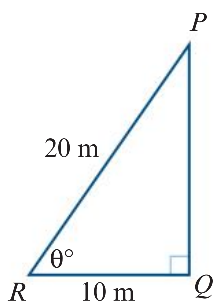
4 An equilateral triangle has altitudes of length 20 cm. Determine the length of one side.

5 A rope is tied to the top of a flagpole. When it hangs straight down, it is 2 m longer than the pole. When the rope is pulled tight with the lower end on the ground, it makes an angle of 60° to the horizontal. How tall is the flagpole?

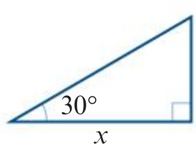
6 This figure shows a vertical mast PQ, which stands on horizontal ground. A straight wire 20 m long runs from P at the top of the mast to a point R on the ground, which is 10 m from the foot of the mast.

a Determine the angle of inclination, $\theta°$, of the wire to the ground.

b Determine the height of the mast.



7 The triangle shown has perimeter 10. Determine the value of x.



 ISBN 978-1-009-53737-7 Cambridge University Press

3C The sine rule

Learning intentions

- To be able to use the sine rule to solve triangles.

In the previous two sections, we focused on right-angled triangles. In this section and the next, we consider non-right-angled triangles. The **sine rule** is used to determine unknown side lengths or angles of a triangle in the following two situations:

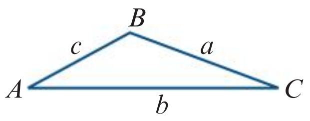
1. one side and two angles are given
2. two sides and a non-included angle are given (that is, the given angle is not 'between' the two given sides).

In the first case, the triangle is uniquely defined up to congruence. In the second case, there may be two triangles.

Labelling triangles

The following convention is used in the remainder of this chapter:

- Interior angles are denoted by uppercase letters.
- The length of the side opposite an angle is denoted by the corresponding lowercase letter.

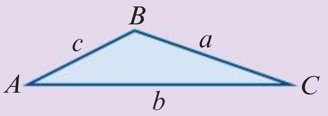


For example, the magnitude of angle BAC is denoted by A, and the length of side BC is denoted by a.

Sine rule

For triangle ABC:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$



Proof We will give a proof for acute-angled triangles. The proof for obtuse-angled triangles is similar.

In triangle ACD:

$$\sin A = \frac{h}{b}$$

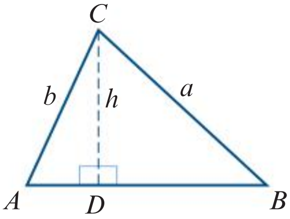
$$\therefore \quad h = b \sin A$$

In triangle BCD:

$$\sin B = \frac{h}{a}$$

$$\therefore \quad a \sin B = b \sin A$$

$$\text{i.e.} \quad \frac{a}{\sin A} = \frac{b}{\sin B}$$



Similarly, starting with a perpendicular from A to BC would give

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$

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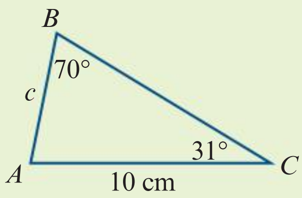
One side and two angles

When one side and two angles are given, this corresponds to the AAS congruence test. The triangle is uniquely defined up to congruence.

Example 5

Use the sine rule to determine the length of AB.



Solution

$$\frac{c}{\sin 31^\circ} = \frac{10}{\sin 70^\circ}$$

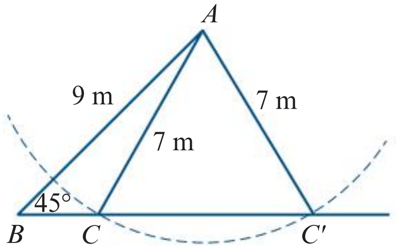
$$\therefore \quad c = \frac{10 \sin 31^\circ}{\sin 70^\circ}$$

$$= 5.4809\ldots$$

The length of AB is 5.48 cm, correct to two decimal places.

Two sides and a non-included angle

Suppose that we are given the two side lengths 7 m and 9 m and a non-included angle of 45°. There are two triangles that satisfy these conditions, as shown in the diagram.



Warning

- When you are given two sides and a non-included angle, you must consider the possibility that there are two such triangles.
- An angle found using the sine rule is possible if the sum of the given angle and the found angle is less than 180°.

Note: If the given angle is obtuse or a right angle, then there is only one such triangle.

The following example illustrates the case where there are two possible triangles.

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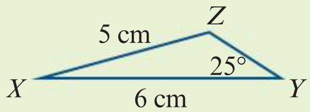
Example 6

Use the sine rule to determine the magnitude of angle XZY in the triangle, given that $Y = 25°$, $y = 5$ cm and $z = 6$ cm.



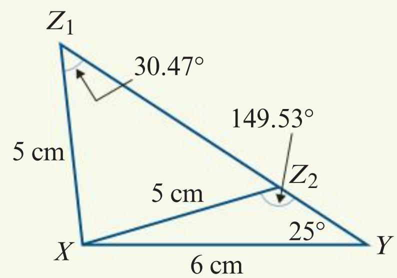
Solution

$$\frac{5}{\sin 25°} = \frac{6}{\sin Z}$$

$$\frac{\sin Z}{6} = \frac{\sin 25°}{5}$$

$$\sin Z = \frac{6 \sin 25°}{5}$$

$$= 0.5071\ldots$$



$\therefore \quad Z = (30.473\ldots)°$ or $Z = (180 - 30.473\ldots)°$

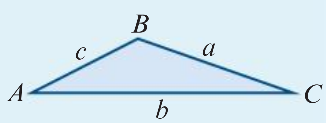
Hence $Z = 30.47°$ or $Z = 149.53°$, correct to two decimal places.

Note: Remember that $\sin(180 - \theta)° = \sin(\theta°)$.

Summary 3C

- **Sine rule** For triangle ABC:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$



- When to use the sine rule:
  - one side and two angles are given (AAS)
  - two sides and a non-included angle are given.

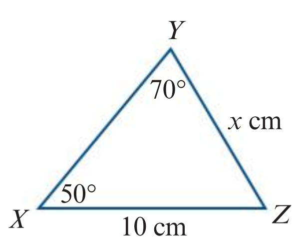
Skill-sheet

Exercise 3C

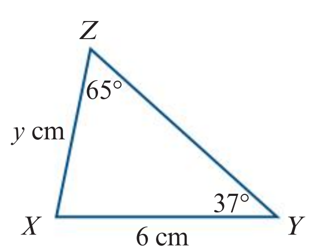
Example 5

1 Determine the value of the pronumeral for each of the following triangles:

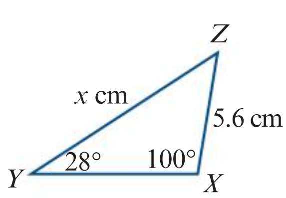
a



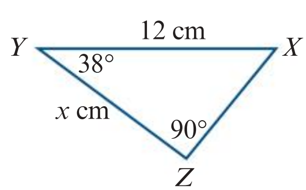
b



c



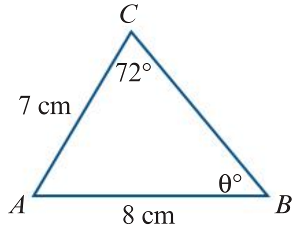
d

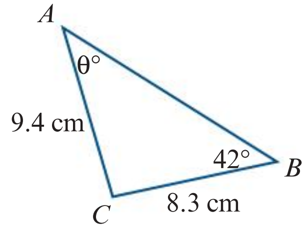


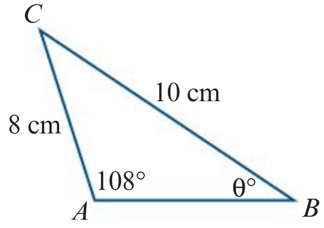
SF

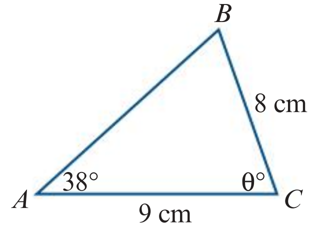
Example 6

2 Determine the value of θ for each of the following triangles:

a


b


c


d


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3 Solve the following triangles (i.e. determine all sides and angles):

a $a = 12,\ B = 59°,\ C = 73°$ **b** $A = 75.3°,\ b = 5.6,\ B = 48.25°$

c $A = 123.2°,\ a = 11.5,\ C = 37°$ **d** $A = 23°,\ a = 15,\ B = 40°$

e $B = 140°,\ b = 20,\ A = 10°$

4 Solve the following triangles (i.e. determine all sides and angles):

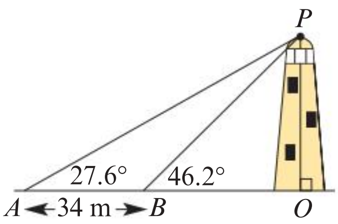
a $b = 17.6,\ C = 48.25°,\ c = 15.3$ **b** $B = 129°,\ b = 7.89,\ c = 4.56$

c $A = 28.35°,\ a = 8.5,\ b = 14.8$

CF

5 A landmark A is observed from two points B and C, which are 400 m apart. The magnitude of angle ABC is measured as 68° and the magnitude of angle ACB as 70°. Determine the distance of A from C.

6 P is a point at the top of a lighthouse. Measurements of the length AB and angles PBO and PAO are as shown in the diagram. Determine the height of the lighthouse.

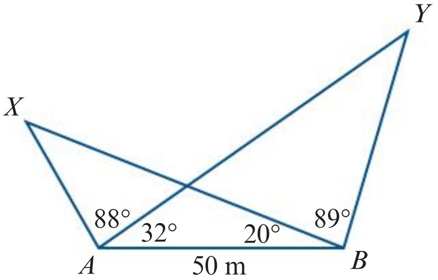


7 A and B are two points on a coastline, and C is a point at sea. The points A and B are 1070 m apart. The angles CAB and CBA have magnitudes of 74° and 69° respectively. Determine the distance of C from A.

8 Determine:

a AX

b AY



3D The cosine rule

Learning intentions

- To be able to use the cosine rule to solve triangles.

The **cosine rule** is used to determine unknown side lengths or angles of a triangle in the following two situations:

1. two sides and the included angle are given
2. three sides are given.

In each case, the triangle is uniquely defined up to congruence.

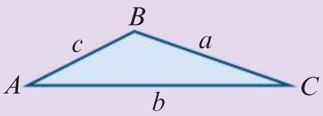
Cosine rule

For triangle ABC:

$$a^2 = b^2 + c^2 - 2bc\cos A$$

or equivalently

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

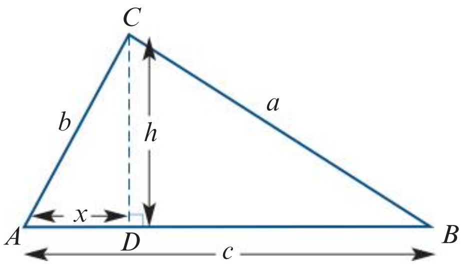


Proof We will give a proof for acute-angled triangles. The proof for obtuse-angled triangles is similar.

In triangle ACD:

$$\cos A = \frac{x}{b}$$

$$\therefore \quad x = b\cos A$$



Using Pythagoras' theorem in triangles ACD and BCD:

$$b^2 = x^2 + h^2$$

$$a^2 = (c - x)^2 + h^2$$

Expanding gives

$$a^2 = c^2 - 2cx + x^2 + h^2$$

$$= c^2 - 2cx + b^2 \qquad (\text{as } b^2 = x^2 + h^2)$$

$$\therefore \quad a^2 = b^2 + c^2 - 2bc\cos A \qquad (\text{as } x = b\cos A)$$

Note: By symmetry, the following results also hold:

$$b^2 = a^2 + c^2 - 2ac\cos B$$

$$c^2 = a^2 + b^2 - 2ab\cos C$$

Two sides and the included angle

When two sides and the included angle are given, this corresponds to the SAS congruence test. The triangle is uniquely defined up to congruence.

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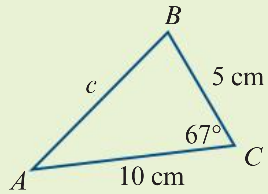
Example 7

For triangle ABC, determine the length of AB in centimetres correct to two decimal places.



Solution

$$c^2 = 5^2 + 10^2 - 2 \times 5 \times 10 \cos 67^\circ$$
$$= 85.9268\ldots$$
$$\therefore \quad c = 9.2696\ldots$$

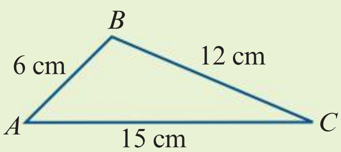
The length of AB is 9.27 cm, correct to two decimal places.

Three sides

When three sides are given, this corresponds to the SSS congruence test. The triangle is uniquely defined up to congruence.

Example 8

Determine the magnitude of angle ABC.



Solution

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$$
$$= \frac{12^2 + 6^2 - 15^2}{2 \times 12 \times 6}$$
$$= -0.3125$$
$$\therefore \quad B = (108.2099\ldots)^\circ$$

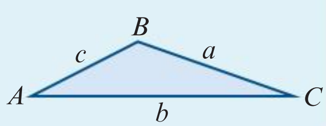
The magnitude of angle ABC is 108.21°, correct to two decimal places.

Summary 3D

- **Cosine rule** For triangle ABC:

$$a^2 = b^2 + c^2 - 2bc\cos A \quad \text{or} \quad \cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

- When to use the cosine rule:
  - two sides and the included angle are given (SAS)
  - three sides are given (SSS).



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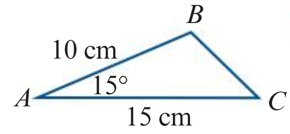
Skill-sheet

Exercise 3D

SF

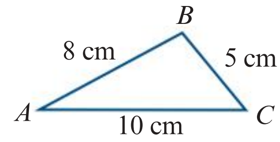
Example 7

1 Determine the length of BC.



Example 8

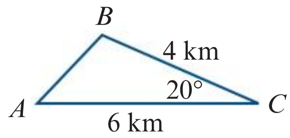
2 Determine the magnitudes of angles ABC and ACB.



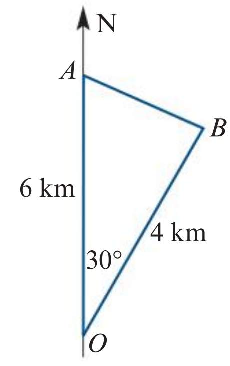
3 For triangle ABC with:

a $A = 60°$ $b = 16$ $c = 30$, determine a
b $a = 14$ $B = 53°$ $c = 12$, determine b
c $a = 27$ $b = 35$ $c = 46$, determine the magnitude of angle ABC
d $a = 17$ $B = 120°$ $c = 63$, determine b
e $a = 31$ $b = 42$ $C = 140°$, determine c
f $a = 10$ $b = 12$ $c = 9$, determine the magnitude of angle BCA
g $a = 11$ $b = 9$ $C = 43.2°$, determine c
h $a = 8$ $b = 10$ $c = 15$, determine the magnitude of angle CBA.

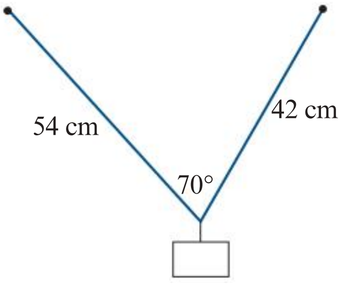
4 A section of an orienteering course is as shown. Determine the length of leg AB.



5 Two ships sail in different directions from a point O. At a particular time, their positions A and B are as shown. Determine the distance between the ships at this time.



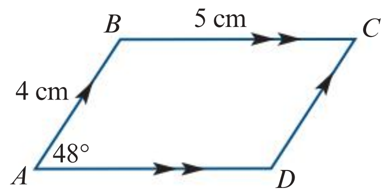
6 A weight is hung from two hooks in a ceiling by strings of length 54 cm and 42 cm, which are inclined at 70° to each other. Determine the distance between the hooks.



7 $ABCD$ is a parallelogram. Determine the lengths of the diagonals:

a AC

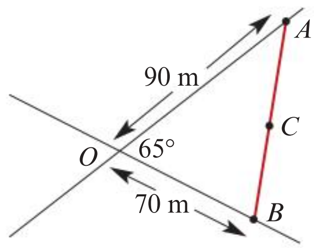
b BD



8 Two straight roads intersect at an angle of 65°. A point A on one road is 90 m from the intersection and a point B on the other road is 70 m from the intersection, as shown.

a Determine the distance of A from B.

b If C is the midpoint of AB, determine the distance of C from the intersection.

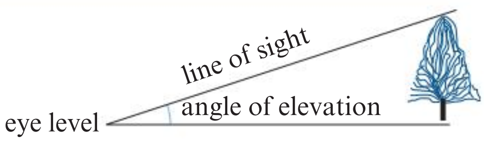


3E Angles of elevation, angles of depression and bearings

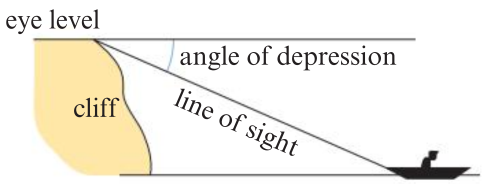
Learning intentions

- To be able to solve problems involving angles of elevation, angles of depression and bearings.

The **angle of elevation** is the angle between the horizontal and a direction above the horizontal.



The **angle of depression** is the angle between the horizontal and a direction below the horizontal.



Example 9

The pilot of a helicopter flying at 400 m observes a small boat at an angle of depression of 1.2°. Calculate the horizontal distance of the boat to the helicopter.

Solution

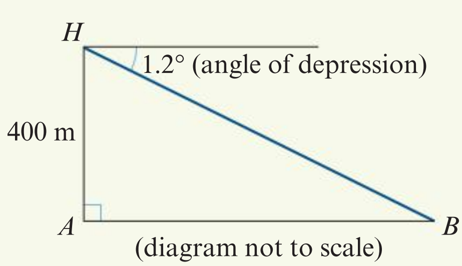
Note that $\angle ABH = 1.2°$, using alternate angles.

Thus

$$\frac{400}{AB} = \tan 1.2°$$

$$\therefore \quad AB = \frac{400}{\tan 1.2°}$$

$$= 19\,095.800\ldots$$



The horizontal distance is 19 100 m, correct to the nearest 10 m.

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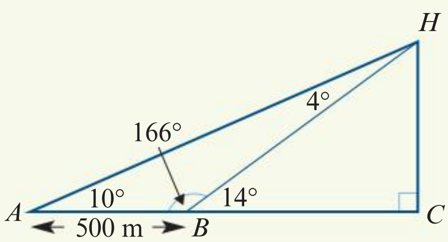
Example 10

From the point A, a man observes that the angle of elevation of the summit of a hill is 10°. He then walks towards the hill for 500 m along flat ground. The summit of the hill is now at an angle of elevation of 14°. Determine the height of the hill above the level of A.

Solution

Magnitude of $\angle HBA = (180 - 14)^\circ = 166^\circ$

Magnitude of $\angle AHB = (180 - (166 + 10))^\circ = 4^\circ$



Using the sine rule in triangle ABH:

$$\frac{500}{\sin 4^\circ} = \frac{HB}{\sin 10^\circ}$$

$$\therefore \quad HB = \frac{500 \sin 10^\circ}{\sin 4^\circ} = 1244.67\ldots$$

In triangle BCH:

$$\frac{HC}{HB} = \sin 14^\circ$$

$$\therefore \quad HC = HB \sin 14^\circ = 301.11\ldots$$

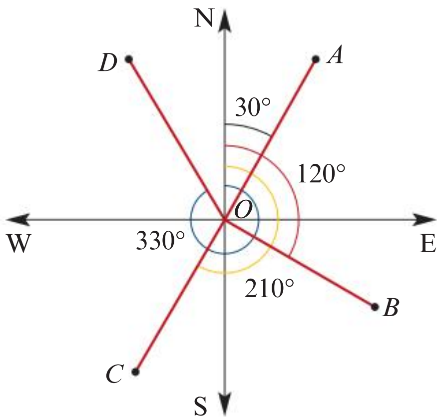
The height of the hill is 301 m, correct to the nearest metre.

Bearings

The **bearing** (or compass bearing) is the direction measured from north clockwise.

For example:

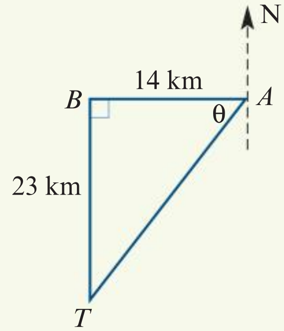
- The bearing of A from O is 030°.
- The bearing of B from O is 120°.
- The bearing of C from O is 210°.
- The bearing of D from O is 330°.



Example 11

The road from town A runs due west for 14 km to town B. A television mast is located due south of B at a distance of 23 km. Calculate the distance and bearing of the mast from the centre of town A.

Solution



$$\tan\theta = \frac{23}{14}$$

$\therefore \quad \theta = 58.67^\circ$ (to two decimal places)

Thus the bearing is

$$180^\circ + (90 - 58.67)^\circ = 211.33^\circ$$

To determine the distance, use Pythagoras' theorem:

$$AT^2 = AB^2 + BT^2$$
$$= 14^2 + 23^2$$
$$= 725$$

$\therefore \quad AT = 26.925\ldots$

The mast is 27 km from the centre of town A (to the nearest kilometre) and on a bearing of 211.33°.

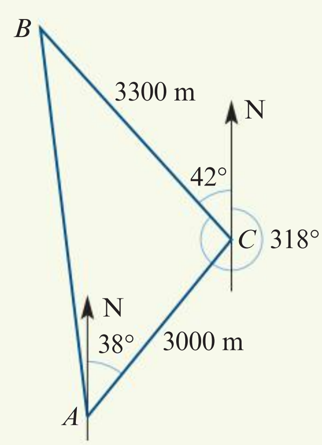
Example 12

A yacht starts from a point A and sails on a bearing of 038° for 3000 m. It then alters its course to a bearing of 318° and after sailing for a further 3300 m reaches a point B. Determine:

a the distance AB

b the bearing of B from A.

Solution



a The magnitude of angle ACB needs to be found so that the cosine rule can be applied in triangle ABC:

$$\angle ACB = (180 - (38 + 42))^\circ = 100^\circ$$

In triangle ABC:

$$AB^2 = 3000^2 + 3300^2 - 2 \times 3000 \times 3300 \cos 100^\circ$$
$$= 23\,328\,233.917\ldots$$

$\therefore \quad AB = 4829.931\ldots$

The distance of B from A is 4830 m (to the nearest metre).

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b To determine the bearing of B from A, the magnitude of angle BAC must first be found. Using the sine rule:

$$\frac{3300}{\sin A} = \frac{AB}{\sin 100^\circ}$$

$$\therefore \quad \sin A = \frac{3300 \sin 100^\circ}{AB}$$

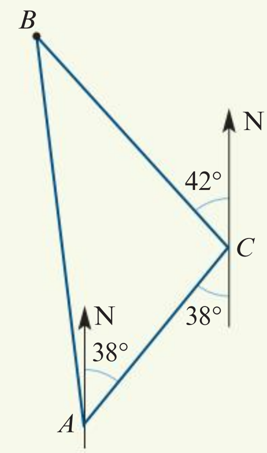
$$= 0.6728\ldots$$

$$\therefore \quad A = (42.288\ldots)^\circ$$

The bearing of B from $A = 360^\circ - (42.29^\circ - 38^\circ)$

$= 355.71^\circ$

The bearing of B from A is 356° to the nearest degree.



Exercise 3E

Example 9

1 From the top of a vertical cliff 130 m high, the angle of depression of a buoy at sea is 18°. What is the distance of the buoy from the foot of the cliff?

SF

2 The angle of elevation of the top of an old chimney stack at a point 40 m from its base is 41°. Determine the height of the chimney.

3 A hiker standing on top of a mountain observes that the angle of depression to the base of a building is 41°. If the height of the hiker above the base of the building is 500 m, determine the horizontal distance from the hiker to the building.

Example 10

4 A person standing on top of a cliff 50 m high is in line with two buoys whose angles of depression are 18° and 20°. Calculate the distance between the buoys.

Example 11

5 A ship sails 10 km north and then sails 15 km east. What is its bearing from the starting point?

6 A ship leaves port A and travels 15 km due east. It then turns and travels 22 km due north.

a What is the bearing of the ship from port A?

b What is the bearing of port A from the ship?

Example 12

7 A yacht sails from point A on a bearing of 035° for 2000 m. It then alters course to a direction with a bearing of 320° and after sailing for 2500 m it reaches point B.

CF

a Determine the distance AB.

b Determine the bearing of B from A.

8 The bearing of a point A from a point B is 207°. What is the bearing of B from A?

SF

9 The bearing of a ship S from a lighthouse A is 055°. A second lighthouse B is due east of A. The bearing of S from B is 302°. Determine the magnitude of angle ASB.

CF

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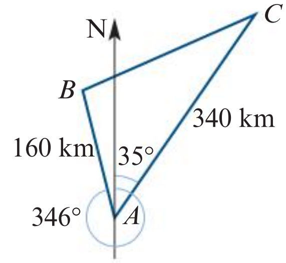
10 A yacht starts from L and sails 12 km due east to M. It then sails 9 km on a bearing of 142° to K. Determine the magnitude of angle MLK.

11 The bearing of C from A is 035°. The bearing of B from A is 346°. The distance of C from A is 340 km. The distance of B from A is 160 km.



a Determine the magnitude of angle BAC.

b Use the cosine rule to determine the distance from B to C.

12 From a ship S, two other ships P and Q are on bearings 320° and 075° respectively. The distance PS is 7.5 km and the distance QS is 5 km. Determine the distance PQ.

3F Angles in the four quadrants

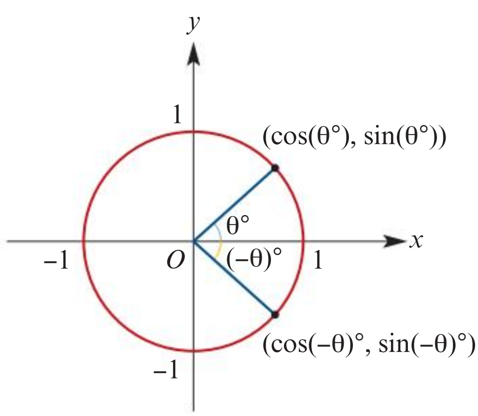
Learning intentions

- To be able to work with angles $-360° < \theta < 360°$

The unit-circle definition of sine and cosine given in Section 3A can be applied to any angle. In this section, we consider the sine and cosine of angles less than 0° and greater than 180°.

Negative angles

Angles formed by moving **anticlockwise** around the unit circle are defined as **positive**; those formed by moving **clockwise** are defined as **negative**.



From the diagram shown, we see that

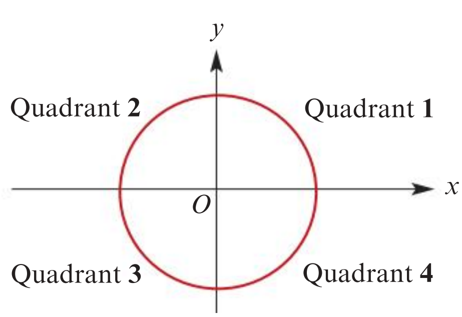
$$\sin(-\theta)^\circ = -\sin(\theta^\circ)$$
$$\cos(-\theta)^\circ = \cos(\theta^\circ)$$

For example:

$$\sin(-60)^\circ = -\sin 60^\circ$$
$$\cos(-60)^\circ = \cos 60^\circ$$

The four quadrants

The coordinate axes divide the unit circle into four quadrants, which are numbered anticlockwise as shown.



You are already familiar with angles in the first and second quadrants. We now consider angles in the third and fourth quadrants.

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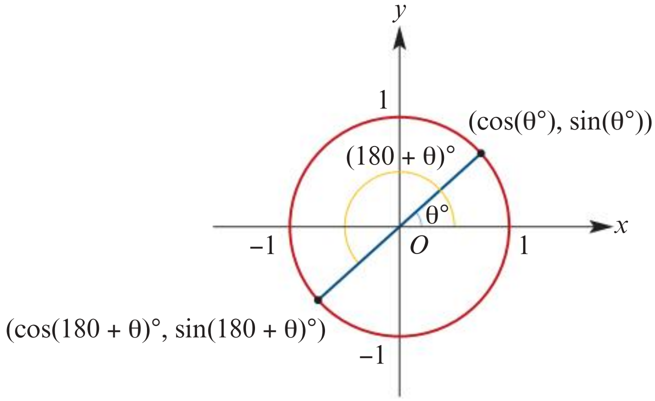
Third quadrant: angles between 180° and 270°

From the unit circle, we see that

$$\sin(180+\theta)^\circ = -\sin(\theta^\circ)$$
$$\cos(180+\theta)^\circ = -\cos(\theta^\circ)$$



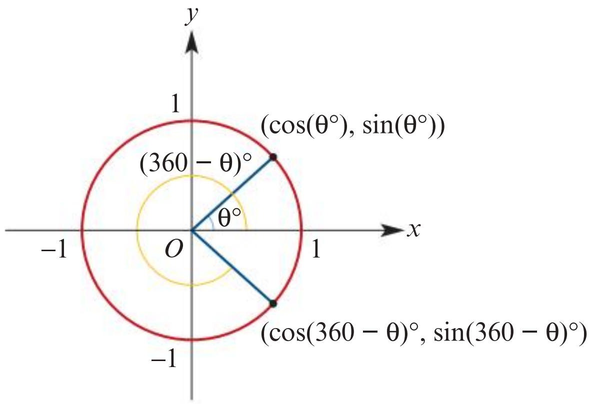
For example:

- $\sin 240^\circ = \sin(180+60)^\circ$
  $= -\sin 60^\circ$
- $\cos 240^\circ = \cos(180+60)^\circ$
  $= -\cos 60^\circ$

Fourth quadrant: angles between 270° and 360°

From the unit circle, we see that

$$\sin(360-\theta)^\circ = -\sin(\theta^\circ)$$
$$\cos(360-\theta)^\circ = \cos(\theta^\circ)$$



For example:

- $\sin 300^\circ = \sin(360-60)^\circ$
  $= -\sin 60^\circ$
- $\cos 300^\circ = \cos(360-60)^\circ$
  $= \cos 60^\circ$

Example 13

Determine the exact value of each of the following:

a $\sin(-45)^\circ$ **b** $\cos(-150)^\circ$

Solution

a $\sin(-45)^\circ = -\sin 45^\circ$

$= -\dfrac{1}{\sqrt{2}}$

b $\cos(-150)^\circ = \cos 150^\circ$

$= \cos(180-30)^\circ$

$= -\cos 30^\circ$

$= -\dfrac{\sqrt{3}}{2}$

Note: Remember that $\sin(180-\theta)^\circ = \sin(\theta^\circ)$ and $\cos(180-\theta)^\circ = -\cos(\theta^\circ)$.

Example 14

If $\cos x^\circ = 0.8$, determine the value of:

a $\cos(180 - x)^\circ$ **b** $\cos(180 + x)^\circ$ **c** $\cos(360 - x)^\circ$ **d** $\cos(-x)^\circ$

Solution

a $\cos(180 - x)^\circ$
$= -\cos x^\circ$
$= -0.8$

b $\cos(180 + x)^\circ$
$= -\cos x^\circ$
$= -0.8$

c $\cos(360 - x)^\circ$
$= \cos x^\circ$
$= 0.8$

d $\cos(-x)^\circ$
$= \cos x^\circ$
$= 0.8$

Summary 3F

- **Negative angles**

$$\sin(-\theta)^\circ = -\sin(\theta^\circ)$$
$$\cos(-\theta)^\circ = \cos(\theta^\circ)$$

- **Second quadrant**

$$\sin(180 - \theta)^\circ = \sin(\theta^\circ)$$
$$\cos(180 - \theta)^\circ = -\cos(\theta^\circ)$$

- **Third quadrant**

$$\sin(180 + \theta)^\circ = -\sin(\theta^\circ)$$
$$\cos(180 + \theta)^\circ = -\cos(\theta^\circ)$$

- **Fourth quadrant**

$$\sin(360 - \theta)^\circ = -\sin(\theta^\circ)$$
$$\cos(360 - \theta)^\circ = \cos(\theta^\circ)$$

Exercise 3F

SF

Example 13

1 Without using a calculator, evaluate the sine and cosine of each of the following angles:

a 120° **b** 135° **c** 180° **d** 210°
e 315° **f** 330° **g** 225° **h** 150°
i 360° **j** −30° **k** −45° **l** −120°
m −330° **n** −225° **o** −315° **p** −360°

Example 14

2 If $\sin x^\circ = 0.4$ and $\cos \theta^\circ = 0.6$, write down the values of:

a $\sin(180 + x)^\circ$ **b** $\cos(180 + \theta)^\circ$ **c** $\sin(360 - x)^\circ$ **d** $\cos(180 - \theta)^\circ$
e $\sin(-x)^\circ$ **f** $\cos(360 - \theta)^\circ$ **g** $\cos(-\theta)^\circ$ **h** $\sin(-x - 180)^\circ$

3 **a** If $\sin x^\circ = \sin 60^\circ$ and $90 < x < 180$, determine the value of x.
b If $\sin x^\circ = -\sin 60^\circ$ and $180 < x < 270$, determine the value of x.
c If $\sin x^\circ = -\sin 60^\circ$ and $-90 < x < 0$, determine the value of x.
d If $\cos x^\circ = -\cos 60^\circ$ and $90 < x < 180$, determine the value of x.
e If $\cos x^\circ = -\cos 60^\circ$ and $180 < x < 270$, determine the value of x.
f If $\cos x^\circ = \cos 60^\circ$ and $270 < x < 360$, determine the value of x.

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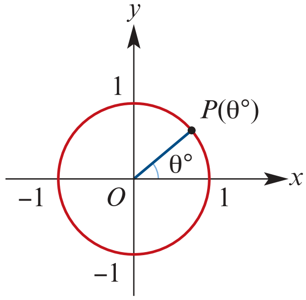
Chapter summary

Trigonometric functions

Sine and cosine

For each angle θ°, there is a point $P(\theta^\circ)$ on the unit circle as shown. The angle is measured *anticlockwise* from the positive direction of the *x*-axis.



- $\cos(\theta^\circ)$ is defined as the *x*-coordinate of the point $P(\theta^\circ)$
- $\sin(\theta^\circ)$ is defined as the *y*-coordinate of the point $P(\theta^\circ)$

Tangent

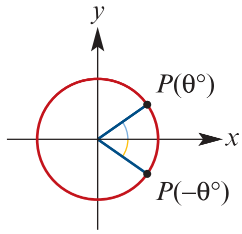
- $\tan(\theta^\circ) = \dfrac{\sin(\theta^\circ)}{\cos(\theta^\circ)}$ for $\cos(\theta^\circ) \neq 0$

Exact values

θ	sin θ	cos θ	tan θ
0°	0	1	0
30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
45°	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
90°	1	0	undefined

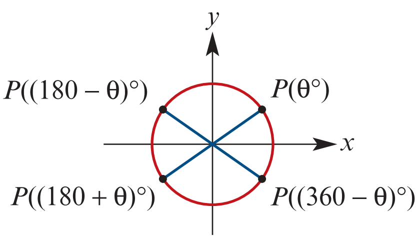
Negative angles

Angles measured *clockwise* from the positive direction of the *x*-axis are defined as *negative*.



$$\sin(-\theta)^\circ = -\sin(\theta^\circ)$$
$$\cos(-\theta)^\circ = \cos(\theta^\circ)$$

Angles in the four quadrants



- Second quadrant:

$$\sin(180 - \theta)^\circ = \sin(\theta^\circ)$$
$$\cos(180 - \theta)^\circ = -\cos(\theta^\circ)$$

- Third quadrant:

$$\sin(180 + \theta)^\circ = -\sin(\theta^\circ)$$
$$\cos(180 + \theta)^\circ = -\cos(\theta^\circ)$$

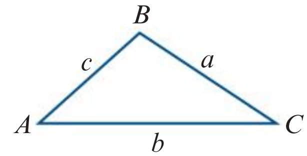
- Fourth quadrant:

$$\sin(360 - \theta)^\circ = -\sin(\theta^\circ)$$
$$\cos(360 - \theta)^\circ = \cos(\theta^\circ)$$

Triangles

■ Labelling triangles

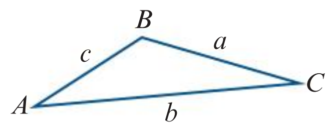
- Interior angles are denoted by uppercase letters.
- The length of the side opposite an angle is denoted by the corresponding lowercase letter.



For example, the magnitude of angle BAC is denoted by A, and the length of side BC by a.

■ Sine rule

For triangle ABC:



$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

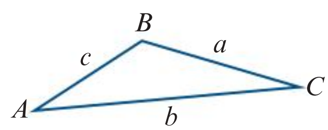
The sine rule is used to determine unknown quantities in a triangle in the following cases:

- one side and two angles are given
- two sides and a non-included angle are given.

In the first case, the triangle is uniquely defined. But in the second case, there may be two triangles.

■ Cosine rule

For triangle ABC:



$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

The symmetrical results also hold:

$$b^2 = a^2 + c^2 - 2ac\cos B$$

$$c^2 = a^2 + b^2 - 2ab\cos C$$

The cosine rule is used to determine unknown quantities in a triangle in the following cases:

- two sides and the included angle are given
- three sides are given.

Skills checklist

Check-list

Download this checklist from the Interactive Textbook, then print it and fill it out to check your skills. ☑

3A **1** I can solve right-angled triangles using sine, cosine and tangent. ☐

See Example 1, Example 2 and Question 3

3B **2** I can determine exact values of trigonometric ratios for certain acute and obtuse angles. ☐

See Example 3, Example 4 and Questions 1 and 2

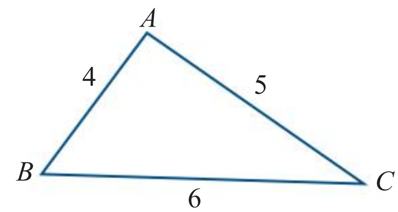
3C	3	I can use the sine rule to solve triangles when one side and two angles are given or two sides and a non-included angle are given.	☐
		See Example 5, Example 6 and Questions 1 and 2	
3D	4	I can use the cosine rule to solve triangles when two sides and an included angle are given or three sides are given.	☐
		See Example 7, Example 8 and Questions 1 and 2	
3E	5	I can solve problems involving angles of elevation and angles of depression.	☐
		See Example 9, Example 10 and Questions 1 and 4	
3E	6	I can solve problems involving bearings.	☐
		See Example 11, Example 12 and Questions 5 and 7	
3F	7	I can understand the relationship between related angles in different quadrants and can use this knowledge to determine exact values of trigonometric functions for certain angles.	☐
		See Example 13, Example 14 and Questions 1 and 2	

Short-response questions

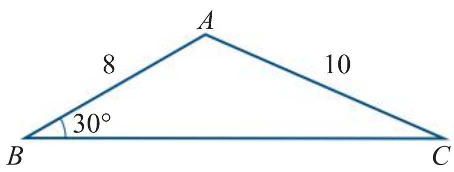
Technology-free short-response questions

SF

1 The triangle shown has sides $AB = 4$, $BC = 6$ and $CA = 5$. Determine $\cos(\angle BAC)$.



2 In triangle ABC, $AB = 8$, $AC = 10$ and $\angle ABC = 30°$. Determine $\sin(\angle ACB)$.



3 Triangle ABC has $AB = BC = 10$ cm and $\angle ABC = 120°$. Determine AC.

4 If $\sin x = \sin 37°$ and x is obtuse, determine x.

CF

5 A point T is 10 km due north of a point S. A point R, which is east of the straight line joining T and S, is 8 km from T and 7 km from S. Calculate the cosine of the bearing of R from S.

6 In $\triangle ABC$, $AB = 5$ cm, $\angle BAC = 60°$ and $AC = 6$ cm. Calculate the sine of $\angle ABC$.

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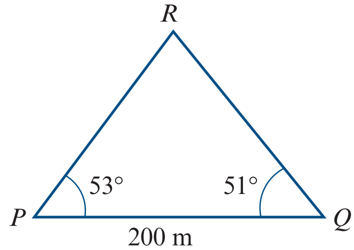
Review

7 A boat sails 11 km from a harbour on a bearing of 220°. It then sails 15 km on a bearing of 340°. How far is the boat from the harbour? CF

8 A helicopter leaves a heliport A and flies 2.4 km on a bearing of 150° to a checkpoint B. It then flies due east to its base C. If the bearing of C from A is 120°, determine the distances AC and BC.

Technology-active short-response questions

9 P and Q are points on the bank of a river. A tree is at a point, R, on the opposite bank such that $\angle QPR$ is 53° and $\angle RQP$ is 51°. CF

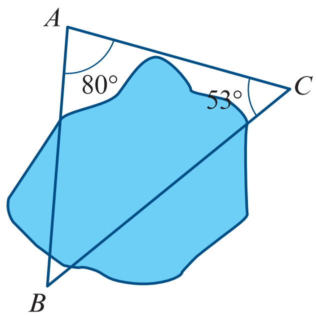


a Determine:

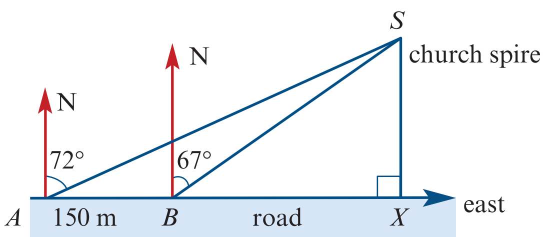
i RP **ii** RQ.

b T is a point between P and Q such that $\angle PTR$ is a right angle. Determine RT and hence the width of the river, correct to two decimal places.

10 Two points, A and B, are on opposite sides of a lake so that the distance between them cannot be measured directly. A third point, C, is chosen at a distance of 300 m from A and with angles BAC and BCA of 80° and 53°, respectively. Calculate the distance between A and B.



11 A man walking due east along a level road observes a church spire from point A. The bearing of the spire from A is 072°. He then walks 150 m to point B where the bearing is 067°.



a Determine the distance of the church spire from B (i.e. BS).

b Determine the distance of the church spire from the road (i.e. SX).

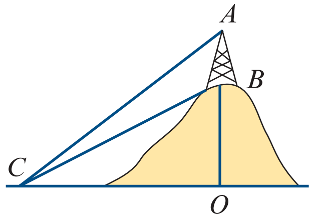
12 From a ship, S, two other ships, P, and Q, are on bearings 320° and 075°, respectively. The distance $PS = 7.5$ km and the distance $QS = 5$ km. Determine the distance PQ.

13 A yacht starts from point A and sails on a bearing of 035° for 2000 m. It then alters its course to one in a direction with a bearing of 320° and after sailing for 2500 m it reaches point B.

a Determine the distance AB.

b Determine the bearing of B from A.

14 AB is a tower 60 m high on top of a hill. The magnitude of $\angle ACO$ is 49° and the magnitude of $\angle BCO$ is 37°.

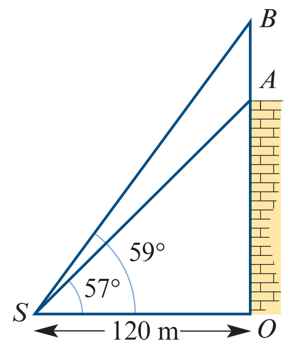


a Determine the magnitudes of $\angle ACB$, $\angle CBO$ and $\angle CBA$.

b Determine the length of BC.

c Determine the height of the hill, i.e. the length of OB.

15 Point S is a distance of 120 m from the base of a building. On the building is an aerial, AB. The angle of elevation from S to A is 57°. The angle of elevation from S to B is 59°. Determine:

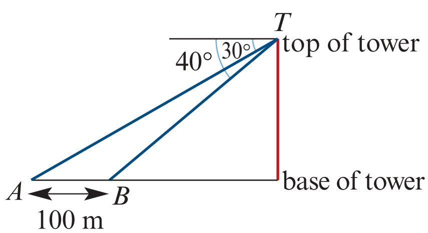


a the distance OA

b the distance OB

c the distance AB.

16 From the top of a communications tower, T, the angles of depression of two points A and B on a horizontal line through the base of the tower are 30° and 40°. The distance between the points is 100 m. Determine:



a the distance AT

b the height of the tower.

Multiple-choice questions

Technology-free multiple-choice questions

1 In triangle ABC, $\sin A = \frac{1}{8}$, $\sin B = \frac{3}{4}$ and $a = 10$. The value of b is.

A 30 **B** 40 **C** 50 **D** 60

2 In a triangle ABC, $a = 30$, $b = 21$ and $\cos C = \frac{5}{7}$. The value of c is

A 18 **B** 19 **C** 20 **D** 21

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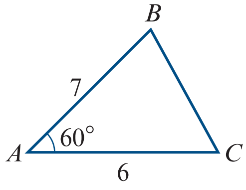
3 From a point on a cliff x m above sea level, the angle of depression to a boat is 60° The distance from the foot of the cliff to the boat, is 20 metres. The value of x is

A 87 m **B** $20\sqrt{3}$ m **C** $\frac{20}{\sqrt{3}}$ m **D** 45 m

4 The square of the length of side BC is:

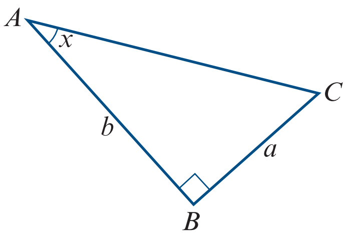


A 85 **B** 49
C 42 **D** 43

5 A boat sails at a bearing of 215° from A to B. The bearing it must take from B to return to A is

A 035° **B** 055° **C** 090° **D** 215°

6 In the triangle ABC, $\cos x =$



A $\frac{a}{\sqrt{a^2 + b^2}}$ **B** $\frac{b}{\sqrt{a^2 + b^2}}$

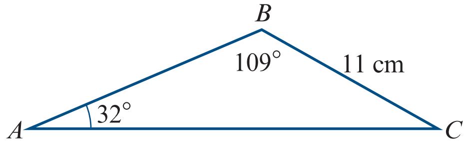
C $\frac{a}{b}$ **D** $\frac{b}{a}$

7 If $\sin(\theta°) = -\frac{\sqrt{3}}{2}$, where $270° < \theta° < 360°$, then the angle $\theta°$ is

A 285° **B** 300° **C** 315° **D** 330°

Technology-active multiple-choice questions

8 The length of AC, correct to one decimal place, is:



A 6.2 cm **B** 16.3 cm
C 19.6 cm **D** 40.4 cm

9 In a triangle XYZ, $x = 21$ cm, $y = 18$ cm and $\angle YXZ = 62°$. The magnitude of $\angle XYZ$, correct to one decimal place, is

A 0.4° **B** 0.8° **C** 1.0° **D** 49.2°

10 In a parallelogram $ABCD$, $AB = CD = 8$ cm and $BC = AD = 12$ cm. If $\angle BCD = 52°$, the length of the diagonal AC, to the nearest centimetre, is:

A 12 cm **B** 18 cm **C** 16 cm **D** 149 cm

11 In a triangle ABC, $a = 30$, $b = 21$ and $\cos C = \frac{51}{53}$. The value of c, to the nearest whole number, is

A 9 **B** 10 **C** 11 **D** 81

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12 In a triangle ABC, $a = 5.2$ cm, $b = 6.8$ cm and $c = 7.3$ cm. The magnitude of $\angle ACB$, correct to the nearest degree, is

A 43° **B** 63° **C** 74° **D** 82°

13 From a point on a cliff 500 m above sea level, the angle of depression to a boat is 20°. The distance from the foot of the cliff to the boat, to the nearest metre, is

A 182 m **B** 193 m **C** 210 m **D** 1374 m

14 A tower 80 m high is 1.3 km away from a point on the ground. The angle of elevation to the top of the tower from this point, correct to the nearest degree, is

A 1° **B** 4° **C** 53° **D** 86°

15 A man walks 5 km due east followed by 7 km due south. The bearing he must take to return to the start is

A 036° **B** 306° **C** 324° **D** 332°

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4

Further algebra

Chapter contents

This chapter is provided to strengthen the student's algebra and prepare for Unit 4.

In this chapter we first consider equating coefficients of polynomial functions, and then apply this technique to establish partial fractions.

In Chapter 1 we added and subtracted algebraic fractions such as

$$\frac{2}{x+3} + \frac{4}{x-3} = \frac{6(x+1)}{x^2-9}$$

In this chapter we learn how to go from right to left in similar equations. This process is sometimes called **partial fraction decomposition**. Another example is

$$\frac{4x^2+2x+6}{(x^2+3)(x-3)} = \frac{2}{x^2+3} + \frac{4}{x-3}$$

This is a useful tool in integral calculus, and partial fractions are applied this way in Specialist Mathematics Units 3 & 4.

This chapter also includes further study of quadratic functions: solving quadratic equations, using the discriminant, applying quadratic functions to problems involving rates and using quadratic equations to determine the intersection of straight lines with parabolas, circles and rectangular hyperbolas.

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4A Polynomial identities

Learning intentions

- To be able to equate coefficients with polynomial identities.

Polynomials are introduced in Mathematical Methods Units 1 & 2.

- A **polynomial function** is a function that can be written in the form

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

  where n is a natural number or zero, and the coefficients $a_0, \dots, a_n$ are real numbers with $a_n \neq 0$.
- The number 0 is called the **zero polynomial**.
- The **leading term**, $a_n x^n$, of a polynomial is the term of highest index among those terms with a non-zero coefficient.
- The **degree of a polynomial** is the index n of the leading term.
- A **monic polynomial** is a polynomial whose leading term has coefficient 1.
- The **constant term** is the term of index 0. (This is the term not involving x.)

Two polynomials are **equal** if they give the same value for all x. It can be proved that, if two polynomials are equal, then they have the same degree and corresponding coefficients are equal. For example:

- If $ax + b = cx^2 + dx + e$, then $c = 0$, $d = a$ and $e = b$.
- If $x^2 - x - 12 = x^2 + (a + b)x + ab$, then $a + b = -1$ and $ab = -12$.

This process is called **equating coefficients**.

Example 1

If the expressions $(a + 2b)x^2 - (a - b)x + 8$ and $3x^2 - 6x + 8$ are equal for all x, determine the values of a and b.

Solution

Assume that

$$(a + 2b)x^2 - (a - b)x + 8 = 3x^2 - 6x + 8 \quad \text{for all } x$$

Then by equating coefficients:

$$a + 2b = 3 \qquad (1)$$

$$-(a - b) = -6 \qquad (2)$$

Solve as simultaneous equations. Add (1) and (2):

$$3b = -3$$

$$\therefore b = -1$$

Substitute into (1):

$$a - 2 = 3$$

$$\therefore a = 5$$

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Example 2

Express x^2 in the form $c(x-3)^2 + a(x-3) + d$.

Solution

Let $x^2 = c(x-3)^2 + a(x-3) + d$

$= c(x^2 - 6x + 9) + a(x-3) + d$

$= cx^2 + (a - 6c)x + 9c - 3a + d$

This implies that

$$c = 1 \qquad (1)$$

$$a - 6c = 0 \qquad (2)$$

$$9c - 3a + d = 0 \qquad (3)$$

From (2): $a = 6$

From (3): $9 - 18 + d = 0$

i.e. $d = 9$

Hence $x^2 = (x-3)^2 + 6(x-3) + 9$.

Example 3

Determine the values of a, b, c and d such that

$$x^3 = a(x+2)^3 + b(x+1)^2 + cx + d \quad \text{for all } x$$

Solution

Expand the right-hand side and collect like terms:

$$x^3 = a(x^3 + 6x^2 + 12x + 8) + b(x^2 + 2x + 1) + cx + d$$

$$= ax^3 + (6a + b)x^2 + (12a + 2b + c)x + (8a + b + d)$$

Equate coefficients:

$$a = 1 \qquad (1)$$

$$6a + b = 0 \qquad (2)$$

$$12a + 2b + c = 0 \qquad (3)$$

$$8a + b + d = 0 \qquad (4)$$

Substituting $a = 1$ into (2) gives

$$6 + b = 0$$

$\therefore$ $b = -6$

Substituting $a = 1$ and $b = -6$ into (3) gives

$$12 - 12 + c = 0$$

$\therefore$ $c = 0$

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Substituting $a = 1$ and $b = -6$ into (4) gives

$$8 - 6 + d = 0$$

$$\therefore \quad d = -2$$

Hence $x^3 = (x + 2)^3 - 6(x + 1)^2 - 2$.

Example 4

Show that $2x^3 - 5x^2 + 4x + 1$ cannot be expressed in the form $a(x + b)^3 + c$.

Solution

Suppose that

$$2x^3 - 5x^2 + 4x + 1 = a(x + b)^3 + c$$

for some constants a, b and c.

Then expanding the right-hand side gives

$$\begin{aligned} 2x^3 - 5x^2 + 4x + 1 &= a(x^3 + 3bx^2 + 3b^2x + b^3) + c \\ &= ax^3 + 3abx^2 + 3ab^2x + ab^3 + c \end{aligned}$$

Equating coefficients:

$$\begin{aligned} a &= 2 \qquad &(1) \\ 3ab &= -5 \qquad &(2) \\ 3ab^2 &= 4 \qquad &(3) \\ ab^3 + c &= 1 \qquad &(4) \end{aligned}$$

From (2), we have $b = -\frac{5}{6}$. But from (3), we have $b = \pm\sqrt{\frac{2}{3}} = \pm\frac{\sqrt{6}}{3}$.

This is a contradiction, and therefore we have shown that $2x^3 - 5x^2 + 4x + 1$ cannot be expressed in the form $a(x + b)^3 + c$.

Summary 4A

- A **polynomial function** can be written in the form

  $$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

  where n is a natural number or zero, and the coefficients $a_0, \ldots, a_n$ are real numbers with $a_n \neq 0$. The **leading term** is $a_n x^n$ (the term of highest index) and the **constant term** is a_0 (the term not involving x).
- The **degree** of a polynomial is the index n of the leading term.
- Equating coefficients

  Two polynomials are equal if they give the same value for all x. If two polynomials are equal, then they have the same degree and corresponding coefficients are equal.

  For example: if $x^2 - x - 12 = x^2 + (a + b)x + ab$, then $a + b = -1$ and $ab = -12$.

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Exercise 4A

SF

1 If $ax^2 + bx + c = 10x^2 - 7$, determine the values of a, b and c.

Example 1 **2** If $(2a - b)x^2 + (a + 2b)x + 8 = 4x^2 - 3x + 8$, determine the values of a and b.

3 If $(2a - 3b)x^2 + (3a + b)x + c = 7x^2 + 5x + 7$, determine the values of a, b and c.

4 If $2x^2 + 4x + 5 = a(x + b)^2 + c$, determine the values of a, b and c.

Example 2 **5** Express x^2 in the form $c(x + 2)^2 + a(x + 2) + d$.

6 Express x^3 in the form $(x + 1)^3 + a(x + 1)^2 + b(x + 1) + c$.

Example 3 **7** Determine the values of a, b and c such that $x^2 = a(x + 1)^2 + bx + c$.

Example 4 **8** **a** Show that $3x^3 - 9x^2 + 8x + 2$ cannot be expressed in the form $a(x + b)^3 + c$. CF

b If $3x^3 - 9x^2 + 9x + 2$ can be expressed in the form $a(x + b)^3 + c$, then determine the values of a, b and c.

9 Show that constants a, b, c and d can be found such that

$$n^3 = a(n + 1)(n + 2)(n + 3) + b(n + 1)(n + 2) + c(n + 1) + d$$

10 **a** Show that no constants a and b can be found such that

$$n^2 = a(n + 1)(n + 2) + b(n + 2)(n + 3)$$

b Express n^2 in the form $a(n + 1)(n + 2) + b(n + 1) + c$.

11 **a** Express $a(x + b)^2 + c$ in expanded form. SF

b Express $ax^2 + bx + c$ in completed-square form.

12 Prove that, if $ax^3 + bx^2 + cx + d = (x - 1)^2(px + q)$, then $b = d - 2a$ and $c = a - 2d$. CF

13 If $3x^2 + 10x + 3 = c(x - a)(x - b)$ for all values of x, determine the values of a, b and c. SF

14 For any number n, show that n^2 can be expressed as $a(n - 1)^2 + b(n - 2)^2 + c(n - 3)^2$, and determine the values of a, b and c. CF

15 If $x^3 + 3x^2 - 9x + c$ can be expressed in the form $(x - a)^2(x - b)$, show that either $c = 5$ or $c = -27$, and determine a and b for each of these cases.

16 A polynomial P is said to be **even** if $P(-x) = P(x)$ for all x. A polynomial P is said to be **odd** if $P(-x) = -P(x)$ for all x.

a Show that, if $P(x) = ax^4 + bx^3 + cx^2 + dx + e$ is even, then $b = d = 0$.

b Show that, if $P(x) = ax^5 + bx^4 + cx^3 + dx^2 + ex + f$ is odd, then $b = d = f = 0$.

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4B Quadratic equations

Learning intentions

- To be able to solve quadratic equations, use the discriminant and apply quadratics to various contexts.

A polynomial function of degree 2 is called a **quadratic function**. The general quadratic function can be written as $P(x) = ax^2 + bx + c$, where $a \neq 0$.

Quadratic functions are studied extensively in Mathematical Methods Units 1 & 2. In this section we provide further practice exercises.

A quadratic equation $ax^2 + bx + c = 0$ may be solved by factorising, by completing the square or by using the general quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The following example demonstrates each method.

Example 5

Solve the following quadratic equations for x:

a $2x^2 + 5x = 12$ **b** $3x^2 + 4x = 2$ **c** $9x^2 + 6x + 1 = 0$

Solution

a

$$2x^2 + 5x - 12 = 0$$

Rearrange the quadratic equation.

$$(2x - 3)(x + 4) = 0$$

Factorise.

$$2x - 3 = 0 \text{ or } x + 4 = 0$$

Use the null factor theorem.

Therefore $x = \frac{3}{2}$ or $x = -4$.

b

$$3x^2 + 4x - 2 = 0$$

Rearrange the quadratic equation.

$$x^2 + \frac{4}{3}x - \frac{2}{3} = 0$$

Divide both sides by 3.

$$x^2 + \frac{4}{3}x + \left(\frac{2}{3}\right)^2 - \left(\frac{2}{3}\right)^2 - \frac{2}{3} = 0$$

Add and subtract $\left(\frac{b}{2}\right)^2$ to 'complete the square'.

$$\left(x + \frac{2}{3}\right)^2 - \frac{4}{9} - \frac{2}{3} = 0$$

$$\left(x + \frac{2}{3}\right)^2 = \frac{10}{9}$$

$$x + \frac{2}{3} = \pm\frac{\sqrt{10}}{3}$$

$$x = -\frac{2}{3} \pm \frac{\sqrt{10}}{3}$$

Therefore $x = \frac{-2 + \sqrt{10}}{3}$ or $x = \frac{-2 - \sqrt{10}}{3}$.

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c If $9x^2 + 6x + 1 = 0$, then

$$x = \frac{-6 \pm \sqrt{6^2 - 4 \times 9 \times 1}}{2 \times 9}$$

$$= \frac{-6 \pm \sqrt{0}}{18}$$

$$= -\frac{1}{3}$$

Use the general quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Alternatively, the equation can be solved by noting that $9x^2 + 6x + 1 = (3x + 1)^2$.

The discriminant: real solutions

The number of solutions to a quadratic equation $ax^2 + bx + c = 0$ can be determined by the **discriminant** Δ, where $\Delta = b^2 - 4ac$.

- If $\Delta > 0$, the equation has two real solutions.
- If $\Delta = 0$, the equation has one real solution.
- If $\Delta < 0$, the equation has no real solutions.

Note: In parts **a** and **b** of Example 5, we have $\Delta > 0$ and so there are two real solutions. In part **c**, we have $\Delta = 6^2 - 4 \times 9 \times 1 = 0$ and so there is only one real solution.

The discriminant: rational solutions

For a quadratic equation $ax^2 + bx + c = 0$ such that a, b and c are rational numbers:

- If Δ is a perfect square and $\Delta \neq 0$, then the equation has two rational solutions.
- If $\Delta = 0$, then the equation has one rational solution.
- If Δ is not a perfect square and $\Delta > 0$, then the equation has two irrational solutions.

Note: In part **a** of Example 5, we have $\Delta = 121$, which is a perfect square.

Example 6

Consider the quadratic equation $x^2 - 4x = t$. Make x the subject and give the values of t for which real solution(s) to the equation can be found.

Solution

$$x^2 - 4x = t$$

$$x^2 - 4x + 4 = t + 4 \qquad \text{(completing the square)}$$

$$(x - 2)^2 = t + 4$$

$$x - 2 = \pm\sqrt{t + 4}$$

$$x = 2 \pm \sqrt{t + 4}$$

For real solutions to exist, we must have $t + 4 \geq 0$, i.e. $t \geq -4$.

Note: In this case the discriminant is $\Delta = 16 + 4t$. There are real solutions when $\Delta \geq 0$.

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Example 7

a Determine the discriminant of the quadratic $x^2 + px - \frac{25}{4}$ in terms of p.

b Solve the quadratic equation $x^2 + px - \frac{25}{4} = 0$ in terms of p.

c Prove that there are two solutions for all values of p.

d Determine the values of p, where p is a non-negative integer, for which the quadratic equation has rational solutions.

Solution

Here we have $a = 1$, $b = p$ and $c = -\frac{25}{4}$.

a $\Delta = b^2 - 4ac = p^2 + 25$

b The quadratic formula gives $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \dfrac{-p \pm \sqrt{p^2 + 25}}{2}$.

c We have $\Delta = p^2 + 25 > 0$, for all values of p. Thus there are always two solutions.

d If there are rational solutions, then $\Delta = p^2 + 25$ is a perfect square. Since p is an integer, we can write $p^2 + 25 = k^2$, where k is an integer with $k \geq 0$.

Rearranging, we have

$$k^2 - p^2 = 25$$

$$\therefore \quad (k - p)(k + p) = 25$$

We can factorise 25 as 5×5 or 1×25.

Note: We do not need to consider negative factors of 25, as p and k are non-negative, and so $k + p \geq 0$. Since p is non-negative, we also know that $k - p \leq k + p$.

The table on the right shows the values of k and p in each of the two cases.

Hence $p = 0$ and $p = 12$ are the only values for which the solutions are rational.

$k - p$	$k + p$	k	p
5	5	5	0
1	25	13	12

Example 8

Solve the equation $x - 4\sqrt{x} - 12 = 0$ for x.

Solution

For $\sqrt{x}$ to be defined, we must have $x \geq 0$.

Let $x = a^2$, where $a \geq 0$. The equation becomes

$$a^2 - 4\sqrt{a^2} - 12 = 0$$

$$a^2 - 4a - 12 = 0$$

$$(a - 6)(a + 2) = 0$$

$$\therefore \quad a = 6 \text{ or } a = -2$$

But $a \geq 0$. Hence $a = 6$ and so $x = 36$.

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Example 9

A rectangle has an area of 288 cm^2. If the width is decreased by 1 cm and the length increased by 1 cm, the area would be decreased by 3 cm^2. Determine the original dimensions of the rectangle.

Solution

Let w and ℓ be the width and length, in centimetres, of the original rectangle.

Then $w\ell = 288 \qquad (1)$

The dimensions of the new rectangle are $w - 1$ and $\ell + 1$, and the area is 285 cm^2.

Thus $(w-1)(\ell+1) = 285 \qquad (2)$

Rearranging (1) to make ℓ the subject and substituting in (2) gives

$$(w-1)\left(\frac{288}{w}+1\right) = 285$$

$$288 - \frac{288}{w} + w - 1 = 285$$

$$w - \frac{288}{w} + 2 = 0$$

$$w^2 + 2w - 288 = 0$$

Using the general quadratic formula gives

$$w = \frac{-2 \pm \sqrt{2^2 - 4 \times (-288)}}{2}$$

$$= \frac{-2 \pm 34}{2}$$

$$= -18 \text{ or } 16$$

But $w > 0$, and so $w = 16$. The original dimensions of the rectangle are 16 cm by 18 cm.

Summary 4B

- Quadratic equations can be solved by completing the square. This method allows us to deal with all quadratic equations, even though some have no solutions.
- To complete the square of $x^2 + bx + c$:
  - Take half the coefficient of x (that is, $\frac{b}{2}$) and add and subtract its square $\frac{b^2}{4}$.
- To complete the square of $ax^2 + bx + c$:
  - First take out a as a factor and then complete the square inside the bracket.
- The solutions of the quadratic equation $ax^2 + bx + c = 0$, where $a \neq 0$, are given by the **quadratic formula**

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

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- The **discriminant** Δ of a quadratic polynomial $ax^2 + bx + c$ is

$$\Delta = b^2 - 4ac$$

For the equation $ax^2 + bx + c = 0$:

  - If $\Delta > 0$, there are two solutions.
  - If $\Delta = 0$, there is one solution.
  - If $\Delta < 0$, there are no real solutions.

- For the equation $ax^2 + bx + c = 0$, where a, b and c are rational numbers:
  - If Δ is a perfect square and $\Delta \neq 0$, there are two rational solutions.
  - If $\Delta = 0$, there is one rational solution.
  - If Δ is not a perfect square and $\Delta > 0$, there are two irrational solutions.

Skill-sheet

Exercise 4B

SF

Example 5

1 Solve the following quadratic equations for x:

a $x^2 + 2x = -1$ **b** $x^2 - 6x + 9 = 0$ **c** $5x^2 - 10x = 1$

d $-2x^2 + 4x = 1$ **e** $2x^2 + 4x = 7$ **f** $6x^2 + 13x + 1 = 0$

CF

2 The following equations have the number of solutions shown in brackets. Determine the possible values of m.

a $x^2 + 3x + m = 0$ (0) **b** $x^2 - 5x + m = 0$ (2) **c** $mx^2 + 5x - 8 = 0$ (1)

d $x^2 + mx + 9 = 0$ (2) **e** $x^2 - mx + 4 = 0$ (0) **f** $4x^2 - mx - m = 0$ (1)

Example 6

3 Make x the subject in each of the following and give the values of t for which real solution(s) to the equation can be found:

a $2x^2 - 4t = x$ **b** $4x^2 + 4x - 4 = t - 2$

c $5x^2 + 4x + 10 = t$ **d** $tx^2 + 4tx + 10 = t$

Example 7

4 **a** Solve the quadratic equation $x^2 + px - 16 = 0$ in terms of p.

b Determine the values of p, where p is an integer with $0 \leq p \leq 10$, for which the quadratic equation in **a** has rational solutions.

5 **a** Show that the solutions of the equation $2x^2 - 3px + (3p - 2) = 0$ are rational for all integer values of p.

b Determine the value of p for which there is only one solution.

c Solve the equation when:

i $p = 1$ **ii** $p = 2$ **iii** $p = -1$

6 **a** Show that the solutions of the equation $4(4p - 3)x^2 - 8px + 3 = 0$ are rational for all integer values of p.

b Determine the value of p for which there is only one solution.

c Solve the equation when:

i $p = 1$ **ii** $p = 2$ **iii** $p = -1$

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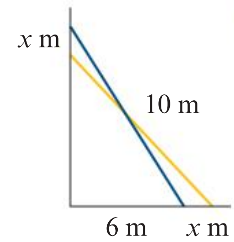
Example 8 **7** Solve each of the following equations for x:

a $x - 8\sqrt{x} + 12 = 0$ **b** $x - 8 = 2\sqrt{x}$ **c** $x - 5\sqrt{x} - 14 = 0$

d $\sqrt[3]{x^2} - 9\sqrt[3]{x} + 8 = 0$ **e** $\sqrt[3]{x^2} - \sqrt[3]{x} - 6 = 0$ **f** $x - 29\sqrt{x} + 100 = 0$

SF

Example 9 **8** A pole 10 m long leans against a wall. The bottom of the pole is 6 m from the wall. If the bottom of the pole is pulled away x m so that the top slides down by the same amount, determine x.



CF

9 A wire of length 200 cm is cut into two parts and each part is bent to form a square. If the area of the larger square is 9 times the area of the smaller square, determine the length of the sides of the larger square.

10 Determine constants a, b and c such that $3x^2 - 5x + 1 = a(x + b)^2 + c$ for all values of x. Hence determine the minimum value of $3x^2 - 5x + 1$.

11 Show that the graphs of $y = 2 - 4x - x^2$ and $y = 24 + 8x + x^2$ do not intersect.

12 Solve the quadratic equation $(b - c)x^2 + (c - a)x + (a - b) = 0$ for x.

13 The two solutions of the equation $2x^2 - 6x - m = 0$ differ by 5. Determine the value of m.

14 For the equation $(b^2 - 2ac)x^2 + 4(a + c)x = 8$:

a Prove that there are always (real) solutions of the equation.

b Determine the conditions for which there is only one solution.

15 The equation $\frac{1}{2} + \frac{1}{x + k} = \frac{1}{x}$ has no solutions. Determine the possible values of k.

16 Determine the smallest positive integer p for which the equation $3x^2 + px + 7 = 0$ has solutions.

4C Applying quadratic equations to rate problems

Learning intentions

- ▶ To be able to apply quadratic equations to rate problems.

A **rate** describes how a certain quantity changes with respect to the change in another quantity (often time). An example of a rate is ‘speed’. A speed of 60 km/h gives us a measure of how fast an object is travelling. A further example is ‘flow’, where a rate of 20 L/min is going to fill an empty swimming pool faster than a rate of 6 L/min. Many problems are solved using rates, which can be expressed as fractions. For example, a speed of 60 km/h can be expressed in fraction form as

$$\frac{\text{distance (km)}}{\text{time taken (h)}} = \frac{60}{1}$$

When solving rate problems, it is often necessary to add two or more fractions with different denominators, as shown in the following examples.

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Example 10

a Express $\frac{6}{x} + \frac{6}{x+8}$ as a single fraction.

b Solve the equation $\frac{6}{x} + \frac{6}{x+8} = 2$ for x.

Solution

a
$$\frac{6}{x} + \frac{6}{x+8} = \frac{6(x+8)}{x(x+8)} + \frac{6x}{x(x+8)}$$
$$= \frac{6x + 48 + 6x}{x(x+8)}$$
$$= \frac{12(x+4)}{x(x+8)}$$

b Since $\frac{6}{x} + \frac{6}{x+8} = \frac{12(x+4)}{x(x+8)}$, we have

$$\frac{12(x+4)}{x(x+8)} = 2$$
$$12(x+4) = 2x(x+8)$$
$$6(x+4) = x(x+8)$$
$$6x + 24 = x^2 + 8x$$
$$x^2 + 2x - 24 = 0$$
$$(x+6)(x-4) = 0$$

Therefore $x = -6$ or $x = 4$.

Example 11

A car travels 500 km at a constant speed. If it had travelled at a speed of 10 km/h less, it would have taken 1 hour more to travel the distance. Determine the speed of the car.

Solution

Let x km/h be the speed of the car.

It takes $\frac{500}{x}$ hours for the journey.

If the speed is 10 km/h less, then the new speed is $(x - 10)$ km/h.

The time taken is $\frac{500}{x} + 1$ hours.

We can now write:

$$500 = (x - 10) \times \left(\frac{500}{x} + 1\right)$$
$$500x = (x - 10)(500 + x)$$
$$500x = 490x - 5000 + x^2$$

Thus

$$x^2 - 10x - 5000 = 0$$

and so

$$x = \frac{10 \pm \sqrt{100 + 4 \times 5000}}{2}$$
$$= 5(1 \pm \sqrt{201})$$

The speed is $5(1 + \sqrt{201}) \approx 75.887$ km/h.

Explanation

For an object travelling at a constant speed in one direction:

$$\text{speed} = \frac{\text{distance travelled}}{\text{time taken}}$$

and so

$$\text{time taken} = \frac{\text{distance travelled}}{\text{speed}}$$

and

$$\text{distance travelled} = \text{speed} \times \text{time taken}$$

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Example 12

A tank is filled by two pipes. The smaller pipe alone will take 24 minutes longer than the larger pipe alone, and 32 minutes longer than when both pipes are used. How long will each pipe take to fill the tank alone? How long will it take for both pipes used together to fill the tank?

Solution

Let C cubic units be the capacity of the tank, and let x minutes be the time it takes for the larger pipe alone to fill the tank.

Then the average rate of flow for the larger pipe is $\frac{C}{x}$ cubic units per minute.

Since the smaller pipe alone takes $(x + 24)$ minutes to fill the tank, the average rate of flow for the smaller pipe is $\frac{C}{x+24}$ cubic units per minute.

The average rate of flow when both pipes are used together is the sum of these two rates:

$$\frac{C}{x} + \frac{C}{x+24} \text{ cubic units per minute}$$

Expressed as a single fraction:

$$\begin{aligned}\frac{C}{x} + \frac{C}{x+24} &= \frac{C(x+24)+Cx}{x(x+24)} \\ &= \frac{2C(x+12)}{x(x+24)}\end{aligned}$$

The time taken to fill the tank using both pipes is

$$\begin{aligned}C \div \frac{2C(x+12)}{x(x+24)} &= C \times \frac{x(x+24)}{2C(x+12)} \\ &= \frac{x(x+24)}{2(x+12)}\end{aligned}$$

Therefore the time taken for the smaller pipe alone to fill the tank can be also be expressed as $\frac{x(x+24)}{2(x+12)} + 32$ minutes.

Thus
$$\begin{aligned}\frac{x(x+24)}{2(x+12)} + 32 &= x + 24 \\ \frac{x(x+24)}{2(x+12)} &= x - 8 \\ x(x+24) &= 2(x+12)(x-8) \\ x^2 + 24x &= 2x^2 + 8x - 192 \\ x^2 - 16x - 192 &= 0 \\ (x-24)(x+8) &= 0\end{aligned}$$

But $x > 0$, and hence $x = 24$.

It takes 24 minutes for the larger pipe alone to fill the tank, 48 minutes for the smaller pipe alone to fill the tank, and 16 minutes for both pipes together to fill the tank.

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Skill-sheet

Exercise 4C

SF

Example 10

1 **a** Express $\frac{6}{x} - \frac{6}{x+3}$ as a single fraction.

b Solve the equation $\frac{6}{x} - \frac{6}{x+3} = 1$ for x.

2 Solve the equation $\frac{300}{x+5} = \frac{300}{x} - 2$ for x.

CF

3 The sum of the reciprocals of two consecutive odd numbers is $\frac{36}{323}$. Form a quadratic equation and hence determine the two numbers.

Example 11

4 A cyclist travels 40 km at a speed of x km/h.

a Determine the time taken in terms of x.

b Determine the time taken when his speed is reduced by 2 km/h.

c If the difference between the times is 1 hour, determine his original speed.

5 A car travels from town A to town B, a distance of 600 km, in x hours. A plane, travelling 220 km/h faster than the car, takes $5\frac{1}{2}$ hours less to cover the same distance.

a Express, in terms of x, the average speed of the car and the average speed of the plane.

b Determine the actual average speed of each of them.

6 A car covers a distance of 200 km at a speed of x km/h. A train covers the same distance at a speed of $(x + 5)$ km/h. If the time taken by the car is 2 hours more than that taken by the train, determine x.

7 A man travels 108 km, and determines that he could have made the journey in $4\frac{1}{2}$ hours less had he travelled at an average speed 2 km/h faster. What was the man's average speed when he made the trip?

8 A bus is due to reach its destination 75 km away at a certain time. The bus usually travels with an average speed of x km/h. Its start is delayed by 18 minutes but, by increasing its average speed by 12.5 km/h, the driver arrives on time.

a Determine x. **b** How long did the journey actually take?

9 Ten minutes after the departure of an express train, a slow train starts, travelling at an average speed of 20 km/h less. The slow train reaches a station 250 km away 3.5 hours after the arrival of the express. Determine the average speed of each of the trains.

10 When the average speed of a car is increased by 10 km/h, the time taken for the car to make a journey of 105 km is reduced by 15 minutes. Determine the original average speed.

11 A tank can be filled with water by two pipes running together in $11\frac{1}{9}$ minutes. If the larger pipe alone takes 5 minutes less to fill the tank than the smaller pipe, determine the time that each pipe will take to fill the tank.

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Example 12 **12** At first two different pipes running together will fill a tank in $\frac{20}{3}$ minutes. The rate that water runs through each of the pipes is then adjusted. If one pipe, running alone, takes 1 minute less to fill the tank at its new rate, and the other pipe, running alone, takes 2 minutes more to fill the tank at its new rate, then the two running together will fill the tank in 7 minutes. Determine in what time the tank will be filled by each pipe running alone at the new rates.

13 The journey between two towns by one route consists of 233 km by rail followed by 126 km by sea. By a second route the journey consists of 405 km by rail followed by 39 km by sea. If the time taken for the first route is 50 minutes longer than for the second route, and travelling by rail is 25 km/h faster than travelling by sea, determine the average speed by rail and the average speed by sea.

14 A sea freighter travelling due north at 12 km/h sights a cruiser straight ahead at an unknown distance and travelling due east at unknown speed. After 15 minutes the vessels are 10 km apart and then, 15 minutes later, they are 13 km apart. (Assume that both travel at constant speeds.) How far apart are the vessels when the cruiser is due east of the freighter?

15 Cask A, which has a capacity of 20 litres, is filled with wine. A certain quantity of wine from cask A is poured into cask B, previously empty, which also has a capacity of 20 litres. Cask B is then filled with water. After this, cask A is filled with some of the mixture from cask B. A further $\frac{20}{3}$ litres of the mixture now in A is poured back into B, and the two casks now have the same amount of wine. How much wine was first taken out of cask A?

16 Two trains travel between two stations 80 km apart. If train A travels at an average speed of 5 km/h faster than train B and completes the journey 20 minutes faster, determine the average speeds of the two trains, giving your answers correct to two decimal places.

17 A tank is filled by two pipes. The smaller pipe running alone will take 24 minutes longer than the larger pipe alone, and a minutes longer than when both pipes are running together.

a Determine, in terms of a, how long each pipe takes to fill the tank.

b Determine how long each pipe takes to fill the tank when:

i $a = 49$ **ii** $a = 32$ **iii** $a = 27$ **iv** $a = 25$

18 Train A leaves Armadale and travels at constant speed to Bundong, which is a town 300 km from Armadale. At the same time, train B leaves Bundong and travels at constant speed to Armadale. They meet at a town Yunga, which is between the two towns. Nine hours after leaving Yunga, train A reaches Bundong, and four hours after leaving Yunga, train B reaches Armadale.

a Determine the distance of Yunga from Armadale.

b Determine the speed of each of the trains.

4D Partial fractions

Learning intentions

- To be able to resolve an algebraic fraction into partial fractions.

A **rational function** is the quotient of two polynomials. If $P(x)$ and $Q(x)$ are polynomials, then $f(x) = \dfrac{P(x)}{Q(x)}$ is a rational function; e.g. $f(x) = \dfrac{4x+2}{x^2-1}$.

- If the degree of $P(x)$ is less than the degree of $Q(x)$, then $f(x)$ is a **proper fraction**.
- If the degree of $P(x)$ is greater than or equal to the degree of $Q(x)$, then $f(x)$ is an **improper fraction**.

By convention, we consider a rational function for its maximal domain. For example, the function $f(x) = \dfrac{4x+2}{x^2-1}$ is only defined for $x \in \mathbb{R} \setminus \{-1, 1\}$.

A rational function may be expressed as a sum of simpler functions by resolving it into what are called **partial fractions**. For example:

$$\frac{4x+2}{x^2-1} = \frac{3}{x-1} + \frac{1}{x+1}$$

This technique can help when sketching the graphs of rational functions or when performing other mathematical procedures such as integration.

Proper fractions

For proper fractions, the technique used for obtaining partial fractions depends on the type of factors in the denominator of the original algebraic fraction. We only consider examples where the denominators have factors that are either degree 1 (linear) or degree 2 (quadratic).

- For every linear factor $ax + b$ in the denominator, there will be a partial fraction of the form $\dfrac{A}{ax+b}$.
- For every repeated linear factor $(cx+d)^2$ in the denominator, there will be partial fractions of the form $\dfrac{B}{cx+d}$ and $\dfrac{C}{(cx+d)^2}$.
- For every irreducible quadratic factor $ax^2 + bx + c$ in the denominator, there will be a partial fraction of the form $\dfrac{Dx+E}{ax^2+bx+c}$.

Note: A quadratic expression is said to be **irreducible** if it cannot be factorised over $\mathbb{R}$. For example, both $x^2 + 1$ and $x^2 + 4x + 10$ are irreducible.

To resolve an algebraic fraction into its partial fractions:

Step 1 Write a statement of identity between the original fraction and a sum of the appropriate number of partial fractions.

Step 2 Express the sum of the partial fractions as a single fraction, and note that the numerators of both sides are equivalent.

Step 3 Determine the values of the introduced constants $A, B, C, \ldots$ by substituting appropriate values for x or by equating coefficients.

Example 13

Resolve $\frac{3x+5}{(x-1)(x+3)}$ into partial fractions.

Solution

Method 1

Let

$$\frac{3x+5}{(x-1)(x+3)} = \frac{A}{x-1} + \frac{B}{x+3} \qquad (1)$$

for all $x \in \mathbb{R} \setminus \{1, -3\}$.

Express the right-hand side as a single fraction:

$$\frac{3x+5}{(x-1)(x+3)} = \frac{A(x+3)+B(x-1)}{(x-1)(x+3)}$$

$$\therefore \quad \frac{3x+5}{(x-1)(x+3)} = \frac{(A+B)x+3A-B}{(x-1)(x+3)}$$

$$\therefore \quad 3x+5 = (A+B)x+3A-B$$

Equate coefficients:

$$A+B=3$$

$$3A-B=5$$

Solving these equations simultaneously gives

$$4A=8$$

and so $A = 2$ and $B = 1$.

Therefore

$$\frac{3x+5}{(x-1)(x+3)} = \frac{2}{x-1} + \frac{1}{x+3}$$

Explanation

Since the denominator has two linear factors, there will be two partial fractions of the form $\frac{A}{x-1}$ and $\frac{B}{x+3}$.

Method 2

From equation (1) we can write:

$$3x+5 = A(x+3)+B(x-1) \qquad (2)$$

Substitute $x = 1$ in equation (2):

$$8 = 4A$$

$$\therefore \quad A = 2$$

Substitute $x = -3$ in equation (2):

$$-4 = -4B$$

$$\therefore \quad B = 1$$

Explanation

We know that equation (2) is true for all $x \in \mathbb{R} \setminus \{1, -3\}$.

But if this is the case, then it also has to be true for $x = 1$ and $x = -3$.

Note: You could substitute any values of x to determine A and B in this way, but these values simplify the calculations.

Example 14

Resolve $\frac{2x+10}{(x+1)(x-1)^2}$ into partial fractions.

Solution

Since the denominator has a repeated linear factor and a single linear factor, there are three partial fractions:

$$\frac{2x+10}{(x+1)(x-1)^2} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

$$\therefore \quad \frac{2x+10}{(x+1)(x-1)^2} = \frac{A(x-1)^2 + B(x+1)(x-1) + C(x+1)}{(x+1)(x-1)^2}$$

This gives the equation

$$2x + 10 = A(x-1)^2 + B(x+1)(x-1) + C(x+1)$$

We will use a combination of methods to determine A, B and C.

Let $x = 1$:

$$\begin{aligned} 2(1) + 10 &= C(1+1) \\ 12 &= 2C \\ \therefore \quad C &= 6 \end{aligned}$$

Let $x = -1$:

$$\begin{aligned} 2(-1) + 10 &= A(-1-1)^2 \\ 8 &= 4A \\ \therefore \quad A &= 2 \end{aligned}$$

Now substitute these values for A and C:

$$\begin{aligned} 2x + 10 &= 2(x-1)^2 + B(x+1)(x-1) + 6(x+1) \qquad (1) \\ &= 2(x^2 - 2x + 1) + B(x^2 - 1) + 6(x+1) \\ &= (2+B)x^2 + 2x + 8 - B \end{aligned}$$

Equate coefficients:

$$2 + B = 0$$

$$8 - B = 10$$

Therefore $B = -2$ and hence

$$\frac{2x+10}{(x+1)(x-1)^2} = \frac{2}{x+1} - \frac{2}{x-1} + \frac{6}{(x-1)^2}$$

Alternatively, the value of B could be found by substituting $x = 0$ into equation (1).

Note: In Exercise 4D, you will show that it is impossible to determine A and C such that

$$\frac{2x+10}{(x+1)(x-1)^2} = \frac{A}{x+1} + \frac{C}{(x-1)^2}$$

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Example 15

Resolve $\dfrac{x^2 + 6x + 5}{(x - 2)(x^2 + x + 1)}$ into partial fractions.

Solution

Since the denominator has a single linear factor and an irreducible quadratic factor (i.e. cannot be reduced to linear factors), there are two partial fractions:

$$\frac{x^2 + 6x + 5}{(x - 2)(x^2 + x + 1)} = \frac{A}{x - 2} + \frac{Bx + C}{x^2 + x + 1}$$

$$\therefore \quad \frac{x^2 + 6x + 5}{(x - 2)(x^2 + x + 1)} = \frac{A(x^2 + x + 1) + (Bx + C)(x - 2)}{(x - 2)(x^2 + x + 1)}$$

This gives the equation

$$x^2 + 6x + 5 = A(x^2 + x + 1) + (Bx + C)(x - 2) \qquad (1)$$

Subsituting $x = 2$:

$$2^2 + 6(2) + 5 = A(2^2 + 2 + 1)$$
$$21 = 7A$$
$$\therefore \quad A = 3$$

We can rewrite equation (1) as

$$\begin{aligned} x^2 + 6x + 5 &= A(x^2 + x + 1) + (Bx + C)(x - 2) \\ &= A(x^2 + x + 1) + Bx^2 - 2Bx + Cx - 2C \\ &= (A + B)x^2 + (A - 2B + C)x + A - 2C \end{aligned}$$

Since $A = 3$, this gives

$$x^2 + 6x + 5 = (3 + B)x^2 + (3 - 2B + C)x + 3 - 2C$$

Equate coefficients:

$3 + B = 1$ and $3 - 2C = 5$

$\therefore \quad B = -2$ $\qquad \therefore \quad C = -1$

Check: $3 - 2B + C = 3 - 2(-2) + (-1) = 6$

Therefore

$$\begin{aligned} \frac{x^2 + 6x + 5}{(x - 2)(x^2 + x + 1)} &= \frac{3}{x - 2} + \frac{-2x - 1}{x^2 + x + 1} \\ &= \frac{3}{x - 2} - \frac{2x + 1}{x^2 + x + 1} \end{aligned}$$

Note: The values of B and C could also be found by substituting $x = 0$ and $x = 1$ in equation (1).

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Improper fractions

Improper algebraic fractions can be expressed as a sum of partial fractions by first dividing the denominator into the numerator to produce a quotient and a proper fraction. This proper fraction can then be resolved into its partial fractions using the techniques just introduced.

Example 16

Express $\dfrac{x^5+2}{x^2-1}$ as partial fractions.

Solution

Dividing through:

$$\begin{array}{r|l} & x^3 + x \\ \hline x^2-1 & x^5 + 2 \\ & x^5 - x^3 \\ \hline & x^3 + 2 \\ & x^3 - x \\ \hline & x + 2 \end{array}$$

Therefore

$$\frac{x^5+2}{x^2-1} = x^3 + x + \frac{x+2}{x^2-1}$$

By expressing $\dfrac{x+2}{x^2-1} = \dfrac{x+2}{(x-1)(x+1)}$ as partial fractions, we obtain

$$\frac{x^5+2}{x^2-1} = x^3 + x - \frac{1}{2(x+1)} + \frac{3}{2(x-1)}$$

Summary 4D

- A rational function may be expressed as a sum of simpler functions by resolving it into **partial fractions**.
- Examples of resolving a proper fraction into partial fractions:

Single linear factors	$\dfrac{3x-4}{(2x-3)(x+5)} = \dfrac{A}{2x-3} + \dfrac{B}{x+5}$
Repeated linear factor	$\dfrac{3x-4}{(2x-3)(x+5)^2} = \dfrac{A}{2x-3} + \dfrac{B}{x+5} + \dfrac{C}{(x+5)^2}$
Irreducible quadratic factor	$\dfrac{3x-4}{(2x-3)(x^2+5)} = \dfrac{A}{2x-3} + \dfrac{Bx+C}{x^2+5}$

- A quadratic polynomial is **irreducible** if it cannot be factorised over $\mathbb{R}$. For example, the quadratics $x^2 + 5$ and $x^2 + 4x + 10$ are irreducible.
- If $f(x) = \dfrac{P(x)}{Q(x)}$ is an improper fraction, i.e. if the degree of $P(x)$ is greater than or equal to the degree of $Q(x)$, then the division must be performed first.

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Exercise 4D

SF

Example 13

1 Resolve the following rational expressions into partial fractions:

a $\dfrac{5x+1}{(x-1)(x+2)}$ **b** $\dfrac{-1}{(x+1)(2x+1)}$ **c** $\dfrac{3x-2}{x^2-4}$

d $\dfrac{4x+7}{x^2+x-6}$ **e** $\dfrac{7-x}{(x-4)(x+1)}$

Example 14

2 Resolve the following rational expressions into partial fractions:

a $\dfrac{2x+3}{(x-3)^2}$ **b** $\dfrac{9}{(1+2x)(1-x)^2}$ **c** $\dfrac{2x-2}{(x+1)(x-2)^2}$

Example 15

3 Resolve the following rational expressions into partial fractions:

a $\dfrac{3x+1}{(x+1)(x^2+x+1)}$ **b** $\dfrac{3x^2+2x+5}{(x^2+2)(x+1)}$ **c** $\dfrac{x^2+2x-13}{2x^3+6x^2+2x+6}$

Example 16

4 Resolve $\dfrac{3x^2-4x-2}{(x-1)(x-2)}$ into partial fractions.

5 Show that it is not possible to determine values of A and C such that

$$\frac{2x+10}{(x+1)(x-1)^2} = \frac{A}{x+1} + \frac{C}{(x-1)^2}$$

6 Express each of the following as partial fractions:

a $\dfrac{1}{(x-1)(x+1)}$ **b** $\dfrac{x}{(x-2)(x+3)}$ **c** $\dfrac{3x+1}{(x-2)(x+5)}$

d $\dfrac{1}{(2x-1)(x+2)}$ **e** $\dfrac{3x+5}{(3x-2)(2x+1)}$ **f** $\dfrac{2}{x^2-x}$

g $\dfrac{3x+1}{x^3+x}$ **h** $\dfrac{3x^2+8}{x(x^2+4)}$ **i** $\dfrac{1}{x^2-4x}$

j $\dfrac{x+3}{x^2-4x}$ **k** $\dfrac{x^3-x^2-1}{x^2-x}$ **l** $\dfrac{x^3-x^2-6}{2x-x^2}$

m $\dfrac{x^2-x}{(x+1)(x^2+2)}$ **n** $\dfrac{x^2+2}{x^3-3x-2}$ **o** $\dfrac{2x^2+x+8}{x(x^2+4)}$

p $\dfrac{1-2x}{2x^2+7x+6}$ **q** $\dfrac{3x^2-6x+2}{(x-1)^2(x+2)}$ **r** $\dfrac{4}{(x-1)^2(2x+1)}$

s $\dfrac{x^3-2x^2-3x+9}{x^2-4}$ **t** $\dfrac{x^3+3}{(x+1)(x-1)}$ **u** $\dfrac{2x-1}{(x+1)(3x+2)}$

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4E Simultaneous equations

Learning intentions

- To be able to solve simultaneous equations involving linear and non-linear functions.

In this section, we look at methods for determining the coordinates of the points of intersection of a linear graph with different non-linear graphs: parabolas, circles and rectangular hyperbolas. We also consider the intersections of two parabolas. These types of graphs are studied further in Mathematical Methods Units 1 & 2.

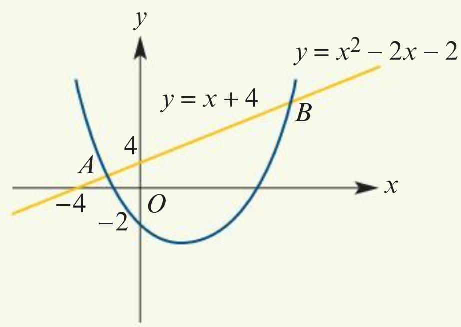
Example 17

Determine the coordinates of the points of intersection of the parabola with equation $y = x^2 - 2x - 2$ and the straight line with equation $y = x + 4$.

Solution

Equate the two expressions for y:

$$x^2 - 2x - 2 = x + 4$$
$$x^2 - 3x - 6 = 0$$
$$\therefore \quad x = \frac{3 \pm \sqrt{9 - 4 \times (-6)}}{2}$$
$$= \frac{3 \pm \sqrt{33}}{2}$$



The points of intersection are $A\left(\frac{3 - \sqrt{33}}{2}, \frac{11 - \sqrt{33}}{2}\right)$ and $B\left(\frac{3 + \sqrt{33}}{2}, \frac{11 + \sqrt{33}}{2}\right)$.

Example 18

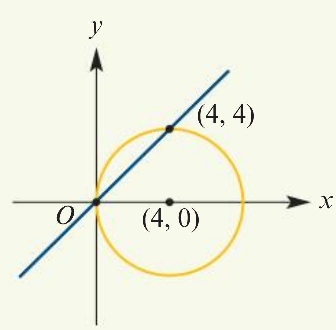
Determine the points of intersection of the circle with equation $(x - 4)^2 + y^2 = 16$ and the line with equation $x - y = 0$.

Solution

Rearrange $x - y = 0$ to make y the subject.

Substitute $y = x$ into the equation of the circle:

$$(x - 4)^2 + x^2 = 16$$
$$x^2 - 8x + 16 + x^2 = 16$$
$$2x^2 - 8x = 0$$
$$2x(x - 4) = 0$$
$$\therefore \quad x = 0 \text{ or } x = 4$$



The points of intersection are $(0, 0)$ and $(4, 4)$.

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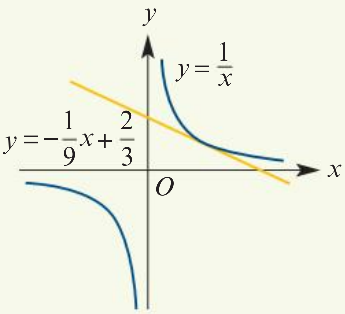
Example 19

Determine the point of contact of the straight line with equation $\frac{1}{9}x + y = \frac{2}{3}$ and the curve with equation $xy = 1$.

Solution

Rewrite the equations as $y = -\frac{1}{9}x + \frac{2}{3}$ and $y = \frac{1}{x}$.



Equate the expressions for y:

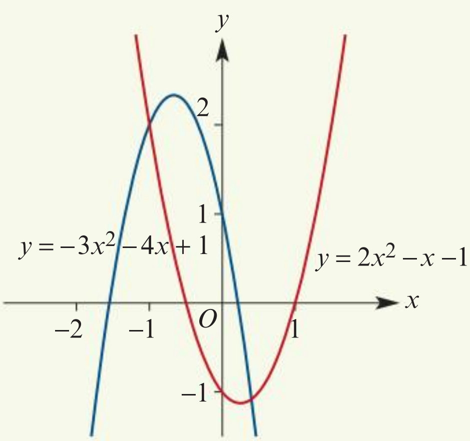
$$-\frac{1}{9}x + \frac{2}{3} = \frac{1}{x}$$
$$-x^2 + 6x = 9$$
$$x^2 - 6x + 9 = 0$$
$$(x-3)^2 = 0$$
$$\therefore \quad x = 3$$

The point of intersection is $\left(3, \frac{1}{3}\right)$.

Example 20

Determine the coordinates of the points of intersection of the graphs of $y = -3x^2 - 4x + 1$ and $y = 2x^2 - x - 1$.

Solution



$$-3x^2 - 4x + 1 = 2x^2 - x - 1$$
$$-5x^2 - 3x + 2 = 0$$
$$5x^2 + 3x - 2 = 0$$
$$(5x-2)(x+1) = 0$$
$$\therefore \quad x = \frac{2}{5} \text{ or } x = -1$$

Substitute in $y = 2x^2 - x - 1$:

When $x = -1$, $y = 2$.

When $x = \frac{2}{5}$, $y = 2 \times \frac{4}{25} - \frac{2}{5} - 1 = -\frac{27}{25}$.

The points of intersection are $(-1, 2)$ and $\left(\frac{2}{5}, -\frac{27}{25}\right)$.

Skill-sheet

Exercise 4E

Example 17

1 Determine the coordinates of the points of intersection for each of the following:

a $y = x^2$, $y = x$

b $y - 2x^2 = 0$, $y - x = 0$

c $y = x^2 - x$, $y = 2x + 1$

SF

Example 18

2 Determine the coordinates of the points of intersection for each of the following:

a $x^2 + y^2 = 178$, $x + y = 16$

b $x^2 + y^2 = 125$, $x + y = 15$

c $x^2 + y^2 = 185$, $x - y = 3$

d $x^2 + y^2 = 97$, $x + y = 13$

e $x^2 + y^2 = 106$, $x - y = 4$

Example 19

3 Determine the coordinates of the points of intersection for each of the following:

a $x + y = 28$, $xy = 187$

b $x + y = 51$, $xy = 518$

c $x - y = 5$, $xy = 126$

CF

4 Determine the coordinates of the points of intersection of the straight line with equation $y = 2x$ and the circle with equation $(x - 5)^2 + y^2 = 25$.

SF

5 Determine the coordinates of the points of intersection of the curves with equations $y = \dfrac{1}{x-2} + 3$ and $y = x$.

6 Determine the coordinates of the points A and B where the line with equation $x - 3y = 0$ meets the circle with equation $x^2 + y^2 - 10x - 5y + 25 = 0$.

7 Determine the coordinates of the points of intersection of the line with equation $\dfrac{y}{4} - \dfrac{x}{5} = 1$ and the circle with equation $x^2 + 4x + y^2 = 12$.

8 Determine the coordinates of the points of intersection of the curve with equation $y = \dfrac{1}{x+2} - 3$ and the line with equation $y = -x$.

9 Determine the point where the line $4y = 9x + 4$ touches the parabola $y^2 = 9x$.

10 Determine the coordinates of the point where the line with equation $y = 2x + 3\sqrt{5}$ touches the circle with equation $x^2 + y^2 = 9$.

11 Determine the coordinates of the point where the straight line with equation $y = \dfrac{1}{4}x + 1$ touches the curve with equation $y = -\dfrac{1}{x}$.

12 Determine the points of intersection of the curve $y = \dfrac{2}{x-2}$ and the line $y = x - 1$.

Example 20

13 Determine the coordinates of the points of intersection of the graphs of the following pairs of quadratic functions:

a $y = 2x^2 - 4x + 1,\ y = 2x^2 - x - 1$

b $y = -2x^2 + x + 1,\ y = 2x^2 - x - 1$

c $y = x^2 + x + 1,\ y = x^2 - x - 2$

d $y = 3x^2 + x + 2,\ y = x^2 - x + 2$

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Chapter summary

Polynomials

- A **polynomial function** can be written in the form

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where $n \in \mathbb{N} \cup \{0\}$ and the coefficients $a_0, \ldots, a_n$ are real numbers with $a_n \neq 0$.
- The **degree** of a polynomial is the index n of the leading term (the term of highest index among those terms with a non-zero coefficient).
- **Equating coefficients**
Two polynomials are equal if they give the same value for all x. If two polynomials are equal, then they have the same degree and corresponding coefficients are equal.
For example: if $x^2 - x - 12 = x^2 + (a + b)x + ab$, then $a + b = -1$ and $ab = -12$.

Quadratics

- A quadratic function can be written in the form $y = ax^2 + bx + c$, where $a \neq 0$.
- A quadratic equation $ax^2 + bx + c = 0$ may be solved by:
  - Factorising
  - Completing the square
  - Using the **general quadratic formula** $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
- The number of solutions of a quadratic equation $ax^2 + bx + c = 0$ can be found from the **discriminant** $\Delta = b^2 - 4ac$:
  - If $\Delta > 0$, the quadratic equation has two real solutions.
  - If $\Delta = 0$, the quadratic equation has one real solution.
  - If $\Delta < 0$, the quadratic equation has no real solutions.

Partial fractions

- A **rational function** has the form $f(x) = \dfrac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials.
For example: $f(x) = \dfrac{2x + 10}{x^3 - x^2 - x + 1}$
- Some rational functions may be expressed as a sum of **partial fractions**:
  - For every linear factor $ax + b$ in the denominator, there will be a partial fraction of the form $\dfrac{A}{ax + b}$.
  - For every repeated linear factor $(cx + d)^2$ in the denominator, there will be partial fractions of the form $\dfrac{B}{cx + d}$ and $\dfrac{C}{(cx + d)^2}$.
  - For every irreducible quadratic factor $ax^2 + bx + c$ in the denominator, there will be a partial fraction of the form $\dfrac{Dx + E}{ax^2 + bx + c}$.

For example: $\dfrac{2x + 10}{(x + 1)(x - 1)^2} = \dfrac{A}{x + 1} + \dfrac{B}{x - 1} + \dfrac{C}{(x - 1)^2}$, where $A = 2$, $B = -2$ and $C = 6$

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Review

Skills checklist

Check-list

Download this checklist from the Interactive Textbook, then print it and fill it out to check your skills.

4A **1** I can use equating coefficients to solve problems. ☐

See Example 1, Example 2, Example 3, Example 4 and Questions 2, 5, 7 and 8

4B **2** I can solve quadratic equations using different techniques. ☐

See Example 5, Example 6, Example 8 and Questions 1, 3 and 7

4B **3** I can use the discriminant to determine properties of quadratic functions. ☐

See Example 7 and Question 4

4B **4** I can use quadratic equations to solve worded problems from a context. ☐

See Example 9 and Question 8

4C **5** I can use quadratic equations to solve rate problems. ☐

See Example 11, Example 12 and Questions 4 and 12

4D **6** I can express suitable rational expressions as partial fractions. ☐

See Example 13, Example 14, Example 15, Example 16 and Questions 1, 2, 3 and 4

4E **7** I can use simultaneous equations to determine the points of intersection of a straight line with circles, parabolas and rectangular hyperbolas. ☐

See Example 17, Example 18, Example 19 and Questions 1, 2 and 3

4E **8** I can use simultaneous equations to determine the points of intersection of two parabolas. ☐

See Example 20 and Question 13

Technology-free questions

Technology-free short-response questions

SF

1 If $(3a + b)x^2 + (a - 2b)x + b + 2c = 11x^2 - x + 4$, determine the values of a, b and c.

2 Express x^3 in the form $(x - 1)^3 + a(x - 1)^2 + b(x - 1) + c$.

3 Prove that, if $ax^3 + bx^2 + cx + d = (x + 1)^2(px + q)$, then $b = 2a + d$ and $c = a + 2d$.

4 Prove that, if $ax^3 + bx^2 + cx + d = (x - 2)^2(px + q)$, then $b = -4a + \frac{1}{4}d$ and $c = 4a - d$.

Review

SF

5 Solve the following quadratic equations for x:

a $x^2 + x = 12$ **b** $x^2 - 2 = x$ **c** $-x^2 + 3x + 11 = 1$

d $2x^2 - 4x + 1 = 0$ **e** $3x^2 - 2x + 5 = t$ **f** $tx^2 + 4 = tx$

6 Solve the equation $\dfrac{2}{x-1} - \dfrac{3}{x+2} = \dfrac{1}{2}$ for x.

7 Express each of the following as partial fractions:

a $\dfrac{-3x+4}{(x-3)(x+2)}$ **b** $\dfrac{7x+2}{x^2-4}$ **c** $\dfrac{7-x}{x^2+2x-15}$

d $\dfrac{3x-9}{x^2-4x-5}$ **e** $\dfrac{3x-4}{(x+3)(x+2)^2}$ **f** $\dfrac{6x^2-5x-16}{(x-1)^2(x+4)}$

g $\dfrac{x^2-6x-4}{(x^2+2)(x+1)}$ **h** $\dfrac{-x+4}{(x-1)(x^2+x+1)}$ **i** $\dfrac{-4x+5}{(x+4)(x-3)}$

8 Express each of the following as partial fractions:

a $\dfrac{14(x-2)}{(x-3)(x^2+x+2)}$ **b** $\dfrac{1}{(x+1)(x^2-x+2)}$ **c** $\dfrac{3x^3}{x^2-5x+4}$

9 Determine the coordinates of the points of intersection for each of the following:

a $y = x^2$, $y = -x$ **b** $x^2 + y^2 = 16$, $x + y = 4$ **c** $x + y = 5$, $xy = 4$

10 Determine the coordinates of the points of intersection of the line with equation $3y - x = 1$ and the circle with equation $x^2 + 2x + y^2 = 9$.

CF

11 A motorist makes a journey of 135 km at an average speed of x km/h.

a Write an expression for the number of hours taken for the journey.

b Owing to road works, on a certain day his average speed for the journey is reduced by 15 km/h. Write an expression for the number of hours taken on that day.

c If the second journey takes 45 minutes longer than the first, form an equation in x and solve it.

d Determine his average speed for each journey.

Technology-active short-response questions

SF

12 For each of the following determine the solutions correct to two decimal places:

a $2.1x^2 - 4.6x - 4.7 = 0$ **b** $-2.2x^2 + 5.7x + 11.4 = 0$

c $-5.6x^2 + 7x = -11$ **d** $5.6(7 - x^2) = 11x$

13 For each of the following determine the coordinates of the points of intersection, giving values correct to two decimal places:

a $y = 3x + 1$ and $y = x^2 - 5x - 6$ **b** $y = 3x + 1$ and $y = \dfrac{2}{4x-1} - 1$

c $y = x^2 - 5x - 6$ and $y = \dfrac{2}{4x-1} - 1$ **d** $y = \dfrac{1}{3x+1}$ and $y = x^2 - 5x - 6$

14 A train completes a journey of 240 km at a constant speed.

a If the train had travelled 4 km/h slower, it would have taken two hours more for the journey. Determine the actual speed of the train.

b If the train had travelled a km/h slower and still taken two hours more for the journey of 240 km, what would have been the actual speed? (Answer in terms of a.) Discuss the practical possible values of a and also the possible values for the speed of the train.

c If the train had travelled a km/h slower and taken a hours more for the journey of 240 km, and if a is an integer and the speed is an integer, Determine the possible values for a and the speed of the train.

15 Two trains are travelling at constant speeds. The slower train takes a hours longer to cover b km. It travels 1 km less than the faster train in c hours.

a What is the speed of the faster train, in terms of a, b and c?

b If a, b, c and the speeds of the trains are all rational numbers, determine five sets of values for a, b and c. Choose and discuss two sensible sets of values.

16 In each of the following, use the discriminant of the resulting quadratic equation:

a Determine the possible values of k for which the straight line $y = k(1 - 2x)$ touches but does not cross the parabola $y = x^2 + 2$.

b Determine the possible values of c for which the line $y = 2x + c$ intersects the circle $x^2 + y^2 = 20$ in two distinct points.

c Determine the value of p for which the line $y = 6$ meets the parabola $y = x^2 + (1 - p)x + 2p$ at only one point.

Multiple-choice questions

Technology-free multiple-choice questions

1 If x^2 is written in the form $(x + 1)^2 + b(x + 1) + c$, then the values of b and c are

A $b = 0,\ c = 0$ **B** $b = -2,\ c = 0$

C $b = -2,\ c = 1$ **D** $b = 1,\ c = 2$

2 If $x^3 = a(x + 2)^3 + b(x + 2)^2 + c(x + 2) + d$, then the values of a, b, c and d are

A $a = 0,\ b = -8,\ c = 10,\ d = -6$ **B** $a = 0,\ b = -6,\ c = 10,\ d = -8$

C $a = 1,\ b = -6,\ c = 12,\ d = -8$ **D** $a = 1,\ b = -8,\ c = 10,\ d = -6$

3 The quadratic equation $3x^2 - 6x + 3 = 0$ has

A two real solutions, $x = \pm 1$ **B** one real solution, $x = -1$

C no real solutions **D** one real solution, $x = 1$

4 A quadratic equation whose solutions are 4 and −6 is

A $(x+4)(x-6)=0$ **B** $x^2-2x-24=0$

C $2x^2+4x=48$ **D** $-x^2+2x-24=0$

5 $\dfrac{3}{x+4}-\dfrac{5}{x-2}$ is equal to

A $\dfrac{2(x+1)}{(x+4)(x-2)}$ **B** $\dfrac{-2(x-7)}{(x+4)(x-2)}$

C $\dfrac{2(4x+13)}{(x+4)(x-2)}$ **D** $\dfrac{-2(x+13)}{(x+4)(x-2)}$

6 $\dfrac{4}{(x+3)^2}+\dfrac{2x}{x+1}$ is equal to

A $\dfrac{2(3x^2+x+18)}{(x+3)^2(x+1)}$ **B** $\dfrac{3x^2+13x+18}{(x+3)^2(x+1)}$

C $\dfrac{2(3x^2+13x+18)}{(x+3)^2(x+1)}$ **D** $\dfrac{2(x^3+6x^2+11x+2)}{(x+3)^2(x+1)}$

7 If $\dfrac{7x^2+13}{(x-1)(x^2+x+2)}$ is expressed in the form $\dfrac{a}{x-1}+\dfrac{bx+c}{x^2+x+2}$, then

A $a=5,\ b=0,\ c=-13$ **B** $a=5,\ b=0,\ c=-10$

C $a=5,\ b=2,\ c=-3$ **D** $a=7,\ b=2,\ c=3$

8 $\dfrac{4x-3}{(x-3)^2}$ is equal to

A $\dfrac{3}{x-3}+\dfrac{1}{x-3}$ **B** $\dfrac{4x}{x-3}-\dfrac{3}{x-3}$

C $\dfrac{9}{x-3}+\dfrac{4}{(x-3)^2}$ **D** $\dfrac{4}{x-3}+\dfrac{9}{(x-3)^2}$

9 $\dfrac{8x+7}{2x^2+5x+2}$ is equal to

A $\dfrac{2}{2x+1}-\dfrac{3}{x+2}$ **B** $\dfrac{2}{2x+1}+\dfrac{3}{x+2}$

C $\dfrac{-4}{2x+2}-\dfrac{1}{x+1}$ **D** $\dfrac{-4}{2x+2}+\dfrac{1}{x+1}$

10 $\dfrac{-3x^2+2x-1}{(x^2+1)(x+1)}$ is equal to

A $\dfrac{2}{x^2+1}+\dfrac{3}{x+1}$ **B** $\dfrac{2}{x^2+1}-\dfrac{3}{x+1}$

C $\dfrac{5}{x^2+1}+\dfrac{2}{x+1}$ **D** $\dfrac{3}{x^2+1}-\dfrac{2}{x+1}$

11 The line $x+y=2k$ touches the circle $x^2+y^2=k$ at just one point, where $k>0$. The value of k is

A $\dfrac{1}{\sqrt{2}}$ **B** $\sqrt{2}$ **C** $\dfrac{1}{2}$ **D** 2

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12 The simultaneous equations $y = x^2 + x$ and $y = bx - 1$ have exactly one real solution, where $b > 0$. The value of b is

A $\frac{5}{2}$ **B** 3 **C** $\frac{7}{2}$ **D** 2

13 If $(bx + c)(2x - 5) = 12x^2 + kx - 10$ for all values of x, then $k =$

A -10 **B** -26 **C** 24 **D** 32

Technology-active multiple-choice questions

14 The parabola $y = x^2 + 2$ intersects the hyperbola $y = \frac{1}{x}$ at the point with coordinates (a, b). The values of a and b correct to two decimal places are

A $a = 0.40, b = 2.20$ **B** $a = 0.45, b = 2.21$

C $a = 0.44, b = 2.9$ **D** $a = 0.35, b = 2.27$

15 The equation $\frac{1}{3} + \frac{1}{x+k} = \frac{1}{x}$ has non-zero integer solutions for

A $k = 1$ **B** $k = 2$ **C** $k = 3$ **D** $k = 4$

16 Consider the quadratic equation $px^2 + qx + 1 = 0$ where both p and q can take values $1, 2, 3, 4$. There are 16 such equations. The number of these equations that have real solutions is

A 5 **B** 6 **C** 7 **D** 8

17 If $b > a$ and the solutions of the equation $(x - a)(x - b) = 1$ are α and β with $\alpha < \beta$ then

A $a < \alpha < b$ and $a < \beta < b$ **B** $\alpha < a$ and $\beta < a$

C $\alpha > b$ and $\beta > b$ **D** $\alpha < a$ and $\beta > b$

18 The graph of $y = \frac{1}{x-3}$ touches the graph of $y = -2\sqrt{x-3} + 3$ at the point where $x = a$. The value of a is

A 1.55 **B** 2.65 **C** 3.98 **D** 4

19 A car travels 600 km at a constant speed. If it had travelled at a speed of 10 km/h less it would have taken 90 minutes more. The speed of the car correct to one decimal place is

A 68.4 km/h **B** 56.6 km/h **C** 62.6 km/h **D** 48.5 km/h

20 If the solutions of the equation $\frac{1}{x+p} + \frac{1}{x+q} = \frac{1}{r}$ are $x = 3.1$ and $x = -3.1$, then $p^2 + q^2$ is equal to

A 19.22 **B** 9.61 **C** 6.2 **D** 36.01

5

Revision of Chapters 1–4

5A Short-response questions

Technology-free short-response questions

SF

1 Rewrite each fraction with an integer denominator:

a $\dfrac{1}{\sqrt{2}-3}$ **b** $\dfrac{3}{\sqrt{5}-1}$ **c** $\dfrac{2}{2\sqrt{2}-1}$

d $\dfrac{3}{\sqrt{5}-\sqrt{3}}$ **e** $\dfrac{1}{\sqrt{7}-\sqrt{2}}$ **f** $\dfrac{1}{2\sqrt{5}-\sqrt{3}}$

2 Write each of the following in the form $\dfrac{m}{n}$, where m and n are integers:

a $0.\dot{5}$ **b** $0.\dot{5}\dot{4}$ **c** $0.\dot{6}0\dot{5}$ **d** $2.\dot{3}$ **e** $4.\dot{3}567\dot{3}$

3 Express each of the following as a product of powers of prime numbers:

a 2002 **b** 555 **c** 7007 **d** 10 000

4 Simplify $\dfrac{5m-2p}{4m^2+mp-3p^2}-\dfrac{1}{4m-3p}$.

5 Expand each of the following and collect like terms:

a $(\sqrt{3}+\sqrt{2})(\sqrt{3}-1)$ **b** $(5\sqrt{3}-\sqrt{6})(2\sqrt{6}+3\sqrt{3})$

c $(2\sqrt{x}-3)^2$ **d** $(\sqrt{x-2}-3)^2$

6 For each of the following, determine the values of A, B and C such that the equation holds for all real numbers x:

a $(Ax+B)(x+1)+C(x^2+3)=x-3$ **b** $16+4x-x^2=A-(B-Cx)^2$

7 Simplify $\dfrac{\sqrt{2}}{\sqrt{3}+\sqrt{2}}$.

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SF

8 Resolve each of the following into partial fractions:

a $\dfrac{2x}{3(x-2)(x+2)}$ **b** $\dfrac{2x+5}{(x+2)(x+3)}$ **c** $\dfrac{5x^2+4x+4}{(x+2)(x^2+4)}$

d $\dfrac{2(x^2-2x-1)}{(x+1)(x-1)^2}$ **e** $\dfrac{2x^2-3x+1}{x^3-3x^2+x-3}$ **f** $\dfrac{3x^2-x+6}{(x^2+4)(x-2)}$

9 Write down the prime factorisation of each of the following numbers, and hence determine the square root of each number:

a 576 **b** 1225 **c** 1936 **d** 1296

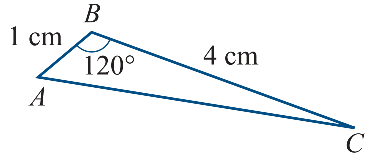
10 Solve the equation $\dfrac{x+b}{x-c} = 1 - \dfrac{x}{x-c}$ for x.

11 Solve the equation $\dfrac{1}{x-a} + \dfrac{1}{x-b} = \dfrac{2}{x}$ for x.

12 The triangle ABC has side lengths $AB = 1$ cm and $BC = 4$ cm. The magnitude of angle ABC is 120°.

a Determine the length of AC.

b Determine the sine of angle BAC.



13 Transpose each of the following to make x the subject:

a $y = 3 + \sqrt{2x-1}$ **b** $y = \dfrac{2}{\sqrt{3x+1}} - 2$

14 Solve each pair of simultaneous equations:

a $\dfrac{x}{3} + \dfrac{y}{4} = 1$

$3x - 4y = 1$

b $\dfrac{x}{a} + \dfrac{y}{b} = 1$

$ax - by = 1$

CF

15 Determine the points of intersection of the circle with equation $(x-9)^2 + y^2 = 25$ and the line with equation $x - 3y = 0$.

16 Determine the points of intersection of the graphs of $y = 2x^2 - 4x - 2$ and $y = -2x^2 - 4x + 2$.

17 Solve the following equation for x:

$$\frac{4}{x^2-x-2} + \frac{3}{x^2-4} = \frac{2}{x^2+3x+2}$$

18 A train travels at a constant speed of 55 km/h for 2 hours and then at a constant speed of 70 km/h for 3 hours. Determine the train's average speed over the 5-hour journey.

19 **a** If the equations $x^2 + x - 1 = 0$ and $x^2 + bx + 1 = 0$ have a common solution, show that $b = \pm\sqrt{5}$.

b Determine the common solution when:

i $b = \sqrt{5}$ **ii** $b = -\sqrt{5}$

Revision

CF

20 Determine constants a, b and c such that $(n+1)(n-7) = a + bn + cn(n-1)$ for all n.

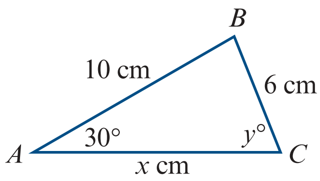
21 Prove that, if $n = \text{HCF}(a, b)$, then n divides $a - b$

22 A triangle ABC satisfies $\sin A : \sin B = 1 : 2$. If $BC = 6$ cm, determine AC.

23 For the triangle shown on the right:

a Determine the two possible values for x.

b Determine the two possible values for y.



24 Determine two sets of values of λ, a, b such that, for all values of x,

$$x^2 - 4x - 8 + \lambda(x^2 - 2x - 5) = a(x-b)^2$$

25 Two bushwalkers are 20 km apart on a track and travelling towards each other. One walks at 3 km/h and the other at 5 km/h. After how many minutes will they meet?

26 Determine the values of m such that the line $y = mx + 6$ touches (but does not cross) the circle $x^2 + y^2 - 6x - 4y + 8 = 0$.

27 Solve the equation $(2x-1)^2 + (x-3)^2 = 36$ for x.

CU

28 A group of 50 students were interviewed about the types of movies that they watch: 25 of the students like action movies, 26 like comedy, 17 like drama, 11 like action and comedy, 5 like action and drama, 8 like comedy and drama, and 3 like all three. How many of these students:

a like none of these types of movies

b like action movies only

c like action and comedy but not drama?

29 Consider the quadratic polynomial $P(x) = ax^2 - 2ax + 1$, where $a \neq 0$.

a Determine the discriminant of $P(x)$.

b Determine the values of a for which the graph of $y = P(x)$:

i touches the x-axis once only

ii crosses the x-axis twice

iii does not intersect the x-axis.

30 Solve for x:

$$\frac{\sqrt{x+1} + \sqrt{x-1}}{\sqrt{x+1} - \sqrt{x-1}} = 3$$

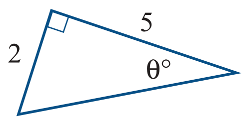
31 A group of 200 Year 11 students each study one or more of the subjects Mathematical Methods, Physics and Chemistry. Suppose that in this group:

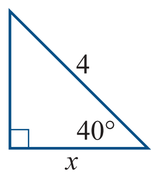
- 40 students study Chemistry only
- 50% more students study Mathematical Methods than study Physics
- 5 students study all three subjects
- 25 students study both Mathematical Methods and Physics
- 8 students study both Mathematical Methods and Chemistry
- 20 students study both Physics and Chemistry.

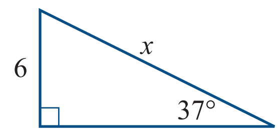
Determine the number of students in the group studying each of the subjects.

Technology-active short-response questions

32 Determine the value of the pronumeral:

a


b


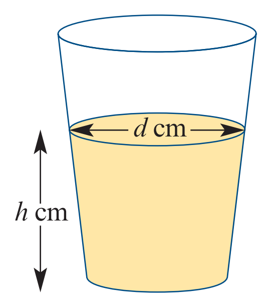
c


33 A triangle ABC has side lengths $AB = 3.6$ cm, $BC = 5.2$ cm and $CA = 4.3$ cm. Determine the magnitudes of the angles of this triangle. (Give answers correct to one decimal place.)

34 In triangle ABC, $AC = 4$ cm, $BC = 5$ cm and $\angle CAB = 42°$. Determine the magnitudes of $\angle ABC$ and $\angle ACB$.

35 The diagram represents a glass containing milk. When the height of the milk in the glass is h cm, the diameter, d cm, of the surface of the milk is given by the formula

$$d = \frac{h}{5} + 6$$



a Determine d when $h = 10$.

b Determine d when $h = 8.5$.

c What is the diameter of the bottom of the glass?

d The diameter of the top of the glass is 9 cm. What is the height of the glass?

36 The cost, $\$C$, of manufacturing each jacket of a particular type is given by the formula SF

$$C = an + b \quad \text{for } 0 < n \le 300$$

where a and b are constants and n is the size of the production run of this type of jacket. For making 100 jackets, the cost is \$108 each. For 120 jackets, the cost is \$100 each.

a Determine the values of a and b.

b Sketch the graph of C against n for $0 < n \le 300$.

c Determine the cost of manufacturing each jacket if 200 jackets are made.

d If the cost of making each jacket is \$48.80, determine the size of the production run.

37 The formula $A = 180 - \dfrac{360}{n}$ gives the size of each interior angle, $A°$, of a regular polygon with n sides. CF

a Determine the value of A when n equals:

i 180 **ii** 360 **iii** 720 **iv** 7200

b As n becomes very large:

i What value does A approach? **ii** What shape does the polygon approach?

c Determine the value of n when $A = 162$.

d Make n the subject of the formula.

e Three regular polygons, two of which are octagons, meet at a point so that they fit together without any gaps. Describe the third polygon.

38 At the beginning of 2007, Andrew and John bought a small catering business. The profit, $\$P$, in a particular year is given by the rule $P = an + b$, where n is the number of years of operation and a and b are constants.

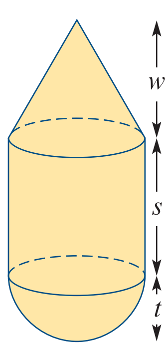
a Given the table, determine the values of a and b.

Year	2007	2011
Number of years of operation (n)	1	5
Profit (P)	−9000	15 000

b Determine the profit when $n = 12$.

c In which year was the profit \$45 000?

39 The figure shows a solid consisting of three parts – a cone, a cylinder and a hemisphere – all of the same base radius. CU



a Determine, in terms of w, s, t and π, the volume of each part.

b **i** If the volume of each of the three parts is the same, determine the ratio $w : s : t$.

ii If also $w + s + t = 11$, determine the total volume in terms of π.

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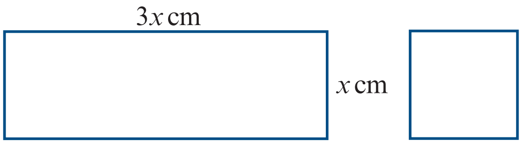
40 A piece of wire 28 cm long is cut into two parts: one to make a rectangle three times as long as it is wide, and the other to make a square.



a What is the perimeter of the rectangle in terms of x?

b What is the perimeter of the square in terms of x?

c What is the length of each side of the square in terms of x?

Let A be the sum of the areas of the two figures.

d Show that $A = 7(x^2 - 4x + 7)$.

e Use a calculator to help sketch the graph of $A = 7(x^2 - 4x + 7)$ for $0 < x < 3.5$.

f Determine the minimum value that A can take and the corresponding value of x.

41 A particular plastic plate manufactured at a factory sells at \$1.50. The cost of production consists of an initial cost of \$3500 and then \$0.50 per plate. Let x be the number of plates produced.

a Let \$$C$ be the cost of production of x plates. Write an expression for C in terms of x.

b Let \$$I$ be the income from selling x plates. Write an expression for I in terms of x.

c On the one set of axes, sketch the graphs of I against x and C against x.

d How many plates must be sold for the income to equal the cost of production?

e How many plates must be sold for a profit of \$2000 to be made?

f Let $P = I - C$. Sketch the graph of P against x. What does P represent?

42 **a** **i** For the equation $\sqrt{7x-5} - \sqrt{2x} = \sqrt{15-7x}$, square both sides to show that this equation implies

$$8x - 10 = \sqrt{14x^2 - 10x}$$

ii Square both sides of this new equation and simplify to form the equation

$$x^2 - 3x + 2 = 0 \qquad (1)$$

iii The solutions to equation (1) are $x = 1$ and $x = 2$. Test these solutions for the equation

$$\sqrt{7x-5} - \sqrt{2x} = \sqrt{15-7x}$$

and hence show that $x = 2$ is the only solution to the original equation.

b Use the techniques of part **a** to solve the equations:

i $\sqrt{x+2} - 2\sqrt{x} = \sqrt{x+1}$

ii $2\sqrt{x+1} + \sqrt{x-1} = 3\sqrt{x}$

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43 Let n be a natural number less than 50 such that $n + 25$ is a perfect square.

a Show that there exists an integer a such that $n = a(a + 10)$.

b Any natural number less than 100 can be written in the form $10p + q$, where p and q are digits. For this representation of n, show that $q = p^2$.

c Give all possible values of n.

44 If an object of mass m kg is at a height of h m above the ground, then its potential energy (P joules) is given by the formula $P = mgh$, where g is a constant.

a **i** Determine the value of the constant g given that $P = 980$ when $h = 20$ and $m = 5$.

ii Sketch the graph of P against h for an object of mass 5 kg.

iii Determine m if $P = 2058$ and $h = 30$.

b **i** What is the effect on the potential energy if the height (h m) is doubled and the mass remains constant?

ii What is the effect on the potential energy if the object has one-quarter of the original height (h m) and double the original mass (m kg)?

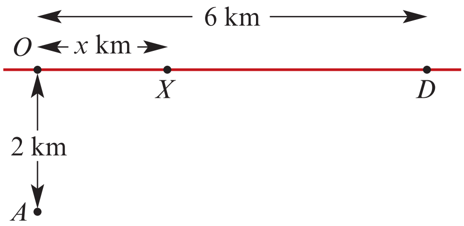
c If an object is dropped from a height (h m) above ground level, its speed (V m/s) when it reaches the ground is given by $V = \sqrt{19.6h}$.

i Determine V when $h = 10$.

ii Determine V when $h = 90$.

d In order to double the speed that a given object has when it hits the ground, by what factor must the height from which it is dropped be increased?

45 The diagram shows a straight road OD, where $OD = 6$ km.



A hiker is at A, which is 2 km from O with OA perpendicular to OD. The hiker walks directly to X and then walks along the road to D. The hiker can walk at 3 km/h off-road, but at 8 km/h along the road.

a If $OX = 3$ km, calculate the total time taken for the hiker to walk from A to D via X in hours and minutes, correct to the nearest minute.

b If the total time taken was $1\frac{1}{2}$ hours, calculate the distance OX in kilometres, correct to one decimal place.

46 Seventy-six photographers submitted work for a photographic exhibition in which they were permitted to enter not more than one photograph in each of three categories: black and white (B), colour prints (C), transparencies (T). Eighteen entrants had all their work rejected, while 30 B, 30 T and 20 C were accepted.

- From the exhibitors, as many showed T only as showed T and C.
- There were three times as many exhibitors showing B only as showing C only.
- Four exhibitors showed B and T but not C.

a Write the last three sentences in symbolic form.

b Draw a Venn diagram representing the information.

c **i** Determine $|B \cap C \cap T|$.

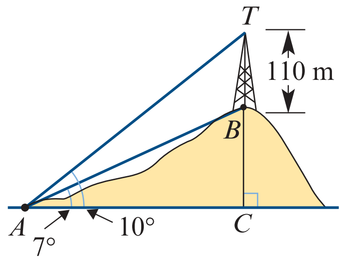
ii Determine $|B \cap C \cap T'|$.

47 A tower 110 m high stands on the top of a hill. From a point A at the foot of the hill, the angle of elevation of the bottom of the tower is 7° and that of the top is 10°.

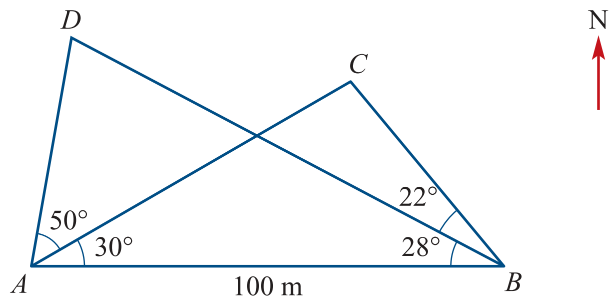
a Determine the magnitudes of angles TAB, ABT and ATB.

b Use the sine rule to determine the length of AB.

c Determine CB, the height of the hill.



48 A triangular survey method was used to calculate the distance from a point A, on land, to two platforms C and D, out to sea. Sightings were made from two accessible points A and B, which are 100 m apart.



a Determine the distance from A to each of the platforms (to the nearest metre).

b Determine the distance between the two platforms (to the nearest metre).

c Determine the bearing of the platform D from the platform C (to the nearest degree).

49 Lighthouse B is 7 km due east of lighthouse A. From A, the bearings of two ships are 031° and 073° respectively. From B, the bearings are 275° and 343° respectively. Determine the distance between the two ships.

50 Evaluate $\left(\sqrt{8+2\sqrt{7}} - \sqrt{8-2\sqrt{7}}\right)^2$.

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CU

51 **a** Fully factorise $n^5 - 5n^3 + 4n$.

b Explain why:

i the product of any five consecutive integers is divisible by 5

ii the product of any three consecutive integers is divisible by 3

iii the product of any four consecutive integers is divisible by 8.

c Explain why $n^5 - 5n^3 + 4n$ is divisible by 120, for each integer n.

52 In order to measure the width of a river with parallel banks, an object C on one bank is sighted from points A and B, which are 60 m apart, on the opposite bank. It is found that $\angle CAB = 40°$ and $\angle CBA = 139°$. Calculate the width of the river.

53 A car travels due north for 30 km and then travels on a bearing of 120° for 50 km. Determine the car's distance and bearing from the starting point.

54 A group of 30 children were surveyed to determine out which of three sports they play: cricket, basketball or volleyball. It was found that:

- 2 children play all three sports
- 16 children play basketball
- 12 children play volleyball
- 3 children do not play any of these sports
- 6 children play basketball and volleyball only
- 3 children play cricket and volleyball only.

Using a Venn diagram, determine how many of the children play cricket only.

55 Determine all ordered pairs of positive integers (x, y) such that $\dfrac{xy}{x+y} = 3$.

56 Express $2^{25} \times 3^{40}$ in the form a^b, where a and b are natural numbers with $b > 1$.

57 A car leaves town A at 10 a.m. and arrives in town B at 11:15 a.m. During the first hour of the journey, the car travels at a constant speed of 80 km/h. The average speed of the car between 10:15 a.m. and 11:15 a.m. is 2 km/h less than the average speed for the whole journey.

a Determine the distance travelled by the car from town A to town B.

b Determine the average speed of the car between 10:15 a.m. and 11:15 a.m.

58 **a** Show that $x^2 + (1-x)^2 = 2\left[\left(x - \dfrac{1}{2}\right)^2 + \dfrac{1}{4}\right]$.

b Hence show that, if $0 \le x \le 1$, then

$$\frac{1}{2} \le x^2 + (1-x)^2 \le 1$$

c A quadrilateral has one vertex on each side of a unit square (that is, a square of side length 1). Show that the side lengths a, b, c and d of the quadrilateral satisfy

$$2 \le a^2 + b^2 + c^2 + d^2 \le 4$$

59 Consider the quadratic expression $x^2 + bx + c$, where b and c are real numbers.

a Given that the equation $x^2 + bx + c + 1 = 0$ has only one solution, determine c in terms of b.

b Given that the expression $x^2 + bx + c - 3$ can be factorised as $(x - k)(x - 2k)$, for some non-zero real number k, determine c in terms of b.

c If the conditions of both parts **a** and **b** are satisfied, determine the possible values of b and c.

60 **a** Determine positive integers m and n such that $\sqrt{9 - 4\sqrt{5}} = \sqrt{m} - n$.

b Given that $x = \sqrt{9 - 4\sqrt{5}}$ is a solution of the quadratic equation

$$x^2 + bx + c = 0 \quad \text{where } b \text{ and } c \text{ are integers}$$

determine the values of b and c.

61 **a** Write down the equation of the line that passes through the point $P(a, b)$ and has gradient m.

b Given that this line touches (but does not cross) the circle with equation $x^2 + y^2 = r^2$, show that

$$(a^2 - r^2)m^2 - 2abm + (b^2 - r^2) = 0$$

c Using part **b**, determine the equations of the two tangents from the point $P(-3, 2)$ to the circle with equation $x^2 + y^2 = 4$.

5B Multiple-choice questions

Technology-free multiple-choice questions

1 In algebraic form, five is seven less than three times one more than x can be written as

A $3x + 1 = 5 - 7$ **B** $(x + 1) - 7 = 5$

C $5 = 7 - 3x + 1$ **D** $5 = 3x - 4$

2 $\dfrac{3}{x-3} - \dfrac{2}{x+3}$ is equal to

A 1 **B** $\dfrac{x+15}{x^2-9}$ **C** $\dfrac{15}{x-9}$ **D** $\dfrac{x-3}{x^2-9}$

3 If $A = \{1, 2, 3, 4\}$, $B = \{2, 3, 4, 5, 6\}$ and $C = \{3, 4, 5, 6, 7\}$, then $A \cap (B \cup C)$ is equal to

A $\{1, 2, 3, 4, 5, 6, 7\}$ **B** $\{1, 2, 3, 4, 5, 6\}$

C $\{2, 3, 4\}$ **D** $\{3, 4\}$

4 The recurring decimal $0.\dot{7}\dot{2}$ is equal to

A $\dfrac{72}{101}$ **B** $\dfrac{72}{100}$ **C** $\dfrac{72}{99}$ **D** $\dfrac{72}{90}$

5 $\dfrac{-4}{x-1} - \dfrac{3}{1-x} + \dfrac{x}{x-1}$ is equal to

A 1 **B** -1 **C** $\dfrac{7x}{x-1}$ **D** $\dfrac{1}{1-x}$

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6 $\dfrac{x+2}{3} - \dfrac{5}{6}$ is equal to

A $\dfrac{x-3}{6}$ **B** $\dfrac{2x+4}{6}$ **C** $\dfrac{2x-1}{6}$ **D** $\dfrac{2x-5}{6}$

7 If $a = 1 + \dfrac{1}{1+b}$, then b equals

A $1 - \dfrac{1}{a-1}$ **B** $1 + \dfrac{1}{a-1}$ **C** $\dfrac{1}{a-1} - 1$ **D** $\dfrac{1}{a+1} + 1$

8 When the repeating decimal $0.\dot{3}\dot{6}$ is written in simplest fractional form, the sum of the numerator and denominator is

A 15 **B** 45 **C** 114 **D** 135

9 If $\dfrac{2x-y}{2x+y} = \dfrac{3}{4}$, then $\dfrac{x}{y}$ equals

A $\dfrac{2}{7}$ **B** $\dfrac{7}{2}$ **C** $\dfrac{3}{4}$ **D** $\dfrac{4}{3}$

10 The coordinates of the point where the lines with equations $3x + y = -7$ and $2x + 5y = 4$ intersect are

A $(3, -16)$ **B** $(-3, 2)$ **C** $(3, -2)$ **D** $(-2, 3)$

11 If $\dfrac{m+2}{4} - \dfrac{2-m}{4} = \dfrac{1}{2}$, then m is equal to

A 1 **B** -1 **C** $\dfrac{1}{2}$ **D** 0

12 The number 46 200 can be written as

A $2 \times 3 \times 5 \times 7 \times 11$ **B** $2^2 \times 3^2 \times 5^2 \times 7 \times 11$

C $2 \times 3^2 \times 5 \times 7^2 \times 11$ **D** $2^3 \times 3 \times 5^2 \times 7 \times 11$

13 If the positive integers $n+1$, $n-1$, $n-6$, $n-5$, $n+4$ are arranged in increasing order of magnitude, then the middle number is

A $n+1$ **B** $n-1$ **C** $n-6$ **D** $n-5$

14 The expression $\dfrac{4}{n+1} + \dfrac{3}{n-1}$ is equal to

A $\dfrac{7n-1}{1-n^2}$ **B** $\dfrac{1-7n}{1-n^2}$ **C** $\dfrac{7n-1}{n^2+1}$ **D** $\dfrac{7}{n^2-1}$

15 The second number is twice the first number; the third number is half the first number; the three numbers sum to 28. These numbers are

A $(8, 16, 4)$ **B** $(2, 3, 12)$ **C** $(7, 9, 11)$ **D** $(6, 8, 16)$

16 $(\sqrt{7}+3)(\sqrt{7}-3)$ is equal to

A -2 **B** 10 **C** $\sqrt{14} - 19$ **D** $2\sqrt{7} - 9$

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17 If $\dfrac{13x - 10}{2x^2 - 9x + 4} = \dfrac{P}{x - 4} + \dfrac{Q}{2x - 1}$, then the values of P and Q are

A $P = 1$ and $Q = 1$ **B** $P = -1$ and $Q = 1$
C $P = 6$ and $Q = 1$ **D** $P = -6$ and $Q = 1$

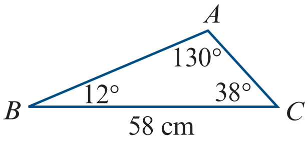
18 If $\dfrac{5x}{(x+2)(x-3)} = \dfrac{P}{x+2} + \dfrac{Q}{x-3}$, then

A $P = 2$ and $Q = 3$ **B** $P = 2$ and $Q = -3$
C $P = -2$ and $Q = 3$ **D** $P = -2$ and $Q = -3$

19 $\sin(180 - \theta)^\circ + \cos(180 - \theta)^\circ$ is equal to

A $\sin(\theta^\circ) + \cos(\theta^\circ)$ **B** $-\sin(\theta^\circ) + \cos(\theta^\circ)$
C $\sin(\theta^\circ) - \cos(\theta^\circ)$ **D** $-\sin(\theta^\circ) - \cos(\theta^\circ)$

20 Which one of the following gives the correct value for c?



A $\dfrac{58\cos 38^\circ}{\cos 130^\circ}$ **B** $\dfrac{58\sin 38^\circ}{\sin 130^\circ}$
C $58\sin 38^\circ$ **D** $\dfrac{58\cos 130^\circ}{\cos 38^\circ}$

21 If the natural number n is a perfect square, then the next perfect square is

A $n^2 + 1$ **B** $n^2 + 2n + 1$ **C** $n^2 + n$ **D** $n + 2\sqrt{n} + 1$

22 Which of the following is *not* a rational number?

A $\dfrac{3}{8}$ **B** $\sqrt{5}$ **C** $\sqrt{16}$ **D** 4.125

23 If $\dfrac{1}{x} = \dfrac{a}{b}$ and $\dfrac{1}{y} = a - b$, then $x + y$ equals

A $\dfrac{2}{a}$ **B** $\dfrac{a^2 - b^2}{a}$ **C** $\dfrac{ba - b^2 + a}{a(a - b)}$ **D** $\dfrac{2a}{a^2 - b^2}$

24 $9x^2 - 4mx + 4$ is a perfect square when m equals

A ± 12 **B** 2 **C** ± 1 **D** ± 3

25 If $x = (n + 1)(n + 2)(n + 3)$, for some positive integer n, then x is not always divisible by

A 2 **B** 3 **C** 5 **D** 6

26 If both n and p are odd numbers, which one of the following numbers must be even?

A $n + p$ **B** $np + 2$ **C** $n + p + 1$ **D** $2n + p$

27 $4a^2b^4 \times 3(ab^3)^{-2}$ is equal to

A $12b^{-2}$ **B** $12ab^{-2}$ **C** $12a^{-4}b^{-2}$ **D** $12a^3b^5$

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28 If $\text{LCM}(12, n) = 60$ and $\text{HCF}(12, n) = 6$, then $n =$

A 10 **B** 15 **C** 20 **D** 30

29 If x^2 is written in the form $(x-2)^2 + b(x-2) + c$, then the values of b and c are

A $b = 2,\ c = 0$ **B** $b = -4,\ c = -4$

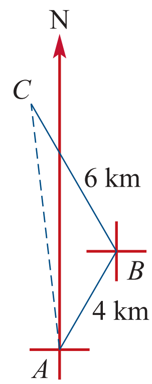
C $b = 4,\ c = 4$ **D** $b = 2,\ c = 2$

30 A car covers a distance of 50 km at an average speed of x km/h. Over the same period of time, a train covers a distance of 70 km at an average speed of $(x + 25)$ km/h. An equation that can be used to determine x is

A $50x = 70(x + 25)$ **B** $70x = 50(x + 25)$

C $50x = 70x + 25$ **D** $70x = 50x + 25$

31 A hiker walks 4 km from point A on a bearing of 030° to point B, and then walks 6 km on a bearing of 330° to point C. The distance AC, in kilometres, is



A $\dfrac{4}{\sin 30^\circ}$

B $\sqrt{6^2 + 4^2 - 48\cos 120^\circ}$

C $\sqrt{6^2 + 4^2 + 48\cos 120^\circ}$

D $6\sin 60^\circ$

32 From a port P, a ship Q is 20 km away on a bearing of 112°, and a ship R is 12 km away on a bearing of 052°. The distance between the two ships is

A 304 km **B** 28 km **C** $4\sqrt{19}$ km **D** 784 km

Technology-active multiple-choice questions

33 A building of height 60 m is 1.1 km away from a point on the ground. The angle of elevation to the top of the building from this point, correct to the nearest degree, is

A 1° **B** 3° **C** 87° **D** 89°

34 $\dfrac{3 \times 10^8}{\sqrt{0.144 \times 10^5}}$ is equal to

A 3.6×10^6 **B** 8×10^3 **C** 8×10^{-3} **D** 2.5×10^6

35 The value of $\dfrac{4.050 \times 2.098}{5.77}$ to three significant figures is

A 1.4726 **B** 1.473 **C** 1.47 **D** 1.48

36 A can of pet food is in the shape of a cylinder. The can has a circumference of 19.25 cm and a volume of 367 cm^3. The height of the can, in centimetres, is closest to

A 14.5 **B** 12.45 **C** 13.06 **D** 22.87

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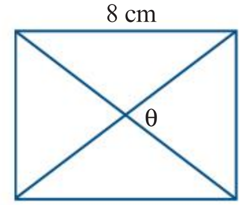
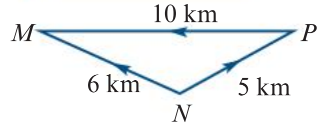
37 A rectangle is 8 cm long and 6 cm wide. The acute angle, θ, between its diagonals, correct to the nearest degree, is:



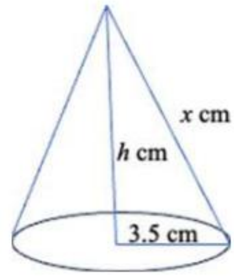
A 37° **B** 41° **C** 49° **D** 74°

38 A yacht follows a triangular course, MNP, as shown. The largest angle between any two legs of the course is closest to:



A 51° **B** 71°
C 121° **D** 131°

39 A cone with a radius $r = 3.5$ cm is shown in the diagram below. The cone has a slant edge of x cm. The volume of this cone is 45 cm^3. The surface area of this cone, including the base, can be found using the rule $surface\ \ area = \pi r(r + x)$. The total surface area of this cone, including the base, in square centimetres, is closest to



A 93 **B** 45 **C** 69 **D** 90

5C Problem-solving and modelling questions

1 There are two ways to express the number 10 as a product of three positive integers:

$$10 = 1 \times 1 \times 10 \quad \text{and} \quad 10 = 1 \times 2 \times 5$$

(Note that the order of the factors does not matter.) How many ways are there to express the number 100 as a product of three positive integers?

2 **a** There is a two-digit number ☐☐ such that, if you add the digit 1 at the front and the back to obtain a four-digit number 1☐☐1, then the new four-digit number is 21 times larger than the original two-digit number. Determine this two-digit number.

b Determine a five-digit number ☐☐ ☐☐☐ such that the six-digit number obtained by adding a 1 at the back, ☐☐☐ ☐☐1, is three times larger than the six-digit number obtained by adding a 1 at the front, 1☐☐ ☐☐☐.

3 Determine the possible integer values of k if:

a the quadratic equation $x^2 + kx - 16 = 0$ has integer solutions

b the quadratic equation $x^2 + kx + 20 = 0$ has integer solutions

c the quadratic equation $x^2 + 12x + k = 0$ has integer solutions (k positive).

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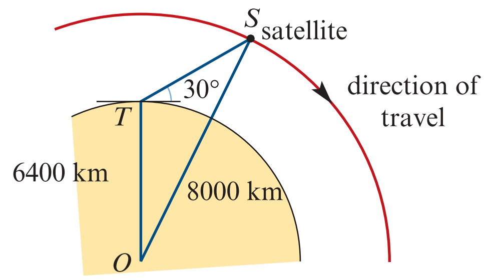
Revision

4 A satellite travelling in a circular orbit 1600 km above the Earth is due to pass directly over a tracking station at 12 p.m. Assume that the satellite takes two hours to make an orbit and that the radius of the Earth is 6400 km.



a If the tracking station antenna is aimed at 30° above the horizon, at what time will the satellite pass through the beam of the antenna?

b Determine the distance between the satellite and the tracking station at 12:06 p.m.

c At what angle above the horizon should the antenna be aimed so that its beam will intercept the satellite at 12:06 p.m.?

5 The factors of 12 are 1, 2, 3, 4, 6, 12.

a How many factors does each of the following numbers have?

i 2^3 **ii** 3^7

b How many factors does 2^n have?

c How many factors does each of the following numbers have?

i $2^3 \cdot 3^7$ **ii** $2^n \cdot 3^m$

d Every natural number greater than 1 may be expressed as a product of powers of primes; this is called prime decomposition. For example: $1080 = 2^3 \times 3^3 \times 5$.
Let x be a natural number greater than 1 and let

$$x = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \dots p_n^{\alpha_n}$$

be its prime decomposition, where each $\alpha_i \in \mathbb{N}$ and each p_i is a prime number. How many factors does x have? (Answer to be given in terms of α_i.)

e Determine the smallest number which has eight factors.

6 **a** Give the prime decompositions of 1080 and 25 200.

b Use your answer to part **a** to determine the lowest common multiple of 1080 and 25 200.

c Carefully explain why, if m and n are natural numbers, then

$$mn = \text{LCM}(m, n) \times \text{HCF}(m, n)$$

d **i** Determine four consecutive even numbers such that the smallest is a multiple of 5, the second a multiple of 7, the third a multiple of 9 and the largest a multiple of 11.

ii Determine four consecutive natural numbers such that the smallest is a multiple of 5, the second a multiple of 7, the third a multiple of 9 and the largest a multiple of 11.

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7 **a** Prove that $a^4 + 4b^4 = (a^2 + 2b^2 + 2ab)(a^2 + 2b^2 - 2ab)$.

Note: This is known as Sophie Germain's identity.

b Use this identity to prove that, if n is an odd number greater than 1, then $n^4 + 4^n$ is not prime.

c Hence show that the number $4^{545} + 545^4$ is not prime.

5D Problem-solving and modelling investigations

For each of the following questions, there are different approaches and directions that you can take. Suggestions are given, but you should develop your solution on an individual basis.

1 **Reciprocals of primes**

a Determine all ordered pairs of positive integers (x, y) such that

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{3}$$

b Determine all ordered pairs of positive integers (x, y) such that

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{11}$$

c Determine all ordered pairs of positive integers (x, y) such that

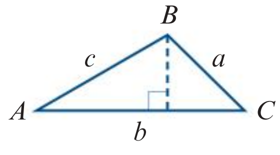
$$\frac{1}{x} + \frac{1}{y} = \frac{1}{p}$$

where p is a prime number. Give your answers with x and y in terms of p.

2 **Area of a triangle**

a Consider a triangle ABC with side lengths a, b and c. Prove that the area of the triangle is given by

$$\text{Area} = \frac{1}{2}bc\sin A$$



b Now let $s = \dfrac{a+b+c}{2}$. Using part **a**, prove that the area of the triangle is given by

$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}$$

This is known as Heron's formula.

Hint: From the unit circle, we have $(\cos A)^2 + (\sin A)^2 = 1$. Use this result and the cosine rule in triangle ABC to show that

$$(\sin A)^2 = \frac{(b+c+a)(b+c-a)(a+b-c)(a-b+c)}{(2bc)^2}$$

c Investigate **Heronian triangles**, which are triangles such that the side lengths and the area are all integers. For example: Explain why there are no equilateral Heronian triangles. Determine an isosceles Heronian triangle. Explain why every right-angled triangle with integer side lengths is Heronian.

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Revision

3 **Filling a tank** A tank can be filled using two pipes. The smaller pipe alone will take a minutes longer than the larger pipe alone to fill the tank. Also, the smaller pipe will take b minutes longer to fill the tank than when both pipes are used.

a In terms of a and b, how long will each of the pipes take to fill the tank?

b If $a = 24$ and $b = 32$, how long will each of the pipes take to fill the tank?

c If a and b are consecutive positive integers, determine five pairs of values of a and b such that $b^2 - ab$ is a perfect square. Interpret these results in the context of this problem.

4 **Applications of partial fractions**

a **i** Show that

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

ii Hence show that

$$\frac{1}{1\times 2} + \frac{1}{2\times 3} + \frac{1}{3\times 4} + \cdots + \frac{1}{99\times 100} = \frac{99}{100}$$

iii Hence evaluate

$$\frac{1}{1} + \frac{1}{1+2} + \frac{1}{1+2+3} + \cdots + \frac{1}{1+2+\cdots+99}$$

b **i** Determine A and B such that

$$\frac{1}{n(n+2)} = \frac{A}{n} + \frac{B}{n+2}$$

ii Hence evaluate

$$\frac{1}{1\times 3} + \frac{1}{3\times 5} + \frac{1}{5\times 7} + \cdots + \frac{1}{99\times 101}$$

iii By using partial fractions in a similar way, evaluate

$$\frac{1}{1\times 6} + \frac{1}{6\times 11} + \frac{1}{11\times 16} + \cdots + \frac{1}{96\times 101}$$

c **i** Show that

$$\frac{1}{n(n+1)(n+2)} = \frac{1}{2}\left(\frac{1}{n} - \frac{1}{n+1}\right) - \frac{1}{2}\left(\frac{1}{n+1} - \frac{1}{n+2}\right)$$

ii Hence evaluate

$$\frac{1}{1\times 2\times 3} + \frac{1}{2\times 3\times 4} + \frac{1}{3\times 4\times 5} + \cdots + \frac{1}{98\times 99\times 100}$$

d **i** By rationalising the denominator of the left-hand side, show that

$$\frac{1}{\sqrt{n+1} + \sqrt{n}} = \sqrt{n+1} - \sqrt{n}$$

ii Hence show that

$$\frac{1}{\sqrt{2}+\sqrt{1}} + \frac{1}{\sqrt{3}+\sqrt{2}} + \frac{1}{\sqrt{4}+\sqrt{3}} + \cdots + \frac{1}{\sqrt{100}+\sqrt{99}} = 9$$

iii By using a similar approach, evaluate

$$\frac{1}{\sqrt{3}+\sqrt{1}} + \frac{1}{\sqrt{5}+\sqrt{3}} + \frac{1}{\sqrt{7}+\sqrt{5}} + \cdots + \frac{1}{\sqrt{121}+\sqrt{119}}$$

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6

Combinatorics

Chapter contents

Take a deck of 52 playing cards. This simple, familiar deck can be arranged in so many ways that if you and every other living human were to shuffle a deck once per second from the beginning of time, then by now only a tiny fraction of all possible arrangements would have been obtained. So, remarkably, every time you shuffle a deck you are likely to be the first person to have created that particular arrangement of cards!

To see this, note that we have 52 choices for the first card, and then 51 choices for the second card, and so on. This gives a total of

$$52 \times 51 \times \cdots \times 2 \times 1 \approx 8.1 \times 10^{67}$$

arrangements. This is quite an impressive number, especially in light of the fact that the universe is estimated to be merely 1.4×10^{10} years old.

Combinatorics is concerned with counting the number of ways of doing something. Our goal is to find thoughtful ways of doing this without explicitly listing all the possibilities. This is particularly important in the study of probability. For instance, we can use combinatorics to explain why certain poker hands are more likely to occur than others without considering all 2 598 960 possible hands.

6A Basic counting methods

Learning intentions

- To be able to use tree diagrams, the addition principle and the multiplication principle to solve counting problems.

Tree diagrams

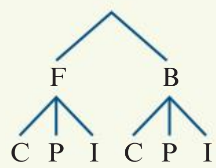
In most combinatorial problems, we are interested in the *number of solutions* to a given problem, rather than the solutions themselves. Nonetheless, for simple counting problems it is sometimes practical to list and then count all the solutions. Tree diagrams provide a systematic way of doing this, especially when the problem involves a small number of steps.

Example 1

A restaurant has a fixed menu, offering a choice of fish or beef for the main meal, and cake, pudding or ice-cream for dessert. How many different meals can be chosen?

Solution

We illustrate the possibilities on a tree diagram:



This gives six different meals, which we can write as

FC, FP, FI, BC, BP, BI

The multiplication principle

In the above example, for each of the two ways of selecting the main meal, there were three ways of selecting the dessert. This gives a total of $2 \times 3 = 6$ ways of choosing a meal. This is an example of the **multiplication principle**, which will be used extensively throughout this chapter.

Multiplication principle

If there are m ways of performing one task and then there are n ways of performing another task, then there are $m \times n$ ways of performing *both* tasks.

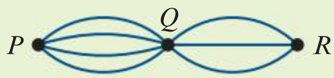
Example 2

Sandra has three different skirts, four different tops and five different pairs of shoes. How many choices does she have for a complete outfit?

Solution	Explanation
$3 \times 4 \times 5 = 60$	Using the multiplication principle, we multiply the number of ways of making each choice.

Example 3

How many paths are there from point P to point R travelling from left to right?



Solution

$4 \times 3 = 12$

Explanation

For each of the four paths from P to Q, there are three paths from Q to R.

The addition principle

In some instances, we have to count the number of ways of choosing between two alternative tasks. In this case, we use the **addition principle**.

Addition principle

Suppose there are m ways of performing one task and n ways of performing another task. If we cannot perform both tasks, then there are $m + n$ ways to perform one of the tasks.

Example 4

To travel from Brisbane to Sydney tomorrow, Kara has a choice between three different flights and two different trains. How many choices does she have?

Solution

$3 + 2 = 5$

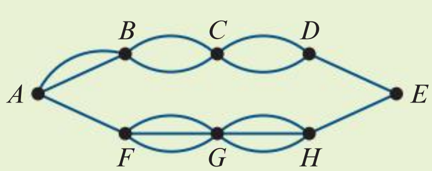
Explanation

The addition principle applies because Kara cannot travel by both plane and train. Therefore, we add the number of ways of making each choice.

Some problems will require use of both the multiplication and the addition principles.

Example 5

How many paths are there from point A to point E travelling from left to right?



Solution

We can take *either* an upper path *or* a lower path:

- Going from A to B to C to D to E there are $2 \times 2 \times 2 \times 1 = 8$ paths.
- Going from A to F to G to H to E there are $1 \times 3 \times 3 \times 1 = 9$ paths.

Using the addition principle, there is a total of $8 + 9 = 17$ paths from A to E.

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Harder problems involving tree diagrams

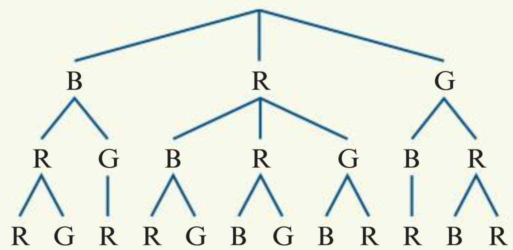
For some problems, a straightforward application of the multiplication and addition principles is not possible.

Example 6

A bag contains one blue token, two red tokens and one green token. Three tokens are removed from the bag and placed in a row. How many arrangements are possible?

Solution

The three tokens are selected without replacement. So once a blue or green token is taken, these cannot appear again. We use a tree diagram to systematically determine every arrangement.



The complete set of possible arrangements can be read by tracing out each path from top to bottom of the diagram. This gives 12 different arrangements:

BRR, BRG, BGR, RBR, RBG, RRB, RRG, RGB, RGR, GBR, GRB, GRR

Summary 6A

Three useful approaches to solving simple counting problems:

- **Tree diagrams**
  These can be used to systematically list all solutions to a problem.
- **Multiplication principle**
  If there are m ways of performing one task and then there are n ways of performing another task, then there are $m \times n$ ways of performing *both* tasks.
- **Addition principle**
  Suppose there are m ways of performing one task and n ways of performing another task. If we cannot perform both tasks, there are $m + n$ ways to perform one of the tasks.

Some problems require use of both the addition and the multiplication principles.

Skill-sheet

Exercise 6A

Example 2

1 Sam has five T-shirts, three pairs of pants and three pairs of shoes. How many different outfits can he assemble using these clothes? SF

SF

Example 4

2 A restaurant offers five beef dishes and three chicken dishes. How many selections of one main meal does a customer have?

3 Each of the 10 boys at a party shakes hands with each of the 12 girls. How many handshakes take place?

4 Draw a tree diagram showing all the two-digit numbers that can be formed using the digits 7, 8 and 9 if each digit:

a cannot be repeated **b** can be repeated.

5 How many different three-digit numbers can be formed using the digits 2, 4 and 6 if each digit can be used:

a as many times as you would like **b** at most once?

CF

6 Jack wants to travel from Sydney to Perth via Adelaide. There are four flights and two trains from Sydney to Adelaide. There are two flights and three trains from Adelaide to Perth. How many ways can Jack travel from Sydney to Perth?

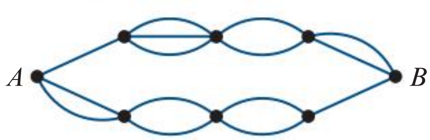
7 Travelling from left to right, how many paths are there from point A to point B in each of the following diagrams?

Example 3

Example 5

a


b


c


d


Example 6

8 A bag contains two blue, one red and two green tokens. Two tokens are removed from the bag and placed in a row. With the help of a tree diagram, list all the different arrangements.

CU

9 How many ways can you make change for 50 cents using 5, 10 and 20 cent pieces?

10 Four teachers decide to swap desks at work. How many ways can this be done if no teacher is to sit at their previous desk?

11 Three runners compete in a race. In how many ways can the runners complete the race assuming:

a there are no tied places **b** the runners can tie places?

12 A six-sided die has faces labelled with the numbers 0, 2, 3, 5, 7 and 11. If the die is rolled twice and the two results are multiplied, how many different answers can be obtained?

6B Factorial notation and permutations

Learning intentions

- To be able to use factorial notation.
- To be able to solve simple permutation problems.

Factorial notation

Factorial notation provides a convenient way of expressing products of consecutive natural numbers. For each natural number n, we define

$$n! = n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot 2 \cdot 1$$

where the notation $n!$ is read as 'n factorial'. We also define $0! = 1$. Although it might seem strange at first, this definition will turn out to be very convenient, as it is compatible with formulas that we will establish shortly. Another very useful identity is

$$n! = n \cdot (n-1)!$$

Example 7

Evaluate:

a $3!$ **b** $\dfrac{50!}{49!}$ **c** $\dfrac{10!}{2!\,8!}$

Solution

a $3! = 3 \cdot 2 \cdot 1$
$= 6$

b $\dfrac{50!}{49!} = \dfrac{50 \cdot 49!}{49!}$
$= 50$

c $\dfrac{10!}{2!\,8!} = \dfrac{10 \cdot 9 \cdot 8!}{2! \cdot 8!}$
$= \dfrac{10 \cdot 9}{2 \cdot 1}$
$= 45$

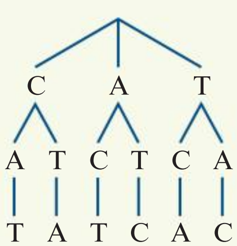
Permutations of n objects

A **permutation** is an ordered arrangement of a collection of objects.

Example 8

Using a tree diagram, list all the permutations of the letters in the word CAT.

Solution



There are six permutations:

CAT, CTA, ACT, ATC, TCA, TAC

Explanation

There are three choices for the first letter. This leaves only two choices for the second letter, and then one for the third.

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Another way to determine the number of permutations for the previous example is to draw three boxes, corresponding to the three positions. In each box, we write the number of choices we have for that position.

- We have 3 choices for the first letter (C, A or T).
- We have 2 choices for the second letter (because we have already used one letter).
- We have 1 choice for the third letter (because we have already used two letters).

3	2	1

By the multiplication principle, the total number of arrangements is

$$3 \times 2 \times 1 = 3!$$

So three objects can be arranged in 3! ways. More generally:

> The number of permutations of n objects is $n!$.

Proof The reason for this is simple:

- The first item can be chosen in n ways.
- The second item can be chosen in $n - 1$ ways, since only $n - 1$ objects remain.
- The third item can be chosen in $n - 2$ ways, since only $n - 2$ objects remain.

⋮

- The last item can be chosen in 1 way, since only 1 object remains.

Therefore, by the multiplication principle, there are

$$n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot 2 \cdot 1 = n!$$

permutations of n objects.

Example 9

How many ways can six different books be arranged on a shelf?

Solution

$$\begin{aligned} 6! &= 6 \times 5 \times 4 \times 3 \times 2 \times 1 \\ &= 720 \end{aligned}$$

Explanation

Six books can be arranged in 6! ways.

Example 10

Using your calculator, determine how many ways 12 students can be lined up in a row.

Using the TI-Nspire CX non-CAS

Evaluate 12! as shown.

Note: The factorial symbol (!) can be accessed using ?!▸, the Symbols palette (ctrl 📖) or menu > **Probability** > **Factorial**.

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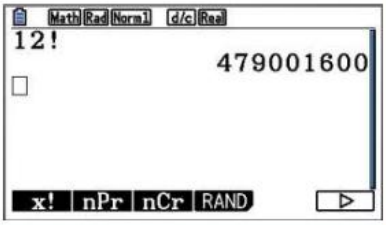
Using the Casio

Evaluate 12! as follows:

- Press MENU 1 to select **Run-Matrix** mode.
- Go to the **Probability** menu OPTN F6 F3.
- Enter 12, select **x!** F1 and press EXE.



Example 11

How many four-digit numbers can be formed using the digits 1, 2, 3 and 4 if:

a they cannot be repeated

b they can be repeated?

Solution

a $4! = 4 \times 3 \times 2 \times 1 = 24$

b $4^4 = 4 \times 4 \times 4 \times 4 = 256$

Explanation

Four numbers can be arranged in 4! ways.

Using the multiplication principle, there are 4 choices for each of the 4 digits.

Permutations of n objects taken r at a time

Imagine a very small country with very few cars. Licence plates consist of a sequence of four digits, and repetitions of the digits are not allowed. How many such licence plates are there?

Here, we are asking for the number of permutations of 10 digits taken four at a time. We will denote this number by ${}^{10}P_4$.

To solve this problem, we draw four boxes. In each box, we write the number of choices we have for that position. For the first digit, we have a choice of 10 digits. Once chosen, we have only 9 choices for the second digit, then 8 choices for the third and 7 choices for the fourth.

10	9	8	7

By the multiplication principle, the total number of licence plates is

$$10 \times 9 \times 8 \times 7$$

There is a clever way of writing this product as a fraction involving factorials:

$$\begin{aligned} {}^{10}P_4 &= 10 \cdot 9 \cdot 8 \cdot 7 \\ &= \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} \\ &= \frac{10!}{6!} \\ &= \frac{10!}{(10-4)!} \end{aligned}$$

We can easily generalise this procedure to give the following result.

Number of permutations

The number of permutations of n objects taken r at a time is denoted by ${}^nP_r$ and is given by the formula

$$ {}^nP_r = \frac{n!}{(n-r)!} $$

Proof To establish this formula we note that:

- The 1st item can be chosen in n ways.
- The 2nd item can be chosen in $n - 1$ ways.

⋮

- The rth item can be chosen in $n - r + 1$ ways.

Therefore, by the multiplication principle, the number of permutations of n objects taken r at a time is

$$ \begin{aligned} {}^nP_r &= n \cdot (n-1) \cdot \cdots \cdot (n-r+1) \\ &= \frac{n \cdot (n-1) \cdot \cdots \cdot (n-r+1) \cdot (n-r) \cdot \cdots \cdot 2 \cdot 1}{(n-r) \cdot \cdots \cdot 2 \cdot 1} \\ &= \frac{n!}{(n-r)!} \end{aligned} $$

Notes:

- If $r = n$, then we have ${}^nP_n$, which is simply the number of permutations of n objects and so must equal $n!$. The formula still works in this instance, since

$$ {}^nP_n = \frac{n!}{(n-n)!} = \frac{n!}{0!} = n! $$

  Note that this calculation depends crucially on our decision to define $0! = 1$.
- If $r = 1$, then we obtain ${}^nP_1 = n$. Given n objects, there are n choices of one object, and each of these can be arranged in just one way.

Example 12

a Using the letters A, B, C, D and E without repetition, how many different two-letter arrangements are there?

b Six runners compete in a race. In how many ways can the gold, silver and bronze medals be awarded?

Solution

a There are five letters to arrange in two positions:

$$ \begin{aligned} {}^5P_2 = \frac{5!}{(5-2)!} &= \frac{5!}{3!} \\ &= \frac{5 \cdot 4 \cdot 3!}{3!} \\ &= 20 \end{aligned} $$

b There are six runners to arrange in three positions:

$$ \begin{aligned} {}^6P_3 = \frac{6!}{(6-3)!} &= \frac{6!}{3!} \\ &= \frac{6 \cdot 5 \cdot 4 \cdot 3!}{3!} \\ &= 120 \end{aligned} $$

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Although the formula developed for ${}^nP_r$ will have an important application later in this chapter, you do not actually have to use it when solving problems. It is often more convenient to simply draw boxes corresponding to the positions, and to write in each box the number of choices for that position.

Example 13

How many ways can seven friends sit along a park bench with space for only four people?

Solution

7	6	5	4

By the multiplication principle, the total number of arrangements is

$$7 \times 6 \times 5 \times 4 = 840$$

Explanation

We draw four boxes, representing the positions to be filled. In each box we write the number of ways we can fill that position.

Using the TI-Nspire CX non-CAS

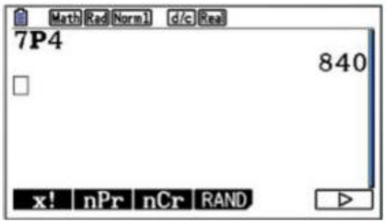
- To evaluate ${}^7P_4$, use menu > **Probability** > **Permutations** as shown.

Note: Alternatively, you can simply type npr(7, 4). The command is not case sensitive.

Using the Casio

To evaluate ${}^7P_4$:

- Press MENU 1 to select **Run-Matrix** mode.
- Go to the **Probability** menu OPTN F6 F3.
- Enter 7, select **nPr** F2, enter 4 and press EXE.



Summary 6B

- $n! = n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot 2 \cdot 1$ and $0! = 1$
- $n! = n \cdot (n-1)!$
- A **permutation** is an ordered arrangement of objects.
- The number of permutations of n objects is $n!$.
- The number of permutations of n objects taken r at a time is given by

$${}^nP_r = \frac{n!}{(n-r)!}$$

Exercise 6B

SF

1 Evaluate $n!$ for $n = 0, 1, 2, \dots, 10$.

Example 7

2 Evaluate each of the following:

a $\dfrac{5!}{4!}$ **b** $\dfrac{10!}{8!}$ **c** $\dfrac{12!}{10!\,2!}$ **d** $\dfrac{100!}{97!\,3!}$

3 Simplify the following expressions:

a $\dfrac{(n+1)!}{n!}$ **b** $\dfrac{(n+2)!}{(n+1)!}$ **c** $\dfrac{n!}{(n-2)!}$ **d** $\dfrac{1}{n!} + \dfrac{1}{(n+1)!}$

4 Evaluate ${}^4P_r$ for $r = 0, 1, 2, 3, 4$.

Example 8

5 Use a tree diagram to determine all the permutations of the letters in the word DOG.

Example 9

6 How many ways can five books on a bookshelf be arranged?

7 How many ways can the letters in the word HYPERBOLA be arranged?

Example 12

8 Write down all the two-letter permutations of the letters in the word FROG.

Example 13

9 How many ways can six students be arranged along a park bench if the bench has:

a six seats **b** five seats **c** four seats?

10 Using the digits 1, 2, 5, 7 and 9 without repetition, how many numbers can you form that have:

a five digits **b** four digits **c** three digits?

11 How many ways can six students be allocated to eight vacant desks?

12 How many ways can three letters be posted in five mailboxes if each mailbox can receive:

a more than one letter **b** at most one letter?

13 Using six differently coloured flags without repetition, how many signals can you make using:

a three flags in a row **b** four flags in a row **c** five flags in a row?

CF

14 You are in possession of four flags, each coloured differently. How many signals can you make using at least two flags arranged in a row?

15 Many Australian car licence plates consist of a sequence of three letters followed by a sequence of three digits.

a How many different car licence plates have letters and numbers arranged this way?

b How many of these have no repeated letters or numbers?

CU

16 Determine all possible values of m and n if $m! \cdot n! = 720$ and $m > n$.

17 Show that $n! = (n^2 - n) \cdot (n-2)!$ for $n \geq 2$.

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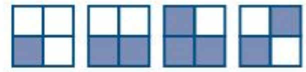
18 **a** The three tiles shown are to be arranged in a row, and can be rotated. How many different ways can this be done?

b The four tiles shown are to be arranged in a row, and can be rotated. How many different ways can this be done?



19 Given six different colours, how many ways can you paint a cube so that all the faces have different colours? Two colourings are considered to be the same when one can be obtained from the other by rotating the cube.

6C Permutations with restrictions

Learning intentions

- To be able to solve problems involving permutations with restrictions.

Suppose we want to know how many three-digit numbers have no repeated digits. The answer is *not* simply ${}^{10}P_3$, the number of permutations of 10 digits taken three at a time. This is because the digit 0 cannot be used in the hundreds place.

- There are 9 choices for the first digit (1, 2, 3, . . . , 9).
- There are 9 choices for the second digit (0 and the eight remaining non-zero digits).
- This leaves 8 choices for the third digit.

100s	10s	units
9	9	8

By the multiplication principle, there are $9 \times 9 \times 8 = 648$ different three-digit numbers.

When considering permutations with restrictions, we deal with the restrictions first.

Example 14

a How many arrangements of the word DARWIN begin and end with a vowel?

b Using the digits 0, 1, 2, 3, 4 and 5 without repetition, how many odd four-digit numbers can you form?

Solution

a We draw six boxes. In each box, we write the number of choices we have for that position. We first consider restrictions. There are two choices of vowel (A or I) for the first letter, leaving only one choice for the last letter.

2					1

This leaves four choices for the second letter, three for the next, and so on.

2	4	3	2	1	1

By the multiplication principle, the number of arrangements is

$$2 \times 4 \times 3 \times 2 \times 1 \times 1 = 48$$

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b We draw four boxes. Again, we first consider restrictions. The last digit must be odd (1, 3 or 5), giving three choices. We cannot use 0 in the first position, so this leaves four choices for that position.

4			3

Once these two digits have been chosen, this leaves four choices and then three choices for the remaining two positions.

4	4	3	3

Thus the number of arrangements is

$$4 \times 4 \times 3 \times 3 = 144$$

Permutations with items grouped together

For some arrangements, we may want certain items to be grouped together. In this case, the trick is to initially treat each group of items as a single object. We then multiply by the numbers of arrangements within each group.

Example 15

a How many arrangements of the word EQUALS are there if the vowels are kept together?

b How many ways can two chemistry, four physics and five biology books be arranged on a shelf if the books of each subject are kept together?

Solution	Explanation
a $4! \times 3!$ $= 144$	We group the three vowels together so that we have four items to arrange: (E, U, A), Q, L, S. They can be arranged in 4! ways. Then the three vowels can be arranged among themselves in 3! ways. We use the multiplication principle.
b $3! \times 2! \times 4! \times 5!$ $= 34\,560$	There are three groups and so they can be arranged in 3! ways. The two chemistry books can be arranged among themselves in 2! ways, the four physics books in 4! ways and the five biology books in 5! ways. We use the multiplication principle.

Summary 6C

- To count permutations that are subject to restrictions, we draw a series of boxes. In each box, we write the number of choices we have for that position. We always consider the restrictions first.
- When items are to be grouped together, we initially treat each group as a single object. We determine the number of arrangements of the groups, and then multiply by the numbers of arrangements within each group.

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Skill-sheet

Exercise 6C

SF

Example 14

1 Using the digits 1, 2, 3, 4 and 5 without repetition, how many five-digit numbers can you form:

a without restriction
b that are odd
c that begin with 5
d that do not begin with 5?

Example 15

2 In how many ways can three girls and two boys be arranged in a row:

a without restriction
b if the two boys sit together
c if the two boys do not sit together
d if girls and boys alternate?

3 How many permutations of the word QUEASY:

a begin with a vowel
b begin and end with a vowel
c keep the vowels together
d keep the vowels and consonants together?

4 How many ways can four boys and four girls be arranged in a row if:

a boys and girls sit in alternate positions
b boys sit together and girls sit together?

5 The digits 0, 1, 2, 3, 4 and 5 can be combined without repetition to form new numbers. In how many ways can you form:

a a six-digit number
b a four-digit number divisible by 5
c a number less than 6000
d an even three-digit number?

6 Two parents and four children are seated in a cinema along six consecutive seats. How many ways can this be done:

a without restriction
b if the two parents sit at either end
c if the children sit together
d if the parents sit together and the children sit together
e if the youngest child must sit between and next to both parents?

CF

7 12321 is a **palindromic number** because it reads the same backwards as forwards. How many palindromic numbers have:

a five digits
b six digits?

8 How many arrangements of the letters in VALUE do not begin and end with a vowel?

9 Using each of the digits 1, 2, 3 and 4 at most once, how many even numbers can you form?

10 How many ways can six girls be arranged in a row so that two of the girls, A and B:

a do not sit together
b have one person between them?

11 How many ways can three girls and three boys be arranged in a row if no two girls sit next to each other?

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6D Permutations of like objects

Learning intentions

- To be able to solve problems involving permutations of like objects.

The name for the Sydney suburb of WOOLLOOMOOLOO has the unusual distinction of having 13 letters in total, of which only four are different. Determining the number of permutations of the letters in this word is not as simple as evaluating 13!. This is because switching like letters does not result in a new permutation.

Our aim is to determine an expression for P, where P is the number of permutations of the letters in the word WOOLLOOMOOLOO. First notice that the word has

1 letter W, 1 letter M, 3 letter Ls, 8 letter Os

Replace the three identical Ls with L_1, L_2 and L_3. These three letters can be arranged in 3! different ways. Therefore, by the multiplication principle, there are now

$P \cdot 3!$ permutations.

Likewise, replace the eight identical Os with $O_1, O_2, \dots, O_8$. These eight letters can be arranged in 8! different ways. Therefore there are now

$P \cdot 3! \cdot 8!$ permutations.

On the other hand, notice that the 13 letters are now distinct, so there are 13! permutations of these letters. Therefore

$$P \cdot 3! \cdot 8! = 13! \quad \text{and so} \quad P = \frac{13!}{3!\,8!}$$

We can easily generalise this procedure to give the following result.

Permutations of like objects

The number of permutations of n objects of which n_1 are alike, n_2 are alike, … and n_r are alike is given by

$$\frac{n!}{n_1!\,n_2!\,\cdots\,n_r!}$$

Example 16

a Determine the number of permutations of the letters in the word RIFFRAFF.

b There are four identical knives, three identical forks and two identical spoons in a drawer. They are taken out of the drawer and lined up in a row. How many ways can this be done?

Solution	Explanation
a $\dfrac{8!}{4!\,2!} = 840$	There are 8 letters of which 4 are alike and 2 are alike.
b $\dfrac{9!}{4!\,3!\,2!} = 1260$	There are 9 items of which 4 are alike, 3 are alike and 2 are alike.

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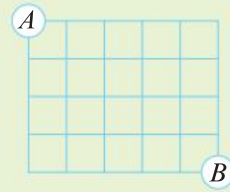
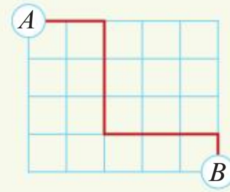
Example 17

The grid shown consists of unit squares. By travelling only right (R) or down (D) along the grid lines, how many paths are there from point A to point B?



Solution

Each path from A to B can be described by a sequence of four Ds and five Rs in some order. For example, the path shown can be described by the sequence RRDDDRRRD.



Since there are 9 letters of which 4 are alike and 5 are alike, the number of permutations of these letters is

$$\frac{9!}{4!\,5!} = 126$$

Summary 6D

- Switching like objects does not give a new arrangement.
- The number of permutations of n objects of which n_1 are alike, n_2 are alike, ... and n_r are alike is given by

$$\frac{n!}{n_1!\,n_2!\,\cdots\,n_r!}$$

Skill-sheet

Exercise 6D

Example 16

1 Ying has four identical 20 cent pieces and three identical 10 cent pieces. How many ways can she arrange these coins in a row?

2 How many ways can the letters in the word MISSISSIPPI be arranged?

3 Determine the number of permutations of the letters in the word WARRNAMBOOL.

4 Using five 9s and three 7s, how many eight-digit numbers can be made?

5 Using three As, four Bs and five Cs, how many sequences of 12 letters can be made?

SF

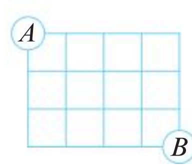
6 How many ways can two red, two black and four blue flags be arranged in a row:

a without restriction
b if the first flag is red
c if the first and last flags are blue
d if every alternate flag is blue
e if the two red flags are adjacent?

CF

Example 17

7 The grid shown consists of unit squares. By travelling only right (R) or down (D) along the grid lines, how many paths are there from point A to point B?



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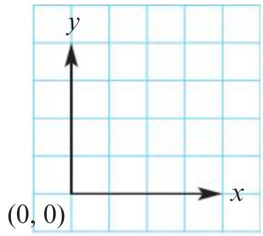
8 The grid shown consists of unit squares. By travelling only along the grid lines, how many paths are there:

a of length 6 from $(0, 0)$ to the point $(2, 4)$

b of length $m + n$ from $(0, 0)$ to the point (m, n), where m and n are natural numbers?



CF

9 Consider a deck of 52 playing cards.

a How many ways can the deck be arranged? Express your answer in the form $a!$.

b If two identical decks are combined, how many ways can the cards be arranged? Express your answer in the form $\frac{a!}{(b!)^c}$.

c If n identical decks are combined, determine an expression for the number of ways that the cards can be arranged.

CU

10 An ant starts at position $(0, 0)$ and walks north, east, south or west, one unit at a time. How many different paths of length 8 units finish at $(0, 0)$?

11 Jessica is about to walk up a flight of 10 stairs. She can take either one or two stairs at a time. How many different ways can she walk up the flight of stairs?

6E Combinations

Learning intentions

- To be able to solve problems involving combinations.

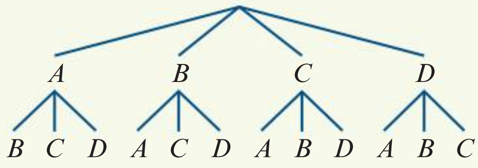
We have seen that a permutation is an ordered arrangement of objects. In contrast, a **combination** is a selection made regardless of order. We use the notation ${}^nP_r$ to denote the number of permutations of n distinct objects taken r at a time. Similarly, we use the notation ${}^nC_r$ to denote the number of combinations of n distinct objects taken r at a time.

Example 18

How many ways can two letters be chosen from the set $\{A, B, C, D\}$?

Solution

The tree diagram below shows the ways that the first and second choices can be made.



This gives 12 arrangements. But there are only six selections, since

$\{A, B\}$ is the same as $\{B, A\}$, $\{A, C\}$ is the same as $\{C, A\}$, $\{A, D\}$ is the same as $\{D, A\}$,

$\{B, C\}$ is the same as $\{C, B\}$, $\{B, D\}$ is the same as $\{D, B\}$ $\{C, D\}$ is the same as $\{D, C\}$

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Suppose we want to count the number of ways that three students can be chosen from a group of seven. Let's label the students with the letters $\{A, B, C, D, E, F, G\}$. One such combination might be BDE. Note that this combination corresponds to 3! permutations:

$$BDE, \quad BED, \quad DBE, \quad DEB, \quad EBD, \quad EDB$$

In fact, each combination of three items corresponds to 3! permutations, and so there are 3! times as many permutations as combinations. Therefore

$$^7P_3 = 3! \times {}^7C_3$$

and so

$$^7C_3 = \frac{^7P_3}{3!}$$

Since we have already established that $^7P_3 = \dfrac{7!}{(7-3)!}$, we obtain

$$^7C_3 = \frac{7!}{3!\,(7-3)!}$$

This argument generalises easily so that we can establish a formula for ^nC_r.

Number of combinations

The number of combinations of n objects taken r at a time is given by the formula

$$^nC_r = \frac{n!}{r!\,(n-r)!}$$

Example 19

a A pizza can have three toppings chosen from nine options. How many different pizzas can be made?

b How many subsets of $\{1, 2, 3, \ldots, 20\}$ have exactly two elements?

Solution

a Three objects are to be chosen from nine options. This can be done in $^9C_3$ ways, and

$$\begin{aligned} ^9C_3 &= \frac{9!}{3!\,(9-3)!} \\ &= \frac{9!}{3!\,6!} \\ &= \frac{9 \cdot 8 \cdot 7 \cdot 6!}{3! \cdot 6!} \\ &= \frac{9 \cdot 8 \cdot 7}{3 \cdot 2} \\ &= 84 \end{aligned}$$

b Two objects are to be chosen from 20 options. This can be done in $^{20}C_2$ ways, and

$$\begin{aligned} ^{20}C_2 &= \frac{20!}{2!\,(20-2)!} \\ &= \frac{20!}{2!\,18!} \\ &= \frac{20 \cdot 19 \cdot 18!}{2! \cdot 18!} \\ &= \frac{20 \cdot 19}{2 \cdot 1} \\ &= 190 \end{aligned}$$

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Example 20

Using your calculator, determine how many ways 10 students can be selected from a class of 20 students.

Using the TI-Nspire CX non-CAS

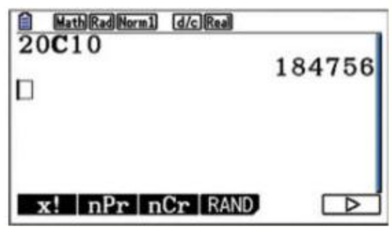
- To evaluate ${}^{20}C_{10}$, use (menu) > **Probability** > **Combinations** as shown.

Note: Alternatively, you can simply type ncr(20, 10). The command is not case sensitive.

Using the Casio

To evaluate ${}^{20}C_{10}$:

- Press (MENU) (1) to select **Run-Matrix** mode.
- Go to the **Probability** menu (OPTN) (F6) (F3).
- Enter 20, select **nCr** (F3), enter 10 and press (EXE).



Example 21

Consider a group of six students. In how many ways can a group of:

a two students be selected

b four students be selected?

Solution

a
$$
\begin{aligned}
{}^{6}C_2 &= \frac{6!}{2!\,(6-2)!} \\
&= \frac{6!}{2!\,4!} \\
&= \frac{6 \cdot 5 \cdot 4!}{2! \cdot 4!} \\
&= \frac{6 \cdot 5}{2 \cdot 1} \\
&= 15
\end{aligned}
$$

b
$$
\begin{aligned}
{}^{6}C_4 &= \frac{6!}{4!\,(6-4)!} \\
&= \frac{6!}{4!\,2!} \\
&= \frac{6 \cdot 5 \cdot 4!}{4! \cdot 2!} \\
&= \frac{6 \cdot 5}{2 \cdot 1} \\
&= 15
\end{aligned}
$$

The fact that parts **a** and **b** of the previous example have the same answer is not a coincidence. Choosing two students out of six is the same as *not choosing* the other four students out of six. Therefore ${}^{6}C_2 = {}^{6}C_4$.

More generally:

$$ {}^{n}C_r = {}^{n}C_{n-r} $$

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Quick calculations

In some instances, you can avoid unnecessary calculations by noting that:

- ${}^nC_0 = 1$, since there is only one way to select no objects from n objects
- ${}^nC_n = 1$, since there is only one way to select n objects from n objects
- ${}^nC_1 = n$, since there are n ways to select one object from n objects
- ${}^nC_{n-1} = n$, since this corresponds to the number of ways of not selecting one object from n objects.

Example 22

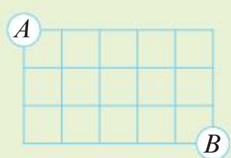
a Six points lie on a circle. How many triangles can you make using these points as the vertices?

b Each of the 20 people at a party shakes hands with every other person. How many handshakes take place?

Solution	Explanation
a ${}^6C_3 = 20$	This is the same as asking how many ways three vertices can be chosen out of six.
b ${}^{20}C_2 = 190$	This is the same as asking how many ways two people can be chosen to shake hands out of 20 people.

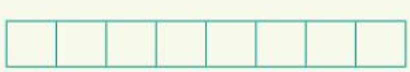
Example 23

The grid shown consists of unit squares. By travelling only right (R) or down (D) along the grid lines, how many paths are there from point A to point B?



Solution

Each path from A to B can be described by a sequence of three Ds and five Rs in some order. Therefore, the number of paths is equal to the number of ways of selecting three of the eight boxes below to be filled with the three Ds. (The rest will be Rs.) This can be done in ${}^8C_3 = 56$ ways.



Alternative notation

We will consistently use the notation ${}^nC_r$ to denote the number of ways of selecting r objects from n objects, regardless of order. However, it is also common to denote this number by $\binom{n}{r}$.

For example:

$$\binom{6}{4} = \frac{6!}{4!\,2!} = 15$$

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Summary 6E

- A **combination** is a selection made regardless of order.
- The number of combinations of n objects taken r at a time is given by

$$^{n}C_r = \frac{n!}{r!\,(n-r)!}$$

Exercise 6E

SF

1 Evaluate $^{5}C_r$ for $r = 0, 1, 2, 3, 4, 5$.

2 Evaluate each of the following without the use of your calculator:

a $^{7}C_1$ **b** $^{6}C_5$ **c** $^{12}C_{10}$ **d** $^{8}C_5$ **e** $^{100}C_{99}$ **f** $^{1000}C_{998}$

3 Simplify each of the following:

a $^{n}C_1$ **b** $^{n}C_2$ **c** $^{n}C_{n-1}$ **d** $^{n+1}C_1$ **e** $^{n+2}C_n$ **f** $^{n+1}C_{n-1}$

Example 19

4 A playlist contains ten of Nandi's favourite songs. How many ways can he:

a arrange three songs in a list **b** select three songs for a list?

5 How many ways can five cards be selected from a deck of 52 playing cards?

6 How many subsets of $\{1, 2, 3, \ldots, 10\}$ contain exactly:

a 1 element **b** 2 elements **c** 8 elements **d** 9 elements?

7 A lottery consists of drawing seven balls out of a barrel of balls numbered from 1 to 45. How many ways can this be done if their order does not matter?

Example 22

8 Eight points lie on a circle. How many triangles can you make using these points as the vertices?

CF

9 **a** In a hockey tournament, each of the 10 teams plays every other team once. How many games take place?

b In another tournament, each team plays every other team once and 120 games take place. How many teams competed?

10 At a party, every person shakes hands with every other person. Altogether there are 105 handshakes. How many people are at the party?

11 Prove that $^{n}C_r = {}^{n}C_{n-r}$.

12 Explain why the number of diagonals in a regular polygon with n sides is $^{n}C_2 - n$.

CU

13 Ten students are divided into two teams of five. Explain why the number of ways of doing this is $\dfrac{^{10}C_5}{2}$.

14 Twelve students are to be divided into two teams of six. In how many ways can this be done? (**Hint:** First complete the previous question.)

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15 Using the formula for ${}^nC_r$, prove that ${}^nC_r = {}^{n-1}C_{r-1} + {}^{n-1}C_r$, where $1 \le r < n$.

16 Consider the 5×5 grid shown.

a How many ways can three dots be chosen?

b How many ways can three dots be chosen so that they lie on a straight line?

c How many ways can three dots be chosen so that they are the vertices of a triangle? (**Hint:** Use parts **a** and **b**.)

6F Combinations with restrictions

Learning intentions

- To be able to solve problems involving combinations with restrictions.

Combinations including specific items

In some problems, we want to determine the number of combinations that include specific items. This reduces both the number of items we have to select and the number of items from which we are selecting.

Example 24

a Grace belongs to a group of eight workers. How many ways can a team of four workers be selected if Grace must be on the team?

b A hand of cards consists of five cards drawn from a deck of 52 playing cards. How many hands contain both the queen and the king of hearts?

Solution	Explanation
a ${}^7C_3 = 35$	Grace must be in the selection. Therefore three more workers are to be selected from the remaining seven workers.
b ${}^{50}C_3 = 19\,600$	The queen and king of hearts must be in the selection. So three more cards are to be selected from the remaining 50 cards.

In some other problems, it can be more efficient to count the selections that we don't want.

Example 25

Four students are to be chosen from a group of eight students for the school tennis team. Two members of the group, Sam and Tess, do not get along and cannot both be on the team. How many ways can the team be selected?

Solution

There are ${}^8C_4$ ways of selecting four students from eight. We then subtract the number of combinations that include both Sam and Tess. If Sam and Tess are on the team, then we can select two more students from the six that remain in ${}^6C_2$ ways. This gives

$${}^8C_4 - {}^6C_2 = 55$$

Combinations from multiple groups

If we are required to make multiple selections from separate groups, then the multiplication principle dictates that we simply multiply the number of ways of performing each task.

Example 26

From seven women and four men in a workplace, how many groups of five can be chosen:

a without restriction

b containing three women and two men

c containing at least one man

d containing at most one man?

Solution

a There are 11 people in total, from which we must select five. This gives

$$^{11}C_5 = 462$$

b There are $^{7}C_3$ ways of selecting three women from seven. There are $^{4}C_2$ ways of selecting two men from four. We then use the multiplication principle to give

$$^{7}C_3 \cdot {}^{4}C_2 = 210$$

c Method 1

If you select at least one man, then you select 1, 2, 3 or 4 men and fill the remaining positions with women. We use the multiplication and addition principles to give

$$^{4}C_1 \cdot {}^{7}C_4 + {}^{4}C_2 \cdot {}^{7}C_3 + {}^{4}C_3 \cdot {}^{7}C_2 + {}^{4}C_4 \cdot {}^{7}C_1 = 441$$

Method 2

It is more efficient to consider all selections of 5 people from 11 and then subtract the number of combinations containing all women. This gives

$$^{11}C_5 - {}^{7}C_5 = 441$$

d If there is at most one man, then either there are no men or there is one man. If there are no men, then there are $^{7}C_5$ ways of selecting all women. If there is one man, then there are $^{4}C_1$ ways of selecting one man and $^{7}C_4$ ways of selecting four women. This gives

$$^{7}C_5 + {}^{4}C_1 \cdot {}^{7}C_4 = 161$$

Permutations and combinations combined

In the following example, we first select the items and then arrange them.

Example 27

a How many arrangements of the letters in the word DUPLICATE can be made that have two vowels and three consonants?

b A president, vice-president, secretary and treasurer are to be chosen from a group containing seven women and six men. How many ways can this be done if exactly two women are chosen?

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Solution	Explanation
a ${}^4C_2 \cdot {}^5C_3 \cdot 5! = 7200$	There are ${}^4C_2$ ways of selecting 2 of 4 vowels and ${}^5C_3$ ways of selecting 3 of 5 consonants. Once chosen, the 5 letters can be arranged in 5! ways.
b ${}^7C_2 \cdot {}^6C_2 \cdot 4! = 7560$	There are ${}^7C_2$ ways of selecting 2 of 7 women and ${}^6C_2$ ways of selecting 2 of 6 men. Once chosen, the 4 people can be arranged into the positions in 4! ways.

Summary 6F

- If a selection must include specific items, then this reduces both the number of items that we have to select and the number of items that we select from.
- If we are required to make multiple selections from separate groups, then we multiply the number of ways of performing each task.
- Some problems will require us to select and then arrange objects.

Skill-sheet

Exercise 6F

SF

Example 24 **1** Jane and Jenny belong to a class of 20 students. How many ways can you select a group of four students from the class if both Jane and Jenny are to be included?

2 How many subsets of $\{1, 2, 3, \ldots, 10\}$ have exactly five elements and contain the number 5?

3 Five cards are dealt from a deck of 52 playing cards. How many hands contain the jack, queen and king of hearts?

CF

Example 25 **4** Six students are to be chosen from a group of 10 students for the school basketball team. Two members of the group, Rachel and Nethra, do not get along and cannot both be on the team. How many ways can the team be selected?

Example 26 **5** From eight girls and five boys, a team of seven is selected for a mixed netball team. How many ways can this be done if:

- **a** there are no restrictions
- **b** there are four girls and three boys on the team
- **c** there must be at least three boys and three girls on the team
- **d** there are at least two boys on the team?

6 There are 10 student leaders at a secondary school. Four are needed for a fundraising committee and three are needed for a social committee. How many ways can the students be selected if they can serve on:

- **a** both committees
- **b** at most one committee?

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7 There are 18 students in a class. Seven are required for a basketball team and eight are required for a netball team. How many ways can the teams be selected if students can play in:

a both teams **b** at most one team?

8 From 10 Labor senators and 10 Liberal senators, a committee of five is formed. How many ways can this be done if:

a there are no restrictions

b there are at least two senators from each political party

c there is at least one Labor senator?

9 Consider the set of numbers $\{1, 2, 3, 4, 5, 6, 7\}$.

a How many subsets have exactly five elements?

b How many five-element subsets contain the numbers 2 and 3?

c How many five-element subsets do not contain both 2 and 3?

10 Four letters are selected from the English alphabet. How many of these selections will contain exactly two vowels?

11 A seven-card hand is dealt from a deck of 52 playing cards. How many distinct hands contain:

a four hearts and three spades **b** exactly two hearts and three spades?

12 A committee of five people is chosen from four doctors, four dentists and three physiotherapists. How many ways can this be done if the committee contains:

a exactly three doctors and one dentist **b** exactly two doctors?

Example 27

13 There are four girls and five boys. Two of each are chosen and then arranged on a bench. How many ways can this be done?

14 A president, vice-president, secretary and treasurer are to be chosen from a group containing six women and five men. How many ways can this be done if exactly two women must be chosen?

15 Using five letters from the word TRAMPOLINE, how many arrangements contain two vowels and three consonants?

16 How many rectangles are there in the grid shown on the right?
Hint: Every rectangle is determined by a choice of two vertical and two horizontal lines.

17 Five cards are dealt from a deck of 52 playing cards. A full house is a hand that contains 3 cards of one rank and 2 cards of another rank (for example, 3 kings and 2 sevens). How many ways can a full house be dealt?

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6G Pascal's triangle (Optional)

Learning intentions

- To construct Pascal's triangle.
- To determine the sum of the entries in each row of this triangle.

The diagram below consists of the binomial coefficients ${}^nC_r$ for $0 \le n \le 5$. They form the first 6 rows of **Pascal's triangle**, named after the seventeenth century French mathematician Blaise Pascal, one of the founders of probability theory.

Interestingly, the triangle was well known to Chinese and Indian mathematicians many centuries earlier.

$n = 0$:	${}^0C_0$	1
$n = 1$:	${}^1C_0$ ${}^1C_1$	1 1
$n = 2$:	${}^2C_0$ ${}^2C_1$ ${}^2C_2$	1 2 1
$n = 3$:	${}^3C_0$ ${}^3C_1$ ${}^3C_2$ ${}^3C_3$	1 3 3 1
$n = 4$:	${}^4C_0$ ${}^4C_1$ ${}^4C_2$ ${}^4C_3$ ${}^4C_4$	1 4 6 4 1
$n = 5$:	${}^5C_0$ ${}^5C_1$ ${}^5C_2$ ${}^5C_3$ ${}^5C_4$ ${}^5C_5$	1 5 10 10 5 1

Pascal's rule

Pascal's triangle has many remarkable properties. Most importantly:

Each entry in Pascal's triangle is the sum of the two entries immediately above.

Pascal's triangle has this property because of the following identity.

Pascal's rule

$${}^nC_r = {}^{n-1}C_{r-1} + {}^{n-1}C_r \qquad \text{where } 1 \le r < n$$

Proof In Question 15 of Exercise 6E, you are asked to prove Pascal's rule using the formula for ${}^nC_r$. However, there is a much nicer argument. The number of subsets of $\{1, 2, \dots, n\}$ containing exactly r elements is ${}^nC_r$. Each of these subsets can be put into one of two groups:

1. those that contain n
2. those that do not contain n.

If the subset contains n, then each of the remaining $r - 1$ elements must be chosen from $\{1, 2, \dots, n - 1\}$. Therefore the first group contains ${}^{n-1}C_{r-1}$ subsets.

If the subset does not contain n, then we still have to choose r elements from $\{1, 2, \dots, n - 1\}$. Therefore the second group contains ${}^{n-1}C_r$ subsets. The two groups together contain all ${}^nC_r$ subsets and so

$${}^nC_r = {}^{n-1}C_{r-1} + {}^{n-1}C_r$$

which establishes Pascal's rule.

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Example 28

Given that ${}^{17}C_2 = 136$ and ${}^{17}C_3 = 680$, evaluate ${}^{18}C_3$.

Solution

$${}^{18}C_3 = {}^{17}C_2 + {}^{17}C_3$$
$$= 136 + 680$$
$$= 816$$

Explanation

We let $n = 18$ and $r = 3$ in Pascal's rule:

$${}^{n}C_r = {}^{n-1}C_{r-1} + {}^{n-1}C_r$$

Example 29

Write down the $n = 6$ row of Pascal's triangle and then write down the value of ${}^{6}C_3$.

Solution

$n = 6$: 1 6 15 **20** 15 6 1

${}^{6}C_3 = 20$

Explanation

Each entry in the $n = 6$ row is the sum of the two entries immediately above.

Note that ${}^{6}C_3$ is the fourth entry in the row, since the first entry corresponds to ${}^{6}C_0$.

Subsets of a set

Suppose your friend says to you: 'I have five books that I no longer need, take any that you want.' How many different selections are possible?

We will look at two solutions to this problem.

Solution 1

You could select none of the books (${}^{5}C_0$ ways), or one out of five (${}^{5}C_1$ ways), or two out of five (${}^{5}C_2$ ways), and so on. This gives the answer

$${}^{5}C_0 + {}^{5}C_1 + {}^{5}C_2 + {}^{5}C_3 + {}^{5}C_4 + {}^{5}C_5 = 32$$

Note that this is simply the sum of the entries in the $n = 5$ row of Pascal's triangle.

Solution 2

For each of the five books we have two options: either accept or reject the book. Using the multiplication principle, we obtain the answer

$$2 \times 2 \times 2 \times 2 \times 2 = 2^5 = 32$$

There are two important conclusions that we can draw from this example.

1. The sum of the entries in row n of Pascal's triangle is 2^n. That is,

$${}^{n}C_0 + {}^{n}C_1 + \cdots + {}^{n}C_{n-1} + {}^{n}C_n = 2^n$$

2. A set of size n has 2^n subsets, including the empty set and the set itself.

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Example 30

a Your friend offers you any of six books that she no longer wants. How many selections are possible assuming that you take at least one book?

b How many subsets of $\{1, 2, 3, \dots, 10\}$ have at least two elements?

Solution	Explanation
a $2^6 - 1 = 63$	There are 2^6 subsets of a set of size 6. We subtract 1 because we discard the empty set of no books.
b $2^{10} - {}^{10}C_1 - {}^{10}C_0$ $= 2^{10} - 10 - 1$ $= 1013$	There are 2^{10} subsets of a set of size 10. There are ${}^{10}C_1$ subsets containing 1 element and ${}^{10}C_0$ subsets containing 0 elements.

Summary 6G

- The values of ${}^nC_r$ can be arranged to give Pascal's triangle.
- Each entry in Pascal's triangle is the sum of the two entries immediately above.
- **Pascal's rule:** ${}^nC_r = {}^{n-1}C_{r-1} + {}^{n-1}C_r$
- The sum of the entries in row n of Pascal's triangle is 2^n. That is,

$$ {}^nC_0 + {}^nC_1 + \cdots + {}^nC_{n-1} + {}^nC_n = 2^n $$

- A set of size n has 2^n subsets, including the empty set and the set itself.

Exercise 6G

SF

Example 28 **1** Evaluate ${}^7C_2$, ${}^6C_2$ and ${}^6C_1$, and verify that the first is the sum of the other two.

Example 29 **2** Write down the $n = 7$ row of Pascal's triangle. Use your answer to write down the values of ${}^7C_2$ and ${}^7C_4$.

3 Write down the $n = 8$ row of Pascal's triangle. Use your answer to write down the values of ${}^8C_4$ and ${}^8C_6$.

Example 30 **4** Your friend offers you any of six different DVDs that he no longer wants. How many different selections are possible?

5 How many subsets does the set $\{A, B, C, D, E\}$ have?

6 How many subsets does the set $\{1, 2, 3, \dots, 10\}$ have?

CF

7 How many subsets of $\{1, 2, 3, 4, 5, 6\}$ have at least one element?

8 How many subsets of $\{1, 2, 3, \dots, 8\}$ have at least two elements?

9 How many subsets of $\{1, 2, 3, \dots, 10\}$ contain the numbers 9 and 10?

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10 You have one 5 cent, one 10 cent, one 20 cent and one 50 cent piece. How many different sums of money can you make assuming that at least one coin is used? CF

11 Let's call a set **selfish** if it contains its size as an element. For example, the set $\{1, 2, 3\}$ is selfish because the set has size 3 and the number 3 belongs to the set. CU

a How many subsets of $\{1, 2, 3, \ldots, 8\}$ are selfish?

b How many subsets of $\{1, 2, 3, \ldots, 8\}$ have the property that both the subset and its complement are selfish?

6H The pigeonhole principle (Optional)

Learning intentions

- To be able to use the pigeonhole principle to solve problems.

The pigeonhole principle is an intuitively obvious counting technique which can be used to prove some remarkably counter-intuitive results. It gets its name from the following simple observation: If $n + 1$ pigeons are placed into n holes, then some hole contains at least two pigeons. Obviously, in most instances we will not be working with pigeons, so we will recast the principle as follows.

Pigeonhole principle

If $n + 1$ or more objects are placed into n holes, then some hole contains at least two objects.

Proof Suppose that each of the n holes contains at most one object. Then the total number of objects is at most n, which is a contradiction.

We are now in a position to prove a remarkable fact: There are at least two people in Australia with the same number of hairs on their head. The explanation is simple. No one has more than 1 million hairs on their head, so let's make 1 million holes labelled with the numbers from 1 to 1 million. We now put each of the 24 million Australians into the hole corresponding to the number of hairs on their head. Clearly, some hole contains at least two people, and all the people in that hole will have the same number of hairs on their head.

Example 31

You have thirteen red, ten blue and eight green socks. How many socks need to be selected at random to ensure that you have a matching pair?

Solution

Label three holes with the colours red, blue and green.

R B G

Selecting just three socks is clearly not sufficient, as you might pick one sock of each colour. Select four socks and place each sock into the hole corresponding to the colour of the sock. As there are four socks and three holes, the pigeonhole principle guarantees that some hole contains at least two socks. This is the required pair.

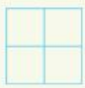
Example 32

a Show that for any five points chosen inside a 2×2 square, at least two of them will be no more than $\sqrt{2}$ units apart.

b Seven football teams play 22 games of football. Show that some pair of teams play each other at least twice.

Solution

a Split the 2×2 square into four unit squares.



Now we have four squares and five points. By the pigeonhole principle, some square contains at least two points. The distance between any two of these points cannot exceed the length of the square's diagonal, $\sqrt{1^2 + 1^2} = \sqrt{2}$.

b There are ${}^7C_2 = 21$ ways that two teams can be chosen to compete from seven. There are 22 games of football, and so some pair of teams play each other at least twice.

The generalised pigeonhole principle

Suppose that 13 pigeons are placed into four holes. By the pigeonhole principle, there is some hole with at least two pigeons. In fact, some hole must contain at least four pigeons. The reason is simple: If each of the four holes contained no more than three pigeons, then there would be no more than 12 pigeons.

This observation generalises as follows.

Generalised pigeonhole principle

If at least $mn + 1$ objects are placed into n holes, then some hole contains at least $m + 1$ objects.

Proof Again, let's suppose that the statement is false. Then each of the n holes contains no more than m objects. However, this means that there are no more than mn objects, which is a contradiction.

Example 33

Sixteen natural numbers are written on a whiteboard. Prove that at least four numbers will leave the same remainder when divided by 5.

Solution

We label five holes with each of the possible remainders on division by 5.

There are 16 numbers to be placed into five holes. Since $16 = 3 \times 5 + 1$, there is some hole with at least four numbers, each of which leaves the same remainder when divided by 5.

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Pigeons in multiple holes

In some instances, objects can be placed into more than one hole.

Example 34

Seven people sit at a round table with 10 chairs. Show that there are three consecutive chairs that are occupied.

Solution

Number the chairs from 1 to 10. There are 10 groups of three consecutive chairs:

$$\{1,2,3\}, \quad \{2,3,4\}, \quad \{3,4,5\}, \quad \{4,5,6\}, \quad \{5,6,7\},$$
$$\{6,7,8\}, \quad \{7,8,9\}, \quad \{8,9,10\}, \quad \{9,10,1\}, \quad \{10,1,2\}$$

Each of the seven people will belong to three of these groups, and so 21 allocations must be made to 10 groups. Since $21 = 2 \times 10 + 1$, the generalised pigeonhole principle guarantees that some group must contain three people.

Summary 6H

- **Pigeonhole principle**
  If $n + 1$ or more objects are placed into n holes, then some hole contains at least two objects.
- **Generalised pigeonhole principle**
  If at least $mn + 1$ objects are placed into n holes, then some hole contains at least $m + 1$ objects.

Exercise 6H

SF

Example 31

1 You have twelve red, eight blue and seven green socks. How many socks need to be selected at random to ensure that you have a matching pair?

2 A sentence contains 27 English words. Show that there are at least two words that begin with the same letter.

3 Show that in any collection of five natural numbers, at least two will leave the same remainder when divided by 4.

4 How many cards need to be dealt from a deck of 52 playing cards to be certain that you will obtain at least two cards of the same:

a colour **b** suit **c** rank?

5 Eleven points on the number line are located somewhere between 0 and 1. Show that there are at least two points no more than 0.1 apart.

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Example 32 **6** An equilateral triangle has side length 2 units. Choose any five points inside the triangle. Prove that there are at least two points that are no more than 1 unit apart.

SF

7 Thirteen points are located inside a rectangle of length 6 and width 8. Show that there are at least two points that are no more than $2\sqrt{2}$ units apart.

CF

8 The **digital sum** of a natural number is defined to be the sum of its digits. For example, the digital sum of 123 is $1 + 2 + 3 = 6$.

a Nineteen two-digit numbers are selected. Prove that at least two of them have the same digital sum.

b Suppose that 82 three-digit numbers are selected. Prove that at least four of them have the same digital sum.

Example 33 **9** Whenever Eva writes down 13 integers, she notices that at least four of them leave the same remainder when divided by 4. Explain why this is always the case.

10 Twenty-nine games of football are played among eight teams. Prove that there is some pair of teams who play each other more than once.

11 A teacher instructs each member of her class to write down a different whole number between 1 and 49. She says that there will be at least one pair of students such that the sum of their two numbers is 50. How many students must be in her class?

Example 34 **12** There are 10 students seated at a round table with 14 chairs. Show that there are three consecutive chairs that are occupied.

13 There are four points on a circle. Show that three of these points lie on a half-circle.
Hint: Pick any one of the four points and draw a diameter through that point.

14 There are 35 players on a football team and each player has a different number chosen from 1 to 99. Prove that there are at least four pairs of players whose numbers have the same sum.

15 Seven boys and five girls sit evenly spaced at a round table. Prove that some pair of boys are sitting opposite each other.

CU

16 There are n guests at a party and some of these guests shake hands when they meet. Use the pigeonhole principle to show that there is a pair of guests who shake hands with the same number of people.
Hint: Place the n guests into holes labelled from 0 to $n - 1$, corresponding to the number of hands that they shake. Why must either the first or the last hole be empty?

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6I The inclusion–exclusion principle

Learning intentions

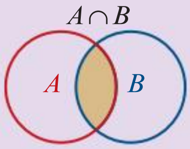
- To be able to use the inclusion-exclusion principle to solve problems.

Basic set theory

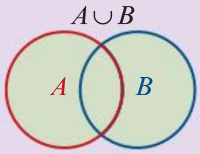
A set is any collection of objects where order is not important. The set with no elements is called the **empty set** and is denoted by $\varnothing$. We say that set B is a **subset** of set A if each element of B is also in A. In this case, we can write $B \subseteq A$. Note that $\varnothing \subseteq A$ and $A \subseteq A$. If A is a finite set, then the number of elements in A will be denoted by $|A|$.

Given any two sets A and B we define two important sets:

1 The **intersection** of sets A and B is denoted by $A \cap B$ and consists of elements belonging to A *and* B.



2 The **union** of sets A and B is denoted by $A \cup B$ and consists of elements belonging to A *or* B.



Note: It is important to realise that $A \cup B$ includes elements belonging to A and B.

Example 35

Consider the three sets of numbers $A = \{2, 3\}$, $B = \{1, 2, 3, 4\}$ and $C = \{3, 4, 5\}$.

a Determine $B \cap C$.
b Determine $A \cup C$.
c Determine $A \cap B \cap C$.
d Determine $A \cup B \cup C$.
e Determine $|A|$.
f List all the subsets of C.

Solution

a $B \cap C = \{3, 4\}$
b $A \cup C = \{2, 3, 4, 5\}$
c $A \cap B \cap C = \{3\}$
d $A \cup B \cup C = \{1, 2, 3, 4, 5\}$
e $|A| = 2$
f $\varnothing$, $\{3\}$, $\{4\}$, $\{5\}$, $\{3, 4\}$, $\{3, 5\}$, $\{4, 5\}$, $\{3, 4, 5\}$

Earlier in the chapter we encountered the addition principle. This principle can be concisely expressed using set notation.

Addition principle

If A and B are two finite sets of objects such that $A \cap B = \varnothing$, then

$$|A \cup B| = |A| + |B|$$

Our aim is to extend this rule for instances where $A \cap B \neq \varnothing$.

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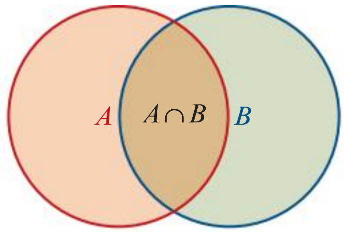
Two sets

To count the number of elements in the set $A \cup B$, we first add (include) $|A|$ and $|B|$. However, this counts the elements in $A \cap B$ twice, and so we subtract (exclude) $|A \cap B|$.



Inclusion–exclusion principle for two sets

If A and B are two finite sets of objects, then

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Example 36

Each of the 25 students in a Year 11 class studies Physics or Chemistry. Of these students, 15 study Physics and 18 study Chemistry. How many students study both subjects?

Solution

$$\begin{aligned} |P \cup C| &= |P| + |C| - |P \cap C| \\ 25 &= 15 + 18 - |P \cap C| \\ 25 &= 33 - |P \cap C| \\ \therefore \quad |P \cap C| &= 8 \end{aligned}$$

Explanation

Let P and C be the sets of students who study Physics and Chemistry respectively.

Since each student studies Physics or Chemistry, we know that $|P \cup C| = 25$.

Example 37

A bag contains 100 balls labelled with the numbers from 1 to 100. How many ways can a ball be chosen that is a multiple of 2 or 5?

Solution

$$\begin{aligned} |A \cup B| &= |A| + |B| - |A \cap B| \\ &= 50 + 20 - 10 \\ &= 60 \end{aligned}$$

Explanation

Within the set of numbers $\{1, 2, 3, \ldots, 100\}$, let A be the set of multiples of 2 and let B be the set of multiples of 5.

Then $A \cap B$ consists of numbers that are multiples of both 2 and 5, that is, multiples of 10.

Therefore $|A| = 50$, $|B| = 20$ and $|A \cap B| = 10$. We then use the inclusion–exclusion principle.

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Example 38

A hand of five cards is dealt from a deck of 52 cards. How many hands contain exactly:

a two clubs
b three spades
c two clubs and three spades
d two clubs or three spades?

Solution

a $^{13}C_2 \cdot {}^{39}C_3 = 712\,842$

Explanation

There are $^{13}C_2$ ways of choosing 2 clubs from 13 and $^{39}C_3$ ways of choosing 3 more cards from the 39 non-clubs.

b $^{13}C_3 \cdot {}^{39}C_2 = 211\,926$

There are $^{13}C_3$ ways of choosing 3 spades from 13 and $^{39}C_2$ ways of choosing 2 more cards from the 39 non-spades.

c $^{13}C_2 \cdot {}^{13}C_3 = 22\,308$

There are $^{13}C_2$ ways of choosing 2 clubs from 13 and $^{13}C_3$ ways of choosing 3 spades from 13.

d $|A \cup B|$

$= |A| + |B| - |A \cap B|$

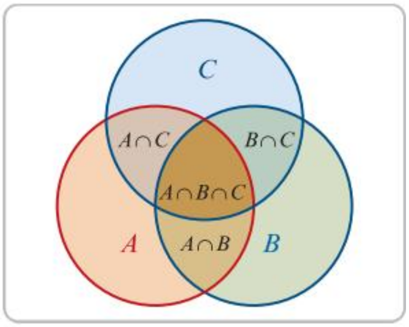
$= 712\,842 + 211\,926 - 22\,308$

$= 902\,460$

We let A be the set of all hands with 2 clubs and let B be the set of all hands with 3 spades. Then $A \cap B$ is the set of all hands with 2 clubs and 3 spades. We use the inclusion–exclusion principle to determine $|A \cup B|$.

Three sets

For three sets A, B and C, the formula for $|A \cup B \cup C|$ is slightly harder to establish.



We first add $|A|$, $|B|$ and $|C|$. However, we have counted the elements in $A \cap B$, $A \cap C$ and $B \cap C$ twice, and the elements in $A \cap B \cap C$ three times.

Therefore we subtract $|A \cap B|$, $|A \cap C|$ and $|B \cap C|$ to compensate. But then the elements in $A \cap B \cap C$ will have been excluded once too often, and so we add $|A \cap B \cap C|$.

Inclusion–exclusion principle for three sets

If A, B and C are three finite sets of objects, then

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

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Example 39

How many integers from 1 to 140 inclusive are not divisible by 2, 5 or 7?

Solution

Let A, B and C be the sets of all integers from 1 to 140 that are divisible by 2, 5 and 7 respectively. We then have

A	multiples of 2	$\lvert A\rvert = 140 \div 2 = 70$
B	multiples of 5	$\lvert B\rvert = 140 \div 5 = 28$
C	multiples of 7	$\lvert C\rvert = 140 \div 7 = 20$
$A \cap B$	multiples of 10	$\lvert A \cap B\rvert = 140 \div 10 = 14$
$A \cap C$	multiples of 14	$\lvert A \cap C\rvert = 140 \div 14 = 10$
$B \cap C$	multiples of 35	$\lvert B \cap C\rvert = 140 \div 35 = 4$
$A \cap B \cap C$	multiples of 70	$\lvert A \cap B \cap C\rvert = 140 \div 70 = 2$

We use the inclusion–exclusion principle to give

$$\begin{aligned}|A \cup B \cup C| &= |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| \\ &= 70 + 28 + 20 - 14 - 10 - 4 + 2 \\ &= 92\end{aligned}$$

Therefore the number of integers not divisible by 2, 5 or 7 is $140 - 92 = 48$.

Summary 6I

- The inclusion–exclusion principle extends the addition principle to instances where the two sets have objects in common.
- The principle works by ensuring that objects belonging to multiple sets are not counted more than once.
- The inclusion–exclusion principles for two sets and three sets:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

Exercise 6I

Example 35

1 Consider the three sets of numbers $A = \{4, 5, 6\}$, $B = \{1, 2, 3, 4, 5\}$ and $C = \{1, 3, 4, 6\}$.

a Determine $B \cap C$.

b Determine $A \cup C$.

c Determine $A \cap B \cap C$.

d Determine $A \cup B \cup C$.

e Determine $|A|$.

f List all the subsets of A.

SF

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Example 36

2 In an athletics team, each athlete competes in track or field events. There are 25 athletes who compete in track events, 23 who compete in field events and 12 who compete in both track and field events. How many athletes are in the team?

SF

3 Fifty patients at a medical clinic are being treated for a disease using two types of medication, A and B. There are 25 patients using medication A and 29 patients using medication B. How many patients are using both types of medication?

Example 37

4 How many integers from 1 to 630 inclusive are multiples of 7 or 9?

5 Consider the integers from 1 to 96 inclusive. How many of these are:

a divisible by 2 or 3

b not divisible by 2 or 3?

Example 38

6 How many five-letter arrangements of the word ALIEN:

CF

a begin with a vowel

b end with a vowel

c begin and end with a vowel

d begin or end with a vowel?

7 **a** How many of the integers from 1 to 100 inclusive are perfect squares or perfect cubes?

b How many of the integers from 1 to 1000 inclusive are perfect squares or perfect cubes?

Example 39

8 How many of the integers from 1 to 120 inclusive are multiples of 2, 3 or 5?

9 How many of the integers from 1 to 220 inclusive are not divisible by 2, 5 or 11?

10 There are 98 Year 11 students at a secondary school and each of them must study at least one of Biology, Physics or Chemistry. There are 36 students who study Biology, 42 who study Physics and 40 who study Chemistry. Moreover, 9 study Biology and Physics, 8 study Biology and Chemistry and 7 study Physics and Chemistry. How many students study all three subjects?

CU

11 A group of six students is selected from four students in Year 10, five in Year 11 and four in Year 12. How many selections have exactly:

CF

a two Year 10 students

b two Year 11 students

c two Year 10 and two Year 11 students

d two Year 10 or two Year 11 students?

12 A hand of five cards is dealt from a deck of 52 cards. How many hands contain exactly one heart or exactly two diamonds?

13 Determine the sum of all the integers from 1 to 100 inclusive that are divisible by 2 or 3.

CU

Hint: Use the formula $1 + 2 + 3 + \cdots + n = \dfrac{n(n+1)}{2}$.

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CU

14 There are seventy Year 11 students at a school and each of them must study at least one of three languages. Thirty are studying French, forty-five are studying Chinese, thirty are studying German and fifteen are studying all three languages. How many students are studying exactly two languages?

6J Applications to probability

Learning intentions

- To be able to apply counting principles to probability.

In this section, we apply our knowledge of permutations and combinations to solving probability problems where there are equally likely outcomes.

Probability is studied in Mathematical Methods Units 1 & 2. For this section, we recall some concepts from your study of probability in previous years.

- The **sample space** for a random experiment is the set of all possible outcomes. For example, the sample space for rolling a die is the set $\{1, 2, 3, 4, 5, 6\}$.
- An **event** is a subset of the sample space. For example, the event of interest when rolling a die may be 'getting an even number', which is described by the set $\{2, 4, 6\}$.

When the sample space is finite, the **probability** of an event is equal to the sum of the probabilities of the outcomes in that event.

For example, when rolling a die, there are six possible outcomes that are all equally likely, so we assign a probability of $\frac{1}{6}$ to each outcome. If A is the event 'getting an even number', then $A = \{2, 4, 6\}$ and so

$$\begin{aligned} P(A) &= P(2) + P(4) + P(6) \\ &= \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2} \end{aligned}$$

Since the outcomes are equally likely, we can calculate this more easily as

$$P(A) = \frac{\text{number of outcomes in } A}{\text{total number of outcomes}} = \frac{3}{6} = \frac{1}{2}$$

More generally:

Equally likely outcomes

If the outcomes of a random experiment are all equally likely, then the probability of an event A is given by

$$P(A) = \frac{\text{number of outcomes in } A}{\text{total number of outcomes}}$$

Establishing the number of outcomes in the event and the total number of outcomes is often achieved by using permutations and combinations.

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Example 40

Four-letter 'words' are to be made by arranging letters of the word SPECIAL. What is the probability that the 'word' will start with a vowel?

Solution

There are 7 letters to be arranged in groups of 4. So the total number of possible arrangements is

$$7 \times 6 \times 5 \times 4 = 840$$

Now consider words which start with a vowel. Since there are three vowels, we have 3 choices for the first letter. Having done this, we have six letters remaining which are to be placed in the three remaining positions, and this can be done in $6 \times 5 \times 4 = 120$ ways.

Thus the number of arrangements which start with a vowel is

$$3 \times 6 \times 5 \times 4 = 360$$

Hence, the probability of the word starting with a vowel is

$$\frac{\text{number of outcomes in the event}}{\text{total number of outcomes}} = \frac{360}{840} = \frac{3}{7}$$

Example 41

Three students are to be chosen to represent the class in a debate. If the class consists of six boys and four girls, what is the probability that the team will contain:

a exactly one girl

b at least two girls?

Solution

Since there is a total of 10 students, the number of possible teams is ${}^{10}C_3 = 120$.

a One girl can be chosen for the team in ${}^{4}C_1 = 4$ different ways. Having placed one girl, the other two places must be filled by boys, and this can be done in ${}^{6}C_2 = 15$ different ways. Thus the total number of teams containing one girl and two boys is $4 \times 15 = 60$, and the probability that the team contains exactly one girl is $\frac{60}{120} = \frac{1}{2}$.

b If the team is to contain at least two girls, then it may contain two *or* three girls. The number of teams containing:

- exactly two girls is ${}^{6}C_1 \times {}^{4}C_2 = 36$
- exactly three girls is ${}^{6}C_0 \times {}^{4}C_3 = 4$

Thus the total number of teams containing at least two girls is 40, and the probability of this is $\frac{40}{120} = \frac{1}{3}$.

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Summary 6J

Using our knowledge of permutations and combinations, we can calculate probabilities for sample spaces with equally like outcomes:

- First determine the total number of possible outcomes.
- Then determine the number of outcomes in the event of interest.
- The required probability is equal to

$$\frac{\text{number of outcomes in the event}}{\text{total number of outcomes}}$$

Skill-sheet

Exercise 6J

SF

1 A four-digit number (with no repetitions) is to be formed from the set of digits $\{1, 2, 3, 4, 5, 6\}$. Determine the probability that the number:

a is even **b** is odd **c** is greater than 4000.

2 Eight people, labelled A, B, C, D, E, F, G and H, are arranged randomly in a line. What is the probability that:

a A and B are next to each other **b** A and B are not next to each other?

Example 40

3 Three-letter 'words' are to be made by arranging the letters of the word COMPUTER. What is the probability that the 'word' will start with a vowel?

4 Three letters are chosen at random from the word HEART and arranged in a row. Determine the probability that:

a the letter H is first **b** the letter H is chosen **c** both vowels are chosen.

5 A class of 30 children contains 18 girls and 12 boys. Four complimentary theatre tickets are distributed at random to the children in the class. What is the probability that:

a all four tickets go to girls **b** two boys and two girls receive tickets?

6 Three men and three women are to be randomly seated in a row. Determine the probability that both the end places will be filled by women.

Example 41

CF

7 A netball team of seven players is to be chosen randomly from six men and seven women. Determine the probability that the selected team contains more men than women.

8 Bill is making a sandwich. He may choose any combination of the following: lettuce, tomato, carrot, cheese, cucumber, beetroot, onion, ham. (Bill's sandwich must contain at least one of these ingredients.) Determine the probability that:

a the sandwich contains ham

b the sandwich contains three ingredients

c the sandwich contains at least three ingredients.

CF

9 A bag contains five white, six red and seven blue balls. If three balls are selected at random, without replacement, determine the probability they are:

a all red **b** all different colours.

10 Susie chooses four pieces of bubble gum from a jar containing five red, two blue and three green pieces of bubble gum. Calculate the probability that Susie selects:

a no green bubble gum

b at least one green bubble gum

c at least one green bubble gum and at least one red bubble gum.

11 A five-digit number is formed using the digits 1, 2, 3, 4 and 5 without repetition. Determine the probability that the number:

a is even **b** is odd **c** is greater than 40 000

d does not have two odd digits next to each other

e does not have the two even digits next to each other.

12 A four-digit number (with no repetitions) is to be formed from the set of digits $\{0, 1, 2, 3, 4, 5, 6, 7\}$. Determine the probability that the number:

a is odd **b** is even **c** is less than 4000.

13 Three numbers are randomly chosen without replacement from the set $\{1, 2, 3, 4, 5, 6\}$. Determine the probability that their sum is greater than 8.

14 A team of four people is chosen at random from a group of eight people. Determine the probability that the team contains:

a the youngest person in the group

b the youngest person in the group and the oldest person in the group.

15 A hand of five cards is dealt from a normal pack of 52 cards. Determine the probability that the hand will contain:

a no aces **b** at least one ace **c** the ace of spades.

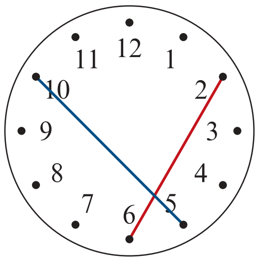
16 A committee of three is to be chosen from a group of four men and five women. Determine the probability that the committee contains:

a all women **b** at least one woman.

CU

17 Three regular dice are rolled and the three outcomes are recorded. Determine the probability that one of the three numbers is the sum of the other two numbers.

18 On a clock face, there are 12 points evenly spaced around a circle. Alice randomly chooses two of these points and Bob does the same. Alice then draws a line segment connecting her two points and Bob does the same. Determine the probability that their line segments do not meet.



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Review

Chapter summary

- The addition and multiplication principles provide efficient methods for counting the number of ways of performing multiple tasks.
- The number of **permutations** (or arrangements) of n objects taken r at a time is given by

$$^nP_r = \frac{n!}{(n-r)!}$$

- The number of **combinations** (or selections) of n objects taken r at a time is given by

$$^nC_r = \frac{n!}{r!\,(n-r)!}$$

- When permutations or combinations involve restrictions, we deal with them first.
- The values of ^nC_r can be arranged to give **Pascal's triangle**, where each entry is the sum of the two entries immediately above.
- The sum of the entries in row n of Pascal's triangle is 2^n. That is,

$$^nC_0 + {}^nC_1 + \cdots + {}^nC_{n-1} + {}^nC_n = 2^n$$

- A set of size n has 2^n subsets.
- The **pigeonhole principle** is used to show that some pair or group of objects have the same property.
- The **inclusion–exclusion principle** allows us to count the number of elements in a union of sets:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

- If the outcomes of a random experiment are all equally likely, then the probability of an event A is given by

$$P(A) = \frac{\text{number of outcomes in } A}{\text{total number of outcomes}}$$

Skills checklist

Download this checklist from the Interactive Textbook, then print it and fill it out to check your skills.

6A **1** I can use the multiplication and addition principles. ☐

See Example 2, Example 4, Example 5, Question 1 and Question 2

6B **2** I can count permutations. ☐

See Example 9, Example 12, Example 13, Question 6, Question 8 and Question 9

6C **3** I can count permutations with restrictions. ☐

See Example 14, Example 15, Question 1 and Question 2

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6D	4	I can count permutations of like objects. See Example 16 and Question 2	☐
6E	5	I can count combinations. See Example 19, Example 21, Question 7 and Question 8	☐
6F	6	I can count combinations with restrictions. See Example 24, Example 25, Example 26, Question 1, Question 4 and Question 5	☐
6G	7	I can use Pascal's rule. (Optional) See Example 29 and Question 3	☐
6G	8	I can count subsets of a set. See Example 30 and Question 5	☐
6H	9	I can use the pigeonhole principle. (Optional) See Example 31, Example 32, Question 1 and Question 3	☐
6I	10	I can use the inclusion-exclusion principle. See Example 37, Example 39, Question 4 and Question 9	☐
6J	11	I can apply counting principles to probability. See Example 40, Example 41, Questions 1, 2, 3 and 7	☐

Short-response questions

Technology-free short-response questions

SF

1 Evaluate:

a $^{6}C_3$ **b** $^{20}C_2$ **c** $^{300}C_1$ **d** $^{100}C_{98}$

2 Determine the value of n if $^{n}C_2 = 55$.

3 How many three-digit numbers can be formed using the digits 1, 2 and 3 if the digits:

a can be repeated

b cannot be repeated?

4 How many ways can six students be arranged on a bench seat with space for three?

5 How many ways can three students be allocated to five vacant desks?

6 There are four Year 11 and three Year 12 students in a school debating club. How many ways can a team of four be selected if two are chosen from each year level?

7 There are three boys and four girls in a group. How many ways can three children be selected if at least one of them is a boy?

8 On a ship's mast are two identical red and three identical black flags that can be arranged to send messages to nearby ships. How many different arrangements using all five flags are possible? SF

9 **Pigeonhole principle** There are 53 English words written on a page. How many are guaranteed to share the same first letter?

10 Each of the twenty students in a class plays netball or basketball. Twelve play basketball and four play both sports. How many students play netball?

11 Six people are to be seated in a row. Calculate the number of ways this can be done so that two particular people, A and B, always have exactly one person between them. CF

12 Two parents and their three children are randomly arranged in a row. Determine the probability that the parents are at the start and end of the row.

13 Three numbers are randomly chosen without replacement from the set $\{1, 2, 3, 4, 5, 6\}$. Determine the probability that the selection includes the number 6.

14 **a** Three letters are randomly placed in three letterboxes. Determine the probability that each letterbox contains exactly one letter. CU

b Four letters are randomly placed in three letterboxes. Determine the probability that each letterbox contains at least one letter.

Technology-active short response questions

15 A six-digit number is formed using the digits 1, 2, 3, 4, 5 and 6 without repetition. How many ways can this be done if: CF

- **a** the first digit is 5
- **b** the first digit is even
- **c** even and odd digits alternate
- **d** the even digits are kept together?

16 Three letters from the word AUNTIE are arranged in a row. How many ways can this be done if:

- **a** the first letter is E
- **b** the first letter is a vowel
- **c** the letter E is used?

17 A student leadership team consists of four boys and six girls. A group of four students is required to organise a social function. How many ways can the group be selected:

- **a** without restriction
- **b** if the school captain is included
- **c** if there are two boys
- **d** if there is at least one boy?

18 Consider the eight letters N, N, J, J, T, T, T, T. How many ways can all eight letters be arranged if:

- **a** there is no restriction
- **b** the first and last letters are both N
- **c** the two Js are adjacent
- **d** no two Ts are adjacent?

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CF

19 In how many ways can a group of four people be chosen from five married couples if:

a there is no restriction

b any two married couples are chosen

c the selected group cannot contain a married couple?

20 A pizza restaurant offers the following toppings: onion, capsicum, mushroom, olives, ham and pineapple.

a How many different kinds of pizza can be ordered with:

i three different toppings

ii three different toppings including ham

iii any number of toppings (between none and all six)?

b Another pizza restaurant boasts that they can make more than 200 varieties of pizza. What is the smallest number of toppings that they could use?

21 The name David Smith has initials DS.

a How many different two-letter initials are possible?

b How many different two-letter initials contain at least one vowel?

c **Pigeonhole principle** Given 50 000 people, how many of them can be guaranteed to share the same two-letter initials?

22 Consider the integers from 1 to 96 inclusive. Let sets A and B consist of those integers that are multiples of 6 and 8 respectively.

a What is the lowest common multiple of 6 and 8?

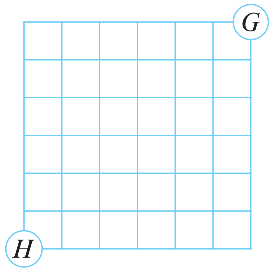
b How many integers belong to $A \cap B$?

c How many integers from 1 to 96 are divisible by 6 or 8?

d An integer from 1 to 96 is chosen at random. What is the probability that it is not divisible by 6 or 8?

CU

23 Every morning, Milly walks from her home $H(0, 0)$ to the gym $G(6, 6)$ along city streets that are laid out in a square grid as shown. She always takes a path of shortest distance.



a How many paths are there from H to G?

b Show that there is some path that she takes at least twice in the course of three years.

c On her way to the gym, she often purchases a coffee at a cafe located at point $C(2, 2)$. How many paths are there from:

i H to C **ii** C to G **iii** H to C to G?

d A new cafe opens up at point $B(4, 4)$. How many paths can Milly take, assuming that she buys coffee at either cafe?

Hint: You will need to use the inclusion–exclusion principle here.

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CU

24 Danica has 10 dice to distribute into three empty boxes. For each of the 10 dice in turn, she randomly chooses one of the three boxes in which to place it. What is the probability that, after she has finished, there is one box containing four dice and there are two boxes containing three dice?

25 An ordinary pack of 52 cards is thoroughly shuffled. The cards are then dealt face up, one at a time, until an ace appears. What is the probability that the first ace appears at:

a the fifth card **b** the kth card **c** the kth card or sooner?

26 A box contains 400 balls, each of which is blue, red, green, yellow or orange. The ratio of blue to red to green balls is 1 : 4 : 2. The ratio of green to yellow to orange balls is 1 : 3 : 6. What is the smallest number of balls that must be drawn to ensure that at least 50 balls of one colour are selected?

27 **a** How many three-digit numbers have digits that are in strictly decreasing order? An example is 752.

b How many four-digit numbers have digits that are in strictly decreasing order? An example is 8750.

c How many four-digit numbers have digits that are in strictly increasing order? An example is 1567.

d Consider the set of ten digits $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. How many ways can you form a ten-digit number whose even digits are in strictly decreasing order, and whose odd digits are in strictly decreasing order? An example is 9867453201. Notice that the odd digits (shown in red) are in decreasing order, as are the even digits (shown in black).

e Consider the same set of ten digits. How many ways can you form a ten-digit number whose odd digits are in strictly decreasing order?

Multiple-choice questions

Technology-free multiple-choice questions

1 Bao plans to study six subjects in Year 12. He has already chosen three subjects and for the remaining three he plans to choose one of four languages, one of three mathematics subjects and one of four science subjects. How many ways can he select his remaining subjects?

A 6 **B** 11 **C** 48 **D** 165

2 There are three flights directly from Melbourne to Brisbane. There are also two flights from Melbourne to Sydney and then four choices of connecting flight from Sydney to Brisbane. How many different paths are there from Melbourne to Brisbane?

A 9 **B** 11 **C** 18 **D** 20

3 In how many ways can 10 people be arranged in a queue at the bank?

A $10!$ **B** 10^{10} **C** 2^{10} **D** ${}^{10}C_2$

4 How many three-digit numbers can be formed using the digits 1, 2, 3, 4, 5 and 6 at most once?

A ${}^{6}C_3$ **B** $3!$ **C** $6!$ **D** $6 \times 5 \times 4$

5 How many permutations of the word UTOPIA begin and end with a vowel?

A 90 **B** 288 **C** 384 **D** 720

6 How many ways can four identical red flags and three identical blue flags be arranged in a row?

A 4×3 **B** $\dfrac{7!}{4! \times 3!}$ **C** $7! \times 3! \times 4!$ **D** $4! \times 3!$

7 How many ways can three DVDs be chosen from a collection of nine different DVDs?

A $3!$ **B** $9 \times 8 \times 7$ **C** ${}^{9}C_3$ **D** $\dfrac{9!}{3!}$

8 The number of subsets of $\{A, B, C, D, E, F\}$ with at least one element is

A ${}^{6}C_2$ **B** ${}^{6}C_2 - 1$ **C** $2^5 - 1$ **D** $2^6 - 1$

9 A class consists of nine girls and eight boys. How many ways can a group of two boys and two girls be chosen?

A $\dfrac{17!}{2!\,2!}$ **B** ${}^{17}C_4$ **C** ${}^{9}C_2 \cdot {}^{8}C_2$ **D** $\dfrac{17!}{9!\,8!}$

10 **Pigeonhole principle** There are six blue balls and five red balls in a bag. How many balls need to be selected at random before you are certain that three will have the same colour?

A 3 **B** 4 **C** 5 **D** 7

11 Each of the 30 students in a class studies French, German or Chinese. Of these students, 15 study French, 17 study German and 15 study Chinese. There are 15 students that study more than one subject. How many students study all three subjects?

A 2 **B** 3 **C** 4 **D** 5

12 If the letters of the word HEADS are arranged in random order, then the probability that the letters are in alphabetical order is

A $\dfrac{1}{120}$ **B** $\dfrac{1}{60}$ **C** $\dfrac{1}{24}$ **D** $\dfrac{1}{5}$

13 A box contains twelve red balls and four green balls. A ball is selected at random from the box and not replaced and then a second ball is drawn. The probability that the two balls are both green is equal to

A $\dfrac{1}{16}$ **B** $\dfrac{3}{64}$ **C** $\dfrac{1}{8}$ **D** $\dfrac{1}{20}$

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Review

Technology-active multiple-choice questions

14 The number of ways that ten people can be arranged along a bench with space for five people is

A 30240 **B** 30420 **C** 40320 **D** 42300

15 An even six-digit number is formed using the digits 1, 2, 3, 4, 5 and 6 without repetition. The number of ways that this can be done is

A 120 **B** 240 **C** 360 **D** 540

16 Tom has a collection of vintage records. He can choose two of these in 210 ways. The number of records in his collection is

A 18 **B** 19 **C** 20 **D** 21

17 Sandra is in possession of a number of identical red tokens, and the same number of identical blue tokens. By arranging these tokens in a line she can form 924 different patterns. The total number of tokens in her possession is

A 6 **B** 8 **C** 10 **D** 12

18 Two boys and three girls are chosen from a group of six boys and seven girls. The number of ways this can be done is

A 50 **B** 100 **C** 525 **D** 1050

19 More than one half of the numbers from the set $\{1, 2, 3, \ldots, 13\}$ are chosen. The number of ways this can be done is

A 512 **B** 1024 **C** 2048 **D** 4096

20 Alice and Ben belong to a group of eight students. The group can be arranged in a line so that Alice or Ben occupy the first or last position. The number of ways this can be done is

A 21870 **B** 12870 **C** 17820 **D** 18720

21 Five letters from the word ALGORITHMS are randomly chosen. The probability that the selection contains no vowels is

A $\frac{1}{9}$ **B** $\frac{1}{12}$ **C** $\frac{1}{15}$ **D** $\frac{1}{16}$

22 If ten people are arranged randomly in a line, then the probability that the tallest two are standing next to each other is

A $\frac{1}{2}$ **B** $\frac{1}{3}$ **C** $\frac{1}{4}$ **D** $\frac{1}{5}$

23 If five people are randomly chosen from amongst five girls and five boys, the probability that at least one girl and one boy are chosen is

A $\frac{124}{125}$ **B** $\frac{125}{126}$ **C** $\frac{24}{25}$ **D** $\frac{25}{26}$

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7

Number and proof

Chapter contents

A **mathematical proof** is an argument that demonstrates the absolute truth of a statement.

It is certainty that makes mathematics different from other sciences. In science, a theory is never proved true. Instead, one aims to prove that a theory is not true. And if such evidence is hard to come by, then this increases the likelihood that a theory is correct, but never provides a guarantee. The possibility of absolute certainty is reserved for mathematics alone.

When writing a proof you should always aim for three things:

- correctness
- clarity
- simplicity.

Perhaps the following proof of Pythagoras' theorem exemplifies these three aims.

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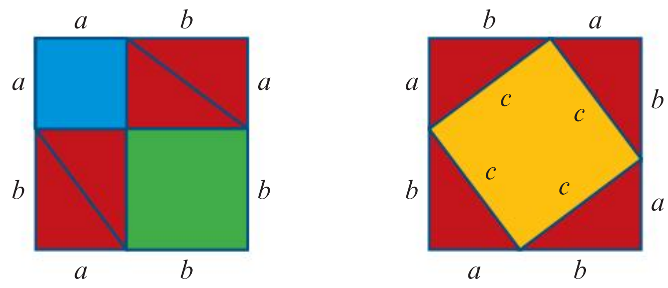
Pythagoras' theorem

Take any triangle with side lengths a, b and c. If the angle between a and b is 90°, then

$$a^2 + b^2 = c^2$$

Proof Consider the two squares shown below.



The two squares each have the same total area. So subtracting four red triangles from each figure will leave the same area. Therefore $a^2 + b^2 = c^2$.

The ideas introduced in this chapter will be used in proofs throughout the rest of this book.

7A Direct proof

Learning intentions

- To be able to prove statements using direct proof.

Conditional statements

Consider the following sentence:

Statement	If	it is raining	then	the grass is wet.

This is called a **conditional statement** and has the form:

Statement	If	P is true	then	Q is true.

This can be abbreviated as

$$P \Rightarrow Q$$

which is read 'P **implies** Q'. We call P the **hypothesis** and Q the **conclusion**.

Not all conditional statements will be true. For example, switching the hypothesis and the conclusion above gives:

Statement	If	the grass is wet	then	it is raining.

Anyone who has seen dewy grass on a cloudless day knows this to be false. In this chapter we will learn how to prove (and disprove) mathematical statements.

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Direct proof

To give a **direct proof** of a conditional statement $P \Rightarrow Q$, we assume that the hypothesis P is true, and then show that the conclusion Q follows.

Example 1

Prove the following statements:

a If a is odd and b is even, then $a + b$ is odd.

b If a is odd and b is odd, then ab is odd.

Solution

a Assume that a is odd and b is even.

Since a is odd, we have $a = 2m + 1$ for some $m \in \mathbb{Z}$. Since b is even, we have $b = 2n$ for some $n \in \mathbb{Z}$. Therefore

$$\begin{aligned} a + b &= (2m + 1) + 2n \\ &= 2m + 2n + 1 \\ &= 2(m + n) + 1 \\ &= 2k + 1 \qquad \text{where } k = m + n \in \mathbb{Z} \end{aligned}$$

Hence $a + b$ is odd.

Note: We must use two different pronumerals m and n here, because these two numbers may be different.

b Assume that both a and b are odd. Then $a = 2m + 1$ and $b = 2n + 1$ for some $m, n \in \mathbb{Z}$. Therefore

$$\begin{aligned} ab &= (2m + 1)(2n + 1) \\ &= 4mn + 2m + 2n + 1 \\ &= 2(2mn + m + n) + 1 \\ &= 2k + 1 \qquad \text{where } k = 2mn + m + n \in \mathbb{Z} \end{aligned}$$

Hence ab is odd.

Example 2

Let $p, q \in \mathbb{Z}$ such that p is divisible by 5 and q is divisible by 3. Prove that pq is divisible by 15.

Solution

Since p is divisible by 5, we have $p = 5m$ for some $m \in \mathbb{Z}$. Since q is divisible by 3, we have $q = 3n$ for some $n \in \mathbb{Z}$. Thus

$$\begin{aligned} pq &= (5m)(3n) \\ &= 15mn \end{aligned}$$

and so pq is divisible by 15.

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Example 3

Let x and y be positive real numbers. Prove that if $x > y$, then $x^2 > y^2$.

Solution

Assume that $x > y$. Then $x - y > 0$.

Since x and y are positive, we also know that $x + y > 0$.

Therefore

$$x^2 - y^2 = \overbrace{(x - y)}^{\text{positive}}\overbrace{(x + y)}^{\text{positive}} > 0$$

Hence $x^2 > y^2$.

Explanation

When trying to prove that $x^2 > y^2$, it is easier to first prove that $x^2 - y^2 > 0$.

Also, note that the product of two positive numbers is positive.

Example 4

Let x and y be any two positive real numbers. Prove that

$$\frac{x + y}{2} \geq \sqrt{xy}$$

Solution

A **false proof** might begin with the statement that we are trying to prove.

$$\begin{aligned} & \frac{x + y}{2} \geq \sqrt{xy} \\ \Rightarrow \quad & x + y \geq 2\sqrt{xy} \\ \Rightarrow \quad & (x + y)^2 \geq 4xy \qquad \text{(using Example 3)} \\ \Rightarrow \quad & x^2 + 2xy + y^2 \geq 4xy \\ \Rightarrow \quad & x^2 - 2xy + y^2 \geq 0 \\ \Rightarrow \quad & (x - y)^2 \geq 0 \end{aligned}$$

Although it is true that $(x - y)^2 \geq 0$, the argument is faulty. We cannot prove that the result is true by assuming that the result is true! However, the above work is not a waste of time. We can correct the proof by reversing the order of the steps shown above.

Note: In the corrected proof, we need to use the fact that $a > b$ implies $\sqrt{a} > \sqrt{b}$ for all positive numbers a and b. This is shown in Question 8 of Exercise 7B.

Breaking a proof into cases

Sometimes it helps to break a problem up into different cases.

Example 5

Every person on an island is either a knight or a knave. Knights always tell the truth, and knaves always lie. Alice and Bob are residents on the island. Alice says: ‘We are both knaves.’ What are Alice and Bob?

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Solution

We will prove that Alice is a knave and Bob is a knight.

Case 1

Suppose Alice is a knight.

$\Rightarrow$ Alice is telling the truth.

$\Rightarrow$ Alice and Bob are both knaves.

$\Rightarrow$ Alice is a knave and a knight.

This is impossible.

Case 2

Suppose Alice is a knave.

$\Rightarrow$ Alice is not telling the truth.

$\Rightarrow$ Alice and Bob are not both knaves.

$\Rightarrow$ Bob is a knight.

Therefore we conclude that Alice must be a knave and Bob must be a knight.

Summary 7A

- A **mathematical proof** establishes the truth of a statement.
- A **conditional statement** has the form: If P is true, then Q is true.
  This can be abbreviated as $P \Rightarrow Q$, which is read 'P **implies** Q'.
- To give a **direct proof** of a conditional statement $P \Rightarrow Q$, we assume that P is true and show that Q follows.

Skill-sheet

Exercise 7A

SF

Example 1

1 Assume that m is even and n is even. Prove that:

a $m + n$ is even

b mn is even.

2 Assume that m is odd and n is odd. Prove that $m + n$ is even.

3 Assume that m is even and n is odd. Prove that mn is even.

Example 2

4 Suppose that m is divisible by 3 and n is divisible by 7. Prove that:

a mn is divisible by 21

b m^2n is divisible by 63.

5 Suppose that m and n are perfect squares. Show that mn is a perfect square.

6 Let m and n be integers. Prove that $(m + n)^2 - (m - n)^2$ is divisible by 4.

7 Suppose that n is an even integer. Prove that $n^2 - 6n + 5$ is odd.

8 Suppose that n is an odd integer. Prove that $n^2 + 8n + 3$ is even.

CF

9 Let $n \in \mathbb{Z}$. Prove that $5n^2 + 3n + 7$ is odd.
Hint: Consider the cases when n is odd and n is even.

Example 3

10 Let x and y be positive real numbers. Show that if $x > y$, then $x^4 > y^4$.

Example 4

11 Let $x, y \in \mathbb{R}$. Show that $x^2 + y^2 \geq 2xy$.

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Example 5

12 Every person on an island is either a knight or a knave. Knights always tell the truth, and knaves always lie. Alice and Bob are residents on the island. Determine whether Alice and Bob are knights or knaves in each of the following separate instances:

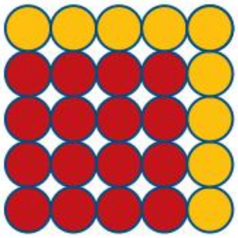
a Alice says: 'We are both knaves.'

b Alice says: 'We are both of the same kind.' Bob says: 'We are of a different kind.'

c Alice says: 'Bob is a knave.' Bob says: 'Neither of us is a knave.'

CF

13 The diagram shows that 9 can be written as the difference of two squares: $9 = 5^2 - 4^2$.



a Draw another diagram to show that 11 can be written as the difference of two squares.

b Prove that every odd number can be written as the difference of two squares.

c Hence, express 101 as the difference of two squares.

CU

14 **a** Consider the numbers $\frac{9}{10}$ and $\frac{10}{11}$. Which is larger?

b Let n be a natural number. Prove that $\frac{n}{n+1} > \frac{n-1}{n}$.

15 **a** Prove that

$$\frac{1}{10} - \frac{1}{11} < \frac{1}{100}$$

b Let $n > 0$. Prove that

$$\frac{1}{n} - \frac{1}{n+1} < \frac{1}{n^2}$$

16 Let $a, b \in \mathbb{R}$. Prove that $\frac{a^2 + b^2}{2} \geq \left(\frac{a+b}{2}\right)^2$.

17 **a** Expand $(x - y)(x^2 + xy + y^2)$.

b Prove that $x^2 + yx + y^2 \geq 0$ for all $x, y \in \mathbb{R}$.

Hint: Complete the square by thinking of y as a constant.

c Hence, prove that if $x \geq y$, then $x^3 \geq y^3$.

18 Sally travels from home to work at a speed of 12 km/h and immediately returns home at a speed of 24 km/h.

a Show that her average speed is 16 km/h.

b Now suppose that Sally travels to work at a speed of a km/h and immediately returns home at a speed of b km/h. Show that her average speed is $\frac{2ab}{a+b}$ km/h.

c Let a and b be any two positive real numbers. Prove that

$$\frac{a+b}{2} \geq \frac{2ab}{a+b}$$

Note: This proves that Sally's average speed for the whole journey can be no greater than the average of her speeds for the two individual legs of the journey.

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7B Proof by contrapositive

Learning intentions

- To be able to prove a statement by proving the contrapositive statement.

The negation of a statement

To **negate** a statement P we write its very opposite, which we call '**not** P'. For example, consider the following four statements and their negations.

P		not P	
The sky is green.	(false)	The sky is not green.	(true)
$1 + 1 = 2$	(true)	$1 + 1 \neq 2$	(false)
All prime numbers are odd.	(false)	There exists an even prime number.	(true)
All triangles have three sides.	(true)	Some triangle does not have three sides.	(false)

Notice that negation turns a true statement into a false statement, and a false statement into a true statement.

Example 6

Write down each statement and its negation. Which of the statement and its negation is true and which is false?

a $2 > 1$

b 5 is divisible by 3

c The sum of any two odd numbers is even.

d There are two primes whose product is 12.

Solution

a P: $2 > 1$ (true)
not P: $2 \leq 1$ (false)

b P: 5 is divisible by 3 (false)
not P: 5 is not divisible by 3 (true)

c P: The sum of any two odd numbers is even. (true)
not P: There are two odd numbers whose sum is odd. (false)

d P: There are two primes whose product is 12. (false)
not P: There are no two primes whose product is 12. (true)

De Morgan's laws

Negating statements that involve 'and' and 'or' requires the use of De Morgan's laws.

De Morgan's laws

not (P *and* Q) is the same as (not P) *or* (not Q)

not (P *or* Q) is the same as (not P) *and* (not Q)

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Example 7

Write down each statement and its negation. Which of the statement and its negation is true and which is false?

a 6 is divisible by 2 and 3 **b** 10 is divisible by 2 or 7

Solution

a P: 6 is divisible by 2 and 6 is divisible by 3 (true)

not P: 6 is not divisible by 2 or 6 is not divisible by 3 (false)

b P: 10 is divisible by 2 or 10 is divisible by 7 (true)

not P: 10 is not divisible by 2 and 10 is not divisible by 7 (false)

Example 8

Every person on an island is either a knight or a knave. Knights always tell the truth, and knaves always lie. Alice and Bob are residents on the island. Alice says: 'I am a knave or Bob is a knight.' What are Alice and Bob?

Solution

We will prove that Alice is a knight and Bob is a knight.

Case 1

Suppose Alice is a knave.

$\Rightarrow$ Alice is not telling the truth.

$\Rightarrow$ Alice is a knight AND Bob is a knave.

$\Rightarrow$ Alice is a knight and a knave.

This is impossible.

Case 2

Suppose Alice is a knight.

$\Rightarrow$ Alice is telling the truth.

$\Rightarrow$ Alice is a knave OR Bob is a knight.

$\Rightarrow$ Bob is a knight.

Therefore we conclude that Alice must be a knight and Bob must be a knight.

Proof by contrapositive

Consider this statement:

Statement	If	it is the end of term	then	the students are happy.

By switching the hypothesis and the conclusion and negating both, we obtain the **contrapositive** statement:

Contrapositive	If	the students are *not* happy	then	it is *not* the end of term.

Note that the original statement and its contrapositive are logically equivalent:

- If the original statement is true, then the contrapositive is true.
- If the original statement is false, then the contrapositive is false.

This means that to prove a conditional statement, we can instead prove its contrapositive. This is helpful, as it is often easier to prove the contrapositive than the original statement.

- The **contrapositive** of $P \Rightarrow Q$ is the statement (not Q) $\Rightarrow$ (not P).
- To prove $P \Rightarrow Q$, we can prove the contrapositive instead.

Example 9

Let $n \in \mathbb{Z}$ and consider this statement: If n^2 is even, then n is even.

a Write down the contrapositive. **b** Prove the contrapositive.

Solution

a If n is odd, then n^2 is odd.

b Assume that n is odd. Then $n = 2m + 1$ for some $m \in \mathbb{Z}$. Squaring n gives

$$\begin{aligned} n^2 &= (2m+1)^2 \\ &= 4m^2 + 4m + 1 \\ &= 2(2m^2 + 2m) + 1 \\ &= 2k + 1 \qquad \text{where } k = 2m^2 + 2m \in \mathbb{Z} \end{aligned}$$

Therefore n^2 is odd.

Note: Although we proved the contrapositive, remember that we have actually proved that if n^2 is even, then n is even.

Example 10

Let $n \in \mathbb{Z}$ and consider this statement: If $n^2 + 4n + 1$ is even, then n is odd.

a Write down the contrapositive. **b** Prove the contrapositive.

Solution

a If n is even, then $n^2 + 4n + 1$ is odd.

b Assume that n is even. Then $n = 2m$ for some $m \in \mathbb{Z}$. Therefore

$$\begin{aligned} n^2 + 4n + 1 &= (2m)^2 + 4(2m) + 1 \\ &= 4m^2 + 8m + 1 \\ &= 2(2m^2 + 4m) + 1 \\ &= 2k + 1 \qquad \text{where } k = 2m^2 + 4m \in \mathbb{Z} \end{aligned}$$

Hence $n^2 + 4n + 1$ is odd.

Example 11

Let x and y be positive real numbers and consider this statement: If $x < y$, then $\sqrt{x} < \sqrt{y}$.

a Write down the contrapositive. **b** Prove the contrapositive.

Solution

a If $\sqrt{x} \geq \sqrt{y}$, then $x \geq y$.

b Assume that $\sqrt{x} \geq \sqrt{y}$. Then $x \geq y$ by Example 3, since $\sqrt{x}$ and $\sqrt{y}$ are positive.

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Summary 7B

- To **negate** a statement we write its opposite.
- For a statement $P \Rightarrow Q$, the **contrapositive** is the statement (not Q) $\Rightarrow$ (not P). That is, we switch the hypothesis and the conclusion and negate both.
- A statement and its contrapositive are logically equivalent.
- Proving the contrapositive of a statement may be easier than giving a direct proof.

Skill-sheet

Exercise 7B

SF

Example 6

1 Write down each statement and its negation. Which of the statement and its negation is true and which is false?

a $1 > 0$

b 4 is divisible by 8

c Each pair of primes has an even sum.

d Some rectangle has four sides of equal length.

Example 7

2 Write down each statement and its negation. Which of the statement and its negation is true and which is false?

a 14 is divisible by 7 and 2

b 12 is divisible by 3 or 4

c 15 is divisible by 3 and 6

d 10 is divisible by 2 or 3

Example 8

3 Every person on an island is either a knight or a knave. Knights always tell the truth, and knaves always lie. Alice and Bob are residents on the island. Alice says: ‘I am a knave and Bob is a knight.’ What are Alice and Bob?

Example 9

4 Write down the contrapositive version of each of these statements:

a If it is raining, then there are clouds in the sky.

b If you are smiling, then you are happy.

c If $x = 1$, then $2x = 2$.

d If $x > y$, then $x^5 > y^5$.

e Let $n \in \mathbb{Z}$. If n^2 is odd, then n is odd.

f Let $m, n \in \mathbb{Z}$. If m and n are odd, then mn is odd.

g Let $m, n \in \mathbb{Z}$. If $m + n$ is even, then m and n are either both even or both odd.

Example 10

5 Let $m, n \in \mathbb{Z}$. For each of the following statements, write down and prove the contrapositive statement:

a If $3n + 5$ is even, then n is odd.

b If n^2 is odd, then n is odd.

c If $n^2 - 8n + 3$ is even, then n is odd.

d If n^2 is not divisible by 3, then n is not divisible by 3.

e If $n^3 + 1$ is even, then n is odd.

f If mn is not divisible by 3, then m is not divisible by 3 and n is not divisible by 3.

g If $m + n$ is odd, then $m \neq n$.

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SF

6 Let $x, y \in \mathbb{R}$. For each of the following statements, write down and prove the contrapositive statement:

a If $x^2 + 3x < 0$, then $x < 0$.

b If $x^3 - x > 0$, then $x > -1$.

c If $x + y \geq 2$, then $x \geq 1$ or $y \geq 1$.

d If $2x + 3y \geq 12$, then $x \geq 3$ or $y \geq 2$.

7 Let $m, n \in \mathbb{Z}$ and consider this statement: If mn and $m + n$ are even, then m and n are even.

a Write down the contrapositive.

b Prove the contrapositive. You will have to consider cases.

CF

Example 11

8 Let x and y be positive real numbers.

a Prove that

$$\sqrt{x} - \sqrt{y} = \frac{x - y}{\sqrt{x} + \sqrt{y}}$$

b Hence, prove that if $x > y$, then $\sqrt{x} > \sqrt{y}$.

c Give a simpler proof by considering the contrapositive.

7C Proof by contradiction

Learning intentions

- To be able to prove a statement by contradiction.

There are various instances when we want to prove mathematically that something cannot be done. To do this, we assume that it can be done, and then show that something goes horribly wrong. Let's first look at a familiar example from geometry.

Example 12

An angle is called **reflex** if it exceeds 180°. Prove that no quadrilateral has more than one reflex angle.

Solution

If there is more than one reflex angle, then the angle sum must exceed $2 \times 180° = 360°$. This contradicts the fact that the angle sum of any quadrilateral is exactly 360°. Therefore there cannot be more than one reflex angle.

The example above is a demonstration of a **proof by contradiction**. The basic outline of a proof by contradiction is:

1 Assume that the statement we want to prove is false.

2 Show that this assumption leads to mathematical nonsense.

3 Conclude that we were wrong to assume that the statement is false.

4 Conclude that the statement must be true.

Example 13

A **Pythagorean triple** consists of three natural numbers (a, b, c) satisfying

$$a^2 + b^2 = c^2$$

Show that if (a, b, c) is a Pythagorean triple, then a, b and c cannot all be odd numbers.

Solution

This will be a proof by contradiction.

Let (a, b, c) be a Pythagorean triple. Then $a^2 + b^2 = c^2$.

Suppose that a, b and c are all odd numbers.
$\Rightarrow$ a^2, b^2 and c^2 are all odd numbers.
$\Rightarrow$ $a^2 + b^2$ is even and c^2 is odd.

Since $a^2 + b^2 = c^2$, this gives a contradiction.

Therefore a, b and c cannot all be odd numbers.

Possibly the most well-known proof by contradiction is the following.

Theorem

$\sqrt{2}$ is irrational.

Proof This will be a proof by contradiction.

Suppose that $\sqrt{2}$ is rational. Then

$$\sqrt{2} = \frac{p}{q} \quad \text{where } p, q \in \mathbb{Z}$$

We can assume that p and q have no common factors (or else they could be cancelled). Then, squaring both sides and rearranging gives

$$\begin{aligned}
& p^2 = 2q^2 && (1) \\
\Rightarrow\ & p^2 \text{ is divisible by } 2 && \\
\Rightarrow\ & p \text{ is divisible by } 2 && \text{(by Example 9)} \\
\Rightarrow\ & p = 2n \text{ for some } n \in \mathbb{Z} && \\
\Rightarrow\ & (2n)^2 = 2q^2 && \text{(substituting into (1))} \\
\Rightarrow\ & q^2 = 2n^2 && \\
\Rightarrow\ & q^2 \text{ is divisible by } 2 && \\
\Rightarrow\ & q \text{ is divisible by } 2 && \text{(by Example 9)}
\end{aligned}$$

Therefore both p and q are divisible by 2, which contradicts the fact that they have no common factors.

Hence $\sqrt{2}$ is irrational.

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Example 14

Suppose x satisfies $5^x = 2$. Show that x is irrational.

Solution

Suppose that x is rational. Since x must be positive, we can write $x = \frac{m}{n}$ where $m, n \in \mathbb{N}$. Therefore

$$\begin{aligned} 5^x = 2 \quad &\Rightarrow \quad 5^{\frac{m}{n}} = 2 \\ &\Rightarrow \quad \left(5^{\frac{m}{n}}\right)^n = 2^n \qquad \text{(raise both sides to the power } n\text{)} \\ &\Rightarrow \quad 5^m = 2^n \end{aligned}$$

The left-hand side of this equation is odd and the right-hand side is even. This gives a contradiction, and so x is not rational.

We finish on a remarkable result, which is attributed to Euclid some 2300 years ago.

Theorem

There are infinitely many prime numbers.

Proof This is a proof by contradiction, so we will suppose that there are only finitely many primes. This means that we can create a list that contains *every* prime number:

$$2,\ 3,\ 5,\ 7,\ \dots,\ p$$

where p is the largest prime number.

Now for the trick. We a create a new number N by multiplying each number in the list and then adding 1:

$$N = 2 \times 3 \times 5 \times 7 \times \cdots \times p + 1$$

The number N is not divisible by any of the primes $2, 3, 5, 7, \dots, p$, since it leaves a remainder of 1 when divided by any of these numbers.

However, every natural number greater than 1 is divisible by a prime number. (This is proved in Question 13 of Exercise 7F.) Therefore N is divisible by some prime number q. But this prime number q is not in the list $2, 3, 5, 7, \dots, p$, contradicting the fact that our list contains every prime number.

Hence there are infinitely many prime numbers.

Summary 7C

- A **proof by contradiction** is used to prove that something cannot be done.
- These proofs always follow the same basic structure:
  1. Assume that the statement we want to prove is false.
  2. Show that this assumption leads to mathematical nonsense.
  3. Conclude that we were wrong to assume that the statement is false.
  4. Conclude that the statement must be true.

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Skill-sheet

Exercise 7C

Example 12

1 Prove that every triangle has some interior angle with a magnitude of at least 60°.

SF

2 Prove that there is no smallest positive rational number.

3 Let p be a prime number. Show that $\sqrt{p}$ is not an integer.

Example 14

4 Suppose that $3^x = 2$. Prove that x is irrational.

5 Prove that $\log_2 5$ is irrational.

6 Suppose that $x > 0$ is irrational. Prove that $\sqrt{x}$ is also irrational.

7 Suppose that a is rational and b is irrational. Prove that $a + b$ is irrational.

CF

8 Suppose that $c^2 - b^2 = 4$. Prove that b and c cannot both be natural numbers.

9 Let a, b and c be real numbers with $a \neq 0$. Prove by contradiction that there is only one solution to the equation $ax + b = c$.

10 **a** Prove that all primes $p > 2$ are odd.

b Hence, prove that there are no two primes whose sum is 1001.

11 **a** Prove that there are no integers a and b for which $42a + 7b = 1$.

Hint: The left-hand side is divisible by 7.

b Prove that there are no integers a and b for which $15a + 21b = 2$.

12 **a** Prove that if n^2 is divisible by 3, then n is divisible by 3.

Hint: Prove the contrapositive by considering two cases.

b Hence, prove that $\sqrt{3}$ is irrational.

CU

13 **a** Prove that if n^3 is divisible by 2, then n is divisible by 2.

Hint: Prove the contrapositive.

b Hence, prove that $\sqrt[3]{2}$ is irrational.

14 Prove that if $a, b \in \mathbb{Z}$, then $a^2 - 4b - 2 \neq 0$.

15 **a** Let $a, b, n \in \mathbb{N}$. Prove that if $n = ab$, then $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$.

b Hence, show that 97 is a prime number.

16 **a** Let m be an integer. Prove that m^2 is divisible by 4 or leaves a remainder of 1.

Hint: Suppose that $m = 4n + r$ and consider m^2 for $r = 0, 1, 2, 3$.

b Let $a, b, c \in \mathbb{Z}$. Prove by contradiction: If $a^2 + b^2 = c^2$, then a is even or b is even.

17 **a** Let $a, b, c, d \in \mathbb{Z}$. Prove that if $a + b\sqrt{2} = c + d\sqrt{2}$, then $a = c$ and $b = d$.

b Hence, determine $c, d \in \mathbb{Z}$ if $\sqrt{3 + 2\sqrt{2}} = c + d\sqrt{2}$. **Hint:** Square both sides.

18 Let $a, b, c \in \mathbb{Z}$. Prove that if a, b and c are all odd, then the equation $ax^2 + bx + c = 0$ cannot have a rational solution.

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7D Equivalent statements

Learning intentions

- To be able to prove equivalent statements.

The converse of a statement

At the beginning of this chapter, we proved Pythagoras' theorem. Consider any triangle with side lengths a, b and c.

Statement	If	the angle between a and b is 90°	then	$a^2 + b^2 = c^2$.

By switching the hypothesis and the conclusion, we obtain the **converse** statement:

Converse	If	$a^2 + b^2 = c^2$	then	the angle between a and b is 90°.

In Chapter 9 we will prove that this is also a true statement.

When we switch the hypothesis and the conclusion of a conditional statement, $P \Rightarrow Q$, we obtain the **converse** statement, $Q \Rightarrow P$.

Note: The converse of a true statement may not be true. For example:

Statement	If it is raining, then there are clouds in the sky.	(true)
Converse	If there are clouds in the sky, then it is raining.	(false)

Example 15

Let x and y be positive real numbers. Consider the statement: If $x < y$, then $x^2 < y^2$.

a Write down the converse of this statement.

b Prove the converse.

Solution

a If $x^2 < y^2$, then $x < y$.

b Assume that $x^2 < y^2$. Then, since both x and y are positive,

$$\begin{aligned} & x^2 - y^2 < 0 && \text{(subtract } y^2\text{)} \\ \Rightarrow\quad & (x-y)(x+y) < 0 && \text{(factorising)} \\ \Rightarrow\quad & x - y < 0 && \text{(divide both sides by } x + y > 0\text{)} \\ \Rightarrow\quad & x < y \end{aligned}$$

as required.

Example 16

Let m and n be integers. Consider the statement: If m and n are even, then $m + n$ is even.

a Write down the converse of this statement.

b Show that the converse is not true.

Solution

a If $m + n$ is even, then m is even and n is even.

b Clearly $1 + 3 = 4$ is even, although 1 and 3 are not.

Equivalent statements

Now consider the following two statements:

P: your heart is beating

Q: you are alive

Notice that both $P \Rightarrow Q$ and its converse $Q \Rightarrow P$ are true statements. In this case, we say that P and Q are **equivalent** statements and we write

$$P \Leftrightarrow Q$$

We will also say that P is true **if and only if** Q is true. So in the above example, we can say

Your heart is beating	if and only if	you are alive.

To prove that two statements P and Q are equivalent, you have to prove two things:

$P \Rightarrow Q$ and $Q \Rightarrow P$

Example 17

Let $n \in \mathbb{Z}$. Prove that n is even if and only if $n + 1$ is odd.

Solution

($\Rightarrow$) Assume that n is even. Then $n = 2m$ for some $m \in \mathbb{Z}$.
Therefore $n + 1 = 2m + 1$, and so $n + 1$ is odd.

($\Leftarrow$) Assume that $n + 1$ is odd. Then $n + 1 = 2m + 1$ for some $m \in \mathbb{Z}$.
Subtracting 1 from both sides gives $n = 2m$. Therefore n is even.

Note: To prove that $P \Leftrightarrow Q$, we have to show that $P \Rightarrow Q$ and $P \Leftarrow Q$. When we are about to prove $P \Rightarrow Q$, we write ($\Rightarrow$). When we are about to prove $P \Leftarrow Q$, we write ($\Leftarrow$).

Summary 7D

- For a statement $P \Rightarrow Q$, the **converse** is the statement $Q \Rightarrow P$.
  That is, we switch the hypothesis and the conclusion.
- If $P \Rightarrow Q$ is true and $Q \Rightarrow P$ is true, then we say that P is **equivalent** to Q, or that P is true **if and only if** Q is true.
- If P and Q are equivalent, we write $P \Leftrightarrow Q$.

Skill-sheet

Exercise 7D

Example 15

1 Write down and prove the converse of each of these statements:

a Let $x \in \mathbb{R}$. If $2x + 3 = 5$, then $x = 1$.

b Let $n \in \mathbb{Z}$. If n is odd, then $n - 3$ is even.

c Let $m \in \mathbb{Z}$. If $m^2 + 2m + 1$ is even, then m is odd.

d Let $n \in \mathbb{Z}$. If n^2 is divisible by 5, then n is divisible by 5.

SF

Example 16

2 Let m and n be integers. Consider the statement: If m and n are even, then mn is a multiple of 4.

a Write down the converse of this statement.

b Show that the converse is not true.

3 Which of these pairs of statements are equivalent?

a P: Vivian is in China.

Q: Vivian is in Asia.

b P: $2x = 4$

Q: $x = 2$

c P: $x > 0$ and $y > 0$

Q: $xy > 0$

d P: m is even or n is even, where $m, n \in \mathbb{Z}$

Q: mn is even, where $m, n \in \mathbb{Z}$

Example 17

CF

4 Let n be an integer. Prove that $n + 1$ is odd if and only if $n + 2$ is even.

5 Let $n \in \mathbb{N}$. Prove that $n^2 - 4$ is a prime number if and only if $n = 3$.

6 Let n be an integer. Prove that n^3 is even if and only if n is even.

7 Let n be an integer. Prove that n is odd if and only if $n = 4k \pm 1$ for some $k \in \mathbb{Z}$.

8 Let $x, y \in \mathbb{R}$. Prove that $(x + y)^2 = x^2 + y^2$ if and only if $x = 0$ or $y = 0$.

9 Let m and n be integers.

a By expanding the right-hand side, prove that $m^3 - n^3 = (m - n)(m^2 + mn + n^2)$.

b Hence, prove that $m - n$ is even if and only if $m^3 - n^3$ is even.

10 Prove that an integer is divisible by 4 if and only if the number formed by its last two digits is divisible by 4. **Hint:** 100 is divisible by 4.

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7E Disproving statements

Learning intentions

- To consider quantification statements and the use of counterexamples.

Quantification using 'for all' and 'there exists'

For all

Universal quantification claims that a property holds for *all* members of a given set. For example, consider this statement:

Statement	For all natural numbers n, we have $2n \geq n + 1$.

To prove that this statement is true, we need to give a general argument that applies to every natural number n.

There exists

Existential quantification claims that a property holds for *at least one* member of a given set. For example, consider this statement:

Statement	There exists an integer m such that $m^2 = 25$.

To prove that this statement is true, we just need to give an example: $5 \in \mathbb{Z}$ with $5^2 = 25$.

Example 18

Rewrite each statement using either 'for all' or 'there exists':

a Some real numbers are irrational.

b Every integer that is divisible by 4 is also divisible by 2.

Solution

a There exists $x \in \mathbb{R}$ such that $x \notin \mathbb{Q}$.

b For all $m \in \mathbb{Z}$, if m is divisible by 4, then m is divisible by 2.

Negating 'for all' and 'there exists'

To negate a statement involving a quantifier, we interchange 'for all' with 'there exists' and then negate the rest of the statement.

Example 19

Write down the negation of each of the following statements:

a For all natural numbers n, we have $2n \geq n + 1$.

b There exists an integer m such that $m^2 = 4$ and $m^3 = -8$.

Solution

a There exists a natural number n such that $2n < n + 1$.

b For all integers m, we have $m^2 \neq 4$ or $m^3 \neq -8$.

Note: For part **b**, we used one of De Morgan's laws.

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Notation

The words 'for all' can be abbreviated using the *turned A* symbol, ∀. The words 'there exists' can be abbreviated using the *turned E* symbol, ∃. For example:

- 'For all natural numbers n, we have $2n \geq n + 1$' can be written as $(\forall n \in \mathbb{N})\ 2n \geq n + 1$.
- 'There exists an integer m such that $m^2 = 25$' can be written as $(\exists m \in \mathbb{Z})\ m^2 = 25$.

Despite the ability of these new symbols to make certain sentences more concise, we do not believe that they make written sentences clearer. Therefore we have avoided using them in this chapter.

Counterexamples

Consider the quadratic function $f(n) = n^2 - n + 11$. Notice how $f(n)$ is a prime number for small natural numbers n:

n	1	2	3	4	5	6	7	8	9	10
$f(n)$	11	13	17	23	31	41	53	67	83	101

From this, we might be led to believe that the following statement is true:

Statement	For all natural numbers n, the number $f(n)$ is prime.

We call this a **universal statement**, because it asserts the truth of a statement without exception. So to disprove a universal statement, we need only show that it is not true in some particular instance. For our example, we need to find $n \in \mathbb{N}$ such that $f(n)$ is not prime. Luckily, we do not have to look very hard.

Example 20

Let $f(n) = n^2 - n + 11$. Disprove this statement: For all $n \in \mathbb{N}$, the number $f(n)$ is prime.

Solution

When $n = 11$, we obtain

$$f(11) = 11^2 - 11 + 11 = 11^2$$

Therefore $f(11)$ is not prime.

To disprove a statement of the form $P \Rightarrow Q$, we simply need to give one example for which P is true and Q is not true. Such an example is called a **counterexample**.

Example 21

Find a counterexample to disprove this statement: For all $x, y \in \mathbb{R}$, if $x > y$, then $x^2 > y^2$.

Solution

Let $x = 1$ and $y = -2$. Clearly $1 > -2$, but $1^2 = 1 \leq 4 = (-2)^2$.

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Disproving existence statements

Consider this statement:

Statement	There exists $n \in \mathbb{N}$ such that $n^2 + 3n + 2$ is a prime number.

We call this an **existence statement**, because it claims the existence of an object possessing a particular property. To show that such a statement is false, we prove that its negation is true:

Negation	For all $n \in \mathbb{N}$, the number $n^2 + 3n + 2$ is not a prime number.

This is easy to prove, as

$$n^2 + 3n + 2 = (n + 1)(n + 2)$$

is clearly a composite number for each $n \in \mathbb{N}$.

Example 22

Disprove this statement: There exists $n \in \mathbb{N}$ such that $n^2 + 13n + 42$ is a prime number.

Solution

We need to prove that, for all $n \in \mathbb{N}$, the number $n^2 + 13n + 42$ is not prime.

This is true, since

$$n^2 + 13n + 42 = (n + 6)(n + 7)$$

is clearly a composite number for each $n \in \mathbb{N}$.

Example 23

Show that this statement is false: There exists some real number x such that $x^2 = -1$.

Solution

We have to prove that the negation is true: For *all* real numbers x, we have $x^2 \neq -1$.

This is easy to prove since, for any real number x, we have $x^2 \geq 0$ and so $x^2 \neq -1$.

Summary 7E

- A **universal statement** claims that a property holds for all members of a given set. Such a statement can be written using the quantifier **'for all'**.
- An **existence statement** claims that a property holds for some member of a given set. Such a statement can be written using the quantifier **'there exists'**.
- A universal statement of the form $P \Rightarrow Q$ can be disproved by giving one example of an instance when P is true but Q is not.
- Such an example is called a **counterexample**.
- To disprove an existence statement, we prove that its negation is true.

Skill-sheet

Exercise 7E

SF

Example 18

1 Which of the following are universal statements ('for all') and which are existence statements ('there exists')?

a For each $n \in \mathbb{N}$, the number $5n^2 + 3n + 7$ is odd.
b There is an even prime number.
c Every natural number greater than 1 has a prime factorisation.
d All triangles have three sides.
e Some natural numbers are primes.
f At least one real number x satisfies the equation $x^2 - x - 1 = 0$.
g Any positive real number has a square root.
h The angle sum of a triangle is 180°.

2 Which of the following statements are true and which are false?

a There exists a real number x such that $x^2 = 2$.
b There exists a real number x such that $x^2 < 0$.
c For all natural numbers n, the number $2n - 1$ is odd.
d There exists $n \in \mathbb{N}$ such that $2n$ is odd.
e For all $x \in \mathbb{R}$, we have $x^3 \geq 0$.

Example 19

3 Write down the negation of each of the following statements:

a There exists $x \in \mathbb{R}$ such that $2 + x^2 = 1 - x^2$.
b For every $n \in \mathbb{N}$, $2^n + 1$ is a prime number.
c There exists $x, y \in \mathbb{R}$ such that $x < y$ and $x^2 > y^2$.
d For all $x, y, z \in \mathbb{R}$, if $x < y$ and $y < z$, then $x + y < z$.
e There exists $m, n \in \mathbb{N}$ for which mn is a prime number.
f For every $n \in \mathbb{N}$, the number $7^n - 2^n$ is divisible by 5.

4 Prove that each of the following statements is false by finding a counterexample:

Example 20

a For every natural number n, the number $2n^2 - 4n + 31$ is prime.
b If $x, y \in \mathbb{R}$, then $(x + y)^2 = x^2 + y^2$.

Example 21

c For all $x \in \mathbb{R}$, we have $x^2 > x$.
d Let $n \in \mathbb{Z}$. If $n^3 - n$ is even, then n is even.
e If $m, n \in \mathbb{N}$, then $m + n \leq mn$.
f Let $m, n \in \mathbb{Z}$. If 6 divides mn, then 6 divides m or 6 divides n.

5 Show that each of the following existence statements is false:

Example 22

a There exists $n \in \mathbb{N}$ such that $9n^2 - 1$ is a prime number.
b There exists $n \in \mathbb{N}$ such that $n^2 + 5n + 6$ is a prime number.

Example 23

c There exists $x \in \mathbb{R}$ such that $2 + x^2 = 1 - x^2$.

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CF

6 Provide a counterexample to disprove each of the following statements.
Hint: The number $\sqrt{2}$ might come in handy.

- **a** If a is irrational and b is irrational, then ab is irrational.
- **b** If a is irrational and b is irrational, then $a + b$ is irrational.
- **c** If a is irrational and b is irrational, then $\frac{a}{b}$ is irrational.

7 Let $a \in \mathbb{Z}$.

- **a** Prove that if a is divisible by 4, then a^2 is divisible by 4.
- **b** Prove that the converse is not true.

8 Let $a, b \in \mathbb{Z}$.

- **a** Prove that if $a - b$ is divisible by 3, then $a^2 - b^2$ is divisible by 3.
- **b** Prove that the converse is not true.

9 Prove that each of the following statements is false:

- **a** There exist real numbers a and b such that $a^2 - 2ab + b^2 = -1$.
- **b** There exists some real number x such that $x^2 - 4x + 5 = \frac{3}{4}$.

CU

10 The numbers $\{1, 2, \ldots, 8\}$ can be paired so that the sum of each pair is a square number:

$$1 + 8 = 9, \quad 2 + 7 = 9, \quad 3 + 6 = 9, \quad 4 + 5 = 9$$

- **a** Prove that you can also do this with the numbers $\{1, 2, \ldots, 16\}$.
- **b** Prove that you cannot do this with the numbers $\{1, 2, \ldots, 12\}$.

11 Let $f(n) = an^2 + bn + c$ be a quadratic function, where a, b, c are natural numbers and $c \geq 2$. Show that there is an $n \in \mathbb{N}$ such that $f(n)$ is not a prime number.

7F Mathematical induction

Learning intentions

- ▶ To be able to prove statements by mathematical induction.

Consider the sum of the first n odd numbers:

$$\begin{aligned} 1 &= 1 = 1^2 \\ 1 + 3 &= 4 = 2^2 \\ 1 + 3 + 5 &= 9 = 3^2 \\ 1 + 3 + 5 + 7 &= 16 = 4^2 \end{aligned}$$

From this limited number of examples, we could make the following proposition $P(n)$ about the number n: the sum of the first n odd numbers is n^2. Since the nth odd number is $2n - 1$, we can write this proposition as

$$P(n): \quad 1 + 3 + 5 + \cdots + (2n - 1) = n^2$$

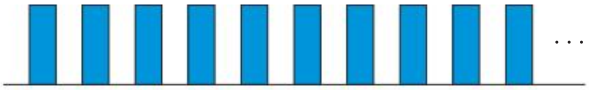
However, we have to be careful here: Just because something looks true does not mean that it is true. In this section, we will learn how to prove statements like the one above.

The principle of mathematical induction

Imagine a row of dominoes extending infinitely to the right. Each of these dominoes can be knocked over provided two conditions are met:

1 The first domino is knocked over.

2 Each domino is sufficiently close to the next domino.



This scenario provides an accurate physical model of the following proof technique.

Principle of mathematical induction

Let $P(n)$ be some proposition about the natural number n.

We can prove that $P(n)$ is true for every natural number n as follows:

a Show that $P(1)$ is true.

b Show that, for every natural number k, if $P(k)$ is true, then $P(k+1)$ is true.

The idea is simple: Condition **a** tells us that $P(1)$ is true. But then condition **b** means that $P(2)$ will also be true. However, if $P(2)$ is true, then condition **b** also guarantees that $P(3)$ is true, and so on. This process continues indefinitely, and so $P(n)$ is true for all $n \in \mathbb{N}$.

$$P(1) \text{ is true} \quad \Rightarrow \quad P(2) \text{ is true} \quad \Rightarrow \quad P(3) \text{ is true} \quad \Rightarrow \quad \cdots$$

Let's see how mathematical induction is used in practice.

Example 24

Prove that

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2$$

for all $n \in \mathbb{N}$.

Solution

For each natural number n, let $P(n)$ be the proposition:

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2$$

Step 1 $P(1)$ is the proposition $1 = 1^2$, that is, $1 = 1$. Therefore $P(1)$ is true.

Step 2 Let k be any natural number, and assume $P(k)$ is true. That is,

$$1 + 3 + 5 + \cdots + (2k - 1) = k^2$$

Step 3 We now have to prove that $P(k+1)$ is true, that is,

$$1 + 3 + 5 + \cdots + (2k - 1) + (2k + 1) = (k + 1)^2$$

Notice that we have written the last and the second-last term in the summation. This is so we can easily see how to use our assumption that $P(k)$ is true.

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We have

$$\begin{aligned} \text{LHS of } P(k+1) &= 1 + 3 + 5 + \cdots + (2k-1) + (2k+1) \\ &= k^2 + (2k+1) \qquad \text{(using } P(k)\text{)} \\ &= (k+1)^2 \\ &= \text{RHS of } P(k+1) \end{aligned}$$

We have proved that if $P(k)$ is true, then $P(k+1)$ is true, for every natural number k.

By the principle of mathematical induction, it follows that $P(n)$ is true for every natural number n.

While mathematical induction is good for proving that formulas are true, it rarely indicates why they should be true in the first place. The formula $1 + 3 + 5 + \cdots + (2n-1) = n^2$ can be discovered in the diagram shown on the right.

Example 25

Prove by induction that $7^n - 4$ is divisible by 3 for all $n \in \mathbb{N}$.

Solution

For each natural number n, let $P(n)$ be the proposition:

$7^n - 4$ is divisible by 3

Step 1 $P(1)$ is the proposition $7^1 - 4 = 3$ is divisible by 3. Clearly, $P(1)$ is true.

Step 2 Let k be any natural number, and assume $P(k)$ is true. That is,

$$7^k - 4 = 3m$$

for some $m \in \mathbb{Z}$.

Step 3 We now have to prove that $P(k+1)$ is true, that is, $7^{k+1} - 4$ is divisible by 3. We have

$$\begin{aligned} 7^{k+1} - 4 &= 7 \times 7^k - 4 \\ &= 7(3m+4) - 4 \qquad \text{(using } P(k)\text{)} \\ &= 21m + 28 - 4 \\ &= 21m + 24 \\ &= 3(7m+8) \end{aligned}$$

Therefore $7^{k+1} - 4$ is divisible by 3.

We have proved that if $P(k)$ is true, then $P(k+1)$ is true, for every natural number k.

Therefore $P(n)$ is true for all $n \in \mathbb{N}$, by the principle of mathematical induction.

Proving inequalities

Induction can be used to prove certain inequalities.

For example, consider this table of values:

n	1	2	3	4	5
3^n	3	9	27	81	243
3×2^n	6	12	24	48	96

From the table, it certainly looks as though

$$3^n > 3 \times 2^n \quad \text{for all } n \geq 3$$

We will prove this formally using induction; this time starting with the proposition $P(3)$ instead of $P(1)$.

Example 26

Prove that

$$3^n > 3 \times 2^n$$

for every natural number $n \geq 3$.

Solution

For each natural number $n \geq 3$, let $P(n)$ be the proposition:

$$3^n > 3 \times 2^n$$

Step 1 $P(3)$ is the proposition $3^3 > 3 \times 2^3$, that is, $27 > 24$. Therefore $P(3)$ is true.

Step 2 Let k be a natural number with $k \geq 3$, and assume $P(k)$ is true. That is,

$$3^k > 3 \times 2^k$$

Step 3 We now have to prove that $P(k+1)$ is true, that is,

$$3^{k+1} > 3 \times 2^{k+1}$$

We have

$$\begin{aligned} \text{LHS of } P(k+1) &= 3^{k+1} \\ &= 3 \times 3^k \\ &> 3 \times 3 \times 2^k && \text{(using } P(k)\text{)} \\ &> 3 \times 2 \times 2^k \\ &= 3 \times 2^{k+1} \\ &= \text{RHS of } P(k+1) \end{aligned}$$

We have proved that if $P(k)$ is true, then $P(k+1)$ is true, for every natural number $k \geq 3$.

By the principle of mathematical induction, it follows that $P(n)$ is true for every natural number $n \geq 3$.

Applications to sequences

Sequences are studied in Mathematical Methods Units 1 & 2. A **sequence** is a list of numbers, with order being important. An example is the sequence of odd numbers:

$$1, 3, 5, 7, 9, \ldots$$

The nth term of a sequence is denoted by t_n. So, for the sequence of odd numbers, we have $t_1 = 1$, $t_2 = 3$ and $t_3 = 5$. In general, we have $t_n = 2n - 1$ for all $n \in \mathbb{N}$.

Some sequences can be defined by a rule that enables each subsequent term to be found from previous terms. We can define the sequence of odd numbers by $t_1 = 1$ and $t_{n+1} = t_n + 2$. This type of rule is called a **recurrence relation**.

Induction proofs are frequently used in the study of sequences. As an example, we will consider the sequence defined by the recurrence relation

$$t_1 = 11 \quad \text{and} \quad t_{n+1} = 10t_n - 9$$

The first five terms of this sequence are listed in the following table.

n	1	2	3	4	5
t_n	11	101	1001	10 001	100 001

Notice that each of these terms is one more than a power of 10. Let's see if we can prove that this is true for *every* term in the sequence.

Example 27

Given $t_1 = 11$ and $t_{n+1} = 10t_n - 9$, prove that $t_n = 10^n + 1$.

Solution

For each natural number n, let $P(n)$ be the proposition: $t_n = 10^n + 1$.

Step 1 Since $t_1 = 11$ and $10^1 + 1 = 11$, it follows that $P(1)$ is true.

Step 2 Let k be any natural number, and assume $P(k)$ is true. That is,

$$t_k = 10^k + 1$$

Step 3 We now have to prove that $P(k + 1)$ is true, that is,

$$t_{k+1} = 10^{k+1} + 1$$

We have

$$\begin{aligned}
\text{LHS of } P(k+1) &= t_{k+1} \\
&= 10t_k - 9 \\
&= 10 \times (10^k + 1) - 9 \qquad (\text{using } P(k)) \\
&= 10^{k+1} + 1 \\
&= \text{RHS of } P(k+1)
\end{aligned}$$

We have proved that if $P(k)$ is true, then $P(k + 1)$ is true, for each $k \in \mathbb{N}$.

By the principle of mathematical induction, it follows that $P(n)$ is true for all $n \in \mathbb{N}$.

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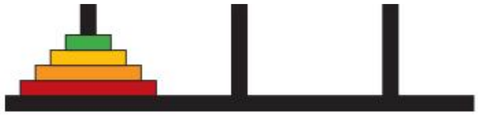
Tower of Hanoi

You have three pegs and a collection of n discs of different sizes. Initially, all the discs are stacked in size order on the left-hand peg. Discs can be moved one at a time from one peg to any other peg, provided that a larger disc never rests on a smaller one. The aim of the puzzle is to transfer all the discs to another peg using the smallest possible number of moves.



Example 28

Let a_n be the minimum number of moves needed to solve the Tower of Hanoi with n discs.

a Determine a formula for a_{n+1} in terms of a_n.

b Evaluate a_n for $n = 1, 2, 3, 4, 5$. Guess a formula for a_n in terms of n.

c Confirm your formula for a_n using mathematical induction.

d If $n = 20$, how many days are needed to transfer all the discs to another peg, assuming that one disc can be moved per second?

Solution

a Suppose there are $n + 1$ discs on the left-hand peg.

If we want to be able to move the largest disc to the right-hand peg, then first we must transfer the other n discs to the centre peg. This takes a minimum of a_n moves.

It takes 1 move to transfer the largest disc to the right-hand peg. Now we can complete the puzzle by transferring the n discs on the centre peg to the right-hand peg. This takes a minimum of a_n moves.

Hence the minimum number of moves required to transfer all the discs is

$$\begin{aligned} a_{n+1} &= a_n + 1 + a_n \\ &= 2a_n + 1 \end{aligned}$$

b We have $a_1 = 1$, since one disc can be moved in one move. Using the recurrence relation from part **a**, we determine that

$$a_2 = 2a_1 + 1 = 2 \times 1 + 1 = 3$$

$$a_3 = 2a_2 + 1 = 2 \times 3 + 1 = 7$$

Continuing in this way, we obtain the following table.

n	1	2	3	4	5
a_n	1	3	7	15	31

It seems as though every term is one less than a power of 2. We guess that

$$a_n = 2^n - 1$$

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c For each natural number n, let $P(n)$ be the proposition:

$$a_n = 2^n - 1$$

Step 1 The minimum number of moves required to solve the Tower of Hanoi puzzle with one disc is 1. Since $2^1 - 1 = 1$, it follows that $P(1)$ is true.

Step 2 Let k be any natural number, and assume $P(k)$ is true. That is,

$$a_k = 2^k - 1$$

Step 3 We now wish to prove that $P(k+1)$ is true, that is,

$$a_{k+1} = 2^{k+1} - 1$$

We have

$$\begin{aligned} \text{LHS of } P(k+1) &= a_{k+1} \\ &= 2a_k + 1 && \text{(using part } \mathbf{a}) \\ &= 2 \times (2^k - 1) + 1 && \text{(using } P(k)) \\ &= 2^{k+1} - 2 + 1 \\ &= 2^{k+1} - 1 \\ &= \text{RHS of } P(k+1) \end{aligned}$$

We have proved that if $P(k)$ is true, then $P(k+1)$ is true, for every natural number k.

By the principle of mathematical induction, it follows that $P(n)$ is true for all $n \in \mathbb{N}$. Hence we have shown that $a_n = 2^n - 1$ for all $n \in \mathbb{N}$.

d A puzzle with 20 discs requires a minimum of $2^{20} - 1$ seconds.

Since there are $60 \times 60 \times 24 = 86\,400$ seconds in a day, it will take

$$\frac{2^{20} - 1}{86\,400} \approx 12.14 \text{ days}$$

to complete the puzzle.

Summary 7F

The basic outline of a proof by mathematical induction is:

0 Define the proposition $P(n)$ for $n \in \mathbb{N}$.

1 Show that $P(1)$ is true.

2 Assume that $P(k)$ is true for some $k \in \mathbb{N}$.

3 Show that $P(k+1)$ is true.

4 Conclude that $P(n)$ is true for all $n \in \mathbb{N}$.

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Skill-sheet

Exercise 7F

SF

Example 24

1 Prove each of the following by mathematical induction:

a $1 + 2 + \cdots + n = \dfrac{n(n+1)}{2}$

b $1 + x + x^2 + \cdots + x^n = \dfrac{1 - x^{n+1}}{1 - x}$, where $x \neq 1$

c $1^2 + 2^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$

d $1 \cdot 2 + 2 \cdot 3 + \cdots + n \cdot (n+1) = \dfrac{n(n+1)(n+2)}{3}$

e $\dfrac{1}{1 \times 3} + \dfrac{1}{3 \times 5} + \cdots + \dfrac{1}{(2n-1)(2n+1)} = \dfrac{n}{2n+1}$

f $\left(1 - \dfrac{1}{2^2}\right)\left(1 - \dfrac{1}{3^2}\right)\cdots\left(1 - \dfrac{1}{n^2}\right) = \dfrac{n+1}{2n}$, for $n \geq 2$

Example 25

2 Prove each of the following divisibility statements by mathematical induction:

a $11^n - 1$ is divisible by 10 for all $n \in \mathbb{N}$

b $3^{2n} + 7$ is divisible by 8 for all $n \in \mathbb{N}$

c $7^n - 3^n$ is divisible by 4 for all $n \in \mathbb{N}$

d $5^n + 6 \times 7^n + 1$ is divisible by 4 for all $n \in \mathbb{N}$

CF

Example 26

3 Prove each of the following inequalities by mathematical induction:

a $4^n > 10 \times 2^n$ for all integers $n \geq 4$

b $3^n > 5 \times 2^n$ for all integers $n \geq 5$

c $2^n > 2n$ for all integers $n \geq 3$

d $n! > 2^n$ for all integers $n \geq 4$

Example 27

4 Prove each of the following statements by mathematical induction:

a If $a_{n+1} = 2a_n - 1$ and $a_1 = 3$, then $a_n = 2^n + 1$.

b If $a_{n+1} = 5a_n + 4$ and $a_1 = 4$, then $a_n = 5^n - 1$.

c If $a_{n+1} = 2a_n - n + 1$ and $a_1 = 3$, then $a_n = 2^n + n$.

5 Prove that 3^n is odd for every $n \in \mathbb{N}$.

6 **a** Prove by mathematical induction that $n^2 - n$ is even for all $n \in \mathbb{N}$.

b Find an easier proof by factorising $n^2 - n$.

7 **a** Prove by mathematical induction that $n^3 - n$ is divisible by 6 for all $n \in \mathbb{N}$.

b Find an easier proof by factorising $n^3 - n$.

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8 Consider the sequence defined by $a_{n+1} = 10a_n + 9$ where $a_1 = 9$.

a Determine a_n for $n = 1, 2, 3, 4, 5$.

b Guess a formula for a_n in terms of n.

c Confirm that your formula is valid by using mathematical induction.

9 The Fibonacci numbers are defined by $f_1 = 1$, $f_2 = 1$ and $f_{n+1} = f_n + f_{n-1}$.

a Determine f_n for $n = 1, 2, \dots, 10$.

b Prove that $f_1 + f_2 + \cdots + f_n = f_{n+2} - 1$.

c Evaluate $f_1 + f_3 + \cdots + f_{2n-1}$ for $n = 1, 2, 3, 4$.

d Try to find a formula for the above expression.

e Confirm that your formula works using mathematical induction.

f Using induction, prove that every third Fibonacci number, f_{3n}, is even.

10 Prove that $4^n + 5^n$ is divisible by 9 for all odd integers n.

11 Prove by induction that, for all $n \in \mathbb{N}$, every set of numbers S with exactly n elements has a largest element.

12 Standing around a circle, there are n friends and n thieves. You begin with no money, but as you go around the circle clockwise, each friend will give you \$1 and each thief will steal \$1. Prove that no matter where the friends and thieves are placed, it is possible to walk once around the circle without going into debt, provided you start at the correct point.

13 Prove by induction that every natural number $n \geq 2$ is divisible by some prime number.
Hint: Let $P(n)$ be the statement that every integer j such that $2 \leq j \leq n$ is divisible by some prime number.

14 If n straight lines are drawn across a sheet of paper, they will divide the paper into regions. Show that it is always possible to colour each region black or white, so that no two adjacent regions have the same colour.

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Chapter summary

- A **conditional statement** has the form: If P is true, then Q is true.
  This can be abbreviated as $P \Rightarrow Q$, which is read 'P implies Q'.
- To give a **direct proof** of a conditional statement $P \Rightarrow Q$, we assume that P is true and show that Q follows.
- The **converse** of $P \Rightarrow Q$ is $Q \Rightarrow P$.
- Statements P and Q are **equivalent** if $P \Rightarrow Q$ and $Q \Rightarrow P$. We write $P \Leftrightarrow Q$.
- The **contrapositive** of $P \Rightarrow Q$ is (not Q) $\Rightarrow$ (not P).
- Proving the contrapositive of a statement may be easier than giving a direct proof.
- A **proof by contradiction** begins by assuming the negation of what is to be proved.
- A **universal statement** claims that a property holds for all members of a given set. Such a statement can be written using the quantifier **'for all'**.
- An **existence statement** claims that a property holds for some member of a given set. Such a statement can be written using the quantifier **'there exists'**.
- **Counterexamples** can be used to demonstrate that a universal statement is false.
- **Mathematical induction** is used to prove that a statement is true for all natural numbers.

Skills checklist

Download this checklist from the Interactive Textbook, then print it and fill it out to check your skills. ☑

7A **1** I can give direct proofs of conditional statements. ☐

See Example 1, Example 2, Question 1 and Question 4

7B **2** I can negate a statement. ☐

See Example 6, Example 7, Question 1 and Question 2

7B **3** I can write down and prove a contrapositive statement. ☐

See Example 9, Example 10, and Question 5

7C **4** I can give a proof by contradiction. ☐

See Example 12, Example 14, Question 1 and Question 4

7D **5** I can write down and prove a converse statement. ☐

See Example 16 and Question 2

7D **6** I can show when two statements are equivalent. ☐

See Example 17, Question 4 and Question 5

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Review

7E **7** I can disprove both universal and existence statements. ☐

See Example 20, Example 22 and Question 4

7F **8** I can give a proof by mathematical induction. ☐

See Example 24, Example 25, Question 1 and Question 2

Short-response questions

Technology-free short-response questions

SF

1 For each of the following statements, if the statement is true, then prove it, and otherwise give a counterexample to show that it is false:

a The sum of any three consecutive integers is divisible by 3.

b The sum of any four consecutive integers is divisible by 4.

2 Assume that n is even. Prove that $n^2 - 3n + 1$ is odd.

CF

3 Let $n \in \mathbb{Z}$. Consider the statement: If n^3 is even, then n is even.

a Write down the contrapositive of this statement.

b Prove the contrapositive.

c Hence, prove by contradiction that $\sqrt[3]{6}$ is irrational.

4 **a** Show that one of three consecutive integers is always divisible by 3.

b Hence, prove that $n^3 + 3n^2 + 2n$ is divisible by 3 for all $n \in \mathbb{Z}$.

5 **a** Suppose that both m and n are divisible by d. Prove that $m - n$ is divisible by d.

b Hence, prove that the highest common factor of two consecutive integers is 1.

c Determine the highest common factor of 1002 and 999.

6 A student claims that $\sqrt{x+y} = \sqrt{x} + \sqrt{y}$, for all $x \geq 0$ and $y \geq 0$.

a Using a counterexample, prove that the equation is not always true.

b Prove that $\sqrt{x+y} = \sqrt{x} + \sqrt{y}$ if and only if $x = 0$ or $y = 0$.

7 Let $n \in \mathbb{Z}$. Prove that $n^2 + 3n + 4$ is even.

Hint: Consider the cases when n is odd and n is even.

8 Suppose that a, b, c and d are positive integers.

a Provide a counterexample to disprove the equation

$$\frac{a}{b} + \frac{c}{d} = \frac{a+c}{b+d}$$

b Now suppose that $\frac{c}{d} > \frac{a}{b}$. Prove that

$$\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$$

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Review

SF

9 Prove by mathematical induction that:

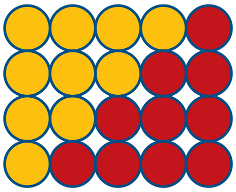
a $6^n + 4$ is divisible by 10 for all $n \in \mathbb{N}$

b $1^2 + 3^2 + \cdots + (2n-1)^2 = \dfrac{n(2n-1)(2n+1)}{3}$ for all $n \in \mathbb{N}$

CU

10 **a** Use the diagram on the right to deduce the equation

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2} \qquad (1)$$



b Using equation (1), prove that the sum $1 + 2 + \cdots + 99$ is divisible by 99.

c Using equation (1), prove that if n is odd, then the sum of any n consecutive odd natural numbers is divisible by n.

d With the help of equation (1) and mathematical induction, prove that

$$1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2 \quad \text{for all } n \in \mathbb{N}$$

11 Define $n! = n \times (n-1) \times \cdots \times 2 \times 1$.

a Prove that $10! + 2, 10! + 3, \ldots, 10! + 10$ are nine consecutive composite numbers.
Hint: The first number is divisible by 2.

b Find a sequence of ten consecutive composite numbers.

12 We call (a, b, c) a Pythagorean triple if a, b, c are natural numbers such that $a^2 + b^2 = c^2$.

a Let $n \in \mathbb{N}$. Prove that if (a, b, c) is a Pythagorean triple, then so is (na, nb, nc).

b Prove that there is only one Pythagorean triple (a, b, c) of consecutive natural numbers.

c Prove that there is no Pythagorean triple (a, b, c) containing the numbers 1 or 2.

13 Let a be an integer that is not divisible by 3. We know that $a = 3k + 1$ or $a = 3k + 2$, for some $k \in \mathbb{Z}$.

a Show that a^2 must leave a remainder of 1 when divided by 3.

b Hence, prove that if (a, b, c) is any Pythagorean triple, then a or b is divisible by 3.

14 **a** Prove by mathematical induction that $n^2 + n$ is even for all $n \in \mathbb{N}$.

b Find an easier proof by factorising $n^2 + n$.

c Hence, prove that if n is odd, then there exists an integer k such that $n^2 = 8k + 1$.

CF

15 Let $n \in \mathbb{Z}$ and consider the statement: If n is divisible by 8, then n^2 is divisible by 8.

a Prove the statement.

b Write down the converse of the statement.

c If the converse is true, prove it. Otherwise, give a counterexample.

Review

CU

16 Goldbach's conjecture is that every even integer greater than 2 can be expressed as the sum of two primes. To date, no one has been able to prove this, although it has been verified for all integers less than 4×10^{18}.

a Express 100 and 102 as the sum of two prime numbers.

b Prove that 101 cannot be written as the sum of two prime numbers.

c Express 101 as the sum of three prime numbers.

d Assuming that Goldbach's conjecture is true, prove that every odd integer greater than 5 can be written as the sum of three prime numbers.

17 **a** Simplify the expression $\dfrac{1}{n-1} - \dfrac{1}{n}$.

b Hence, show that

$$\frac{1}{2 \times 1} + \frac{1}{3 \times 2} + \frac{1}{4 \times 3} + \cdots + \frac{1}{n(n-1)} = 1 - \frac{1}{n}$$

c Give another proof of the above equation using mathematical induction.

d Using the above equation, prove that

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} < 2 \quad \text{for all } n \in \mathbb{N}$$

18 **a** Let $x \geq 0$ and $y \geq 0$. Prove that

$$\frac{x+y}{2} \geq \sqrt{xy}$$

by substituting $x = a^2$ and $y = b^2$ into $\dfrac{x+y}{2} - \sqrt{xy}$.

b Using the above inequality, or otherwise, prove each of the following:

i If $a > 0$, then $a + \dfrac{1}{a} \geq 2$.

ii If a, b and c are positive real numbers, then $(a+b)(b+c)(c+a) \geq 8abc$.

iii If a, b and c are positive real numbers, then $a^2 + b^2 + c^2 \geq ab + bc + ca$.

c Take any rectangle of length x and width y. Prove that a square with the same perimeter has an area greater than or equal to that of the rectangle.

19 Exactly one of the following three people is lying. Who is the liar?

- Jay says: 'Kaye is lying.'
- Kaye say: 'Elle is lying.'
- Elle says: 'I am not lying.'

20 There are four sentences written below. Which of them is true?

- Exactly one of these statements is false.
- Exactly two of these statements are false.
- Exactly three of these statements are false.
- Exactly four of these statements are false.

21 We will say that a set of numbers can be **split** if it can be divided into two groups so that no two numbers appear in the same group as their sum. For example, the set $\{1, 2, 3, 4, 5, 6\}$ can be split into the two groups $\{1, 2, 4\}$ and $\{3, 5, 6\}$.

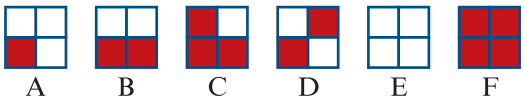
a Prove that the set $\{1, 2, \dots, 8\}$ can be split.

b Hence, explain why the set $\{1, 2, \dots, n\}$ can be split, where $1 \le n \le 8$.

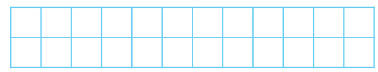
c Prove that it is impossible to split the set $\{1, 2, \dots, 9\}$.

d Hence, prove that it is impossible to split the set $\{1, 2, \dots, n\}$, where $n \ge 9$.

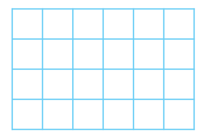
22 Consider the set of six 2×2 square tiles shown below.



a Tile the 2×12 grid shown using all six tiles, so that neighbouring squares have matching colours along the boundaries between tiles. Tiles can be rotated.



b Prove that there are only four ways to tile the 4×6 grid shown using all six tiles, so that neighbouring squares have matching colours along the boundaries between tiles. Tiles can be rotated.



Multiple-choice questions

Technology-free multiple-choice questions

1 If m is even and n is odd, then which of these statements is true?

A $m + n$ is even
B $m \times n$ is odd
C $m^2 - n^2$ is even
D $m - 3n$ is odd

2 If m is divisible by 6 and n is divisible by 15, then which of these statements might be false?

A $m \times n$ is divisible by 30
B $m \times n$ is divisible by 15
C $m + n$ is divisible by 3
D $m + n$ is divisible by 15

3 The contrapositive of $P \Rightarrow Q$ is

A $Q \Rightarrow P$
B $(\text{not } P) \Rightarrow (\text{not } Q)$
C $(\text{not } Q) \Rightarrow (\text{not } P)$
D $Q \Leftrightarrow P$

4 The converse of $P \Rightarrow Q$ is

A $(\text{not } Q) \Rightarrow (\text{not } P)$
B $Q \Rightarrow P$
C $Q \Leftrightarrow Q$
D $(\text{not } P) \Leftrightarrow (\text{not } Q)$

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5 The number of pairs of integers (m, n) that satisfy $m + n = mn$ is

A 0 **B** 1 **C** 2 **D** 3

6 If a, b and c are any real numbers with $a > b$, the statement that must be true is

A $\frac{1}{a} > \frac{1}{b}$ **B** $\frac{1}{a} < \frac{1}{b}$ **C** $ac > bc$ **D** $a + c > b + c$

7 If $n = (m-1)(m-2)(m-3)$ where m is an integer, then n will not always be divisible by

A 1 **B** 2 **C** 3 **D** 5

8 Let $m, n \in \mathbb{Z}$. Which of the following statements is false?

A n is even if and only if $n + 1$ is odd
B $m + n$ is odd if and only if $m - n$ is odd
C $m + n$ is even if and only if m and n are even
D m and n are odd if and only if mn is odd

9 The number n is neither even nor a square. Which of the following statements is logically equivalent?

A The number n is not even or not a square.
B The number n is not even and not a square.
C The number n is even or a square.
D The number n is not even.

10 Let m and n be integers and consider the statement:

If m is divisible by 2 and n is divisible by 5, then mn is divisible by 10.

The contrapositive of this statement is:

A If mn is not divisible by 10, then m is not divisible by 2 or n is not divisible by 5.
B If mn is not divisible by 10, then m is divisible by 2 or n is divisible by 5.
C If mn is not divisible by 10, then m is not divisible by 2 or n is not divisible by 5.
D If mn is divisible by 10, then m is divisible by 2 and n is divisible by 5.

8

Vectors in the plane

Chapter contents

In scientific experiments, some of the things that are measured are completely determined by their magnitude. Mass, length and time are determined by a number and an appropriate unit of measurement.

length	30 cm is the length of the page of a particular book
time	10 s is the time for one athlete to run 100 m

More is required to describe displacement, velocity or force. The direction must be recorded as well as the magnitude.

displacement	30 km in the direction north
velocity	60 km/h in the direction south-east

A quantity that has both a magnitude and a direction is called a **vector**.

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8A Introduction to vectors

Learning intentions

- To be able to represent a vector in two dimensions as a 2×1 column matrix.
- To be able to add and subtract vectors and multiply a vector by a scalar.
- To be able to establish when two vectors are parallel.
- To be able to determine the midpoint of a line segment.

Suppose that you are asked: ‘Where is your school in relation to your house?’

It is not enough to give an answer such as ‘four kilometres’. You need to specify a direction as well as a distance. You could give the answer ‘four kilometres north-east’.

Position is an example of a vector quantity.

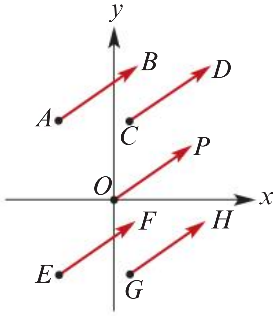
Directed line segments

A quantity that has a direction as well as a magnitude can be represented by an arrow:

- the arrow points in the direction of the action
- the length of the arrow gives the magnitude of the quantity in terms of a suitably chosen unit.

Arrows with the same length and direction are regarded as equivalent. These arrows are **directed line segments** and the sets of equivalent segments are called **vectors**.

The five directed line segments shown all have the same length and direction, and so they are equivalent.



A directed line segment from a point A to a point B is denoted by $\overrightarrow{AB}$.

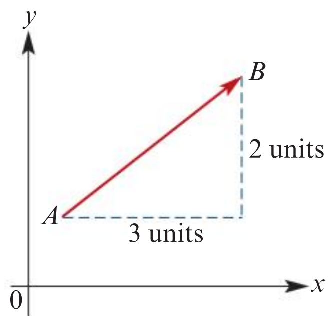
For simplicity of language, this is also called vector $\overrightarrow{AB}$. That is, the set of equivalent segments can be named through one member of the set.

Note: The five directed line segments in the diagram all name the same vector: $\overrightarrow{AB} = \overrightarrow{CD} = \overrightarrow{OP} = \overrightarrow{EF} = \overrightarrow{GH}$.

Column vectors

We can also represent a vector using a column of numbers. The column of numbers corresponds to a set of equivalent directed line segments.

For example, the column $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ corresponds to the directed line segments which go 3 across and 2 up.



Vector notation

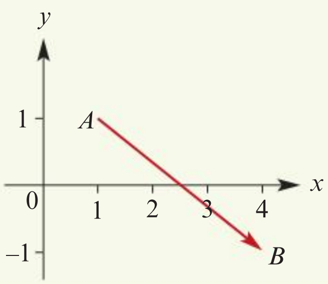
A vector is often denoted by a single bold lowercase letter. The vector from A to B can be denoted by $\overrightarrow{AB}$ or by a single letter $\boldsymbol{v}$. That is, $\boldsymbol{v} = \overrightarrow{AB}$.

When a vector is handwritten, the notation is $\underset{\sim}{v}$.

Example 1

Draw a directed line segment corresponding to $\begin{bmatrix} 3 \\ -2 \end{bmatrix}$.

Solution



Explanation

The vector $\begin{bmatrix} 3 \\ -2 \end{bmatrix}$ is '3 to the right and 2 down'.

Note: Here the segment starts at (1, 1) and goes to (4, −1). It can start at any point.

Example 2

The vector $\boldsymbol{u}$ is defined by the directed line segment from (2, 6) to (3, 1).

If $\boldsymbol{u} = \begin{bmatrix} a \\ b \end{bmatrix}$, determine a and b.

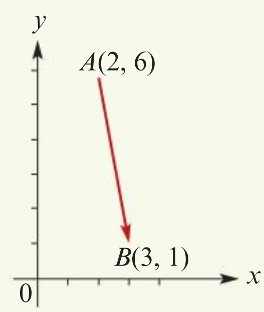
Solution

The vector is

$$\boldsymbol{u} = \begin{bmatrix} 3-2 \\ 1-6 \end{bmatrix} = \begin{bmatrix} 1 \\ -5 \end{bmatrix}$$

Hence $a = 1$ and $b = -5$.

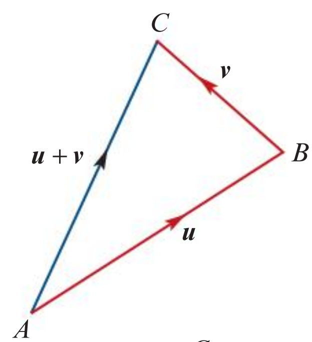
Explanation



Addition of vectors

Adding vectors geometrically

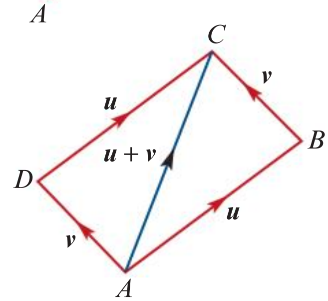
Two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ can be added geometrically by drawing a line segment representing $\boldsymbol{u}$ from A to B and then a line segment representing $\boldsymbol{v}$ from B to C.



The sum $\boldsymbol{u} + \boldsymbol{v}$ is the vector from A to C. That is,

$$\boldsymbol{u} + \boldsymbol{v} = \overrightarrow{AC}$$

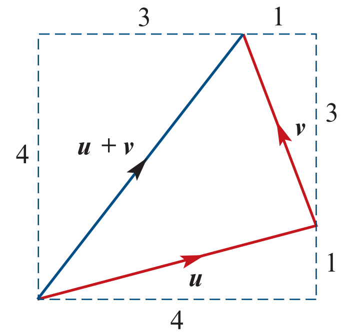
The same result is achieved if the order is reversed. This is represented in the diagram on the right:



$$\begin{aligned} \boldsymbol{u} + \boldsymbol{v} &= \overrightarrow{AC} \\ &= \boldsymbol{v} + \boldsymbol{u} \end{aligned}$$

Adding column vectors

Two vectors can be added using column-vector notation.



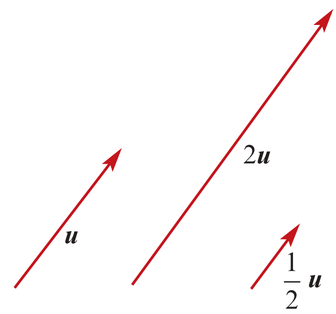
For example, if $\boldsymbol{u} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ and $\boldsymbol{v} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$, then

$$\boldsymbol{u} + \boldsymbol{v} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

Scalar multiplication

Multiplication by a real number (scalar) changes the length of the vector. For example:

- $2\boldsymbol{u}$ is twice the length of $\boldsymbol{u}$
- $\frac{1}{2}\boldsymbol{u}$ is half the length of $\boldsymbol{u}$

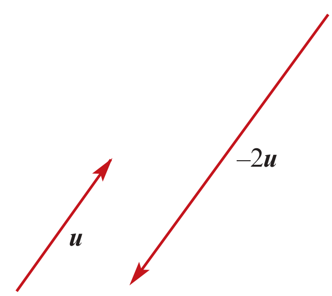


We have $2\boldsymbol{u} = \boldsymbol{u} + \boldsymbol{u}$ and $\frac{1}{2}\boldsymbol{u} + \frac{1}{2}\boldsymbol{u} = \boldsymbol{u}$.

In general, for $k \in \mathbb{R}^+$, the vector $k\boldsymbol{u}$ has the same direction as $\boldsymbol{u}$, but its length is multiplied by a factor of k.

When a vector is multiplied by -2, the vector's direction is reversed and the length is doubled.

When a vector is multiplied by -1, the vector's direction is reversed and the length remains the same.



If $\boldsymbol{u} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, then $-\boldsymbol{u} = \begin{bmatrix} -3 \\ -2 \end{bmatrix}$, $2\boldsymbol{u} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$ and $-2\boldsymbol{u} = \begin{bmatrix} -6 \\ -4 \end{bmatrix}$.

If $\boldsymbol{u} = \overrightarrow{AB}$, then

$$-\boldsymbol{u} = -\overrightarrow{AB} = \overrightarrow{BA}$$

The directed line segment $-\overrightarrow{AB}$ goes from B to A.

Zero vector

The **zero vector** is denoted by **0** and represents a line segment of zero length. The zero vector has no direction.

Subtraction of vectors

To determine $\boldsymbol{u} - \boldsymbol{v}$, we add $-\boldsymbol{v}$ to $\boldsymbol{u}$.

Example 3

For the vectors $\boldsymbol{u} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ and $\boldsymbol{v} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$, determine $2\boldsymbol{u} + 3\boldsymbol{v}$.

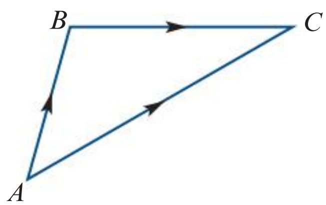
Solution

$$2\boldsymbol{u} + 3\boldsymbol{v} = 2\begin{bmatrix} 3 \\ -1 \end{bmatrix} + 3\begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \end{bmatrix} + \begin{bmatrix} -6 \\ 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$$

Polygons of vectors

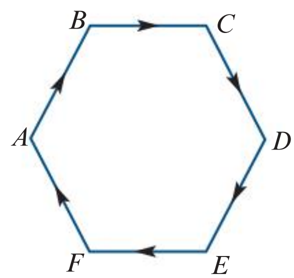
- For two vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$, we have

  $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$



- For a polygon $ABCDEF$, we have

  $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE} + \overrightarrow{EF} + \overrightarrow{FA} = \mathbf{0}$



Parallel vectors

Two parallel vectors have the same direction or opposite directions.

Two non-zero vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are **parallel** if there is some $k \in \mathbb{R} \setminus \{0\}$ such that $\boldsymbol{u} = k\boldsymbol{v}$.

For example, if $\boldsymbol{u} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$ and $\boldsymbol{v} = \begin{bmatrix} -6 \\ 9 \end{bmatrix}$, then the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are parallel as $\boldsymbol{v} = 3\boldsymbol{u}$.

Position vectors

We can use a point O, the origin, as a starting point for a vector to indicate the position of a point A in space relative to O.

In this chapter, we study vectors in two dimensions and the point O is the origin of the Cartesian plane. (Vectors in three dimensions are studied in Specialist Mathematics Units 3 & 4.)

For a point A, the **position vector** is $\overrightarrow{OA}$.

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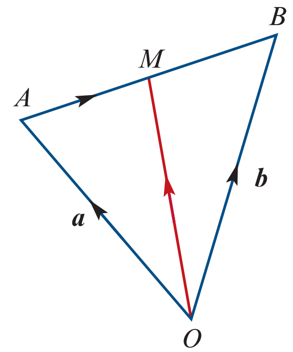
Determining the midpoint of a line segment

Let M be the midpoint of a line segment AB, where the points A and B have position vectors $\boldsymbol{a} = \overrightarrow{OA}$ and $\boldsymbol{b} = \overrightarrow{OB}$ respectively. Then



$$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = -\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{b} - \boldsymbol{a}$$

The position vector of the midpoint M is

$$\begin{aligned}\overrightarrow{OM} &= \overrightarrow{OA} + \frac{1}{2}\overrightarrow{AB} \\ &= \boldsymbol{a} + \frac{1}{2}(\boldsymbol{b} - \boldsymbol{a}) \\ &= \frac{1}{2}(\boldsymbol{a} + \boldsymbol{b})\end{aligned}$$

If M is the midpoint of line segment AB, then

$$\overrightarrow{OM} = \frac{1}{2}(\overrightarrow{OA} + \overrightarrow{OB})$$

Linear combinations of non-parallel vectors

If two non-zero vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are not parallel, then

$$m\boldsymbol{a} + n\boldsymbol{b} = p\boldsymbol{a} + q\boldsymbol{b} \quad \text{implies} \quad m = p \text{ and } n = q$$

Proof Assume that $m\boldsymbol{a} + n\boldsymbol{b} = p\boldsymbol{a} + q\boldsymbol{b}$. Then

$$m\boldsymbol{a} - p\boldsymbol{a} = q\boldsymbol{b} - n\boldsymbol{b}$$

$$\therefore \quad (m - p)\boldsymbol{a} = (q - n)\boldsymbol{b}$$

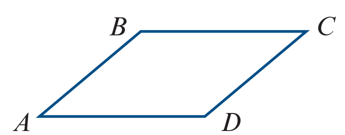
If $m \neq p$ or $n \neq q$, we could therefore write

$$\boldsymbol{a} = \frac{q - n}{m - p}\boldsymbol{b} \quad \text{or} \quad \boldsymbol{b} = \frac{m - p}{q - n}\boldsymbol{a}$$

But this is not possible, as $\boldsymbol{a}$ and $\boldsymbol{b}$ are non-zero vectors that are not parallel. Therefore $m = p$ and $n = q$.

Parallelograms

A parallelogram $ABCD$ is a quadrilateral whose opposite sides are parallel. Therefore $\overrightarrow{DC} = k\overrightarrow{AB}$ and $\overrightarrow{BC} = \ell\overrightarrow{AD}$, for some $k, \ell \in \mathbb{R} \setminus \{0\}$. But we have

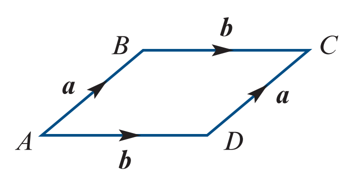


$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AD} + \overrightarrow{DC}$$

$$\therefore \quad \overrightarrow{AB} + \ell\overrightarrow{AD} = \overrightarrow{AD} + k\overrightarrow{AB}$$

So the previous result gives $k = 1$ and $\ell = 1$.

We have proved that every parallelogram is spanned by two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ as shown on the right.



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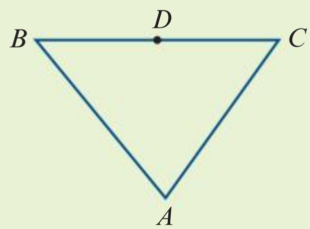
Example 4

Let A, B and C be the vertices of a triangle, and let D be the midpoint of BC.



Let $\boldsymbol{a} = \overrightarrow{AB}$ and $\boldsymbol{b} = \overrightarrow{BC}$.

Determine each of the following in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$:

a $\overrightarrow{BD}$ **b** $\overrightarrow{DC}$ **c** $\overrightarrow{AC}$

d $\overrightarrow{AD}$ **e** $\overrightarrow{CA}$

Solution

a $\overrightarrow{BD} = \frac{1}{2}\overrightarrow{BC} = \frac{1}{2}\boldsymbol{b}$

b $\overrightarrow{DC} = \overrightarrow{BD} = \frac{1}{2}\boldsymbol{b}$

c $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} = \boldsymbol{a} + \boldsymbol{b}$

d $\overrightarrow{AD} = \overrightarrow{AB} + \overrightarrow{BD} = \boldsymbol{a} + \frac{1}{2}\boldsymbol{b}$

e $\overrightarrow{CA} = -\overrightarrow{AC} = -(\boldsymbol{a} + \boldsymbol{b})$

Explanation

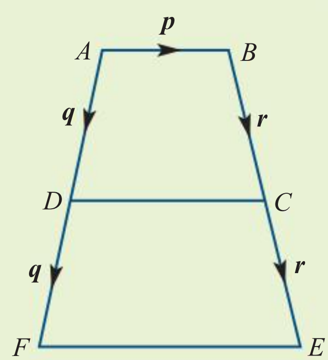
a: same direction and half the length

b: equivalent vectors

e: since $\overrightarrow{CA} + \overrightarrow{AC} = \boldsymbol{0}$

Example 5

In the figure, $\overrightarrow{DC} = k\boldsymbol{p}$ where $k \in \mathbb{R} \setminus \{0\}$.



a Express $\boldsymbol{p}$ in terms of k, $\boldsymbol{q}$ and $\boldsymbol{r}$.

b Express $\overrightarrow{FE}$ in terms of k and $\boldsymbol{p}$ to show that FE is parallel to DC.

c If $\overrightarrow{FE} = 4\overrightarrow{AB}$, determine the value of k.

Solution

a $\boldsymbol{p} = \overrightarrow{AB}$

$= \overrightarrow{AD} + \overrightarrow{DC} + \overrightarrow{CB}$

$= \boldsymbol{q} + k\boldsymbol{p} - \boldsymbol{r}$

Therefore

$(1-k)\boldsymbol{p} = \boldsymbol{q} - \boldsymbol{r}$

$\boldsymbol{p} = \frac{1}{1-k}(\boldsymbol{q} - \boldsymbol{r})$

b $\overrightarrow{FE} = -2\boldsymbol{q} + \boldsymbol{p} + 2\boldsymbol{r}$

$= 2(\boldsymbol{r} - \boldsymbol{q}) + \boldsymbol{p}$

From part **a**, we have

$\boldsymbol{r} - \boldsymbol{q} = k\boldsymbol{p} - \boldsymbol{p}$

$= (k-1)\boldsymbol{p}$

Therefore

$\overrightarrow{FE} = 2(k-1)\boldsymbol{p} + \boldsymbol{p}$

$= 2k\boldsymbol{p} - 2\boldsymbol{p} + \boldsymbol{p}$

$= (2k-1)\boldsymbol{p}$

c $\overrightarrow{FE} = 4\overrightarrow{AB}$

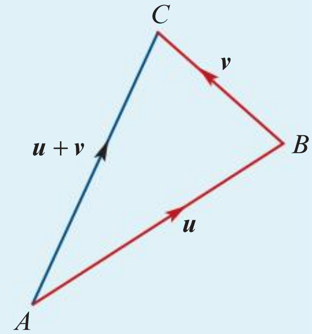
$(2k-1)\boldsymbol{p} = 4\boldsymbol{p}$

$2k - 1 = 4$

$\therefore \quad k = \frac{5}{2}$

Summary 8A

- A **vector** is a set of equivalent **directed line segments**.



- Addition of vectors
  If $\boldsymbol{u} = \overrightarrow{AB}$ and $\boldsymbol{v} = \overrightarrow{BC}$, then $\boldsymbol{u} + \boldsymbol{v} = \overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$.
- Scalar multiplication
  - For $k \in \mathbb{R}^+$, the vector $k\boldsymbol{u}$ has the same direction as $\boldsymbol{u}$, but its length is multiplied by a factor of k.
  - If $\boldsymbol{u} = \overrightarrow{AB}$, then $-\boldsymbol{u} = -\overrightarrow{AB} = \overrightarrow{BA}$.
- Zero vector
  The **zero vector**, denoted by $\mathbf{0}$, has zero length and has no direction.
- Subtraction of vectors
  $\boldsymbol{u} - \boldsymbol{v} = \boldsymbol{u} + (-\boldsymbol{v})$

- Parallel vectors
  Two non-zero vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are **parallel** if there is some $k \in \mathbb{R} \setminus \{0\}$ such that $\boldsymbol{u} = k\boldsymbol{v}$.
- Midpoint of a line segment
  If M is the midpoint of line segment AB, then $\overrightarrow{OM} = \frac{1}{2}(\overrightarrow{OA} + \overrightarrow{OB})$.

Skill-sheet

Exercise 8A

SF

Example 1

1 On the same graph, draw arrows which represent the following vectors:

a $\begin{bmatrix} 1 \\ 5 \end{bmatrix}$ **b** $\begin{bmatrix} 0 \\ -2 \end{bmatrix}$ **c** $\begin{bmatrix} -1 \\ -2 \end{bmatrix}$ **d** $\begin{bmatrix} -4 \\ 3 \end{bmatrix}$

Example 2

2 The vector $\boldsymbol{u}$ is defined by the directed line segment from $(1, 5)$ to $(6, 6)$.
If $\boldsymbol{u} = \begin{bmatrix} a \\ b \end{bmatrix}$, determine a and b.

3 The vector $\boldsymbol{v}$ is defined by the directed line segment from $(-1, 5)$ to $(2, -10)$.
If $\boldsymbol{v} = \begin{bmatrix} a \\ b \end{bmatrix}$, determine a and b.

4 Let $A = (1, -2)$, $B = (3, 0)$ and $C = (2, -3)$ and let O be the origin.
Express each of the following vectors in the form $\begin{bmatrix} a \\ b \end{bmatrix}$:

a $\overrightarrow{OA}$ **b** $\overrightarrow{AB}$ **c** $\overrightarrow{BC}$ **d** $\overrightarrow{CO}$ **e** $\overrightarrow{CB}$

Example 3

5 Let $\boldsymbol{a} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\boldsymbol{b} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ and $\boldsymbol{c} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

a Determine:

i $\boldsymbol{a} + \boldsymbol{b}$ **ii** $2\boldsymbol{c} - \boldsymbol{a}$ **iii** $\boldsymbol{a} + \boldsymbol{b} - \boldsymbol{c}$

b Show that $\boldsymbol{a} + \boldsymbol{b}$ is parallel to $\boldsymbol{c}$.

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SF

6 If $A = (2, -3)$, $B = (4, 0)$, $C = (1, -4)$ and O is the origin, sketch the following vectors:

a $\overrightarrow{OA}$ **b** $\overrightarrow{AB}$ **c** $\overrightarrow{BC}$ **d** $\overrightarrow{CO}$ **e** $\overrightarrow{CB}$

7 On graph paper, sketch the vectors joining the following pairs of points in the direction indicated:

a $(0, 0) \to (2, 1)$ **b** $(3, 4) \to (0, 0)$ **c** $(1, 3) \to (3, 4)$
d $(2, 4) \to (4, 3)$ **e** $(-2, 2) \to (5, -1)$ **f** $(-1, -3) \to (3, 0)$

8 Identify vectors from Question 7 which are parallel to each other.

CF

9 **a** Plot the points $A(-1, 0)$, $B(1, 4)$, $C(4, 3)$ and $D(2, -1)$ on a set of coordinate axes.
b Sketch the vectors $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{AD}$ and $\overrightarrow{DC}$.
c Show that:
i $\overrightarrow{AB} = \overrightarrow{DC}$ **ii** $\overrightarrow{BC} = \overrightarrow{AD}$
d Describe the shape of the quadrilateral $ABCD$.

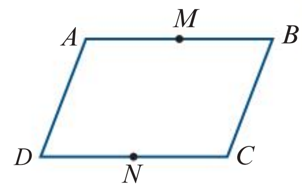
SF

10 Determine the values of m and n such that $m\begin{bmatrix} 3 \\ -3 \end{bmatrix} + n\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} -19 \\ 61 \end{bmatrix}$

CF

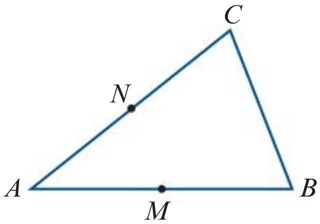
Example 4

11 Points A, B, C, D are the vertices of a parallelogram, and M and N are the midpoints of AB and DC respectively. Let $\boldsymbol{a} = \overrightarrow{AB}$ and $\boldsymbol{b} = \overrightarrow{AD}$.



a Express the following in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$:
i $\overrightarrow{MD}$ **ii** $\overrightarrow{MN}$
b Determine the relationship between $\overrightarrow{MN}$ and $\overrightarrow{AD}$.

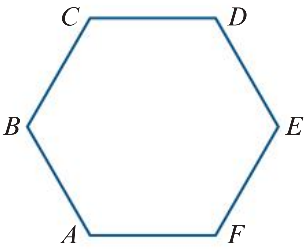
12 The figure represents the triangle ABC, where M and N are the midpoints of AB and AC respectively. Let $\boldsymbol{a} = \overrightarrow{AB}$ and $\boldsymbol{b} = \overrightarrow{AC}$.



a Express $\overrightarrow{CB}$ and $\overrightarrow{MN}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$.
b Hence describe the relation between the two vectors (or directed line segments).

Example 5

13 The figure shows a regular hexagon $ABCDEF$. Let $\boldsymbol{a} = \overrightarrow{AF}$ and $\boldsymbol{b} = \overrightarrow{AB}$. Express the following vectors in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$:



a $\overrightarrow{CD}$ **b** $\overrightarrow{ED}$ **c** $\overrightarrow{BE}$ **d** $\overrightarrow{FC}$
e $\overrightarrow{FA}$ **f** $\overrightarrow{FB}$ **g** $\overrightarrow{FE}$

14 In parallelogram $ABCD$, let $\boldsymbol{a} = \overrightarrow{AB}$ and $\boldsymbol{b} = \overrightarrow{BC}$. Express each of the following vectors in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$:

a $\overrightarrow{DC}$ **b** $\overrightarrow{DA}$ **c** $\overrightarrow{AC}$ **d** $\overrightarrow{CA}$ **e** $\overrightarrow{BD}$

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15 In triangle OAB, let $\boldsymbol{a} = \overrightarrow{OA}$ and $\boldsymbol{b} = \overrightarrow{OB}$. The point P on AB is such that $\overrightarrow{AP} = 2\overrightarrow{PB}$ and the point Q is such that $\overrightarrow{OP} = 3\overrightarrow{PQ}$. Express each of the following vectors in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$:

a $\overrightarrow{BA}$ **b** $\overrightarrow{PB}$ **c** $\overrightarrow{OP}$ **d** $\overrightarrow{PQ}$ **e** $\overrightarrow{BQ}$

16 $PQRS$ is a quadrilateral in which $\overrightarrow{PQ} = \boldsymbol{u}$, $\overrightarrow{QR} = \boldsymbol{v}$ and $\overrightarrow{RS} = \boldsymbol{w}$. Express each of the following vectors in terms of $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$:

a $\overrightarrow{PR}$ **b** $\overrightarrow{QS}$ **c** $\overrightarrow{PS}$

17 $OABC$ is a parallelogram. Let $\boldsymbol{u} = \overrightarrow{OA}$ and $\boldsymbol{v} = \overrightarrow{OC}$. Let M be the midpoint of AB.

a Express $\overrightarrow{OB}$ and $\overrightarrow{OM}$ in terms of $\boldsymbol{u}$ and $\boldsymbol{v}$.

b Express $\overrightarrow{CM}$ in terms of $\boldsymbol{u}$ and $\boldsymbol{v}$.

c If P is a point on CM and $\overrightarrow{CP} = \frac{2}{3}\overrightarrow{CM}$, express $\overrightarrow{CP}$ in terms of $\boldsymbol{u}$ and $\boldsymbol{v}$.

d Determine $\overrightarrow{OP}$ and hence show that P lies on the line segment OB.

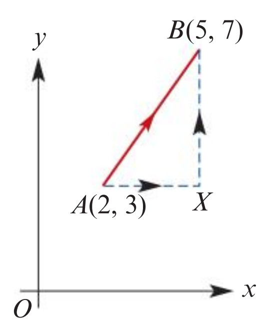
e Determine the ratio $OP : PB$.

8B Components of vectors

Learning intentions

- To be able to represent a vector in component form using unit vectors.
- To be able to determine the magnitude of a vector.

The vector $\overrightarrow{AB}$ in the diagram is described by the column vector $\begin{bmatrix}3\\4\end{bmatrix}$.



From the diagram, we see that the vector $\overrightarrow{AB}$ can also be expressed as the sum

$$\overrightarrow{AB} = \overrightarrow{AX} + \overrightarrow{XB}$$

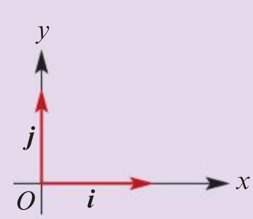
Using column-vector notation:

$$\begin{bmatrix}3\\4\end{bmatrix} = \begin{bmatrix}3\\0\end{bmatrix} + \begin{bmatrix}0\\4\end{bmatrix}$$

This suggests the introduction of two important vectors.

Standard unit vectors in two dimensions

- Let $\hat{\boldsymbol{i}}$ be the vector of unit length in the positive direction of the x-axis.
- Let $\hat{\boldsymbol{j}}$ be the vector of unit length in the positive direction of the y-axis.



Using column-vector notation, we have $\hat{\boldsymbol{i}} = \begin{bmatrix}1\\0\end{bmatrix}$ and $\hat{\boldsymbol{j}} = \begin{bmatrix}0\\1\end{bmatrix}$.

Sometimes we write $\hat{\boldsymbol{i}}$ and $\hat{\boldsymbol{j}}$ as $\boldsymbol{\imath}$ and $\boldsymbol{\jmath}$.

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For the example at the start of this section, we have $\overrightarrow{AX} = 3\hat{i}$ and $\overrightarrow{XB} = 4\hat{j}$. Therefore

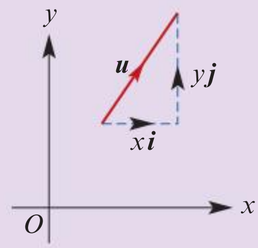
$$\begin{aligned}\overrightarrow{AB} &= \overrightarrow{AX} + \overrightarrow{XB} \\ &= 3\hat{i} + 4\hat{j}\end{aligned}$$

It is possible to describe any two-dimensional vector in this way.

Component form

- We can write the vector $\boldsymbol{u} = \begin{bmatrix} x \\ y \end{bmatrix}$ as $\boldsymbol{u} = x\hat{i} + y\hat{j}$.

  We say that $\boldsymbol{u}$ is the sum of the two **components** $x\hat{i}$ and $y\hat{j}$.
- The **magnitude** of vector $\boldsymbol{u} = x\hat{i} + y\hat{j}$ is denoted by $|\boldsymbol{u}|$ and is given by $|\boldsymbol{u}| = \sqrt{x^2 + y^2}$.



Operations with vectors now look more like basic algebra:

- $(x\hat{i} + y\hat{j}) + (m\hat{i} + n\hat{j}) = (x + m)\hat{i} + (y + n)\hat{j}$
- $k(x\hat{i} + y\hat{j}) = kx\hat{i} + ky\hat{j}$

Two vectors are equal if and only if their components are equal:

$$x\hat{i} + y\hat{j} = m\hat{i} + n\hat{j} \quad \text{if and only if} \quad x = m \text{ and } y = n$$

Example 6

Let A and B be points in the Cartesian plane such that $\overrightarrow{OA} = 3\hat{i}$ and $\overrightarrow{OB} = 2\hat{i} - \hat{j}$.

a Determine $\overrightarrow{AB}$.

b Determine $\overrightarrow{OM}$, where M is the midpoint of AB.

Solution

a
$$\begin{aligned}\overrightarrow{AB} &= \overrightarrow{AO} + \overrightarrow{OB} \\ &= -\overrightarrow{OA} + \overrightarrow{OB} \\ &= -3\hat{i} + (2\hat{i} - \hat{j}) \\ &= -\hat{i} - \hat{j}\end{aligned}$$

b
$$\begin{aligned}\overrightarrow{OM} &= \frac{1}{2}(\overrightarrow{OA} + \overrightarrow{OB}) \\ &= \frac{1}{2}(3\hat{i} + 2\hat{i} - \hat{j}) \\ &= \frac{5}{2}\hat{i} - \frac{1}{2}\hat{j}\end{aligned}$$

Example 7

Determine the magnitude of the vector $2\hat{i} - 3\hat{j}$.

Solution

$$\begin{aligned}|2\hat{i} - 3\hat{j}| &= \sqrt{2^2 + (-3)^2} \\ &= \sqrt{4 + 9} \\ &= \sqrt{13}\end{aligned}$$

Example 8

Let A and B be points in the Cartesian plane such that $\overrightarrow{OA} = 2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}$ and $\overrightarrow{OB} = \hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}}$.

Determine $\overrightarrow{AB}$ and $|\overrightarrow{AB}|$.

Solution

$$\begin{aligned}\overrightarrow{AB} &= \overrightarrow{AO} + \overrightarrow{OB}\\ &= -\overrightarrow{OA} + \overrightarrow{OB}\end{aligned}$$

$$\therefore \quad \begin{aligned}\overrightarrow{AB} &= -(2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}) + \hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}}\\ &= -\hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}}\end{aligned}$$

$$\therefore \quad |\overrightarrow{AB}| = \sqrt{1+16} = \sqrt{17}$$

Unit vectors

A **unit vector** is a vector of length one unit. For example, both $\hat{\boldsymbol{i}}$ and $\hat{\boldsymbol{j}}$ are unit vectors.

The unit vector in the direction of $\boldsymbol{a}$ is denoted by $\hat{\boldsymbol{a}}$. (We say 'a hat'.)

Since $|\hat{\boldsymbol{a}}| = 1$, we have

$$|\boldsymbol{a}|\,\hat{\boldsymbol{a}} = \boldsymbol{a}$$

$$\therefore \quad \hat{\boldsymbol{a}} = \frac{1}{|\boldsymbol{a}|}\boldsymbol{a}$$

Example 9

Let $\boldsymbol{a} = 3\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}}$.

Determine $|\boldsymbol{a}|$, the magnitude of $\boldsymbol{a}$, and hence find the unit vector in the direction of $\boldsymbol{a}$.

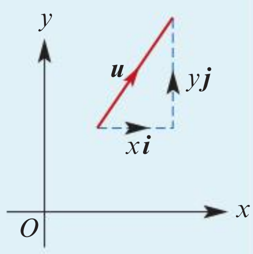
Solution

$$|\boldsymbol{a}| = \sqrt{9+16} = 5$$

$$\therefore \quad \hat{\boldsymbol{a}} = \frac{1}{|\boldsymbol{a}|}\boldsymbol{a} = \frac{1}{5}(3\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}})$$

Summary 8B

- A **unit vector** is a vector of length one unit.
- Each vector $\boldsymbol{u}$ in the plane can be written in **component form** as $\boldsymbol{u} = x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}}$, where
  - $\hat{\boldsymbol{i}}$ ($\boldsymbol{i}$) is the unit vector in the positive direction of the x-axis
  - $\hat{\boldsymbol{j}}$ ($\boldsymbol{j}$) is the unit vector in the positive direction of the y-axis.



- The **magnitude** of vector $\boldsymbol{u} = x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}}$ is given by $|\boldsymbol{u}| = \sqrt{x^2 + y^2}$.
- The unit vector in the direction of vector $\boldsymbol{a}$ is given by $\hat{\boldsymbol{a}} = \dfrac{1}{|\boldsymbol{a}|}\boldsymbol{a}$.

Skill-sheet

Exercise 8B

Example 6a

1 If A and B are points in the plane such that $\overrightarrow{OA} = \hat{\imath} + 2\hat{\jmath}$ and $\overrightarrow{OB} = 3\hat{\imath} - 5\hat{\jmath}$, determine $\overrightarrow{AB}$.

2 $OAPB$ is a rectangle with $\overrightarrow{OA} = 5\hat{\imath}$ and $\overrightarrow{OB} = 6\hat{\jmath}$. Express each of the following vectors in terms of $\hat{\imath}$ and $\hat{\jmath}$:

a $\overrightarrow{OP}$ **b** $\overrightarrow{AB}$ **c** $\overrightarrow{BA}$

Example 6b

3 Points A and B have position vectors $\overrightarrow{OA} = 10\hat{\imath}$ and $\overrightarrow{OB} = 4\hat{\imath} + 5\hat{\jmath}$. If M is the midpoint of AB, determine $\overrightarrow{OM}$ in terms of $\hat{\imath}$ and $\hat{\jmath}$.

Example 7

4 Determine the magnitude of each of the following vectors:

a $5\hat{\imath}$ **b** $-2\hat{\jmath}$ **c** $3\hat{\imath} + 4\hat{\jmath}$ **d** $-5\hat{\imath} + 12\hat{\jmath}$

5 The vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are given by $\boldsymbol{u} = 7\hat{\imath} + 8\hat{\jmath}$ and $\boldsymbol{v} = 2\hat{\imath} - 4\hat{\jmath}$.

a Determine $|\boldsymbol{u} - \boldsymbol{v}|$.

b Determine constants x and y such that $x\boldsymbol{u} + y\boldsymbol{v} = 44\hat{\jmath}$.

6 $OPAQ$ is a rectangle with $\overrightarrow{OP} = 2\hat{\imath}$ and $\overrightarrow{OQ} = \hat{\jmath}$. Let M be the point on OP such that $OM = \frac{1}{5}OP$ and let N be the point on MQ such that $MN = \frac{1}{6}MQ$.

a Determine each of the following vectors in terms of $\hat{\imath}$ and $\hat{\jmath}$:

i $\overrightarrow{OM}$ **ii** $\overrightarrow{MQ}$ **iii** $\overrightarrow{MN}$ **iv** $\overrightarrow{ON}$ **v** $\overrightarrow{OA}$

b **i** Hence show that N is on the diagonal OA.

ii State the ratio of the lengths $ON : NA$.

7 The position vectors of A and B are given by $\overrightarrow{OA} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\overrightarrow{OB} = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$. Determine the distance between A and B.

8 Determine the pronumerals in the following equations:

a $\hat{\imath} + 3\hat{\jmath} = 2(\ell\hat{\imath} + k\hat{\jmath})$ **b** $(x-1)\hat{\imath} + y\hat{\jmath} = 5\hat{\imath} + (x-4)\hat{\jmath}$

c $(x+y)\hat{\imath} + (x-y)\hat{\jmath} = 6\hat{\imath}$ **d** $k(\hat{\imath} + \hat{\jmath}) = 3\hat{\imath} - 2\hat{\jmath} + \ell(2\hat{\imath} - \hat{\jmath})$

Example 8

9 Let $A = (2, 3)$ and $B = (5, 1)$. Determine $\overrightarrow{AB}$ and $|\overrightarrow{AB}|$.

10 Let $\overrightarrow{OA} = 3\hat{\imath}$, $\overrightarrow{OB} = \hat{\imath} + 4\hat{\jmath}$ and $\overrightarrow{OC} = -3\hat{\imath} + \hat{\jmath}$. Determine:

a $\overrightarrow{AB}$ **b** $\overrightarrow{AC}$ **c** $|\overrightarrow{BC}|$

11 Let $A = (5, 1)$, $B = (0, 4)$ and $C = (-1, 0)$. Determine:

a D such that $\overrightarrow{AB} = \overrightarrow{CD}$

b F such that $\overrightarrow{AF} = \overrightarrow{BC}$

c G such that $\overrightarrow{AB} = 2\overrightarrow{GC}$

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SF

12 Let $\boldsymbol{a} = \hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}}$ and $\boldsymbol{b} = -2\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}}$. Points A, B and C are such that $\overrightarrow{AO} = \boldsymbol{a}$, $\overrightarrow{OB} = \boldsymbol{b}$ and $\overrightarrow{BC} = 2\boldsymbol{a}$, where O is the origin. Determine the coordinates of A, B and C.

CF

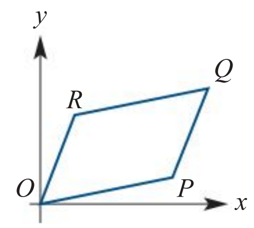
13 A, B, C and D are the vertices of a parallelogram and O is the origin. $A = (2, -1)$, $B = (-5, 4)$ and $C = (1, 7)$.

a Determine:

i $\overrightarrow{OA}$ **ii** $\overrightarrow{OB}$ **iii** $\overrightarrow{OC}$ **iv** $\overrightarrow{BC}$ **v** $\overrightarrow{AD}$

b Hence determine the coordinates of D.

14 The diagram shows a parallelogram $OPQR$. The points P and Q have coordinates $(12, 5)$ and $(18, 13)$ respectively.



a Determine $\overrightarrow{OP}$ and $\overrightarrow{PQ}$.

b Determine $|\overrightarrow{RQ}|$ and $|\overrightarrow{OR}|$.

15 $A(1, 6)$, $B(3, 1)$ and $C(13, 5)$ are the vertices of a triangle ABC.

a Determine:

i $|\overrightarrow{AB}|$ **ii** $|\overrightarrow{BC}|$ **iii** $|\overrightarrow{CA}|$

b Hence show that ABC is a right-angled triangle.

16 $A(4, 4)$, $B(3, 1)$ and $C(7, 3)$ are the vertices of a triangle ABC.

a Determine the vectors:

i $\overrightarrow{AB}$ **ii** $\overrightarrow{BC}$ **iii** $\overrightarrow{CA}$

b Determine:

i $|\overrightarrow{AB}|$ **ii** $|\overrightarrow{BC}|$ **iii** $|\overrightarrow{CA}|$

c Hence show that triangle ABC is a right-angled isosceles triangle.

17 Consider points $A(-3, 2)$ and $B(0, 7)$ in the Cartesian plane. Let O be the origin and let M be the point on the line segment AB such that $AM = \frac{1}{3}AB$.

a Determine:

i $\overrightarrow{OA}$ **ii** $\overrightarrow{OB}$ **iii** $\overrightarrow{BA}$ **iv** $\overrightarrow{BM}$

b Hence determine the coordinates of M. (**Hint:** $\overrightarrow{OM} = \overrightarrow{OB} + \overrightarrow{BM}$.)

SF

Example 9

18 Determine the unit vector in the direction of each of the following vectors:

a $\boldsymbol{a} = 3\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}}$ **b** $\boldsymbol{b} = 3\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}}$ **c** $\boldsymbol{c} = -\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}$

d $\boldsymbol{d} = \hat{\boldsymbol{i}} - \hat{\boldsymbol{j}}$ **e** $\boldsymbol{e} = \frac{1}{2}\hat{\boldsymbol{i}} + \frac{1}{3}\hat{\boldsymbol{j}}$ **f** $\boldsymbol{f} = 6\hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}}$

8C Polar form of a vector

Learning intentions

- To be able to represent a vector in polar form.

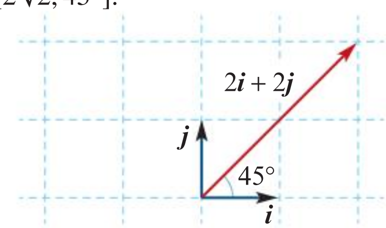
So far in this chapter, we have represented vectors using directed line segments, column vectors and component form. In this section, we introduce another way to describe vectors.

Each vector $\boldsymbol{v}$ in the plane can be written in **polar form** as $\boldsymbol{v} = [r, \theta]$, where

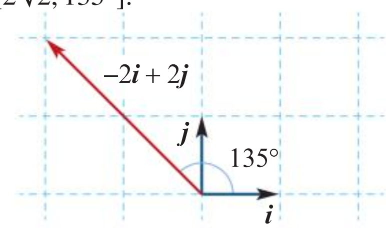
- r is the magnitude of $\boldsymbol{v}$
- θ describes the angle that $\boldsymbol{v}$ makes with the $\hat{\boldsymbol{i}}$-direction, measured anticlockwise.

For example:

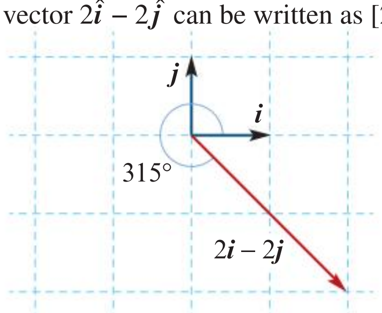
The vector $2\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}}$ can be written as $[2\sqrt{2}, 45°]$.

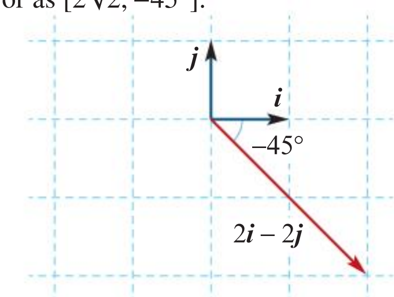


The vector $-2\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}}$ can be written as $[2\sqrt{2}, 135°]$.



The vector $2\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}}$ can be written as $[2\sqrt{2}, 315°]$ or as $[2\sqrt{2}, -45°]$.





Note: From the example $2\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}}$ above, we see that the same vector can have different representations in polar form. Generally, we will prefer to choose $-180° < \theta \le 180°$, so that the representation is unique.

We can convert between polar form and component form using basic trigonometry.

- **Converting from polar form to component form**

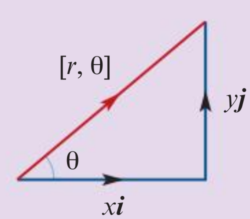
  Given $\boldsymbol{v} = [r, \theta]$, we can write $\boldsymbol{v} = x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}}$, where

  $$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta$$

- **Converting from component form to polar form**

  Given $\boldsymbol{v} = x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}}$, we can write $\boldsymbol{v} = [r, \theta]$, where

  $$r = \sqrt{x^2 + y^2}, \quad \cos\theta = \frac{x}{r} \quad \text{and} \quad \sin\theta = \frac{y}{r}$$



Example 10

Write each of the following vectors in component form:

a $[3, 30°]$ **b** $[4, -60°]$ **c** $[2, 50°]$

Solution

a $x = 3\cos 30°$

$y = 3\sin 30°$

$\therefore [3, 30°] = \frac{3\sqrt{3}}{2}\hat{\boldsymbol{i}} + \frac{3}{2}\hat{\boldsymbol{j}}$

b $x = 4\cos(-60°)$

$y = 4\sin(-60°)$

$\therefore [4, -60°] = 2\hat{\boldsymbol{i}} - 2\sqrt{3}\hat{\boldsymbol{j}}$

c $x = 2\cos 50°$

$y = 2\sin 50°$

$\therefore [2, 50°] \approx 1.29\hat{\boldsymbol{i}} + 1.53\hat{\boldsymbol{j}}$

Example 11

Write each of the following vectors in polar form:

a $-4\sqrt{3}\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}}$ **b** $-4\hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}}$

Solution

a We have $x = -4\sqrt{3}$ and $y = 4$, giving

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ &= \sqrt{48 + 16} \\ &= 8 \end{aligned}$$

The vector $-4\sqrt{3}\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}}$ points to the 2nd quadrant, and so $90° < \theta < 180°$. We know that

$$\cos\theta = \frac{x}{r} = -\frac{\sqrt{3}}{2}$$

and $\quad \sin\theta = \frac{y}{r} = \frac{1}{2}$

Hence $\theta = 150°$ and therefore

$$-4\sqrt{3}\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}} = [8, 150°]$$

b We have $x = -4$ and $y = -4$, giving

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ &= \sqrt{16 + 16} \\ &= 4\sqrt{2} \end{aligned}$$

The vector $-4\hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}}$ points to the 3rd quadrant, and so $-180° < \theta < -90°$. We know that

$$\cos\theta = \frac{x}{r} = -\frac{1}{\sqrt{2}}$$

and $\quad \sin\theta = \frac{y}{r} = -\frac{1}{\sqrt{2}}$

Hence $\theta = -135°$ and therefore

$$-4\hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}} = [4\sqrt{2}, -135°]$$

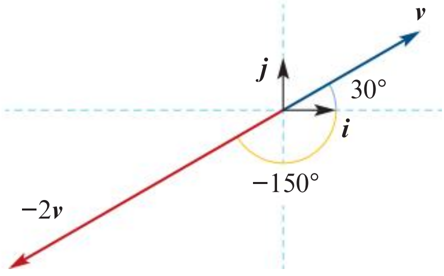
Basic operations on vectors in polar form

We can easily perform scalar multiplication for vectors written in polar form.

For example, if $\boldsymbol{v} = [3, 30°]$, then

$$\begin{aligned} -2\boldsymbol{v} &= [2 \times 3, (30 - 180)°] \\ &= [6, -150°] \end{aligned}$$

Here the magnitude is doubled and the direction is reversed.



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The next example demonstrates two different methods for determining the sum of two vectors given in polar form.

Example 12

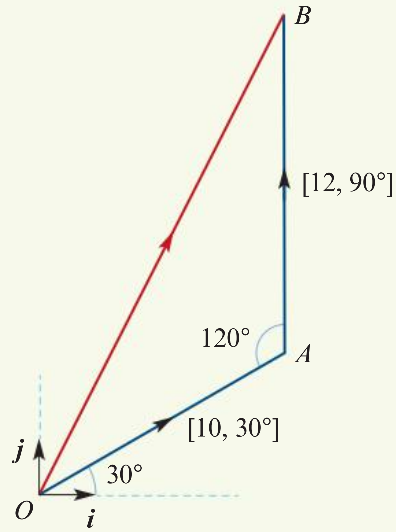
Determine the sum of the vectors $\boldsymbol{u} = [10, 30°]$ and $\boldsymbol{v} = [12, 90°]$.

Solution

Method 1: Using trigonometry

In the diagram, we have $\boldsymbol{u} = \overrightarrow{OA}$ and $\boldsymbol{v} = \overrightarrow{AB}$.

So $\boldsymbol{u} + \boldsymbol{v} = \overrightarrow{OB}$.



Use the cosine rule to determine OB:

$$\begin{aligned} OB^2 &= 10^2 + 12^2 - 2 \times 10 \times 12 \cos 120° \\ &= 100 + 144 - 240 \times \left(-\frac{1}{2}\right) \\ &= 364 \end{aligned}$$

$$\therefore \quad \begin{aligned} OB &= \sqrt{364} \\ &= 2\sqrt{91} \end{aligned}$$

Now use the sine rule to determine $\angle BOA$:

$$\frac{12}{\sin(\angle BOA)} = \frac{2\sqrt{91}}{\sin 120°}$$

$$\sin(\angle BOA) = \frac{12 \sin 120°}{2\sqrt{91}}$$

$$\therefore \quad \angle BOA \approx 33.004°$$

The angle that $\overrightarrow{OB}$ makes with the $\hat{\boldsymbol{i}}$-direction is approximately 63.004°.

Hence $\boldsymbol{u} + \boldsymbol{v} = \overrightarrow{OB} \approx [2\sqrt{91}, 63.004°]$.

Method 2: Using components

We have $\quad \boldsymbol{u} = 10 \cos 30° \, \hat{\boldsymbol{i}} + 10 \sin 30° \, \hat{\boldsymbol{j}}$

$\quad = 5\sqrt{3}\hat{\boldsymbol{i}} + 5\hat{\boldsymbol{j}}$

and $\quad \boldsymbol{v} = 12\hat{\boldsymbol{j}}$

Hence $\quad \boldsymbol{u} + \boldsymbol{v} = 5\sqrt{3}\hat{\boldsymbol{i}} + 17\hat{\boldsymbol{j}}$

Note: If required, we can convert $\boldsymbol{u} + \boldsymbol{v}$ into polar form $[r, \theta]$ using

$$r = \sqrt{(5\sqrt{3})^2 + 17^2} = 2\sqrt{91} \quad \text{and} \quad \theta = \tan^{-1}\left(\frac{17}{5\sqrt{3}}\right) \approx 63.004°$$

However, the answer is neater in component form.

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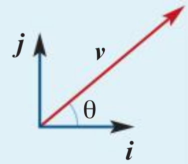
Summary 8C

- Each vector $\boldsymbol{v}$ in the plane can be written in **polar form** as $\boldsymbol{v} = [r, \theta]$, where
  - r is the magnitude of $\boldsymbol{v}$
  - θ describes the angle that $\boldsymbol{v}$ makes with the $\hat{\boldsymbol{i}}$-direction, measured anticlockwise.



- Converting from polar form to component form

  If $\boldsymbol{v} = [r, \theta]$, then $\boldsymbol{v} = x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}}$, where

  $$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta$$

- Converting from component form to polar form

  If $\boldsymbol{v} = x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}}$, then $\boldsymbol{v} = [r, \theta]$, where

  $$r = \sqrt{x^2 + y^2}, \quad \cos\theta = \frac{x}{r} \quad \text{and} \quad \sin\theta = \frac{y}{r}$$

Exercise 8C

SF

1 For each of the following, draw a sketch to illustrate the vector and write the vector in component form:

a $[2, 30°]$ **b** $[4, -30°]$ **c** $[2, 135°]$ **d** $[4, -120°]$ **e** $[4, 150°]$

Example 10

2 Write each of the following vectors in component form:

a $[8, 60°]$ **b** $[10, 30°]$ **c** $[6, 150°]$ **d** $[8, -45°]$ **e** $[12, -150°]$

Example 11

3 Write each of the following vectors in polar form:

a $4\hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}}$ **b** $-3\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}$ **c** $-2\sqrt{3}\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}}$ **d** $3\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}}$ **e** $-5\hat{\boldsymbol{i}} - 12\hat{\boldsymbol{j}}$

4 Write each of the following vectors in polar form:

a $\begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix}$ **b** $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ **c** $\begin{bmatrix} -1 \\ -1 \end{bmatrix}$ **d** $\begin{bmatrix} \sqrt{3} \\ -1 \end{bmatrix}$ **e** $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

5 Write each of the following vectors in component form, giving values correct to two decimal places:

a $[7, 40°]$ **b** $[10, 35°]$ **c** $[11, 155°]$ **d** $[9, -35°]$ **e** $[12, -125°]$

6 Express each of the following vectors in polar form:

a $2\boldsymbol{u}$, where $\boldsymbol{u} = [10, 60°]$ **b** $-\frac{1}{2}\boldsymbol{v}$, where $\boldsymbol{v} = [2, 15°]$ **c** $-3\boldsymbol{w}$, where $\boldsymbol{w} = [6, -20°]$

Example 12

7 Determine each vector sum in polar form:

a $[10, 60°] + [15, 90°]$ **b** $[10, 60°] + [10, 45°]$ **c** $[8, -60°] + [8, 50°]$

Example 12

8 Determine each vector sum in component form:

a $[8, 30°] + [12, 60°]$ **b** $[10, 45°] + [10, -90°]$ **c** $[4, 20°] + [6, 80°]$

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8D Scalar product of vectors

Learning intentions

- To be able to determine scalar products and utilise the properties of scalar products.

The scalar product is an operation that takes two vectors and gives a real number.

Definition of the scalar product

We define the **scalar product** of two vectors $\boldsymbol{a} = a_1\hat{\boldsymbol{i}} + a_2\hat{\boldsymbol{j}}$ and $\boldsymbol{b} = b_1\hat{\boldsymbol{i}} + b_2\hat{\boldsymbol{j}}$ by

$$\boldsymbol{a} \cdot \boldsymbol{b} = a_1 b_1 + a_2 b_2$$

For example:

$$(2\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}) \cdot (\hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}}) = 2 \times 1 + 3 \times (-4) = -10$$

The scalar product is often called the **dot product**.

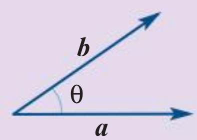
Note: If $\boldsymbol{a} = \boldsymbol{0}$ or $\boldsymbol{b} = \boldsymbol{0}$, then $\boldsymbol{a} \cdot \boldsymbol{b} = 0$.

Geometric description of the scalar product

For vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, we have

$$\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}|\,|\boldsymbol{b}| \cos\theta$$

where θ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$.



Proof Let $\boldsymbol{a} = a_1\hat{\boldsymbol{i}} + a_2\hat{\boldsymbol{j}}$ and $\boldsymbol{b} = b_1\hat{\boldsymbol{i}} + b_2\hat{\boldsymbol{j}}$. Then using the cosine rule in $\triangle OAB$ gives

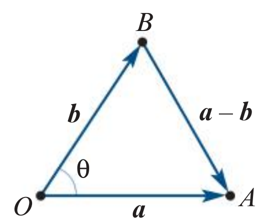
$$|\boldsymbol{a}|^2 + |\boldsymbol{b}|^2 - 2|\boldsymbol{a}|\,|\boldsymbol{b}| \cos\theta = |\boldsymbol{a} - \boldsymbol{b}|^2$$

$$(a_1^2 + a_2^2) + (b_1^2 + b_2^2) - 2|\boldsymbol{a}|\,|\boldsymbol{b}| \cos\theta = (a_1 - b_1)^2 + (a_2 - b_2)^2$$

$$2(a_1 b_1 + a_2 b_2) = 2|\boldsymbol{a}|\,|\boldsymbol{b}| \cos\theta$$

$$a_1 b_1 + a_2 b_2 = |\boldsymbol{a}|\,|\boldsymbol{b}| \cos\theta$$

$$\therefore \quad \boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}|\,|\boldsymbol{b}| \cos\theta$$



Example 13

a If $|\boldsymbol{a}| = 4$, $|\boldsymbol{b}| = 5$ and the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ is 30°, determine $\boldsymbol{a} \cdot \boldsymbol{b}$.

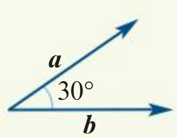
b If $|\boldsymbol{a}| = 4$, $|\boldsymbol{b}| = 5$ and the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ is 150°, determine $\boldsymbol{a} \cdot \boldsymbol{b}$.

Solution

a $\boldsymbol{a} \cdot \boldsymbol{b} = 4 \times 5 \times \cos 30^\circ$

$= 20 \times \dfrac{\sqrt{3}}{2}$

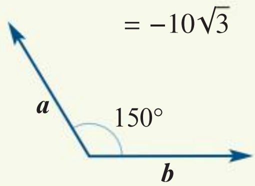
$= 10\sqrt{3}$



b $\boldsymbol{a} \cdot \boldsymbol{b} = 4 \times 5 \times \cos 150^\circ$

$= 20 \times \left(-\dfrac{\sqrt{3}}{2}\right)$

$= -10\sqrt{3}$



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Properties of the scalar product

- $\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{b} \cdot \boldsymbol{a}$
- $k(\boldsymbol{a} \cdot \boldsymbol{b}) = (k\boldsymbol{a}) \cdot \boldsymbol{b} = \boldsymbol{a} \cdot (k\boldsymbol{b})$
- $\boldsymbol{a} \cdot \boldsymbol{0} = 0$
- $\boldsymbol{a} \cdot (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \cdot \boldsymbol{c}$
- $\boldsymbol{a} \cdot \boldsymbol{a} = |\boldsymbol{a}|^2$
- If the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular, then $\boldsymbol{a} \cdot \boldsymbol{b} = 0$.
- If $\boldsymbol{a} \cdot \boldsymbol{b} = 0$ for non-zero vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, then the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular.
- For parallel vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, we have

$$\boldsymbol{a} \cdot \boldsymbol{b} = \begin{cases} |\boldsymbol{a}|\,|\boldsymbol{b}| & \text{if } \boldsymbol{a} \text{ and } \boldsymbol{b} \text{ are parallel and in the same direction} \\ -|\boldsymbol{a}|\,|\boldsymbol{b}| & \text{if } \boldsymbol{a} \text{ and } \boldsymbol{b} \text{ are parallel and in opposite directions} \end{cases}$$

determining the magnitude of the angle between two vectors

To determine the angle θ between two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, we can use the two different forms of the scalar product:

$$\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}|\,|\boldsymbol{b}| \cos\theta \quad \text{and} \quad \boldsymbol{a} \cdot \boldsymbol{b} = a_1 b_1 + a_2 b_2$$

Therefore

$$\cos\theta = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{a}|\,|\boldsymbol{b}|} = \frac{a_1 b_1 + a_2 b_2}{|\boldsymbol{a}|\,|\boldsymbol{b}|}$$

Example 14

A, B and C are points defined by the position vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ respectively, where

$$\boldsymbol{a} = \hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}, \quad \boldsymbol{b} = 2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} \quad \text{and} \quad \boldsymbol{c} = \hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}}$$

Determine the magnitude of $\angle ABC$.

Solution

$\angle ABC$ is the angle between vectors $\overrightarrow{BA}$ and $\overrightarrow{BC}$.

$$\overrightarrow{BA} = \boldsymbol{a} - \boldsymbol{b} = -\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}}$$

$$\overrightarrow{BC} = \boldsymbol{c} - \boldsymbol{b} = -\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}}$$

We will apply the scalar product:

$$\overrightarrow{BA} \cdot \overrightarrow{BC} = |\overrightarrow{BA}|\,|\overrightarrow{BC}| \cos(\angle ABC)$$

We have

$$\overrightarrow{BA} \cdot \overrightarrow{BC} = (-\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}}) \cdot (-\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}}) = 1 - 6 = -5$$

$$|\overrightarrow{BA}| = \sqrt{1 + 4} = \sqrt{5}$$

$$|\overrightarrow{BC}| = \sqrt{1 + 9} = \sqrt{10}$$

Therefore

$$\cos(\angle ABC) = \frac{\overrightarrow{BA} \cdot \overrightarrow{BC}}{|\overrightarrow{BA}|\,|\overrightarrow{BC}|} = \frac{-5}{\sqrt{5}\sqrt{10}} = \frac{-1}{\sqrt{2}}$$

Hence $\angle ABC = 135°$.

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Scalar product of vectors in polar form

If $\boldsymbol{a}_1 = [r_1, \theta_1]$ and $\boldsymbol{a}_2 = [r_2, \theta_2]$, then the geometric description of the scalar product gives

$$\boldsymbol{a}_1 \cdot \boldsymbol{a}_2 = r_1 r_2 \cos(\theta_1 - \theta_2)$$

Note: In this chapter, we restrict to $\theta_1, \theta_2 \in (-180°, 180°]$ to ensure $\theta_1 - \theta_2 \in (-360°, 360°)$. We will consider the cosine of angles outside this range in Chapter 13.

Example 15

Determine the scalar product of each of the following pairs of vectors:

a $[4, 60°]$ and $[5, 30°]$ **b** $[7, -30°]$ and $[6, 120°]$ **c** $[11, 60°]$ and $[8, -20°]$

Solution

a $[4, 60°] \cdot [5, 30°]$
$= 4 \times 5 \cos(60 - 30)°$
$= 20 \cos 30°$
$= 10\sqrt{3}$

b $[7, -30°] \cdot [6, 120°]$
$= 7 \times 6 \cos(-30 - 120)°$
$= 42 \cos(-150)°$
$= 42 \cos 150°$
$= -21\sqrt{3}$

c $[11, 60°] \cdot [8, -20°]$
$= 11 \times 8 \cos(60 + 20)°$
$= 88 \cos 80°$
≈ 15.28

Summary 8D

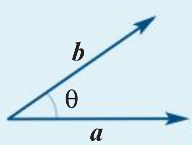
- The **scalar product** of vectors $\boldsymbol{a} = a_1\hat{\boldsymbol{i}} + a_2\hat{\boldsymbol{j}}$ and $\boldsymbol{b} = b_1\hat{\boldsymbol{i}} + b_2\hat{\boldsymbol{j}}$ is given by

  $$\boldsymbol{a} \cdot \boldsymbol{b} = a_1 b_1 + a_2 b_2$$

- The scalar product can be described geometrically by

  $$\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}|\,|\boldsymbol{b}| \cos\theta$$

  where θ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$.



- Therefore $\boldsymbol{a} \cdot \boldsymbol{a} = |\boldsymbol{a}|^2$.
- Two non-zero vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are **perpendicular** if and only if $\boldsymbol{a} \cdot \boldsymbol{b} = 0$.
- For vectors in polar form $\boldsymbol{a}_1 = [r_1, \theta_1]$ and $\boldsymbol{a}_2 = [r_2, \theta_2]$, the scalar product is given by

  $$\boldsymbol{a}_1 \cdot \boldsymbol{a}_2 = r_1 r_2 \cos(\theta_1 - \theta_2)$$

Skill-sheet

Exercise 8D

SF

1 Let $\boldsymbol{a} = \hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}}$, $\boldsymbol{b} = 2\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}$ and $\boldsymbol{c} = -2\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}}$. Determine:

a $\boldsymbol{a} \cdot \boldsymbol{a}$ **b** $\boldsymbol{b} \cdot \boldsymbol{b}$ **c** $\boldsymbol{c} \cdot \boldsymbol{c}$ **d** $\boldsymbol{a} \cdot \boldsymbol{b}$
e $\boldsymbol{a} \cdot (\boldsymbol{b} + \boldsymbol{c})$ **f** $(\boldsymbol{a} + \boldsymbol{b}) \cdot (\boldsymbol{a} + \boldsymbol{c})$ **g** $(\boldsymbol{a} + 2\boldsymbol{b}) \cdot (3\boldsymbol{c} - \boldsymbol{b})$

2 Let $\boldsymbol{a} = 2\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}}$, $\boldsymbol{b} = 3\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}}$ and $\boldsymbol{c} = -\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}$. Determine:

a $\boldsymbol{a} \cdot \boldsymbol{a}$ **b** $\boldsymbol{b} \cdot \boldsymbol{b}$ **c** $\boldsymbol{a} \cdot \boldsymbol{b}$
d $\boldsymbol{a} \cdot \boldsymbol{c}$ **e** $\boldsymbol{a} \cdot (\boldsymbol{a} + \boldsymbol{b})$

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SF

Example 13

3 **a** If $|\boldsymbol{a}| = 5$, $|\boldsymbol{b}| = 6$ and the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ is 45°, determine $\boldsymbol{a} \cdot \boldsymbol{b}$.

b If $|\boldsymbol{a}| = 5$, $|\boldsymbol{b}| = 6$ and the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ is 135°, determine $\boldsymbol{a} \cdot \boldsymbol{b}$.

4 Expand and simplify:

a $(\boldsymbol{a} + 2\boldsymbol{b}) \cdot (\boldsymbol{a} + 2\boldsymbol{b})$

b $|\boldsymbol{a} + \boldsymbol{b}|^2 - |\boldsymbol{a} - \boldsymbol{b}|^2$

c $\boldsymbol{a} \cdot (\boldsymbol{a} + \boldsymbol{b}) - \boldsymbol{b} \cdot (\boldsymbol{a} + \boldsymbol{b})$

d $\dfrac{\boldsymbol{a} \cdot (\boldsymbol{a} + \boldsymbol{b}) - \boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{a}|}$

5 If A and B are points defined by the position vectors $\boldsymbol{a} = 2\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}}$ and $\boldsymbol{b} = -\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}$ respectively, determine:

a $\overrightarrow{AB}$ **b** $|\overrightarrow{AB}|$ **c** the magnitude of the angle between vectors $\overrightarrow{AB}$ and $\boldsymbol{a}$.

6 Let C and D be points with position vectors $\boldsymbol{c}$ and $\boldsymbol{d}$ respectively. If $|\boldsymbol{c}| = 5$, $|\boldsymbol{d}| = 7$ and $\boldsymbol{c} \cdot \boldsymbol{d} = 4$, determine $|\overrightarrow{CD}|$.

7 Solve each of the following equations:

a $(\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}}) \cdot (5\hat{\boldsymbol{i}} + x\hat{\boldsymbol{j}}) = -6$

b $(x\hat{\boldsymbol{i}} + 7\hat{\boldsymbol{j}}) \cdot (-4\hat{\boldsymbol{i}} + x\hat{\boldsymbol{j}}) = 10$

c $(x\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}) \cdot (-2\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}}) = x$

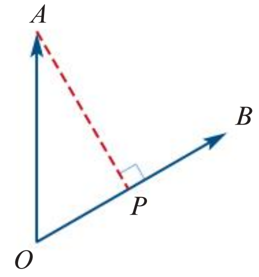
d $x(2\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}) \cdot (\hat{\boldsymbol{i}} + x\hat{\boldsymbol{j}}) = 6$

8 Points A and B are defined by the position vectors $\boldsymbol{a} = \hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}}$ and $\boldsymbol{b} = 2\hat{\boldsymbol{i}} + 5\hat{\boldsymbol{j}}$. Let P be the point on OB such that AP is perpendicular to OB. Then $\overrightarrow{OP} = q\boldsymbol{b}$, for a constant q.

a Express $\overrightarrow{AP}$ in terms of q, $\boldsymbol{a}$ and $\boldsymbol{b}$.

b Use the fact that $\overrightarrow{AP} \cdot \overrightarrow{OB} = 0$ to determine the value of q.

c Determine the coordinates of the point P.



Example 14

9 Determine the angle, in degrees, between each of the following pairs of vectors, correct to two decimal places:

a $\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}}$ and $\hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}}$

b $-2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}$ and $-2\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}}$

c $2\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}}$ and $4\hat{\boldsymbol{i}}$

d $7\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}$ and $-\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}$

10 Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be non-zero vectors such that $\boldsymbol{a} \cdot \boldsymbol{b} = 0$. Use the geometric description of the scalar product to show that $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular vectors.

Example 15

11 Determine the scalar product of each of the following pairs of vectors:

a [10, 45°] and [5, 15°] **b** [8, 70°] and [5, 10°] **c** [12, −60°] and [10, −30°]

d [4, 120°] and [9, −60°] **e** [7, 40°] and [2, 70°] **f** [10, −25°] and [4, 35°]

For Questions 12–13, *determine the angles in degrees correct to two decimal places.*

12 Let A and B be the points defined by the position vectors $\boldsymbol{a} = \hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}$ and $\boldsymbol{b} = 2\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}}$ respectively. Let M be the midpoint of AB. Determine:

a $\overrightarrow{OM}$ **b** $\angle AOM$ **c** $\angle BMO$

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SF

13 Let A, B and C be the points defined by the position vectors $3\hat{\boldsymbol{i}}$, $4\hat{\boldsymbol{j}}$ and $-2\hat{\boldsymbol{i}} + 6\hat{\boldsymbol{j}}$ respectively. Let M and N be the midpoints of $\overrightarrow{AB}$ and $\overrightarrow{AC}$ respectively. Determine:

a **i** $\overrightarrow{OM}$ **ii** $\overrightarrow{ON}$ **b** $\angle MON$ **c** $\angle MOC$

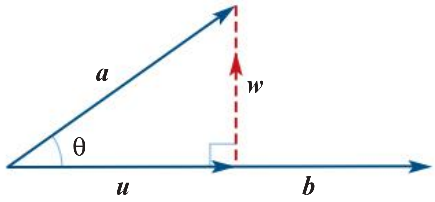
8E Vector projections

Learning intentions

- To be able to determine the projection of one vector on another.

It is often useful to decompose a vector $\boldsymbol{a}$ into a sum of two vectors, one parallel to a given vector $\boldsymbol{b}$ and the other perpendicular to $\boldsymbol{b}$.

From the diagram, it can be seen that



$$\boldsymbol{a} = \boldsymbol{u} + \boldsymbol{w}$$

where $\boldsymbol{u} = k\boldsymbol{b}$ and so $\boldsymbol{w} = \boldsymbol{a} - \boldsymbol{u} = \boldsymbol{a} - k\boldsymbol{b}$.

For $\boldsymbol{w}$ to be perpendicular to $\boldsymbol{b}$, we must have

$$\begin{aligned} \boldsymbol{w} \cdot \boldsymbol{b} &= 0 \\ (\boldsymbol{a} - k\boldsymbol{b}) \cdot \boldsymbol{b} &= 0 \\ \boldsymbol{a} \cdot \boldsymbol{b} - k(\boldsymbol{b} \cdot \boldsymbol{b}) &= 0 \end{aligned}$$

Hence $k = \dfrac{\boldsymbol{a} \cdot \boldsymbol{b}}{\boldsymbol{b} \cdot \boldsymbol{b}}$ and therefore $\boldsymbol{u} = \dfrac{\boldsymbol{a} \cdot \boldsymbol{b}}{\boldsymbol{b} \cdot \boldsymbol{b}}\boldsymbol{b}$.

This vector $\boldsymbol{u}$ is called the **vector projection** (or **vector resolute**) of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$.

Vector resolute

The **vector resolute** of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$ can be expressed in any one of the following equivalent forms:

$$\boldsymbol{u} = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{\boldsymbol{b} \cdot \boldsymbol{b}}\boldsymbol{b} = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{b}|^2}\boldsymbol{b} = \left(\boldsymbol{a} \cdot \frac{\boldsymbol{b}}{|\boldsymbol{b}|}\right)\left(\frac{\boldsymbol{b}}{|\boldsymbol{b}|}\right) = (\boldsymbol{a} \cdot \hat{\boldsymbol{b}})\,\hat{\boldsymbol{b}}$$

Note: The quantity $\boldsymbol{a} \cdot \hat{\boldsymbol{b}} = \dfrac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{b}|}$ is the 'signed length' of the vector resolute $\boldsymbol{u}$ and is called the **scalar resolute** of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$.

Note that, from our previous calculation, we have

$$\boldsymbol{w} = \boldsymbol{a} - \boldsymbol{u} = \boldsymbol{a} - \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{\boldsymbol{b} \cdot \boldsymbol{b}}\boldsymbol{b}$$

Expressing $\boldsymbol{a}$ as the sum of the two components, the first parallel to $\boldsymbol{b}$ and the second perpendicular to $\boldsymbol{b}$, gives

$$\boldsymbol{a} = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{\boldsymbol{b} \cdot \boldsymbol{b}}\boldsymbol{b} + \left(\boldsymbol{a} - \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{\boldsymbol{b} \cdot \boldsymbol{b}}\boldsymbol{b}\right)$$

This is sometimes described as resolving the vector $\boldsymbol{a}$ into **rectangular components**, one parallel to $\boldsymbol{b}$ and the other perpendicular to $\boldsymbol{b}$.

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Example 16

Let $\boldsymbol{a} = \hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}$ and $\boldsymbol{b} = \hat{\boldsymbol{i}} - \hat{\boldsymbol{j}}$. Determine the vector resolute of:

a $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$

b $\boldsymbol{b}$ in the direction of $\boldsymbol{a}$.

Solution

a $\boldsymbol{a} \cdot \boldsymbol{b} = 1 - 3 = -2$

$\boldsymbol{b} \cdot \boldsymbol{b} = 1 + 1 = 2$

The vector resolute of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$ is

$$\begin{aligned}\frac{\boldsymbol{a} \cdot \boldsymbol{b}}{\boldsymbol{b} \cdot \boldsymbol{b}}\boldsymbol{b} &= \frac{-2}{2}(\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}}) \\ &= -1(\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}}) \\ &= -\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}\end{aligned}$$

b $\boldsymbol{b} \cdot \boldsymbol{a} = \boldsymbol{a} \cdot \boldsymbol{b} = -2$

$\boldsymbol{a} \cdot \boldsymbol{a} = 1 + 9 = 10$

The vector resolute of $\boldsymbol{b}$ in the direction of $\boldsymbol{a}$ is

$$\begin{aligned}\frac{\boldsymbol{b} \cdot \boldsymbol{a}}{\boldsymbol{a} \cdot \boldsymbol{a}}\boldsymbol{a} &= \frac{-2}{10}(\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}) \\ &= -\frac{1}{5}(\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}})\end{aligned}$$

Example 17

Resolve $\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}$ into rectangular components, one of which is parallel to $2\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}}$.

Solution

Let $\boldsymbol{a} = \hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}$ and $\boldsymbol{b} = 2\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}}$.

The vector resolute of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$ is given by $\dfrac{\boldsymbol{a} \cdot \boldsymbol{b}}{\boldsymbol{b} \cdot \boldsymbol{b}}\boldsymbol{b}$.

We have

$$\boldsymbol{a} \cdot \boldsymbol{b} = 2 - 6 = -4$$

$$\boldsymbol{b} \cdot \boldsymbol{b} = 4 + 4 = 8$$

Therefore the vector resolute is

$$\begin{aligned}\frac{-4}{8}(2\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}}) &= -\frac{1}{2}(2\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}}) \\ &= -\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}\end{aligned}$$

The perpendicular component is

$$\begin{aligned}\boldsymbol{a} - (-\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}) &= (\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}) - (-\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}) \\ &= 2\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}}\end{aligned}$$

Hence we can write

$$\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} = (-\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}) + (2\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}})$$

Check: We can check our calculation by verifying that the second component is indeed perpendicular to $\boldsymbol{b}$. We have $(2\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}}) \cdot (2\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}}) = 4 - 4 = 0$, as expected.

Example 18

Determine the scalar resolute of $\boldsymbol{a} = 2\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}}$ in the direction of $\boldsymbol{b} = -\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}$.

Solution

$\boldsymbol{a} \cdot \boldsymbol{b} = -2 + 6 = 4$

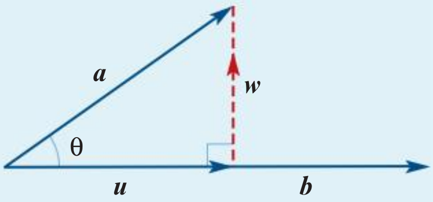
$|\boldsymbol{b}| = \sqrt{1+9} = \sqrt{10}$

The scalar resolute of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$ is

$$\frac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{b}|} = \frac{4}{\sqrt{10}} = \frac{2\sqrt{10}}{5}$$

Summary 8E

- Resolving a vector $\boldsymbol{a}$ into rectangular components is expressing the vector $\boldsymbol{a}$ as a sum of two vectors, one parallel to a given vector $\boldsymbol{b}$ and the other perpendicular to $\boldsymbol{b}$.
- The **vector resolute** of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$ is given by $\boldsymbol{u} = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{\boldsymbol{b} \cdot \boldsymbol{b}}\boldsymbol{b}$.
- The **scalar resolute** of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$ is the 'signed length' of the vector resolute $\boldsymbol{u}$ and is given by $\frac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{b}|}$.



Exercise 8E

SF

1. Points A and B are defined by the position vectors $\boldsymbol{a} = \hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}$ and $\boldsymbol{b} = 2\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}}$.
   - **a** Determine $\hat{\boldsymbol{a}}$.
   - **b** Determine $\hat{\boldsymbol{b}}$.
   - **c** determine $\hat{\boldsymbol{c}}$, where $\boldsymbol{c} = \overrightarrow{AB}$.

2. Let $\boldsymbol{a} = 3\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}}$ and $\boldsymbol{b} = \hat{\boldsymbol{i}} - \hat{\boldsymbol{j}}$.
   - **a** Determine $\hat{\boldsymbol{a}}$ and $|\boldsymbol{b}|$.
   - **b** Determine the vector with the same magnitude as $\boldsymbol{b}$ and with the same direction as $\boldsymbol{a}$.

CF

3. Points A and B are defined by the position vectors $\boldsymbol{a} = 3\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}}$ and $\boldsymbol{b} = 5\hat{\boldsymbol{i}} + 12\hat{\boldsymbol{j}}$.
   - **a** Determine $\hat{\boldsymbol{a}}$ and $\hat{\boldsymbol{b}}$.
   - **b** Determine the unit vector which bisects $\angle AOB$.

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Example 16

4. For each pair of vectors, determine the vector resolute of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$:
   - **a** $\boldsymbol{a} = \hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}$ and $\boldsymbol{b} = \hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}}$
   - **b** $\boldsymbol{a} = \hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}}$ and $\boldsymbol{b} = \hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}}$
   - **c** $\boldsymbol{a} = 4\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}}$ and $\boldsymbol{b} = 4\hat{\boldsymbol{i}}$

Example 17

5. For each of the following pairs of vectors, determine the resolution of the vector $\boldsymbol{a}$ into rectangular components, one of which is parallel to $\boldsymbol{b}$:
   - **a** $\boldsymbol{a} = 2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}},\ \boldsymbol{b} = 5\hat{\boldsymbol{i}}$
   - **b** $\boldsymbol{a} = 3\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}},\ \boldsymbol{b} = \hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}$
   - **c** $\boldsymbol{a} = -\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}},\ \boldsymbol{b} = 2\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}}$

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Example 18

6 For each of the following pairs of vectors, determine the scalar resolute of the first vector in the direction of the second vector:

a $\boldsymbol{a} = 2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}$ and $\boldsymbol{b} = \hat{\boldsymbol{i}}$
b $\boldsymbol{a} = 3\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}$ and $\boldsymbol{c} = \hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}}$
c $\boldsymbol{b} = 2\hat{\boldsymbol{j}}$ and $\boldsymbol{a} = 2\hat{\boldsymbol{i}} + \sqrt{3}\hat{\boldsymbol{j}}$
d $\boldsymbol{b} = \hat{\boldsymbol{i}} - \sqrt{5}\hat{\boldsymbol{j}}$ and $\boldsymbol{c} = -\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}}$

SF

7 Let A and B be the points defined by the position vectors $\boldsymbol{a} = \hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}$ and $\boldsymbol{b} = \hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}$ respectively. Determine:

a the vector resolute of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$
b a unit vector perpendicular to OB

CF

8 Let A and B be the points defined by the position vectors $\boldsymbol{a} = 4\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}$ and $\boldsymbol{b} = \hat{\boldsymbol{i}} - \hat{\boldsymbol{j}}$ respectively. Determine:

a the vector resolute of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$
b the vector component of $\boldsymbol{a}$ perpendicular to $\boldsymbol{b}$
c the shortest distance from A to line OB

9 Points A, B and C have position vectors $\boldsymbol{a} = \hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}}$, $\boldsymbol{b} = 2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}$ and $\boldsymbol{c} = 2\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}}$. Determine:

a **i** $\overrightarrow{AB}$ **ii** $\overrightarrow{AC}$
b the vector resolute of $\overrightarrow{AB}$ in the direction of $\overrightarrow{AC}$
c the shortest distance from B to line AC
d the area of triangle ABC

8F Applications of vectors: displacement and velocity

Learning intentions

► To be able to apply vectors to the study of displacement and velocity.

For the remainder of this chapter, we will be working with vector and scalar quantities:

- A **vector quantity** has both magnitude and direction. We will introduce the vector quantities displacement, velocity and force.
- A **scalar quantity** has only magnitude. We will use the scalar quantities distance, time, speed and mass.

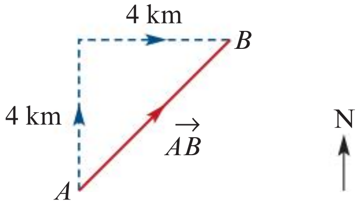
Displacement

We have been describing points in the plane using position vectors. Points A and B have position vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$ respectively.

If an object moves from point A to point B, then the **displacement** of the object is the change in position of the object; it is described by the vector $\overrightarrow{AB}$.

For example, suppose that a person walks 4 km north and then 4 km east.

The person's displacement is $4\sqrt{2}$ km north-east.



Note: The total distance that the person has walked is 8 km, which is not equal to the magnitude of the displacement vector.

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Example 19

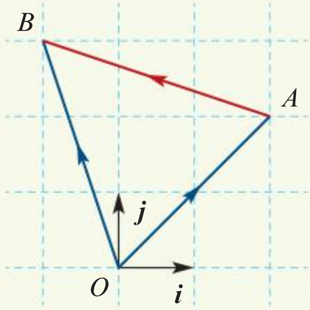
A particle moves from point $A(2, 2)$ to point $B(-1, 3)$. Express the displacement vector of the particle in component form.

Solution

We have $\overrightarrow{OA} = 2\hat{i} + 2\hat{j}$ and $\overrightarrow{OB} = -\hat{i} + 3\hat{j}$.

The displacement vector is

$$\begin{aligned}\overrightarrow{AB} &= \overrightarrow{AO} + \overrightarrow{OB}\\ &= -(2\hat{i} + 2\hat{j}) + (-\hat{i} + 3\hat{j})\\ &= -3\hat{i} + \hat{j}\end{aligned}$$



Displacement problems involving bearings can be solved using the trigonometric techniques demonstrated in Example 12 and in Section 3E.

Velocity

Velocity is the rate of change of position with respect to time.

Velocity is a vector quantity; it has magnitude and direction. The units of velocity which will be used in this chapter are metres per second (m/s) and kilometres per hour (km/h).

Some examples of velocity vectors are:

- 80 km/h in the direction north
- 10 km/h on a bearing of 080°
- $3\hat{i} + 4\hat{j}$ m/s

The first two vectors have magnitudes 80 km/h and 10 km/h respectively. The third vector has magnitude $|3\hat{i} + 4\hat{j}| = \sqrt{3^2 + 4^2} = 5$ m/s. The magnitude of velocity is called **speed**.

Motion with constant velocity

In this chapter, we only deal with constant velocity (that is, the velocity does not change over a particular time interval). Consider the following two examples:

- If a car travels for 2 hours with a constant velocity of 80 km/h north, then its displacement is $2 \times 80 = 160$ km north.
- If a particle starts at the origin and moves with a velocity of $3\hat{i} + 4\hat{j}$ m/s for 2 seconds, then its position is $2(3\hat{i} + 4\hat{j}) = 6\hat{i} + 8\hat{j}$ m.

If an object moves with a constant velocity of $\boldsymbol{v}$ m/s for t seconds, then its displacement vector, $\boldsymbol{s}$ m, is given by

$$\boldsymbol{s} = t\boldsymbol{v}$$

Note: Here $\boldsymbol{s}$ and $\boldsymbol{v}$ are vector quantities and t is a scalar quantity. So this is an example of scalar multiplication.

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Example 20

A particle starts at the point A with position vector $\overrightarrow{OA} = \hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}$, where the unit is metres. The particle begins moving with a constant velocity of $2\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}}$ m/s. Determine the position vector of the particle after:

a 5 seconds

b t seconds.

Solution

a Let P be the point that the particle reaches after 5 seconds. Then

$$\begin{aligned}\overrightarrow{OP} &= \overrightarrow{OA} + 5(2\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}})\\ &= \hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} + 10\hat{\boldsymbol{i}} + 20\hat{\boldsymbol{j}}\\ &= 11\hat{\boldsymbol{i}} + 23\hat{\boldsymbol{j}}\end{aligned}$$

b Let Q be the point that the particle reaches after t seconds. Then

$$\begin{aligned}\overrightarrow{OQ} &= \overrightarrow{OA} + t(2\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}})\\ &= \hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}} + 2t\hat{\boldsymbol{i}} + 4t\hat{\boldsymbol{j}}\\ &= (1 + 2t)\hat{\boldsymbol{i}} + (3 + 4t)\hat{\boldsymbol{j}}\end{aligned}$$

Direction of motion

The velocity vector is in the direction of motion. We often use the unit vector of the velocity vector to describe the direction of motion.

For example, if $\boldsymbol{v} = 3\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}}$, then the unit vector $\hat{\boldsymbol{v}} = \frac{1}{5}(3\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}})$ is in the direction of motion.

Example 21

Particle A starts moving from point O with a constant velocity of $\boldsymbol{v}_A = 3\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}}$ m/s. Three seconds later, particle B starts from O and moves in the same direction as A with a constant speed of 7 m/s. When and where will B catch up to A?

Solution

At time t seconds, particle A is at the point with position vector

$$\overrightarrow{OP_A} = t(3\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}})$$

At time t seconds, for $t \geq 3$, particle B has been moving for $t - 3$ seconds and is at the point with position vector

$$\overrightarrow{OP_B} = \frac{7(t-3)}{5}(3\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}})$$

The two particles are at the same point when

$$\begin{aligned}\frac{7(t-3)}{5} &= t\\ 7(t-3) &= 5t\\ 2t &= 21\\ \therefore \quad t &= \frac{21}{2}\end{aligned}$$

Particle B catches up to particle A at time $t = 10.5$ seconds.

At this time, both particles have position vector $31.5\hat{\boldsymbol{i}} + 42\hat{\boldsymbol{j}}$.

Example 22

A particle starts from O with a constant velocity of $\boldsymbol{v}_1 = 3\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}}$ m/s. At the same time, a second particle starts moving with constant velocity from point B, where $\overrightarrow{OB} = 25\hat{\boldsymbol{j}}$.

Given that the two particles meet and their paths are at right angles, determine:

a the position vector of the point where they meet

b the velocity of the second particle.

Solution

a Assume that the particles meet at the point P at time t seconds. Since their paths are at right angles, we have

$$\overrightarrow{OP} \cdot \overrightarrow{BP} = 0$$

At time t seconds, the position vector of the first particle is

$$\overrightarrow{OP} = t(3\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}}) = 3t\hat{\boldsymbol{i}} + 4t\hat{\boldsymbol{j}}$$

Therefore

$$\begin{aligned}\overrightarrow{BP} &= \overrightarrow{BO} + \overrightarrow{OP} \\ &= -25\hat{\boldsymbol{j}} + (3t\hat{\boldsymbol{i}} + 4t\hat{\boldsymbol{j}}) \\ &= 3t\hat{\boldsymbol{i}} + (4t - 25)\hat{\boldsymbol{j}}\end{aligned}$$

Since $\overrightarrow{OP} \cdot \overrightarrow{BP} = 0$, we obtain

$$\begin{aligned}(3t\hat{\boldsymbol{i}} + 4t\hat{\boldsymbol{j}}) \cdot (3t\hat{\boldsymbol{i}} + (4t - 25)\hat{\boldsymbol{j}}) &= 0 \\ 9t^2 + 4t(4t - 25) &= 0 \\ 25t^2 - 100t &= 0 \\ \therefore \quad t(t - 4) &= 0\end{aligned}$$

The particles do not meet at time 0 s, so they meet at time $t = 4$ s.

The position vector of the point where they meet is

$$\overrightarrow{OP} = 4(3\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}}) = 12\hat{\boldsymbol{i}} + 16\hat{\boldsymbol{j}}$$

b Let $\boldsymbol{v}$ m/s be the velocity of the second particle. We use the formula $\boldsymbol{s} = t\boldsymbol{v}$.

At time $t = 4$, the displacement of the second particle is $\overrightarrow{BP}$. Therefore

$$\begin{aligned}\overrightarrow{BP} &= 4\boldsymbol{v} \\ \overrightarrow{BO} + \overrightarrow{OP} &= 4\boldsymbol{v} \\ -25\hat{\boldsymbol{j}} + (12\hat{\boldsymbol{i}} + 16\hat{\boldsymbol{j}}) &= 4\boldsymbol{v} \\ 12\hat{\boldsymbol{i}} - 9\hat{\boldsymbol{j}} &= 4\boldsymbol{v}\end{aligned}$$

Hence the velocity of the second particle is $\boldsymbol{v} = 3\hat{\boldsymbol{i}} - \frac{9}{4}\hat{\boldsymbol{j}}$ m/s.

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Summary 8F

- The **displacement** of a particle is the change in its position. If a particle moves from point A to point B, then its displacement is $\overrightarrow{AB}$.
- The **velocity** of a particle is the rate of change of its position with respect to time.
- Displacement and velocity are vector quantities. The magnitude of velocity is **speed**.
- **Motion with constant velocity** If a particle moves with a constant velocity of $\boldsymbol{v}$ m/s for t seconds, then its displacement vector, $\boldsymbol{s}$ m, is given by $\boldsymbol{s} = t\boldsymbol{v}$.

Exercise 8F

SF

Example 19

1 For each of the following, determine the displacement vector in component form for a particle that moves from point A to point B:

a $A(3, 7)$, $B(2, -4)$ **b** $A(-2, 4)$, $B(3, -2)$ **c** $A(3, 1)$, $B(4, 6)$

d $A(3, 7)$, $B(3, -4)$ **e** $A(-2, -7)$, $B(2, -7)$ **f** $A(5, -6)$, $B(11, 5)$

2 From the point O, a hiker walks 5 km north and then 8 km on a bearing of 330°, finishing at a point A. Describe the displacement vector $\overrightarrow{OA}$ by giving a distance and a bearing.

3 From the point O, a yacht sails 3 km east and then 5 km on a bearing of 060°, finishing at a point A. Describe the displacement vector $\overrightarrow{OA}$ by giving a distance and a bearing.

4 Give the corresponding speed for each of the following velocity vectors:

a $5\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}}$ m/s **b** $3\hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}}$ m/s **c** $-\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}}$ m/s

d $-2\hat{\boldsymbol{i}} - 6\hat{\boldsymbol{j}}$ m/s **e** $5\hat{\boldsymbol{i}} - 12\hat{\boldsymbol{j}}$ m/s **f** $-7\hat{\boldsymbol{i}} + 11\hat{\boldsymbol{j}}$ m/s

In each of the following questions, the unit of distance is metres.

CF

Example 20

5 A particle starts from the point A with position vector $\overrightarrow{OA} = -\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}}$ and moves with a constant velocity of $5\hat{\boldsymbol{i}} + 12\hat{\boldsymbol{j}}$ m/s. Determine the position vector of the particle after:

a 5 seconds **b** t seconds.

6 An object takes 5 seconds to move with constant velocity from point A to point B, where $\overrightarrow{OA} = 5\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}}$ and $\overrightarrow{OB} = -15\hat{\boldsymbol{i}} + 24\hat{\boldsymbol{j}}$. Determine the velocity of the object.

7 A particle starts from the point B with position vector $\overrightarrow{OB} = -2\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}$ and moves with a constant velocity of $7\hat{\boldsymbol{i}} + 24\hat{\boldsymbol{j}}$ m/s.

a Determine the position vector of the particle after:

i 4 seconds **ii** t seconds.

b Determine the particle's distance from the origin after:

i 4 seconds **ii** t seconds.

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8 Let O be the origin and let A and B be the points with $\overrightarrow{OA} = 5\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}}$ and $\overrightarrow{OB} = -5\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}}$. A particle moves with constant velocity from A to B in 10 seconds. Determine:

a the velocity of the particle **b** the speed of the particle.

Example 21 **9** Particle A starts moving from point O with a constant velocity of $\boldsymbol{v}_A = \hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}}$ m/s. Two seconds later, particle B starts from O and moves in the same direction as A with a constant speed of 6 m/s. When and where will B catch up to A?

Example 22 **10** A particle starts from O with a constant velocity of $\boldsymbol{v}_1 = 2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}$ m/s. At the same time, a second particle starts moving with constant velocity from point B, where $\overrightarrow{OB} = 20\hat{\boldsymbol{j}}$. Given that the two particles meet and their paths are at right angles, determine:

a the position vector of the point where they meet

b the velocity of the second particle.

11 Points A and B have position vectors $\overrightarrow{OA} = 10\hat{\boldsymbol{j}}$ and $\overrightarrow{OB} = 20\hat{\boldsymbol{i}}$. A particle starts moving from point A with a constant velocity of $\boldsymbol{v}_1 = 2\hat{\boldsymbol{i}}$ m/s. At the same time, a second particle starts moving from point B with constant velocity. Given that the two particles meet and their paths are at right angles, determine:

a the position vector of the point where they meet

b the velocity of the second particle.

8G Applications of vectors: relative velocity

Learning intentions

- To be able to apply vectors to the study of relative and resultant velocities.

Resultant velocity

If two or more velocity vectors are added, then the sum is called a **resultant velocity**.

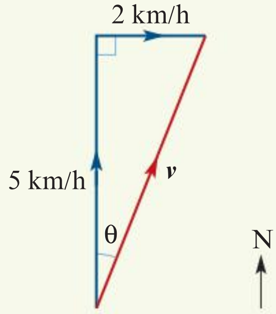
Example 23

A river is flowing north at 5 km/h. Mila can swim at 2 km/h in still water. She dives in from the west bank of the river and swims towards the opposite bank.

a In which direction does she travel? **b** What is her actual speed?

Solution

The swimmer's actual velocity, $\boldsymbol{v}$, is the vector sum of her velocity relative to the water (2 km/h east) and the water's velocity (5 km/h north).



a From the diagram, we have

$$\tan\theta = \frac{2}{5}$$

$$\therefore \quad \theta \approx 21.8^\circ$$

She is travelling on a bearing of 022°.

b Her actual speed is

$$|\boldsymbol{v}| = \sqrt{2^2 + 5^2} \approx 5.39 \text{ km/h}$$

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Relative velocity

In the previous example, the velocity of the water is given *relative to the bank* and the velocity of the swimmer is given *relative to the water*. The velocity of the swimmer *relative to the bank* is found by taking the vector sum. That is:

Velocity of swimmer relative to bank	=	Velocity of swimmer relative to water	+	Velocity of water relative to bank

The **relative velocity** of an object A with respect to another object B is the velocity that object A would appear to have to an observer moving along with object B.

Consider another example: A train is travelling north at 60 km/h, and a passenger walks at 3 km/h along the corridor towards the back of the train.

Velocity of passenger relative to Earth	=	Velocity of passenger relative to train	+	Velocity of train relative to Earth

The passenger is moving with a velocity of 57 km/h north relative to Earth.

In general, if an object A is in motion relative to another object B, then we can determine the velocity of A using a vector sum:

Velocity of A relative to Earth	=	Velocity of A relative to B	+	Velocity of B relative to Earth

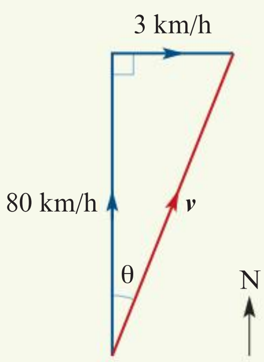
Velocities measured relative to Earth are often called **true velocities** or **actual velocities**.

Example 24

A train is moving with a constant velocity of 80 km/h north. A passenger walks straight across a carriage from the west side to the east side at 3 km/h. What is the true velocity of the passenger?

Solution

The passenger's true velocity, $\boldsymbol{v}$, is the vector sum of his velocity relative to the train (3 km/h east) and the train's velocity (80 km/h north).



Speed:

$$|\boldsymbol{v}| = \sqrt{80^2 + 3^2} = \sqrt{6409} \approx 80.06 \text{ km/h}$$

Direction: $\tan\theta = \dfrac{3}{80}$

$\therefore \quad \theta \approx 2.15°$

The passenger's true velocity is 80.06 km/h on a bearing of 002°.

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Example 25

Car A is moving with a velocity of 50 km/h due north, while car B is moving with a velocity of 120 km/h due west. What is the velocity of car A relative to car B?

Solution

Let $\mathbf{v}_A$ be the velocity of car A, and let $\mathbf{v}_B$ be the velocity of car B.

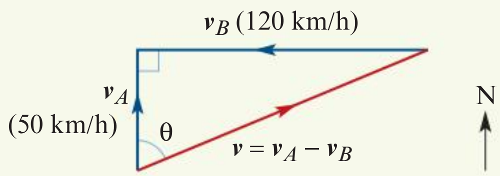
The velocity of car A relative to car B is given by $\mathbf{v} = \mathbf{v}_A - \mathbf{v}_B$.

Speed: $|\mathbf{v}| = \sqrt{50^2 + 120^2}$

$= 130$ km/h

Direction: $\theta = \tan^{-1}\left(\frac{12}{5}\right)$

$\approx 67.38°$



The velocity of car A relative to car B is 130 km/h on a bearing of 067°.

Wind effect on flight paths

The **airspeed** of an aircraft is its speed relative to air. In the next example, we see how the wind affects the actual velocity of an aircraft.

Example 26

A light aircraft has an airspeed of 250 km/h. The pilot sets a course due north. If the wind is blowing from the north-west at 80 km/h, what is the true speed and direction of the aircraft?

Solution

We can use the cosine rule to determine the true speed:

$$|\mathbf{v}| = \sqrt{250^2 + 80^2 - 2 \times 250 \times 80 \cos 45°}$$

$$= 201.5334\ldots \text{ km/h}$$

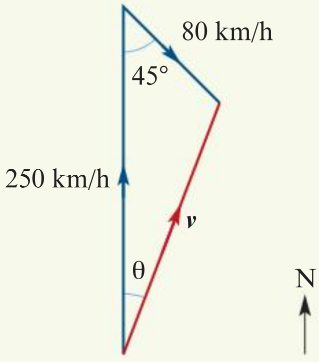
We can now use the sine rule to determine the angle θ:

$$\frac{80}{\sin\theta} = \frac{|\mathbf{v}|}{\sin 45°}$$

$$\sin\theta = \frac{80 \sin 45°}{|\mathbf{v}|}$$

$$= 0.2806\ldots$$

$\therefore \quad \theta \approx 16.30°$



The aircraft is flying at 201.53 km/h on a bearing of 016°.

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To fly an aircraft in a given direction, the pilot must compensate for the effect of the wind.

Example 27

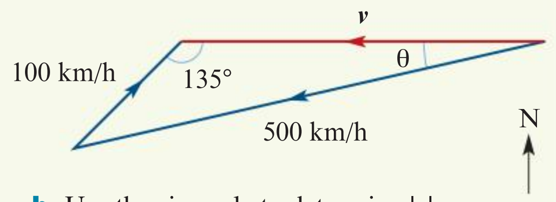
An aeroplane is scheduled to travel from a point P to a point Q, which is 1000 km due west of P. The aeroplane's airspeed is 500 km/h and the wind is blowing from the south-west at 100 km/h.

a In which direction should the pilot set the course?

b How long will the flight take?

Solution

We want to ensure that the plane's true velocity, $\boldsymbol{v}$, is due west.



a Use the sine rule to determine θ:

$$\frac{500}{\sin 135^\circ} = \frac{100}{\sin\theta}$$

$$\sin\theta = \frac{100\sin 135^\circ}{500}$$

$$= 0.1414\ldots$$

$$\therefore \quad \theta = (8.130\ldots)^\circ$$

The pilot should head on a bearing of 262°.

b Use the sine rule to determine $|\boldsymbol{v}|$:

$$\frac{500}{\sin 135^\circ} = \frac{|\boldsymbol{v}|}{\sin(36.869\ldots)^\circ}$$

$$\therefore \quad |\boldsymbol{v}| = \frac{500\sin(36.869\ldots)^\circ}{\sin 135^\circ}$$

$$= 424.264\ldots$$

$$\approx 424.26 \text{ km/h}$$

The plane's speed relative to the ground is approximately 424 km/h. The flight will take approximately 2.4 hours.

Summary 8G

- If two or more velocity vectors are added, then the sum is called a **resultant velocity**.
- The **relative velocity** of an object A with respect to another object B is the velocity that object A would appear to have to an observer moving along with object B.
- If an object A is in motion relative to another object B, we can determine the velocity of A using a vector sum:

Velocity of A relative to Earth	=	Velocity of A relative to B	+	Velocity of B relative to Earth

Exercise 8G

Example 23 **1** A river is flowing south at 4 km/h. Max can swim at 3 km/h in still water. He dives in from the west bank of the river and swims towards the opposite bank.

a In which direction does he travel? **b** What is his actual speed?

CF

Example 24

2 A train is moving due north at 100 km/h. A passenger walks straight across a carriage from the east side to the west side at 4 km/h. What is the true velocity of the passenger?

CF

3 Cars A and B are driving along a straight level road that runs east–west.

a If car A has a velocity of 100 km/h west and car B has a velocity of 80 km/h west, what is the velocity of car A relative to car B?

b If car A has a velocity of 100 km/h west and car B has a velocity of 80 km/h east, what is the velocity of car A relative to car B?

4 A cricketer is on a moving walkway which runs from south to north at 2 m/s. He bowls his fastest delivery, which is 45 m/s, again in a direction north. What is the velocity of the ball (relative to Earth)?

5 A ship is moving in a straight line at 15 m/s. A bird flies horizontally from the front of the ship towards the back of the ship at a speed of 5 m/s relative to the ship. What is the speed of the bird relative to the sea?

6 Car A is travelling north at 60 km/h along a straight level road. Car B is on the same road travelling north at 40 km/h. Determine:

a the velocity of car A relative to car B **b** the velocity of car B relative to car A.

7 A plane is heading due north, its airspeed is 240 km/h and there is an 80 km/h wind blowing from west to east. What is the velocity of the plane relative to Earth?

Example 25

8 Car A is moving with a velocity of 60 km/h due north, while car B is moving with a velocity of 80 km/h due west. What is the velocity of car A relative to car B?

9 A glider P is travelling due north at 60 km/h, and another glider Q is travelling north-west at 40 km/h. Determine the velocity of P relative to Q.

10 Two particles, A and B, are moving with constant velocities of $\boldsymbol{v}_A = 4\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}}$ m/s and $\boldsymbol{v}_B = 5\hat{\boldsymbol{i}} - 7\hat{\boldsymbol{j}}$ m/s respectively.

a Determine the velocity of B relative to A.

b Determine the magnitude of this relative velocity.

11 A ship is moving in a straight line at 15 m/s. A bird flies at an angle of 18° to the horizontal from the front of the ship towards the back of the ship at a speed of 5 m/s relative to the ship. What is the speed of the bird relative to the sea?

Example 26

12 A light aircraft has an airspeed of 240 km/h. The pilot sets a course due north. The wind is blowing from the north-east at 70 km/h. What is the true speed and direction of the aircraft?

Example 27

13 An aeroplane with an airspeed of 200 km/h is flying to an airport south-west of its present position. There is a wind blowing at 70 km/h from the east.

a Determine the course that the pilot must set.

b Determine the speed of the aeroplane relative to the ground.

14 A canoeist can paddle at 2 m/s in still water. He wishes to go straight across a river so that his path is at right angles to the banks of the river. The river is flowing at 1.5 m/s.

a Determine the direction in which he must paddle.

b If the river is 60 m wide, how long will it take him to cross the river?

8H Applications of vectors: forces and equilibrium

Learning intentions

- To be able to apply vectors to the study of forces acting on a particle.

A **force** is a measure of the strength of a *push* or *pull*. Forces can start motion, stop motion, make objects move faster or slower, and change the direction of motion.

Force can be defined as the physical quantity that causes a change in motion.

We will focus on situations where the forces 'cancel each other out'. For example, if an inflatable raft is floating in a swimming pool, then the water is exerting an upwards force on the raft (called the buoyant force) that cancels out the downwards force of gravity.

Introduction to forces

A force has both magnitude and direction – it may be represented by a vector.

When considering the forces that act on an object, it is convenient to treat the forces as acting on a single particle. The single particle may be thought of as a point at which the entire mass of the object is concentrated.

Weight and units of force

Every object near the surface of the Earth is subject to the force of gravity. We refer to this force as the **weight** of the object. Weight is a force that acts vertically downwards on an object (actually towards the centre of the Earth).

The unit of force used in this section is the **kilogram weight** (kg wt). If an object has a **mass** of 1 kg, then the force due to gravity acting on the object is 1 kg wt.

This unit is convenient for objects near the Earth's surface. An object with a mass of 1 kg would have a different weight on the moon.

Note: The standard unit of force is the newton (N). At the Earth's surface, a mass of m kg has a force of m kg wt = mg N acting on it, where g is the acceleration due to gravity (g m/s^2 ≈ 9.8 m/s^2).

Resultant force and equilibrium

When a number of forces act simultaneously on an object, their combined effect is called the **resultant force**. The resultant force is the vector sum of the forces acting on the particle.

If the resultant force acting on an object is zero, the object will remain at rest or continue moving with constant velocity. The object is said to be in **equilibrium**.

Note: Planet Earth is moving and our galaxy is moving, but we use Earth as our frame of reference and so our observation of an object being at rest is determined in this way.

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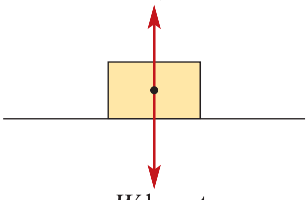
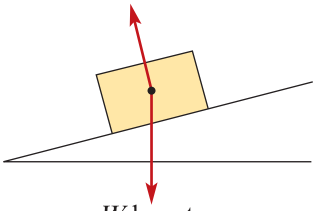
Normal force

Any mass placed on a surface, either horizontal or inclined, experiences a force perpendicular to the surface. This force is referred to as a **normal force**.



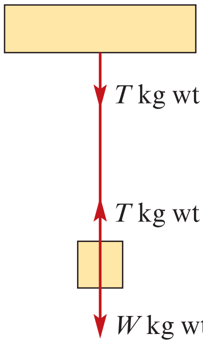


For example, a book sitting on a table is obviously being subjected to a force due to gravity. But the fact that it does not fall to the ground indicates that there must be a second force on the book. The table is exerting a force on the book equal in magnitude to gravity, but in the opposite direction. Hence the book remains at rest; it is in **equilibrium**.

Tension force

The diagram shows a string attached to the ceiling supporting a mass, which is at rest. The force of gravity, W kg wt, acts downwards on the mass and the string exerts an equal force, T kg wt, upwards on the mass. The force exerted by the string is called the **tension force**.

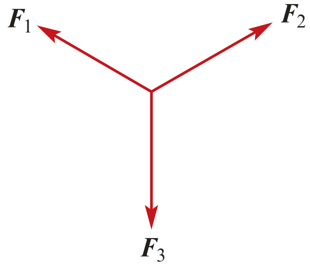
Note that there is a force, equal in magnitude but opposite in direction, acting on the ceiling at the point of contact.

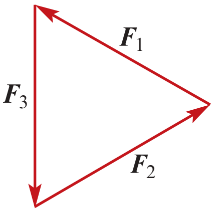


Triangle of forces

If three forces are acting on a point in equilibrium, then they can be represented by three vectors forming a triangle.

Suppose that three forces $\boldsymbol{F}_1$, $\boldsymbol{F}_2$ and $\boldsymbol{F}_3$ are acting on a particle in equilibrium, as shown in the diagram on the left. Since the particle is in equilibrium, we must have $\boldsymbol{F}_1 + \boldsymbol{F}_2 + \boldsymbol{F}_3 = \mathbf{0}$. Therefore the three forces can be rearranged into a triangle as shown on the right.





The magnitudes of the forces and the angles between the forces can now be found using trigonometric ratios (if the triangle contains a right angle) or using the sine or cosine rule.

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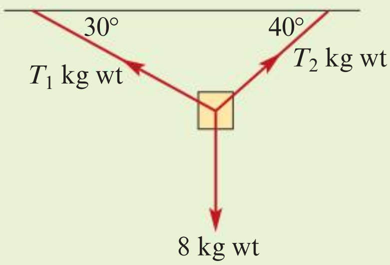
In the following examples and exercise, strings and ropes are considered to have negligible mass. A smooth light pulley is considered to have negligible mass and the friction between a rope and pulley is considered to be negligible.

Example 28

A particle of mass 8 kg is suspended by two strings attached to two points in the same horizontal plane. If the two strings make angles of 30° and 40° to the horizontal, determine the tension in each string.



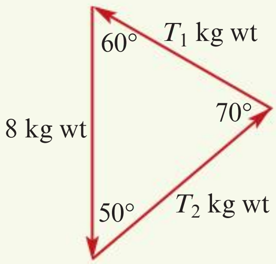
Solution

Represent the forces in a triangle. The sine rule gives

$$\frac{T_1}{\sin 50^\circ} = \frac{T_2}{\sin 60^\circ} = \frac{8}{\sin 70^\circ}$$

$$T_1 = \frac{8 \sin 50^\circ}{\sin 70^\circ} \approx 6.52 \text{ kg wt}$$

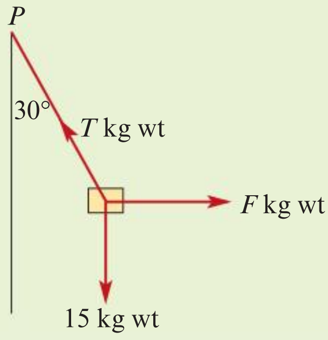
$$T_2 = \frac{8 \sin 60^\circ}{\sin 70^\circ} \approx 7.37 \text{ kg wt}$$



Example 29

A particle of mass 15 kg is suspended vertically from a point P by a string. The particle is pulled horizontally by a force of F kg wt so that the string makes an angle of 30° with the vertical.

Determine the value of F and the tension in the string.



Solution

Representing the forces in a triangle gives

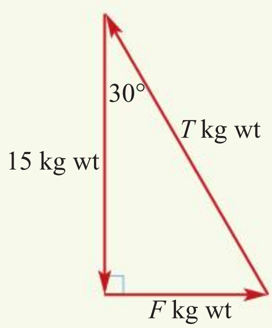
$$\frac{F}{15} = \tan 30^\circ$$

$$F = 15 \tan 30^\circ = 5\sqrt{3}$$

and $$\frac{15}{T} = \cos 30^\circ$$

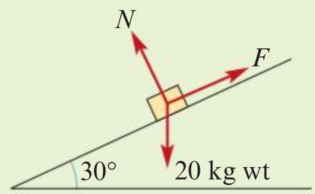
$$T = \frac{15}{\cos 30^\circ} = 10\sqrt{3}$$

The tension in the string is $10\sqrt{3}$ kg wt.



Example 30

A body of mass 20 kg is placed on a smooth plane inclined at 30° to the horizontal. A string is attached to a point further up the plane which prevents the body from moving. Determine the tension in the string and the magnitude of the force exerted on the body by the plane.

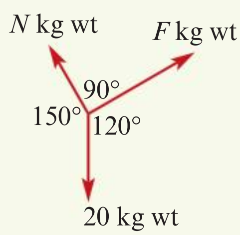


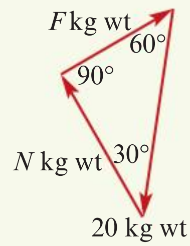
Solution

The three forces form a triangle (as the body is in equilibrium). Therefore

$$F = 20\sin 30^\circ = 10 \text{ kg wt}$$

$$N = 20\cos 30^\circ = 10\sqrt{3} \text{ kg wt}$$





Note: Force is a vector quantity, but it is often useful to employ only the magnitude of a force in calculations, and the direction is evident from the context. In this section, and in particular in diagrams, we often denote the magnitude of a force (for example, $\boldsymbol{F}$) by the same unbolded letter (in this case, F).

Resolution of forces

Obviously there are many situations where more than three forces (or in fact only two forces) will be acting on a body. An alternative method is required to solve such problems.

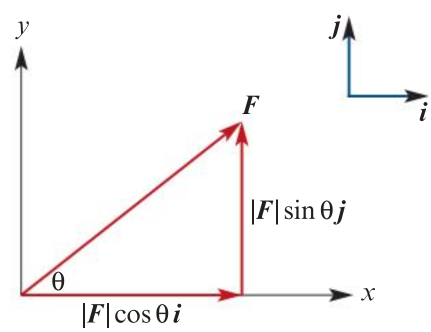
If all forces under consideration are acting in the same plane, then these forces and the resultant force can each be expressed as a sum of its $\hat{\boldsymbol{i}}$- and $\hat{\boldsymbol{j}}$-components.

If a force $\boldsymbol{F}$ acts at an angle of θ to the x-axis, then $\boldsymbol{F}$ can be written as the sum of two forces, one 'horizontal' and the other 'vertical':

$$\boldsymbol{F} = |\boldsymbol{F}|\cos\theta\,\hat{\boldsymbol{i}} + |\boldsymbol{F}|\sin\theta\,\hat{\boldsymbol{j}}$$

The force $\boldsymbol{F}$ is **resolved** into two components:

- the $\hat{\boldsymbol{i}}$-component is parallel to the x-axis
- the $\hat{\boldsymbol{j}}$-component is parallel to the y-axis.



For a particle that is in equilibrium, if all the forces acting on the particle are resolved into their $\hat{\boldsymbol{i}}$- and $\hat{\boldsymbol{j}}$-components, then:

- the sum of all the $\hat{\boldsymbol{i}}$-components is zero
- the sum of all the $\hat{\boldsymbol{j}}$-components is zero.

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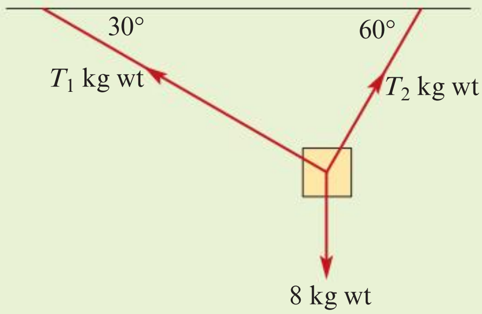
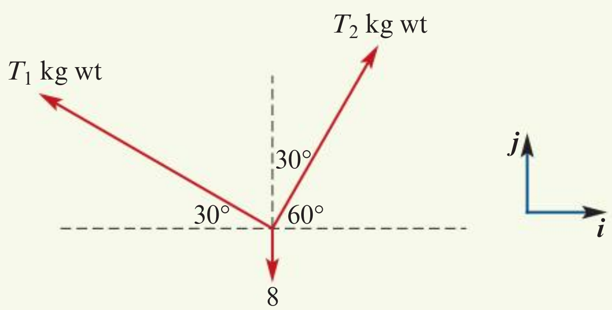
Example 31

A particle of mass 8 kg is suspended by two strings attached to two points in the same horizontal plane. If the two strings make angles of 30° and 60° to the horizontal, determine the tension in each string.



Solution



Resolution in the $\hat{j}$-direction:

$$T_1 \sin 30^\circ + T_2 \sin 60^\circ - 8 = 0$$

$$T_1\left(\frac{1}{2}\right) + T_2\left(\frac{\sqrt{3}}{2}\right) - 8 = 0 \qquad (1)$$

Resolution in the $\hat{i}$-direction:

$$-T_1 \cos 30^\circ + T_2 \cos 60^\circ = 0$$

$$-T_1\left(\frac{\sqrt{3}}{2}\right) + T_2\left(\frac{1}{2}\right) = 0 \qquad (2)$$

From (2): $\sqrt{3}\,T_1 = T_2$

Substituting in (1) gives

$$T_1\left(\frac{1}{2}\right) + \sqrt{3}\,T_1\left(\frac{\sqrt{3}}{2}\right) - 8 = 0$$

$$4T_1 = 16$$

$$\therefore \quad T_1 = 4$$

Hence $T_1 = 4$ and $T_2 = 4\sqrt{3}$.

The tensions in the strings are 4 kg wt and $4\sqrt{3}$ kg wt.

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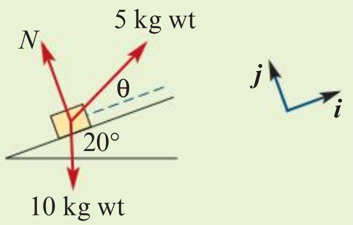
Example 32

A body of mass 10 kg is held at rest on a smooth plane inclined at 20° by a string with tension 5 kg wt as shown.

Determine the angle between the string and the inclined plane.



Solution

We resolve the forces parallel and perpendicular to the plane. Then the normal force $\boldsymbol{N}$ has no parallel component, since it is perpendicular to the plane.

Resolving in the $\hat{\boldsymbol{i}}$-direction:

$$5\cos\theta - 10\sin 20^\circ = 0$$

$$\cos\theta = \frac{10\sin 20^\circ}{5}$$

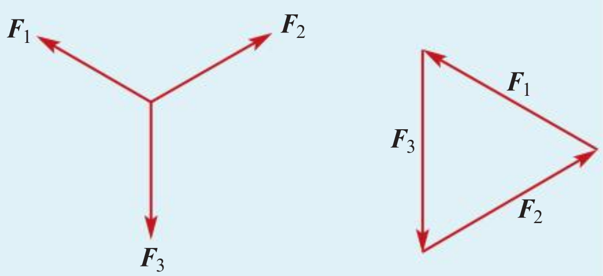
$$= 0.6840\ldots$$

$$\therefore \quad \theta \approx 46.84^\circ$$

The angle between the string and the inclined plane is 46.84°.

Summary 8H

- **Force** is a vector quantity.
- The magnitude of a force can be measured using **kilogram weight** (kg wt). If an object near the surface of the Earth has a mass of 1 kg, then the force due to gravity acting on the object is 1 kg wt.
- Triangle of forces
  If three forces are acting on a point in equilibrium, then they can be represented by three vectors forming a triangle.



- Resolution of forces
  - A force $\boldsymbol{F}$ is **resolved** into components by writing it in the form $\boldsymbol{F} = x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}}$.
  - If forces are acting on a particle that is in equilibrium, then:
    - the sum of the $\hat{\boldsymbol{i}}$-components of all the forces is zero
    - the sum of the $\hat{\boldsymbol{j}}$-components of all the forces is zero.

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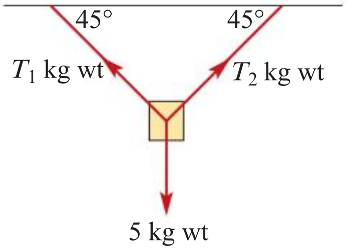
Skill-sheet

Exercise 8H

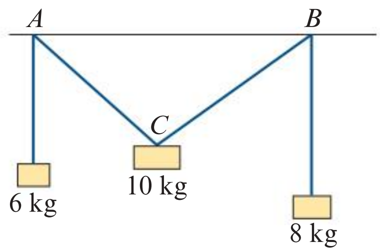
Complete Questions 1–10 *using triangles of forces.*

Example 28

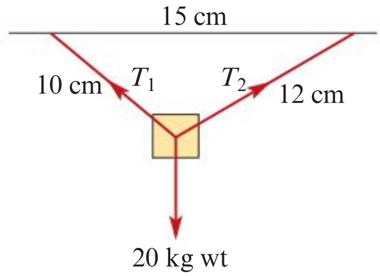
1 A particle of mass 5 kg is suspended by two strings attached to two points in the same horizontal plane. If the two strings make angles of 45° with the horizontal, Determine the tension in each string.



2 Using strings and pulleys, three weights of mass 6 kg, 8 kg and 10 kg are suspended in equilibrium as shown. Calculate the magnitude of the angle ACB.



3 A mass of 20 kg is suspended from two strings of length 10 cm and 12 cm, the ends of the strings being attached to two points in a horizontal line, 15 cm apart. Determine the tension in each string.

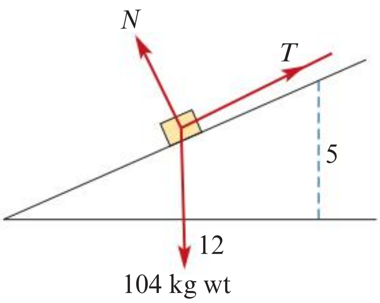


Example 29

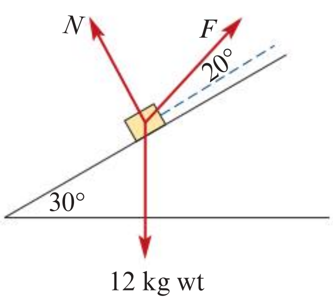
4 A boat is being pulled by a force of 40 kg wt towards the east and by a force of 30 kg wt towards the north-west. What third force must be acting on the boat if it remains stationary? Give the magnitude and direction.

Example 30

5 A body of mass 104 kg is placed on a smooth inclined plane which rises 5 cm vertically for every 12 cm horizontally. A string is attached to a point further up the plane which prevents the body from moving. Determine the tension in the string and the magnitude of the force exerted on the body by the plane.



6 A body of mass 12 kg is kept at rest on a smooth inclined plane of 30° by a force acting at an angle of 20° to the plane. Determine the magnitude of the force.



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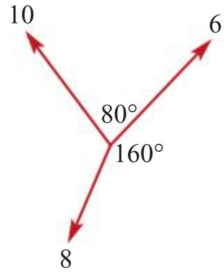
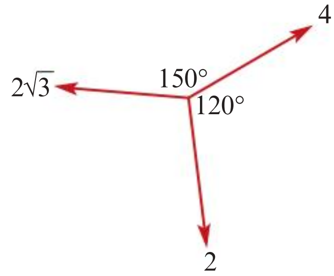
7 In each of the following cases, determine whether the particle is in equilibrium:

a 

b 

8 Three forces of magnitude 4 kg wt, 7 kg wt and 10 kg wt are in equilibrium. Determine the magnitudes of the angles between the forces.

9 A mass of 15 kg is maintained at rest on a smooth inclined plane by a string that is parallel to the plane. Determine the tension in the string if:

a the plane is at 30° to the horizontal

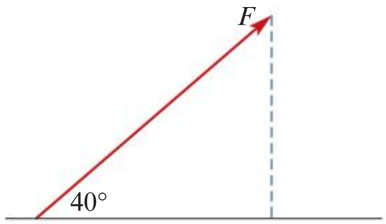
b the plane is at 40° to the horizontal

c the plane is at 30° to the horizontal, but the string is held at an angle of 10° to the plane.

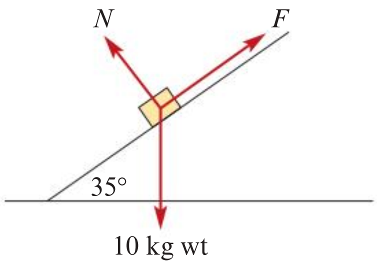
10 A string is connected to two points A and D in a horizontal line and masses of 12 kg and W kg are attached at points B and C with C lower than B. If AB, BC and CD make angles of 40°, 20° and 50° respectively with the horizontal, calculate the tensions in the string and the value of W.

Complete Questions 11–19 *using resolution of forces.*

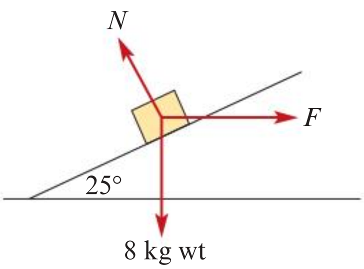
11 A force of F kg wt makes an angle of 40° with the horizontal. If its horizontal component is a force of 10 kg wt, determine the value of F.



Example 31 **12** Determine the magnitude of the force, acting on a smooth inclined plane of angle 35°, required to support a mass of 10 kg resting on the plane.



13 A body of mass 8 kg rests on a smooth inclined plane of angle 25° under the action of a horizontal force. Determine the magnitude of the force and the reaction of the plane on the body.



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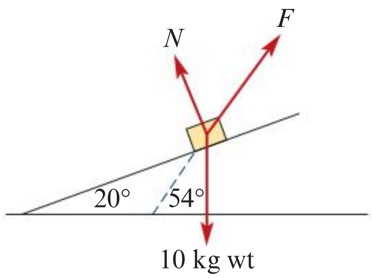
Example 32

14 A body of mass 10 kg rests on a smooth inclined plane of angle 20°. Determine the force that will keep it in equilibrium when it acts at an angle of 54° with the horizontal.

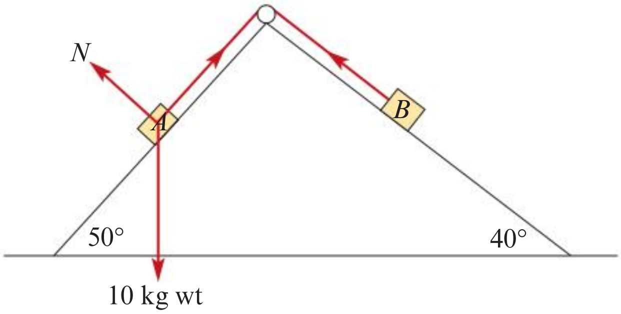


15 If a body of mass 12 kg is suspended by a string, determine the horizontal force required to hold it at an angle of 30° from the vertical.

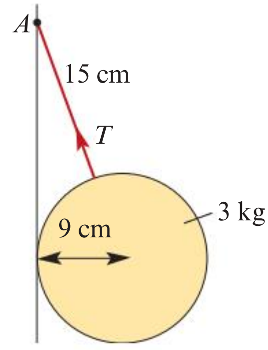
16 A force of 20 kg wt acting directly up a smooth plane inclined at an angle of 40° maintains a body in equilibrium on the plane. Calculate the mass of the body and the force it exerts on the plane.

17 Two men are supporting a block by ropes. One exerts a force of 20 kg wt, his rope making an angle of 35° with the vertical, and the other exerts a force of 30 kg wt. Determine the mass of the block and the angle of direction of the second rope.

18 A body A of mass 10 kg is supported against a smooth plane of angle 50°. Determine the force that the body exerts on the plane and the tension in the string, which is parallel to the slope. A body B on a plane of angle 40° is connected to A by a string passing over a smooth pulley on the ridge. If the system is in equilibrium, what is the mass of B?



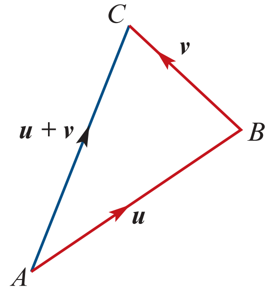
19 A sphere of radius 9 cm is attached to a point A on a vertical wall by a string of length 15 cm. If the mass of the sphere is 3 kg and the string meets the sphere at right angles determine the tension in the string.



Chapter summary

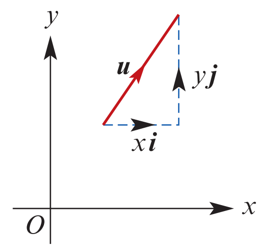
- A **vector** is a set of equivalent **directed line segments**.
- A directed line segment from a point A to a point B is denoted by $\overrightarrow{AB}$.
- The **position vector** of a point A is the vector $\overrightarrow{OA}$, where O is the origin.
- A vector can be written as a column of numbers. The vector $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ is '2 across and 3 up'.

Basic operations on vectors



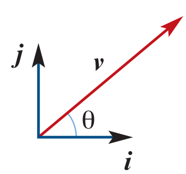
- **Addition**
  - If $\boldsymbol{u} = \begin{bmatrix} a \\ b \end{bmatrix}$ and $\boldsymbol{v} = \begin{bmatrix} c \\ d \end{bmatrix}$, then $\boldsymbol{u} + \boldsymbol{v} = \begin{bmatrix} a + c \\ b + d \end{bmatrix}$.
  - The sum $\boldsymbol{u} + \boldsymbol{v}$ can also be obtained geometrically as shown.
- **Scalar multiplication**
  - For $k \in \mathbb{R}^+$, the vector $k\boldsymbol{u}$ has the same direction as $\boldsymbol{u}$, but its length is multiplied by a factor of k.
  - The vector $-\boldsymbol{v}$ has the same length as $\boldsymbol{v}$, but the opposite direction.
  - Two non-zero vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are **parallel** if there exists $k \in \mathbb{R} \setminus \{0\}$ such that $\boldsymbol{u} = k\boldsymbol{v}$.
- **Subtraction** $\boldsymbol{u} - \boldsymbol{v} = \boldsymbol{u} + (-\boldsymbol{v})$

Component form



- In two dimensions, each vector $\boldsymbol{u}$ can be written in the form $\boldsymbol{u} = x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}}$, where
  - $\hat{\boldsymbol{i}}$ ($\hat{\imath}$) is the unit vector in the positive direction of the x-axis
  - $\hat{\boldsymbol{j}}$ ($\hat{\jmath}$) is the unit vector in the positive direction of the y-axis.
- The **magnitude** of vector $\boldsymbol{u} = x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}}$ is given by $|\boldsymbol{u}| = \sqrt{x^2 + y^2}$.
- The unit vector in the direction of vector $\boldsymbol{a}$ is given by $\hat{\boldsymbol{a}} = \dfrac{1}{|\boldsymbol{a}|}\boldsymbol{a}$.

Polar form

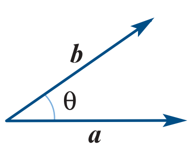


- In two dimensions, each vector $\boldsymbol{v}$ can be written in the form $\boldsymbol{v} = [r, \theta]$, where
  - r is the magnitude of $\boldsymbol{v}$
  - θ describes the angle that $\boldsymbol{v}$ makes with the $\hat{\boldsymbol{i}}$-direction, measured anticlockwise.

Scalar product

- The **scalar product** of vectors $\boldsymbol{a} = a_1\hat{\boldsymbol{i}} + a_2\hat{\boldsymbol{j}}$ and $\boldsymbol{b} = b_1\hat{\boldsymbol{i}} + b_2\hat{\boldsymbol{j}}$ is given by

$$\boldsymbol{a} \cdot \boldsymbol{b} = a_1 b_1 + a_2 b_2$$



- The scalar product is described geometrically by $\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}|\,|\boldsymbol{b}| \cos\theta$, where θ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$.
- Therefore $\boldsymbol{a} \cdot \boldsymbol{a} = |\boldsymbol{a}|^2$.
- Two non-zero vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are **perpendicular** if and only if $\boldsymbol{a} \cdot \boldsymbol{b} = 0$.

Vector projections

- Resolving a vector $\boldsymbol{a}$ into rectangular components is expressing the vector $\boldsymbol{a}$ as a sum of two vectors, one parallel to a given vector $\boldsymbol{b}$ and the other perpendicular to $\boldsymbol{b}$.
- The **vector resolute** of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$ is $\dfrac{\boldsymbol{a} \cdot \boldsymbol{b}}{\boldsymbol{b} \cdot \boldsymbol{b}}\boldsymbol{b}$.
- The **scalar resolute** of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$ is $\dfrac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{b}|}$.

Displacement and velocity

- The **displacement** of a particle is the change in its position. If a particle moves from point A to point B, then its displacement is $\overrightarrow{AB}$.
- The **velocity** of a particle is the rate of change of its position with respect to time.
- **Motion with constant velocity** If a particle moves with a constant velocity of $\boldsymbol{v}$ m/s for t seconds, then its displacement vector, $\boldsymbol{s}$ m, is given by $\boldsymbol{s} = t\boldsymbol{v}$.

Relative velocity

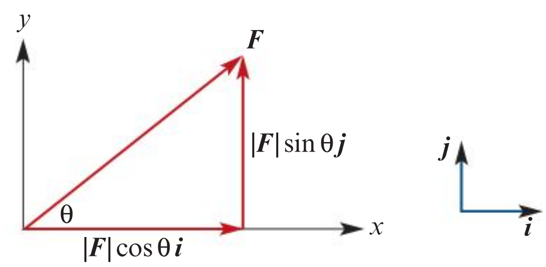
- The **relative velocity** of an object A with respect to another object B is the velocity that object A would appear to have to an observer moving along with object B.
- If an object A is in motion relative to another object B, we can determine the velocity of A using a vector sum:

Velocity of A relative to Earth	=	Velocity of A relative to B	+	Velocity of B relative to Earth

Forces and equilibrium

- **Resultant force** When a number of forces act simultaneously on an object, their combined effect is called the **resultant force**.
- **Equilibrium** If the resultant force acting on an object is zero, then the object is said to be in **equilibrium**; it will remain at rest or continue moving with constant velocity.
- **Triangle of forces** If three forces are acting on a particle in equilibrium, then the vectors representing the forces may be arranged to form a triangle. The magnitudes of the forces and the angles between them can be found using trigonometric ratios (if the triangle contains a right angle) or using the sine or cosine rule.
- **Resolution of forces**
  If all forces on a particle are acting in two dimensions, then each force can be expressed in terms of its components in the $\hat{\boldsymbol{i}}$- and $\hat{\boldsymbol{j}}$-directions:

$$\boldsymbol{F} = |\boldsymbol{F}|\cos\theta\,\hat{\boldsymbol{i}} + |\boldsymbol{F}|\sin\theta\,\hat{\boldsymbol{j}}$$



For the particle to be in equilibrium, the sum of all the $\hat{\boldsymbol{i}}$-components must be zero and the sum of all the $\hat{\boldsymbol{j}}$-components must be zero.

Skills checklist

Check-list

Download this checklist from the Interactive Textbook, then print it and fill it out to check your skills. ✓

8A **1** I can represent a vector as a directed line segment. ☐

See Example 1 and Question 1

8A **2** I can determine the vector represented by a directed line segment. ☐

See Example 2 and Question 2

8A **3** I can calculate the scalar multiple of a column vector and add column vectors. ☐

See Example 3 and Question 5

8A **4** I can use vectors in geometric contexts including midpoints and parallel vectors. ☐

See Example 4, Example 5 and Questions 11 and 13

8B **5** I can write a vector in component form and carry out addition and scalar multiplication of vectors expressed in this form. ☐

See Example 6 and Questions 1 and 3

8B **6** I can calculate the magnitude of a vector. ☐

See Example 7, Example 8 and Questions 4 and 9

8B **7** I can calculate the unit vector in the direction of a given vector. ☐

See Example 9 and Question 18

8C **8** I can write a vector written in polar form in component form ☐

See Example 10 and Question 2

8C **9** I can write a vector written in component form in polar form ☐

See Example 11 and Question 3

8C **10** I can carry out operations on vectors written in polar form ☐

See Example 12 and Questions 7 and 8

8D **11** I can use the scalar product to determine the angle between two vectors. ☐

See Example 13, Example 14 and Questions 3 and 9

8D **12** I can determine the scalar product of two vectors written in polar form ☐

See Example 15 and Question 11

Review

8E **13** I can determine the vector resolute of one vector in the direction of another. ☐

See Example 16 and Question 4

8E **14** I can resolve a vector into rectangular components, one of which is parallel to the vector in which the resolution is defined. ☐

See Example 17 and Question 5

8E **15** I can determine the scalar resolute of one vector in the direction of another. ☐

See Example 18 and Question 6

8F **16** Given that a particle moves from point *A* to point *B*, both with given coordinates, I can determine the displacement vector in component form. ☐

See Example 19 and Question 1

8F **17** Given that a particle starts from a given point and moves in a straight line with constant velocity, I can determine the position of the particle at a given time. ☐

See Example 20 and Question 5

8F **18** I can investigate whether two particles each travelling with constant velocity meet, if given sufficient information. ☐

See Example 21, Example 22 and Questions 9 and 10

8G **19** I can determine the resultant velocity of a particle by adding two velocity vectors. ☐

See Example 23 and Question 1

8G **20** I can determine the relative velocity of one particle with respect to another if both particles have constant velocity. ☐

See Example 24, Example 25 and Questions 2 and 8

8G **21** I can model the motion of an aeroplane using resultant velocity. ☐

See Example 26, Example 27 and Questions 12 and 13

8H **22** I can use a triangle of forces to consider forces acting on a particle which is in equilibrium. ☐

See Example 28, Example 29, Example 30 and Questions 1, 4 and 5

8H **23** I can use resolution of forces to consider forces acting on a particle which is in equilibrium. ☐

See Example 31, Example 32, and Questions 12 and 14

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Review

SF

Technology-free questions

Technology-free short-response questions

1 Let $A = (2, -3)$, $B = (2, 0)$ and $C = (4, -2)$ and let O be the origin.

Express each of the following vectors in the form $\begin{bmatrix} a \\ b \end{bmatrix}$:

a $\overrightarrow{OA}$ **b** $\overrightarrow{AB}$ **c** $\overrightarrow{BC}$ **d** $\overrightarrow{CO}$ **e** $\overrightarrow{CB}$

2 Let $\boldsymbol{a} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$ and $\boldsymbol{b} = \begin{bmatrix} -1 \\ -3 \end{bmatrix}$.

a Determine $|\boldsymbol{a}|$ and $|\boldsymbol{b}|$.

b Determine $\boldsymbol{a} \cdot \boldsymbol{b}$.

c Determine $\hat{\boldsymbol{a}}$, the unit vector in the direction of $\boldsymbol{a}$.

d Write each of the following in the form $\begin{bmatrix} m \\ n \end{bmatrix}$, where m and n are integers:

i $3\boldsymbol{a}$ **ii** $2\boldsymbol{a} + 3\boldsymbol{b}$ **iii** $2\boldsymbol{a} - \boldsymbol{b}$

3 Given that $\boldsymbol{a} = 7\hat{\boldsymbol{i}} + 6\hat{\boldsymbol{j}}$ and $\boldsymbol{b} = 2\hat{\boldsymbol{i}} + x\hat{\boldsymbol{j}}$, determine the values of x for which:

a $\boldsymbol{a}$ is parallel to $\boldsymbol{b}$

b $\boldsymbol{a}$ and $\boldsymbol{b}$ have the same magnitude.

4 $ABCD$ is a parallelogram where $\overrightarrow{OA} = 2\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}}$, $\overrightarrow{AB} = 3\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}}$ and $\overrightarrow{AD} = -2\hat{\boldsymbol{i}} + 5\hat{\boldsymbol{j}}$. Determine the coordinates of the four vertices of the parallelogram.

5 Let $\boldsymbol{a} = 2\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}}$ and $\boldsymbol{b} = 2\hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}}$. Determine the value of k such that $\boldsymbol{a} + k\boldsymbol{b}$ is parallel to the x-axis.

6 The position vectors of P and Q are $2\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}}$ and $3\hat{\boldsymbol{i}} - 7\hat{\boldsymbol{j}}$ respectively.

a Determine $|\overrightarrow{PQ}|$.

b Determine the unit vector in the direction of $\overrightarrow{PQ}$.

7 If $\boldsymbol{a} = 5\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}}$ and $\boldsymbol{b} = 3\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}}$, determine:

a $\boldsymbol{a} + 2\boldsymbol{b}$ **b** $|\boldsymbol{a}|$ **c** $\hat{\boldsymbol{a}}$ **d** $\boldsymbol{a} - \boldsymbol{b}$

8 The vector $\boldsymbol{v}$ is defined by the directed line segment from $(2\sqrt{3}, 1)$ to $(0, 3)$. Express $\boldsymbol{v}$ in polar form.

9 Let $\boldsymbol{a} = 2\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}}$, $\boldsymbol{b} = -\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}$ and $\boldsymbol{c} = -2\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}}$. Determine:

a $\boldsymbol{a} \cdot \boldsymbol{a}$ **b** $\boldsymbol{b} \cdot \boldsymbol{b}$ **c** $\boldsymbol{c} \cdot \boldsymbol{c}$ **d** $\boldsymbol{a} \cdot \boldsymbol{b}$

e $\boldsymbol{a} \cdot (\boldsymbol{b} + \boldsymbol{c})$ **f** $(\boldsymbol{a} + \boldsymbol{b}) \cdot (\boldsymbol{a} + \boldsymbol{c})$ **g** $(\boldsymbol{a} + 2\boldsymbol{b}) \cdot (3\boldsymbol{c} - \boldsymbol{b})$

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SF

10 Let $\boldsymbol{a} = \begin{bmatrix} 14 \\ 4 \end{bmatrix}$, $\boldsymbol{b} = \begin{bmatrix} m \\ -6 \end{bmatrix}$ and $\boldsymbol{c} = \begin{bmatrix} 10 \\ n \end{bmatrix}$.
If $\boldsymbol{a} + 2\boldsymbol{b} = \boldsymbol{c}$ determine the values of m and n.

11 Let $\boldsymbol{a} = \begin{bmatrix} 6 \\ 11 \end{bmatrix}$, $\boldsymbol{b} = \begin{bmatrix} k \\ 7 \end{bmatrix}$ and $\boldsymbol{c} = \begin{bmatrix} 8 \\ -10 \end{bmatrix}$.
If $\boldsymbol{a} - \boldsymbol{b}$ is parallel to $\boldsymbol{b} + \boldsymbol{c}$ determine the value of k.

12 Given that $\boldsymbol{p} = 3\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}$ and $\boldsymbol{q} = -2\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}}$, determine:

a $|\boldsymbol{p} - \boldsymbol{q}|$ **b** $|\boldsymbol{p}| - |\boldsymbol{q}|$ **c** $\boldsymbol{r}$ such that $\boldsymbol{p} + 2\boldsymbol{q} + \boldsymbol{r} = \boldsymbol{0}$

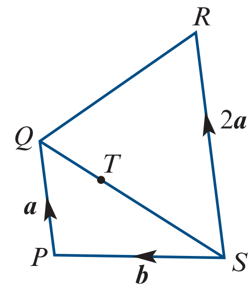
CF

13 $\overrightarrow{OA} = 4\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}$ and C is a point on OA such that $|\overrightarrow{OC}| = \dfrac{16}{5}$.

a Determine the unit vector in the direction of $\overrightarrow{OA}$.

b Hence determine $\overrightarrow{OC}$.

14 In the diagram, $ST = 2TQ$, $\overrightarrow{PQ} = \boldsymbol{a}$, $\overrightarrow{SR} = 2\boldsymbol{a}$ and $\overrightarrow{SP} = \boldsymbol{b}$.



a Determine each of the following in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$:

i $\overrightarrow{SQ}$ **ii** $\overrightarrow{TQ}$ **iii** $\overrightarrow{RQ}$ **iv** $\overrightarrow{PT}$ **v** $\overrightarrow{TR}$

b Prove that T lies on the line PR by showing that PT and TR are parallel.

15 Let M be the midpoint of the line segment AB, where $\overrightarrow{OA} = 5\hat{\boldsymbol{i}} + s\hat{\boldsymbol{j}}$ and $\overrightarrow{OB} = t\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}$. Given that $\overrightarrow{OM} = 7\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}}$, determine s and t.

16 The vector $\boldsymbol{p}$ has magnitude 7 units and bearing 050° and the vector $\boldsymbol{q}$ has magnitude 12 units and bearing 170°. (These are compass bearings on the horizontal plane.) Draw a diagram (not to scale) showing $\boldsymbol{p}$, $\boldsymbol{q}$ and $\boldsymbol{p} + \boldsymbol{q}$. Calculate the magnitude of $\boldsymbol{p} + \boldsymbol{q}$.

17 Let O, A and B be the points $(0, 0)$, $(3, 4)$ and $(4, -6)$ respectively.

a If C is the point such that $\overrightarrow{OA} = \overrightarrow{OC} + \overrightarrow{OB}$, determine the coordinates of C.

b If D is the point $(1, 24)$ and $\overrightarrow{OD} = h\overrightarrow{OA} + k\overrightarrow{OB}$, determine the values of h and k.

18 The points A, B and C have position vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ relative to an origin O. Write down an equation connecting $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ for each of the following cases:

a $OABC$ is a parallelogram

b B divides AC in the ratio $3 : 2$. That is, $AB : BC = 3 : 2$.

19 Points A, B and C have position vectors $\boldsymbol{a} = 4\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}$, $\boldsymbol{b} = 3\hat{\boldsymbol{i}} + 5\hat{\boldsymbol{j}}$ and $\boldsymbol{c} = -5\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}$ respectively. Evaluate $\overrightarrow{AB} \cdot \overrightarrow{BC}$ and hence show that $\triangle ABC$ is right-angled at B.

20 Given the vectors $\boldsymbol{p} = 5\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}$ and $\boldsymbol{q} = 2\hat{\boldsymbol{i}} + t\hat{\boldsymbol{j}}$, determine the values of t for which:

a $\boldsymbol{p} + \boldsymbol{q}$ is parallel to $\boldsymbol{p} - \boldsymbol{q}$ **b** $\boldsymbol{p} - 2\boldsymbol{q}$ is perpendicular to $\boldsymbol{p} + 2\boldsymbol{q}$ **c** $|\boldsymbol{p} - \boldsymbol{q}| = |\boldsymbol{q}|$

21 Points A, B and C have position vectors $\boldsymbol{a} = 2\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}}$, $\boldsymbol{b} = \hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}}$ and $\boldsymbol{c} = 2\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}}$. Determine:

a **i** $\overrightarrow{AB}$ **ii** $\overrightarrow{AC}$

b the vector resolute of $\overrightarrow{AB}$ in the direction of $\overrightarrow{AC}$

c the shortest distance from B to the line AC.

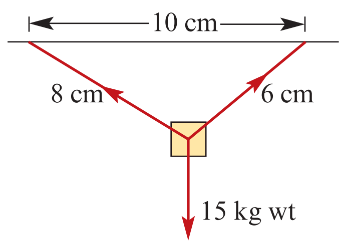
22 Priya can swim at a speed of 1.6 m/s in still water. She swims across a river that is 48 metres wide and flows at 1.2 m/s between parallel banks.

a Determine the speed of the swimmer relative to the river bank.

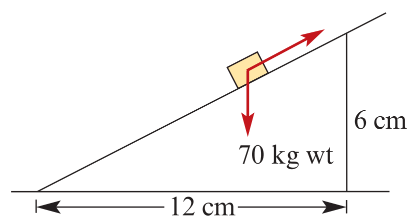
b Determine the time that it takes her to cross the river.

c Describe the position at which she arrives on the opposite bank.

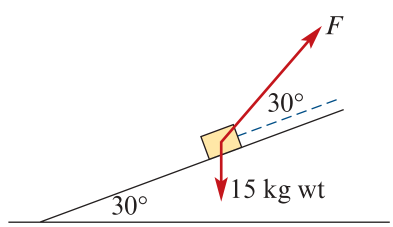
23 A mass of 15 kg is suspended from two strings of length 6 cm and 8 cm, the ends of the strings being attached to two points in a horizontal line, 10 cm apart. Determine the tension in each string.



24 A body of mass 70 kg is placed on a smooth inclined plane which rises 6 cm vertically for every 12 cm horizontally. A string is attached to a point further up the plane which prevents the body from moving. Determine the tension in the string and the magnitude of the force exerted on the body by the plane.



25 A body of mass 15 kg is kept at rest on a smooth inclined plane of 30° by a force acting at an angle of 30° to the plane. Determine the magnitude of the force.



26 The quadrilateral $PQRS$ is a parallelogram. The point P has coordinates $(5, 8)$, the point R has coordinates $(32, 17)$ and the vector $\overrightarrow{PQ}$ is given by $\overrightarrow{PQ} = \begin{bmatrix} 20 \\ -15 \end{bmatrix}$.

a Determine the coordinates of Q and write down the vector $\overrightarrow{QR}$.

b Write down the vector $\overrightarrow{RS}$ and show that the coordinates of S are $(12, 32)$.

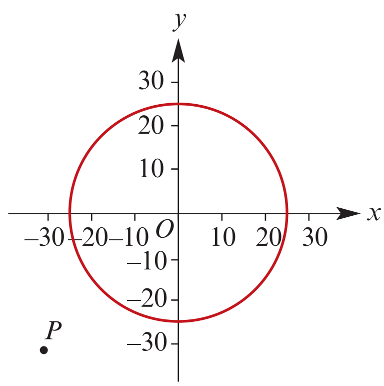
Review

27 Let $\begin{bmatrix}1\\0\end{bmatrix}$ represent a displacement 1 km due east.

Let $\begin{bmatrix}0\\1\end{bmatrix}$ represent a displacement 1 km due north.

The diagram shows a circle of radius 25 km with centre at $O(0, 0)$. A lighthouse entirely surrounded by sea is located at O. The lighthouse is not visible from points outside the circle.

A ship is initially at point P, which is 31 km west and 32 km south of the lighthouse.



CU

a Write down the vector $\overrightarrow{OP}$.

The ship is travelling in the direction of vector $\boldsymbol{u} = \begin{bmatrix}4\\3\end{bmatrix}$ with speed 20 km/h.

An hour after leaving P, the ship is at point R.

b Show that $\overrightarrow{PR} = \begin{bmatrix}16\\12\end{bmatrix}$ and hence determine the vector $\overrightarrow{OR}$.

c Show that the lighthouse first becomes visible when the ship reaches R.

Technology-active short-response questions

28 For each of the following determine the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$, in degrees, correct to one decimal place. SF

a $\boldsymbol{a} = 5\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}$, $\boldsymbol{b} = 3\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}}$.

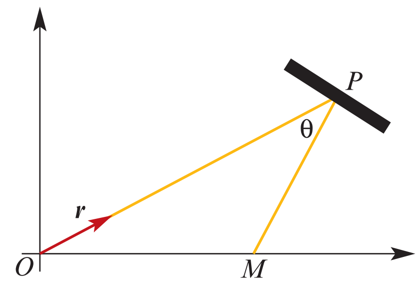
b $\boldsymbol{a} = -3\hat{\boldsymbol{i}} + 5\hat{\boldsymbol{j}}$, $\boldsymbol{b} = -7\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}}$.

29 A car A is moving at a velocity of 40 km/h due south while car B is moving with a velocity of 100 km/h due east. Determine the velocity of car A relative to car B. CF

30 Two forces $\boldsymbol{P}$ and $\boldsymbol{Q}$ of magnitudes $P = 6$ kg wt and $Q = 2$ kg wt acting on a particle have a resultant force of magnitude 5 kg wt. Determine the angle between forces $\boldsymbol{P}$ and $\boldsymbol{Q}$.

31 The diagram shows the path of a light beam from its source at O in the direction of the vector $\boldsymbol{r} = \begin{bmatrix}3\\1\end{bmatrix}$. CU

At point P, the beam is reflected by an adjustable mirror and meets the x-axis at M. The position of M varies, depending on the adjustment of the mirror at P.



a Given that $\overrightarrow{OP} = 4\boldsymbol{r}$, determine the coordinates of P.

b The point M has coordinates $(k, 0)$. Find an expression, in terms of k, for vector $\overrightarrow{PM}$.

c Determine the magnitudes of vectors $\overrightarrow{OP}$, $\overrightarrow{OM}$ and $\overrightarrow{PM}$, and hence determine the value of k for which θ is equal to 90°.

d Determine the value of θ for which M has coordinates $(9, 0)$.

32 A small boat S, drifting in the sea, is modelled as a particle moving in a straight line at constant speed. At 9:00 am, S is at a point with position vector $5\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}$ km relative to a fixed origin O, where $\hat{\boldsymbol{i}}$ and $\hat{\boldsymbol{j}}$ are unit vectors due east and due north respectively. At 10 am, S is at the point with position vector $6\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}}$ km. At time t hours after 9:00 am, S is at the point with position vector $\boldsymbol{s}$ km.

a Calculate the bearing on which S is drifting.

b Determine an expression for $\boldsymbol{s}$ in terms of t.

At 10:30 am a boat M leaves O and travels with constant velocity $p\hat{\boldsymbol{i}} + q\hat{\boldsymbol{j}}$ km/h. Given that M intercepts S at 11:00 am:

c calculate the value of p and the value of q.

33 A helicopter can fly at 150 km/h in still air. The wind is blowing at 30 km/h from the east.

a How long in total would it take the helicopter to fly directly to a point 180 km due east and back again?

b On what bearing should the helicopter head in order to fly directly to a point 90 km due north? How long would this take?

c On what bearing should the helicopter head in order to fly due south?

34 The unit vectors $\hat{\boldsymbol{i}}$ and $\hat{\boldsymbol{j}}$ represent 1 km east and 1 km north respectively. Two motor boats, A and B, are moving with velocities $\boldsymbol{v}_A = 12\hat{\boldsymbol{i}} + 16\hat{\boldsymbol{j}}$ km/h and $\boldsymbol{v}_B = 8\hat{\boldsymbol{i}} + \alpha\hat{\boldsymbol{j}}$ km/h, where α is a real number.

a Determine an expression for the velocity of boat A relative to boat B.

b When the two boats first sight each other, boat A is 10 km due west of boat B.

i Determine the value of α for which the two boats would collide if they maintained their current velocities.

ii Determine the time between the boats first sighting each other and the collision.

Multiple-choice questions

Technology-free multiple-choice questions

1 The vector $\boldsymbol{v}$ is defined by the directed line segment from (1, 1) to (3, 5). If $\boldsymbol{v} = a\hat{\boldsymbol{i}} + b\hat{\boldsymbol{j}}$, then

A $a = 3$ and $b = 5$
B $a = -2$ and $b = -4$
C $a = 2$ and $b = 4$
D $a = 2$ and $b = 3$

2 If vector $\overrightarrow{AB} = \boldsymbol{u}$ and vector $\overrightarrow{AC} = \boldsymbol{v}$, then vector $\overrightarrow{CB}$ is equal to

A $\boldsymbol{u} + \boldsymbol{v}$
B $\boldsymbol{v} - \boldsymbol{u}$
C $\boldsymbol{u} - \boldsymbol{v}$
D $\boldsymbol{u} \times \boldsymbol{v}$

3 If vector $\boldsymbol{a} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ and vector $\boldsymbol{b} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$, then $2\boldsymbol{a} - 3\boldsymbol{b} =$

A $\begin{bmatrix} 9 \\ -13 \end{bmatrix}$ **B** $\begin{bmatrix} 9 \\ 7 \end{bmatrix}$ **C** $\begin{bmatrix} 9 \\ -7 \end{bmatrix}$ **D** $\begin{bmatrix} 3 \\ -13 \end{bmatrix}$

4 $PQRS$ is a parallelogram. If $\overrightarrow{PQ} = \boldsymbol{p}$ and $\overrightarrow{QR} = \boldsymbol{q}$, then $\overrightarrow{SQ}$ is equal to

A $\boldsymbol{p} + \boldsymbol{q}$ **B** $\boldsymbol{p} - \boldsymbol{q}$ **C** $\boldsymbol{q} - \boldsymbol{p}$ **D** $2\boldsymbol{q}$

5 $|3\hat{\boldsymbol{i}} - 5\hat{\boldsymbol{j}}| =$

A 2 **B** $\sqrt{34}$ **C** 34 **D** 8

6 If $\overrightarrow{OA} = 2\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}$ and $\overrightarrow{OB} = \hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}}$, then $\overrightarrow{AB}$ equals

A $-\hat{\boldsymbol{i}} - 5\hat{\boldsymbol{j}}$ **B** $-\hat{\boldsymbol{i}} + 5\hat{\boldsymbol{j}}$ **C** $-\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}}$ **D** $-\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}$

7 If $\overrightarrow{OA} = 2\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}$ and $\overrightarrow{OB} = \hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}}$, then $|\overrightarrow{AB}|$ equals

A 6 **B** 26 **C** $\sqrt{26}$ **D** $\sqrt{24}$

8 If $\boldsymbol{a} = 2\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}$, then the unit vector in the direction of $\boldsymbol{a}$ is

A $2\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}$ **B** $\frac{1}{13}(2\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}})$

C $\frac{1}{\sqrt{5}}(2\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}})$ **D** $\frac{1}{\sqrt{13}}(2\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}})$

9 The vector $[\sqrt{6}, -45^\circ]$ corresponds to

A $\begin{bmatrix} \sqrt{6} \\ -\sqrt{6} \end{bmatrix}$ **B** $\begin{bmatrix} -\sqrt{6} \\ \sqrt{6} \end{bmatrix}$ **C** $\begin{bmatrix} -\sqrt{3} \\ \sqrt{3} \end{bmatrix}$ **D** $\begin{bmatrix} \sqrt{3} \\ -\sqrt{3} \end{bmatrix}$

10 An aircraft has an airspeed of 100 km/h. The aircraft is heading in the direction $3\hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}}$ and the wind is blowing with a velocity of $-5\hat{\boldsymbol{i}} + 20\hat{\boldsymbol{j}}$ km/h. The velocity of the aircraft relative to the ground (in km/h) is

A $55\hat{\boldsymbol{i}} - 60\hat{\boldsymbol{j}}$ **B** $65\hat{\boldsymbol{i}} - 60\hat{\boldsymbol{j}}$ **C** $305\hat{\boldsymbol{i}} - 420\hat{\boldsymbol{j}}$ **D** $60\hat{\boldsymbol{i}} - 40\hat{\boldsymbol{j}}$

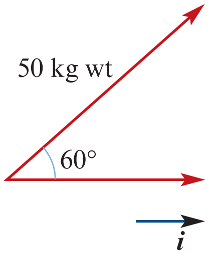
11 The velocity of a ship is $20\hat{\boldsymbol{i}}$ km/h and the velocity of the wind is $-4\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}$ km/h. The direction of the smoke trail coming from the ship's funnel is given by

A $16\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}$ **B** $-24\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}$ **C** $-4\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}$ **D** $-16\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}}$

12 The magnitude of the component of force $\boldsymbol{F}$ in the $\hat{\boldsymbol{i}}$-direction is

A 50 kg wt **B** 40 kg wt

C 20 kg wt **D** 25 kg wt



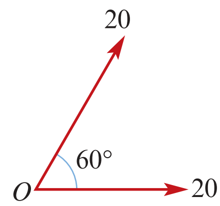
13 Two perpendicular forces have magnitudes 5 kg wt and 4 kg wt. The magnitude of the resultant force is

A 3 kg wt **B** $\sqrt{11}$ kg wt **C** $\sqrt{41}$ kg wt **D** 1 kg wt

14 A particle is acted on by two forces: $\boldsymbol{F}_1 = [7, 45°]$ kg wt and $\boldsymbol{F}_2 = [a, -45°]$ kg wt. If the resultant force $\boldsymbol{F}_1 + \boldsymbol{F}_2$ has magnitude 9 kg wt, then the value of a must be

A 2 **B** $4\sqrt{2}$ **C** $\sqrt{130}$ **D** 16

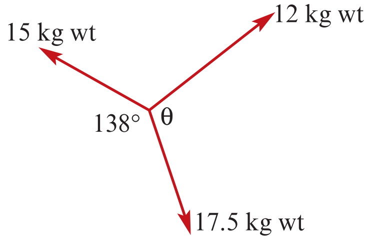
15 Two forces of magnitude 20 kg wt act on a particle at O as shown. The magnitude of the resultant force (in kg wt) is



A 40 **B** $20\sqrt{3}$
C 0 **D** 20

Technology-active multiple-choice questions

16 If this system of forces is in equilibrium, then θ is approximately



A 138° **B** 130°
C 123° **D** 100°

17 Two forces, each of magnitude F kg wt, act on a body. The resultant force has magnitude $\frac{F}{3}$ kg wt. The angle, in degrees correct to one decimal place, between the two forces is

A 160.1.2° **B** 48.2° **C** 65.1° **D** 45.0°

18 Consider two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ with $|\boldsymbol{a}| = |\boldsymbol{b}| = 6$ and $\boldsymbol{a}.\boldsymbol{b} = 10$. The angle, in degrees correct to one decimal place, between $\boldsymbol{a}$ and $\boldsymbol{b}$ is

A 48.1° **B** 75.1° **C** 46.9° **D** 73.9°

19 The scalar resolute of vector $\boldsymbol{a}$ in the direction of vector $\boldsymbol{b}$ is -8. The magnitude of vector $\boldsymbol{a}$ is 10. The angle, in degrees correct to one decimal place, between $\boldsymbol{a}$ and $\boldsymbol{b}$ is

A 73.1° **B** 125.6° **C** 110.8° **D** 143.1°

20 A boat moves at 1.85 m/s in still water. The boat crosses a river which is 110 m wide pointed directly across the river.. The river flows at 1.20 m/s. The angle in degrees, correct to two decimal places, that the boat's path makes with respect to the direction directly across the river is

A 22.97° **B** 57.03° **C** 64.78° **D** 32.97°

21 A boat moves at 5 m/s in still water. The boat is to cross a river that flows at 3 m/s. At what angle to the bank, in degrees correct to two decimal places, should the boat head upstream in order to land directly across the river?

A 42.66° **B** 25.66° **C** 53.13° **D** 38.87°

9 Matrices

Chapter contents

A **matrix** is a rectangular array of numbers. An example of a matrix is

$$\begin{bmatrix} 2 & 3 & 1 \\ 2 & 4 & -1 \\ 1 & 2 & -2 \end{bmatrix}$$

Matrix algebra was first studied in England in the middle of the nineteenth century. Matrices are now used in many areas of science: for example, in physics, medical research, encryption and internet search engines.

In this chapter we will show how addition and multiplication of matrices can be defined and how matrices can be used to solve simultaneous linear equations. In Chapter 17 we will see how they can be used to study transformations of the plane.

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9A Matrix notation

Learning intentions

- To be able to determine when two matrices are equal.

A **matrix** is a rectangular array of numbers. The numbers in the array are called the **entries** of the matrix.

The following are examples of matrices:

$$\begin{bmatrix} -1 & 2 \\ -3 & 4 \\ 5 & 6 \end{bmatrix} \qquad \begin{bmatrix} 2 & 1 & 5 & 6 \end{bmatrix} \qquad \begin{bmatrix} \sqrt{2} & \pi & 3 \\ 0 & 0 & 1 \\ \sqrt{2} & 0 & \pi \end{bmatrix} \qquad \begin{bmatrix} 5 \end{bmatrix}$$

The dimension of a matrix

Matrices vary in dimension. The **dimension** of the matrix is described by specifying the number of **rows** (horizontal lines) and **columns** (vertical lines) that occur in the matrix.

The dimensions of the above matrices are, in order:

$$3 \times 2, \quad 1 \times 4, \quad 3 \times 3, \quad 1 \times 1$$

The first number represents the number of rows, and the second the number of columns.

An $m \times n$ matrix has m rows and n columns.

Example 1

Write down the dimensions of the following matrices:

a $\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 0 \end{bmatrix}$ **b** $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ **c** $\begin{bmatrix} 2 & 2 & 3 \end{bmatrix}$

Solution

a 2×3 **b** 4×1 **c** 1×3

Storing information in matrices

The use of matrices to store information is demonstrated by the following example.

Four exporters A, B, C and D sell refrigerators (r), dishwashers (d), microwave ovens (m) and televisions (t). The sales in a particular month can be represented by a 4×4 array of numbers. This array of numbers is called a matrix.

	r	d	m	t	
A	120	95	370	250	row 1
B	430	380	950	900	row 2
C	60	50	150	100	row 3
D	200	100	470	50	row 4
	column 1	column 2	column 3	column 4	

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From this matrix it can be seen that:

- Exporter A sold 120 refrigerators, 95 dishwashers, 370 microwave ovens, 250 televisions.
- Exporter B sold 430 refrigerators, 380 dishwashers, 950 microwave ovens, 900 televisions.

The entries for the sales of refrigerators are in column 1.

The entries for the sales of exporter A are in row 1.

Example 2

A minibus has four rows of seats, with three seats in each row. If 0 indicates that a seat is vacant and 1 indicates that a seat is occupied, write down a matrix to represent:

a the 1st and 3rd rows are occupied, but the 2nd and 4th rows are vacant

b only the seat at the front-left corner of the minibus is occupied.

Solution

a $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

b $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Example 3

There are four clubs in a local football league:

- Club A has 2 senior teams and 3 junior teams.
- Club B has 2 senior teams and 4 junior teams.
- Club C has 1 senior team and 2 junior teams.
- Club D has 3 senior teams and 3 junior teams.

Represent this information in a matrix.

Solution

$$\begin{bmatrix} 2 & 3 \\ 2 & 4 \\ 1 & 2 \\ 3 & 3 \end{bmatrix}$$

Explanation

The rows represent clubs A, B, C, D and the columns represent the number of senior and junior teams.

Entries and equality

We will use uppercase letters $\mathbf{A}, \mathbf{B}, \mathbf{C}, \dots$ to denote matrices.

If $\mathbf{A}$ is a matrix, then a_{ij} will be used to denote the entry that occurs in row i and column j of $\mathbf{A}$. Thus a 3×4 matrix may be written as

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$

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Two matrices **A** and **B** are **equal**, and we can write **A** = **B**, when:

- they have the same number of rows and the same number of columns, and
- they have the same entry at corresponding positions.

For example:

$$\begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1+1 & 1 & -1 \\ 1-1 & 1 & \frac{6}{2} \end{bmatrix}$$

Example 4

If matrices **A** and **B** are equal, determine the values of x and y.

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ x & 4 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 2 & 1 \\ -3 & y \end{bmatrix}$$

Solution

$x = -3$ and $y = 4$

Although a matrix is made from a set of numbers, it is important to think of a matrix as a single entity, somewhat like a 'super number'.

Summary 9A

- A **matrix** is a rectangular array of numbers. The numbers in the array are called the **entries** of the matrix.
- The **dimension** of a matrix is described by specifying the number of rows and the number of columns. An $m \times n$ matrix has m rows and n columns.
- Two matrices **A** and **B** are equal when:
  - they have the same number of rows and the same number of columns, and
  - they have the same entry at corresponding positions.

Exercise 9A

SF

Example 1 **1** Write down the dimensions of the following matrices:

a $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ **b** $\begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 3 \end{bmatrix}$ **c** $\begin{bmatrix} a & b & c & d \end{bmatrix}$ **d** $\begin{bmatrix} p \\ q \\ r \\ s \end{bmatrix}$

Example 2 **2** There are 25 seats arranged in five rows and five columns. Using 0 to indicate that a seat is vacant and 1 to indicate that a seat is occupied, write down a matrix to represent the situation when:

a only the seats on the two diagonals are occupied

b all seats are occupied.

SF

3 Seating arrangements are again represented by matrices, as in Question 2. Describe the seating arrangement represented by each of the following matrices:

a the entry a_{ij} is 1 if $i = j$, but 0 if $i \neq j$

b the entry a_{ij} is 1 if $i > j$, but 0 if $i \leq j$

c the entry a_{ij} is 1 if $i = j + 1$, but 0 otherwise.

Example 3

4 At a certain school there are 200 girls and 110 boys in Year 7. The numbers of girls and boys in the other year levels are 180 and 117 in Year 8, 135 and 98 in Year 9, 110 and 89 in Year 10, 56 and 53 in Year 11, and 28 and 33 in Year 12. Summarise this information in a matrix.

Example 4

5 From the following, select those pairs of matrices which could be equal, and write down the values of x and y which would make them equal:

a $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 0 \\ x \end{bmatrix}$, $\begin{bmatrix} 0 & x \end{bmatrix}$, $\begin{bmatrix} 0 & 4 \end{bmatrix}$

b $\begin{bmatrix} 4 & 7 \\ 1 & -2 \end{bmatrix}$, $\begin{bmatrix} 1 & -2 \\ 4 & x \end{bmatrix}$, $\begin{bmatrix} x & 7 \\ 1 & -2 \end{bmatrix}$, $\begin{bmatrix} 4 & x & 1 & -2 \end{bmatrix}$

c $\begin{bmatrix} 2 & x & 4 \\ -1 & 10 & 3 \end{bmatrix}$, $\begin{bmatrix} y & 0 & 4 \\ -1 & 10 & 3 \end{bmatrix}$, $\begin{bmatrix} 2 & 0 & 4 \\ -1 & 10 & 3 \end{bmatrix}$

6 Determine the values of the pronumerals so that matrices **A** and **B** are equal:

a $\mathbf{A} = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 3 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} x & 1 & -1 \\ 0 & 1 & y \end{bmatrix}$

b $\mathbf{A} = \begin{bmatrix} x \\ 2 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 3 \\ y \end{bmatrix}$

c $\mathbf{A} = \begin{bmatrix} -3 & x \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} y & 4 \end{bmatrix}$

d $\mathbf{A} = \begin{bmatrix} 1 & y \\ 4 & 3 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 1 & -2 \\ 4 & x \end{bmatrix}$

7 The statistics for five members of a basketball team are recorded as follows:

Player A points 21, rebounds 5, assists 5

Player B points 8, rebounds 2, assists 3

Player C points 4, rebounds 1, assists 1

Player D points 14, rebounds 8, assists 60

Player E points 0, rebounds 1, assists 2

Express this information in a 5×3 matrix.

9B Addition, subtraction and multiplication by a real number

Learning intentions

- To be able to add and subtract two matrices of the same dimension.

Addition of matrices

If **A** and **B** are two matrices of the same dimension, then the sum **A** + **B** is the matrix obtained by adding together the corresponding entries of the two matrices.

For example:

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 0 & -3 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 4 & 3 \end{bmatrix}$$

and

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \\ a_{31}+b_{31} & a_{32}+b_{32} \end{bmatrix}$$

Multiplication of a matrix by a real number

If **A** is any matrix and k is a real number, then the product k**A** is the matrix obtained by multiplying each entry of **A** by k.

For example:

$$3\begin{bmatrix} 2 & -2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 6 & -6 \\ 0 & 3 \end{bmatrix}$$

Note: If a matrix is added to itself, then the result is twice the matrix, i.e. **A** + **A** = 2**A**. Similarly, for any natural number n, the sum of n matrices each equal to **A** is n**A**.

If **B** is any matrix, then −**B** denotes the product (−1)**B**.

Subtraction of matrices

If **A** and **B** are matrices of the same dimension, then **A** − **B** is defined to be the sum

$$\mathbf{A} + (-\mathbf{B}) = \mathbf{A} + (-1)\mathbf{B}$$

For two matrices **A** and **B** of the same dimension, the difference **A** − **B** can be found by subtracting corresponding entries.

Example 5

Determine:

a $\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} - \begin{bmatrix} 2 & -1 \\ -4 & 1 \end{bmatrix}$

b $\begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}$

Solution

a $\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} - \begin{bmatrix} 2 & -1 \\ -4 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 6 & -1 \end{bmatrix}$

b $\begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

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Zero matrix

The $m \times n$ matrix with all entries equal to zero is called the **zero matrix**, and will be denoted by **O**.

For any $m \times n$ matrix **A** and the $m \times n$ zero matrix **O**, we have

$$\mathbf{A} + \mathbf{O} = \mathbf{A} \qquad \text{and} \qquad \mathbf{A} + (-\mathbf{A}) = \mathbf{O}$$

Example 6

Let $\mathbf{X} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$, $\mathbf{Y} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$, $\mathbf{A} = \begin{bmatrix} 2 & 0 \\ -1 & 2 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 5 & 0 \\ -2 & 4 \end{bmatrix}$.

Determine $\mathbf{X} + \mathbf{Y}$, $2\mathbf{X}$, $4\mathbf{Y} + \mathbf{X}$, $\mathbf{X} - \mathbf{Y}$, $-3\mathbf{A}$ and $-3\mathbf{A} + \mathbf{B}$.

Solution

$$\mathbf{X} + \mathbf{Y} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} + \begin{bmatrix} 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \end{bmatrix}$$

$$2\mathbf{X} = 2\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$$

$$4\mathbf{Y} + \mathbf{X} = 4\begin{bmatrix} 3 \\ 6 \end{bmatrix} + \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 12 \\ 24 \end{bmatrix} + \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 14 \\ 28 \end{bmatrix}$$

$$\mathbf{X} - \mathbf{Y} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} - \begin{bmatrix} 3 \\ 6 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$$

$$-3\mathbf{A} = -3\begin{bmatrix} 2 & 0 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -6 & 0 \\ 3 & -6 \end{bmatrix}$$

$$-3\mathbf{A} + \mathbf{B} = \begin{bmatrix} -6 & 0 \\ 3 & -6 \end{bmatrix} + \begin{bmatrix} 5 & 0 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1 & -2 \end{bmatrix}$$

Example 7

If $\mathbf{A} = \begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 0 & -4 \\ -2 & 8 \end{bmatrix}$, determine the matrix **X** such that $2\mathbf{A} + \mathbf{X} = \mathbf{B}$.

Solution

If $2\mathbf{A} + \mathbf{X} = \mathbf{B}$, then $\mathbf{X} = \mathbf{B} - 2\mathbf{A}$. Therefore

$$\begin{aligned} \mathbf{X} &= \begin{bmatrix} 0 & -4 \\ -2 & 8 \end{bmatrix} - 2\begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 - 2 \times 3 & -4 - 2 \times 2 \\ -2 - 2 \times (-1) & 8 - 2 \times 1 \end{bmatrix} \\ &= \begin{bmatrix} -6 & -8 \\ 0 & 6 \end{bmatrix} \end{aligned}$$

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TI-84

Using the TI-Nspire CX non-CAS

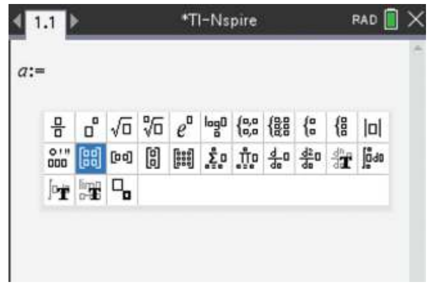
The matrix template

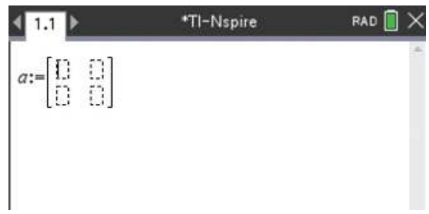
Matrices can be assigned (or stored) as variables for further computations.

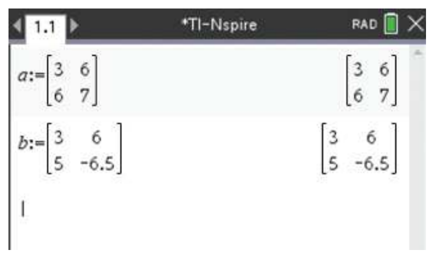
Assign matrix $\mathbf{A} = \begin{bmatrix} 3 & 6 \\ 6 & 7 \end{bmatrix}$ as follows:

- In a **Calculator** page, type $a :=$ and then enter the matrix. (Access the assignment symbol := using (ctrl)(templates key).)
- The simplest way to enter a 2×2 matrix is using the 2×2 matrix template as shown. (Access the templates using either (templates key) or (ctrl)(menu) > **Math Templates**.)
- Notice that there is also a template for entering $m \times n$ matrices.
- Use the touchpad arrows (or (tab)) to move between the entries of the matrix.

Assign the matrix $\mathbf{B} = \begin{bmatrix} 3 & 6 \\ 5 & -6.5 \end{bmatrix}$ similarly.

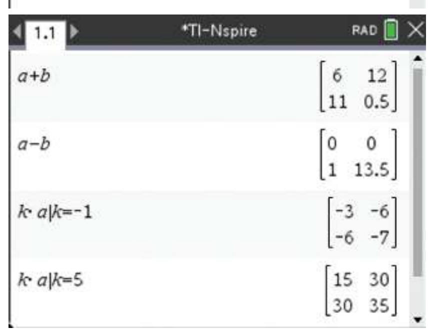






Operations on matrices

Once **A** and **B** are defined as above, the matrices **A** + **B**, **A** – **B** and k**A** can easily be determined.



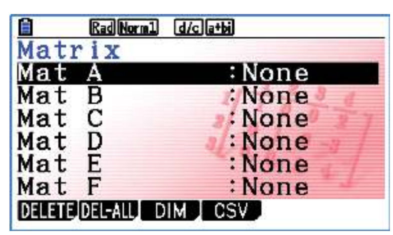
Using the Casio

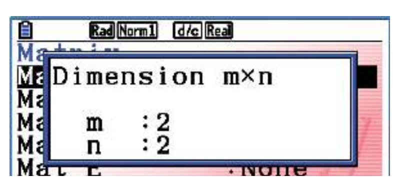
Storing matrices

To store matrices $\mathbf{A} = \begin{bmatrix} 3 & 6 \\ 6 & 7 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 3 & 6 \\ 5 & -6.5 \end{bmatrix}$:

- In **Run-Matrix** mode, go to the Matrix Editor screen by selecting ▸ **Mat** (F3).
- Press (EXE) to select Matrix A.
- Specify the dimension of Matrix A:

  (2) (EXE) (2) (EXE) (EXE)





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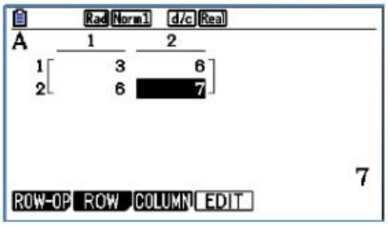
- Input the entries of Matrix A:

  3 EXE 6 EXE 6 EXE 7 EXE

- Press EXIT to return to the Matrix Editor screen.
- Use the cursor key ▼ to move down to Matrix B, and then store the matrix **B** similarly.
- Press EXIT.

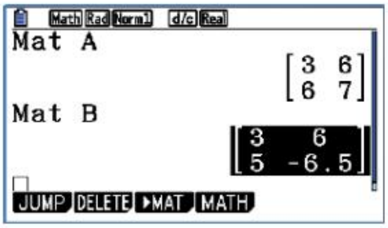


Accessing stored matrices

To access a stored matrix, use **Mat** (SHIFT 2) followed by the matrix name.

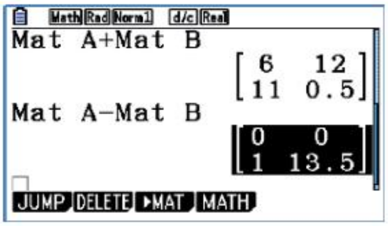
- For example, to view Matrix A, press:

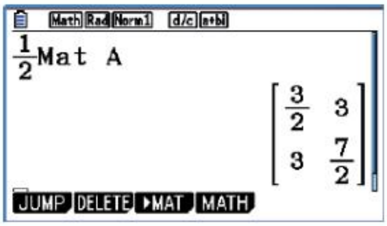
  SHIFT 2 ALPHA X,θ,T EXE



Operations on stored matrices

The matrices **A** + **B**, **A** − **B** and $\frac{1}{2}$**A** can now be found as shown.





Summary 9B

- If **A** and **B** are matrices of the same dimension, then:
  - the matrix **A** + **B** is obtained by adding the corresponding entries of **A** and **B**
  - the matrix **A** − **B** is obtained by subtracting the corresponding entries of **A** and **B**.
- If **A** is any matrix and k is a real number, then the matrix k**A** is obtained by multiplying each entry of **A** by k.

Exercise 9B

Example 6

1 Let $\mathbf{X} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$, $\mathbf{Y} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$, $\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 4 & 0 \\ -1 & 2 \end{bmatrix}$.

Determine $\mathbf{X} + \mathbf{Y}$, $2\mathbf{X}$, $4\mathbf{Y} + \mathbf{X}$, $\mathbf{X} - \mathbf{Y}$, $-3\mathbf{A}$ and $-3\mathbf{A} + \mathbf{B}$.

SF

2 Let $\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}$. Determine $2\mathbf{A}$, $-3\mathbf{A}$ and $-6\mathbf{A}$.

3 For $m \times n$ matrices **A**, **B** and **C**, is it always true that:

a $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$

b $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$?

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4 Let $\mathbf{A} = \begin{bmatrix} 3 & 2 \\ -2 & -2 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 0 & -3 \\ 4 & 1 \end{bmatrix}$. Calculate:

a 2A **b** 3B **c** 2A + 3B **d** 3B − 2A

5 Let $\mathbf{P} = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$, $\mathbf{Q} = \begin{bmatrix} -1 & 1 \\ 2 & 0 \end{bmatrix}$ and $\mathbf{R} = \begin{bmatrix} 0 & 4 \\ 1 & 1 \end{bmatrix}$. Calculate:

a P + Q **b** P + 3Q **c** 2P − Q + R

Example 7

6 If $\mathbf{A} = \begin{bmatrix} 3 & 1 \\ -1 & 4 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 0 & -10 \\ -2 & 17 \end{bmatrix}$, Determine matrices **X** and **Y** such that 2**A** − 3**X** = **B** and 3**A** + 2**Y** = 2**B**.

7 Matrices **X** and **Y** show the production of four models of cars *a*, *b*, *c*, *d* at two factories *P*, *Q* in successive weeks. Determine **X** + **Y** and describe what this sum represents.

Week 1: $\mathbf{X} = \begin{array}{c} \\ P \\ Q \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 150 & 90 & 100 & 50 \\ 100 & 0 & 75 & 0 \end{bmatrix} \end{array}$

Week 2: $\mathbf{Y} = \begin{array}{c} \\ P \\ Q \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 160 & 90 & 120 & 40 \\ 100 & 0 & 50 & 0 \end{bmatrix} \end{array}$

9C Multiplication of matrices

Learning intentions

- To be able to multiply suitable matrices.

Multiplication of a matrix by a real number has been discussed in the previous section. The definition for multiplication of matrices is less straightforward. The procedure for multiplying two 2 × 2 matrices is shown first.

Let $\mathbf{A} = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 5 & 1 \\ 6 & 3 \end{bmatrix}$.

Then
$$\begin{aligned} \mathbf{AB} &= \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} 5 & 1 \\ 6 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 1\times5+3\times6 & 1\times1+3\times3 \\ 4\times5+2\times6 & 4\times1+2\times3 \end{bmatrix} \\ &= \begin{bmatrix} 23 & 10 \\ 32 & 10 \end{bmatrix} \end{aligned}$$

and
$$\begin{aligned} \mathbf{BA} &= \begin{bmatrix} 5 & 1 \\ 6 & 3 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 5\times1+1\times4 & 5\times3+1\times2 \\ 6\times1+3\times4 & 6\times3+3\times2 \end{bmatrix} \\ &= \begin{bmatrix} 9 & 17 \\ 18 & 24 \end{bmatrix} \end{aligned}$$

Note that **AB** ≠ **BA**.

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If **A** is an $m \times n$ matrix and **B** is an $n \times r$ matrix, then the product **AB** is the $m \times r$ matrix whose entries are determined as follows:

> To determine the entry in row i and column j of **AB**, single out row i in matrix **A** and column j in matrix **B**. Multiply the corresponding entries from the row and column and then add up the resulting products.

Note: The product **AB** is defined only if the number of columns of **A** is the same as the number of rows of **B**.

Example 8

For $\mathbf{A} = \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$, determine **AB**.

Solution

A is a 2×2 matrix and **B** is a 2×1 matrix. Therefore the product **AB** is defined and will be a 2×1 matrix.

$$\mathbf{AB} = \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix}\begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \times 5 + 4 \times 3 \\ 3 \times 5 + 6 \times 3 \end{bmatrix} = \begin{bmatrix} 22 \\ 33 \end{bmatrix}$$

Example 9

Matrix **X** shows the number of cars of models a and b bought by four dealers A, B, C, D. Matrix **Y** shows the cost in dollars of cars a and b. Determine **XY** and explain what it represents.

$$\mathbf{X} = \begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\!\!\begin{array}{c} \begin{array}{cc} a & b \end{array} \\ \begin{bmatrix} 3 & 1 \\ 2 & 2 \\ 1 & 4 \\ 1 & 1 \end{bmatrix} \end{array} \qquad \mathbf{Y} = \begin{bmatrix} 26\,000 \\ 32\,000 \end{bmatrix}\begin{array}{c} a \\ b \end{array}$$

Solution

X is a 4×2 matrix and **Y** is a 2×1 matrix. Therefore **XY** is a 4×1 matrix.

$$\mathbf{XY} = \begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\!\!\begin{array}{c} \begin{array}{cc} a & b \end{array} \\ \begin{bmatrix} 3 & 1 \\ 2 & 2 \\ 1 & 4 \\ 1 & 1 \end{bmatrix} \end{array}\begin{bmatrix} 26\,000 \\ 32\,000 \end{bmatrix}\begin{array}{c} a \\ b \end{array}$$

$$= \begin{bmatrix} 3 \times 26\,000 + 1 \times 32\,000 \\ 2 \times 26\,000 + 2 \times 32\,000 \\ 1 \times 26\,000 + 4 \times 32\,000 \\ 1 \times 26\,000 + 1 \times 32\,000 \end{bmatrix} = \begin{bmatrix} 110\,000 \\ 116\,000 \\ 154\,000 \\ 58\,000 \end{bmatrix}$$

The matrix **XY** shows that dealer A spent \$110 000, dealer B spent \$116 000, dealer C spent \$154 000 and dealer D spent \$58 000.

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Example 10

For $\mathbf{A} = \begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 4 & 0 \\ 1 & 2 \\ 0 & 3 \end{bmatrix}$, determine $\mathbf{AB}$.

Solution

$\mathbf{A}$ is a 2×3 matrix and $\mathbf{B}$ is a 3×2 matrix. Therefore $\mathbf{AB}$ is a 2×2 matrix.

$$\mathbf{AB} = \begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 1 & 2 \\ 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \times 4 + 3 \times 1 + 4 \times 0 & 2 \times 0 + 3 \times 2 + 4 \times 3 \\ 5 \times 4 + 6 \times 1 + 7 \times 0 & 5 \times 0 + 6 \times 2 + 7 \times 3 \end{bmatrix} = \begin{bmatrix} 11 & 18 \\ 26 & 33 \end{bmatrix}$$

Summary 9C

- If $\mathbf{A}$ is an $m \times n$ matrix and $\mathbf{B}$ is an $n \times r$ matrix, then the product $\mathbf{AB}$ is the $m \times r$ matrix whose entries are determined as follows:

  To determine the entry in row i and column j of $\mathbf{AB}$, single out row i in matrix $\mathbf{A}$ and column j in matrix $\mathbf{B}$. Multiply the corresponding entries from the row and column and then add up the resulting products.

- The product $\mathbf{AB}$ is defined only if the number of columns of $\mathbf{A}$ is the same as the number of rows of $\mathbf{B}$.

Skill-sheet

Exercise 9C

SF

Example 8, 10

1 Let $\mathbf{X} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $\mathbf{Y} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, $\mathbf{A} = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$, $\mathbf{C} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ and $\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Determine the products $\mathbf{AX}$, $\mathbf{BX}$, $\mathbf{AY}$, $\mathbf{IX}$, $\mathbf{AC}$, $\mathbf{CA}$, $(\mathbf{AC})\mathbf{X}$, $\mathbf{C}(\mathbf{BX})$, $\mathbf{AI}$, $\mathbf{IB}$, $\mathbf{AB}$, $\mathbf{BA}$, $\mathbf{A}^2$, $\mathbf{B}^2$, $\mathbf{A}(\mathbf{CA})$ and $\mathbf{A}^2\mathbf{C}$.

2 Which of the following products of matrices from Question 1 are defined?
$\mathbf{AY}$, $\mathbf{YA}$, $\mathbf{XY}$, $\mathbf{X}^2$, $\mathbf{CI}$, $\mathbf{XI}$

3 If $\mathbf{A} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 0 & 0 \\ -3 & 2 \end{bmatrix}$, determine $\mathbf{AB}$.

4 Let $\mathbf{A}$ and $\mathbf{B}$ be 2×2 matrices and let $\mathbf{O}$ be the 2×2 zero matrix. Is the following argument correct?

'If $\mathbf{AB} = \mathbf{O}$ and $\mathbf{A} \neq \mathbf{O}$, then $\mathbf{B} = \mathbf{O}$.'

5 Find a matrix $\mathbf{A}$ such that $\mathbf{A} \neq \mathbf{O}$ but $\mathbf{A}^2 = \mathbf{O}$.

6 If $\mathbf{L} = \begin{bmatrix} 2 & -1 \end{bmatrix}$ and $\mathbf{X} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$, determine $\mathbf{LX}$ and $\mathbf{XL}$.

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7 Assume that both **A** and **B** are $m \times n$ matrices. Are **AB** and **BA** defined and, if so, how many rows and columns do they have?

8 Suppose that $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

a Show that $ad - bc = 1$.

b What is the product matrix if the order of multiplication on the left-hand side is reversed?

9 Using the result of Question 8, write down a pair of matrices **A** and **B** such that $\mathbf{AB} = \mathbf{BA} = \mathbf{I}$, where $\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

10 Determine matrices **A** and **B** such that $(\mathbf{A} + \mathbf{B})^2 \neq \mathbf{A}^2 + 2\mathbf{AB} + \mathbf{B}^2$.

Example 9 **11** It takes John 5 minutes to drink a milk shake which costs \$2.50, and 12 minutes to eat a banana split which costs \$3.00.

a Determine the product $\begin{bmatrix} 5 & 12 \\ 2.50 & 3.00 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and interpret the result in fast-food economics.

b Two friends join John. Determine $\begin{bmatrix} 5 & 12 \\ 2.50 & 3.00 \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 1 \end{bmatrix}$ and interpret the result.

12 Let $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$. Determine $\mathbf{A}^2$ and use your answer to determine $\mathbf{A}^4$ and $\mathbf{A}^8$.

13 Let $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Determine $\mathbf{A}^2$, $\mathbf{A}^3$ and $\mathbf{A}^4$. Write down a formula for $\mathbf{A}^n$.

9D Identities, inverses and determinants for 2 × 2 matrices

Learning intentions

- To be able to calculate the determinant of a 2×2 matrix and to determine the inverse of a 2×2 matrix if the determinant is non-zero.

Identities

A matrix with the same number of rows and columns is called a **square matrix**. For square matrices of a given dimension (e.g. 2×2), a multiplicative identity **I** exists.

For 2×2 matrices, the **identity matrix** is $\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

For example, if $\mathbf{A} = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$, then $\mathbf{AI} = \mathbf{A} = \mathbf{IA}$, and this result holds for any square matrix multiplied by the appropriate multiplicative identity.

For 3×3 matrices, the identity matrix is $\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Inverses

Given a 2×2 matrix $\mathbf{A}$, is there a matrix $\mathbf{B}$ such that $\mathbf{AB} = \mathbf{I} = \mathbf{BA}$?

For example, consider $\mathbf{A} = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$ and let $\mathbf{B} = \begin{bmatrix} x & y \\ u & v \end{bmatrix}$.

Then $\mathbf{AB} = \mathbf{I}$ implies

$$\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} x & y \\ u & v \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

i.e. $$\begin{bmatrix} 2x + 3u & 2y + 3v \\ x + 4u & y + 4v \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\therefore \quad 2x + 3u = 1 \quad$ and $\quad 2y + 3v = 0$

$\quad\quad x + 4u = 0 \quad\quad\quad\quad y + 4v = 1$

These simultaneous equations can be solved to determine x, y, u, v and hence $\mathbf{B}$.

$$\mathbf{B} = \begin{bmatrix} 0.8 & -0.6 \\ -0.2 & 0.4 \end{bmatrix}$$

In general:

> If $\mathbf{A}$ is a square matrix and if a matrix $\mathbf{B}$ can be found such that
>
> $$\mathbf{AB} = \mathbf{I} = \mathbf{BA}$$
>
> then $\mathbf{A}$ is said to be **non-singular** and $\mathbf{B}$ is called the **inverse** of $\mathbf{A}$.

We leave it as an exercise to show that the inverse of a non-singular matrix is unique.

We will denote the inverse of $\mathbf{A}$ by $\mathbf{A}^{-1}$.

> For a non-singular matrix $\mathbf{A}$, we have
>
> $$\mathbf{AA}^{-1} = \mathbf{I} = \mathbf{A}^{-1}\mathbf{A}$$
>
> A matrix which does not have an inverse is called **singular**

The inverse of a general 2 × 2 matrix

Now consider $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and let $\mathbf{B} = \begin{bmatrix} x & y \\ u & v \end{bmatrix}$.

Then $\mathbf{AB} = \mathbf{I}$ implies

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x & y \\ u & v \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

i.e. $$\begin{bmatrix} ax + bu & ay + bv \\ cx + du & cy + dv \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\therefore \quad ax + bu = 1 \quad$ and $\quad ay + bv = 0$

$\quad\quad cx + du = 0 \quad\quad\quad\quad cy + dv = 1$

These form two pairs of simultaneous equations, the first for x, u and the second for y, v.

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The first pair of equations gives

$$(ad - bc)x = d \qquad \text{(eliminating } u\text{)}$$

$$(bc - ad)u = c \qquad \text{(eliminating } x\text{)}$$

These two equations can be solved for x and u provided $ad - bc \neq 0$:

$$x = \frac{d}{ad - bc} \qquad \text{and} \qquad u = \frac{c}{bc - ad} = \frac{-c}{ad - bc}$$

In a similar way, we obtain

$$y = \frac{-b}{ad - bc} \qquad \text{and} \qquad v = \frac{-a}{bc - ad} = \frac{a}{ad - bc}$$

We have established the following result.

Inverse of a 2 × 2 matrix

If $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then the inverse of $\mathbf{A}$ is given by

$$\mathbf{A}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \qquad \text{(provided } ad - bc \neq 0\text{)}$$

The determinant

The quantity $ad - bc$ that appears in the formula for $\mathbf{A}^{-1}$ has a name: the **determinant** of $\mathbf{A}$. This is denoted $\det(\mathbf{A})$.

Determinant of a 2 × 2 matrix

If $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $\det(\mathbf{A}) = ad - bc$.

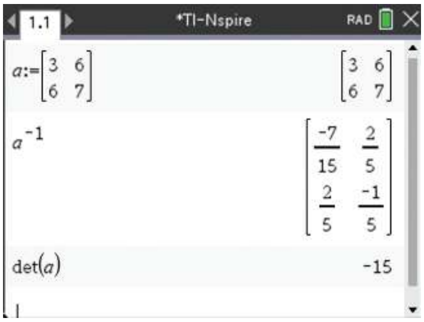
A 2×2 matrix $\mathbf{A}$ has an inverse only if $\det(\mathbf{A}) \neq 0$.

TI-84

Using the TI-Nspire CX non-CAS

- The inverse of a matrix is obtained by raising the matrix to the power of -1.
- The determinant command (menu > **Matrix & Vector** > **Determinant**) is used as shown.

Hint: You can also type in $\det(a)$.



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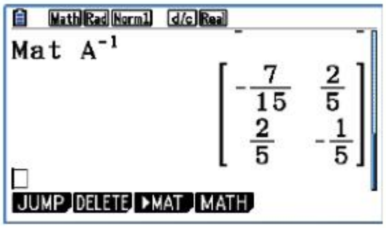
Using the Casio

- To find the inverse of a stored matrix, raise it to the power of −1. For example, to determine $\mathbf{A}^{-1}$:

  SHIFT 2 ALPHA X,θ,T SHIFT) EXE

- To find the determinant of a stored matrix, use the **Matrix operations** menu OPTN F2. Select **Det** F3; select **Mat** F1; then enter the matrix name.

(Here $\mathbf{A} = \begin{bmatrix} 3 & 6 \\ 6 & 7 \end{bmatrix}$ has been stored as Matrix A.)



Example 11

For the matrix $\mathbf{A} = \begin{bmatrix} 5 & 2 \\ 3 & 1 \end{bmatrix}$, determine:

a $\det(\mathbf{A})$

b $\mathbf{A}^{-1}$

Solution

a $\det(\mathbf{A}) = 5 \times 1 - 2 \times 3 = -1$

b $\mathbf{A}^{-1} = \dfrac{1}{-1}\begin{bmatrix} 1 & -2 \\ -3 & 5 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 3 & -5 \end{bmatrix}$

Example 12

For the matrix $\mathbf{A} = \begin{bmatrix} 3 & 2 \\ 1 & 6 \end{bmatrix}$, determine:

a $\det(\mathbf{A})$

b $\mathbf{A}^{-1}$

c $\mathbf{X}$, if $\mathbf{AX} = \begin{bmatrix} 5 & 6 \\ 7 & 2 \end{bmatrix}$

d $\mathbf{Y}$, if $\mathbf{YA} = \begin{bmatrix} 5 & 6 \\ 7 & 2 \end{bmatrix}$

Solution

a $\det(\mathbf{A}) = 3 \times 6 - 2 = 16$

b $\mathbf{A}^{-1} = \dfrac{1}{16}\begin{bmatrix} 6 & -2 \\ -1 & 3 \end{bmatrix}$

c $\mathbf{AX} = \begin{bmatrix} 5 & 6 \\ 7 & 2 \end{bmatrix}$

Multiply both sides (on the left) by $\mathbf{A}^{-1}$.

$$\mathbf{A}^{-1}\mathbf{AX} = \mathbf{A}^{-1}\begin{bmatrix} 5 & 6 \\ 7 & 2 \end{bmatrix}$$

$$\therefore \quad \mathbf{IX} = \mathbf{X} = \frac{1}{16}\begin{bmatrix} 6 & -2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 5 & 6 \\ 7 & 2 \end{bmatrix}$$

$$= \frac{1}{16}\begin{bmatrix} 16 & 32 \\ 16 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$$

d $\mathbf{YA} = \begin{bmatrix} 5 & 6 \\ 7 & 2 \end{bmatrix}$

Multiply both sides (on the right) by $\mathbf{A}^{-1}$.

$$\mathbf{YAA}^{-1} = \begin{bmatrix} 5 & 6 \\ 7 & 2 \end{bmatrix}\mathbf{A}^{-1}$$

$$\therefore \quad \mathbf{YI} = \mathbf{Y} = \frac{1}{16}\begin{bmatrix} 5 & 6 \\ 7 & 2 \end{bmatrix}\begin{bmatrix} 6 & -2 \\ -1 & 3 \end{bmatrix}$$

$$= \frac{1}{16}\begin{bmatrix} 24 & 8 \\ 40 & -8 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{5}{2} & -\frac{1}{2} \end{bmatrix}$$

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Summary 9D

- For a 2×2 matrix $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$:
  - the inverse of $\mathbf{A}$ is given by

$$\mathbf{A}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \qquad (\text{if } ad - bc \neq 0)$$

  - the determinant of $\mathbf{A}$ is given by

$$\det(\mathbf{A}) = ad - bc$$

- A 2×2 matrix $\mathbf{A}$ has an inverse only if $\det(\mathbf{A}) \neq 0$.

Skill-sheet

Exercise 9D

SF

Example 11

1 For the matrices $\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} -2 & -2 \\ 3 & 2 \end{bmatrix}$, determine:

a $\det(\mathbf{A})$ **b** $\mathbf{A}^{-1}$ **c** $\det(\mathbf{B})$ **d** $\mathbf{B}^{-1}$

2 Determine the inverse of each of the following non-singular matrices (where k is any non-zero real number):

a $\begin{bmatrix} 3 & -1 \\ 4 & -1 \end{bmatrix}$ **b** $\begin{bmatrix} 3 & 1 \\ -2 & 4 \end{bmatrix}$ **c** $\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$

3 If the matrix $\mathbf{A}$ is non-singular, show that the inverse is unique.

4 Let $\mathbf{A}$ and $\mathbf{B}$ be the non-singular matrices $\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$.

a Determine $\mathbf{A}^{-1}$ and $\mathbf{B}^{-1}$.

b Determine $\mathbf{AB}$ and hence determine, if possible, $(\mathbf{AB})^{-1}$.

c From $\mathbf{A}^{-1}$ and $\mathbf{B}^{-1}$, determine the products $\mathbf{A}^{-1}\mathbf{B}^{-1}$ and $\mathbf{B}^{-1}\mathbf{A}^{-1}$. What do you notice?

Example 12

5 Let $\mathbf{A} = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$.

a Determine $\mathbf{A}^{-1}$. **b** If $\mathbf{AX} = \begin{bmatrix} 3 & 4 \\ 1 & 6 \end{bmatrix}$, determine $\mathbf{X}$. **c** If $\mathbf{YA} = \begin{bmatrix} 3 & 4 \\ 1 & 6 \end{bmatrix}$, find $\mathbf{Y}$.

6 Let $\mathbf{A} = \begin{bmatrix} 3 & 2 \\ 1 & 6 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 4 & -1 \\ 2 & 2 \end{bmatrix}$ and $\mathbf{C} = \begin{bmatrix} 3 & 4 \\ 2 & 6 \end{bmatrix}$.

a Determine $\mathbf{X}$ such that $\mathbf{AX} + \mathbf{B} = \mathbf{C}$. **b** Determine $\mathbf{Y}$ such that $\mathbf{YA} + \mathbf{B} = \mathbf{C}$.

7 Assume that $\mathbf{A}$ is a 2×2 matrix such that $a_{12} = a_{21} = 0$, $a_{11} \neq 0$ and $a_{22} \neq 0$. Show that $\mathbf{A}$ is non-singular and determine $\mathbf{A}^{-1}$.

8 Let $\mathbf{A}$ be a non-singular 2×2 matrix, let $\mathbf{B}$ be a 2×2 matrix and assume that $\mathbf{AB} = \mathbf{O}$. Show that $\mathbf{B} = \mathbf{O}$.

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9 Determine all 2×2 matrices such that $\mathbf{A}^{-1} = \mathbf{A}$.

10 For what values of a does the matrix $\mathbf{A} = \begin{bmatrix} a & 1 \\ 2 & a \end{bmatrix}$ not have an inverse?

11 Let n be a natural number and let

$$\mathbf{A} = \begin{bmatrix} \frac{1}{n} & \frac{1}{n+1} \\ \frac{1}{n+1} & \frac{1}{n+2} \end{bmatrix}$$

Show that all the entries of the inverse matrix $\mathbf{A}^{-1}$ are integers.

SF

CF

9E The algebra of matrices

Learning intentions

- To be able to work with operations on matrices.

In the previous sections we have looked at the operations of addition and subtraction of matrices, multiplication of a matrix by a real number, multiplication of matrices and forming the inverse of suitable matrices. In this section we list and name many of the properties that we have worked with. We first recall:

- **Addition of matrices** If $\mathbf{A}$ and $\mathbf{B}$ are two matrices of the same dimension, then the sum $\mathbf{A} + \mathbf{B}$ is the matrix obtained by adding together the corresponding entries of the two matrices.
  Note: The addition $\mathbf{A} + \mathbf{B}$ is defined only if $\mathbf{A}$ and $\mathbf{B}$ are of the same dimension.
- If $\mathbf{A}$ is an $m \times n$ matrix and $\mathbf{B}$ is an $n \times r$ matrix, then the product $\mathbf{AB}$ is an $m \times r$ matrix.
  Note: The product $\mathbf{AB}$ is defined only if the number of columns of $\mathbf{A}$ is the same as the number of rows of $\mathbf{B}$.

We refer the reader to section **4B** for the definition of the zero matrix and to section **4D** for the definition of multiplicative identity.

$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$	(commutative law for addition)
$\mathbf{A} + 0 = \mathbf{A}$	(additive identity)
$\mathbf{A} + (-\mathbf{A}) = 0$	(additive inverse)
$\mathbf{AI} = \mathbf{A} = \mathbf{IA}$	(multiplicative identity for square matrices)
$\mathbf{AA}^{-1} = \mathbf{I} = \mathbf{A}^{-1}\mathbf{A}$	(multiplicative inverse for non-singular matrices)
$\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$	(left distributive law)
$(\mathbf{B} + \mathbf{C})\mathbf{A} = \mathbf{BA} + \mathbf{CA}$	(right distributive law)
$(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$	(associative law of addition)
$(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$	(associative law of multiplication)

Note: We again observe that in general $\mathbf{AB} \neq \mathbf{BA}$

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Exercise 9E

SF

1 Let $\mathbf{A} = \begin{bmatrix} 3 & 4 & 5 \\ 2 & 6 & 6 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 3 & 4 \\ 1 & 6 \\ 2 & 3 \end{bmatrix}$ and $\mathbf{C} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$,

a State the dimension of each matrix.

b Evaluate the products **AB** and **BC**.

c Evaluate the products (**AB**)**C** and **A**(**BC**).

2 **a** Determine the additive inverse of $\mathbf{C} = \begin{bmatrix} 3 & 4 & 5 \\ 1 & 2 & -3 \end{bmatrix}$.

b Determine the multiplicative inverse of the the non-singular matrix $\mathbf{B} = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$.

3 Let $\mathbf{A} = \begin{bmatrix} 3 & 4 \\ 2 & 6 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} -2 & 4 \\ 3 & 1 \end{bmatrix}$ and $\mathbf{C} = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$.

a Solve the equation $\mathbf{AX} + \mathbf{BX} = \mathbf{C}$ for **X**.

b Solve the equation $\mathbf{XA} + \mathbf{XB} = \mathbf{B}$ for **X**.

4 Let $\mathbf{A} = \begin{bmatrix} -3 & 2 \\ 1 & 5 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} -2 & -1 \\ 3 & -4 \end{bmatrix}$ and $\mathbf{C} = \begin{bmatrix} 6 & 2 \\ 2 & -4 \end{bmatrix}$

a Solve the equation $\mathbf{AX} + \mathbf{BX} = \mathbf{C}$ for **X**.

b Solve the equation $\mathbf{XA} + \mathbf{XB} = \mathbf{C}$ for **X**.

5 Let $\mathbf{A} = \begin{bmatrix} -3 \\ 6 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} -7 \\ 3 \end{bmatrix}$ and $\mathbf{C} = \begin{bmatrix} 16 \\ 24 \end{bmatrix}$.

Solve the equation $\mathbf{X} + \mathbf{A} + \mathbf{B} = \mathbf{C}$ for **X**.

CF

6 Let $\mathbf{A} = \begin{bmatrix} 1 & 4 \\ 3 & 0 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} p & q \\ r & 0 \end{bmatrix}$ where p, q and r are non-zero real numbers. Determine possible values of p, q and r such that $\mathbf{AB}^2 = \mathbf{BA}^2$.

7 Let $\mathbf{A} = \begin{bmatrix} 1 & m \\ n & 0 \end{bmatrix}$ where m and n are non-zero real numbers. Determine $\mathbf{A}^{-1}$.

8 If the matrix **A** is a square non-singular matrix prove that the inverse of **A** is unique.

9 If the matrices **A** and **B** are square and non-singular prove that the inverse of **AB** is $\mathbf{B}^{-1}\mathbf{A}^{-1}$.

10 Prove that for a square non-singular matrix **A**, $\left(\mathbf{A}^2\right)^{-1} = \left(\mathbf{A}^{-1}\right)^2$.

11 Prove that for a square non-singular matrix **A** and non-zero real number k, $(k\mathbf{A})^{-1} = \frac{1}{k}\mathbf{A}^{-1}$.

12 Let $\mathbf{A} = \begin{bmatrix} p & q \\ 3 & 0 \end{bmatrix}$. If $\mathbf{A}^2 = \mathbf{A}$ determine the values of p and q.

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9F Solution of simultaneous equations using matrices

Learning intentions

- To be able to solve simultaneous linear equations with two variables using matrices.

Simultaneous equations with a unique solution

For example, consider the pair of simultaneous equations

$$3x - 2y = 5$$
$$5x - 3y = 9$$

This can be written as a matrix equation:

$$\begin{bmatrix} 3 & -2 \\ 5 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 9 \end{bmatrix}$$

Let $\mathbf{A} = \begin{bmatrix} 3 & -2 \\ 5 & -3 \end{bmatrix}$. The determinant of $\mathbf{A}$ is $3(-3) - (-2)5 = 1$.

Since the determinant is non-zero, the inverse matrix exists:

$$\mathbf{A}^{-1} = \begin{bmatrix} -3 & 2 \\ -5 & 3 \end{bmatrix}$$

Now multiply both sides of the original matrix equation on the left by $\mathbf{A}^{-1}$:

$$\begin{bmatrix} 3 & -2 \\ 5 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 9 \end{bmatrix}$$

$$\mathbf{A}^{-1}\mathbf{A}\begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{A}^{-1}\begin{bmatrix} 5 \\ 9 \end{bmatrix}$$

$$\mathbf{I}\begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{A}^{-1}\begin{bmatrix} 5 \\ 9 \end{bmatrix} \qquad \text{since } \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$$

$$\therefore \quad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ -5 & 3 \end{bmatrix}\begin{bmatrix} 5 \\ 9 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

This is the solution to the simultaneous equations. Check by substituting $x = 3$ and $y = 2$ into the two equations.

Simultaneous equations without a unique solution

If a pair of simultaneous linear equations in two variables corresponds to two parallel lines, then a singular matrix results.

For example, the following pair of simultaneous equations has no solution:

$$x + 2y = 3$$
$$-2x - 4y = 6$$

The associated matrix equation is

$$\begin{bmatrix} 1 & 2 \\ -2 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

The determinant of the matrix $\begin{bmatrix} 1 & 2 \\ -2 & -4 \end{bmatrix}$ is $1(-4) - 2(-2) = 0$, so the matrix has no inverse.

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Example 13

Let $\mathbf{A} = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$ and $\mathbf{K} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$. Solve the system $\mathbf{AX} = \mathbf{K}$, where $\mathbf{X} = \begin{bmatrix} x \\ y \end{bmatrix}$.

Solution

If $\mathbf{AX} = \mathbf{K}$, then

$$\mathbf{X} = \mathbf{A}^{-1}\mathbf{K}$$
$$= \frac{1}{5}\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Example 14

Solve the following simultaneous equations:

$$3x - 2y = 6$$
$$7x + 4y = 7$$

Solution

The matrix equation is

$$\begin{bmatrix} 3 & -2 \\ 7 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 7 \end{bmatrix}$$

Let $\mathbf{A} = \begin{bmatrix} 3 & -2 \\ 7 & 4 \end{bmatrix}$. Then $\mathbf{A}^{-1} = \frac{1}{26}\begin{bmatrix} 4 & 2 \\ -7 & 3 \end{bmatrix}$.

Therefore

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{26}\begin{bmatrix} 4 & 2 \\ -7 & 3 \end{bmatrix}\begin{bmatrix} 6 \\ 7 \end{bmatrix} = \frac{1}{26}\begin{bmatrix} 38 \\ -21 \end{bmatrix}$$

Skill-sheet

Exercise 9F

Example 13

1 Let $\mathbf{A} = \begin{bmatrix} 3 & -1 \\ 4 & -1 \end{bmatrix}$ and $\mathbf{X} = \begin{bmatrix} x \\ y \end{bmatrix}$.

Solve the system $\mathbf{AX} = \mathbf{K}$, where:

a $\mathbf{K} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ **b** $\mathbf{K} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$

Example 14

2 Use matrices to solve each of the following pairs of simultaneous equations:

a $-2x + 4y = 6$, $3x + y = 1$

b $-x + 2y = -1$, $-x + 4y = 2$

c $3x - 5y = 9$, $2x - 3y = 12$

d $3x + 2y = 2$, $2x + 5y = 4$

e $3x + 5y = 6$, $2x + 4y = 3$

f $3x + 4.5y = 9$, $2x + 3y = 4$

9F

3 Use matrices to determine the point of intersection of the lines given by the equations $2x - 3y = 7$ and $3x + y = 5$.

4 Solve each of the following matrix equations for the matrix $\begin{bmatrix} x \\ y \end{bmatrix}$:

a $\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ **b** $\begin{bmatrix} 1 & -1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ **c** $\begin{bmatrix} 4 & 3 \\ 5 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

5 Two children spend their pocket money buying some books and some DVDs. One child spends \$120 and buys four books and four DVDs. The other child spends \$114 and buys three DVDs and five books. Set up a system of simultaneous equations and use matrices to determine the cost of a single book and a single DVD.

6 A manufacturer makes two sorts of orange-flavoured chocolates: House Brand and Orange Delights. The number of kilograms of House Brand, x kg and the number of kilograms of Orange Delights, y kg that can be made from 80 kg of chocolate and 120 kg of orange filling can be found by solving the following pairs of equations:

$$0.3x + 0.5y = 80$$
$$0.7x + 0.5y = 120$$

Solve for x and y using matrix methods.

7 Consider the system

$$2x - 3y = 3$$
$$4x - 6y = 6$$

a Write this system in matrix form, as $\mathbf{AX} = \mathbf{K}$.
b Is $\mathbf{A}$ a non-singular matrix?
c Can any solutions be found for this system of equations?
d How many pairs does the solution set contain?

8 Consider the system

$$2x - 3y = 7$$
$$4x - 6y = 6$$

a Write this system in matrix form, as $\mathbf{AX} = \mathbf{K}$.
b Is $\mathbf{A}$ a non-singular matrix?
c Can any solutions be found for this system of equations?

9 Suppose that $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{X}$ are 2×2 matrices and that both $\mathbf{A}$ and $\mathbf{B}$ are non-singular. Solve the following for $\mathbf{X}$:

a $\mathbf{AX} = \mathbf{C}$ **b** $\mathbf{ABX} = \mathbf{C}$
c $\mathbf{AXB} = \mathbf{C}$ **d** $\mathbf{A}(\mathbf{X} + \mathbf{B}) = \mathbf{C}$
e $\mathbf{AX} + \mathbf{B} = \mathbf{C}$ **f** $\mathbf{XA} + \mathbf{B} = \mathbf{A}$

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9G Inverses and determinants for $n \times n$ matrices

Learning intentions

- To be able to determine the determinant and inverse of a 3×3 matrix.

In the next two sections, we see how the theory that has been developed for 2×2 matrices can be extended to $n \times n$ matrices, where $n \geq 3$. It provides an introduction for the study of matrices in Specialist Mathematics 3&4. Much of the work in these two sections will be completed with the use of technology.

An $n \times n$ matrix $\mathbf{A}$ can be written as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{bmatrix}$$

Here a_{ij} is the entry in row i and column j of $\mathbf{A}$.

We will concentrate on 3×3 matrices, but the techniques used for larger square matrices are similar.

Identities

For 3×3 matrices, the identity matrix is

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

For each 3×3 matrix $\mathbf{A}$, we have $\mathbf{AI} = \mathbf{A} = \mathbf{IA}$.

Similarly, for 4×4 matrices, the identity matrix is

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

In general, the $n \times n$ identity matrix has 1s along the main diagonal (top-left to bottom-right) and 0s everywhere else.

Inverses

Recall that, if $\mathbf{A}$ is a square matrix and there exists a matrix $\mathbf{B}$ such that

$$\mathbf{AB} = \mathbf{I} = \mathbf{BA}$$

then $\mathbf{B}$ is called the inverse of $\mathbf{A}$. When it exists, the inverse of a square matrix $\mathbf{A}$ is unique and is denoted by $\mathbf{A}^{-1}$.

We can use the following useful fact to help determine inverses.

Let $\mathbf{A}$ and $\mathbf{B}$ be $n \times n$ matrices. If $\mathbf{AB} = \mathbf{I}$, then it follows that $\mathbf{BA} = \mathbf{I}$ and so $\mathbf{B} = \mathbf{A}^{-1}$.

In this course, you are expected to use technology to find the inverse of a 3×3 matrix. However, the inverse can also be found by hand, as shown in the next example. (There are more efficient methods for finding inverses, but they are beyond the scope of this course.)

Example 15

Without using a calculator, determine the inverse of the matrix $\mathbf{A} = \begin{bmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{bmatrix}$.

Solution

We want to determine a matrix $\mathbf{B} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ such that $\mathbf{AB} = \mathbf{I}$.

That is:

$$\begin{bmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{bmatrix}\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} a - 3d + 2g & b - 3e + 2h & c - 3f + 2i \\ -3a + 3d - g & -3b + 3e - h & -3c + 3f - i \\ 2a - d & 2b - e & 2c - f \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We first solve the three equations from the left-hand columns for a, d and g:

$$a - 3d + 2g = 1 \qquad (1)$$

$$-3a + 3d - g = 0 \qquad (2)$$

$$2a - d = 0 \qquad (3)$$

From (3), we have $d = 2a$. Substitute into (1) and (2):

$$-5a + 2g = 1 \qquad (1')$$

$$3a - g = 0 \qquad (2')$$

We obtain $a = 1$, $d = 2$ and $g = 3$.

Solving the three equations from the middle columns gives $b = 2$, $e = 4$ and $h = 5$.
Solving the three equations from the right-hand columns gives $c = 3$, $f = 5$ and $i = 6$.

We obtain $\mathbf{A}^{-1} = \mathbf{B} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$.

Check: $\mathbf{AA}^{-1} = \begin{bmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{I}$

Not every 3×3 matrix has an inverse. For example, the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ is singular.

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Example 16

Let $\mathbf{A} = \begin{bmatrix} 8 & 8 & 7 \\ 1 & 0 & 1 \\ 9 & 9 & 8 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 9 & 1 & -8 \\ -1 & -1 & 1 \\ -9 & 0 & 8 \end{bmatrix}$.

Determine the product **AB**, and hence determine $\mathbf{A}^{-1}$.

Solution

$$\mathbf{AB} = \begin{bmatrix} 8 & 8 & 7 \\ 1 & 0 & 1 \\ 9 & 9 & 8 \end{bmatrix}\begin{bmatrix} 9 & 1 & -8 \\ -1 & -1 & 1 \\ -9 & 0 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{I}$$

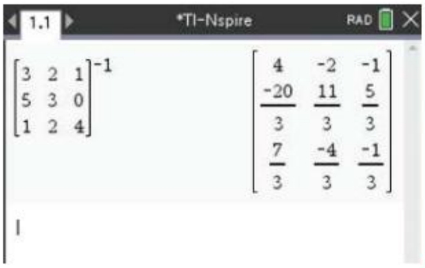
Hence $\mathbf{A}^{-1} = \mathbf{B}$.

Example 17

Using your calculator, determine the inverse of the matrix $\begin{bmatrix} 3 & 2 & 1 \\ 5 & 3 & 0 \\ 1 & 2 & 4 \end{bmatrix}$.

Using the TI-Nspire CX non-CAS

The inverse of a matrix is obtained by raising the matrix to the power of −1.

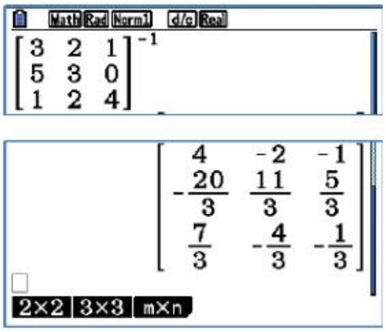


Using the Casio

- In **Run-Matrix** mode, you can access a template for entering a 3 × 3 matrix by selecting **Math** F4, **Matrix** F1, then **3×3** F2.
- Input the entries of the matrix:

  3 ► 2 ► 1 ►
  5 ► 3 ► 0 ►
  1 ► 2 ► 4 ►
- Raise the matrix to the power of −1:

  SHIFT) EXE



Note: Matrices entered in this way are not stored for future calculations.

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The determinant

In Section 9D, we defined the determinant of a 2×2 matrix. The definition was motivated by the formula for the inverse of a 2×2 matrix. We saw that a 2×2 matrix has an inverse if and only if its determinant is non-zero.

In fact, the determinant is defined for all square matrices. In this course, you are expected to use technology to determine the determinant of an $n \times n$ matrix when $n \geq 3$. However, we will also see how to determine the determinant of a 3×3 matrix by hand.

The determinant of a 3×3 matrix

Consider a 3×3 matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

The determinant of **A** can be defined as follows:

$$\begin{aligned} \det(\mathbf{A}) &= a_{11} \det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12} \det \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13} \det \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \\ &= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}) \end{aligned}$$

This formula comes from working through the first row of **A**:

- The first 2×2 matrix is obtained by deleting the row and column containing a_{11}.
- The second 2×2 matrix is obtained by deleting the row and column containing a_{12}.
- The third 2×2 matrix is obtained by deleting the row and column containing a_{13}.

Example 18

Determine the determinant of $\mathbf{A} = \begin{bmatrix} 3 & 2 & 0 \\ 3 & 4 & 1 \\ 2 & 1 & 2 \end{bmatrix}$.

Solution

$$\begin{aligned} \det(\mathbf{A}) &= 3 \times \det \begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix} - 2 \times \det \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix} + 0 \times \det \begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix} \\ &= 3(4 \times 2 - 1 \times 1) - 2(3 \times 2 - 1 \times 2) + 0 \\ &= 3 \times 7 - 2 \times 4 \\ &= 13 \end{aligned}$$

We can obtain equivalent formulas for the determinant by using any row or column of **A** in a similar way. For example, working through the first column of **A**:

$$\det(\mathbf{A}) = a_{11} \det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{21} \det \begin{bmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{bmatrix} + a_{31} \det \begin{bmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{bmatrix}$$

Check that this formula gives the same result.

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The sign of the a_{ij} term in a formula for the determinant is determined by $(-1)^{i+j}$.
For example:

- For a_{11}, the sign is given by $(-1)^{1+1} = 1$.
- For a_{12}, the sign is given by $(-1)^{1+2} = -1$.

These signs can also be determined using the following array:

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

For example, working through the second row of **A**:

$$\det(\mathbf{A}) = -a_{21}\det\begin{bmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{bmatrix} + a_{22}\det\begin{bmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{bmatrix} - a_{23}\det\begin{bmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{bmatrix}$$

Note: When determining the determinant of a 3×3 matrix **A** by hand, it helps to work through the row or column of **A** that has the most 0 entries.

The determinant of an $n \times n$ matrix

We have seen that the determinant of a 3×3 matrix is defined using 2×2 matrices. Similarly, the determinant of a 4×4 matrix is defined using 3×3 matrices, and so on. You can use your calculator to determine the determinant of large square matrices.

The determinant has the following important property.

Determinant of an $n \times n$ matrix

An $n \times n$ matrix **A** has an inverse if and only if $\det(\mathbf{A}) \neq 0$.

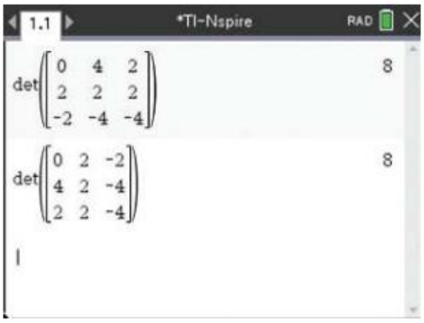
Example 19

Using a calculator, determine the determinant of:

a $\begin{bmatrix} 0 & 4 & 2 \\ 2 & 2 & 2 \\ -2 & -4 & -4 \end{bmatrix}$ **b** $\begin{bmatrix} 0 & 2 & -2 \\ 4 & 2 & -4 \\ 2 & 2 & -4 \end{bmatrix}$

Using the TI-Nspire CX non-CAS

The determinant command (menu > **Matrix and Vector** > **Determinant**) is used as shown. Alternatively, type det(.



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Using the Casio

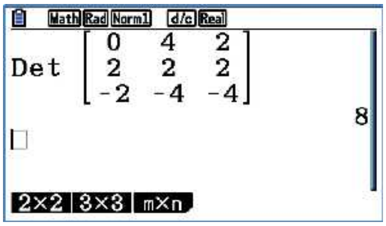
a ■ In **Run-Matrix** mode, go to the **Matrix operations** menu (OPTN) (F2).

■ Select **Det** (F3). Press (EXIT) twice.

■ To access the 3 × 3 matrix template, select **Math** (F4), **Matrix** (F1), then **3×3** (F2).

■ Input the entries of the matrix as shown:

(0) ▶ (4) ▶ (2) ▶
(2) ▶ (2) ▶ (2) ▶
((−)) (2) ▶ ((−)) (4) ▶ ((−)) (4) (EXE)



b Similarly, we can determine $\det\begin{bmatrix} 0 & 2 & -2 \\ 4 & 2 & -4 \\ 2 & 2 & -4 \end{bmatrix} = 8$.

Summary 9G

Identity matrix

■ For each natural number n, there is an $n \times n$ identity matrix $\mathbf{I}$. This matrix satisfies $\mathbf{AI} = \mathbf{A} = \mathbf{IA}$, for all $n \times n$ matrices $\mathbf{A}$.

■ The 3 × 3 identity matrix is $\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Inverse matrices

■ If $\mathbf{A}$ is a square matrix and there exists a matrix $\mathbf{B}$ such that $\mathbf{AB} = \mathbf{I} = \mathbf{BA}$, then $\mathbf{B}$ is called the inverse of $\mathbf{A}$.

■ When it exists, the inverse of a square matrix $\mathbf{A}$ is unique and is denoted by $\mathbf{A}^{-1}$.

■ Let $\mathbf{A}$ and $\mathbf{B}$ be $n \times n$ matrices. If $\mathbf{AB} = \mathbf{I}$, then it follows that $\mathbf{BA} = \mathbf{I}$ and so $\mathbf{B} = \mathbf{A}^{-1}$.

Determinant

■ The determinant is defined for all square matrices.

■ A square matrix has an inverse if and only if its determinant is non-zero.

■ For a 3 × 3 matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

the determinant of $\mathbf{A}$ can be defined by

$$\det(\mathbf{A}) = a_{11}\det\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12}\det\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13}\det\begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

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Exercise 9G

SF

Example 15

1 Without using a calculator, determine the inverse matrix of each of the following:

a $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ **b** $\begin{bmatrix} 1 & 4 & 5 \\ 0 & 2 & 3 \\ 0 & 0 & 5 \end{bmatrix}$

2 Show that the inverse of $\mathbf{A} = \begin{bmatrix} 1 & 3 & 2 \\ 4 & 7 & 5 \\ 1 & 0 & 0 \end{bmatrix}$ is the matrix $\mathbf{B} = \begin{bmatrix} 0 & 0 & 1 \\ 5 & -2 & 3 \\ -7 & 3 & -5 \end{bmatrix}$.

Example 16

3 Let $\mathbf{A} = \begin{bmatrix} 1 & 2 & -3 \\ 2 & -1 & -4 \\ -2 & 5 & 1 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 19 & -17 & -11 \\ 6 & -5 & -2 \\ 8 & -9 & -5 \end{bmatrix}$.

Determine the product $\mathbf{AB}$, and hence determine $\mathbf{A}^{-1}$.

4 Let $\mathbf{A} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 3 \\ 0 & 3 & 0 \end{bmatrix}$. Determine $\mathbf{A}^2$, and hence determine $\mathbf{A}^{-1}$.

5 Let $\mathbf{A} = \begin{bmatrix} 0 & 4 & 2 \\ 2 & 2 & 2 \\ -2 & -4 & -4 \end{bmatrix}$. Determine $\mathbf{A}^2$, and hence determine $\mathbf{A}^{-1}$.

Example 17

6 Use your calculator to determine the inverse of each of the following matrices:

a $\begin{bmatrix} 5 & 2 & 3 \\ 1 & 6 & 4 \\ 1 & 1 & 1 \end{bmatrix}$ **b** $\begin{bmatrix} 5 & 8 & 3 \\ 3 & 6 & 4 \\ 2 & 1 & 2 \end{bmatrix}$ **c** $\begin{bmatrix} 9 & 1 & 2 & 5 \\ 1 & 2 & 3 & 1 \\ 3 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}$ **d** $\begin{bmatrix} 9 & 1 & 3 & 0 \\ 1 & 2 & 2 & 0 \\ 2 & 3 & 1 & 1 \\ 5 & 1 & 0 & 2 \end{bmatrix}$

Example 18

7 Without using a calculator, determine the determinant of:

a $\begin{bmatrix} 9 & 1 & 3 \\ 1 & 2 & 2 \\ 2 & 3 & 1 \end{bmatrix}$ **b** $\begin{bmatrix} 1 & 3 & 2 \\ 4 & 7 & 5 \\ 1 & 0 & 0 \end{bmatrix}$

Example 19

8 Using a calculator, determine the determinant of:

a **i** $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 2 \\ 4 & 2 & 1 \end{bmatrix}$ **ii** $\begin{bmatrix} 1 & 2 & 4 \\ 2 & 2 & 2 \\ 3 & 2 & 1 \end{bmatrix}$

b **i** $\begin{bmatrix} 2 & 4 & 8 \\ 2 & 2 & 2 \\ 3 & 2 & 1 \end{bmatrix}$ **ii** $\begin{bmatrix} 2 & 4 & 8 \\ 4 & 4 & 4 \\ 6 & 4 & 2 \end{bmatrix}$

9 **a** Without using a calculator, find the determinant of $\mathbf{A} = \begin{bmatrix} 1 & 2 & p \\ 2 & 2 & 2 \\ 3 & 2 & 1 \end{bmatrix}$ in terms of p.

b Hence determine the value of p for which $\mathbf{A}$ does not have an inverse.

10 **a** Without using a calculator, find the determinant of $\mathbf{A} = \begin{bmatrix} 1 & 2 & p \\ 2 & 2 & 2 \\ p & 2 & p \end{bmatrix}$ in terms of p.

b Hence determine the values of p for which $\mathbf{A}$ does not have an inverse.

CF

9H Simultaneous linear equations with more than two variables

Learning intentions

- To be able to use inverse matrices to solve systems of simultaneous linear equations.

In Specialist Mathematics Units 3 & 4, you will learn a general technique that can be used to solve any system of simultaneous linear equations.

Linear equations in three variables

Consider the general system of three linear equations in three variables:

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

This can be written as a matrix equation:

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

Define the matrices

$$\mathbf{A} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, \quad \mathbf{X} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

Then the matrix equation becomes

$$\mathbf{AX} = \mathbf{B}$$

If the inverse matrix $\mathbf{A}^{-1}$ exists, we can multiply both sides on the left by $\mathbf{A}^{-1}$:

$$\mathbf{A}^{-1}(\mathbf{AX}) = \mathbf{A}^{-1}\mathbf{B}$$
$$(\mathbf{A}^{-1}\mathbf{A})\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$$
$$\mathbf{IX} = \mathbf{A}^{-1}\mathbf{B} \qquad \text{(where } \mathbf{I} \text{ is the } 3 \times 3 \text{ identity matrix)}$$
$$\therefore \quad \mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$$

Hence, if the inverse matrix $\mathbf{A}^{-1}$ exists, then the system of simultaneous equations has a unique solution given by $\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$. You can use your calculator to determine the inverse matrix $\mathbf{A}^{-1}$.

Example 20

Use matrix methods to solve the following system of three equations in three variables:

$$\begin{aligned} 2x + y + z &= -1 \\ 3y + 4z &= -7 \\ 6x + z &= 8 \end{aligned}$$

Solution

Define the matrices

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 3 & 4 \\ 6 & 0 & 1 \end{bmatrix}, \quad \mathbf{X} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} -1 \\ -7 \\ 8 \end{bmatrix}$$

Then the system of equations can be written as a matrix equation:

$$\mathbf{AX} = \mathbf{B}$$

Multiply both sides on the left by $\mathbf{A}^{-1}$:

$$\begin{aligned} \mathbf{A}^{-1}\mathbf{AX} &= \mathbf{A}^{-1}\mathbf{B} \\ \mathbf{IX} &= \mathbf{A}^{-1}\mathbf{B} \\ \therefore \quad \mathbf{X} &= \mathbf{A}^{-1}\mathbf{B} \end{aligned}$$

Use your calculator to determine $\mathbf{A}^{-1}\mathbf{B}$:

$$\mathbf{X} = \mathbf{A}^{-1}\mathbf{B} = \begin{bmatrix} 1 \\ -5 \\ 2 \end{bmatrix}$$

The solution is $x = 1$, $y = -5$ and $z = 2$.

You can also use your calculator to solve a system of three linear equations directly, without determining an inverse matrix.

Example 21

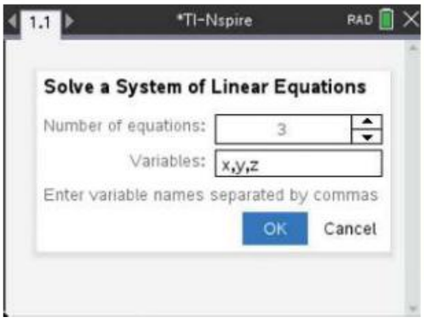
Solve the following simultaneous linear equations for x, y and z:

$$x - y + z = 6, \qquad 2x + z = 4, \qquad 3x + 2y - z = 6$$

Using the TI-Nspire CX non-CAS

Simultaneous linear equations can be solved in a **Calculator** application.

- Use menu > **Algebra** > **Solve System of Linear Equations**.
- Complete the pop-up screen as shown.



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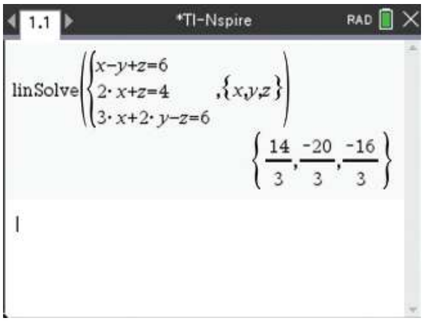
- Enter the three equations:

$$x - y + z = 6$$
$$2x + z = 4$$
$$3x + 2y - z = 6$$

- Hence $x = \frac{14}{3}$, $y = -\frac{20}{3}$ and $z = -\frac{16}{3}$.



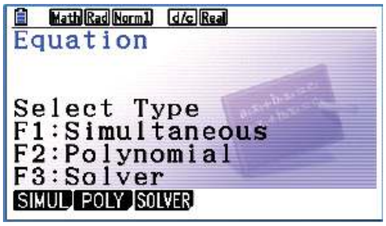
Using the Casio

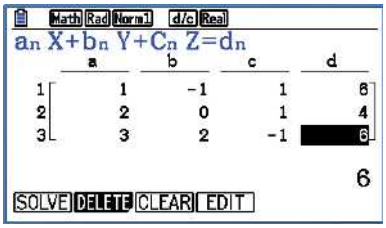
- In **Equation** mode, select **Simultaneous** F1, then select three unknowns F2.
- Enter the coefficients of the three equations

$$x - y + z = 6$$
$$2x + z = 4$$
$$3x + 2y - z = 6$$

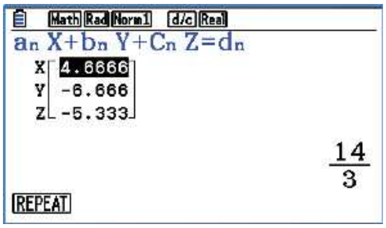
as shown:

1 EXE (−) 1 EXE 1 EXE 6 EXE
2 EXE 0 EXE 1 EXE 4 EXE
3 EXE 2 EXE (−) 1 EXE 6 EXE





- Select **Solve** F1.
- Use the cursor key ▼ to move down the solution matrix and view the exact values.
- The solution is $x = \frac{14}{3}$, $y = -\frac{20}{3}$ and $z = -\frac{16}{3}$.



Simultaneous equations without a unique solution

Just as for two linear equations in two variables, there is a geometric interpretation for three linear equations in three variables. There is only a unique solution if the three equations represent three planes intersecting at a point.

There are three possible cases for a system of three linear equations in three variables:

- a unique solution
- no solution
- infinitely many solutions.

These three cases and their geometric interpretations are studied in Specialist Mathematics Units 3 & 4.

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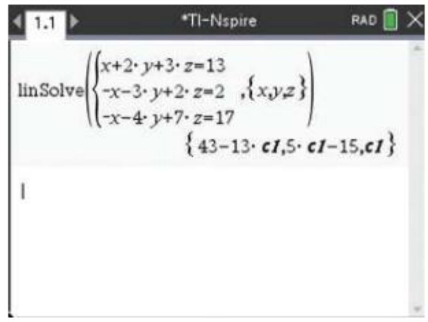
Example 22

Use your calculator to solve the simultaneous equations

$$x + 2y + 3z = 13, \qquad -x - 3y + 2z = 2, \qquad -x - 4y + 7z = 17$$

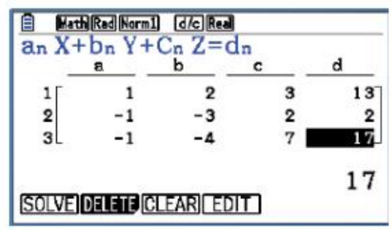
Using the TI-Nspire CX non-CAS

- In a **Calculator** application, use menu > **Algebra** > **Solve System of Linear Equations**.
- Complete the pop-up screen and then enter the three equations as shown.
- The solutions are described in terms of a parameter $c1$. Using λ for the parameter, we can write the solutions as $x = 43 - 13\lambda$, $y = 5\lambda - 15$ and $z = \lambda$, for $\lambda \in \mathbb{R}$.

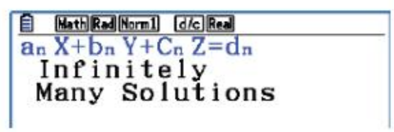


Using the Casio

- In **Equation** mode, select **Simultaneous** F1, then select three unknowns F2.
- Enter the coefficients of the three equations.
- Select **Solve** F1.



- The system of simultaneous equations has infinitely many solutions.



Linear equations in more than three variables

More generally, we can consider a system of n linear equations in n variables:

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & \ddots & & \vdots & & \vdots \\ a_{n1}x_1 & + & a_{n2}x_2 & + & \cdots & + & a_{nn}x_n & = & b_n \end{array}$$

Such a system of equations can be written as a matrix equation:

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

If the $n \times n$ matrix has an inverse, then the system of equations has a unique solution.

Summary 9H

Matrix method for solving simultaneous linear equations

- A system of three linear equations in three variables has the form:

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

- This can be written as a matrix equation $\mathbf{AX} = \mathbf{B}$, where

$$\mathbf{A} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, \quad \mathbf{X} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

- If the inverse matrix $\mathbf{A}^{-1}$ exists, then the system of simultaneous equations has a unique solution given by $\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$.

Exercise 9H

Example 20

SF

1 Use matrix methods to solve each of the following systems of simultaneous equations:

a $2x + 3y - z = 12$
$2y + z = 7$
$2y - z = 5$

b $x + 2y + 3z = 13$
$-x - y + 2z = 2$
$-x + 3y + 4z = 26$

c $x + y = 5$
$y + z = 7$
$z + x = 12$

d $x - y - z = 0$
$5x + 20z = 50$
$10y - 20z = 30$

e $x + y - z = 3$
$x - z + w = 0$
$2x - y - z + 3w = 1$
$-4x + 2y + 3z - 4w = 0$

CF

2 Consider the following system of simultaneous equations:

$$x + 2y + 3z = 13 \qquad (1)$$
$$-x - 3y + 2z = 2 \qquad (2)$$
$$-x - 4y + 7z = 17 \qquad (3)$$

a Write this system as a matrix equation $\mathbf{AX} = \mathbf{B}$.

b Determine det($\mathbf{A}$). Is the matrix $\mathbf{A}$ non-singular?

c This system of simultaneous equations has infinitely many solutions. Express the solutions in terms of a parameter λ by following these steps:

i Add equation (2) to equation (1) and subtract equation (2) from equation (3).

ii Comment on the equations obtained in part **i**.

iii Let $z = \lambda$ and determine y in terms of λ.

iv Substitute for z and y in terms of λ in equation (1) to determine x in terms of λ.

Chapter summary

- A **matrix** is a rectangular array of numbers.
- Two matrices **A** and **B** are equal when:
  - they have the same number of rows and the same number of columns, and
  - they have the same entry at corresponding positions.
- The **dimension** of a matrix is described by specifying the number of rows and the number of columns. An $m \times n$ matrix has m rows and n columns.
- Addition is defined for two matrices only when they have the same dimension. The sum is found by adding corresponding entries.

  $$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix}$$

  Subtraction is performed in a similar way.
- If **A** is any matrix and k is a real number, then the matrix $k\mathbf{A}$ is obtained by multiplying each entry of **A** by k.

  $$k\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}$$

- If **A** is an $m \times n$ matrix and **B** is an $n \times r$ matrix, then the product **AB** is the $m \times r$ matrix whose entries are determined as follows:

  To determine the entry in row i and column j of **AB**, single out row i in matrix **A** and column j in matrix **B**. Multiply the corresponding entries from the row and column and then add up the resulting products.

  Note that the product **AB** is defined only if the number of columns of **A** is the same as the number of rows of **B**.
- If **A** is a square matrix and if a matrix **B** can be found such that $\mathbf{AB} = \mathbf{I} = \mathbf{BA}$, then **A** is said to be **non-singular** and **B** is called the **inverse** of **A**.
- For a 2×2 matrix $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$:
  - the inverse of **A** is given by

    $$\mathbf{A}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \qquad (\text{if } ad - bc \neq 0)$$

  - the determinant of **A** is given by

    $$\det(\mathbf{A}) = ad - bc$$

- A square matrix **A** has an inverse if and only if $\det(\mathbf{A}) \neq 0$.
- Simultaneous equations can sometimes be solved using inverse matrices. For example, the system of equations

  $$ax + by = c$$
  $$dx + ey = f$$

  can be written as $\begin{bmatrix} a & b \\ d & e \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c \\ f \end{bmatrix}$ and solved using $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ d & e \end{bmatrix}^{-1}\begin{bmatrix} c \\ f \end{bmatrix}$.

Laws of matrices

$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$	(commutative law for addition)
$\mathbf{A} + 0 = \mathbf{A}$	(additive identity)
$\mathbf{A} + (-\mathbf{A}) = 0$	(additive inverse)
$\mathbf{AI} = \mathbf{A} = \mathbf{IA}$	(multiplicative identity for square matrices)
$\mathbf{AA}^{-1} = \mathbf{I} = \mathbf{A}^{-1}\mathbf{A}$	(multiplicative inverse for non-singular matrices)
$\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$	(left distributive law)
$(\mathbf{B} + \mathbf{C})\mathbf{A} = \mathbf{BA} + \mathbf{CA}$	(right distributive law)
$(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$	(associative law of addition)
$(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$	(associative law of multiplication)

We note that to properly state these laws the dimensions of the matrices should be given. One could add further to this list. For example, the properties when matrices are multiplied by a real number.

Skills checklist

Check-list

Download this checklist from the Interactive Textbook, then print it and fill it out to check your skills. ☑

9A **1** I can determine the dimension of a matrix. ☐

See Example 1 and Question 1

9A **2** I can represent information in a matrix. ☐

See Example 2, Example 3 and Questions 2 and 4

9A **3** I can determine when two matrices are equal. ☐

See Example 4 and Question 5

9B **4** I can add and subtract matrices of the same dimension. ☐

See Example 5 and Question 1

9B **5** I can multiply a matrix by a real number. ☐

See Example 6, Example 7 and Questions 2 and 7

9C **6** I can multiply two matrices. ☐

See Example 8, Example 10 and Question 1

9C **7** I can apply multiplication of matrices in a practical situation. ☐

See Example 9 and Question 12

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Review

9D **8** I can evaluate the determinant and determine the inverse of a 2×2 matrix. ☐

See Example 11 and Question 1

9D **9** I can solve simple matrix equations. ☐

See Example 12 and Question 5

9F **10** I can solve simultaneous equations with 2 variables using matrix techniques. ☐

See Example 13, Example 14 and Questions 1 and 2

9G **11** I can determine the inverse of a non-singular $n \times n$ matrix using a calculator. ☐

See Example 17 and Question 6

9G **12** I can determine the determinant of an $n \times n$ matrix using a calculator. ☐

See Example 19 and Question 8

9H **13** I can solve simultaneous linear equations with more than two variables, by matrix methods, using a calculator. ☐

See Example 20, Example 21 and Question 1

Short-response questions

Technology-free short-response questions

SF

1 If $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$, determine:

a $(\mathbf{A} + \mathbf{B})(\mathbf{A} - \mathbf{B})$ **b** $\mathbf{A}^2 - \mathbf{B}^2$

2 Determine all possible matrices **A** which satisfy the equation $\begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix}\mathbf{A} = \begin{bmatrix} 8 \\ 16 \end{bmatrix}$.

3 Let $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 3 & -1 & 2 \end{bmatrix}$, $\mathbf{C} = \begin{bmatrix} 6 \\ 1 \end{bmatrix}$, $\mathbf{D} = \begin{bmatrix} 2 & 4 \end{bmatrix}$ and $\mathbf{E} = \begin{bmatrix} 5 \\ 0 \\ 2 \end{bmatrix}$.

a State whether or not each of the following products exists: **AB**, **AC**, **CD**, **BE**.

b Determine **DA** and $\mathbf{A}^{-1}$.

4 If $\mathbf{A} = \begin{bmatrix} 1 & -2 & 1 \\ -5 & 1 & 2 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 1 & -4 \\ 1 & -6 \\ 3 & -8 \end{bmatrix}$ and $\mathbf{C} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, determine **AB** and $\mathbf{C}^{-1}$.

5 Determine the 2×2 matrix **A** such that $\mathbf{A}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 12 & 14 \end{bmatrix}$.

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Review

CF

6 If $\mathbf{A} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix}$, determine $\mathbf{A}^2$ and hence determine $\mathbf{A}^{-1}$.

7 If the matrix $\begin{bmatrix} 1 & 2 \\ 4 & x \end{bmatrix}$ does not have an inverse, determine the value of x.

8 **a** If $\mathbf{M} = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$, determine:

i $\mathbf{MM} = \mathbf{M}^2$ **ii** $\mathbf{MMM} = \mathbf{M}^3$ **iii** $\mathbf{M}^{-1}$

b Determine x and y, given that $\mathbf{M}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$.

9 Let $\mathbf{A} = \begin{bmatrix} a & 4 \\ -2 & -3 \end{bmatrix}$. Determine the value of a if $\mathbf{A}^{-1} = \mathbf{A}$.

10 **a** Consider the system of equations

$$2x - 3y = 3$$
$$4x + y = 5$$

i Write this system in matrix form, as $\mathbf{AX} = \mathbf{K}$.

ii Determine $\det(\mathbf{A})$ and $\mathbf{A}^{-1}$.

iii Solve the system of equations.

iv Interpret your solution geometrically.

b Consider the system of equations

$$2x + y = 3$$
$$4x + 2y = 8$$

i Write this system in matrix form, as $\mathbf{AX} = \mathbf{K}$.

ii Determine $\det(\mathbf{A})$ and explain why $\mathbf{A}^{-1}$ does not exist.

c Interpret your findings in part **b** geometrically.

CU

11 Suppose that $\mathbf{A}$ and $\mathbf{B}$ are 2×2 matrices.

a Prove that $\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B})$.

b Hence prove that if both $\mathbf{A}$ and $\mathbf{B}$ are non-singular, then $\mathbf{AB}$ is non-singular.

Technology-active short-response questions

SF

12 Let $\mathbf{A} = \begin{bmatrix} 2 & 3 & -4 \\ 1 & 6 & 2 \\ 1 & 2 & 0 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 2 \\ -1 & 2 & 0 \end{bmatrix}$. Use your calculator to evaluate

a $\mathbf{A} - \mathbf{B}$ **b** $\mathbf{AB}$ **c** $\mathbf{BA}$

d $\mathbf{A}^3$ **e** $2\mathbf{A}^2\mathbf{B}^3$ **f** $\mathbf{AB} - \mathbf{BA}$

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Review

SF

13 Let $\mathbf{A} = \begin{bmatrix} 2 & 3 \\ 1 & 6 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} -2 & 4 \\ 4 & 6 \end{bmatrix}$ and $\mathbf{C} = \begin{bmatrix} -1 & 5 \\ 2 & 7 \end{bmatrix}$. Using your calculator, determine the 2×2 matrices $\mathbf{X}$ and $\mathbf{Y}$ such that

a $\mathbf{XA} + \mathbf{B} = \mathbf{C}$

b $\mathbf{AY} + \mathbf{B} = \mathbf{C}$

14 Let $\mathbf{A} = \begin{bmatrix} 2 & 3 \\ 1 & 6 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$ and $\mathbf{C} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$. Using your calculator, determine the 2×1 matrix $\mathbf{X}$ such that $\mathbf{AX} + \mathbf{B} = \mathbf{C}$.

15 **a** Let $\mathbf{A} = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$. Determine each of the following

i $\mathbf{A}^3$

ii $\mathbf{A}^6$

iii $\mathbf{A}^9$

iv $\mathbf{A}^{12}$

b Let $\mathbf{A} = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$. Determine each of the following

i $\mathbf{A}^2$

ii $\mathbf{A}^3$

iii $\mathbf{A}^4$

iv $\mathbf{A}^6$

CF

16 The final grades for Physics and Chemistry are made up of three components: tests, practical work and exams. Each semester, an integer mark out of 100 is awarded for each component. Wendy scored the following marks in the three components for Physics:

Semester 1 tests 79, practical work 78, exam 80

Semester 2 tests 80, practical work 78, exam 82

a Represent this information in a 2×3 matrix.

To calculate the final grade for each semester, the three components are weighted: tests are worth 20%, practical work is worth 30% and the exam is worth 50%.

b Represent this information in a 3×1 matrix.

c Calculate Wendy's final grade for Physics in each semester.

Wendy also scored the following marks in the three components for Chemistry:

Semester 1 tests 86, practical work 82, exam 84

Semester 2 tests 81, practical work 80, exam 70

d Calculate Wendy's final grade for Chemistry in each semester.

Students who gain a total score of 320 or more for Physics and Chemistry over the two semesters are awarded a Certificate of Merit in Science.

e Will Wendy be awarded a Certificate of Merit in Science?

She asks her teacher to re-mark her Semester 2 Chemistry exam, hoping that she will gain the necessary marks to be awarded a Certificate of Merit.

f How many extra marks on the exam does she need?

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CU

17 A company runs computing classes and employs full-time and part-time teaching staff, as well as technical staff, catering staff and cleaners. The number of staff employed depends on demand from term to term.
In one year the company employed the following teaching staff:

Term 1 full-time 10, part-time 2
Term 2 full-time 8, part-time 4
Term 3 full-time 8, part-time 8
Term 4 full-time 6, part-time 10

a Represent this information in a 4×2 matrix.

Full-time teachers are paid \$70 per hour and part-time teachers are paid \$60 per hour.

b Represent this information in a 2×1 matrix.

c Calculate the cost per hour to the company for teaching staff for each term.

In the same year the company also employed the following support staff:

Term 1 technical 2, catering 2, cleaning 1
Term 2 technical 2, catering 2, cleaning 1
Term 3 technical 3, catering 4, cleaning 2
Term 4 technical 3, catering 4, cleaning 2

d Represent this information in a 4×3 matrix.

Technical staff are paid \$60 per hour, catering staff are paid \$55 per hour and cleaners are paid \$40 per hour.

e Represent this information in a 3×1 matrix.

f Calculate the cost per hour to the company for support staff for each term.

g Calculate the total cost per hour to the company for teaching and support staff for each term.

18 Bronwyn and Noel have a clothing warehouse in Summerville. They are supplied by three contractors: Brad, Flynn and Lina.
The matrix shows the number of dresses, pants and shirts that one worker, for each of the contractors, can produce in a week.

	Brad	Flynn	Lina
Dresses	5	6	10
Pants	3	4	5
Shirts	2	6	5

The number produced varies because of the different equipment used by the contractors.
The warehouse requires 310 dresses, 175 pants and 175 shirts in a week. How many workers should each contractor employ to meet the requirement exactly?

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Review

Multiple-choice questions

Technology-free multiple-choice questions

1 The matrix $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 2 & -1 \\ -2 & 3 \\ 3 & 0 \end{bmatrix}$ has dimension

A 8 **B** 4×2 **C** 2×4 **D** 1×4

2 Matrix $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \\ 6 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 0 & 6 \end{bmatrix}$. Matrix $Q = A \times B$. The element in row i and column j of matrix Q is q_{ij}. Element q_{33} is determined by the calculation

A $2 \times 4 + 3 \times 2$ **B** $1 \times 4 + 4 \times 6$
C $6 \times 3 + 2 \times 0$ **D** $6 \times 4 + 2 \times 6$

3 If $\mathbf{A} = \begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 1 & -3 & 4 \\ -1 & -3 & -1 \end{bmatrix}$, then $\mathbf{A} + \mathbf{B} =$

A $\begin{bmatrix} 3 & -3 \\ -2 & 0 \end{bmatrix}$ **B** $\begin{bmatrix} 3 & 4 \\ -2 & 2 \end{bmatrix}$ **C** $\begin{bmatrix} -1 & 2 \\ 2 & 3 \end{bmatrix}$ **D** undefined

4 If $\mathbf{C} = \begin{bmatrix} 2 & -3 & 1 \\ 1 & 0 & -2 \end{bmatrix}$ and $\mathbf{D} = \begin{bmatrix} 1 & -3 & 1 \\ 2 & 3 & -1 \end{bmatrix}$, then $\mathbf{D} - \mathbf{C} =$

A $\begin{bmatrix} 1 & 0 & 0 \\ -1 & -3 & -1 \end{bmatrix}$ **B** $\begin{bmatrix} 2 & -6 & 4 \\ -2 & 0 & -4 \end{bmatrix}$

C $\begin{bmatrix} -1 & 0 & 0 \\ 1 & 3 & 1 \end{bmatrix}$ **D** $\begin{bmatrix} 1 & -6 & 0 \\ 1 & 3 & 1 \end{bmatrix}$

5 If $\mathbf{M} = \begin{bmatrix} -4 & 0 \\ -2 & -6 \end{bmatrix}$, then $-\mathbf{M} =$

A $\begin{bmatrix} -4 & 0 \\ -2 & -6 \end{bmatrix}$ **B** $\begin{bmatrix} 0 & -4 \\ -6 & -2 \end{bmatrix}$ **C** $\begin{bmatrix} 4 & 0 \\ -2 & -6 \end{bmatrix}$ **D** $\begin{bmatrix} 4 & 0 \\ 2 & 6 \end{bmatrix}$

6 If $\mathbf{M} = \begin{bmatrix} 0 & 2 \\ -3 & 1 \end{bmatrix}$ and $\mathbf{N} = \begin{bmatrix} 0 & 4 \\ 3 & 0 \end{bmatrix}$, then $2\mathbf{M} - 2\mathbf{N} =$

A $\begin{bmatrix} 0 & 0 \\ -9 & 2 \end{bmatrix}$ **B** $\begin{bmatrix} 0 & -2 \\ -6 & 1 \end{bmatrix}$ **C** $\begin{bmatrix} 0 & -4 \\ -12 & 2 \end{bmatrix}$ **D** $\begin{bmatrix} 0 & 4 \\ 12 & -2 \end{bmatrix}$

7 If both $\mathbf{A}$ and $\mathbf{B}$ are $m \times n$ matrices, where $m \neq n$, then $\mathbf{A} + \mathbf{B}$ is

A an $m \times n$ matrix **B** an $m \times m$ matrix
C an $n \times n$ matrix **D** not defined

8 If **P** is an $m \times n$ matrix and **Q** is an $n \times p$ matrix, where $m \neq p$, then **QP** is

A an $n \times n$ matrix
B an $m \times p$ matrix
C an $n \times p$ matrix
D not defined

9 The determinant of the matrix $\begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}$ is

A 4 **B** 0 **C** −4 **D** 1

10 The inverse of the matrix $\begin{bmatrix} 1 & -1 \\ 1 & -2 \end{bmatrix}$ is

A $\begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$ **B** $\begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$ **C** $\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$ **D** $\begin{bmatrix} 2 & -1 \\ 1 & -1 \end{bmatrix}$

11 If $\mathbf{M} = \begin{bmatrix} 0 & -2 \\ -3 & 1 \end{bmatrix}$ and $\mathbf{N} = \begin{bmatrix} 0 & 2 \\ 3 & 1 \end{bmatrix}$, then $\mathbf{NM} =$

A $\begin{bmatrix} 0 & -4 \\ -9 & 1 \end{bmatrix}$ **B** $\begin{bmatrix} -4 & -2 \\ 2 & -8 \end{bmatrix}$ **C** $\begin{bmatrix} 0 & 4 \\ 9 & 1 \end{bmatrix}$ **D** $\begin{bmatrix} -6 & 2 \\ -3 & -5 \end{bmatrix}$

Technology-active multiple-choice questions

12 The determinant of the matrix $\begin{bmatrix} 2.1 & 6.2 \\ 3.5 & 9 \end{bmatrix}$ is

A 0 **B** 2.4 **C** −2.2 **D** −2.8

13 Consider the matrix equation

$$3 \times \begin{bmatrix} 4 & 2 \\ 6 & 7 \end{bmatrix} + X = \begin{bmatrix} 14 & 12 \\ 18 & 22 \end{bmatrix}$$

Which one of the following is the matrix X?

A $\begin{bmatrix} 4 & 2 \\ 6 & 7 \end{bmatrix}$ **B** $\begin{bmatrix} 2 & 5 \\ 6 & 7 \end{bmatrix}$ **C** $\begin{bmatrix} 2 & 5 \\ 0 & 2 \end{bmatrix}$ **D** $\begin{bmatrix} 2 & 6 \\ 0 & 1 \end{bmatrix}$

14 Let $\mathbf{A} = 3.2\begin{bmatrix} 4.5 & 6 \\ 6.4 & 4.2 \end{bmatrix}$. Then $\mathbf{A}^{-1}$ is equal to

A $\begin{bmatrix} 14.4 & 19.2 \\ 20.48 & 13.44 \end{bmatrix}$
B $\begin{bmatrix} 72 & 96 \\ 512 & 336 \end{bmatrix}$
C $\dfrac{1}{208}\begin{bmatrix} -14 & 20 \\ \frac{64}{3} & -15 \end{bmatrix}$
D $\dfrac{1}{104}\begin{bmatrix} -14 & 20 \\ 64 & -15 \end{bmatrix}$

15 If $\begin{bmatrix} 2p+q & p-2q \\ 5r-s & 4r+3s \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ 11 & 24 \end{bmatrix}$ then value of $2pqrs$ is

A 6 **B** 48 **C** 12 **D** 24

Revision

10 Revision of Unit 1

10A Short-response questions

Technology-free short-response questions

SF

1 Express each of the following as a single matrix:

a $\begin{bmatrix} 3 \\ -2 \end{bmatrix} - 4\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ **b** $\begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ **c** $\begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} -1 & 4 \\ 2 & 4 \end{bmatrix} + \begin{bmatrix} 0 & 4 \\ 2 & 0 \end{bmatrix}$

2 Given that $\mathbf{A} = \begin{bmatrix} w & 2w+5 \\ -1 & w+1 \end{bmatrix}$ and $\det(\mathbf{A}) = 15$, determine the possible values of w.

3 Determine the value of x if

$$\begin{bmatrix} 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ 5 \end{bmatrix} = \begin{bmatrix} x & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

4 Determine the values of a and b if

$$\begin{bmatrix} a & b \\ -b & a \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$$

5 Let $\boldsymbol{a} = 3\hat{\boldsymbol{i}} + 5\hat{\boldsymbol{j}}$ and $\boldsymbol{b} = 2\hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}}$. Calculate:

a $\boldsymbol{a} + \boldsymbol{b}$ **b** $\boldsymbol{a} - \boldsymbol{b}$ **c** $3\boldsymbol{a} - 4\boldsymbol{b}$ **d** $\boldsymbol{a} \cdot \boldsymbol{a}$ **e** $|\boldsymbol{a}|$ **f** $\boldsymbol{a} \cdot \boldsymbol{b}$

6 For non-zero vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ that are not parallel, determine the values of s and t such that $2\boldsymbol{b} + t(\boldsymbol{a} + \boldsymbol{b}) = 2\boldsymbol{a} + \boldsymbol{b} + s(\boldsymbol{a} + 3\boldsymbol{b})$.

7 Resolve $4\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}$ into rectangular components, one of which is parallel to $\sqrt{3}\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}}$.

8 A particle starts at the point A with position vector $\overrightarrow{OA} = 3\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}}$, where the unit is metres. The particle begins moving with a constant velocity of $\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}$ m/s. determine the position vector of the particle after:

a 5 seconds **b** t seconds.

9 Let $\boldsymbol{a} = \sqrt{3}\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}$.

a determine the magnitude of $\boldsymbol{a}$.

b determine the unit vector in the direction of $\boldsymbol{a}$.

c determine the vector of magnitude 30 in the direction of $\boldsymbol{a}$.

10 How many ways can:

a five children be arranged in a row

b five children be arranged on a bench with space for three

c three children be selected from a group of five?

11 How many ways can two adults and three children be arranged along a park bench so that the three children are sitting together?

12 How many ways can the letters in the word COOLANGATTA be arranged?

13 How many ways can four letters be chosen from the set {A, B, C, D, E, F, G, H, I}:

a without restriction

b if the letter B must be chosen

c if at least one vowel must be chosen?

14 **a** Prove that there are two pairs of prime numbers (p, q) for which

$$15p^2 - 19pq + 6q^2 = 0$$

b Prove that there is one pair of prime numbers (p, q) for which

$$10p^2 - 9pq + 2q^2 = 0$$

c Prove that there are no pairs of prime numbers (p, q) for which

$$6p^2 - 5pq + q^2 = 0$$

15 Let $\boldsymbol{a} = 2\hat{\boldsymbol{i}} + 6\hat{\boldsymbol{j}}$.

a determine $|\boldsymbol{a}|$.

b determine the unit vector in the direction of $\boldsymbol{a}$.

c Write down a vector of magnitude 8 that has the same direction as $\boldsymbol{a}$.

d Write down a vector of magnitude 2 that has the opposite direction to $\boldsymbol{a}$.

16 Let $\boldsymbol{a} = 2\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}}$, $\boldsymbol{b} = -2\hat{\boldsymbol{i}} + 3\hat{\boldsymbol{j}}$ and $\boldsymbol{c} = -3\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}}$. determine:

a $\boldsymbol{a} \cdot \boldsymbol{a}$ **b** $\boldsymbol{b} \cdot \boldsymbol{b}$ **c** $\boldsymbol{c} \cdot \boldsymbol{c}$ **d** $\boldsymbol{a} \cdot \boldsymbol{b}$

e $\boldsymbol{a} \cdot (\boldsymbol{b} + \boldsymbol{c})$ **f** $(\boldsymbol{a} + \boldsymbol{b}) \cdot (\boldsymbol{a} + \boldsymbol{c})$ **g** $(\boldsymbol{a} + 2\boldsymbol{b}) \cdot (3\boldsymbol{c} - \boldsymbol{b})$

17 The points A, B, C and D have position vectors $\overrightarrow{OA} = 4\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}}$, $\overrightarrow{OB} = -\hat{\boldsymbol{i}} + 7\hat{\boldsymbol{j}}$, $\overrightarrow{OC} = 8\hat{\boldsymbol{i}} + 6\hat{\boldsymbol{j}}$ and $\overrightarrow{OD} = p\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}}$.

a determine the values of m and n such that $m\overrightarrow{OA} + n\overrightarrow{BC} = 2\hat{\boldsymbol{i}} + 10\hat{\boldsymbol{j}}$.

b determine the value of p such that $\overrightarrow{OB}$ is perpendicular to $\overrightarrow{CD}$.

c determine the values of p such that $|\overrightarrow{AD}| = \sqrt{17}$.

Revision

SF

18 Let $\mathbf{A} = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 0 & -1 & -2 \end{bmatrix}$, $\mathbf{C} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$, $\mathbf{D} = \begin{bmatrix} -2 & 4 \end{bmatrix}$ and $\mathbf{E} = \begin{bmatrix} -5 \\ 5 \\ 0 \end{bmatrix}$.

a State whether or not each of the following products exists: **AB**, **AC**, **CD**, **BE**

b Evaluate **DA** and $\mathbf{A}^{-1}$.

19 If $\mathbf{A} = \begin{bmatrix} 1 & -2 \\ 2 & 4 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} -2 & 3 \\ 1 & 5 \end{bmatrix}$, determine:

a $(\mathbf{A} + \mathbf{B})(\mathbf{A} - \mathbf{B})$ **b** $\mathbf{A}^2 - \mathbf{B}^2$

20 If the matrix $\begin{bmatrix} 1 & 2 \\ 4 & x \end{bmatrix}$ is non-invertible, determine the value of x.

21 determine all possible matrices **A** which satisfy the equation $\begin{bmatrix} 3 & -1 \\ -6 & 2 \end{bmatrix} \mathbf{A} = \begin{bmatrix} 5 \\ -10 \end{bmatrix}$.

22 If $\mathbf{A} = \begin{bmatrix} -1 & -2 & 3 \\ -5 & -1 & 2 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 2 & -4 \\ -1 & -6 \\ -3 & -8 \end{bmatrix}$ and $\mathbf{C} = \begin{bmatrix} -1 & 2 \\ 3 & -4 \end{bmatrix}$, evaluate **AB** and $\mathbf{C}^{-1}$.

23 Suppose that **A**, **B**, **C** and **X** are $n \times n$ matrices and that both **A** and **B** are invertible. Denote the $n \times n$ zero matrix by **O** and the $n \times n$ identity matrix by **I**. Solve each of the following matrix equations for **X**:

a $\mathbf{B} + \mathbf{XA} = \mathbf{C}$ **b** $\mathbf{B}(\mathbf{X} + \mathbf{A}) = \mathbf{C}$ **c** $\mathbf{AX} + \mathbf{BA} = \mathbf{A}$

d $\mathbf{X} + \mathbf{A} = \mathbf{O}$ **e** $2\mathbf{X} - \mathbf{B} = \mathbf{O}$ **f** $\mathbf{AX} + \mathbf{I} = \mathbf{A}$

24 A particle starts at a point A. It moves with velocity $\boldsymbol{v}_1 = [4, 30°]$ m/s for 2 seconds and then moves with velocity $\boldsymbol{v}_2 = [6, -60°]$ m/s for 3 seconds, finishing at the point B.

a Express the velocity vectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ in component form.

b Express the particle's displacement vector $\overrightarrow{AB}$ in component form.

25 How many ways can four different books be arranged on a shelf?

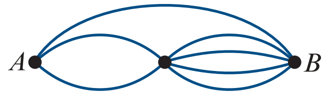
26 How many ways can three teachers and three students be arranged in a row if a teacher must be at the start of the row?

27 How many different three-digit numbers can be formed using the digits 1, 3, 5, 7 and 9:

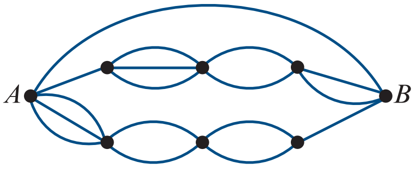
a as many times as you would like **b** at most once?

28 Travelling from left to right, how many paths are there from point A to point B in each of the following diagrams?

a



b



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29 Evaluate each of the following:

a $4!$ **b** $\dfrac{6!}{4!}$ **c** $\dfrac{8!}{6!\,2!}$ **d** ${}^{10}C_2$

30 How many ways can five children be arranged on a bench with space for:

a four children **b** five children?

31 Suppose that n is odd. Prove that $n^2 + n$ is even.

32 Prove that if m and n are consecutive integers, then $n^2 - m^2 = n + m$.

33 Let $n \in \mathbb{Z}$. Consider the statement: If $5n + 3$ is even, then n is odd.

a Write down the converse statement. **b** Prove the converse.

c Write down the contrapositive statement. **d** Prove the contrapositive.

34 Let $\mathbf{A} = \begin{bmatrix} 1 & -2 \\ 0 & a \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 1 & 4 \\ b & 2 \end{bmatrix}$.

a determine **AB**.

b If $\mathbf{B} = \mathbf{A}^{-1}$, determine the values of a and b.

35 determine all triples of real numbers (a, b, c) such that $\mathbf{AB} = \mathbf{O}$, where

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ a & 1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} a & b \\ c & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{O} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

36 Asha has three identical 20 cent pieces and two identical 10 cent pieces. How many ways can she arrange these coins in a row?

37 How many ways can you select:

a three children from a group of six **b** two letters from the alphabet

c four numbers from the set $\{1, 2, \ldots, 10\}$ **d** three sides of an octagon?

38 Consider the set of numbers $X = \{1, 2, \ldots, 8\}$.

a How many subsets of X have exactly two elements?

b How many subsets of X have exactly three elements, one of which is the number 8?

c determine the total number of subsets of X.

39 A team of five students is selected randomly from a group of five boys and four girls. What is the probability that the team consists of three boys and two girls?

40 There are four Labor and five Liberal parliamentarians, from which four are to be selected to form a committee. If the committee must include at least one member from each party, how many ways can this be done?

41 There are 10 blue, 11 green and 12 red balls in a bag. How many balls must be chosen at random to be sure that at least three will have the same colour?

42 How many different natural numbers from 1 to 99 inclusive must be chosen at random to be sure there will be at least one pair of numbers that sums to 100?

43 How many integers from 1 to 120 inclusive are divisible by 2 or 3?

44 Suppose the number x is irrational. Prove by contradiction that $x + 1$ is also irrational.

45 Prove by contradiction that 6 cannot be written as the difference of two perfect squares.

46 Let $n \in \mathbb{Z}$. Prove that $3n + 1$ is even if and only if n is odd.

47 Prove that each of the following statements is false by determineing a counterexample:

a The sum of two prime numbers cannot be a prime number.

b For all $x \in \mathbb{R}$, we have $x^3 > x^2$.

48 Show this statement is false: There exists $n \in \mathbb{N}$ such that $25n^2 - 9$ is a prime number.

49 Prove by mathematical induction that:

a $2 + 4 + \cdots + 2n = n(n + 1)$

b $11^n - 6$ is divisible by 5, for all $n \in \mathbb{N}$

50 Prove that there are no real numbers a and b such that $\dfrac{a+b}{b} = \dfrac{2a}{a+b}$.

51 Let $\overrightarrow{OA} = \hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}$, $\overrightarrow{OB} = 2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}$ and $\overrightarrow{OP} = \overrightarrow{OA} + t\overrightarrow{OB}$, where t is a real number. Determine the value of t such that:

a $\overrightarrow{OP}$ is perpendicular to $\overrightarrow{AB}$

b $|\overrightarrow{OP}| = |\overrightarrow{AP}|$

52 How many five-digit even numbers greater than 30 000 can be formed using the digits 0, 1, 2, 3 and 4 if:

a the digits can be repeated

b the digits cannot be repeated?

53 Consider the letters in the word AUSTRALIA.

a How many arrangements of these letters are there?

b How many arrangements begin and end with a consonant?

c How many arrangements have all the vowels together?

54 Determine how many three-digit numbers have both of the following properties:

- none of the digits is 1
- one digit is the product of the other two digits.

55 Let a and b be integers.

a Show that if d is a factor of both a and b, then d is also a factor of $a - b$.

b Hence, explain why the highest common factor of 1 000 001 and 999 999 is 1.

c Now assume that a and b are odd numbers that differ by a power of 2. Show that the highest common factor of a and b is 1.

CF

56 Consider the sum

$$S_n = \frac{1}{4 \cdot 1^2 - 1} + \frac{1}{4 \cdot 2^2 - 1} + \cdots + \frac{1}{4 \cdot n^2 - 1}$$

a Evaluate S_n for:

i $n = 1$ **ii** $n = 2$ **iii** $n = 3$

b Hence, guess a formula for S_n in terms of n.

c Prove that your formula is correct using mathematical induction.

57 Assume that **A** and **B** are invertible $n \times n$ matrices. Prove that $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$.

58 Two forces of equal magnitude F kg wt act on a particle and they have a resultant force of magnitude 6 kg wt. When one of the forces is doubled in magnitude, the resultant force is 11 kg wt. determine the value of F and the cosine of the angle between the two forces.

59 A particle in equilibrium is being acted on by three forces $\boldsymbol{F}_1$, $\boldsymbol{F}_2$ and $\boldsymbol{F}_3$ in the same plane. Given that $\boldsymbol{F}_1 = [6, 45°]$ kg wt and $\boldsymbol{F}_2 = [8, -90°]$ kg wt, determine the square of the magnitude of $\boldsymbol{F}_3$.

60 A block of mass 10 kg is maintained at rest on a smooth plane inclined at 30° to the horizontal by a string. Calculate the tension in the string and the reaction of the plane if:

a the string is parallel to the plane **b** the string is horizontal.

61 A mass of 10 kg is suspended by two strings of lengths 5 cm and 12 cm that are attached to fixed points on the same horizontal level 13 cm apart. determine the tension in each string.

CU

62 Points A and B have position vectors $\boldsymbol{a} = \hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}$ and $\boldsymbol{b} = 4\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}}$ respectively, relative to an origin O.

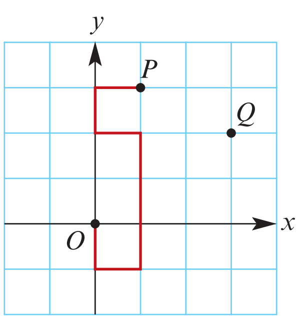
a Let P be a point on the line AB.

i Explain why the position vector of P can be written as $\overrightarrow{OP} = \boldsymbol{a} + t(\boldsymbol{b} - \boldsymbol{a})$, for some real number t.

ii Show that, if $\overrightarrow{OP}$ makes an angle of 60° with $\overrightarrow{OA}$, then $3t^2 - 1 = 0$.

b Let P_1 and P_2 be two points on the line AB such that both $\overrightarrow{OP_1}$ and $\overrightarrow{OP_2}$ make an angle of 60° with $\overrightarrow{OA}$. Use part **a** to determine $\overrightarrow{OP_1} \cdot \overrightarrow{OP_2}$.

63 An ant moves around a unit grid by walking, one unit at a time, in any one of the four directions: north (N), east (E), south (S) or west (W). Consider the three points $O(0, 0)$, $P(1, 3)$ and $Q(3, 2)$. The path from O to P shown in the diagram has length 8 and can be described by the string of letters SENNNWNE.



determine the number of paths of length 8 from O to:

a O **b** P **c** Q

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Technology-active short-response questions

SF

64 For each of the following determine the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$, in degrees, correct to one decimal place.

a $\boldsymbol{a} = 3\hat{\boldsymbol{i}} + 7\hat{\boldsymbol{j}}$, $\boldsymbol{b} = -3\hat{\boldsymbol{i}} + 11\hat{\boldsymbol{j}}$. **b** $\boldsymbol{a} = -7\hat{\boldsymbol{i}} + 11\hat{\boldsymbol{j}}$, $\boldsymbol{b} = -11\hat{\boldsymbol{i}} - 5\hat{\boldsymbol{j}}$.

65 A hiker goes from point O to point A by first walking east for 10 km and then walking on a bearing of 030° for 8 km. Describe the hiker's displacement vector $\overrightarrow{OA}$ by giving a distance and a bearing.

66 A car A is moving at a velocity of 70 km/h due south while car B is moving with a velocity of 110 km/h due east. Determine the velocity of car A relative to car B.

67 A motorboat heads due east at 16 m/s across a river that flows due north at 9 m/s.

a What is the resultant velocity of the boat?

b If the river is 136 m wide, how long does it take the boat to cross the river?

c How far downstream is the boat when it reaches the other side of the river?

68 A bookshelf has three different mathematics books and two different physics books. How many ways can these books be arranged:

a without restriction **b** if the mathematics books are kept together?

69 Using the digits 0, 1, 2, 3 and 4 without repetition, how many five-digit numbers can you form:

a without restriction **b** that are divisible by 10

c that are greater than 20 000 **d** that are even?

CF

70 $\overrightarrow{OA} = \boldsymbol{a}$, $\overrightarrow{OB} = \boldsymbol{b}$, $\overrightarrow{OP} = \frac{4}{5}\overrightarrow{OA}$ and Q is the midpoint of AB.

a Express $\overrightarrow{AB}$ and $\overrightarrow{PQ}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$.

b The line segments PQ and OB are extended to meet at R, with $\overrightarrow{QR} = n\overrightarrow{PQ}$ and $\overrightarrow{BR} = k\boldsymbol{b}$. Express the vector $\overrightarrow{QR}$ in terms of:

i n, $\boldsymbol{a}$ and $\boldsymbol{b}$

ii k, $\boldsymbol{a}$ and $\boldsymbol{b}$

c Determine the values of n and k.

71 **a** A man walks north at a rate of 4 km/h and notices that the wind *appears* to blow from the west. He doubles his speed and now the wind appears to blow from the north-west. What is the velocity of the wind?

Note: Both the direction and the magnitude must be given.

b A river 400 m wide flows from east to west at a steady speed of 1 km/h. A swimmer, whose speed in still water is 2 km/h, starts from the south bank and heads north across the river. Determine the swimmer's speed over the river bed and how far downstream he is when he reaches the north bank.

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CF

c To a motorcyclist travelling due north at 50 km/h, the wind appears to come from the north-west at 60 km/h. What is the true velocity of the wind?

d A dinghy in distress is 6 km on a bearing of 230° from a lifeboat and is drifting in a direction of 150° at 5 km/h. In what direction should the lifeboat travel to reach the dinghy as quickly as possible if the maximum speed of the lifeboat is 35 km/h?

72 A five-digit number is formed using the digits 0, 1, 2, 3, 4, 5 and 6 without repetition. How many ways can this be done:

a without restriction

b if the number is divisible by 10

c if the number is odd

d if the number is even?

73 Mike and Sonia belong to a group of eight coworkers. There are three men and five women in this group. A team of four workers is required to complete a project. How many ways can the team be selected:

a without restriction

b if it must contain two men and two women

c if it must contain both Mike and Sonia

d if it must not contain both Mike and Sonia?

74 A sailing boat has three identical black flags and three identical red flags. The boat can send signals to nearby boats by arranging flags along its mast.

a How many ways can all six flags be arranged in a row?

b How many ways can all six flags be arranged in a row if no two black flags are adjacent?

c Using at least one flag, how many arrangements in a row are possible?

75 Consider the letters in the word BAGGAGE.

a How many arrangements of these letters are there?

b How many arrangements begin and end with a vowel?

c How many arrangements begin and end with a consonant?

d How many arrangements have all vowels together and all consonants together?

CU

76 There are 25 people at a party.

a If every person shakes hands with every other person, what is the total number of handshakes?

b In fact, there are two rival groups at the party, so everyone only shakes hands with every other person in their group. If there are 150 handshakes, how many people are in each of the rival groups?

c At another party, there are 23 guests. Explain why it is not possible for each person to shake hands with exactly three other guests.

CU

77 On a clock's face, twelve points are evenly spaced around a circle.

a How many ways can you select four of these points?

b How many ways can you select two points that are not diametrically opposite?

c For every selection of two points that are not diametrically opposite, you can draw one rectangle on the face that has these two points as vertices. What are the other two vertices?

d How many ways can you select four points that are the vertices of a rectangle?
Hint: Why must you divide the answer to part **b** by 4?

e Four points are randomly selected. What is the probability that the four points are the vertices of a rectangle?

78 Let a, b and c be integers. Suppose you know that $a + b$ is even and $b + c$ is odd.

a Is it possible to work out whether a, b and c are even or odd?

b What if you also know that $a + b + c$ is even?

79 **a** Determine all positive integer values of a, b and c such that $a < b < c$ and

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1$$

b Determine all positive integer values of a, b, c and d such that $a < b < c < d$ and

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} > 2$$

80 Let a, b and c be positive real numbers. Prove that if $b > a$, then $\dfrac{a+c}{b+c} > \dfrac{a}{b}$.

81 **a** Determine the smallest value of $n \in \mathbb{N}$ such that $2^n > 10^3$.

b Hence, prove that 2^{100} has at least 31 digits.

c Hence, explain why some digit in the decimal expansion of 2^{100} occurs at least four times. (**Hint:** There are 10 different digits: $0, 1, \dots, 9$.)

82 Take a close look at the following square numbers:

$$15^2 = 225, \quad 25^2 = 625, \quad 35^2 = 1225, \quad 45^2 = 2025, \quad 55^2 = 3025, \quad 65^2 = 4225$$

a Determine and describe the pattern that you see in these square numbers.

b Confirm that your pattern works for the number 75.

c Prove that your pattern actually works. (**Hint:** Each number is of the form $10n + 5$.)

83 Heidi has 10 wooden cubes, with edges of length 1 cm through to 10 cm.

a Using all the cubes, can she build two towers of the same height?

b Now Heidi has n wooden cubes, with edges of length 1 through to n. For what values of n can Heidi use all the cubes to build two towers of the same height?

84 **a** Suppose that a is odd and b is odd. Prove that ab is odd.

b Suppose that a is odd and $n \in \mathbb{N}$. Prove by induction that a^n is odd.

c Hence, prove that if x satisfies $3^x = 2$, then x is irrational.

85 **a** If $n^4 + 6n^3 + 11n^2 + 6n + 1 = (an^2 + bn + c)^2$, determine the positive values of a, b and c.

b Hence, prove that when 1 is added to the product of four consecutive integers, the result is always a perfect square.

c Hence, write the number $5 \times 6 \times 7 \times 8 + 1$ as a product of prime numbers.

86 Let $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Consider the matrix $\mathbf{A} - m\mathbf{I}$, where $\mathbf{I}$ is the 2×2 identity matrix.

a Determine $\det(\mathbf{A} - m\mathbf{I})$, writing your answer as a quadratic polynomial in m.

b If $m = \lambda_1$ and $m = \lambda_2$ are the solutions of the quadratic equation $\det(\mathbf{A} - m\mathbf{I}) = 0$, show that $\lambda_1 + \lambda_2 = a + d$ and $\lambda_1\lambda_2 = \det(\mathbf{A})$.

c Suppose that $a + b = c + d = 1$. Show that $m = 1$ is a solution of the quadratic equation, and determine the other solution in terms of a and c.

d **i** Suppose that $\mathbf{A} = \begin{bmatrix} -3 & 4 \\ 6 & -5 \end{bmatrix}$. Solve the equation $\det(\mathbf{A} - m\mathbf{I}) = 0$ for m.

ii The equation $\begin{bmatrix} -3 & 4 \\ 6 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$ has infinitely many solutions. Describe them.

iii If $m = 1$ and $m = \lambda_2$ are the two solutions from part **i**, describe the solutions of the equation $\begin{bmatrix} -3 & 4 \\ 6 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \lambda_2 \begin{bmatrix} x \\ y \end{bmatrix}$.

e Now consider examples of matrices $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $a + b = c + d = 6$.

More generally, consider examples with $a + b = c + d = k$, where k is an integer.

87 **a** Let a and b be real numbers and let x and y be positive real numbers. Prove that

$$\frac{(a+b)^2}{x+y} \le \frac{a^2}{x} + \frac{b^2}{y}$$

b Now let a, b and c be real numbers and let x, y and z be positive real numbers. Use part **a** to prove that

$$\frac{(a+b+c)^2}{x+y+z} \le \frac{a^2}{x} + \frac{b^2}{y} + \frac{c^2}{z}$$

c Let x, y and z be positive real numbers for which $x + y + z = 3$. Prove that

$$\frac{1}{x} + \frac{4}{y} + \frac{9}{z} \ge 12$$

88 Show that $\dfrac{(n+1)! + n!}{n! + (n-1)!} = \dfrac{n(n+2)}{n+1}$ for each $n \in \mathbb{N}$.

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89 **a** Show that $k \cdot {}^{n}C_{k} = n \cdot {}^{n-1}C_{k-1}, \quad n \geq k \geq 1.$

b In Section 6G we showed that

$$ {}^{n}C_{0} + {}^{n}C_{1} + {}^{n}C_{2} + \cdots + {}^{n}C_{n-1} + {}^{n}C_{n} = 2^{n} $$

Using this result and part **a**, prove that

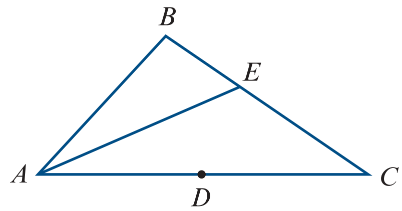
$$ 0 \cdot {}^{n}C_{0} + 1 \cdot {}^{n}C_{1} + 2 \cdot {}^{n}C_{2} + \cdots + (n-1) \cdot {}^{n}C_{n-1} + n \cdot {}^{n}C_{n} = n \cdot 2^{n-1} $$

90 Prove by mathematical induction that, for every $n \in \mathbb{N}$,

$$ \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} \geq \sqrt{n} $$

91 The sum of a set of positive integers is 20. Determine the largest possible value of their product, and prove that you can't do any better than this value.

92 In the diagram, D is the midpoint of AC and E is the point on BC such that $BE : EC = 1 : t$, where $t > 0$. Suppose that DE is extended to a point F such that $DE : EF = 1 : 7$.



Let $\boldsymbol{a} = \overrightarrow{AD}$ and $\boldsymbol{b} = \overrightarrow{AB}$.

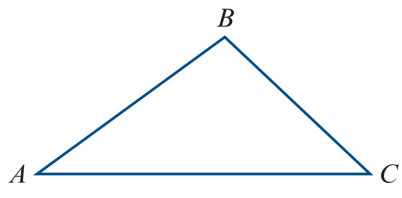
a Express $\overrightarrow{AE}$ in terms of t, $\boldsymbol{a}$ and $\boldsymbol{b}$.

b Express $\overrightarrow{AE}$ in terms of $\boldsymbol{a}$ and $\overrightarrow{AF}$.

c Show that $\overrightarrow{AF} = \dfrac{9-7t}{1+t}\boldsymbol{a} + \dfrac{8t}{1+t}\boldsymbol{b}$.

d If A, B and F are collinear, determine the value of t.

93 The vertices A, B and C of a triangle have position vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ respectively, relative to an origin in the plane ABC.



a Let P be an arbitrary point on the line segment AB. Show that the position vector of P can be written in the form

$$ m\boldsymbol{a} + n\boldsymbol{b}, \quad \text{where } m \geq 0,\ n \geq 0 \text{ and } m + n = 1 $$

Hint: Assume that P divides AB in the ratio $x : y$.

b Determine $\overrightarrow{PC}$ in terms of $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$.

c Let Q be an arbitrary point on the line segment PC. Show that the position vector of Q can be written in the form

$$ \lambda\boldsymbol{a} + \mu\boldsymbol{b} + \gamma\boldsymbol{c}, \quad \text{where } \lambda \geq 0,\ \mu \geq 0,\ \gamma \geq 0 \text{ and } \lambda + \mu + \gamma = 1 $$

Note: The triple of numbers (λ, μ, γ) are known as the **barycentric coordinates** of the point Q in the triangle ABC.

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10B Multiple-choice questions

Technology-free multiple-choice questions

1 If $\mathbf{P}^2 = 4\mathbf{I}$, then $\mathbf{P}^{-1}$ equals

A $\frac{1}{4}\mathbf{P}$ **B** $\frac{1}{2}\mathbf{P}$ **C** $-\frac{1}{2}\mathbf{P}$ **D** $2\mathbf{P}$

2 If $\mathbf{R} = \begin{bmatrix} 5 & 3 & 1 \end{bmatrix}$ and $\mathbf{S} = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$, then $\mathbf{RS}$ is

A undefined **B** $\begin{bmatrix} -1 \end{bmatrix}$ **C** $\begin{bmatrix} 0 & 0 & 0 \\ -5 & -3 & -1 \\ 10 & 6 & 2 \end{bmatrix}$ **D** $\begin{bmatrix} 0 & -3 & 2 \end{bmatrix}$

3 If $\mathbf{A} = \begin{bmatrix} 9 & 8 \\ -11 & 5 \end{bmatrix}$, then $\det(\mathbf{A})$ equals

A $-\frac{1}{43}$ **B** $\frac{1}{333}$ **C** 17 **D** 133

4 If $\mathbf{A} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} -2 & 6 & 4 \end{bmatrix}$, then $\mathbf{BA}$ has size

A 1×1 **B** 3×1 **C** 1×3 **D** 3×3

5 Let $\mathbf{A} = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 2 & -1 \\ 6 & 7 \end{bmatrix}$ and $\mathbf{C} = \begin{bmatrix} 5 & 4 \\ 8 & 9 \end{bmatrix}$. If $\mathbf{AX} + \mathbf{B} = \mathbf{C}$, then $\mathbf{X}$ equals

A $\frac{1}{20}\begin{bmatrix} -2 & 19 \\ -2 & 6 \end{bmatrix}$ **B** $\begin{bmatrix} -1 & 1 \\ 4 & 0 \end{bmatrix}$

C $\begin{bmatrix} -2 & 19 \\ -2 & 6 \end{bmatrix}$ **D** $\begin{bmatrix} 3 & -10 \\ -4 & 10 \end{bmatrix}$

6 Let $\mathbf{P} = \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}$, $\mathbf{Q} = \begin{bmatrix} 4 & 2 \\ 6 & 5 \end{bmatrix}$ and $\mathbf{R} = \begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix}$.

If $\mathbf{X} = \mathbf{PQR}$, then the number of zero entries of $\mathbf{X}$ is

A 0 **B** 1 **C** 2 **D** 3

7 If $\mathbf{X} = \begin{bmatrix} 3 & 5 \\ -1 & -2 \end{bmatrix}$, then $\mathbf{X}^{-1}$ is

A $\begin{bmatrix} 2 & 5 \\ -1 & -3 \end{bmatrix}$ **B** $\begin{bmatrix} -2 & -5 \\ 1 & 3 \end{bmatrix}$ **C** $\begin{bmatrix} \frac{1}{3} & \frac{1}{5} \\ -1 & -\frac{1}{2} \end{bmatrix}$ **D** $\begin{bmatrix} -3 & -1 \\ 5 & 2 \end{bmatrix}$

8 The determinant of the matrix $\begin{bmatrix} 4 & 6 \\ 2 & 4 \end{bmatrix}$ is

A 16 **B** 4 **C** −16 **D** $\frac{1}{4}$

9 If $\mathbf{S} = \begin{bmatrix} 5 & 7 \\ 2 & 2 \end{bmatrix}$, then $\mathbf{S}^{-1}$ is

A $-\begin{bmatrix} 5 & 7 \\ 2 & 2 \end{bmatrix}$ **B** $\begin{bmatrix} 5 & -7 \\ -2 & 5 \end{bmatrix}$

C $-\frac{1}{4}\begin{bmatrix} -2 & 7 \\ 2 & -5 \end{bmatrix}$ **D** $\frac{1}{4}\begin{bmatrix} -2 & 7 \\ 2 & -5 \end{bmatrix}$

10 The magnitude of vector $\boldsymbol{a} = 5\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}}$ is

A $\sqrt{29}$ **B** 29 **C** 21 **D** $\sqrt{21}$

11 The unit vector in the direction of vector $\boldsymbol{a} = 3\hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}}$ is

A $\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}}$ **B** $\frac{1}{5}(3\hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}})$ **C** $\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}$ **D** $\frac{1}{25}(3\hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}})$

12 If $\overrightarrow{OA} = 2\hat{\boldsymbol{i}} - 4\hat{\boldsymbol{j}}$ and $\overrightarrow{OB} = 3\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}}$, then $\overrightarrow{AB}$ equals

A $5\hat{\boldsymbol{i}}$ **B** $-\hat{\boldsymbol{i}} - 8\hat{\boldsymbol{j}}$ **C** $-5\hat{\boldsymbol{i}}$ **D** $\hat{\boldsymbol{i}} + 8\hat{\boldsymbol{j}}$

13 If $\boldsymbol{a} = 2\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}}$ and $\boldsymbol{b} = 3\hat{\boldsymbol{i}} - 2\hat{\boldsymbol{j}}$, then $\boldsymbol{a} - 2\boldsymbol{b}$ equals

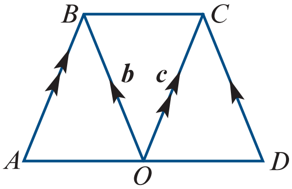
A $8\hat{\boldsymbol{i}}$ **B** $-4\hat{\boldsymbol{i}} + 8\hat{\boldsymbol{j}}$ **C** $-\hat{\boldsymbol{i}} + 6\hat{\boldsymbol{j}}$ **D** $5\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}}$

14 In the diagram, AB is parallel to OC, DC is parallel to OB, $\boldsymbol{b} = \overrightarrow{OB}$, $\boldsymbol{c} = \overrightarrow{OC}$ and $AB = OB = OC = DC$.

Vector $\overrightarrow{AD}$ is equal to

A $\boldsymbol{b} + \boldsymbol{c}$ **B** $2(\boldsymbol{c} - \boldsymbol{b})$

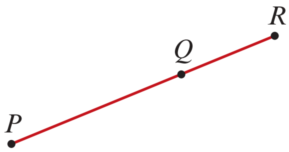
C $2(\boldsymbol{b} - \boldsymbol{c})$ **D** $2\boldsymbol{b} + 2\boldsymbol{c}$



15 PQR is a straight line and $PQ = 2QR$. If $\overrightarrow{OQ} = 2\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}}$ and $\overrightarrow{OR} = \hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}}$, then $\overrightarrow{OP}$ could be equal to

A $4\hat{\boldsymbol{i}} - 13\hat{\boldsymbol{j}}$ **B** $3\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}}$

C $2\hat{\boldsymbol{i}} - 10\hat{\boldsymbol{j}}$ **D** $3\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}}$



16 Let $\boldsymbol{u} = a\hat{\boldsymbol{i}} - 5\hat{\boldsymbol{j}}$ and $\boldsymbol{v} = -3\hat{\boldsymbol{i}} + 6\hat{\boldsymbol{j}}$. Vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are parallel when

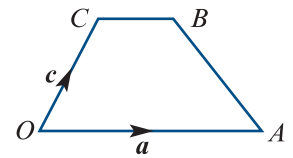
A $a = -3$ **B** $a = \frac{5}{2}$ **C** $a = 3$ **D** $a = -\frac{5}{6}$

17 Let $\boldsymbol{a} = 3\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}}$, $\boldsymbol{b} = 2\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}}$ and $\boldsymbol{x} = \hat{\boldsymbol{i}} + 5\hat{\boldsymbol{j}}$. If $\boldsymbol{x} = s\boldsymbol{a} + t\boldsymbol{b}$, then the scalars s and t are given by

A $s = -1$ and $t = -1$ **B** $s = -1$ and $t = 1$

C $s = 1$ and $t = -1$ **D** $s = 1$ and $t = 1$

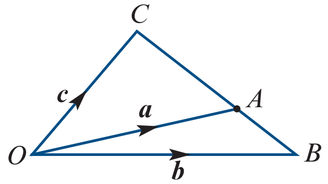
18 In this diagram, $OABC$ is a trapezium.
If $\boldsymbol{a} = \overrightarrow{OA}$, $\boldsymbol{c} = \overrightarrow{OC}$ and $\overrightarrow{OA} = 3\overrightarrow{CB}$, then $\overrightarrow{AB}$ equals



A $3\boldsymbol{c}$ **B** $\boldsymbol{c} - \frac{2}{3}\boldsymbol{a}$

C $3\boldsymbol{c} - 2\boldsymbol{a}$ **D** $\frac{2}{3}\boldsymbol{a} - \boldsymbol{c}$

19 In this diagram, $\boldsymbol{a} = \overrightarrow{OA}$, $\boldsymbol{b} = \overrightarrow{OB}$, $\boldsymbol{c} = \overrightarrow{OC}$ and $AC : AB = 2 : 1$. The vector $\boldsymbol{c}$ is equal to



A $\boldsymbol{a} + 2\boldsymbol{b}$ **B** $3\boldsymbol{a} - 2\boldsymbol{b}$

C $2\boldsymbol{a} + \boldsymbol{b}$ **D** $2\boldsymbol{a} - \boldsymbol{b}$

20 A bus and a car are on a straight level road that runs east–west. The bus is moving east at 20 m/s and the car is moving west at 20 m/s. If a man walks from the back to the front of the bus at 2 m/s, what is the velocity of the man relative to the car?

A 38 m/s east **B** 38 m/s west **C** 42 m/s east **D** 42 m/s west

21 How many ways can five people be arranged in a line?

A $5!$ **B** 2^5 **C** 5^5 **D** ${}^5C_1$

22 The letters of the word CAIRNS are arranged in a random order. What is the probability that the arrangement begins and ends with a vowel?

A $\frac{1}{3}$ **B** $\frac{1}{15}$ **C** $\frac{1}{30}$ **D** $\frac{1}{48}$

23 How many four-digit numbers can be formed using the digits 1, 2, 3, 4 and 5 at most once?

A ${}^5C_4$ **B** $5 + 4 + 3 + 2$ **C** $5 \times 4 \times 3 \times 2$ **D** $4!$

24 The number of arrangements of the digits in the number 111222 is

A $\frac{6!}{3! \times 3!}$ **B** $6!$ **C** $\frac{6!}{3!}$ **D** $3! \times 3! \times 3!$

25 Sam has n identical 10 cent pieces and n identical 20 cent pieces. How many ways can these coins be arranged in a row?

A $n! \times n!$ **B** $\frac{(2n)!}{(n!)^2}$ **C** $\frac{(n!)^2}{(2n)!}$ **D** $(2n)!$

26 There are 10 flavours of ice-cream at a shop. Mark will select three flavours for his cone, one of which must be chocolate. The total number of different selections is

A ${}^{10}C_3$ **B** ${}^{10}C_2$ **C** ${}^9C_3$ **D** ${}^9C_2$

27 There are four Labor and five Liberal parliamentarians, from which two of each are to be selected to form a committee. How many ways can this be done?

A ${}^9C_2$ **B** ${}^9C_4$ **C** ${}^9C_2 \times {}^9C_2$ **D** ${}^4C_2 \times {}^5C_2$

Revision

28 From 10 friends, you can invite any number of them to the movies. Assuming that you invite at least one friend, how many different selections can you make?

A 2^9 **B** $2^9 - 1$ **C** 2^{10} **D** $2^{10} - 1$

29 An untidy kitchen drawer has a jumbled collection of eight knives, six forks and ten spoons. What is the smallest number of items that must be randomly chosen to ensure that at least four items of the same type are selected?

A 10 **B** 11 **C** 12 **D** 13

30 Whenever n integers are written on a whiteboard, at least six of them leave the same remainder when divided by 3. What is the smallest possible value of n?

A 4 **B** 7 **C** 15 **D** 16

31 How many integers from 1 to 60 inclusive are multiples of 2 or 5?

A 32 **B** 36 **C** 40 **D** 44

32 Suppose that both m and n are odd. Which of the following statements is false?

A $m - n$ is even **B** $3m + 5n$ is even **C** $2m + n$ is odd **D** $mn + 1$ is odd

33 Suppose that m is divisible by 4 and n is divisible by 12. Which of the following statements might be false?

A $m \times n$ is divisible by 48 **B** $m + n$ is divisible by 4
C m^2n is divisible by 48 **D** n is divisible by m

34 Let m and n be integers. Which of the following statements is always true?

A If mn is even, then m is even.
B The number $m + n$ is even if and only if both m and n are even.
C If $m + n$ is odd, then mn is odd.
D If mn is odd, then $m + n$ is even.

35 Consider the statement: If n is even, then $n + 3$ is odd. The converse of this statement is

A If $n + 3$ is even, then n is even. **B** If n is odd, then $n + 3$ is even.
C If $n + 3$ is odd, then n is even. **D** If $n + 3$ is odd, then n is odd.

36 Assume that a and b are positive real numbers with $a > b$. Which of the following might be false?

A $\frac{a}{b} - \frac{b}{a} > 0$ **B** $a + b > 2b$ **C** $a + 3 > b + 2$ **D** $2a > 3b$

37 The number of pairs of integers (m, n) that satisfy $mn - n = 12$ is

A 3 **B** 4 **C** 6 **D** 12

38 Suppose that n is a positive integer. For how many values of n is the number $9n^2 - 4$ a prime?

A 0 **B** 1 **C** 2 **D** 3

39 If a, b, c and d are consecutive integers, then which of the following statements may be false?

A $a + b + c + d$ is divisible by 2 **B** $a + b + c + d$ is divisible by 4
C $a \times b \times c \times d$ is divisible by 3 **D** $a \times b \times c \times d$ is divisible by 8

Technology-active multiple-choice questions

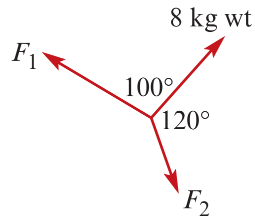
40 A small deck of five cards are numbered 1 to 5. First one card and then a second card are selected at random, with replacement. What is the probability that the sum of the values on the cards is a prime number?

A $\frac{9}{25}$ **B** $\frac{10}{25}$ **C** $\frac{11}{25}$ **D** $\frac{8}{25}$

41 How many different rearrangements are there of the letters in the word TATARS if the two A's are never adjacent?

A 24 **B** 60 **C** 120 **D** 180

Questions 42–43 *refer to this system of forces, which is in equilibrium.*



42 The magnitude of force $\boldsymbol{F}_1$ is approximately

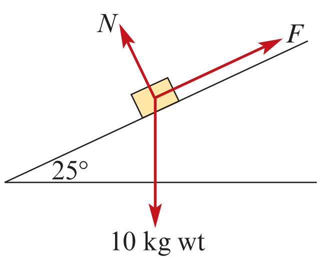
A 10.78 kg wt **B** 5.94 kg wt
C 9.10 kg wt **D** 12.26 kg wt

43 The magnitude of force $\boldsymbol{F}_2$ is approximately

A 10.78 kg wt **B** 5.94 kg wt **C** 9.10 kg wt **D** 12.26 kg wt

Questions 44–45 *refer to the following information:*

A 10 kg block is resting on a smooth plane inclined at 25° to the horizontal and is prevented from slipping down the plane by a string, as shown in the diagram.



44 The magnitude, N, of the normal reaction force is approximately

A 4.23 kg wt **B** 9.06 kg wt **C** 8.19 kg wt **D** 2.59 kg wt

45 The magnitude, F, of the tension in the string is approximately

A 4.23 kg wt **B** 9.06 kg wt **C** 8.19 kg wt **D** 2.59 kg wt

46 Expressing the vector $[7, 20°]$ in component form, with values correct to two decimal places, gives

A $7\hat{\boldsymbol{i}} + 20\hat{\boldsymbol{j}}$ **B** $6.58\hat{\boldsymbol{i}} + 2.39\hat{\boldsymbol{j}}$
C $2.39\hat{\boldsymbol{i}} + 6.58\hat{\boldsymbol{j}}$ **D** $0.94\hat{\boldsymbol{i}} + 0.34\hat{\boldsymbol{j}}$

Revision

10C Problem-solving and modelling questions

1 A square matrix $\mathbf{A}$ is said to be **idempotent** if $\mathbf{A}^2 = \mathbf{A}$.

a Show that each of the following 2×2 matrices is idempotent:

i $\begin{bmatrix} -3 & -6 \\ 2 & 4 \end{bmatrix}$ **ii** $\begin{bmatrix} 1 & 5 \\ 0 & 0 \end{bmatrix}$ **iii** $\begin{bmatrix} 0 & 3 \\ 0 & 1 \end{bmatrix}$ **iv** $\frac{1}{2}\begin{bmatrix} 1-\cos\theta & \sin\theta \\ \sin\theta & 1+\cos\theta \end{bmatrix}$

b Show that each of the following 3×3 matrices is idempotent:

i $\frac{1}{2}\begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ -1 & -2 & -1 \end{bmatrix}$ **ii** $\frac{1}{6}\begin{bmatrix} 5 & -1 & 2 \\ -1 & 5 & 2 \\ 2 & 2 & 2 \end{bmatrix}$

c Show that a product of two idempotent matrices is not necessarily idempotent.

d Show that for every idempotent matrix $\mathbf{A}$, either $\det(\mathbf{A}) = 0$ or $\det(\mathbf{A}) = 1$.

Note: You can use the result that $\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B})$.

e Show that if an idempotent matrix $\mathbf{A}$ has an inverse, then $\mathbf{A} = \mathbf{I}$.

f Show that for every idempotent matrix $\mathbf{A}$, the matrix $\mathbf{I} - \mathbf{A}$ is also idempotent.

g Describe all 2×2 idempotent matrices.

2 The Fibonacci sequence $1, 1, 2, 3, 5, 8, \dots$ is defined by the recurrence relation

$$f_{n+2} = f_n + f_{n+1} \quad \text{and} \quad f_1 = f_2 = 1$$

a Let $\mathbf{Q} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$.

Determine $\mathbf{Q}^2$, $\mathbf{Q}^3$ and $\mathbf{Q}^4$. Deduce the entries of $\mathbf{Q}^n$, for $n \geq 2$. If you have studied mathematical induction, then you can prove your claim.

b Determine $\det(\mathbf{Q})$, $\det(\mathbf{Q}^2)$, $\det(\mathbf{Q}^3)$ and $\det(\mathbf{Q}^4)$. Use these to deduce the result

$$f_{n+1}f_{n-1} - (f_n)^2 = (-1)^n$$

Note: You can use the result that $\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B})$.

c From the equation $\mathbf{Q}^{n+1}\mathbf{Q}^n = \mathbf{Q}^{2n+1}$, prove that $(f_{n+1})^2 + (f_n)^2 = f_{2n+1}$.

d From the equation $\mathbf{Q}^m\mathbf{Q}^{n-1} = \mathbf{Q}^{m+n-1}$, prove that $f_{m+n} = f_{m+1}f_n + f_m f_{n-1}$.

e Solve the equation $\det(\mathbf{Q} - x\mathbf{I}) = 0$ for x.

3 A square matrix $\mathbf{A}$ is said to be **involutory** if $\mathbf{A}^2 = \mathbf{I}$, that is, if it is its own inverse.

a Show that each of the following matrices is involutory:

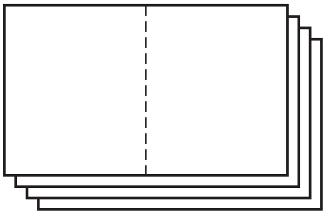
i $\begin{bmatrix} 1 & 5 \\ 0 & -1 \end{bmatrix}$ **ii** $\begin{bmatrix} -3 & -2 \\ 4 & 3 \end{bmatrix}$ **iii** $\begin{bmatrix} 0 & 2 & 1 \\ 1 & 1 & 1 \\ -1 & -2 & -2 \end{bmatrix}$

b Show that for every involutory matrix $\mathbf{A}$, either $\det(\mathbf{A}) = 1$ or $\det(\mathbf{A}) = -1$.

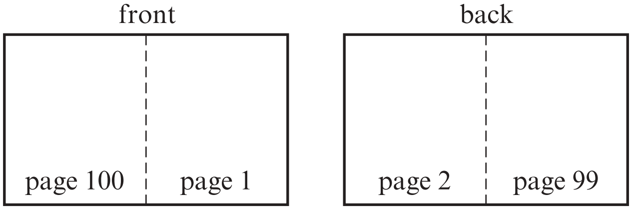
c Describe all 2×2 involutory matrices.

d Let $\mathbf{A}$ be a square matrix. Prove that $\mathbf{A}$ is involutory if and only if the matrix $\frac{1}{2}(\mathbf{A} + \mathbf{I})$ is idempotent. (See Question **1** for the definition of idempotent.)

4 A stack of paper, printed on both sides, is folded in the middle to make a newspaper.



Each sheet contains four pages. The page numbers on the top sheet of Monday's newspaper are 1, 2, 99 and 100.



a What are the page numbers on the bottom sheet of Monday's stack?

b One of the sheets in Monday's newspaper has page numbers 7 and 8. What are its other two page numbers?

c Suppose that a newspaper is made from n sheets of paper. Prove that the sum of the four page numbers on each sheet is a constant.

d Tuesday's newspaper has a sheet whose pages are numbered 11, 12, 33 and 34. How many pages does this newspaper have?

5 Sam has 20 one-dollar coins and seven pockets. He wants to put coins into his pockets so that each pocket contains a different number of coins. (The number 0 is allowed.)

a Prove that this is impossible.

b What is the minimum number of coins Sam would need to do this?

c If Sam had 50 one-dollar coins, determine the maximum number of pockets that he could fill, each with a different number of coins.

6 Each of the following is true:

- $1 = 1^2$
- $2 + 3 + 4 = 3^2$
- $3 + 4 + 5 + 6 + 7 = 5^2$

Generalise these results and prove the generalisation.

7 Let p be a prime number with $p \geq 5$.

a Prove that when p is divided by 6, the remainder is either 1 or 5.

b Hence, prove that when p^2 is divided by 24, the remainder is 1.

c Hence, prove that $p^2 + 2$ is divisible by 3.

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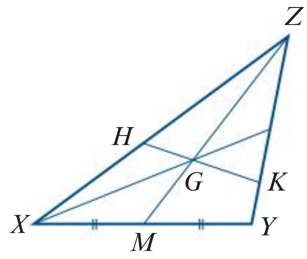
8 **a** Let points O, A, B and C be coplanar and let $\boldsymbol{a} = \overrightarrow{OA}$, $\boldsymbol{b} = \overrightarrow{OB}$ and $\boldsymbol{c} = \overrightarrow{OC}$. Assume that $\boldsymbol{a}$ and $\boldsymbol{b}$ are not parallel. If points A, B and C are collinear with

$$\boldsymbol{c} = \alpha\boldsymbol{a} + \beta\boldsymbol{b} \quad \text{where } \alpha, \beta \in \mathbb{R}$$

show that $\alpha + \beta = 1$.

b In the figure, the point G is the centroid of a triangle (i.e. the point where the lines joining each vertex to the midpoint of the opposite side meet).
A line passing through G meets ZX and ZY at points H and K respectively, with $ZH = hZX$ and $ZK = kZY$.



i Prove that $\overrightarrow{ZG} = \frac{2}{3}\overrightarrow{ZM}$.

ii Express $\overrightarrow{ZG}$ in terms of h, k, $\overrightarrow{ZH}$ and $\overrightarrow{ZK}$.

iii Determine the value of $\frac{1}{h} + \frac{1}{k}$. (Use the result from **a**.)

iv If $h = k$, determine the value of h and describe geometrically what this implies.

v If the area of triangle XYZ is 1 cm^2, what is the area of triangle HKZ when $h = k$?

vi If $k = 2h$, determine the value of h and describe geometrically what this implies.

vii Describe the restrictions on h and k, and sketch the graph of h against k for suitable values of k.

viii Investigate the area, A cm^2, of triangle HKZ as a ratio with respect to the area of triangle XYZ, as k varies. Sketch the graph of A against k. Be careful with the domain.

10D Problem-solving and modelling investigations

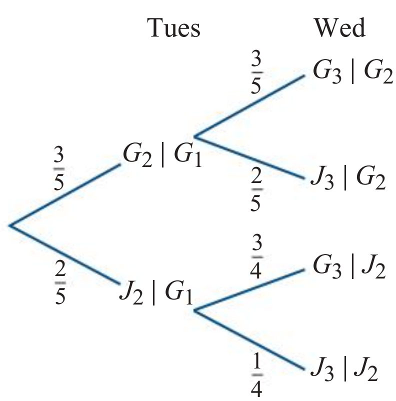
For each of the following questions, there are different approaches and directions that you can take. Suggestions are given, but you should develop your solution on an individual basis.

1 **Transition matrices** Olivia drinks either green tea or jasmine tea every day. If she drinks green tea one day, then she drinks jasmine tea the next day with probability $\frac{2}{5}$. If she drinks jasmine tea one day, she drinks green tea the next day with probability $\frac{3}{4}$.

a Olivia drinks green tea on Monday (day 1).

Let G_n be the event 'Green tea on day n'.
Let J_n be the event 'Jasmine tea on day n'.

Using the tree diagram, determine $P(G_3)$. That is, determine the probability that Olivia will drink green tea on Wednesday.



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We can represent the probabilities in this example using a **transition matrix**, $\mathbf{T}$, and a sequence of **state vectors**, $\mathbf{S}_1, \mathbf{S}_2, \mathbf{S}_3, \dots$, defined as follows:

$$\mathbf{T} = \begin{bmatrix} P(G_n \mid G_{n-1}) & P(G_n \mid J_{n-1}) \\ P(J_n \mid G_{n-1}) & P(J_n \mid J_{n-1}) \end{bmatrix} = \begin{bmatrix} \frac{3}{5} & \frac{3}{4} \\ \frac{2}{5} & \frac{1}{4} \end{bmatrix} \quad \text{and} \quad \mathbf{S}_n = \begin{bmatrix} P(G_n) \\ P(J_n) \end{bmatrix}$$

b **i** Explain why $\mathbf{S}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. **ii** Explain why $\mathbf{S}_n = \mathbf{TS}_{n-1}$ for all $n \geq 2$.

c Use part **b** to determine $\mathbf{S}_2$ and $\mathbf{S}_3$. Check against your answer to part **a**.

d Explain why $\mathbf{S}_n = \mathbf{T}^{n-1}\mathbf{S}_1$ for all $n \geq 2$.

e Use your calculator to determine $\mathbf{S}_{20}$. Hence determine the probability that Olivia will drink green tea on day 20.

f Use your calculator to determine $\mathbf{S}_{200}$. Hence determine the probability that Olivia will drink green tea on day 200.

g Determine $\mathbf{S} = \begin{bmatrix} a \\ b \end{bmatrix}$ such that $a + b = 1$ and $\mathbf{S} = \mathbf{TS}$. Compare with your answer for part **f**.

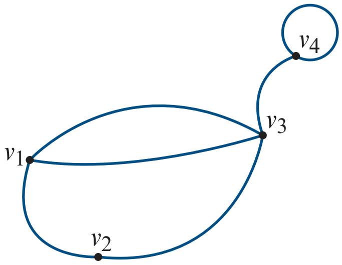
Now consider another example. A computer system operates in two different modes. Every hour, it remains in the same mode or switches to the other mode, according to the following transition matrix:

$$\mathbf{T} = \begin{bmatrix} \frac{2}{5} & \frac{3}{5} \\ \frac{3}{5} & \frac{2}{5} \end{bmatrix}$$

h If the system is in mode 1 at 5:30 p.m., what is the probability that it will be in mode 1 at 8:30 p.m. on the same day?

i Construct similar examples based on switching between two alternatives. Investigate what happens to the state vector $\mathbf{S}_n$ as n gets larger and larger.

2 **Matrix powers and walks** The graph G on the right represents towns (shown as vertices) and the routes between the towns (shown as edges).



For example, there are two direct routes from v_1 to v_3. Each of these is a **walk of length 1**. You can also go from v_1 to v_3 via v_2. This is a **walk of length 2**.

We start by counting the number of different walks of length 2 from v_1 to v_1.

- You can go from v_1 to v_3 in two ways, and then from v_3 to v_1 in two ways. Therefore there are four walks from v_1 to v_1 via v_3.
- Alternatively, you can go from v_1 to v_2 to v_1.
- In total, there are five walks of length 2 from v_1 to v_1.

The **adjacency matrix**, **A**, describes the graph G by giving the number of edges between each pair of vertices:

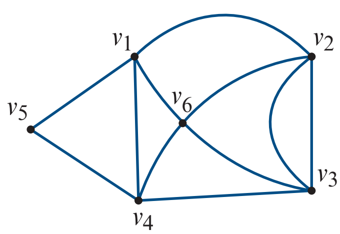
$$\mathbf{A} = \begin{matrix} & v_1 & v_2 & v_3 & v_4 \\ v_1 & 0 & 1 & 2 & 0 \\ v_2 & 1 & 0 & 1 & 0 \\ v_3 & 2 & 1 & 0 & 1 \\ v_4 & 0 & 0 & 1 & 1 \end{matrix} \quad \text{and} \quad \mathbf{A}^2 = \begin{matrix} & v_1 & v_2 & v_3 & v_4 \\ v_1 & 5 & 2 & 1 & 2 \\ v_2 & 2 & 2 & 2 & 1 \\ v_3 & 1 & 2 & 6 & 1 \\ v_4 & 2 & 1 & 1 & 2 \end{matrix}$$

You can see that, when you multiply the first row of **A** by the first column of **A**, you get

$$0 \times 0 + 1 \times 1 + 2 \times 2 + 0 \times 0 = 5$$

This replicates our calculation above for the number of walks of length 2 from v_1 to v_1. The entries of $\mathbf{A}^2$ give the number of walks of length 2 between each pair of vertices. More generally:

> If **A** is the adjacency matrix of a graph G, then the entry of $\mathbf{A}^n$ in row i and column j gives the number of walks of length n from vertex v_i to vertex v_j.



a Give the adjacency matrix of the graph H on the right.

b Determine the number of walks of length 2 between:

i v_1, v_2 **ii** v_3, v_4 **iii** v_1, v_1 **iv** v_5, v_6

c Determine the number of walks of length 3 between:

i v_1, v_2 **ii** v_3, v_4 **iii** v_1, v_1 **iv** v_5, v_6

d A **triangle** in a graph is a cycle of three edges that join three vertices. For example, in the graph H shown above, there are two triangles with vertices v_2, v_3 and v_6.

i List all the triangles in the graph H.

ii Draw a graph with five vertices and five edges that has one triangle.

iii Draw a graph with five vertices and five edges that has no triangles.

iv Assume that a graph G has no loops (that is, no edges joining a vertex to itself). How could you determine the number of triangles in G by looking at $\mathbf{A}^3$, where **A** is the adjacency matrix of G?

3 **Motion of two particles** In this question, the unit of distance is metres and the unit of velocity is metres per second. Relative to the origin O, particles A and B have initial positions $0\hat{\boldsymbol{i}} + 0\hat{\boldsymbol{j}}$ (the origin) and $10\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}}$ respectively. Both particles start moving at the same time. Particle A moves with constant velocity $3\hat{\boldsymbol{i}} + 4\hat{\boldsymbol{j}}$, and particle B moves with constant velocity $a\hat{\boldsymbol{i}} + b\hat{\boldsymbol{j}}$.

a Determine the position vectors of particles A and B at time t seconds after the start of motion.

b **i** Determine a in terms of b given that the particles collide at some time $t \geq 0$.

ii Investigate collisions for different values of a and b.

c Determine a condition for which the paths of A and B cross (collision not necessary).

d Determine the values of a and b such that the particles collide and their paths are at right angles at the point of collision.

e Investigate the velocity of A relative to B.

f Investigate different starting points and constant velocities.

4 **Round-trip flight** An aircraft flies with speed v in still air. The aircraft starts at a point A, flies directly to a point B and then returns to point A. Throughout the flight, the wind is blowing with speed w at an angle of $\theta°$ to $\overrightarrow{AB}$.

a Show that, for both the outbound trip and the return trip, the pilot must steer the aircraft at an angle of $\alpha°$ to the line AB, where $v\sin(\alpha°) = w\sin(\theta°)$.

b Let d denote the distance AB, and let t_1 and t_2 denote the lengths of time taken for the outbound and return trips, respectively. Show that:

i $d(t_1 + t_2) = 2vt_1t_2\cos(\alpha°)$

ii $d^2 = t_1t_2(v^2 - w^2)$

c Complete several calculations for specific values of v, w, θ and d.

5 **Stars and bars** The technique of using stars and bars for combinatorics problems was introduced by William Feller (1906–1970).

a Suppose that 10 identical chocolates are to be distributed among three children, Amy, Ben and Clara. We will investigate the number of ways that this can be done.

Let a, b and c denote the number of chocolates given to Amy, Ben and Clara respectively. We can represent allocations of the 10 chocolates using *stars and bars*. For example:

- The allocation $(a, b, c) = (2, 3, 5)$ is represented as

  $* \; * \; | \; * \; * \; * \; | \; * \; * \; * \; * \; *$

- The allocation $(a, b, c) = (3, 0, 7)$ is represented as

  $* \; * \; * \; | \; | \; * \; * \; * \; * \; * \; * \; *$

i Using stars and bars, represent each of the following allocations:

- $(a, b, c) = (4, 5, 1)$
- $(a, b, c) = (0, 6, 4)$
- $(a, b, c) = (0, 0, 10)$

ii Notice that each allocation is represented by some arrangement of 10 stars and 2 bars. How many different ways can you allocate the 10 chocolates to the three children?

iii Using a similar technique, determine the number of ways that eight chocolates can be distributed among four children.

iv How many ways can you distribute n chocolates among k children?

v If Amy, Ben and Clara are each to receive at least one of the 10 chocolates, how many ways can this be done?

vi How many ways can you distribute n chocolates among k children if each child is to receive at least one chocolate?

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All of the following problems can be solved using this technique:

b How many ways can you distribute three identical balls into three different boxes?

c How many sequences of four non-negative integers are there that sum to 10?

d How many sequences of three odd positive integers are there that sum to 17?

e How many paths are there from the top-left corner to the bottom-right corner of an $m \times n$ grid if you can only travel right or down along the grid lines?

f List some other situations that can be considered in this way and analyse them using the technique of stars and bars.

11

Trigonometric identities

Chapter contents

In this chapter we build on our study of trigonometric functions from Mathematical Methods Units 1&2. Further material is available in Appendix A.

There are many interesting and useful relationships between the trigonometric functions. The most fundamental is the Pythagorean identity:

$$\sin^2 A + \cos^2 A = 1$$

Some of these identities were discovered a very long time ago. For example, the following two results were discovered by the Indian mathematician Bhāskara II in the twelfth century:

$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$
$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$

They are of great importance in many areas of mathematics, including calculus.

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11A Reciprocal trigonometric functions and the Pythagorean identity

Learning intentions

- To be able to solve equations involving the reciprocal trigonometric functions.
- To be able to prove identities involving the reciprocal trigonometric functions.

In Chapter 13 we will look at the graphs of the reciprocal trigonometric functions. Here we use these functions in alternative forms of the Pythagorean identity.

Reciprocal trigonometric functions

Recall that the trigonometric functions sine, cosine and tangent are used to form three new functions, called the reciprocal trigonometric functions.

Secant, cosecant and cotangent

- $\sec\theta = \dfrac{1}{\cos\theta}$ (for $\cos\theta \neq 0$)
- $\operatorname{cosec}\theta = \dfrac{1}{\sin\theta}$ (for $\sin\theta \neq 0$)
- $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$ (for $\sin\theta \neq 0$)

Note: For $\cos\theta \neq 0$ and $\sin\theta \neq 0$, we have

$$\cot\theta = \frac{1}{\tan\theta} \quad \text{and} \quad \tan\theta = \frac{1}{\cot\theta}$$

Example 1

Determine the exact value of each of the following:

a $\sec\left(\dfrac{2\pi}{3}\right)$ **b** $\cot\left(\dfrac{5\pi}{4}\right)$ **c** $\operatorname{cosec}\left(\dfrac{7\pi}{4}\right)$

Solution

a
$$\begin{aligned}\sec\left(\frac{2\pi}{3}\right) &= \frac{1}{\cos\left(\frac{2\pi}{3}\right)} \\ &= \frac{1}{\cos\left(\pi - \frac{\pi}{3}\right)} \\ &= \frac{1}{-\cos\left(\frac{\pi}{3}\right)} \\ &= 1 \div \left(-\frac{1}{2}\right) \\ &= -2\end{aligned}$$

b
$$\begin{aligned}\cot\left(\frac{5\pi}{4}\right) &= \frac{\cos\left(\frac{5\pi}{4}\right)}{\sin\left(\frac{5\pi}{4}\right)} \\ &= \frac{\cos\left(\pi + \frac{\pi}{4}\right)}{\sin\left(\pi + \frac{\pi}{4}\right)} \\ &= \frac{-1}{\sqrt{2}} \div \left(\frac{-1}{\sqrt{2}}\right) \\ &= 1\end{aligned}$$

c
$$\begin{aligned}\operatorname{cosec}\left(\frac{7\pi}{4}\right) &= \frac{1}{\sin\left(2\pi - \frac{\pi}{4}\right)} \\ &= \frac{1}{-\sin\left(\frac{\pi}{4}\right)} \\ &= 1 \div \left(-\frac{1}{\sqrt{2}}\right) \\ &= -\sqrt{2}\end{aligned}$$

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Example 2

Determine the values of x between 0 and 2π for which:

a $\sec x = -2$ **b** $\cot x = -1$

Solution

a $\sec x = -2$

$$\frac{1}{\cos x} = -2$$

$$\cos x = \frac{-1}{2}$$

Since $\cos x$ is negative, the point $P(x)$ is in the 2nd or 3rd quadrant.

$$\therefore \quad x = \pi - \frac{\pi}{3} \quad \text{or} \quad x = \pi + \frac{\pi}{3}$$

$$\therefore \quad x = \frac{2\pi}{3} \quad \text{or} \quad x = \frac{4\pi}{3}$$

b $\cot x = -1$

$\tan x = -1$

Since $\tan x$ is negative, the point $P(x)$ is in the 2nd or 4th quadrant.

$$\therefore \quad x = \pi - \frac{\pi}{4} \quad \text{or} \quad x = 2\pi - \frac{\pi}{4}$$

$$\therefore \quad x = \frac{3\pi}{4} \quad \text{or} \quad x = \frac{7\pi}{4}$$

The Pythagorean identity

We introduced the Pythagorean identity in Section 11B. We can now derive two other forms of this identity.

Pythagorean identity

$$\cos^2\theta + \sin^2\theta = 1$$

$$1 + \tan^2\theta = \sec^2\theta$$

$$\cot^2\theta + 1 = \operatorname{cosec}^2\theta$$

Proof Divide both sides of the Pythagorean identity by $\cos^2\theta$:

$$\frac{\cos^2\theta}{\cos^2\theta} + \frac{\sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

$$\therefore \quad 1 + \tan^2\theta = \sec^2\theta$$

Divide both sides of the Pythagorean identity by $\sin^2\theta$:

$$\frac{\cos^2\theta}{\sin^2\theta} + \frac{\sin^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta}$$

$$\therefore \quad \cot^2\theta + 1 = \operatorname{cosec}^2\theta$$

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Example 3

a If $\operatorname{cosec} x = \frac{7}{4}$, determine $\cos x$.

b If $\sec x = -\frac{3}{2}$ and $\frac{\pi}{2} \le x \le \pi$, determine $\sin x$.

Solution

a Since $\operatorname{cosec} x = \frac{7}{4}$, we have $\sin x = \frac{4}{7}$.

Now
$$\cos^2 x + \sin^2 x = 1$$
$$\cos^2 x + \frac{16}{49} = 1$$
$$\cos^2 x = \frac{33}{49}$$
$$\therefore \quad \cos x = \pm\frac{\sqrt{33}}{7}$$

b Since $\sec x = -\frac{3}{2}$, we have $\cos x = -\frac{2}{3}$.

Now
$$\cos^2 x + \sin^2 x = 1$$
$$\frac{4}{9} + \sin^2 x = 1$$
$$\therefore \quad \sin x = \pm\frac{\sqrt{5}}{3}$$

But $\sin x$ is positive for $P(x)$ in the 2nd quadrant, and so $\sin x = \frac{\sqrt{5}}{3}$.

Example 4

Prove the identity $\dfrac{1}{1-\cos\theta} + \dfrac{1}{1+\cos\theta} = 2\operatorname{cosec}^2\theta$.

Solution

$$\begin{aligned}
\text{LHS} &= \frac{1}{1-\cos\theta} + \frac{1}{1+\cos\theta} \\
&= \frac{1+\cos\theta+1-\cos\theta}{1-\cos^2\theta} \\
&= \frac{2}{1-\cos^2\theta} \\
&= \frac{2}{\sin^2\theta} \\
&= 2\operatorname{cosec}^2\theta \\
&= \text{RHS}
\end{aligned}$$

Summary 11A

- **Reciprocal trigonometric functions**

$$\sec\theta = \frac{1}{\cos\theta} \quad (\text{for } \cos\theta \neq 0)$$
$$\operatorname{cosec}\theta = \frac{1}{\sin\theta} \quad (\text{for } \sin\theta \neq 0)$$
$$\cot\theta = \frac{\cos\theta}{\sin\theta} \quad (\text{for } \sin\theta \neq 0)$$

- **Pythagorean identity**

$$\cos^2\theta + \sin^2\theta = 1$$
$$1 + \tan^2\theta = \sec^2\theta$$
$$\cot^2\theta + 1 = \operatorname{cosec}^2\theta$$

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Exercise 11A

Example 1

1 Determine the exact value of each of the following:

a $\cot\left(\frac{3\pi}{4}\right)$ **b** $\operatorname{cosec}\left(\frac{5\pi}{4}\right)$ **c** $\sec\left(\frac{5\pi}{6}\right)$

d $\operatorname{cosec}\left(\frac{\pi}{2}\right)$ **e** $\sec\left(\frac{4\pi}{3}\right)$ **f** $\operatorname{cosec}\left(\frac{13\pi}{6}\right)$

g $\cot\left(\frac{7\pi}{3}\right)$ **h** $\sec\left(\frac{5\pi}{3}\right)$

2 Without using a calculator, write down the exact value of each of the following:

a $\cot 135°$ **b** $\sec 150°$ **c** $\operatorname{cosec} 90°$

d $\cot 240°$ **e** $\operatorname{cosec} 225°$ **f** $\sec 330°$

g $\cot 315°$ **h** $\operatorname{cosec} 300°$ **i** $\cot 420°$

Example 2

3 Determine the values of x between 0 and 2π for which:

a $\operatorname{cosec} x = 2$ **b** $\cot x = \sqrt{3}$

c $\sec x + \sqrt{2} = 0$ **d** $\operatorname{cosec} x = \sec x$

Example 3

4 If $\sec\theta = \frac{-17}{8}$ and $\frac{\pi}{2} < \theta < \pi$, determine:

a $\cos\theta$ **b** $\sin\theta$ **c** $\tan\theta$

5 If $\tan\theta = \frac{-7}{24}$ and $\frac{3\pi}{2} < \theta < 2\pi$, determine $\cos\theta$ and $\sin\theta$.

6 Determine the value of $\sec\theta$ if $\tan\theta = 0.4$ and θ is not in the 1st quadrant.

7 If $\tan\theta = \frac{4}{3}$ and $\pi < \theta < \frac{3\pi}{2}$, evaluate $\frac{\sin\theta - 2\cos\theta}{\cot\theta - \sin\theta}$.

8 If $\cos\theta = \frac{2}{3}$ and θ is in the 4th quadrant, express $\frac{\tan\theta - 3\sin\theta}{\cos\theta - 2\cot\theta}$ in simplest surd form.

Example 4

9 Prove each of the following identities for suitable values of θ and φ:

a $(1 - \cos^2\theta)(1 + \cot^2\theta) = 1$ **b** $\cos^2\theta\,\tan^2\theta + \sin^2\theta\,\cot^2\theta = 1$

c $\frac{\tan\theta}{\tan\varphi} = \frac{\tan\theta + \cot\varphi}{\cot\theta + \tan\varphi}$ **d** $(\sin\theta + \cos\theta)^2 + (\sin\theta - \cos\theta)^2 = 2$

e $\frac{1 + \cot^2\theta}{\cot\theta\,\operatorname{cosec}\theta} = \sec\theta$ **f** $\sec\theta + \tan\theta = \frac{\cos\theta}{1 - \sin\theta}$

SF

CF

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11B Angle sum and difference identities

Learning intentions

- To be able to work with the angle sum and angle difference identities for sine and cosine.
- To be able to work with the double angle identities for sine and cosine.

Angle sum and difference identities for cosine

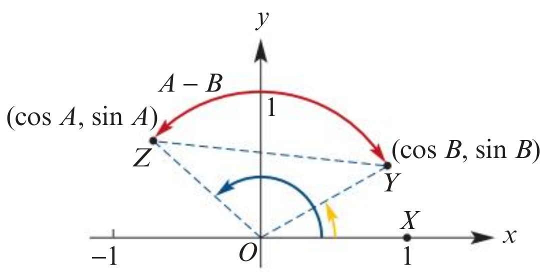
1. $\cos(A + B) = \cos A \cos B - \sin A \sin B$
2. $\cos(A - B) = \cos A \cos B + \sin A \sin B$

Proof Consider a unit circle as shown:

arc length $XY = B$ units

arc length $XZ = A$ units

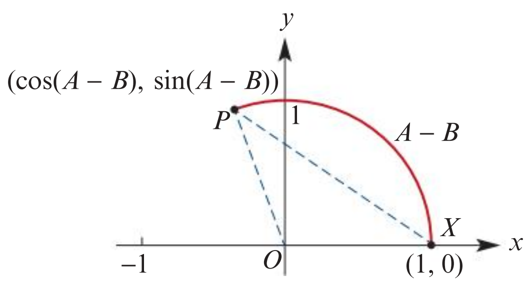
arc length $YZ = A - B$ units



Rotate ΔOZY so that Y is coincident with X. Then Z is moved to

$$P(\cos(A - B), \sin(A - B))$$

Since the triangles ZYO and PXO are congruent, we have $ZY = PX$.



Using the coordinate distance formula:

$$ZY^2 = (\cos A - \cos B)^2 + (\sin A - \sin B)^2$$
$$= 2 - 2(\cos A \cos B + \sin A \sin B)$$

$$PX^2 = (\cos(A - B) - 1)^2 + (\sin(A - B) - 0)^2$$
$$= 2 - 2\cos(A - B)$$

Since $ZY = PX$, this gives

$$2 - 2\cos(A - B) = 2 - 2(\cos A \cos B + \sin A \sin B)$$
$$\therefore \quad \cos(A - B) = \cos A \cos B + \sin A \sin B$$

We can now obtain the first identity from the second by replacing B with $-B$:

$$\cos(A + B) = \cos(A - (-B))$$
$$= \cos A \cos(-B) + \sin A \sin(-B)$$
$$= \cos A \cos B - \sin A \sin B$$

Note: Here we used $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$.

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We can use these two identities to determine exact values for cosine.

Example 5

Evaluate:

a $\cos 75°$ **b** $\cos 15°$

Solution

a
$$\begin{aligned}\cos 75° &= \cos(45° + 30°)\\ &= \cos 45° \cos 30° - \sin 45° \sin 30°\\ &= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \times \frac{1}{2}\\ &= \frac{\sqrt{3}-1}{2\sqrt{2}}\\ &= \frac{\sqrt{3}-1}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}\\ &= \frac{\sqrt{6}-\sqrt{2}}{4}\end{aligned}$$

b
$$\begin{aligned}\cos 15° &= \cos(45° - 30°)\\ &= \cos 45° \cos 30° + \sin 45° \sin 30°\\ &= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \times \frac{1}{2}\\ &= \frac{\sqrt{3}+1}{2\sqrt{2}}\\ &= \frac{\sqrt{3}+1}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}\\ &= \frac{\sqrt{6}+\sqrt{2}}{4}\end{aligned}$$

Angle sum and difference identities for sine

1. $\sin(A + B) = \sin A \cos B + \cos A \sin B$
2. $\sin(A - B) = \sin A \cos B - \cos A \sin B$

Proof We use the symmetry properties $\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$ and $\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right)$:

$$\begin{aligned}\sin(A + B) &= \cos\left(\frac{\pi}{2} - (A + B)\right)\\ &= \cos\left(\left(\frac{\pi}{2} - A\right) - B\right)\\ &= \cos\left(\frac{\pi}{2} - A\right)\cos B + \sin\left(\frac{\pi}{2} - A\right)\sin B\\ &= \sin A \cos B + \cos A \sin B\end{aligned}$$

We can now obtain the second identity from the first by replacing B with $-B$:

$$\begin{aligned}\sin(A - B) &= \sin(A + (-B))\\ &= \sin A \cos(-B) + \cos A \sin(-B)\\ &= \sin A \cos B - \cos A \sin B\end{aligned}$$

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Example 6

Evaluate:

a $\sin 75^\circ$

b $\sin 15^\circ$

Solution

a
$$\begin{aligned}\sin 75^\circ &= \sin(30^\circ + 45^\circ)\\ &= \sin 30^\circ \cos 45^\circ + \cos 30^\circ \sin 45^\circ\\ &= \frac{1}{2} \times \frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}}\\ &= \frac{1+\sqrt{3}}{2\sqrt{2}}\\ &= \frac{1+\sqrt{3}}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}\\ &= \frac{\sqrt{2}+\sqrt{6}}{4}\end{aligned}$$

b
$$\begin{aligned}\sin 15^\circ &= \sin(45^\circ - 30^\circ)\\ &= \sin 45^\circ \cos 30^\circ - \cos 45^\circ \sin 30^\circ\\ &= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \times \frac{1}{2}\\ &= \frac{\sqrt{3}-1}{2\sqrt{2}}\\ &= \frac{\sqrt{3}-1}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}\\ &= \frac{\sqrt{6}-\sqrt{2}}{4}\end{aligned}$$

Example 7

Determine $\sin(A - B)$ given that A and B are acute angles with $\cos A = \frac{3}{5}$ and $\cos B = \frac{12}{13}$.

Solution

Using the Pythagorean identity, we have

$$\begin{aligned}\sin^2 A &= 1 - \cos^2 A & \text{and} \qquad \sin^2 B &= 1 - \cos^2 B\\ &= 1 - \frac{9}{25} & &= 1 - \frac{144}{169}\\ &= \frac{16}{25} & &= \frac{25}{169}\end{aligned}$$

So $\sin A = \frac{4}{5}$ and $\sin B = \frac{5}{13}$, since A and B are acute. Hence

$$\begin{aligned}\sin(A - B) &= \sin A \cos B - \cos A \sin B\\ &= \frac{4}{5} \times \frac{12}{13} - \frac{3}{5} \times \frac{5}{13}\\ &= \frac{48 - 15}{65}\\ &= \frac{33}{65}\end{aligned}$$

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Double-angle identities

Using the angle sum identities for sine and cosine, we can easily derive useful expressions for $\sin(2A)$ and $\cos(2A)$.

Double-angle identities for cosine

$$\begin{aligned}\cos(2A) &= \cos^2 A - \sin^2 A \\ &= 2\cos^2 A - 1 \qquad (\text{since } \sin^2 A = 1 - \cos^2 A) \\ &= 1 - 2\sin^2 A \qquad (\text{since } \cos^2 A = 1 - \sin^2 A)\end{aligned}$$

Proof $\cos(A + A) = \cos A \cos A - \sin A \sin A$

$$= \cos^2 A - \sin^2 A$$

Double-angle identity for sine

$$\sin(2A) = 2\sin A \cos A$$

Proof $\sin(A + A) = \sin A \cos A + \cos A \sin A$

$$= 2\sin A \cos A$$

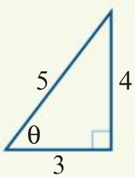
Example 8

If $\tan\theta = \frac{4}{3}$ and $0 < \theta < \frac{\pi}{2}$, evaluate:

a $\sin(2\theta)$ **b** $\cos(2\theta)$ **c** $\tan(2\theta)$

Solution

Since θ is acute, we have $\sin\theta = \frac{4}{5}$ and $\cos\theta = \frac{3}{5}$.



a $$\begin{aligned}\sin(2\theta) &= 2\sin\theta\cos\theta \\ &= 2 \times \frac{4}{5} \times \frac{3}{5} \\ &= \frac{24}{25}\end{aligned}$$

b $$\begin{aligned}\cos(2\theta) &= \cos^2\theta - \sin^2\theta \\ &= \frac{9}{25} - \frac{16}{25} \\ &= -\frac{7}{25}\end{aligned}$$

c $$\begin{aligned}\tan(2\theta) &= \frac{\sin(2\theta)}{\cos(2\theta)} \\ &= \frac{24}{25} \times \left(-\frac{25}{7}\right) \\ &= -\frac{24}{7}\end{aligned}$$

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Example 9

Prove each of the following identities:

a $\dfrac{2\sin\theta\cos\theta}{\cos^2\theta - \sin^2\theta} = \tan(2\theta)$

b $\dfrac{\sin\theta}{\sin\varphi} + \dfrac{\cos\theta}{\cos\varphi} = \dfrac{2\sin(\theta+\varphi)}{\sin(2\varphi)}$

c $\dfrac{1}{\cos\theta + \sin\theta} + \dfrac{1}{\cos\theta - \sin\theta} = \tan(2\theta)\operatorname{cosec}\theta$

Solution

a
$$\begin{aligned}\text{LHS} &= \frac{2\sin\theta\cos\theta}{\cos^2\theta - \sin^2\theta}\\ &= \frac{\sin(2\theta)}{\cos(2\theta)}\\ &= \tan(2\theta)\\ &= \text{RHS}\end{aligned}$$

Note: Identity holds when $\cos(2\theta) \neq 0$.

b
$$\begin{aligned}\text{LHS} &= \frac{\sin\theta}{\sin\varphi} + \frac{\cos\theta}{\cos\varphi}\\ &= \frac{\sin\theta\cos\varphi + \cos\theta\sin\varphi}{\sin\varphi\cos\varphi}\\ &= \frac{\sin(\theta+\varphi)}{\frac{1}{2}\sin(2\varphi)}\\ &= \frac{2\sin(\theta+\varphi)}{\sin(2\varphi)}\\ &= \text{RHS}\end{aligned}$$

Note: Identity holds when $\sin(2\varphi) \neq 0$.

c
$$\begin{aligned}\text{LHS} &= \frac{1}{\cos\theta + \sin\theta} + \frac{1}{\cos\theta - \sin\theta}\\ &= \frac{\cos\theta - \sin\theta + \cos\theta + \sin\theta}{\cos^2\theta - \sin^2\theta}\\ &= \frac{2\cos\theta}{\cos(2\theta)}\\ &= \frac{2\cos\theta}{\cos(2\theta)} \times \frac{\sin\theta}{\sin\theta}\\ &= \frac{\sin(2\theta)}{\cos(2\theta)\sin\theta}\\ &= \frac{\tan(2\theta)}{\sin\theta}\\ &= \tan(2\theta)\operatorname{cosec}\theta\\ &= \text{RHS}\end{aligned}$$

Note: Identity holds when $\cos(2\theta) \neq 0$ and $\sin\theta \neq 0$.

Sometimes the easiest way to prove that two expressions are equal is to simplify each of them separately. This is demonstrated in the following example.

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Example 10

Prove that $(\sec A - \cos A)(\operatorname{cosec} A - \sin A) = \dfrac{1}{\tan A + \cot A}$.

Solution

$$\begin{aligned}\text{LHS} &= (\sec A - \cos A)(\operatorname{cosec} A - \sin A)\\ &= \left(\frac{1}{\cos A} - \cos A\right)\left(\frac{1}{\sin A} - \sin A\right)\\ &= \frac{1-\cos^2 A}{\cos A} \times \frac{1-\sin^2 A}{\sin A}\\ &= \frac{\sin^2 A \cos^2 A}{\cos A \sin A}\\ &= \cos A \sin A\end{aligned}$$

$$\begin{aligned}\text{RHS} &= \frac{1}{\tan A + \cot A}\\ &= \frac{1}{\frac{\sin A}{\cos A} + \frac{\cos A}{\sin A}}\\ &= \frac{\cos A \sin A}{\sin^2 A + \cos^2 A}\\ &= \cos A \sin A\end{aligned}$$

We have shown that LHS = RHS.

Summary 11B

- **Angle sum and difference identities**

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$
$$\cos(A-B) = \cos A \cos B + \sin A \sin B$$
$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$
$$\sin(A-B) = \sin A \cos B - \cos A \sin B$$

- **Double-angle identities**

$$\begin{aligned}\cos(2A) &= \cos^2 A - \sin^2 A\\ &= 2\cos^2 A - 1\\ &= 1 - 2\sin^2 A\end{aligned}$$
$$\sin(2A) = 2\sin A \cos A$$

Note: Further multi-angle identities are investigated in the following Exercise and Section 11D.

Exercise 11B

Skill-sheet

Example 5, 6

1 Using angle sum and difference identities, determine exact values for the following:

a $\cos 165°$ **b** $\cos 105°$ **c** $\sin 165°$ **d** $\sin 105°$

SF

2 Determine the exact value of:

a $\cos\left(\dfrac{5\pi}{12}\right)$ **b** $\sin\left(\dfrac{11\pi}{12}\right)$

Example 7

3 If $\sin u = \dfrac{12}{13}$ and $\sin v = \dfrac{3}{5}$, evaluate $\sin(u+v)$. (**Note:** There is more than one answer.)

4 Simplify the following:

a $\sin\left(\theta + \dfrac{\pi}{6}\right)$ **b** $\cos\left(\varphi - \dfrac{\pi}{4}\right)$ **c** $\cos\left(\theta + \dfrac{\pi}{3}\right)$ **d** $\sin\left(\theta - \dfrac{\pi}{4}\right)$

5 Simplify:

CF

a $\cos(u-v)\sin v + \sin(u-v)\cos v$ **b** $\sin(u+v)\sin v + \cos(u+v)\cos v$

Example 8

6 If $\sin\theta = \frac{-3}{5}$, with θ in the 3rd quadrant, and $\cos\varphi = \frac{-5}{13}$, with φ in the 2nd quadrant, evaluate each of the following without using a calculator:

a $\cos(2\varphi)$ **b** $\sin(2\theta)$ **c** $\cos(2\theta)$ **d** $\sec(2\varphi)$
e $\sin(\theta + \varphi)$ **f** $\cos(\theta - \varphi)$ **g** $\operatorname{cosec}(\theta + \varphi)$ **h** $\cot(2\theta)$

CF

7 If $\sin\alpha = \frac{3}{5}$ and $\sin\beta = \frac{24}{25}$, with $\frac{\pi}{2} < \beta < \alpha < \pi$, evaluate:

a $\cos(2\alpha)$ **b** $\sin(\alpha - \beta)$ **c** $\cos(\alpha + \beta)$ **d** $\sin(2\beta)$

8 If $\sin\theta = -\frac{\sqrt{3}}{2}$ and $\cos\theta = \frac{1}{2}$, evaluate:

a $\sin(2\theta)$ **b** $\cos(2\theta)$

9 Simplify each of the following expressions:

a $(\sin\theta - \cos\theta)^2$ **b** $\cos^4\theta - \sin^4\theta$

Example 9, 10

10 Prove the following identities:

a $\sqrt{2}\sin\left(\theta - \frac{\pi}{4}\right) = \sin\theta - \cos\theta$ **b** $\cos\left(\theta - \frac{\pi}{3}\right) + \cos\left(\theta + \frac{\pi}{3}\right) = \cos\theta$

c $\cos\left(\theta + \frac{\pi}{6}\right) + \sin\left(\theta + \frac{\pi}{3}\right) = \sqrt{3}\cos\theta$ **d** $\frac{\sin(u + v)}{\cos u \cos v} = \tan v + \tan u$

e $\frac{\tan u + \tan v}{\tan u - \tan v} = \frac{\sin(u + v)}{\sin(u - v)}$ **f** $\frac{1 - \sin(2\theta)}{\sin\theta - \cos\theta} = \sin\theta - \cos\theta$

11 Angle sum and difference identities for tangent

a Show that

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

b Hence show that

$$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B} \quad \text{and} \quad \tan(2A) = \frac{2\tan A}{1 - \tan^2 A}$$

c Determine the exact value of each of the following:

i $\tan 15°$ **ii** $\tan 75°$ **iii** $\tan 105°$

12 **Triple-angle identities** Prove each of the following:

a $\cos(3\theta) = 4\cos^3\theta - 3\cos\theta$ **b** $\sin(3\theta) = 3\sin\theta - 4\sin^3\theta$

13 **Quadruple-angle identities** Prove each of the following:

a $\cos(4\theta) = 8\cos^4\theta - 8\cos^2\theta + 1$ **b** $\cos(4\theta) = 1 - 8\sin^2\theta + 8\sin^4\theta$
c $\sin(4\theta) = 4\sin\theta\cos\theta - 8\sin^3\theta\cos\theta$ **d** $\sin(4\theta) = 4\sin\theta\cos^3\theta - 4\cos\theta\sin^3\theta$

CU

14 Prove the following identities:

a $8\cos^4\theta = 3 + 4\cos(2\theta) + \cos(4\theta)$ **b** $8\sin^4\theta = 3 - 4\cos(2\theta) + \cos(4\theta)$

15 Prove the following identities:

a $\operatorname{cosec} 2x - \cot 2x = \tan x$ **b** $\operatorname{cosec} 2x + \cot 2x = \cot x$ **c** $2\operatorname{cosec} 2x = \cot x + \tan x$

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11C Simplifying $a \cos x + b \sin x$

Learning intentions

- To be able to rewrite the function $f(x) = a\cos x + b\sin x$ in terms of a single trigonometric function.

$$a\cos x + b\sin x = r\cos(x - \alpha) \quad \text{where } r = \sqrt{a^2 + b^2},\ \cos\alpha = \frac{a}{r} \text{ and } \sin\alpha = \frac{b}{r}$$

Proof Let $r = \sqrt{a^2 + b^2}$. Consider the point $P\left(\frac{a}{r}, \frac{b}{r}\right)$ and its distance from the origin O:

$$OP^2 = \left(\frac{a}{r}\right)^2 + \left(\frac{b}{r}\right)^2 = \frac{a^2}{a^2 + b^2} + \frac{b^2}{a^2 + b^2} = 1$$

Therefore the point P is on the unit circle, and so there is an angle α such that

$$\frac{a}{r} = \cos\alpha \quad \text{and} \quad \frac{b}{r} = \sin\alpha$$

We can now write

$$\begin{aligned} a\cos x + b\sin x &= r\left(\frac{a}{r}\cos x + \frac{b}{r}\sin x\right) \\ &= r(\cos\alpha\ \cos x + \sin\alpha\ \sin x) \\ &= r\cos(x - \alpha) \end{aligned}$$

Similarly, it may be shown that

$$a\cos x + b\sin x = r\sin(x + \beta) \quad \text{where } r = \sqrt{a^2 + b^2},\ \sin\beta = \frac{a}{r},\ \cos\beta = \frac{b}{r}$$

Example 11

a Express $\cos x - \sqrt{3}\sin x$ in the form $r\cos(x - \alpha)$.

b Hence determine the range of the function f with rule $f(x) = \cos x - \sqrt{3}\sin x$, and determine the maximum and minimum values of f.

Solution

a Here $a = 1$ and $b = -\sqrt{3}$. Therefore

$$r = \sqrt{1 + 3} = 2, \quad \cos\alpha = \frac{a}{r} = \frac{1}{2} \quad \text{and} \quad \sin\alpha = \frac{b}{r} = \frac{-\sqrt{3}}{2}$$

We see that $\alpha = -\frac{\pi}{3}$ and so $\cos x - \sqrt{3}\sin x = 2\cos\left(x + \frac{\pi}{3}\right)$

b From part **a** we have

$$f(x) = 2\cos\left(x + \frac{\pi}{3}\right)$$

Thus the range of f is $[-2, 2]$, the maximum value is 2 and the minimum value is -2.

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Example 12

Solve $\cos x - \sqrt{3}\sin x = 1$ for $x \in [0, 2\pi]$.

Solution

From Example 11, we have

$$\cos x - \sqrt{3}\sin x = 2\cos\left(x + \frac{\pi}{3}\right)$$

$$\therefore \quad 2\cos\left(x + \frac{\pi}{3}\right) = 1$$

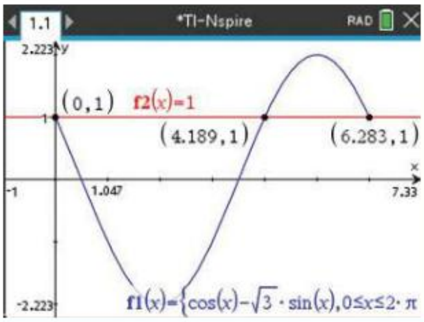
$$\cos\left(x + \frac{\pi}{3}\right) = \frac{1}{2}$$

$$x + \frac{\pi}{3} = \frac{\pi}{3}, \frac{5\pi}{3} \text{ or } \frac{7\pi}{3}$$

$$x = 0, \frac{4\pi}{3} \text{ or } 2\pi$$

Using the TI-Nspire CX non-CAS

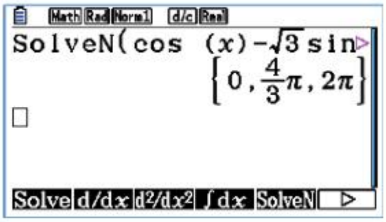
Since trigonometric equations often have multiple solutions, they are best solved graphically. Only approximate solutions are displayed.



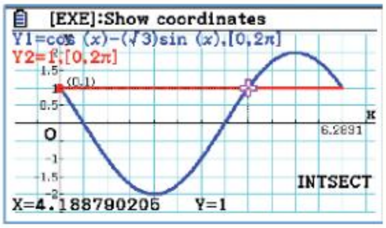
Using the Casio

- Ensure that the angle setting is Radians.
- In **Run-Matrix** mode, select the numerical solver (**Calculation** OPTN F4, **SolveN** F5).
- Complete the equation and domain by entering:

  $\cos(x) - \sqrt{3}\sin(x) = 1, x, 0, 2\pi)$



- Approximate solutions can also be found graphically.



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Example 13

Express $\sqrt{3}\sin(2x) - \cos(2x)$ in the form $r\sin(2x + \alpha)$.

Solution

A slightly different technique is used. Assume that

$$\begin{aligned}\sqrt{3}\sin(2x) - \cos(2x) &= r\sin(2x+\alpha)\\ &= r\,(\sin(2x)\cos\alpha + \cos(2x)\sin\alpha)\end{aligned}$$

This is to hold for all x.

For $x = \frac{\pi}{4}$: $\quad \sqrt{3} = r\cos\alpha \quad (1)$

For $x = 0$: $\quad -1 = r\sin\alpha \quad (2)$

Squaring and adding (1) and (2) gives

$$\begin{aligned} r^2\cos^2\alpha + r^2\sin^2\alpha &= 4\\ r^2 &= 4\\ \therefore \quad r &= \pm 2\end{aligned}$$

We take the positive solution. Substituting in (1) and (2) gives

$$\frac{\sqrt{3}}{2} = \cos\alpha \quad \text{and} \quad -\frac{1}{2} = \sin\alpha$$

Thus $\alpha = -\frac{\pi}{6}$ and hence

$$\sqrt{3}\sin(2x) - \cos(2x) = 2\sin\left(2x - \frac{\pi}{6}\right)$$

Check: Expand the new expression using the angle difference identity for sine:

$$\begin{aligned} 2\sin\left(2x - \frac{\pi}{6}\right) &= 2\left(\sin(2x)\cos\left(\frac{\pi}{6}\right) - \cos(2x)\sin\left(\frac{\pi}{6}\right)\right)\\ &= 2\left(\sin(2x)\times\frac{\sqrt{3}}{2} - \cos(2x)\times\frac{1}{2}\right)\\ &= \sqrt{3}\sin(2x) - \cos(2x)\end{aligned}$$

Summary 11C

- $a\cos x + b\sin x = r\cos(x - \alpha)$ where $r = \sqrt{a^2+b^2}$, $\cos\alpha = \frac{a}{r}$, $\sin\alpha = \frac{b}{r}$
- $a\cos x + b\sin x = r\sin(x + \beta)$ where $r = \sqrt{a^2+b^2}$, $\sin\beta = \frac{a}{r}$, $\cos\beta = \frac{b}{r}$

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Skill-sheet

Exercise 11C

SF

Example 11

1 Determine the maximum and minimum values of the following:

a $4\cos x + 3\sin x$ **b** $\sqrt{3}\cos x + \sin x$ **c** $\cos x - \sin x$

d $\cos x + \sin x$ **e** $3\cos x + \sqrt{3}\sin x$ **f** $\sin x - \sqrt{3}\cos x$

g $\cos x - \sqrt{3}\sin x + 2$ **h** $5 + 3\sin x - 2\cos x$

Example 12

2 Solve each of the following for $x \in [0, 2\pi]$ or for $\theta \in [0, 360]$:

a $\sin x - \cos x = 1$ **b** $\sqrt{3}\sin x + \cos x = 1$

c $\sin x - \sqrt{3}\cos x = -1$ **d** $3\cos x - \sqrt{3}\sin x = 3$

e $4\sin\theta° + 3\cos\theta° = 5$ **f** $2\sqrt{2}\sin\theta° - 2\cos\theta° = 3$

3 Write $\sqrt{3}\cos(2x) - \sin(2x)$ in the form $r\cos(2x + \alpha)$.

Example 13

4 Write $\cos(3x) - \sin(3x)$ in the form $r\sin(3x - \alpha)$.

5 Sketch the graph of each of the following, showing one cycle:

a $f(x) = \sin x - \cos x$ **b** $f(x) = \sqrt{3}\sin x + \cos x$

c $f(x) = \sin x + \cos x$ **d** $f(x) = \sin x - \sqrt{3}\cos x$

11D Sums and products of sines and cosines

Learning intentions

▶ To be able to rewrite products of sines and cosines as sums or differences and vice versa.

In Section 11B, we derived the angle sum and difference identities for sine and cosine. We use them in this section to obtain new identities.

Expressing products as sums or differences

Product-to-sum identities

$$2\cos A\cos B = \cos(A - B) + \cos(A + B)$$
$$2\sin A\sin B = \cos(A - B) - \cos(A + B)$$
$$2\sin A\cos B = \sin(A + B) + \sin(A - B)$$

Proof We use the angle sum and difference identities for sine and cosine:

$$\cos(A + B) = \cos A\cos B - \sin A\sin B \qquad (1)$$
$$\cos(A - B) = \cos A\cos B + \sin A\sin B \qquad (2)$$
$$\sin(A + B) = \sin A\cos B + \cos A\sin B \qquad (3)$$
$$\sin(A - B) = \sin A\cos B - \cos A\sin B \qquad (4)$$

The first product-to-sum identity is obtained by adding (2) and (1), the second identity is obtained by subtracting (1) from (2), and the third by adding (3) and (4).

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Example 14

Express each of the following products as sums or differences:

a $2\sin(3\theta)\cos(\theta)$

b $2\sin 50^\circ \cos 60^\circ$

c $2\cos\left(\theta + \frac{\pi}{4}\right)\cos\left(\theta - \frac{\pi}{4}\right)$

Solution

a Use the third product-to-sum identity:

$$\begin{aligned} 2\sin(3\theta)\cos(\theta) &= \sin(3\theta + \theta) + \sin(3\theta - \theta) \\ &= \sin(4\theta) + \sin(2\theta) \end{aligned}$$

b Use the third product-to-sum identity:

$$\begin{aligned} 2\sin 50^\circ \cos 60^\circ &= \sin 110^\circ + \sin(-10)^\circ \\ &= \sin 110^\circ - \sin 10^\circ \end{aligned}$$

c Use the first product-to-sum identity:

$$\begin{aligned} 2\cos\left(\theta + \frac{\pi}{4}\right)\cos\left(\theta - \frac{\pi}{4}\right) &= \cos\left(\frac{\pi}{2}\right) + \cos(2\theta) \\ &= \cos(2\theta) \end{aligned}$$

Expressing sums and differences as products

Sum-to-product identities

$$\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

$$\cos A - \cos B = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

$$\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

$$\sin A - \sin B = 2\sin\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right)$$

Proof Using the first product-to-sum identity, we have

$$\begin{aligned} 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) &= \cos\left(\frac{A+B}{2} - \frac{A-B}{2}\right) + \cos\left(\frac{A+B}{2} + \frac{A-B}{2}\right) \\ &= \cos B + \cos A \\ &= \cos A + \cos B \end{aligned}$$

The other three sum-to-product identities can be obtained similarly.

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Example 15

Express each of the following as products:

a $\sin 36° + \sin 10°$

b $\cos 36° + \cos 10°$

c $\sin 36° - \sin 10°$

d $\cos 36° - \cos 10°$

Solution

a $\sin 36° + \sin 10° = 2\sin 23° \cos 13°$

b $\cos 36° + \cos 10° = 2\cos 23° \cos 13°$

c $\sin 36° - \sin 10° = 2\cos 23° \sin 13°$

d $\cos 36° - \cos 10° = -2\sin 23° \sin 13°$

Example 16

Prove that

$$\frac{\cos(\theta) - \cos(3\theta)}{\sin(3\theta) - \sin(\theta)} = \tan(2\theta)$$

Solution

$$\begin{aligned}\text{LHS} &= \frac{\cos(\theta) - \cos(3\theta)}{\sin(3\theta) - \sin(\theta)} \\ &= \frac{-2\sin(2\theta)\sin(-\theta)}{2\sin(\theta)\cos(2\theta)} \\ &= \frac{2\sin(2\theta)\sin(\theta)}{2\sin(\theta)\cos(2\theta)} \\ &= \tan(2\theta) \\ &= \text{RHS}\end{aligned}$$

Example 17

Solve the equation $\sin(3x) + \sin(11x) = 0$ for $x \in [0, \pi]$.

Solution

$$\begin{aligned}& \sin(3x) + \sin(11x) = 0 \\ \Leftrightarrow\ & 2\sin(7x)\cos(4x) = 0 \\ \Leftrightarrow\ & \sin(7x) = 0 \quad \text{or} \quad \cos(4x) = 0 \\ \Leftrightarrow\ & 7x = 0, \pi, 2\pi, 3\pi, 4\pi, 5\pi, 6\pi, 7\pi \quad \text{or} \quad 4x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2} \\ \Leftrightarrow\ & x = 0, \frac{\pi}{7}, \frac{2\pi}{7}, \frac{3\pi}{7}, \frac{4\pi}{7}, \frac{5\pi}{7}, \frac{6\pi}{7}, \pi, \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8} \text{ or } \frac{7\pi}{8}\end{aligned}$$

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Summary 11D

- **Product-to-sum identities**

$$2\cos A\ \cos B = \cos(A - B) + \cos(A + B)$$
$$2\sin A\ \sin B = \cos(A - B) - \cos(A + B)$$
$$2\sin A\ \cos B = \sin(A + B) + \sin(A - B)$$

- **Sum-to-product identities**

$$\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$
$$\cos A - \cos B = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$
$$\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$
$$\sin A - \sin B = 2\sin\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right)$$

Exercise 11D

SF

Example 14

1 Express each of the following products as sums or differences:

a $2\sin(3\pi t)\cos(2\pi t)$

b $\sin 20^\circ \cos 30^\circ$

c $2\cos\left(\frac{\pi x}{4}\right)\sin\left(\frac{3\pi x}{4}\right)$

d $2\sin\left(\frac{A+B+C}{2}\right)\cos\left(\frac{A-B-C}{2}\right)$

2 Express $2\sin(3\theta)\sin(2\theta)$ as a difference of cosines.

3 Use a product-to-sum identity to derive the expression for $2\sin\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right)$ as a difference of sines.

4 Show that $\sin 75^\circ \sin 15^\circ = \frac{1}{4}$.

Example 15

5 Express each of the following as products:

a $\sin 56^\circ + \sin 22^\circ$

b $\cos 56^\circ + \cos 22^\circ$

c $\sin 56^\circ - \sin 22^\circ$

d $\cos 56^\circ - \cos 22^\circ$

6 Express each of the following as products:

a $\sin(6A) + \sin(2A)$

b $\cos(x) + \cos(2x)$

c $\sin(4x) - \sin(3x)$

d $\cos(3A) - \cos(A)$

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Example 16

7 Show that $\sin(A) + 2\sin(3A) + \sin(5A) = 4\cos^2(A)\sin(3A)$.

CF

8 For any three angles α, β and γ, show that

$$\sin(\alpha + \beta)\sin(\alpha - \beta) + \sin(\beta + \gamma)\sin(\beta - \gamma) + \sin(\gamma + \alpha)\sin(\gamma - \alpha) = 0$$

9 Show that $\cos 70^\circ + \sin 40^\circ = \cos 10^\circ$.

10 Show that $\cos 20^\circ + \cos 100^\circ + \cos 140^\circ = 0$.

Example 17

11 Solve each of the following equations for $x \in [-\pi, \pi]$:

a $\cos(5x) + \cos(x) = 0$

b $\cos(5x) - \cos(x) = 0$

c $\sin(5x) + \sin(x) = 0$

d $\sin(5x) - \sin(x) = 0$

12 Solve each of the following equations for $\theta \in [0, \pi]$:

a $\cos(2\theta) - \sin(\theta) = 0$

b $\sin(5\theta) - \sin(3\theta) + \sin(\theta) = 0$

c $\sin(7\theta) - \sin(\theta) = \sin(3\theta)$

d $\cos(3\theta) - \cos(5\theta) + \cos(7\theta) = 0$

13 Prove that $\dfrac{\sin A + \sin B}{\cos A + \cos B} = \tan\left(\dfrac{A + B}{2}\right)$.

14 Prove the identity:

$$4\sin(A + B)\sin(B + C)\sin(C + A) = \sin(2A) + \sin(2B) + \sin(2C) - \sin(2A + 2B + 2C)$$

15 Prove that $\dfrac{\cos(2A) - \cos(2B)}{\sin(2A - 2B)} = -\dfrac{\sin(A + B)}{\cos(A - B)}$.

16 Prove each of the following identities:

a $\dfrac{\sin(A) + \sin(3A) + \sin(5A)}{\cos(A) + \cos(3A) + \cos(5A)} = \tan(3A)$

b $\cos^2(A) + \cos^2(B) - 1 = \cos(A + B)\cos(A - B)$

c $\cos^2(A - B) - \cos^2(A + B) = \sin(2A)\sin(2B)$

d $\cos^2(A - B) - \sin^2(A + B) = \cos(2A)\cos(2B)$

CU

17 Determine the sum

$$\sin(x) + \sin(3x) + \sin(5x) + \cdots + \sin(99x)$$

Hint: First multiply this sum by $2\sin(x)$.

Chapter summary

Reciprocal trigonometric functions

$$\sec\theta = \frac{1}{\cos\theta} \qquad \operatorname{cosec}\theta = \frac{1}{\sin\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{1}{\tan\theta} \quad (\text{if } \cos\theta \neq 0)$$

Pythagorean identity

$$\cos^2\theta + \sin^2\theta = 1$$
$$1 + \tan^2\theta = \sec^2\theta$$
$$\cot^2\theta + 1 = \operatorname{cosec}^2\theta$$

Angle sum and difference identities

$$\cos(A+B) = \cos A\cos B - \sin A\sin B$$
$$\cos(A-B) = \cos A\cos B + \sin A\sin B$$
$$\sin(A+B) = \sin A\cos B + \cos A\sin B$$
$$\sin(A-B) = \sin A\cos B - \cos A\sin B$$

Double-angle identities

$$\cos(2A) = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$$
$$\sin(2A) = 2\sin A\cos A$$

Linear combinations

$$a\cos x + b\sin x = r\cos(x-\alpha) \quad \text{where } r = \sqrt{a^2+b^2}, \quad \cos\alpha = \frac{a}{r}, \quad \sin\alpha = \frac{b}{r}$$

$$a\cos x + b\sin x = r\sin(x+\beta) \quad \text{where } r = \sqrt{a^2+b^2}, \quad \sin\beta = \frac{a}{r}, \quad \cos\beta = \frac{b}{r}$$

Product-to-sum identities

$$2\cos A\cos B = \cos(A-B) + \cos(A+B)$$
$$2\sin A\sin B = \cos(A-B) - \cos(A+B)$$
$$2\sin A\cos B = \sin(A+B) + \sin(A-B)$$

Sum-to-product identities

$$\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$
$$\cos A - \cos B = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$
$$\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$
$$\sin A - \sin B = 2\sin\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right)$$

Review

Skills checklist

Check-list

Download this checklist from the Interactive Textbook, then print it and fill it out to check your skills. ✓

11A **1** I can determine the exact value of a reciprocal trigonometric function for certain integer multiples of $\frac{\pi}{6}$ *and* $\frac{\pi}{4}$. ☐

See Example 1 and Question 1

11A **2** I can solve equations involving the the reciprocal trigonometric functions. ☐

See Example 2 and Question 3

11A **3** I can use the Pythagorean identities to help determine values of trigonometric expressions. ☐

See Example 3 and Question 4

11A **4** I can prove identities involving the reciprocal trigonometric functions. ☐

See Example 4 and Question 9

11B **5** I can use the sum and difference identities to help evaluate trigonometric expressions for suitable values. ☐

See Example 5, Example 6, Example 7 and Questions 1, 2 and 3

11B **6** I can use the double angle identities to evaluate trigonometric expressions. ☐

See Example 8 and Question 6

11B **7** I can prove identities involving the sum and difference and double angle identities ☐

See Example 9, Example 10 and Question 10

11C **8** I can write expressions of the form $a \cos x + b \sin x$ in terms of a function involving only cosine or only sine. ☐

See Example 11, Example 12, Example 13 and Questions 1, 2 and 4

11D **9** I can use the products to sums identities. ☐

See Example 14 and Question 1

11D **10** I can use the sums to products identities. ☐

See Example 15 and Question 5

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Short-response questions

Technology-free short-response questions

1 Determine the maximum and minimum values of each of the following:

a $3 + 2\sin\theta$ **b** $1 - 3\cos\theta$ **c** $4\sin\left(\dfrac{3\theta}{2}\right)$ **d** $2\sin^2\left(\dfrac{\theta}{2}\right)$ **e** $\dfrac{1}{2 + \cos\theta}$

2 Determine the values of $\theta \in [0, 2\pi]$ for which:

a $\tan^2\theta = \dfrac{1}{3}$ **b** $\tan(2\theta) = -1$ **c** $\sin(3\theta) = -1$ **d** $\sec(2\theta) = \sqrt{2}$

3 Prove each of the following identities:

a $\sec\theta + \operatorname{cosec}\theta\cot\theta = \sec\theta\operatorname{cosec}^2\theta$ **b** $\sec\theta - \sin\theta = \dfrac{\tan^2\theta + \cos^2\theta}{\sec\theta + \sin\theta}$

4 If $\sin A = \dfrac{5}{13}$ and $\sin B = \dfrac{8}{17}$, where A and B are acute, determine:

a $\cos(A + B)$ **b** $\sin(A + B)$ **c** $\tan(A + B)$

5 Determine $\cos 80^\circ \cos 20^\circ + \sin 80^\circ \sin 20^\circ$.

6 Sketch the graph of:

a $y = 2\cos^2 x$ **b** $y = 2\sin^2 x$

7 If $A + B = \dfrac{\pi}{2}$, determine the value of:

a $\sin A\cos B + \cos A\sin B$ **b** $\cos A\cos B - \sin A\sin B$

8 Prove each of the following:

a $\sin^2 A\cos^2 B - \cos^2 A\sin^2 B = \sin^2 A - \sin^2 B$

b $\dfrac{\sin\theta}{1 + \cos\theta} + \dfrac{1 + \cos\theta}{\sin\theta} = \dfrac{2}{\sin\theta}$ **c** $\dfrac{\sin\theta - 2\sin^3\theta}{2\cos^3\theta - \cos\theta} = \tan\theta$

9 Given that $\sin A = \dfrac{\sqrt{5}}{3}$ and that A is obtuse, determine the value of:

a $\cos(2A)$ **b** $\sin(2A)$ **c** $\sin(4A)$

10 Prove:

a $\dfrac{1 - \tan^2 A}{1 + \tan^2 A} = \cos(2A)$ **b** $\sqrt{2r^2(1 - \cos\theta)} = 2r\sin\left(\dfrac{\theta}{2}\right)$ for $r > 0$ and θ acute

11 Determine $\tan 15^\circ$ in simplest surd form.

Review

CF

12 Solve each of the following equations for $x \in [0, 2\pi]$:

a $\sin x + \cos x = 1$

b $\sin\left(\frac{1}{2}x\right)\cos\left(\frac{1}{2}x\right) = -\frac{1}{4}$

c $3\tan(2x) = 2\tan x$

d $\sin^2 x = \cos^2 x + 1$

e $\sin(3x)\cos x - \cos(3x)\sin x = \frac{\sqrt{3}}{2}$

f $2\cos\left(2x - \frac{\pi}{3}\right) = -\sqrt{3}$

13 Given that $\cos A = \frac{4}{5}$ and $\cos(A + \theta) = \frac{5}{13}$, where A and θ are acute, determine the exact value of $\cos\theta$.

14 **a** Express $2\cos\theta + 9\sin\theta$ in the form $r\cos(\theta - \alpha)$, where $r > 0$ and $0 < \alpha < \frac{\pi}{2}$.

b **i** Give the maximum value of $2\cos\theta + 9\sin\theta$.

ii Give the cosine of θ for which this maximum occurs.

iii Determine the smallest positive solution of the equation $2\cos\theta + 9\sin\theta = 1$.

15 Solve each of the following equations for $\theta \in [0, \pi]$:

a $\sin(4\theta) + \sin(2\theta) = 0$

b $\sin(2\theta) - \sin(\theta) = 0$

16 Prove that $\dfrac{\cos A - \cos B}{\sin A + \sin B} = \tan\left(\dfrac{B - A}{2}\right)$.

17 **a** Prove each of the identities:

i $\cos\theta = \dfrac{1 - \tan^2(\frac{1}{2}\theta)}{1 + \tan^2(\frac{1}{2}\theta)}$

ii $\sin\theta = \dfrac{2\tan(\frac{1}{2}\theta)}{1 + \tan^2(\frac{1}{2}\theta)}$

b Use the results of **a** to determine the value of $\tan(\frac{1}{2}\theta)$, given that $8\cos\theta - \sin\theta = 4$.

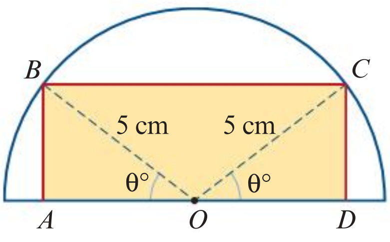
18 Solve the equation $\sqrt{2}(\sin 2\theta + \cos\theta) + \cos 3\theta = \sin 2\theta + \cos\theta$, for $0 \le \theta \le 2\pi$.

19 Solve the equation $\sin^2 x + \cos^2 x + \tan^2 x + \cot^2 x + \sec^2 x + \operatorname{cosec}^2 x = 31$, for $0 \le \theta \le \frac{\pi}{2}$.

Technology-active short-response questions

CF

20 The diagram shows a rectangle $ABCD$ inside a semicircle, centre O and radius 5 cm, with $\angle BOA = \angle COD = \theta°$.



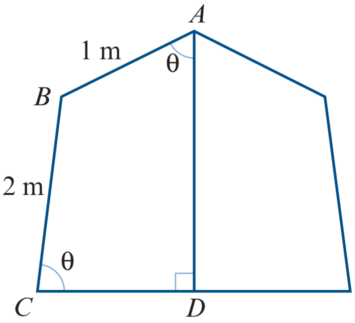
a Show that the perimeter, P cm, of the rectangle is given by $P = 20\cos\theta + 10\sin\theta$.

b Express P in the form $r\cos(\theta - \alpha)$ and hence determine the value of θ for which $P = 16$.

c Determine the value of k for which the area of the rectangle is $k\sin(2\theta)$ cm^2.

d Determine the value of θ for which the area is a maximum.

21 The diagram shows a vertical section through a tent in which $AB = 1$ m, $BC = 2$ m and $\angle BAD = \angle BCD = \theta$. The line CD is horizontal, and the diagram is symmetrical about the vertical AD.



a Obtain an expression for AD in terms of θ.

b Express AD in the form $r\cos(\theta - \alpha)$, where r is positive.

c State the maximum length of AD and the corresponding value of θ.

d Given that $AD = 2.15$ m, determine the value of θ for which $\theta > \alpha$.

22 **a** Prove the identity

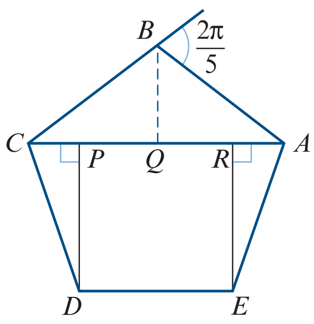
$$\cos(2\theta) = \frac{1 - \tan^2\theta}{1 + \tan^2\theta}$$

b **i** Use the result of **a** to show that $1 + x^2 = \sqrt{2}x^2 - \sqrt{2}$, where $x = \tan(67\frac{1}{2})^\circ$.

ii Hence determine the values of integers a and b such that $\tan(67\frac{1}{2})^\circ = a + b\sqrt{2}$.

c Determine the value of $\tan(7\frac{1}{2})^\circ$.

23 $ABCDE$ is a regular pentagon with side length one unit. The exterior angles of a regular pentagon each have magnitude $\frac{2\pi}{5}$.



a **i** Show that the magnitude of $\angle BCA$ is $\frac{\pi}{5}$.

ii Determine the length of CA.

b **i** Show the magnitude of $\angle DCP$ is $\frac{2\pi}{5}$.

ii Use the fact that $AC = 2CQ = 2CP + PR$ to show that $2\cos\left(\frac{\pi}{5}\right) = 2\cos\left(\frac{2\pi}{5}\right) + 1$.

iii Use the identity $\cos(2\theta) = 2\cos^2\theta - 1$ to form a quadratic equation in terms of $\cos\left(\frac{\pi}{5}\right)$.

iv Determine the exact value of $\cos\left(\frac{\pi}{5}\right)$.

Multiple-choice questions

1 $\operatorname{cosec} x - \sin x$ is equal to

A $\cos x \cot x$ **B** $\operatorname{cosec} x \tan x$

C $1 - \sin^2 x$ **D** $\sin x \operatorname{cosec} x$

Review

2 If $\cos x = \dfrac{-1}{3}$, then the possible values of $\sin x$ are

A $\dfrac{-2\sqrt{2}}{3}, \dfrac{2\sqrt{2}}{3}$ **B** $\dfrac{-2}{3}, \dfrac{2}{3}$ **C** $\dfrac{-8}{9}, \dfrac{8}{9}$ **D** $\dfrac{-\sqrt{2}}{3}, \dfrac{\sqrt{2}}{3}$

3 If $\cos\theta = \dfrac{a}{b}$ and $0 < \theta < \dfrac{\pi}{2}$, then $\tan\theta$ is equal to

A $\dfrac{\sqrt{a^2+b^2}}{b}$ **B** $\dfrac{\sqrt{b^2-a^2}}{a}$ **C** $\dfrac{a}{\sqrt{b^2-a^2}}$ **D** $\dfrac{a}{\sqrt{b^2+a^2}}$

4 $\sin(\alpha+\beta)\cos(\beta) - \cos(\alpha+\beta)\sin(\beta)$ is equal to

A $\sin(\alpha)\cos^2(\beta) - \sin(\alpha)\sin^2(\beta)$ **B** $\sin(\alpha)$

C $2\sin(\alpha)\cos(\alpha)\cos(\beta)$ **D** $2\sin(\beta)\cos(\beta)(\sin(\alpha) - \cos(\alpha))$

5 If $\dfrac{\pi}{2} < A < \pi$ and $\pi < B < \dfrac{3\pi}{2}$ with $\cos A = t$ and $\sin B = t$, then $\sin(B+A)$ equals

A 0 **B** 1 **C** $2t^2 - 1$ **D** $1 - 2t^2$

6 $\dfrac{\sin(2A)}{\cos(2A)-1}$ is equal to

A $\sin(2A) + \sec(2A)$ **B** $\dfrac{\sin A}{\cos A - 1}$

C $\sin(2A) - \tan(2A)$ **D** $-\cot A$

7 $(1+\cot x)^2 + (1-\cot x)^2$ is equal to

A 2 **B** $-4\cot x$ **C** $2 + \cot(2x)$ **D** $2\operatorname{cosec}^2 x$

8 If $\sin(2A) = m$ and $\cos A = n$, then $\tan A$ is equal to

A $\dfrac{m}{2n^2}$ **B** $\dfrac{n}{m}$ **C** $\dfrac{2n}{m^2}$ **D** $\dfrac{2n}{m}$

9 Expressing $-\cos x + \sin x$ in the form $r\sin(x+\alpha)$, where $r > 0$, gives

A $\sqrt{2}\sin\left(x + \dfrac{\pi}{4}\right)$ **B** $-\sin\left(x + \dfrac{\pi}{4}\right)$

C $\sqrt{2}\sin\left(x + \dfrac{5\pi}{4}\right)$ **D** $\sqrt{2}\sin\left(x + \dfrac{7\pi}{4}\right)$

10 The product $\sin 25^\circ \cos 75^\circ$ can be rewritten as

A $2(\sin 100^\circ + \sin 50^\circ)$ **B** $2(\sin 100^\circ - \sin 50^\circ)$

C $\dfrac{1}{2}(\sin 100^\circ + \sin 50^\circ)$ **D** $\dfrac{1}{2}(\sin 100^\circ - \sin 50^\circ)$

11 The greatest value of $5\cos\theta - 4\sin\theta$ is

A 1 **B** 9 **C** $\sqrt{41}$ **D** 41

12 Which expression is equivalent to $\dfrac{\tan 2x - \tan x}{1 + \tan 2x \tan x}$?

A $\tan x$ **B** $\sec^2 x$ **C** $\tan 4x$ **D** $\cot x$

12

Complex numbers

Chapter contents

In this chapter we introduce a new set of numbers, called *complex numbers*. These numbers first arose in the search for solutions to polynomial equations.

In the sixteenth century, mathematicians including Girolamo Cardano began to consider square roots of negative numbers. Although these numbers were regarded as 'impossible', they arose in calculations to determine real solutions of cubic equations.

For example, the cubic equation $x^3 - 15x - 4 = 0$ has three real solutions. Cardano's formula gives the solution

$$x = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}}$$

which you can show equals 4.

Today complex numbers are widely used in physics and engineering, such as in the study of aerodynamics.

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12A Building the complex numbers

Learning intentions

- To determine the **real part** and the **imaginary part** of a complex number.
- To be able to perform **addition** and **subtraction** of complex numbers.

Mathematicians in the eighteenth century introduced the imaginary number i with the property that

$$i^2 = -1$$

The equation $x^2 = -1$ has two solutions, namely i and $-i$.

By declaring that $i = \sqrt{-1}$, we can determine square roots of all negative numbers.

For example:

$$\begin{aligned}\sqrt{-4} &= \sqrt{4 \times (-1)}\\ &= \sqrt{4} \times \sqrt{-1}\\ &= 2i\end{aligned}$$

Note: The identity $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$ holds for positive real numbers a and b, but does not hold when both a and b are negative. In particular, $\sqrt{-1} \times \sqrt{-1} \neq \sqrt{(-1) \times (-1)}$.

Now consider the equation $x^2 + 2x + 3 = 0$. Using the quadratic formula gives

$$\begin{aligned}x &= \frac{-2 \pm \sqrt{4 - 12}}{2}\\ &= \frac{-2 \pm \sqrt{-8}}{2}\\ &= -1 \pm \sqrt{-2}\end{aligned}$$

This equation has no real solutions. However, using complex numbers we obtain solutions

$$x = -1 \pm \sqrt{2}i$$

The set of complex numbers

A **complex number** is an expression of the form $a + bi$, where a and b are real numbers.

The set of all complex numbers is denoted by $\mathbb{C}$. That is,

$$\mathbb{C} = \{ a + bi : a, b \in \mathbb{R} \}$$

The letter z is often used to denote a complex number.

Therefore if $z \in \mathbb{C}$, then $z = a + bi$ for some $a, b \in \mathbb{R}$.

- If $a = 0$, then $z = bi$ is said to be an **imaginary number**.
- If $b = 0$, then $z = a$ is a **real number**.

The real numbers and the imaginary numbers are subsets of $\mathbb{C}$.

We can now extend the diagram from Chapter 2 to include the complex numbers.

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Real and imaginary parts

For a complex number $z = a + bi$, we define

$$\operatorname{Re}(z) = a \quad \text{and} \quad \operatorname{Im}(z) = b$$

where $\operatorname{Re}(z)$ is called the **real part** of z and $\operatorname{Im}(z)$ is called the **imaginary part** of z.

For example, for the complex number $z = 2 + 5i$, we have $\operatorname{Re}(z) = 2$ and $\operatorname{Im}(z) = 5$.

Note: Both $\operatorname{Re}(z)$ and $\operatorname{Im}(z)$ are real numbers. That is, $\operatorname{Re}\colon \mathbb{C} \to \mathbb{R}$ and $\operatorname{Im}\colon \mathbb{C} \to \mathbb{R}$.

Equality of complex numbers

Two complex numbers are defined to be **equal** if both their real parts and their imaginary parts are equal:

$$a + bi = c + di \quad \text{if and only if} \quad a = c \text{ and } b = d$$

Example 1

If $4 - 3i = 2a + bi$, determine the real values of a and b.

Solution

$$2a = 4 \quad \text{and} \quad b = -3$$

$$\Rightarrow \quad a = 2 \quad \text{and} \quad b = -3$$

Example 2

Determine the real values of a and b such that $(2a + 3b) + (a - 2b)i = -1 + 3i$.

Solution

$$2a + 3b = -1 \qquad (1)$$

$$a - 2b = 3 \qquad (2)$$

Multiply (2) by 2:

$$2a - 4b = 6 \qquad (3)$$

Subtract (3) from (1):

$$7b = -7$$

Therefore $b = -1$ and $a = 1$.

Operations on complex numbers

Addition and subtraction

Addition of complex numbers

If $z_1 = a + bi$ and $z_2 = c + di$, then

$$z_1 + z_2 = (a + c) + (b + d)i$$

The **zero** of the complex numbers can be written as $0 = 0 + 0i$.

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If $z = a + bi$, then we define $-z = -a - bi$.

Subtraction of complex numbers

If $z_1 = a + bi$ and $z_2 = c + di$, then

$$z_1 - z_2 = z_1 + (-z_2) = (a - c) + (b - d)i$$

It is easy to check that the following familiar properties of the real numbers extend to the complex numbers:

- $z_1 + z_2 = z_2 + z_1$
- $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$
- $z + 0 = z$
- $z + (-z) = 0$

Example 3

If $z_1 = 2 - 3i$ and $z_2 = -4 + 5i$, determine:

a $z_1 + z_2$

b $z_1 - z_2$

Solution

a $z_1 + z_2 = (2 - 3i) + (-4 + 5i)$
$= (2 + (-4)) + (-3 + 5)i$
$= -2 + 2i$

b $z_1 - z_2 = (2 - 3i) - (-4 + 5i)$
$= (2 - (-4)) + (-3 - 5)i$
$= 6 - 8i$

Multiplication by a real constant

If $z = a + bi$ and $k \in \mathbb{R}$, then

$$kz = k(a + bi) = ka + kbi$$

For example, if $z = 3 - 6i$, then $3z = 9 - 18i$.

Powers of *i*

Successive multiplication by i gives the following:

- $i^0 = 1$
- $i^1 = i$
- $i^2 = -1$
- $i^3 = -i$
- $i^4 = (-1)^2 = 1$
- $i^5 = i$
- $i^6 = -1$
- $i^7 = -i$

In general, for $n = 0, 1, 2, 3, \ldots$

- $i^{4n} = 1$
- $i^{4n+1} = i$
- $i^{4n+2} = -1$
- $i^{4n+3} = -i$

Example 4

Simplify:

a i^{13}

b $3i^4 \times (-2i)^3$

Solution

a $i^{13} = i^{4\times3+1}$
$= i$

b $3i^4 \times (-2i)^3 = 3 \times (-2)^3 \times i^4 \times i^3$
$= -24i^7$
$= 24i$

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Summary 12A

- The imaginary number i satisfies $i^2 = -1$.
- If a is a positive real number, then $\sqrt{-a} = \sqrt{a} \cdot i$.
- The set of **complex numbers** is $\mathbb{C} = \{ a + bi : a, b \in \mathbb{R} \}$.
- For a complex number $z = a + bi$:
  - the **real part** of z is $\text{Re}(z) = a$
  - the **imaginary part** of z is $\text{Im}(z) = b$.
- Equality of complex numbers:

$$a + bi = c + di \quad \text{if and only if} \quad a = c \text{ and } b = d$$

- If $z_1 = a + bi$ and $z_2 = c + di$, then

$$z_1 + z_2 = (a + c) + (b + d)i \quad \text{and} \quad z_1 - z_2 = (a - c) + (b - d)i$$

- When simplifying powers of i, remember that $i^4 = 1$.

Exercise 12A

SF

1 State the values of $\text{Re}(z)$ and $\text{Im}(z)$ for each of the following:

a $z = 2 + 3i$ **b** $z = 4 + 5i$ **c** $z = \frac{1}{2} - \frac{3}{2}i$

d $z = -4$ **e** $z = 3i$ **f** $z = \sqrt{2} - 2\sqrt{2}i$

Example 1, 2

2 Determine the real values of a and b in each of the following:

a $2a - 3bi = 4 + 6i$ **b** $a + b - 2abi = 5 - 12i$

c $2a + bi = 10$ **d** $3a + (a - b)i = 2 + i$

Example 3

3 Simplify:

a $(2 - 3i) + (4 - 5i)$ **b** $(4 + i) + (2 - 2i)$

c $(-3 - i) - (3 + i)$ **d** $(2 - \sqrt{2}i) + (5 - \sqrt{8}i)$

e $(1 - i) - (2i + 3)$ **f** $(2 + i) - (-2 - i)$

g $4(2 - 3i) - (2 - 8i)$ **h** $-(5 - 4i) + (1 + 2i)$

i $5(i + 4) + 3(2i - 7)$ **j** $\frac{1}{2}(4 - 3i) - \frac{3}{2}(2 - i)$

Example 4

4 Simplify:

a $\sqrt{-16}$ **b** $2\sqrt{-9}$ **c** $\sqrt{-2}$

d i^3 **e** i^{14} **f** i^{20}

g $-2i \times i^3$ **h** $4i^4 \times 3i^2$ **i** $\sqrt{8}i^5 \times \sqrt{-2}$

5 Simplify:

a $i(2 - i)$ **b** $i^2(3 - 4i)$

c $\sqrt{2}i(i - \sqrt{2})$ **d** $-\sqrt{3}(\sqrt{-3} + \sqrt{2})$

CF

6 Evaluate $i + i^2 + i^3 + \cdots + i^{99}$.

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12B Multiplication and division of complex numbers

Learning intentions

- To be able to perform **multiplication** and **division** of complex numbers.
- To be able to determine the conjugate and modulus of a complex number.

In the previous section, we defined addition and subtraction of complex numbers. We begin this section by defining multiplication.

Multiplication of complex numbers

Let $z_1 = a + bi$ and $z_2 = c + di$ (where $a, b, c, d \in \mathbb{R}$). Then

$$\begin{aligned} z_1 \times z_2 &= (a + bi)(c + di) \\ &= ac + adi + bci + bdi^2 \\ &= (ac - bd) + (ad + bc)i \qquad (\text{since } i^2 = -1) \end{aligned}$$

We carried out this calculation with an assumption that we are in a system where all the usual rules of algebra apply. However, it should be understood that the following is a *definition* of multiplication for $\mathbb{C}$.

Multiplication of complex numbers

Let $z_1 = a + bi$ and $z_2 = c + di$. Then

$$z_1 \times z_2 = (ac - bd) + (ad + bc)i$$

The multiplicative identity for $\mathbb{C}$ is $1 = 1 + 0i$.

It is easy to check that the following familiar properties of the real numbers extend to the complex numbers:

- $z_1 z_2 = z_2 z_1$
- $z \times 1 = z$
- $(z_1 z_2) z_3 = z_1 (z_2 z_3)$
- $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$

Example 5

If $w = 3 - 2i$ and $z = 1 + i$, determine wz.

Solution

$$\begin{aligned} wz &= (3 - 2i)(1 + i) \\ &= 3 + 3i - 2i - 2i^2 \\ &= 5 + i \end{aligned}$$

Explanation

Expand the brackets in the usual way.

Remember that $i^2 = -1$.

The conjugate of a complex number

Let $z = a + bi$. The **conjugate** of z is denoted by $\bar{z}$ and is given by

$$\bar{z} = a - bi, \quad a, b \in \mathbb{R}$$

For example, the conjugate of $-4 + 3i$ is $-4 - 3i$, and vice versa.

Example 6

If $w = 2 - 3i$ and $z = -1 + 2i$, determine:

a $\overline{w + z}$ and $\overline{w} + \overline{z}$

b $\overline{w \cdot z}$ and $\overline{w} \cdot \overline{z}$

Solution

We have $\overline{w} = 2 + 3i$ and $\overline{z} = -1 - 2i$.

a

$$\begin{aligned} w + z &= (2 - 3i) + (-1 + 2i) \\ &= 1 - i \end{aligned}$$

$$\overline{w + z} = 1 + i$$

$$\begin{aligned} \overline{w} + \overline{z} &= (2 + 3i) + (-1 - 2i) \\ &= 1 + i \end{aligned}$$

b

$$\begin{aligned} w \cdot z &= (2 - 3i)(-1 + 2i) \\ &= -2 + 4i + 3i - 6i^2 \\ &= 4 + 7i \end{aligned}$$

$$\overline{w \cdot z} = 4 - 7i$$

$$\begin{aligned} \overline{w} \cdot \overline{z} &= (2 + 3i)(-1 - 2i) \\ &= -2 - 4i - 3i - 6i^2 \\ &= 4 - 7i \end{aligned}$$

- The conjugate of a sum is equal to the sum of the conjugates:

$$\overline{w + z} = \overline{w} + \overline{z}$$

- The conjugate of a product is equal to the product of the conjugates:

$$\overline{w \cdot z} = \overline{w} \cdot \overline{z}$$

The modulus of a complex number

For a complex number $z = a + bi$, we have

$$\begin{aligned} z\overline{z} &= (a + bi)(a - bi) \\ &= a^2 - abi + abi - b^2i^2 \\ &= a^2 + b^2 \qquad \text{where } a^2 + b^2 \text{ is a real number} \end{aligned}$$

The **modulus** of the complex number $z = a + bi$ is denoted by $|z|$ and is given by

$$|z| = \sqrt{a^2 + b^2}$$

The calculation above shows that

$$z\overline{z} = |z|^2$$

Note: In the case that z is a real number, this definition of $|z|$ agrees with the definition of the modulus of a real number given in Chapter 13.

In Section 12F, we will see the geometric interpretation of the modulus of a complex number.

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Example 7

If $w = 2 + 3i$ and $z = 1 - 2i$, determine:

a $|wz|$

b $|w|\,|z|$

Solution

a We first determine that

$$\begin{aligned} wz &= (2 + 3i)(1 - 2i) \\ &= 2 - 4i + 3i - 6i^2 \\ &= 8 - i \end{aligned}$$

Therefore

$$|wz| = \sqrt{8^2 + (-1)^2} = \sqrt{65}$$

b We first determine that

$$|w| = \sqrt{2^2 + 3^2} = \sqrt{13}$$

$$|z| = \sqrt{1^2 + (-2)^2} = \sqrt{5}$$

Therefore

$$|w|\,|z| = \sqrt{13}\sqrt{5} = \sqrt{65}$$

The modulus of a product is equal to the product of the moduli:

$$|wz| = |w|\,|z|$$

Division of complex numbers

Multiplicative inverse

We begin with some familiar algebra that will motivate the definition:

$$\begin{aligned} \frac{1}{a + bi} &= \frac{1}{a + bi} \times \frac{a - bi}{a - bi} \\ &= \frac{a - bi}{(a + bi)(a - bi)} \\ &= \frac{a - bi}{a^2 + b^2} \end{aligned}$$

We can see that

$$(a + bi) \times \frac{a - bi}{a^2 + b^2} = 1$$

Although we have carried out this arithmetic, we have not yet defined what $\dfrac{1}{a + bi}$ means.

Multiplicative inverse of a complex number

If $z = a + bi$ with $z \neq 0$ and $a, b \in \mathbb{R}$, then

$$z^{-1} = \frac{a - bi}{a^2 + b^2} = \frac{\bar{z}}{|z|^2}$$

Note: We can check that $(wz)^{-1} = w^{-1}z^{-1}$.

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Division

The formal definition of division in the complex numbers is via the multiplicative inverse:

Division of complex numbers

$$\frac{z_1}{z_2} = z_1 z_2^{-1} = \frac{z_1 \overline{z_2}}{|z_2|^2} \qquad (\text{for } z_2 \neq 0)$$

Here is the procedure that is used in practice:

Assume that $z_1 = a + bi$ and $z_2 = c + di$ (where $a, b, c, d \in \mathbb{R}$). Then

$$\frac{z_1}{z_2} = \frac{a + bi}{c + di}$$

Multiply the numerator and denominator by the conjugate of z_2:

$$\frac{z_1}{z_2} = \frac{a + bi}{c + di} \times \frac{c - di}{c - di}$$
$$= \frac{(a + bi)(c - di)}{c^2 + d^2}$$

We complete the division by simplifying.

This procedure is demonstrated in the next example.

Example 8

a Determine $\dfrac{2 - i}{3 + 2i}$.

b Determine z if $(2 + 3i)z = -1 - 2i$.

Solution

a
$$\frac{2 - i}{3 + 2i} = \frac{2 - i}{3 + 2i} \times \frac{3 - 2i}{3 - 2i}$$
$$= \frac{6 - 4i - 3i + 2i^2}{3^2 + 2^2}$$
$$= \frac{4 - 7i}{13}$$
$$= \frac{1}{13}(4 - 7i)$$

b $(2 + 3i)z = -1 - 2i$
$$\Rightarrow \quad z = \frac{-1 - 2i}{2 + 3i}$$
$$= \frac{-1 - 2i}{2 + 3i} \times \frac{2 - 3i}{2 - 3i}$$
$$= \frac{-2 + 3i - 4i + 6i^2}{2^2 + 3^2}$$
$$= \frac{-8 - i}{13}$$
$$= -\frac{1}{13}(8 + i)$$

There is an obvious similarity between the process for expressing a complex number with a real denominator and the process for rationalising the denominator of a surd expression.

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Example 9

If $z = 2 - 5i$, determine z^{-1} and express with a real denominator.

Solution

$$\begin{aligned} z^{-1} = \frac{1}{z} &= \frac{1}{2-5i} \\ &= \frac{1}{2-5i} \times \frac{2+5i}{2+5i} \\ &= \frac{2+5i}{29} \\ &= \frac{1}{29}(2+5i) \end{aligned}$$

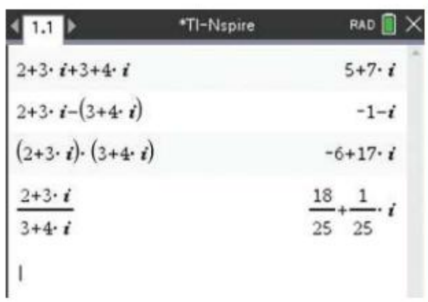
Using the TI-Nspire CX non-CAS

Set to complex mode using (on) > **Settings** > **Document Settings**. Select **Rectangular** from the **Real or Complex** field.

Note: The square root of a negative number can be found only in complex mode. But most computations with complex numbers can also be performed in real mode.

Arithmetic operations

The results of the arithmetic operations +, −, × and ÷ are illustrated using the two complex numbers $2 + 3i$ and $3 + 4i$.

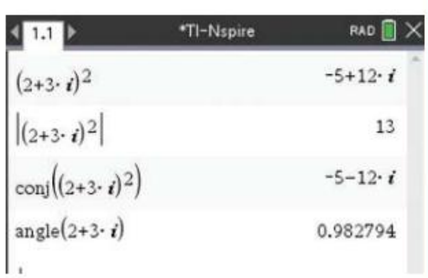


Note: Do not use the text i for the imaginary constant. The symbol i is found using (π▸) or the Symbols palette ((ctrl) (📖)).

Complex number tools

Special operations on complex numbers can be accessed using (menu) > **Number** > **Complex Number Tools**:

- Select **Real Part** to determine the real part.
- Select **Magnitude** to determine the modulus.
- Select **Complex Conjugate** to determine the conjugate.
- Select **Polar Angle** to determine the argument. (The argument of a complex number is introduced in Section 12F.)



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Using the Casio

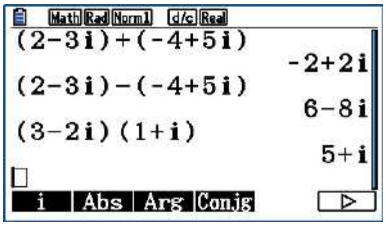
Calculations with complex numbers can be performed in **Run-Matrix** mode using the **Complex numbers** menu (OPTN) (F3).

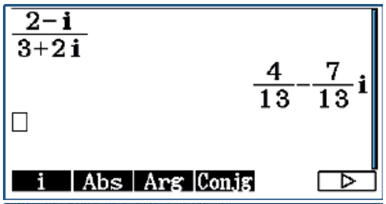
Arithmetic operations

To perform arithmetic calculations with complex numbers:

- Go to the **Complex numbers** menu (OPTN) (F3).
- Enter the complex numbers as shown. For the symbol i, select **i** (F1).

Note: You can also enter i by pressing (SHIFT) (0).

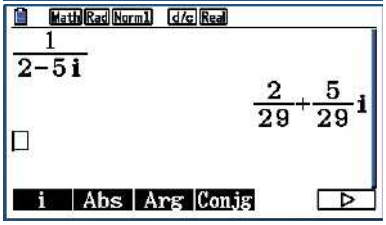




To determine the multiplicative inverse of a complex number:

- Go to the **Complex numbers** menu (OPTN) (F3).
- Enter the reciprocal number as shown.

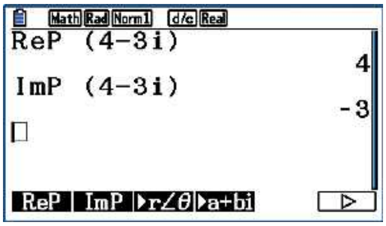
Note: Entering $(2 - 5i)^{-1}$ will give a decimal approximation only.



Complex number operations

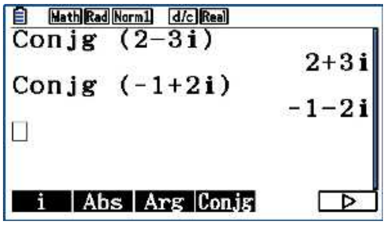
To determine the real and imaginary parts of a complex number:

- Go to the **Complex numbers** menu (OPTN) (F3).
- For the real part, select **ReP** (F6) (F1) and enter the complex number as shown. (For the symbol i, select **i** (F6) (F1).)
- For the imaginary part, use **ImP** (F6) (F2).



To determine the conjugate of a complex number:

- Go to the **Complex numbers** menu (OPTN) (F3).
- Select **Conjg** (F4) and enter the complex number as shown.



Summary 12B

- **Multiplication** To determine a product $(a + bi)(c + di)$, expand the brackets in the usual way, remembering that $i^2 = -1$.
- **Conjugate** If $z = a + bi$, then $\overline{z} = a - bi$.
- **Modulus** If $z = a + bi$, then $|z| = \sqrt{a^2 + b^2}$.
- **Division** To perform a division, start with

$$\frac{a+bi}{c+di} = \frac{a+bi}{c+di} \times \frac{c-di}{c-di} = \frac{(a+bi)(c-di)}{c^2+d^2}$$

and then simplify.
- **Multiplicative inverse** To determine z^{-1}, calculate $\frac{1}{z}$.

Exercise 12B

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Example 5

1 Expand and simplify:

a $(4 + i)^2$ **b** $(2 - 2i)^2$ **c** $(3 + 2i)(2 + 4i)$
d $(-1 - i)^2$ **e** $(\sqrt{2} - \sqrt{3}i)(\sqrt{2} + \sqrt{3}i)$ **f** $(5 - 2i)(-2 + 3i)$

2 a If $w = 3 + 2i$ and $z = 2 + 4i$, determine $\text{Re}(wz)$.
b If $w = 4 + 5i$ and $z = 3 - 2i$, determine $\text{Im}(wz)$.

3 Write down the conjugate of each of the following complex numbers:

a $2 - 5i$ **b** $-1 + 3i$ **c** $\sqrt{5} - 2i$ **d** $-5i$

4 Evaluate $z \cdot \overline{z}$ for each of the following:

a $z = 3 + 4i$ **b** $z = 1 + i$ **c** $z = 2 - 3i$ **d** $z = \sqrt{2} + \sqrt{3}i$

Example 6

5 If $z_1 = 2 - i$ and $z_2 = -3 + 2i$, determine:

a $\overline{z_1}$ **b** $\overline{z_2}$ **c** $z_1 \cdot z_2$ **d** $\overline{z_1 \cdot z_2}$
e $\overline{z_1} \cdot \overline{z_2}$ **f** $z_1 + z_2$ **g** $\overline{z_1 + z_2}$ **h** $\overline{z_1} + \overline{z_2}$

Example 7

6 If $w = 1 + i$ and $z = 3 - 4i$, determine:

a $|wz|$ **b** $|w|\,|z|$ **c** $|w + z|$ **d** $|3w - 2z|$

7 If $z = 2 - 4i$, express each of the following in the form $x + yi$:

a $\overline{z}$ **b** $z\overline{z}$ **c** $z + \overline{z}$ **d** $z(z + \overline{z})$

Example 9

e $z - \overline{z}$ **f** $i(z - \overline{z})$ **g** z^{-1} **h** $\frac{z}{i}$

8 Determine the real values of a and b such that $(a + bi)(2 + 5i) = 3 - i$.

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Example 8a

9 Determine each of the following, expressing your answer in the form $x + yi$:

a $\frac{2-i}{4+i}$ **b** $\frac{3+2i}{2-3i}$ **c** $\frac{4+3i}{1+i}$ **d** $\frac{2-2i}{4i}$ **e** $\frac{1}{2-3i}$ **f** $\frac{i}{2+6i}$

10 Determine the real values of a and b if $(3-i)(a+bi) = 6-7i$.

Example 8b

11 Solve each of the following for z:

a $(2-i)z = 42i$ **b** $(1+3i)z = -2-i$ **c** $(3i+5)z = 1+i$

d $2(4-7i)z = 5+2i$ **e** $z(1+i) = 4$

SF

12 If $a, b \in \mathbb{R}$ and $(a+bi)^2 = -5+12i$, determine a and b.

13 If $a \in \mathbb{R}$ and $\frac{1}{a+3i} + \frac{1}{a-3i} = \frac{4}{13}$, determine a.

14 Let $z = a+bi$ be a complex number.

a If $z = \bar{z}$, prove that z is a real number. That is, prove that $\text{Im}(z) = 0$.

b Prove that $z + \bar{z}$ is a real number.

c For $z \neq 0$, prove that $\frac{1}{z} + \frac{1}{\bar{z}}$ is a real number.

15 Let $z = \frac{a+bi}{a-bi}$, where $a, b \in \mathbb{R}$. Prove that $\frac{z^2+1}{2z}$ is a real number.

CU

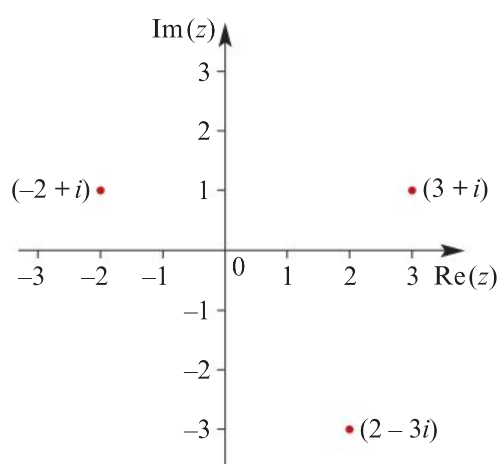
12C Argand diagrams

Learning intentions

- To be able to represent complex numbers on an Argand diagram.

An **Argand diagram** is a geometric representation of the set of complex numbers. A complex number has two dimensions: the real part and the imaginary part. Therefore a plane is required to represent $\mathbb{C}$.

An Argand diagram is drawn with two perpendicular axes. The horizontal axis represents $\text{Re}(z)$, for $z \in \mathbb{C}$, and the vertical axis represents $\text{Im}(z)$, for $z \in \mathbb{C}$.



Each point on an Argand diagram represents a complex number. The complex number $a + bi$ is situated at the point (a, b) on the equivalent Cartesian axes, as shown by the examples in this figure.

A complex number written as $a + bi$ is said to be in **Cartesian form**.

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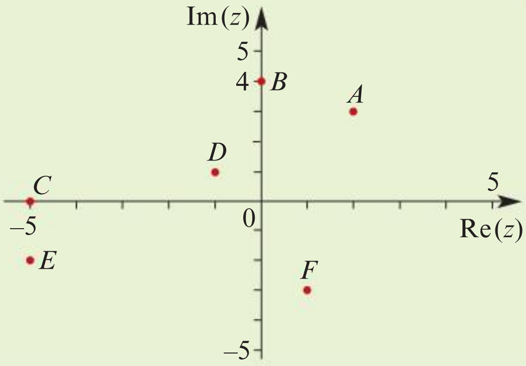
Example 10

Write down the complex number represented by each of the points shown on this Argand diagram.



Solution

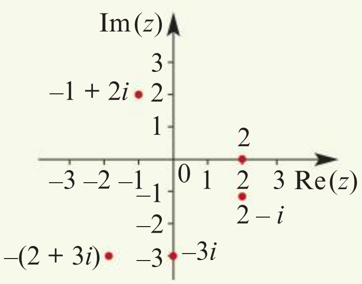
A $2 + 3i$ **B** $4i$ **C** -5 **D** $-1 + i$ **E** $-5 - 2i$ **F** $1 - 3i$

Example 11

Represent the following complex numbers as points on an Argand diagram:

a 2 **b** $-3i$ **c** $2 - i$ **d** $-(2 + 3i)$ **e** $-1 + 2i$

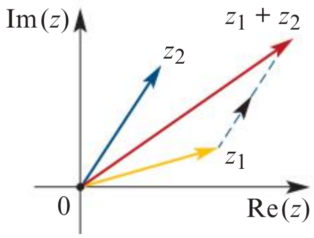
Solution

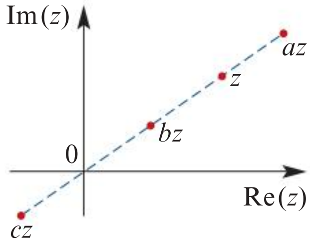


Geometric representation of the basic operations on complex numbers

In an Argand diagram, the sum of two complex numbers z_1 and z_2 can be found geometrically by placing the 'tail' of z_2 on the 'tip' of z_1, as shown in the diagram on the left.

When a complex number is multiplied by a real constant, it maintains the same 'direction', but its distance from the origin is scaled. This is shown in the diagram on the right.





$a > 1$

$0 < b < 1$

$c < 0$

The difference $z_1 - z_2$ is represented by the sum $z_1 + (-z_2)$.

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Example 12

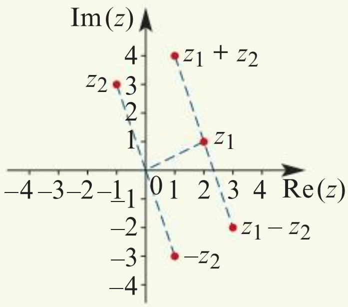
Let $z_1 = 2 + i$ and $z_2 = -1 + 3i$.

Represent the complex numbers z_1, z_2, $z_1 + z_2$ and $z_1 - z_2$ on an Argand diagram and show the geometric interpretation of the sum and difference.

Solution

$$z_1 + z_2 = (2 + i) + (-1 + 3i)$$
$$= 1 + 4i$$

$$z_1 - z_2 = (2 + i) - (-1 + 3i)$$
$$= 3 - 2i$$

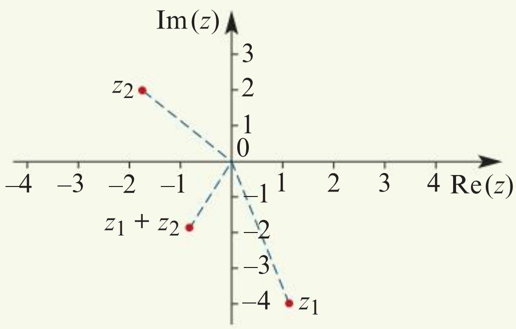


Example 13

Let $z_1 = 1 - 4i$ and $z_2 = -2 + 2i$. Determine $z_1 + z_2$ algebraically and illustrate $z_1 + z_2$ on an Argand diagram.

Solution

$$z_1 + z_2 = (1 - 4i) + (-2 + 2i)$$
$$= -1 - 2i$$

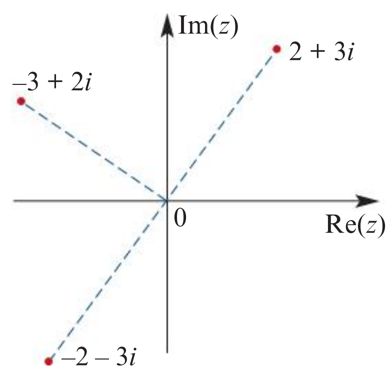


Rotation about the origin

When the complex number $2 + 3i$ is multiplied by -1, the result is $-2 - 3i$. This is achieved through a rotation of 180° about the origin.

When the complex number $2 + 3i$ is multiplied by i, we obtain

$$i(2 + 3i) = 2i + 3i^2$$
$$= 2i - 3$$
$$= -3 + 2i$$



The result is achieved through a rotation of 90° anticlockwise about the origin.

If $-3 + 2i$ is multiplied by i, the result is $-2 - 3i$. This is again achieved through a rotation of 90° anticlockwise about the origin.

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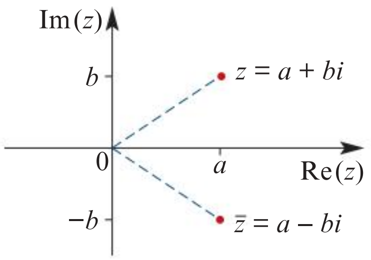
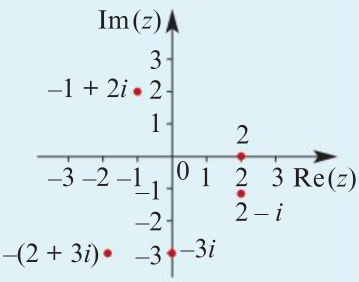
Reflection in the horizontal axis

The conjugate of a complex number $z = a + bi$ is $\bar{z} = a - bi$. Therefore $\bar{z}$ is the reflection of z in the horizontal axis of an Argand diagram.



Summary 12C

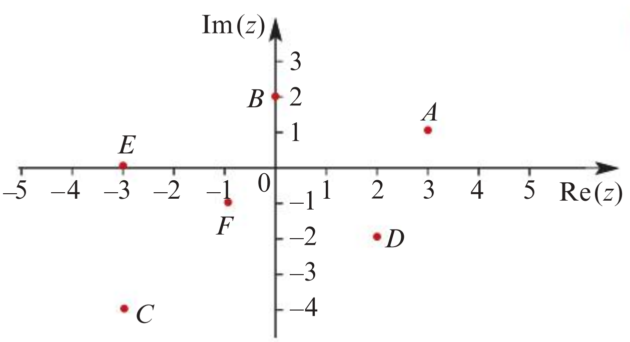
- An **Argand diagram** is a geometric representation of the set of complex numbers.
- The horizontal axis represents Re(z) and the vertical axis represents Im(z), for $z \in \mathbb{C}$.
- The operations of addition, subtraction and multiplication by a real constant all have geometric interpretations on an Argand diagram.
- Multiplication of a complex number by i corresponds to a rotation of 90° anticlockwise about the origin.
- Complex conjugate corresponds to reflection in the horizontal axis.



Exercise 12C

Example 10

1 Write down the complex numbers represented on this Argand diagram.



Example 11

2 Represent each of the following complex numbers as points on an Argand diagram:

a $3 - 4i$ **b** $-4 + i$ **c** $4 + i$ **d** -3 **e** $-2i$ **f** $-5 - 2i$

Example 12, 13

3 If $z_1 = 6 - 5i$ and $z_2 = -3 + 4i$, represent each of the following on an Argand diagram:

a $z_1 + z_2$ **b** $z_1 - z_2$

4 If $z = 1 + 3i$, represent each of the following on an Argand diagram:

a z **b** $\bar{z}$ **c** z^2 **d** $-z$ **e** $\dfrac{1}{z}$

5 If $z = 2 - 5i$, represent each of the following on an Argand diagram:

a z **b** zi **c** zi^2 **d** zi^3 **e** zi^4

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12D Solving quadratic equations over the complex numbers

Learning intentions

- To be able to solve a quadratic equation over the complex numbers.

Quadratic equations with a negative discriminant have no real solutions. The introduction of complex numbers enables us to solve such quadratic equations.

Sum of two squares

Since $i^2 = -1$, we can rewrite a sum of two squares as a difference of two squares:

$$\begin{aligned} z^2 + a^2 &= z^2 - (ai)^2 \\ &= (z + ai)(z - ai) \end{aligned}$$

This allows us to solve equations of the form $z^2 + a^2 = 0$.

Example 14

Solve the following equations over $\mathbb{C}$:

a $z^2 + 16 = 0$ **b** $2z^2 + 6 = 0$

Solution

a

$$\begin{aligned} z^2 + 16 &= 0 \\ z^2 - 16i^2 &= 0 \\ (z + 4i)(z - 4i) &= 0 \\ \therefore \quad z &= \pm 4i \end{aligned}$$

b

$$\begin{aligned} 2z^2 + 6 &= 0 \\ z^2 + 3 &= 0 \\ z^2 - 3i^2 &= 0 \\ (z + \sqrt{3}i)(z - \sqrt{3}i) &= 0 \\ \therefore \quad z &= \pm\sqrt{3}i \end{aligned}$$

Solution of quadratic equations

To solve a quadratic equation with a negative discriminant, we can either complete the square or use the quadratic formula.

Example 15

a Solve $z^2 + 6z + 11 = 0$ over $\mathbb{C}$ by completing the square.

b Solve $3z^2 + 5z + 3 = 0$ over $\mathbb{C}$ by using the quadratic formula.

Solution

a

$$\begin{aligned} z^2 + 6z + 11 &= 0 \\ (z^2 + 6z + 9) - 9 + 11 &= 0 \\ (z + 3)^2 + 2 &= 0 \\ (z + 3)^2 - 2i^2 &= 0 \\ (z + 3 + \sqrt{2}i)(z + 3 - \sqrt{2}i) &= 0 \\ \therefore \quad z &= -3 \pm \sqrt{2}i \end{aligned}$$

b Using the quadratic formula:

$$\begin{aligned} z &= \frac{-5 \pm \sqrt{25 - 36}}{6} \\ &= \frac{-5 \pm \sqrt{-11}}{6} \\ &= \frac{1}{6}(-5 \pm \sqrt{11}i) \end{aligned}$$

Example 16

Let $b, c \in \mathbb{R}$. If the quadratic equation $z^2 + bz + c = 0$ has solutions $z = 2 - 3i$ and $z = 2 + 3i$, determine the values of b and c.

Solution

The quadratic has factors $z - 2 + 3i$ and $z - 2 - 3i$. Multiplying them together gives

$$\begin{aligned}(z - 2 + 3i)(z - 2 - 3i) &= ((z - 2) + 3i)((z - 2) - 3i)\\ &= (z - 2)^2 - (3i)^2\\ &= z^2 - 4z + 4 + 9\\ &= z^2 - 4z + 13\end{aligned}$$

Therefore $b = -4$ and $c = 13$.

Summary 12D

- Quadratic equations can be solved over the complex numbers by completing the square or by using the quadratic formula.
- Two properties of complex numbers that are useful when solving equations:
  - $z^2 + a^2 = z^2 - (ai)^2 = (z + ai)(z - ai)$
  - $\sqrt{-a} = \sqrt{a} \cdot i$, where a is a positive real number.

Skill-sheet

Exercise 12D

SF

Example 14

1 Solve the following equations over $\mathbb{C}$:

a $z^2 + 1 = 0$ **b** $z^2 + 9 = 0$ **c** $z^2 = -16$
d $4z^2 = -25$ **e** $z^2 = -2$ **f** $2z^2 + 8 = 0$
g $3z^2 + 75 = 0$ **h** $4z^2 + 1 = 0$ **i** $16z^2 + 9 = 0$
j $z^2 + 3 = 0$ **k** $2z^2 + 10 = 0$ **l** $(z + 1)^2 + 1 = 0$
m $(z - 2)^2 + 5 = 0$ **n** $(z + 3)^2 + 3 = 0$ **o** $(z - 2)^2 = -4$

Example 15a

2 Solve the following quadratic equations over $\mathbb{C}$ by completing the square:

a $z^2 + 2z + 3 = 0$ **b** $z^2 - 4z + 5 = 0$
c $z^2 + 6z + 12 = 0$ **d** $2z^2 - 8z + 10 = 0$
e $3z^2 + 2z + 1 = 0$ **f** $2z^2 + 2z + 1 = 0$

Example 15b

3 Solve the following quadratic equations over $\mathbb{C}$ by using the quadratic formula:

a $z^2 + 3z + 3 = 0$ **b** $z^2 - 4z + 5 = 0$
c $z^2 + 6z + 12 = 0$ **d** $z^2 - 4z + 8 = 0$
e $3z^2 + 2z + 1 = 0$ **f** $2z^2 - \sqrt{2}z + 1 = 0$

Example 16

4 Determine the values of $b, c \in \mathbb{R}$ if the quadratic equation $z^2 + bz + c = 0$ has solutions:

a $z = 1 + i$ and $z = 1 - i$ **b** $z = -2 - 5i$ and $z = -2 + 5i$

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SF

5 Solve the following equations over $\mathbb{C}$ using any method:

a $z^2 + 4 = 0$
b $2z^2 + 18 = 0$
c $3z^2 = -15$
d $(z - 2)^2 + 16 = 0$
e $(z + 1)^2 = -49$
f $z^2 - 2z + 3 = 0$
g $z^2 + 3z + 3 = 0$
h $2z^2 + 5z + 4 = 0$
i $3z^2 = z - 2$
j $2z = z^2 + 5$
k $2z^2 - 6z = -10$
l $z^2 - 6z = -14$

CU

6 Consider the quadratic equation $az^2 + bz + c = 0$, where a, b and c are consecutive positive integers. Show that the solutions of this equation are not real numbers.

7 Let z_1 and z_2 be the solutions of $2z^2 + 2z + c = 0$ where $c > \frac{1}{2}$ is a real number. Determine c given that z_1, z_2 and 0 are the vertices of an equilateral triangle on an Argand diagram.

12E Solving polynomial equations over the complex numbers (Optional)

Learning intentions

- ▶ To be able to solve suitable polynomial equations over the complex numbers..

Note: The material in this section goes beyond what is expected in Specialist Mathematics Units 1 & 2. However, early exposure might help your studies for next year.

In Mathematical Methods Units 1 & 2, you have seen the correspondence between the linear factors of a polynomial $P(x)$ and the solutions of the equation $P(x) = 0$. This correspondence extends to the complex numbers.

Factor theorem

Let $\alpha \in \mathbb{C}$. Then $z - \alpha$ is a factor of a polynomial $P(z)$ if and only if $P(\alpha) = 0$.

Every quadratic equation has two solutions over the complex numbers, if we count repeated solutions twice. For example, the equation $(z - 3)^2 = 0$ has a repeated solution $z = 3$. We say that this solution has a **multiplicity** of 2.

Likewise, every cubic equation has three solutions over the complex numbers, counting multiplicity. More generally, we have the following important theorem.

Fundamental theorem of algebra

For $n \geq 1$, every polynomial of degree n can be expressed as a product of n linear factors over the complex numbers. Therefore every polynomial equation of degree n has n solutions (counting multiplicity).

Note: This theorem applies to polynomials with real or complex coefficients. The proof of the theorem is surprisingly difficult and beyond the scope of this book.

Example 17

Show that $z = 1$ is a solution of $z^3 + z^2 + 3z - 5 = 0$, and then determine the other two solutions.

Solution

Let $P(z) = z^3 + z^2 + 3z - 5$. Since $P(1) = 1^3 + 1^2 + 3 - 5 = 0$ we see that $z = 1$ is a solution of $P(z) = 0$. Therefore $z - 1$ is a factor of $P(z)$.

By inspection (or polynomial division), we can determine the other factors:

$$\begin{aligned} z^3 + z^2 + 3z - 5 &= (z-1)(z^2 + 2z + 5) \\ &= (z-1)((z^2 + 2z + 1) - 1 + 5) \\ &= (z-1)((z+1)^2 + 4) \\ &= (z-1)((z+1)^2 - (2i)^2) \\ &= (z-1)(z+1+2i)(z+1-2i) \end{aligned}$$

Therefore the remaining two solutions are $z = -1 - 2i$ and $z = -1 + 2i$.

Notice in the previous example that the solutions $z = -1 - 2i$ and $z = -1 + 2i$ are conjugates of each other. This is not a coincidence.

Conjugate root theorem

Let $P(z)$ be a polynomial with real coefficients. If $a + bi$ is a solution of the equation $P(z) = 0$, with a and b real numbers, then the complex conjugate $a - bi$ is also a solution.

You will prove this theorem in Exercise 12E. Note that the theorem does not hold without the assumption that $P(z)$ has real coefficients. For example, the linear equation $z - i = 0$ has just one solution, $z = i$, and its conjugate is clearly not a solution.

Example 18

Let $P(z) = z^3 + 4z^2 + 6z + 4$. Given that $z = -1 + i$ is a solution of the equation $P(z) = 0$, determine all three solutions.

Solution

Note that the polynomial $P(z)$ has real coefficients. Since $-1 + i$ is a solution of $P(z) = 0$, its conjugate $-1 - i$ is also a solution.

We now have two monic linear factors $z + 1 - i$ and $z + 1 + i$ of $P(z)$. Their product is also a factor:

$$\begin{aligned} (z+1-i)(z+1+i) &= ((z+1) - i)((z+1) + i) \\ &= (z+1)^2 - i^2 \\ &= z^2 + 2z + 1 + 1 \\ &= z^2 + 2z + 2 \end{aligned}$$

The remaining factor can be found by inspection or polynomial division. This gives

$$P(z) = z^3 + 4z^2 + 6z + 4 = (z+2)(z^2 + 2z + 2)$$

Therefore the three solutions are $z = -2$, $z = -1 + i$ and $z = -1 - i$.

Summary 12E

The following three theorems are useful when solving polynomial equations over $\mathbb{C}$:

- **Factor theorem**
  Let $\alpha \in \mathbb{C}$. Then $z - \alpha$ is a factor of a polynomial $P(z)$ if and only if $P(\alpha) = 0$.
- **Fundamental theorem of algebra**
  For $n \geq 1$, every polynomial of degree n can be expressed as a product of n linear factors over the complex numbers. Therefore every polynomial equation of degree n has n solutions (counting multiplicity).
- **Conjugate root theorem**
  Let $P(z)$ be a polynomial with real coefficients. If $a + bi$ is a solution of $P(z) = 0$, with a and b real numbers, then the complex conjugate $a - bi$ is also a solution.

Exercise 12E

SF

Example 17

1 Show that $z = 2$ is a solution of the cubic equation $z^3 + 2z^2 - 3z - 10 = 0$, and then determine the other two solutions.

2 Determine a real solution of $z^3 + 3z^2 + 4z + 2 = 0$, and then determine the other two solutions.

Example 18

3 Given that $z = 3 - 2i$ is a solution of $z^3 - 9z^2 + 31z - 39 = 0$, determine all three solutions.

4 Given that $z = 1 - \sqrt{2}i$ is a solution of $z^3 - 4z^2 + 7z - 6 = 0$, determine all three solutions.

5 Show that $z = 2i$ is a solution of the cubic equation $z^3 - 3z^2 + 4z - 12 = 0$, and then determine the other two solutions.

6 Show that $z = 3i$ is a solution of the quartic equation $z^4 + z^3 + 7z^2 + 9z - 18 = 0$, and then determine the other three solutions.

7 Solve each of the following cubic equations over $\mathbb{C}$:

a $z^3 - z^2 + z - 1 = 0$

b $z^3 - z^2 + 3z + 5 = 0$

c $z^3 - 2z + 4 = 0$

d $z^3 + 3z^2 - 6z - 36 = 0$

CF

8 Let $a, b, c \in \mathbb{R}$. If $z = 1 + i$ and $z = 3$ are solutions of the equation $z^3 + az^2 + bz + c = 0$, determine the values of a, b and c.

9 Let $c \in \mathbb{R}$. If $z = 1 - 2i$ is a solution of $2z^3 - 5z^2 + cz - 5 = 0$, determine the value of c.

10 Let $P(z) = z^3 + bz^2 + cz + d$ be a cubic polynomial with real coefficients. Given that $P(3) = 0$ and $P(3 - i) = 0$, determine b, c and d.

11 Is it possible to determine a cubic polynomial $P(z)$ with real coefficients and three distinct imaginary solutions? If yes, give an example, if no, explain why not.

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12 Consider the polynomial $P(z)$. Suppose $P(z) = 0$ has solutions $z = 1, 2 + 3i$ and $4 + 5i$.

a What is the minimum degree of this polynomial?

b If $P(z)$ has real coefficients, what is the minimum degree of this polynomial?

13 Give an example of a quartic polynomial $P(z)$ with real coefficients such that the equation $P(z) = 0$ has:

a four distinct complex solutions

b two imaginary and two real solutions

14 For a quadratic polynomial $P(z)$ with real coefficients, there are three possible cases for the equation $P(z) = 0$: two distinct real solutions, one real solution of multiplicity 2, or two conjugate complex solutions.

a What are the possible cases for a cubic polynomial $P(z)$ with real coefficients?

b Give an example for each case.

15 Solve $3z^4 - z^3 - 7z^2 + 49z - 60 = 0$ if $z = 1 + 2i$ is one solution.

16 In this question, you will prove the conjugate root theorem.

a If $z = a + bi$ and $w = c + di$ are complex numbers, prove that $\overline{z + w} = \overline{z} + \overline{w}$.

b If $z = a + bi$ and $w = c + di$ are complex numbers, prove that $\overline{zw} = \overline{z}\,\overline{w}$.

c If $z = a + bi$ is a complex number and c is a real number, prove that $\overline{cz} = c\,\overline{z}$.

d Let z be a complex number. Using mathematical induction, prove that $\overline{z^n} = \overline{z}^{\,n}$.

e Consider a polynomial equation $a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 = 0$, where all the coefficients are real. Let z be a solution of this equation. Show that $\overline{z}$ is also a solution. (**Hint:** Take the complex conjugate of both sides of the equation.)

17 **a** Let $P(z) = z^3 + bz^2 + cz + d$ be any cubic. Suppose $P(z) = 0$ has complex solutions u, v and w, Prove that

$$\begin{aligned} u + v + w &= -b, \\ uv + uw + vw &= c, \\ uvw &= -d. \end{aligned}$$

b Suppose $P(z) = 0$ has solutions $z = 2$ and $z = 2 + 3i$. Use the previous result to (quickly!) determine the values of b, c and d.

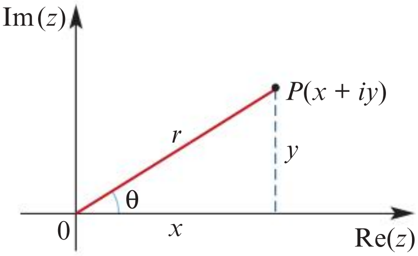
12F Polar form of a complex number

Learning intentions

- To be able to write complex numbers in polar form.
- To be able to multiply and divide complex numbers written in polar form.

Each complex number may be described by an angle and a distance from the origin. In this section, we will see that this is a very useful way to describe complex numbers.

The diagram shows the point P corresponding to the complex number $z = x + yi$. We see that $x = r\cos\theta$ and $y = r\sin\theta$, and so we can write



$$\begin{aligned} z &= x + yi \\ &= r\cos\theta + (r\sin\theta)\,i \\ &= r(\cos\theta + i\sin\theta) \end{aligned}$$

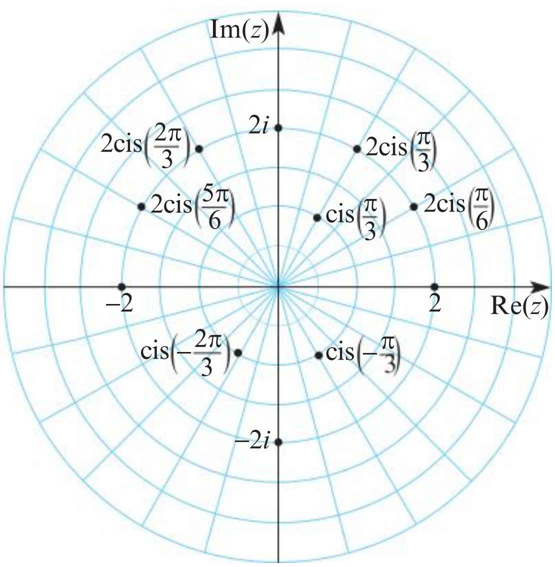
This is called the **polar form** of the complex number. The polar form is abbreviated to

$$z = r\operatorname{cis}\theta$$

- The distance $r = \sqrt{x^2 + y^2}$ is called the **modulus** of z and is denoted by $|z|$.
- The angle θ, measured anticlockwise from the horizontal axis, is called an **argument** of z.

Polar form for complex numbers is also called **modulus–argument form**.

This Argand diagram uses a polar grid with rays at intervals of $\frac{\pi}{12} = 15°$.



Non-uniqueness of polar form

Each complex number has more than one representation in polar form.

Since $\cos\theta = \cos(\theta + 2n\pi)$ and $\sin\theta = \sin(\theta + 2n\pi)$, for all $n \in \mathbb{Z}$, we can write

$$z = r\operatorname{cis}\theta = r\operatorname{cis}(\theta + 2n\pi) \quad \text{for all } n \in \mathbb{Z}$$

The convention is to use the angle θ such that $-\pi < \theta \le \pi$. This value of θ is called the **principal value** of the argument of z and is denoted by $\operatorname{Arg} z$. That is,

$$-\pi < \operatorname{Arg} z \le \pi$$

Note: The principal value of the argument is not defined for the complex number 0, since it can be written in polar form as $0\operatorname{cis}\theta$ for any angle θ.

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Example 19

Express each of the following complex numbers in polar form:

a $z = 1 + \sqrt{3}i$

b $z = 2 - 2i$

Solution

a We have $x = 1$ and $y = \sqrt{3}$, giving

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ &= \sqrt{1+3} \\ &= 2 \end{aligned}$$

The point $z = 1 + \sqrt{3}i$ is in the 1st quadrant, and so $0 < \theta < \frac{\pi}{2}$.

We know that

$$\cos\theta = \frac{x}{r} = \frac{1}{2}$$

and $\quad \sin\theta = \frac{y}{r} = \frac{\sqrt{3}}{2}$

Hence $\theta = \frac{\pi}{3}$ and therefore

$$\begin{aligned} z &= 1 + \sqrt{3}i \\ &= 2\operatorname{cis}\left(\frac{\pi}{3}\right) \end{aligned}$$

b We have $x = 2$ and $y = -2$, giving

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ &= \sqrt{4+4} \\ &= 2\sqrt{2} \end{aligned}$$

The point $z = 2 - 2i$ is in the 4th quadrant, and so $-\frac{\pi}{2} < \theta < 0$.

We know that

$$\cos\theta = \frac{x}{r} = \frac{1}{\sqrt{2}}$$

and $\quad \sin\theta = \frac{y}{r} = -\frac{1}{\sqrt{2}}$

Hence $\theta = -\frac{\pi}{4}$ and therefore

$$\begin{aligned} z &= 2 - 2i \\ &= 2\sqrt{2}\operatorname{cis}\left(-\frac{\pi}{4}\right) \end{aligned}$$

Example 20

Express $z = 2\operatorname{cis}\left(-\frac{2\pi}{3}\right)$ in Cartesian form.

Solution

$$\begin{aligned} x &= r\cos\theta \\ &= 2\cos\left(-\frac{2\pi}{3}\right) \\ &= 2 \times \left(-\frac{1}{2}\right) \\ &= -1 \end{aligned} \qquad \begin{aligned} y &= r\sin\theta \\ &= 2\sin\left(-\frac{2\pi}{3}\right) \\ &= 2 \times \left(-\frac{\sqrt{3}}{2}\right) \\ &= -\sqrt{3} \end{aligned}$$

Hence $z = 2\operatorname{cis}\left(-\frac{2\pi}{3}\right) = -1 - \sqrt{3}i$.

Multiplication and division in polar form

We can give a simple geometric interpretation of multiplication and division of complex numbers in polar form.

If $z_1 = r_1 \operatorname{cis} \theta_1$ and $z_2 = r_2 \operatorname{cis} \theta_2$, then

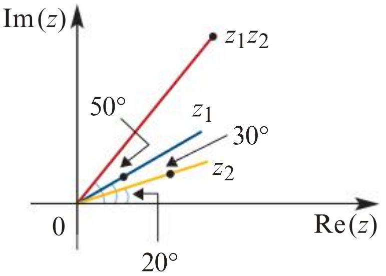
$$z_1 z_2 = r_1 r_2 \operatorname{cis}(\theta_1 + \theta_2) \qquad \text{(multiply the moduli and add the angles)}$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} \operatorname{cis}(\theta_1 - \theta_2) \qquad \text{(divide the moduli and subtract the angles)}$$

For example, if $z_1 = 2 \operatorname{cis} 30°$ and $z_2 = 4 \operatorname{cis} 20°$, then

$$\begin{aligned} z_1 z_2 &= r_1 r_2 \operatorname{cis}(\theta_1 + \theta_2) \\ &= (2 \cdot 4) \operatorname{cis}(30° + 20°) \\ &= 8 \operatorname{cis} 50° \end{aligned}$$

You will prove this result in Exercise 12F.



When we multiply a complex number z by $r \operatorname{cis} \theta$, the effect on the point representing z in an Argand diagram is a dilation of factor r from the origin followed by a rotation about the origin by angle θ anticlockwise.

Example 21

Let $z_1 = 3 \operatorname{cis}\left(\frac{\pi}{2}\right)$ and $z_2 = 2 \operatorname{cis}\left(\frac{5\pi}{6}\right)$. Find the product $z_1 z_2$ and express in Cartesian form.

Solution

$$\begin{aligned} z_1 z_2 &= r_1 r_2 \operatorname{cis}(\theta_1 + \theta_2) \\ &= 6 \operatorname{cis}\left(\frac{\pi}{2} + \frac{5\pi}{6}\right) \\ &= 6 \operatorname{cis}\left(\frac{4\pi}{3}\right) \end{aligned}$$

$$\therefore \quad z_1 z_2 = 6 \operatorname{cis}\left(-\frac{2\pi}{3}\right) \qquad \text{since } -\pi < \operatorname{Arg} z \le \pi$$

Expressing this in Cartesian form, we determine that

$$\begin{aligned} z_1 z_2 &= 6 \operatorname{cis}\left(-\frac{2\pi}{3}\right) \\ &= 6 \cos\left(-\frac{2\pi}{3}\right) + 6 \sin\left(-\frac{2\pi}{3}\right) i \\ &= 6\left(-\frac{1}{2}\right) + 6\left(-\frac{\sqrt{3}}{2}\right) i \\ &= -3 - 3\sqrt{3} i \end{aligned}$$

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Example 22

Let $z_1 = 2\operatorname{cis}\left(\frac{5\pi}{6}\right)$ and $z_2 = 4\operatorname{cis}\left(\frac{\pi}{6}\right)$. Determine the quotient $\frac{z_1}{z_2}$ and express in Cartesian form.

Solution

We determine that

$$\begin{aligned}\frac{z_1}{z_2} &= \frac{r_1}{r_2}\operatorname{cis}(\theta_1 - \theta_2)\\ &= \frac{2}{4}\operatorname{cis}\left(\frac{5\pi}{6} - \frac{\pi}{6}\right)\\ &= \frac{1}{2}\operatorname{cis}\left(\frac{2\pi}{3}\right)\end{aligned}$$

Expressing this in Cartesian form, we determine that

$$\begin{aligned}\frac{z_1}{z_2} &= \frac{1}{2}\cos\left(\frac{2\pi}{3}\right) + \frac{1}{2}\sin\left(\frac{2\pi}{3}\right)i\\ &= \frac{1}{2}\left(-\frac{1}{2}\right) + \frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)i\\ &= -\frac{1}{4}(1 - \sqrt{3}i)\end{aligned}$$

Example 23

The point (2, 3) is rotated about the origin by angle $\frac{\pi}{6}$ anticlockwise. By multiplying two complex numbers, determine the image of the point.

Solution

The point (2, 3) corresponds to the complex number $z = 2 + 3i$.

To rotate z about the origin by $\frac{\pi}{6}$ anticlockwise, we multiply z by $\operatorname{cis}\left(\frac{\pi}{6}\right)$.

This gives

$$\begin{aligned}(2 + 3i)\operatorname{cis}\left(\frac{\pi}{6}\right) &= (2 + 3i)\left(\cos\left(\frac{\pi}{6}\right) + \sin\left(\frac{\pi}{6}\right)i\right)\\ &= (2 + 3i)\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)\\ &= \sqrt{3} + i + \frac{3\sqrt{3}}{2}i + \frac{3}{2}i^2\\ &= \frac{2\sqrt{3} - 3}{2} + \left(\frac{3\sqrt{3} + 2}{2}\right)i\end{aligned}$$

Therefore the image is the point $\left(\frac{2\sqrt{3} - 3}{2}, \frac{3\sqrt{3} + 2}{2}\right)$.

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Summary 12F

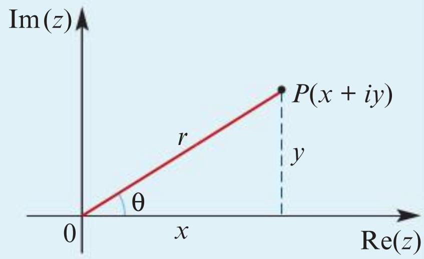
- **Polar form**

A complex number in Cartesian form

$$z = x + yi$$

can be written in polar form as

$$z = r(\cos\theta + i\sin\theta)$$
$$= r\operatorname{cis}\theta$$



  - The distance $r = \sqrt{x^2 + y^2}$ is called the **modulus** of z and is denoted by $|z|$.
  - The angle θ, measured anticlockwise from the horizontal axis, is called an **argument** of z.

- The polar form of a complex number is not unique. For a non-zero complex number z, the argument θ of z such that $-\pi < \theta \le \pi$ is called the **principal value** of the argument of z and is denoted by $\operatorname{Arg} z$.

- **Multiplication and division in polar form**

If $z_1 = r_1 \operatorname{cis}\theta_1$ and $z_2 = r_2 \operatorname{cis}\theta_2$, then

$$z_1 z_2 = r_1 r_2 \operatorname{cis}(\theta_1 + \theta_2) \quad \text{and} \quad \frac{z_1}{z_2} = \frac{r_1}{r_2}\operatorname{cis}(\theta_1 - \theta_2)$$

Skillsheet

Exercise 12F

SF

Example 19

1 Express each of the following in polar form $r\operatorname{cis}\theta$ with $-\pi < \theta \le \pi$:

a $1 + \sqrt{3}i$ **b** $1 - i$ **c** $-2\sqrt{3} + 2i$

d $-4 - 4i$ **e** $12 - 12\sqrt{3}i$ **f** $-\frac{1}{2} + \frac{1}{2}i$

Example 20

2 Express each of the following in the form $x + yi$:

a $3\operatorname{cis}\left(\frac{\pi}{2}\right)$ **b** $\sqrt{2}\operatorname{cis}\left(\frac{\pi}{3}\right)$ **c** $2\operatorname{cis}\left(\frac{\pi}{6}\right)$ **d** $5\operatorname{cis}\left(\frac{3\pi}{4}\right)$

e $12\operatorname{cis}\left(\frac{5\pi}{6}\right)$ **f** $3\sqrt{2}\operatorname{cis}\left(-\frac{\pi}{4}\right)$ **g** $5\operatorname{cis}\left(\frac{4\pi}{3}\right)$ **h** $5\operatorname{cis}\left(-\frac{2\pi}{3}\right)$

CF

3 Simplify the following and express the answers in Cartesian form:

Example 21

a $2\operatorname{cis}\left(\frac{\pi}{6}\right) \cdot 3\operatorname{cis}\left(\frac{\pi}{12}\right)$ **b** $4\operatorname{cis}\left(\frac{\pi}{12}\right) \cdot 3\operatorname{cis}\left(\frac{\pi}{4}\right)$ **c** $\operatorname{cis}\left(\frac{\pi}{4}\right) \cdot 5\operatorname{cis}\left(\frac{5\pi}{12}\right)$

d $12\operatorname{cis}\left(-\frac{\pi}{3}\right) \cdot 3\operatorname{cis}\left(\frac{2\pi}{3}\right)$ **e** $12\operatorname{cis}\left(\frac{5\pi}{6}\right) \cdot 3\operatorname{cis}\left(\frac{\pi}{2}\right)$ **f** $(\sqrt{2}\operatorname{cis}\pi) \cdot \sqrt{3}\operatorname{cis}\left(-\frac{3\pi}{4}\right)$

Example 22

g $\dfrac{10\operatorname{cis}\left(\frac{\pi}{4}\right)}{5\operatorname{cis}\left(\frac{\pi}{12}\right)}$ **h** $\dfrac{12\operatorname{cis}\left(-\frac{\pi}{3}\right)}{3\operatorname{cis}\left(\frac{2\pi}{3}\right)}$ **i** $\dfrac{12\sqrt{8}\operatorname{cis}\left(\frac{3\pi}{4}\right)}{3\sqrt{2}\operatorname{cis}\left(\frac{\pi}{12}\right)}$ **j** $\dfrac{20\operatorname{cis}\left(-\frac{\pi}{6}\right)}{8\operatorname{cis}\left(\frac{5\pi}{6}\right)}$

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Example 23

4 For each of the following, multiply two complex numbers to determine the image of the point under the rotation about the origin:

a $(5, 2)$ is rotated by $\frac{\pi}{3}$ anticlockwise **b** $(3, 2)$ is rotated by $\frac{\pi}{4}$ clockwise

c (x, y) is rotated by θ anticlockwise

5 Let $z_1 = r_1 \operatorname{cis} \theta_1$ and $z_2 = r_2 \operatorname{cis} \theta_2$. Use the compound angle formulas for sine and cosine to prove that $z_1 z_2 = r_1 r_2 \operatorname{cis}(\theta_1 + \theta_2)$.

6 Simplify $\dfrac{\cos(3\theta) + i\sin(3\theta)}{\cos(2\theta) + i\sin(2\theta)}$, writing your answer in polar form.

7 **a** Show that $\operatorname{cis}(-\theta) = \cos(\theta) - i\sin(\theta)$.

b Hence, show that $\cos\theta = \frac{1}{2}(\operatorname{cis}(\theta) + \operatorname{cis}(-\theta))$.

c Determine a similar formula for $\sin\theta$ in terms of $\operatorname{cis}(\theta)$ and $\operatorname{cis}(-\theta)$

8 Suppose $z = r\operatorname{cis}\theta$ where $r \neq 0$. Show that $\frac{1}{z} = \frac{1}{r}\operatorname{cis}(-\theta)$.

9 Prove that $\dfrac{2}{1 + \operatorname{cis}\theta} = 1 - i\tan\frac{\theta}{2}$.

CF

CU

12G De Moivre's theorem

Learning intentions

- To be able to use De Moivre's theorem to simplify integer powers of complex numbers.

De Moivre's theorem allows us to readily simplify expressions of the form z^n when z is expressed in polar form.

De Moivre's theorem

$(r\operatorname{cis}\theta)^n = r^n \operatorname{cis}(n\theta)$, where $n \in \mathbb{Z}$

Proof This result is usually proved by mathematical induction, but can be explained by a simple inductive argument.

Let $z = \operatorname{cis}\theta$

Then $z^2 = \operatorname{cis}\theta \times \operatorname{cis}\theta = \operatorname{cis}(2\theta)$ by the multiplication rule

$z^3 = z^2 \times \operatorname{cis}\theta = \operatorname{cis}(3\theta)$

$z^4 = z^3 \times \operatorname{cis}\theta = \operatorname{cis}(4\theta)$

Continuing in this way, we see that $(\operatorname{cis}\theta)^n = \operatorname{cis}(n\theta)$, for each positive integer n.

To obtain the result for negative integers, again let $z = \operatorname{cis}\theta$. Then

$$z^{-1} = \frac{1}{z} = \bar{z} = \operatorname{cis}(-\theta)$$

For $k \in \mathbb{N}$, we have, $z^{-k} = (z^{-1})^k = (\operatorname{cis}(-\theta))^k = \operatorname{cis}(-k\theta)$

using the result for positive integers.

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Example 24

Simplify:

a $\left(\operatorname{cis}\left(\frac{\pi}{3}\right)\right)^9$

b $\dfrac{\operatorname{cis}\left(\frac{7\pi}{4}\right)}{\left(\operatorname{cis}\left(\frac{\pi}{3}\right)\right)^7}$

Solution

a

$$\begin{aligned}\left(\operatorname{cis}\left(\frac{\pi}{3}\right)\right)^9 &= \operatorname{cis}\left(9 \times \frac{\pi}{3}\right)\\ &= \operatorname{cis}(3\pi)\\ &= \operatorname{cis}\pi\\ &= \cos\pi + i\sin\pi\\ &= -1\end{aligned}$$

b

$$\begin{aligned}\frac{\operatorname{cis}\left(\frac{7\pi}{4}\right)}{\left(\operatorname{cis}\left(\frac{\pi}{3}\right)\right)^7} &= \operatorname{cis}\left(\frac{7\pi}{4}\right)\left(\operatorname{cis}\left(\frac{\pi}{3}\right)\right)^{-7}\\ &= \operatorname{cis}\left(\frac{7\pi}{4}\right)\operatorname{cis}\left(\frac{-7\pi}{3}\right)\\ &= \operatorname{cis}\left(\frac{7\pi}{4} - \frac{7\pi}{3}\right)\\ &= \operatorname{cis}\left(\frac{-7\pi}{12}\right)\end{aligned}$$

Example 25

Simplify $\dfrac{(1+i)^3}{(1-\sqrt{3}i)^5}$.

Solution

First convert to polar form:

$$1 + i = \sqrt{2}\operatorname{cis}\left(\frac{\pi}{4}\right)$$

$$1 - \sqrt{3}i = 2\operatorname{cis}\left(\frac{-\pi}{3}\right)$$

Therefore

$$\begin{aligned}\frac{(1+i)^3}{(1-\sqrt{3}i)^5} &= \frac{\left(\sqrt{2}\operatorname{cis}\left(\frac{\pi}{4}\right)\right)^3}{\left(2\operatorname{cis}\left(\frac{-\pi}{3}\right)\right)^5}\\ &= \frac{2\sqrt{2}\operatorname{cis}\left(\frac{3\pi}{4}\right)}{32\operatorname{cis}\left(\frac{-5\pi}{3}\right)} \qquad \text{by De Moivre's theorem}\\ &= \frac{\sqrt{2}}{16}\operatorname{cis}\left(\frac{3\pi}{4} - \left(\frac{-5\pi}{3}\right)\right)\\ &= \frac{\sqrt{2}}{16}\operatorname{cis}\left(\frac{29\pi}{12}\right)\\ &= \frac{\sqrt{2}}{16}\operatorname{cis}\left(\frac{5\pi}{12}\right)\end{aligned}$$

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Summary 12G

De Moivre's theorem

We can use De Moivre's theorem to simplify powers of complex numbers expressed in polar form:

$$(r\operatorname{cis}\theta)^n = r^n\operatorname{cis}(n\theta) \quad \text{where } n \in \mathbb{Z}$$

Skill-sheet

Exercise 12G

SF

1 Simplify each of the following using De Moivre's theorem. Give your answers in both polar form and Cartesian form.

a $(1+i)^2$ **b** $(1+i)^3$ **c** $(1+i)^4$ **d** $(1+i)^5$ **e** $(1+i)^8$

2 Simplify each of the following using De Moivre's theorem. Give your answers in both polar form and Cartesian form.

a $(1+\sqrt{3}i)^2$ **b** $(1+\sqrt{3}i)^3$ **c** $(1+\sqrt{3}i)^4$ **d** $(1+\sqrt{3}i)^5$

e $(1+\sqrt{3}i)^6$ **f** $(1+\sqrt{3}i)^9$ **g** $(1+\sqrt{3}i)^{12}$

3 Simplify each of the following using De Moivre's theorem. Give your answers in both polar form and Cartesian form.

a $\left(2\operatorname{cis}\left(\frac{5\pi}{6}\right)\right)^2$ **b** $\left(3\operatorname{cis}\left(\frac{2\pi}{3}\right)\right)^3$ **c** $\left(3\operatorname{cis}\left(-\frac{2\pi}{3}\right)\right)^4$ **d** $\left(2\operatorname{cis}\left(-\frac{3\pi}{4}\right)\right)^4$

CF

Example 24

4 Simplify each of the following:

a $2\operatorname{cis}\left(\frac{5\pi}{6}\right)\times\left(\sqrt{2}\operatorname{cis}\left(\frac{7\pi}{8}\right)\right)^4$ **b** $\dfrac{1}{\left(\frac{3}{2}\operatorname{cis}\left(\frac{5\pi}{8}\right)\right)^3}$

c $\left(\operatorname{cis}\left(\frac{\pi}{6}\right)\right)^8\times\left(\sqrt{3}\operatorname{cis}\left(\frac{\pi}{4}\right)\right)^6$ **d** $\left(\frac{1}{2}\operatorname{cis}\left(\frac{\pi}{2}\right)\right)^{-5}$

e $\left(2\operatorname{cis}\left(\frac{3\pi}{2}\right)\times 3\operatorname{cis}\left(\frac{\pi}{6}\right)\right)^3$ **f** $\left(\frac{1}{2}\operatorname{cis}\left(\frac{\pi}{8}\right)\right)^{-6}\times\left(4\operatorname{cis}\left(\frac{\pi}{3}\right)\right)^2$ **g** $\dfrac{\left(6\operatorname{cis}\left(\frac{2\pi}{5}\right)\right)^3}{\left(\frac{1}{2}\operatorname{cis}\left(\frac{-\pi}{4}\right)\right)^{-5}}$

Example 25

5 Simplify each of the following, giving your answer in polar form $r\operatorname{cis}\theta$, with $r>0$ and $\theta\in(-\pi,\pi]$:

a $(1+\sqrt{3}i)^6$ **b** $(1-i)^{-5}$ **c** $i(\sqrt{3}-i)^7$

d $(-3+\sqrt{3}i)^{-3}$ **e** $\dfrac{(1+\sqrt{3}i)^3}{i(1-i)^5}$ **f** $\dfrac{(-1+\sqrt{3}i)^4(-\sqrt{2}-\sqrt{2}i)^3}{\sqrt{3}-3i}$

g $(-1+i)^5\left(\frac{1}{2}\operatorname{cis}\left(\frac{\pi}{4}\right)\right)^3$ **h** $\dfrac{\left(\operatorname{cis}\left(\frac{2\pi}{5}\right)\right)^3}{(1-\sqrt{3}i)^2}$ **i** $\left((1-i)\operatorname{cis}\left(\frac{2\pi}{3}\right)\right)^7$

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12H Sketching subsets of the complex plane

Learning intentions

- To be able to sketch sets of complex numbers on an Argand diagram.

We have already seen how complex numbers can be plotted on an Argand diagram (also called the **complex plane**). In this section, we treat complex numbers as points in the complex plane, and therefore we can illustrate sets of complex numbers.

Distance in the complex plane

Recall that, if $z = x + yi$ is a complex number, then its modulus

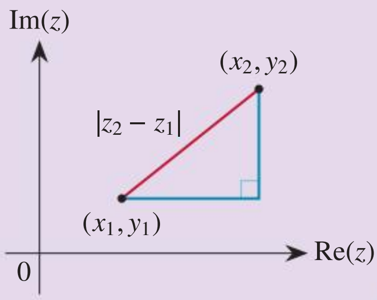
$$|z| = \sqrt{x^2 + y^2}$$

is equal to its distance from the origin in the complex plane. More generally:

Distance between two complex numbers

For complex numbers $z_1 = x_1 + y_1 i$ and $z_2 = x_2 + y_2 i$, the distance between z_1 and z_2 in the complex plane is equal to

$$|z_2 - z_1| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$



Example 26

Determine the distance between $z_1 = -1 + 4i$ and $z_2 = 3 + 2i$ in the complex plane.

Solution

The distance can be found by evaluating

$$\begin{aligned} |z_2 - z_1| &= |(3 + 2i) - (-1 + 4i)| \\ &= |4 - 2i| \\ &= \sqrt{4^2 + (-2)^2} \\ &= \sqrt{20} = 2\sqrt{5} \end{aligned}$$

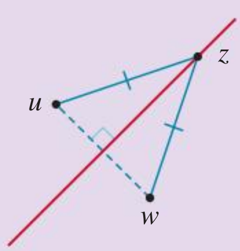
Lines in the complex plane

Equation of a line in the complex plane

Let u and w be fixed complex numbers. Then the equation

$$|z - u| = |z - w|$$

defines the set of all points z that are equal distance from u and w. This set is a straight line.



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Example 27

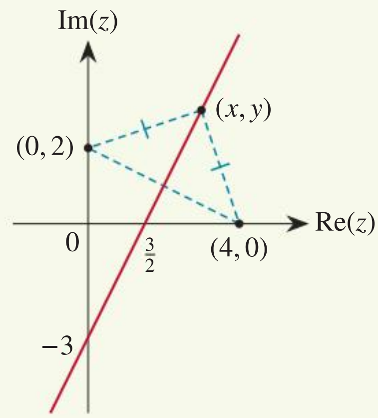
Determine the Cartesian equation for the set of points z such that $|z - 2i| = |z - 4|$.

Solution

Note that the point z is equidistant from $2i$ and 4. Therefore the set of points is the straight line that is the perpendicular bisector of the line segment between $(0, 2)$ and $(4, 0)$.

Letting $z = x + yi$, we can determine the Cartesian equation algebraically as follows:

$$
\begin{aligned}
|z - 2i| &= |z - 4| \\
|x + yi - 2i| &= |x + yi - 4| \\
|x + (y - 2)i| &= |(x - 4) + yi| \\
\sqrt{x^2 + (y - 2)^2} &= \sqrt{(x - 4)^2 + y^2} \\
x^2 + y^2 - 4y + 4 &= x^2 - 8x + 16 + y^2 \\
-4y + 4 &= -8x + 16 \\
y &= 2x - 3
\end{aligned}
$$



The set of points is the straight line with Cartesian equation $y = 2x - 3$.

Rays in the complex plane

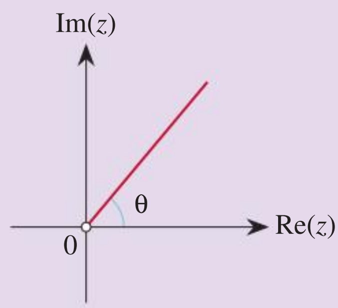
Equation of a ray starting at the origin

Let θ be a fixed angle. Then the equation

$$\operatorname{Arg} z = \theta$$

defines a ray extending from the origin at an angle of θ measured anticlockwise from the horizontal axis.

Note: The origin is not included in the set of points, as the principal argument is not defined for the complex number 0.

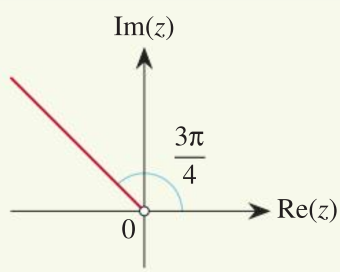


Example 28

Sketch the subset of the complex plane defined by $\operatorname{Arg} z = \frac{3\pi}{4}$.

Solution

The equation defines the set of complex numbers with a principal argument of $\frac{3\pi}{4}$.



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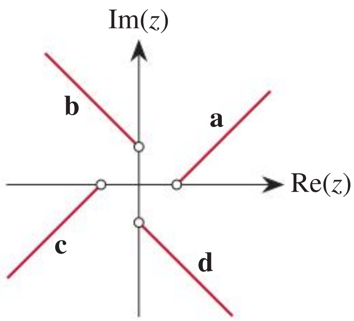
By applying a translation, we can describe rays that do not start at the origin. The diagram on the right shows the rays with equations:

a $\text{Arg}(z - 1) = \frac{\pi}{4}$ **b** $\text{Arg}(z - i) = \frac{3\pi}{4}$

c $\text{Arg}(z + 1) = -\frac{3\pi}{4}$ **d** $\text{Arg}(z + i) = -\frac{\pi}{4}$



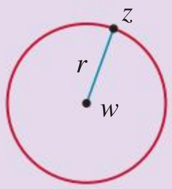
Circles in the complex plane

Equation of a circle in the complex plane

Let w be a fixed complex number and let $r > 0$. Then the equation

$$|z - w| = r$$

defines a circle with centre w and radius r.



Example 29

a Determine the Cartesian equation for the set of points z such that $|z - (2 - 3i)| = 2$.

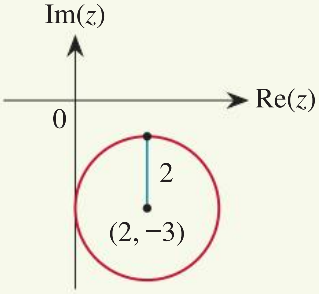
b Determine the shortest distance from a point in this set to the point corresponding to the complex number $w = 4 + i$.

Solution

a Note that the point z is a distance of 2 units from $2 - 3i$. Therefore the set of points is the circle of radius 2 centred at $(2, -3)$.

Letting $z = x + yi$, we can determine the Cartesian equation algebraically as follows:

$$\begin{aligned} |z - (2 - 3i)| &= 2 \\ |x + yi - (2 - 3i)| &= 2 \\ |(x - 2) + (y + 3)i| &= 2 \\ \sqrt{(x - 2)^2 + (y + 3)^2} &= 2 \\ (x - 2)^2 + (y + 3)^2 &= 4 \end{aligned}$$



b The point $w = 4 + i$ corresponds to the point $P(4, 1)$. Draw a line from the point $P(4, 1)$ to the centre $C(2, -3)$ of the circle. The distance from C to P is

$$\begin{aligned} CP &= \sqrt{(4 - 2)^2 + (1 - (-3))^2} \\ &= \sqrt{20} \\ &= 2\sqrt{5}. \end{aligned}$$

Now subtract the radius of the circle to determine the shortest distance: $2\sqrt{5} - 2$.

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Example 30

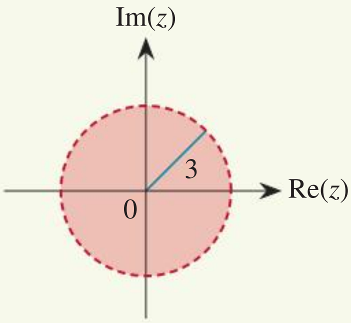
Sketch the set of complex numbers z described by each rule:

a $|z| < 3$

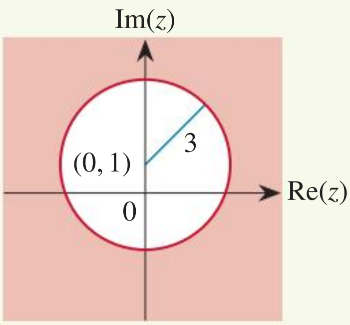
b $|z - i| \geq 3$

Solution

a If $|z| < 3$, then z is less than 3 units from the origin. Therefore z is inside the circle of radius 3 centred at the origin.



b If $|z - i| \geq 3$, then z is at least 3 units from i. Therefore z lies on or outside the circle of radius 3 centred at (0, 1).



Other subsets of the complex plane

Sometimes the subset of the complex plane defined by a particular rule is not obvious until we determine a Cartesian description for the set.

Example 31

Consider the set of points z in the complex plane such that

$$2|z - 2| = |z - \bar{z} + 2i|$$

Determine the Cartesian equation that describes this set.

Solution

Let $z = x + yi$. Then

$$\begin{aligned}
2|z - 2| &= |z - \bar{z} + 2i| \\
2|x + yi - 2| &= |x + yi - (x - yi) + 2i| \\
2|(x - 2) + yi| &= |(2y + 2)i| \\
|(x - 2) + yi| &= |(y + 1)i| \\
\sqrt{(x - 2)^2 + y^2} &= \sqrt{(y + 1)^2} \\
(x - 2)^2 + y^2 &= y^2 + 2y + 1 \\
y &= \tfrac{1}{2}(x - 2)^2 - \tfrac{1}{2}
\end{aligned}$$

This set of points is a parabola in the complex plane.

In the next example, we look at combining regions of the complex plane using union and intersection.

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Example 32

Define sets S and T of complex numbers by

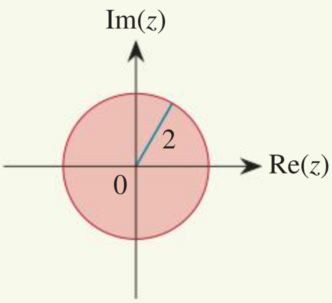
$$S = \{ z : |z| \le 2 \} \quad \text{and} \quad T = \left\{ z : -\frac{\pi}{4} \le \operatorname{Arg} z \le \frac{\pi}{4} \right\}$$

Sketch the following regions of the complex plane:

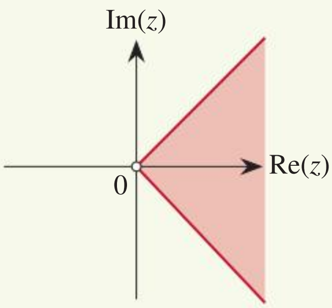
a S **b** T **c** $S \cap T$ **d** $S \cup T$

Solution

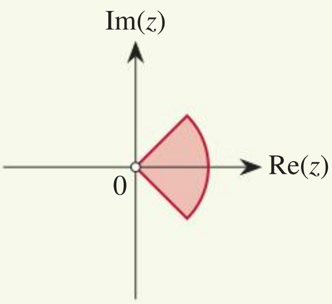
a Region S is the set of points at most 2 units from the origin. This is a disc of radius 2 that includes its boundary.



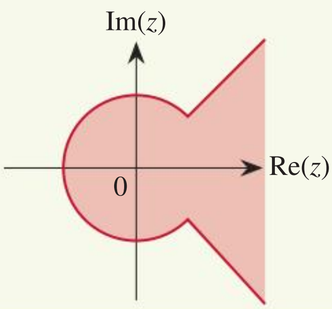
b Region T is the set of points with principal argument between $-\frac{\pi}{4}$ and $\frac{\pi}{4}$. So T is the wedge shown below.



c The intersection consists of the points in common to S and T.



d The union consists of all points in S or in T (or both).



Summary 12H

- **Distance in the complex plane**
  - For $z \in \mathbb{C}$, the distance of z from the origin is equal to $|z|$.
  - For $z_1, z_2 \in \mathbb{C}$, the distance between z_1 and z_2 is equal to $|z_2 - z_1|$.
- **Subsets of the complex plane**
  - The equation $|z - w| = r$ defines the circle with centre w and radius r.
  - The equation $|z - u| = |z - w|$ defines a line.
  - The equation $\operatorname{Arg} z = \theta$ defines the ray extending from the origin at angle θ. (The origin is not included.)

Skill-sheet

Exercise 12H

SF

Example 26

1 For each of the following, determine the distance between z and w in the complex plane:

a $z = 1 + i,\ w = 4 + 5i$ **b** $z = 3 - 4i,\ w = 2 - 3i$
c $z = 4 - 6i,\ w = -1 + 6i$ **d** $z = 2,\ w = -2i$
e $z = 10i,\ w = -3i$ **f** $z = \sqrt{2} + i,\ w = 2i$

Example 27

2 For each of the following, determine the Cartesian equation of the line described by the rule and sketch the line on an Argand diagram:

a $\mathrm{Re}(z) = 2$ **b** $\mathrm{Im}(z) = -1$
c $\mathrm{Im}(z) = 3\,\mathrm{Re}(z)$ **d** $3\,\mathrm{Re}(z) + 4\,\mathrm{Im}(z) = 12$
e $|z - 1| = |z - i|$ **f** $|z - (1 + i)| = |z + 1|$
g $z + \bar{z} = 6$ **h** $z - \bar{z} = 4i$

Example 28

3 Sketch the subsets of the complex plane described by the following rules:

a $\mathrm{Arg}\, z = \frac{\pi}{4}$ **b** $\mathrm{Arg}\, z = -\frac{5\pi}{6}$ **c** $0 \le \mathrm{Arg}\, z \le \frac{\pi}{2}$
d $\mathrm{Arg}(z - 1) = \frac{3\pi}{4}$ **e** $\mathrm{Arg}(z + i) = -\frac{\pi}{4}$ **f** $\mathrm{Arg}(z - 1 + i) = \pi$

Example 29

4 Consider the set of points $z \in \mathbb{C}$ for which $|z - 2| = 1$. By letting $z = x + yi$, show algebraically that this corresponds to the circle with equation $(x - 2)^2 + y^2 = 1$.

5 Consider the set of points $z \in \mathbb{C}$ for which $|z| = |z - 2 - 2i|$. By letting $z = x + yi$, show algebraically that this corresponds to the straight line with equation $y = 2 - x$.

6 Sketch the set of points $z \in \mathbb{C}$ that are distance 2 from the point $w = 2 + 2i$.

Example 30

7 By interpreting $|z - w|$ as the distance from z to w, sketch the set of complex numbers z described each rule. (You do *not* have to determine the Cartesian equation algebraically.)

a $|z| = 3$ **b** $|z| \le 2$ **c** $|z| > 2$
d $|z - 1| = 2$ **e** $|z - i| < 2$ **f** $|z + 2| \ge 3$
g $|z + 2i| < 2$ **h** $|z - (1 + i)| > 3$ **i** $|z + 1 - 2i| \le 3$

CF

Example 32

8 Define three sets of complex numbers by

$$R = \{z : |z| \le 3\},\quad S = \{z : \mathrm{Re}(z) \ge 0\}\quad \text{and}\quad T = \left\{z : \frac{\pi}{4} < \mathrm{Arg}\, z \le \frac{3\pi}{4}\right\}$$

Sketch each of the following regions of the complex plane:

a R **b** S **c** T **d** $R \cap S$
e $R \cap T$ **f** $S \cap T$ **g** $R \cup S$ **h** $R \cap S \cap T$

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9 Define sets S and T of complex numbers by

$$S = \{ z : z + \bar{z} \le |z|^2 \} \quad \text{and} \quad T = \left\{ z : \frac{\pi}{4} \le \operatorname{Arg} z \le \frac{\pi}{2} \right\}$$

a By letting $z = x + yi$, determine a Cartesian description for the set S.
b Sketch S in the complex plane.
c Sketch T in the complex plane.
d Sketch $S \cap T$.

10 Show that the equation $|z + 2i| = |2iz - 1|$ defines a circle in the complex plane. Determine its centre and radius.

Example 31 **11** Consider the set of points z in the complex plane such that

$$2|z - i| = |z + \bar{z} + 2|$$

Determine the Cartesian equation that describes this set.

12 Define the set $S = \{ z \in \mathbb{C} : |z + 16| = 4|z + 1| \}$.
a Prove that $S = \{ z \in \mathbb{C} : |z| = 4 \}$.
b Hence, sketch the set S in the complex plane.

13 Show that the equation $|z| = 3|z + 8|$ defines a circle in the complex plane. Determine its centre and radius.

14 Let $S = \{ z \in \mathbb{C} : |z - 1| = 1 \}$.
a Sketch the set S in the complex plane.
b Hence, sketch each of the following subsets of the complex plane:
i $T = \{ z + 1 : z \in S \}$
ii $U = \{ z + i : z \in S \}$
iii $V = \{ 2z : z \in S \}$
iv $W = \{ iz : z \in S \}$

Hint: Multiplication by i corresponds to a rotation.

CF
CU

Review

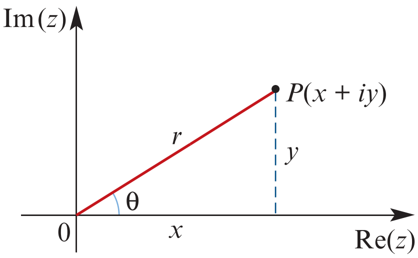
Chapter summary

Complex numbers

- The imaginary number i has the property $i^2 = -1$.
- The set of **complex numbers** is $\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$.
- For a complex number $z = a + bi$ the **real part** of z is $\text{Re}(z) = a$ and the **imaginary part** of z is $\text{Im}(z) = b$.
- Complex numbers z_1 and z_2 are equal if and only if $\text{Re}(z_1) = \text{Re}(z_2)$ and $\text{Im}(z_1) = \text{Im}(z_2)$.
- An **Argand diagram** is a geometric representation of $\mathbb{C}$.
- The **modulus** of z, denoted by $|z|$, is the distance from the origin to the point representing z in an Argand diagram. Thus $|a + bi| = \sqrt{a^2 + b^2}$.
- The complex number $z = x + yi$ can be expressed in **polar form** as

$$z = r(\cos\theta + i\sin\theta)$$
$$= r\operatorname{cis}\theta$$

where $r = |z| = \sqrt{x^2 + y^2}$, $x = r\cos\theta$, $y = r\sin\theta$. This is also called modulus–argument form.



- The angle θ, measured anticlockwise from the horizontal axis, is called an **argument** of z.
- For a non-zero complex number z, the argument θ of z such that $-\pi < \theta \leq \pi$ is called the **principal value** of the argument of z and is denoted by $\operatorname{Arg} z$.

Operations on complex numbers

- The **complex conjugate** of $z = a + bi$ is given by $\bar{z} = a - bi$. Note that $z\bar{z} = |z|^2$.
- Division of complex numbers:

$$\frac{z_1}{z_2} = \frac{z_1}{z_2} \times \frac{\overline{z_2}}{\overline{z_2}} = \frac{z_1\overline{z_2}}{|z_2|^2}$$

- Multiplication and division in polar form:
  Let $z_1 = r_1\operatorname{cis}\theta_1$ and $z_2 = r_2\operatorname{cis}\theta_2$. Then

$$z_1 z_2 = r_1 r_2 \operatorname{cis}(\theta_1 + \theta_2) \quad \text{and} \quad \frac{z_1}{z_2} = \frac{r_1}{r_2}\operatorname{cis}(\theta_1 - \theta_2)$$

- **De Moivre's theorem**

$$(r\operatorname{cis}\theta)^n = r^n \operatorname{cis}(n\theta) \quad \text{where } n \in \mathbb{Z}$$

Polynomial equations over the complex numbers

- **Factor theorem**
  A polynomial $P(z)$ has $z - \alpha$ as a factor if and only if $P(\alpha) = 0$.
- **Fundamental theorem of algebra**
  For $n \geq 1$, every polynomial of degree n can be expressed as a product of n linear factors over the complex numbers. Therefore every polynomial equation of degree n has n solutions (counting multiplicity).

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Review

- **Conjugate root theorem**

  Let $P(z)$ be a polynomial with real coefficients. If $a + bi$ is a solution of $P(z) = 0$, with a and b real numbers, then the complex conjugate $a - bi$ is also a solution.

Subsets of the complex plane

- For $z_1, z_2 \in \mathbb{C}$, the distance between z_1 and z_2 is equal to $|z_2 - z_1|$.
- The equation $|z - w| = r$ defines the circle with centre w and radius r.
- The equation $|z - u| = |z - w|$ defines a line.
- The equation $\operatorname{Arg} z = \theta$ defines the ray extending from the origin at angle θ. (The origin is not included.)

Skills checklist

Check-list

Download this checklist from the Interactive Textbook, then print it and fill it out to check your skills.

12A **1** I can identify the real and imaginary parts of a complex number. ☐

See Question 1

12A **2** I can add and subtract complex numbers. ☐

See Example 3 and Question 3

12A **3** I can determine powers of *i*. ☐

See Example 4, Question 4 and Question 5

12B **4** I can multiply two complex numbers. ☐

See Example 5 and Question 1

12B **5** I can determine the conjugate of a complex number. ☐

See Example 6, Question 3 and Question 4

12B **6** I can determine the modulus of a complex number. ☐

See Example 7 and Question 6

12B **7** I can divide two complex numbers. ☐

See Example 8, Example 9, Question 9, Question 10 and Question 11

12C **8** I can represent complex numbers on an Argand diagram. ☐

See Example 11, Example 12, Question 1, Question 2 and Question 3

12D **9** I can solve quadratic equations and factorise over the complex numbers. ☐

See Example 14, Example 15, Question 1, Question 2 and Question 3

12E **10** I can solve cubic equations over the complex numbers. ☐

See Example 17, Example 18, Question 1, Question 2 and Question 3

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Review

12F **11** I can convert between polar form and Cartesian form. ☐

See Example 19, Example 20, Question 1 and Question 2

12F **12** I can multiply and divide complex numbers in polar form. ☐

See Example 21, Example 22, Example 23 and Question 3

12G **13** I can use De Moivre's theorem to simplify powers of complex numbers. ☐

See Example 24, Example 25, Question 1, Question 2, and Question 3

12H **14** I can sketch subsets of the complex plane including lines and circles. ☐

See Example 27, Example 29, Question 2, Question 4, and Question 5

Short-response questions

Technology-free short-response questions

SF

1 For $z_1 = m + ni$ and $z_2 = p + qi$, express each of the following in the form $a + bi$:

a $2z_1 + 3z_2$ **b** $\overline{z_2}$ **c** $z_1\overline{z_2}$

d $\dfrac{z_1}{z_2}$ **e** $z_1 + \overline{z_1}$ **f** $(z_1 + z_2)(z_1 - z_2)$

g $\dfrac{1}{z_1}$ **h** $\dfrac{z_2}{z_1}$ **i** $\dfrac{3z_1}{z_2}$

2 Let $z = 1 - \sqrt{3}i$. For each of the following, express in the form $a + bi$ and mark on an Argand diagram:

a z **b** z^2 **c** z^3 **d** $\dfrac{1}{z}$ **e** $\overline{z}$ **f** $\dfrac{1}{\overline{z}}$

3 Write each of the following in polar form:

a $1 + i$ **b** $1 - \sqrt{3}i$ **c** $2\sqrt{3} + i$

d $3\sqrt{2} + 3\sqrt{2}i$ **e** $-3\sqrt{2} - 3\sqrt{2}i$ **f** $\sqrt{3} - i$

4 Write each of the following in Cartesian form:

a $-2\operatorname{cis}\left(\dfrac{\pi}{3}\right)$ **b** $3\operatorname{cis}\left(\dfrac{\pi}{4}\right)$ **c** $3\operatorname{cis}\left(\dfrac{3\pi}{4}\right)$

d $-3\operatorname{cis}\left(-\dfrac{3\pi}{4}\right)$ **e** $3\operatorname{cis}\left(-\dfrac{5\pi}{6}\right)$ **f** $\sqrt{2}\operatorname{cis}\left(-\dfrac{\pi}{4}\right)$

5 Let $z = \operatorname{cis}\left(\dfrac{\pi}{3}\right)$. On an Argand diagram, carefully plot:

a z^2 **b** $\overline{z}$ **c** $\dfrac{1}{z}$ **d** $\operatorname{cis}\left(\dfrac{2\pi}{3}\right)$

SF

6 Let $z = \text{cis}\left(\frac{\pi}{4}\right)$. On an Argand diagram, carefully plot:

a iz **b** $\bar{z}$ **c** $\frac{1}{z}$ **d** $-iz$

7 Solve each of the following quadratic equations over $\mathbb{C}$:

a $z^2 + 4 = 0$ **b** $3z^2 + 9 = 0$

c $z^2 + 4z + 5 = 0$ **d** $2z^2 - 3z + 4 = 0$

8 Show that $z = 2$ is a solution of the equation $z^3 - 2z^2 + 4z - 8 = 0$, and then determine the other two solutions.

CF

9 **a** Show that $z = i$ is a solution of the equation $12z^3 - 11z^2 + 12z - 11 = 0$.

b Hence, determine the other two solutions.

c Now consider the equation $nz^3 - (n-1)z^2 + nz - (n-1) = 0$, where n is an integer. Show that there are only two values of n such that the equation has an integer solution.

SF

10 Sketch the following subsets of the complex plane:

a $\{z : |z - 1| \le 3\}$ **b** $\{z : z + \bar{z} = 4\}$ **c** $\left\{z : \text{Arg}\, z = -\frac{3\pi}{4}\right\}$

11 Express $\sqrt{3} + i$ in polar form. Hence determine $(\sqrt{3} + i)^7$ and express your answer in Cartesian form.

12 Express $z = 1 + i$ and $w = 1 - i$ in polar form. Hence determine $\frac{z^4}{w^5}$ and express your answer in Cartesian form.

Technology-active short-response questions

CU

13 **a** Let $z = x + yi$. Express z^2 in Cartesian form.

b Sketch the subset of the complex plane defined by $\text{Re}(z^2) = 1$.

c Sketch the subset of the complex plane defined by $\text{Im}(z^2) = 1$.

14 **a** Determine the exact solutions in $\mathbb{C}$ for the equation $z^2 - 2\sqrt{3}z + 4 = 0$.

b **i** Plot the two solutions from part **a** on an Argand diagram.

ii Determine the equation of the circle, with centre the origin, which passes through these two points.

iii Determine the value of $a \in \mathbb{Z}$ such that the circle passes through $(0, \pm a)$.

15 Let z be a complex number with $|z| = 6$. Let A be the point representing z and let B be the point representing $(1 + i)z$.

a Determine:

i $|(1 + i)z|$ **ii** $|(1 + i)z - z|$

b Prove that OAB is a right-angled isosceles triangle.

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Review

CU

16 Let $z = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$.

a On an Argand diagram, the points O, A, Z, P and Q represent the complex numbers 0, 1, z, $1 + z$ and $1 - z$ respectively. Show these points on a diagram.

b Prove that the magnitude of $\angle POQ$ is $\frac{\pi}{2}$. Determine the ratio $\frac{OP}{OQ}$.

17 Let z_1 and z_2 be two complex numbers. Prove the following:

a $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + z_1\overline{z_2} + \overline{z_1}z_2$

b $|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - (z_1\overline{z_2} + \overline{z_1}z_2)$

c $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2|z_1|^2 + 2|z_2|^2$

State a geometric theorem from the result of **c**.

18 Let z_1 and z_2 be two complex numbers.

a Prove the following:

i $\overline{\overline{z_1}z_2} = z_1\overline{z_2}$

ii $z_1\overline{z_2} + \overline{z_1}z_2$ is a real number

iii $z_1\overline{z_2} - \overline{z_1}z_2$ is an imaginary number or zero

iv $(z_1\overline{z_2} + \overline{z_1}z_2)^2 - (z_1\overline{z_2} - \overline{z_1}z_2)^2 = 4|z_1 z_2|^2$

b Use the results from part **a** and Question 17 to prove that $|z_1 + z_2| \leq |z_1| + |z_2|$.
Hint: Show that $(|z_1| + |z_2|)^2 - |z_1 + z_2|^2 \geq 0$.

c Hence prove that $|z_1 - z_2| \geq |z_1| - |z_2|$.

19 Assume that $|z| = 1$ and that the argument of z is θ, where $0 < \theta < \pi$. Determine the modulus and argument of:

a $z + 1$ **b** $z - 1$ **c** $\frac{z - 1}{z + 1}$

20 The quadratic expression $ax^2 + bx + c$ has real coefficients.

a Determine the discriminant of $ax^2 + bx + c$.

b Determine the condition in terms of a, b and c for which the equation $ax^2 + bx + c = 0$ has no real solutions.

c If this condition is fulfilled, let z_1 and z_2 be the complex solutions of the equation and let P_1 and P_2 be the corresponding points on an Argand diagram.

i Determine $z_1 + z_2$ and $|z_1|$ in terms of a, b and c.

ii Determine $\cos(\angle P_1OP_2)$ in terms of a, b and c.

21 Let z_1 and z_2 be the solutions of the quadratic equation $z^2 + z + 1 = 0$.

a Determine z_1 and z_2.

b Prove that $z_1 = z_2^2$ and $z_2 = z_1^2$.

c Determine the modulus and the principal value of the argument of z_1 and z_2.

d Let P_1 and P_2 be the points on an Argand diagram corresponding to z_1 and z_2. Determine the area of triangle P_1OP_2.

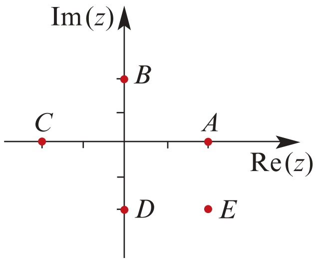
Multiple-choice questions

Technology-free multiple-choice questions

1 If $u = 1 + i$, then $\dfrac{1}{2-u}$ is equal to

A $-\frac{1}{2} - \frac{1}{2}i$ **B** $\frac{1}{5} + \frac{2}{5}i$ **C** $\frac{1}{2} + \frac{1}{2}i$ **D** $-\frac{1}{2} + \frac{1}{5}i$

2 The point C on the Argand diagram represents the complex number z. Which point represents the complex number $i \times z$?



A A **B** B
C C **D** D

3 If $|z| = 5$, then $\left|\dfrac{1}{z}\right|$ is equal to

A $\frac{1}{\sqrt{5}}$ **B** $-\frac{1}{\sqrt{5}}$ **C** $\frac{1}{5}$ **D** $-\frac{1}{5}$

4 If $(x + yi)^2 = -32i$ for real values of x and y, then

A $x = 4, y = 4$ **B** $x = -4, y = 4$
C $x = 4, y = -4$ **D** $x = 4, y = -4$ or $x = -4, y = 4$

5 The linear factors of $z^2 + 6z + 10$ over $\mathbb{C}$ are

A $(z + 3 + i)^2$ **B** $(z + 3 - i)^2$
C $(z + 3 + i)(z - 3 + i)$ **D** $(z + 3 - i)(z + 3 + i)$

6 Let $z = \dfrac{1}{1-i}$. If $r = |z|$ and $\theta = \operatorname{Arg} z$, then

A $r = \frac{1}{2}$ and $\theta = \frac{\pi}{4}$ **B** $r = \sqrt{2}$ and $\theta = -\frac{\pi}{4}$
C $r = \frac{1}{\sqrt{2}}$ and $\theta = -\frac{\pi}{4}$ **D** $r = \frac{1}{\sqrt{2}}$ and $\theta = \frac{\pi}{4}$

7 The solution of the equation $\dfrac{z - 2i}{z - (3 - 2i)} = 2$, where $z \in \mathbb{C}$, is

A $z = 6 + 2i$ **B** $z = 6 - 2i$
C $z = -6 - 6i$ **D** $z = 6 - 6i$

8 Let $z = a + bi$, where $a, b \in \mathbb{R}$. If $z^2(1 + i) = 2 - 2i$, then the Cartesian form of one value of z could be

A $\sqrt{2}i$ **B** $-\sqrt{2}i$ **C** $-1 - i$ **D** $-1 + i$

9 The value of the discriminant for the quadratic expression $(2 + 2i)z^2 + 8iz - 4(1 - i)$ is

A -32 **B** 0 **C** 64 **D** 32

10 If $\text{Arg}(ai + 1) = \frac{\pi}{6}$, then the real number a is

A $\sqrt{3}$ **B** $-\sqrt{3}$ **C** 1 **D** $\frac{1}{\sqrt{3}}$

Technology-active multiple-choice questions

11 The expression $(1 + i)(2 + 2i)(3 + 3i)(4 + 4i)(5 + 5i)$ is equal to

A $240(1 + i)$ **B** $480(1 - i)$ **C** $-480(1 + i)$ **D** -240

12 If $z(1 + 2i) = 3 + 4i$, then z is equal to

A $\frac{11}{5} + \frac{2}{4}i$ **B** $\frac{11}{5} - \frac{2}{5}i$ **C** $\frac{2}{5} + \frac{11}{5}i$ **D** $\frac{2}{5} - \frac{11}{5}i$

13 The Cartesian equation corresponding to the set of points z for which $|z - 1| = |z + i|$ is

A $y = 0$ **B** $x = 0$ **C** $y = x$ **D** $y = -x$

14 A region in the complex plane is defined by the equations $|z| < 8$ and $-\frac{\pi}{4} < \text{Arg}\, z < \frac{\pi}{4}$. The area of this region is

A 2π **B** 4π **C** 8π **D** 16π

15 If $a \in \mathbb{R}$, then the modulus of the complex number $\frac{a + i}{a - i}$ is

A $\frac{1}{4}$ **B** $\frac{1}{2}$ **C** 1 **D** 2

16 Consider all complex numbers z and w satisfying $|z| = 12$ and $|w - 3 - 4i| = 5$. The minimum value of $|z - w|$ is

A 1 **B** 2 **C** 3 **D** 4

17 If $|2z - 1| = 2|z|$, then z lies on

A a circle centred at $(\frac{1}{2}, 0)$ **B** a circle centred at $(0, \frac{1}{2})$
C the line $x = \frac{1}{4}$ **D** the line $y = \frac{1}{4}$

18 If $z = \left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)^5 + \left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right)^5$, then

A $z = 0$ **B** $\text{Re}(z) = 0$
C $\text{Im}(z) = 0$ **D** $\text{Re}(z) > 0,\ \text{Im}(z) > 0$

19 Consider complex numbers z and w for which $|z| = 1$ and $|w| = 1$. If the real part of z is positive and the imaginary part of w is negative, then zw **cannot** be equal to

A 0 **B** 1 **C** -1 **D** i

13

Graphing techniques

Chapter contents

In this chapter we build on the study of functions in Mathematical Methods Units 1 & 2, extending the range of functions for which we can sketch graphs.

The modulus function

The modulus function is useful in many contexts. For example, it enables us to talk about the distance between two points on the real number line. We gave a definition of the modulus function for complex numbers in Chapter 12.

Reciprocal functions

Given the graph of a function $y = f(x)$, we will learn how to sketch the graph of $y = \dfrac{1}{f(x)}$. This function is called the reciprocal of f.

Reciprocal trigonometric function functions

The same techniques can be used to graph the reciprocal trigonometric functions, defined by

$$\sec x = \frac{1}{\cos x}, \operatorname{cosec} x = \frac{1}{\sin x} \text{ and } \cot x = \frac{\cos x}{\sin x}.$$

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13A The modulus function

Learning intentions

- To be able to sketch graphs and solve equations involving the modulus function

The **modulus** or **absolute value** of a real number x is denoted by $|x|$ and is defined by

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

It may also be defined as $|x| = \sqrt{x^2}$. For example: $|5| = 5$ and $|-5| = 5$.

Example 1

Evaluate each of the following:

a **i** $|-3 \times 2|$ **ii** $|-3| \times |2|$

b **i** $\left|\frac{-4}{2}\right|$ **ii** $\frac{|-4|}{|2|}$

c **i** $|-6 + 2|$ **ii** $|-6| + |2|$

Solution

a **i** $|-3 \times 2| = |-6| = 6$ **ii** $|-3| \times |2| = 3 \times 2 = 6$ Note: $|-3 \times 2| = |-3| \times |2|$

b **i** $\left|\frac{-4}{2}\right| = |-2| = 2$ **ii** $\frac{|-4|}{|2|} = \frac{4}{2} = 2$ Note: $\left|\frac{-4}{2}\right| = \frac{|-4|}{|2|}$

c **i** $|-6 + 2| = |-4| = 4$ **ii** $|-6| + |2| = 6 + 2 = 8$ Note: $|-6 + 2| \neq |-6| + |2|$

Properties of the modulus function

- $|ab| = |a|\,|b|$ and $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$
- $|x| = a$ implies $x = a$ or $x = -a$
- $|a + b| \leq |a| + |b|$
- If a and b are both positive or both negative, then $|a + b| = |a| + |b|$.
- If $a \geq 0$, then $|x| \leq a$ is equivalent to $-a \leq x \leq a$.
- If $a \geq 0$, then $|x - k| \leq a$ is equivalent to $k - a \leq x \leq k + a$.

The following example uses the second property listed above to solve simple equations.

Example 2

Solve each of the following equations for x:

a $|x - 4| = 6$ **b** $|2x - 4| = 16$

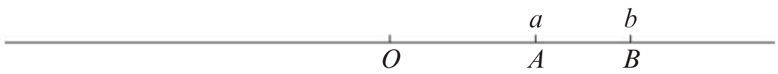
Solution

a
$$|x - 4| = 6$$
$$\Rightarrow \quad x - 4 = 6 \quad \text{or} \quad x - 4 = -6$$
$$\Rightarrow \quad x = 10 \quad \text{or} \quad x = -2$$

b
$$|2x - 4| = 16$$
$$\Rightarrow \quad 2x - 4 = 16 \quad \text{or} \quad 2x - 4 = -16$$
$$\Rightarrow \quad x = 10 \quad \text{or} \quad x = -6$$

The modulus function as a measure of distance

Consider two points A and B on a number line:



On a number line, the distance between points A and B is $|a - b| = |b - a|$.

For example:

- The statement $|x - 2| \leq 3$ can be read as 'on the number line, the distance of x from 2 is less than or equal to 3'.
- The statement $|x| \leq 3$ can be read as 'on the number line, the distance of x from the origin is less than or equal to 3'.

Note that $|x| \leq 3$ is equivalent to $-3 \leq x \leq 3$ or $x \in [-3, 3]$.

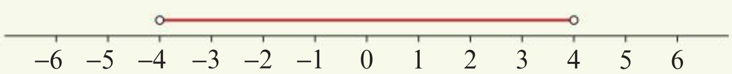
Example 3

Illustrate each of the following sets on a number line and represent the sets using interval notation:

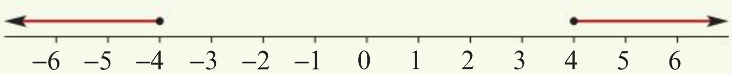
a $\{ x : |x| < 4 \}$ **b** $\{ x : |x| \geq 4 \}$ **c** $\{ x : |x - 1| \leq 4 \}$

Solution

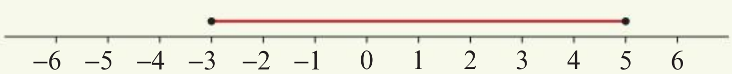
a $(-4, 4)$



b $(-\infty, -4] \cup [4, \infty)$



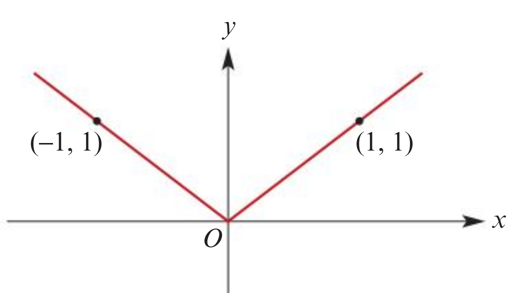
c $[-3, 5]$



The graph of $y = |x|$

The graph of the function $y = |x|$ is shown on the right.

This graph is symmetric about the y-axis, since $|x| = |-x|$.



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Example 4

For each of the following functions, sketch the graph and state the range:

a $f(x) = |x - 3| + 1$

b $f(x) = -|x - 3| + 1$

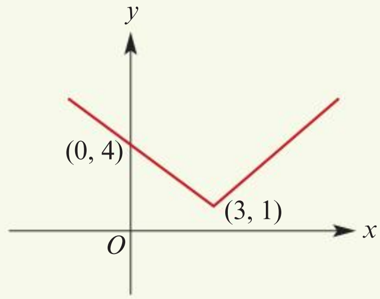
Solution

Note that $|a - b| = a - b$ if $a \geq b$, and $|a - b| = b - a$ if $b \geq a$.

a $f(x) = |x - 3| + 1 = \begin{cases} x - 3 + 1 & \text{if } x \geq 3 \\ 3 - x + 1 & \text{if } x < 3 \end{cases}$

$= \begin{cases} x - 2 & \text{if } x \geq 3 \\ 4 - x & \text{if } x < 3 \end{cases}$

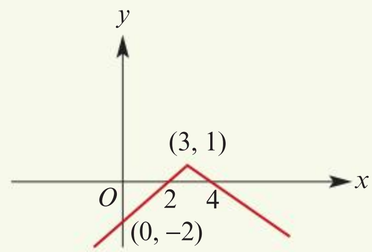
Range $= [1, \infty)$



b $f(x) = -|x - 3| + 1 = \begin{cases} -(x - 3) + 1 & \text{if } x \geq 3 \\ -(3 - x) + 1 & \text{if } x < 3 \end{cases}$

$= \begin{cases} -x + 4 & \text{if } x \geq 3 \\ -2 + x & \text{if } x < 3 \end{cases}$

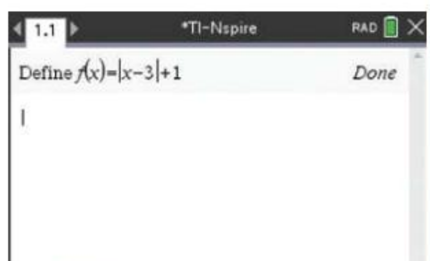
Range $= (-\infty, 1]$



Using the TI-Nspire CX non-CAS

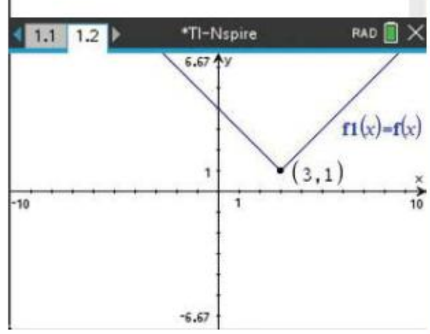
- Use (menu) > **Actions** > **Define** to define the function $f(x) = \text{abs}(x - 3) + 1$.

Note: The absolute value function can be obtained by typing **abs( )** or using the 2D-template palette.



- Open a **Graphs** application ((ctrl)(I) > **Graphs**) and let $f1(x) = f(x)$.
- Press (enter) to obtain the graph.

Note: The expression $\text{abs}(x - 3) + 1$ could have been entered directly for $f1(x)$.



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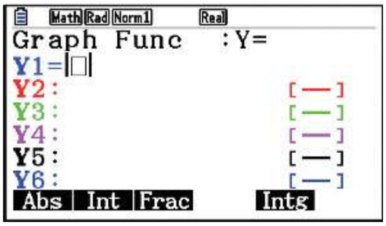
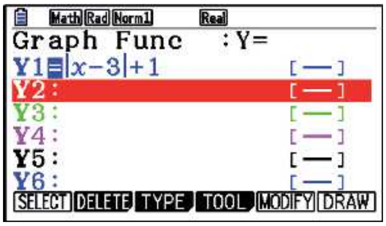
Using the Casio

- Press (MENU) (5) to select **Graph** mode.
- To obtain the absolute value function, go to the **Numeric** menu (OPTN) (F5), then select **Abs** (F1).



- Finish entering the rule $y = |x - 3| + 1$ in $Y1$:

  (X,θ,T) (−) (3) ▶ (+) (1) (EXE)

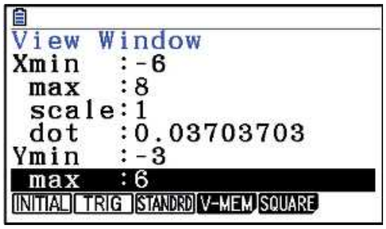


- To view the graph, select **Draw** (F6).
- Adjust the View Window:

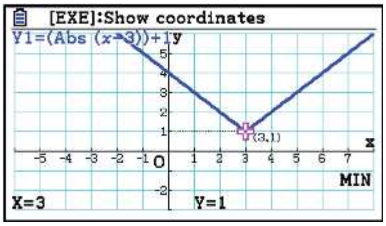
  (SHIFT) (F3)

  ((−)) (6) (EXE) (8) (EXE) (1) (EXE) ▼

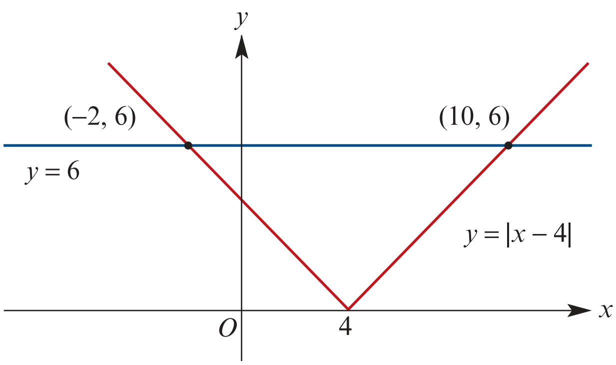
  ((−)) (3) (EXE) (6) (EXE) (1) (EXE)



- To return to the graph, press (EXIT).
- To determine the coordinates of the minimum point, go to the **G-Solve** menu (SHIFT) (F5), then select **Minimum** (F3).



Note: The solution of equations can be illustrated graphically. For example, the following graph shows the solutions of the equation $|x - 4| = 6$ from Example 2a.



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Functions with rules of the form $y = |f(x)|$ and $y = f(|x|)$

If the graph of $y = f(x)$ is known, then we can sketch the graph of $y = |f(x)|$ using the following observation:

$$|f(x)| = f(x) \text{ if } f(x) \geq 0 \quad \text{and} \quad |f(x)| = -f(x) \text{ if } f(x) < 0$$

Example 5

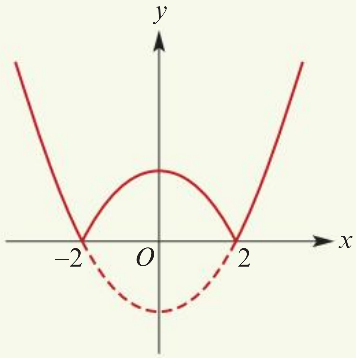
Sketch the graph of each of the following:

a $y = |x^2 - 4|$

b $y = |2^x - 1|$

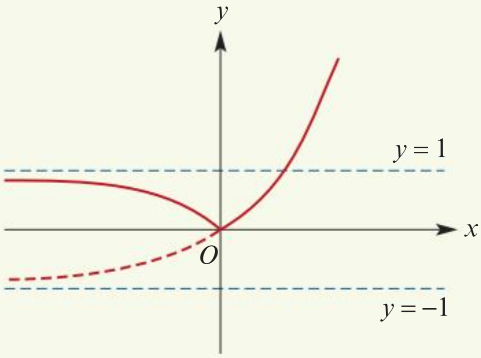
Solution

a



The graph of $y = x^2 - 4$ is drawn and the negative part reflected in the x-axis.

b



The graph of $y = 2^x - 1$ is drawn and the negative part reflected in the x-axis.

The graph of $y = f(|x|)$, for $x \in \mathbb{R}$, is sketched by reflecting the graph of $y = f(x)$, for $x \geq 0$, in the y-axis.

Example 6

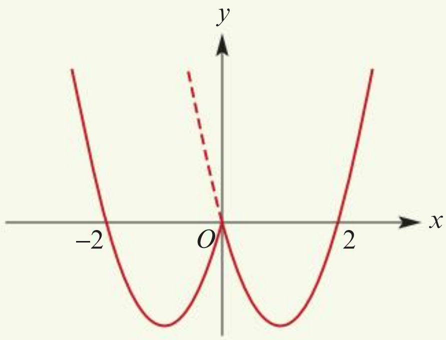
Sketch the graph of each of the following:

a $y = |x|^2 - 2|x|$

b $y = 2^{|x|}$

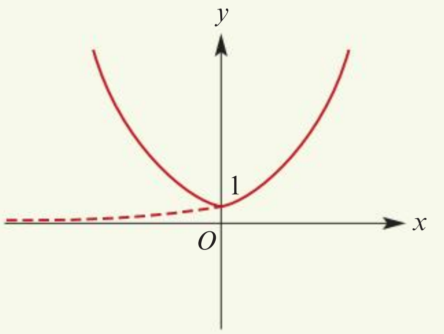
Solution

a



The graph of $y = x^2 - 2x$, $x \geq 0$, is reflected in the y-axis.

b



The graph of $y = 2^x$, $x \geq 0$, is reflected in the y-axis.

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Example 7

a Solve the equation $|x^2 - 4x| = 3$ for x.

b Illustrate the solutions by graphing $y = |x^2 - 4x|$ and $y = 3$ on the same set of axes.

Solution

a
$$|x^2 - 4x| = 3$$
$$\Rightarrow \quad x^2 - 4x = 3 \quad \text{or} \quad x^2 - 4x = -3$$
$$\Rightarrow \quad x^2 - 4x - 3 = 0 \quad \text{or} \quad x^2 - 4x + 3 = 0$$
$$\Rightarrow \quad x = 2 \pm \sqrt{7} \quad \text{or} \quad x = 1 \text{ or } x = 3$$

Therefore $x = 2 + \sqrt{7}$, $x = 2 - \sqrt{7}$, $x = 1$ or $x = 3$.

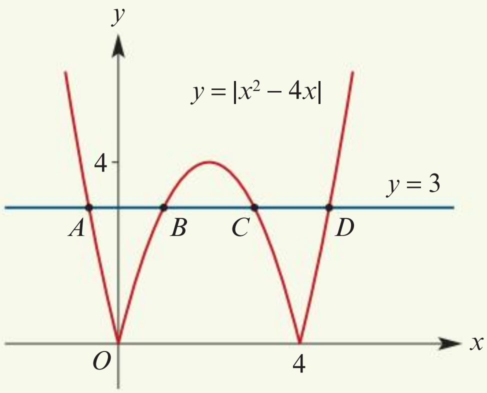
b The solutions correspond to the points of intersection of the two graphs:

$A(2 - \sqrt{7}, 3)$

$B(1, 3)$

$C(3, 3)$

$D(2 + \sqrt{7}, 3)$



Note: We can see from the graph of $y = |x^2 - 4x|$ that the equation $|x^2 - 4x| = 4$ has three solutions and the equation $|x^2 - 4x| = 5$ has two solutions.

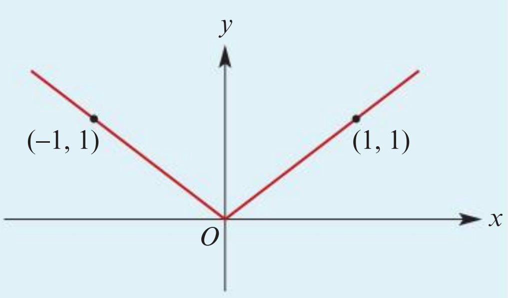
Summary 13A

- The **modulus** or **absolute value** of a real number x is

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

For example: $|5| = 5$ and $|-5| = 5$.

- The graph of the modulus function $y = |x|$ is shown on the right.



- On the number line, the distance between a and b is given by $|a - b| = |b - a|$. For example: $|x - 2| < 5$ can be read as 'the distance of x from 2 is less than 5'.

Skill-sheet

Exercise 13A

SF

Example 1

1 Evaluate each of the following:

a $|-5| + 3$ **b** $|-5| + |-3|$ **c** $|-5| - |-3|$
d $|-5| - |-3| - 4$ **e** $|-5| - |-3| - |-4|$ **f** $|-5| + |-3| - |-4|$

Example 2

2 Solve each of the following equations for x:

a $|x - 1| = 2$ **b** $|2x - 3| = 4$ **c** $|5x - 3| = 9$ **d** $|x - 3| - 9 = 0$
e $|3 - x| = 4$ **f** $|3x + 4| = 8$ **g** $|5x + 11| = 9$

Example 3

3 For each of the following, illustrate the set on a number line and represent the set using interval notation:

a $\{ x : |x| < 3 \}$ **b** $\{ x : |x| \geq 5 \}$ **c** $\{ x : |x - 2| \leq 1 \}$
d $\{ x : |x - 2| < 3 \}$ **e** $\{ x : |x + 3| \geq 5 \}$ **f** $\{ x : |x + 2| \leq 1 \}$

Example 4

4 For each of the following functions, sketch the graph and state the range:

a $f(x) = |x - 4| + 1$ **b** $f(x) = -|x + 3| + 2$
c $f(x) = |x + 4| - 1$ **d** $f(x) = 2 - |x - 1|$

5 Solve each of the following inequalities for x:

a $|x| \leq 5$ **b** $|x| \geq 2$ **c** $|2x - 3| \leq 1$
d $|5x - 2| < 3$ **e** $|-x + 3| \geq 7$ **f** $|-x + 2| \leq 1$

6 Let $f(x) = 4 - x$.

a Sketch the graphs of $y = |f(x)|$ and $y = f(|x|)$.
b State the set of values of x for which $|f(x)| = f(|x|)$.

Example 5

7 Sketch the graph of each of the following:

a $y = |x^2 - 9|$ **b** $y = |3^x - 3|$ **c** $y = |x^2 - x - 12|$
d $y = |x^2 - 3x - 40|$ **e** $y = |x^2 - 2x - 8|$ **f** $y = |2^x - 4|$

CF

Example 6

8 Sketch the graph of each of the following:

a $y = |x|^2 - 4|x|$ **b** $y = 3^{|x|}$ **c** $y = |x|^2 - 7|x| + 12$
d $y = |x|^2 - |x| - 12$ **e** $y = |x|^2 + |x| - 12$ **f** $y = -3^{|x|} + 1$

Example 7

9 Solve each of the following equations for x:

a $|x^2 - 2x| = \frac{1}{2}$ **b** $|x^2 - 2x| = 1$ **c** $|x^2 - 2x| = 8$
d $|x^2 - 6x| = 8$ **e** $|x^2 - 6x| = 16$ **f** $|x^2 - 6x| = 9$

10 Solve each of the following equations for x:

a $|x - 4| - |x + 2| = 6$ **b** $|2x - 5| - |4 - x| = 10$ **c** $|2x - 1| + |4 - 2x| = 10$

11 If $f(x) = |x - a| + b$ with $f(3) = 3$ and $f(-1) = 3$, determine the values of a and b.

CU

12 Prove each of the following for all $x, y, z \in \mathbb{R}$:

a $|x - y| \leq |x| + |y|$ **b** $|x| - |y| \leq |x - y|$ **c** $|x + y + z| \leq |x| + |y| + |z|$

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13B Graphs of reciprocal functions

Learning intentions

- To be able to sketch the graphs of the reciprocals of polynomial and other functions.

Reciprocals of polynomials

You have learned in previous years that the **reciprocal** of a non-zero number a is $\frac{1}{a}$. Likewise, we have the following definition.

If $y = f(x)$ is a polynomial function, then its **reciprocal function** is defined by the rule

$$y = \frac{1}{f(x)}$$

For example, the reciprocal of the function $y = x^3$ is $y = \frac{1}{x^3}$.

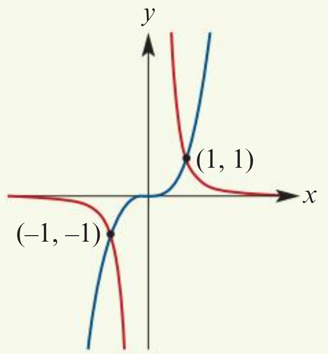
In this section, we will determine relationships between the graph of a function and the graph of its reciprocal. Let's consider some specific examples, from which we will draw general conclusions.

Example 8

Sketch the graphs of $y = x^3$ and $y = \frac{1}{x^3}$ on the same set of axes.

Solution

We first sketch the graph of $y = x^3$. This is shown in blue.



Horizontal asymptotes

If $x \to \pm\infty$, then $\frac{1}{x^3} \to 0$. Therefore the line $y = 0$ is a horizontal asymptote of the reciprocal function.

Vertical asymptotes

Notice that $x^3 = 0$ when $x = 0$.

If x is a small positive number, then $\frac{1}{x^3}$ is a large positive number.

If x is a small negative number, then $\frac{1}{x^3}$ is a large negative number.

Therefore the line $x = 0$ is a vertical asymptote of the reciprocal function.

Observations from the example

This example highlights behaviour typical of reciprocal functions:

- If $y = f(x)$ is a non-zero polynomial function, then the graph of $y = \frac{1}{f(x)}$ will have vertical asymptotes where $f(x) = 0$.
- The graphs of a function and its reciprocal are always on the same side of the x-axis.
- If the graphs of a function and its reciprocal intersect, then it must be where $f(x) = \pm 1$.

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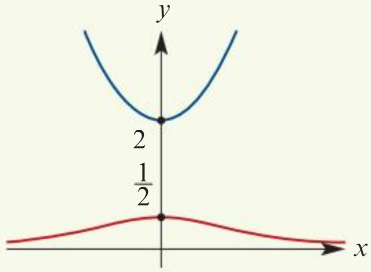
The following example is perhaps easier, because the reciprocal graph has no vertical asymptotes. This time we are interested in turning points.

Example 9

Consider the function $f(x) = x^2 + 2$. Sketch the graphs of $y = f(x)$ and $y = \frac{1}{f(x)}$ on the same set of axes.

Solution

We first sketch $y = x^2 + 2$. This is shown in blue.



Horizontal asymptotes If $x \to \pm\infty$, then $\frac{1}{f(x)} \to 0$. Therefore the line $y = 0$ is a horizontal asymptote of the reciprocal function.

Vertical asymptotes There are no vertical asymptotes, as there is no solution to the equation $f(x) = 0$.

Turning points Notice that the graph of $y = x^2 + 2$ has a minimum at $(0, 2)$. The reciprocal function therefore has a maximum at $(0, \frac{1}{2})$.

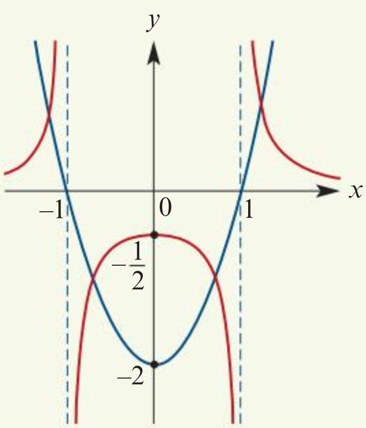
- If the graph of $y = f(x)$ has a local minimum at $x = a$, then the graph of $y = \frac{1}{f(x)}$ will have a local maximum at $x = a$.
- If the graph of $y = f(x)$ has a local maximum at $x = a$, then the graph of $y = \frac{1}{f(x)}$ will have a local minimum at $x = a$.

Example 10

Consider the function $f(x) = 2(x - 1)(x + 1)$. Sketch the graphs of $y = f(x)$ and $y = \frac{1}{f(x)}$ on the same set of axes.

Solution

We first sketch $y = 2(x - 1)(x + 1)$. This is shown in blue.



Horizontal asymptotes If $x \to \pm\infty$, then $\frac{1}{f(x)} \to 0$. Therefore the line $y = 0$ is a horizontal asymptote of the reciprocal function.

Vertical asymptotes We have $f(x) = 0$ when $x = -1$ or $x = 1$. Therefore the lines $x = -1$ and $x = 1$ are vertical asymptotes of the reciprocal function.

Turning points The graph of $y = f(x)$ has a minimum at $(0, -2)$. Therefore the reciprocal has a local maximum at $(0, -\frac{1}{2})$.

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Reciprocals of further functions

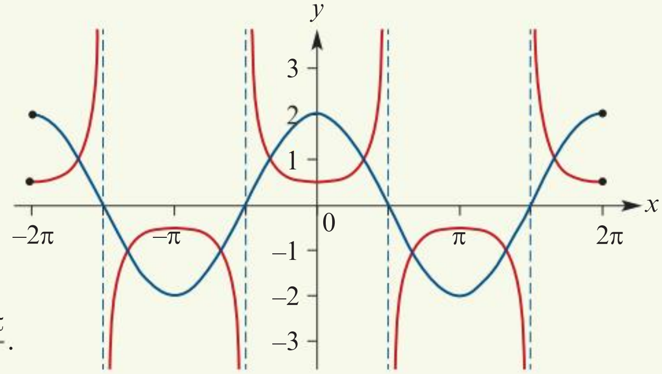
The techniques used to sketch graphs of the reciprocals of polynomial functions can also be used for the reciprocals of other functions. We give two examples here.

Example 11

Let $f(x) = 2\cos x$ for $-2\pi \le x \le 2\pi$. Sketch the graphs of $y = f(x)$ and $y = \dfrac{1}{f(x)}$ on the same set of axes.

Solution

We first sketch $y = 2\cos x$ for $x \in [-2\pi, 2\pi]$. This is shown in blue.



Vertical asymptotes

Vertical asymptotes of the reciprocal function will occur when $f(x) = 0$.

These are given by $x = \pm\dfrac{\pi}{2}, \pm\dfrac{3\pi}{2}$.

Turning points

The points $(0, 2)$ and $(\pm 2\pi, 2)$ are local maximums of $y = f(x)$. Therefore the points $(0, \frac{1}{2})$ and $(\pm 2\pi, \frac{1}{2})$ are local minimums of the reciprocal.

The points $(\pm\pi, -2)$ are local minimums of $y = f(x)$. Therefore the points $(\pm\pi, -\frac{1}{2})$ are local maximums of the reciprocal.

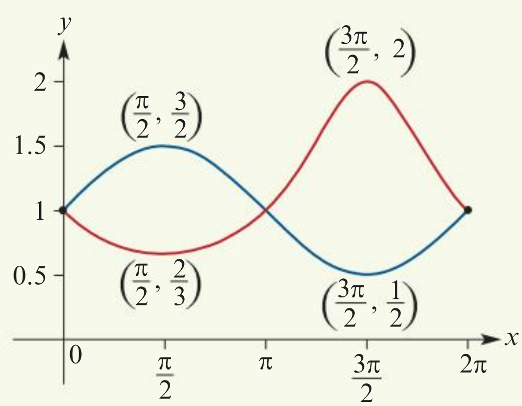
The graph of the next function has no x-axis intercepts, and so its reciprocal has no vertical asymptotes.

Example 12

Let $f(x) = 0.5\sin x + 1$ for $0 \le x \le 2\pi$. Sketch the graphs of $y = f(x)$ and $y = \dfrac{1}{f(x)}$ on the same set of axes.

Solution

We first sketch $y = 0.5\sin x + 1$ for $x \in [0, 2\pi]$. This is shown in blue.



Turning points

The point $(\frac{\pi}{2}, \frac{3}{2})$ is a local maximum of $y = f(x)$. Therefore the point $(\frac{\pi}{2}, \frac{2}{3})$ is a local minimum of the reciprocal.

The point $(\frac{3\pi}{2}, \frac{1}{2})$ is a local minimum of $y = f(x)$. Therefore the point $(\frac{3\pi}{2}, 2)$ is a local maximum of the reciprocal.

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Summary 13B

Given the graph of a continuous function $y = f(x)$, we can sketch the graph of $y = \dfrac{1}{f(x)}$ with the help of the following observations:

Function $y = f(x)$	Reciprocal function $y = \dfrac{1}{f(x)}$
x-axis intercept at $x = a$	vertical asymptote $x = a$
local maximum at $x = a$	local minimum at $x = a$
local minimum at $x = a$	local maximum at $x = a$
above the x-axis	above the x-axis
below the x-axis	below the x-axis
increasing over an interval	decreasing over the interval
decreasing over an interval	increasing over the interval
values approach ∞	values approach 0 from above
values approach $-\infty$	values approach 0 from below
values approach 0 from above	values approach ∞
values approach 0 from below	values approach $-\infty$

Skill-sheet

Exercise 13B

SF

Example 8, 9

1 For each of the following functions, sketch the graphs of $y = f(x)$ and $y = \dfrac{1}{f(x)}$ on the same set of axes:

- **a** $f(x) = x + 3$
- **b** $f(x) = x^2$
- **c** $f(x) = x^2 + 4$
- **d** $f(x) = (x - 1)(x + 1)$
- **e** $f(x) = 4 - x^2$
- **f** $f(x) = (x - 1)^2 - 1$
- **g** $f(x) = x^2 - 2x - 3$
- **h** $f(x) = -x^2 - 2x + 3$
- **i** $f(x) = x^3 + 1$

Example 11, 12

2 For each of the following functions, sketch the graphs of $y = f(x)$ and $y = \dfrac{1}{f(x)}$ on the same set of axes. Label asymptotes, turning points and endpoints.

- **a** $f(x) = \sin x$ for $0 \le x \le 2\pi$
- **b** $f(x) = \cos x$ for $0 \le x \le 2\pi$
- **c** $f(x) = -2\cos x$ for $-\pi \le x \le \pi$
- **d** $f(x) = \cos x + 1$ for $0 \le x \le 4\pi$
- **e** $f(x) = -\sin x - 1$ for $-2\pi \le x \le 2\pi$
- **f** $f(x) = \cos x - 2$ for $0 \le x \le 2\pi$
- **g** $f(x) = -\sin x + 2$ for $0 \le x \le 2\pi$
- **h** $f(x) = -2\cos x + 3$ for $-\pi \le x \le \pi$

3 Consider the quadratic function $f(x) = x^2 + 2x + 2$.

- **a** By completing the square, determine the turning point of the graph of $y = f(x)$.
- **b** Hence sketch the graphs of $y = f(x)$ and $y = \dfrac{1}{f(x)}$ on the same set of axes.

4 Consider the quadratic function $f(x) = 5x(1 - x)$.

- **a** Sketch the graphs of $y = f(x)$ and $y = \dfrac{1}{f(x)}$ on the same set of axes.
- **b** Determine the points of intersection of the two graphs by solving $f(x) = 1$ and $f(x) = -1$.

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5 Sketch the graphs of $y = 2\sin^2 x$ and $y = \dfrac{1}{2\sin^2 x}$ on the same set of axes, over the interval $0 \le x \le 2\pi$.

CF

6 Consider the function $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = 2^x - 1$. Sketch the graphs of $y = f(x)$ and $y = \dfrac{1}{f(x)}$ on the same set of axes.

7 Let $k \in \mathbb{R}$ and consider the function $f(x) = x^2 + 2kx + 1$.

a By completing the square, show that the graph of $y = f(x)$ has a minimum turning point at $(-k, 1 - k^2)$.

b For what values of k does the graph of $y = f(x)$ have:

i no x-axis intercept **ii** one x-axis intercept **iii** two x-axis intercepts?

c Sketch the graphs of $y = f(x)$ and $y = \dfrac{1}{f(x)}$ when the graph of $y = f(x)$ has:

i no x-axis intercept **ii** one x-axis intercept **iii** two x-axis intercepts.

Hint: It helps to ignore the y-axis.

8 The graph of $f(x) = \dfrac{1}{x^2 + bx + c}$ has vertical asymptotes at $x = 3$ and $x = -2$.

Determine the values of b and c.

9 The graph of $f(x) = \dfrac{1}{x^2 + bx + c}$ has a vertical asymptote at $x = 3$ and passes through the point $(1, 2)$. Determine the values of b and c.

10 Let $c \in \mathbb{R}$. The range of $f(x) = \dfrac{1}{x^2 + 2x + c}$ is $(0, 2]$. Determine the value of c.

11 Let $h \in \mathbb{R}$. The graph of $f(x) = \dfrac{1}{(x+1)(x-3)} + h$ has exactly one x-intercept. Determine the value of h.

12 Let $c \in \mathbb{R}$ and define $f(x) = x^3 + x^2 + cx$. Determine the values of c for which the graph of $y = \dfrac{1}{f(x)}$ has three vertical asymptotes.

13 Consider the two functions $f(x) = \dfrac{1}{x^2 + 4}$ and $g(x) = \dfrac{1}{x^2 - 4}$.

CU

a Determine the range of $y = f(x)$.

b Determine the range of $y = g(x)$.

c Determine the range of $y = g(f(x))$.

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13C Graphing the reciprocal trigonometric functions

Learning intentions

- To be able to sketch graphs of the reciprocals of trigonometric functions.

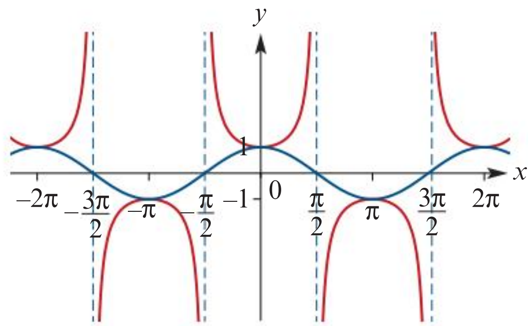
The reciprocal circular functions were introduced in Chapter 17. In this section, we consider the graphs of these functions and basic transformations of these functions.

The secant function

The secant function is defined by

$$\sec x = \frac{1}{\cos x}$$

provided $\cos x \neq 0$.



The graphs of $y = \cos x$ and $y = \sec x$ are shown here on the same axes.

The significant features of the two graphs are listed in the following table.

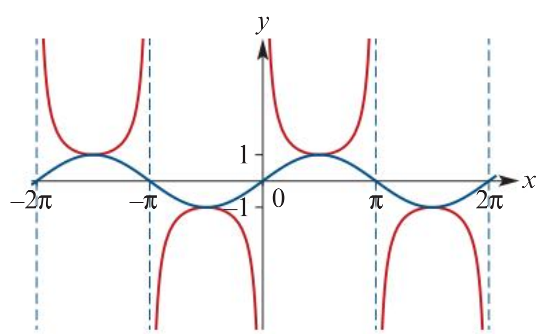
Function $y = \cos x$	Reciprocal function $y = \sec x$
x-axis intercepts at $x = \frac{(2n+1)\pi}{2},\ n \in \mathbb{Z}$	vertical asymptotes at $x = \frac{(2n+1)\pi}{2},\ n \in \mathbb{Z}$
domain $= \mathbb{R}$	domain $= \mathbb{R} \setminus \left\{ \frac{(2n+1)\pi}{2} : n \in \mathbb{Z} \right\}$
local maximums at $(2n\pi, 1),\ n \in \mathbb{Z}$ local minimums at $((2n+1)\pi, -1),\ n \in \mathbb{Z}$ range $= [-1, 1]$	local minimums at $(2n\pi, 1),\ n \in \mathbb{Z}$ local maximums at $((2n+1)\pi, -1),\ n \in \mathbb{Z}$ range $= (-\infty, -1] \cup [1, \infty)$

The cosecant function

The cosecant function is defined by

$$\operatorname{cosec} x = \frac{1}{\sin x}$$

provided $\sin x \neq 0$.



The graphs of $y = \sin x$ and $y = \operatorname{cosec} x$ are shown here on the same axes.

Function $y = \sin x$	Reciprocal function $y = \operatorname{cosec} x$
x-axis intercepts at $x = n\pi,\ n \in \mathbb{Z}$ domain $= \mathbb{R}$	vertical asymptotes at $x = n\pi,\ n \in \mathbb{Z}$ domain $= \mathbb{R} \setminus \{ n\pi : n \in \mathbb{Z} \}$
local maximums at $\left(2n\pi + \frac{\pi}{2}, 1\right),\ n \in \mathbb{Z}$	local minimums at $\left(2n\pi + \frac{\pi}{2}, 1\right),\ n \in \mathbb{Z}$
local minimums at $\left(2n\pi - \frac{\pi}{2}, -1\right),\ n \in \mathbb{Z}$	local maximums at $\left(2n\pi - \frac{\pi}{2}, -1\right),\ n \in \mathbb{Z}$
range $= [-1, 1]$	range $= (-\infty, -1] \cup [1, \infty)$

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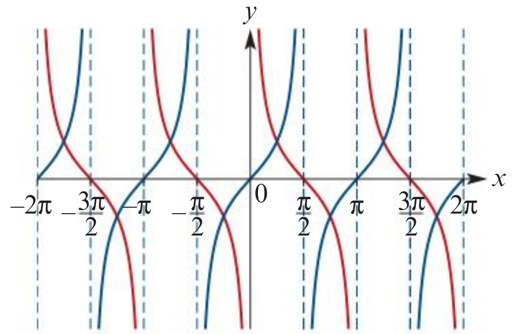
The cotangent function

The cotangent function is defined by

$$\cot x = \frac{\cos x}{\sin x}$$

provided $\sin x \neq 0$.

This diagram shows the graph of $y = \tan x$ in blue and the graph of $y = \cot x$ in red.



Function $y = \tan x$	Function $y = \cot x$
x-axis intercepts at $x = n\pi,\ n \in \mathbb{Z}$	vertical asymptotes at $x = n\pi,\ n \in \mathbb{Z}$
vertical asymptotes at $x = \dfrac{(2n+1)\pi}{2},\ n \in \mathbb{Z}$	x-axis intercepts at $x = \dfrac{(2n+1)\pi}{2},\ n \in \mathbb{Z}$
domain $= \mathbb{R} \setminus \left\{ \dfrac{(2n+1)\pi}{2} : n \in \mathbb{Z} \right\}$	domain $= \mathbb{R} \setminus \{ n\pi : n \in \mathbb{Z} \}$
range $= \mathbb{R}$	range $= \mathbb{R}$

Note the similarity between the graphs of $y = \cot x$ and $y = \tan x$. Using the complementary relationship between sine and cosine, we have

$$\cot x = \frac{\cos x}{\sin x} = \frac{\sin\left(\frac{\pi}{2} - x\right)}{\cos\left(\frac{\pi}{2} - x\right)} = \tan\left(\frac{\pi}{2} - x\right) = \tan\left(-\left(x - \frac{\pi}{2}\right)\right) = -\tan\left(x - \frac{\pi}{2}\right)$$

Therefore the graph of $y = \cot x$ is obtained from the graph of $y = \tan x$ by a reflection in the x-axis followed by a translation of $\frac{\pi}{2}$ units in the positive direction of the x-axis.

Transformations of the reciprocal circular functions

We now look at dilations, reflections and translations of the reciprocal circular functions.

Example 13

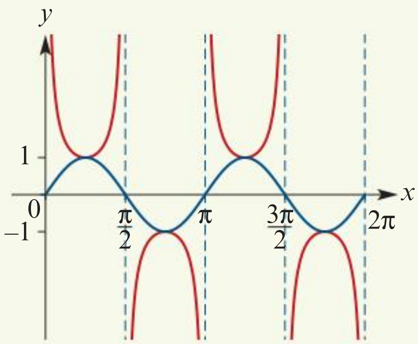
Sketch the graph of each of the following over the interval $[0, 2\pi]$:

a $y = \operatorname{cosec}(2x)$ **b** $y = 2\sec x$ **c** $y = -\cot x$

Solution

a The graph of $y = \operatorname{cosec}(2x)$ is obtained from the graph of $y = \operatorname{cosec} x$ by a dilation of factor $\frac{1}{2}$ from the y-axis.

It is helpful to draw the graph of $y = \sin(2x)$ on the same axes.



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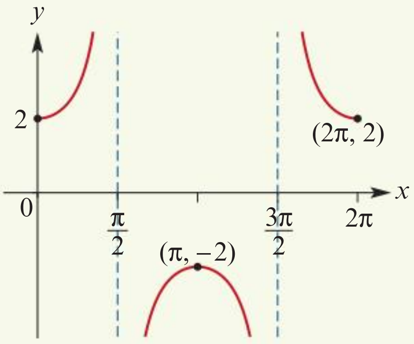
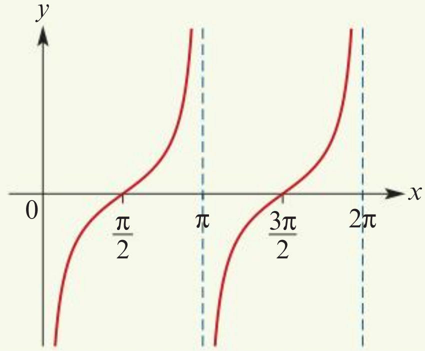
b The graph of $y = 2\sec x$ is obtained from the graph of $y = \sec x$ by a dilation of factor 2 from the x-axis.



c The graph of $y = -\cot x$ is obtained from the graph of $y = \cot x$ by a reflection in the x-axis.



Example 14

Sketch the graph of each of the following over the interval $[0, 2\pi]$:

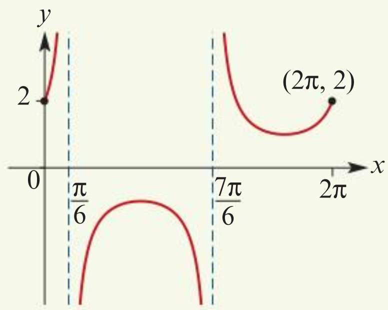
a $y = \sec\left(x + \frac{\pi}{3}\right)$ **b** $y = \operatorname{cosec}(x) - 2$ **c** $y = \cot\left(x - \frac{\pi}{4}\right)$

Solution

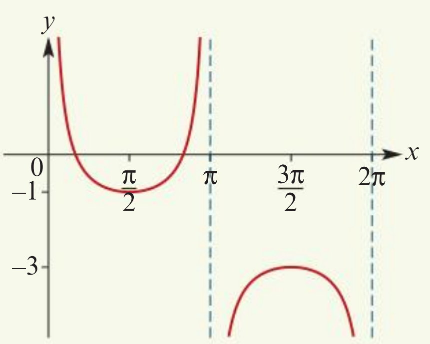
a The graph of $y = \sec\left(x + \frac{\pi}{3}\right)$ can be obtained from the graph of $y = \sec x$ by a translation of $\frac{\pi}{3}$ units in the negative direction of the x-axis.

The y-axis intercept is $\sec\left(\frac{\pi}{3}\right) = 2$.

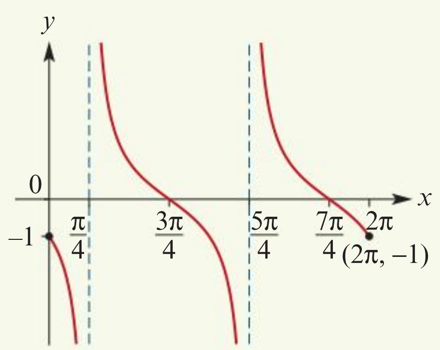
The asymptotes are $x = \frac{\pi}{6}$ and $x = \frac{7\pi}{6}$.



b The graph of $y = \operatorname{cosec}(x) - 2$ is obtained from the graph of $y = \operatorname{cosec} x$ by a translation of 2 units in the negative direction of the y-axis.



c The graph of $y = \cot\left(x - \frac{\pi}{4}\right)$ is obtained from the graph of $y = \cot x$ by a translation of $\frac{\pi}{4}$ units in the positive direction of the x-axis.



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Example 15

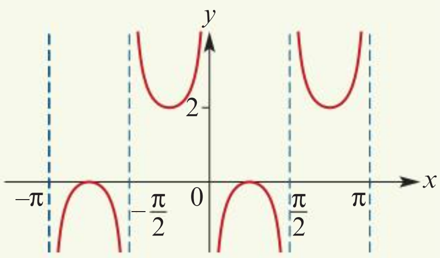
Describe a sequence of transformations that will take the graph of $y = \sec x$ to the graph of $y = -\sec\left(2x - \frac{\pi}{2}\right) + 1$. Sketch the transformed graph over the interval $[-\pi, \pi]$.

Solution

It helps to write the equation of the transformed graph as $y = -\sec\left(2\left(x - \frac{\pi}{4}\right)\right) + 1$.

An appropriate sequence is:

1. reflection in the x-axis
2. dilation of factor $\frac{1}{2}$ from the y-axis
3. translation of $\frac{\pi}{4}$ units to the right and 1 unit up.



Summary 13C

Reciprocal circular functions

- $\sec x = \dfrac{1}{\cos x}$ (provided $\cos x \neq 0$)
- $\operatorname{cosec} x = \dfrac{1}{\sin x}$ (provided $\sin x \neq 0$)
- $\cot x = \dfrac{\cos x}{\sin x}$ (provided $\sin x \neq 0$)

Exercise 13C

SF

Example 13
1 Sketch the graph of each of the following over the interval $[0, 2\pi]$:

a $y = \sec(2x)$ **b** $y = \cot(2x)$ **c** $y = 3\sec x$
d $y = 2\operatorname{cosec} x$ **e** $y = -\operatorname{cosec} x$ **f** $y = -2\sec x$

Example 14
2 Sketch the graph of each of the following over the interval $[0, 2\pi]$:

a $y = \sec\left(x - \frac{\pi}{2}\right)$ **b** $y = \cot\left(x + \frac{\pi}{4}\right)$ **c** $y = -\operatorname{cosec}\left(x + \frac{\pi}{2}\right)$
d $y = 1 + \sec x$ **e** $y = 2 - \operatorname{cosec} x$ **f** $y = 1 + \cot\left(x + \frac{\pi}{4}\right)$

Example 15
3 Describe a sequence of transformations that will take the graph of $y = \sec x$ to the graph of $y = -2\sec\left(x - \frac{\pi}{2}\right)$. Sketch the transformed graph over the interval $[-\pi, \pi]$.

4 Describe a sequence of transformations that will take the graph of $y = \operatorname{cosec} x$ to the graph of $y = \operatorname{cosec}(-2x) + 1$. Sketch the transformed graph over the interval $[0, 2\pi]$.

5 Describe a sequence of transformations that will take the graph of $y = \cot x$ to the graph of $y = -\cot\left(2x - \frac{\pi}{2}\right) - 1$. Sketch the transformed graph over the interval $[0, 2\pi]$.

6 On the one set of axes, sketch the graphs of $y = \sec x$ and $y = \operatorname{cosec} x$ over the interval $[0, 2\pi]$. Determine and label the points of intersection.

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Review

Chapter summary

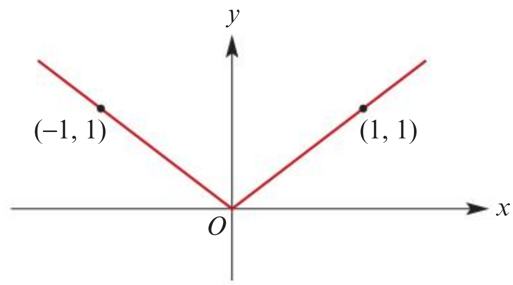
The modulus function

- The **modulus** or **absolute value** of a real number x is

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

For example: $|5| = 5$ and $|-5| = 5$.

- The graph of the modulus function $y = |x|$ is shown on the right.



- On the number line, the distance between two numbers a and b is given by $|a - b| = |b - a|$. For example: $|x - 2| < 5$ can be read as 'the distance of x from 2 is less than 5'.
- To sketch the graph of $y = |f(x)|$, first draw the graph of $y = f(x)$. Then reflect the sections of the graph that are below the x-axis so that they are above the x-axis.
- To sketch the graph of $y = f(|x|)$, first draw the graph of $y = f(x)$ for $x \geq 0$. Then reflect the graph across the y-axis to obtain the graph for $x \leq 0$.

Reciprocal functions

- If $y = f(x)$ is a function, then the **reciprocal function** is defined by the rule $y = \dfrac{1}{f(x)}$.
- To sketch the graph of $y = \dfrac{1}{f(x)}$, we first sketch the graph of $y = f(x)$.
- The x-axis intercepts of $y = f(x)$ will become vertical asymptotes of $y = \dfrac{1}{f(x)}$.
- Local maximums of $y = f(x)$ will become local minimums of $y = \dfrac{1}{f(x)}$, and vice versa.

Reciprocal circular functions

- $\sec x = \dfrac{1}{\cos x}$ (provided $\cos x \neq 0$)
- $\operatorname{cosec} x = \dfrac{1}{\sin x}$ (provided $\sin x \neq 0$)
- $\cot x = \dfrac{\cos x}{\sin x}$ (provided $\sin x \neq 0$)

Skills checklist

Check-list

Download this checklist from the Interactive Textbook, then print it and fill it out to check your skills.

13A **1** I can evaluate expressions using the modulus function. ☐

See Example 1 and Question 1

13A **2** I can solve equations and sketch intervals defined by the modulus function. ☐

See Example 2 and Question 2

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13A **3** I can sketch the graph of the modulus function and its transformations. ☐

See Example 4 and Question 4

13A **4** I can sketch graphs of functions of the form $y = f(|x|)$ and $y = |f(x)|$. ☐

See Example 5, Example 6 and Questions 6, 7 and 8

13B **5** I can sketch graphs of functions of the form $y = \dfrac{1}{f(x)}$. ☐

See Example 9, Example 10 and Questions 1 and 2

13C **6** I can sketch and transform the graphs of $y = \sec x$, $y = \operatorname{cosec} x$ and $y = \cot x$. ☐

See Example 13, Example 14 and Questions 1 and 2

Short-response questions

Technology-free short-response questions

SF

1 State the value of each of the following without using the absolute value function in your answer:

a $|-9|$ **b** $\left|-\dfrac{1}{400}\right|$ **c** $|9-5|$ **d** $|5-9|$ **e** $|\pi-3|$ **f** $|\pi-4|$

2 Let $f(x) = |x^2 - 3x|$. Solve the equation $f(x) = x$.

3 For each of the following, sketch the graph of $y = f(x)$ and state the range of f:

a $f(x) = |x^2 - 4x|$ **b** $f(x) = |x^2 - 4x| - 3$ **c** $f(x) = 3 - |x^2 - 4x|$

4 **a** Determine the four integer values of n such that $|n^2 - 9|$ is a prime number.

b Solve each equation for x:

i $x^2 + 5|x| - 6 = 0$ **ii** $x + |x| = 0$

c Solve the inequality $5 - |x| < 4$ for x.

5 For each of the following, sketch the graphs of $y = f(x)$ and $y = \dfrac{1}{f(x)}$ on the same axes:

a $f(x) = \dfrac{1}{2}(x^2 - 4)$ **b** $f(x) = (x+1)^2 + 1$

c $f(x) = \cos(x) + 1,\ x \in [0, 2\pi]$ **d** $f(x) = \sin(x) + 2,\ x \in [0, 2\pi]$

6 Sketch the graph of each of the following functions over the interval $[0, 2\pi]$:

a $y = 2\sec(x)$ **b** $y = -\operatorname{cosec}(x - \pi) + 1$ **c** $y = -\cot(2x)$

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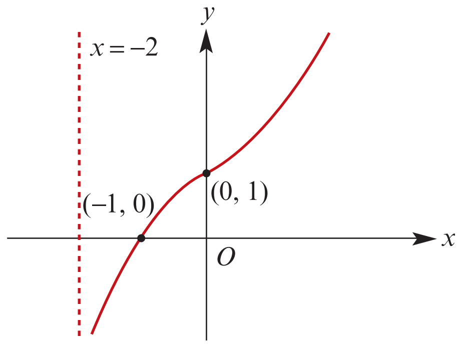
Review

7 This is the graph of $y = f(x)$.

Sketch the graph of $y = \dfrac{1}{f(x)}$.



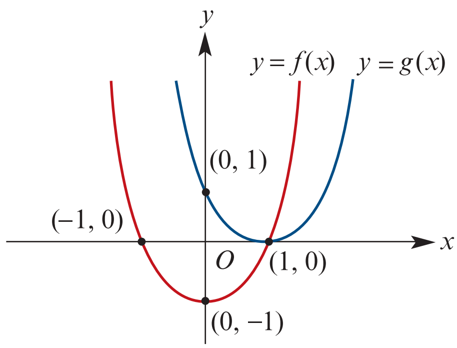
8 Let $g(x) = x^4 - x^2$.

a The graph of $y = g(x)$ has local minimums at the points $\left(\frac{1}{\sqrt{2}}, -\frac{1}{4}\right)$ and $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{4}\right)$. Use this information to help sketch the graph of $y = g(x)$.

b Sketch the graph of $y = \dfrac{1}{g(x)}$.

Technology-active short-response questions

9 These are the graphs of $y = f(x)$ and $y = g(x)$, where f and g are quadratic functions.



a Sketch the graphs of:

i $y = f(x) + g(x)$

ii $y = \dfrac{1}{f(x) + g(x)}$

iii $y = \dfrac{1}{f(x)} + \dfrac{1}{g(x)}$

b Use the points given to determine the rules $y = f(x)$ and $y = g(x)$.

c Hence determine, in simplest form, the rules:

i $y = f(x) + g(x)$ **ii** $y = \dfrac{1}{f(x) + g(x)}$ **iii** $y = \dfrac{1}{f(x)} + \dfrac{1}{g(x)}$

10 Let $f(x) = |mx + 2|$, where $m > 0$.

a Determine the x-axis intercept of the graph of f in terms of m.

b For what values of m is the x-axis intercept less than -2?

c **i** Determine the equation of the line ℓ that is perpendicular to the graph of f at the point with coordinates $(0, 2)$.

ii For $m > 1$, determine the coordinates of the other point of intersection of the line ℓ with the graph of f.

iii What happens for $m = 1$?

iv For what value of m does the line ℓ meet the graph of f where $x = -\frac{3}{2}$?

d Solve the equation $f(x) = \dfrac{1}{f(x)}$ for x in terms of m.

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11 **a** Consider the function with rule $f(x) = |x^2 - ax|$, where a is a constant.

i Sketch the graph of $y = f(x)$ for $a = 2$.

ii For $a \neq 0$, determine the x-axis intercepts of the graph of $y = f(x)$.

iii For $a \neq 0$, determine the coordinates of the local maximum on the graph of $y = f(x)$.

iv Determine the values of a for which the point $(-1, 4)$ lies on the graph of $y = f(x)$.

b Consider the function with rule $g(x) = |x|^2 - a|x|$, where a is a constant.

i Sketch the graph of $y = g(x)$ for $a = 2$.

ii For $a > 0$, determine the x-axis intercepts of the graph of $y = g(x)$.

iii For $a > 0$, determine the coordinates of the local minimums on the graph of $y = g(x)$.

iv Determine the value of a for which the point $(-1, 4)$ lies on the graph of $y = g(x)$.

c For $a > 0$, determine the values of x such that $f(x) = g(x)$.

d For $a < 0$, determine the values of x such that $f(x) = g(x)$.

e Determine the rule for the function $h(x) = f(x) + g(x)$.

f Sketch the graph of $y = h(x)$ for $a = 2$.

g Sketch the graphs of $y = \dfrac{1}{f(x)}$ and $y = \dfrac{1}{g(x)}$ for $a = 2$.

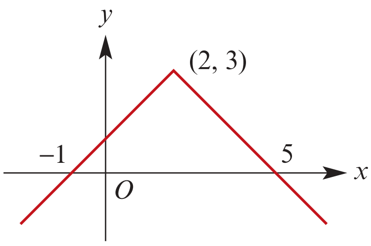
Multiple-choice questions

Technology-free multiple-choice questions

1 The rule for the graph shown is

A $y = |x - 2| + 3$ **B** $y = |-x + 2| + 3$

C $y = |x + 2| + 3$ **D** $y = -|x - 2| + 3$



2 The graph of $y = \dfrac{1}{x^2 - 4}$ has

A a local minimum at $(0, -\frac{1}{4})$ **B** a local maximum at $(\frac{1}{4}, 0)$

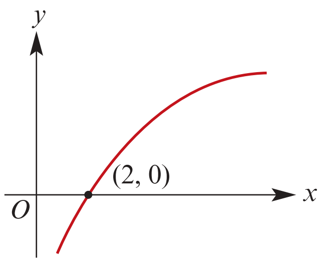
C a local maximum at $(0, -4)$ **D** a local maximum at $(0, -\frac{1}{4})$

3 The graph of $y = \dfrac{1}{x^2 + 8x + k}$ will have two vertical asymptotes provided

A $k = 16$ **B** $k < 16$ **C** $k > 16$ **D** $k < -4$ or $k > 4$

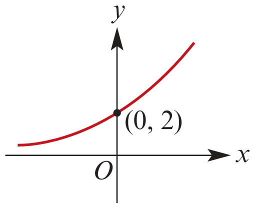
Review

4 The graph of $y = f(x)$ is shown on the right.

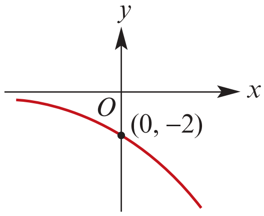


Which one of the following best represents the graph of $y = \dfrac{1}{f(x)}$?

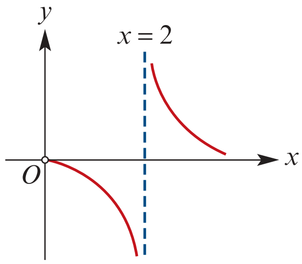
A



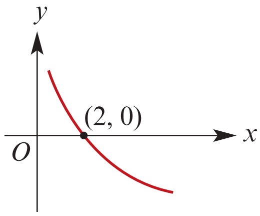
B



C



D



5 Let $f(x) = 2x^2 + 3x - 20$. The graph of $y = \dfrac{1}{f(x)}$ has

A vertical asymptotes at $x = \dfrac{5}{2}$ and $x = 4$

B vertical asymptotes at $x = -\dfrac{5}{2}$ and $x = 4$

C a local minimum at the point $\left(-\dfrac{3}{4}, -\dfrac{169}{8}\right)$

D a local maximum at the point $\left(-\dfrac{3}{4}, -\dfrac{8}{169}\right)$

6 Consider the equation $|x^2 + bx - c| = d$, where b, c and d are positive constants. This equation has four solutions when

A $d > \dfrac{b^2}{4} + c$

B $d > b^2 + 4c$

C $d < 0$

D $0 < d < \dfrac{b^2}{4} + c$

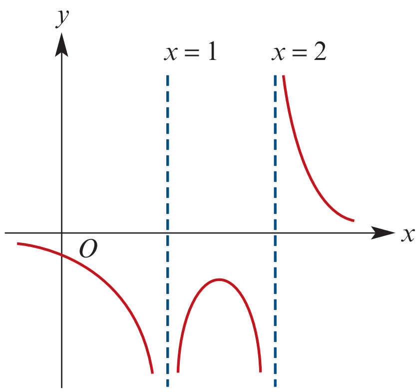
7 This is the graph of

A $y = \dfrac{1}{(x-1)(x-2)}$

B $y = \dfrac{x}{(x-1)(x-2)}$

C $y = \dfrac{(x-1)(x-2)}{x}$

D $y = \dfrac{1}{(x-2)(x-1)^2}$



8 The graphs of $y = -|x-1| + 1$ and $y = |x-2|$ intersect at every point in the interval

A $[1, 2]$ **B** $[-1, -2]$ **C** $[1, -2]$ **D** $[-1, 2]$

9 If $y = 2\cos x$ and $y = \sec x$ are graphed over the interval $[0, 2\pi]$, then these graphs will intersect at

A $x = \dfrac{\pi}{6}, \dfrac{5\pi}{6}, \dfrac{7\pi}{6}, \dfrac{11\pi}{6}$ **B** $x = \dfrac{\pi}{4}, \dfrac{3\pi}{4}, \dfrac{5\pi}{4}, \dfrac{7\pi}{4}$

C $x = \dfrac{\pi}{4}, \pi, \dfrac{7\pi}{4}$ **D** $x = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$

10 If $y = \sec(2x)$ is graphed over the interval $[0, \pi]$, then there will be vertical asymptotes at

A $x = \dfrac{\pi}{4}, \pi, \dfrac{3\pi}{4}$ **B** $x = \dfrac{\pi}{4}, \dfrac{3\pi}{4}$

C $x = \dfrac{\pi}{3}, \dfrac{2\pi}{3}$ **D** $x = 0, \pi$

Technology-active multiple-choice questions

11 Consider the function $f(x) = \operatorname{cosec}(x)$ defined on the interval $[0, \pi]$. The horizontal line $y = c$ cuts the graph of f at points A and B. The length of AB is 2. The value of c, correct to 2 decimal places, is

A 1.55 **B** 1.65 **C** 1.75 **D** 1.85

12 The maximum value of the function $f(x) = \dfrac{1}{2x^2 + 5x + 7}$ occurs where x is equal to

A −1.25 **B** −1.20 **C** 1.25 **D** 1.20

13 The sum of all solutions of the equation $|x^2 - 4x| = 3$ is equal to

A 4 **B** 5 **C** 7 **D** 8

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14 Let $f(x) = x^2 - 4$. The graphs of $y = f(x)$ and $y = \frac{1}{f(x)}$ intersect at n points, where n is equal to

A 1 **B** 2 **C** 3 **D** 4

15 The graph of $y = \frac{1}{x^2 + bx + c}$ has an asymptote at $x = 2$. The graph also passes through the point $\left(1, -\frac{1}{2}\right)$. The value of c is

A -2 **B** -1 **C** 0 **D** 1

16 Let $f(x) = |2x - 3| - 4$. The graph of $y = \frac{1}{f(x)}$ will have vertical asymptotes at

A $x = -\frac{1}{2}$ and $x = \frac{7}{2}$
B $x = \frac{1}{2}$ and $x = -\frac{3}{2}$
C $x = -\frac{3}{2}$ and $x = \frac{5}{2}$
D $x = \frac{1}{2}$ and $x = -\frac{3}{2}$

17 Which of the following functions will have exactly three x-intercepts?

A $f(x) = |x^2 - 1|$
B $f(x) = |x^2 + 1|$
C $f(x) = |x|(|x| + 1)$
D $f(x) = |x|(|x| - 1)$

18 Which of the following functions will have a range equal to $[-2, \infty)$?

A $f(x) = |x|^2 + |x| + 2$
B $f(x) = |x|^2 - |x| - 2$
C $f(x) = |x|^2 + 2|x| - 2$
D $f(x) = |x|^2 - 2|x| - 2$

19 If the range of $y = f(x)$ is $[-2, 3]$, then the range of $y = \frac{1}{f(x)}$ is

A $[2, 3]$
B $\left[-\frac{1}{2}, \frac{1}{3}\right]$
C $(-\infty, -2] \cup [3, \infty)$
D $\left(-\infty, -\frac{1}{2}\right] \cup \left[\frac{1}{3}, \infty\right)$

20 If $|x^2 - 2x - 2| = a$ has three solutions, then a is equal to

A 1.5 **B** 2 **C** 2.5 **D** 3

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14

Review of geometry and vector proof

Chapter contents

There are three main reasons for the study of geometry at school.

The first reason is that the properties of figures in two and three dimensions are helpful in other areas of mathematics. The second reason is that the subject provides a good setting to show how a large body of results may be deduced from a small number of assumptions. The third reason is that it gives you, the student, the opportunity to practise writing coherent, logical mathematical arguments.

Thi chapter provides a review of geometry from Years 9 and 10 and is included as a preparation for the following chapter and also to provide background and definitions for the section on vector proof.

You may choose to omit sections 14*A* – 14*F and only refer to them when needed for* 14*G and Chapter* 15

14A Points, lines and angles

Learning intentions

- To be able to determine unknown angles associated with parallel lines.

In this section we do not pretend to be fully rigorous, but aim to make you aware that assumptions are being made and that we base the proofs of the results on these assumptions. The assumptions do seem obvious to us, but there are ways of making the study of geometry even more rigorous. However, whatever we do, we will need to accept a set of results as our starting point.

Points, lines and planes

We begin with a few basic concepts. No formal definitions are given.

Point In geometry, a point is used to indicate position.

Line In the physical world, we may illustrate the idea of a line as a tightly stretched wire or a fold in a piece of paper. A line has no width and is infinite in length.

Plane A plane has no thickness and it extends infinitely in all directions.

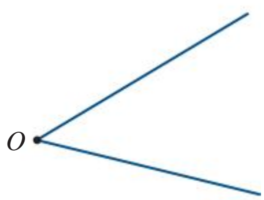
We make the following assumptions about points and lines:

- Given a point and a line, the point may or may not lie on the line.
- Two distinct points are contained in exactly one line.
- Two distinct lines do not have more than one point in common.

Angles

A **ray** is a portion of a line consisting of a point O and all the points on one side of O.

An **angle** is the figure formed by two distinct rays which have a common endpoint O. The common endpoint is called the **vertex** of the angle.



- If the two rays are part of one straight line, the angle is called a **straight angle** and measures 180°.

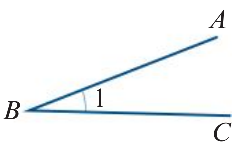
- A **right angle** is an angle of 90°.
- An **acute angle** is an angle which is less than 90°.
- An **obtuse angle** is an angle which is greater than 90° and less than 180°.
- **Supplementary angles** are two angles whose sum is 180°.
- **Complementary angles** are two angles whose sum is 90°.

Naming angles

The convention for naming an angle is to fully describe the rays of the angle and the endpoint where the rays meet.

The marked angle is denoted by $\angle ABC$.

When there is no chance of ambiguity, it can be written as $\angle B$.



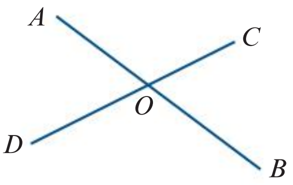
Sometimes an angle can simply be numbered as shown, and in a proof we refer to the angle as $\angle 1$.

The important thing is that the writing of your argument must be clear and unambiguous. With complicated diagrams, the $\angle ABC$ notation is safest.

Theorem

If two straight lines intersect, then the opposite angles are equal in pairs.

Such angles are said to be **vertically opposite**.



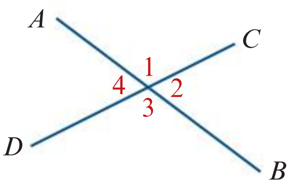
Proof using angle names

$\angle AOC$ and $\angle COB$ are supplementary.
That is, $\angle AOC + \angle COB = 180°$.

Also, $\angle COB$ and $\angle BOD$ are supplementary.
That is, $\angle COB + \angle BOD = 180°$.

Hence $\angle AOC = \angle BOD$.

The proof can also be presented with the labelling technique.



Proof using number labels

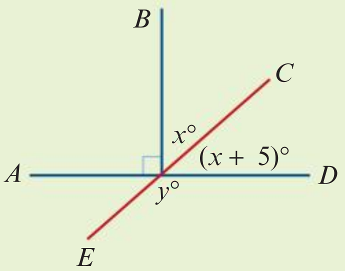
$\angle 1 + \angle 2 = 180°$ (supplementary angles)

$\angle 2 + \angle 3 = 180°$ (supplementary angles)

$\therefore \quad \angle 1 = \angle 3$

Example 1

Determine the values of x and y in the diagram.



Solution

$x + (x + 5) = 90$ (complementary angles)

$2x = 85$

$\therefore \quad x = 42.5$

$y + (x + 5) = 180$ (supplementary angles)

$y + 47.5 = 180$

$\therefore \quad y = 132.5$

Parallel lines

Given two distinct lines ℓ_1 and ℓ_2 in the plane, either the lines intersect in a single point or the lines have no point in common. In the latter case, the lines are said to be **parallel**. We can write this as $\ell_1 \parallel \ell_2$.

Here is another important assumption.

Playfair's axiom

Given any point P not on a line ℓ, there is only one line through P parallel to ℓ.

From this we have the following results for three distinct lines ℓ_1, ℓ_2 and ℓ_3 in the plane:

- If $\ell_1 \parallel \ell_2$ and $\ell_2 \parallel \ell_3$, then $\ell_1 \parallel \ell_3$.
- If $\ell_1 \parallel \ell_2$ and ℓ_3 intersects ℓ_1, then ℓ_3 also intersects ℓ_2.

We prove the first of these and leave the other as an exercise. The proof is by contradiction.

Proof Let ℓ_1, ℓ_2 and ℓ_3 be three distinct lines in the plane such that $\ell_1 \parallel \ell_2$ and $\ell_2 \parallel \ell_3$. Now suppose that ℓ_1 is not parallel to ℓ_3. Then ℓ_1 and ℓ_3 meet at a point P. But by Playfair's axiom, there is only one line parallel to ℓ_2 passing through P. Therefore $\ell_1 = \ell_3$. But this gives a contradiction, as ℓ_1 and ℓ_3 are distinct by assumption.

Corresponding, alternate and co-interior angles

The following types of pairs of angles play an important role in considering parallel lines.

In the diagram, the lines ℓ_1 and ℓ_2 are crossed by a **transversal** ℓ_3.

Corresponding angles:

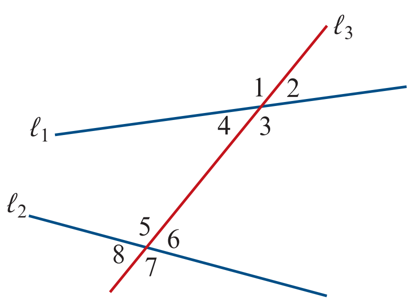
- Angles 1 and 5
- Angles 2 and 6
- Angles 3 and 7
- Angles 4 and 8

Alternate angles:

- Angles 3 and 5
- Angles 4 and 6

Co-interior angles:

- Angles 3 and 6
- Angles 4 and 5



The following result is easy to prove, and you should complete it as an exercise.

Theorem

When two lines are crossed by a transversal, any one of the following three conditions implies the other two:

- a pair of alternate angles are equal
- a pair of corresponding angles are equal
- a pair of co-interior angles are supplementary.

The next result is important as it gives us the ability to establish properties of the angles associated with parallel lines crossed by a transversal, and it also gives us an easily applied method for proving that two lines are parallel.

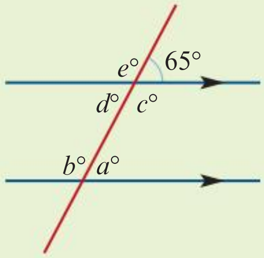
Theorem

- If two parallel lines are crossed by a transversal, then alternate angles are equal.
- Conversely, if two lines crossed by a transversal form an equal pair of alternate angles, then the two lines are parallel.

Example 2

Determine the values of the pronumerals.

Note: The arrows indicate that the two lines are parallel.



Solution

$a = 65$ (corresponding)

$d = 65$ (alternate with a)

$b = 115$ (co-interior with d)

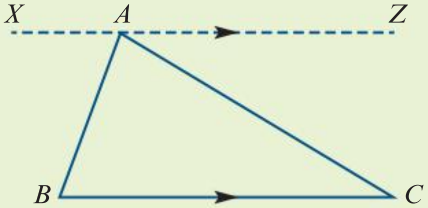
$e = 115$ (corresponding with b)

$c = 115$ (vertically opposite e)

Example 3

For $\triangle ABC$ shown in the diagram, the line XAZ is drawn through vertex A parallel to BC.

Use this construction to prove that the sum of the interior angles of a triangle is a straight angle (180°).



Solution

$\angle ABC = \angle XAB$ (alternate angles)

$\angle ACB = \angle ZAC$ (alternate angles)

$\angle XAB + \angle ZAC + \angle BAC$ is a straight angle.

Therefore $\angle ABC + \angle ACB + \angle BAC = 180°$.

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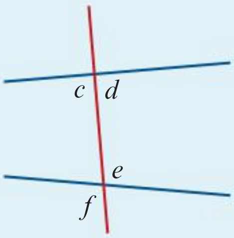
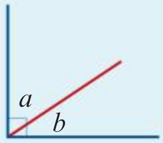
Summary 14A

- **Pairs of angles**
  - complementary (a and b)
  - supplementary (c and d)
  - vertically opposite (e and f)
  - alternate (c and e)
  - corresponding (c and f)
  - co-interior (d and e)





- Parallel lines

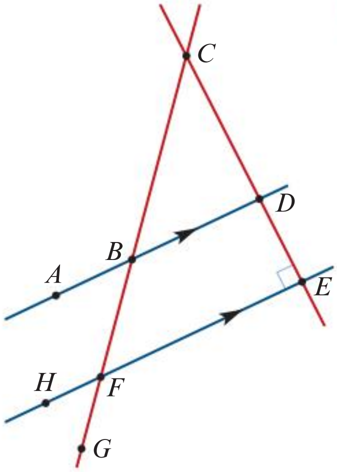
  If two parallel lines are crossed by a transversal, then:
  - alternate angles are equal
  - corresponding angles are equal
  - co-interior angles are supplementary.

  If two lines crossed by a transversal form an equal pair of alternate angles, then the two lines are parallel.

Exercise 14A

SF

1 Consider the diagram shown.



a State whether each of the following angles is acute, obtuse, right or straight:

i $\angle ABC$ **ii** $\angle HFE$ **iii** $\angle CBD$ **iv** $\angle FED$

b State which angle is:

i corresponding to $\angle ABC$

ii alternate to $\angle ABF$

iii vertically opposite $\angle BFE$

iv co-interior to $\angle DBF$

c State which angles are:

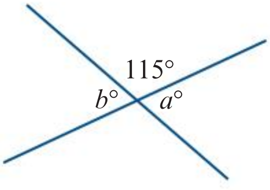
i complementary to $\angle BCD$

ii supplementary to $\angle CBD$

Example 1, 2

2 Calculate the values of the unknowns for each of the following. Give reasons.

a



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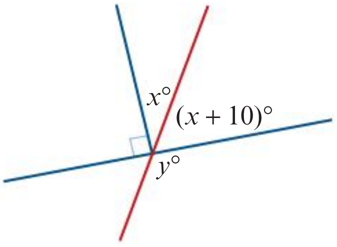

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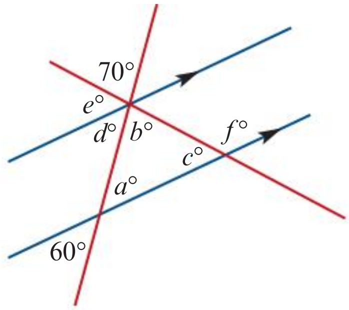
c



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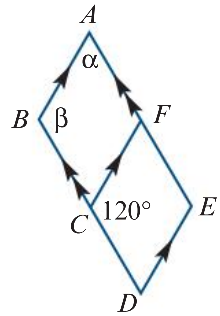
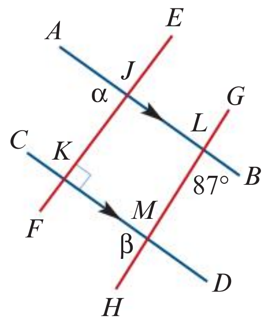


e



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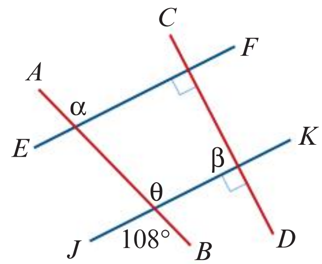
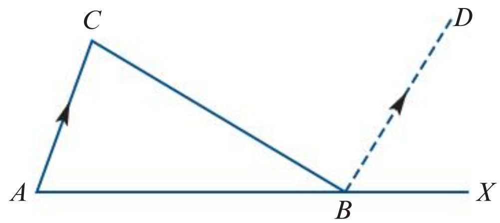


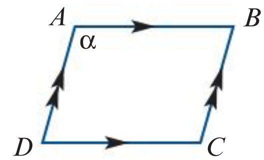
Example 3

3 Side AB of $\triangle ABC$ is extended to point X and line BD is drawn parallel to side AC. Prove that the sum of two interior angles of a triangle is equal to the opposite exterior angle.

Hint: Using the diagram, this means showing that $\angle CAB + \angle ACB = \angle CBX$.



4 Recall that a parallelogram is a quadrilateral whose opposite sides are parallel. A parallelogram $ABCD$ is shown on the right. Let $\angle A = \alpha$.

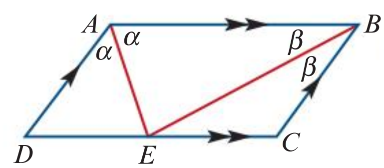


a Determine the sizes of $\angle B$ and $\angle D$ in terms of α.

b Hence determine the size of $\angle C$ in terms of α.

5 Prove the converse of the result in Question 4. That is, prove that if the opposite angles of a quadrilateral are equal, then the quadrilateral is a parallelogram.

6 Prove that AE is perpendicular to EB.



7 The lines PQ and RS are parallel. A transversal meets PQ at X and RS at Y. Lines XA and YB are bisectors of the angles PXY and XYS. Prove that XA is parallel to YB.

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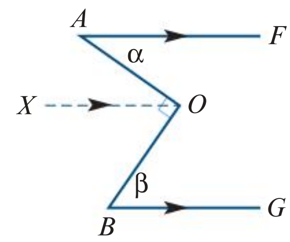
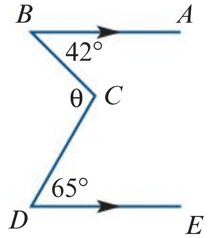
8 For the diagram on the right, show that $\alpha + \beta = 90°$.

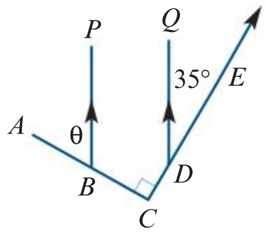


9 For each of the following, use a construction line to determine the angle marked θ:

a



b



14B Triangles and polygons

Learning intentions

To be able to use the properties of polygons and in particular triangles to determine unknown angles and side lengths.

We first define polygons.

A **line segment** AB is a portion of a line consisting of two distinct points A and B and all the points between them.

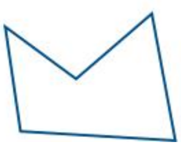
If distinct points $A_1, A_2, \ldots, A_n$ in the plane are connected in order by the line segments $A_1A_2, A_2A_3, \ldots, A_nA_1$, then the figure formed is a **polygon**. The points $A_1, A_2, \ldots, A_n$ are the vertices of the polygon, and the line segments $A_1A_2, A_2A_3, \ldots, A_nA_1$ are its sides.

Types of polygons

A **simple polygon** is a polygon such that no two sides have a point in common except a vertex.

A **convex polygon** is a polygon that contains each line segment connecting any pair of points on its boundary. For example, the left-hand figure is convex, while the right-hand figure is not.

A convex polygon



A non-convex polygon

Note: In this chapter we will always assume that the polygons being considered are convex.

A **regular polygon** is a polygon in which all the angles are equal and all the sides are equal.

Names of polygons

- triangle (3 sides)
- quadrilateral (4 sides)
- pentagon (5 sides)
- hexagon (6 sides)
- heptagon (7 sides)
- octagon (8 sides)
- nonagon (9 sides)
- decagon (10 sides)
- dodecagon (12 sides)

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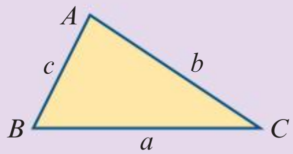
Triangles

A **triangle** is a figure formed by three line segments determined by a set of three points not on one line. If the three points are A, B and C, then the figure is called triangle ABC and commonly written $\triangle ABC$. The points A, B and C are called the vertices of the triangle.

Triangle inequality

An important property of a triangle is that any side is shorter than the sum of the other two.

In $\triangle ABC$: $a < b + c$, $b < c + a$ and $c < a + b$.



Note: For $\triangle ABC$ labelled as shown, we have $c < b < a$ if and only if $\angle C < \angle B < \angle A$.

The following two results have been proved in Example 3 and in Question 3 of Exercise 14A.

Angles of a triangle

- The sum of the three interior angles of a triangle is $180°$.
- The sum of two interior angles of a triangle is equal to the opposite exterior angle.

Classification of triangles

Equilateral triangle	a triangle in which all three sides are equal
Isosceles triangle	a triangle in which two sides are equal
Scalene triangle	a triangle in which all three sides are unequal

Important lines in a triangle

Median A **median** of a triangle is a line segment from a vertex to the midpoint of the opposite side.

Altitude An **altitude** of a triangle is a line segment from a vertex to the opposite side (possibly extended) which forms a right angle where it meets the opposite side.

Example 4

The sides of a triangle are $6 - x$, $4x + 1$ and $2x + 3$. Determine the value of x for which the triangle is isosceles, and show that if it is isosceles, then it is equilateral.

Solution

$$6 - x = 4x + 1$$
$$\Rightarrow \quad 5x = 5$$
$$\Rightarrow \quad x = 1$$

Similarly, if $6 - x = 2x + 3$ or if $4x + 1 = 2x + 3$, we can show that $x = 1$

When $x = 1$, we have $6 - x = 5$, $4x + 1 = 5$ and $2x + 3 = 5$. Hence the triangle is equilateral with each side of length 5 units.

Explanation

We want to show that if any two side lengths are equal, then the third length is the same.

Because any two distinct lines can intersect at most once, there is no other x-value where two sides are equal but not the third.

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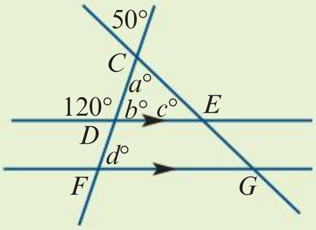
Example 5

Determine the values of a, b, c and d, giving reasons.



Solution

$a = 50$ (vertically opposite angles)

$b = 60$ (supplementary angles)

$c = 180 - (50 + 60) = 70$ (angle sum of a triangle)

$d = 60$ (corresponding angles $DE \parallel FG$)

Angle sum of a polygon

If a convex polygon has n sides, then we can draw $n - 3$ diagonals from a vertex. In this way, we can divide the polygon into $n - 2$ triangles, each with an angle sum of 180°. We have drawn a hexagon to illustrate this, but we could have used any polygon.

For a concave polygon, it may not always be possible to draw $n - 3$ diagonals from a single vertex, but it can be shown that the polygon can still be divided into $n - 2$ triangles with the right choice of diagonals.

Angle sum of a polygon

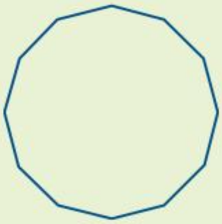
- The sum of the interior angles of an n-sided polygon is $(n - 2)180°$.
- Each interior angle of a regular n-sided polygon has size $\dfrac{(n-2)}{n}180°$.

Example 6

A regular dodecagon is shown to the right.

a Determine the sum of the interior angles of a dodecagon.

b Determine the size of each interior angle of a regular dodecagon.



Solution

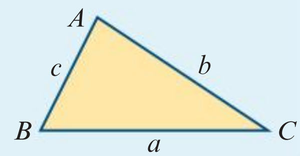
a The angle sum of a polygon with n sides is $(n - 2)180°$.
Therefore the angle sum of a dodecagon is 1800°.

b Each of the interior angles is $\dfrac{1800}{12} = 150°$.

Summary 14B

- **Polygons**
  - The sum of the interior angles of an n-sided polygon is $(n - 2)180°$.
  - In a **regular polygon**, all the angles are equal and all the sides are equal.
    Each interior angle of a regular n-sided polygon has size $\frac{(n-2)}{n}180°$.
- **Triangles**
  - An **equilateral triangle** is a triangle in which all three sides are equal.
  - An **isosceles triangle** is a triangle in which two sides are equal.
  - A **scalene triangle** is a triangle in which all three sides are unequal.
  - The sum of the three interior angles of a triangle is 180°.
  - The sum of two interior angles of a triangle is equal to the opposite exterior angle.

  In $\triangle ABC$:
  - $a < b + c, \; b < c + a$ and $c < a + b$
  - $c < b < a$ if and only if $\angle C < \angle B < \angle A$



Exercise 14B

SF

1. Is it possible for a triangle to have sides of lengths:
   a 12 cm, 9 cm, 20 cm **b** 24 cm, 24 cm, 40 cm
   c 5 cm, 5 cm, 5 cm **d** 12 cm, 9 cm, 2 cm?
2. Describe each of the triangles in Question 1.
3. If a triangle has sides 10 cm and 20 cm, what can be said about the third side?
4. The sides of a triangle are $2n - 1$, $n + 5$ and $3n - 8$.
   a Determine the value(s) of n for which the triangle is isosceles.
   b Is there a value of n which makes the triangle equilateral?

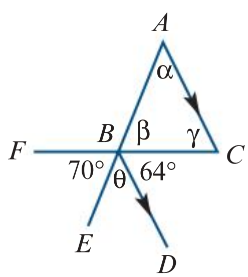
Example 4

5. The sides of a triangle are $2n - 1$, $n + 7$ and $3n - 9$. Prove that if the triangle is isosceles, then it is equilateral.

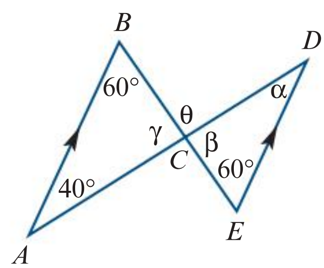
Example 5

6. Calculate the value of the unknowns for each of the following. Give reasons.

a



b



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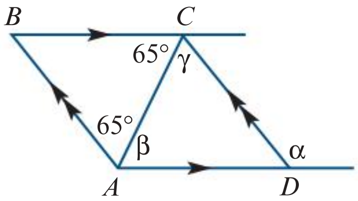
SF

c



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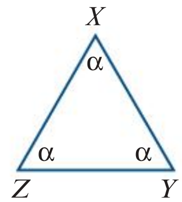

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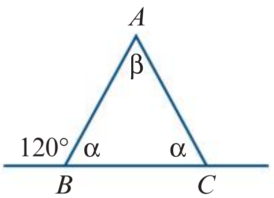

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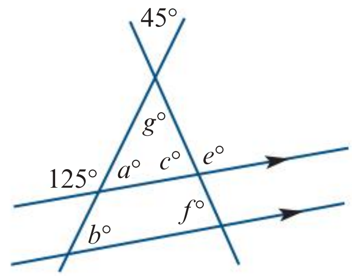

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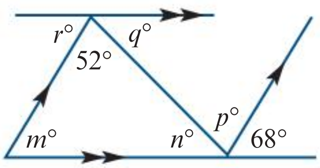


Example 6

7 Determine the interior-angle sum and the size of each angle of a regular polygon with:

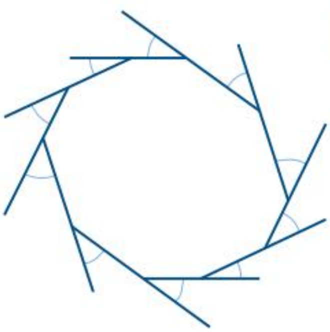
a 6 sides **b** 12 sides **c** 20 sides

CF

8 In the decagon shown on the right, each side has been extended to form an exterior angle.

a Explain why the sum of the interior angles plus the sum of the exterior angles is 1800°.

b Hence determine the sum of the decagon's 10 exterior angles.



9 Prove that the sum of the exterior angles of any polygon is 360°.

10 If the sum of the interior angles of a polygon is four times the sum of the exterior angles, how many sides does the polygon have?

11 Assume that the sum of the interior angles of a polygon is k times the sum of the exterior angles (where $k \in \mathbb{N}$). Prove that the polygon has $2(k + 1)$ sides.

14C Congruence and proofs

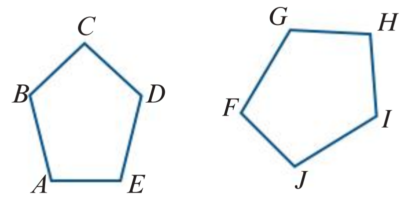
Learning intentions

- To be able to use congruence to establish results involving triangles.

Two plane figures are called **congruent** if one figure can be moved on top of the other figure, by a sequence of translations, rotations and reflections, so that they coincide exactly.

Congruent figures have exactly the same shape and size. For example, the two figures shown are congruent. We can write:

$$\text{pentagon } ABCDE \equiv \text{pentagon } FGHIJ$$



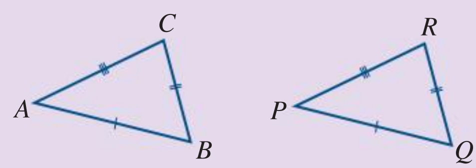
When two figures are congruent, we can determine a transformation that pairs up every part of one figure with the corresponding part of the other, so that:

- paired angles have the same size
- paired line segments have the same length
- paired regions have the same area.

Congruent triangles

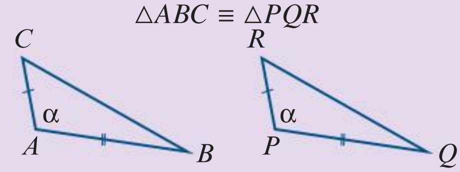
We used the four standard congruence tests for triangles in Chapter 3.

- **The SSS congruence test**
  If the three sides of one triangle are respectively equal to the three sides of another triangle, then the two triangles are congruent.



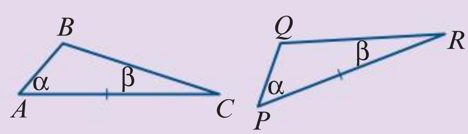
$\triangle ABC \equiv \triangle PQR$

- **The SAS congruence test**
  If two sides and the included angle of one triangle are respectively equal to two sides and the included angle of another triangle, then the two triangles are congruent.



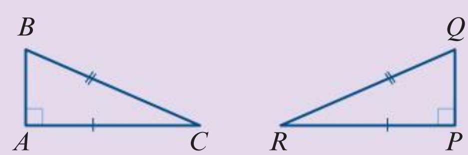
$\triangle ABC \equiv \triangle PQR$

- **The AAS congruence test**
  If two angles and one side of one triangle are respectively equal to two angles and the matching side of another triangle, then the two triangles are congruent.



$\triangle ABC \equiv \triangle PQR$

- **The RHS congruence test**
  If the hypotenuse and one side of one right-angled triangle are respectively equal to the hypotenuse and one side of another right-angled triangle, then the two triangles are congruent.



$\triangle ABC \equiv \triangle PQR$

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Classification of quadrilaterals

Trapezium	a quadrilateral with at least one pair of opposite sides parallel
Parallelogram	a quadrilateral with both pairs of opposite sides parallel
Rhombus	a parallelogram with a pair of adjacent sides equal
Rectangle	a quadrilateral in which all angles are right angles
Square	a quadrilateral that is both a rectangle and a rhombus

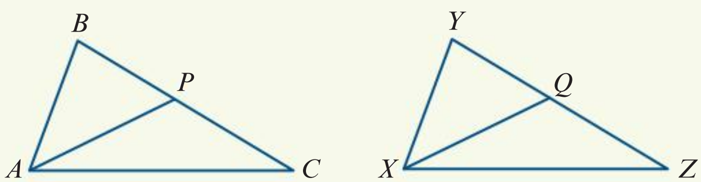
Proofs using congruence

Example 7

Let $\triangle ABC$ and $\triangle XYZ$ be such that $\angle BAC = \angle YXZ$, $AB = XY$ and $AC = XZ$.

If P and Q are the midpoints of BC and YZ respectively, prove that $AP = XQ$.

Solution



From the given conditions, we have $\triangle ABC \equiv \triangle XYZ$ (SAS).

Therefore $\angle ABP = \angle XYQ$ and $BC = YZ$.

Thus $BP = YQ$, as P and Q are the midpoints of BC and YZ respectively.

Hence $\triangle ABP \equiv \triangle XYQ$ (SAS) and so $AP = XQ$.

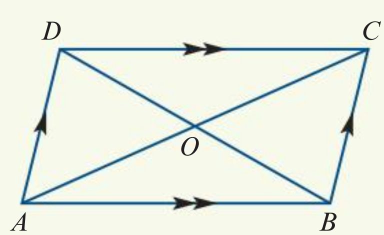
Example 8

a Prove that, in a parallelogram, the diagonals bisect each other.

b Prove that if the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram.

Solution

a Note that opposite sides of a parallelogram are equal. (See Question 8 of Exercise 14C.)



In triangles DOC and BOA:

$\angle ODC = \angle OBA$	(alternate angles $CD \parallel AB$)
$\angle OCD = \angle OAB$	(alternate angles $CD \parallel AB$)
$\angle AOB = \angle DOC$	(vertically opposite)
$AB = CD$	(opposite sides of parallelogram are equal)
$\triangle DOC \equiv \triangle BOA$	(AAS)

Hence $AO = OC$ and $DO = OB$.

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b $OD = OB$ (diagonals bisect each other)

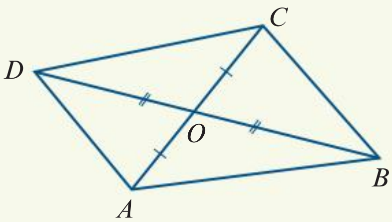
$OA = OC$ (diagonals bisect each other)

$\angle AOB = \angle DOC$ (vertically opposite)

$\angle DOA = \angle COB$ (vertically opposite)

$\triangle DOC \equiv \triangle BOA$ (SAS)

$\triangle DOA \equiv \triangle BOC$ (SAS)



Therefore $\angle ODC = \angle OBA$ and so $CD \parallel AB$, since alternate angles are equal. Similarly, we have $AD \parallel BC$. Hence $ABCD$ is a parallelogram.

Example 9

Prove that the triangle formed by joining the midpoints of the three sides of an isosceles triangle (with the midpoints as the vertices of the new triangle) is also isosceles.

Solution

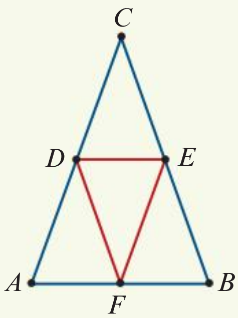
Assume $\triangle ABC$ is isosceles with $CA = CB$ and $\angle CAB = \angle CBA$. (See Question 3 of Exercise 14C.)

Then we have $DA = EB$, where D and E are the midpoints of CA and CB respectively.

We also have $AF = BF$, where F is the midpoint of AB.

Therefore $\triangle DAF \equiv \triangle EBF$ (SAS).

Hence $DF = EF$ and so $\triangle DEF$ is isosceles.



Summary 14C

- **Congruent figures** have exactly the same shape and size.
- If triangle ABC is congruent to triangle XYZ, this can be written as $\triangle ABC \equiv \triangle XYZ$.
- Two triangles are congruent provided any one of the following four conditions holds:
  - **SSS** the three sides of one triangle are equal to the three sides of the other triangle
  - **SAS** two sides and the included angle of one triangle are equal to two sides and the included angle of the other triangle
  - **AAS** two angles and one side of one triangle are equal to two angles and the matching side of the other triangle
  - **RHS** the hypotenuse and one side of a right-angled triangle are equal to the hypotenuse and one side of another right-angled triangle.

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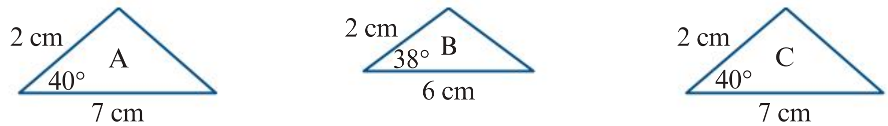
Exercise 14C

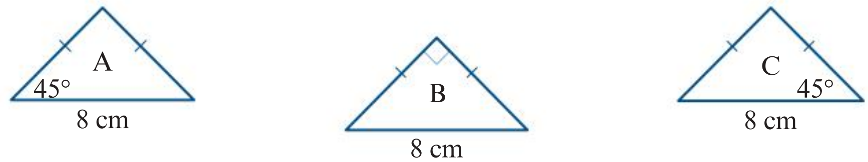
SF

1 In each part, determine pairs of congruent triangles. State the congruence tests used.

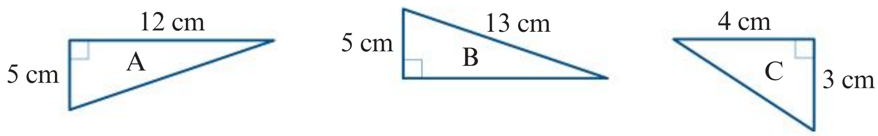
a



b

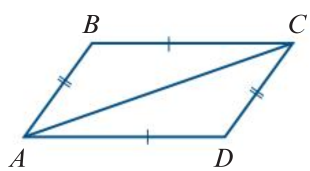


c

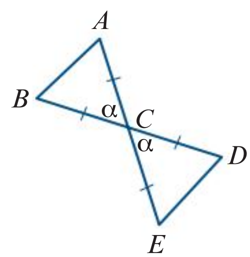


2 Name the congruent triangles and state the congruence test used:

a



b



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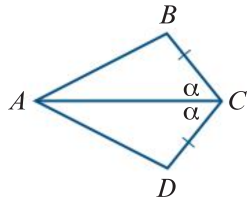
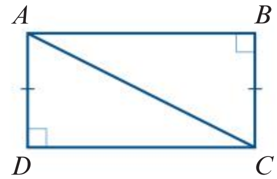
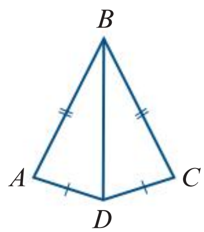


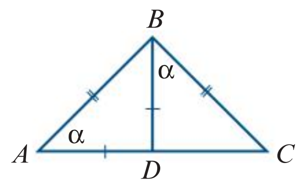
d



e



f



Example 7

3 Prove that if $\triangle ABC$ is isosceles with $AB = AC$, then $\angle ABC = \angle ACB$.

4 Prove that if $\triangle ABC$ is such that $\angle ABC = \angle ACB$, then $\triangle ABC$ is isosceles. (This is the converse of Question 3.)

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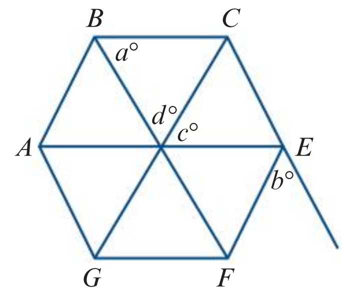
SF

5 For the quadrilateral shown, prove that $AB \parallel CD$.

6 $ABCEFG$ is a regular hexagon.

a Determine the values of a, b, c and d.

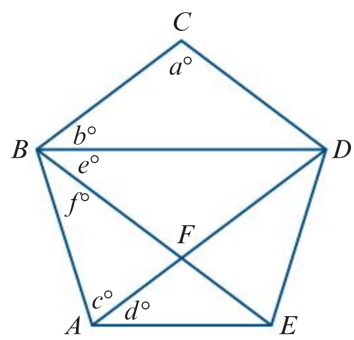
b Prove that $AE \parallel BC$ and $CG \parallel BA$.



7 $ABCDE$ is a regular pentagon.

a Determine the values of a, b, c, d, e and f.

b Prove that $AE \parallel BD$ and $BE \parallel CD$.



CF

Example 8

8 **Proofs involving parallelograms** Prove each of the following:

a In a parallelogram, opposite sides are equal and opposite angles are equal.

b If each side of a quadrilateral is equal to the opposite side, then the quadrilateral is a parallelogram.

c If each angle of a quadrilateral is equal to the opposite angle, then the quadrilateral is a parallelogram.

d If one side of a quadrilateral is equal and parallel to the opposite side, then the quadrilateral is a parallelogram.

9 Let $ABCD$ be a parallelogram and let P and Q be the midpoints of AB and DC respectively. Prove that $APCQ$ is a parallelogram.

10 Let $PQRS$ be a parallelogram whose diagonals meet at O. Let X, Y, Z and W be the midpoints of PO, QO, RO and SO respectively. Prove that $XYZW$ is a parallelogram.

11 **Proofs involving rhombuses** Prove each of the following:

a The diagonals of a rhombus bisect each other at right angles.

b The diagonals of a rhombus bisect the vertex angles through which they pass.

c If the diagonals of a quadrilateral bisect each other at right angles, then the quadrilateral is a rhombus.

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CF

12 **Proofs involving rectangles** Prove each of the following:

a The diagonals of a rectangle are equal and bisect each other.

b A parallelogram with one right angle is a rectangle.

c If the diagonals of a quadrilateral are equal and bisect each other, then the quadrilateral is a rectangle.

Example 9

13 $ABCDE$ is a pentagon in which all the sides are equal and diagonal AC is equal to diagonal AD. Prove that $\angle ABC = \angle AED$.

14 $\triangle ABC$ is equilateral and sides BC, CA and AB respectively are extended to points X, Y and Z so that AY, BZ and CX are all equal in length to the sides of $\triangle ABC$. Prove that $\triangle XYZ$ is also equilateral.

15 $ABCD$ is a quadrilateral in which $AB = BC$ and $AD = DC$. The diagonal BD is extended to a point K. Prove that $AK = CK$.

16 Prove that if the angle C of a triangle ABC is equal to the sum of the other two angles, then the length of side AB is equal to twice the length of the line segment joining C with the midpoint of AB.

17 Prove that if NO is the base of isosceles triangle MNO and if the perpendicular from N to MO meets MO at A, then angle ANO is equal to half of angle NMO.

18 If a median of a triangle is drawn, prove that the perpendiculars from the other vertices upon this median are equal. (The median may be extended.)

14D Pythagoras' theorem

Learning intentions

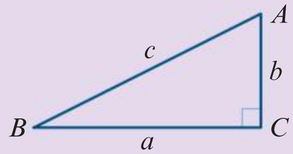
- To be able to use Pythagoras' theorem and its converse to solve problems.

Pythagoras' theorem

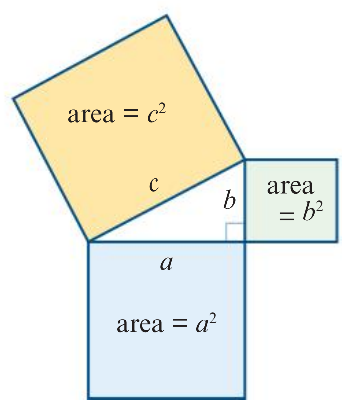
Let ABC be a triangle with side lengths a, b and c.

If $\angle C$ is a right angle, then

$$a^2 + b^2 = c^2$$



Pythagoras' theorem can be illustrated by the diagram shown here. The sum of the areas of the two smaller squares is equal to the area of the square on the longest side (hypotenuse).



There are many different proofs of Pythagoras' theorem. Here we give a proof due to James A. Garfield, the 20th President of the United States.

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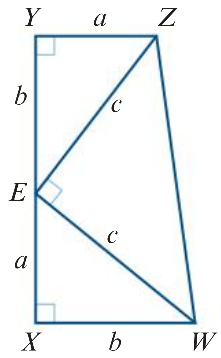
Proof The proof is based on the diagram shown on the right.



Area of trapezium $XYZW = \frac{1}{2}(a+b)(a+b)$

Area of $\triangle EWX$ + area of $\triangle EYZ$ + area of $\triangle EWZ$

$$= \tfrac{1}{2}ab + \tfrac{1}{2}ab + \tfrac{1}{2}c^2$$
$$= ab + \tfrac{1}{2}c^2$$

Thus $\quad \frac{1}{2}(a+b)(a+b) = ab + \frac{1}{2}c^2$

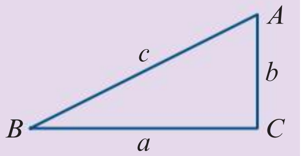
$$a^2 + 2ab + b^2 = 2ab + c^2$$

Hence $\quad a^2 + b^2 = c^2$

Converse of Pythagoras' theorem

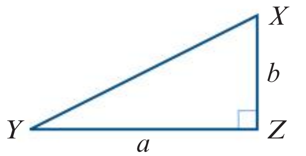
Let ABC be a triangle with side lengths a, b and c.

If $a^2 + b^2 = c^2$, then $\angle C$ is a right angle.



Proof Assume $\triangle ABC$ has side lengths $a = BC$, $b = CA$ and $c = AB$ such that $a^2 + b^2 = c^2$.

Construct a second triangle $\triangle XYZ$ with $YZ = a$ and $ZX = b$ such that $\angle XZY$ is a right angle.



By Pythagoras' theorem, the length of the hypotenuse of $\triangle XYZ$ is

$$\sqrt{a^2 + b^2} = \sqrt{c^2}$$
$$= c$$

Therefore $\triangle ABC \equiv \triangle XYZ$ (SSS).

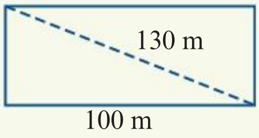
Hence $\angle C$ is a right angle.

Example 10

The diagonal of a soccer field is 130 m and the length of the long side of the field is 100 m. Determine the length of the short side, correct to the nearest centimetre.

Solution

Let x be the length of the short side. Then



$$x^2 + 100^2 = 130^2$$
$$x^2 = 130^2 - 100^2$$
$$\therefore \quad x = \sqrt{6900}$$

Correct to the nearest centimetre, the length of the short side is 83.07 m.

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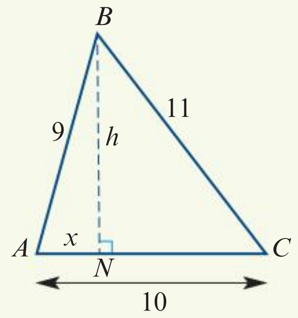
Example 11

Consider $\triangle ABC$ with $AB = 9$ cm, $BC = 11$ cm and $AC = 10$ cm. Determine the length of the altitude of $\triangle ABC$ on AC.

Solution



Let BN be the altitude on AC as shown, with $BN = h$ cm.

Let $AN = x$ cm. Then $CN = (10 - x)$ cm.

In $\triangle ABN$: $\quad x^2 + h^2 = 81 \quad (1)$

In $\triangle CBN$: $\quad (10 - x)^2 + h^2 = 121 \quad (2)$

Expanding in equation (2) gives

$$100 - 20x + x^2 + h^2 = 121$$

Substituting for $x^2 + h^2$ from (1) gives

$$100 - 20x + 81 = 121$$

$$\therefore \quad x = 3$$

Substituting in (1), we have

$$9 + h^2 = 81$$

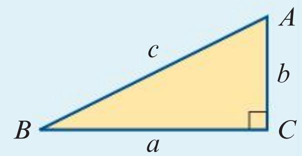
$$h^2 = 72$$

$$\therefore \quad h = 6\sqrt{2}$$

The length of altitude BN is $6\sqrt{2}$ cm.

Summary 14D

Pythagoras' theorem and its converse



Let ABC be a triangle with side lengths a, b and c.

- If $\angle C$ is a right angle, then $a^2 + b^2 = c^2$.
- If $a^2 + b^2 = c^2$, then $\angle C$ is a right angle.

Exercise 14D

SF

1 An 18 m ladder is 7 m away from the bottom of a vertical wall. How far up the wall does it reach?

Example 10

2 Determine the length of the diagonal of a rectangle with dimensions 40 m by 9 m.

3 In a circle of centre O, a chord AB is of length 4 cm. The radius of the circle is 14 cm. Determine the distance of the chord from O.

SF

4 A square has an area of 169 cm^2. What is the length of the diagonal?

5 Determine the area of a square with a diagonal of length:

a 10 cm

b 8 cm

6 $ABCD$ is a square of side length 2 cm. If E is a point on AB extended and $CA = CE$, determine the length of DE.

7 In a square of side length 2 cm, the midpoints of each side are joined to form a new square. Determine the area of the new square.

Example 11

8 Consider $\triangle ABC$ with $AB = 7$ cm, $BC = 6$ cm and $AC = 5$ cm. Determine the length of AN, the altitude on BC.

9 Which of the following are the three side lengths of a right-angled triangle?

a 5 cm, 6 cm, 7 cm

b 3.9 cm, 3.6 cm, 1.5 cm

c 2.4 cm, 2.4 cm, 4 cm

d 82 cm, 18 cm, 80 cm

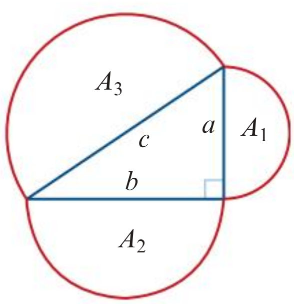
10 Prove that a triangle with sides lengths $x^2 - 1$, $2x$ and $x^2 + 1$ is a right-angled triangle.

11 Consider $\triangle ABC$ such that $AB = 20$ cm, $AC = 15$ cm and the altitude AN has length 12 cm. Prove that $\triangle ABC$ is a right-angled triangle.

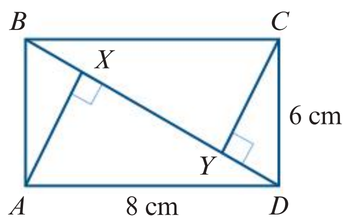
12 Determine the length of an altitude in an equilateral triangle with side length 16 cm.

CF

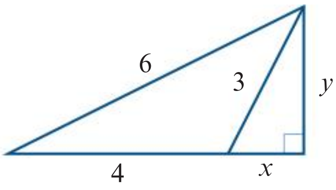
13 Three semicircles are drawn on the sides of this right-angled triangle. Let A_1, A_2 and A_3 be the areas of these semicircles. Prove that $A_3 = A_1 + A_2$.



14 Rectangle $ABCD$ has $CD = 6$ cm and $AD = 8$ cm. Line segments CY and AX are drawn such that points X and Y lie on BD and $\angle AXD = \angle CYD = 90°$. Determine the length of XY.



15 Determine the values of x and y.



16 If P is a point in rectangle $ABCD$ such that $PA = 3$ cm, $PB = 4$ cm and $PC = 5$ cm, determine the length of PD.

17 Let AQ be an altitude of $\triangle ABC$, where Q lies between B and C. Let P be the midpoint of BC. Prove that $AB^2 + AC^2 = 2PB^2 + 2AP^2$.

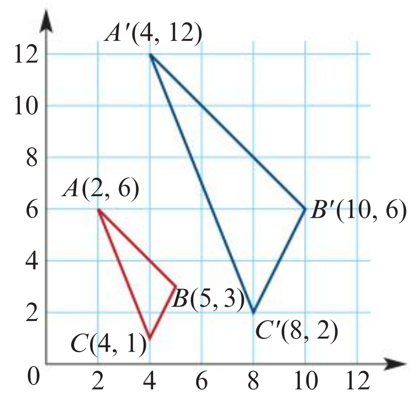
18 For a parallelogram $ABCD$, prove that $2AB^2 + 2BC^2 = AC^2 + BD^2$.

14E Introduction to similarity

Learning intentions

- To be able to use similarity to establish results involving triangles and apply to practical contexts.

The two triangles ABC and $A'B'C'$ shown in the diagram are similar.



Note: $OA' = 2OA$, $OB' = 2OB$, $OC' = 2OC$

Triangle $A'B'C'$ can be considered as the image of triangle ABC under a mapping of the plane in which the coordinates are multiplied by 2.

This mapping is called an **expansion** from the origin of factor 2. From now on we will call this factor the **similarity factor**.

The rule for this mapping can be written in transformation notation as $(x, y) \to (2x, 2y)$.

There is also a mapping from $\triangle A'B'C'$ to $\triangle ABC$, which is an expansion from the origin of factor $\frac{1}{2}$. The rule for this is $(x, y) \to (\frac{1}{2}x, \frac{1}{2}y)$.

Two figures are called **similar** if we can enlarge one figure so that its enlargement is congruent to the other figure.

- Matching lengths of similar figures are in the same ratio.
- Matching angles of similar figures are equal.

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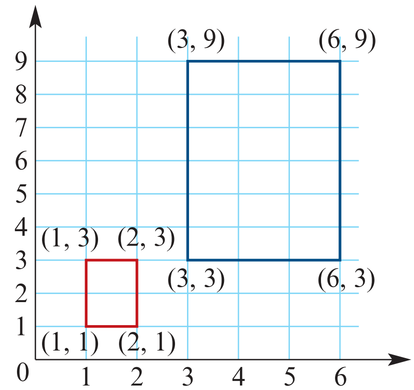
For example, the rectangle with side lengths 1 and 2 is similar to the rectangle with side lengths 3 and 6.

Here the similarity factor is 3 and the rule for the mapping is $(x, y) \to (3x, 3y)$.



Notes:

- Any two circles are similar.
- Any two squares are similar.
- Any two equilateral triangles are similar.

Similar triangles

If triangle ABC is similar to triangle $A'B'C'$, we can write this as

$$\triangle ABC \sim \triangle A'B'C'$$

The triangles are named so that angles of equal magnitude hold the same position. That is, A matches to A', B matches to B' and C matches to C'. So we have

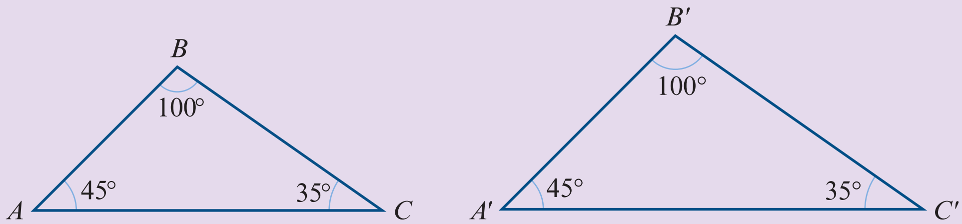
$$\frac{A'B'}{AB} = \frac{B'C'}{BC} = \frac{C'A'}{CA} = k$$

where k is the **similarity factor**.

There are four standard tests for two triangles to be similar.

- The AAA similarity test

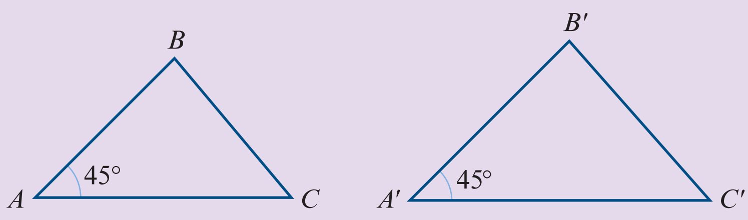
  If two angles of one triangle are respectively equal to two angles of another triangle, then the two triangles are similar.



- The SAS similarity test

  If the ratios of two pairs of matching sides are equal and the included angles are equal, then the two triangles are similar.

$$\frac{A'B'}{AB} = \frac{A'C'}{AC}$$



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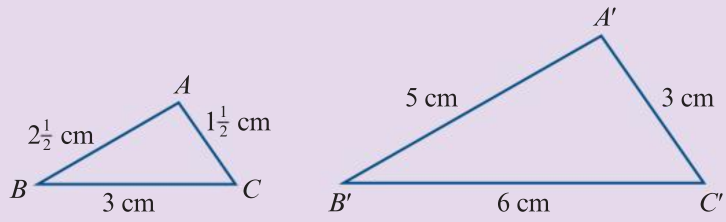
- **The SSS similarity test**

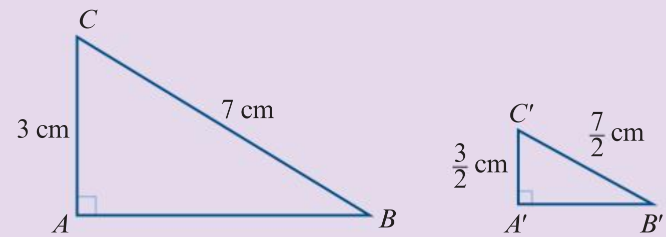
  If the sides of one triangle can be matched up with the sides of another triangle so that the ratio of matching lengths is constant, then the two triangles are similar.

$$\frac{A'B'}{AB} = \frac{B'C'}{BC} = \frac{A'C'}{AC}$$



- **The RHS similarity test**

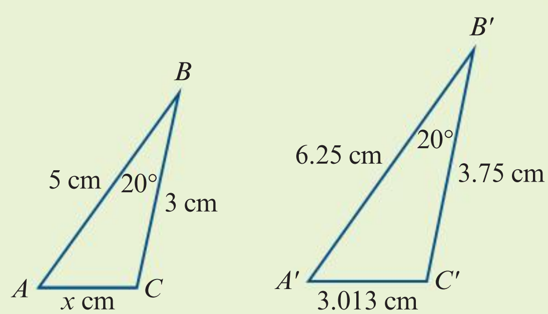
  If the ratio of the hypotenuses of two right-angled triangles equals the ratio of another pair of sides, then the two triangles are similar.



Example 12

a Give the reason for triangle ABC being similar to triangle $A'B'C'$.

b Determine the value of x.



Solution

a Triangle ABC is similar to triangle $A'B'C'$ by SAS, since

$$\frac{5}{6.25} = 0.8 = \frac{3}{3.75}$$

and $\angle ABC = 20^\circ = \angle A'B'C'$

b
$$\frac{x}{3.013} = \frac{5}{6.25}$$

$$\therefore \quad x = \frac{5}{6.25} \times 3.013$$

$$= 2.4104$$

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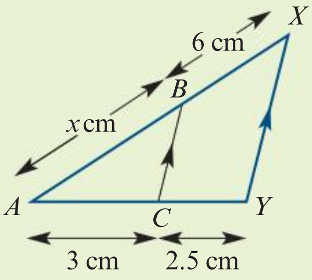
Example 13

a Give the reason for triangle ABC being similar to triangle AXY.

b Determine the value of x.



Solution

a Corresponding angles are of equal magnitude (AAA).

b
$$\frac{AB}{AX} = \frac{AC}{AY}$$
$$\frac{x}{x+6} = \frac{3}{5.5}$$
$$5.5x = 3(x+6)$$
$$2.5x = 18$$
$$\therefore \quad x = 7.2$$

Summary 14E

- Two figures are **similar** if we can enlarge one figure so that its enlargement is congruent to the other figure.
  - Matching lengths of similar figures are in the same ratio.
  - Matching angles of similar figures are equal.
- Two triangles are similar provided any one of the following four conditions holds:

  AAA two angles of one triangle are equal to two angles of the other triangle

  SAS the ratios of two pairs of matching sides are equal and the included angles are equal

  SSS the sides of one triangle can be matched up with the sides of the other triangle so that the ratio of matching lengths is constant

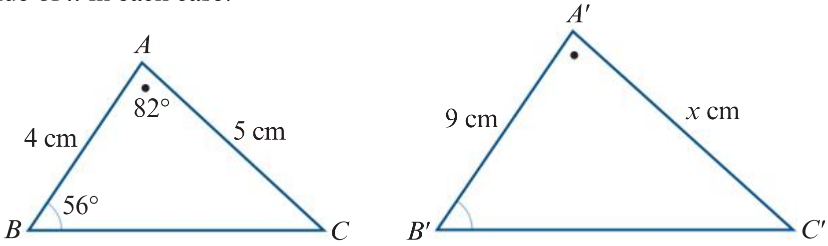
  RHS the ratio of the hypotenuses of two right-angled triangles equals the ratio of another pair of sides.

Exercise 14E

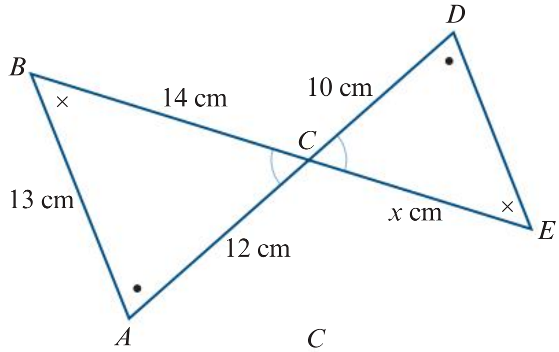
Example 12

1 Give reasons why each of the following pairs of triangles are similar and determine the value of x in each case:

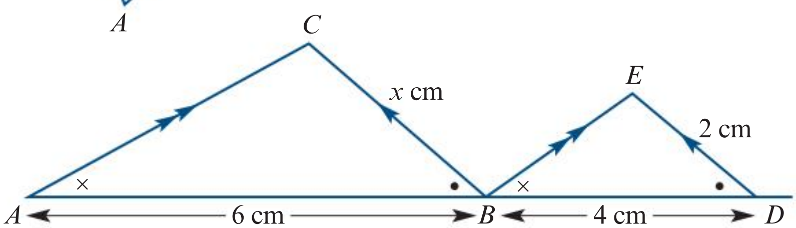
a



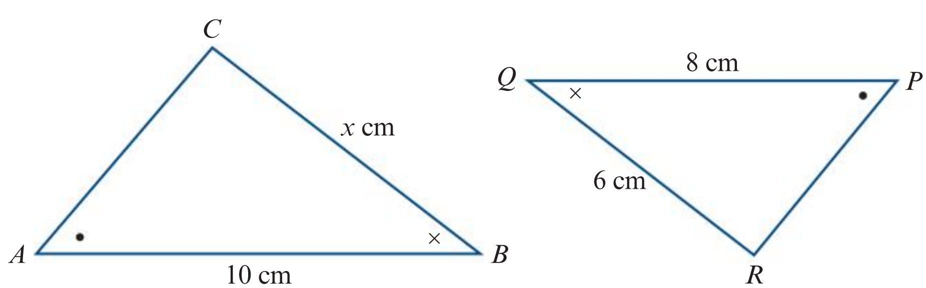
b



c



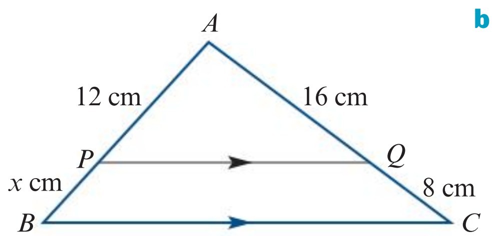
d



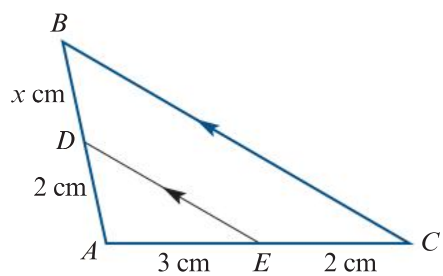
Example 13

2 Give reasons why each of the following pairs of triangles are similar and determine the value of x in each case:

a



b



SF

14F Proofs involving similarity

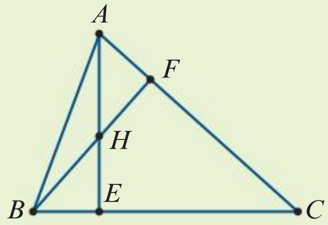
Learning intentions

- To be able to use similarity of triangles to prove results.

Example 14

The altitudes AE and BF of $\triangle ABC$ intersect at H. Prove that

$$\frac{AE}{BF} = \frac{AC}{BC}$$



Solution

$\angle CEA = \angle CFB$ (AE and BF are altitudes)

$\angle ACE = \angle BCF$ (common)

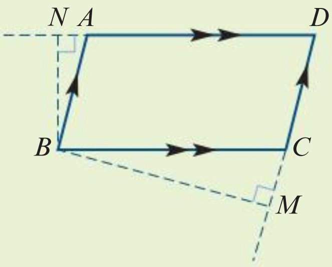
$\therefore \quad \triangle CAE \sim \triangle CBF$ (AAA)

$\therefore \quad \dfrac{AE}{BF} = \dfrac{AC}{BC}$

Example 15

$ABCD$ is a parallelogram with $\angle ABC$ acute. BM is perpendicular to DC extended, and BN is perpendicular to DA extended. Prove that

$$DC \times CM = DA \times AN$$



Solution

$\angle BCM = \angle ABC$ (alternate angles $AB \parallel CD$)

$\angle BAN = \angle ABC$ (alternate angles $BC \parallel AD$)

$\therefore \quad \angle BCM = \angle BAN$

$\angle BNA = \angle BMC = 90^\circ$ (given)

$\therefore \quad \triangle BCM \sim \triangle BAN$ AAA

Hence $\dfrac{CM}{AN} = \dfrac{BC}{AB}$

But $AB = CD$ and $BC = AD$, giving

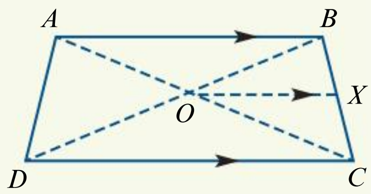
$$\frac{CM}{AN} = \frac{AD}{CD}$$

Hence $DC \times CM = DA \times AN$.

Example 16

$ABCD$ is a trapezium with diagonals intersecting at O. A line through O, parallel to the base CD, meets BC at X. Prove that $BX \times DC = XC \times AB$.

Solution



$\triangle ABC \sim \triangle OXC$ (AAA)

$\triangle DCB \sim \triangle OXB$ (AAA)

Thus $\dfrac{BX}{BC} = \dfrac{OX}{DC}$ (1)

and $\dfrac{XC}{BC} = \dfrac{OX}{AB}$ (2)

Divide (1) by (2):

$$\frac{BX}{XC} = \frac{AB}{DC}$$

$\therefore \quad BX \times DC = XC \times AB$

Skill-sheet

Exercise 14F

Example 14

1 Let M be the midpoint of a line segment AB. Assume that AXB and MYB are equilateral triangles on opposite sides of AB and that XY cuts AB at Z. Prove that $\triangle AXZ \sim \triangle BYZ$ and hence prove that $AZ = 2ZB$.

Example 15

2 $ABCD$ is a rectangle. Assume that P, Q and R are points on AB, BC and CD respectively such that $\angle PQR$ is a right angle. Prove that $BQ \times QC = PB \times CR$.

3 **a** AC is a diagonal of a regular pentagon $ABCDE$. Determine the sizes of $\angle BAC$ and $\angle CAE$.

b AC, AD and BD are diagonals of a regular pentagon $ABCDE$, with AC and BD meeting at X. Prove that $(AB)^2 = BX \times BD$.

4 $\triangle ABC$ has a right angle at A, and AD is the altitude to BC.

a Prove that $AD \times BC = AB \times AC$.

b Prove that $(DA)^2 = DB \times DC$.

c Prove that $(BA)^2 = BD \times BC$.

Example 16

5 $ABCD$ is a trapezium with AB one of the parallel sides. The diagonals meet at O. OX is the perpendicular from O to AB, and XO extended meets CD at Y. Prove that $\dfrac{OX}{OY} = \dfrac{OA}{OC} = \dfrac{AB}{CD}$.

6 P is the point on side AB of $\triangle ABC$ such that $AP : AB = 1 : 3$, and Q is the point on BC such that $CQ : CB = 1 : 3$. The line segments AQ and CP intersect at X. Prove that $AX : AQ = 3 : 5$.

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CF

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7 P and Q are points on sides AB and AC respectively of $\triangle ABC$ such that $PQ \parallel BC$. The median AD meets PQ at M. Prove that $PM = MQ$.

8 $ABCD$ is a straight line and $AB = BC = CD$. An equilateral triangle $\triangle BCP$ is drawn with base BC. Prove that $(AP)^2 = AB \times AD$.

9 $ABCD$ is a quadrilateral such that $\angle BAD = \angle DBC$ and $\dfrac{DA}{AB} = \dfrac{DB}{BC}$. Prove that DB bisects $\angle ADC$.

10 $\triangle ABC$ has a right angle at C. The bisector of $\angle BCA$ meets AB at D, and DE is the perpendicular from D to AC. Prove that $\dfrac{1}{BC} + \dfrac{1}{AC} = \dfrac{1}{DE}$.

11 **Proportions in a right-angled triangle**

- **a** Prove that, for a right-angled triangle, the altitude on its hypotenuse forms two triangles which are similar to the original triangle, and hence to each other.
- **b** Prove Pythagoras' theorem by using part **a** (or by using similar triangles directly).

CF

CU

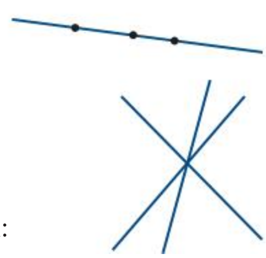
14G Geometric proofs using vectors

Learning intentions

- ▶ To be able to use vector methods to establish geometric results.

In this section we see how vectors can be used as a tool for proving geometric results.

We require the following two definitions.



Collinear points Three or more points are collinear if they all lie on a single line.

Concurrent lines Three or more lines are concurrent if they all pass through a single point.

Here are some properties of vectors from Chapter 7 that will be useful:

Parallel vectors

- For $k \in \mathbb{R}^+$, the vector $k\boldsymbol{a}$ is in the same direction as $\boldsymbol{a}$ and has magnitude $k|\boldsymbol{a}|$, and the vector $-k\boldsymbol{a}$ is in the opposite direction to $\boldsymbol{a}$ and has magnitude $k|\boldsymbol{a}|$.
- Two non-zero vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel if and only if $\boldsymbol{b} = k\boldsymbol{a}$ for some $k \in \mathbb{R} \setminus \{0\}$.
- If $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel with at least one point in common, then $\boldsymbol{a}$ and $\boldsymbol{b}$ lie on the same straight line. For example, if $\overrightarrow{AB} = k\overrightarrow{BC}$ for some $k \in \mathbb{R} \setminus \{0\}$, then A, B and C are collinear.

Scalar product

- Two non-zero vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular if and only if $\boldsymbol{a} \cdot \boldsymbol{b} = 0$.
- $\boldsymbol{a} \cdot \boldsymbol{a} = |\boldsymbol{a}|^2$

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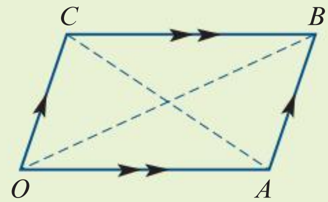
Linear combinations of non-parallel vectors

- For two non-zero vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ that are not parallel, if $m\boldsymbol{a} + n\boldsymbol{b} = p\boldsymbol{a} + q\boldsymbol{b}$, then $m = p$ and $n = q$.

Example 17

Suppose that $OABC$ is a parallelogram. Let $\boldsymbol{a} = \overrightarrow{OA}$ and $\boldsymbol{c} = \overrightarrow{OC}$.



a Express each of the following in terms of $\boldsymbol{a}$ and $\boldsymbol{c}$:

i $\overrightarrow{OB}$ **ii** $\overrightarrow{CA}$

b Determine in terms of $\boldsymbol{a}$ and $\boldsymbol{c}$:

i $|\overrightarrow{OB}|^2$ **ii** $|\overrightarrow{CA}|^2$

c Hence, prove that if the diagonals of a parallelogram are of equal length, then the parallelogram is a rectangle.

Solution

a **i** $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB}$
$= \overrightarrow{OA} + \overrightarrow{OC}$
$= \boldsymbol{a} + \boldsymbol{c}$

ii $\overrightarrow{CA} = \overrightarrow{CB} + \overrightarrow{BA}$
$= \overrightarrow{OA} - \overrightarrow{OC}$
$= \boldsymbol{a} - \boldsymbol{c}$

b **i** $|\overrightarrow{OB}|^2 = \overrightarrow{OB} \cdot \overrightarrow{OB}$
$= (\boldsymbol{a} + \boldsymbol{c}) \cdot (\boldsymbol{a} + \boldsymbol{c})$
$= \boldsymbol{a} \cdot \boldsymbol{a} + \boldsymbol{a} \cdot \boldsymbol{c} + \boldsymbol{c} \cdot \boldsymbol{a} + \boldsymbol{c} \cdot \boldsymbol{c}$
$= |\boldsymbol{a}|^2 + 2\boldsymbol{a} \cdot \boldsymbol{c} + |\boldsymbol{c}|^2$

ii $|\overrightarrow{CA}|^2 = \overrightarrow{CA} \cdot \overrightarrow{CA}$
$= (\boldsymbol{a} - \boldsymbol{c}) \cdot (\boldsymbol{a} - \boldsymbol{c})$
$= \boldsymbol{a} \cdot \boldsymbol{a} - \boldsymbol{a} \cdot \boldsymbol{c} - \boldsymbol{c} \cdot \boldsymbol{a} + \boldsymbol{c} \cdot \boldsymbol{c}$
$= |\boldsymbol{a}|^2 - 2\boldsymbol{a} \cdot \boldsymbol{c} + |\boldsymbol{c}|^2$

c Assume that the diagonals of the parallelogram $OABC$ are of equal length. Then $|\overrightarrow{OB}| = |\overrightarrow{CA}|$. This implies that

$$|\overrightarrow{OB}|^2 = |\overrightarrow{CA}|^2$$
$$|\boldsymbol{a}|^2 + 2\boldsymbol{a} \cdot \boldsymbol{c} + |\boldsymbol{c}|^2 = |\boldsymbol{a}|^2 - 2\boldsymbol{a} \cdot \boldsymbol{c} + |\boldsymbol{c}|^2$$
$$4\boldsymbol{a} \cdot \boldsymbol{c} = 0$$
$$\therefore \quad \boldsymbol{a} \cdot \boldsymbol{c} = 0$$

We have shown that $\overrightarrow{OA} \cdot \overrightarrow{OC} = 0$. So $\angle COA = 90°$. Hence the parallelogram $OABC$ is a rectangle.

Example 18

Three points P, Q and R have position vectors $\boldsymbol{p}$, $\boldsymbol{q}$ and $k(2\boldsymbol{p} + \boldsymbol{q})$ respectively, relative to a fixed origin O. The points O, P and Q are not collinear. Determine the value of k if:

a $\overrightarrow{QR}$ is parallel to $\boldsymbol{p}$ **b** $\overrightarrow{PR}$ is parallel to $\boldsymbol{q}$ **c** P, Q and R are collinear.

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Solution

a $\overrightarrow{QR} = \overrightarrow{QO} + \overrightarrow{OR}$

$= -\boldsymbol{q} + k(2\boldsymbol{p} + \boldsymbol{q})$

$= 2k\boldsymbol{p} + (k-1)\boldsymbol{q}$

If $\overrightarrow{QR}$ is parallel to $\boldsymbol{p}$, then there is some $\lambda \in \mathbb{R} \setminus \{0\}$ such that

$$2k\boldsymbol{p} + (k-1)\boldsymbol{q} = \lambda\boldsymbol{p}$$

This implies that

$$2k = \lambda \quad \text{and} \quad k - 1 = 0$$

Hence $k = 1$.

b $\overrightarrow{PR} = \overrightarrow{PO} + \overrightarrow{OR}$

$= -\boldsymbol{p} + k(2\boldsymbol{p} + \boldsymbol{q})$

$= (2k-1)\boldsymbol{p} + k\boldsymbol{q}$

If $\overrightarrow{PR}$ is parallel to $\boldsymbol{q}$, then there is some $m \in \mathbb{R} \setminus \{0\}$ such that

$$(2k-1)\boldsymbol{p} + k\boldsymbol{q} = m\boldsymbol{q}$$

This implies that

$$2k - 1 = 0 \quad \text{and} \quad k = m$$

Hence $k = \frac{1}{2}$.

Note: Since points O, P and Q are not collinear, the vectors $\boldsymbol{p}$ and $\boldsymbol{q}$ are not parallel.

c If points P, Q and R are collinear, then there exists $n \in \mathbb{R} \setminus \{0\}$ such that

$$n\overrightarrow{PQ} = \overrightarrow{QR}$$

$$\therefore \quad n(-\boldsymbol{p} + \boldsymbol{q}) = 2k\boldsymbol{p} + (k-1)\boldsymbol{q}$$

This implies that

$$-n = 2k \quad \text{and} \quad n = k - 1$$

Therefore $3k - 1 = 0$ and so $k = \frac{1}{3}$.

Skill-sheet

Exercise 14G

You may have proved some of the results of Questions 1 to 7 by other techniques previously. In this exercise vector techniques are to be used.

Example 17

1 Suppose that $OABC$ is a parallelogram. Let $\boldsymbol{a} = \overrightarrow{OA}$ and $\boldsymbol{c} = \overrightarrow{OC}$.

a Express each of the following vectors in terms of $\boldsymbol{a}$ and $\boldsymbol{c}$:

i $\overrightarrow{AB}$ **ii** $\overrightarrow{OB}$ **iii** $\overrightarrow{AC}$

b Determine $\overrightarrow{OB} \cdot \overrightarrow{AC}$.

c Hence, prove that the diagonals of a parallelogram intersect at right angles if and only if it is a rhombus.

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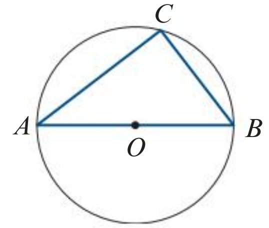
2 The figure shows a triangle ABC inscribed in a circle with centre O, where AB is a diameter of the circle. Let $\boldsymbol{a} = \overrightarrow{OA}$ and $\boldsymbol{c} = \overrightarrow{OC}$.



a Express the vectors $\overrightarrow{AC}$ and $\overrightarrow{BC}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{c}$.

b Determine $\overrightarrow{AC} \cdot \overrightarrow{BC}$.

c Hence, show that the angle subtended by a diameter of a circle at the circumference is a right angle.

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SF

3 Prove that the midpoints of any quadrilateral are the vertices of a parallelogram.

4 Let OAB be an isosceles triangle with $OA = OB$ and let M be the midpoint of AB. Prove that OM is perpendicular to AB.

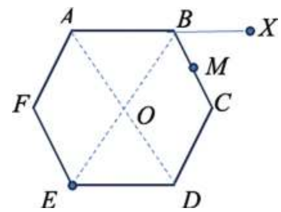
5 Prove that the diagonals of a square are of equal length and bisect each other.

6 **a** Prove that, in a parallelogram, the diagonals bisect each other.
b Prove that if the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram.

7 Prove that if one side of a quadrilateral is equal and parallel to the opposite side, then the quadrilateral is a parallelogram.

8 Prove that if the diagonals of a quadrilateral are equal and bisect each other, then the quadrilateral is a rectangle.

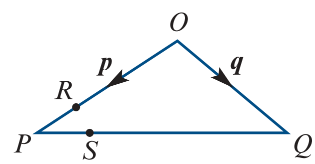
9 $ABCDEF$ is a regular hexagon with centre O. Let M be the midpoint of BC and let X be a point on AB produced such that $\overrightarrow{BX} = \frac{2}{3}\overrightarrow{AB}$. Let $\overrightarrow{OA} = \boldsymbol{a}$ and $\overrightarrow{OB} = \boldsymbol{b}$.



a Determine $\overrightarrow{EX}$ and $\overrightarrow{MX}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$.
b Hence show that E, M and X are collinear.

CF

10 In the diagram, $OR = \frac{4}{5}OP$, $\boldsymbol{p} = \overrightarrow{OP}$, $\boldsymbol{q} = \overrightarrow{OQ}$ and $PS : SQ = 1 : 4$.



a Express each of the following in terms of $\boldsymbol{p}$ and $\boldsymbol{q}$:
i $\overrightarrow{OR}$ **ii** $\overrightarrow{RP}$ **iii** $\overrightarrow{PO}$ **iv** $\overrightarrow{PS}$ **v** $\overrightarrow{RS}$
b What can be said about line segments RS and OQ?
c What type of quadrilateral is $ORSQ$?
d The area of triangle PRS is 5 cm^2. What is the area of $ORSQ$?

11 The position vectors of three points A, B and C relative to an origin O are $\boldsymbol{a}$, $\boldsymbol{b}$ and $k\boldsymbol{a}$ respectively. The point P lies on the line segment BC and is such that $AP = 2PB$. The point Q lies on BC and is such that $CQ = 6QB$.
a Determine in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$:
i the position vector of P **ii** the position vector of Q
b Given that OPQ is a straight line, determine:
i the value of k **ii** the ratio $\frac{OP}{PQ}$
c The position vector of a point R is $\frac{7}{3}\boldsymbol{a}$. Show that PR is parallel to BC.

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Example 18

12 The position vectors of two points A and B relative to an origin O are $3\boldsymbol{i} + 3.5\boldsymbol{j}$ and $6\boldsymbol{i} - 1.5\boldsymbol{j}$ respectively.

a **i** Given that $\overrightarrow{OD} = \frac{1}{3}\overrightarrow{OB}$ and $\overrightarrow{AE} = \frac{1}{4}\overrightarrow{AB}$, write down the position vectors of D and E.

ii Hence determine $|\overrightarrow{ED}|$.

b Given that OE and AD intersect at X and that $\overrightarrow{OX} = p\overrightarrow{OE}$ and $\overrightarrow{XD} = q\overrightarrow{AD}$, determine the position vector of X in terms of:

i p **ii** q

c Hence determine the values of p and q.

13 Suppose that $ORST$ is a parallelogram, where O is the origin. Let U be the midpoint of RS and let V be the midpoint of ST. Denote the position vectors of R, S, T, U and V by $\boldsymbol{r}$, $\boldsymbol{s}$, $\boldsymbol{t}$, $\boldsymbol{u}$ and $\boldsymbol{v}$ respectively.

a Express $\boldsymbol{s}$ in terms of $\boldsymbol{r}$ and $\boldsymbol{t}$.

b Express $\boldsymbol{v}$ in terms of $\boldsymbol{s}$ and $\boldsymbol{t}$.

c Hence, or otherwise, show that $4(\boldsymbol{u} + \boldsymbol{v}) = 3(\boldsymbol{r} + \boldsymbol{s} + \boldsymbol{t})$.

14 Apollonius' theorem

For $\triangle OAB$, the point C is the midpoint of side AB. Prove that:

a $4\overrightarrow{OC} \cdot \overrightarrow{OC} = OA^2 + OB^2 + 2\overrightarrow{OA} \cdot \overrightarrow{OB}$

b $4\overrightarrow{AC} \cdot \overrightarrow{AC} = OA^2 + OB^2 - 2\overrightarrow{OA} \cdot \overrightarrow{OB}$

c $2OC^2 + 2AC^2 = OA^2 + OB^2$

15 If P is any point in the plane of rectangle $ABCD$, prove that

$$PA^2 + PC^2 = PB^2 + PD^2$$

16 Prove that the medians bisecting the equal sides of an isosceles triangle are equal.

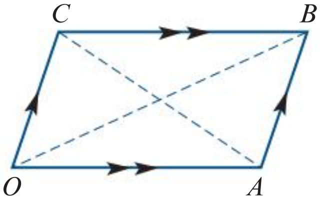
17 **a** Prove that if $(\boldsymbol{c} - \boldsymbol{b}) \cdot \boldsymbol{a} = 0$ and $(\boldsymbol{c} - \boldsymbol{a}) \cdot \boldsymbol{b} = 0$, then $(\boldsymbol{b} - \boldsymbol{a}) \cdot \boldsymbol{c} = 0$.

b Use part **a** to prove that the altitudes of a triangle meet at a point.

18 For a parallelogram $OABC$, prove that

$$OB^2 + AC^2 = 2OA^2 + 2OC^2$$

That is, prove that the sum of the squares of the lengths of the diagonals of a parallelogram is equal to the sum of the squares of the lengths of the sides.



Review

Chapter summary

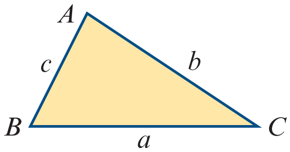
Parallel lines

- If two parallel lines are crossed by a transversal, then:
  - alternate angles are equal
  - corresponding angles are equal
  - co-interior angles are supplementary.
- If two lines crossed by a transversal form an equal pair of alternate angles, then the two lines are parallel.

Polygons

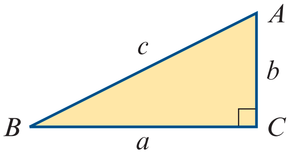
- The sum of the interior angles of an n-sided polygon is $(n - 2)180°$.
- A **regular polygon** is a polygon in which all angles are equal and all sides are equal.
- **Triangle inequality**

  In $\triangle ABC$: $a < b + c$, $b < c + a$ and $c < a + b$.



- **Pythagoras' theorem and its converse**

  Let ABC be a triangle with side lengths a, b and c.
  - If $\angle C$ is a right angle, then $a^2 + b^2 = c^2$.
  - If $a^2 + b^2 = c^2$, then $\angle C$ is a right angle.



- Classification of quadrilaterals:
  - A **trapezium** is a quadrilateral with at least one pair of opposite sides parallel.
  - A **parallelogram** is a quadrilateral with both pairs of opposite sides parallel.
  - A **rhombus** is a parallelogram with a pair of adjacent sides equal.
  - A **rectangle** is a parallelogram in which one angle is a right angle.
  - A **square** is a rectangle with a pair of adjacent sides equal.

Congruence

- **Congruent figures** have exactly the same shape and size.
- If triangle ABC is congruent to triangle XYZ, this can be written as $\triangle ABC \equiv \triangle XYZ$.
- Two triangles are congruent provided any one of the following four conditions holds:

  SSS the three sides of one triangle are equal to the three sides of the other triangle

  SAS two sides and the included angle of one triangle are equal to two sides and the included angle of the other triangle

  AAS two angles and one side of one triangle are equal to two angles and the matching side of the other triangle

  RHS the hypotenuse and one side of a right-angled triangle are equal to the hypotenuse and one side of another right-angled triangle.

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Similarity

- Two figures are **similar** if we can enlarge one figure so that its enlargement is congruent to the other figure.
  - Matching lengths of similar figures are in the same ratio.
  - Matching angles of similar figures are equal.
- If triangle ABC is similar to triangle XYZ, this can be written as $\triangle ABC \sim \triangle XYZ$.
- Two triangles are similar provided any one of the following four conditions holds:

  AAA two angles of one triangle are equal to two angles of the other triangle

  SAS the ratios of two pairs of matching sides are equal and the included angles are equal

  SSS the sides of one triangle can be matched up with the sides of the other triangle so that the ratio of matching lengths is constant

  RHS the ratio of the hypotenuses of two right-angled triangles equals the ratio of another pair of sides.

Vector proof

Parallel vectors

- For $k \in \mathbb{R}^+$, the vector $k\boldsymbol{a}$ is in the same direction as $\boldsymbol{a}$ and has magnitude $k|\boldsymbol{a}|$, and the vector $-k\boldsymbol{a}$ is in the opposite direction to $\boldsymbol{a}$ and has magnitude $k|\boldsymbol{a}|$.
- Two non-zero vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel if and only if $\boldsymbol{b} = k\boldsymbol{a}$ for some $k \in \mathbb{R} \setminus \{0\}$.
- If $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel with at least one point in common, then $\boldsymbol{a}$ and $\boldsymbol{b}$ lie on the same straight line. For example, if $\overrightarrow{AB} = k\overrightarrow{BC}$ for some $k \in \mathbb{R} \setminus \{0\}$, then A, B and C are collinear.

Scalar product

- Two non-zero vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular if and only if $\boldsymbol{a} \cdot \boldsymbol{b} = 0$.
- $\boldsymbol{a} \cdot \boldsymbol{a} = |\boldsymbol{a}|^2$

Linear combinations of non-parallel vectors

- For two non-zero vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ that are not parallel, if $m\boldsymbol{a} + n\boldsymbol{b} = p\boldsymbol{a} + q\boldsymbol{b}$, then $m = p$ and $n = q$.

Skills checklist

Check-list

Download this checklist from the Interactive Textbook, then print it and fill it out to check your skills. ☑

14A **1** I can use the definitions of supplementary and complementary angles to determine the values of angles. ☐

See Example 1 and Question 1

14A **2** I can use the properties of corresponding, alternate and cointerior angles to determine values of angles ☐

See Example 2 and Question 1

14A **3** I can prove that the angle sum of a triangle is 180°. ☐

See Example 3 and Question 3

Review

14B **4** I can work with the definitions of isosceles and equilateral triangles to determine values. ☐

See Example 4 and Question 4

14B **5** I can work with properties of triangles and parallel lines to determine values. ☐

See Example 5 and Question 6

14C **6** I can prove results about triangles and quadrilaterals using congruence of triangles. ☐

See Example 7, Example 8, Example 9 and Questions 3, 8 and 13

14D **7** I can apply Pythagoras' theorem to determine lengths in right-angled triangles. ☐

See Example 10, Example 11 and Questions 2 and 8

14E **8** I can determine if two triangles are similar or not and determine lengths and angles of similar triangles. ☐

See Example 12, Example 13 and Questions 1 and 2

14F **9** I can prove results about triangles using the similarity conditions. ☐

See Example 14, Example 15, Example 16 and Questions 1, 2 and 5

14G **10** I can use vector methods to prove geometric results. ☐

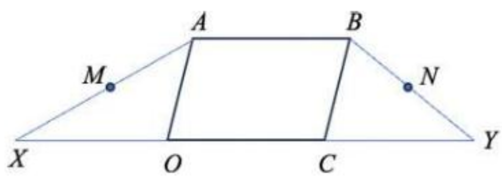
See Example 17, Example 18 and Questions 1 and 10

Short-response questions

Vector geometry proof short-response questions

SF

1 $OABC$ is a parallelogram. Let $\overrightarrow{OA} = \boldsymbol{a}$ and $\overrightarrow{OC} = \boldsymbol{c}$. Points X and Y lie on OC extended in both directions so that $OX = OC = CY$.



Points M and N are the midpoints of XA and YB respectively.

a Determine $\overrightarrow{OM}$ and $\overrightarrow{ON}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{c}$.

b Use these to determine $\overrightarrow{MN}$.

2 The points P and R have position vectors $\overrightarrow{OP} = 2\boldsymbol{a} + \boldsymbol{b}$ and $\overrightarrow{OR} = \boldsymbol{a} - 3\boldsymbol{b}$ respectively. Prove that if $OPQR$ is a square, then $|\boldsymbol{a}|^2 = 2|\boldsymbol{b}|^2$.

3 AB and DC are parallel sides of a trapezium and $DC = 3AB$. The diagonals AC and DB intersect at O. Prove that $AO = \frac{1}{4}AC$.

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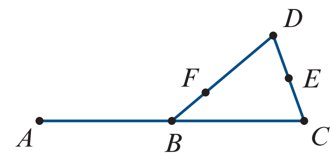
SF

4 In the diagram, points A, B and C lie on a straight line and $AB = BC = BD$. Point E is the midpoint of CD and $BF = \frac{1}{3}BD$.

Use vector methods to prove that points A, F and E are collinear and determine the ratio $AF : FE$.



5 $ABCD$ is a parallelogram. If L and M are the midpoints of BC and CD respectively:

a Determine $\overrightarrow{AL}$ and $\overrightarrow{AM}$ in terms of $\overrightarrow{AB}$ and $\overrightarrow{AD}$.

b Prove that $\overrightarrow{AL} + \overrightarrow{AM} = \frac{3}{2}\overrightarrow{AC}$.

6 If $ABCD$ is a parallelogram and E is the midpoint of AB, and M is the point of intersection of DE and AC, prove that $\overrightarrow{AM} = \frac{1}{3}\overrightarrow{AC}$ and $\overrightarrow{DM} = \frac{2}{3}\overrightarrow{DE}$.

7 $ABCD$ is a rhombus with $AB = 16$ cm. The midpoints of its sides are joined to form a quadrilateral.

a Describe the quadrilateral formed.

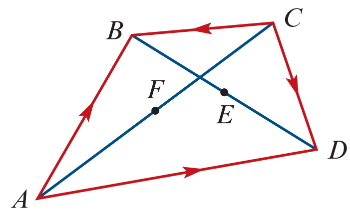
b What is the length of the diagonal of this quadrilateral?

CF

8 If the position vectors of 4 points A, B, C and D with respect to a point O are such that $\overrightarrow{OA} + \overrightarrow{OC} = \overrightarrow{OB} + \overrightarrow{OD}$ prove that $ABCD$ is a parallelogram.

9 $ABCD$ is a quadrilateral, and E and F are the midpoints of BD and AC respectively. Prove that

$$\overrightarrow{AB} + \overrightarrow{CB} + \overrightarrow{CD} + \overrightarrow{AD} = 4\overrightarrow{FE}$$



10 Points P and Q have position vectors $\boldsymbol{p}$ and $\boldsymbol{q}$, with reference to an origin O, and M is the point on PQ such that

$$\beta\overrightarrow{PM} = \alpha\overrightarrow{MQ}$$

a Prove that the position vector of M is given by $\boldsymbol{m} = \dfrac{\beta\boldsymbol{p} + \alpha\boldsymbol{q}}{\alpha + \beta}$.

b Write the position vectors of P and Q as $\boldsymbol{p} = k\boldsymbol{a}$ and $\boldsymbol{q} = \ell\boldsymbol{b}$, where k and ℓ are positive real numbers and $\boldsymbol{a}$ and $\boldsymbol{b}$ are unit vectors.

i Prove that the position vector of any point on the internal bisector of $\angle POQ$ has the form $\lambda(\boldsymbol{a} + \boldsymbol{b})$.

ii If M is the point where the internal bisector of $\angle POQ$ meets PQ, show that

$$\frac{\alpha}{\beta} = \frac{k}{\ell}$$

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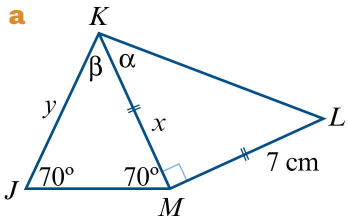
Review of geometry short-response questions

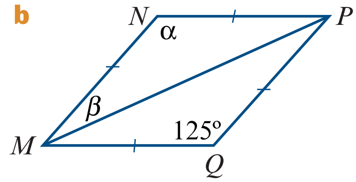
11 Determine the side length of a rhombus whose diagonals are 6 cm and 10 cm.

12 Determine the values of the unknowns (x, y, α, β, γ and θ) for each of the following:

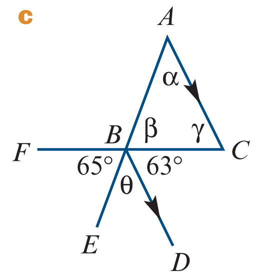
a



b

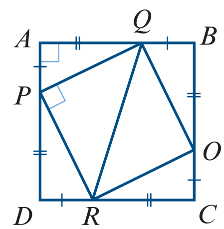


c



13 **a** Prove that $\triangle PAQ \equiv \triangle QBO$.

b Prove that $\triangle PQR \equiv \triangle ORQ$.



14 Let XYZ be a triangle with a point P on XY and a point Q on XZ such that PQ is parallel to YZ.

a Show that the two triangles XYZ and XPQ are similar.

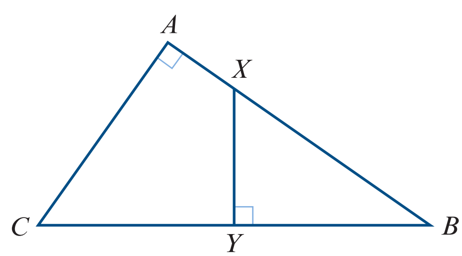
b If $XY = 36$ cm, $XZ = 30$ cm and $XP = 24$ cm, determine:

i XQ **ii** QZ

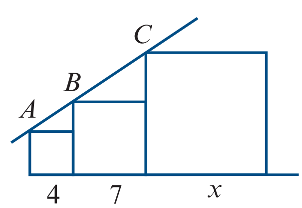
c Write down the values of $XP : PY$ and $PQ : YZ$.

15 If a 1 m stake casts a shadow 2.3 m long, determine the height (in metres) of a tree which casts a shadow 21 m long.

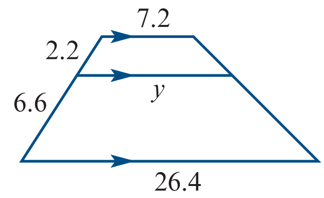
16 ABC is a right-angled triangle with $AB = 4$ and $AC = 3$. If the triangle is folded along the line XY, then vertex C coincides with vertex B. Determine the length of XY.



17 Points A, B and C lie on a straight line. The squares are adjacent and have side lengths 4, 7 and x. Determine the value of x.



18 Determine the value of y in the diagram on the right.

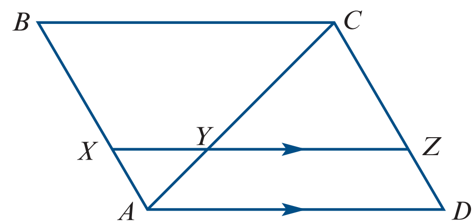


SF

19 AC is the diagonal of a rhombus $ABCD$. The line XYZ is parallel to AD, and $AX = 3$ cm and $AB = 9$ cm. Determine:

a $\dfrac{XY}{BC}$ **b** $\dfrac{AY}{AC}$ **c** $\dfrac{CY}{AC}$

d $\dfrac{YZ}{AD}$



CF

20 Triangles ABC and PQR are similar. The medians AX and PY are drawn, where X is the midpoint of BC and Y is the midpoint of QR. Prove that:

a triangles ABX and PQY are similar **b** $\dfrac{AX}{PY} = \dfrac{BC}{QR}$

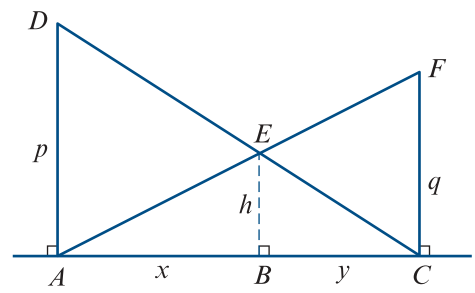
21 **a** In this diagram, which other triangle is similar to $\triangle DAC$?

b Explain why $\dfrac{h}{p} = \dfrac{y}{x+y}$.

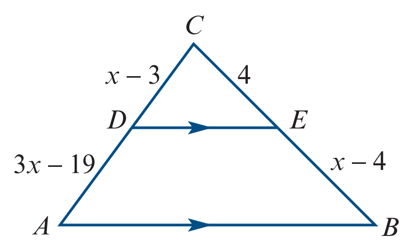
c Use another pair of similar triangles to write down an expression for $\dfrac{h}{q}$ in terms of x and y.

d Explain why $h \cdot \left(\dfrac{1}{p} + \dfrac{1}{q}\right) = 1$.

e Calculate h when $p = 4$ and $q = 5$.



22 Place conditions upon x such that DE is parallel to AB given that $CD = x - 3$, $DA = 3x - 19$, $CE = 4$ and $EB = x - 4$.

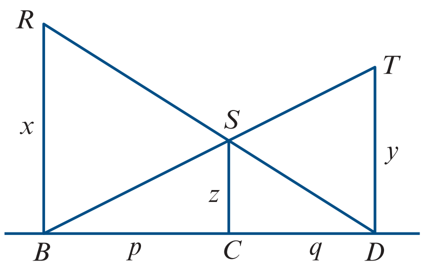


23 **a** If BR, CS and DT are perpendicular to BD, name the pairs of similar triangles.

b Which of the following is correct?

$$\frac{z}{y} = \frac{p}{q} \quad \text{or} \quad \frac{z}{y} = \frac{p}{p+q}$$

c Show that $\dfrac{1}{x} + \dfrac{1}{y} = \dfrac{1}{z}$.



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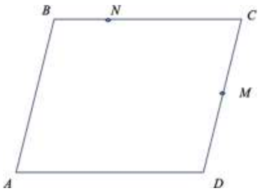
Review

Multiple-choice questions

Vector geometry proof multiple-choice questions

1 $ABCD$ is a parallelogram. Let $\overrightarrow{AB} = \boldsymbol{a}$ and $\overrightarrow{AD} = \boldsymbol{b}$. The points M and N lie on CD and BC respectively such that $\overrightarrow{CM} = \overrightarrow{MD}$ and $\overrightarrow{CN} = 2\overrightarrow{NB}$. The vector $\overrightarrow{MN}$ is equal to:



A $\frac{1}{2}\boldsymbol{a} + \frac{1}{3}\boldsymbol{b}$ **B** $\frac{1}{3}(\boldsymbol{a} - \boldsymbol{b})$

C $\frac{1}{2}\boldsymbol{a} - \frac{2}{3}\boldsymbol{b}$ **D** $\frac{1}{2}\boldsymbol{a} + \frac{1}{3}\boldsymbol{b}$

2 For triangle OAB, let $\overrightarrow{OA} = \boldsymbol{a}$ and $\overrightarrow{OB} = \boldsymbol{b}$ and P be a point on AB such that $AP : PB = 1 : 3$. The vector $\overrightarrow{OP}$ is equal to

A $\frac{1}{2}(\boldsymbol{a} + \boldsymbol{b})$ **B** $\frac{1}{3}(\boldsymbol{a} - \boldsymbol{b})$ **C** $\frac{1}{4}(\boldsymbol{a} - \boldsymbol{b})$ **D** $\frac{1}{4}(3\boldsymbol{a} + \boldsymbol{b})$

3 If $\boldsymbol{a}$ and $\boldsymbol{b}$ are any two non-zero vectors such that $|\boldsymbol{a} - \boldsymbol{b}| = |\boldsymbol{a} + \boldsymbol{b}|$ then which of the following is always true?

A $\boldsymbol{a} \cdot \boldsymbol{b} = 0$ **B** $\boldsymbol{a} = \boldsymbol{b}$ **C** $\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{0}$ **D** $\boldsymbol{a}$ is parallel to $\boldsymbol{b}$

4 If $\boldsymbol{a}$ and $\boldsymbol{b}$ are two unit vectors and the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ is α, then if $\boldsymbol{a} + \boldsymbol{b}$ is a unit vector, α is equal to

A $\frac{\pi}{2}$ **B** $\frac{\pi}{6}$ **C** $\frac{5\pi}{6}$ **D** $\frac{2\pi}{3}$

5 The points A, B and C have position vectors $\overrightarrow{OA} = \boldsymbol{p} + \boldsymbol{q}$, $\overrightarrow{OB} = 3\boldsymbol{p} - 2\boldsymbol{q}$ and $\overrightarrow{OC} = 6\boldsymbol{p} + m\boldsymbol{q}$ where $\boldsymbol{p}$ and $\boldsymbol{q}$ are non-zero and not parallel and m is a non-zero real number. If A, B and C are collinear the value of m is

A 2 **B** -3 **C** $-\frac{13}{2}$ **D** $-\frac{5}{2}$

6 The points A and B have position vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ respectively, relative to an origin O. The point P divides OA in the ratio $2 : 3$ and the point R divides OB in the ratio $4 : 1$. Given that $RPBQ$ is a parallelogram, $\overrightarrow{OQ}$ is equal to

A $\frac{1}{5}(2\boldsymbol{a} + 4\boldsymbol{b})$ **B** $\frac{2}{3}\boldsymbol{a} + \boldsymbol{b}$ **C** $\frac{1}{5}(-2\boldsymbol{a} + 9\boldsymbol{b})$ **D** $\frac{1}{5}(2\boldsymbol{a} + \boldsymbol{b})$

7 In rhombus $OABC$, let $\overrightarrow{OA} = \boldsymbol{a}$ and $\overrightarrow{OC} = \boldsymbol{c}$. Then $\overrightarrow{OB} \cdot \overrightarrow{AC}$ is equal to

A 0 **B** $|\boldsymbol{a}|^2 + |\boldsymbol{c}|^2$ **C** $|\boldsymbol{a}| + |\boldsymbol{c}|$ **D** $|\boldsymbol{a} - \boldsymbol{c}|^2$

8 If $ABCD$ is a trapezium with AB parallel to DC with $\overrightarrow{AB} = 2\boldsymbol{a} - 8\boldsymbol{b}$ and $\overrightarrow{DC} = m\boldsymbol{a} - 40\boldsymbol{b}$, where $\boldsymbol{a}$ and $\boldsymbol{b}$ are non-parallel vectors, the value of m is

A 5 **B** 10 **C** $\frac{15}{2}$ **D** $-\frac{5}{2}$

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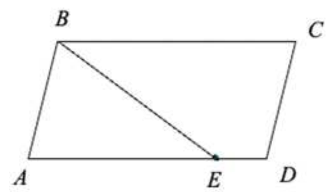
9 If $OABC$ is a rhombus with $\overrightarrow{OA} = \boldsymbol{a}$, $\overrightarrow{OC} = \boldsymbol{c}$ and M the midpoint of AC which one of the following is **not** true?

A $\overrightarrow{OB} = \boldsymbol{a} + \boldsymbol{c}$ **B** $\overrightarrow{AM} = \frac{1}{2}\overrightarrow{AC}$ **C** $\boldsymbol{a} = \boldsymbol{c}$ **D** $\overrightarrow{OM} = \frac{1}{2}\overrightarrow{OB}$

10 If $\boldsymbol{a}$ and $\boldsymbol{b}$ are unit vectors and θ, $0° < \theta < 180°$, the angle between them, then $|\boldsymbol{a} - \boldsymbol{b}|$ is equal to

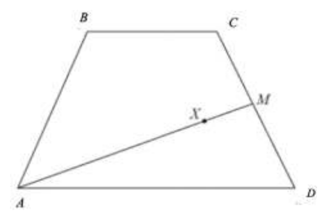
A $2\cos\left(\frac{\theta}{2}\right)$ **B** $2\sin\theta$ **C** $2\sin\left(\frac{\theta}{2}\right)$ **D** $2\sec\left(\frac{\theta}{2}\right)$

11 $ABCD$ is a parallelogram and E is a point on AD such that $AE : ED = 9 : 2$. Let $\overrightarrow{BC} = \boldsymbol{a}$ and $\overrightarrow{AB} = \boldsymbol{b}$. Then $\overrightarrow{BE}$ is equal to



A $\frac{11}{9}\boldsymbol{a} - 2\boldsymbol{b}$ **B** $\frac{2}{9}(\boldsymbol{a} - \boldsymbol{b})$
C $\frac{9}{2}\boldsymbol{a} - \frac{1}{2}\boldsymbol{b}$ **D** $\frac{9}{11}\boldsymbol{a} - \boldsymbol{b}$

12 $ABCD$ is a quadrilateral and M is the midpoint of CD. If $\overrightarrow{AB} = \overrightarrow{BC} - \overrightarrow{CD}$ and BDX is a straight line, where X is a point on AM such that $AX : XM = n : 1$ then the value of n is



A $\frac{1}{3}$ **B** 2 **C** 3 **D** 4

Review of geometry multiple-choice questions

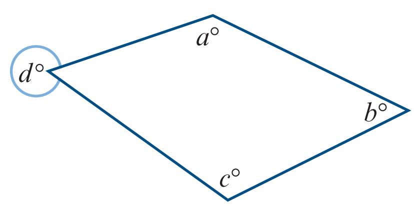
13 One angle of a triangle is twice the size of the second angle, and the third angle is 66°. The smallest angle is

A 66° **B** 53° **C** 38° **D** 24°

14 Three of the angles of a pentagon are right angles. The other two angles are of equal size $x°$. The angle $x°$ is equal to

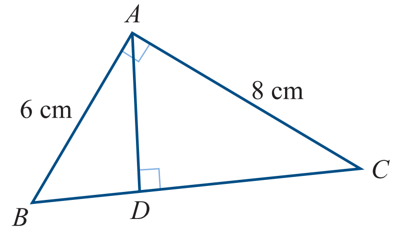
A 45° **B** 135° **C** 120° **D** 150°

15 d is equal to



A $360 - (a + b + c)$ **B** $a + b + c$
C $b - a + c$ **D** $a - b + c$

16 In the figure, $\angle BAC$ is a right angle and AD is perpendicular to BC. If $AB = 6$ cm and $AC = 8$ cm, then the length of AD is



A 4 cm **B** $\frac{24}{5}$ cm
C $\frac{17}{3}$ cm **D** $\frac{13}{2}$ cm

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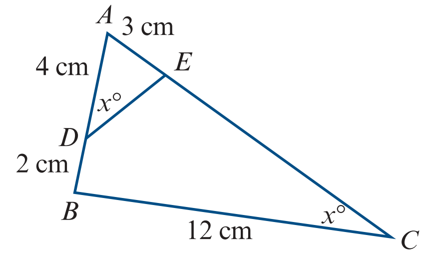
Review

17 Two sides of a triangle have lengths 14 and 18. Which of the following *cannot* be the length of the third side?

A 2 **B** 6 **C** 7 **D** 28

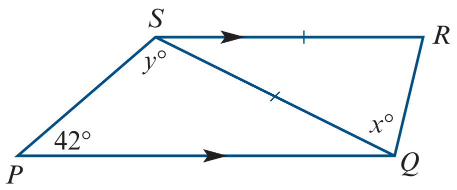
18 D and E are points on AB and AC, respectively. $AD = 4$ cm, $DB = 2$ cm, $AE = 3$ cm and $BC = 12$ cm. If $\angle ADE = \angle ACB$, then the length DE, in centimetres, is

A 6 **B** $\frac{9}{2}$ **C** 9 **D** 10



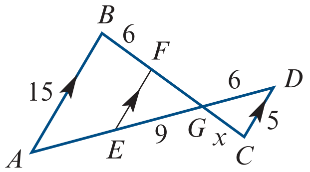
19 In this diagram, PQ and SR are parallel and $SR = SQ$. The angles x and y satisfy the equation

A $x + y = 138$ **B** $2x + y = 42$
C $x = y + 42$ **D** $2x - y = 42$



20 In the figure, $AB = 15$, $CD = 5$, $BF = 6$, $GD = 6$ and $EG = 9$. The length x is equal to

A 3 **B** 4 **C** 4.5 **D** 4.75



15

Circle geometry

Chapter contents

The two basic figures of geometry in the plane are the triangle and the circle. We have considered triangles and their properties in Chapter 14, and we will use the results of that chapter in establishing results involving circles.

A **circle** is the set of all points in the plane at a fixed distance r from a point O. Circles with the same radius are congruent to each other (and are said to be equal circles). We have seen in the previous chapter that all circles are similar to each other.

You may have come across the Cartesian equation of the circle in Mathematical Methods Units 1 & 2. For example, the circle with radius 1 and centre the origin has equation $x^2 + y^2 = 1$. In this chapter we take a different approach to the study of circles.

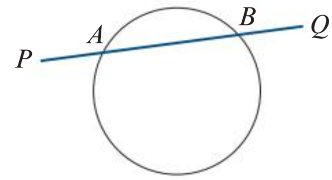
The theorems and related results in this chapter can be investigated through a geometry package such as GeoGebra or Cabri Geometry.

15A Angle properties of circles

Learning intentions

- To prove the following results and obtain further results using them:
  - The angle at the centre of a circle is twice the angle at the circumference subtended by the same arc.
  - Angles in the same segment of a circle are equal.
  - The angle subtended by a diameter at the circumference is equal to a right angle (90°).
  - A quadrilateral is cyclic if and only if the sum of each pair of opposite angles is 180°.

A line segment joining two points on a circle is called a **chord**. A line that cuts a circle at two distinct points is called a **secant**.



For example, in the diagram, the line PQ is a secant and the line segment AB is a chord.

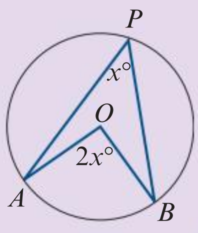
Suppose that we have a line segment or an arc AB and a point P not on AB. Then $\angle APB$ is the angle **subtended** by AB at the point P.

You should prove the following two results. The first proof uses the SSS congruence test and the second uses the SAS congruence test.

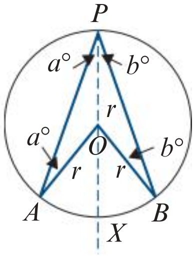
- Equal chords of a circle subtend equal angles at the centre.
- If two chords subtend equal angles at the centre, then the chords are equal.

Theorem 1

The angle at the centre of a circle is twice the angle at the circumference subtended by the same arc.



Proof Join points P and O and extend the line through O, as shown in the diagram on the right.



Note that $AO = BO = PO = r$, the radius of the circle. Therefore triangles PAO and PBO are isosceles.

Let $\angle APO = \angle PAO = a^\circ$ and $\angle BPO = \angle PBO = b^\circ$.

Then angle AOX is $2a^\circ$ (exterior angle of a triangle) and angle BOX is $2b^\circ$ (exterior angle of a triangle).

Hence $\angle AOB = 2a^\circ + 2b^\circ = 2(a + b)^\circ = 2\angle APB$.

Note: In this proof, the centre O and point P are on the same side of chord AB. Slight variations of this proof can be used for other cases. The result is always true.

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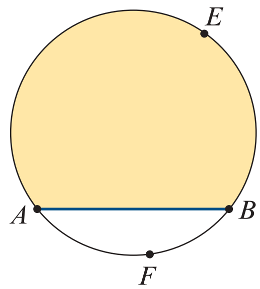
Converse of Theorem 1 Let A and B be points on a circle, centre O, and let P be a point on the same side of AB as O. If the angle APB is half the angle AOB, then P lies on the circle.

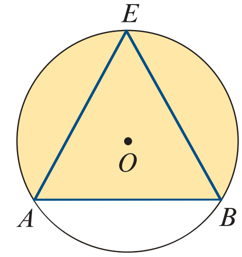
Segments



A **segment** of a circle is the part of the plane bounded by an arc and its chord. For example, in the diagram:

- Arc AEB and chord AB define a **major segment** (shaded).
- Arc AFB and chord AB define a **minor segment** (unshaded).

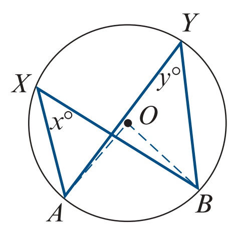
Angles in a segment



$\angle AEB$ is said to be an angle in segment AEB.

Theorem 2: Angles in the same segment

Angles in the same segment of a circle are equal.



Proof Let $\angle AXB = x°$ and $\angle AYB = y°$.

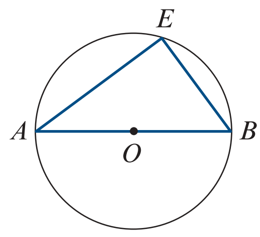
Then, by Theorem 1, $\angle AOB = 2x° = 2y°$.

Therefore $x = y$.

Note: A converse of this result is proved later in this section.

Theorem 3: Angle subtended by a diameter

The angle subtended by a diameter at the circumference is equal to a right angle (90°).



Proof The angle subtended at the centre is 180°, and so the result follows from Theorem 1.

We can give a straightforward proof of a converse of this result.

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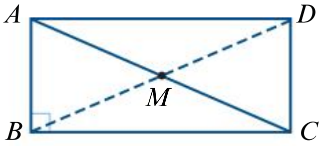
Converse of Theorem 3 The circle whose diameter is the hypotenuse of a right-angled triangle passes through all three vertices of the triangle.

Proof In the diagram, triangle ABC has a right angle at B. Let M be the midpoint of the hypotenuse AC.



We need to prove that $MA = MB = MC$.

Complete the rectangle $ABCD$.

In Question 12 of Exercise 9C, you proved that the diagonals of a rectangle are equal and bisect each other.

Hence $AC = BD$ and M is also the midpoint of BD. It follows that $MA = MB = MC$.

Cyclic polygons

- A set of points is said to be **concyclic** if they all lie on a common circle.
- A polygon is said to be **inscribed in a circle** if all its vertices lie on the circle. This implies that no part of the polygon lies outside the circle.
- A quadrilateral that can be inscribed in a circle is called a **cyclic quadrilateral**.

Theorem 4

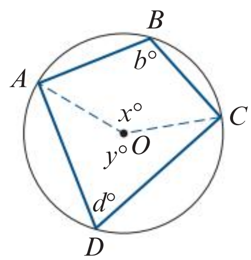
The opposite angles of a quadrilateral inscribed in a circle sum to 180°.

That is, the opposite angles of a cyclic quadrilateral are supplementary.

Proof In the diagram, the quadrilateral $ABCD$ is inscribed in a circle with centre O.

By Theorem 1, we have $x = 2d$ and $y = 2b$.

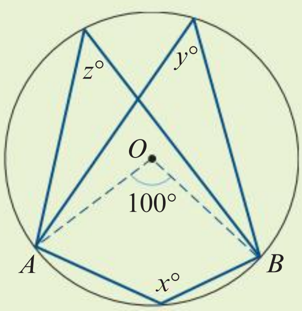
Now	$x + y = 360$
and so	$2b + 2d = 360$
Hence	$b + d = 180$



Converse of Theorem 4 If opposite angles of a quadrilateral are supplementary, then the quadrilateral is cyclic.

Example 1

Determine the value of each of the pronumerals in the diagram, where O is the centre of the circle and $\angle AOB = 100°$.



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Solution

Theorem 1 gives $y = z = 50$.

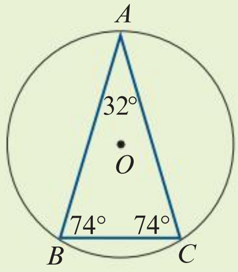
The value of x can be found by observing either of the following:

1 Reflex angle AOB is 260°.
Therefore $x = 130$ (by Theorem 1).

2 We have $x + y = 180$ (by Theorem 4).
Therefore $x = 180 - 50 = 130$.

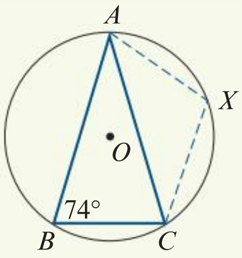
Example 2

An isosceles triangle is inscribed in a circle as shown. Determine the angles in the three minor segments of the circle cut off by the sides of this triangle.



Solution

To determine $\angle AXC$, form the cyclic quadrilateral $AXCB$. Then $\angle AXC$ and $\angle ABC$ are supplementary. Therefore $\angle AXC = 106°$, and so all angles in the minor segment formed by AC have magnitude 106°.



Similarly, it can be shown that all angles in the minor segment formed by AB have magnitude 106°, and that all angles in the minor segment formed by BC have magnitude 148°.

Example 3

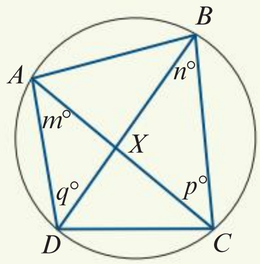
A, B, C and D are points on a circle. The diagonals of quadrilateral $ABCD$ meet at X. Prove that triangles ADX and BCX are similar.

Solution

$\angle DAC$ and $\angle DBC$ are in the same segment. Therefore $m = n$.

$\angle ADB$ and $\angle ACB$ are in the same segment. Therefore $q = p$.

Hence triangles ADX and BCX are similar (AAA).



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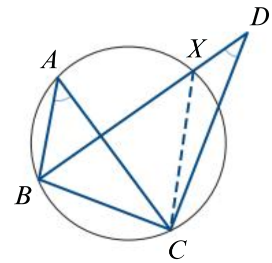
The converse theorems

We only prove a converse of Theorem 2 here, but the proofs of the converses of Theorems 1 and 4 use similar techniques. Try them for yourself.

Converse of Theorem 2 If a line segment subtends equal angles at two points on the same side of the line segment, then the two points and the endpoints of the line segment are concyclic.

Proof A circle is drawn through points A, B and C. (This can be done with any three non-collinear points.)



Assume that $\angle BAC = \angle BDC$ and that D lies outside the circle. (There is another case to consider when D is inside, but the proof is similar. If D lies on the circle, then we are finished.)

Let X be the point of intersection of line BD with the circle. Then, by Theorem 2, $\angle BAC = \angle BXC$ and so $\angle BDC = \angle BXC$. But this is impossible. (You can use the equality of the angle sums of $\triangle BXC$ and $\triangle BDC$ to show this.)

Hence D lies on the same circle as A, B and C.

Summary 15A

- Equal chords of a circle subtend equal angles at the centre.
- If two chords subtend equal angles at the centre, then the chords are equal.
- **Theorem 1** The angle at the centre of a circle is twice the angle at the circumference subtended by the same arc.
- **Theorem 2** Angles in the same segment of a circle are equal.
- **Theorem 3** The angle subtended by a diameter at the circumference is 90°.
- **Theorem 4** Opposite angles of a cyclic quadrilateral sum to 180°.

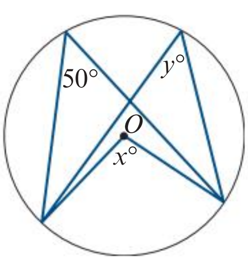
Skill-sheet

Exercise 15A

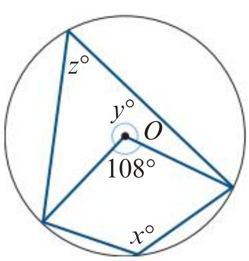
Example 1

1 Determine the values of the pronumerals for each of the following, where O denotes the centre of the given circle: SF

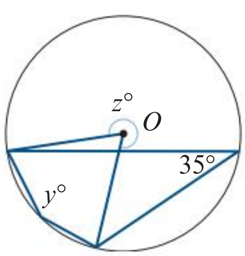
a



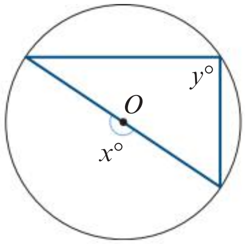
b



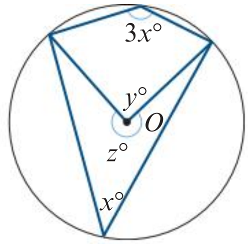
c



d

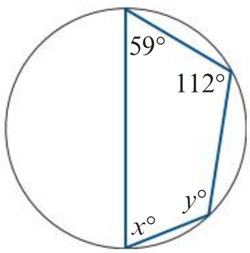


e

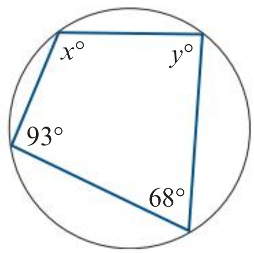


2 Determine the values of the pronumerals for each of the following:

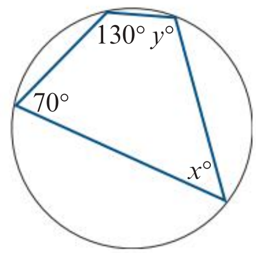
a



b

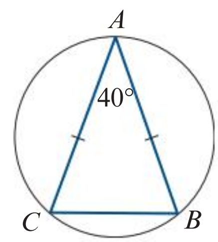


c



Example 2

3 An isosceles triangle ABC is inscribed in a circle. (Inscribed means that all the vertices of the triangle lie on the circle.) What are the angles in the three minor segments cut off by the sides of this triangle?



4 $ABCDE$ is a pentagon inscribed in a circle. If $AE = DE$, $\angle BDC = 20°$, $\angle CAD = 28°$ and $\angle ABD = 70°$, determine all the interior angles of the pentagon.

Example 3

5 Prove that if two opposite sides of a cyclic quadrilateral are equal, then the other two sides are parallel.

6 $ABCD$ is a parallelogram. The circle through the points A, B and C cuts CD (extended if necessary) at E. Prove that $AE = AD$.

7 $ABCD$ is a cyclic quadrilateral and O is the centre of the circle through A, B, C and D. If $\angle AOC = 120°$, determine the magnitude of $\angle ADC$.

8 $PQRS$ is a cyclic quadrilateral with $\angle SQR = 36°$, $\angle PSQ = 64°$ and $\angle RSQ = 42°$. Determine the interior angles of the quadrilateral.

9 Prove that if a parallelogram can be inscribed in a circle, then it must be a rectangle.

10 Let $ABCD$ be a quadrilateral, and let lines lA, lB, lC and lD be the interior bisectors of angles A, B, C and D respectively. Let P be the intersection of lA and lB, Q the intersection of lB and lC, R the intersection of lC and lD, and S the intersection of lD and lA. If P, Q, R and S are all distinct, prove that they all lie on a single circle.

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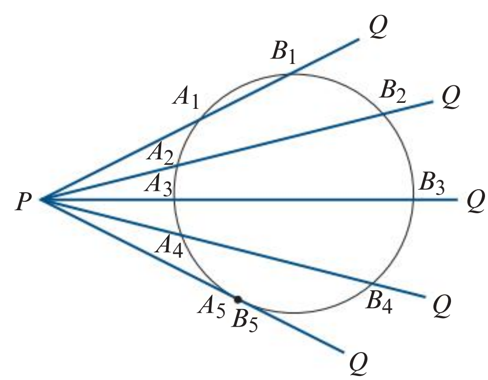
15B Tangents

Learning intentions

- To prove the following results and obtain further results using them:
  - A tangent to a circle is perpendicular to the radius drawn from the point of contact.
  - The two tangents drawn from an external point to a circle are the same length.
  - The angle between a tangent and a chord drawn from the point of contact is equal to any angle in the alternate segment.

Consider a point P outside a circle, as shown in the diagram. By rotating the secant PQ, with P as the pivot point, we obtain a sequence of pairs of points on the circle. As PQ moves towards the edge of the circle, the pairs of points become closer together, until they eventually coincide.



When PQ is in this final position (i.e. when the intersection points A and B coincide), it is called a **tangent** to the circle.

A tangent touches the circle at only one point, and this point is called the **point of contact**.

The **length of a tangent** from a point P outside the circle is the distance between P and the point of contact.

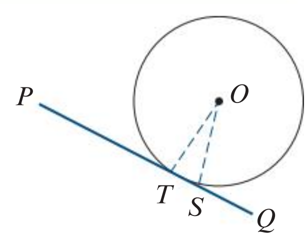
Theorem 5: Tangent is perpendicular to radius

A tangent to a circle is perpendicular to the radius drawn from the point of contact.

Proof This will be a proof by contradiction.

Let T be the point of contact of tangent PQ and suppose that $\angle OTP$ is not a right angle.

Let S be the point on PQ, not T, such that OSP is a right angle. Then triangle OST has a right angle at S.



Therefore $OT > OS$, as OT is the hypotenuse of triangle OST.

This implies that S is inside the circle, as OT is a radius.

Thus the line through T and S must cut the circle again. But PQ is a tangent, and so this is a contradiction. Hence we have shown that $\angle OTP$ is a right angle.

Theorem 6: Two tangents from the same point

The two tangents drawn from an external point to a circle are the same length.

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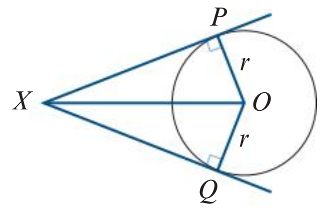
Proof We can see that $\triangle XPO$ is congruent to $\triangle XQO$ using the RHS test, as $\angle XPO = \angle XQO = 90^\circ$, the side XO is common and $OP = OQ$ (radii).

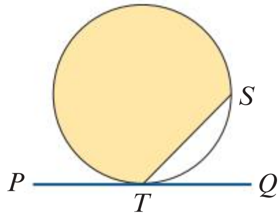
Therefore $XP = XQ$.



The alternate segment theorem

In the diagram:

- The shaded segment is called the **alternate segment** in relation to $\angle STQ$.
- The unshaded segment is alternate to $\angle STP$.



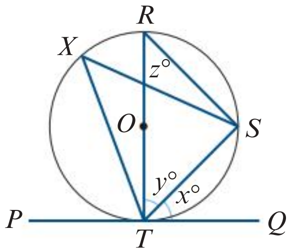
Theorem 7: Alternate segment theorem

The angle between a tangent and a chord drawn from the point of contact is equal to any angle in the alternate segment.

Proof Let $\angle STQ = x^\circ$, $\angle RTS = y^\circ$ and $\angle TRS = z^\circ$, where RT is a diameter.

Then $\angle RST = 90^\circ$ (Theorem 3, angle subtended by a diameter), and therefore $y + z = 90$.

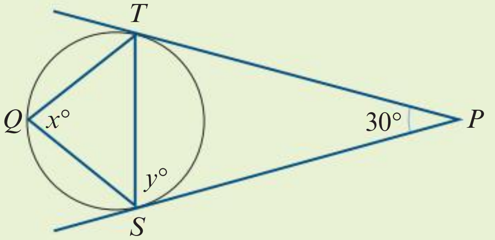
Also $\angle RTQ = 90^\circ$ (Theorem 5, tangent is perpendicular to radius), and therefore $x + y = 90$.



Thus $x = z$. But $\angle TXS$ is in the same segment as $\angle TRS$ and so $\angle TXS = x^\circ$.

Example 4

Determine the magnitudes of the angles x and y in the diagram.



Solution

Triangle PST is isosceles (Theorem 6, two tangents from the same point).

Therefore $\angle PST = \angle PTS$ and so $y = 75$.

The alternate segment theorem gives $x = y = 75$.

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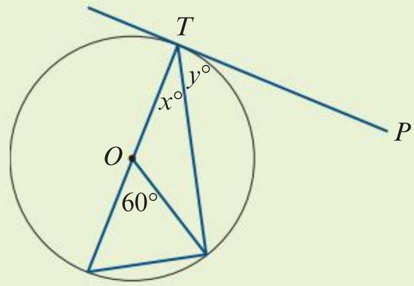
Example 5

determine the values of x and y, where PT is tangent to the circle centre O.

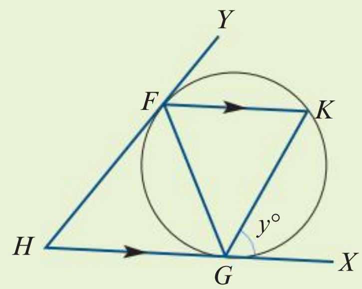


Solution

$x = 30$ as the angle at the circumference is half the angle subtended at the centre, and so $y = 60$ as $\angle OTP$ is a right angle.

Example 6

The tangents to a circle at F and G meet at H. A chord FK is drawn parallel to HG. Prove that triangle FGK is isosceles.



Solution

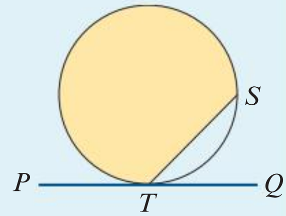
Let $\angle XGK = y°$.

Then $\angle GFK = y°$ (alternate segment theorem) and $\angle GKF = y°$ (alternate angles).

Therefore triangle FGK is isosceles with $FG = KG$.

Summary 15B

- A tangent to a circle is perpendicular to the radius drawn from the point of contact.
- The two tangents drawn from an external point to a circle are the same length.
- In the diagram, the **alternate segment** to $\angle STQ$ is shaded, and the alternate segment to $\angle STP$ is unshaded.
- **Alternate segment theorem** The angle between a tangent and a chord drawn from the point of contact is equal to any angle in the alternate segment.



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Skill-sheet

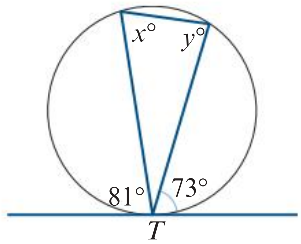
Exercise 15B

SF

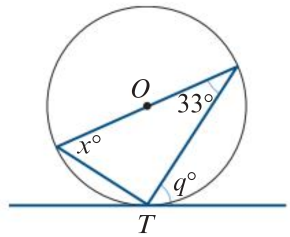
Example 4

1 Determine the values of the pronumerals for each of the following, where T is the point of contact of the tangent and O is the centre of the circle:

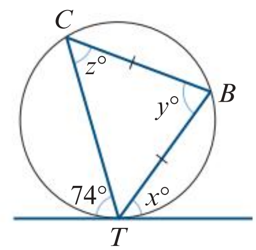
a



b

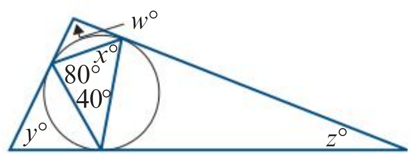


c

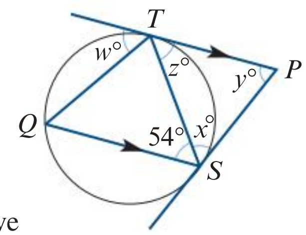


$BC = BT$

d



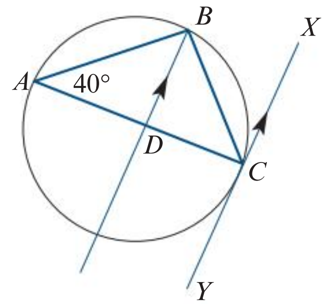
e



Note: In the diagram for part **e**, the two tangents from P have points of contact at S and T, and TP is parallel to QS.

Example 5

2 A triangle ABC is inscribed in a circle, and the tangent to the circle at C is parallel to the bisector of angle ABC.



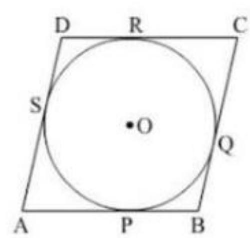
a Determine the magnitude of $\angle BCX$.

b Determine the magnitude of $\angle CBD$, where D is the point of intersection of the bisector of angle ABC with AC.

c Determine the magnitude of $\angle ABC$.

3 Assume that AB and AC are two tangents to a circle, touching the circle at B and C, and that $\angle BAC = 116°$. Determine the magnitudes of the angles in the two segments into which BC divides the circle.

4 Prove that if a parallelogram circumscribes a circle as shown in the diagram then the parallelogram is a rhombus.



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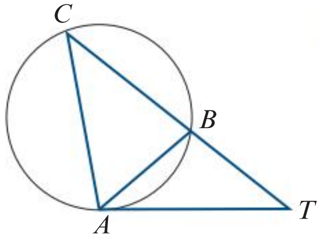
CF

5 AT is a tangent at A and TBC is a secant to the circle. Given that $\angle CTA = 30°$ and $\angle CAT = 110°$, determine the magnitude of angles ACB, ABC and BAT.



Example 6

6 From a point A outside a circle, a secant ABC is drawn cutting the circle at B and C, and a tangent AD touching it at D. A chord DE is drawn equal in length to chord DB. Prove that triangles ABD and CDE are similar.

7 AD is a diameter of a circle. B is a point external to the circle and the line AB is tangent to the circle at A.The line BD meets the circle again at E. If $DE = EB$ prove that $\angle EAB = 45°$.

8 The point P is external to a circle. The line PA is tangent to the circle, touching the circle at A. The point P is joined to the centre of the circle C. The line segment PC meets the circle at Q. The point D lies on PC such that AD is perpendicular to PC. Prove that AQ bisects the angle PAD.

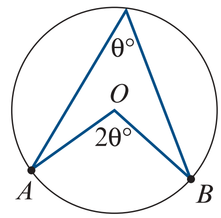
9 Assume that AB is a chord of a circle and that CT, the tangent at C, is parallel to AB. Prove that $CA = CB$.

10 Through a point T, a tangent TA and a secant TPQ are drawn to a circle APQ. The chord AB is drawn parallel to PQ. Prove that the triangles PAT and BAQ are similar.

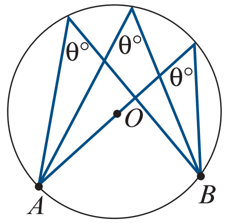
11 PQ is a diameter of a circle and AB is a perpendicular chord cutting it at N. Prove that PN is equal in length to the perpendicular from P onto the tangent at A.

Chapter summary

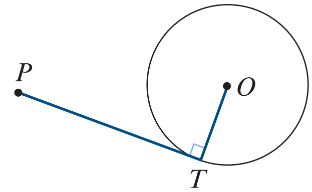
- The angle at the centre of a circle is twice the angle at the circumference subtended by the same arc.



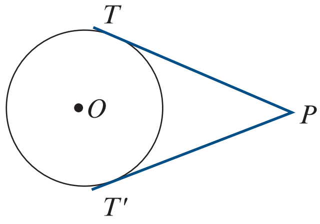
- Angles in the same segment of a circle are equal.



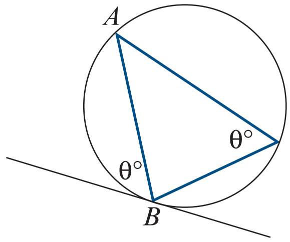
- A tangent to a circle is perpendicular to the radius drawn from the point of contact.



- The two tangents drawn from an external point are the same length, i.e. $PT = PT'$.



- The angle between a tangent and a chord drawn from the point of contact is equal to any angle in the alternate segment.



- A quadrilateral is cyclic if and only if the sum of each pair of opposite angles is 180°.

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Skills checklist

Check-list

Download this checklist from the Interactive Textbook, then print it and fill it out to check your skills. ☑

15A **1** I can apply the theorem: The angle at the centre of a circle is twice the angle at the circumference subtended by the same arc. ☐

See Example 1 and Question 1

15A **2** I can apply the theorem: Angles in the same segment of a circle are equal. ☐

See Example 2, Example 3 and Questions 3 and 5

15B **3** I can apply the theorem: The two tangents from an external point are the same length. ☐

See Example 4 and Question 1

15B **4** I can apply the alternate segment theorem. ☐

See Example 4, Example 6 and Questions 1 and 6

15B **5** I can apply the theorem: A tangent to a circle is perpendicular to the radius drawn from the point of contact. ☐

See Example 5 and Question 2

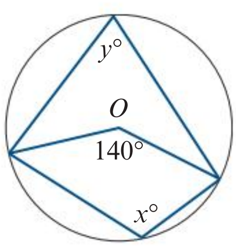
Short-response questions

SF

1 $\triangle ABC$ has $\angle A = 36°$ and $\angle C = 90°$. M is the midpoint of AB and CN is the altitude on AB. Determine the size of $\angle MCN$.

2 Determine the values of the pronumerals in each of the following:

a



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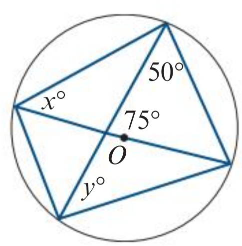
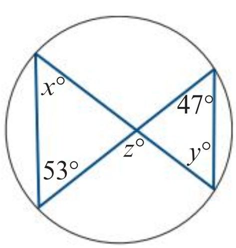
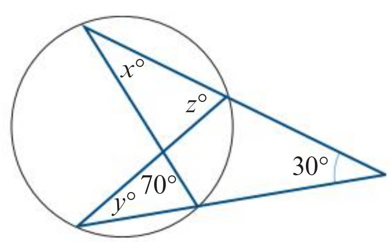


c



d



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3 Let OP be a radius of a circle with centre O. A chord BA is drawn parallel to OP. The lines OA and BP intersect at C. Prove that:

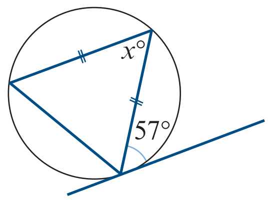
a $\angle CAB = 2\angle CBA$ **b** $\angle PCA = 3\angle PBA$

4 A chord AB of a circle, centre O, is extended to C. The straight line bisecting $\angle OAB$ meets the circle at E. Prove that EB bisects $\angle OBC$.

5 Two circles intersect at A and B. The tangent at B to one circle meets the second again at D, and a straight line through A meets the first circle at P and the second at Q. Prove that BP is parallel to DQ.

6 Determine the values of the pronumerals for each of the following:

a



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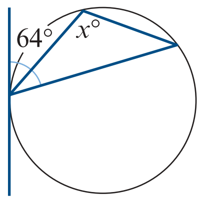
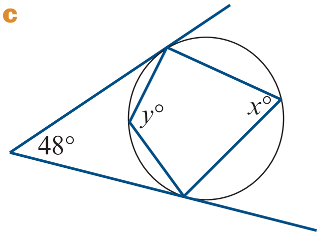


c

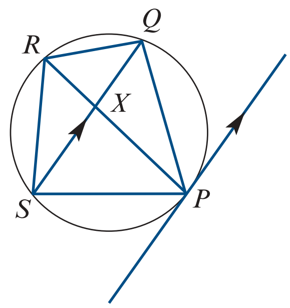


7 Two circles intersect at M and N. The tangent to the first circle at M meets the second circle at P, while the tangent to the second at N meets the first at Q. Prove that $MN^2 = NP \cdot QM$.

8 The diagonals PR and QS of a cyclic quadrilateral $PQRS$ intersect at X. The tangent at P is parallel to QS. Prove that:

a $PQ = PS$

b PR bisects $\angle QRS$.



9 *If line segments AB and CD intersect at a point M and $AM \cdot BM = CM \cdot DM$, then the points A, B, C and D are concyclic.*

To prove this claim, show that:

a $\dfrac{AM}{CM} = \dfrac{DM}{BM}$ **b** $\triangle AMC \sim \triangle DMB$

c $\angle CAM = \angle BDM$ **d** $ABCD$ is cyclic.

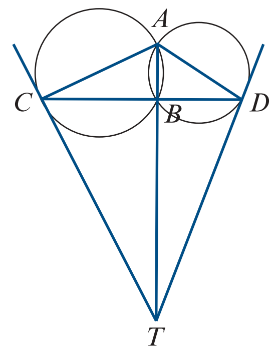
Review

CU

10 Two circles intersect at A and B. The tangents at C and D intersect at T on the extension of AB. Prove that, if CBD is a straight line, then:

a $TCAD$ is a cyclic quadrilateral

b $\angle TAC = \angle TAD$.

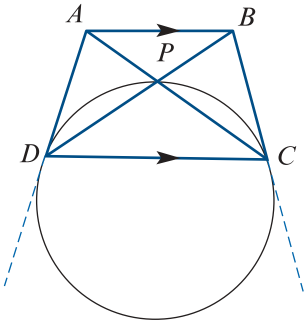


11 $ABCD$ is a trapezium in which AB is parallel to DC and the diagonals meet at P. The circle through D, P and C touches AD and BC at D and C respectively.
Prove that:

a $\angle BAC = \angle ADB$

b the circle through A, P and D touches BA at A

c $ABCD$ is a cyclic quadrilateral.



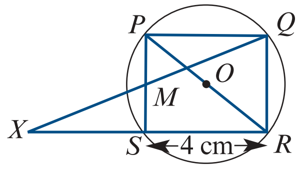
12 $PQRS$ is a square of side length 4 cm inscribed in a circle with centre O. The midpoint of the side PS is M. The line segments QM and RS are extended to meet at X.

a Prove that:

i $\triangle XPR$ is isosceles

ii PX is the tangent to the circle at P.

b Calculate the area of trapezium $PQRX$.



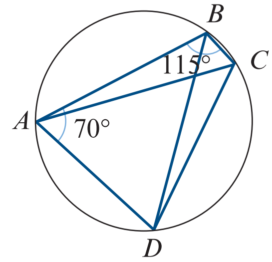
13 An isosceles triangle ABC, with $AB = AC$, is inscribed in a circle. A chord AD intersects BC at E. Prove that

$$AB^2 - AE^2 = BE \cdot CE$$

Multiple-choice questions

1 In the diagram, the points A, B, C and D lie on a circle, $\angle ABC = 115°$, $\angle BAD = 70°$ and $AB = AD$. The magnitude of $\angle ACD$ is

A 45° **B** 55° **C** 40° **D** 70°



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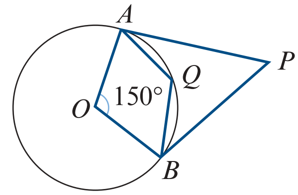
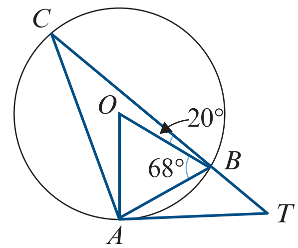
2 In the diagram, PA and PB are tangents to the circle centre O. Given that Q is a point on the minor arc AB and that $\angle AOB = 150°$, the magnitudes of $\angle APB$ and $\angle AQB$ are

A $\angle APB = 30°$ and $\angle AQB = 105°$
B $\angle APB = 25°$ and $\angle AQB = 105°$
C $\angle APB = 30°$ and $\angle AQB = 110°$
D $\angle APB = 25°$ and $\angle AQB = 100°$



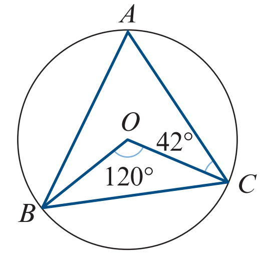
3 A circle with centre O passes through A, B and C. The line AT is the tangent to the circle at A, and CBT is a straight line. Given that $\angle ABO = 68°$ and $\angle OBC = 20°$, the magnitude of $\angle ATB$ is

A 64° **B** 65° **C** 70° **D** 66°



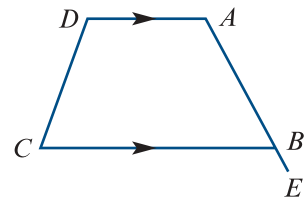
4 In the diagram, the points A, B and C lie on a circle with centre O. If $\angle BOC = 120°$ and $\angle ACO = 42°$, then the magnitude of $\angle ABO$ is

A 18° **B** 22° **C** 24° **D** 26°



5 $ABCD$ is a cyclic quadrilateral with AD parallel to BC, and $\angle DCB = 65°$. The magnitude of $\angle CBE$ is

A 110° **B** 115° **C** 120° **D** 122°

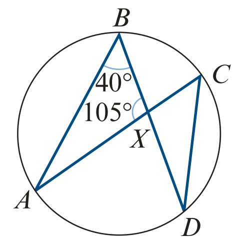


6 A chord AB of a circle subtends an angle of 50° at a point on the circumference of the circle. The acute angle between the tangents at A and B has magnitude

A 80° **B** 75° **C** 85° **D** 82°

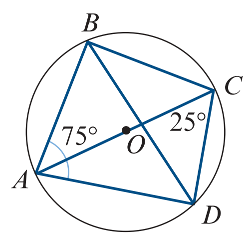
7 A, B, C and D are points on a circle, with $\angle ABD = 40°$ and $\angle AXB = 105°$. The magnitude of $\angle XDC$ is

A 35° **B** 45° **C** 50° **D** 55°



8 A, B, C and D are points on a circle, centre O, such that AC is a diameter of the circle. If $\angle BAD = 75°$ and $\angle ACD = 25°$, then the magnitude of $\angle BDC$ is

A 10° **B** 15° **C** 25° **D** 30°



16

Transformations of the plane

Chapter contents

Modern animations are largely created with the use of computers. Many basic visual effects can be understood in terms of simple transformations of the plane.

For example, suppose that an animator wants to give the car below a sense of movement. This can be achieved by gradually tilting the car so that it leans forwards. We will see later how this can easily be done using a transformation called a **shear**.

Aside from computer graphics, linear transformations play an important role in many diverse fields such as mathematics, physics, engineering and economics.

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16A Linear transformations

Learning intentions

- To be able to represent linear transformations by 2×2 matrices.

Each point in the plane can be denoted by an ordered pair (x, y). The set of all ordered pairs is often called the **Cartesian plane**: $\mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}$. A **transformation** of the plane maps each point (x, y) in the plane to a new point (x', y'). We say that (x', y') is the **image** of (x, y).

We will mainly be concerned with **linear transformations**, which have rules of the form

$$(x, y) \to (ax + by, cx + dy)$$

Example 1

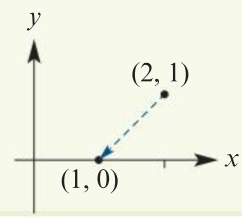
Determine the image of the point $(2, 1)$ under the transformation with rule

$$(x, y) \to (3x - 5y, 2x - 4y)$$

Solution

We let $x = 2$ and $y = 1$, giving

$$(2, 1) \to (3 \times 2 - 5 \times 1, 2 \times 2 - 4 \times 1) = (1, 0)$$



Matrices and linear transformations

Each ordered pair can also be written as a 2×1 matrix, which we call a **column vector**:

$$(x, y) = \begin{bmatrix} x \\ y \end{bmatrix}$$

This is a very useful observation, since we can now easily perform the linear transformation $(x, y) \to (ax + by, cx + dy)$ by using matrix multiplication:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

Example 2

a Determine the matrix of the linear transformation with rule $(x, y) \to (x - 2y, 3x + y)$.

b Use the matrix to determine the image of the point $(2, 3)$ under the transformation.

Solution

a $\begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix}$

b $\begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \times 2 - 2 \times 3 \\ 3 \times 2 + 1 \times 3 \end{bmatrix} = \begin{bmatrix} -4 \\ 9 \end{bmatrix}$

Therefore the image of $(2, 3)$ is $(-4, 9)$.

Explanation

The rows of the matrix are given by the coefficients of x and y.

We write the point $(2, 3)$ as a column vector and multiply by the transformation matrix.

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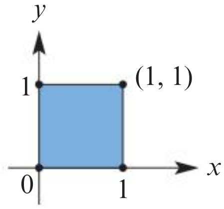
Transforming the unit square

The **unit square** is the square with vertices $(0, 0)$, $(1, 0)$, $(0, 1)$ and $(1, 1)$. The effect of a linear transformation can often be demonstrated by studying its effect on the unit square.



Example 3

A linear transformation is represented by the matrix $\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$.

a Determine the image of the unit square under this transformation.

b Sketch the unit square and its image.

Solution

a We could determine the images of the four vertices of the square one at a time:

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

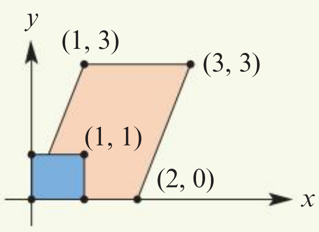
However, this can be done in a single step by multiplying the transformation matrix by a rectangular matrix whose columns are the coordinates of each vertex of the square:

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 1 & 3 \\ 0 & 0 & 3 & 3 \end{bmatrix}$$

The columns of the result give the images of the vertices:

$(0, 0)$, $(2, 0)$, $(1, 3)$, $(3, 3)$

b The unit square is shown in blue and its image in red.



Mapping the standard unit vectors

Let's express the points $(1, 0)$ and $(0, 1)$ as column vectors:

$$(1, 0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad (0, 1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

These are called the **standard unit vectors** in $\mathbb{R}^2$.

We now consider the images of these points under the transformation with matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

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We have

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix} = \text{first column of the matrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix} = \text{second column of the matrix}$$

To determine the matrix of a linear transformation:

- The first column is the image of $(1, 0)$, written as a column vector.
- The second column is the image of $(0, 1)$, written as a column vector.

This observation allows us to write down the matrix of a linear transformation given just two pieces of information.

Example 4

A linear transformation maps the points $(1, 0)$ and $(0, 1)$ to the points $(1, 1)$ and $(-2, 3)$ respectively.

a Determine the matrix of the transformation.

b Determine the image of the point $(-3, 4)$.

Solution

a $\begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}$

b $\begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} -3 \\ 4 \end{bmatrix} = \begin{bmatrix} -11 \\ 9 \end{bmatrix}$

Therefore $(-3, 4) \to (-11, 9)$.

Explanation

The image of $(1, 0)$ is $(1, 1)$, and the image of $(0, 1)$ is $(-2, 3)$. We write these images as the columns of a matrix.

Write the point $(-3, 4)$ as a column vector and multiply by the transformation matrix.

Summary 16A

- A **transformation** maps each point (x, y) in the plane to a new point (x', y').
- A **linear transformation** is defined by a rule of the form $(x, y) \to (ax + by, cx + dy)$.
- Linear transformations can be represented using matrix multiplication:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

- The **unit square** has vertices $(0, 0)$, $(1, 0)$, $(0, 1)$ and $(1, 1)$. The effect of a linear transformation can be seen by looking at the image of the unit square.
- In the matrix of a linear transformation:
  - the first column is the image of $(1, 0)$, written as a column vector
  - the second column is the image of $(0, 1)$, written as a column vector.

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Skill-sheet

Exercise 16A

SF

Example 1

1 Determine the image of the point $(2, -4)$ under the transformation with rule:

a $(x, y) \to (x + y, x - y)$
b $(x, y) \to (2x + 3y, 3x - 4y)$
c $(x, y) \to (3x - 5y, x)$
d $(x, y) \to (y, -x)$

Example 2

2 Determine the image of the point $(2, 3)$ under the linear transformation with matrix:

a $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
b $\begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix}$
c $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$
d $\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$

3 Determine the matrix of the linear transformation defined by the rule:

a $(x, y) \to (2x + 3y, 4x + 5y)$
b $(x, y) \to (11x - 3y, 3x - 8y)$
c $(x, y) \to (2x, x - 3y)$
d $(x, y) \to (y, -x)$

Example 3

4 Determine and sketch the image of the unit square under the linear transformation represented by the matrix:

a $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$
b $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$
c $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$
d $\begin{bmatrix} -1 & 3 \\ 2 & -1 \end{bmatrix}$

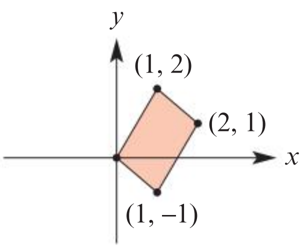
5 Determine the image of the triangle with vertices $(1, 1)$, $(1, 2)$ and $(2, 1)$ under the linear transformation represented by the matrix $\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$.

Example 4

6 Determine the matrix of the linear transformation that maps the points $(1, 0)$ and $(0, 1)$ to the points $(3, 4)$ and $(5, 6)$ respectively. Hence determine the image of the point $(-2, 4)$.

7 Determine the matrix of the linear transformation that maps the points $(1, 0)$ and $(0, 1)$ to the points $(-3, 2)$ and $(1, -1)$ respectively. Hence determine the image of the point $(2, 3)$.

8 Find a matrix that transforms the unit square to each of the following parallelograms.
Note: There are two possible answers for each part.

a


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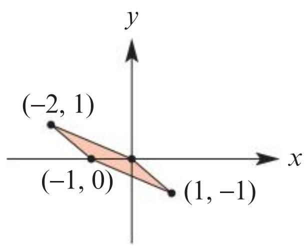
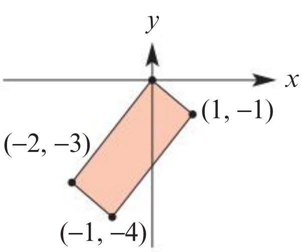


c


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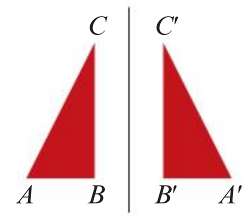
16B Geometric transformations

Learning intentions

- To be able to define and apply reflections, dilations, shears, projections and translations.

We now look at various important transformations that are geometric in nature.

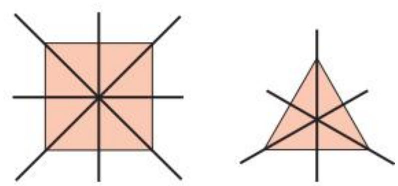
Reflections



A **reflection** in a line ℓ maps each point in the plane to its mirror image on the other side of the line. The point A and its image A' are the same distance from ℓ and the line AA' is perpendicular to ℓ.

These transformations are important for studying figures with **reflective symmetry**, that is, figures that look the same when reflected in a **line of symmetry**.

A square has four lines of symmetry, while an equilateral triangle has just three.



Note: A reflection is an example of a transformation that does not change lengths. Such a transformation is called an **isometry**.

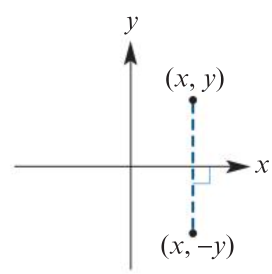
Reflection in the x-axis

A reflection in the x-axis is defined by

$$(x, y) \to (x, -y)$$

So if (x', y') is the image of the point (x, y), then

$$x' = x \quad \text{and} \quad y' = -y$$



This transformation can also be represented using matrix multiplication:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

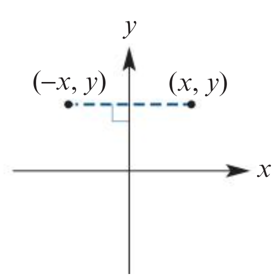
Reflection in the y-axis

A reflection in the y-axis is defined by

$$(x, y) \to (-x, y)$$

So if (x', y') is the image of the point (x, y), then

$$x' = -x \quad \text{and} \quad y' = y$$



Once again, this transformation can be represented using matrix multiplication:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

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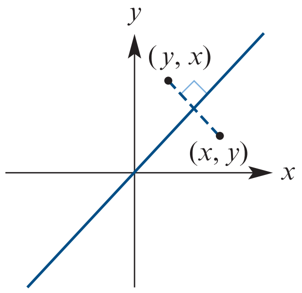
Reflection in the line $y = x$



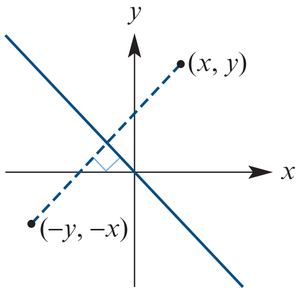
If the point (x, y) is reflected in the line $y = x$, then it is mapped to the point (y, x). So if (x', y') is the image of (x, y), then

$$x' = y \quad \text{and} \quad y' = x$$

Expressing this using matrix multiplication gives

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Reflection in the line $y = -x$



If the point (x, y) is reflected in the line $y = -x$, it is mapped to $(-y, -x)$. So if (x', y') is the image of (x, y), then

$$x' = -y \quad \text{and} \quad y' = -x$$

Expressing this using matrix multiplication gives

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Transformation	Rule	Matrix
Reflection in the x-axis	$x' = 1x + 0y$ $y' = 0x - 1y$	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
Reflection in the y-axis	$x' = -1x + 0y$ $y' = 0x + 1y$	$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$
Reflection in the line $y = x$	$x' = 0x + 1y$ $y' = 1x + 0y$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
Reflection in the line $y = -x$	$x' = 0x - 1y$ $y' = -1x + 0y$	$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

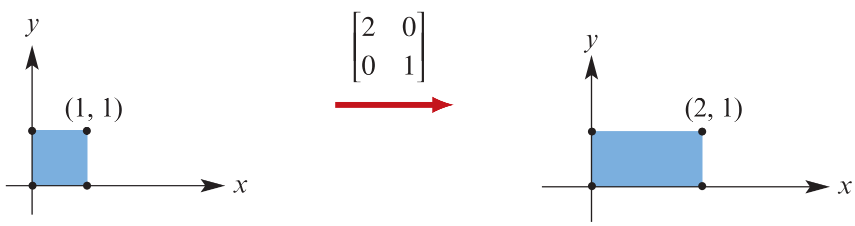
Dilations

Dilation from the y-axis

A dilation from the y-axis is a transformation of the form

$$(x, y) \to (cx, y)$$

where $c > 0$. The x-coordinate is scaled by a factor of c, but the y-coordinate is unchanged.



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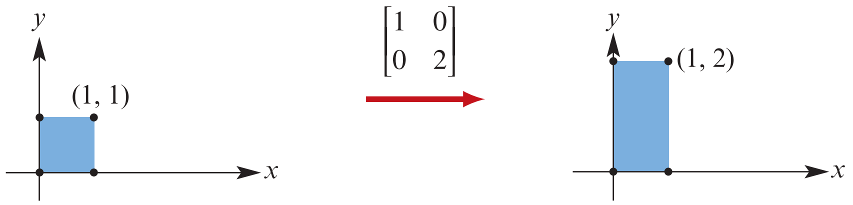
Dilation from the x-axis

Likewise, a dilation from the x-axis is a transformation of the form

$$(x, y) \to (x, cy)$$

where $c > 0$. The y-coordinate is scaled by a factor of c, but the x-coordinate is unchanged.



Dilation from the x- and y-axes

We can also simultaneously scale along the x- and y-axes using the transformation

$$(x, y) \to (cx, dy)$$

with scale factors $c > 0$ and $d > 0$.

Transformation	Rule	Matrix
Dilation from the y-axis	$x' = cx + 0y$ $y' = 0x + 1y$	$\begin{bmatrix} c & 0 \\ 0 & 1 \end{bmatrix}$
Dilation from the x-axis	$x' = 1x + 0y$ $y' = 0x + cy$	$\begin{bmatrix} 1 & 0 \\ 0 & c \end{bmatrix}$
Dilation from the x- and y-axes	$x' = cx + 0y$ $y' = 0x + dy$	$\begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix}$

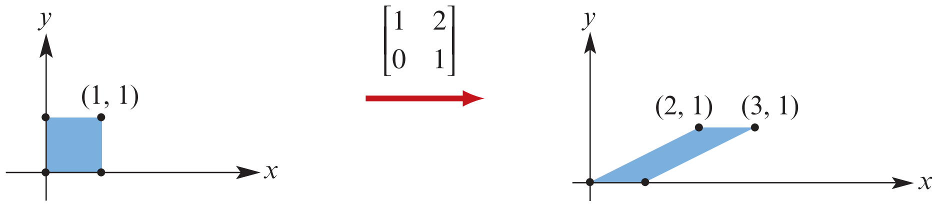
Shears

Shear parallel to the x-axis

A shear parallel to the x-axis is a transformation of the form

$$(x, y) \to (x + cy, y)$$

Notice that each point is moved in the x-direction by an amount proportional to the distance from the x-axis. This means that the unit square is tilted in the x-direction.



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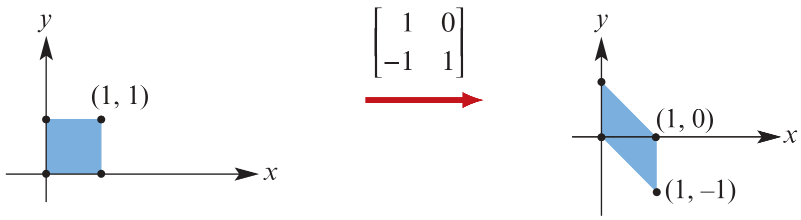
Shear parallel to the y-axis

A shear parallel to the y-axis is a transformation of the form

$$(x, y) \to (x, cx + y)$$

Here, each point is moved in the y-direction by an amount proportional to the distance from the y-axis. Now the unit square is tilted in the y-direction.

Note that if $c < 0$, then we obtain a shear in the negative direction.



Transformation	Rule	Matrix
Shear parallel to the x-axis	$x' = 1x + cy$ $y' = 0x + 1y$	$\begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}$
Shear parallel to the y-axis	$x' = 1x + 0y$ $y' = cx + 1y$	$\begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix}$

Projections

The transformation defined by

$$(x, y) \to (x, 0)$$

will project the point (x, y) onto the x-axis.

Likewise, the transformation defined by

$$(x, y) \to (0, y)$$

will project the point (x, y) onto the y-axis.

Transformation	Rule	Matrix
Projection onto the x-axis	$x' = 1x + 0y$ $y' = 0x + 0y$	$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$
Projection onto the y-axis	$x' = 0x + 0y$ $y' = 0x + 1y$	$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

Projections are an important class of transformations. For example, the image on a television screen is the projection of a three-dimensional scene onto a two-dimensional surface.

Example 5

Determine the image of the point (3, 4) under each of the following transformations:

a reflection in the y-axis

b dilation of factor 2 from the y-axis

c shear of factor 4 parallel to the x-axis

d projection onto the y-axis

Solution

a $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$

$(3, 4) \to (-3, 4)$

b $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$

$(3, 4) \to (6, 4)$

c $\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 19 \\ 4 \end{bmatrix}$

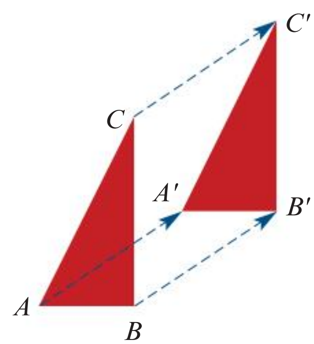
$(3, 4) \to (19, 4)$

d $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$

$(3, 4) \to (0, 4)$

Translations

A **translation** moves a figure so that every point in the figure moves in the same direction and over the same distance.



A translation of a units in the x-direction and b units in the y-direction is defined by the rule

$$(x, y) \to (x + a, y + b)$$

This can be expressed using vector addition:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix}$$

Note: Translations cannot be represented using matrix multiplication. To see this, note that matrix multiplication will always map the point (0, 0) to itself. Therefore, there is no matrix that will translate the point (0, 0) to (a, b), unless $a = b = 0$.

Example 6

Determine the rule for a translation of 2 units in the x-direction and −1 units in the y-direction, and sketch the image of the unit square under this translation.

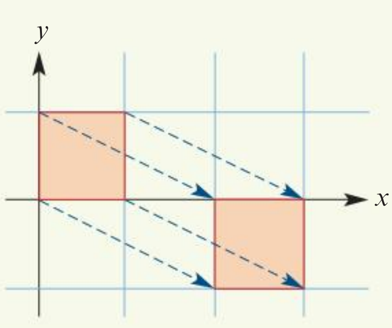
Solution

Using vector addition, this translation can be defined by the rule

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} x + 2 \\ y - 1 \end{bmatrix}$$

or equivalently

$$x' = x + 2 \quad \text{and} \quad y' = y - 1$$



Summary 16B

- Important geometric transformation matrices are summarised in the table below.

Transformation	Matrix	Transformation	Matrix
Reflection in the x-axis	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$	Reflection in the y-axis	$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$
Reflection in the line $y = x$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	Reflection in the line $y = -x$	$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$
Dilation from the y-axis	$\begin{bmatrix} c & 0 \\ 0 & 1 \end{bmatrix}$	Dilation from the x-axis	$\begin{bmatrix} 1 & 0 \\ 0 & c \end{bmatrix}$
Shear parallel to the x-axis	$\begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}$	Shear parallel to the y-axis	$\begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix}$
Projection onto the x-axis	$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$	Projection onto the y-axis	$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

- A translation of a units in the x-direction and b units in the y-direction is defined by the rule $(x, y) \to (x + a, y + b)$. This can be expressed using vector addition:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix}$$

Skill-sheet

Exercise 16B

SF

Example 5

1 For each of the transformations described below:

i Determine the matrix of the transformation

ii sketch the image of the unit square under this transformation.

a dilation of factor 2 from the x-axis

b dilation of factor 3 from the y-axis

c shear of factor 3 parallel to the x-axis

d shear of factor -1 parallel to the y-axis

e reflection in the x-axis

f reflection in the line $y = -x$

Example 6

2 For each of the translations described below:

i Determine the rule for the translation using column vectors

ii sketch the image of the unit square under this translation.

a translation of 2 units in the x-direction

b translation of -3 units in the y-direction

c translation of -2 units in the x-direction and -4 units in the y-direction

d translation by the vector $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$

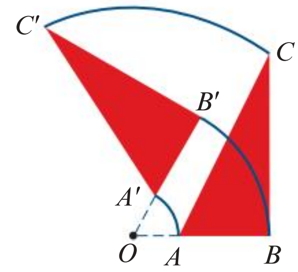
e translation by the vector $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$

16C Rotations and general reflections

Learning intentions

- To be able to define and apply rotations and general reflections.

Rotations



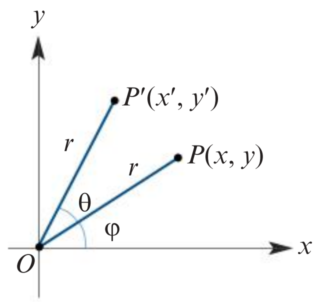
A **rotation** turns an object about a point, but keeps its distance to the point fixed. A rotation does not change lengths, and so is another example of an isometry.

Rotations are important for studying figures with **rotational symmetry**, that is, figures that look the same when rotated through a certain angle.

These two figures have rotational symmetry, but no reflective symmetry.

Determining the rotation matrix

Consider the transformation that rotates each point in the plane about the origin by angle θ anticlockwise. We will show that this is a linear transformation and determine its matrix.



Let O be the origin and let $P(x, y)$ be a point in the plane.

Then we can write

$$x = r\cos\varphi \quad \text{and} \quad y = r\sin\varphi$$

where r is the distance OP and φ is the angle between OP and the positive direction of the x-axis.

Now let $P'(x', y')$ be the image of $P(x, y)$ under a rotation about O by angle θ anticlockwise.

As $OP' = r$, we can use the angle sum identities to show that

$$\begin{aligned} x' &= r\cos(\varphi + \theta) \\ &= r\cos\varphi\cos\theta - r\sin\varphi\sin\theta \\ &= x\cos\theta - y\sin\theta \end{aligned}$$

and

$$\begin{aligned} y' &= r\sin(\varphi + \theta) \\ &= r\sin\varphi\cos\theta + r\cos\varphi\sin\theta \\ &= y\cos\theta + x\sin\theta \end{aligned}$$

Writing this using matrix multiplication gives

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

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Example 7

Determine the matrix that represents a rotation of the plane about the origin by:

a 90° anticlockwise

b 45° clockwise.

Solution

a $\begin{bmatrix} \cos 90° & -\sin 90° \\ \sin 90° & \cos 90° \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

b $\begin{bmatrix} \cos(-45°) & -\sin(-45°) \\ \sin(-45°) & \cos(-45°) \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

Explanation

An anticlockwise rotation means that we let $\theta = 90°$ in the formula for the rotation matrix.

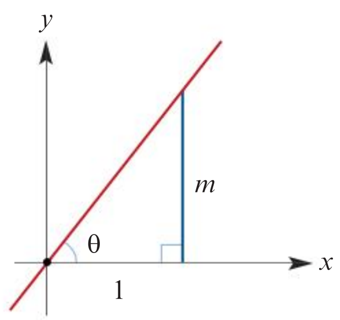
A clockwise rotation means that we let $\theta = -45°$ in the formula for the rotation matrix.

Reflection in the line $y = mx$

Reflection in a line that passes through the origin is also a linear transformation. We will determine the matrix that will reflect the point (x, y) in the line $y = mx$.

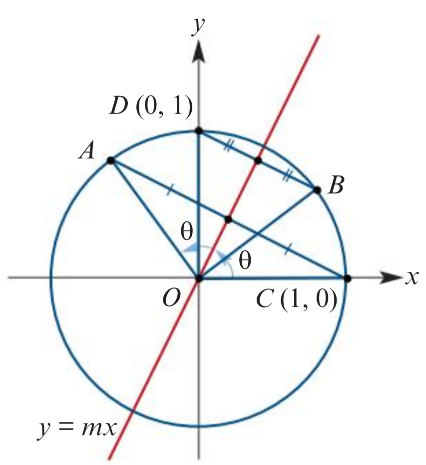
Let's suppose that the angle between the positive direction of the x-axis and the line $y = mx$ is θ. Then $\tan\theta = m$ and so

$$y = mx = x\tan\theta$$



Determining the reflection matrix

We will use the fact that the first column of the required matrix will be the image A of $C(1, 0)$, written as a column vector, and the second column will be the image B of $D(0, 1)$, written as a column vector.



Since $\angle AOC = 2\theta$, we have

$$(1, 0) \to (\cos(2\theta), \sin(2\theta))$$

Moreover, since $\angle BOC = 2\theta - 90°$, we have

$$\begin{aligned}(0, 1) &\to (\cos(2\theta - 90°), \sin(2\theta - 90°)) \\ &= (\sin(2\theta), -\cos(2\theta))\end{aligned}$$

Writing these images as column vectors gives the reflection matrix:

$$\begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$$

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Example 8

a Determine the matrix that will reflect the point (x, y) in the line through the origin at an angle of 30° to the positive direction of the x-axis.

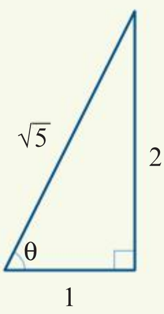
b Determine the matrix that will reflect the point (x, y) in the line $y = 2x$.

Solution

a We simply let $\theta = 30°$, and so the required reflection matrix is

$$\begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix} = \begin{bmatrix} \cos 60° & \sin 60° \\ \sin 60° & -\cos 60° \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{-1}{2} \end{bmatrix}$$

b Since $\tan\theta = 2 = \frac{2}{1}$, we draw a right-angled triangle with opposite and adjacent lengths 2 and 1 respectively.



Pythagoras' theorem gives the hypotenuse as $\sqrt{5}$. Therefore

$$\cos\theta = \frac{1}{\sqrt{5}} \quad \text{and} \quad \sin\theta = \frac{2}{\sqrt{5}}$$

We then use the double-angle identities to show that

$$\cos(2\theta) = 2\cos^2\theta - 1 = 2\left(\frac{1}{\sqrt{5}}\right)^2 - 1 = \frac{2}{5} - 1 = -\frac{3}{5}$$

$$\sin(2\theta) = 2\sin\theta\cos\theta = 2 \times \frac{2}{\sqrt{5}} \times \frac{1}{\sqrt{5}} = \frac{4}{5}$$

Therefore the required reflection matrix is

$$\begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix} = \begin{bmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}$$

Summary 16C

Rotation matrix

The matrix that will rotate the plane about the origin by angle θ anticlockwise is

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Reflection matrix

The matrix that will reflect the plane in the line $y = mx = x\tan\theta$ is

$$\begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$$

Exercise 16C

Example 7

1 Determine the matrix for each of the following rotations about the origin:

a 270° anticlockwise

b 30° anticlockwise

c 60° clockwise

d 135° clockwise

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SF

2 Determine the image of the point $(2, 3)$ under a rotation about the origin by:

a 90° anticlockwise **b** 45° clockwise.

Example 8a

3 Determine the matrix that will reflect the point (x, y) in the line through the origin that makes an angle with the positive direction of the x-axis of:

a 45° **b** 60° **c** -30° **d** 15°

Example 8b

4 Determine the matrix that will reflect the point (x, y) in the line:

a $y = 3x$ **b** $y = 5x$ **c** $y = \frac{2x}{3}$ **d** $y = -3x$

5 **a** Determine a formula (in terms of m) for the matrix that will reflect the point (x, y) in the line $y = mx$.

b Hence determine the image of $(1, 1)$ when reflected in the line $y = 6x$.

6 The unit square is rotated about the origin by 45° anticlockwise.

a Determine the matrix of this transformation.

b Draw the unit square and its image on the same set of axes.

c determine the area of the overlapping region.

7 The point $A(1, 0)$ is rotated to point B through angle 120° anticlockwise. The point $A(1, 0)$ is also rotated to point C through angle 240° anticlockwise.

a Determine the coordinates of B and C.

b What sort of triangle is ABC?

c Triangle ABC has three lines of symmetry. Determine the equations of all three lines.

16D Composition of transformations

Learning intentions

- To be able to use matrix arithmetic for composition of transformations.

Composition as matrix multiplication

The matrix representation of linear transformations makes it easy to determine the effect of one transformation followed by another. Suppose that we would like to:

1 reflect the point (x, y) in the y-axis

2 then dilate the result by a factor of 2 from the x-axis.

To reflect the point (x, y), we would first evaluate

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

To dilate the result by a factor of 2 from the x-axis, we would then multiply

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

Therefore the matrix that will achieve both transformations is the product

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$$

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If matrices **A** and **B** correspond to two different linear transformations, then:

- **AB** is the matrix of transformation **B** followed by **A**
- **BA** is the matrix of transformation **A** followed by **B**.

Example 9

Determine the matrix that corresponds to:

a a reflection in the x-axis and then a rotation about the origin by 90° anticlockwise

b a rotation about the origin by 90° anticlockwise and then a reflection in the x-axis.

Solution

a $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

b $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

Explanation

A reflection in the x-axis has matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

An anticlockwise rotation by 90° has matrix

$$\begin{bmatrix} \cos 90° & -\sin 90° \\ \sin 90° & \cos 90° \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

We then multiply these two matrices together in the correct order.

Note: In this example, we get a different matrix when the same two transformations take place in reverse order. This should not be a surprise, as matrix multiplication is not commutative in general.

Compositions involving translations

Example 10

a Determine the rule for the transformation that will reflect (x, y) in the x-axis and then translate the result by the vector $\begin{bmatrix} -3 \\ 4 \end{bmatrix}$.

b Determine the rule for the transformation if the translation takes place before the reflection.

Solution

a
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -3 \\ 4 \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix} + \begin{bmatrix} -3 \\ 4 \end{bmatrix} = \begin{bmatrix} x-3 \\ -y+4 \end{bmatrix}$$

Therefore the transformation is $(x, y) \to (x-3, -y+4)$.

b
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -3 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x-3 \\ y+4 \end{bmatrix} = \begin{bmatrix} x-3 \\ -y-4 \end{bmatrix}$$

Therefore the transformation is $(x, y) \to (x-3, -y-4)$.

Summary 16D

If matrices **A** and **B** correspond to two different linear transformations, then:

- **AB** is the matrix of transformation **B** followed by **A**
- **BA** is the matrix of transformation **A** followed by **B**.

The order is important, as matrix multiplication is not commutative in general.

Skill-sheet

Exercise 16D

SF

Example 9

1 Determine the matrix that represents a reflection in the y-axis followed by a dilation of factor 3 from the x-axis.

2 Determine the matrix that represents a rotation about the origin by 90° anticlockwise followed by a reflection in the x-axis.

3 **a** Determine the matrix that represents a reflection in the x-axis followed by a reflection in the y-axis.

b Show that this matrix corresponds to a rotation about the origin by 180°.

4 Consider these two transformations:

- T_1: A reflection in the x-axis.
- T_2: A dilation of factor 2 from the y-axis.

a Find the matrix of T_1 followed by T_2. **b** Find the matrix of T_2 followed by T_1.

c Does the order of transformation matter in this instance?

5 Consider these two transformations:

- T_1: A rotation about the origin by 90° clockwise.
- T_2: A reflection in the line $y = x$.

a Find the matrix of T_1 followed by T_2. **b** Find the matrix of T_2 followed by T_1.

c Does the order of transformation matter in this instance?

Example 10

6 Consider these two transformations:

- T_1: A reflection in the y-axis.
- T_2: A translation of −3 units in the x-direction and 5 units in the y-direction.

a Determine the rule for T_1 followed by T_2. **b** Determine the rule for T_2 followed by T_1.

c Does the order of transformation matter in this instance?

7 Express each of the following transformation matrices as the product of a dilation matrix and a reflection matrix:

a $\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$ **b** $\begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix}$ **c** $\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$ **d** $\begin{bmatrix} 0 & -2 \\ -1 & 0 \end{bmatrix}$

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8 **a** Determine the matrix for the transformation that is a reflection in the x-axis followed by a reflection in the line $y = x$.
b Show that these two reflections can be achieved with one rotation.

SF

9 Suppose that matrix $\mathbf{A}$ gives a rotation about the origin by angle θ anticlockwise and that matrix $\mathbf{B}$ gives a reflection in the line $y = x$. If $\mathbf{AB} = \mathbf{BA}$, determine the angle θ.

CF

10 Suppose that matrix $\mathbf{A}$ rotates the plane about the origin by angle θ anticlockwise.
a Through what angle will the matrix $\mathbf{A}^2$ rotate the plane?
b Evaluate $\mathbf{A}^2$.
c Hence determine formulas for $\cos(2\theta)$ and $\sin(2\theta)$.

11 A transformation T consists of a reflection in the line $y = x$ followed by a translation by the vector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.
a Determine the rule for the transformation T.
b Show that the transformation T can also be obtained by a translation and then a reflection in the line $y = x$. determine the translation vector.

12 **a** Determine the rotation matrix for an angle of 60° anticlockwise.
b Determine the rotation matrix for an angle of 45° clockwise.
c By multiplying these two matrices, determine the rotation matrix for an angle of 15° anticlockwise.
d Hence write down the exact values of sin 15° and cos 15°.

13 A transformation consists of a reflection in the line $y = x\tan\varphi$ and then in the line $y = x\tan\theta$. Show that this is equivalent to a single rotation.

CU

16E Inverse transformations

Learning intentions

- To be able to use the inverse of a matrix to describe the inverse of a transformation.

If transformation T maps the point (x, y) to the point (x', y'), then the **inverse transformation** T^{-1} maps the point (x', y') to the point (x, y).

For a linear transformation T, we can write

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\mathbf{X}' = \mathbf{AX}$$

If the inverse matrix $\mathbf{A}^{-1}$ exists, then we have

$$\mathbf{A}^{-1}\mathbf{X}' = \mathbf{A}^{-1}\mathbf{AX}$$

$$\mathbf{A}^{-1}\mathbf{X}' = \mathbf{IX}$$

$$\mathbf{X} = \mathbf{A}^{-1}\mathbf{X}'$$

Therefore $\mathbf{A}^{-1}$ is the matrix of the inverse transformation T^{-1}.

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If the matrix of a linear transformation is

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

then the matrix of the inverse transformation is

$$\mathbf{A}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

The inverse exists if and only if $\det(\mathbf{A}) = ad - bc \neq 0$.

Example 11

Determine the inverse of the transformation with rule $(x, y) \to (3x + 2y, 5x + 4y)$.

Solution

Since the matrix of this linear transformation is

$$\mathbf{A} = \begin{bmatrix} 3 & 2 \\ 5 & 4 \end{bmatrix}$$

the inverse transformation will have matrix

$$\begin{aligned} \mathbf{A}^{-1} &= \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \\ &= \frac{1}{3 \times 4 - 2 \times 5}\begin{bmatrix} 4 & -2 \\ -5 & 3 \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} 4 & -2 \\ -5 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 2 & -1 \\ -\frac{5}{2} & \frac{3}{2} \end{bmatrix} \end{aligned}$$

Therefore the rule of the inverse transformation is $(x, y) \to (2x - y, -\frac{5}{2}x + \frac{3}{2}y)$.

Example 12

Determine the matrix of the linear transformation such that $(4, 3) \to (9, 10)$ and $(2, 1) \to (5, 6)$.

Solution

We need to determine a matrix $\mathbf{A}$ such that

$$\mathbf{A}\begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 9 \\ 10 \end{bmatrix} \quad \text{and} \quad \mathbf{A}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

This can be written as a single equation:

$$\mathbf{A}\begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 5 \\ 10 & 6 \end{bmatrix}$$

Therefore

$$\mathbf{A} = \begin{bmatrix} 9 & 5 \\ 10 & 6 \end{bmatrix}\begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 3 & -1 \\ 4 & -2 \end{bmatrix}$$

Inverses of important transformations

For important geometric transformations, it is often obvious what the inverse transformation should be.

Example 13

Let **R** be the matrix corresponding to a rotation of the plane by angle θ anticlockwise. Show that $\mathbf{R}^{-1}$ corresponds to a rotation by angle θ clockwise.

Solution

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}^{-1} = \frac{1}{\cos^2\theta + \sin^2\theta}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix}$$

This matrix corresponds to a rotation of the plane by angle θ clockwise.

Explanation

We determine the inverse matrix using the formula

$$\mathbf{A}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

We also use the symmetry properties:

$$\sin(-\theta) = -\sin\theta$$

$$\cos(-\theta) = \cos\theta$$

The following table summarises the important geometric transformations along with their inverses. You will demonstrate some of these results in the exercises.

Transformation	Matrix **A**	Inverse matrix $\mathbf{A}^{-1}$	Inverse transformation
Dilation from the y-axis	$\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} \frac{1}{k} & 0 \\ 0 & 1 \end{bmatrix}$	Dilation from the y-axis
Dilation from the x-axis	$\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{k} \end{bmatrix}$	Dilation from the x-axis
Shear parallel to the x-axis	$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & -k \\ 0 & 1 \end{bmatrix}$	Shear parallel to the x-axis
Rotation by θ anticlockwise	$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$	$\begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix}$	Rotation by θ clockwise
Reflection in the x-axis	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$	Reflection in the x-axis
Reflection in the y-axis	$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$	Reflection in the y-axis
Reflection in the line $y = mx$	$\begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$	$\begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$	Reflection in the line $y = mx$

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Summary 16E

If the matrix of a linear transformation is

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

then the matrix of the inverse transformation is

$$\mathbf{A}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Exercise 16E

SF

1 Determine the inverse matrix of each of the following transformation matrices:

a $\begin{bmatrix} 4 & 1 \\ 3 & 1 \end{bmatrix}$ **b** $\begin{bmatrix} 3 & 1 \\ 2 & -4 \end{bmatrix}$

c $\begin{bmatrix} 0 & 3 \\ -2 & 4 \end{bmatrix}$ **d** $\begin{bmatrix} -1 & 3 \\ -4 & 5 \end{bmatrix}$

Example 11

2 For each of the following transformations, determine the rule for their inverse:

a $(x, y) \to (5x - 2y, 2x - y)$ **b** $(x, y) \to (x - y, x)$

3 Determine the point (x, y) that is mapped to $(1, 1)$ by the transformation with matrix:

a $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ **b** $\begin{bmatrix} 4 & 3 \\ 2 & 2 \end{bmatrix}$

Example 12

4 Determine the matrix of the linear transformation such that $(1, 2) \to (2, 1)$ and $(2, 3) \to (1, 1)$.

5 Determine the vertices of the paralleogram that is mapped to the unit square by the transformation with matrix $\begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$.

Example 13

6 Consider a dilation of factor k from the y-axis, where $k > 0$.

a Write down the matrix of this transformation.

b Show that the inverse matrix corresponds to a dilation of factor $\frac{1}{k}$ from the y-axis.

7 Consider a shear of factor k parallel to the x-axis.

a Write down the matrix of this transformation.

b Show that the inverse matrix corresponds to a shear of factor $-k$ parallel to the x-axis.

8 Consider the transformation that reflects each point in the x-axis.

a Write down the matrix $\mathbf{A}$ of this transformation.

b Show that $\mathbf{A}^{-1} = \mathbf{A}$, and explain why you should expect this result.

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SF

9 Consider the transformation that reflects each point in the line $y = mx = x\tan\theta$.

a Write down the matrix **B** of this transformation in terms of θ.

b Show that $\mathbf{B}^{-1} = \mathbf{B}$, and explain why you should expect this result.

16F Transformations of straight lines and other graphs

Learning intentions

- To be able to apply transformations to straight lines and other graphs.

We have considered the effect of various transformations on points and figures in the plane. We will now turn our attention to graphs.

Here, we will aim to determine the equations of transformed graphs. We will also investigate the effects of linear transformations on straight lines.

Linear transformations of straight lines

We will first investigate the effect of linear transformations on straight lines.

Example 14

Determine the equation of the image of the line $y = 2x + 3$ under a reflection in the x-axis followed by a dilation of factor 2 from the y-axis.

Solution

The matrix of the combined transformation is

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$$

If (x', y') are the coordinates of the image of (x, y), then

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ -y \end{bmatrix}$$

Therefore

$$x' = 2x \quad \text{and} \quad y' = -y$$

Rearranging gives

$$x = \frac{x'}{2} \quad \text{and} \quad y = -y'$$

Therefore the equation $y = 2x + 3$ becomes

$$-y' = 2\left(\frac{x'}{2}\right) + 3$$

$$-y' = x' + 3$$

$$y' = -x' - 3$$

We now ignore the dashes, and so the equation of the image is simply $y = -x - 3$.

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Example 15

Consider the graph of $y = x + 1$. Determine the equation of its image under the linear transformation $(x, y) \to (x + 2y, y)$.

Solution

Let (x', y') be the coordinates of the image of (x, y). Then this transformation can be written in matrix form as

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Therefore

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x' - 2y' \\ y' \end{bmatrix}$$

and so $x = x' - 2y'$ and $y = y'$.

The equation $y = x + 1$ becomes

$$\begin{aligned} y' &= x' - 2y' + 1 \\ 3y' &= x' + 1 \\ y' &= \frac{x'}{3} + \frac{1}{3} \end{aligned}$$

The equation of the image is $y = \dfrac{x}{3} + \dfrac{1}{3}$.

In the previous two examples, you will have noticed that the image of each straight line was another straight line. In fact, linear transformations get their name in part from the following fact, which is proved in the exercises.

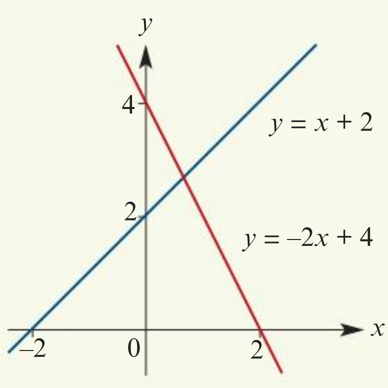
The image of any straight line under an invertible linear transformation is a straight line.

Example 16

Determine a matrix that transforms the line $y = x + 2$ to the line $y = -2x + 4$.

Solution

Let's determine the matrix that maps the x-axis intercept of the first line to the x-axis intercept of the second line, and likewise for the y-axis intercepts.



We want

$$(-2, 0) \to (2, 0) \quad \text{and} \quad (0, 2) \to (0, 4)$$

This can be achieved by a reflection in the y-axis and then a dilation of factor 2 from the x-axis.

This transformation has the matrix $\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$.

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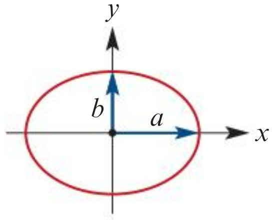
Ellipses as transformations of circles

If the unit circle is dilated by factor a from the x-axis and factor b from the y-axis, we obtain the curve with equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$



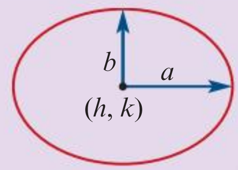
This is an **ellipse** with x-axis intercepts $\pm a$ and y-axis intercepts $\pm b$.

Applying the translation defined by $(x, y) \to (x + h, y + k)$, we can see the following result:

The graph of

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

is an ellipse centred at the point (h, k).



Note: Ellipses are studied further in Appendix B.

Example 17

Sketch the graph of each ellipse:

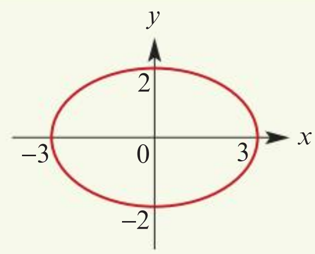
a $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$ **b** $4x^2 + 9y^2 = 1$ **c** $\dfrac{(x-1)^2}{4} + \dfrac{(y+2)^2}{9} = 1$

Solution

a The equation can be written as

$$\frac{x^2}{3^2} + \frac{y^2}{2^2} = 1$$

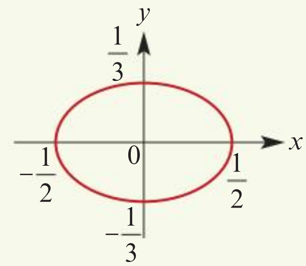
This is an ellipse with centre $(0, 0)$ and axis intercepts at $x = \pm 3$ and $y = \pm 2$.



b The equation can be written as

$$\frac{x^2}{(\frac{1}{2})^2} + \frac{y^2}{(\frac{1}{3})^2} = 1$$

This is an ellipse with centre $(0, 0)$ and axis intercepts at $x = \pm\frac{1}{2}$ and $y = \pm\frac{1}{3}$.



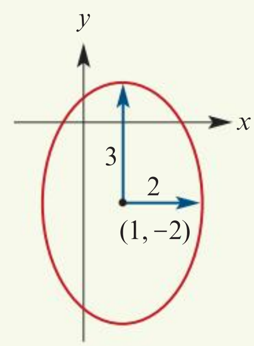
c This is an ellipse with centre $(1, -2)$.

To determine the x-axis intercepts, let $y = 0$. Then solving for x gives

$$x = \frac{3 \pm 2\sqrt{5}}{3}$$

Likewise, to determine the y-axis intercepts, let $x = 0$. This gives

$$y = \frac{-4 \pm 3\sqrt{3}}{2}$$



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Example 18

Find the image of the unit circle, $x^2 + y^2 = 1$, under a dilation of factor 2 from the y-axis and then a rotation about the origin by 90° anticlockwise. Sketch the circle and its image.

Solution

The dilation matrix is $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$.

The rotation matrix is $\begin{bmatrix} \cos 90° & -\sin 90° \\ \sin 90° & \cos 90° \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

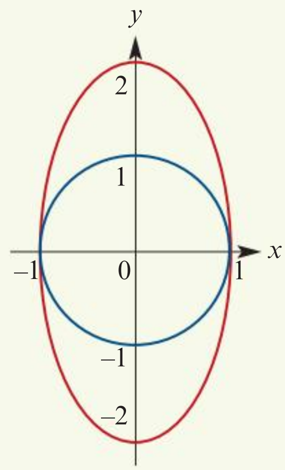
Let (x', y') be the image of (x, y). Then the transformation is given by

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ 2x \end{bmatrix}$$

Thus $x' = -y$ and $y' = 2x$, giving $y = -x'$ and $x = \dfrac{y'}{2}$.

The equation $x^2 + y^2 = 1$ becomes $\left(\dfrac{y'}{2}\right)^2 + (-x')^2 = 1$.

Hence the image is the ellipse with equation $x^2 + \dfrac{y^2}{2^2} = 1$.



Transformations of other graphs

The method for determining the image of a straight line or circle can be used for other graphs.

Example 19

Determine the image of the graph of $y = x^2 + 1$ under a translation by the vector $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ followed by a reflection in the y-axis.

Solution

Let (x', y') be the image of (x, y). Then the transformation is given by

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \end{bmatrix}\right)$$

$$= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x + 2 \\ y - 1 \end{bmatrix} = \begin{bmatrix} -x - 2 \\ y - 1 \end{bmatrix}$$

Therefore $x' = -x - 2$ and $y' = y - 1$.

This gives $x = -x' - 2$ and $y = y' + 1$.

The equation $y = x^2 + 1$ becomes

$$\begin{aligned} y' + 1 &= (-x' - 2)^2 + 1 \\ y' &= (-x' - 2)^2 \\ &= (x' + 2)^2 \end{aligned}$$

The equation of the image is $y = (x + 2)^2$.

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Exercise 16F

Example 14 **1** Determine the equation of the image of the graph of $y = 3x + 1$ under:

- **a** a reflection in the x-axis
- **b** a dilation of factor 2 from the y-axis
- **c** a dilation of factor 3 from the x-axis and factor 2 from the y-axis
- **d** a reflection in the x-axis and then in the y-axis
- **e** a reflection in the y-axis and then a dilation of factor 3 from the x-axis
- **f** a rotation about the origin by 90° anticlockwise
- **g** a rotation about the origin by 90° clockwise and then a reflection in the x-axis.

Example 15 **2** Determine the image of $y = 2 - 3x$ under each of the following transformations:

- **a** $(x, y) \to (2x, 3y)$
- **b** $(x, y) \to (-y, x)$
- **c** $(x, y) \to (x - 2y, y)$
- **d** $(x, y) \to (3x + 5y, x + 2y)$

Example 16 **3** Determine a matrix that transforms the line $x + y = 1$ to the line $x + y = 2$.

4 Determine a matrix that transforms the line $y = x + 1$ to the line $y = 6 - 2x$.

Example 17 **5** Sketch the graph of each ellipse, labelling the centre and the axis intercepts:

- **a** $\dfrac{x^2}{9} + \dfrac{y^2}{64} = 1$
- **b** $\dfrac{x^2}{100} + \dfrac{y^2}{25} = 1$
- **c** $25x^2 + 9y^2 = 225$
- **d** $\dfrac{(x-3)^2}{9} + \dfrac{(y-4)^2}{16} = 1$
- **e** $\dfrac{(x+3)^2}{9} + \dfrac{(y+4)^2}{25} = 1$

Example 18 **6** Determine the image of the unit circle, $x^2 + y^2 = 1$, under a dilation of factor 3 from the x-axis and then a rotation about the origin by 90° anticlockwise. Sketch the circle and its image.

Example 19 **7** Determine the equation of the image of the graph of $y = x^2 - 1$ under a translation by the vector $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ and then a reflection in the x-axis.

8 Determine the equation of the image of the graph of $y = (x - 1)^2$ under a reflection in the y-axis and then a translation by the vector $\begin{bmatrix} 2 \\ -3 \end{bmatrix}$.

9 Consider any invertible linear transformation

$$(x, y) \to (ax + by, cx + dy)$$

Show that the image of the straight line $px + qy = r$ is a straight line.

10 Rotate the graph of $y = \dfrac{1}{x}$ by 45° anticlockwise. Show that the equation of the image is $y^2 - x^2 = 2$.

Note: This shows that the two curves are congruent hyperbolas.

SF

CF

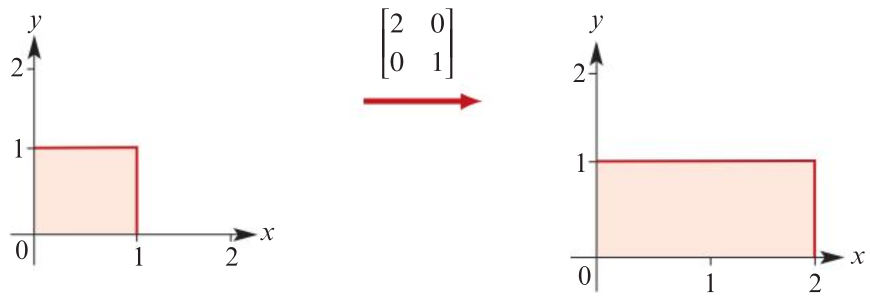
16G Area and determinant

Learning intentions

- To be able to use the determinant of a transformation matrix to determine the area of a transformed region of the plane.

If we apply a linear transformation to some region of the plane, then the area may change.

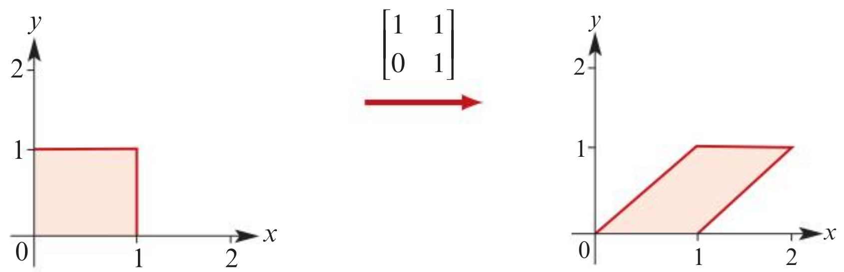
For example, if we dilate the unit square by a factor of 2 from the y-axis, then the area increases by a factor of 2.



Notice that this increase corresponds to the determinant of the transformation matrix:

$$\det\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = 2$$

On the other hand, if we shear the unit square by a factor of 1 parallel to the x-axis, then the area is unchanged.



Notice that the determinant of this transformation matrix is

$$\det\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = 1$$

More generally, we can prove the following remarkable result.

If a region of the plane is transformed by matrix $\mathbf{B}$, then

$$\text{Area of image} = |\det(\mathbf{B})| \times \text{Area of region}$$

Proof We will prove the result when the unit square is transformed by matrix $\mathbf{B} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

The result can be extended to other regions by approximating them by squares.

We will assume that a, b, c and d are all positive and that $\det(\mathbf{B}) > 0$. The proof can easily be adapted if we relax these assumptions.

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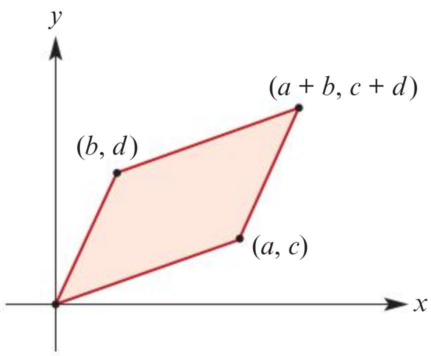
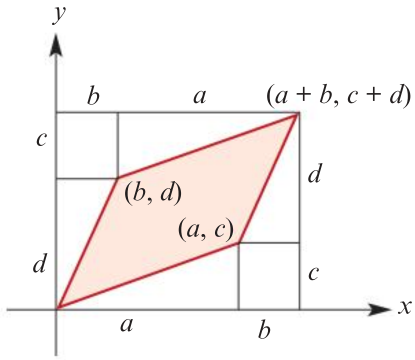
The image of the unit square under transformation **B** is a parallelogram.





To determine the area of the image, we draw a rectangle around it as shown, and subtract the area of the two small rectangles and four triangles from the total area:

$$\begin{aligned}\text{Area of image} &= (a+b)(c+d) - bc - bc - \frac{ac}{2} - \frac{ac}{2} - \frac{bd}{2} - \frac{bd}{2}\\ &= (a+b)(c+d) - 2bc - ac - bd\\ &= ac + ad + bc + bd - 2bc - ac - bd\\ &= ad - bc\end{aligned}$$

This is equal to the determinant of matrix **B**.

Example 20

The triangular region with vertices (1, 1), (2, 1) and (1, 2) is transformed by the rule $(x, y) \to (-x + 2y, 2x + y)$.

a Determine the matrix of the linear transformation.

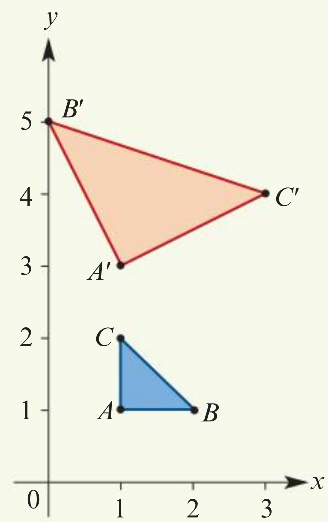
b On the same set of axes, sketch the region and its image.

c Determine the area of the image.

Solution

a The matrix is given by $\mathbf{B} = \begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}$.

b The region is shown in blue and its image in red.



c The area of the original region is $\frac{1}{2}$.

The determinant of the transformation matrix is

$$\det\begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix} = (-1) \times 1 - 2 \times 2 = -5$$

Therefore

$$\begin{aligned}\text{Area of image} &= |\det(\mathbf{B})| \times \text{Area of region}\\ &= |-5| \times \frac{1}{2} = \frac{5}{2}\end{aligned}$$

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Example 21

The unit square is mapped to a parallelogram of area 3 by the matrix

$$\mathbf{B} = \begin{bmatrix} m & 2 \\ m & m \end{bmatrix}$$

Determine the possible values of m.

Solution

The original area is 1. Therefore

$$\begin{aligned} \text{Area of image} &= |\det(\mathbf{B})| \times \text{Area of region} \\ 3 &= |ad - bc| \times 1 \\ 3 &= |m^2 - 2m| \end{aligned}$$

Therefore either $m^2 - 2m = 3$ or $m^2 - 2m = -3$.

Case 1:

$$\begin{aligned} m^2 - 2m &= 3 \\ m^2 - 2m - 3 &= 0 \\ (m + 1)(m - 3) &= 0 \\ m = -1 &\text{ or } m = 3 \end{aligned}$$

Case 2:

$$\begin{aligned} m^2 - 2m &= -3 \\ m^2 - 2m + 3 &= 0 \end{aligned}$$

This quadratic equation has no solutions, since the discriminant is

$$\begin{aligned} \Delta &= b^2 - 4ac \\ &= (-2)^2 - 4(1)(3) = 4 - 12 < 0 \end{aligned}$$

The connection between area and determinant has many important applications. In the next example, we see how it can be used to determine the area of an ellipse. Alternative approaches to determining this area are much more sophisticated.

Example 22

The circle with equation $x^2 + y^2 = 1$ is mapped to an ellipse by the rule $(x, y) \to (ax, by)$, where both a and b are positive.

a Determine the equation of the ellipse and sketch its graph.

b Determine the area of the ellipse.

Solution

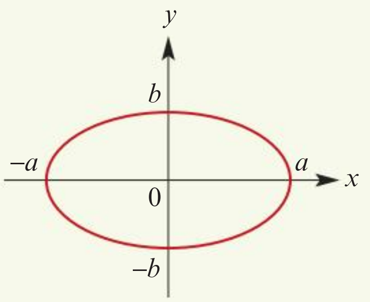
a We have $x' = ax$ and $y' = by$.

This gives $x = \dfrac{x'}{a}$ and $y = \dfrac{y'}{b}$.

The equation $x^2 + y^2 = 1$ becomes

$$\left(\frac{x'}{a}\right)^2 + \left(\frac{y'}{b}\right)^2 = 1$$

Hence the equation of the ellipse is $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.



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b The area of the original circle of radius 1 is π.

The determinant of the transformation matrix is

$$\det\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = a \times b - 0 \times 0 = ab$$

Therefore the area of the ellipse is πab.

Note: When $a = b = r$, this formula gives the area of a circle of radius r.

Summary 16G

If a region of the plane is transformed by matrix **B**, then

Area of image $= |\det(\mathbf{B})| \times$ Area of region

Exercise 16G

SF

1 Each of the following matrices maps the unit square to a parallelogram. Sketch each parallelogram and determine its area.

a $\begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}$ **b** $\begin{bmatrix} -1 & 1 \\ 1 & 3 \end{bmatrix}$ **c** $\begin{bmatrix} 1 & -1 \\ -2 & 1 \end{bmatrix}$ **d** $\begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$

Example 20

2 The matrix $\begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$ maps the triangle with vertices $(0, 1)$, $(1, 1)$ and $(0, 0)$ to a new triangle.

a Sketch the original triangle and its image.

b Determine the areas of both triangles.

3 The matrix $\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$ maps the triangle with vertices $(-1, 1)$, $(1, 1)$ and $(1, 0)$ to a new triangle.

a Sketch the original triangle and its image.

b Determine the areas of both triangles.

Example 21

CF

4 The matrix $\begin{bmatrix} m & 2 \\ -1 & m \end{bmatrix}$ maps the unit square to a parallelogram of area 6 square units. Determine the value(s) of m.

5 The matrix $\begin{bmatrix} m & m \\ 1 & m \end{bmatrix}$ maps the unit square to a parallelogram of area 2 square units. Determine the value(s) of m.

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CF

6 **a** By evaluating a determinant, show that each of the following transformations will not change the area of any region:

i a shear of factor k parallel to the x-axis

ii an anticlockwise rotation about the origin by angle θ

iii a reflection in any straight line through the origin

Note: We say that each of these transformations **preserves area**.

b Let $k > 0$. A linear transformation has matrix $\begin{bmatrix} k & 0 \\ 0 & \frac{1}{k} \end{bmatrix}$.

i Describe the geometric effect of the transformation.

ii Show that this transformation preserves area.

7 For each $x \in \mathbb{R}$, the matrix $\begin{bmatrix} x & 1 \\ -2 & x+2 \end{bmatrix}$ maps the unit square to a parallelogram.

a Show that the area of the parallelogram is $(x+1)^2 + 1$.

b For what value of x is the area of the parallelogram a minimum?

8 For what values of m does the matrix $\begin{bmatrix} m & 2 \\ 3 & 4 \end{bmatrix}$ map the unit square to a parallelogram of area greater than 2?

9 Determine all matrices that will map the unit square to a rhombus of area $\frac{1}{2}$ with one vertex at $(0,0)$ and another at $(1,0)$.

CU

10 **a** Determine a matrix that transforms the triangle with vertices $(0,0)$, $(1,0)$ and $(0,1)$ to the triangle with vertices $(0,0)$, (a,c) and (b,d).

b Hence show that the area of the triangle with vertices $(0,0)$, (a,c) and (b,d) is given by the formula

$$A = \frac{1}{2}|ad - bc|$$

c Hence prove that if a, b, c and d are rational numbers, then the area of this triangle is rational.

d A **rational point** has coordinates (x,y) such that both x and y are rational numbers. Prove that no equilateral triangle can be drawn in the Cartesian plane so that all three of its vertices are rational points.

Hint: You can assume that the vertices of the triangle are $(0,0)$, (a,c) and (b,d). Determine another expression for the area of the triangle using Pythagoras' theorem. You can also assume that $\sqrt{3}$ is irrational.

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16H General transformations

Learning intentions

- To be able to determine a sequence of transformations which determines a rotation about any given point, or a reflection in any given straight line.

Earlier in this chapter we considered rotations about the origin. But what if we want to rotate a figure about a point that is not the origin? In this section we will see how a more complicated transformation can be achieved by a sequence of simpler transformations.

Rotation about the point (a, b)

If we want to rotate the plane about the point (a, b) by angle θ anticlockwise, we can do this in a sequence of three steps:

Step 1 Translate the plane so that the centre of rotation is now the origin, by adding $\begin{bmatrix} -a \\ -b \end{bmatrix}$.

Step 2 Rotate the plane through angle θ anticlockwise, by multiplying by $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

Step 3 Translate the plane back to its original position, by adding $\begin{bmatrix} a \\ b \end{bmatrix}$.

Chaining these three transformations together gives the overall transformation

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x-a \\ y-b \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix}$$

Example 23

a Determine the transformation that rotates the plane by 90° anticlockwise about the point (1, 1).

b Check your answer by showing that (0, 1) is mapped to the correct point.

Solution

a We do this in a sequence of three steps, starting with the initial point (x, y):

Initial point	Translate	Rotate 90° anticlockwise	Translate back
$\begin{bmatrix} x \\ y \end{bmatrix}$	$\begin{bmatrix} x-1 \\ y-1 \end{bmatrix}$	$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x-1 \\ y-1 \end{bmatrix}$	$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x-1 \\ y-1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

This gives the overall transformation

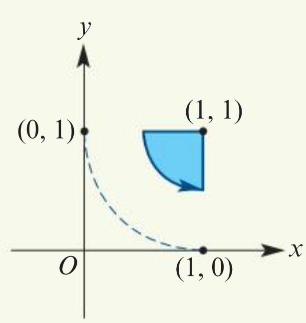
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x-1 \\ y-1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -y+2 \\ x \end{bmatrix}$$

b We check our answer by determining the image of (0, 1).

Let $x = 0$ and $y = 1$. Then

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1+2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Therefore $(0, 1) \to (1, 0)$, as expected.



Reflection in the line $y = x\tan\theta + c$

To reflect the plane in a line $y = x\tan\theta + c$ that does not go through the origin, we can also do this in a sequence of three steps:

Step 1 Translate the plane so that the line passes through the origin, by adding $\begin{bmatrix} 0 \\ -c \end{bmatrix}$.

Step 2 Reflect the plane in the line $y = x\tan\theta$, by multiplying by $\begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$.

Step 3 Translate the plane back to its original position, by adding $\begin{bmatrix} 0 \\ c \end{bmatrix}$.

Example 24

a Determine the transformation that reflects the plane in the line $y = -x + 1$.

b Check your answer by determining the image of the point $(1, 1)$.

Solution

a We do this in a sequence of three steps, starting with the initial point (x, y). The first step translates the line $y = -x + 1$ so that it passes through the origin.

Initial point	Translate	Reflect in line $y = -x$	Translate back
$\begin{bmatrix} x \\ y \end{bmatrix}$	$\begin{bmatrix} x \\ y-1 \end{bmatrix}$	$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y-1 \end{bmatrix}$	$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y-1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

This gives the overall transformation

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y-1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

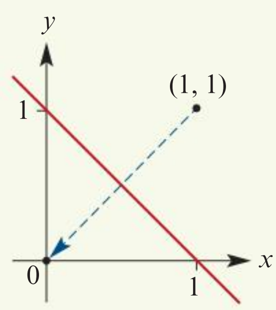
$$= \begin{bmatrix} -y+1 \\ -x+1 \end{bmatrix}$$

b We check our answer by determining the image of $(1, 1)$.

Let $x = 1$ and $y = 1$. Then

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1+1 \\ -1+1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Therefore $(1, 1) \rightarrow (0, 0)$, as expected.



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Summary 16H

More difficult transformations can be achieved by combining simpler transformations.

- To rotate the plane about the point (a, b):
  1. Translate the plane so that the origin is the centre of rotation.
  2. Rotate the plane about the origin.
  3. Translate the plane back to its original position.
- To reflect the plane in the line $y = mx + c$:
  1. Translate the plane so that the line passes through the origin.
  2. Reflect the plane in the line $y = mx$.
  3. Translate the plane back to its original position.

Exercise 16H

SF

Example 23

1 Determine the transformation that rotates the plane by 90° clockwise about the point $(2, 2)$. Check your answer by showing that the point $(2, 1)$ is mapped to the correct point.

2 Determine the transformation that rotates the plane by 180° anticlockwise about the point $(-1, 1)$. Check your answer by showing that the point $(-1, 0)$ is mapped to the correct point.

Example 24

3 Determine the transformation that reflects the plane in each of the following lines. Check your answer by showing that the point $(0, 0)$ is mapped to the correct point.

a $y = x - 1$ **b** $y = -x - 1$

c $y = 1$ **d** $x = -2$

4 **a** Write down the matrix **A** for a rotation about the origin by angle θ clockwise.

b Write down the matrix **B** for a dilation of factor k from the x-axis.

c Write down the matrix **C** for a rotation about the origin by angle θ anticlockwise.

d Hence determine the matrix that increases the perpendicular distance from the line $y = x \tan\theta$ by a factor of k.

5 Determine the transformation matrix that projects the point (x, y) onto the line $y = x \tan\theta$.

Hint: First rotate the plane clockwise by angle θ.

6 Consider these two transformations:

- T_1: A reflection in the line $y = x + 1$.
- T_2: A reflection in the line $y = x$.

Show that T_1 followed by T_2 is a translation.

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Review

Chapter summary

- A **linear transformation** is defined by a rule of the form $(x, y) \to (ax + by, cx + dy)$.
- Linear transformations can be represented using matrix multiplication:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

  The point (x', y') is called the **image** of the point (x, y).
- The matrix of a composition of two linear transformations can be found by multiplying the two transformation matrices in the correct order.
- If **A** is the matrix of a linear transformation, then $\mathbf{A}^{-1}$ is the matrix of the inverse transformation.
- If a region of the plane is transformed by matrix **B**, then

$$\text{Area of image} = |\det(\mathbf{B})| \times \text{Area of region}$$

- Difficult transformations can be achieved by combining simpler transformations.

Transformation	Matrix	Transformation	Matrix
Reflection in the x-axis	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$	Reflection in the y-axis	$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$
Reflection in the line $y = x$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	Reflection in the line $y = -x$	$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$
Dilation from the y-axis	$\begin{bmatrix} c & 0 \\ 0 & 1 \end{bmatrix}$	Dilation from the x-axis	$\begin{bmatrix} 1 & 0 \\ 0 & c \end{bmatrix}$
Shear parallel to the x-axis	$\begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}$	Shear parallel to the y-axis	$\begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix}$
Projection onto the x-axis	$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$	Projection onto the y-axis	$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$
Rotation by θ anticlockwise	$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$	Reflection in the line $y = x\tan\theta$	$\begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$

Skills checklist

Download this checklist from the Interactive Textbook, then print it and fill it out to check your skills.

16A **1** I can determine the images of points under a linear transformation. ☐

See Example 1, Example 2, Example 3 and Questions 1, 2 and 4

16B **2** I can determine the matrix for a reflection, dilation, shear and projection. ☐

See Example 5 and Question 1

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16B **3** I can represent translations using vector addition. ☐

See Example 6 and Question 2

16C **4** I can describe reflection and rotation matrices. ☐

See Example 7, Example 8 and Questions 1 and 3

16D **5** I can determine the matrix for a composition of transformations. ☐

See Example 9, Example 10 and Questions 1 and 2

16E **6** I can determine the matrix for the inverse of a linear transformation. ☐

See Example 11, Example 13 and Questions 1 and 2

16F **7** I can determine the image of a graph under a linear transformation. ☐

See Example 14, Example 15 and Questions 1 and 2

16G **8** I can determine the area of an image by calculating a determinant. ☐

See Example 20 and Questions 1 and 2

16H **9** I can build complex transformations by combining simpler transformations. ☐

See Example 23, Example 24 and Questions 1 and 2

Short-response questions

Technology-free short-response questions

SF

1 The rule for a transformation is $(x, y) \to (2x + y, -x + 2y)$.

a Determine the image of the point $(2, 3)$.

b Determine the matrix of this transformation.

c Sketch the image of the unit square and determine its area.

d Determine the rule for the inverse transformation.

2 Determine the matrix corresponding to each of the following linear transformations:

a reflection in the y-axis

b dilation of factor 5 from the x-axis

c shear of factor -3 parallel to the x-axis

d projection onto the x-axis

e rotation by 30° anticlockwise

f reflection in the line $y = x$

3 **a** Determine the matrix that will reflect the plane in the line $y = 3x$.

b Determine the image of the point $(2, 4)$ under this transformation.

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4 Determine the transformation matrix that corresponds to: SF

a a reflection in the x-axis and then a reflection in the line $y = -x$

b a rotation about the origin by 90° anticlockwise and then a dilation of factor 2 from the x-axis

c a reflection in the line $y = x$ and then a shear of factor 2 parallel to the y-axis.

5 **a** Determine the rule for the transformation that will reflect (x, y) in the x-axis and then translate the result by the vector $\begin{bmatrix} -3 \\ 4 \end{bmatrix}$.

b Determine the rule for the transformation if the translation takes place before the reflection.

6 **a** Write down the matrix for a shear of factor k parallel to the y-axis.

b Show that the inverse matrix corresponds to a shear of factor $-k$ parallel to the y-axis.

7 Each of the following matrices maps the unit square to a parallelogram. Sketch each parallelogram and determine its area.

a $\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}$ **b** $\begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}$

8 **a** determine the rule for the transformation that rotates the plane about the point $(1, -1)$ by 90° anticlockwise. (**Hint:** Translate the point $(1, -1)$ to the origin, rotate the plane, and then translate the point back to its original position.) CF

b Determine the image of the point $(2, -1)$ under this transformation.

c Sketch the unit square and its image under this transformation.

Technology-active short-response questions

9 **a** determine the matrix that will rotate the plane by 45° anticlockwise. CF

b Determine the matrix that will rotate the plane by 30° anticlockwise.

c Hence determine the matrix that will rotate the plane by 75° anticlockwise.

d Hence deduce exact values for $\cos 75°$ and $\sin 75°$.

10 The triangle with vertices $(0, 0)$, $(2, 0)$ and $(0, 2)$ is transformed by the matrix $\begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$.

a Sketch the triangle and its image on the same set of axes.

b Determine the area of the triangle and its image.

c The image of the triangle is revolved around the y-axis to create a three-dimensional solid. Determine the volume of this solid.

11 Consider the transformation with rule $(x, y) \to (x + y, y)$.

a Write down the matrix of this transformation.

b What name is given to this type of transformation?

c Find the images of the points $(-1, 1)$, $(0, 0)$ and $(1, 1)$ under this transformation.

d Hence sketch the graph of $y = x^2$ and its image under this transformation.

12 A square with vertices $(\pm 1, \pm 1)$ is rotated about the origin by 45° anticlockwise.

a Determine the coordinates of the vertices of its image.

b Sketch the square and its image on the same set of axes.

c When these two squares are combined, the resulting figure is called a Star of Lakshmi. Determine its area.

13 In this chapter we investigated two important transformation matrices. These were the rotation and reflection matrices, which we will now denote by

$$\text{Rot}(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad \text{and} \quad \text{Ref}(\theta) = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$$

a Using matrix multiplication and an application of trigonometric identities, prove the following four matrix equations:

i $\text{Rot}(\theta)\,\text{Rot}(\varphi) = \text{Rot}(\theta + \varphi)$

ii $\text{Ref}(\theta)\,\text{Ref}(\varphi) = \text{Rot}(2\theta - 2\varphi)$

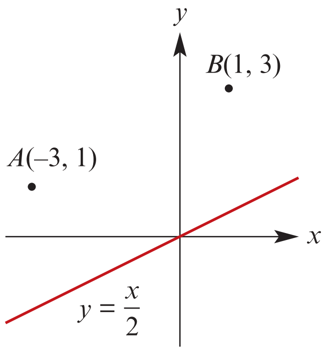
iii $\text{Rot}(\theta)\,\text{Ref}(\varphi) = \text{Ref}(\varphi + \frac{1}{2}\theta)$

iv $\text{Ref}(\theta)\,\text{Rot}(\varphi) = \text{Ref}(\theta - \frac{1}{2}\varphi)$

b Explain in words what each of the above four equations shows.

c Using these identities, determine the matrix $\text{Rot}(60°)\,\text{Ref}(60°)\,\text{Ref}(60°)\,\text{Rot}(60°)$.

14 An ant is at point $A(-3, 1)$. His friend is at point $B(1, 3)$. The ant wants to walk from A to B, but first wants to visit the straight line $y = \frac{1}{2}x$. Being an economical ant, he wants the total length of his path to be as short as possible.



a Determine the matrix that will reflect the plane in the line $y = \frac{1}{2}x$.

b Determine the image A' of the point A when reflected in the line $y = \frac{1}{2}x$.

c Determine the distance from point A' to point B.

d The straight line $A'B$ intersects the line $y = \frac{1}{2}x$ at the point C. What type of triangle is ACA'?

e Suppose that D is any other point on the line $y = \frac{1}{2}x$. Show that

$$AD + DB \geq AC + CB$$

f Hence determine the shortest possible distance travelled by the ant.

15 A rectangle R_1 has vertices $(0, 0)$, $(a, 0)$, $(0, b)$ and (a, b), where a and b are positive real numbers.

a Sketch the rectangle R_1.

b The rectangle R_1 is rotated about the origin by angle θ anticlockwise, where $0° \leq \theta \leq 90°$. The image is another rectangle R_2. determine the coordinates of the vertices of rectangle R_2 in terms of a, b and θ.

Review

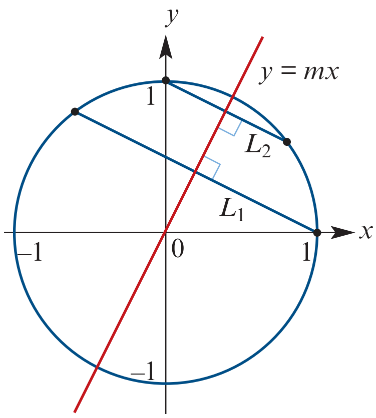
CU

c The vertices of R_2 lie on another rectangle R_3 that has edges parallel to the coordinate axes. Show that the area of rectangle R_3 is

$$A = \frac{1}{2}(a^2 + b^2)\sin(2\theta) + ab$$

d Hence show that the maximum area of rectangle R_3 is $\frac{1}{2}(a+b)^2$, which occurs when $\theta = 45°$.

16 The graphs of the unit circle $x^2 + y^2 = 1$ and the line $y = mx$ are shown. Lines L_1 and L_2 are perpendicular to the line $y = mx$ and go through the points $(1, 0)$ and $(0, 1)$ respectively.



a Determine the equation of the line L_1, and determine where it intersects the unit circle in terms of m.

b Determine the equation of the line L_2, and determine where it intersects the unit circle in terms of m.

c Hence deduce the formula for the matrix that reflects the point (x, y) in the line $y = mx$.

Hint: Recall that the columns of the matrix will be the images of the standard unit vectors.

Multiple-choice questions

Technology-free multiple-choice questions

1 The image of the point $(2, -1)$ under the transformation $(x, y) \to (2x - 3y, -x + 4y)$ is

A $(1, -6)$ **B** $(7, -6)$ **C** $(7, 6)$ **D** $(7, 2)$

2 The matrix that will reflect the plane in the line $y = -x$ is

A $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ **B** $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ **C** $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ **D** $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

3 The matrix that corresponds to a dilation of factor 2 from the y-axis followed by a reflection in the x-axis is

A $\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$ **B** $\begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}$ **C** $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ **D** $\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$

4 The matrix $\begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{-1}{2} \end{bmatrix}$ will

A rotate the plane by 30° **B** rotate the plane by −30°

C rotate the plane by −60° **D** reflect the plane in the line $y = x\tan 30°$

5 The transformation that translates (x, y) by the vector $\begin{bmatrix} 2 \\ -3 \end{bmatrix}$ and then rotates the result about the origin by 90° anticlockwise is given by

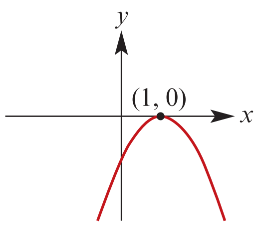
A $(x, y) \to (y - 3, x + 2)$ **B** $(x, y) \to (-y + 3, -x - 2)$
C $(x, y) \to (-y + 3, x + 2)$ **D** $(x, y) \to (-x - 2, -y + 3)$

6 The matrix $\begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$ corresponds to

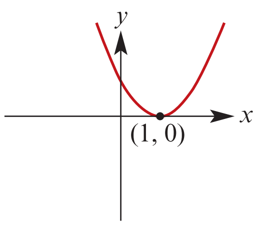
A a rotation by 180° and then a dilation of factor 2 from the x-axis
B a rotation by 90° clockwise and then a dilation of factor 2 from the x-axis
C a rotation by 180° and then a dilation of factor 2 from the y-axis
D a reflection in the y-axis and then a dilation of factor 2 from the y-axis

7 Which of the following shows the image of the graph of $y = (x - 1)^2$ under the linear transformation with matrix $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$?

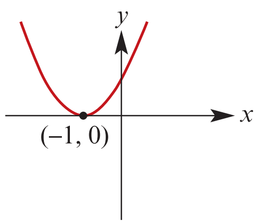
A



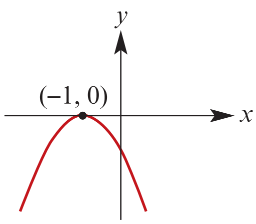
B



C



D



8 Which of these matrices maps the unit square to a parallelogram of area 2 square units?

A $\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$ **B** $\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$ **C** $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ **D** $\begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}$

9 The matrix **R** will rotate the plane through angle 40°. The smallest value of m such that $\mathbf{R}^m = \mathbf{I}$, where **I** is the identity matrix, is

A 6 **B** 7 **C** 8 **D** 9

Technology-active multiple-choice questions

10 The point mapped to $(2, 4)$ by the linear transformation $(x, y) \to (4x - 3y, 4x + 2y)$ is

A $(-4, 16)$ **B** $(4, 16)$ **C** $(4, -16)$ **D** $\left(\frac{4}{5}, \frac{2}{5}\right)$

11 The inverse of the linear transformation $(x, y) \to (5x + 8y, 8x + 13y)$ is

A $(x, y) \to (13x + 8y, -8x - 5y)$ **B** $(x, y) \to (13x - 8y, -8x + 5y)$
C $(x, y) \to (-13x + 8y, -8x + 5y)$ **D** $(x, y) \to (13x + 8y, 8x - 5y)$

Review

12 The graph of $2x + 3y = 4$ is first reflected in the line $y = x$ and then translated by the vector $\begin{bmatrix} 5 \\ 6 \end{bmatrix}$. The equation of its image is

A $2x + 3y = 21$ **B** $2y + 3x = 21$ **C** $2y - 3x = 21$ **D** $3x + 2y = 31$

13 A linear transformation consists of a rotation by 90° anticlockwise and then a dilation by a factor of 2 from the x-axis. The matrix corresponding to this transformation is

A $\begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix}$ **B** $\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$ **C** $\begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix}$ **D** $\begin{bmatrix} 0 & -1 \\ 2 & 0 \end{bmatrix}$

14 A linear transformation maps the point $(1, 2)$ to $(2, 1)$ and the point $(2, 1)$ to $(1, 2)$. The same transformation maps $(1, 1)$ to the point

A $(1, 1)$ **B** $(2, 2)$ **C** $(-1, -1)$ **D** $(-1, 1)$

15 Which of these matrices will map a square of area 1 to a parallelogram of area 1?

A $\begin{bmatrix} 3 & 4 \\ 5 & 4 \end{bmatrix}$ **B** $\begin{bmatrix} 3 & 4 \\ 6 & 5 \end{bmatrix}$ **C** $\begin{bmatrix} 3 & 4 \\ 4 & 3 \end{bmatrix}$ **D** $\begin{bmatrix} 3 & 4 \\ 4 & 5 \end{bmatrix}$

16 Let $c \in \mathbb{R}$. A triangle has vertices at $(0, 0)$, $(2, 0)$ and $(2, 2)$. Its image under the linear transformation $(x, y) \to (cx - 2y, 3x + cy)$ is a triangle of area 18. The value of c is

A ± 1 **B** $\pm\sqrt{2}$ **C** $\pm\sqrt{3}$ **D** ± 2

17 Consider the transformation

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos 20^\circ & -\sin 20^\circ \\ \sin 20^\circ & \cos 20^\circ \end{bmatrix} \left(\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right) + \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

This transformation can be described as

A an anticlockwise rotation by 20° about the point $(-1, 2)$

B an anticlockwise rotation by 20° about the point $(1, -2)$

C an anticlockwise rotation by 20° about the point $(-2, 1)$

D a clockwise rotation by 20° about the point $(1, -2)$

18 The linear transformation $(x, y) \to (-x + 2y, y)$ can be described as

A a shear parallel to the y-axis, then a reflection in the y-axis

B a shear parallel to the y-axis, then a reflection in the y-axis

C a shear parallel to the y-axis, then a reflection in the x-axis

D a reflection in the y-axis, then a shear parallel to the x-axis

17

Revision of Unit 2

Revision

17A Short-response questions

Technology-free short-response questions

SF

1 Let $z = 1 + 2i$ and $w = 3 - 5i$. Express each of the following complex numbers in Cartesian form:

a $z + w$ **b** $z - w$ **c** $2z - w$ **d** zw

e $\overline{zw}$ **f** $\dfrac{z}{w}$ **g** $\dfrac{1}{\overline{z}}$

2 Solve the equation $z^2 - 2z + 10 = 0$ over $\mathbb{C}$.

3 Let $f(x) = x^2 - 4$. Determine the range of the function given by $g(x) = \dfrac{1}{f(x)}$ for $-2 < x < 2$.

4 Write each of the following complex numbers in polar form:

a $3\sqrt{2} + 3\sqrt{2}i$ **b** $3 - 3\sqrt{3}i$ **c** $5\sqrt{3} - 5i$

5 Solve the equation $2\cos(3x) = \sqrt{3}$ for $0 \le x \le \pi$.

6 Solve the equation $\dfrac{3z - (1 + i)}{z - 4} = 2$ for z.

7 Solve each of the following equations for $0 \le \theta \le 2\pi$:

a $1 + \cos\theta = \cos\left(\dfrac{\theta}{2}\right)$ **b** $\sin^2\theta + 2\cos^2\theta = \sin\theta$

c $(5 + \cos\theta)\cot\theta + \sin\theta = 0$

8 Let $f(x) = \sin x + \dfrac{1}{2}\cos x$. Rewrite this rule in the form $f(x) = r\sin(x + \alpha)$. Hence, determine the maximum and minimum values of $f(x)$ for $0 \le x \le \dfrac{\pi}{2}$.

Revision

SF

9 Sketch the graph of each of the following functions:

a $y = 2|3 - 4x|$ **b** $y = |x^2 - 9x|$ **c** $y = |\cos x|$

10 **a** Write $1 + \sqrt{3}i$ and $1 - i$ in polar form.

b Hence, express $\dfrac{(1 + \sqrt{3}i)^4}{(1 - i)^6}$ in Cartesian form.

11 **a** Represent $z_1 = -\sqrt{3} + i$ and $z_2 = \sqrt{2} + \sqrt{2}i$ on an Argand diagram.

b Determine $z_1 z_2$ and $\dfrac{z_1}{z_2}$ in polar form, and represent them on the same diagram.

12 Determine the matrix corresponding to each of the following linear transformations:

a dilation of factor 4 from the x-axis **b** dilation of factor $\frac{1}{2}$ from the y-axis

c rotation by 60° clockwise **d** projection onto the x-axis

e shear of factor $\frac{1}{2}$ parallel to the y-axis **f** reflection in the line $y = \dfrac{1}{\sqrt{3}}x$

13 **a** Determine the single transformation matrix that represents a reflection in the y-axis followed by a dilation of factor 2 from the x-axis.

b Determine the single transformation matrix that represents a rotation by 90° anti-clockwise followed by a reflection in the line $y = x$.

14 The unit square is mapped to a parallelogram by the linear transformation with matrix $\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$. Determine the area of the parallelogram.

15 Given that $\sin A = \frac{3}{5}$, where A is acute, and that $\cos B = -\frac{1}{2}$, where B is obtuse, determine the exact values of:

a $\sec A$ **b** $\cot A$ **c** $\cot B$ **d** $\operatorname{cosec} B$

16 Given that $\cos A = \frac{1}{3}$, determine the possible values of $\cos\left(\frac{A}{2}\right)$.

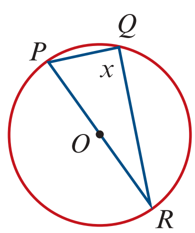
17 If $w = 3 + 2i$ and $z = 3 - 2i$, express each of the following in the form $a + bi$, where a and b are real numbers:

a $w + z$ **b** $w - z$ **c** wz **d** $w^2 + z^2$

e $(w + z)^2$ **f** $(w - z)^2$ **g** $w^2 - z^2$ **h** $(w - z)(w + z)$

18 Determine the values of x and y in each of the following diagrams, giving reasons.

a



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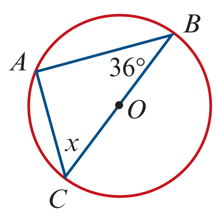

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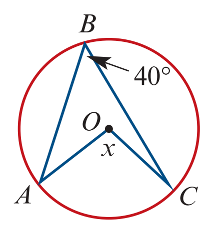
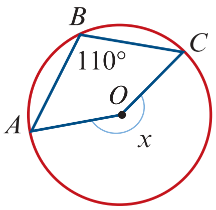
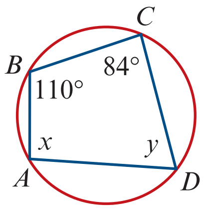
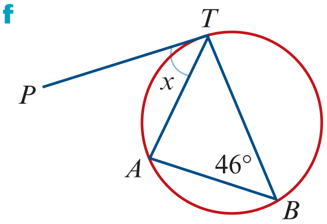
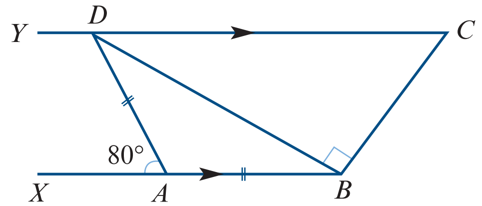


d 

e 

f 

19 In the diagram, XB and YC are parallel. Given that $AD = AB$, $\angle XAD = 80°$ and $\angle DBC = 90°$, determine:



a $\angle ABD$ **b** $\angle BDY$ **c** $\angle BCD$

20 If $w = 1 - 2i$ and $z = 2 - 3i$, express each of the following in the form $a + bi$, where a and b are real numbers:

a $w + z$ **b** $w - z$ **c** wz **d** $\dfrac{w}{z}$ **e** iw **f** $\dfrac{i}{w}$

g $\dfrac{w}{i}$ **h** $\dfrac{z}{w}$ **i** $\dfrac{w}{w+z}$ **j** $(1+i)w$ **k** $\dfrac{w}{1+i}$ **l** w^2

21 Write each polynomial as a product of linear factors:

a $z^2 + 49$ **b** $z^2 - 2z + 10$ **c** $9z^2 - 6z + 5$ **d** $4z^2 + 12z + 13$

22 For each of the following functions, sketch the graphs of $y = f(x)$ and $y = \dfrac{1}{f(x)}$ on the same set of axes:

a $f(x) = x^2 + 3x + 2$ **b** $f(x) = (x-1)^2 + 1$

c $f(x) = \sin(x) + 1,\ x \in [0, 2\pi]$ **d** $f(x) = \cos(x) + 2,\ x \in [0, 2\pi]$

23 Solve each of the following equations for x:

a $|3x - 7| = 11$ **b** $|6x + 4| = 36$ **c** $|x^2 - 4x| = 21$

d $|x - 2| + |x - 3| = 5$ **e** $|x^2 - 6x| = 16$ **f** $|2x^2 - 2x - 8| = 2$

24 **a** Determine the rule for the transformation that will reflect (x, y) in the y-axis then translate the result by the vector $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

b Determine the rule for the transformation if the translation takes place before the reflection.

25 Each of the following matrices maps the unit square to a parallelogram. Sketch each parallelogram and determine its area.

a $\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$ **b** $\begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}$

26 For each of the following, sketch the graphs of $y = f(x)$ and $y = \dfrac{1}{f(x)}$ on the same axes:

a $f(x) = \cos(x) + 1,\ x \in [0, 2\pi]$ **b** $f(x) = \sin(x) + 2,\ x \in [0, 2\pi]$

Revision

SF

27 Sketch the graph of each of the following over the interval $[0, 2\pi]$:

a $y = \cot(4x)$ **b** $y = 2\operatorname{cosec}(2x)$ **c** $y = -3\sec(2x)$

d $y = \operatorname{cosec}\left(x - \frac{\pi}{3}\right)$ **e** $y = \sec(x) - 3$ **f** $y = \cot\left(x + \frac{\pi}{3}\right)$

28 Describe a sequence of transformations that will take the graph of $y = \operatorname{cosec} x$ to the graph of $y = -3\operatorname{cosec}(2x) + 1$. Sketch the transformed graph over the interval $[-\pi, \pi]$.

CF

29 **a** Determine the rule for the transformation that will reflect the plane in the line $y = x - 1$.

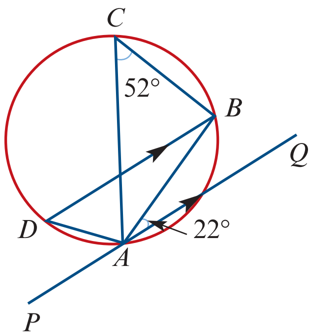
Hint: Translate the plane 1 unit in the y-direction, reflect in the line $y = x$, and then translate the plane back to its original position.

b Determine the image of the point $(0, 0)$ under this transformation.

c Sketch the unit square and its image under this transformation.

30 In the diagram, AC is a diameter of the circle, $PQ \parallel DB$, $\angle BCA = 52°$ and $\angle BAQ = 22°$. Determine:

a $\angle CAB$ **b** $\angle PAD$ **c** $\angle CBD$



31 **a** Determine the two square roots of $3 + 4i$.

b Use the quadratic formula to solve $(2 - i)z^2 + (4 + 3i)z + (-1 + 3i) = 0$ for z.

32 Sketch the graph of each of the following functions:

a $y = |x|^2 - 2|x| - 4$ **b** $y = |x^2 - 2x - 4|$ **c** $y = |x - 1| + |x - 2|$

33 Prove that $\dfrac{(\sin x - 2\cos x)^2}{(1 - \sin x)(1 + \sin x)} = 3 - 4\tan x + \sec^2 x$.

34 Solve the following equation for $0 \le x \le 2\pi$:

$$\sin x + \cos x + \sec x + \operatorname{cosec} x = 0$$

35 Prove the identity $\dfrac{1}{1 + \sin A} + \dfrac{1}{1 - \sin A} = 2\sec^2 A$.

36 Prove each of the following:

a $\dfrac{\sin(3x) + \sin(x)}{\cos(3x) + \cos(x)} = \tan(2x)$ **b** $\dfrac{\sin(x) + \sin(2x)}{1 + \cos(x) + \cos(2x)} = \tan(x)$

37 Prove each of the following identities:

a $\dfrac{\tan(2\theta) + \sec(2\theta) - 1}{\tan(2\theta) - \sec(2\theta) + 1} = \dfrac{1 + \tan\theta}{1 - \tan\theta}$ **b** $\dfrac{\cos(4\theta) + \cos(2\theta)}{\sin(4\theta) - \sin(2\theta)} + \tan\theta = 2\operatorname{cosec}(2\theta)$

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Revision

SF

Technology-active short-response questions

38 A transformation has rule $(x, y) \to (2x + y, -x - 2y)$.

a Determine the image of the point $(2, 3)$.

b Determine the matrix of this transformation.

c Sketch the image of the unit square and determine its area.

d Determine the rule for the inverse transformation.

39 Determine the matrix corresponding to each of the following linear transformations:

a reflection in the x-axis

b dilation of factor 3 from the y-axis

c shear of factor 2 parallel to the y-axis

d projection onto the y-axis

e rotation by 45° anticlockwise

f rotation by 30° clockwise

g reflection in the line $y = -x$

h reflection in the line $y = x \tan 30°$

40 **a** Write the product $\cos(3x)\sin(x)$ as a sum or a difference.

b Determine the general solution of the equation $\cos(3\theta) + \cos(\theta) = 0$.

41 State the range of the function $f(x) = 3\operatorname{cosec}(4x) + 3$ defined for the implied domain.

42 Sketch the graph of $y = 2\sec(2x) + 1$ for $0 \le x \le 2\pi$.

43 Solve the equation $2\sec\left(x - \frac{\pi}{4}\right) = 4$ for $0 \le x \le 2\pi$.

CF

44 **a** Determine the matrix that will reflect the plane in the line $y = 4x$.

b Determine the image of the point $(2, 4)$ under this transformation.

45 Solve the equation $3z^2 - (1 + i)z - i = 0$ for z.

46 **a** Solve each of the following equations for x:

i $|2x - 1| + |4 - x| = 5$ **ii** $|2x - 1| - |4 - x| = 7$ **iii** $|2x - 1| - |4 - x| = -\frac{7}{2}$

b State the range of the function given by $f(x) = |2x - 1| - |4 - x|$ for $x \in \mathbb{R}$.

47 **a** Write $4x^2 - 24x + 32$ in the form $a(x - h)^2 + k$.

b Sketch the graph of $y = \dfrac{1}{4x^2 - 24x + 32}$.

48 Suppose that $z = 1 + \sqrt{5}i$ is a solution of the equation $z^2 - 2z + a = 0$, where a is a real number. determine the value of a.

49 Solve the equation $z^2 = 21 - 20i$.

50 Determine the values of x that satisfy the inequality $|x + 3a| > |3x - 2a|$, where a is a positive constant.

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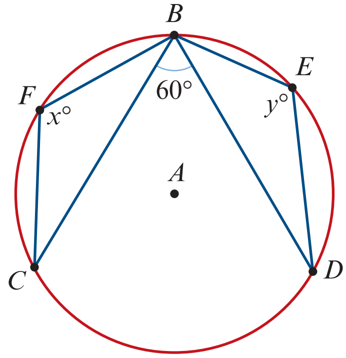
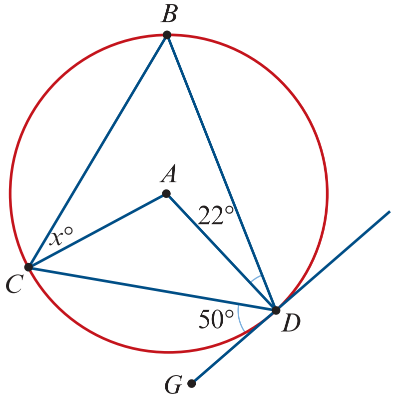
Revision

GF

51 **a** determine the value of $x + y$. Give reasons.



b In the diagram, the circle has centre A and line GD is tangent to the circle at D. Determine the value of x.



52 **a** Describe in words a sequence of transformations of the plane that will take the graph of $y = x^2$ to the graph of $y = p(x - h)^2 + k$, where $p > 0$.

b Determine a rule for a single transformation T that will achieve the same result. Write the rule for T in the form

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} e \\ f \end{bmatrix}$$

c Now determine a rule for a transformation S that will take the graph of $y = q(x - m)^2 + n$ to the graph of $y = x^2$, where $q > 0$. Write the rule for S in the form

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} e \\ f \end{bmatrix} \right)$$

d Finally, determine a rule for a single transformation of the plane that will take the graph of $y = q(x - m)^2 + n$ to the graph of $y = p(x - h)^2 + k$, where $p, q > 0$.

53 In this question, we consider shears of the plane parallel to a line through the origin.

a Write down the matrix that represents each of the following transformations:

i T_1: A rotation by angle θ clockwise.

ii T_2: A shear of factor k parallel to the x-axis.

iii T_3: A rotation by angle θ anticlockwise.

b Hence, determine the single transformation matrix that represents T_1 followed by T_2 followed by T_3.

c Give a brief geometric description of the transformation represented by this matrix.

54 Consider the function $f(x) = 2\sin(x) + k$ for $x \in [0, 2\pi]$, where k is a real number.

a Sketch the graphs of $y = f(x)$ and $y = \dfrac{1}{f(x)}$ when:

i $k = 1$ **ii** $k = 3$

b For what positive value of k does the graph of $y = \dfrac{1}{f(x)}$ have only one vertical asymptote?

c For this value of k, sketch the graphs of $y = f(x)$ and $y = \dfrac{1}{f(x)}$.

55 Solve the equation $3\tan^3 x - 3\tan^2 x - \tan x + 1 = 0$ for $0 < x < \pi$.

56 Solve the equation $z^2 = \overline{z}$ for z.

57 Suppose that $z = 5 - i$ is a solution of the equation $z^2 - (a + bi)z + 4 + 20i = 0$, where a and b are real numbers.

a Determine the values of a and b.

b Determine the other solution of this equation given that it is of the form $z = ki$, where k is a real number.

58 Show that if $\cos(2A) = \tan^2 B$, then $\cos(2B) = \tan^2 A$.

59 For real numbers α and β, let $k = \dfrac{\cos(\frac{1}{2}(\alpha + \beta))}{\cos(\frac{1}{2}(\alpha - \beta))}$. Prove that $\dfrac{1 - k^2}{1 + k^2} = \dfrac{\sin\alpha\,\sin\beta}{1 + \cos\alpha\,\cos\beta}$.

60 **a** Prove the identity

$$(\cot x + \operatorname{cosec} x)^2 = \frac{1 + \cos x}{1 - \cos x}$$

b Obtain a similar identity by putting $x = \dfrac{\pi}{2} + y$.

61 Let $z = \operatorname{cis}\theta$. Show that $\dfrac{z - 1}{z + 1} = i\tan\left(\dfrac{\theta}{2}\right)$.

Revision

CU

62 Let z_1 and z_2 be the solutions of the quadratic equation $z^2 + 4z + 16 = 0$.

a Determine z_1 and z_2.

b Write z_1 and z_2 in polar form.

c Calculate $(z_1)^3$ and $(z_2)^3$.

d Plot z_1, z_2 and $z_3 = 4 + 0i$ on an Argand diagram.

e Calculate $|z_1 - z_3|$ and $|z_2 - z_3|$. Comment on your answer.

f Investigate the quadratic equation $z^2 - 4z + 16 = 0$ in the same way.

g Investigate the family of quadratic equations $z^2 + az + a^2 = 0$, where a is an integer.

63 Consider triples of real numbers (a, b, c) such that

$$a \le b \le c, \quad |a| + |b| + |c| = 14, \quad |a + b + c| = 2 \quad \text{and} \quad |abc| = 72$$

a Explain why such triples must satisfy

$$a \le 0 \le b \le c \quad \text{or} \quad a \le b \le 0 \le c$$

b Determine all such triples.

64 In this question, we consider the geometry of complex numbers represented on an Argand diagram.

a For a complex number z, explain why $|z| = 1$ if and only if z lies on the circle of radius 1 centred at the origin.

b Similarly, explain why $|z| = 2$ if and only if z lies on the circle of radius 2 centred at the origin.

c Explain why $|z_1 - z_2|$ is the distance between the complex numbers z_1 and z_2.

d Determine the distance between z_1 and z_2 if:

i $z_1 = 2 + i$ and $z_2 = 3 - 4i$ **ii** $z_1 = 2 - i$ and $z_2 = 4 - 4i$

iii $z_1 = 2 + 3i$ and $z_2 = 4 + 5i$ **iv** $z_1 = -2 - 3i$ and $z_2 = -4 + i$

e Determine the complex number z, with $1 \le |z| \le 2$, such that:

i the distance $|z - (-1 - 3i)|$ is minimised

ii the distance $|z - (1 + 3i)|$ is minimised

iii the distance $\left|z - \left(\frac{1}{2} + \frac{1}{2}i\right)\right|$ is minimised.

65 Let $z = \frac{1}{2}(1 + \sqrt{3}i)$.

a Calculate z^2, z^3, z^4, z^5 and z^6.

b Show that $z + z^2 + z^3 + z^4 + z^5 + z^6 = 0$.

66 The motion of a particle moving in a straight line is modelled by the equation

$$x = 5 - 2\cos^2 t \quad \text{for } t \geq 0$$

where x m is the distance of the particle from a point O on the line at time t seconds.

a Determine the particle's distance from O when:

i $t = 0$ **ii** $t = \frac{\pi}{2}$ **iii** $t = \pi$ **iv** $t = 2\pi$

b Write the equation of motion in the form $x = a + b\cos(2t)$.

c Determine:

i the centre of the motion

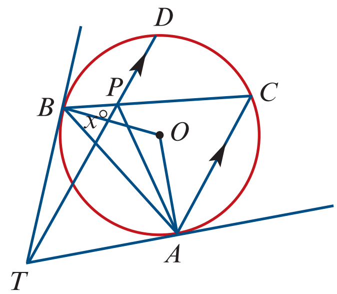
ii the period of the motion

iii the amplitude of the motion.

d Determine the first time that the particle is 5 m from O.

e Determine the first time that the particle is 3.5 m from O.

67 In the figure, O is the centre of a circle. TD and AC are parallel. TA and TB are tangents to the circle. Let $\angle BPT = x°$.

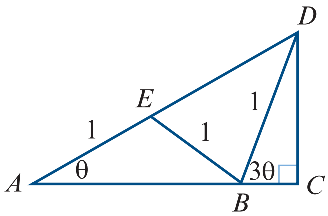


a Prove that $TBOA$ is a cyclic quadrilateral.

b Determine $\angle BCA$, $\angle BOA$, $\angle TAB$ and $\angle TBA$ in terms of x.

Triple-angle identities for sine and cosine

68 In the figure, $AE = BE = BD = 1$ unit and $\angle BCD$ is a right angle.



a Show that the magnitude of $\angle BDE$ is 2θ.

b Use the cosine rule in triangle BDE to show that $DE = 2\cos(2\theta)$.

c Show that:

i $DC = \sin(3\theta)$ **ii** $AD = \dfrac{\sin(3\theta)}{\sin\theta}$

d Use the results of **b** and **c** to show that $\sin(3\theta) = 3\sin\theta - 4\sin^3\theta$.

69 **a** Use the angle sum and double-angle identities to prove each of the following:

i $\cos(3\theta) = 4\cos^3\theta - 3\cos\theta$ **ii** $\sin(3\theta) = 3\sin\theta - 4\sin^3\theta$

Now let $\alpha = \cos\left(\frac{\pi}{9}\right)$.

b Use an identity from part **a** to show that $x = \alpha$ is a solution of the equation $f(x) = 0$, where $f(x) = x^3 - \frac{3}{4}x - \frac{1}{8}$.

c Use your calculator to plot the graph of $y = f(x)$. Hence determine that $x = \alpha$ is the unique positive real solution of the cubic equation $f(x) = 0$.

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70 For a cubic equation of the form $z^3 + pz + q = 0$, we can use Cardano's formula to determine a solution over $\mathbb{C}$:

$$z = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

Use Cardano's formula to determine a solution of the equation $f(z) = 0$ over $\mathbb{C}$ where $f(z)$ is the function defined in Question **69 b**. Express your answer in the form $z = a(\sqrt[3]{1 + bi} + \sqrt[3]{1 - bi})$, for $a, b \in \mathbb{R}^+$.

Note: It can be shown that an expression of the form obtained must be a positive real number. Thus this expression is equal to $\alpha = \cos\left(\frac{\pi}{9}\right)$.

71 **a** Use the rule for multiplying complex numbers in polar form to show that:

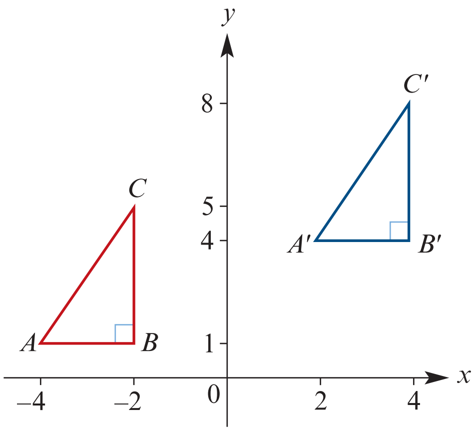
i $(\text{cis}\,\theta)^2 = \text{cis}(2\theta)$ **ii** $(\text{cis}\,\theta)^3 = \text{cis}(3\theta)$

b From part **a ii**, we have the equation

$$(\cos\theta + i\sin\theta)^3 = \cos(3\theta) + i\sin(3\theta)$$

Expand and simplify the left-hand side of this equation. Then, by equating real and imaginary parts, derive the two triple-angle identities from Question 69.

72 The coordinates of A, B and C are $(-4, 1)$, $(-2, 1)$ and $(-2, 5)$ respectively.



a Determine the rule of the transformation that maps triangle ABC to triangle $A'B'C'$.

b On graph paper, draw triangle ABC and its image under a reflection in the x-axis.

c On the same set of axes, draw the image of ABC under a dilation of factor 2 from the y-axis.

d Determine the image of the parabola $y = x^2$ under a dilation of factor 2 from the x-axis followed by a translation by the vector $\begin{bmatrix} -3 \\ 2 \end{bmatrix}$.

e Determine the rule for the transformation that maps the graph of $y = x^2$ to the graph of $y = -2(x - 3)^2 + 4$.

f Let $f(x) = x^3 - 2x$. Use a calculator to help sketch the graph of $y = 3f(x - 2) + 4$.

73 Let transformation D be a dilation of factor 4 from the y-axis.

a Determine the image of the point $(1, 1)$ under dilation D.

b **i** Describe the image of the square with vertices $A(0, 0)$, $B(0, 1)$, $C(1, 1)$, $E(1, 0)$ under the dilation D.

ii Determine the area of the square $ABCE$.

iii Determine the area of the image of $ABCE$.

iv If the dilation had been of factor k, what would be the area of the image?

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c State the rule for the dilation D.

d **i** Determine the equation of the image of the graph of $y = x^2$ under the dilation D.

ii Determine the equation of the image of the graph of $y = x^2$ under the dilation D followed by the translation by the vector $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

iii Sketch the graph of $y = x^2$ and its image defined in **d ii** on the one set of axes. State the coordinates of the vertex and the axis intercepts of the image.

e State the rule for the transformation that maps the graph of $y = 5(x + 2)^2 - 3$ to the graph of $y = x^2$.

74 A linear transformation is represented by the matrix

$$\mathbf{M} = \begin{bmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}$$

a Show that this transformation is a rotation.

b Let C be the circle that passes through the origin and has its centre at $(0, 1)$.

i Determine the equation of C.

ii Determine the equation of C', the image of C under the transformation defined by $\mathbf{M}$.

c Determine the coordinates of the points of intersection of C and C'.

75 A linear transformation is represented by the matrix

$$\mathbf{M} = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$$

a Determine the image of the point $(-2, 5)$ under this transformation.

b Determine the inverse of $\mathbf{M}$.

c Given that the point $(11, 13)$ is the image of the point (a, b), determine the values of a and b.

d Determine the coordinates of the image of the point (a, a) in terms of a.

e Given that $a \neq 0$ and $b \neq 0$ with

$$\mathbf{M}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \lambda a \\ \lambda b \end{bmatrix}$$

Determine the possible values of λ and the relationship between a and b for each of these values.

76 Let $\mathbf{R}$ be the transformation matrix for a rotation about the origin by $\frac{\pi}{4}$ anticlockwise.

a Give the 2×2 matrix $\mathbf{R}$.

b Determine the inverse of this matrix.

c If the image of (a, b) is $(1, 1)$, determine the values of a and b.

d If the image of (c, d) is $(1, 2)$, determine the values of c and d.

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e **i** If $(x, y) \to (x', y')$ under this transformation, use the result of **b** to determine x and y in terms of x' and y'.

ii Determine the image of $y = x^2$ under this transformation.

77 Consider lines $y = x$ and $y = 2x$.

a Sketch these two lines on the same set of axes.

b The acute angle between the two lines, θ radians, can be written in the form $\theta = \tan^{-1}(a) - b$. What are the values of a and b?

c Hence determine a rotation matrix that will rotate the line $y = x$ to the line $y = 2x$. You will need to use the angle difference identities for sine and cosine.

78 Let M be the transformation that reflects the plane in the line $y = x$.

a **i** Determine the image of the point $A(1, 3)$ under this transformation.

ii The image of the triangle with vertices $A(1, 3)$, $B(1, 5)$ and $C(3, 3)$ is another triangle. Determine the coordinates of the vertices of the image.

iii Sketch triangle ABC and its image on a set of axes, with both axes from −5 to 5.

b **i** Show that the equation of the image of the graph of $y = x^2 - 2$ under the transformation M is $x = y^2 - 2$.

ii Determine the coordinates of the points of intersection of $y = x^2 - 2$ and the line $y = x$.

iii Show that the x-coordinates of the points of intersection of $y = x^2 - 2$ and its image may be determined by the equation $x^4 - 4x^2 - x + 2 = 0$.

iv Two solutions of the equation $x^4 - 4x^2 - x + 2 = 0$ are

$$x = \frac{1}{2}(-1 + \sqrt{5}) \quad \text{and} \quad x = \frac{1}{2}(-1 - \sqrt{5})$$

Use this result and the result of **b ii** to determine the coordinates of the points of intersection of $y = x^2 - 2$ and its image under M.

17B Multiple-choice questions

Technology-free multiple-choice questions

1 If $\sin A = \frac{5}{13}$ and $\sin B = \frac{8}{17}$, where A and B are acute, then $\sin(A - B)$ is given by

A $-\frac{21}{221}$ **B** $\frac{220}{221}$ **C** $-\frac{107}{140}$ **D** $\frac{107}{140}$

2 If $\sin A = \frac{5}{13}$ and $\sin B = \frac{8}{17}$, where A and B acute, then $\cos(A - B)$ is given by

A $-\frac{21}{221}$ **B** $\frac{220}{221}$ **C** $-\frac{171}{140}$ **D** $\frac{171}{140}$

3 If $\cos\theta = c$ and θ is acute, then $\cot\theta$ can be expressed in terms of c as

A $\sqrt{1 - c^2}$ **B** $\frac{1}{\sqrt{1 - c^2}}$ **C** $\frac{c}{\sqrt{1 - c^2}}$ **D** $2c\sqrt{1 - c^2}$

4 If $A + B = \frac{\pi}{2}$, then the value of $\cos A \cos B - \sin A \sin B$ is

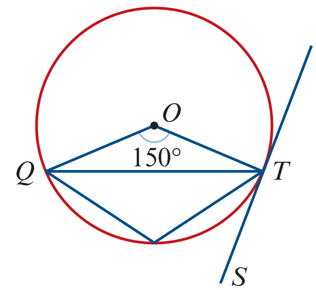
A 1 **B** -1 **C** 0 **D** 2

5 Given that $\sin A = \frac{\sqrt{5}}{3}$ and that A is obtuse, the value of $\sin(2A)$ is

A $-\frac{1}{9}$ **B** $-\frac{8\sqrt{5}}{27}$ **C** $\frac{5}{9}$ **D** $-\frac{4\sqrt{5}}{9}$

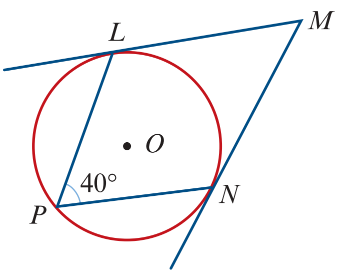
6 ST is a tangent at T to the circle with centre O.
If $\angle QOT = 150°$, then the magnitude of $\angle QTS$ is

A 70° **B** 75°
C 95° **D** 105°



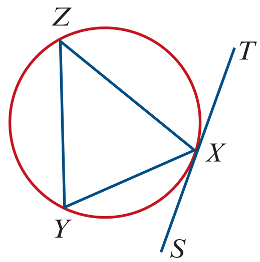
7 ML and MN are tangents to the circle at L and N.
The magnitude of angle LMN is

A 80° **B** 90°
C 100° **D** 110°



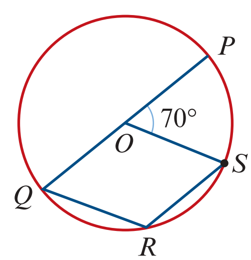
8 TS is a tangent at X and ZX bisects angle TXY. Given these facts, it can be proved that

A $YZ = XT$
B $YZ = XZ$
C $\angle YZX = \angle ZXT$
D $\angle SXY = \angle ZXY$



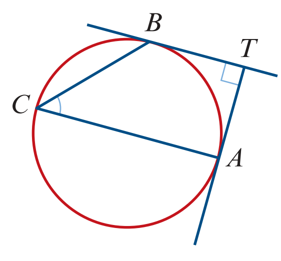
9 POQ is a diameter of the circle centre O. The size of angle QRS is

A 90° **B** 100°
C 110° **D** 125°



10 In the figure, TA and TB are tangents to the circle.
If TA is perpendicular to TB and TA is perpendicular to AC, then the magnitude of $\angle BCA$ is

A 30° **B** 40°
C 45° **D** 55°



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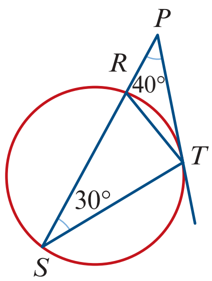
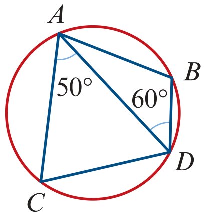
11 R, S and T are three points on the circumference of a circle, with $\angle RST$ equal to $30°$. The tangent to the circle at T meets the line SR at P, and $\angle RPT$ is equal to $40°$. The magnitude of $\angle RTS$ is



A $70°$ **B** $80°$
C $90°$ **D** $100°$

12 If $AB = AC$, $\angle ADB = 60°$ and $\angle CAD = 50°$, then $\angle ABD$ is equal to



A $80°$ **B** $90°$
C $100°$ **D** $110°$

13 If $\cos\theta = c$ and θ is acute, then $\sin(2\theta)$ can be expressed in terms of c as

A $\sqrt{1-c^2}$ **B** $\dfrac{1}{\sqrt{1-c^2}}$ **C** $\dfrac{c}{\sqrt{1-c^2}}$ **D** $2c\sqrt{1-c^2}$

14 The expression $8\sin\theta\cos^3\theta - 8\sin^3\theta\cos\theta$ is equal to

A $8\sin\theta\cos\theta$ **B** $\sin(8\theta)$
C $2\sin(4\theta)$ **D** $4\cos(2\theta)$

15 The complex number $2\operatorname{cis}\left(\dfrac{2\pi}{3}\right)$ is written in Cartesian form as

A $\sqrt{3} - i$ **B** $-\sqrt{3} + i$ **C** $1 - \sqrt{3}i$ **D** $-1 + \sqrt{3}i$

16 If $z = -\dfrac{\sqrt{3}}{2} - \dfrac{1}{2}i$, then $\operatorname{Arg} z$ is equal to

A $\dfrac{7\pi}{6}$ **B** $-\dfrac{\pi}{6}$ **C** $-\dfrac{2\pi}{3}$ **D** $-\dfrac{5\pi}{6}$

17 The imaginary part of the complex number $-2 - 3i$ is

A -3 **B** $-3i$ **C** 3 **D** -2

18 If $u = 3\operatorname{cis}\left(\dfrac{\pi}{2}\right)$ and $v = 5\operatorname{cis}\left(\dfrac{2\pi}{3}\right)$, then uv is equal to

A $15\operatorname{cis}\left(\dfrac{\pi}{3}\right)$ **B** $15\operatorname{cis}\left(\dfrac{\pi^2}{3}\right)$ **C** $15\operatorname{cis}\left(\dfrac{-5\pi}{6}\right)$ **D** $8\operatorname{cis}\left(\dfrac{\pi^2}{3}\right)$

19 The modulus of $12 - 5i$ is

A 169 **B** 7 **C** 13 **D** $\sqrt{119}$

20 Let $z = x + yi$, where x and y are real numbers which are not both zero. Which one of the following expressions does not necessarily represent a real number?

A z^2 **B** $z\bar{z}$ **C** $z^{-1}z$ **D** $\operatorname{Im}(z)$

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21 If $z = -14 - 7i$, then the complex conjugate of z is equal to

A $7 - 14i$ **B** $14 + 7i$ **C** $-14 + 7i$ **D** $14 - 7i$

22 The expression $3z^2 + 9$ is factorised over $\mathbb{C}$. Which one of the following is a factor?

A $z + 3$ **B** $z + 3i$ **C** $z - 3i$ **D** $z + \sqrt{3}i$

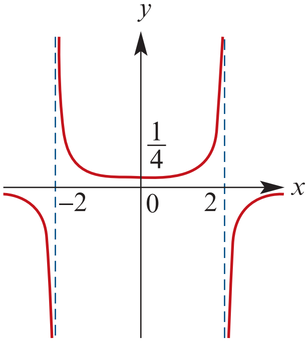
23 $(1 + 2i)^2$ is equal to

A -3 **B** $-3 + 2i$ **C** $-3 + 4i$ **D** $-1 + 4i$

24 Which of the following equations has the graph shown?

A $y = \dfrac{1}{4 - x^2}$ **B** $y = \dfrac{1}{x^2 - 4}$

C $y = \dfrac{1}{2 - x^2}$ **D** $y = \dfrac{1}{x^2 - 2}$



25 If a and b are positive real numbers, then the graph of the reciprocal of $y = a\sin(x) + b$, where $0 \le x \le 2\pi$, will have two vertical asymptotes provided

A $b > a$ **B** $a > b$ **C** $b > -a$ **D** $a > -b$

26 The graph of $f(x) = \sec(2x)$, for $-\pi \le x \le \pi$, has its local minimum points at

A $x = 0$ **B** $x = -\pi, \pi$

C $x = -\pi, 0, \pi$ **D** $x = -\dfrac{\pi}{2}, \dfrac{\pi}{2}$

27 The inequality $|3 - 2x| > 5$ holds for

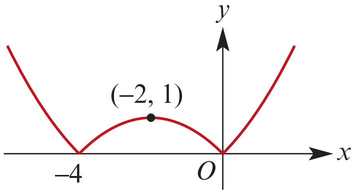
A $-1 < x < 4$ **B** $-1 < x < 1$

C $x < -4$ or $x > -1$ **D** $x < -1$ or $x > 4$

28 The rule for the graph shown could be

A $y = |x(x + 4)|$ **B** $y = |x|\,(|x| + 4)$

C $y = \left|\frac{1}{4}x^2 + x\right|$ **D** $y = \frac{1}{4}|x|^2 + |x|$



29 The point (a, b) is reflected in the line with equation $x = m$. The image has coordinates

A $(2m - a, b)$ **B** $(a, 2m - b)$ **C** $(a - m, b)$ **D** $(a, b - m)$

30 The image of the line $\{(x, y) : x + y = 4\}$ under a dilation of factor $\dfrac{1}{2}$ from the y-axis followed by a reflection in the line $x = 4$ is

A $\{(x, y) : y + 2 = 0\}$ **B** $\{(x, y) : y + 2x - 16 = 0\}$

C $\{(x, y) : x + y = 0\}$ **D** $\{(x, y) : y = 2x - 12\}$

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31 The image of $\{(x, y) : y = x^2\}$ under a translation by the vector $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ followed by a reflection in the x-axis is

A $\{(x, y) : y = (x - 3)^2 + 2\}$ **B** $\{(x, y) : -(x - 3)^2 = y + 2\}$
C $\{(x, y) : y = (x + 3)^2 + 2\}$ **D** $\{(x, y) : -y + 2 = (x - 3)^2\}$

32 Consider these two transformations:

- T_1: A reflection in the line with equation $x = 2$.
- T_2: A translation by the vector $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

The rule for T_1 followed by T_2 is given by

A $(x, y) \to (2 - x, y + 3)$ **B** $(x, y) \to (-x, y + 3)$
C $(x, y) \to (x + 2, y + 3)$ **D** $(x, y) \to (6 - x, y + 3)$

33 The image of the graph of $y = 2^x$ under a dilation of factor 2 from the x-axis followed by a dilation of factor $\frac{1}{3}$ from the y-axis has the equation

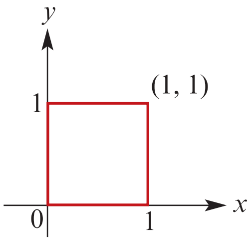
A $y = \frac{1}{3} \times 2^{3x}$ **B** $y = 3 \times 2^{\frac{x}{2}}$ **C** $y = 2 \times 2^{3x}$ **D** $y = 2 \times 2^{\frac{x}{3}}$

34 A transformation has rule $(x, y) \to (4x + 3y, 5x + 4y)$. The rule for the inverse transformation is

A $(x, y) \to (3x + 4y, 5x + 4y)$ **B** $(x, y) \to (3x - 4y, 5x - 4y)$
C $(x, y) \to (4x + 3y, 5x + 4y)$ **D** $(x, y) \to (4x - 3y, -5x + 4y)$

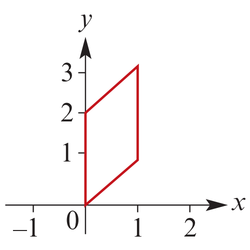
35 The unit square is subject to two successive linear transformations:

- the first transformation has matrix $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$
- the second transformation has matrix $\begin{bmatrix} 0 & -1 \\ -2 & 1 \end{bmatrix}$.

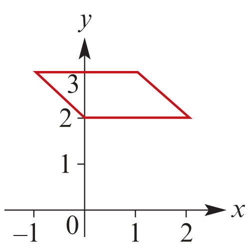


Which one of the following shows the image of the unit square under these two transformations?

A



B



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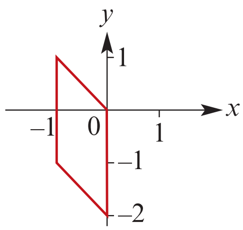
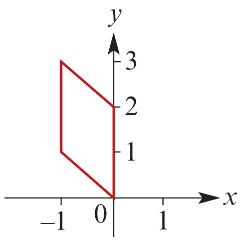
C



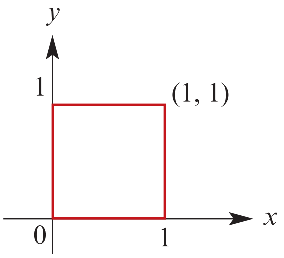
D



36 The linear transformation determined by the matrix $\begin{bmatrix} \cos 40° & -\sin 40° \\ \sin 40° & \cos 40° \end{bmatrix}$ is

A clockwise rotation by 40°
B anticlockwise rotation by 40°
C reflection in the line $y = x \tan 40°$
D reflection in the line $y = x \tan 20°$

37 The unit square is transformed by the linear transformation with matrix $\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$. The area of the image is



A 1 **B** 2 **C** 3 **D** 4

Technology-active multiple-choice questions

38 The angles between 0° and 360° which satisfy the equation $4 \cos x - 3 \sin x = 1$, given correct to two decimal places, are

A 53.13° and 126.87°
B 48.41° and 205.33°
C 41.59° and 244.67°
D 131.59° and 334.67°

39 If v, w and z are complex numbers such that $v = 4 \operatorname{cis}(-0.3\pi)$, $w = 5 \operatorname{cis}(0.6\pi)$ and $z = v\overline{w}$, then $\operatorname{Arg} z$ is equal to

A 0.9π **B** -0.9π **C** 0.3π **D** -0.3π

40 Transformation T rotates the plane about the origin by 35° clockwise.
Transformation S rotates the plane about the origin by 15° anticlockwise.
The matrix of T followed by S is

A $\begin{bmatrix} \cos 50° & -\sin 50° \\ \sin 50° & \cos 50° \end{bmatrix}$
B $\begin{bmatrix} \cos 20° & -\sin 20° \\ \sin 20° & \cos 20° \end{bmatrix}$
C $\begin{bmatrix} \cos 50° & \sin 50° \\ -\sin 50° & \cos 50° \end{bmatrix}$
D $\begin{bmatrix} \cos 20° & \sin 20° \\ -\sin 20° & \cos 20° \end{bmatrix}$

41 The point $(5, -2)$ is reflected in the line $y = x$. The coordinates of its image are

A $(5, -2)$ **B** $(-5, 2)$ **C** $(2, -5)$ **D** $(-2, 5)$

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42 The point $(2, -6)$ is reflected in the line $y = -x$. The coordinates of its image are

A $(2, -6)$ **B** $(-2, 6)$ **C** $(6, -2)$ **D** $(-6, 2)$

43 Let P be the point $(5, -4)$. After a translation by the vector $\begin{bmatrix} 2 \\ -3 \end{bmatrix}$ and then a reflection in the line $y = 1$, the coordinates of the image of P are

A $(7, 7)$ **B** $(7, 9)$ **C** $(-5, -7)$ **D** $(7, 11)$

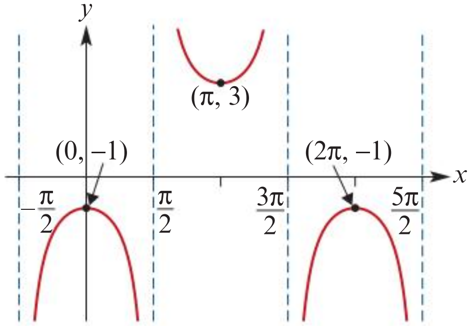
44 An equation for the graph shown is

A $y = -2\operatorname{cosec}(x) + 1$

B $y = -2\sec(x) + 1$

C $y = 2\operatorname{cosec}(x) + 1$

D $y = 2\sec(x) + 1$



17C Problem-solving and modelling questions

1 A **glide reflection** is a transformation of the plane that consists of a reflection in a mirror line ℓ followed by a translation parallel to the mirror line.

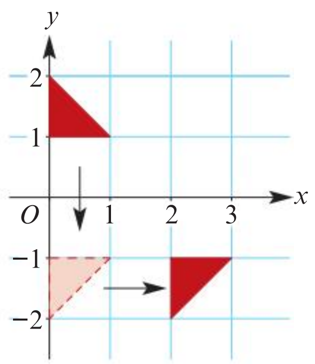
For example, the glide reflection shown on the right consists of:

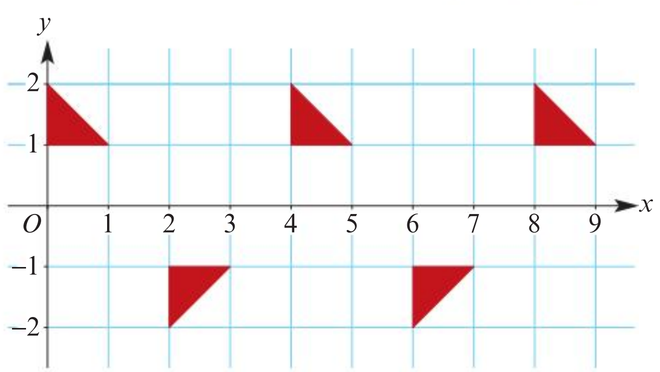
- a reflection in the x-axis
- followed by a translation of 2 units in the x-direction.

This glide reflection can be described by the rule

$$(x, y) \to (x + 2, -y)$$

If this glide reflection is applied many times, then the pattern shown below is formed.



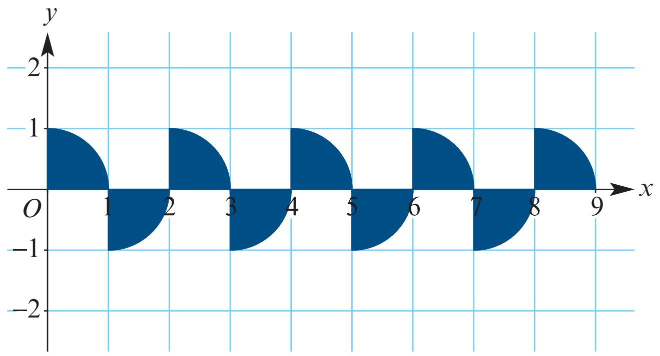


a Determine the rule for the glide reflection that consists of:

- a reflection in the y-axis
- followed by a translation of 3 units in the y-direction.

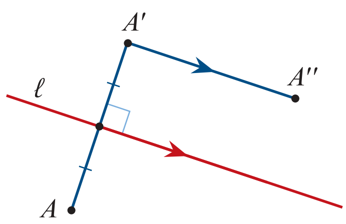
b Describe in words the glide reflection that will generate the pattern shown below, starting from the blue quarter disc at the origin.

c Determine the rule for the glide reflection described in part **b**.



d Determine the rule for the glide reflection that is obtained by reflecting in the line $y = x$ and then translating by the vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

2 A glide reflection reflects a point A in a mirror line ℓ to point A' and then translates the result parallel to the line ℓ to the point A''.



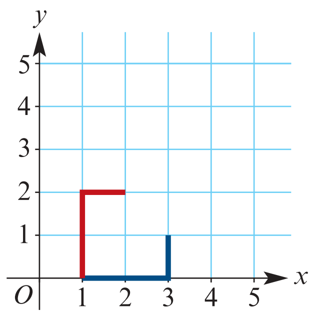
a Let M be the point where the line segment AA'' intersects ℓ. Show that M is the midpoint of AA''.

Hint: If AA' intersects ℓ at B, then $AB = A'B$.

b Suppose that a glide reflection maps the point $A(1, 2)$ to the point $A''(3, 4)$, and maps the point $C(2, 7)$ to the point $C''(6, 5)$. Determine the equation of the mirror line ℓ for this glide reflection.

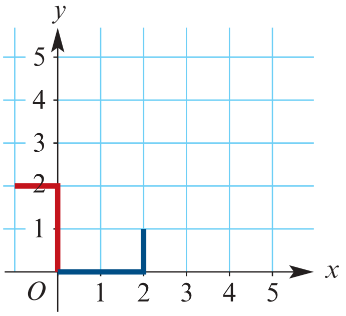
Hint: Using part **a**, we know that the midpoint of AA'' lies on the line ℓ, and likewise for CC''.

3 **a** For the diagram on the right, determine the equation of the mirror line through which the blue figure is reflected to obtain the red figure.



b In the diagram on the right, the blue figure can be transformed to the red figure by a sequence of two reflections. Determine the equations of the two mirror lines.

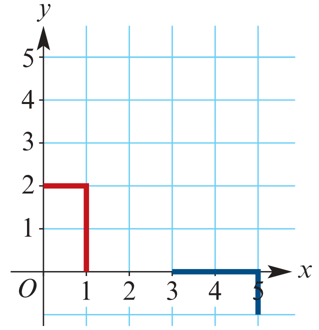
Note: There is more than one possible answer.



c Briefly explain why the blue figure in part **b** cannot be transformed to the red figure using only one reflection.

d In the diagram on the right, the blue figure can be transformed to the red figure by a sequence of three reflections. Determine the equations of the three mirror lines.

Note: There is more than one possible answer.



e Briefly explain why the blue figure in part **d** cannot be transformed to the red figure using only two reflections.

4 Let $u = a + bi$ and $v = c + di$ be complex numbers, where $a, b, c, d \in \mathbb{R}$.

a Show that $\overline{u + v} = \overline{u} + \overline{v}$.

b Show that $\overline{u \cdot v} = \overline{u} \cdot \overline{v}$.

c Show that $\overline{k \cdot u} = k \cdot \overline{u}$ for each real number k.

d **Conjugate root theorem** Consider a cubic polynomial $P(z) = az^3 + bz^2 + cz + d$ such that the coefficients a, b, c and d are all real numbers. Using the previous results, prove that if u is a complex number with $P(u) = 0$, then $P(\overline{u}) = 0$.

e Show that the result from part **d** may not be true if we do not assume that the coefficients a, b, c and d are all real numbers.

5 **a** Let w and z be complex numbers such that the angles $\text{Arg}(w)$ and $\text{Arg}(z)$ are acute. Using the rule for multiplying complex numbers in polar form, show that

$$\text{Arg}(wz) = \text{Arg}(w) + \text{Arg}(z)$$

b Determine the exact values of:

i $\text{Arg}(2 + i)$ **ii** $\text{Arg}(3 + i)$ **iii** $\text{Arg}(5 + 5i)$

c Show that $(2 + i)(3 + i) = 5 + 5i$.

d By taking the argument of both sides of the equation from part **c**, prove that

$$\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{3}\right) = \frac{\pi}{4}$$

e By first evaluating $(3 + i)^2(7 + i)$, prove that

$$2\tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{7}\right) = \frac{\pi}{4}$$

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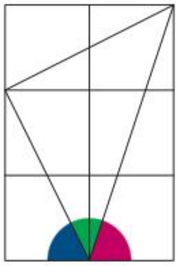
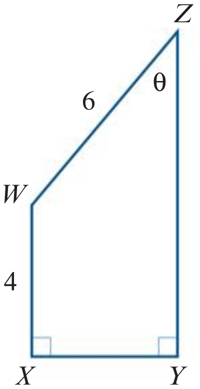
f By first evaluating $(1 + i)(1 + 2i)(1 + 3i)$, prove that

$$\tan^{-1}(1) + \tan^{-1}(2) + \tan^{-1}(3) = \pi$$

g Explain how the result from part **f** can also be shown using the diagram on the right. The grid is composed of unit-length squares.



6 The diagram shows a trapezium $XYZW$ such that $XW = 4$, $WZ = 6$, $\angle YXW = \angle XYZ = \frac{\pi}{2}$ and $\angle YZW = \theta$, where $0 < \theta < \frac{\pi}{2}$.



a Show that the area, A, of the trapezium is given by

$$A = 24\sin\theta + 9\sin(2\theta)$$

b Show that the perimeter, P, of the trapezium is given by

$$P = 14 + 6(\sin\theta + \cos\theta)$$

c Determine the maximum value of P and the corresponding value of θ.

d Determine the values of θ for which $P = 21$.

e Determine the exact value of A when $\theta = \frac{\pi}{4}$.

f Use your calculator to determine the maximum value of A (correct to two decimal places) and the value of θ for which this occurs (correct to one decimal place).

g Sketch the graph of A against θ.

h **Change of variable** Let $x = XY$. Show that

$$A = 4x + \frac{1}{2}x\sqrt{36 - x^2}$$

Use your calculator to determine the maximum value of A by graphing A against x.

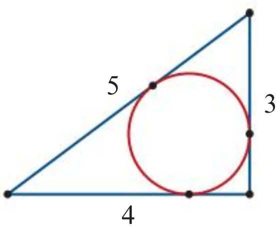
17D Problem-solving and modelling investigations

For each of the following questions, there are different approaches and directions that you can take. Suggestions are given, but you should develop your solution on an individual basis.

1 Circumcircles and incircles of right-angled triangles

a Determine the radius of the circumcircle of any right-angled triangle with hypotenuse of length c.

b A right-angled triangle with sides 3, 4 and 5 is shown with its incircle. Determine the radius of the circle.



c Determine the radius of the incircle of a right-angled triangle with sides 5, 12 and 13.

d Determine the radius of the incircle of a right-angled triangle with sides 7, 24 and 25.

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e Generalise these results for any Pythagorean triple (a, b, c) such that $c = b + 1$, where c is the length of the hypotenuse.

2 Graphs involving the modulus function

a **i** Sketch the graphs of the functions $y = |x - 2| + |x - 3|$ and $y = |2x - 3| + |x - 4|$.

ii Investigate graphs of the form $y = |ax + b| + |cx + d|$ and $y = |ax + b| - |cx + d|$.

b **i** Sketch the graph of the relation $|2x + y| = 2$.

ii Investigate graphs of the form $|ax + by| = c$.

c **i** Sketch the region of the plane defined by $|x + y| \leq 2$.

ii Investigate graphs of the form $|ax + by| \leq c$.

d **i** Sketch the graph of the relation $|x| + |y| = 2$. Determine the area of the enclosed region.

ii Investigate graphs of the form $|ax| + |by| = c$ and the corresponding areas.

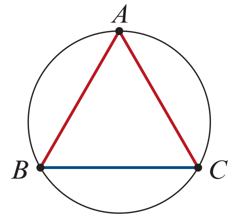
e Sketch the region of the plane defined by $|x| + |y| + |x + y| \leq 2$ and determine the area of this region.

f Investigate other families of graphs involving the modulus function. For example, consider the graphs of $|xy| = 1$ and $|xy| \leq 2$.

3 Products of lengths

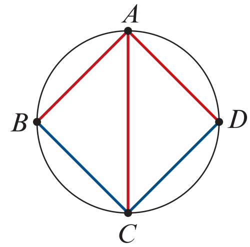
a The vertices A, B and C of an equilateral triangle lie on a circle of radius 1. Determine the value of

$$AB \times AC$$



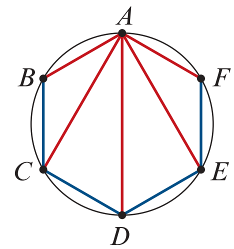
b The vertices A, B, C and D of a square lie on a circle of radius 1. Determine the value of

$$AB \times AC \times AD$$



c The vertices A, B, C, D, E and F of a regular hexagon lie on a circle of radius 1. Determine the value of

$$AB \times AC \times AD \times AE \times AF$$



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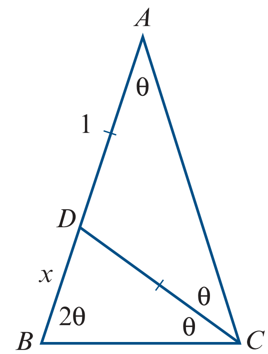
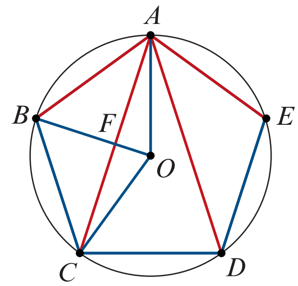
d We now aim to prove the corresponding result for a regular pentagon. In the figure on the right, $\triangle BAC$ is isosceles. Moreover, $\angle A = \theta$, $\angle B = 2\theta$ and $AD = DC = 1$.



- **i** Show that $BC = 1$.
  Hint: First determine $\angle BDC$ in terms of θ.
- **ii** Prove that $\triangle BAC$ is similar to $\triangle DCB$.
- **iii** Let $x = BD$. Use similar triangles to show that $x^2 + x - 1 = 0$.
- **iv** Hence, show that $x = \dfrac{-1 + \sqrt{5}}{2}$.
- **v** Explain why $\theta = \dfrac{\pi}{5}$.
- **vi** Let E be the midpoint of BC. Using $\triangle BAE$, show that $\cos\left(\dfrac{2\pi}{5}\right) = \dfrac{\sqrt{5} - 1}{4}$.
- **vii** Hence, prove that $\sin\left(\dfrac{2\pi}{5}\right) = \dfrac{\sqrt{10 + 2\sqrt{5}}}{4}$.

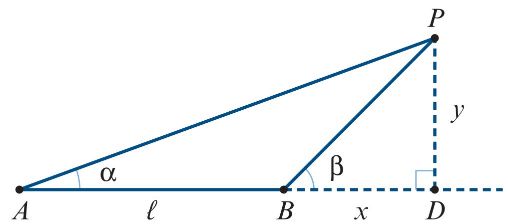
e The vertices A, B, C, D and E of a regular pentagon lie on a circle of radius 1 and centre O.



- **i** Determine the length of diagonal AC.
  Hint: First determine $\angle FOA$ and then determine AF.
- **ii** Hence, determine the value of

$$AB \times AC \times AD \times AE$$

4 Measiurement errors A practical use of trigonometry is in determining the position of an inaccessible object. The technique used in this activity is called *triangulation*.



In this diagram: The point P is inaccessible. The distance $\ell = AB$ is known. The distances $x = BD$ and $y = PD$ are unknown. The angles α and β are measured.

a Note that $\tan\beta = \dfrac{y}{x}$ and $\tan\alpha = \dfrac{y}{x + \ell}$. Show that $x = \dfrac{\ell\tan\alpha}{\tan\beta - \tan\alpha}$.

b Hence, show that

$$x = \frac{\ell\sin\alpha\,\cos\beta}{\sin(\beta - \alpha)} \quad \text{and} \quad y = \frac{\ell\sin\alpha\,\sin\beta}{\sin(\beta - \alpha)}$$

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c Investigate the effect that measurement errors in the angles α and β have on the values of x and y calculated using the formulas from part **b**. In particular, consider the effect for different magnitudes of $\beta - \alpha$. For example, what is the effect of a 1° error in α when the measured values are $\alpha = 40^\circ$ and $\beta = 45^\circ$?

d Take actual measurements in real-world situations and investigate your errors.

5 Complex quadratics In this question, we use the quadratic formula to help solve polynomial equations over the complex numbers. Recall that the solutions of a quadratic equation $az^2 + bz + c = 0$ are given by

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

a **i** Solve the quadratic equation $z^2 + az + a^2 = 0$, where a is a positive real number.

ii Hence, solve the cubic equation $z^3 = a^3$, where a is a positive real number.
Hint: Use the factorisation $z^3 - a^3 = (z - a)(z^2 + az + a^2)$.

iii For the case $a = 1$, write the solutions of the cubic equation in polar form. Plot them on an Argand diagram with the unit circle.

iv Add the solutions for the cases $a = 2$ and $a = 3$ to your Argand diagram.

v Summarise what you have found.

b Repeat part **a** for $z^2 - az + a^2 = 0$ and $z^3 = -a^3$, where a is a positive real number.
Hint: Use the factorisation $z^3 + a^3 = (z + a)(z^2 - az + a^2)$.

c Repeat part **a** for $z^2 + aiz - a^2 = 0$ and $z^3 = -a^3 i$, where a is a positive real number.

d Repeat part **a** for $z^2 - aiz - a^2 = 0$ and $z^3 = a^3 i$, where a is a positive real number.

e There is more to consider in this way. For example, look at the equation $z^4 + 1 = 0$. Factorise the left-hand side by considering $z^4 + 2z^2 + 1 - 2z^2$.

6 Reciprocals of quadratics Consider the family of functions with rules of the form

$$f(x) = \frac{1}{ax^2 + bx + c}$$

where a, b and c are real constants with $a \neq 0$. It is a good idea to graph some examples of functions from this family before starting the following.

a Determine the coordinates of the turning point on the graph of $y = f(x)$, and state whether it is a local maximum or a local minimum. Consider the two cases $a > 0$ and $a < 0$ separately.

b For $a > 0$, sketch the graph of $y = f(x)$, giving the equations of all asymptotes. Consider the following three cases separately:

i $b^2 - 4ac < 0$ **ii** $b^2 - 4ac = 0$ **iii** $b^2 - 4ac > 0$

c For $a < 0$, sketch the graph of $y = f(x)$, giving the equations of all asymptotes. Consider the following three cases separately:

i $b^2 - 4ac < 0$ **ii** $b^2 - 4ac = 0$ **iii** $b^2 - 4ac > 0$

7 Graphing functions without calculus

Consider:

$$\begin{aligned} f(x) &= x^2 + \frac{1}{x^2} \\ &= \frac{x^4 + 1}{x^2} \\ &= \frac{x^4 - 2x^2 + 1 + 2x^2}{x^2} \\ &= \frac{(x^2 - 1)^2}{x^2} + 2 \end{aligned}$$

Therefore the graph of $y = f(x)$ has local minimums at $(1, 2)$ and $(-1, 2)$.

a Sketch the graph of $f(x) = x^2 + \dfrac{1}{x^2}$.

b Consider the family of curves with equations

$$y = x^{2m} + \frac{a}{x^{2m}} \qquad \text{where } m \in \mathbb{N} \text{ and } a > 0$$

State the equations of the asymptotes and the coordinates of the turning points.

c Consider the family of curves with equations

$$y = \frac{x^{2m}}{a + x^{4m}} \qquad \text{where } m \in \mathbb{N} \text{ and } a > 0$$

Sketch the curve for $m = a = 1$. Describe the properties of this family of curves.

d Investigate $y = x^{2m-1} + \dfrac{a}{x^{2m-1}}$, where $m \in \mathbb{N}$ and $a > 0$.

e Investigate $y = \dfrac{x^{2m-1}}{a + x^{4m-2}}$, where $m \in \mathbb{N}$ and $a > 0$.

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Appendix

A

Trigonometric functions

Chapter contents

This chapter reviews the study of the three trigonometric functions – sine, cosine and tangent – introduced in Mathematical Methods Unit 1.

An important property of the trigonometric functions is that they are periodic. That is, they each repeat their values in regular intervals or periods. In general, a function f is **periodic** if there is a positive constant a such that $f(x + a) = f(x)$. The sine and cosine functions each have period 2π, while the tangent function has period π.

The sine and cosine functions are used to model wave motion, and are therefore central to the application of mathematics to any problem in which periodic motion is involved – from the motion of the tides and ocean waves to sound waves and modern telecommunications.

We continued our study of trigonometric functions in Chapter 11, where we derived and applied several important trigonometric identities.

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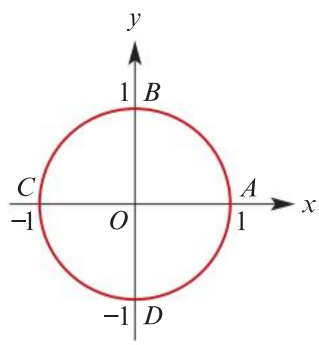
A1 Defining the trigonometric functions

Learning intentions

- To be able to convert degrees to radians and vice versa.
- To define the sine and cosine functions.

Measuring angles in degrees and radians

The diagram shows a **unit circle**, i.e. a circle of radius 1 unit.



The circumference of the unit circle $= 2\pi \times 1$

$= 2\pi$ units

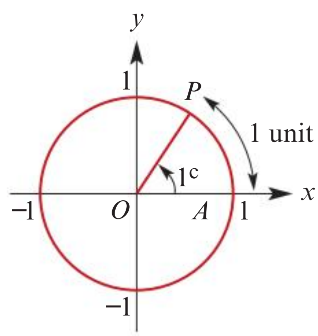
Thus, the distance in an anticlockwise direction around the circle from

A to $B = \frac{\pi}{2}$ units $\quad$ A to $C = \pi$ units

A to $D = \frac{3\pi}{2}$ units

Definition of a radian

In moving around the circle a distance of 1 unit from A to P, the angle POA is defined. The measure of this angle is 1 radian.



> One **radian** (written 1^c) is the angle subtended at the centre of the unit circle by an arc of length 1 unit.

Note: Angles formed by moving **anticlockwise** around the unit circle are defined as **positive**; those formed by moving **clockwise** are defined as **negative**.

Converting between degrees and radians

The angle, in radians, swept out in one revolution of a circle is $2\pi^c$.

$2\pi^c = 360° \Rightarrow \pi^c = 180°$

$\therefore \quad 1^c = \frac{180°}{\pi} \quad \text{or} \quad 1° = \frac{\pi^c}{180}$

Example 1

a Convert 135° to radians. **b** Convert $\frac{\pi^c}{6}$ to degrees.

Solution

a $135° = \frac{135 \times \pi^c}{180} = \frac{3\pi^c}{4}$

b $\frac{\pi^c}{6} = \frac{\pi}{6} \times \frac{180°}{\pi} = 30°$

Note: Usually the symbol for radians, ^{c}, is omitted. Any angle is assumed to be measured in radians unless indicated otherwise.

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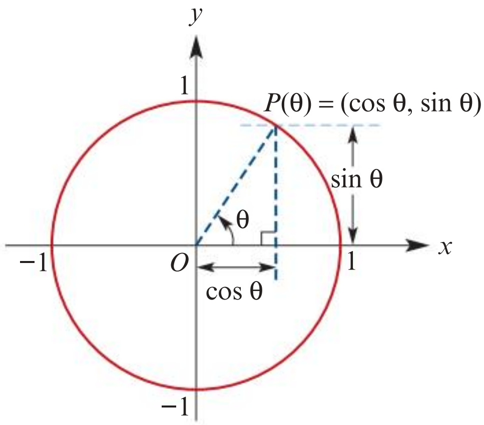
Defining sine and cosine

Let $P(\theta)$ denote the point on the unit circle corresponding to an angle θ. Then:

- $\cos\theta$ is the x-coordinate of $P(\theta)$
- $\sin\theta$ is the y-coordinate of $P(\theta)$



Hence the coordinates of the point $P(\theta)$ are $(\cos\theta, \sin\theta)$.

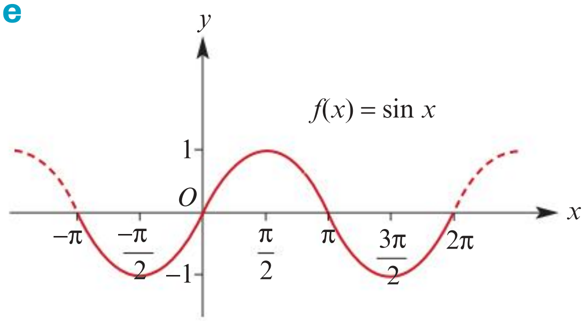
Note: Adding 2π to the angle results in a return to the same point on the unit circle. Thus $\cos(2\pi + \theta) = \cos\theta$ and $\sin(2\pi + \theta) = \sin\theta$.

The graphs of sine and cosine

A sketch of the graph of

$$f(x) = \sin x, \quad x \in \mathbb{R}$$

is shown opposite.

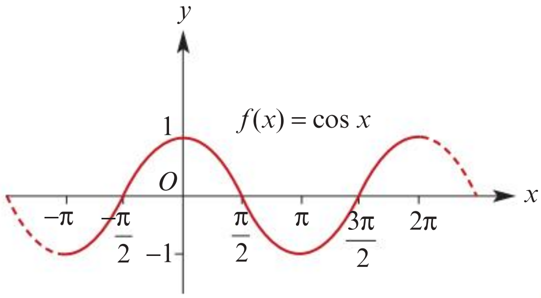


As $\sin(x + 2\pi) = \sin x$ for all $x \in \mathbb{R}$, the sine function is **periodic**.
The period is 2π. The amplitude is 1.

A sketch of the graph of

$$f(x) = \cos x, \quad x \in \mathbb{R}$$

is shown opposite.



The period of the cosine function is 2π.
The amplitude is 1.

Example 2

Evaluate each of the following:

a $\sin\left(\frac{3\pi}{2}\right)$ **b** $\cos\left(-\frac{\pi}{2}\right)$ **c** $\cos\left(\frac{21\pi}{2}\right)$

Solution

a $\sin\left(\frac{3\pi}{2}\right) = -1$

b $\cos\left(-\frac{\pi}{2}\right) = 0$

c $\cos\left(\frac{21\pi}{2}\right) = \cos\left(10\pi + \frac{\pi}{2}\right) = 0$

Explanation

since $P\left(\frac{3\pi}{2}\right)$ has coordinates $(0, -1)$.

since $P\left(-\frac{\pi}{2}\right)$ has coordinates $(0, -1)$.

since $P\left(\frac{\pi}{2}\right)$ has coordinates $(0, 1)$.

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Defining tangent

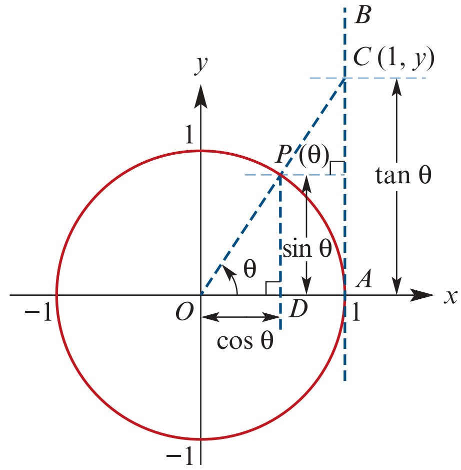
Again consider the unit circle.

If we draw a tangent to the unit circle at A, then the y-coordinate of C, the point of intersection of the line OP and the tangent, is called **tangent** θ (abbreviated to $\tan\theta$).

By considering the similar triangles OPD and OCA:

$$\frac{\tan\theta}{1} = \frac{\sin\theta}{\cos\theta}$$

$$\therefore \quad \tan\theta = \frac{\sin\theta}{\cos\theta}$$



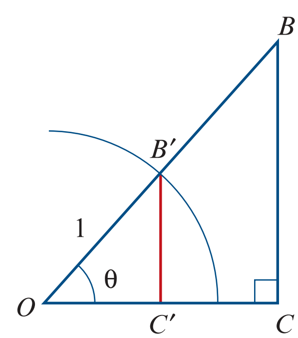
Note that $\tan\theta$ is undefined when $\cos\theta = 0$. So $\tan\theta$ is undefined for $\theta = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \dots$

Note: Adding π to the angle does not change the line OP. Thus $\tan(\pi + \theta) = \tan\theta$.

The trigonometric ratios

We have seen in Chapter 3 that, for acute angles, the unit-circle definition of sine and cosine is equivalent to the ratio definition.

For a right-angled triangle OBC, we can construct a similar triangle $OB'C'$ that lies in the unit circle.



From the diagram:

$$B'C' = \sin\theta \quad \text{and} \quad OC' = \cos\theta$$

The similarity factor is the length OB, giving

$$BC = OB\sin\theta \quad \text{and} \quad OC = OB\cos\theta$$

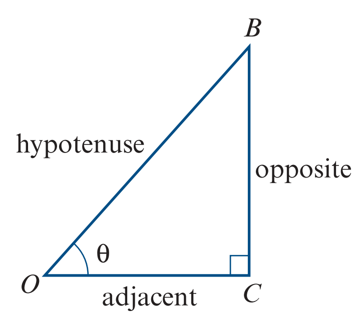
$$\therefore \quad \frac{BC}{OB} = \sin\theta \quad \text{and} \quad \frac{OC}{OB} = \cos\theta$$

This gives the ratio definition of sine and cosine for a right-angled triangle. The naming of sides with respect to an angle θ is as shown.

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$



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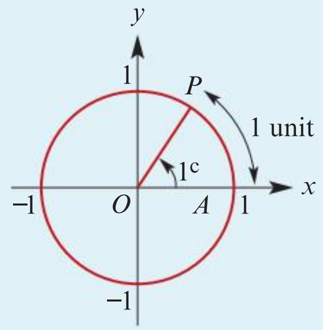
Summary A1

- **Definition of a radian**

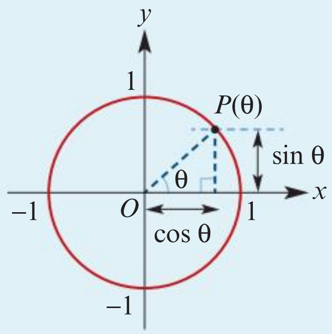
  One radian (written 1^c) is the angle formed at the centre of the unit circle by an arc of length 1 unit:

  $$1^c = \frac{180^\circ}{\pi} \quad \text{and} \quad 1^\circ = \frac{\pi^c}{180}$$



- **Sine and cosine functions**

  The point $P(\theta)$ on the unit circle corresponding to an angle θ has coordinates $(\cos\theta, \sin\theta)$.



Exercise A1

Example 1a

1 Convert the following angles from degrees to exact values in radians:

a 720° **b** 540° **c** -450°
d 15° **e** -10° **f** -315°

Example 1b

2 Convert the following angles from radians to degrees:

a $\frac{5\pi}{4}$ **b** $-\frac{2\pi}{3}$ **c** $\frac{7\pi}{12}$
d $-\frac{11\pi}{6}$ **e** $\frac{13\pi}{9}$ **f** $-\frac{11\pi}{12}$

Example 2

3 Determine the exact value of each of the following:

a $\cos\left(\frac{3\pi}{2}\right)$ **b** $\sin\left(-\frac{\pi}{2}\right)$ **c** $\cos(6\pi)$ **d** $\sin\left(\frac{15\pi}{2}\right)$

4 Determine the exact value of each of the following:

a $\sin(270^\circ)$ **b** $\cos(-540^\circ)$ **c** $\sin(450^\circ)$ **d** $\cos(720^\circ)$

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A2 Symmetry properties and the Pythagorean identity

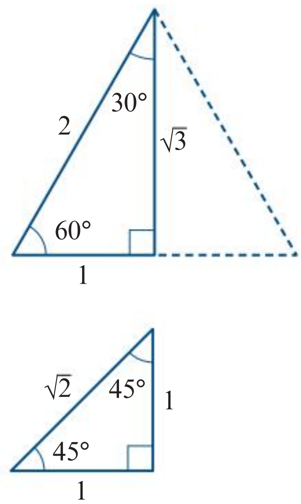
Learning intentions

- ▶ To be able to use symmetry properties and the Pythagorean identity to determine exact values for the trigonometric functions.

Exact values of trigonometric functions

The values in this table for 30°, 45° and 60° can be determined from the two triangles shown.

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
0	0	1	0
$\frac{\pi}{6}$ (30°)	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
$\frac{\pi}{4}$ (45°)	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
$\frac{\pi}{3}$ (60°)	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$\frac{\pi}{2}$ (90°)	1	0	undefined

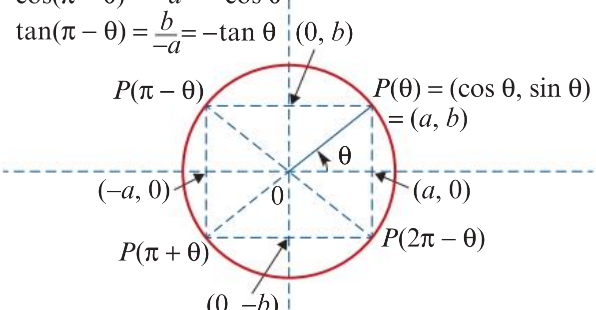


Symmetry properties

The coordinate axes divide the unit circle into four quadrants, numbered anticlockwise from the positive direction of the x-axis. Using symmetry, we can determine relationships between the trigonometric functions for angles in different quadrants:

Quadrant 2
By symmetry
$\sin(\pi - \theta) = b = \sin\theta$
$\cos(\pi - \theta) = -a = -\cos\theta$
$\tan(\pi - \theta) = \frac{b}{-a} = -\tan\theta$

Quadrant 1



Quadrant 3
$\sin(\pi + \theta) = -b = -\sin\theta$
$\cos(\pi + \theta) = -a = -\cos\theta$
$\tan(\pi + \theta) = \frac{-b}{-a} = \tan\theta$

Quadrant 4
$\sin(2\pi - \theta) = -b = -\sin\theta$
$\cos(2\pi - \theta) = a = \cos\theta$
$\tan(2\pi - \theta) = \frac{-b}{a} = -\tan\theta$

Note: $\sin(2\pi + \theta) = \sin\theta$
$\cos(2\pi + \theta) = \cos\theta$
$\tan(2\pi + \theta) = \tan\theta$

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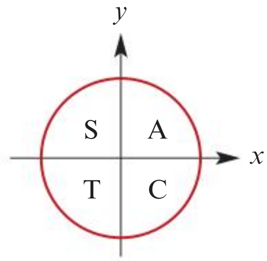
Signs of trigonometric functions

Using these symmetry properties, the signs of sin, cos and tan for the four quadrants can be summarised as follows:

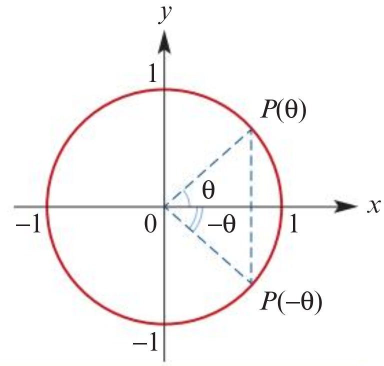
1st quadrant	all are positive	(A)
2nd quadrant	sin is positive	(S)
3rd quadrant	tan is positive	(T)
4th quadrant	cos is positive	(C)



Negative angles

By symmetry:

$$\sin(-\theta) = -\sin\theta$$
$$\cos(-\theta) = \cos\theta$$
$$\tan(-\theta) = \frac{-\sin\theta}{\cos\theta} = -\tan\theta$$



Example 3

Determine the exact value of:

a $\sin\left(\frac{11\pi}{6}\right)$

b $\cos\left(\frac{-5\pi}{4}\right)$

Solution

a $\sin\left(\frac{11\pi}{6}\right) = \sin\left(2\pi - \frac{\pi}{6}\right)$
$= -\sin\left(\frac{\pi}{6}\right)$
$= -\frac{1}{2}$

b $\cos\left(\frac{-5\pi}{4}\right) = \cos\left(\frac{5\pi}{4}\right)$
$= \cos\left(\pi + \frac{\pi}{4}\right)$
$= -\cos\left(\frac{\pi}{4}\right)$
$= -\frac{1}{\sqrt{2}}$

Example 4

Determine the exact value of:

a $\sin(-150^\circ)$

b $\cos 585^\circ$

Solution

a $\sin(-150^\circ) = -\sin 150^\circ$
$= -\sin(180^\circ - 30^\circ)$
$= -\sin 30^\circ$
$= -\frac{1}{2}$

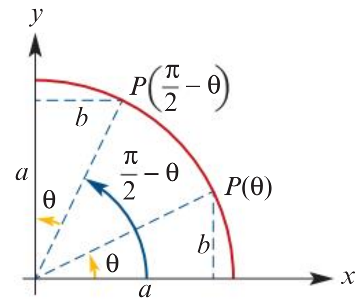
b $\cos 585^\circ = \cos(585^\circ - 360^\circ)$
$= \cos 225^\circ$
$= -\cos 45^\circ$
$= -\frac{1}{\sqrt{2}}$

Complementary relationships

From the diagram to the right:

$$\sin\left(\frac{\pi}{2} - \theta\right) = a = \cos\theta$$

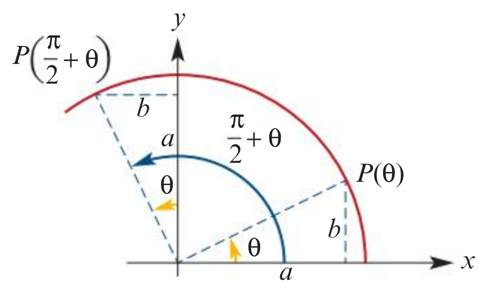
$$\cos\left(\frac{\pi}{2} - \theta\right) = b = \sin\theta$$



From the diagram to the right:

$$\sin\left(\frac{\pi}{2} + \theta\right) = a = \cos\theta$$

$$\cos\left(\frac{\pi}{2} + \theta\right) = -b = -\sin\theta$$



Example 5

If $\sin\theta = 0.4$ and $\cos\alpha = 0.8$, determine the value of:

a $\sin\left(\frac{\pi}{2} - \alpha\right)$ **b** $\cos\left(\frac{\pi}{2} + \theta\right)$ **c** $\sin(-\theta)$

Solution

a $\sin\left(\frac{\pi}{2} - \alpha\right) = \cos\alpha$
$= 0.8$

b $\cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta$
$= -0.4$

c $\sin(-\theta) = -\sin\theta$
$= -0.4$

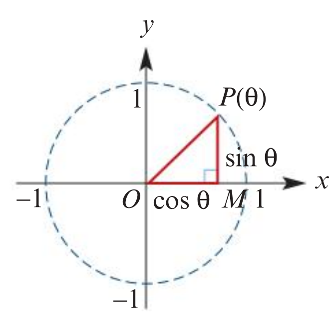
The Pythagorean identity

Consider a point, $P(\theta)$, on the unit circle.

By Pythagoras' theorem:

$$OP^2 = OM^2 + MP^2$$

$$\therefore \quad 1 = (\cos\theta)^2 + (\sin\theta)^2$$



Since this is true for all values of θ, it is called an identity.

We can write $(\cos\theta)^2$ and $(\sin\theta)^2$ as $\cos^2\theta$ and $\sin^2\theta$, and therefore we obtain:

Pythagorean identity

$$\cos^2\theta + \sin^2\theta = 1$$

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Example 6

If $\sin x = 0.3$ and $0 < x < \frac{\pi}{2}$, determine:

a $\cos x$

b $\tan x$

Solution

a $\cos^2 x + \sin^2 x = 1$

$\cos^2 x + 0.09 = 1$

$\cos^2 x = 0.91$

$\therefore \quad \cos x = \pm\sqrt{0.91}$

Since the point $P(x)$ is in the 1st quadrant, this gives

$$\cos x = \sqrt{0.91} = \sqrt{\frac{91}{100}} = \frac{\sqrt{91}}{10}$$

b $$\tan x = \frac{\sin x}{\cos x} = \frac{0.3}{\sqrt{0.91}} = \frac{3}{\sqrt{91}} = \frac{3\sqrt{91}}{91}$$

Summary A2

- **Exact values**

θ		$\sin\theta$	$\cos\theta$	$\tan\theta$
0		0	1	0
$\frac{\pi}{6}$	(30°)	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
$\frac{\pi}{4}$	(45°)	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
$\frac{\pi}{3}$	(60°)	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$\frac{\pi}{2}$	(90°)	1	0	undefined

- **Complementary relationships**

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$$

$$\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$$

$$\sin\left(\frac{\pi}{2} + \theta\right) = \cos\theta$$

$$\cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta$$

- **Pythagorean identity**

$$\cos^2\theta + \sin^2\theta = 1$$

Exercise A2

Example 3

1 Evaluate each of the following:

a $\cos\left(\frac{3\pi}{4}\right)$ **b** $\sin\left(\frac{5\pi}{4}\right)$ **c** $\sin\left(\frac{25\pi}{2}\right)$ **d** $\sin\left(\frac{15\pi}{6}\right)$ **e** $\cos\left(\frac{17\pi}{4}\right)$

f $\sin\left(-\frac{15\pi}{4}\right)$ **g** $\sin(27\pi)$ **h** $\sin\left(-\frac{17\pi}{3}\right)$ **i** $\cos\left(\frac{75\pi}{6}\right)$ **j** $\cos\left(-\frac{15\pi}{6}\right)$

k $\sin\left(-\frac{35\pi}{2}\right)$ **l** $\cos\left(-\frac{45\pi}{6}\right)$ **m** $\cos\left(\frac{16\pi}{3}\right)$ **n** $\sin\left(-\frac{105\pi}{2}\right)$ **o** $\cos(1035\pi)$

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Example 4 **2** Determine the exact value of each of the following:

a $\sin(135°)$ **b** $\cos(-300°)$ **c** $\sin(480°)$
d $\cos(240°)$ **e** $\sin(-225°)$ **f** $\sin(420°)$

Example 5 **3** If $\sin x = 0.3$ and $\cos \alpha = 0.6$, determine the value of:

a $\cos(-\alpha)$ **b** $\sin\left(\frac{\pi}{2} + \alpha\right)$ **c** $\cos\left(\frac{\pi}{2} - x\right)$ **d** $\sin(-x)$

e $\cos\left(\frac{\pi}{2} + x\right)$ **f** $\sin\left(\frac{\pi}{2} - \alpha\right)$ **g** $\sin\left(\frac{3\pi}{2} + \alpha\right)$ **h** $\cos\left(\frac{3\pi}{2} - x\right)$

Example 6 **4** If $\sin x = 0.5$ and $\frac{\pi}{2} < x < \pi$, determine $\cos x$ and $\tan x$.

5 If $\cos x = -0.7$ and $\pi < x < \frac{3\pi}{2}$, determine $\sin x$ and $\tan x$.

6 If $\sin x = -0.5$ and $\pi < x < \frac{3\pi}{2}$, determine $\cos x$ and $\tan x$.

7 If $\sin x = -0.3$ and $\frac{3\pi}{2} < x < 2\pi$, determine $\cos x$ and $\tan x$.

A3 Solution of equations involving sine and cosine

Learning intentions

- To be able to solve trigonometric equations.

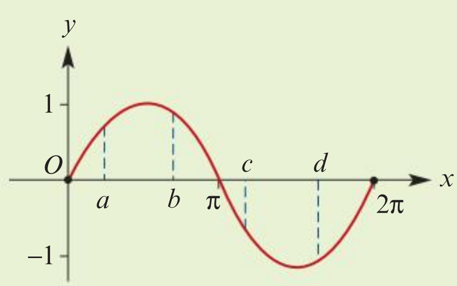
If a trigonometric equation has a solution, then it will have a corresponding solution in each 'cycle' of its domain. Such an equation is solved by using the symmetry of the graph to obtain solutions within one 'cycle' of the function. Other solutions may be obtained by adding multiples of the period to these solutions.

Example 7

The graph shown is

$$y = \sin x, \quad 0 \le x \le 2\pi$$

For each pronumeral marked on the x-axis, determine the other x-value which has the same y-value.



Solution

For $x = a$, the other value is $\pi - a$. For $x = b$, the other value is $\pi - b$.

For $x = c$, the other value is $2\pi - (c - \pi) = 3\pi - c$.

For $x = d$, the other value is $\pi + (2\pi - d) = 3\pi - d$.

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Example 8

Solve the equation $\sin\left(2x + \frac{\pi}{3}\right) = \frac{1}{2}$ for $x \in [0, 2\pi]$.

Solution

Let $\theta = 2x + \frac{\pi}{3}$. Note that

$$\begin{aligned} 0 \le x \le 2\pi \quad &\Leftrightarrow \quad 0 \le 2x \le 4\pi \\ &\Leftrightarrow \quad \frac{\pi}{3} \le 2x + \frac{\pi}{3} \le \frac{13\pi}{3} \\ &\Leftrightarrow \quad \frac{\pi}{3} \le \theta \le \frac{13\pi}{3} \end{aligned}$$

To solve $\sin\left(2x + \frac{\pi}{3}\right) = \frac{1}{2}$ for $x \in [0, 2\pi]$, we first solve $\sin\theta = \frac{1}{2}$ for $\frac{\pi}{3} \le \theta \le \frac{13\pi}{3}$.

Consider $\sin\theta = \frac{1}{2}$.

$$\therefore \quad \theta = \frac{\pi}{6} \text{ or } \frac{5\pi}{6} \text{ or } 2\pi + \frac{\pi}{6} \text{ or } 2\pi + \frac{5\pi}{6} \text{ or } 4\pi + \frac{\pi}{6} \text{ or } 4\pi + \frac{5\pi}{6} \text{ or } \ldots$$

The solutions $\frac{\pi}{6}$ and $\frac{29\pi}{6}$ are not required, as they lie outside the restricted domain for θ.

For $\frac{\pi}{3} \le \theta \le \frac{13\pi}{3}$:

$$\theta = \frac{5\pi}{6} \text{ or } \frac{13\pi}{6} \text{ or } \frac{17\pi}{6} \text{ or } \frac{25\pi}{6}$$

$$\therefore \quad 2x + \frac{2\pi}{6} = \frac{5\pi}{6} \text{ or } \frac{13\pi}{6} \text{ or } \frac{17\pi}{6} \text{ or } \frac{25\pi}{6}$$

$$\therefore \quad 2x = \frac{3\pi}{6} \text{ or } \frac{11\pi}{6} \text{ or } \frac{15\pi}{6} \text{ or } \frac{23\pi}{6}$$

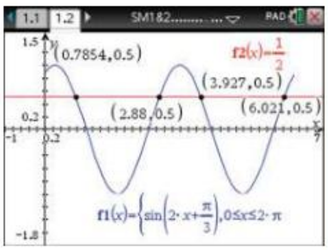
$$\therefore \quad x = \frac{\pi}{4} \text{ or } \frac{11\pi}{12} \text{ or } \frac{5\pi}{4} \text{ or } \frac{23\pi}{12}$$

Using the TI-Nspire CX non-CAS

- Ensure that the calculator is in radian mode.

In a **Graphs** application, plot the graphs of:

- $f1(x) = \sin\left(2x + \frac{\pi}{3}\right) \mid 0 \le x \le 2\pi$
- $f2(x) = \frac{1}{2}$



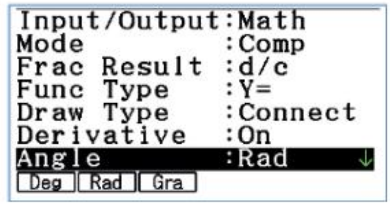
- Use (menu) > **Geometry** > **Points & Lines** > **Intersection Point(s)**.

Note: To change the mode, go to (on) > **Settings** > **Document Settings**. The **Graphs** application has its own settings, which are accessed from a **Graphs** page using (menu) > **Settings**.

Using the Casio

Ensure that the angle setting is Radians:

- In **Run-Matrix** mode, go to the set-up screen by pressing SHIFT MENU.
- Use the cursor key ▼ to move down to **Angle**.
- Press F2 to change to **Radians**. Then press EXIT.



Method 1: Using the numerical solver

- In **Run-Matrix** mode, select the numerical solver (**Calculation** OPTN F4, **SolveN** F5).
- Complete the equation and domain by entering:

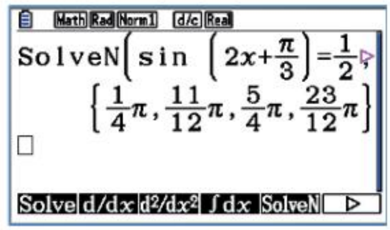
$$\sin\left(2x + \frac{\pi}{3}\right) = \frac{1}{2}, x, 0, 2\pi\right)$$

sin (2 X,θ,T + $a\,^b/_c$ SHIFT EXP ▼ 3 ►

) SHIFT • $a\,^b/_c$ 1 ▼ 2 ► ,

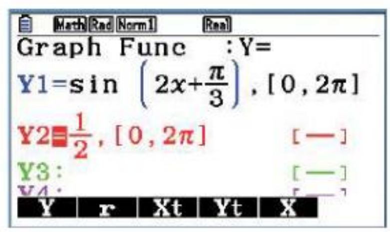
X,θ,T , 0 , 2 SHIFT EXP) EXE

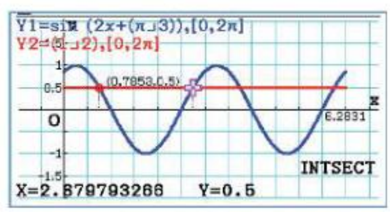
EXIT



Method 2: Using Graph mode

- In **Graph** mode, enter the rules for the curve and the line with the restricted domain, as shown.
- Adjust the View Window SHIFT F3 for $-1 \leq x \leq 7$ and $-2 \leq y \leq 2$.
- Select **Draw** F6.
- To determine the intersection points, go to the **G-Solve** menu SHIFT F5 and select **Intersection** F5.
  Use the cursor key ► to determine the next point.





Note: Exact solutions cannot be obtained using the graphical method.

Skill-sheet

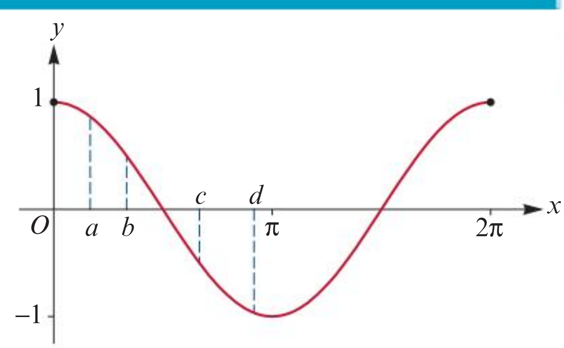
Exercise A3

Example 7

1 The graph shown is

$$y = \cos x, \quad 0 \leq x \leq 2\pi$$

For each pronumeral marked on the x-axis, determine the other x-value which has the same y-value.



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Example 8

2 Solve each of the following for $x \in [0, 2\pi]$:

a $\sin x = -\frac{\sqrt{3}}{2}$ **b** $\sin(2x) = -\frac{\sqrt{3}}{2}$ **c** $2\cos(2x) = -1$

d $\sin\left(x + \frac{\pi}{3}\right) = -\frac{1}{2}$ **e** $2\cos\left(2\left(x + \frac{\pi}{3}\right)\right) = -1$ **f** $2\sin\left(2x + \frac{\pi}{3}\right) = -\sqrt{3}$

3 Solve each of the following for $x \in [-\pi, \pi]$:

a $\sin x + \frac{1}{2} = 0$ **b** $\sin(3x) = 0$ **c** $\cos\left(\frac{x}{2}\right) = 1$

d $\sin\left(x - \frac{\pi}{6}\right) = -\frac{1}{2}$ **e** $2\cos\left(2\left(x + \frac{\pi}{6}\right)\right) = -1$

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A4 Transformations of the graphs of sine and cosine

Learning intentions

▶ To be able to apply transformations to the graphs of the trigonometric functions.

The graphs of functions with rules of the form

$$f(x) = a\sin(n(x + \varepsilon)) + b \quad \text{and} \quad f(x) = a\cos(n(x + \varepsilon)) + b$$

can be obtained from the graphs of $y = \sin x$ and $y = \cos x$ by transformations.

Example 9

Sketch the graph of the function

$$h(x) = 3\cos\left(2x + \frac{\pi}{3}\right) + 1, \quad x \in [0, 2\pi]$$

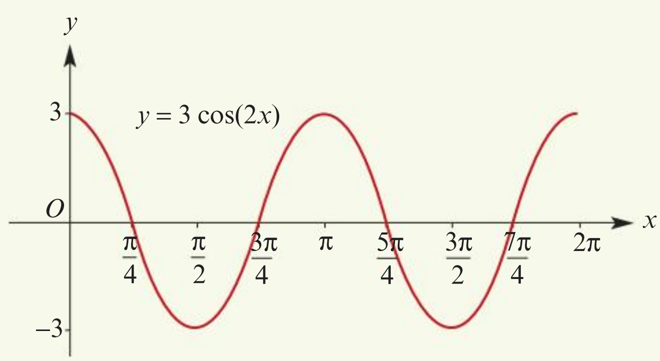
Solution

We can write $h(x) = 3\cos\left(2\left(x + \frac{\pi}{6}\right)\right) + 1$.

The graph of $y = h(x)$ is obtained from the graph of $y = \cos x$ by:

- a dilation of factor $\frac{1}{2}$ from the y-axis
- a dilation of factor 3 from the x-axis
- a translation of $\frac{\pi}{6}$ units in the negative direction of the x-axis
- a translation of 1 unit in the positive direction of the y-axis.

First apply the two dilations to the graph of $y = \cos x$.



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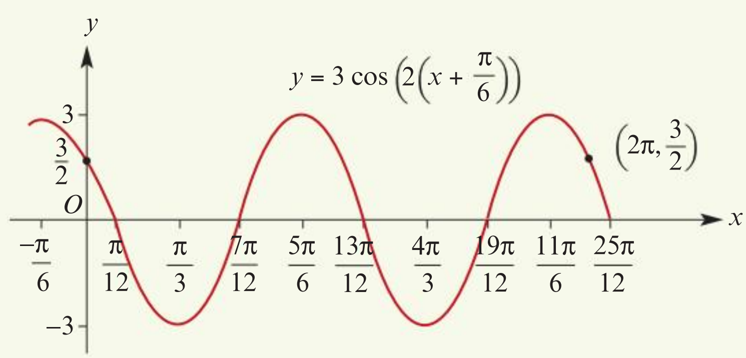
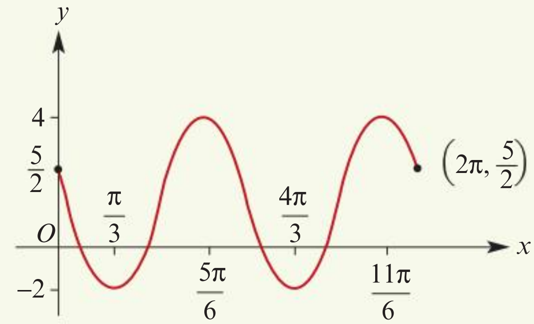
Next apply the translation $\frac{\pi}{6}$ units in the negative direction of the x-axis.



Apply the final translation and restrict the graph to the required domain.



Summary A4

For the graphs of $y = a\sin(nx)$ and $y = a\cos(nx)$, where $a > 0$ and $n > 0$:

- Period $= \frac{2\pi}{n}$
- Amplitude $= a$
- Range $= [-a, a]$

Skill-sheet

Exercise A4

Example 9

1 Sketch the graph of each of the following for the stated domain:

a $f(x) = \sin(2x),\ x \in [0, 2\pi]$

b $f(x) = \cos\left(x + \frac{\pi}{3}\right),\ x \in \left[\frac{-\pi}{3}, \pi\right]$

c $f(x) = \cos\left(2\left(x + \frac{\pi}{3}\right)\right),\ x \in [0, \pi]$

d $f(x) = 2\sin(3x) + 1,\ x \in [0, \pi]$

e $f(x) = 2\sin\left(x - \frac{\pi}{4}\right) + \sqrt{3},\ x \in [0, 2\pi]$

f $f(x) = \cos(2x),\ x \in [-\pi, \pi]$

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2 Sketch the graph of each of the following for the stated domain:

a $f(x) = \sin\left(x + \frac{\pi}{6}\right),\ x \in [-\pi, \pi]$

b $f(x) = \sin\left(2\left(x + \frac{\pi}{4}\right)\right),\ x \in [-\pi, \pi]$

c $f(x) = 2\cos\left(\frac{x}{3}\right) + 1,\ x \in [0, 6\pi]$

d $f(x) = 2\cos\left(x - \frac{\pi}{3}\right) + \sqrt{3},\ x \in [0, 2\pi]$

e $f(x) = \cos(\pi x),\ x \in [-2, 2]$

f $f(x) = \cos\left(\frac{\pi x}{6}\right),\ x \in [-12, 12]$

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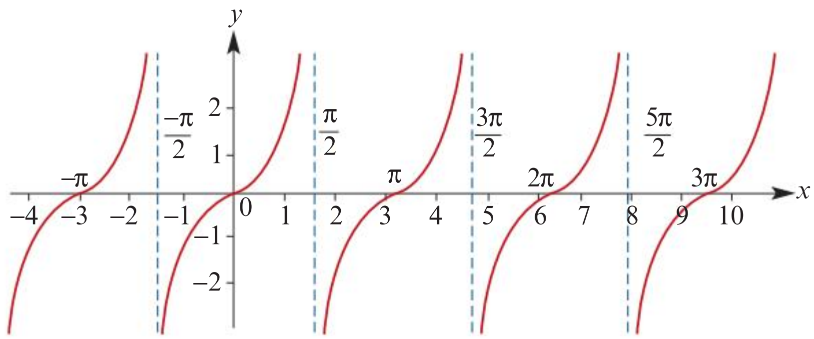
A5 The tangent function

Learning intentions

- To be able to graph the tangent function and transformations of this graph.
- To be able to solve equations involving the tangent function.

Recall that the tangent function is given by $\tan x = \dfrac{\sin x}{\cos x}$ for $\cos x \neq 0$.

The graph of $y = \tan x$ is shown below.



Properties of the tangent function

- The graph repeats itself every π units, i.e. the period of tan is π.
- The range of tan is $\mathbb{R}$.
- The vertical asymptotes have equations $x = \dfrac{(2k+1)\pi}{2}$ where $k \in \mathbb{Z}$.
- The axis intercepts are at $x = k\pi$ where $k \in \mathbb{Z}$.

Symmetry properties of tangent

Using symmetry properties of sine and cosine, we have

$$\tan(\pi - \theta) = \frac{\sin(\pi - \theta)}{\cos(\pi - \theta)} = \frac{\sin\theta}{-\cos\theta} = -\tan\theta$$

Similarly, we obtain:

- $\tan(\pi + \theta) = \tan\theta$
- $\tan(2\pi - \theta) = -\tan\theta$
- $\tan(-\theta) = -\tan\theta$
- $\tan\left(\dfrac{\pi}{2} - \theta\right) = \dfrac{\cos\theta}{\sin\theta}$
- $\tan\left(\dfrac{\pi}{2} + \theta\right) = -\dfrac{\cos\theta}{\sin\theta}$

Example 10

Determine the exact value of:

a $\tan\left(\dfrac{4\pi}{3}\right)$

b $\tan 330°$

Solution

a
$$\tan\left(\frac{4\pi}{3}\right) = \tan\left(\pi + \frac{\pi}{3}\right) = \tan\left(\frac{\pi}{3}\right) = \sqrt{3}$$

b
$$\tan 330° = \tan(360° - 30°) = -\tan 30° = -\frac{1}{\sqrt{3}}$$

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Solution of equations involving the tangent function

We now consider the solution of equations involving the tangent function, which can be applied to determining the x-axis intercepts for graphs of the tangent function.

The method is similar to that used for solving equations involving sine and cosine, except that only one solution needs to be found and then all other solutions are one period length apart.

Example 11

Solve the equation $3\tan(2x) = \sqrt{3}$ for $x \in (0, 2\pi)$.

Solution

$$3\tan(2x) = \sqrt{3}$$

$$\tan(2x) = \frac{\sqrt{3}}{3} = \frac{1}{\sqrt{3}}$$

$$\therefore \quad 2x = \frac{\pi}{6} \text{ or } \frac{7\pi}{6} \text{ or } \frac{13\pi}{6} \text{ or } \frac{19\pi}{6}$$

$$x = \frac{\pi}{12} \text{ or } \frac{7\pi}{12} \text{ or } \frac{13\pi}{12} \text{ or } \frac{19\pi}{12}$$

Explanation

Since we want solutions for x in $(0, 2\pi)$, we determine solutions for $2x$ in $(0, 4\pi)$.

Once we have found one solution for $2x$, we can obtain all other solutions by adding and subtracting multiples of π.

Example 12

Solve the equation $\tan\left(\frac{1}{2}\left(x - \frac{\pi}{4}\right)\right) = -1$ for $x \in [-2\pi, 2\pi]$.

Solution

Let $\theta = \frac{1}{2}\left(x - \frac{\pi}{4}\right)$. Note that

$$-2\pi \le x \le 2\pi \quad \Leftrightarrow \quad -\frac{9\pi}{4} \le x - \frac{\pi}{4} \le \frac{7\pi}{4}$$

$$\Leftrightarrow \quad -\frac{9\pi}{8} \le \frac{1}{2}\left(x - \frac{\pi}{4}\right) \le \frac{7\pi}{8}$$

$$\Leftrightarrow \quad -\frac{9\pi}{8} \le \theta \le \frac{7\pi}{8}$$

We first solve the equation $\tan\theta = -1$ for $-\frac{9\pi}{8} \le \theta \le \frac{7\pi}{8}$:

$$\theta = \frac{-\pi}{4} \text{ or } \frac{3\pi}{4}$$

$$\frac{1}{2}\left(x - \frac{\pi}{4}\right) = \frac{-\pi}{4} \text{ or } \frac{3\pi}{4}$$

$$x - \frac{\pi}{4} = \frac{-\pi}{2} \text{ or } \frac{3\pi}{2}$$

$$\therefore \quad x = \frac{-\pi}{4} \text{ or } \frac{7\pi}{4}$$

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Graphing the tangent function

When graphing a transformation of the tangent function:

- Determine the period.
- Determine the equations of the asymptotes.
- Determine the intercepts with the axes.

Example 13

Sketch the graph of each of the following for $x \in [-\pi, \pi]$:

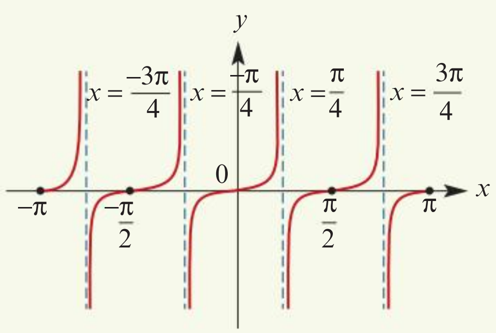
a $y = 3\tan(2x)$ **b** $y = -2\tan(3x)$

Solution

a Period $= \frac{\pi}{n} = \frac{\pi}{2}$

Asymptotes: $x = \frac{(2k+1)\pi}{4}, \ k \in \mathbb{Z}$

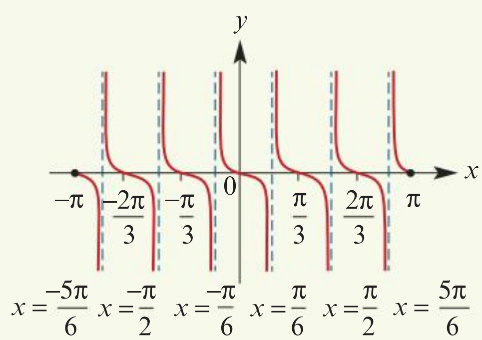
Axis intercepts: $x = \frac{k\pi}{2}, \ k \in \mathbb{Z}$



b Period $= \frac{\pi}{n} = \frac{\pi}{3}$

Asymptotes: $x = \frac{(2k+1)\pi}{6}, \ k \in \mathbb{Z}$

Axis intercepts: $x = \frac{k\pi}{3}, \ k \in \mathbb{Z}$



Summary A5

- The tangent function is given by $\tan\theta = \frac{\sin\theta}{\cos\theta}$ for $\cos\theta \neq 0$.
- The graph of $y = a\tan(nx)$, where $a \neq 0$ and $n > 0$:
  - The period is $\frac{\pi}{n}$.
  - The vertical asymptotes have equations $x = \frac{(2k+1)\pi}{2n}$ where $k \in \mathbb{Z}$.
  - The axis intercepts are at $x = \frac{k\pi}{n}$ where $k \in \mathbb{Z}$.
- Useful symmetry properties:
  - $\tan(\pi + \theta) = \tan\theta$
  - $\tan(-\theta) = -\tan\theta$

Exercise A5

SF

Example 10

1 Determine the exact value of each of the following:

a $\tan\left(\frac{5\pi}{4}\right)$ **b** $\tan\left(-\frac{2\pi}{3}\right)$ **c** $\tan\left(-\frac{29\pi}{6}\right)$

2 Determine the exact value of each of the following:

a $\tan 240°$ **b** $\tan(-150°)$ **c** $\tan 315°$

3 If $\tan x = \frac{1}{4}$ and $\pi \le x \le \frac{3\pi}{2}$, determine the exact value of:

a $\sin x$ **b** $\cos x$ **c** $\tan(-x)$ **d** $\tan(\pi - x)$

4 If $\tan x = -\frac{\sqrt{3}}{2}$ and $\frac{\pi}{2} \le x \le \pi$, determine the exact value of:

a $\sin x$ **b** $\cos x$ **c** $\tan(-x)$ **d** $\tan(x - \pi)$

Example 11

5 Solve each of the following equations for x in the stated interval:

a $\tan x = -1,\ x \in (0, 2\pi)$ **b** $\tan x = \sqrt{3},\ x \in (0, 2\pi)$

c $\tan x = \frac{1}{\sqrt{3}},\ x \in (0, 2\pi)$ **d** $\tan(2x) = 1,\ x \in (-\pi, \pi)$

e $\tan(2x) = \sqrt{3},\ x \in (-\pi, \pi)$ **f** $\tan(2x) = -\frac{1}{\sqrt{3}},\ x \in (-\pi, \pi)$

CF

Example 12

6 Solve each of the following equations for x in the stated interval:

a $\tan\left(2\left(x - \frac{\pi}{4}\right)\right) = 1,\ x \in (0, 2\pi)$ **b** $\tan\left(2\left(x - \frac{\pi}{4}\right)\right) = -1,\ x \in (-\pi, \pi)$

c $\tan\left(3\left(x - \frac{\pi}{6}\right)\right) = \sqrt{3},\ x \in (-\pi, \pi)$ **d** $\tan\left(\frac{1}{2}\left(x - \frac{\pi}{6}\right)\right) = -\frac{1}{\sqrt{3}},\ x \in (-\pi, \pi)$

Example 13

7 Sketch the graph of each of the following:

a $y = \tan(2x)$ **b** $y = \tan(3x)$ **c** $y = -\tan(2x)$

d $y = 3\tan x$ **e** $y = \tan\left(\frac{x}{2}\right)$ **f** $y = 2\tan\left(x + \frac{\pi}{4}\right)$

g $y = 3\tan x + 1$ **h** $y = 2\tan\left(x + \frac{\pi}{2}\right) + 1$ **i** $y = 3\tan\left(2\left(x - \frac{\pi}{4}\right)\right) - 2$

8 Sketch the graph of each of the following for the stated domain:

a $y = \tan\left(x + \frac{\pi}{3}\right) + \sqrt{3}$ for $x \in [0, 2\pi]$ **b** $y = \tan\left(\frac{x}{2}\right)$ for $x \in [0, 4\pi]$

c $y = \tan\left(\frac{\pi x}{2}\right)$ for $x \in [0, 4]$

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A6 General solution of trigonometric equations

Learning intentions

- To be able to obtain general solutions for trigonometric equations.

We have seen how to solve equations involving trigonometric functions over a restricted domain. We now consider the general solutions of such equations over the maximal domain for each function.

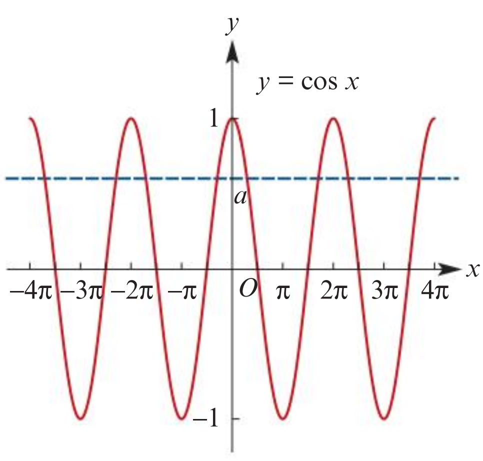
By convention:

- $\cos^{-1}$ has range $[0, \pi]$
- $\sin^{-1}$ has range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
- $\tan^{-1}$ has range $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

For example:

$$\cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}, \qquad \cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}, \qquad \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}, \qquad \sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$$

If an equation involving a trigonometric function has one or more solutions in one 'cycle', then it will have corresponding solutions in each 'cycle' of its domain, i.e. there will be infinitely many solutions.



For example, consider the equation

$$\cos x = a$$

for some fixed $a \in [-1, 1]$. The solution in the interval $[0, \pi]$ is given by

$$x = \cos^{-1}(a)$$

By the symmetry properties of the cosine function, the other solutions are given by

$$-\cos^{-1}(a),\ \pm 2\pi + \cos^{-1}(a),\ \pm 2\pi - \cos^{-1}(a),\ \pm 4\pi + \cos^{-1}(a),\ \pm 4\pi - \cos^{-1}(a),\ \dots$$

In general, we have the following:

- For $a \in [-1, 1]$, the general solution of the equation $\cos x = a$ is
  $$x = 2n\pi \pm \cos^{-1}(a), \quad \text{where } n \in \mathbb{Z}$$
- For $a \in \mathbb{R}$, the general solution of the equation $\tan x = a$ is
  $$x = n\pi + \tan^{-1}(a), \quad \text{where } n \in \mathbb{Z}$$
- For $a \in [-1, 1]$, the general solution of the equation $\sin x = a$ is
  $$x = 2n\pi + \sin^{-1}(a) \quad \text{or} \quad x = (2n+1)\pi - \sin^{-1}(a), \quad \text{where } n \in \mathbb{Z}$$

Note: An alternative and more concise way to express the general solution of $\sin x = a$ is $x = n\pi + (-1)^n \sin^{-1}(a)$, where $n \in \mathbb{Z}$.

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Example 14

Determine the general solution of each of the following equations:

a $\cos x = 0.5$ **b** $\sqrt{3}\tan(3x) = 1$ **c** $2\sin x = \sqrt{2}$

Solution

a $\cos x = 0.5$

$$x = 2n\pi \pm \cos^{-1}(0.5)$$
$$= 2n\pi \pm \frac{\pi}{3}$$
$$= \frac{(6n \pm 1)\pi}{3}, \quad n \in \mathbb{Z}$$

b $\tan(3x) = \frac{1}{\sqrt{3}}$

$$3x = n\pi + \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$
$$= n\pi + \frac{\pi}{6}$$
$$= \frac{(6n + 1)\pi}{6}$$
$$x = \frac{(6n + 1)\pi}{18}, \quad n \in \mathbb{Z}$$

c $\sin x = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$

$$x = 2n\pi + \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) \quad \text{or} \quad x = (2n + 1)\pi - \sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$$
$$= 2n\pi + \frac{\pi}{4} \qquad = (2n + 1)\pi - \frac{\pi}{4}$$
$$= \frac{(8n + 1)\pi}{4}, \quad n \in \mathbb{Z} \qquad = \frac{(8n + 3)\pi}{4}, \quad n \in \mathbb{Z}$$

Example 15

Determine the first three positive solutions of each of the following equations:

a $\cos x = 0.5$ **b** $\sqrt{3}\tan(3x) = 1$ **c** $2\sin x = \sqrt{2}$

Solution

a The general solution (from Example 14a) is given by $x = \frac{(6n \pm 1)\pi}{3}$, $n \in \mathbb{Z}$.

When $n = 0$, $x = \pm\frac{\pi}{3}$, and when $n = 1$, $x = \frac{5\pi}{3}$ or $x = \frac{7\pi}{3}$.

Thus the first three positive solutions of $\cos x = 0.5$ are $x = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}$.

b The general solution (from Example 14b) is given by $x = \frac{(6n + 1)\pi}{18}$, $n \in \mathbb{Z}$.

When $n = 0$, $x = \frac{\pi}{18}$, and when $n = 1$, $x = \frac{7\pi}{18}$, and when $n = 2$, $x = \frac{13\pi}{18}$.

Thus the first three positive solutions of $\sqrt{3}\tan(3x) = 1$ are $x = \frac{\pi}{18}, \frac{7\pi}{18}, \frac{13\pi}{18}$.

c The general solution (from Example 14c) is $x = \frac{(8n+1)\pi}{4}$ or $x = \frac{(8n+3)\pi}{4}$, $n \in \mathbb{Z}$.

When $n = 0$, $x = \frac{\pi}{4}$ or $x = \frac{3\pi}{4}$, and when $n = 1$, $x = \frac{9\pi}{4}$ or $x = \frac{11\pi}{4}$.

Thus the first three positive solutions of $2\sin x = \sqrt{2}$ are $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{9\pi}{4}$.

Example 16

Determine the general solution for each of the following:

a $\sin\left(x - \frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$

b $\tan\left(2x - \frac{\pi}{3}\right) = 1$

Solution

a $\sin\left(x - \frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$

$$x - \frac{\pi}{3} = n\pi + (-1)^n \sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$$

$$\therefore \quad x = n\pi + (-1)^n\left(\frac{\pi}{3}\right) + \frac{\pi}{3}, \quad n \in \mathbb{Z}$$

The solutions are $x = \frac{(3n+2)\pi}{3}$ for n even and $x = n\pi$ for n odd.

b $\tan\left(2x - \frac{\pi}{3}\right) = 1$

$$2x - \frac{\pi}{3} = n\pi + \frac{\pi}{4}$$

$$2x = n\pi + \frac{\pi}{4} + \frac{\pi}{3}$$

$$\therefore \quad x = \frac{1}{2}\left(n\pi + \frac{7\pi}{12}\right)$$

$$= \frac{(12n+7)\pi}{24}, \quad n \in \mathbb{Z}$$

Summary A6

- For $a \in [-1, 1]$, the general solution of the equation $\cos x = a$ is
  $$x = 2n\pi \pm \cos^{-1}(a), \quad \text{where } n \in \mathbb{Z}$$
- For $a \in \mathbb{R}$, the general solution of the equation $\tan x = a$ is
  $$x = n\pi + \tan^{-1}(a), \quad \text{where } n \in \mathbb{Z}$$
- For $a \in [-1, 1]$, the general solution of the equation $\sin x = a$ is
  $$x = 2n\pi + \sin^{-1}(a) \quad \text{or} \quad x = (2n+1)\pi - \sin^{-1}(a), \quad \text{where } n \in \mathbb{Z}$$

Skill-sheet

Exercise A6

SF

1 Evaluate each of the following for **i** $n = 1$ **ii** $n = 2$ **iii** $n = -2$:

a $2n\pi \pm \cos^{-1}(1)$ **b** $2n\pi \pm \cos^{-1}\left(-\frac{1}{2}\right)$

Example 14

2 Determine the general solution of each of the following equations:

a $\cos x = \frac{\sqrt{3}}{2}$ **b** $2\sin(3x) = \sqrt{3}$ **c** $\sqrt{3}\tan x = 3$

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Example 15

3 Determine the first two positive solutions of each of the following equations:

a $\sin x = 0.5$ **b** $2\cos(2x) = \sqrt{3}$ **c** $\sqrt{3}\tan(2x) = -3$

4 Given that a trigonometric equation has general solution $x = n\pi + (-1)^n \sin^{-1}\left(\frac{1}{2}\right)$, where $n \in \mathbb{Z}$, Determine the solutions of the equation in the interval $[-2\pi, 2\pi]$.

5 Given that a trigonometric equation has general solution $x = 2n\pi \pm \cos^{-1}\left(\frac{1}{2}\right)$, where $n \in \mathbb{Z}$, determine the solutions of the equation in the interval $[-\pi, 2\pi]$.

Example 16

6 Determine the general solution for each of the following:

a $\cos\left(2\left(x + \frac{\pi}{3}\right)\right) = \frac{1}{2}$ **b** $2\tan\left(2\left(x + \frac{\pi}{4}\right)\right) = 2\sqrt{3}$ **c** $2\sin\left(x + \frac{\pi}{3}\right) = -1$

7 Determine the general solution of $2\cos\left(2x + \frac{\pi}{4}\right) = \sqrt{2}$ and hence determine all the solutions for x in the interval $(-2\pi, 2\pi)$.

8 Determine the general solution of $\sqrt{3}\tan\left(\frac{\pi}{6} - 3x\right) - 1 = 0$ and hence determine all the solutions for x in the interval $[-\pi, 0]$.

9 Determine the general solution of $2\sin(4\pi x) + \sqrt{3} = 0$ and hence determine all the solutions for x in the interval $[-1, 1]$.

SF

A7 Applications of trigonometric functions

Learning intentions

- To be able to apply trigonometric functions to practical situations.

A **sinusoidal function** has a rule of the form $y = a\sin(nt + \varepsilon) + b$ or, equivalently, of the form $y = a\cos(nt + \varepsilon) + b$. Such functions can be used to model periodic phenomena.

Example 17

The height, $h(t)$ metres, of the tide above mean sea level at a harbour entrance over one day is given by the rule $h(t) = 3\sin\left(\frac{\pi t}{6}\right)$, where t is the number of hours after midnight.

a Draw the graph of $y = h(t)$ for $0 \le t \le 24$.

b When was high tide?

c What was the height of the high tide?

d What was the height of the tide at 8 a.m.?

e A boat can only enter the harbour when the tide is at least 1 metre above mean sea level. When could the boat enter the harbour during this particular day?

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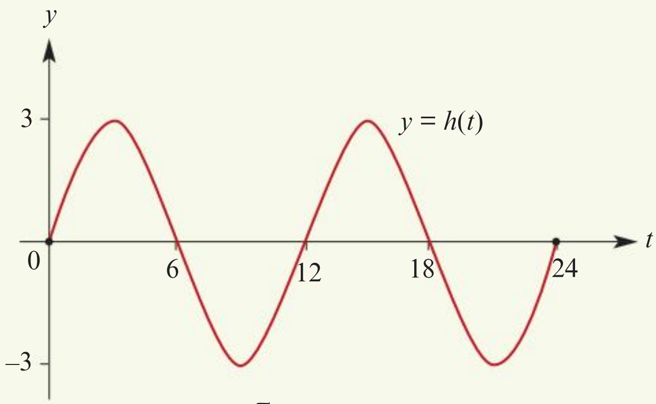
Solution

a



Note: Period $= 2\pi \div \frac{\pi}{6} = 12$

b High tide occurs when $h(t) = 3$:

$$3\sin\left(\frac{\pi t}{6}\right) = 3$$

$$\sin\left(\frac{\pi t}{6}\right) = 1$$

$$\frac{\pi t}{6} = \frac{\pi}{2}, \frac{5\pi}{2}$$

$$\therefore \quad t = 3, \ 15$$

i.e. high tide occurs at 03:00 and 15:00 (3 p.m.).

c The high tide has height 3 metres above the mean height.

d $h(8) = 3\sin\left(\frac{8\pi}{6}\right) = 3\sin\left(\frac{4\pi}{3}\right) = 3 \times \frac{-\sqrt{3}}{2} = -\frac{3\sqrt{3}}{2}$

The water is $\frac{3}{2}\sqrt{3}$ metres below the mean height at 8 a.m.

e First consider when $h(t) = 1$:

$$3\sin\left(\frac{\pi t}{6}\right) = 1$$

$$\sin\left(\frac{\pi t}{6}\right) = \frac{1}{3}$$

$$\therefore \quad t = 0.649, \ 5.351, \ 12.649, \ 17.351$$

i.e. the water is at height 1 metre at 00:39, 05:21, 12:39, 17:21.
Thus the boat can enter the harbour between 00:39 and 05:21, and between 12:39 and 17:21.

Exercise A7

Example 17

1 The number of hours of daylight at a point on the Arctic Circle is given approximately by $d = 12 - 12\cos\left(\frac{\pi}{6}\left(t + \frac{1}{3}\right)\right)$, where t is the number of months which have elapsed since 1 January.

a **i** Determine d on 21 December ($t \approx 11.7$).

ii Determine d on 21 June ($t \approx 5.7$).

b When will there be 5 hours of daylight?

2 The depth, $D(t)$ metres, of water at the entrance to a harbour at t hours after midnight on a particular day is given by $D(t) = 12 + 4\sin\left(\frac{\pi t}{6}\right)$, $0 \le t \le 24$.

a Sketch the graph of $D(t)$ for $0 \le t \le 24$.

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b Determine the values of t for which $D(t) \geq 12$.

c Boats which need a depth of w metres are permitted to enter the harbour only if the depth of the water at the entrance is at least w metres for a continuous period of 1 hour. Determine, correct to one decimal place, the largest value of w which satisfies this condition.

3 A particle moves on a straight line, OX, and its distance x metres from O at time t seconds is given by $x = 4 + 3\sin(2\pi t)$.

a Determine its greatest distance from O.

b Determine its least distance from O.

c Determine the times at which it is 7 metres from O for $0 \leq t \leq 2$.

d Determine the times at which it is 1 metre from O for $0 \leq t \leq 2$.

e Describe the motion of the particle.

4 The temperature, A°C, inside a house at t hours after 4 a.m. is given by the rule $A = 21 - 3\cos\left(\frac{\pi t}{12}\right)$, for $0 \leq t \leq 24$. The temperature, B°C, outside the house at the same time is given by $B = 22 - 5\cos\left(\frac{\pi t}{12}\right)$, for $0 \leq t \leq 24$.

a Determine the temperature inside the house at 8 a.m.

b Write down an expression for $D = A - B$, the difference between the inside and outside temperatures.

c Sketch the graph of D for $0 \leq t \leq 24$.

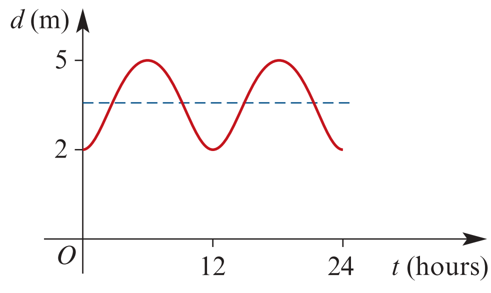
d Determine when the inside temperature is less than the outside temperature.

5 The water level on a beach wall is given by $d(t) = 6 + 4\cos\left(\frac{\pi}{6}t - \frac{\pi}{3}\right)$, where t is the number of hours after midnight and d is the depth of the water in metres.

a What is the earliest time of day at which the water is at its highest?

b When is the water 2 m up the wall?

6 The graph shows the distance, $d(t)$, of the tip of the hour hand of a large clock from the ceiling at time t hours.



a The function d is sinusoidal. Determine:

i the amplitude

ii the period

iii the rule for $d(t)$

iv the length of the hour hand.

b At what times is the distance less than 3.5 metres from the ceiling?

CF

7 In a tidal river, the time between high tide and low tide is 8 hours. The average depth of water at a point in the river is 4 metres; at high tide the depth is 5 metres.

a Sketch the graph of the depth of water at the point for the time interval from 0 to 24 hours if the relationship between time and depth is sinusoidal and there is a high tide at noon.

b If a boat requires a depth of 4 metres of water in order to sail, at what time before noon can it enter the point in the river and by what time must it leave if it is not to be stranded?

c If a boat requires a depth of 3.5 metres of water in order to sail, at what time before noon can it enter the point in the river and by what time must it leave if it is not to be stranded?

8 The population of a particular species of ant varies with time. The population, $N(t)$, at time t weeks after 1 January 2019 is given by

$$N(t) = 3000 \sin\left(\frac{\pi(t - 10)}{26}\right) + 4000$$

a For the rule $N(t)$, state:

i the period **ii** the amplitude **iii** the range.

b **i** State the values of $N(0)$ and $N(100)$.

ii Sketch the graph of $y = N(t)$ for $t \in [0, 100]$.

c Determine the values of $t \in [0, 100]$ for which the population is:

i 7000 **ii** 1000

d Determine the values of $t \in [0, 100]$ for which $N(t) > 5500$. That is, determine the intervals of time during the first 100 weeks for which the population of ants is greater than 5500.

e A second population of ants also varies with time. The rule for the population, $M(t)$, at time t weeks after 1 January 2019 is of the form

$$M(t) = a \sin\left(\frac{\pi(t - c)}{b}\right) + d$$

where a, b, c and d are positive constants. Determine a set of possible values for the constants a, b, c and d given that the population has the following properties:

- the maximum population is 40 000 and occurs at $t = 10$
- the minimum population is 10 000 and occurs at $t = 20$
- the maximum and minimum values do not occur between $t = 10$ and $t = 20$.

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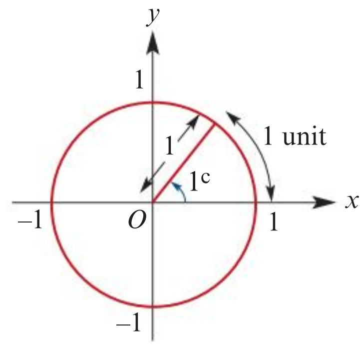
Chapter summary

■ Definition of a radian

One radian (written 1^c) is the angle formed at the centre of the unit circle by an arc of length 1 unit.

$$1^c = \frac{180^\circ}{\pi} \qquad 1^\circ = \frac{\pi^c}{180}$$



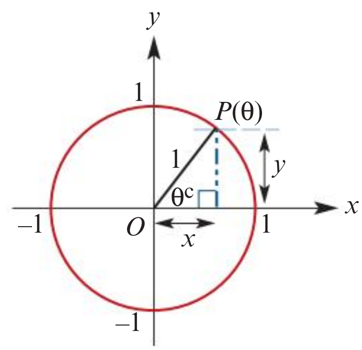
■ Sine and cosine functions

x-coordinate of $P(\theta)$ on unit circle:

$$x = \cos\theta, \qquad \theta \in \mathbb{R}$$

y-coordinate of $P(\theta)$ on unit circle:

$$y = \sin\theta, \qquad \theta \in \mathbb{R}$$



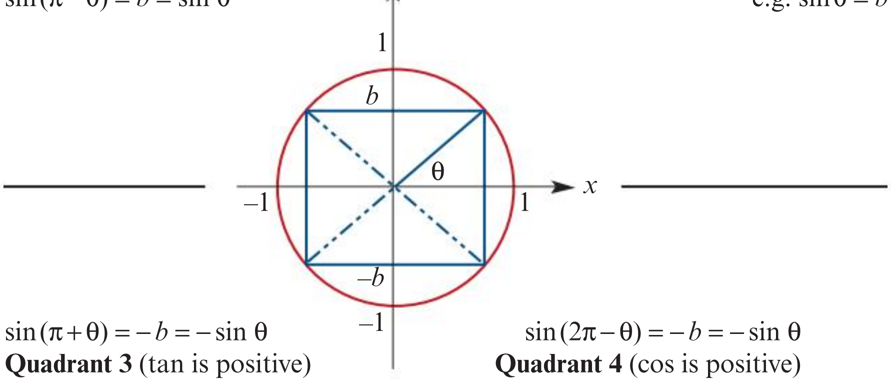
■ Tangent function

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \quad \text{for } \cos\theta \neq 0$$

■ Symmetry properties of trigonometric functions

Quadrant 2 (sin is positive)
$\sin(\pi - \theta) = b = \sin\theta$

Quadrant 1 (all functions are positive)
e.g. $\sin\theta = b$



$\sin(\pi + \theta) = -b = -\sin\theta$
Quadrant 3 (tan is positive)

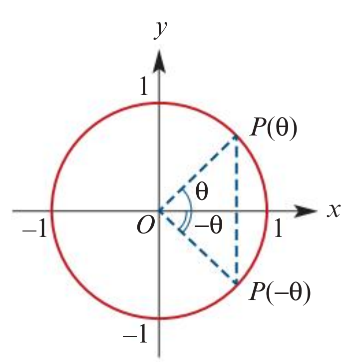
$\sin(2\pi - \theta) = -b = -\sin\theta$
Quadrant 4 (cos is positive)

Negative angles:

$$\sin(-\theta) = -\sin\theta$$

$$\cos(-\theta) = \cos\theta$$

$$\tan(-\theta) = \frac{-\sin\theta}{\cos\theta} = -\tan\theta$$



Review

Complementary relationships:

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta, \qquad \sin\left(\frac{\pi}{2} + \theta\right) = \cos\theta$$

$$\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta, \qquad \cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta$$

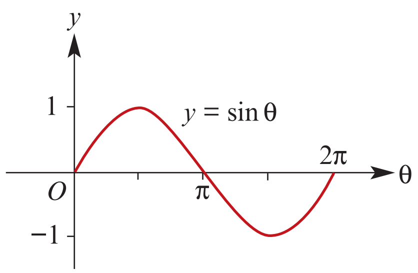
- **Pythagorean identity**

$$\cos^2\theta + \sin^2\theta = 1$$

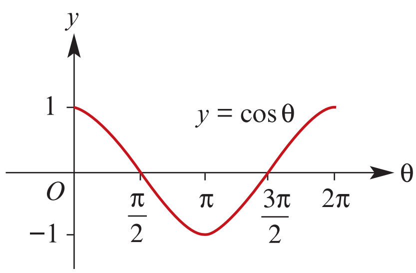
- **Exact values of trigonometric functions**

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
0	0	1	0
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$\frac{\pi}{2}$	1	0	undefined

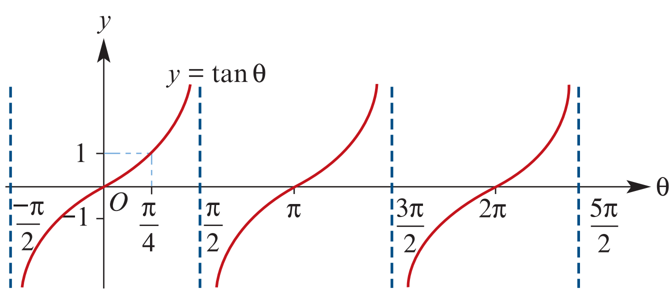
- **Graphs of trigonometric functions**



Amplitude = 1
Period = 2π



Amplitude = 1
Period = 2π



Amplitude is undefined
Period = π

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Review

Transformations of the graphs of trigonometric functions

For the graphs of $y = a\sin(n(x + \varepsilon)) + b$ and $y = a\cos(n(x + \varepsilon)) + b$, where $a, n \in \mathbb{R}^+$:

- Period $= \dfrac{2\pi}{n}$
- Amplitude $= a$
- Range $= [-a + b, a + b]$

For the graph of $y = a\tan(n(x + \varepsilon)) + b$, where $n \in \mathbb{R}^+$:

- Period $= \dfrac{\pi}{n}$
- Asymptotes: $x = \dfrac{(2k+1)\pi}{2n} - \varepsilon$, where $k \in \mathbb{Z}$

General solution of trigonometric equations

- For $a \in [-1, 1]$, the general solution of the equation $\cos x = a$ is

$$x = 2n\pi \pm \cos^{-1}(a), \quad \text{where } n \in \mathbb{Z}$$

- For $a \in \mathbb{R}$, the general solution of the equation $\tan x = a$ is

$$x = n\pi + \tan^{-1}(a), \quad \text{where } n \in \mathbb{Z}$$

- For $a \in [-1, 1]$, the general solution of the equation $\sin x = a$ is

$$x = 2n\pi + \sin^{-1}(a) \quad \text{or} \quad x = (2n+1)\pi - \sin^{-1}(a), \quad \text{where } n \in \mathbb{Z}$$

Skills checklist

Check-list

Download this checklist from the Interactive Textbook, then print it and fill it out to check your skills. ✓

A1 **1** I can convert radians to degrees and vice versa. ☐

See Example 1 and Questions 1 and 2

A1 **2** I can determine exact values of sine and cosine for integer multiples of $\frac{\pi}{2}$. ☐

See Example 2 and Question 3

A2 **3** I can determine exact values of sine and cosine functions for integer multiples of $\frac{\pi}{6}$ and $\frac{\pi}{4}$. ☐

See Example 3, Example 4 and Questions 1 and 2

A2 **4** I can use symmetry properties to determine exact values of trigonometric functions. ☐

See Example 5, Example 6 and Questions 3 and 4

A3 **5** I can solve trigonometric equations. ☐

See Example 7, Example 8 and Questions 1 and 2

A4 **6** I can apply transformations to the graphs of the sine and cosine functions. ☐

See Example 9 and Question 1

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Review

A5 **7** I can determine exact values of the tangent function for integer multiples of $\frac{\pi}{6}$ and $\frac{\pi}{4}$. ☐

See Example 10 and Question 1

A5 **8** I can solve trigonometric equations involving the tangent function. ☐

See Example 11, Example 12 and Questions 5 and 6

A5 **9** I can sketch the graphs of transformations of the graph of the tangent function. ☐

See Example 11, Example 13 and Question 7

A6 **10** I can determine the general solution of trigonometric functions. ☐

See Example 14, Example 15, Example 16 and Questions 2, 3 and 6

A7 **11** I can use trigonometric functions in the modelling of practical situations. ☐

See Example 17 and Question 1

Short-response questions

Technology-free short-response questions

SF

1 Convert each of the following to radian measure in terms of π:

a $390°$ **b** $840°$ **c** $1110°$ **d** $1065°$ **e** $165°$

f $450°$ **g** $420°$ **h** $390°$ **i** $40°$

2 Convert each of the following to degree measure:

a $\frac{11\pi}{6}$ **b** $\frac{17\pi}{4}$ **c** $\frac{9\pi}{4}$ **d** $\frac{7\pi}{12}$ **e** $\frac{17\pi}{2}$

f $-\frac{11\pi}{4}$ **g** $-\frac{5\pi}{4}$ **h** $-\frac{13\pi}{4}$ **i** $\frac{23\pi}{4}$

3 Give the exact value of each of the following:

a $\sin\left(\frac{9\pi}{4}\right)$ **b** $\cos\left(\frac{-5\pi}{4}\right)$ **c** $\sin\left(\frac{3\pi}{2}\right)$ **d** $\cos\left(\frac{-3\pi}{2}\right)$

e $\cos\left(\frac{11\pi}{6}\right)$ **f** $\sin\left(\frac{21\pi}{6}\right)$ **g** $\tan\left(\frac{-25\pi}{3}\right)$ **h** $\tan\left(\frac{-15\pi}{4}\right)$

4 State the amplitude and period of each of the following:

a $4\sin\left(\frac{\theta}{2}\right)$ **b** $-5\sin(6\theta)$ **c** $\frac{1}{3}\sin(4\theta)$

d $-2\cos(5x)$ **e** $-7\sin\left(\frac{\pi x}{4}\right)$ **f** $\frac{2}{3}\sin\left(\frac{2\pi x}{3}\right)$

5 Determine the maximum and minimum values of the function with rule:

a $3 + 2\sin\theta$ **b** $4 - 5\cos\theta$

6 Sketch the graph of each of the following (showing one cycle):

a $y = 2\cos(2x)$ **b** $y = -3\sin\left(\frac{x}{3}\right)$ **c** $y = -2\cos(3x)$

d $y = 2\cos\left(\frac{x}{3}\right)$ **e** $y = \cos\left(x - \frac{\pi}{4}\right)$ **f** $y = \cos\left(x + \frac{2\pi}{3}\right)$

g $y = 2\sin\left(x - \frac{5\pi}{6}\right)$ **h** $y = -3\sin\left(x + \frac{5\pi}{6}\right)$

7 Solve each of the following equations:

a $\cos\theta = -\frac{\sqrt{3}}{2}$ for $\theta \in [-\pi, \pi]$ **b** $\cos(2\theta) = -\frac{\sqrt{3}}{2}$ for $\theta \in [-\pi, \pi]$

c $\cos\left(\theta - \frac{\pi}{3}\right) = -\frac{1}{2}$ for $\theta \in [0, 2\pi]$ **d** $\cos\left(\theta + \frac{\pi}{3}\right) = -1$ for $\theta \in [0, 2\pi]$

e $\cos\left(\frac{\pi}{3} - \theta\right) = -\frac{1}{2}$ for $\theta \in [0, 2\pi]$

8 Sketch the graph of each of the following for $x \in [-\pi, 2\pi]$:

a $f(x) = 2\cos(2x) + 1$ **b** $f(x) = 1 - 2\cos(2x)$ **c** $f(x) = 3\sin\left(x + \frac{\pi}{3}\right)$

d $f(x) = 2 - \sin\left(x + \frac{\pi}{3}\right)$ **e** $f(x) = 1 - 2\cos(3x)$

9 Solve each of the following for $x \in [0, 2\pi]$:

a $\tan x = -\sqrt{3}$ **b** $\tan\left(3x - \frac{\pi}{6}\right) = \frac{\sqrt{3}}{3}$

c $2\tan\left(\frac{x}{2}\right) + 2 = 0$ **d** $3\tan\left(\frac{\pi}{2} + 2x\right) = -3$

10 Sketch the graph of each of the following for $x \in [0, \pi]$, clearly labelling all intercepts with the axes and all asymptotes:

a $f(x) = \tan(2x)$ **b** $f(x) = \tan\left(x - \frac{\pi}{3}\right)$

c $f(x) = 2\tan\left(2x + \frac{\pi}{3}\right)$ **d** $f(x) = 2\tan\left(2x + \frac{\pi}{3}\right) - 2$

11 Determine the values of $\theta \in [0, 2\pi]$ for which:

a $\sin^2\theta = \frac{1}{4}$ **b** $\sin(2\theta) = \frac{1}{2}$ **c** $\cos(3\theta) = \frac{\sqrt{3}}{2}$ **d** $\sin^2(2\theta) = 1$

12 Solve the equation $\tan(\theta^\circ) = 2\sin(\theta^\circ)$ for values of θ° from 0° to 360°.

13 Determine the general solution for each of the following:

a $\sin(2x) = -1$ **b** $\cos(3x) = 1$ **c** $\tan x = -1$

Technology-active short-response questions

14 The temperature, T°C, in a small town in the mountains over a day is modelled by the function with rule

$$T = 15 - 8\cos\left(\frac{\pi t}{12} + 6\right), \quad 0 \le t \le 24$$

where t is the time in hours after midnight.

a What is the temperature at midnight, correct to two significant figures?

b What are the maximum and minimum temperatures reached?

c At what times of the day, to the nearest minute, are temperatures warmer than 20°C?

d Sketch the graph for the temperatures over a day.

15 A particle oscillates back and forth, in a straight line, between points A and B about a point O. Its position, $x(t)$ metres, relative to O at time t seconds is given by the rule $x(t) = 3\sin(2\pi t - a)$. The position of the particle when $t = 1$ is $x = -1.5$.

a If $a \in \left[0, \frac{\pi}{2}\right]$, determine the value of a.

b Sketch the graph of $x(t)$ against t for $t \in [0, 2]$. Label the maximum and minimum points, the axis intercepts and the endpoints with their coordinates.

c How far from O is point A?

d At what time does the particle first pass through A?

e How long is it before the particle returns to A?

f How long does it take for the particle to go from A to O?

g How far does the particle travel in:

i the first 2 seconds of its motion

ii the first 2.5 seconds of its motion?

16 The depth, D metres, of sea water in a bay t hours after midnight on a particular day may be represented by a function with rule

$$D(t) = a + b\cos\left(\frac{2\pi t}{k}\right)$$

where a, b and k are real numbers with $k > 0$. The water is at a maximum depth of 15.4 metres at midnight and noon, and is at a minimum depth of 11.4 metres at 6 a.m. and 6 p.m.

a Determine the value of:

i a **ii** b **iii** k

b Determine the times when the depth of the water is 13.4 metres.

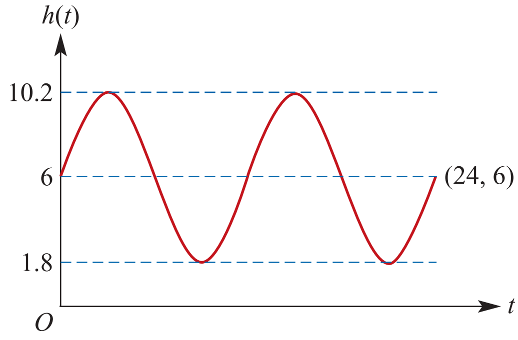
c Determine the values of t for which the depth of the water is less than 14.4 metres.

CF

17 The depth of water, $h(t)$ m, at a particular jetty in a harbour at time t hours after midnight is given by the rule

$$h(t) = p + q\sin\left(\frac{\pi t}{6}\right)$$

for constants p and q. The graph of $h(t)$ against t for $t \in [0, 24]$ is shown. The maximum depth is 10.2 m and the minimum depth is 1.8 m.



a Determine the values of p and q.

b At what times during the time interval $[0, 24]$ is the depth of water at a maximum?

c What is the average depth of the water over the time interval $[0, 24]$?

d At what times during the time interval $[0, 24]$ is the depth of the water 3.9 m?

e For how long during the 24-hour period from midnight is the depth of the water more than 8.1 m?

18 Consider the function with rule $f(x) = 2\sin(3x) + 1$ for $x \in [0, 2\pi]$.

a Determine the values of k such that the equation $f(x) = k$ has:

i six solutions for $x \in [0, 2\pi]$

ii three solutions for $x \in [0, 2\pi]$

iii no solutions for $x \in [0, 2\pi]$.

b Determine a sequence of transformations which takes the graph of $y = f(x)$ to the graph of $y = \sin x$.

c Determine the values of $h \in [0, 2\pi]$ such that the graph of $y = f(x + h)$ has:

i a maximum at the point $\left(\frac{\pi}{3}, 3\right)$

ii a minimum at the point $\left(\frac{\pi}{3}, -1\right)$.

CU

19 The population, N, of a particular species of ant varies with the seasons. The population is modelled by the equation

$$N = 3000\sin\left(\frac{\pi(t-1)}{6}\right) + 4000$$

where t is the number of months after 1 January in a given year. The population, M, of a second species of ant also varies with time. Its population is modelled by the equation

$$M = 3000\sin\left(\frac{\pi(t-3.5)}{5}\right) + 5500$$

where t is again the number of months after 1 January in a given year.

Use your calculator to plot the graphs of both equations over a period of one year on the same axes, and hence:

a Determine the maximum and minimum populations of both species and the months in which those maximums and minimums occur

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Review

CU

b Determine the months of the year during which the populations of both species are equal and determine the population of each species at that time

c by formulating a third equation, determine when the combined population of species N and M is at a maximum and determine this maximum value

d by formulating a fourth equation, determine when the difference between the two populations is at a maximum.

20 Passengers on a ferris wheel access their seats from a platform 5 m above the ground. As each seat is filled, the ferris wheel moves around so that the next seat can be filled. Once all seats are filled, the ride begins and lasts for 6 minutes. The height, h m, of Isobel's seat above the ground t seconds after the ride has begun is given by $h = 15\sin(10t - 45)^\circ + 16.5$.

a Use a calculator to sketch the graph of h against t for the first 2 minutes of the ride.

b How far above the ground is Isobel's seat at the commencement of the ride?

c After how many seconds does Isobel's seat pass the access platform?

d How many times will her seat pass the access platform in the first 2 minutes?

e How many times will her seat pass the access platform during the entire ride?

Due to a malfunction, the ferris wheel stops abruptly 1 minute 40 seconds into the ride.

f How far above the ground is Isobel stranded?

g If Isobel's brother Hamish had a seat 1.5 m above the ground at the commencement of the ride, how far above the ground is Hamish stranded?

Multiple-choice questions

Technology-free multiple-choice questions

1 The period of the graph of $y = 2\sin(3x - \pi) + 4$ is

A $\dfrac{2\pi}{3}$ **B** 2 **C** 3 **D** π

2 The amplitude of the graph of $y = -5\cos(5x) + 3$ is

A -5 **B** -2 **C** 2 **D** 5

3 The number of solutions of $5\sin(2x - \pi) + 2 = 0$ in the interval $[0, 2\pi]$ is

A 1 **B** 2 **C** 3 **D** 4

4 The solutions of $2\sin(3x) + \sqrt{2} = 0$ in the interval $\left(\dfrac{5\pi}{12}, \dfrac{23\pi}{12}\right)$ are

A $\dfrac{5\pi}{12}, \dfrac{7\pi}{12}, \dfrac{13\pi}{12}, \dfrac{5\pi}{4}, \dfrac{7\pi}{4}, \dfrac{23\pi}{4}$

B 1.83, 3.40, 3.93, 5.50

C $\dfrac{4\pi}{9}, \dfrac{5\pi}{9}, \dfrac{10\pi}{9}, \dfrac{11\pi}{9}, \dfrac{16\pi}{9}, \dfrac{17\pi}{9}$

D $\dfrac{7\pi}{12}, \dfrac{13\pi}{12}, \dfrac{5\pi}{4}, \dfrac{7\pi}{4}$

5 $\cos\left(-\frac{13\pi}{6}\right)$ is equal to

A $-\frac{\sqrt{3}}{2}$ **B** $\cos\left(-\frac{7\pi}{6}\right)$ **C** $-\frac{1}{2}$ **D** $\sin\left(\frac{2\pi}{3}\right)$

6 $\tan(180 - \theta)^\circ$ is equal to

A $\frac{\cos(180 - \theta)^\circ}{\sin(180 - \theta)^\circ}$ **B** $\frac{\sin(90 - \theta)^\circ}{\cos(90 + \theta)^\circ}$ **C** $\frac{\cos(90 - \theta)^\circ}{\sin(90 + \theta)^\circ}$ **D** $\frac{\cos(90 + \theta)^\circ}{\sin(90 - \theta)^\circ}$

7 $\sin\left(\frac{\pi}{2} - x\right)$ is not equal to

A $\cos(2\pi - x)$ **B** $-\sin\left(\frac{3\pi}{2} + x\right)$ **C** $\sin x$ **D** $\cos(-x)$

8 Given that $\sin a^\circ = b$, which of the following is equal to b?

A $\sin(90 + a)^\circ$ **B** $\sin(90 - a)^\circ$ **C** $\sin(180 + a)^\circ$ **D** $\sin(180 - a)^\circ$

Technology-active multiple-choice questions

9 The period of the graph of $f(x) = 4\sin(3\pi x) - 3\cos(2\pi x)$ is

A 1 **B** 2 **C** 3 **D** 4

10 A solution of the equation $\sin(2\theta) - a\cos(2\theta) = 0$ is $\frac{\pi}{12}$. The value of a is

A $\frac{\sqrt{3}}{3}$ **B** 1 **C** $\sqrt{3}$ **D** $\frac{2\sqrt{3}}{3}$

11 The sum of the solutions of the equation $6\sin(2x) = 3$, where $0 \le x \le 2\pi$, is

A $\frac{\pi}{2}$ **B** 4π **C** π **D** 3π

12 The sum of the solutions of the equation $\sin(3x) = -\frac{1}{2}$, where $0 \le x \le 2\pi$, is

A 7π **B** 4π **C** 5π **D** 3π

13 Let $n \in \mathbb{Z}$. The general solution of the equation $\cos 2x = \frac{\sqrt{3}}{2}$ is

A $\frac{(12n \pm 1)\pi}{12}$ **B** $\frac{(3n \pm 1)\pi}{3}$ **C** $\frac{(6n \pm 1)\pi}{6}$ **D** $(2n \pm 1)\pi$

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Appendix B

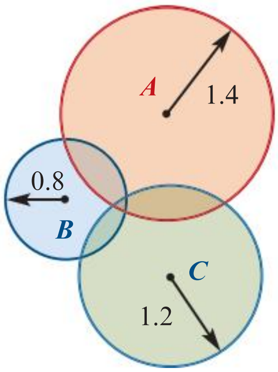
Further graphing techniques

Chapter contents

This chapter is provided as preparation for Units 3 *and* 4.

The extensive use of mobile phones has led to an increased awareness of potential threats to the privacy of their users. For example, a little basic mathematics can be employed to track the movements of someone in possession of a mobile phone.

Suppose that there are three transmission towers within range of your mobile phone. By measuring the time taken for signals to travel between your phone and each transmission tower, it is possible to estimate the distance from your phone to each tower.



In the diagram, there are transmission towers at points A, B and C. If it is estimated that a person is no more than 1.4 km from A, no more than 0.8 km from B and no more than 1.2 km from C, then the person can be located in the intersection of the three circles.

In this chapter, we will look at different ways of describing circles and various other figures.

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B1 Locus of points

Learning intentions

- To be able to describe algebraically a set of points defined by a geometric condition.

Until now, all the curves we have studied have been described by an algebraic relationship between the x- and y-coordinates, such as $y = x^2 + 1$. In this section, we are interested in sets of points described by a geometric condition. A set described in this way is often called a **locus**. Many of these descriptions will give curves that are already familiar.

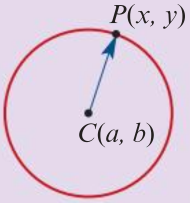
Circles

Circles have a very simple geometric description.

Locus definition of a circle

A **circle** is the locus of a point $P(x, y)$ that moves so that its distance from a fixed point $C(a, b)$ is constant.

Note: The constant distance is called the **radius** and the fixed point $C(a, b)$ is called the **centre** of the circle.



This definition can be used to determine the equation of a circle.

Recall that the distance between points $A(x_1, y_1)$ and $B(x_2, y_2)$ is given by

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Let r be the radius of the circle. Then

$$\begin{aligned} CP &= r \\ \sqrt{(x-a)^2 + (y-b)^2} &= r \\ (x-a)^2 + (y-b)^2 &= r^2 \end{aligned}$$

The circle with radius r and centre $C(a, b)$ has equation

$$(x-a)^2 + (y-b)^2 = r^2$$

Example 1

a Determine the locus of points $P(x, y)$ whose distance from $C(2, -1)$ is 3.

b Determine the centre and radius of the circle with equation $x^2 + 2x + y^2 - 4y = 1$.

Solution

a We know that the point $P(x, y)$ satisfies

$$\begin{aligned} CP &= 3 \\ \sqrt{(x-2)^2 + (y+1)^2} &= 3 \\ (x-2)^2 + (y+1)^2 &= 3^2 \end{aligned}$$

This is a circle with centre $(2, -1)$ and radius 3.

b We must complete the square in both variables. This gives

$$x^2 + 2x + y^2 - 4y = 1$$
$$(x^2 + 2x + 1) - 1 + (y^2 - 4y + 4) - 4 = 1$$
$$(x + 1)^2 + (y - 2)^2 = 6$$

Therefore the centre of the circle is $(-1, 2)$ and its radius is $\sqrt{6}$.

Straight lines

You have learned in previous years that a straight line is the set of points (x, y) satisfying

$$ax + by = c$$

for some constants a, b, c with $a \neq 0$ or $b \neq 0$.

Lines can also be described geometrically as follows.

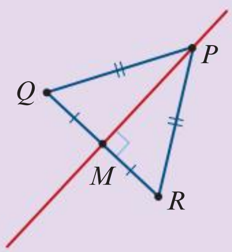
Locus definition of a straight line

Suppose that points Q and R are fixed.

A **straight line** is the locus of a point P that moves so that its distance from Q is the same as its distance from R. That is,

$$QP = RP$$

We can say that point P is **equidistant** from points Q and R.



Note: This straight line is the **perpendicular bisector** of line segment QR. To see this, we note that the midpoint M of QR is on the line. If P is any other point on the line, then

$$QP = RP, \quad QM = RM \quad \text{and} \quad MP = MP$$

and so $\triangle QMP$ is congruent to $\triangle RMP$. Therefore $\angle QMP = \angle RMP = 90°$.

Example 2

a Determine the locus of points $P(x, y)$ that are equidistant from the points $Q(1, 1)$ and $R(3, 5)$.

b Show that this is the perpendicular bisector of line segment QR.

Solution

a We know that the point $P(x, y)$ satisfies

$$QP = RP$$
$$\sqrt{(x-1)^2 + (y-1)^2} = \sqrt{(x-3)^2 + (y-5)^2}$$
$$(x-1)^2 + (y-1)^2 = (x-3)^2 + (y-5)^2$$
$$x + 2y = 8$$
$$y = -\tfrac{1}{2}x + 4$$

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b This line has gradient $-\frac{1}{2}$. The line through $Q(1, 1)$ and $R(3, 5)$ has gradient $\dfrac{5-1}{3-1} = 2$.

Because the product of the two gradients is -1, the two lines are perpendicular.

We also need to check that the line $y = -\frac{1}{2}x + 4$ passes through the midpoint of QR, which is $(2, 3)$. When $x = 2$, $y = -\frac{1}{2} \times 2 + 4 = 3$. Thus $(2, 3)$ is on the line.

Summary B1

- A **locus** is the set of points described by a geometric condition.
- A **circle** is the locus of a point P that moves so that its distance from a fixed point C is constant.
- A **straight line** is the locus of a point P that moves so that it is equidistant from two fixed points Q and R.

Skill-sheet

Exercise B1

Example 1

1 Determine the locus of points $P(x, y)$ whose distance from $Q(1, -2)$ is 4. SF

2 Determine the locus of points $P(x, y)$ whose distance from $Q(-4, 3)$ is 5.

Example 2

3 **a** Determine the locus of points $P(x, y)$ that are equidistant from $Q(-1, -1)$ and $R(1, 1)$.
b Show that this is the perpendicular bisector of line segment QR.

4 **a** Determine the locus of points $P(x, y)$ that are equidistant from $Q(0, 2)$ and $R(1, 0)$.
b Show that this is the perpendicular bisector of line segment QR.

5 Point P is equidistant from points $Q(0, 1)$ and $R(2, 3)$. Moreover, its distance from point $S(3, 3)$ is 3. Determine the possible coordinates of P.

6 Point P is equidistant from points $Q(0, 1)$ and $R(2, 0)$. Moreover, it is also equidistant from points $S(-1, 0)$ and $T(0, 2)$. Determine the coordinates of P.

7 A valuable item is buried in a forest. It is 10 metres from a tree stump located at coordinates $T(0, 0)$ and 2 metres from a rock at coordinates $R(6, 10)$. determine the possible coordinates of the buried item.

8 Consider the three points $R(4, 5)$, $S(6, 1)$ and $T(1, -4)$. CF
a Determine the locus of points $P(x, y)$ that are equidistant from the points R and S.
b Determine the locus of points $P(x, y)$ that are equidistant from the points S and T.
c Hence determine the point that is equidistant from the points R, S and T.
d Hence determine the equation of the circle through the points R, S and T.

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9 Given two fixed points $A(0, 1)$ and $B(2, 5)$, determine the locus of P if the gradient of AB equals that of BP.

10 A triangle OAP has vertices $O(0, 0)$, $A(4, 0)$ and $P(x, y)$, where $y > 0$. The triangle has area 12 square units. Determine the locus of P.

11 **a** Determine the locus of a point $P(x, y)$ that moves so that its distance from the origin is equal to the sum of its x- and y-coordinates.

b Determine the locus of a point $P(x, y)$ that moves so that the *square* of its distance from the origin is equal to the sum of its x- and y-coordinates.

12 $A(0, 0)$ and $B(3, 0)$ are two vertices of a triangle ABP. The third vertex P is such that $AP : BP = 3$. Determine the locus of P.

13 Determine the locus of the point P that moves so that its distance from the line $y = 3$ is always 2 units.

14 A steel pipe is too heavy to drag, but can be lifted at one end and rotated about its opposite end. How many moves are required to rotate the pipe into the parallel position indicated by the dotted line? The distance between the parallel lines is less than the length of the pipe.

B2 Parabolas

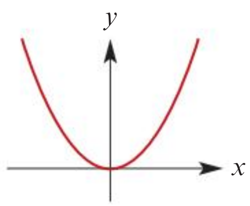
Learning intentions

- To be able to work with the locus definition of a parabola.

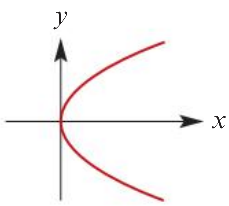
The parabola has been studied since antiquity and is admired for its range of applications, one of which we will explore at the end of this section.

The standard form of a parabola is $y = ax^2$.

Rotating the figure by 90° gives a parabola with equation $x = ay^2$.



$y = ax^2$



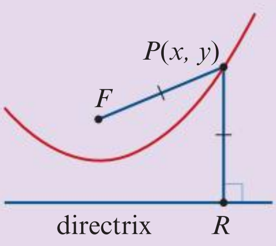
$x = ay^2$

The parabola can also be defined geometrically.

Locus definition of a parabola

A **parabola** is the locus of a point P that moves so that its distance from a fixed point F is equal to its perpendicular distance from a fixed line.

Note: The fixed point is called the **focus** and the fixed line is called the **directrix**.

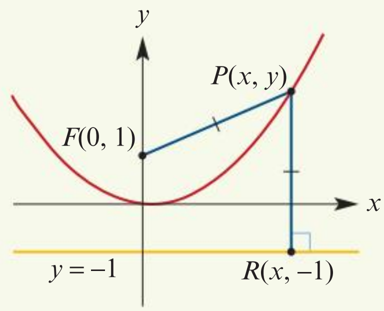


Example 3

Verify that the set of all points $P(x, y)$ that are equidistant from the point $F(0, 1)$ and the line $y = -1$ is a parabola.

Solution

We know that the point $P(x, y)$ satisfies



$$
\begin{aligned}
FP &= RP \\
\sqrt{x^2 + (y-1)^2} &= \sqrt{(y-(-1))^2} \\
x^2 + (y-1)^2 &= (y+1)^2 \\
x^2 + y^2 - 2y + 1 &= y^2 + 2y + 1 \\
x^2 - 2y &= 2y \\
x^2 &= 4y \\
y &= \frac{x^2}{4}
\end{aligned}
$$

Therefore the set of points is the parabola with equation $y = \dfrac{x^2}{4}$.

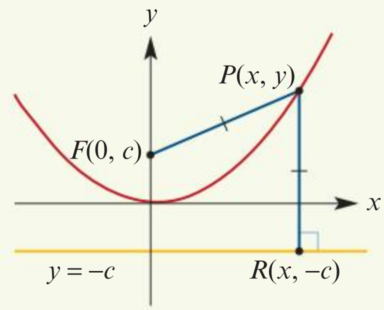
Example 4

a Determine the equation of the parabola with focus $F(0, c)$ and directrix $y = -c$.

b Hence determine the focus of the parabola with equation $y = 2x^2$.

Solution

a A point $P(x, y)$ on the parabola satisfies



$$
\begin{aligned}
FP &= RP \\
\sqrt{x^2 + (y-c)^2} &= \sqrt{(y-(-c))^2} \\
x^2 + (y-c)^2 &= (y+c)^2 \\
x^2 + y^2 - 2cy + c^2 &= y^2 + 2cy + c^2 \\
x^2 - 2cy &= 2cy \\
x^2 &= 4cy
\end{aligned}
$$

The parabola has equation $4cy = x^2$.

b Since $\dfrac{y}{2} = x^2$, we solve $\dfrac{1}{2} = 4c$, giving $c = \dfrac{1}{8}$.

Hence the focus is $F\left(0, \dfrac{1}{8}\right)$.

In the previous example, we proved the following result:

The parabola with focus $F(0, c)$ and directrix $y = -c$ has equation $4cy = x^2$.

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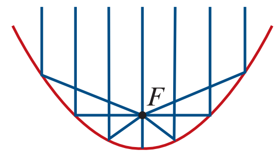
A remarkable application

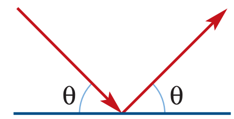
Parabolas have a remarkable property that makes them extremely useful. Light travelling parallel to the axis of symmetry of a reflective parabola is always reflected to its focus.



Parabolas can therefore be used to make reflective telescopes. Low intensity signals from outer space will reflect off the dish and converge at a receiver located at the focus.

To see how this works, we require a simple law of physics:

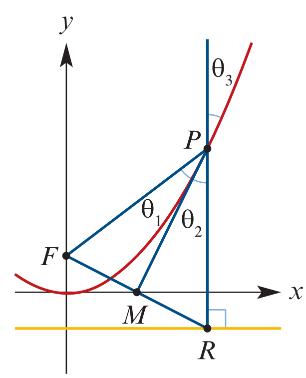
- When light is reflected off a surface, the angle between the ray and the tangent to the surface is preserved after reflection.



Reflective property of the parabola

Any ray of light parallel to the axis of symmetry of the parabola that reflects off the parabola at point P will pass through the focus at F.

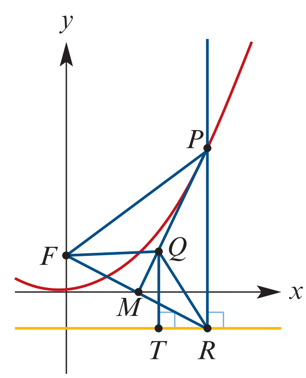
Proof Since point P is on the parabola, the distance to the focus F is the same as the distance to the directrix. Therefore $FP = RP$, and so $\triangle FPR$ is isosceles.



Let M be the midpoint of FR. Then $\triangle FMP$ is congruent to $\triangle RMP$ (by SSS). Therefore MP is the perpendicular bisector of FR and

$$\begin{aligned} \theta_1 &= \theta_2 && (\text{as } \triangle FMP \equiv \triangle RMP) \\ &= \theta_3 && (\text{vertically opposite angles}) \end{aligned}$$

However, we also need to ensure that line MP is tangent to the parabola. To see this, we will show that point P is the only point common to the parabola and line MP.



Take any other point Q on line MP. Suppose that point T is the point on the directrix closest to Q. Then

$$FQ = RQ > TQ$$

and so point Q is not on the parabola.

Summary B2

- A **parabola** is the locus of a point P that moves so that its distance from a fixed point F is equal to its perpendicular distance from a fixed line.
- The fixed point is called the **focus** and the fixed line is called the **directrix**.
- The parabola with equation $4cy = x^2$ has focus $F(0, c)$ and directrix $y = -c$.

Exercise B2

Example 3 **1** Determine the equation of the locus of points $P(x, y)$ whose distance to the point $F(0, 3)$ is equal to the perpendicular distance to the line with equation $y = -3$.

2 Find the equation of the locus of points $P(x, y)$ whose distance to the point $F(0, -4)$ is equal to the perpendicular distance to the line with equation $y = 2$.

3 Determine the equation of the locus of points $P(x, y)$ whose distance to the point $F(2, 0)$ is equal to the perpendicular distance to the line with equation $x = -4$.

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Example 4 **4** **a** Determine the equation of the parabola with focus $F(c, 0)$ and directrix $x = -c$.
b Hence determine the focus of the parabola with equation $x = 3y^2$.

5 **a** Determine the equation of the locus of points $P(x, y)$ whose distance to the point $F(a, b)$ is equal to the perpendicular distance to the line with equation $y = c$.
b Hence determine the equation of the parabola with focus $(1, 2)$ and directrix $y = 3$.

6 A parabola goes through the point $P(7, 9)$ and its focus is $F(1, 1)$. The axis of symmetry of the parabola is $x = 1$. Determine the equation of its directrix.
Hint: The directrix will be a horizontal line, $y = c$. Expect to determine two answers.

7 A parabola goes through the point $(1, 1)$, its axis of symmetry is the line $x = 2$ and its directrix is the line $y = 3$. Determine the coordinates of its focus.
Hint: The focus must lie on the axis of symmetry.

CF

B3 Ellipses

Learning intentions

- To be able to work with the locus definition of an ellipse.

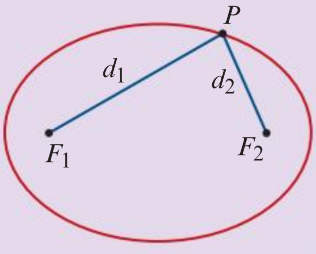
A ball casts a shadow that looks like a squashed circle. This figure – called an ellipse – is of considerable geometric significance. For instance, the planets in our solar system have elliptic orbits.

Locus definition of an ellipse

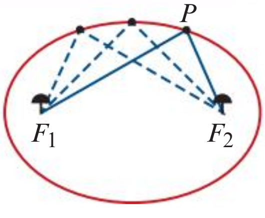
An **ellipse** is the locus of a point P that moves so that the sum of its distances from two fixed points F_1 and F_2 is a constant. That is,

$$F_1P + F_2P = k$$

Note: Points F_1 and F_2 are called the **foci** of the ellipse.

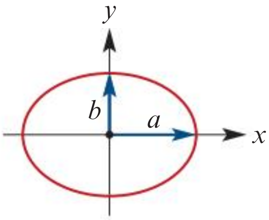


Drawing an ellipse An ellipse can be drawn by pushing two pins into paper. These will be the foci. A string of length k is tied to each of the two pins and the tip of a pen is used to pull the string taut and form a triangle. The pen will trace an ellipse if it is moved around the pins while keeping the string taut.



Cartesian equations of ellipses

In Chapter 18, we obtained ellipses as transformations of circles. The standard form of the Cartesian equation of an ellipse centred at the origin is



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

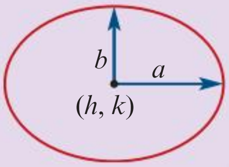
This ellipse has x-axis intercepts $\pm a$ and y-axis intercepts $\pm b$.

Applying the translation defined by $(x, y) \to (x + h, y + k)$, we obtain:

The graph of

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

is an ellipse centred at the point (h, k).



Example 5

Sketch the graph of each ellipse:

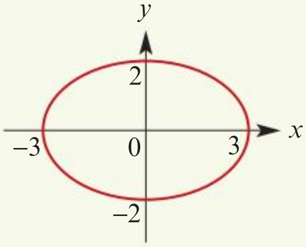
a $\frac{x^2}{9} + \frac{y^2}{4} = 1$ **b** $4x^2 + 9y^2 = 1$ **c** $\frac{(x-1)^2}{4} + \frac{(y+2)^2}{9} = 1$

Solution

a The equation can be written as

$$\frac{x^2}{3^2} + \frac{y^2}{2^2} = 1$$

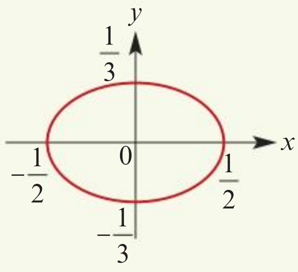
This is an ellipse with centre $(0, 0)$ and axis intercepts at $x = \pm 3$ and $y = \pm 2$.



b The equation can be written as

$$\frac{x^2}{(\frac{1}{2})^2} + \frac{y^2}{(\frac{1}{3})^2} = 1$$

This is an ellipse with centre $(0, 0)$ and axis intercepts at $x = \pm\frac{1}{2}$ and $y = \pm\frac{1}{3}$.

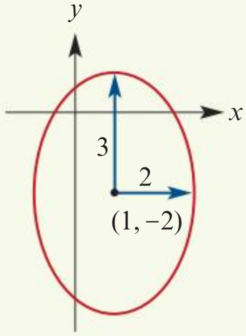


c This is an ellipse with centre $(1, -2)$.
To determine the x-axis intercepts, let $y = 0$. Then solving for x gives

$$x = \frac{3 \pm 2\sqrt{5}}{3}$$

Likewise, to determine the y-axis intercepts, let $x = 0$. This gives

$$y = \frac{-4 \pm 3\sqrt{3}}{2}$$



Using the locus definition

Example 6

Consider points $A(-2, 0)$ and $B(2, 0)$. Determine the equation of the locus of points P satisfying $AP + BP = 8$.

Solution

Let (x, y) be the coordinates of point P. If $AP + BP = 8$, then

$$\sqrt{(x+2)^2 + y^2} + \sqrt{(x-2)^2 + y^2} = 8$$

and so $$\sqrt{(x+2)^2 + y^2} = 8 - \sqrt{(x-2)^2 + y^2}$$

Square both sides, then expand and simplify:

$$(x+2)^2 + y^2 = 64 - 16\sqrt{(x-2)^2 + y^2} + (x-2)^2 + y^2$$
$$x^2 + 4x + 4 + y^2 = 64 - 16\sqrt{(x-2)^2 + y^2} + x^2 - 4x + 4 + y^2$$
$$x - 8 = -2\sqrt{(x-2)^2 + y^2}$$

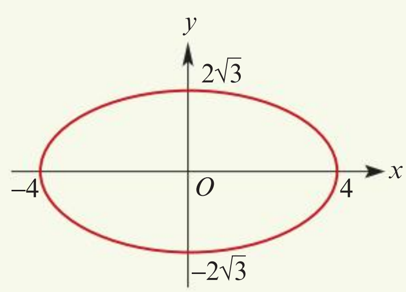
Square both sides again:

$$x^2 - 16x + 64 = 4(x^2 - 4x + 4 + y^2)$$

Simplifying yields

$$3x^2 + 4y^2 = 48$$

i.e. $$\frac{x^2}{16} + \frac{y^2}{12} = 1$$



This is an ellipse with centre the origin and axis intercepts at $x = \pm 4$ and $y = \pm 2\sqrt{3}$.

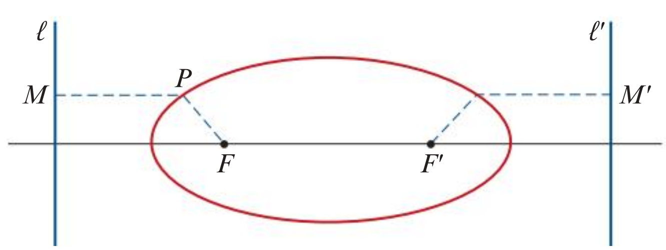
Every point P on the ellipse satisfies $AP + BP = 8$.

Note: You might like to consider the general version of this example with $A(-c, 0)$, $B(c, 0)$ and $AP + BP = 2a$, where $a > c > 0$.

It can also be shown that an ellipse is the locus of points $P(x, y)$ satisfying

$$FP = eMP$$

where F is a fixed point, $0 < e < 1$ and MP is the perpendicular distance from P to a fixed line ℓ. From the symmetry of the ellipse, it is clear that there is a second point F' and a second line ℓ' such that $F'P = eM'P$ defines the same locus, where $M'P$ is the perpendicular distance from P to ℓ'.



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Example 7

Determine the equation of the locus of points $P(x, y)$ if the distance from P to the point $F(1, 0)$ is half the distance MP, the perpendicular distance from P to the line with equation $x = -2$. That is, $FP = \frac{1}{2}MP$.

Solution

Let (x, y) be the coordinates of point P.

If $FP = \frac{1}{2}MP$, then

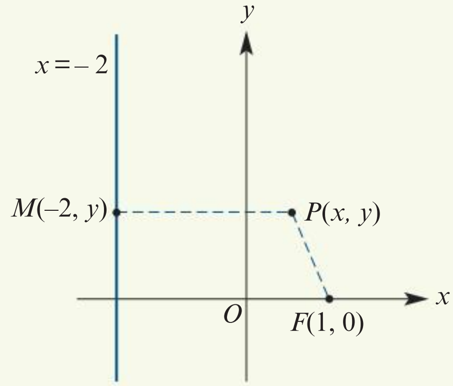
$$\sqrt{(x-1)^2 + y^2} = \frac{1}{2}\sqrt{(x+2)^2}$$

Square both sides:

$$(x-1)^2 + y^2 = \frac{1}{4}(x+2)^2$$

$$4(x^2 - 2x + 1) + 4y^2 = x^2 + 4x + 4$$

$$3x^2 - 12x + 4y^2 = 0$$



Complete the square:

$$3(x^2 - 4x + 4) + 4y^2 = 12$$

$$3(x-2)^2 + 4y^2 = 12 \quad \text{or equivalently} \quad \frac{(x-2)^2}{4} + \frac{y^2}{3} = 1$$

This is an ellipse with centre $(2, 0)$.

Summary B3

- An **ellipse** is the locus of a point P that moves so that the sum of its distances d_1 and d_2 from two fixed points F_1 and F_2 (called the **foci**) is equal to a fixed positive constant.
- The graph of

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

is an ellipse centred at the point (h, k).

Skill-sheet

Exercise B3

Example 5

1 Sketch the graph of each ellipse, labelling the axis intercepts:

a $\frac{x^2}{9} + \frac{y^2}{64} = 1$ **b** $\frac{x^2}{100} + \frac{y^2}{25} = 1$ **c** $\frac{y^2}{9} + \frac{x^2}{64} = 1$ **d** $25x^2 + 9y^2 = 225$

2 Sketch the graph of each ellipse, labelling the centre and the axis intercepts:

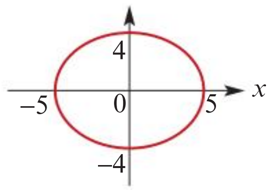
a $\frac{(x-3)^2}{9} + \frac{(y-4)^2}{16} = 1$ **b** $\frac{(x+3)^2}{9} + \frac{(y+4)^2}{25} = 1$

c $\frac{(y-3)^2}{16} + \frac{(x-2)^2}{4} = 1$ **d** $25(x-5)^2 + 9y^2 = 225$

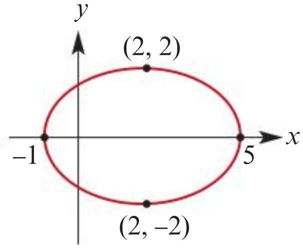
CF

3 Determine the Cartesian equations of the following ellipses:

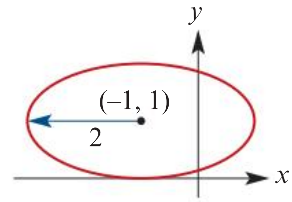
a



b



c



Example 6

4 Determine the locus of the point P as it moves such that the sum of its distances from two fixed points $A(1, 0)$ and $B(-1, 0)$ is 4 units.

5 Determine the locus of the point P as it moves such that the sum of its distances from two fixed points $A(0, 2)$ and $B(0, -2)$ is 6 units.

Example 7

6 Determine the equation of the locus of points $P(x, y)$ such that the distance from P to the point $F(2, 0)$ is half the distance MP, the perpendicular distance from P to the line with equation $x = -4$. That is, $FP = \frac{1}{2}MP$.

7 A circle has equation $x^2 + y^2 = 1$. It is then dilated by a factor of 3 from the x-axis and by a factor of 5 from the y-axis. Find the equation of the image and sketch its graph.

B4 Hyperbolas

Learning intentions

- To be able to work with the locus definition of a hyperbola.

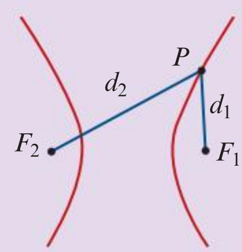
Hyperbolas are defined analogously to ellipses, but using the difference instead of the sum.

Locus definition of a hyperbola

A **hyperbola** is the locus of a point P that moves so that the difference between its distances from two fixed points F_1 and F_2 is a constant. That is,

$$|F_2P - F_1P| = k$$

Note: Points F_1 and F_2 are called the **foci** of the hyperbola.



The standard form of the Cartesian equation of a hyperbola centred at the origin is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Applying the translation defined by $(x, y) \to (x + h, y + k)$, we can see the following result:

The graph of

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

is a hyperbola centred at the point (h, k).

Note: Interchanging x and y in this equation produces another hyperbola (rotated by 90°).

Asymptotes of the hyperbola

We now investigate the behaviour of the hyperbola as $x \to \pm\infty$. We first show that the hyperbola with equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

has asymptotes

$$y = \frac{b}{a}x \quad \text{and} \quad y = -\frac{b}{a}x$$

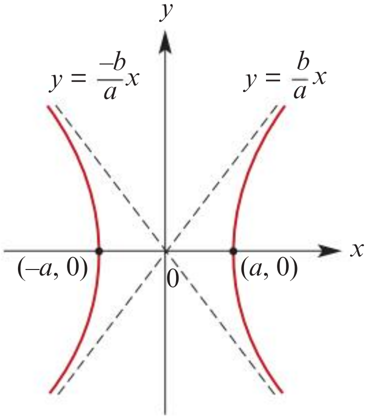
To see why this should be the case, we rearrange the equation of the hyperbola as follows:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$\frac{y^2}{b^2} = \frac{x^2}{a^2} - 1$$

$$y^2 = \frac{b^2x^2}{a^2} - b^2$$

$$= \frac{b^2x^2}{a^2}\left(1 - \frac{a^2}{x^2}\right)$$



If $x \to \pm\infty$, then $\frac{a^2}{x^2} \to 0$. This suggests $y^2 \to \frac{b^2x^2}{a^2}$ as $x \to \pm\infty$. That is,

$$y \to \pm\frac{bx}{a} \quad \text{as} \quad x \to \pm\infty$$

Applying the translation defined by $(x, y) \to (x + h, y + k)$, we obtain the following result:

The hyperbola with equation

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

has asymptotes given by

$$y - k = \pm\frac{b}{a}(x - h)$$

Example 8

For each of the following equations, sketch the graph of the corresponding hyperbola. Give the coordinates of the centre, the axis intercepts and the equations of the asymptotes.

a $\frac{x^2}{9} - \frac{y^2}{4} = 1$

b $\frac{y^2}{9} - \frac{x^2}{4} = 1$

c $(x-1)^2 - (y+2)^2 = 1$

d $\frac{(y-1)^2}{4} - \frac{(x+2)^2}{9} = 1$

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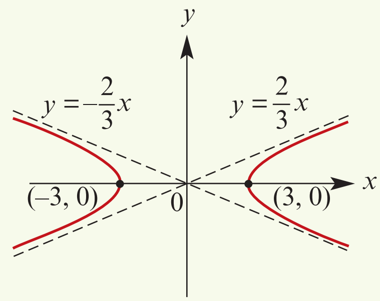
Solution

a Since $\frac{x^2}{9} - \frac{y^2}{4} = 1$, we have

$$y^2 = \frac{4x^2}{9}\left(1 - \frac{9}{x^2}\right)$$

Thus the equations of the asymptotes are $y = \pm\frac{2}{3}x$.

If $y = 0$, then $x^2 = 9$ and so $x = \pm 3$. The x-axis intercepts are $(3, 0)$ and $(-3, 0)$. The centre is $(0, 0)$.

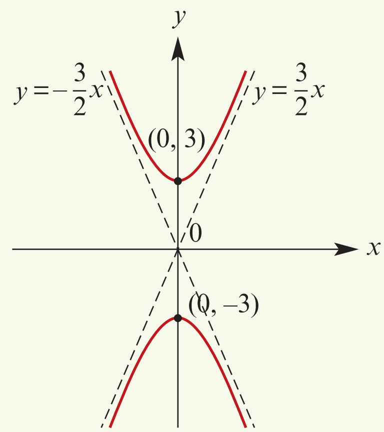


b Since $\frac{y^2}{9} - \frac{x^2}{4} = 1$, we have

$$y^2 = \frac{9x^2}{4}\left(1 + \frac{4}{x^2}\right)$$

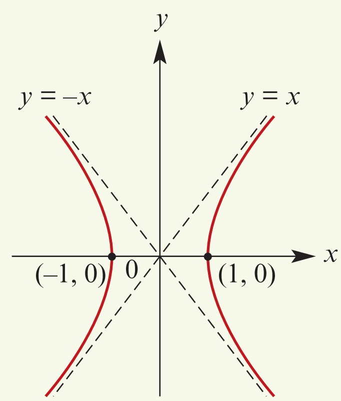
Thus the equations of the asymptotes are $y = \pm\frac{3}{2}x$.

The y-axis intercepts are $(0, 3)$ and $(0, -3)$.
The centre is $(0, 0)$.



c First sketch the graph of $x^2 - y^2 = 1$. The asymptotes are $y = x$ and $y = -x$. The centre is $(0, 0)$ and the axis intercepts are $(1, 0)$ and $(-1, 0)$.

Note: This hyperbola is called a **rectangular hyperbola**, as its asymptotes are perpendicular.



Now to sketch the graph of

$$(x - 1)^2 - (y + 2)^2 = 1$$

we apply the translation $(x, y) \to (x + 1, y - 2)$.

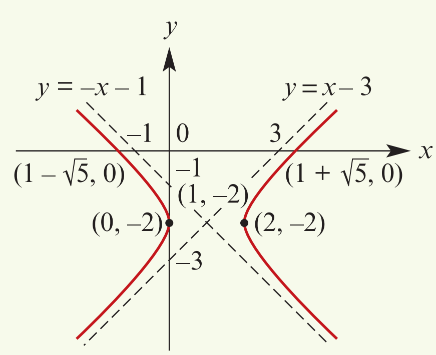
The new centre is $(1, -2)$ and the asymptotes have equations $y + 2 = \pm(x - 1)$. That is, $y = x - 3$ and $y = -x - 1$.

Axis intercepts

If $x = 0$, then $y = -2$.

If $y = 0$, then $(x - 1)^2 = 5$ and so $x = 1 \pm \sqrt{5}$.

Therefore the axis intercepts are $(0, -2)$ and $(1 \pm \sqrt{5}, 0)$.



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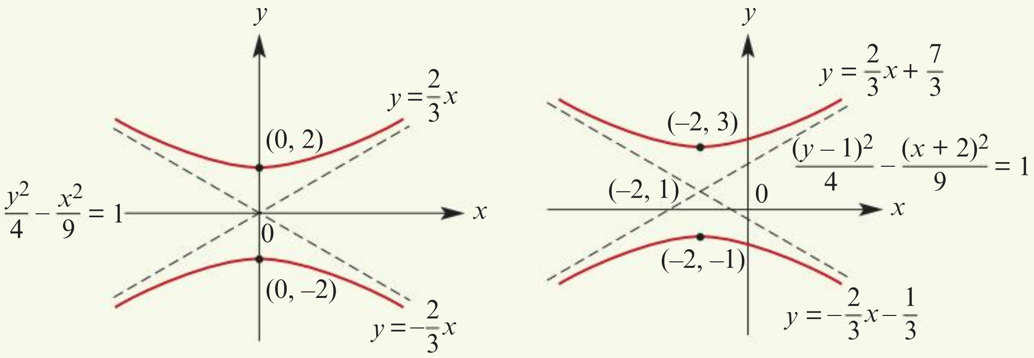
d The graph of $\frac{(y-1)^2}{4} - \frac{(x+2)^2}{9} = 1$ is obtained from the hyperbola $\frac{y^2}{4} - \frac{x^2}{9} = 1$ through the translation $(x, y) \to (x-2, y+1)$. Its centre will be $(-2, 1)$.



Using the locus definition

Example 9

Consider the points $A(-2, 0)$ and $B(2, 0)$. Determine the equation of the locus of points P satisfying $AP - BP = 3$.

Solution

Let (x, y) be the coordinates of point P.

If $AP - BP = 3$, then

$$\sqrt{(x+2)^2 + y^2} - \sqrt{(x-2)^2 + y^2} = 3$$

and so $$\sqrt{(x+2)^2 + y^2} = 3 + \sqrt{(x-2)^2 + y^2}$$

Square both sides, then expand and simplify:

$$(x+2)^2 + y^2 = 9 + 6\sqrt{(x-2)^2 + y^2} + (x-2)^2 + y^2$$
$$x^2 + 4x + 4 + y^2 = 9 + 6\sqrt{(x-2)^2 + y^2} + x^2 - 4x + 4 + y^2$$
$$8x - 9 = 6\sqrt{(x-2)^2 + y^2}$$

Note that this only holds if $x \geq \frac{9}{8}$. Squaring both sides again gives

$$64x^2 - 144x + 81 = 36(x^2 - 4x + 4 + y^2)$$
$$28x^2 - 36y^2 = 63$$
$$\frac{4x^2}{9} - \frac{4y^2}{7} = 1 \quad \text{for } x \geq \frac{3}{2}$$

This is the right branch of a hyperbola with centre the origin and x-axis intercept $\frac{3}{2}$.

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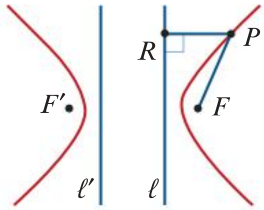
It can also be shown that a hyperbola is the locus of points $P(x, y)$ satisfying

$$FP = eRP$$

where F is a fixed point, $e > 1$ and RP is the perpendicular distance from P to a fixed line ℓ.



From the symmetry of the hyperbola, it is clear that there is a second point F' and a second line ℓ' such that $F'P = eR'P$ defines the same locus, where $R'P$ is the perpendicular distance from P to ℓ'.

Example 10

Determine the equation of the locus of points $P(x, y)$ that satisfy the property that the distance from P to the point $F(1, 0)$ is twice the distance MP, the perpendicular distance from P to the line with equation $x = -2$. That is, $FP = 2MP$.

Solution

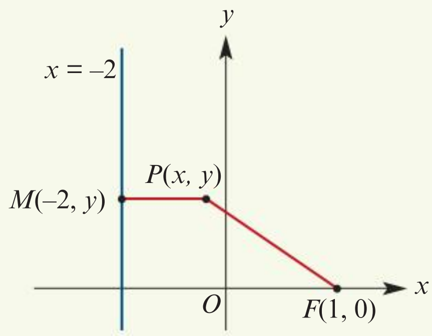
Let (x, y) be the coordinates of point P.

If $FP = 2MP$, then

$$\sqrt{(x-1)^2 + y^2} = 2\sqrt{(x+2)^2}$$

Squaring both sides gives

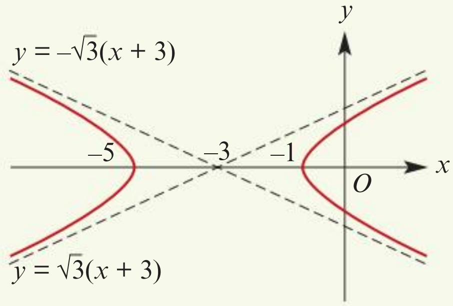
$$(x-1)^2 + y^2 = 4(x+2)^2$$
$$x^2 - 2x + 1 + y^2 = 4(x^2 + 4x + 4)$$
$$3x^2 + 18x - y^2 + 15 = 0$$



By completing the square, we obtain

$$3(x^2 + 6x + 9) - 27 - y^2 + 15 = 0$$
$$3(x+3)^2 - y^2 = 12$$
$$\frac{(x+3)^2}{4} - \frac{y^2}{12} = 1$$

This is a hyperbola with centre $(-3, 0)$.



Summary B4

- A **hyperbola** is the locus of a point P that moves so that the difference between its distances from two fixed points F_1 and F_2 (called the **foci**) is a constant. That is, $|F_2P - F_1P| = k$.
- The graph of
$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$
is a hyperbola centred at the point (h, k). The asymptotes are $y - k = \pm\frac{b}{a}(x - h)$.

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Skill-sheet

Exercise B4

Example 8

1 Sketch the graph of each of the following hyperbolas. Label axis intercepts and give the equations of the asymptotes.

a $\dfrac{x^2}{4} - \dfrac{y^2}{9} = 1$ **b** $x^2 - \dfrac{y^2}{4} = 1$

c $\dfrac{y^2}{25} - \dfrac{x^2}{100} = 1$ **d** $25x^2 - 9y^2 = 225$

2 Sketch the graph of each of the following hyperbolas. State the centre and label axis intercepts and asymptotes.

a $(x - 1)^2 - (y + 2)^2 = 1$ **b** $\dfrac{(x + 1)^2}{4} - \dfrac{(y - 2)^2}{16} = 1$

c $\dfrac{(y - 3)^2}{9} - (x - 2)^2 = 1$ **d** $25(x - 4)^2 - 9y^2 = 225$

e $x^2 - 4y^2 - 4x - 8y - 16 = 0$ **f** $9x^2 - 25y^2 - 90x + 150y = 225$

Example 9

3 Consider the points $A(4, 0)$ and $B(-4, 0)$. Determine the equation of the locus of points P satisfying $AP - BP = 6$.

4 Determine the equation of the locus of points $P(x, y)$ satisfying $AP - BP = 4$, given coordinates $A(-3, 0)$ and $B(3, 0)$.

Example 10

5 Determine the equation of the locus of points $P(x, y)$ that satisfy the property that the distance to P from the point $F(5, 0)$ is twice the distance MP, the perpendicular distance to P from the line with equation $x = -1$. That is, $FP = 2MP$.

6 Determine the equation of the locus of points $P(x, y)$ that satisfy the property that the distance to P from the point $F(0, -1)$ is twice the distance MP, the perpendicular distance to P from the line with equation $y = -4$. That is, $FP = 2MP$.

CF

B5 Parametric equations

Learning intentions

- To be able to work with parametric equations.

A **parametric curve** in the plane is a pair of functions

$$x = f(t) \quad \text{and} \quad y = g(t)$$

The variable t is called the **parameter**, and for each choice of t we get a point in the plane $(f(t), g(t))$. The set of all such points will be a curve in the plane.

It is sometimes useful to think of t as being *time*, so that the equations $x = f(t)$ and $y = g(t)$ give the position of an object at time t. Points on the curve can be plotted by substituting various values of t into the two equations.

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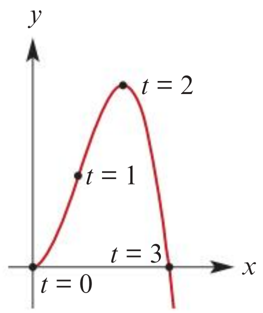
For instance, we can plot points on the curve defined by the parametric equations

$$x = t \quad \text{and} \quad y = 3t^2 - t^3$$

by letting $t = 0, 1, 2, 3$.

t	0	1	2	3
x	0	1	2	3
y	0	2	4	0



In this instance, it is possible to eliminate the parameter t to obtain a Cartesian equation in x and y alone. Substituting $t = x$ into the second equation gives $y = 3x^2 - x^3$.

Lines

Example 11

a Determine the Cartesian equation for the curve defined by the parametric equations

$$x = t + 2 \quad \text{and} \quad y = 2t - 3$$

b Determine parametric equations for the line through the points $A(2, 3)$ and $B(4, 7)$.

Solution

a Substitute $t = x - 2$ into the second equation to give

$$\begin{aligned} y &= 2(x - 2) - 3 \\ &= 2x - 7 \end{aligned}$$

Thus every point lies on the straight line with equation $y = 2x - 7$.

b The gradient of the straight line through points $A(2, 3)$ and $B(4, 7)$ is

$$m = \frac{7 - 3}{4 - 2} = 2$$

Therefore the line has equation

$$\begin{aligned} y - 3 &= 2(x - 2) \\ y &= 2x - 1 \end{aligned}$$

We can simply let $x = t$ and so $y = 2t - 1$.

Note: There are infinitely many pairs of parametric equations that describe the same curve. In part **b**, we could also let $x = 2t$ and $y = 4t - 1$. These parametric equations describe exactly the same set of points. As t increases, the point moves along the same line twice as fast.

Parabolas

Example 12

Determine the Cartesian equation of the parabola defined by the parametric equations

$$x = t - 1 \quad \text{and} \quad y = t^2 + 1$$

Solution

Substitute $t = x + 1$ into the second equation to give $y = (x + 1)^2 + 1$.

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Circles

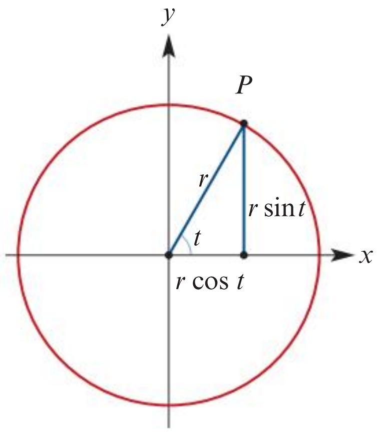
We have seen that the circle with radius r and centre at the origin can be written in Cartesian form as

$$x^2 + y^2 = r^2$$

We now introduce the parameter t and let

$$x = r\cos t \quad \text{and} \quad y = r\sin t$$

As t increases from 0 to 2π, the point $P(x, y)$ travels from $(r, 0)$ anticlockwise around the circle and returns to its original position.



To demonstrate that this parameterises the circle, we evaluate

$$\begin{aligned} x^2 + y^2 &= r^2\cos^2 t + r^2\sin^2 t \\ &= r^2(\cos^2 t + \sin^2 t) \\ &= r^2 \end{aligned}$$

where we have used the Pythagorean identity $\cos^2 t + \sin^2 t = 1$.

Example 13

a Determine the Cartesian equation of the circle defined by the parametric equations

$$x = \cos t + 1 \quad \text{and} \quad y = \sin t - 2$$

b Determine parametric equations for the circle with Cartesian equation

$$(x + 1)^2 + (y + 3)^2 = 4$$

Solution

a We rearrange each equation to isolate $\cos t$ and $\sin t$ respectively. This gives

$$x - 1 = \cos t \quad \text{and} \quad y + 2 = \sin t$$

Using the Pythagorean identity:

$$(x - 1)^2 + (y + 2)^2 = \cos^2 t + \sin^2 t = 1$$

So every point on the graph lies on the circle with equation $(x - 1)^2 + (y + 2)^2 = 1$.

b We let

$$\cos t = \frac{x + 1}{2} \quad \text{and} \quad \sin t = \frac{y + 3}{2}$$

giving

$$x = 2\cos t - 1 \quad \text{and} \quad y = 2\sin t - 3$$

We can easily check that these equations parameterise the given circle.

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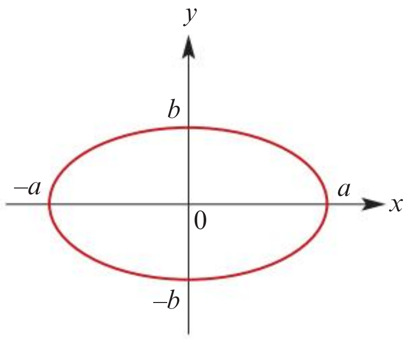
Ellipses

An ellipse can be thought of as a squashed circle.

This is made apparent from the parametric equations for an ellipse:

$$x = a\cos t \quad \text{and} \quad y = b\sin t$$



As with the circle, we see the sine and cosine functions, but these are now scaled by different constants, giving different dilations from the x- and y-axes.

We can turn this pair of parametric equations into one Cartesian equation as follows:

$$\frac{x}{a} = \cos t \quad \text{and} \quad \frac{y}{b} = \sin t$$

giving

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \cos^2 t + \sin^2 t = 1$$

which is the standard form of an ellipse centred at the origin with axis intercepts at $x = \pm a$ and $y = \pm b$.

Example 14

a Determine the Cartesian equation of the ellipse defined by the parametric equations

$$x = 3\cos t + 1 \quad \text{and} \quad y = 2\sin t - 1$$

b Determine parametric equations for the ellipse with Cartesian equation

$$\frac{(x-1)^2}{4} + \frac{(y+2)^2}{16} = 1$$

Solution

a We rearrange each equation to isolate $\cos t$ and $\sin t$ respectively. This gives

$$\frac{x-1}{3} = \cos t \quad \text{and} \quad \frac{y+1}{2} = \sin t$$

Using the Pythagorean identity:

$$\left(\frac{x-1}{3}\right)^2 + \left(\frac{y+1}{2}\right)^2 = \cos^2 t + \sin^2 t = 1$$

So every point on the graph lies on the ellipse with equation $\dfrac{(x-1)^2}{3^2} + \dfrac{(y+1)^2}{2^2} = 1$.

b We let

$$\cos t = \frac{x-1}{2} \quad \text{and} \quad \sin t = \frac{y+2}{4}$$

giving

$$x = 2\cos t + 1 \quad \text{and} \quad y = 4\sin t - 2$$

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Hyperbolas

We can parameterise a hyperbola using the equations

$$x = a\sec t \quad \text{and} \quad y = b\tan t$$

From these two equations, we can determine the more familiar Cartesian equation:

$$\frac{x}{a} = \sec t \quad \text{and} \quad \frac{y}{b} = \tan t$$

giving

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \sec^2 t - \tan^2 t = 1$$

which is the standard form of a hyperbola centred at the origin.

Example 15

a Determine the Cartesian equation of the hyperbola defined by the parametric equations

$$x = 3\sec t - 1 \quad \text{and} \quad y = 2\tan t + 2$$

b Determine parametric equations for the hyperbola with Cartesian equation

$$\frac{(x+2)^2}{4} - \frac{(y-3)^2}{16} = 1$$

Solution

a We rearrange each equation to isolate $\sec t$ and $\tan t$ respectively. This gives

$$\frac{x+1}{3} = \sec t \quad \text{and} \quad \frac{y-2}{2} = \tan t$$

and therefore

$$\left(\frac{x+1}{3}\right)^2 - \left(\frac{y-2}{2}\right)^2 = \sec^2 t - \tan^2 t = 1$$

So each point on the graph lies on the hyperbola with equation $\dfrac{(x+1)^2}{3^2} - \dfrac{(y-2)^2}{2^2} = 1$.

b We let

$$\sec t = \frac{x+2}{2} \quad \text{and} \quad \tan t = \frac{y-3}{4}$$

giving

$$x = 2\sec t - 2 \quad \text{and} \quad y = 4\tan t + 3$$

Parametric equations with restricted domains

Example 16

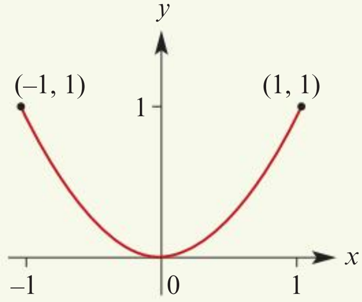
Eliminate the parameter to determine the graph of the parameterised curve

$$x = t - 1, \quad y = t^2 - 2t + 1 \qquad \text{for } 0 \le t \le 2$$

Solution

Substitute $t = x + 1$ from the first equation into the second equation, giving

$$y = (x+1)^2 - 2(x+1) + 1$$
$$= x^2 + 2x + 1 - 2x - 2 + 1$$
$$= x^2$$



Since $0 \le t \le 2$, it follows that $-1 \le x \le 1$.

Therefore, as t increases from 0 to 2, the point travels along the parabola $y = x^2$ from $(-1, 1)$ to $(1, 1)$.

Intersections of curves defined parametrically

It is often difficult to determine the intersection of two curves defined parametrically. This is because, although the curves may intersect, they might do so for different values of the parameter t.

In many instances, it is easiest to determine the points of intersection using the Cartesian equations for the two curves.

Example 17

Determine the points of intersection of the circle and line defined by the parametric equations:

circle $x = 5\cos t$ and $y = 5\sin t$

line $x = t - 3$ and $y = 2t - 8$

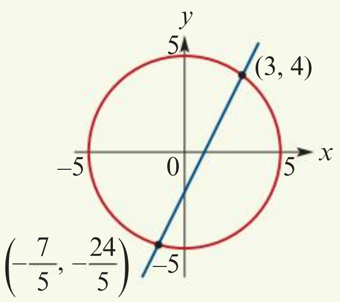
Solution

The Cartesian equation of the circle is $x^2 + y^2 = 25$.

The Cartesian equation of the line is $y = 2x - 2$.

Substituting the second equation into the first gives

$$x^2 + (2x-2)^2 = 25$$
$$x^2 + 4x^2 - 8x + 4 = 25$$
$$5x^2 - 8x - 21 = 0$$
$$(x-3)(5x+7) = 0$$



This gives solutions $x = 3$ and $x = -\frac{7}{5}$.

Substituting these into the equation $y = 2x - 2$ gives $y = 4$ and $y = -\frac{24}{5}$ respectively.

The points of intersection are $(3, 4)$ and $\left(-\frac{7}{5}, -\frac{24}{5}\right)$.

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Using a graphics calculator with parametric equations

Example 18

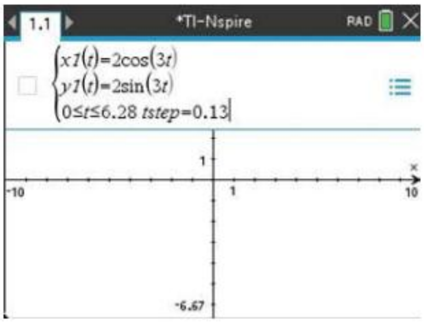
Plot the graph of the parametric curve given by

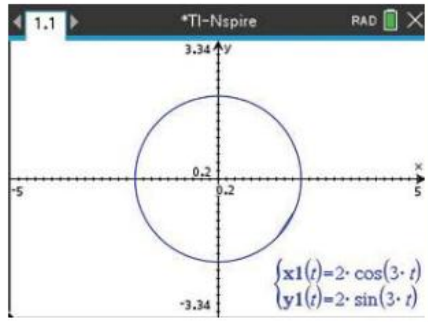
$$x = 2\cos(3t) \quad \text{and} \quad y = 2\sin(3t)$$

TI-84

Using the TI-Nspire CX non-CAS

- Open a **Graphs** application (on > **New Document** > **Add Graphs**).
- Use menu > **Graph Entry/Edit** > **Parametric** to show the entry line for parametric equations.
- Enter $x1(t) = 2\cos(3t)$ and $y1(t) = 2\sin(3t)$ as shown.





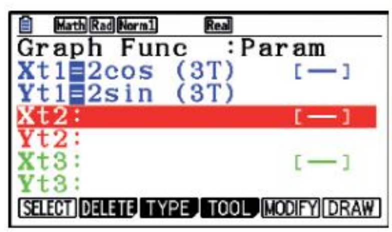
Using the Casio

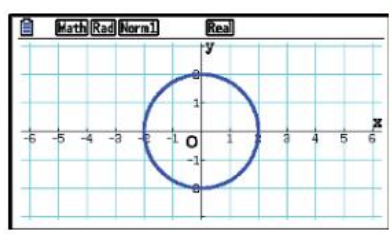
- Press MENU 5 to select **Graph** mode.
- To specify the graph type, go to **Type** F3 and choose **Parametric** F3.
- Enter the rule $x = 2\cos(3t)$ in $Xt1$:

  2 cos (3 X,θ,T) EXE
- Enter the rule $y = 2\sin(3t)$ in $Yt1$:

  2 sin (3 X,θ,T) EXE
- Adjust the View Window:

  SHIFT F3 F1 EXE
- Select **Draw** F6.





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Summary B5

- A **parametric curve** in the plane is a pair of functions

$$x = f(t) \quad \text{and} \quad y = g(t)$$

where t is called the **parameter** of the curve. For example:

	Cartesian equation	Parametric equations
Circle	$x^2 + y^2 = r^2$	$x = r\cos t$ and $y = r\sin t$
Ellipse	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	$x = a\cos t$ and $y = b\sin t$
Hyperbola	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$x = a\sec t$ and $y = b\tan t$

- We can sometimes determine the Cartesian equation of a parametric curve by eliminating t and solving for y in terms of x.

Skill-sheet

Exercise B5

SF

Example 11

1 Consider the parametric equations

$$x = t - 1 \quad \text{and} \quad y = t^2 - 1$$

a Determine the Cartesian equation of the curve described by these equations.

b Sketch the curve and label the points on the curve corresponding to $t = 0, 1, 2$.

Example 12

2 For each of the following pairs of parametric equations, determine the Cartesian equation and sketch the curve:

a $x = t + 1$ and $y = 2t + 1$

b $x = t - 1$ and $y = 2t^2 + 1$

c $x = t^3$ and $y = t^9$

d $x = t + 2$ and $y = \frac{1}{t+1}$

CF

Example 13

3 **a** Determine the Cartesian equation of the circle defined by the parametric equations

$$x = 2\cos t \quad \text{and} \quad y = 2\sin t$$

Example 14

b Determine the Cartesian equation of the ellipse defined by the parametric equations

$$x = 3\cos t - 1 \quad \text{and} \quad y = 2\sin t + 2$$

c Determine parametric equations for the circle with Cartesian equation

$$(x + 3)^2 + (y - 2)^2 = 9$$

d Determine parametric equations for the ellipse with Cartesian equation

$$\frac{(x + 2)^2}{9} + \frac{(y - 1)^2}{4} = 1$$

4 Determine parametric equations for the line through the points $A(-1, -2)$ and $B(1, 4)$.

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Example 16

5 **a** Eliminate the parameter t to determine the equation of the parameterised curve

$$x = t - 1 \quad \text{and} \quad y = -2t^2 + 4t - 2 \qquad \text{for } 0 \le t \le 2$$

b Sketch the graph of this curve over an appropriate domain.

Example 17

6 Find the points of intersection of the circle and line defined by the parametric equations:

circle $x = \cos t$ and $y = \sin t$

line $x = 3t + 6$ and $y = 4t + 8$

7 A curve is parameterised by the equations

$$x = \sin t \quad \text{and} \quad y = 2\sin^2 t + 1 \qquad \text{for } 0 \le t \le 2\pi$$

a Find the curve's Cartesian equation.
b What is the domain of the curve?
c What is the range of the curve?
d Sketch the graph of the curve.

8 A curve is parameterised by the equations

$$x = 2^t \quad \text{and} \quad y = 2^{2t} + 1 \qquad \text{for } t \in \mathbb{R}$$

a Determine the curve's Cartesian equation.
b What is the domain of the curve?
c What is the range of the curve?
d Sketch the graph of the curve.

9 Eliminate the parameter to determine the graph of the parameterised curve

$$x = \cos t \quad \text{and} \quad y = 1 - 2\sin^2 t \qquad \text{for } 0 \le t \le 2\pi$$

10 Consider the parametric equations

$$x = 2^t + 2^{-t} \quad \text{and} \quad y = 2^t - 2^{-t}$$

a Show that the Cartesian equation of the curve is $\frac{x^2}{4} - \frac{y^2}{4} = 1$ for $x \ge 2$.
b Sketch the graph of the curve.

11 Consider the circle with Cartesian equation $x^2 + (y - 1)^2 = 1$.

a Sketch the graph of the circle.
b Show that the parametric equations $x = \cos t$ and $y = \sin t + 1$ define the same circle.
c A different parameterisation of the circle can be found without the use of the cosine and sine functions. Suppose that t is any real number and let $P(x, y)$ be the point of intersection of the line $y = 2 - tx$ with the circle. Solve for x and y in terms of t, assuming that $x \ne 0$.
d Verify that the equations found in part **c** parameterise the same circle, excluding the point (0,0).

12 The curve with parametric equations $x = \frac{t}{2\pi}\cos t$ and $y = \frac{t}{2\pi}\sin t$ is called an **Archimedean spiral**.

a With the help of your calculator, sketch the curve over the interval $0 \le t \le 6\pi$.
b Label the points on the curve corresponding to $t = 0, \pi, 2\pi, 3\pi, 4\pi, 5\pi, 6\pi$.

Review

Chapter summary

Parabolas, ellipses and hyperbolas

- A **locus** is the set of points described by a geometric condition.
- A **circle** is the locus of a point P that moves so that its distance from a fixed point C is constant.
- A **straight line** is the locus of a point P that moves so that it is equidistant from two fixed points Q and R.
- A **parabola** is the locus of a point P that moves so that its distance from a fixed point F is equal to its perpendicular distance from a fixed line.
- An **ellipse** is the locus of a point P that moves so that the sum of its distances d_1 and d_2 from two fixed points F_1 and F_2 is a constant.
- The graph of

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

is an ellipse centred at the point (h, k).
- A **hyperbola** is the locus of a point P that moves so that the difference between its distances from two fixed points F_1 and F_2 is a constant.
- The graph of

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

is a hyperbola centred at the point (h, k). The asymptotes are $y - k = \pm\frac{b}{a}(x - h)$.

Parametric curves

- A **parametric curve** in the plane is a pair of functions

$$x = f(t) \quad \text{and} \quad y = g(t)$$

where t is called the **parameter** of the curve.
- It can be helpful to think of the parameter t as describing time. Parametric curves are then useful for describing the motion of an object.
- We can sometimes determine the Cartesian equation of a parametric curve by eliminating t and solving for y in terms of x.

Skills checklist

Check-list

Download this checklist from the Interactive Textbook, then print it and fill it out to check your skills.

B1 **1** I can determine the equation of the locus of points for a circle. ☐

See Example 1a and Question 1

Review

B1 **2** I can determine the centre and radius of a circle given the Cartesian equation of the circle. ☐

See Example 1b

B1 **3** I can determine the equation of the locus of points for a straight line. ☐

See Example 2 and Question 3

B2 **4** I can determine the equation of the locus of points for a parabola. ☐

See Example 3, Example 4 and Questions 1 and 4

B3 **5** I can sketch the graph of an ellipse given its Cartesian equation. ☐

See Example 3 and Question 1

B3 **6** I can determine the equation of the locus of points for an ellipse. ☐

See Example 6, Example 7 and Questions 4 and 6

B4 **7** I can sketch the graph of a hyperbola given its Cartesian equation. ☐

See Example 8 and Question 1

B4 **8** I can determine the equation of the locus of points for a hyperbola. ☐

See Example 9, Example 10 and Questions 3 and 5

B5 **9** I can determine the Cartesian equation of a line given its parametric equations. ☐

See Example 11a and Question 2a

B5 **10** I can determine the parametric equations of a line given two points through which the line passes. ☐

See Example 11b and Question 4

B5 **11** I can determine the Cartesian equation of a parabola given its parametric equations. ☐

See Example 12 and Question 1

B5 **12** I can determine the Cartesian equation of a circle given its parametric equations. ☐

See Example 13 and Question 3

B5 **13** I can determine the Cartesian equation of an ellipse given its parametric equations. ☐

See Example 14 and Question 4

B5 **14** I can determine the Cartesian equation of a hyperbola given its parametric equations. ☐

See Example 15

B5 **15** I can determine the domain and equation of a curve given the parametric equations and restrictions of the parameter. ☐

See Example 16 and Question 5

B5 **16** I can determine the intersection of the graphs of two relations defined parametrically ☐

See Example 17 and Question 6

B5 **17** I can use a calculator to plot the graphs of a relation defined parametrically ☐

See Example 18 and Question 12

Short-response questions

Technology-free short-response questions

SF

1 The point $P(x, y)$ moves so that it is equidistant from $Q(2, -1)$ and $R(1, 2)$. Determine the locus of the point P.

2 Determine the locus of the point $P(x, y)$ such that $AP = 6$, given point $A(3, 2)$.

CF

3 A circle has equation $x^2 + 4x + y^2 - 8y = 0$. Determine the coordinates of the centre and the radius of the circle.

4 Sketch the graph of each ellipse and determine the coordinates of its axis intercepts:

a $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$ **b** $\dfrac{(x-2)^2}{4} + \dfrac{(y+1)^2}{9} = 1$

5 An ellipse has equation $x^2 + 4x + 2y^2 = 0$. Determine the coordinates of the centre and the axis intercepts of the ellipse.

6 Sketch the graph of each hyperbola and write down the equations of its asymptotes:

a $x^2 - \dfrac{y^2}{4} = 1$ **b** $\dfrac{(y-1)^2}{16} - \dfrac{(x+2)^2}{4} = 1$

7 A point $P(x, y)$ moves so that its distance from the point $K(-2, 5)$ is twice its distance from the line $x = 1$. Determine its locus.

8 For each of the following pairs of parametric equations, determine the corresponding Cartesian equation:

a $x = 2t - 1$ and $y = 6 - 4t$ **b** $x = 2\cos t$ and $y = 2\sin t$

c $x = 3\cos t + 1$ and $y = 5\sin t - 1$ **d** $x = \cos t$ and $y = 3\sin^2 t - 2$

9 A curve has parametric equations $x = t - 1$ and $y = 2t^2 - 1$ for $0 \le t \le 2$.

a Determine the curve's Cartesian equation. **b** What is the domain of the curve?

c What is the range of the curve? **d** Sketch the graph of the curve.

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Review

Technology-active short-response questions

SF

10 Consider points $A(0, 3)$ and $B(6, 0)$. determine the locus of the point $P(x, y)$ given that:

a $AP = BP$

b $AP = 2BP$

11 Determine the equation of the locus of points $P(x, y)$ which satisfy the property that the distance to P from the point $F(0, 4)$ is equal to:

a MP, the perpendicular distance from the line with equation $y = -2$

b half the distance MP, the perpendicular distance from the line $y = -2$

c twice the distance MP, the perpendicular distance from the line $y = -2$.

CF

12 A ball is thrown into the air. The position of the ball at time $t \geq 0$ is given by the parametric equations $x = 10t$ and $y = 20t - 5t^2$.

a Determine the Cartesian equation of the ball's flight.

b Sketch the graph of the ball's path.

c What is the maximum height reached by the ball?

A second ball is thrown into the air. Its position at time $t \geq 0$ is given by the parametric equations $x = 60 - 10t$ and $y = 20t - 5t^2$.

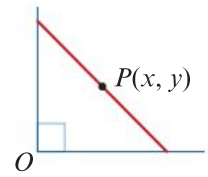
d Determine the Cartesian equation of the second ball's flight.

e Sketch the graph of the second ball's path on the same set of axes.

f Determine the points of intersection of the two paths.

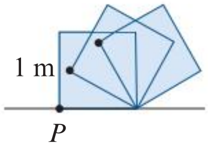
g Do the balls collide?

13 A ladder of length 6 metres stands against a vertical wall. The ladder then slides along on the floor until it lies flat. Show that the midpoint $P(x, y)$ of the ladder moves along a circular path.



CU

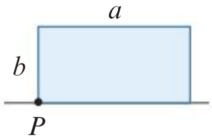
14 A square box of side length 1 metre is too heavy to lift, but can be rolled along the flat ground, using each edge as a pivot. The box is rolled one full revolution.



a Sketch the full path of the point P.

b Determine the total distance travelled by the point P.

c A second rectangular box has length a metres and width b metres. Sketch the path taken by the point P when the box is rolled one full revolution, and determine the total distance travelled by this point.



d For the second box, determine the area between the path taken by P and the ground.

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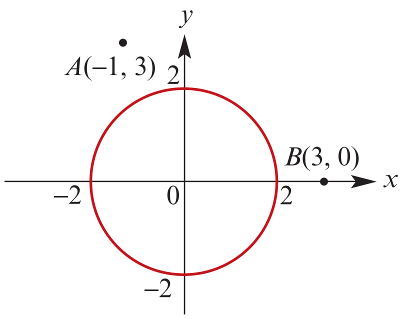
15 The circle with equation $x^2 + y^2 = 4$ is shown. We will say that point A is **visible** to point B if the line AB does not intersect the circle.



a Consider points $A(-1, 3)$ and $B(3, 0)$. Show that the equations

$$x = 4t - 1 \quad \text{and} \quad y = 3 - 3t$$

parameterise the line AB.

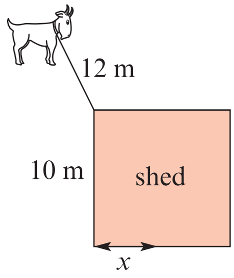
b Show that A is not visible to B by showing that there are two values of t for which the line AB intersects the circle.

c Determine parametric equations for the line that goes through points $C(-1, 4)$ and $B(3, 0)$.

d Show that C is visible to B by showing that there is no value of t for which the line CB intersects the circle.

e Determine the range of values k for which the point $D(-1, k)$ is visible to B.

16 A shed has a square base of side length 10 metres. A goat is tied to a corner of the shed by a rope of length 12 metres. As the goat pulls tightly on the rope and walks around the shed in both directions, a path is traced by the goat.



a Sketch the shed and the path described above.

b Determine the size of the area over which the goat can walk.

c The goat is now tied to a point on the shed x metres from the corner, where $0 \le x \le 5$. Determine a formula for the area A over which the goat can walk, in terms of x.
Hint: Consider the two cases $0 \le x \le 2$ and $2 < x \le 5$.

d Sketch the graph of A against x for $0 \le x \le 5$.

e Where should the goat be tied if the area is to be:

i a maximum **ii** a minimum?

Multiple-choice questions

Technology-free multiple-choice questions

1 The locus of points $P(x, y)$ which satisfy the property that $AP = BP$, given points $A(2, -5)$ and $B(-4, 1)$, is described by the equation

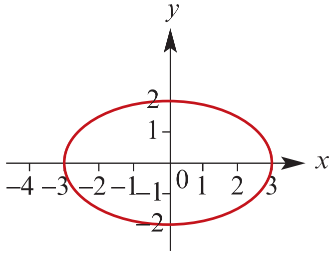
A $y = x - 1$ **B** $y = x - 6$ **C** $y = -x - 3$ **D** $y = x + 1$

2 A parabola has focus $(0, 2)$ and directrix $y = -2$. Which of the following is not true about the parabola?

A Its axis of symmetry is the line $x = 0$. **B** It passes through the origin.
C It contains no point below the x-axis. **D** The point $(2, 1)$ lies on the parabola.

Review

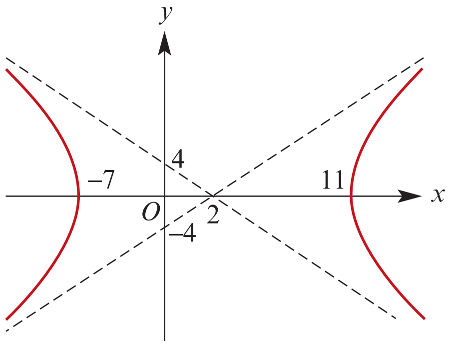
3 The equation of the graph shown is



A $\frac{x^2}{3} + \frac{y^2}{2} = 1$ **B** $\frac{x^2}{9} - \frac{y^2}{4} = 1$

C $\frac{x^2}{9} + \frac{y^2}{4} = 1$ **D** $\frac{x^2}{3} - \frac{y^2}{2} = 1$

4 The equation of the graph shown is



A $\frac{(x+2)^2}{27} - \frac{y^2}{108} = 1$

B $\frac{(x-2)^2}{9} - \frac{y^2}{34} = 1$

C $\frac{(x+2)^2}{81} - \frac{y^2}{324} = 1$

D $\frac{(x-2)^2}{81} - \frac{y^2}{324} = 1$

5 The asymptotes of the hyperbola with equation $\frac{(y-2)^2}{9} - \frac{(x+3)^2}{4} = 1$ intersect at the point

A $(3, 2)$ **B** $(3, -2)$ **C** $(-3, 2)$ **D** $(2, -3)$

6 An ellipse is parameterised by the equations $x = 4\cos t + 1$ and $y = 2\sin t - 1$. The coordinates of its x-axis intercepts are

A $(1 - 3\sqrt{2}, 0),\ (1 + 3\sqrt{2}, 0)$ **B** $(-3, 0),\ (5, 0)$

C $(1 - 2\sqrt{3}, 0),\ (1 + 2\sqrt{3}, 0)$ **D** $(0, -3),\ (0, 5)$

Technology-active multiple-choice questions

7 A curve is parameterised by the equations $x = 2t - 3$ and $y = t^2 - 3t$. Which of the following points does the curve pass through?

A $(5, 1)$ **B** $(5, 2)$ **C** $(5, 3)$ **D** $(5, 4)$

8 The coordinates of the y-axis intercepts of the graph of the ellipse with equation $\frac{x^2}{9} + \frac{(y+2)^2}{4} = 1$ are

A $(-2, 0)$ and $(2, 0)$ **B** $(-4, 0)$ and $(4, 0)$

C $(0, -4)$ and $(0, 4)$ **D** $(0, 0)$ and $(0, -4)$

9 The graph of the equation $ax^2 + by^2 = 8$ has y axis intercept 2 and passes through the point with coordinates $\left(1, \frac{1}{2}\sqrt{10}\right)$. Then

A $a = 2$ and $b = 3$ **B** $a = 4$ and $b = 3$

C $a = \sqrt{3}$ and $b = 2$ **D** $a = 3$ and $b = 2$

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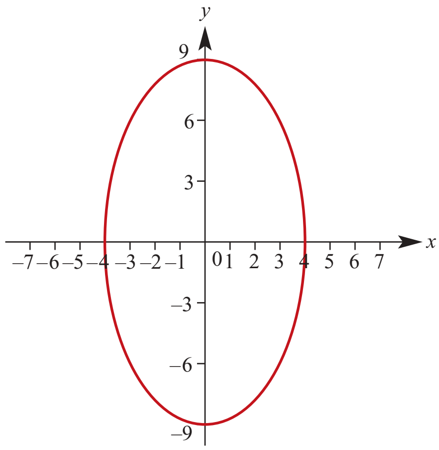
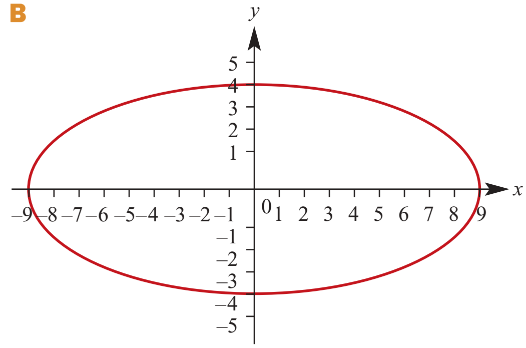
10 The graph of the ellipse with equation $\dfrac{x^2}{81} + \dfrac{y^2}{16} = 1$ is

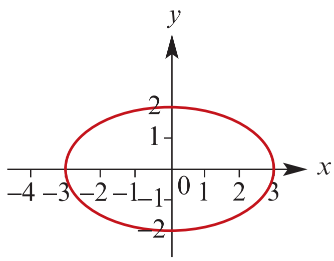
A



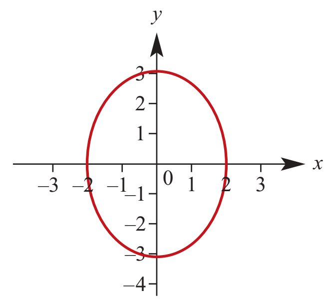
B



C



D



11 Consider the curve defined by the parametric equations $x = t^2 + 2$ and $y = 6 - t^3$ for $t \in \mathbb{R}$. The point on the curve where $t = 2$ is

A $(10, 0)$ **B** $(6, 14)$ **C** $(6, -2)$ **D** $(10, -6)$

12 A curve is defined parametrically by the equations $x = 2\sec t$ and $y = 3\tan t$. The point on the curve where $t = -\dfrac{\pi}{3}$ is

A $(4, 3\sqrt{3})$ **B** $(4, -3\sqrt{3})$ **C** $(3\sqrt{3}, -4)$ **D** $(-4, -3\sqrt{3})$

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Appendix

C

The problem-solving and modelling task

Joel Speranza

Chapter contents

C1 About the problem-solving and modelling task

Mathematisation is the process of taking a real-world problem, translating it to a mathematically purposeful representation, and solving that problem. In Specialist Mathematics you will be assessed on your ability to do this through an assessment item called a problem-solving and modelling task (PSMT).

This chapter outlines how to plan, solve and present your PSMT at a high level and provides real examples of high-level student work. Each section of this chapter is accompanied by a video lesson with additional advice, accessible via the included QR codes.

How is the PSMT marked?

Before you begin any assessment, you should consider how teachers will make judgements of your work. Your teacher will use the Instrument-specific Marking Guide (ISMG) from the syllabus to determine your mark. The ISMG is broken into four criteria (Formulate, Solve, Evaluate and Communicate) and each criterion is assessed using three to five descriptors. Throughout this chapter, we will be focusing on the top descriptor in each criterion. These top descriptors will be displayed in this appendix where needed. The full ISMG with all descriptors can accessed through the Interactive Textbook.

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C2 A content guide for a PSMT report

A suggested set of headings for writing the PSMT report has been provided in a downloadable Word document in the Interactive Textbook. The rest of this appendix provides notes on what to include under these headings, which are reproduced here in black ('Introduction to the task', 'Formulating a solution', 'Developing a solution', and so on). Extracts from the ISMG display the criteria and descriptors covered by each heading of the report.

Video C1

You also need to adhere to the required word limit and conventions of the mathematical report genre. You may have seen another report genre, the scientific report, and the two have similarities.

Word length guides of each section of the report are indicative only and will be dependent on your specific PSMT. Setting out your PSMT in this way is not compulsory but does help you to achieve this highlighted descriptor from the ISMG:

Communicate	Marks
The student response has the following characteristics:	
• correct use of appropriate mathematical language • logical organisation of the response, which can be read independently of the task sheet • justification of decisions using mathematical reasoning	3–4

1. Introduction to the task (200 words)

Video C2

These are the criteria and descriptors to be covered by the introduction:

Formulate	Marks
The student response has the following characteristics:	
• justified statements of important assumptions • justified statements of important observations • justified mathematical translation of important aspects of the task	3–4

Communicate	Marks
The student response has the following characteristics:	
• correct use of appropriate mathematical language • logical organisation of the response, which can be read independently of the task sheet • justification of decisions using mathematical reasoning	3–4

Note that, as long as you understand the task, and have written down what you need to do, you do not have to finalise the introduction at the beginning of the process. You may find it better to write the introduction alongside writing the conclusion, ensuring that the conclusion addresses the goal of the task which was stated in the introduction.

The purpose of the introduction is twofold:

1 To introduce a report which 'can be read independently' of the task sheet. This means that a person who has never seen the task sheet will be able to understand what the task is,

simply by reading your introduction. No reference to the task sheet should be made here. You should also begin with providing some context on what the problem is to be solved, and why solving the problem is important.

2 To show 'justified mathematical translation of important aspects of the task'. Mathematical translation is the process of taking a real-world problem and moving it into the mathematical world – a process known as **mathematisation**. In this part of an introduction, you should give a brief description of the mathematical techniques you will use to solve the problem. This description can include:

- Applicable mathematical or statistical principles
- Mathematical concepts and techniques
- Technology that you will use throughout the task

The student sample below shows:

- how the task can be read independently of the task sheet
- justified mathematical translation of important aspects of the task.

Introduction

As the population of Australia continues to grow, it is crucial that we plan ahead. Important aspects in life such as resources, housing, and medical products and services need to account for this growing population before the demand for necessary requirements of life have such a demand that cannot be catered for (Sommerfeld, 2018). By knowing predictions for a future population count, we can plan ahead and make choices that will benefit the future now.

The purpose of this task is to determine a prediction for the rate of population growth in Australia for 2061 by exploring multiple mathematical models. A multitude of mathematical techniques will be utilised throughout this task, including; modelling of exponential, logarithmic, logistic, and sinusoidal functions, as well as deriving to find a rate of change, determining percentage error, and using technology such as GeoGebra, a TI-84 Plus CE Graphics Calculator and Excel to model equations and manage data.

Student sample taken from the 2020 Mathematical Methods subject report

Using mathematical language?

In the introduction and throughout the PSMT, you must demonstrate correct use of mathematical language.

Video C3

These are the criteria and descriptors to be covered:

Communicate	Marks
The student response has the following characteristics:	
• correct use of appropriate mathematical language • logical organisation of the response, which can be read independently of the task sheet • justification of decisions using mathematical reasoning	3–4

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The syllabus objectives elaborate on mathematical language as ‘terminology, symbols, conventions and representations’. What follows is a non-exhaustive list of ways to demonstrate each aspect of mathematical language.

Terminology

- Procedural mathematical language (mathematical verbs) e.g. determine, solve, verify, calculate integrate, derive.
- Technical mathematical language (mathematical nouns) e.g. search the unit outline for terminology specific to the content being assessed.

Conventions

- Equations have a left and right hand side
- Equal signs are aligned
- Define variables before using them

Symbols

- Use mathematical symbols correctly
- If typing mathematics, use equation editor

Representations: Graphs

- An appropriate title
- Axes labelled with appropriate units
- An appropriate scale for each axis
- A legend if appropriate

Representations: Diagrams

- An appropriate title
- Labelled
- Drawn to scale or a label indicating otherwise
- Vertices of shapes labelled with capital letters
- Angles labelled with Greek letters

2. Formulating a solution (400 words)

In the section titled ‘Formulating a solution’, you are aiming to demonstrate the descriptors of the ISMG shown below.

Formulate	Marks
The student response has the following characteristics:	
• justified statements of important assumptions • justified statements of important observations • justified mathematical translation of important aspects of the task	3–4

It is vital that your assumptions are both important and expressed as justified statements. This is the most important section of the PSMT, as many sections of the PSMT cannot be completed to a high-standard without them. This is demonstrated in the following flowchart.

Observations are considered and assumptions made in order to 'mathematise' the problem.

If the solution does not rely on these, then they are not 'important' and therefore do not meet the criteria. If they are not backed by justified statements, then the evidence for your solution being valid is low.

Once a solution is found, 'justified statements about the reasonableness of the solution by considering observations and assumptions' must be made. If these observations and assumptions are poorly justified or are not important, then it will be difficult to do this.

At the end of the PMST, 'justified statements of relevant strengths and limitations of the solution' are made. These are often found by examining the assumptions and observations that have been made. If assumptions and observations are not important, this section becomes more difficult to complete.

Observations vs assumptions

Students can often be confused about the difference between an observation and an assumption. Both are vital to creating a mathematical model but serve different purposes. The table below outlines the key differences between them.

Aspect	Observations	Assumptions
Definition	Data or information required to solve a mathematical problem and/or develop a mathematical model.	Conditions that are stated to be true when beginning to solve a mathematical problem and/or develop a mathematical model.
Nature and verifiability	Factual, based on actual data or information, and can be empirically verified.	While not directly verifiable, are rational and necessary for solving the problem. They are based on plausible reasoning or existing knowledge.
Role in modelling	Provides data and empirical evidence for creation of the model.	Simplifies complexity and fills data gaps. Dictates the strengths, limitations, and applicability of the model.
Flexibility	Generally rigid; they are facts that can't be altered. However, the interpretation of observations can evolve with new data.	More flexible and can be adjusted or replaced as new information becomes available or as the model evolves.

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Aspect	Observations	Assumptions
Examples	**1** Measuring the temperature and humidity levels in various regions over a year. **2** Recording the frequency and intensity of rainfall in a specific area. **3** The observable fact that warm air rises and cool air sinks, affecting weather patterns.	**1** Assuming that future weather patterns will reflect past trends due to climate consistency. **2** Assuming a certain level of accuracy in satellite data used for cloud cover analysis. **3** Assuming that ocean currents will remain relatively stable over the short-term forecasting period.
Role in validation	Used for validating the model by comparing its predictions with actual observations.	Validated indirectly through the model's performance and its ability to make accurate predictions within the defined framework of these assumptions.

2.1 Observations

Video C4

To ensure that your observations reach the level of 'justified statements' they should include:

- a discussion of how the observations affect the mathematical model/solution
- in-text referencing to a reputable source (while it is possible to justify statements without this, consider the inclusion of a reference the 'gold-standard').

The following student sample demonstrates justified statements of observations. Each observation contains:

- a statement of what the observation is
- a reference that provides justification for the observation being made
- justification for how this observation impacts the mathematical model/solution.

Structuring your observations in this way provide a high likelihood of them being considered justified statements on the ISMG.

Observations

It was observed that the male-to-female ratio in Western Australia is 102 males for every 100 females (McCrindle, 2014). This observation directly impacts the mathematical model as the initial total population must divide by this ratio only to consider the female Western Australian population. This is relevant as males do not reproduce any offspring and cannot be factored into the Leslie matrix.

It was observed that Western Australia takes 30% of immigrants into Australia each year (Australian Bureau of Statistic, 2021). Further, 12,706 to 18,200 immigrants settled in Australia during 2018 and 2020. (Lawrence, 2018). These observations impacted the mathematical model as an increase in immigrants will impact the projected populations, causing an increase or decrease in total population growth rate.

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Humans live to approximately 100 years (Vaupel, 2010); therefore, a 21×21 matrix is reasonable. From this, the age classes for the Leslie matrix were split into 5-year categories. Hence, each new generation produced by the Leslie matrix represents a 5-year gap. This observation is relevant as it impacts the scope mathematical model by investigating only eight new generations.

Student sample taken from the 2022 Specialist Mathematics subject report

2.2 Assumptions

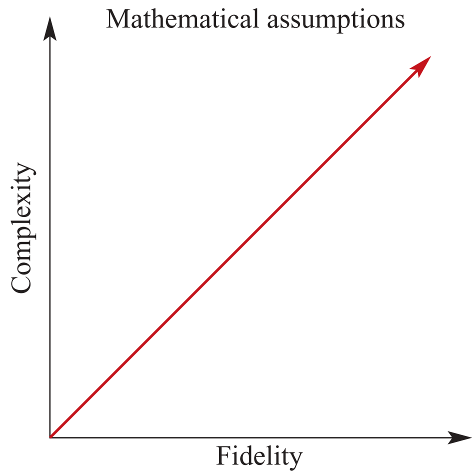
Video C5

As with observations, to justify the assumptions you make you should include discussion of how the assumptions affect the mathematical model/solution; and in-text referencing to a reputable source. While it is possible to justify statements without this, it is considered best practice to include one.

Referencing should be used to justify an assumption and it can be done in two different ways:

- use a source that brings your mathematical model closer to representing the real world
- use a source that takes your mathematical model further from representing the real world but is required to reduce the complexity of the model.

To illustrate this further, each assumption can be thought of as existing on the graph shown below, with mathematical complexity increasing as fidelity (how closely the assumption models the real world) increases. Therefore, each assumption can be thought of as a trade-off between complexity and fidelity. Adopting this approach from the beginning makes future sections of the PSMT (evaluating the reasonableness of the solution and strengths and limitations) easier to complete.



The student sample which follows demonstrates justified statements of assumptions. Each assumption contains:

- a statement of what the assumption is
- a reference that provides justification for the assumption being made
- justification for how the assumption impacts the mathematical model/solution.

Assumptions

It was assumed that birth rates would only impact women aged 15–44 as in Australia. The average woman's reproductive years are between ages 15 and 44 (Watson, 2018). This assumption restricts birth rates to only six of twenty-one age classes. The assumption was made to reduce the anomalies to develop clean data.

It was assumed that the new immigrant population introduced into Western Australia was equally divided into the age classes from 18 to 34. Most migrants to Australia are young adults, with 61.2% aged between 18 and 34 years (abs.gov.au, 2018). Further, 12 706 to 18 200 immigrants settled in Australia during 2018 and 2020. (Lawrence, 2018). This assumption was made to create a realistic data spread that included the possible impact immigrants' survival and or birth rate would have on the total population.

It was assumed that the investigation started during 2016 as the female population data collected was from 2016 (abs.gov.au, 2016); therefore, a 21×21 matrix is reasonable. This assumption impacts the investigation as the potential increasing or decreasing growth rate and total population can be compared to secondary data to determine the model's validity.

Student sample taken from the 2022 Specialist Mathematics subject report.

3. Developing a Solution (700 words)

Video C6

In this section of the PSMT, you are attempting to demonstrate almost all the descriptors of the ISMG, as shown in the following example. This is also a section where students can get a little confused about how to set things out. Below is a flowchart you can use throughout this section to ensure that you demonstrate the full range of ISMG descriptors.

STEP 1 Justify the 'mathematical decision' you have made or the 'mathematical translation' you have performed.

→

STEP 2 Do some mathematics using:

- Equations
- Graphs
- Tables
- Technology
- Diagrams

→

STEP 3 Make a 'justified statement' of one of the following:

- Observation
- Assumption
- The reasonableness of the solution

OR

- Verify your result.

(→ back to STEP 1)

The sample following is a simple example of how this flowchart can be put into practice. It is from a PSMT in which the task is to develop a flying fox. The mathematics has been simplified to allow us to focus on the structure of the response, rather than the mathematics. To guide you, the example has been annotated with:

- in the left margin, the numbered steps from the flowchart above
- in the right margin, the matching ISMG descriptors from the syllabus.

Use them to guide your solution, but do not annotate your own report with them.

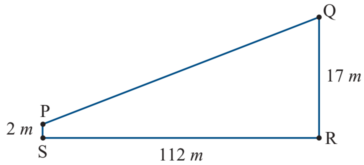
3.1 Calculating the length of the cable

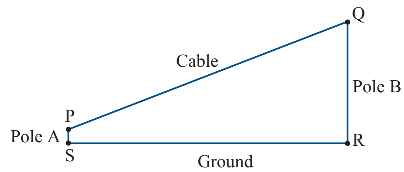
STEP 1

The flying fox cable, two supporting poles and the ground can be modelled as the quadrilateral, *PQRS*, as shown below.

> **Formulate**
> - Justified mathematical translation of important aspects of the task

STEP 2





> **Communicate**
> - Correct use of appropriate mathematical language

STEP 3

It is assumed that the cable is perfectly taut and has no sag. While the cable in a real flying fox will have a sag (Evans, 2017), this assumption allows the cable's length to be calculated.

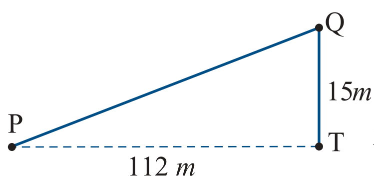
> **Formulate**
> - Justified statements of important assumptions

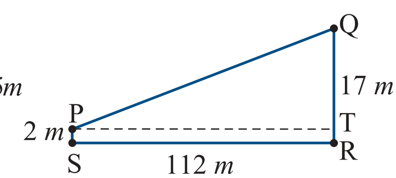
STEP 1

A horizontal line drawn through point *P* creates a right triangle *PQT*. Creating a right triangle will allow the length of the cable to be calculated using Pythagoras' theorem.

> **Communicate**
> - Justification of decisions using mathematical reasoning

STEP 2





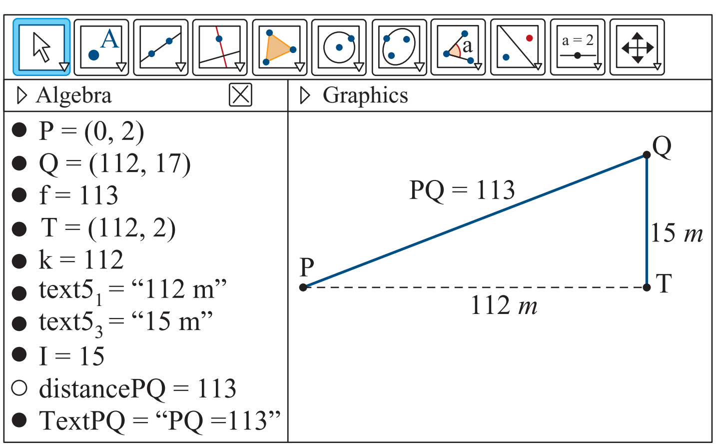
$$\text{Length of cable} = \sqrt{112^2 + 15^2}$$
$$= \sqrt{12544 + 225}$$
$$= \sqrt{12769}$$
$$= 113$$

> **Solve**
> - Accurate use of mathematical knowledge for important aspects of the task

STEP 3

The length of the cable is calculated to be 113 metres.

This result can be verified using a scale diagram drawn in Geogebra, as shown in this screenshot.



> **Solve**
> - Efficient use of technology

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3.2 Calculating the total cost of the cable

STEP 1

The total cost of cable can now be calculated. In construction, it is common to assume that you require an additional 10% of materials to allow for wastage (Jones, 2017). 8 mm aircraft-grade galvanised cable is perfect for ziplines up to 150 metres in length and has a current cost of \$8 per metre. (cable-ride.com, n.d)

Formulate
- Justified statements of important assumptions
- Justified statements of important observations

STEP 2

$$\text{Total cost} = 113 \times 8 \times 1.1$$
$$= \$994.40$$

Solve
- Accurate use of mathematical knowledge for important aspects of the task

Adding an additional 10% to materials is a reasonable solution when considering the assumption made that there is no sag in the cable. In reality there will be sag, and this will increase the amount of the cable required.

Evaluate
- Justified statements about the reasonableness of the solution by considering the assumptions

STEP 3

The \$994.40 total cost for the cable can be verified by comparing it to a 90 metre zipline kit available online for \$1387. (cable-ride.com, n.d). While this is \$392.60 more expensive than our cable, it is for a complete kit rather than just a cable. These two prices are close enough to support our result.

Evaluate
- Verified results

Efficient use of technology

Video C7

Solve
The student response has the following characteristics:
• Efficient use of technology

Throughout your assignment, you should be looking for opportunities to demonstrate the efficient use of technology, as required by the ISMG descriptor shown. The key descriptor here is ‘efficient’, which can be understood as using technology in a way that minimises wasted effort and/or time. An example of efficiency would be using Excel formulas to perform repeated calculations, rather than performing each calculation by hand or on a calculator.

It is often the case that while students use quite a bit of technology throughout their PSMT, they often don’t provide the evidence that they have used it. Here are some types of technology students use and the evidence that they can provide to demonstrate that they have used it.

Types of technology	Evidence provided
Spreadsheet software (e.g. Microsoft Excel)	■ Screenshots of graphs or tables ■ Samples of spreadsheet formulas used
Calculator	■ Photos or screenshots of the calculator screen ■ Descriptions of how the calculator was used
Graphing software (e.g. Desmos or Geogebra)	■ Screenshots ■ Descriptions of how the software was used
Logging software (e.g. data collectors)	■ Sample of the data collected ■ Screenshots

The following examples which evidence efficient use of technology are taken from the 2022 Mathematics subject reports.

To verify this a larger Excel spreadsheet was created to solve the parametric equations for a variety of t values (also extending to ten years):

t	Xleft	Xright	Difference	Intersect?	Yleft	Yright	Difference	Intersect?	Zleft	Zright	Difference	Intersect?
0.5	0.999963	1.999997	1.0000337	FALSE	0.0086	0.001805	0.0067944	FALSE	0	-2	1.999997	FALSE
1	0.999852	1.999987	1.0001349	FALSE	0.017199	0.003611	0.0135882	FALSE	0	-1.99999	1.999987	FALSE

Figure 12: Spreadsheet

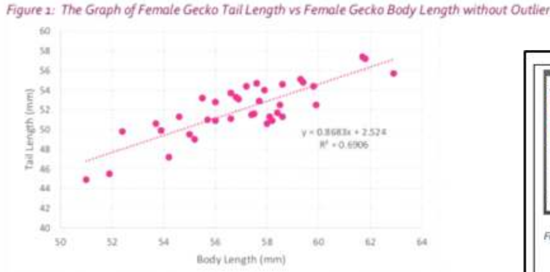
The Excel equation inserted in the 'Intersect?' column (Figure 13), calculated if the difference between the two sides of the parametric equation was less than 4.2587×10^{-5}, the conversion of Earth's radius to AU, to ensure constant units.

```
=IF(D2<(0.0000425875),"YES")
```

Student sample taken from the 2022 Specialist Mathematics subject report

The data without outliers was utilised for further analysis and extrapolation.

Figure 1: The Graph of Female Gecko Tail Length vs Female Gecko Body Length without Outlier



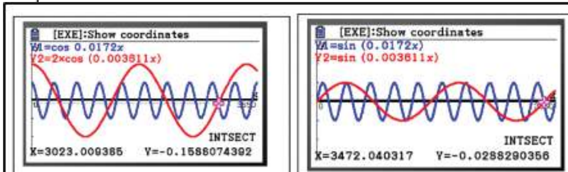


Figure 9: $x: \cos(0.0172t) = 2 \times \cos(0.003611t)$

Figure 10: $y: \sin(0.0172t) = \sin(0.003611t)$

Student sample taken from the 2022 General Mathematics subject report

Student sample taken from the 2022 Specialist Mathematics subject report

4. Evaluating and verifying the solution (600 words)

Video C8

4.1 Verifying results

In this section you are aiming to demonstrate the descriptor shown here, verifying the overall solution. If you have been following the advice from the previous section, you will have been verifying some of your results in the process of coming to your solution. Below, we look at four techniques students can use for verifying results.

Evaluate
The student response has the following characteristics:
• Verified results

1 **Verifying through estimation:** This method simplifies the problem before calculating an approximate solution. For instance, if you have calculated the area of a composite shape, you can verify your solution by simplifying it into a basic rectangle and recalculating the area. If the estimated area is close to your calculated area, it helps confirm the accuracy of your solution.

2 **Verifying through research:** This method involves comparing results with reliable sources. This method is particularly useful for tasks related to historical data or correlation studies. Findings can be matched against data from textbooks, academic journals, or credible online sources. For example, in a task exploring the correlation between car weight and fuel consumption, you can verify the solution against existing research on this topic.

3 **Verifying through technology:** Using technology to redo algebraic calculations can aid in verifying results. For example, when calculating the area under a curve, employing graphing calculators or software for an approximation and comparing it to manual calculations can serve as a verification method.

4 **Verifying through an alternative method:** This method is effective in problems with multiple solutions and involves employing various techniques to solve the same issue. For example, you may initially solve an algebraic equation by factoring, then recheck the solution using the quadratic formula. If both methods produce identical results, it strongly suggests that the solution is correct.

4.2 Evaluating reasonableness of the solution

In this section you are aiming to demonstrate the content descriptor below:

Video C9

Evaluate	Marks
The student response has the following characteristics:	
• verified results • justified statements about the reasonableness of the solution by considering the assumptions • justified statements about the reasonableness of the solution by considering the observations • justified statements of relevant strengths of the solution • justified statements of relevant limitations of the solution	4–5

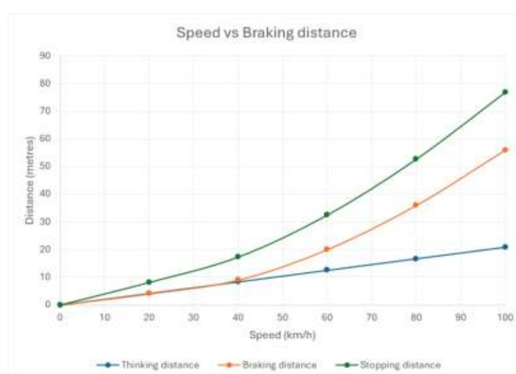
When evaluating the reasonableness of your solution by considering assumptions and observations you should:

- consider the solution found
- consider how it is affected by the observation or assumption
- consider how the solution might be different if the observation or assumption was altered (often with some mathematical working included).

Below is an example of evaluating the reasonableness of a solution by considering an assumption from a PSMT investigating the braking distance of a car dependent on the speed at which it is travelling.

Example

The distance calculated for a car to come to a complete stop is underestimated because of the assumption that driver's response times were instantaneous. If driver response time were factored in, the braking distance would be greater than the distances calculated in this report.



Assuming average driver response time of 1.5 seconds (Muttart, 2004), we can see how the solution would change if this time was taken into account in the graph pictured. At the top speed of 100 km/h, braking distance is increased from the initial solution of 56 m to 77 m, an increase of 37.5%.

Note: Students often make the mistake of evaluating the reasonableness of their assumption, rather than evaluating the reasonableness of their solution by considering their assumption. The distinction is subtle but important. The justified statements made must refer to the solution and how it is affected by the assumption, not just the assumption itself.
Correct: The solution is reasonable because...

Incorrect: The assumption is reasonable because...

A subtle but important difference.

4.3 Strengths and limitations of the solution

In this section you are aiming to demonstrate the content descriptor below:

Video C10

Evaluate	Marks
The student response has the following characteristics:	
• verified results • justified statements about the reasonableness of the solution by considering the assumptions • justified statements about the reasonableness of the solution by considering the observations • justified statements of relevant strengths of the solution • justified statements of relevant limitations of the solution	4–5

Strengths and limitations of the solution can be thought of in the following way.

- Strengths – aspects of the model that make it useful
- Weaknesses – aspects of the model that limit its usefulness

A series of questions to help identify these strengths and limitations is below. Do not aim to answer all these questions but use them as prompts to generate ideas.

Strengths	**Limitations**
What assumptions were made that closely align to the real world?	What assumptions were made that do not align closely with the real world?
What aspects of the real world does the solution consider?	What aspects of the real world does the solution not consider?
Could the method used to create this solution be easily adapted and used to solve other, related problems?	Are there other, related problems that the method used could not be easily adapted to solve?
What aspects of the solution can be verified using other observational data?	What aspects of the solution cannot be verified using observational data?
What are the potential, positive consequences of using this solution in the real world?	What are the potential, negative consequences of using this solution in the real world?
What aspects of the solution will continue to be accurate into the future?	What aspects of the solution will cease to be accurate into the future?

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To make justified statements of strengths/limitations:

- state the strength/limitation
- justify why it is a strength/limitation.

The student sample of limitations below provides examples of this.

The following limitations were observed

There was a limited amount of data points that were used as a sample. This means the findings were less reliable as it may not be an accurate representation of all rugby games.

Another limitation is that when using extrapolation with regards to the regression line, it may not be accurate to predict further outcomes because the prediction is outside the sample data range.

One final limitation is that the R2 value found is not considerably strong, therefore a smaller percentage of the points scored per game can be attributed to the line breaks achieved per game, decreasing the reliability of the study.

Student sample taken from the 2022 General Mathematics subject report

5. Conclusion (100 words)

The conclusion is another opportunity to show logical organisation of your response. In the conclusion you should:

- restate the purpose of the mathematical report
- provide a summary of your solution, stating an appropriate answer to the task.

Conclusion

The purpose of this report was to use functions and derivatives to create a reasonable prediction for the rate of change for the Australian population in 2061. It was found that the most reasonable model was solution three as it produces a reasonable population and somewhat reasonable rates of change. Therefore, using this logistic function, it is predicted that the population will reach approximately 39.6 million in 2061, with a percentage rate of change of 0.95% and an instantaneous rate of change of approximately 370 000 addition people per year.

Student sample taken from the 2022 Mathematical Methods subject report

6. Reference List (Not included in page or word count)

Use a standard referencing style. Ask your teacher for guidance on this if you need it.

7. Appendix (Not included in page or word count)

An appendix is for supporting material such as data, diagrams, calculations and screenshots or print-outs from technology, that don't form a direct part of the solution or evaluation. The appendix is not marked so don't include important items that you want a mark for. If you haven't already included your use of technology in the report, you could put a small sample into the body of your assignment to get marks for efficient use of technology.

Glossary

A

Absolute value function [p. 420] The absolute value of a real number x is defined by

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Also called the *modulus function*

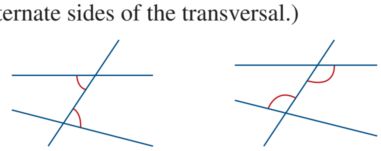
Addition of complex numbers [p. 377] If $z_1 = a + bi$ and $z_2 = c + di$, then $z_1 + z_2 = (a + c) + (b + d)i$.

Addition of vectors [p. 231] If $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j}$ and $\boldsymbol{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j}$, then $\boldsymbol{a} + \boldsymbol{b} = (a_1 + b_1)\boldsymbol{i} + (a_2 + b_2)\boldsymbol{j}$.

Addition principle [p. 147] Suppose there are m ways of performing one task and n ways of performing another task. If we cannot perform both tasks, then there are $m + n$ ways to perform one of the tasks.

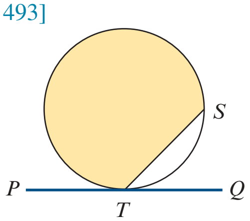
Airspeed [p. 261] the speed of an aircraft relative to air. The actual velocity of an aircraft is the vector sum of the velocity of the aircraft relative to air and the velocity of the wind.

Alternate angles [p. 446] In each diagram, the two marked angles are alternate angles. (They are on alternate sides of the transversal.)



Alternate segment [p. 493]

The alternate segment to $\angle STQ$ is shaded, and the alternate segment to $\angle STP$ is unshaded.



Altitude of a triangle [p. 451] a line segment from a vertex to the opposite side (possibly extended) which forms a right angle where it meets the opposite side

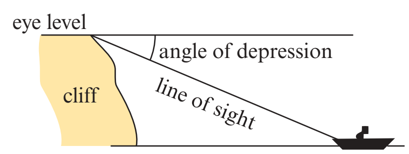
Amplitude of trigonometric functions [p. 570] The distance between the mean position and the maximum position is called the amplitude. The graph of $y = a \sin x$ has an amplitude of $|a|$.

Angle between two vectors [p. 248] can be found using the scalar product:

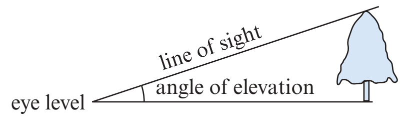
$\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}|\,|\boldsymbol{b}| \cos\theta$

where θ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$

Angle of depression [p. 84] the angle between the horizontal and a direction below the horizontal



Angle of elevation [p. 84] the angle between the horizontal and a direction above the horizontal



Angle sum and difference identities [p. 354]

- $\cos(A + B) = \cos A \cos B - \sin A \sin B$
- $\cos(A - B) = \cos A \cos B + \sin A \sin B$
- $\sin(A + B) = \sin A \cos B + \cos A \sin B$
- $\sin(A - B) = \sin A \cos B - \cos A \sin B$

Angle sum of a polygon [p. 452] The sum of the interior angles of a simple n-sided polygon is $(n - 2)180°$.

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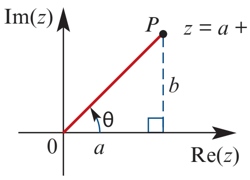
Arc [p. 486] Two points on a circle divide the circle into arcs; the shorter is the *minor arc*, and the longer is the *major arc*.

Area of image [p. 528] If a linear transformation (with matrix **B**) is applied to a region of the plane, then Area of image = |det(**B**)| × Area of region.

Argand diagram [p. 387] a geometric representation of the set of complex numbers



Argument of a complex number [p. 397]

- The argument of z is an angle θ from the positive direction of the x-axis to the line joining the origin to z.
- The *principal value* of the argument, denoted by $\operatorname{Arg} z$, is the angle in the interval $(-\pi, \pi]$.

Arrangement [p. 150] *see* permutation

Asymptote A straight line or curve is an asymptote of the graph of a function $y = f(x)$ if the graph of $y = f(x)$ gets arbitrarily close to a straight line. A straight-line asymptote can be vertical, horizontal or oblique.

Asymptotes of hyperbolas [p. 614] The hyperbola with equation

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

has asymptotes given by

$$y - k = \pm\frac{b}{a}(x-h)$$

Asymptotes of rational functions The rule of a rational function can be written in quotient–remainder form as $f(x) = q(x) + \dfrac{r(x)}{b(x)}$.

- The vertical asymptotes occur where $b(x) = 0$.
- The non-vertical asymptote is given by $y = q(x)$.

B

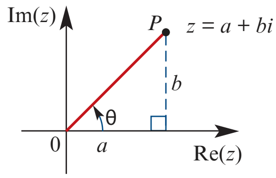
Bearing [p. 85] the compass bearing; the direction measured from north clockwise

C

ℂ [p. 376] the set of complex numbers: $\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$

Cartesian equation [p. 603] an equation that describes a curve in the plane by giving the relationship between the x- and y-coordinates of the points on the curve; e.g. $y = x^2 + 1$

Cartesian form of a complex number [p. 376] A complex number is expressed in Cartesian form as $z = a + bi$, where a is the real part of z and b is the imaginary part of z.

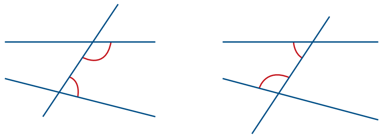


Chord [p. 486] a line segment with endpoints on a circle

Circle, general Cartesian equation [p. 603] The circle with radius r and centre (h, k) has equation $(x-h)^2 + (y-k)^2 = r^2$.

cis θ [p. 397] $\cos\theta + i\sin\theta$

Co-interior angles [p. 446] In each diagram, the two marked angles are co-interior angles. (They are on the same side of the transversal.)



Collinear points [p. 471] Three or more points are collinear if they all lie on a single line.

Column vector [pp. 230, 503] an $n \times 1$ matrix. A column vector $\begin{bmatrix} a \\ b \end{bmatrix}$ can be used to represent a vector in the plane, an ordered pair, a point in the Cartesian plane or a translation of the plane.

Combination [p. 161] a selection where order is not important. The number of combinations of n objects taken r at a time is given by

$${}^nC_r = \frac{n!}{r!\,(n-r)!}$$

An alternative notation for ${}^nC_r$ is $\dbinom{n}{r}$.

Compass bearing [p. 85] the direction measured from north clockwise

Complement of a set [p. 39] The complement of a set A, written A', is the set of all elements of ξ that are not elements of A.

Complementary angles [p. 444] two angles whose sum is 90°

Complementary relationships [p. 575]

- $\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$
- $\sin\left(\frac{\pi}{2} + \theta\right) = \cos\theta$
- $\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$
- $\cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta$

Complex conjugate, $\bar{z}$ [p. 380]

- If $z = a + bi$, then $\bar{z} = a - bi$.
- If $z = r\,\text{cis}\,\theta$, then $\bar{z} = r\,\text{cis}(-\theta)$.

Complex conjugate, properties [p. 381]

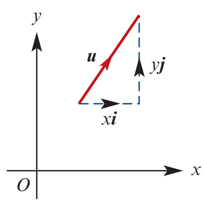
- $z + \bar{z} = 2\,\text{Re}(z)$
- $z\bar{z} = |z|^2$
- $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$
- $\overline{z_1 \cdot z_2} = \overline{z_1} \cdot \overline{z_2}$

Complex number [p. 376] an expression of the form $a + bi$, where a and b are real numbers

Complex plane [p. 387] *see* Argand diagram

Component form [p. 239] Each vector $\boldsymbol{u}$ in the plane can be written in component form as $\boldsymbol{u} = x\boldsymbol{i} + y\boldsymbol{j}$, where

- $\boldsymbol{i}$ is the unit vector in the positive direction of the x-axis
- $\boldsymbol{j}$ is the unit vector in the positive direction of the y-axis.



Composite [p. 50] A natural number m is a composite number if it can be written as a product $m = a \times b$, where a and b are natural numbers greater than 1 and less than m.

Concurrent lines [p. 471] Three or more lines are concurrent if they all pass through a single point.

Conditional statement [p. 194] a statement of the form 'If P is true, then Q is true', which can be abbreviated to $P \Rightarrow Q$

Congruence tests [pp. 68, 455] Two triangles are congruent if one of the following conditions holds:

- **SSS** the three sides of one triangle are equal to the three sides of the other triangle
- **SAS** two sides and the included angle of one triangle are equal to two sides and the included angle of the other triangle
- **AAS** two angles and one side of one triangle are equal to two angles and the matching side of the other triangle
- **RHS** the hypotenuse and one side of a right-angled triangle are equal to the hypotenuse and one side of another right-angled triangle.

Congruent figures [pp. 68, 455] have exactly the same shape and size

Constant velocity [p. 255] If a particle moves with a constant velocity of $\boldsymbol{v}$ m/s for t seconds, then its displacement vector, $\boldsymbol{s}$ m, is given by $\boldsymbol{s} = t\boldsymbol{v}$.

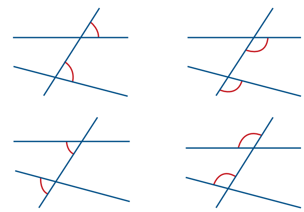
Contrapositive [p. 200] The contrapositive of $P \Rightarrow Q$ is the statement (not Q) $\Rightarrow$ (not P). The contrapositive is equivalent to the original statement.

Converse [p. 207] The converse of $P \Rightarrow Q$ is the statement $Q \Rightarrow P$.

Conversion between Cartesian and polar forms [pp. 243, 397]

$x = r\cos\theta, \quad y = r\sin\theta, \quad r^2 = x^2 + y^2$

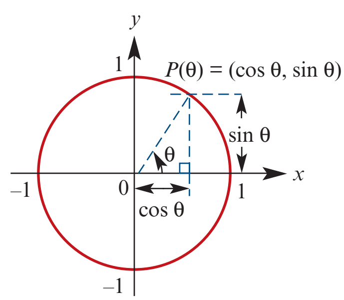
Corresponding angles [p. 446] In each diagram, the two marked angles are corresponding angles.



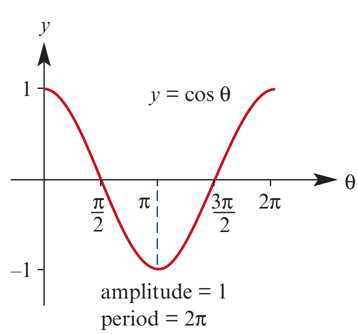
Cosecant function [pp. 432, 350]

$\text{cosec}\,\theta = \dfrac{1}{\sin\theta}$ for $\sin\theta \neq 0$

Cosine function [p. 570] cosine θ is defined as the x-coordinate of the point P on the unit circle where OP forms an angle of θ radians with the positive direction of the x-axis.



Cosine function, graph [p. 570]



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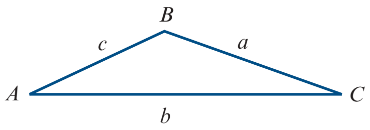
Cosine rule [p. 81] For triangle ABC:
$a^2 = b^2 + c^2 - 2bc\cos A$



Cotangent function [pp. 433, 350]
$\cot\theta = \dfrac{\cos\theta}{\sin\theta}$ for $\sin\theta \neq 0$

Counterexample [p. 211] an example that shows that a universal statement is false. For example, the number 2 is a counterexample to the claim 'Every prime number is odd.'

Cyclic quadrilateral [p. 488] a quadrilateral such that all its vertices lie on a circle. Opposite angles of a cyclic quadrilateral are supplementary.

D

De Moivre's theorem [p. 402]
$(r\,\text{cis}\,\theta)^n = r^n\,\text{cis}(n\theta)$, where $n \in \mathbb{Z}$

De Morgan's laws [p. 199]
- 'not (P and Q)' is '(not P) or (not Q)'
- 'not (P or Q)' is '(not P) and (not Q)'

Degree of a polynomial [p. 99] given by the highest power of x with a non-zero coefficient; e.g. the polynomial $2x^5 - 7x^2 + 4$ has degree 5

Determinant of a matrix [pp. 298, 309] Associated with each square matrix $\mathbf{A}$, there is a real number called the determinant of $\mathbf{A}$, which is denoted by $\det(\mathbf{A})$. A square matrix $\mathbf{A}$ has an inverse if and only if $\det(\mathbf{A}) \neq 0$.

If $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $\det(\mathbf{A}) = ad - bc$.

Diameter [p. 486] a chord of a circle that passes through the centre

Dilation [p. 508] A dilation scales the x- or y-coordinate of each point in the plane.
- Dilation from the x-axis: $(x, y) \to (x, cy)$
- Dilation from the y-axis: $(x, y) \to (cx, y)$

Direct proof [p. 195] To give a direct proof of a conditional statement $P \Rightarrow Q$, we assume that P is true and show that Q follows.

Discriminant, Δ, of a quadratic [p. 104] the expression $b^2 - 4ac$, which is part of the quadratic formula. For the quadratic equation $ax^2 + bx + c = 0$:
- If $b^2 - 4ac > 0$, there are two real solutions.
- If $b^2 - 4ac = 0$, there is one real solution.
- If $b^2 - 4ac < 0$, there are no real solutions.

Disjoint sets [p. 38] Sets A and B are said to be disjoint if they have no elements in common, i.e. if $A \cap B = \emptyset$.

Displacement [p. 254] the change in position. If a particle moves from point A to point B, then its displacement is described by the vector $\overrightarrow{AB}$.

Division of complex numbers [pp. 382, 399]
$\dfrac{z_1}{z_2} = \dfrac{z_1}{z_2} \times \dfrac{\overline{z_2}}{\overline{z_2}} = \dfrac{z_1\overline{z_2}}{|z_2|^2}$
If $z_1 = r_1\,\text{cis}\,\theta_1$ and $z_2 = r_2\,\text{cis}\,\theta_2$, then
$\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2}\,\text{cis}(\theta_1 - \theta_2)$

Dot product [p. 247] *see* scalar product

Double-angle identities [p. 357]
- $\cos(2A) = \cos^2 A - \sin^2 A$
  $= 2\cos^2 A - 1$
  $= 1 - 2\sin^2 A$
- $\sin(2A) = 2\sin A\,\cos A$

E

Ellipse [pp. 525, 609] The graph of the equation
$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$
is an ellipse centred at the point (h, k).

Equality of complex numbers [p. 377]
$a + bi = c + di$ if and only if $a = c$ and $b = d$

Equally likely outcomes [p. 182] If the outcomes of a random experiment are all equally likely, then the probability of an event A is given by
$$P(A) = \frac{\text{number of outcomes in } A}{\text{total number of outcomes}}$$

Equilibrium [p. 264] A particle is said to be in equilibrium if the resultant force acting on it is zero; the particle will remain at rest or continue moving with constant velocity.

Equivalence of vectors [p. 239]
Let $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j}$ and $\boldsymbol{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j}$. If $\boldsymbol{a} = \boldsymbol{b}$, then $a_1 = b_1$ and $a_2 = b_2$.

Equivalent statements [p. 208]
Statements P and Q are equivalent if $P \Rightarrow Q$ and $Q \Rightarrow P$; this is abbreviated to $P \Leftrightarrow Q$.
For equivalent statements P and Q, we also say 'P is true *if and only if* Q is true'.

Event [p. 182] a subset of the sample space. It may consist of a single outcome, or it may consist of several outcomes.

Existence statement [pp. 210, 212] a statement claiming that a property holds for some member of a given set. Such a statement can be written using the quantifier 'there exists'.

F

Factor [p. 50] A natural number a is a factor of a natural number b if there exists a natural number k such that $b = ak$.

Factorial notation [p. 150] The notation $n!$ (read as 'n factorial') is an abbreviation for the product of all the integers from n down to 1:
$n! = n \times (n-1) \times (n-2) \times (n-3) \times \cdots \times 2 \times 1$

Force [p. 264] causes a change in motion; e.g. gravitational force, tension force, normal reaction force. Force is a vector quantity.

Formula [p. 19] an equation containing symbols that states a relationship between two or more quantities; e.g. $A = \ell w$ (area = length × width). The value of A, the subject of the formula, can be found by substituting given values of ℓ and w.

H

Highest common factor [p. 52] The highest common factor of two natural numbers a and b, denoted by HCF(a, b), is the largest natural number that is a factor of both a and b.

Hyperbola [p. 613] The graph of the equation

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

is a hyperbola centred at the point (h, k); the asymptotes are given by

$$y - k = \pm\frac{b}{a}(x-h)$$

I

Imaginary number i [p. 376] $i^2 = -1$

Imaginary part of a complex number [p. 377] If $z = a + bi$, then $\text{Im}(z) = b$. Note that $\text{Im}(z)$ is a real number.

Implication [p. 194] *see* conditional statement

Inclusion–exclusion principle [p. 177] allows us to count the number of elements in a union of sets. In the case of two sets:
$|A \cup B| = |A| + |B| - |A \cap B|$

Index laws [p. 2]

- $a^m \times a^n = a^{m+n}$
- $a^m \div a^n = a^{m-n}$
- $(a^m)^n = a^{mn}$
- $(ab)^n = a^n b^n$
- $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$
- $a^{-n} = \frac{1}{a^n}$
- $\frac{1}{a^{-n}} = a^n$
- $a^0 = 1$
- $a^{\frac{1}{n}} = \sqrt[n]{a}$
- $a^{\frac{m}{n}} = \left(a^{\frac{1}{n}}\right)^m$

Integers [p. 41] the elements of
$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$

Intersection of sets [p. 39] The intersection of two sets A and B, written $A \cap B$, is the set of all elements common to A and B.

Interval [p. 44] a subset of the real numbers of the form $[a, b]$, $[a, b)$, (a, ∞), etc.

Irrational number [p. 41] a real number that is not rational; e.g. π and $\sqrt{2}$

K

Kilogram weight, kg wt [p. 264] a unit of force. If an object on the surface of the Earth has a mass of 1 kg, then the gravitational force acting on this object is 1 kg wt.

L

Like surds [p. 47] surds with the same irrational factor; e.g. $2\sqrt{7}$ and $9\sqrt{7}$

Linear equation [p. 8] a polynomial equation of degree 1; e.g. $2x + 1 = 0$

Linear transformation [p. 503] a transformation of the plane with a rule of the form
$(x, y) \to (ax + by, cx + dy)$
Each linear transformation can be represented by a 2×2 matrix:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Linear transformation, inverse [p. 519] If $\mathbf{A}$ is the matrix of a linear transformation and $\mathbf{A}$ is invertible, then $\mathbf{A}^{-1}$ is the matrix of the inverse transformation.

Linear transformations, composition [p. 516] If $\mathbf{A}$ and $\mathbf{B}$ are the matrices of two different linear transformations, then the product $\mathbf{BA}$ is the matrix of the transformation $\mathbf{A}$ followed by $\mathbf{B}$.

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Literal equation [p. 25] an equation for the variable x in which the coefficients of x, including the constants, are pronumerals; e.g. $ax + b = c$

Locus [p. 603] a set of points described by a geometric condition; e.g. the locus of points P that satisfy $PO = 3$, where O is the origin, is the circle of radius 3 centred at the origin

Lowest common multiple [p. 53] The lowest common multiple of two natural numbers a and b, denoted by LCM(a, b), is the smallest natural number that is a multiple of both a and b.

M

Magnitude of a vector [p. 239] the length of a directed line segment corresponding to the vector. If $\boldsymbol{u} = x\boldsymbol{i} + y\boldsymbol{j}$, then $|\boldsymbol{u}| = \sqrt{x^2 + y^2}$.

Mass [p. 264] The mass of an object is the amount of matter it contains, and can be measured in kilograms. Mass is not the same as weight.

Mathematical induction [p. 214] a proof technique for showing that a statement is true for all natural numbers; uses the *principle of mathematical induction*

Matrices, addition [p. 289] Addition is defined for two matrices of the same size (same number of rows and same number of columns). The sum is found by adding corresponding entries. For example:

$$\begin{bmatrix}1 & 0\\0 & 2\end{bmatrix} + \begin{bmatrix}0 & -3\\4 & 1\end{bmatrix} = \begin{bmatrix}1 & -3\\4 & 3\end{bmatrix}$$

Matrices, equal [p. 287] Two matrices $\mathbf{A}$ and $\mathbf{B}$ are equal, and we write $\mathbf{A} = \mathbf{B}$, when:

- they have the same number of rows and the same number of columns, and
- they have the same entry at corresponding positions.

Matrices, multiplication [p. 293] The product of two matrices $\mathbf{A}$ and $\mathbf{B}$ is only defined if the number of columns of $\mathbf{A}$ is the same as the number of rows of $\mathbf{B}$. If $\mathbf{A}$ is an $m \times n$ matrix and $\mathbf{B}$ is an $n \times r$ matrix, then the product $\mathbf{AB}$ is the $m \times r$ matrix whose entries are determined as follows:

> To find the entry in row i and column j of $\mathbf{AB}$, single out row i in matrix $\mathbf{A}$ and column j in matrix $\mathbf{B}$. Multiply the corresponding entries from the row and column and then add up the resulting products.

Matrix [p. 285] a rectangular array of numbers

Matrix, dimensions [p. 285] A matrix that has m rows and n columns is said to be an $m \times n$ matrix.

Matrix, identity [pp. 296, 306] For square matrices of a given size (e.g. 2×2), a multiplicative identity $\mathbf{I}$ exists.

For 2×2 matrices, the identity is $\mathbf{I} = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$ and $\mathbf{AI} = \mathbf{A} = \mathbf{IA}$ for each 2×2 matrix $\mathbf{A}$.

Matrix, inverse [pp. 297, 306] If $\mathbf{A}$ is a square matrix and there exists a matrix $\mathbf{B}$ such that $\mathbf{AB} = \mathbf{I} = \mathbf{BA}$, then $\mathbf{B}$ is called the inverse of $\mathbf{A}$. When it exists, the inverse of a square matrix $\mathbf{A}$ is unique and is denoted by $\mathbf{A}^{-1}$.

If $\mathbf{A} = \begin{bmatrix}a & b\\c & d\end{bmatrix}$, then $\mathbf{A}^{-1} = \dfrac{1}{ad - bc}\begin{bmatrix}d & -b\\-c & a\end{bmatrix}$ provided $ad - bc \neq 0$.

Matrix, multiplication by a scalar [p. 289] If $\mathbf{A}$ is an $m \times n$ matrix and k is a real number, then $k\mathbf{A}$ is an $m \times n$ matrix whose entries are k times the corresponding entries of $\mathbf{A}$. For example:

$$3\begin{bmatrix}2 & -2\\0 & 1\end{bmatrix} = \begin{bmatrix}6 & -6\\0 & 3\end{bmatrix}$$

Matrix, non-singular [p. 297] A square matrix is said to be non-singular if the determinant of the matrix is not zero 0.

Matrix, singular [p. 297] A square matrix is said to be singular if it does not have an inverse.

Matrix, size [p. 285] A matrix that has m rows and n columns is said to be an $m \times n$ matrix.

Matrix, square [p. 296] A matrix with the same number of rows and columns is called a square matrix; e.g. a 2×2 matrix.

Matrix, zero [p. 290] The $m \times n$ matrix with all entries equal to zero is called the zero matrix and is usually denoted by $\mathbf{O}$.

Matrix algebra [pp. 289–296] Some properties of arithmetic operations on $n \times n$ matrices:

- $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$ — commutative law
- $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$ — associative law
- $\mathbf{A} + \mathbf{O} = \mathbf{A}$ — zero matrix
- $\mathbf{A} + (-\mathbf{A}) = \mathbf{O}$ — additive inverse
- $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$ — associative law
- $\mathbf{AI} = \mathbf{A} = \mathbf{IA}$ — identity matrix
- $\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$ — distributive law
- $(\mathbf{B} + \mathbf{C})\mathbf{A} = \mathbf{BA} + \mathbf{CA}$ — distributive law

Note: Matrix multiplication is not commutative.

Median of a triangle [p. 451] a line segment from a vertex to the midpoint of the opposite side

Midpoint of a line segment [p. 234] If M is the midpoint of line segment AB, then

$$\overrightarrow{OM} = \frac{1}{2}(\overrightarrow{OA} + \overrightarrow{OB})$$

Modulus–argument form of a complex number [p. 397] *see* polar form of a complex number

Modulus function [p. 420] The modulus of a real number x is defined by

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Also called the *absolute value function*

Modulus of a complex number, $|z|$ [p. 381] the distance of the complex number from the origin. If $z = a + bi$, then $|z| = \sqrt{a^2 + b^2}$.

Modulus, properties [p. 399] For complex numbers z_1 and z_2:

- $|z_1 z_2| = |z_1|\,|z_2|$ (the modulus of a product is the product of the moduli)
- $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$ (the modulus of a quotient is the quotient of the moduli)

Multiple [p. 53] A natural number a is a multiple of a natural number b if there exists a natural number k such that $a = kb$.

Multiplication of a complex number by i [pp. 389, 399] corresponds to a rotation about the origin by 90° anticlockwise. If $z = a + bi$, then $iz = i(a + bi) = -b + ai$.

Multiplication of a vector by a scalar [p. 232] If $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j}$ and $m \in \mathbb{R}$, then $m\boldsymbol{a} = ma_1\boldsymbol{i} + ma_2\boldsymbol{j}$.

Multiplication of complex numbers [pp. 380, 399] If $z_1 = a + bi$ and $z_2 = c + di$, then

$z_1 z_2 = (ac - bd) + (ad + bc)i$

If $z_1 = r_1 \operatorname{cis} \theta_1$ and $z_2 = r_2 \operatorname{cis} \theta_2$, then

$z_1 z_2 = r_1 r_2 \operatorname{cis}(\theta_1 + \theta_2)$

Multiplication principle [p. 146] If there are m ways of performing one task and then there are n ways of performing another task, then there are $m \times n$ ways of performing *both* tasks.

N

$n!$ [p. 150] *see* factorial notation

Natural numbers [p. 41] the elements of $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$

Negation [p. 199] The negation of a statement P is the opposite statement, called 'not P'. For example, the negation of $x \geq 1$ is $x < 1$.

Normal reaction force [p. 265] A mass placed on a surface (horizontal or inclined) experiences a force perpendicular to the surface, called the normal force.

O

Ordered pair [p. 42] An ordered pair, denoted (x, y), is a pair of elements x and y in which x is considered to be the first coordinate and y the second coordinate.

P

Parallelogram [pp. 234, 456] a quadrilateral with both pairs of opposite sides parallel

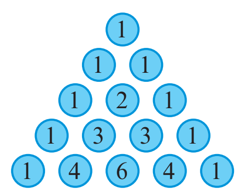
Parametric equations [p. 618] a pair of equations $x = f(t)$ and $y = g(t)$ describing a curve in the plane, where t is called the *parameter* of the curve

Partial fractions [p. 113] Some rational functions may be expressed as a sum of partial fractions; e.g.

$$\frac{A}{ax + b} + \frac{B}{cx + d} + \frac{C}{(cx + d)^2} + \frac{Dx + E}{ex^2 + fx + g}$$

Particle model [p. 264] an object is considered as a point. This can be done when the size of the object can be neglected in comparison with other lengths in the problem being considered, or when rotational motion effects can be ignored.

Pascal's triangle [p. 170] a triangular pattern of numbers formed by the binomial coefficients ${}^nC_r$. Each entry of Pascal's triangle is the sum of the two entries immediately above.



Period of a function [p. 570] A function f with domain $\mathbb{R}$ is periodic if there is a positive constant a such that $f(x + a) = f(x)$ for all x. The smallest such a is called the period of f. For example, the period of the sine function is 2π, as $\sin(x + 2\pi) = \sin x$.

Permutation [p. 150] an ordered arrangement of objects. The number of permutations of n objects taken r at a time is given by

$${}^nP_r = \frac{n!}{(n - r)!}$$

Pigeonhole principle [p. 173] If $n + 1$ or more objects are placed into n holes, then some hole contains at least two objects.

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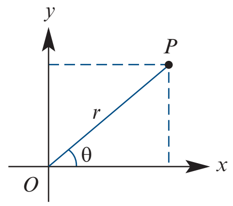
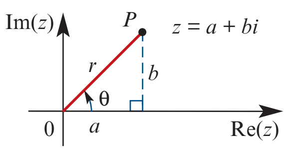
Polar coordinates A point P in the plane has polar coordinates (r, θ), where

- r is the distance from the origin O to P
- θ is the angle between the positive direction of the x-axis and the ray OP.

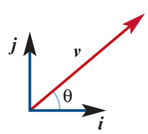


Polar form of a complex number [p. 397] A complex number is expressed in polar form as $z = r \operatorname{cis} \theta$, where r is the modulus of z and θ is an argument of z. This is also called *modulus–argument form*.



Polar form of a vector [p. 243] Each vector $\boldsymbol{v}$ in the plane can be written in polar form as $\boldsymbol{v} = [r, \theta]$, where

- r is the magnitude of $\boldsymbol{v}$
- θ describes the angle that $\boldsymbol{v}$ makes with the $\boldsymbol{i}$-direction, measured anticlockwise.



Polynomial function [p. 99] A polynomial has a rule of the type

$$y = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0, \quad n \in \mathbb{N} \cup \{0\}$$

where $a_0, a_1, \ldots, a_n$ are numbers called coefficients.

Position vector [p. 233] A position vector, $\overrightarrow{OP}$, indicates the position in space of the point P relative to the origin O.

Prime [p. 50] A natural number greater than 1 is a prime number if its only factors are itself and 1.

Prime decomposition [p. 51] expressing a composite number as a product of powers of prime numbers; e.g. $500 = 2^2 \times 5^3$

Principle of mathematical induction [p. 214] used to prove that a statement is true for all natural numbers

Probability [p. 182] a numerical value assigned to the likelihood of an event occurring. If the event A is impossible, then $P(A) = 0$; if the event A is certain, then $P(A) = 1$; otherwise $0 < P(A) < 1$.

Product-to-sum identities [p. 364]

- $2 \cos A \cos B = \cos(A - B) + \cos(A + B)$
- $2 \sin A \sin B = \cos(A - B) - \cos(A + B)$
- $2 \sin A \cos B = \sin(A + B) + \sin(A - B)$

Projection [p. 510] A projection maps each point in the plane onto an axis.

- Projection onto the x-axis: $(x, y) \to (x, 0)$
- Projection onto the y-axis: $(x, y) \to (0, y)$

Proof by contradiction [p. 203] a proof that begins by assuming the negation of what is to be proved

Pythagoras' theorem [p. 460] For a right-angled triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides:

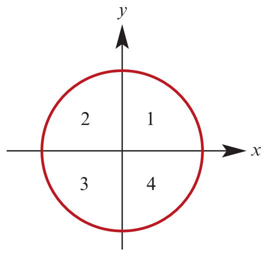
$(\text{hyp})^2 = (\text{opp})^2 + (\text{adj})^2$

Pythagorean identity [pp. 575, 351]

- $\cos^2 \theta + \sin^2 \theta = 1$
- $1 + \tan^2 \theta = \sec^2 \theta$
- $\cot^2 \theta + 1 = \operatorname{cosec}^2 \theta$

Q

Quadrants [p. 88] The coordinate axes divide the unit circle into four quadrants, which are numbered anticlockwise as shown.



Quadratic formula [p. 103] An equation of the form $az^2 + bz + c = 0$, with $a \neq 0$, may be solved quickly by using the quadratic formula:

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Quadratic function [p. 103] A quadratic has a rule of the form $y = ax^2 + bx + c$, where a, b and c are constants and $a \neq 0$.

Quadratic surd [p. 46] a number of the form $\sqrt{a}$, where a is a rational number which is not the square of another rational number

Quantifier [p. 210] *see* existence statement, universal statement

Glossary Q

R

$\mathbb{R}^+$ [p. 45] $\{ x : x > 0 \}$, positive real numbers

$\mathbb{R}^-$ [p. 45] $\{ x : x < 0 \}$, negative real numbers

$\mathbb{R} \setminus \{0\}$ [p. 45] the set of real numbers excluding 0

$\mathbb{R}^2$ [p. 42] $\{ (x, y) : x, y \in \mathbb{R} \}$; i.e. $\mathbb{R}^2$ is the set of all ordered pairs of real numbers

Radian [p. 569] One radian (written 1^c) is the angle subtended at the centre of the unit circle by an arc of length 1 unit:

$$1^c = \frac{180^\circ}{\pi} \quad \text{and} \quad 1^\circ = \frac{\pi^c}{180}$$

Random experiment [p. 182] an experiment, such as the rolling of a die, in which the outcome of a single trial is uncertain but observable

Rate [p. 108] describes how a certain quantity changes with respect to the change in another quantity (often time)

Rational function [pp. 113] a function with a rule of the form

$$f(x) = \frac{P(x)}{Q(x)}$$

where $P(x)$ and $Q(x)$ are polynomials

Rational number [p. 41] a number that can be written as $\frac{p}{q}$, for some integers p and q with $q \neq 0$

Real part of a complex number [p. 377] If $z = a + bi$, then $\text{Re}(z) = a$.

Reciprocal function [p. 427] The reciprocal of the function $y = f(x)$ is defined by $y = \frac{1}{f(x)}$.

Reciprocal trigonometric functions [pp. 432, 350] the secant, cosecant and cotangent functions

Recurrence relation [p. 218] a rule which enables each subsequent term of a sequence to be found from previous terms; e.g. $t_1 = 1$, $t_n = t_{n-1} + 2$

Reflection [p. 507] A reflection in a line ℓ maps each point in the plane to its mirror image on the other side of the line.

- Reflection in the x-axis: $(x, y) \to (x, -y)$
- Reflection in the y-axis: $(x, y) \to (-x, y)$
- Reflection in the line $y = x$: $(x, y) \to (y, x)$
- Reflection in the line $y = -x$: $(x, y) \to (-y, -x)$

Reflection matrix [p. 514] A reflection in the line $y = mx = x \tan\theta$ is expressed using matrix multiplication as

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Regular polygon [p. 450] a polygon in which all the angles are equal and all the sides are equal

Resultant force [p. 264] the vector sum of the forces acting at a point

Rhombus [p. 456] a parallelogram with all sides of equal length

Rotation matrix [p. 513] A rotation about the origin by angle θ anticlockwise is expressed using matrix multiplication as

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

S

Sample space [p. 182] the set of all possible outcomes for a random experiment

Scalar [p. 232] a real number; name used when working with vectors or matrices

Scalar product [p. 247] The scalar product of two vectors $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j}$ and $\boldsymbol{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j}$ is given by

$\boldsymbol{a} \cdot \boldsymbol{b} = a_1 b_1 + a_2 b_2$

Scalar product, properties [p. 248]

- $\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{b} \cdot \boldsymbol{a}$
- $k(\boldsymbol{a} \cdot \boldsymbol{b}) = (k\boldsymbol{a}) \cdot \boldsymbol{b} = \boldsymbol{a} \cdot (k\boldsymbol{b})$
- $\boldsymbol{a} \cdot \boldsymbol{0} = 0$
- $\boldsymbol{a} \cdot (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \cdot \boldsymbol{c}$
- $\boldsymbol{a} \cdot \boldsymbol{a} = |\boldsymbol{a}|^2$

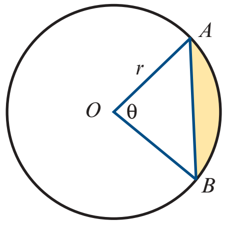
Scalar quantity [p. 254] a quantity determined by its magnitude; e.g. distance, time, speed, mass

Scientific notation [p. 5] A number is in standard form when written as a product of a number between 1 and 10 and an integer power of 10; e.g. 6.626×10^{-34}.

Secant function [pp. 432, 350] $\sec\theta = \frac{1}{\cos\theta}$ for $\cos\theta \neq 0$

Secant of a circle [p. 486] a line that cuts a circle at two distinct points

Segment [p. 487] Every chord divides the interior of a circle into two regions called segments; the smaller is the *minor segment* (shaded), and the larger is the *major segment*.



Selection [p. 161] *see* combination

Sequence [p. 218] a list of numbers, with the order being important; e.g. 1, 1, 2, 3, 5, 8, 13, . . . The numbers of a sequence are called its *terms*, and the *n*th term is often denoted by t_n.

Set notation [p. 38]

$\in$	means 'is an element of'
$\notin$	means 'is not an element of'
$\subseteq$	means 'is a subset of'
$\cap$	means 'intersection'
$\cup$	means 'union'
$\emptyset$	is the empty set, containing no elements
ξ	is the universal set, containing all elements being considered
A'	is the complement of a set A
$\lvert A\rvert$	is the number of elements in a finite set A

Sets of numbers [pp. 41, 376]

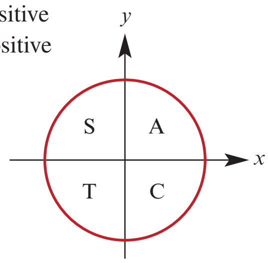
$\mathbb{N}$	is the set of natural numbers
$\mathbb{Z}$	is the set of integers
$\mathbb{Q}$	is the set of rational numbers
$\mathbb{R}$	is the set of real numbers
$\mathbb{C}$	is the set of complex numbers

Shear [p. 509] A shear moves each point in the plane by an amount proportional to its distance from an axis.

- Shear parallel to the x-axis: $(x, y) \to (x + cy, y)$
- Shear parallel to the y-axis: $(x, y) \to (x, cx + y)$

Signs of trigonometric functions [p. 574]

1st quadrant	all are positive
2nd quadrant	sin is positive
3rd quadrant	tan is positive
4th quadrant	cos is positive



Similar figures [p. 464] Two figures are similar if we can enlarge one figure so that its enlargement is congruent to the other figure.

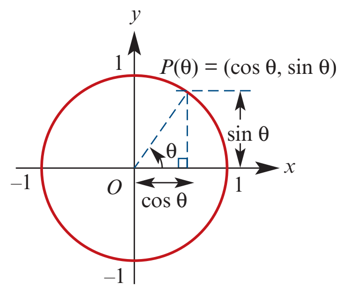
Similarity tests [p. 465] Two triangles are similar if one of the following conditions holds:

- **AAA** two angles of one triangle are equal to two angles of the other triangle
- **SAS** the ratios of two pairs of matching sides are equal and the included angles are equal
- **SSS** the ratios of matching sides are equal
- **RHS** the ratio of the hypotenuses of two right-angled triangles equals the ratio of another pair of sides.

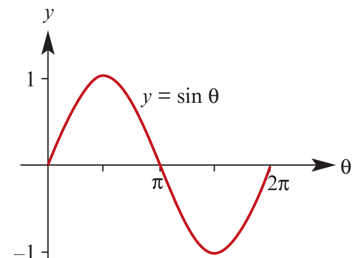
Simplest form [p. 46] a surd $\sqrt{a}$ is in simplest form if the number under the square root has no factors which are squares of a rational number

Simultaneous equations [pp. 8, 119, 303] equations of two or more lines or curves in the Cartesian plane, the solutions of which are the points of intersection of the lines or curves

Sine function [p. 570] sin θ is defined as the y-coordinate of the point P on the unit circle where OP forms an angle of θ radians with the positive direction of the x-axis.



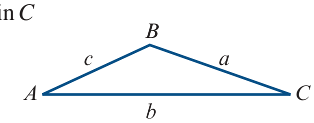
Sine function, graph [p. 570]



amplitude = 1
period = 2π

Sine rule [p. 77] For triangle ABC:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$



Speed [p. 255] the magnitude of velocity

Standard form [p. 5] *see* scientific notation

Strictly decreasing A function f is strictly decreasing on an interval if $x_2 > x_1$ implies $f(x_2) < f(x_1)$.

Strictly increasing A function f is strictly increasing on an interval if $x_2 > x_1$ implies $f(x_2) > f(x_1)$.

Subtraction of complex numbers [p. 378] If $z_1 = a + bi$ and $z_2 = c + di$, then $z_1 - z_2 = (a - c) + (b - d)i$.

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Subtraction of vectors [p. 232]
If $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j}$ and $\boldsymbol{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j}$, then $\boldsymbol{a} - \boldsymbol{b} = (a_1 - b_1)\boldsymbol{i} + (a_2 - b_2)\boldsymbol{j}$.

Sum-to-product identities [p. 365]

- $\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$
- $\cos A - \cos B = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$
- $\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$
- $\sin A - \sin B = 2\sin\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right)$

Supplementary angles [p. 444] two angles whose sum is 180°

Surd of order *n* [p. 46] a number of the form $\sqrt[n]{a}$, where a is a rational number which is not a perfect nth power

Surd, quadratic [p. 46] a number of the form $\sqrt{a}$, where a is a rational number which is not the square of another rational number

T

Tangent function [pp. 571, 582] $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ for $\cos\theta \neq 0$

Tangent to a circle [p. 492] a line that touches the circle at exactly one point, called the *point of contact*

Tension force [p. 265] the pulling force exerted by a string that connects two objects. The forces at each end of the string have equal magnitude.

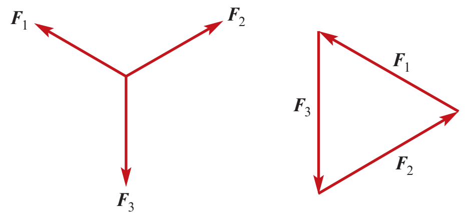
Transformation [p. 503] A transformation of the plane maps each point (x, y) in the plane to a new point (x', y'). We say that (x', y') is the *image* of (x, y).

Translation [p. 511] a transformation that moves each point in the plane in the same direction and over the same distance: $(x, y) \to (x + a, y + b)$

Trapezium [p. 456] a quadrilateral with at least one pair of opposite sides parallel

Triangle inequality [p. 451] If a, b and c are the side lengths of a triangle, then $a < b + c$, $b < c + a$ and $c < a + b$.

Triangle of forces [p. 265] If three forces are acting on a point in equilibrium, then they can be represented by three vectors forming a triangle.



Trigonometric equations [p. 586]

- For $a \in [-1, 1]$, the general solution of the equation $\cos(x) = a$ is $x = 2n\pi \pm \cos^{-1}(a)$, where $n \in \mathbb{Z}$.
- For $a \in [-1, 1]$, the general solution of the equation $\sin(x) = a$ is $x = 2n\pi + \sin^{-1}(a)$ or $x = (2n + 1)\pi - \sin^{-1}(a)$, where $n \in \mathbb{Z}$.
- For $a \in \mathbb{R}$, the general solution of the equation $\tan(x) = a$ is $x = n\pi + \tan^{-1}(a)$, where $n \in \mathbb{Z}$.

Trigonometric functions [pp. 570, 571] the sine, cosine and tangent functions

Trigonometric functions, exact values [pp. 74, 573]

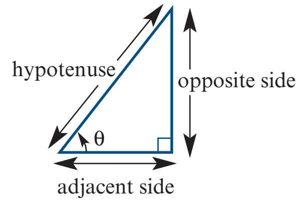
θ^c	$\theta°$	$\sin\theta$	$\cos\theta$	$\tan\theta$
0	0°	0	1	0
$\frac{\pi}{6}$	30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
$\frac{\pi}{4}$	45°	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
$\frac{\pi}{3}$	60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$\frac{\pi}{2}$	90°	1	0	undefined

Trigonometric ratios [p. 69]

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$



U

Union of sets [p. 38] The union of two sets A and B, written $A \cup B$, is the set of all elements which are in A or B or both.

Unit vector [p. 240] a vector of magnitude 1. The unit vectors in the positive directions of the x- and y-axes are $\boldsymbol{i}$ and $\boldsymbol{j}$ respectively. The unit vector in the direction of $\boldsymbol{a}$ is given by

$$\hat{\boldsymbol{a}} = \frac{1}{|\boldsymbol{a}|}\boldsymbol{a}$$

Universal statement [pp. 210, 211] a statement claiming that a property holds for all members of a given set. Such a statement can be written using the quantifier 'for all'.

V

Vector [p. 230] a set of equivalent directed line segments

Vector quantity [p. 254] a quantity determined by its magnitude and direction; e.g. position, displacement, velocity, force

Vectors, parallel [p. 233] Two non-zero vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel if and only if $\boldsymbol{a} = k\boldsymbol{b}$ for some $k \in \mathbb{R} \setminus \{0\}$.

Vectors, perpendicular [p. 248] Two non-zero vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular if and only if $\boldsymbol{a} \cdot \boldsymbol{b} = 0$.

Vectors, properties [pp. 230–232]

- $\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{b} + \boldsymbol{a}$ commutative law
- $(\boldsymbol{a} + \boldsymbol{b}) + \boldsymbol{c} = \boldsymbol{a} + (\boldsymbol{b} + \boldsymbol{c})$ associative law
- $\boldsymbol{a} + \boldsymbol{0} = \boldsymbol{a}$ zero vector
- $\boldsymbol{a} + (-\boldsymbol{a}) = \boldsymbol{0}$ additive inverse
- $m(\boldsymbol{a} + \boldsymbol{b}) = m\boldsymbol{a} + m\boldsymbol{b}$ distributive law

Vectors, resolution [p. 251] A vector $\boldsymbol{a}$ is resolved into rectangular components by writing it as a sum of two vectors, one parallel to a given vector $\boldsymbol{b}$ and the other perpendicular to $\boldsymbol{b}$.

The *vector resolute* of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$ is given by

$$\boldsymbol{u} = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{\boldsymbol{b} \cdot \boldsymbol{b}}\boldsymbol{b}$$

Velocity [p. 255] the rate of change of position with respect to time. Velocity is a vector quantity.

Velocity, relative [p. 260] The relative velocity of an object A with respect to another object B is the velocity that object A would appear to have to an observer moving along with object B.

Velocity, resultant [p. 259] the sum of two or more velocity vectors. For example, if a train is travelling north at 60 km/h and a passenger walks at 3 km/h towards the back of the train, then the passenger's resultant velocity is 57 km/h north.

Velocity, true [p. 260] the velocity of an object measured relative to Earth

W

Weight [p. 264] On the Earth's surface, a mass of m kg has a force of m kg wt acting on it; this force is known as the weight.

Z

Zero vector, 0 [p. 232] a line segment of zero length with no direction

Answers

Chapter 1

Exercise 1A

1 a x^7 b a^2 c x^3 d y^{-4}
e x^{12} f p^{-7} g $a^{-\frac{1}{6}}$ h a^{-8}
i y^{14} j x^{15} k a^{-12} l x^2
m n^2 n $8x^{\frac{7}{2}}$ o a p x^4
q $\frac{1}{2n^6}$ r $-8x^2$ s $a^{-2}b^5$ t 1

2 a 5 b 4 c $\frac{4}{3}$ d $\frac{1}{4}$
e $\frac{6}{7}$ f 3 g 12 h 16
i 27 j $\frac{3}{2}$ k 1 l 8

3 a 18.92 b 79.63 c 5.89
d 125 000 e 0.90 f 1.23
g 0.14 h 1.84 i 0.40

4 a a^4b^7 b $64a^4b^7$ c b
d a^6b^9 e $2a^4b^7$ f $\frac{a^2b^5}{128}$

5 2^{2n-4}

6 6^{3x}

7 a $\left(\frac{1}{2}\right)^{\frac{1}{6}}$ b $a^{\frac{11}{20}}$ c $2^{\frac{5}{6}}$
d $2^{\frac{19}{6}}$ e $2^{\frac{3}{5}}$

8 a $a^{\frac{1}{3}}b$ b $a^{\frac{5}{2}}b^{\frac{1}{2}}$ c $ab^{\frac{1}{5}}$
d $\left(\frac{b}{a}\right)^{\frac{1}{2}}$ e $a^{\frac{5}{2}}b^{\frac{1}{2}}c^{-4}$ f $a^{\frac{1}{5}}b^{\frac{3}{5}}$
g $a^{-4}b^{\frac{7}{2}}c^5$

Exercise 1B

1 a 4.78×10 b 6.728×10^3
c 7.923×10 d 4.358×10^4
e 2.3×10^{-3} f 5.6×10^{-7}
g $1.200\,034 \times 10$ h 5.0×10^7
i 2.3×10^{10} j 1.3×10^{-9}
k 1.65×10^5 l 1.4567×10^{-5}

2 a 2.99×10^{-23} b 3.1536×10^7
c 3.057×10^{21}

3 a 1 390 000 000 b 0.000 0075
c 0.000 000 000 000 0056

4 a 4.569×10^2 b 3.5×10^4
c 5.6791×10^3 d 4.5×10^{-2}
e 9.0×10^{-2} f 4.5682×10^3

5 a 1 b 4.8×10^{-2}
c 2.16×10^{11} d 2.5×10^{-3}
e 2×10^5 f 2×10^7

6 a 11.8 b 4.76×10^7

Exercise 1C

1 a $x = \frac{8}{3}$ b $x = 48$ c $x = -\frac{20}{3}$
d $x = 63$ e $x = -0.7$ f $x = 2.4$
g $x = 0.3$ h $x = -6$ i $x = -\frac{15}{92}$
j $x = -\frac{21}{17}$

2 a $x = \frac{160}{9}$ b $x = 19.2$ c $x = -4$
d $x = \frac{80}{51}$ e $x = 6.75$ f $x = -\frac{85}{38}$
g $x = \frac{487}{13}$ h $x = \frac{191}{91}$

3 a $x = \frac{18}{13},\ y = -\frac{14}{13}$ **b** $x = \frac{16}{11},\ y = -\frac{18}{11}$
c $x = 12,\ y = 17$ **d** $x = 8,\ y = 2$
e $x = 0,\ y = 2$ **f** $x = 1,\ y = 6$

Exercise 1D

1 a $4(x-2) = 60;\ x = 17$
b $\left(\frac{2x+7}{4}\right)^2 = 49;\ x = 10.5$
c $x - 5 = 2(12 - x);\ x = \frac{29}{3}$ **d** $y = 6x - 4$
e $Q = np$ **f** $R = 1.1pS$ **g** $\frac{60n}{5} = 2400$
h $a = \frac{\pi}{3}(x+3)$
2 \$2500
3 Eight dresses and three handbags
4 8.375 m by 25.125 m
5 \$56.50
6 Nine
7 20, 34 and 17
8 Annie 165, Belinda 150, Cassie 189
9 15 km/h
10 2.04×10^{-23} g
11 30 pearls
12 Oldest \$48, Middle \$36, Youngest \$12
13 98%
14 25 students
15 After 20 minutes
16 a 40 minutes **b** 90 minutes **c** 20 minutes
17 200 km
18 39 km/h

Exercise 1E

1 140.625 km **2** 51
3 10 000 adults **4** Men \$220; boys \$160
5 127 and 85
6 500 adults, 1100 students
7 252 litres 40% and 448 litres 15%
8 120 and 100; 60 **9** \$370 588

Exercise 1F

1 a 25 **b** 330 **c** 340.47 **d** 1653.48
e 612.01 **f** 77.95 **g** 2.42 **h** 2.1
i ±9.43 **j** ±9.54
2 a $a = \frac{v-u}{t}$ **b** $\ell = \frac{2S}{n} - a$ **c** $b = \frac{2A}{h}$
d $I = \pm\sqrt{\frac{P}{R}}$ **e** $a = \frac{2(s-ut)}{t^2}$
f $v = \pm\sqrt{\frac{2E}{m}}$ **g** $h = \frac{Q^2}{2g}$ **h** $x = \frac{-z}{y}$
i $x = \frac{-b(c+y)}{a-c}$ **j** $x = \frac{-b(c+1)}{m-c}$

3 a 82.4°F **b** $C = \frac{5(F-32)}{9}$; 57.22°C
4 a 1080° **b** $n = \frac{S}{180} + 2$; 9 sides
5 a 115.45 cm^3 **b** 12.53 cm **c** 5.00 cm
6 a 66.5 **b** 4 **c** 11

Exercise 1G

1 a $\frac{13x}{6}$ **b** $\frac{5a}{4}$ **c** $\frac{-h}{8}$ **d** $\frac{5x-2y}{12}$
e $\frac{3y+2x}{xy}$ **f** $\frac{7x-2}{x(x-1)}$
g $\frac{5x-1}{(x-2)(x+1)}$ **h** $\frac{-7x^2-36x+27}{2(x+3)(x-3)}$
i $\frac{4x+7}{(x+1)^2}$ **j** $\frac{5a^2+8a-16}{8a}$
k $\frac{4(x^2+1)}{5x}$ **l** $\frac{2x+5}{(x+4)^2}$
m $\frac{3x+14}{(x-1)(x+4)}$ **n** $\frac{x+14}{(x-2)(x+2)}$
o $\frac{7x^2+28x+16}{(x-2)(x+2)(x+3)}$ **p** $\frac{(x-y)^2-1}{x-y}$
q $\frac{4x+3}{x-1}$ **r** $\frac{3-2x}{x-2}$

2 a $2xy^2$ **b** $\frac{xy}{8}$ **c** $\frac{2}{x}$ **d** $\frac{x}{y^2}$
e $\frac{a}{3}$ **f** $\frac{1}{2x}$ **g** $\frac{x-1}{x+4}$ **h** $x+2$
i $\frac{x-1}{x}$ **j** $\frac{a}{4b}$ **k** $\frac{2x}{x+2}$ **l** $\frac{x-1}{4x}$
m $\frac{x+1}{2x}$ **n** $\frac{1}{3}x(x+3)$
o $\frac{x-2}{3x(3x-2)(x+5)}$

3 a $\frac{3}{x-3}$ **b** $\frac{4x-14}{x^2-7x+12}$
c $\frac{5x-1}{x^2+x-12}$ **d** $\frac{2x^2+10x-6}{x^2+x-12}$
e $\frac{2x-9}{x^2-10x+25}$ **f** $\frac{5x-8}{(x-4)^2}$
g $\frac{1}{3-x}$ **h** $\frac{23-3x}{x^2+x-12}$
i $\frac{5x^2-3x}{x^2-9}$ **j** $\frac{11-2x}{x^2-10x+25}$
k $\frac{12}{(x-6)^3}$ **l** $\frac{9x-25}{x^2-7x+12}$

4 a $\frac{3-x}{\sqrt{1-x}}$ **b** $\frac{2\sqrt{x-4}+6}{3\sqrt{x-4}}$ **c** $\frac{5}{\sqrt{x+4}}$
d $\frac{x+7}{\sqrt{x+4}}$ **e** $-\frac{12x^2}{\sqrt{x+4}}$ **f** $\frac{9x^2(x+2)}{2\sqrt{x+3}}$

5 a $\frac{6x-4}{(6x-3)^{\frac{2}{3}}}$ **b** $\frac{3}{(2x+3)^{\frac{2}{3}}}$ **c** $\frac{3-3x}{(x-3)^{\frac{2}{3}}}$

Exercise 1H

1 a $x = \frac{m-n}{a}$ b $x = \frac{b}{b-a}$ c $x = -\frac{bc}{a}$
d $x = \frac{5}{p-q}$ e $x = \frac{m+n}{n-m}$ f $x = \frac{ab}{1-b}$
g $x = 3a$ h $x = -mn$ i $x = \frac{a^2-b^2}{2ab}$
j $x = \frac{p-q}{p+q}$ k $x = \frac{3ab}{b-a}$ l $x = \frac{1}{3a-b}$
m $x = \frac{p^2+p^2t+t^2}{q(p+t)}$ n $x = -\frac{5a}{3}$

4 a $x = \frac{d-bc}{1-ab}, y = \frac{c-ad}{1-ab}$
b $x = \frac{a^2+ab+b^2}{a+b}, y = \frac{ab}{a+b}$
c $x = \frac{t+s}{2a}, y = \frac{t-s}{2b}$ d $x = a+b, y = a-b$
e $x = c, y = -a$ f $x = a+1, y = a-1$

5 a $s = a(2a+1)$ b $s = \frac{2a^2}{1-a}$
c $s = \frac{a^2+a+1}{a(a+1)}$ d $s = \frac{a}{(a-1)^2}$
e $s = 3a^3(3a+1)$ f $s = \frac{3a}{a+2}$
g $s = 2a^2 - 1 + \frac{1}{a^2}$ h $s = \frac{5a^2}{a^2+6}$

Chapter 1 review

Short-response questions

Technology-free

1 a x^{12} b y^{-9} c $-15x^{\frac{11}{2}}$ d x^{-1}

2 3.22×10^{11}

3 a $\frac{2x+y}{10}$ b $\frac{4y-7x}{xy}$
c $\frac{7x-1}{(x+2)(x-1)}$ d $\frac{7x+20}{(x+2)(x+4)}$
e $\frac{13x^2+2x+40}{2(x+4)(x-2)}$ f $\frac{3(x-4)}{(x-2)^2}$

4 a $\frac{2}{x}$ b $\frac{x-4}{4x}$ c $\frac{x^2-4}{3}$ d $4x^2$

5 a 2×10^6 photos, i.e. 2 million photos
b 2×10^5 seconds ($\approx$ 55.6 hours)

6 12

7 88 classical, 80 blues, 252 heavy metal

8 a 300π cm^3 b $h = \frac{V}{\pi r^2}$; $\frac{117}{5\pi}$ cm
c $r = \sqrt{\frac{V}{\pi h}}$; $\sqrt{\frac{128}{\pi}}$ cm $= \frac{8\sqrt{2}}{\sqrt{\pi}}$ cm

9 a $x = \frac{b}{a+y}$ b $x = \frac{a+b}{c}$
c $x = \frac{2ab}{b-a}$ d $x = \frac{ab+b^2d-d^2}{d(a+b)}$

10 a $\frac{p^2+q^2}{p^2-q^2}$ b $\frac{x+y}{x(y-x)}$
c $(x-2)(2x-1)$ d $\frac{2}{a}$

11 A 36; B 12; C 2

12 a $a = 8, b = 18$ b $x = p+q, y = 2q$

13 $x = 3.5$

14 a $4n^2k^2$ b $\frac{40cx^2}{ab^2}$

15 $x = -1$

16 a $\frac{b(a+b)}{a(b-a)}$ b $-\frac{c}{d}$ c $\frac{y+1}{y-1}$
d $-\frac{1}{p+q}$

17 $\frac{ab}{4a+3b}$

18 a Thomas a; George $\frac{3a}{2}$; Sally $a-18$; Zeb $\frac{a}{3}$; Henry $\frac{5a}{6}$
b $\frac{3a}{2} + a - 18 + \frac{a}{3} = a + \frac{5a}{6} + 6$
c $a = 24$; Thomas 24; Henry 20; George 36; Sally 6; Zeb 8

19 $\frac{10\sqrt{3}}{3}$ cm

20 -40

21 $\frac{240}{11}$ km/h

Technology-active

22 0.38 23 \$71 894.95 24 30.9 m/s

25 7.16

26 a 2.78×10^{-1} b 6.23×10^{-4}
c 1.65×10^{-2} d 4.43×10^{-15}

27 5.07×10^2 seconds

28 length = 17.5 cm, width = 9.3 cm

29 a $\frac{5x}{4}$ hours b $\frac{4x}{7}$ hours c $\frac{19x}{28}$ hours
d i $x = \frac{14}{19} \approx 0.737$
ii Jack $\frac{140}{19} \approx 7$ km; Benny $\frac{560}{19} \approx 29$ km

30 a 18 000 cm^3 per minute b $V = 18\,000t$
c $h = \frac{45t}{4\pi}$ d After

31 a 1.9×10^{-8} N b $m_1 = \frac{Fr^2 10^{11}}{6.67m_2}$
c 9.8×10^{24} kg

32 a $V = (1.8 \times 10^7)d$ b 5.4×10^8 m^3
c $k = 9.81 \times 10^3$ d 1.325×10^{15} J
e 1202 days (to the nearest day)

33 a $h = 20 - r$
b i $V = \left(20r^2 - \frac{r^3}{3}\right)\pi$
ii $r = 5.94$ cm; $h = 14.06$ cm

34 a $\frac{2}{3}$ litre from A; $\frac{1}{3}$ litre from B

b 600 mL from A; 400 mL from B

c $\frac{(p-q)(n+m)}{2(np-qm)}$ litres from A,

$\frac{(n-m)(p+q)}{2(np-qm)}$ litres from B, where

$\frac{n}{m} \neq \frac{q}{p}$ and one of $\frac{n}{m}$ or $\frac{q}{p}$ is ≥ 1 and the other is ≤ 1

35 a $h = 2(10 - r)$ **b** $V = 2\pi r^2(10 - r)$

c $r = 3.4986$, $h = 13.0029$ or $r = 9.0224$, $h = 1.9551$

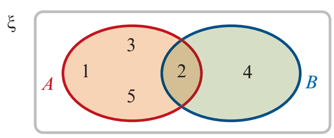
Multiple-choice questions

1 A **2** A **3** C **4** A **5** B
6 D **7** B **8** B **9** B **10** B
11 A **12** C **13** D **14** B **15** B
16 D **17** A

Chapter 2

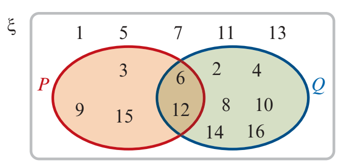
Exercise 2A

1



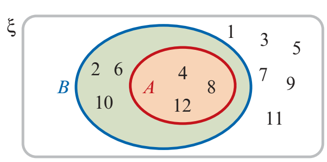
a $\{4\}$ **b** $\{1, 3, 5\}$ **c** $\{1, 2, 3, 4, 5\} = \xi$
d $\emptyset$ **e** $\emptyset$

2



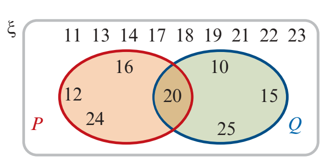
a $\{1, 2, 4, 5, 7, 8, 10, 11, 13, 14, 16\}$
b $\{1, 3, 5, 7, 9, 11, 13, 15\}$
c $\{2, 3, 4, 6, 8, 9, 10, 12, 14, 15, 16\}$
d $\{1, 5, 7, 11, 13\}$ **e** $\{1, 5, 7, 11, 13\}$

3



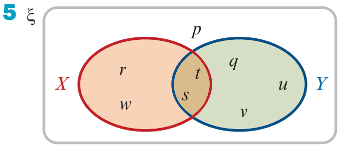
a $\{1, 2, 3, 5, 6, 7, 9, 10, 11\}$
b $\{1, 3, 5, 7, 9, 11\}$ **c** $\{2, 4, 6, 8, 10, 12\}$
d $\{1, 3, 5, 7, 9, 11\}$ **e** $\{1, 3, 5, 7, 9, 11\}$

4



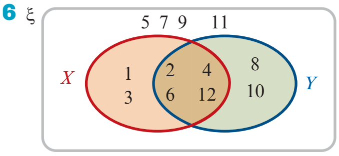
a $\{10, 11, 13, 14, 15, 17, 18, 19, 21, 22, 23, 25\}$
b $\{11, 12, 13, 14, 16, 17, 18, 19, 21, 22, 23, 24\}$
c $\{10, 12, 15, 16, 20, 24, 25\}$
d $\{11, 13, 14, 17, 18, 19, 21, 22, 23\}$
e $\{11, 13, 14, 17, 18, 19, 21, 22, 23\}$

5



a $\{p, q, u, v\}$ **b** $\{p, r, w\}$ **c** $\{p\}$
d $\{p, q, r, u, v, w\}$ **e** $\{q, r, s, t, u, v, w\}$ **f** $\{p\}$

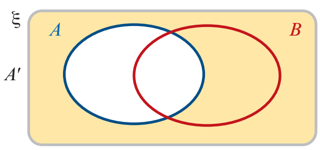
6



a $\{5, 7, 8, 9, 10, 11\}$ **b** $\{1, 3, 5, 7, 9, 11\}$
c $\{1, 3, 5, 7, 8, 9, 10, 11\}$
d $\{1, 3, 5, 7, 8, 9, 10, 11\}$
e $\{1, 2, 3, 4, 6, 8, 10, 12\}$ **f** $\{5, 7, 9, 11\}$

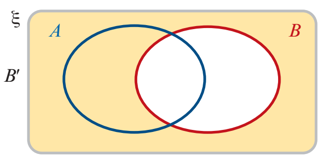
7 a

A'



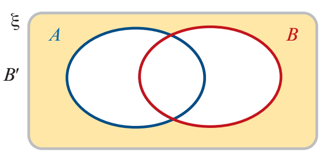
b

B'



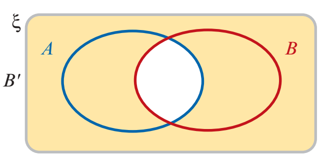
c

$A' \cap B'$



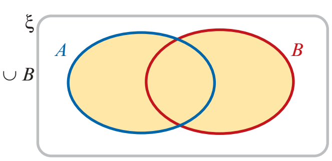
d

$A' \cup B'$



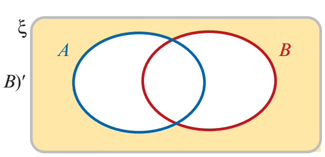
e

$A \cup B$



f

$(A \cup B)'$



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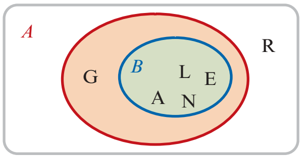
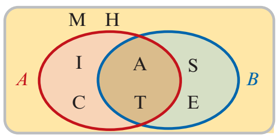
8 ξ



a {R} **b** {G, R} **c** {L, E, A, N}
d {A, N, G, E, L} **e** {R} **f** {G, R}

9 ξ



a {E, H, M, S} **b** {C, H, I, M}
c {A, T} **d** {H, M} **e** {C, E, H, I, M, S}
f {H, M}

Exercise 2B

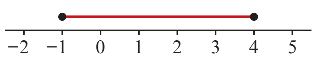
1 a Yes **b** Yes **c** Yes

2 a No **b** No **c** No

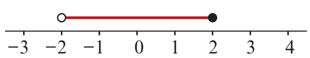
3 a $\frac{9}{20}$ **b** $\frac{3}{11}$ **c** $\frac{3}{25}$ **d** $\frac{2}{7}$ **e** $\frac{4}{11}$ **f** $\frac{2}{9}$

4 a $0.\dot{2}8571\dot{4}$ **b** $0.4\dot{5}$ **c** 0.35
d $0.\dot{3}0769\dot{2}$ **e** $0.\dot{0}58823529411764\dot{7}$

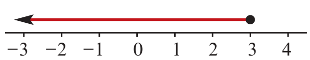
5 a



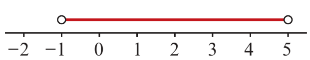
b



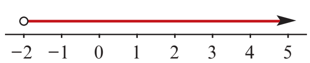
c



d



e



6 a $(-\infty, 3)$ **b** $[-3, \infty)$ **c** $(-\infty, -3]$
d $(5, \infty)$ **e** $[-2, 3)$ **f** $[-2, 3]$
g $(-2, 3]$ **h** $(-5, 3)$

Exercise 2C

1 a $2\sqrt{2}$ **b** $2\sqrt{3}$ **c** $3\sqrt{3}$ **d** $5\sqrt{2}$
e $3\sqrt{5}$ **f** $11\sqrt{10}$ **g** $7\sqrt{2}$ **h** $6\sqrt{3}$
i 5 **j** $5\sqrt{3}$ **k** $16\sqrt{2}$

2 a $3\sqrt{2}$ **b** $6\sqrt{3}$ **c** $4\sqrt{7}$
d $5\sqrt{10}$ **e** $28\sqrt{2}$ **f** 0

3 a $11\sqrt{3} + \sqrt{14}$ **b** $5\sqrt{7}$
c 0 **d** $\sqrt{2} + \sqrt{3}$
e $5\sqrt{2} + 15\sqrt{3}$ **f** $\sqrt{2} + \sqrt{5}$

4 a $\frac{\sqrt{5}}{5}$ **b** $\frac{\sqrt{7}}{7}$ **c** $\frac{-\sqrt{2}}{2}$
d $\frac{2\sqrt{3}}{3}$ **e** $\frac{\sqrt{6}}{2}$ **f** $\frac{\sqrt{2}}{4}$
g $\sqrt{2} - 1$ **h** $2 + \sqrt{3}$ **i** $\frac{4 + \sqrt{10}}{6}$
j $\sqrt{6} - 2$ **k** $\frac{\sqrt{5} + \sqrt{3}}{2}$ **l** $3(\sqrt{6} + \sqrt{5})$
m $3 + 2\sqrt{2}$

5 a $6 + 4\sqrt{2}$ **b** $9 + 4\sqrt{5}$ **c** $-1 + \sqrt{2}$
d $4 - 2\sqrt{3}$ **e** $\frac{2\sqrt{3}}{9}$ **f** $\frac{8 + 5\sqrt{3}}{11}$
g $\frac{3 + \sqrt{5}}{2}$ **h** $\frac{6 + 5\sqrt{2}}{14}$

6 a $4a - 4\sqrt{a} + 1$
b $3 + 2x + 2\sqrt{(x+1)(x+2)}$

7 $3\sqrt{5}, 4\sqrt{3}, 7, 5\sqrt{2}$

8 a $5 - 3\sqrt{2}$ **b** $7 - 2\sqrt{6}$

9 a $\frac{3}{\sqrt{2}}$ **b** $\frac{\sqrt{5}}{2}$ **c** $\frac{\sqrt{5}}{5}$ **d** $\frac{8}{\sqrt{3}}$

10 a $b = 0,\ c = -3$ **b** $b = 0,\ c = -12$
c $b = -2,\ c = -1$ **d** $b = -4,\ c = 1$
e $b = -6,\ c = 1$
f $b = -7 + 5\sqrt{5},\ c = -58 - 13\sqrt{5}$

11 $\frac{3\sqrt{2} + 2\sqrt{3} - \sqrt{30}}{12}$

12 b $-1 - 2^{\frac{1}{3}} - 2^{\frac{2}{3}}$

13 3

Exercise 2D

1 a $2^2 \times 3 \times 5$ **b** $2^2 \times 13^2$
c $2^2 \times 3 \times 19$ **d** $2^2 \times 3^2 \times 5^2$
e $2^2 \times 3^2 \times 7$ **f** $2^2 \times 3^2 \times 5^2 \times 7$
g $2^5 \times 3 \times 5 \times 11 \times 13$
h $2^5 \times 3 \times 7 \times 11 \times 13$
i $2^5 \times 7 \times 11 \times 13$
j $2^5 \times 7 \times 11 \times 13 \times 17$

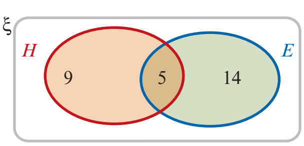
2 a 1 **b** 27 **c** 5 **d** 31 **e** 6

3 a 18: 1, 2, 3, 6, 9, 18; 36: 1, 2, 3, 4, 6, 9, 18, 36
b 36 is a square number ($36 = 6 \times 6$)
c 121 has factors 1, 11 and 121

4 5, 14 and 15 **5** 121
6 15 **7** 105
8 8 **9** 4
10 1:12 p.m.
11 600 and 108 000; 2400 and 27 000; 3000 and 21 600; 5400 and 12 000

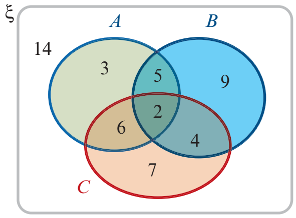
Exercise 2E

1 a ξ



b i 19 **ii** 9 **iii** 23

2 a ξ



b i 23 **ii** 37 **iii** 9

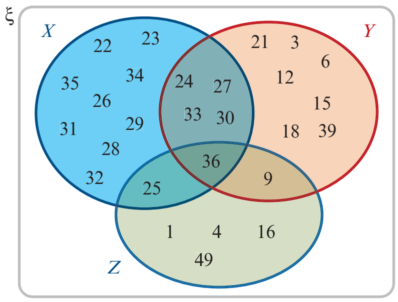
3 20%

4 a 5 **b** 10

5 45

6 a $x = 5$ **b** 16 **c** 0

7 a ξ



b i $X \cap Y \cap Z = \{36\}$ **ii** $|X \cap Y| = 5$

8 31 students; 15 black, 12 green, 20 red

9 $|M \cap F| = 11$ **10** 1

11 $x = 6$; 16 students **12** 102 students

Chapter 2 review

Short-response questions

Technology-free

1 a $\frac{7}{90}$ **b** $\frac{5}{11}$ **c** $\frac{1}{200}$

d $\frac{81}{200}$ **e** $\frac{4}{15}$ **f** $\frac{6}{35}$

2 $2^3 \times 3^2 \times 7$

3 a $\frac{2\sqrt{6} - \sqrt{2}}{2}$ **b** $4\sqrt{5} + 9$ **c** $2\sqrt{6} + 5$

4 $-23 - 12\sqrt{3}$

5 a $2\sqrt{6} + 6$ **b** $\frac{a - \sqrt{a^2 - b^2}}{b}$

6 a 15 **b** 15

7 a 1 **b** 22 **c** 22

8 5 **9** 2 cm^2

10 $-15\sqrt{7}$ **11** $x = \pm 2$

12 $\sqrt{5} - \sqrt{6}$ **13** $\frac{51\sqrt{3}}{5}$

14 a 57 **b** 3 **c** 32

15 $2\sqrt{2} + 3$

17 Tom is 36 and Fred is 27

18 a $a = 6,\ b = 5$ **b** $p = 26,\ q = 16$

c $a = -1,\ b = \frac{2}{3}$

d i $\frac{12\sqrt{3} - 19}{71}$ **ii** $3 \pm \sqrt{3}$ **iii** $\frac{1 \pm \sqrt{3}}{2}$

e $\mathbb{Q} = \{a + 0\sqrt{3} : a \in \mathbb{Q}\}$

19 d $x = \pm 2$

21 a $b = -4,\ c = 1$ **b** $2 + \sqrt{3}$

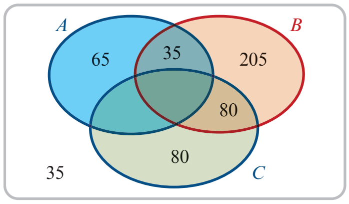
22 a i Region 8

ii Male, red hair, blue eyes

iii Male, not red hair, blue eyes

b i 5 **ii** 182

23 a



$|A \cap C| = 0$

b 160 **c** 65 **d** 0

Technology-active

24 a HCF $= 2^5 \times 3^5 \times 5 \times 13 = 505\,440$,

LCM $= 2^6 \times 3^5 \times 5^3 \times 11^2 \times 13$

$= 3\,057\,912\,000$

b HCF $= 2^5 \times 3^5 \times 5 \times 13 = 505\,440$,

LCM $= 2^7 \times 3^5 \times 5^3 \times 7 \times 13$

$= 353\,808\,000$

25 505 440

26 140

27 6

28 $2^3 \times 3 \times 5^2 \times 7^2 \times 11$; $70\sqrt{66}$

29 a $\sqrt{11} + \sqrt{3}$

b $2\sqrt{2} - \sqrt{7}$ or $\sqrt{7} - 2\sqrt{2}$

c $3\sqrt{3} - 2\sqrt{6}$ or $2\sqrt{6} - 3\sqrt{3}$

30 a i $(20, 21, 29)$ **ii** $(70, 24, 74)$

b i $m = 8, n = 3$ **ii** $m = 9, n = 4$

Multiple-choice questions

1 A **2** C **3** D **4** C **5** C **6** D

7 B **8** B **9** C **10** A **11** D **12** B

13 B **14** D **15** B **16** C **17** D **18** C

Chapter 3

Exercise 3A

1 a A and C (SAS) **b** All of them (AAS)

c A and B (SSS)

2 a Yes (SAS) **b** Yes (SSS) **c** No

d Yes (RHS) **e** No **f** Yes (AAS)

3 a 4.10 **b** 0.87 **c** 2.94

d 4.08 **e** 33.69° **f** 11.92

4 23 m

5 a 9.59° **b** $\sqrt{35}$ m

6 a 6.84 m **b** 6.15 m

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7 12.51° **8** 182.7 m **9** 1451 m
10 66.42°, 66.42°, 47.16°
11 a $5\sqrt{2}$ cm **b** 90°
12 3.07 cm **13** 37.8 cm **14** 4.38 m
15 $x = 4.08,\ y = 3.43,\ z = 5.33$

Exercise 3B

1 a $-\frac{1}{\sqrt{2}}$ **b** $\frac{1}{\sqrt{2}}$ **c** $-\frac{1}{2}$ **d** $\frac{1}{2}$
2 a $x = 5\sqrt{3},\ y = 5$ **b** $x = 8\sqrt{3},\ y = 16$
c $x = 6,\ y = 3\sqrt{3}$ **d** $x = y = 4\sqrt{2}$
3 $\frac{100\sqrt{3}}{3}$ m **4** $\frac{40\sqrt{3}}{3}$ cm
5 $(6 + 4\sqrt{3})$ m
6 a 60° **b** $10\sqrt{3}$ m
7 $x = 5(\sqrt{3} - 1)$

Exercise 3C

1 a 8.15 **b** 3.98 **c** 11.75 **d** 9.46
2 a 56.32° **b** 36.22° **c** 49.54°
d 98.16° or 5.84°
3 a $A = 48°,\ b = 13.84,\ c = 15.44$
b $a = 7.26,\ C = 56.45°,\ c = 6.26$
c $B = 19.8°,\ b = 4.66,\ c = 8.27$
d $C = 117°,\ b = 24.68,\ c = 34.21$
e $C = 30°,\ a = 5.40,\ c = 15.56$
4 a $B = 59.12°,\ A = 72.63°,\ a = 19.57$ or $B = 120.88°,\ A = 10.87°,\ a = 3.87$
b $C = 26.69°,\ A = 24.31°,\ a = 4.18$
c $B = 55.77°,\ C = 95.88°,\ c = 17.81$ or $B = 124.23°,\ C = 27.42°,\ c = 8.24$
5 554.26 m **6** 35.64 m
7 1659.86 m
8 a 26.60 m **b** 75.12 m

Exercise 3D

1 5.93 cm
2 $\angle ABC = 97.90°,\ \angle ACB = 52.41°$
3 a 26 **b** 11.74 **c** 49.29° **d** 73
e 68.70 **f** 47.22° **g** 7.59 **h** 38.05°
4 2.626 km **5** 3.23 km **6** 55.93 cm
7 a 8.23 cm **b** 3.77 cm
8 a 87.61 m **b** 67.7 m

Exercise 3E

1 400.10 m **2** 34.77 m **3** 575.18 m
4 16.51 m **5** 056°
6 a 034° **b** 214°
7 a 3583.04 m **b** 353°
8 027° **9** 113° **10** 22.01°
11 a $\angle BAC = 49°$ **b** 264.24 km
12 10.63 km

Exercise 3F

1

	sin	cos		sin	cos
a	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	**b**	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$
c	0	−1	**d**	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$
e	$-\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	**f**	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
g	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	**h**	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$
i	0	1	**j**	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
k	$-\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	**l**	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$
m	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	**n**	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$
o	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	**p**	0	1

2 a −0.4 **b** −0.6 **c** −0.4 **d** −0.6
e −0.4 **f** 0.6 **g** 0.6 **h** 0.4
3 a 120 **b** 240 **c** −60 **d** 120
e 240 **f** 300

Chapter 3 review

Short-response questions

Technology-free

1 $\frac{1}{8}$ **2** $\frac{2}{5}$ **3** $10\sqrt{3}$ cm
4 143° **5** $\frac{17}{28}$ **6** $\frac{3\sqrt{93}}{31}$
7 $\sqrt{181}$ km
8 $AC = \frac{12\sqrt{3}}{5}$ km, $BC = 2.4$ km

Technology-active

9 a i 160.19 m **ii** 164.62 m
b 127.93 m
10 327.6 m
11 a 531.8 m **b** 208 m
12 10.6 km
13 a 3583 m **b** 353°
14 a $\angle ACB = 12°,\ \angle CBO = 53°,\ \angle CBA = 127°$
b 189.33 m **c** 113.94 m
15 a 184.78 m **b** 199.71 m **c** 14.93 m
16 a 370.17 m **b** 287.94 m **c** 185.08 m

Multiple-choice questions

1 D **2** D **3** B **4** D **5** A
6 B **7** B **8** C **9** D **10** B
11 C **12** C **13** D **14** B **15** C

Chapter 4

Exercise 4A

1 $a = 10,\ b = 0,\ c = -7$

2 $a = 1,\ b = -2$

3 $a = 2,\ b = -1,\ c = 7$

4 $a = 2,\ b = 1,\ c = 3$

5 $(x+2)^2 - 4(x+2) + 4$

6 $(x+1)^3 - 3(x+1)^2 + 3(x+1) - 1$

7 $a = 1,\ b = -2,\ c = -1$

8 a It is impossible to find a, b and c such that $a = 3$, $3ab = -9$, $3ab^2 = 8$ and $ab^3 + c = 2$

b $a = 3,\ b = -1,\ c = 5$

9 $a = 1,\ b = -6,\ c = 7,\ d = -1$

10 a If $a = -\frac{5}{3}b$ and $a = -3b$, then both a and b are zero, but then $a + b = 1$ is not satisfied

b $(n+1)(n+2) - 3(n+1) + 1$

11 a $ax^2 + 2abx + ab^2 + c$

b $a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}$

13 $a = -3,\ b = -\frac{1}{3},\ c = 3$ or

$a = -\frac{1}{3},\ b = -3,\ c = 3$

14 $a = 3,\ b = -3,\ c = 1$

15 If $c = 5$, then $a = 1$ and $b = -5$;
if $c = -27$, then $a = -3$ and $b = 3$

Exercise 4B

1 a $x = -1$ **b** $x = 3$

c $x = 1 \pm \frac{\sqrt{30}}{5}$ **d** $x = 1 \pm \frac{\sqrt{2}}{2}$

e $x = -1 \pm \frac{3\sqrt{2}}{2}$ **f** $x = \frac{-13 \pm \sqrt{145}}{12}$

2 a $m > \frac{9}{4}$ **b** $m < \frac{25}{4}$ **c** $m = -\frac{25}{32}$

d $m < -6$ or $m > 6$ **e** $-4 < m < 4$

f $m = 0$ or $m = -16$

3 a $x = \frac{1 \pm \sqrt{32t+1}}{4},\ t \geq \frac{-1}{32}$

b $x = \frac{-1 \pm \sqrt{t+3}}{2},\ t \geq -3$

c $x = \frac{-2 \pm \sqrt{5t-46}}{5},\ t \geq \frac{46}{5}$

d $x = -2 \pm \frac{\sqrt{5t(t-2)}}{t},\ t < 0$ or $t \geq 2$

4 a $x = \frac{-p \pm \sqrt{p^2+64}}{2}$ **b** $p = 0, 6$

5 a $\Delta = (3p-4)^2$ **b** $p = \frac{4}{3}$

c i $x = 1$ or $x = \frac{1}{2}$ **ii** $x = 1$ or $x = 2$

iii $x = 1$ or $x = -\frac{5}{2}$

6 a $\Delta = 16(2p-3)^2$ **b** $p = \frac{3}{2}$

c i $x = \frac{3}{2}$ or $x = \frac{1}{2}$ **ii** $x = \frac{1}{2}$ or $x = \frac{3}{10}$

iii $x = \frac{1}{2}$ or $x = -\frac{3}{14}$

7 a $x = 4$ or $x = 36$ **b** $x = 16$

c $x = 49$ **d** $x = 1$ or $x = 512$

e $x = 27$ or $x = -8$ **f** $x = 16$ or $x = 625$

8 $x = 2$

9 Side length 37.5 cm

10 $a = 3,\ b = -\frac{5}{6},\ c = -\frac{13}{12}$; Minimum $-\frac{13}{12}$

12 $x = 1$ or $x = \frac{a-b}{b-c}$

13 $m = 8$

14 a $\Delta = 16((a-c)^2 + 2b^2) \geq 0$

b $a = c$ and $b = 0$

15 $-8 < k < 0$

16 $p = 10$

Exercise 4C

1 a $\frac{18}{x(x+3)}$ **b** $x = -6$ or $x = 3$

2 $x = -30$ or $x = 25$

3 17 and 19

4 a $\frac{40}{x}$ hours **b** $\frac{40}{x-2}$ hours **c** 10 km/h

5 a Car $\frac{600}{x}$ km/h; Plane $\left(\frac{600}{x} + 220\right)$ km/h

b Car 80 km/h; Plane 300 km/h

6 $x = 20$

7 6 km/h

8 a $x = 50$ **b** 72 minutes

9 Slow train 30 km/h; Express train 50 km/h

10 60 km/h

11 Small pipe 25 minutes; Large pipe 20 minutes

12 Each pipe running alone takes 14 minutes

13 Rail 43 km/h; Sea 18 km/h

14 22 km

15 10 litres

16 32.23 km/h, 37.23 km/h

17 a $a + \sqrt{a^2 - 24a}$ minutes,
$a - 24 + \sqrt{a^2 - 24a}$ minutes

b i 84 minutes, 60 minutes

ii 48 minutes, 24 minutes

iii 36 minutes, 12 minutes

iv 30 minutes, 6 minutes

18 a 120 km **b** 20 km/h, 30 km/h

Exercise 4D

1 a $\frac{2}{x-1} + \frac{3}{x+2}$ **b** $\frac{1}{x+1} - \frac{2}{2x+1}$

c $\frac{2}{x+2} + \frac{1}{x-2}$ **d** $\frac{1}{x+3} + \frac{3}{x-2}$

e $\frac{3}{5(x-4)} - \frac{8}{5(x+1)}$

2 a $\frac{2}{x-3} + \frac{9}{(x-3)^2}$

b $\frac{4}{1+2x} + \frac{2}{1-x} + \frac{3}{(1-x)^2}$

c $\frac{-4}{9(x+1)} + \frac{4}{9(x-2)} + \frac{2}{3(x-2)^2}$

3 a $\frac{-2}{x+1} + \frac{2x+3}{x^2+x+1}$ **b** $\frac{x+1}{x^2+2} + \frac{2}{x+1}$

c $\frac{x-2}{x^2+1} - \frac{1}{2(x+3)}$

4 $3 + \frac{3}{x-1} + \frac{2}{x-2}$

5 It is impossible to find A and C such that $A = 0$, $C - 2A = 2$ and $A + C = 10$

6 a $\frac{1}{2(x-1)} - \frac{1}{2(x+1)}$

b $\frac{2}{5(x-2)} + \frac{3}{5(x+3)}$

c $\frac{1}{x-2} + \frac{2}{x+5}$

d $\frac{2}{5(2x-1)} - \frac{1}{5(x+2)}$

e $\frac{3}{3x-2} - \frac{1}{2x+1}$ **f** $\frac{2}{x-1} - \frac{2}{x}$

g $\frac{1}{x} + \frac{3-x}{x^2+1}$ **h** $\frac{2}{x} + \frac{x}{x^2+4}$

i $\frac{1}{4(x-4)} - \frac{1}{4x}$ **j** $\frac{7}{4(x-4)} - \frac{3}{4x}$

k $x + \frac{1}{x} - \frac{1}{x-1}$ **l** $-x - 1 - \frac{3}{x} - \frac{1}{2-x}$

m $\frac{2}{3(x+1)} + \frac{x-4}{3(x^2+2)}$

n $\frac{2}{3(x-2)} + \frac{1}{3(x+1)} - \frac{1}{(x+1)^2}$

o $\frac{2}{x} + \frac{1}{x^2+4}$ **p** $\frac{8}{2x+3} - \frac{5}{x+2}$

q $\frac{26}{9(x+2)} + \frac{1}{9(x-1)} - \frac{1}{3(x-1)^2}$

r $\frac{16}{9(2x+1)} - \frac{8}{9(x-1)} + \frac{4}{3(x-1)^2}$

s $x - 2 + \frac{1}{4(x+2)} + \frac{3}{4(x-2)}$

t $x - \frac{1}{x+1} + \frac{2}{x-1}$ **u** $\frac{3}{x+1} - \frac{7}{3x+2}$

Exercise 4E

1 a $(1, 1), (0, 0)$ **b** $(0, 0), \left(\frac{1}{2}, \frac{1}{2}\right)$

c $\left(\frac{3+\sqrt{13}}{2}, 4+\sqrt{13}\right), \left(\frac{3-\sqrt{13}}{2}, 4-\sqrt{13}\right)$

2 a $(13, 3), (3, 13)$ **b** $(10, 5), (5, 10)$

c $(-8, -11), (11, 8)$ **d** $(9, 4), (4, 9)$

e $(9, 5), (-5, -9)$

3 a $(11, 17), (17, 11)$ **b** $(37, 14), (14, 37)$

c $(14, 9), (-9, -14)$

4 $(0, 0), (2, 4)$

5 $\left(\frac{5+\sqrt{5}}{2}, \frac{5+\sqrt{5}}{2}\right), \left(\frac{5-\sqrt{5}}{2}, \frac{5-\sqrt{5}}{2}\right)$

6 $\left(\frac{15}{2}, \frac{5}{2}\right), (3, 1)$

7 $\left(\frac{-130+80\sqrt{2}}{41}, \frac{60+64\sqrt{2}}{41}\right)$, $\left(\frac{-130-80\sqrt{2}}{41}, \frac{60-64\sqrt{2}}{41}\right)$

8 $\left(\frac{1+\sqrt{21}}{2}, \frac{-1-\sqrt{21}}{2}\right), \left(\frac{1-\sqrt{21}}{2}, \frac{-1+\sqrt{21}}{2}\right)$

9 $\left(\frac{4}{9}, 2\right)$

10 $\left(\frac{-6\sqrt{5}}{5}, \frac{3\sqrt{5}}{5}\right)$

11 $\left(-2, \frac{1}{2}\right)$

12 $(0, -1), (3, 2)$

13 a $\left(\frac{2}{3}, -\frac{7}{9}\right)$ **b** $\left(-\frac{1}{2}, 0\right), (1, 0)$

c $\left(-\frac{3}{2}, \frac{7}{4}\right)$ **d** $(-1, 4), (0, 2)$

Chapter 4 review

Short-response questions

Technology-free

1 $a = 3,\ b = 2,\ c = 1$

2 $(x-1)^3 + 3(x-1)^2 + 3(x-1) + 1$

5 a $x = -4$ or $x = 3$ **b** $x = -1$ or $x = 2$

c $x = -2$ or $x = 5$ **d** $x = \frac{2 \pm \sqrt{2}}{2}$

e $x = \frac{1 \pm \sqrt{3t-14}}{3}$ **f** $x = \frac{t \pm \sqrt{t^2-16t}}{2t}$

6 $x = \frac{-3 \pm \sqrt{73}}{2}$

7 a $\frac{-1}{x-3} - \frac{2}{x+2}$ **b** $\frac{3}{x+2} + \frac{4}{x-2}$

c $\frac{1}{2(x-3)} - \frac{3}{2(x+5)}$

d $\frac{1}{x-5} + \frac{2}{x+1}$

e $\frac{13}{x+2} - \frac{13}{x+3} - \frac{10}{(x+2)^2}$

f $\frac{4}{x+4}+\frac{2}{x-1}-\frac{3}{(x-1)^2}$

g $\frac{1}{x+1}-\frac{6}{x^2+2}$ **h** $\frac{1}{x-1}-\frac{x+3}{x^2+x+1}$

i $\frac{1}{3-x}-\frac{3}{x+4}$

8 a $\frac{1}{x-3}-\frac{x-10}{x^2+x+2}$

b $\frac{1}{4(x+1)}-\frac{x-2}{4(x^2-x+2)}$

c $3x+15+\frac{64}{x-4}-\frac{1}{x-1}$

9 a $(0,0), (-1,1)$ **b** $(0,4), (4,0)$
c $(1,4), (4,1)$

10 $(-4,-1), (2,1)$

11 a $t=\frac{135}{x}$ **b** $t=\frac{135}{x-15}$ **c** $x=60$
d 60 km/h, 45 km/h

Technology-active

12 a $x=-0.76$ or $x=2.95$
b $x=-1.32$ or $x=3.91$
c $x=-0.91$ or $x=2.16$
d $x=-3.80$ or $x=1.84$

13 a $(-0.80,-1.39)$ and $(8.80, 27.39)$
b $(-0.82,-1.47)$ and $(0.41, 2.22)$
c $(-0.78,-1.48)$ and $(-0.16,-6.79)$ and $(5.87,-0.91)$
d $(-0.92,-0.57)$ and $(-0.42,-3.70)$ and $(6.01, 0.05)$

14 a 24 km/h

b Speed $=\frac{a+\sqrt{a(a+480)}}{2}$, $a>0$;
When $a=60$, speed $=120$ km/h, which is a very fast constant speed for a train. If we choose this as the upper limit for the speed, then $0<a<60$ and $0<$ speed <120.

c

a	1	8	14	22	34	43	56	77	118	239
speed	16	20	24	30	40	48	60	80	120	240

15 a $\frac{a+\sqrt{a^2+4abc}}{2ac}$

b e.g. $a=3$, $b=1$, $c=\frac{4}{3}$

16 a $k=-2$, $k=1$ **b** $-10<c<10$
c $p=5$

Multiple-choice questions

1 C **2** C **3** D **4** C **5** D
6 D **7** C **8** D **9** B **10** B
11 C **12** B **13** B **14** B **15** D
16 C **17** D **18** D **19** A **20** A

Chapter 5

Short-response questions

Technology-free

1 a $-\frac{\sqrt{2}+3}{7}$ **b** $\frac{3(\sqrt{5}+1)}{4}$ **c** $\frac{4\sqrt{2}+2}{7}$
d $\frac{3(\sqrt{5}+\sqrt{3})}{2}$ **e** $\frac{\sqrt{7}+\sqrt{2}}{5}$ **f** $\frac{2\sqrt{5}+\sqrt{3}}{17}$

2 a $\frac{5}{9}$ **b** $\frac{6}{11}$ **c** $\frac{605}{999}$ **d** $\frac{7}{3}$ **e** $\frac{145\,223}{33\,333}$

3 a $2\times7\times11\times13$ **b** $3\times5\times37$
c $7^2\times11\times13$ **d** $2^4\times5^4$

4 $\frac{1}{m+p}$

5 a $\sqrt{6}-\sqrt{3}-\sqrt{2}+3$ **b** $21\sqrt{2}+33$
c $4x-12\sqrt{x}+9$ **d** $-6\sqrt{x-2}+x+7$

6 a $A=1,\ B=0,\ C=-1$
b $A=20,\ B=2$ and $C=1$, or $A=20,\ B=-2$ and $C=-1$

7 $\sqrt{6}-2$

8 a $\frac{1}{3(x-2)}+\frac{1}{3(x+2)}$

b $\frac{1}{x+2}+\frac{1}{x+3}$ **c** $\frac{2}{x+2}+\frac{3x-2}{x^2+4}$

d $\frac{1}{x+1}+\frac{1}{x-1}-\frac{2}{(x-1)^2}$

e $\frac{x}{x^2+1}+\frac{1}{x-3}$ **f** $\frac{x+1}{x^2+4}+\frac{2}{x-2}$

9 a $576=2^6\times3^2$, $\sqrt{576}=24$
b $1225=5^2\times7^2$, $\sqrt{1225}=35$
c $1936=2^4\times11^2$, $\sqrt{1936}=44$
d $1296=6^4$, $\sqrt{1296}=36$

10 $x=-b-c$

11 $x=\frac{2ab}{a+b}$

12 a $\sqrt{21}$ cm **b** $\frac{2\sqrt{7}}{7}$

13 a $x=\frac{(y-3)^2+1}{2}$ **b** $x=\frac{1}{3}\left(\frac{4}{(y+2)^2}-1\right)$

14 a $x=\frac{51}{25}$, $y=\frac{32}{25}$

b $x=\frac{a(b^2+1)}{a^2+b^2}$, $y=\frac{b(a^2-1)}{a^2+b^2}$

15 $\left(\frac{21}{5},\frac{7}{5}\right)$, $(12,4)$ **16** $(1,-4), (-1,4)$

17 $x=-3$ **18** 64 km/h

19 b i $x=\frac{-1-\sqrt{5}}{2}$ **ii** $x=\frac{-1+\sqrt{5}}{2}$

20 $a=-7,\ b=-5,\ c=1$

22 12 cm

23 a $5\sqrt{3}\pm\sqrt{11}$ **b** $\sin^{-1}\left(\frac{5}{6}\right)$ or $180^\circ-\sin^{-1}\left(\frac{5}{6}\right)$

Answers 4 review → 5 revision

24 $a = -\frac{1}{3},\ b = -2,\ \lambda = -\frac{4}{3};$

$a = -\frac{1}{2},\ b = -1,\ \lambda = -\frac{3}{2}$

25 150 minutes

26 $m = \frac{-1}{2}$ or $m = -\frac{11}{2}$

27 $x = \frac{5 \pm \sqrt{155}}{5}$

28 **a** 3 **b** 12 **c** 8

29 **a** $\Delta = 4a(a-1)$

b **i** $a = 1$ **ii** $a > 1$ or $a < 0$

iii $0 < a < 1$

30 $x = \frac{5}{3}$

31 Maths 111, Physics 74, Chemistry 63

Technolgy-active

32 **a** 21.80° **b** 3.06 **c** 9.97

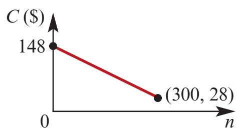
33 $\angle A = 81.8°,\ \angle B = 54.9°,\ \angle C = 43.3°$

34 $\angle ACB = 32.4°,\ \angle ABC = 105.6°$

35 **a** 8 **b** 7.7 **c** 6 cm **d** 15 cm

36 **a** $a = -0.4,\ b = 148$

b



c \$68 **d** 248

37 **a** **i** 178 **ii** 179 **iii** 179.5 **iv** 179.95

b **i** 180 **ii** Circle

c 20 **d** $n = \frac{360}{180 - A}$ **e** Square

38 **a** $a = 6000,\ b = -15\,000$ **b** \$57 000

c 2016

39 **a** Volume of hemisphere $= \frac{2}{3}\pi t^3$,

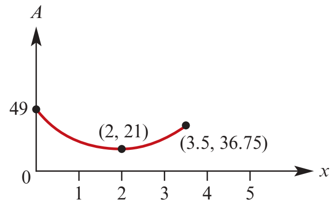
Volume of cylinder $= \pi t^2 s$,

Volume of cone $= \frac{1}{3}\pi t^2 w$

b **i** 6 : 2 : 3 **ii** 54π cubic units

40 **a** $8x$ cm **b** $28 - 8x$ cm **c** $7 - 2x$ cm

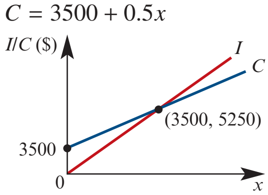
e



f $A = 21$ when $x = 2$

41 **a** $C = 3500 + 0.5x$ **b** $I = 1.5x$

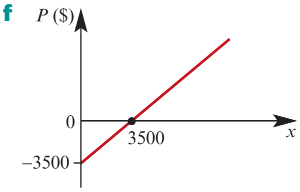
c



d 3500

e 5500

f

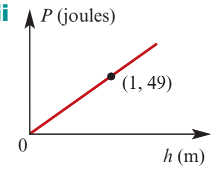


P represents profit

42 **b** **i** $x = \frac{1}{24}$ **ii** $x = \frac{25}{24}$

43 **c** 11, 24 and 39

44 **a** **i** $g = 9.8$ **ii**



iii 7 kg

b **i** Doubled **ii** Halved

c **i** 14 m/s **ii** 42 m/s

d 4

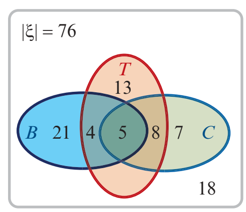
45 **a** 1 hour 35 minutes **b** 2.5 km

46 **a** $|B' \cap C' \cap T| = |C \cap T|$,

$|B \cap C' \cap T'| = 3|B' \cap C \cap T'|$,

$|B \cap C' \cap T| = 4$

b



c **i** 5 **ii** 0

47 **a** $\angle TAB = 3°,\ \angle ABT = 97°,\ \angle ATB = 80°$

b 2069.87 m **c** 252.25 m

48 **a** 78 m to C, 49 m to D

b 60 m **c** 279°

49 6.21 km

50 4

51 **a** $n(n-2)(n-1)(n+1)(n+2)$

52 1449.8 m

53 43.59 km on a bearing of 083.4°

54 7 children

55 (4, 12), (6, 6), (12, 4)

56 $(2^5 \times 3^8)^5$

57 **a** 90 km **b** 70 km/h

59 **a** $c = \frac{b^2 - 4}{4}$ **b** $c = \frac{2b^2}{9} + 3$

c $b = \pm 12,\ c = 35$

60 **a** $m = 5,\ n = 2$ **b** $b = 4,\ c = -1$

61 **a** $y - b = m(x - a)$

c $y = 2,\ y = -\frac{12}{5}x - \frac{26}{5}$

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Multiple-choice questions

1 D	**2** B	**3** C	**4** C	**5** A
6 C	**7** C	**8** A	**9** B	**10** B
11 A	**12** D	**13** B	**14** B	**15** A
16 A	**17** C	**18** A	**19** C	**20** B
21 D	**22** B	**23** C	**24** D	**25** C
26 A	**27** A	**28** D	**29** C	**30** B
31 B	**32** C	**33** B	**34** D	**35** C
36 B	**37** D	**38** D	**39** A	

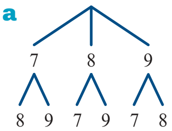
Problem-solving and modelling

See solutions supplement

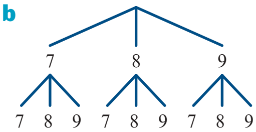
Chapter 6

Exercise 6A

1 45
2 8
3 120
4 a



b



5 a 27 **b** 6
6 30
7 a 6 **b** 18 **c** 20 **d** 15
8 BB, BR, BG, RB, RG, GB, GR, GG
9 12
10 9
11 a 6 **b** 13
12 16

Exercise 6B

1 1, 1, 2, 6, 24, 120, 720, 5040, 40 320, 362 880, 3 628 800
2 a 5 **b** 90 **c** 66 **d** 161 700
3 a $n+1$ **b** $n+2$ **c** $n(n-1)$ **d** $\dfrac{n+2}{(n+1)!}$
4 1, 4, 12, 24, 24
5 DOG, DGO, ODG, OGD, GOD, GDO
6 120
7 362 880
8 FR, FO, FG, RF, RO, RG, OF, OR, OG, GF, GR, GO
9 a 720 **b** 720 **c** 360
10 a 120 **b** 120 **c** 60
11 20 160
12 a 125 **b** 60
13 a 120 **b** 360 **c** 720
14 60
15 a 17 576 000 **b** 11 232 000
16 $(m, n) = (6, 0), (6, 1), (5, 3)$
17 $(n^2 - n) \cdot (n-2)! = n \cdot (n-1) \cdot (n-2)! = n!$
18 a 384 **b** 3072
19 30

Exercise 6C

1 a 120 **b** 72 **c** 24 **d** 96
2 a 120 **b** 48 **c** 72 **d** 12
3 a 360 **b** 144 **c** 144 **d** 72
4 a 1152 **b** 1152
5 a 600 **b** 108 **c** 431 **d** 52
6 a 720 **b** 48 **c** 144 **d** 96 **e** 48
7 a 900 **b** 900
8 84
9 32
10 a 480 **b** 192
11 144

Exercise 6D

1 35
2 34 650
3 4 989 600
4 56
5 27 720
6 a 420 **b** 105 **c** 90 **d** 12 **e** 105
7 35
8 a 15 **b** $\dfrac{(m+n)!}{m! \cdot n!}$
9 a 52! **b** $\dfrac{104!}{(2!)^{52}}$ **c** $\dfrac{(52n)!}{(n!)^{52}}$
10 4900
11 89

Exercise 6E

1 1, 5, 10, 10, 5, 1
2 a 7 **b** 6 **c** 66 **d** 56 **e** 100 **f** 499 500
3 a n **b** $\dfrac{n(n-1)}{2}$ **c** n **d** $n+1$ **e** $\dfrac{(n+2)(n+1)}{2}$ **f** $\dfrac{n(n+1)}{2}$
4 a 720 **b** 120
5 2 598 960
6 a 10 **b** 45 **c** 45 **d** 10
7 45 379 620
8 56
9 a 45 **b** 16

10 15

11 $${}^nC_{n-r} = \frac{n!}{(n-r)!\,(n-(n-r))!} = \frac{n!}{(n-r)!\,r!} = {}^nC_r$$

12 Each diagonal is obtained by choosing 2 vertices from n vertices. This can be done in ${}^nC_2$ ways. But n of these choices define a side of the polygon, not a diagonal. Therefore there are ${}^nC_2 - n$ diagonals.

13 There are ${}^{10}C_5$ ways to choose 5 students for team A. The remaining 5 students will belong to team B. However, the labelling of the teams does not matter, so we must divide by 2.

14 462

15 ${}^{n-1}C_{r-1} + {}^{n-1}C_r$

$$= \frac{(n-1)!}{(r-1)!\,(n-1-(r-1))!} + \frac{(n-1)!}{r!\,(n-1-r)!}$$
$$= \frac{(n-1)!}{(r-1)!\,(n-r)!} + \frac{(n-1)!}{r!\,(n-r-1)!}$$
$$= \frac{(n-1)!}{(r-1)!\,(n-r-1)!}\left(\frac{1}{n-r} + \frac{1}{r}\right)$$
$$= \frac{(n-1)!}{(r-1)!\,(n-r-1)!} \cdot \frac{n}{r(n-r)}$$
$$= \frac{n!}{r!\,(n-r)!} = {}^nC_r$$

16 **a** 2300 **b** 152 **c** 2148

Exercise 6F

1 153 **2** 126
3 1176 **4** 140
5 **a** 1716 **b** 700 **c** 980 **d** 1568
6 **a** 25 200 **b** 4200
7 **a** 1 392 554 592 **b** 5 250 960
8 **a** 15 504 **b** 10 800 **c** 15 252
9 **a** 21 **b** 10 **c** 11
10 2100
11 **a** 204 490 **b** 7 250 100
12 **a** 48 **b** 210
13 1440 **14** 3600
15 14 400 **16** 150
17 3744

Exercise 6G

1 ${}^7C_2 = 21$, ${}^6C_2 = 15$, ${}^6C_1 = 6$
2 1, 7, 21, 35, 35, 21, 7, 1; ${}^7C_2 = 21$, ${}^7C_4 = 35$
3 1, 8, 28, 56, 70, 56, 28, 8, 1; ${}^8C_4 = 70$, ${}^8C_6 = 28$
4 $2^6 = 64$
5 $2^5 = 32$
6 $2^{10} = 1024$
7 $2^6 - 1 = 63$
8 $2^8 - {}^8C_1 - {}^8C_0 = 247$
9 $2^8 = 256$
10 $2^4 - 1 = 15$
11 **a** 128 **b** 44

Exercise 6H

1 4

2 Label 26 holes from A to Z. Put each of the 27 words into the hole labelled by its first letter. Some hole contains at least two words.

3 Label 4 holes by 0, 1, 2, 3. Put each of the 5 numbers into the hole labelled by its remainder when divided by 4. Some hole contains at least two numbers.

4 **a** 3 **b** 5 **c** 14

5 Divide [0, 1] into 10 subintervals: [0, 0.1), [0.1, 0.2), . . . , [0.9, 1]. Some interval contains at least two of the 11 numbers.

6 Divide into 4 equilateral triangles of side length 1 unit as shown. Some triangle contains at least two of the 5 points.

7 Divide the rectangle into squares of size 2×2. There are 12 squares and 13 points, so some square contains at least two points.
The distance between two points in the same square cannot exceed the length of the square's diagonal, $\sqrt{2^2 + 2^2} = 2\sqrt{2}$.

8 **a** For two-digit numbers, the possible digital sums are 1, 2, . . . , 18. Since $19 > 18$, some digital sum occurs at least twice.
b For three-digit numbers, the possible digital sums are 1, 2, . . . , 27. Since $82 = 3 \times 27 + 1$, some digital sum occurs at least 4 times.

9 Label 4 holes by 0, 1, 2, 3. Place each number into the hole labelled by its remainder when divided by 4. Since $13 = 3 \times 4 + 1$, some hole contains at least 4 numbers.

10 Two teams can be chosen in ${}^8C_2 = 28$ ways. Since there are 29 games, some pair of teams play each other at least twice.

11 At least 26 students. To show that 26 numbers suffice, label 25 holes by (1 or 49), (2 or 48), . . . , (24 or 26), (25). To show that 25 numbers do not, consider 1, 2, 3, . . . , 25.

12 Label the chairs 1, 2, . . . , 14. There are 14 groups of three consecutive chairs:
{1, 2, 3}, {2, 3, 4}, . . . , {13, 14, 1}, {14, 1, 2}
Each of the 10 people belongs to 3 groups, so there are 30 allocations to be made to 14 groups. Since $30 \ge 2 \times 14 + 1$, some group contains at least 3 people.

13 Draw a diameter through one of the 4 points. This creates 2 half circles. One half circle contains at least two of the 3 remaining points (and the chosen point).

14 There are 195 possible sums: $3, 4, \ldots, 197$. There are ${}^{35}C_2 = 595$ ways to choose a pair of players. Since $595 \geq 3 \times 195 + 1$, at least 4 pairs have the same sum.

15 Label the chairs $1, 2, \ldots, 12$. There are 6 pairs of opposite seats:

$\{1, 7\}, \{2, 8\}, \{3, 9\}, \{4, 10\}, \{5, 11\}, \{6, 12\}$

Some pair contains two of the 7 boys.

16 Label n holes by $0, 1, 2, \ldots, n - 1$. Place each guest in the hole labelled by the number of hands they shake. The first or last hole must be empty. (If a guest shakes 0 hands, then no guest shakes n hands. If a guest shakes n hands, then no guest shakes 0 hands.) This leaves $n - 1$ holes, so some hole contains at least two guests.

Exercise 6I

1 a $\{1, 3, 4\}$ **b** $\{1, 3, 4, 5, 6\}$ **c** $\{4\}$
d $\{1, 2, 3, 4, 5, 6\}$ **e** 3
f $\emptyset, \{4\}, \{5\}, \{6\}, \{4, 5\}, \{4, 6\}, \{5, 6\}, \{4, 5, 6\}$

2 36 **3** 4

4 150

5 a 64 **b** 32

6 a 72 **b** 72 **c** 36 **d** 108

7 a 12 **b** 38

8 88 **9** 80

10 4

11 a 756 **b** 700 **c** 360 **d** 1096

12 1 452 555 **13** 3417

14 5

Exercise 6J

1 a $\frac{1}{2}$ **b** $\frac{1}{2}$ **c** $\frac{1}{2}$

2 a $\frac{1}{4}$ **b** $\frac{3}{4}$

3 $\frac{3}{8}$

4 a $\frac{1}{5}$ **b** $\frac{3}{5}$ **c** $\frac{3}{10}$

5 a $\frac{68}{609}$ **b** $\frac{374}{1015}$

6 $\frac{1}{5}$

7 $\frac{329}{858}$

8 a $\frac{2^7}{2^8 - 1} \approx 0.502$ **b** $\frac{56}{255}$ **c** $\frac{73}{85}$

9 a $\frac{5}{204}$ **b** $\frac{35}{136}$

10 a $\frac{1}{6}$ **b** $\frac{5}{6}$ **c** $\frac{17}{21}$

11 a $\frac{2}{5}$ **b** $\frac{3}{5}$ **c** $\frac{2}{5}$ **d** $\frac{1}{10}$ **e** $\frac{3}{5}$

12 a $\frac{24}{49}$ **b** $\frac{25}{49}$ **c** $\frac{3}{7}$

13 $\frac{4}{5}$

14 a $\frac{1}{2}$ **b** $\frac{3}{14}$

15 a 0.659 **b** 0.341 **c** 0.096

16 a $\frac{5}{42}$ **b** $\frac{20}{21}$

17 $\frac{5}{24}$

18 $\frac{5}{11}$

Chapter 6 review

Short-response questions

Technology-free

1 a 20 **b** 190 **c** 300 **d** 4950

2 11

3 a 27 **b** 6

4 120 **5** 60

6 18 **7** 31

8 10 **9** 3

10 12 **11** 192

12 $\frac{1}{10}$ **13** $\frac{1}{2}$

14 a $\frac{2}{9}$ **b** $\frac{4}{9}$

Technology-active

15 a 120 **b** 360 **c** 72 **d** 144

16 a 20 **b** 80 **c** 60

17 a 210 **b** 84 **c** 90 **d** 195

18 a 420 **b** 15 **c** 105 **d** 30

19 a 210 **b** 10 **c** 80

20 a i 20 **ii** 10 **iii** 64
b 8

21 a 676 **b** 235 **c** 74

22 a 24 **b** 4 **c** 24 **d** $\frac{3}{4}$

23 a 924
b There are at least $365 \times 3 = 1095$ days in three years and there are 924 different paths, so some path is taken at least twice.
c i 6 **ii** 70 **iii** 420
d 624

24 $\frac{1400}{6561}$

25 a $\frac{4 \cdot 48!\,47!}{52!\,44!}$ **b** $\frac{4 \cdot 48!\,(52-k)!}{52!\,(49-k)!}$

c $1 - \frac{48!\,(52-k)!}{52!\,(48-k)!}$

26 196

27 a 120 **b** 210 **c** 252 **d** 27 216

Multiple-choice questions

1 C	**2** B	**3** A	**4** D	**5** B
6 B	**7** C	**8** D	**9** C	**10** C
11 A	**12** A	**13** D	**14** A	**15** C
16 D	**17** D	**18** C	**19** D	**20** D
21 B	**22** D	**23** B		

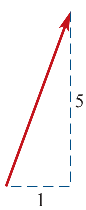
Chapter 7

See solutions supplement

Chapter 8

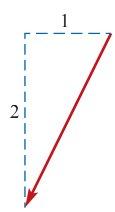
Exercise 8A

1 a

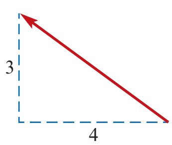


b

c



d



2 $a = 5,\ b = 1$

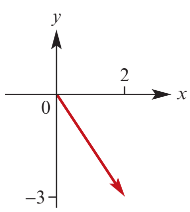
3 $a = 3,\ b = -15$

4 a $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ **b** $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$ **c** $\begin{bmatrix} -1 \\ -3 \end{bmatrix}$ **d** $\begin{bmatrix} -2 \\ 3 \end{bmatrix}$ **e** $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$

5 a i $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ **ii** $\begin{bmatrix} -5 \\ 0 \end{bmatrix}$ **iii** $\begin{bmatrix} 4 \\ -2 \end{bmatrix}$

b $\boldsymbol{a} + \boldsymbol{b} = -\boldsymbol{c}$

6 a



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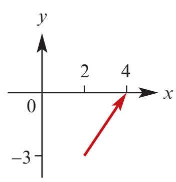

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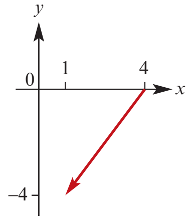

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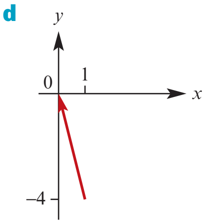

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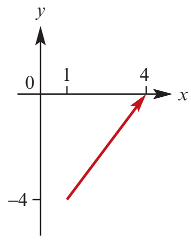
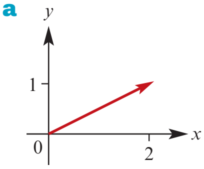


7 a



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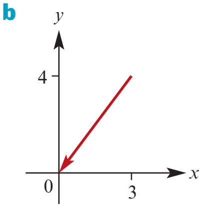

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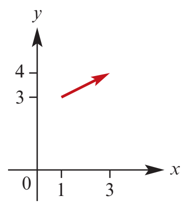

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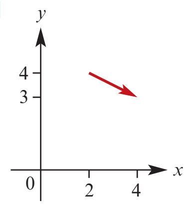
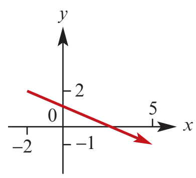
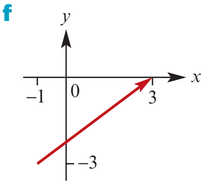


e

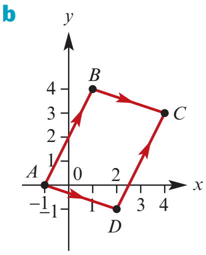


f



8 a and **c**

9 a b



d Parallelogram

10 $m = -11,\ n = 7$

11 a i $\boldsymbol{b} - \frac{1}{2}\boldsymbol{a}$ **ii** $\boldsymbol{b}$

b $\overrightarrow{MN} = \overrightarrow{AD}$

12 a $\overrightarrow{CB} = \boldsymbol{a} - \boldsymbol{b},\ \overrightarrow{MN} = \frac{1}{2}(\boldsymbol{b} - \boldsymbol{a})$

b $\overrightarrow{CB} = -2\overrightarrow{MN}$

13 a $\boldsymbol{a}$ **b** $\boldsymbol{b}$ **c** $2\boldsymbol{a}$ **d** $2\boldsymbol{b}$
e $-\boldsymbol{a}$ **f** $\boldsymbol{b}-\boldsymbol{a}$ **g** $\boldsymbol{a}+\boldsymbol{b}$

14 a $\boldsymbol{a}$ **b** $-\boldsymbol{b}$ **c** $\boldsymbol{a}+\boldsymbol{b}$
d $-\boldsymbol{a}-\boldsymbol{b}$ **e** $\boldsymbol{b}-\boldsymbol{a}$

15 a $\boldsymbol{a}-\boldsymbol{b}$ **b** $\frac{1}{3}(\boldsymbol{b}-\boldsymbol{a})$ **c** $\frac{1}{3}(\boldsymbol{a}+2\boldsymbol{b})$
d $\frac{1}{9}(\boldsymbol{a}+2\boldsymbol{b})$ **e** $\frac{1}{9}(4\boldsymbol{a}-\boldsymbol{b})$

16 a $\boldsymbol{u}+\boldsymbol{v}$ **b** $\boldsymbol{v}+\boldsymbol{w}$ **c** $\boldsymbol{u}+\boldsymbol{v}+\boldsymbol{w}$

17 a $\overrightarrow{OB}=\boldsymbol{u}+\boldsymbol{v},\ \overrightarrow{OM}=\boldsymbol{u}+\frac{1}{2}\boldsymbol{v}$ **b** $\boldsymbol{u}-\frac{1}{2}\boldsymbol{v}$
c $\frac{2}{3}\left(\boldsymbol{u}-\frac{1}{2}\boldsymbol{v}\right)$
d $\overrightarrow{OP}=\frac{2}{3}(\boldsymbol{u}+\boldsymbol{v})=\frac{2}{3}\overrightarrow{OB}$ **e** $2:1$

Exercise 8B

1 $2\boldsymbol{i}-7\boldsymbol{j}$

2 a $5\boldsymbol{i}+6\boldsymbol{j}$ **b** $-5\boldsymbol{i}+6\boldsymbol{j}$ **c** $5\boldsymbol{i}-6\boldsymbol{j}$

3 $7\boldsymbol{i}+\frac{5}{2}\boldsymbol{j}$

4 a 5 **b** 2 **c** 5 **d** 13

5 a 13 **b** $x=2,\ y=-7$

6 a i $\frac{2}{5}\boldsymbol{i}$ **ii** $-\frac{2}{5}\boldsymbol{i}+\boldsymbol{j}$ **iii** $\frac{1}{6}\left(-\frac{2}{5}\boldsymbol{i}+\boldsymbol{j}\right)$
iv $\frac{1}{3}\boldsymbol{i}+\frac{1}{6}\boldsymbol{j}$ **v** $2\boldsymbol{i}+\boldsymbol{j}$
b i $\overrightarrow{ON}=\frac{1}{6}\overrightarrow{OA}$ **ii** $1:5$

7 $4\sqrt{2}$ units

8 a $k=\frac{3}{2},\ \ell=\frac{1}{2}$ **b** $x=6,\ y=2$
c $x=3,\ y=3$ **d** $k=-\frac{1}{3},\ \ell=-\frac{5}{3}$

9 $3\boldsymbol{i}-2\boldsymbol{j},\ \sqrt{13}$

10 a $-2\boldsymbol{i}+4\boldsymbol{j}$ **b** $-6\boldsymbol{i}+\boldsymbol{j}$ **c** 5

11 a $D(-6,3)$ **b** $F(4,-3)$ **c** $G\left(\frac{3}{2},-\frac{3}{2}\right)$

12 $A(-1,-4),\ B(-2,2),\ C(0,10)$

13 a i $2\boldsymbol{i}-\boldsymbol{j}$ **ii** $-5\boldsymbol{i}+4\boldsymbol{j}$ **iii** $\boldsymbol{i}+7\boldsymbol{j}$
iv $6\boldsymbol{i}+3\boldsymbol{j}$ **v** $6\boldsymbol{i}+3\boldsymbol{j}$
b $D(8,2)$

14 a $\overrightarrow{OP}=12\boldsymbol{i}+5\boldsymbol{j},\ \overrightarrow{PQ}=6\boldsymbol{i}+8\boldsymbol{j}$ **b** 13, 10

15 a i $\sqrt{29}$ **ii** $\sqrt{116}$ **iii** $\sqrt{145}$
b $(\sqrt{29})^2+(\sqrt{116})^2=(\sqrt{145})^2$

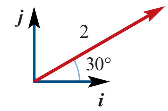
16 a i $-\boldsymbol{i}-3\boldsymbol{j}$ **ii** $4\boldsymbol{i}+2\boldsymbol{j}$ **iii** $-3\boldsymbol{i}+\boldsymbol{j}$
b i $\sqrt{10}$ **ii** $2\sqrt{5}$ **iii** $\sqrt{10}$

17 a i $-3\boldsymbol{i}+2\boldsymbol{j}$ **ii** $7\boldsymbol{j}$
iii $-3\boldsymbol{i}-5\boldsymbol{j}$ **iv** $-2\boldsymbol{i}-\frac{10}{3}\boldsymbol{j}$
b $M\left(-2,\frac{11}{3}\right)$

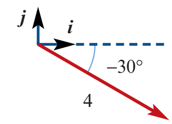
18 a $\frac{1}{5}(3\boldsymbol{i}+4\boldsymbol{j})$ **b** $\frac{1}{\sqrt{10}}(3\boldsymbol{i}-\boldsymbol{j})$
c $\frac{1}{\sqrt{2}}(-\boldsymbol{i}+\boldsymbol{j})$ **d** $\frac{1}{\sqrt{2}}(\boldsymbol{i}-\boldsymbol{j})$
e $\frac{6}{\sqrt{13}}\left(\frac{1}{2}\boldsymbol{i}+\frac{1}{3}\boldsymbol{j}\right)$ **f** $\frac{1}{\sqrt{13}}(3\boldsymbol{i}-2\boldsymbol{j})$

Exercise 8C

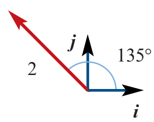
1 a $\sqrt{3}\boldsymbol{i}+\boldsymbol{j}$



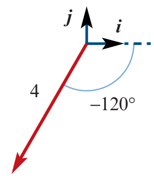
b $2\sqrt{3}\boldsymbol{i}-2\boldsymbol{j}$



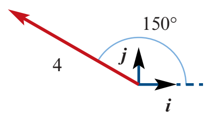
c $-\sqrt{2}\boldsymbol{i}+\sqrt{2}\boldsymbol{j}$



d $-2\boldsymbol{i}-2\sqrt{3}\boldsymbol{j}$



e $-2\sqrt{3}\boldsymbol{i}+2\boldsymbol{j}$



2 a $4\boldsymbol{i}+4\sqrt{3}\boldsymbol{j}$ **b** $5\sqrt{3}\boldsymbol{i}+5\boldsymbol{j}$
c $-3\sqrt{3}\boldsymbol{i}+3\boldsymbol{j}$ **d** $4\sqrt{2}\boldsymbol{i}-4\sqrt{2}\boldsymbol{j}$
e $-6\sqrt{3}\boldsymbol{i}-6\boldsymbol{j}$

3 a $[4\sqrt{2},-45^\circ]$ **b** $[3\sqrt{2},135^\circ]$
c $[4,150^\circ]$ **d** $[5,53.13^\circ]$
e $[13,-112.62^\circ]$

4 a $[2,30^\circ]$ **b** $[\sqrt{5},63.43^\circ]$
c $[\sqrt{2},-135^\circ]$ **d** $[2,-30^\circ]$
e $[\sqrt{2},135^\circ]$

5 a $5.36\boldsymbol{i}+4.50\boldsymbol{j}$ **b** $8.19\boldsymbol{i}+5.74\boldsymbol{j}$
c $-9.97\boldsymbol{i}+4.65\boldsymbol{j}$ **d** $7.37\boldsymbol{i}-5.16\boldsymbol{j}$
e $-6.88\boldsymbol{i}-9.83\boldsymbol{j}$

6 a $[20,60^\circ]$ **b** $[1,-165^\circ]$
c $[18,160^\circ]$

7 a $[24.18,78.07^\circ]$ **b** $[19.83,52.50^\circ]$
c $[9.18,-5.00^\circ]$

8 a $(6+4\sqrt{3})\boldsymbol{i}+(4+6\sqrt{3})\boldsymbol{j}$
b $5\sqrt{2}\boldsymbol{i}+(5\sqrt{2}-10)\boldsymbol{j}$ **c** $4.80\boldsymbol{i}+7.28\boldsymbol{j}$

Exercise 8D

1 a 17 **b** 13 **c** 8 **d** −10
e −4 **f** 3 **g** −58

2 a 5 **b** 13 **c** 8
d −5 **e** 13

3 a $15\sqrt{2}$ **b** $-15\sqrt{2}$

4 a $|\boldsymbol{a}|^2 + 4|\boldsymbol{b}|^2 + 4\boldsymbol{a}\cdot\boldsymbol{b}$ **b** $4\boldsymbol{a}\cdot\boldsymbol{b}$
c $|\boldsymbol{a}|^2 - |\boldsymbol{b}|^2$ **d** $|\boldsymbol{a}|$

5 a $-3\boldsymbol{i} + \boldsymbol{j}$ **b** $\sqrt{10}$ **c** $116.57°$

6 $\sqrt{66}$

7 a $-\dfrac{11}{2}$ **b** $\dfrac{10}{3}$ **c** -1
d $\dfrac{-2 \pm \sqrt{76}}{6}$

8 a $-\boldsymbol{a} + q\boldsymbol{b}$ **b** $\dfrac{22}{29}$ **c** $\left(\dfrac{44}{29}, \dfrac{110}{29}\right)$

9 a $139.40°$ **b** $71.57°$ **c** $26.57°$
d $126.87°$

11 a $25\sqrt{3}$ **b** 20 **c** $60\sqrt{3}$
d -36 **e** $7\sqrt{3}$ **f** 20

12 a $\dfrac{3}{2}\boldsymbol{i}$ **b** $45°$ **c** $116.57°$

13 a i $\dfrac{3}{2}\boldsymbol{i} + 2\boldsymbol{j}$ **ii** $\dfrac{1}{2}\boldsymbol{i} + 3\boldsymbol{j}$
b $27.41°$
c $55.30°$

Exercise 8E

1 a $\dfrac{1}{\sqrt{10}}(\boldsymbol{i} + 3\boldsymbol{j})$ **b** $\dfrac{1}{\sqrt{2}}(\boldsymbol{i} + \boldsymbol{j})$ **c** $\dfrac{1}{\sqrt{2}}(\boldsymbol{i} - \boldsymbol{j})$

2 a $\hat{\boldsymbol{a}} = \dfrac{1}{5}(3\boldsymbol{i} + 4\boldsymbol{j}),\ |\boldsymbol{b}| = \sqrt{2}$
b $\dfrac{\sqrt{2}}{5}(3\boldsymbol{i} + 4\boldsymbol{j})$

3 a $\hat{\boldsymbol{a}} = \dfrac{1}{5}(3\boldsymbol{i} + 4\boldsymbol{j}),\ \hat{\boldsymbol{b}} = \dfrac{1}{13}(5\boldsymbol{i} + 12\boldsymbol{j})$
b $\dfrac{1}{\sqrt{65}}(4\boldsymbol{i} + 7\boldsymbol{j})$

4 a $-\dfrac{11}{17}(\boldsymbol{i} - 4\boldsymbol{j})$ **b** $\dfrac{13}{17}(\boldsymbol{i} - 4\boldsymbol{j})$ **c** $4\boldsymbol{i}$

5 a $\boldsymbol{a} = \boldsymbol{u} + \boldsymbol{w}$ where $\boldsymbol{u} = 2\boldsymbol{i}$ and $\boldsymbol{w} = \boldsymbol{j}$
b $\boldsymbol{a} = \boldsymbol{u} + \boldsymbol{w}$ where $\boldsymbol{u} = 2\boldsymbol{i} + 2\boldsymbol{j}$ and $\boldsymbol{w} = \boldsymbol{i} - \boldsymbol{j}$
c $\boldsymbol{a} = \boldsymbol{u} + \boldsymbol{w}$ where $\boldsymbol{u} = \boldsymbol{0}$ and $\boldsymbol{w} = -\boldsymbol{i} + \boldsymbol{j}$

6 a 2 **b** $\dfrac{1}{\sqrt{5}}$ **c** $\dfrac{2\sqrt{3}}{\sqrt{7}}$ **d** $\dfrac{-1 - 4\sqrt{5}}{\sqrt{17}}$

7 a $2\boldsymbol{i} + 2\boldsymbol{j}$ **b** $\pm\dfrac{1}{\sqrt{2}}(-\boldsymbol{i} + \boldsymbol{j})$

8 a $\dfrac{3}{2}(\boldsymbol{i} - \boldsymbol{j})$ **b** $\dfrac{5}{2}(\boldsymbol{i} + \boldsymbol{j})$ **c** $\dfrac{5\sqrt{2}}{2}$

9 a i $\boldsymbol{i} - \boldsymbol{j}$ **ii** $\boldsymbol{i} - 5\boldsymbol{j}$
b $\dfrac{3}{13}(\boldsymbol{i} - 5\boldsymbol{j})$ **c** $\dfrac{2\sqrt{26}}{13}$ **d** 2

Exercise 8F

1 a $-\boldsymbol{i} - 11\boldsymbol{j}$ **b** $5\boldsymbol{i} - 6\boldsymbol{j}$ **c** $\boldsymbol{i} + 5\boldsymbol{j}$
d $-11\boldsymbol{j}$ **e** $4\boldsymbol{i}$ **f** $6\boldsymbol{i} + 11\boldsymbol{j}$

2 12.58 km on a bearing of 341.46°

3 7.74 km on a bearing of 071.17°

4 a $\sqrt{41}$ m/s **b** 5 m/s **c** $\sqrt{17}$ m/s
d $2\sqrt{10}$ m/s **e** 13 m/s **f** $\sqrt{170}$ m/s

5 a $24\boldsymbol{i} + 62\boldsymbol{j}$ **b** $(5t - 1)\boldsymbol{i} + (12t + 2)\boldsymbol{j}$

6 $-4\boldsymbol{i} + 4\boldsymbol{j}$ m/s

7 a i $26\boldsymbol{i} + 99\boldsymbol{j}$
ii $(7t - 2)\boldsymbol{i} + (24t + 3)\boldsymbol{j}$
b i 102.36 m
ii $\sqrt{(7t - 2)^2 + (24t + 3)^2}$ m

8 a $-\boldsymbol{i} - \dfrac{1}{2}\boldsymbol{j}$ m/s **b** $\dfrac{\sqrt{5}}{2}$ m/s

9 After $\dfrac{12(6 + \sqrt{5})}{31}$ seconds;
position vector $\dfrac{12(6 + \sqrt{5})}{31}(\boldsymbol{i} + 2\boldsymbol{j})$

10 a $8\boldsymbol{i} + 4\boldsymbol{j}$ **b** $2\boldsymbol{i} - 4\boldsymbol{j}$ m/s

11 a $20\boldsymbol{i} + 10\boldsymbol{j}$ **b** $\boldsymbol{j}$ m/s

Exercise 8G

1 a On a bearing of 143.13°
b 5 km/h

2 100.08 km/h on a bearing of 357.71°

3 a 20 km/h west **b** 180 km/h west

4 47 m/s north

5 10 m/s

6 a 20 km/h north **b** 20 km/h south

7 252.98 km/h on a bearing of 018.43°

8 100 km/h on a bearing of 053.13°

9 42.5 km/h on a bearing of 41.73°

10 a $\boldsymbol{i} - 4\boldsymbol{j}$ m/s **b** 4.12 m/s

11 10.36 m/s

12 196.83 km/h on a bearing of 345.44°

13 a Bearing 210.67°
b 243.28 km/h

14 a Upstream at an angle of 48.59° to his desired path
b 45.36 seconds

Exercise 8H

1 $T_1 = T_2 = \dfrac{5\sqrt{2}}{2}$ kg wt

2 90°

3 $T_1 = 14.99$ kg wt, $T_2 = 12.10$ kg wt

4 28.34 kg wt, 221.5°S

5 $T = 40$ kg wt, $N = 96$ kg wt

6 $F = 6.39$ kg wt

7 a No **b** Yes

8 146.88°, 51.32°, 161.8°

9 a 7.5 kg wt **b** 9.64 kg wt **c** 7.62 kg wt

10 32.97 kg wt, 26.88 kg wt, 39.29 kg wt,
$W = 39.29$ kg

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11 13.05 kg wt
12 5.74 kg wt
13 3.73 kg wt, 8.83 kg wt
14 4.13 kg wt
15 6.93 kg wt
16 31.11 kg, 23.84 kg wt
17 44.10 kg, 22.48° to the vertical
18 6.43 kg wt, 7.66 kg wt, 11.92 kg
19 3.24 kg wt

Chapter 8 review

Short-response questions

Technology-free

1 a $\overrightarrow{OA} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$ **b** $\overrightarrow{AB} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$
c $\overrightarrow{BC} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$ **d** $\overrightarrow{CO} = \begin{bmatrix} -4 \\ 2 \end{bmatrix}$
e $\overrightarrow{CB} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$

2 a $|\boldsymbol{a}| = \sqrt{13}$, $|\boldsymbol{b}| = \sqrt{10}$ **b** $\boldsymbol{a} \cdot \boldsymbol{b} = 7$
c $\begin{bmatrix} \frac{2}{\sqrt{13}} \\ -\frac{3}{\sqrt{13}} \end{bmatrix}$
d i $3\boldsymbol{a} = \begin{bmatrix} 6 \\ -9 \end{bmatrix}$ **ii** $2\boldsymbol{a} + 3\boldsymbol{b} = \begin{bmatrix} 1 \\ -15 \end{bmatrix}$
iii $2\boldsymbol{a} - \boldsymbol{b} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$

3 a $\frac{12}{7}$ **b** ± 9
4 $A(2, -1)$, $B(5, 3)$, $C(3, 8)$, $D(0, 4)$
5 $k = -\frac{3}{4}$
6 a $\sqrt{26}$ **b** $\frac{1}{\sqrt{26}}(\boldsymbol{i} - 5\boldsymbol{j})$
7 a $11\boldsymbol{i} - 2\boldsymbol{j}$ **b** $\sqrt{29}$
c $\frac{1}{\sqrt{29}}(5\boldsymbol{i} + 2\boldsymbol{j})$ **d** $2\boldsymbol{i} + 4\boldsymbol{j}$
8 $[4, 150°]$
9 a 13 **b** 10 **c** 8 **d** −11
e −9 **f** 0 **g** −27
10 $m = -2, n = -8$
11 $k = -50$
12 a $\sqrt{34}$ **b** $\sqrt{10} - \sqrt{20}$ **c** $\boldsymbol{r} = \boldsymbol{i} - 9\boldsymbol{j}$
13 a $\frac{1}{5}(4\boldsymbol{i} + 3\boldsymbol{j})$ **b** $\frac{16}{25}(4\boldsymbol{i} + 3\boldsymbol{j})$
14 a i $\boldsymbol{a} + \boldsymbol{b}$ **ii** $\frac{1}{3}(\boldsymbol{a} + \boldsymbol{b})$ **iii** $\boldsymbol{b} - \boldsymbol{a}$
iv $\frac{1}{3}(2\boldsymbol{a} - \boldsymbol{b})$ **v** $\frac{2}{3}(2\boldsymbol{a} - \boldsymbol{b})$
b $\overrightarrow{TR} = 2\overrightarrow{PT}$
15 $s = -7$, $t = 9$
16 $\sqrt{109}$ units
17 a $(-1, 10)$ **b** $h = 3$, $k = -2$
18 a $\boldsymbol{b} = \boldsymbol{a} + \boldsymbol{c}$ **b** $\boldsymbol{b} = \frac{2}{5}\boldsymbol{a} + \frac{3}{5}\boldsymbol{c}$
20 a $\frac{6}{5}$ **b** $\pm\frac{3}{\sqrt{2}}$ **c** $\frac{7}{3}$
21 a i $\overrightarrow{AB} = -\boldsymbol{i}$ **ii** $\overrightarrow{AC} = -5\boldsymbol{j}$
b $\boldsymbol{0}$
c 1
22 a 2 m/s **b** 30 seconds
c 36 m downstream of her starting point
23 9 kg wt, 12 kg wt
24 $14\sqrt{5}$ kg wt, $28\sqrt{5}$ kg wt
25 $5\sqrt{3}$ kg wt
26 a $(25, -7)$, $\begin{bmatrix} 7 \\ 24 \end{bmatrix}$ **b** $\begin{bmatrix} -20 \\ 15 \end{bmatrix}$
27 a $\begin{bmatrix} -31 \\ -32 \end{bmatrix}$ **b** $\begin{bmatrix} -15 \\ -20 \end{bmatrix}$ **c** $|\overrightarrow{OR}| = 25$

Technology-active

28 a 64.7° **b** 67.2°
29 107.7 km/h, bearing 248.2°
30 128.68°
31 a $(12, 4)$ **b** $\begin{bmatrix} k-12 \\ -4 \end{bmatrix}$
c $\sqrt{160}$, k, $\sqrt{(k-12)^2 + 16}$, $k = \frac{40}{3}$
d 34.7°
32 a 135° **b** $\boldsymbol{s} = 5\boldsymbol{i} + 3\boldsymbol{j} + t(\boldsymbol{i} - \boldsymbol{j})$
c $p = 14$, $q = 2$
33 a 2.5 hours
b 011.54°, 36.7 minutes
c 168.46°
34 a $4\boldsymbol{i} + (16 - \alpha)\boldsymbol{j}$ km/h
b i $\alpha = 16$ **ii** 2.5 hours

Multiple-choice questions

1 C **2** C **3** A **4** B **5** B **6** A
7 C **8** D **9** D **10** A **11** B **12** D
13 C **14** B **15** B **16** C **17** A **18** D
19 D **20** D **21** C

Chapter 9

Exercise 9A

1 a 2×2 **b** 2×3
c 1×4 **d** 4×1
2 a $\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}$ **b** $\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$

3 a $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ **b** $\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}$

c $\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$

4 $\begin{bmatrix} 200 & 180 & 135 & 110 & 56 & 28 \\ 110 & 117 & 98 & 89 & 53 & 33 \end{bmatrix}$

5 a $\begin{bmatrix} 0 & x \end{bmatrix} = \begin{bmatrix} 0 & 4 \end{bmatrix}$ if $x = 4$

b $\begin{bmatrix} 4 & 7 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} x & 7 \\ 1 & -2 \end{bmatrix}$ if $x = 4$

c $\begin{bmatrix} 2 & x & 4 \\ -1 & 10 & 3 \end{bmatrix} = \begin{bmatrix} y & 0 & 4 \\ -1 & 10 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 4 \\ -1 & 10 & 3 \end{bmatrix}$ if $x = 0$, $y = 2$

6 a $x = 2$, $y = 3$ **b** $x = 3$, $y = 2$
c $x = 4$, $y = -3$ **d** $x = 3$, $y = -2$

7 $\begin{bmatrix} 21 & 5 & 5 \\ 8 & 2 & 3 \\ 4 & 1 & 1 \\ 14 & 8 & 60 \\ 0 & 1 & 2 \end{bmatrix}$

Exercise 9B

1 $\mathbf{X} + \mathbf{Y} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$ $2\mathbf{X} = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$ $4\mathbf{Y} + \mathbf{X} = \begin{bmatrix} 13 \\ -2 \end{bmatrix}$

$\mathbf{X} - \mathbf{Y} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$ $-3\mathbf{A} = \begin{bmatrix} -3 & 3 \\ -6 & -9 \end{bmatrix}$

$-3\mathbf{A} + \mathbf{B} = \begin{bmatrix} 1 & 3 \\ -7 & -7 \end{bmatrix}$

2 $2\mathbf{A} = \begin{bmatrix} 2 & -2 \\ 0 & 4 \end{bmatrix}$ $-3\mathbf{A} = \begin{bmatrix} -3 & 3 \\ 0 & -6 \end{bmatrix}$

$-6\mathbf{A} = \begin{bmatrix} -6 & 6 \\ 0 & -12 \end{bmatrix}$

3 a Yes **b** Yes

4 a $\begin{bmatrix} 6 & 4 \\ -4 & -4 \end{bmatrix}$ **b** $\begin{bmatrix} 0 & -9 \\ 12 & 3 \end{bmatrix}$

c $\begin{bmatrix} 6 & -5 \\ 8 & -1 \end{bmatrix}$ **d** $\begin{bmatrix} -6 & -13 \\ 16 & 7 \end{bmatrix}$

5 a $\begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}$ **b** $\begin{bmatrix} -2 & 3 \\ 6 & 3 \end{bmatrix}$ **c** $\begin{bmatrix} 3 & 3 \\ -1 & 7 \end{bmatrix}$

6 $\mathbf{X} = \begin{bmatrix} 2 & 4 \\ 0 & -3 \end{bmatrix}$, $\mathbf{Y} = \begin{bmatrix} \frac{-9}{2} & \frac{-23}{2} \\ \frac{-1}{2} & 11 \end{bmatrix}$

7 $\mathbf{X} + \mathbf{Y} = \begin{bmatrix} 310 & 180 & 220 & 90 \\ 200 & 0 & 125 & 0 \end{bmatrix}$
represents the total production at two factories in two successive weeks

Exercise 9C

1 $\mathbf{AX} = \begin{bmatrix} 4 \\ -5 \end{bmatrix}$ $\mathbf{BX} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ $\mathbf{AY} = \begin{bmatrix} -5 \\ 8 \end{bmatrix}$

$\mathbf{IX} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ $\mathbf{AC} = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}$

$\mathbf{CA} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ $(\mathbf{AC})\mathbf{X} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\mathbf{C}(\mathbf{BX}) = \begin{bmatrix} 9 \\ 5 \end{bmatrix}$ $\mathbf{AI} = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$

$\mathbf{IB} = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$ $\mathbf{AB} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\mathbf{BA} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\mathbf{A}^2 = \begin{bmatrix} 3 & -8 \\ -4 & 11 \end{bmatrix}$

$\mathbf{B}^2 = \begin{bmatrix} 11 & 8 \\ 4 & 3 \end{bmatrix}$ $\mathbf{A}(\mathbf{CA}) = \begin{bmatrix} 1 & -3 \\ -1 & 4 \end{bmatrix}$

$\mathbf{A}^2\mathbf{C} = \begin{bmatrix} -2 & -5 \\ 3 & 7 \end{bmatrix}$

2 Defined: **AY**, **CI**;
Not defined: **YA**, **XY**, $\mathbf{X}^2$, **XI**

3 $\mathbf{AB} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

4 No

5 One possible answer is $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$

6 $\mathbf{LX} = \begin{bmatrix} 7 \end{bmatrix}$, $\mathbf{XL} = \begin{bmatrix} 4 & -2 \\ -6 & 3 \end{bmatrix}$

7 **AB** and **BA** are not defined unless $m = n$

8 b $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

9 One possible answer is
$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} -2 & 1 \\ 1.5 & -0.5 \end{bmatrix}$

10 For example: $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$

11 a $\begin{bmatrix} 29 \\ 8.50 \end{bmatrix}$, John took 29 minutes to eat food costing \$8.50

b $\begin{bmatrix} 29 & 22 & 12 \\ 8.50 & 8.00 & 3.00 \end{bmatrix}$,
John's friends took 22 and 12 minutes to eat food costing \$8.00 and \$3.00 respectively

12 $\mathbf{A}^2 = \begin{bmatrix} -3 & 4 \\ -4 & -3 \end{bmatrix}$, $\mathbf{A}^4 = \begin{bmatrix} -7 & -24 \\ 24 & -7 \end{bmatrix}$,

$\mathbf{A}^8 = \begin{bmatrix} -527 & 336 \\ -336 & -527 \end{bmatrix}$

13 $\mathbf{A}^2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, $\mathbf{A}^3 = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$, $\mathbf{A}^4 = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$,

$\mathbf{A}^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$

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Exercise 9D

1 a 1 **b** $\begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$ **c** 2 **d** $\frac{1}{2}\begin{bmatrix} 2 & 2 \\ -3 & -2 \end{bmatrix}$

2 a $\begin{bmatrix} -1 & 1 \\ -4 & 3 \end{bmatrix}$ **b** $\begin{bmatrix} \frac{2}{7} & \frac{-1}{14} \\ \frac{1}{7} & \frac{3}{14} \end{bmatrix}$ **c** $\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{k} \end{bmatrix}$

4 a $\mathbf{A}^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & -1 \end{bmatrix}$, $\mathbf{B}^{-1} = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$

b $\mathbf{AB} = \begin{bmatrix} 5 & 1 \\ -3 & -1 \end{bmatrix}$, $(\mathbf{AB})^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{3}{2} & -\frac{5}{2} \end{bmatrix}$

c $\mathbf{A}^{-1}\mathbf{B}^{-1} = \begin{bmatrix} -1 & \frac{1}{2} \\ 3 & -1 \end{bmatrix}$,

$\mathbf{B}^{-1}\mathbf{A}^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{3}{2} & -\frac{5}{2} \end{bmatrix}$, $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$

5 a $\begin{bmatrix} -\frac{1}{2} & \frac{3}{2} \\ 1 & -2 \end{bmatrix}$ **b** $\begin{bmatrix} 0 & 7 \\ 1 & -8 \end{bmatrix}$ **c** $\begin{bmatrix} \frac{5}{2} & -\frac{7}{2} \\ \frac{11}{2} & -\frac{21}{2} \end{bmatrix}$

6 a $\begin{bmatrix} \frac{-3}{8} & \frac{11}{8} \\ \frac{1}{16} & \frac{7}{16} \end{bmatrix}$ **b** $\begin{bmatrix} \frac{-11}{16} & \frac{17}{16} \\ \frac{-1}{4} & \frac{3}{4} \end{bmatrix}$

7 $\begin{bmatrix} \frac{1}{a_{11}} & 0 \\ 0 & \frac{1}{a_{22}} \end{bmatrix}$

9 $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix},$

$\begin{bmatrix} 1 & 0 \\ k & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ k & 1 \end{bmatrix}, \begin{bmatrix} 1 & k \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & k \\ 0 & 1 \end{bmatrix}, k \in \mathbb{R},$

$\begin{bmatrix} a & b \\ \frac{1-a^2}{b} & -a \end{bmatrix}, b \neq 0$

10 $a = \pm\sqrt{2}$

Exercise 9E

1 a $\mathbf{A} : 2 \times 3$, $\mathbf{B} : 3 \times 2$ $\mathbf{C} : 2 \times 1$

b $\mathbf{AB} = \begin{bmatrix} 23 & 51 \\ 24 & 62 \end{bmatrix}$, $\mathbf{BC} = \begin{bmatrix} 13 \\ 9 \\ 9 \end{bmatrix}$

c $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC}) = \begin{bmatrix} 120 \\ 134 \end{bmatrix}$

2 a $\begin{bmatrix} -3 & -4 & -5 \\ -1 & -2 & 3 \end{bmatrix}$ **b** $\begin{bmatrix} 1 & -2 \\ 1 & 3 \\ -\frac{1}{2} & \frac{3}{2} \end{bmatrix}$

3 a $\begin{bmatrix} -\frac{26}{33} \\ \frac{28}{33} \end{bmatrix}$ **b** $\begin{bmatrix} \frac{34}{33} & -\frac{20}{33} \\ -\frac{16}{33} & \frac{23}{33} \end{bmatrix}$

4 a $\begin{bmatrix} -\frac{4}{9} & -\frac{2}{3} \\ \frac{34}{9} & -\frac{4}{3} \end{bmatrix}$ **b** $\begin{bmatrix} \frac{2}{9} & \frac{16}{9} \\ -2 & -2 \end{bmatrix}$

5 $\begin{bmatrix} 26 \\ 15 \end{bmatrix}$

6 $p = 1, q = 4, r = 3$

7 $\begin{bmatrix} 0 & \frac{1}{n} \\ \frac{1}{m} & -\frac{1}{mn} \end{bmatrix}$

12 $p = 1, q = 0$

Exercise 9F

1 a $\begin{bmatrix} 3 \\ 10 \end{bmatrix}$ **b** $\begin{bmatrix} 5 \\ 17 \end{bmatrix}$

2 a $x = -\frac{1}{7}$, $y = \frac{10}{7}$ **b** $x = 4$, $y = 1.5$

c $x = 33$, $y = 18$ **d** $x = \frac{2}{11}$, $y = \frac{8}{11}$

e $x = \frac{9}{2}$, $y = -\frac{3}{2}$ **f** No solution

3 $(2, -1)$

4 a $\begin{bmatrix} 5 \\ -3 \end{bmatrix}$ **b** $\begin{bmatrix} -3 \\ -5 \end{bmatrix}$ **c** $\begin{bmatrix} 5 \\ -6 \end{bmatrix}$

5 Book \$12, DVD \$18

6 $x = 100$, $y = 100$

7 a $\begin{bmatrix} 2 & -3 \\ 4 & -6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$ **b** $\begin{bmatrix} 2 & -3 \\ 4 & -6 \end{bmatrix}$ is singular

c System has solutions (not a unique solution)

d Solution set contains infinitely many pairs

8 a $\begin{bmatrix} 2 & -3 \\ 4 & -6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$ **b** $\begin{bmatrix} 2 & -3 \\ 4 & -6 \end{bmatrix}$ is singular

c System has no solutions

9 a $\mathbf{A}^{-1}\mathbf{C}$ **b** $\mathbf{B}^{-1}\mathbf{A}^{-1}\mathbf{C}$ **c** $\mathbf{A}^{-1}\mathbf{C}\mathbf{B}^{-1}$

d $\mathbf{A}^{-1}\mathbf{C} - \mathbf{B}$ **e** $\mathbf{A}^{-1}(\mathbf{C} - \mathbf{B})$

f $(\mathbf{A} - \mathbf{B})\mathbf{A}^{-1} = \mathbf{I} - \mathbf{B}\mathbf{A}^{-1}$

Exercise 9G

1 a $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{5} \end{bmatrix}$ **b** $\begin{bmatrix} 1 & -2 & \frac{1}{5} \\ 0 & \frac{1}{2} & \frac{-3}{10} \\ 0 & 0 & \frac{1}{5} \end{bmatrix}$

3 $\mathbf{AB} = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix}$; $\mathbf{A}^{-1} = \frac{1}{7}\mathbf{B}$

4 $\mathbf{A}^2 = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$; $\mathbf{A}^{-1} = \frac{1}{9}\mathbf{A}$

5 $\mathbf{A}^2 = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$; $\mathbf{A}^{-1} = \frac{1}{4}\mathbf{A}$

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6 a $\begin{bmatrix} 2 & 1 & -10 \\ 3 & 2 & -17 \\ -5 & -3 & 28 \end{bmatrix}$

b $\dfrac{1}{29}\begin{bmatrix} 8 & -13 & 14 \\ 2 & 4 & -11 \\ -9 & 11 & 6 \end{bmatrix}$

c $\dfrac{1}{37}\begin{bmatrix} 6 & 4 & -7 & -17 \\ -13 & -21 & 46 & 43 \\ 8 & 30 & -34 & -35 \\ -4 & -15 & 17 & 36 \end{bmatrix}$

d $\dfrac{1}{37}\begin{bmatrix} 6 & -13 & 8 & -4 \\ 4 & -21 & 30 & -15 \\ -7 & 46 & -34 & 17 \\ -17 & 43 & -35 & 36 \end{bmatrix}$

7 a -36 **b** 1

8 a i -2 **ii** -2
b i -4 **ii** -16

9 a $\det(\mathbf{A}) = -2p + 6$ **b** $p = 3$

10 a $\det(\mathbf{A}) = -2(p-2)(p-1)$
b $p = 2$ or $p = 1$

Exercise 9H

1 a $x = 2, y = 3, z = 1$ **b** $x = -3, y = 5, z = 2$
c $x = 5, y = 0, z = 7$ **d** $x = 6, y = 5, z = 1$
e $x = 5, y = 2, z = 4, w = -1$

2 a $\begin{bmatrix} 1 & 2 & 3 \\ -1 & -3 & 2 \\ -1 & -4 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 13 \\ 2 \\ 17 \end{bmatrix}$

b $\det(\mathbf{A}) = 0$, so $\mathbf{A}$ is non-invertible
c i $-y + 5z = 15,\ -y + 5z = 15$
ii The two equations are the same
iii $y = 5\lambda - 15$
iv $x = 43 - 13\lambda$

Chapter 9 review

Short-response questions

Technology-free

1 a $\begin{bmatrix} 0 & 0 \\ 12 & 8 \end{bmatrix}$ **b** $\begin{bmatrix} 0 & 0 \\ 8 & 8 \end{bmatrix}$

2 $\begin{bmatrix} a \\ 2 - \frac{3}{4}a \end{bmatrix},\ a \in \mathbb{R}$

3 a Exist: **AC**, **CD**, **BE**; Does not exist: **AB**
b $\mathbf{DA} = \begin{bmatrix} 14 & 0 \end{bmatrix},\quad \mathbf{A}^{-1} = \dfrac{1}{7}\begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}$

4 $\mathbf{AB} = \begin{bmatrix} 2 & 0 \\ 2 & -2 \end{bmatrix},\quad \mathbf{C}^{-1} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$

5 $\begin{bmatrix} -1 & 2 \\ -3 & 5 \end{bmatrix}$

6 $\mathbf{A}^2 = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix},\quad \mathbf{A}^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{bmatrix}$

7 8

8 a i $\begin{bmatrix} 3 & -5 \\ 5 & 8 \end{bmatrix}$ **ii** $\begin{bmatrix} 1 & -18 \\ 18 & 19 \end{bmatrix}$ **iii** $\dfrac{1}{7}\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$
b $x = 2,\ y = 1$

9 $a = 3$

10 a i $\begin{bmatrix} 2 & -3 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$
ii $\det(\mathbf{A}) = 14,\ \mathbf{A}^{-1} = \dfrac{1}{14}\begin{bmatrix} 1 & 3 \\ -4 & 2 \end{bmatrix}$
iii $\dfrac{1}{7}\begin{bmatrix} 9 \\ -1 \end{bmatrix}$
iv $\left(\dfrac{9}{7}, -\dfrac{1}{7}\right)$ is the point of intersection of the two lines
b i $\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 8 \end{bmatrix}$
ii $\det(\mathbf{A}) = 0$, so $\mathbf{A}$ is non-invertible
c Equations of two parallel lines

Technology-active

12 a $\begin{bmatrix} 0 & 3 & -4 \\ 0 & 4 & 0 \\ 2 & 0 & 0 \end{bmatrix}$ **b** $\begin{bmatrix} 11 & -2 & 6 \\ 6 & 16 & 12 \\ 4 & 4 & 4 \end{bmatrix}$

c $\begin{bmatrix} 4 & 6 & -8 \\ 6 & 19 & 0 \\ 0 & 9 & 8 \end{bmatrix}$ **d** $\begin{bmatrix} 20 & 101 & 20 \\ 71 & 304 & 46 \\ 23 & 102 & 14 \end{bmatrix}$

e $\begin{bmatrix} 304 & 704 & 480 \\ 848 & 2320 & 1504 \\ 304 & 720 & 480 \end{bmatrix}$ **f** $\begin{bmatrix} 7 & -8 & 14 \\ 0 & -3 & 12 \\ 4 & -5 & -4 \end{bmatrix}$

13 a $\dfrac{1}{9}\begin{bmatrix} 5 & -1 \\ -13 & 8 \end{bmatrix}$ **b** $\dfrac{1}{9}\begin{bmatrix} 12 & 3 \\ -5 & 1 \end{bmatrix}$

14 $\dfrac{1}{9}\begin{bmatrix} 12 \\ -5 \end{bmatrix}$

15 a i $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ **ii** $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ **iii** $\begin{bmatrix} 0 & -1 \\ 1 & -0 \end{bmatrix}$
iv $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

b i $\begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$ **ii** $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

iii $\begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$ **iv** $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

16 a $\begin{bmatrix} 79 & 78 & 80 \\ 80 & 78 & 82 \end{bmatrix}$ **b** $\begin{bmatrix} 0.2 \\ 0.3 \\ 0.5 \end{bmatrix}$
c Semester 1: 79.2; Semester 2: 80.4
d Semester 1: 83.8; Semester 2: 75.2
e No, total score is 318.6
f 3 marks

17 a $\begin{bmatrix}10 & 2\\8 & 4\\8 & 8\\6 & 10\end{bmatrix}$ **b** $\begin{bmatrix}70\\60\end{bmatrix}$

c Term 1: \$820; Term 2: \$800; Term 3: \$1040; Term 4: \$1020

d $\begin{bmatrix}2 & 2 & 1\\2 & 2 & 1\\3 & 4 & 2\\3 & 4 & 2\end{bmatrix}$ **e** $\begin{bmatrix}60\\55\\40\end{bmatrix}$

f Term 1: \$270; Term 2: \$270; Term 3: \$480; Term 4: \$480

g Term 1: \$1090; Term 2: \$1070; Term 3: \$1520; Term 4: \$1500

18 Brad 20; Flynn 10; Lina 15

Multiple-choice questions

1 B **2** D **3** D **4** C **5** D
6 C **7** A **8** D **9** A **10** D
11 D **12** D **13** D **14** C **15** B

Chapter 10

Short-response questions

Technology-free

1 a $\begin{bmatrix}-1\\-10\end{bmatrix}$ **b** $\begin{bmatrix}4\\5\end{bmatrix}$ **c** $\begin{bmatrix}4 & 24\\7 & 4\end{bmatrix}$

2 $w = -5$ or $w = 2$

3 $x = 3$

4 $a = 0,\ b = -1$

5 a $5\boldsymbol{i} + \boldsymbol{j}$ **b** $\boldsymbol{i} + 9\boldsymbol{j}$ **c** $\boldsymbol{i} + 31\boldsymbol{j}$ **d** 34
e $\sqrt{34}$ **f** -14

6 $s = \frac{3}{2},\ t = \frac{7}{2}$

7 $4\boldsymbol{i} + 3\boldsymbol{j} = \boldsymbol{u} + \boldsymbol{w}$ where $\boldsymbol{u} = \left(\sqrt{3} - \frac{3}{4}\right)(\sqrt{3}\boldsymbol{i} - \boldsymbol{j})$
and $\boldsymbol{w} = \left(\frac{3\sqrt{3}}{4} + 1\right)\boldsymbol{i} + \left(\sqrt{3} + \frac{9}{4}\right)\boldsymbol{j}$

8 a $8\boldsymbol{i} + 9\boldsymbol{j}$ **b** $(3 + t)\boldsymbol{i} + (4 + t)\boldsymbol{j}$

9 a 2 **b** $\frac{1}{2}(\sqrt{3}\boldsymbol{i} + \boldsymbol{j})$ **c** $15(\sqrt{3}\boldsymbol{i} + \boldsymbol{j})$

10 a 120 **b** 60 **c** 10

11 36

12 1 663 200

13 a 126 **b** 56 **c** 111

15 a $2\sqrt{10}$ **b** $\frac{1}{\sqrt{10}}(\boldsymbol{i} + 3\boldsymbol{j})$
c $\frac{8}{\sqrt{10}}(\boldsymbol{i} + 3\boldsymbol{j})$ **d** $-\frac{2}{\sqrt{10}}(\boldsymbol{i} + 3\boldsymbol{j})$

16 a 13 **b** 13 **c** 13 **d** -13
e -13 **f** 0 **g** -13

17 a $m = \frac{46}{11},\ n = -\frac{18}{11}$ **b** $p = -48$
c $p = 3, 5$

18 a All defined except **AB**
b $\mathbf{DA} = \begin{bmatrix}6 & -12\end{bmatrix}$, $\mathbf{A}^{-1} = \begin{bmatrix}\frac{1}{9} & \frac{4}{9}\\ \frac{2}{9} & -\frac{1}{9}\end{bmatrix}$

19 a $\begin{bmatrix}-2 & 4\\18 & -24\end{bmatrix}$ **b** $\begin{bmatrix}-10 & -19\\7 & -16\end{bmatrix}$

20 8

21 $\mathbf{A} = \begin{bmatrix}t\\3t - 5\end{bmatrix}, t \in \mathbb{R}$

22 $\mathbf{AB} = \begin{bmatrix}-9 & -8\\-15 & 10\end{bmatrix}$, $\mathbf{C}^{-1} = \begin{bmatrix}2 & 1\\ \frac{3}{2} & \frac{1}{2}\end{bmatrix}$

23 a $(\mathbf{C} - \mathbf{B})\mathbf{A}^{-1}$ **b** $\mathbf{B}^{-1}\mathbf{C} - \mathbf{A}$
c $\mathbf{I} - \mathbf{A}^{-1}\mathbf{BA}$ **d** $-\mathbf{A}$
e $\frac{1}{2}\mathbf{B}$ **f** $\mathbf{I} - \mathbf{A}^{-1}$

24 a $\boldsymbol{v}_1 = 2\sqrt{3}\boldsymbol{i} + 2\boldsymbol{j}$ m/s, $\boldsymbol{v}_2 = 3\boldsymbol{i} - 3\sqrt{3}\boldsymbol{j}$ m/s
b $(4\sqrt{3} + 9)\boldsymbol{i} + (4 - 9\sqrt{3})\boldsymbol{j}$

25 24

26 360

27 a 125 **b** 60

28 a 9 **b** 25

29 a 24 **b** 30 **c** 28 **d** 45

30 a 120 **b** 120

33 a If n is odd, then $5n + 3$ is even.
c If n is even, then $5n + 3$ is odd.

34 a $\begin{bmatrix}1 - 2b & 0\\ab & 2a\end{bmatrix}$ **b** $a = \frac{1}{2},\ b = 0$

35 $(0, 0, 0), (1, 0, -1)$

36 10

37 a 20 **b** 325 **c** 210 **d** 56

38 a 28 **b** 21 **c** $2^8 = 256$

39 $\frac{10}{21}$

40 120

41 7

42 51

43 80

51 a $t = -\frac{1}{2}$ **b** $t = -\frac{1}{3}$

52 a 749 **b** 30

53 a 60 480 **b** 10 080 **c** 2400

54 26

56 a i $\frac{1}{3}$ **ii** $\frac{2}{5}$ **iii** $\frac{3}{7}$
b $S_n = \frac{n}{2n + 1}$

58 $F = 7$ kg wt, $\cos\theta = -\frac{31}{49}$

59 $(F_3)^2 = 100 - 48\sqrt{2}$

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60 a $T = 5$ kg wt, $N = 5\sqrt{3}$ kg wt

b $T = \frac{10\sqrt{3}}{3}$ kg wt, $N = \frac{20\sqrt{3}}{3}$ kg wt

61 $\frac{120}{13}$ kg wt, $\frac{50}{13}$ kg wt

62 b -4

63 a 4900 **b** 1568 **c** 0

Technology-active

64 a 38.45° **b** 81.97°

65 $2\sqrt{61}$ km on a bearing of 063.67°

66 $10\sqrt{170}$ km/h on a bearing of 237.529°t

67 a $\sqrt{337}$ m/s on a bearing of $\tan^{-1}\left(\frac{16}{9}\right)$

b 8.5 seconds

c 76.5 m

68 a 120 **b** 36

69 a 96 **b** 24 **c** 72 **d** 60

70 a $\overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a}$, $\overrightarrow{PQ} = \frac{-3}{10}\boldsymbol{a} + \frac{1}{2}\boldsymbol{b}$

b i $n\left(\frac{-3}{10}\boldsymbol{a} + \frac{1}{2}\boldsymbol{b}\right)$ **ii** $\left(k + \frac{1}{2}\right)\boldsymbol{b} - \frac{1}{2}\boldsymbol{a}$

c $n = \frac{5}{3}$, $k = \frac{1}{3}$

71 a $4\sqrt{2}$ km/h blowing from the south-west

b $\sqrt{5}$ km/h; 200 m downstream

c 43.1 km/h on a bearing of 080° **d** 222°

72 a 2160 **b** 360 **c** 900 **d** 1260

73 a 70 **b** 30 **c** 15 **d** 55

74 a 20 **b** 4 **c** 68

75 a 420 **b** 60 **c** 120 **d** 24

76 a 300 **b** 10 and 15

77 a 495 **b** 60

c The two points diametrically opposite

d 15 **e** $\frac{1}{33}$

78 a No

b Yes; both a and b are odd, and c is even

79 a $(a, b, c) = (2, 3, 6)$

b $(a, b, c, d) = (1, 2, 3, 4)$ or $(a, b, c, d) = (1, 2, 3, 5)$

81 a 10

83 a No **b** $n = 4k$ or $n = 4k - 1$

85 a $a = 1$, $b = 3$, $c = 1$ **c** 41^2

86 a $m^2 - (a + d)m + ad - bc$

c $m = a - c$

d i $m = 1$ or $m = -9$ **ii** $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} k \\ k \end{bmatrix}, k \in \mathbb{R}$

iii $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} k \\ -\frac{3k}{2} \end{bmatrix}, k \in \mathbb{R}$

91 1458

92 a $\overrightarrow{AE} = \frac{1}{t+1}(2\boldsymbol{a} + t\boldsymbol{b})$

b $\overrightarrow{AE} = \frac{1}{8}(7\boldsymbol{a} + \overrightarrow{AF})$ **d** $t = \frac{9}{7}$

93 b $(n - 1)\boldsymbol{a} - n\boldsymbol{b} + \boldsymbol{c}$

Multiple-choice questions

1 A	**2** B	**3** D	**4** A	**5** B
6 C	**7** A	**8** B	**9** D	**10** A
11 B	**12** D	**13** B	**14** B	**15** A
16 B	**17** C	**18** B	**19** B	**20** C
21 A	**22** B	**23** C	**24** A	**25** B
26 D	**27** D	**28** D	**29** A	**30** D
31 B	**32** D	**33** D	**34** D	**35** C
36 D	**37** D	**38** B	**39** B	**40** C
41 C	**42** A	**43** D	**44** B	**45** A
46 B				

Problem-solving and modelling

See solutions supplement

Chapter 11

Exercise 11A

1 a -1 **b** $-\sqrt{2}$ **c** $\frac{-2}{\sqrt{3}} = \frac{-2\sqrt{3}}{3}$ **d** 1

e -2 **f** 2 **g** $\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$ **h** 2

2 a -1 **b** $\frac{-2}{\sqrt{3}} = \frac{-2\sqrt{3}}{3}$ **c** 1

d $\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$ **e** $-\sqrt{2}$ **f** $\frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$

g -1 **h** $\frac{-2}{\sqrt{3}} = \frac{-2\sqrt{3}}{3}$ **i** $\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

3 a $\frac{\pi}{6}, \frac{5\pi}{6}$ **b** $\frac{\pi}{6}, \frac{7\pi}{6}$ **c** $\frac{3\pi}{4}, \frac{5\pi}{4}$ **d** $\frac{\pi}{4}, \frac{5\pi}{4}$

4 a $-\frac{8}{17}$ **b** $\frac{15}{17}$ **c** $-\frac{15}{8}$

5 $\cos\theta = \frac{24}{25}$, $\sin\theta = -\frac{7}{25}$

6 $-\frac{\sqrt{29}}{5}$

7 $\frac{8}{31}$

8 $\frac{15}{4(9 + \sqrt{5})} = \frac{15(6 - \sqrt{5})}{124}$

Exercise 11B

1 a $-\frac{\sqrt{2} + \sqrt{6}}{4}$ **b** $\frac{\sqrt{2} - \sqrt{6}}{4}$

c $\frac{\sqrt{6} - \sqrt{2}}{4}$ **d** $\frac{\sqrt{2} + \sqrt{6}}{4}$

2 a $\frac{\sqrt{6} - \sqrt{2}}{4}$ **b** $\frac{\sqrt{6} - \sqrt{2}}{4}$

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3 For $u, v \in \left(0, \frac{\pi}{2}\right)$, $\sin(u+v) = \frac{63}{65}$;

For $u, v \in \left(\frac{\pi}{2}, \pi\right)$, $\sin(u+v) = -\frac{63}{65}$;

For $u \in \left(0, \frac{\pi}{2}\right)$, $v \in \left(\frac{\pi}{2}, \pi\right)$, $\sin(u+v) = -\frac{33}{65}$;

For $u \in \left(\frac{\pi}{2}, \pi\right)$, $v \in \left(0, \frac{\pi}{2}\right)$, $\sin(u+v) = \frac{33}{65}$

4 a $\frac{1}{2}(\sqrt{3}\sin\theta + \cos\theta)$ **b** $\frac{1}{\sqrt{2}}(\cos\varphi + \sin\varphi)$

c $\frac{1}{2}(\cos\theta - \sqrt{3}\sin\theta)$ **d** $\frac{1}{\sqrt{2}}(\sin\theta - \cos\theta)$

5 a $\sin u$ **b** $\cos u$

6 a $-\frac{119}{169}$ **b** $\frac{24}{25}$ **c** $\frac{7}{25}$ **d** $-\frac{169}{119}$

e $-\frac{33}{65}$ **f** $-\frac{16}{65}$ **g** $-\frac{65}{33}$ **h** $\frac{7}{24}$

7 a $\frac{7}{25}$ **b** $\frac{3}{5}$ **c** $-\frac{44}{125}$ **d** $-\frac{336}{625}$

8 a $-\frac{\sqrt{3}}{2}$ **b** $-\frac{1}{2}$

9 a $1 - \sin(2\theta)$ **b** $\cos(2\theta)$

11 c i $2 - \sqrt{3}$ **ii** $2 + \sqrt{3}$ **iii** $-2 - \sqrt{3}$

Exercise 11C

1 a 5, −5 **b** 2, −2 **c** $\sqrt{2}, -\sqrt{2}$

d $\sqrt{2}, -\sqrt{2}$ **e** $2\sqrt{3}, -2\sqrt{3}$

f 2, −2 **g** 4, 0 **h** $5 + \sqrt{13}, 5 - \sqrt{13}$

2 a $\frac{\pi}{2}, \pi$ **b** $0, \frac{2\pi}{3}, 2\pi$

c $\frac{\pi}{6}, \frac{3\pi}{2}$ **d** $0, \frac{5\pi}{3}, 2\pi$

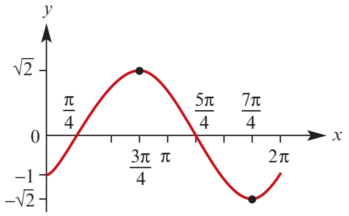
e 53.13° **f** 95.26°, 155.26°

3 $2\cos\left(2x + \frac{\pi}{6}\right)$

4 $\sqrt{2}\sin\left(3x - \frac{5\pi}{4}\right)$

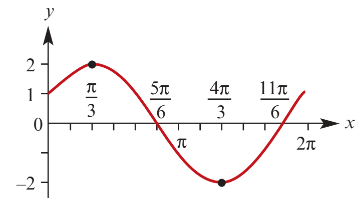
5 a $f(x) = \sin x - \cos x = \sqrt{2}\cos\left(x - \frac{3\pi}{4}\right)$

$= \sqrt{2}\sin\left(x + \frac{7\pi}{4}\right) = \sqrt{2}\sin\left(x - \frac{\pi}{4}\right)$



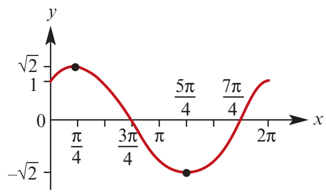
b $f(x) = \sqrt{3}\sin x + \cos x$

$= 2\cos\left(x - \frac{\pi}{3}\right) = 2\sin\left(x + \frac{\pi}{6}\right)$



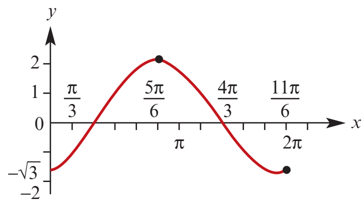
c $f(x) = \sin x + \cos x$

$= \sqrt{2}\cos\left(x - \frac{\pi}{4}\right) = \sqrt{2}\sin\left(x + \frac{\pi}{4}\right)$



d $f(x) = \sin x - \sqrt{3}\cos x = 2\cos\left(x - \frac{5\pi}{6}\right)$

$= 2\sin\left(x + \frac{5\pi}{3}\right) = 2\sin\left(x - \frac{\pi}{3}\right)$



Exercise 11D

1 a $\sin(5\pi t) + \sin(\pi t)$ **b** $\frac{1}{2}(\sin 50° - \sin 10°)$

c $\sin(\pi x) + \sin\left(\frac{\pi x}{2}\right)$ **d** $\sin(A) + \sin(B + C)$

2 $\cos(\theta) - \cos(5\theta)$

3 $\sin A - \sin B$

5 a $2\sin 39° \cos 17°$ **b** $2\cos 39° \cos 17°$

c $2\cos 39° \sin 17°$ **d** $-2\sin 39° \sin 17°$

6 a $2\sin(4A)\cos(2A)$ **b** $2\cos\left(\frac{3x}{2}\right)\cos\left(\frac{x}{2}\right)$

c $2\sin\left(\frac{x}{2}\right)\cos\left(\frac{7x}{2}\right)$ **d** $-2\sin(2A)\sin(A)$

11 a $-\frac{5\pi}{6}, -\frac{3\pi}{4}, -\frac{\pi}{2}, -\frac{\pi}{4}, \frac{\pi}{6}, \frac{\pi}{2}, \frac{3\pi}{4}, \frac{5\pi}{6}$

b $-\pi, -\frac{2\pi}{3}, -\frac{\pi}{2}, -\frac{\pi}{3}, 0, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \pi$

c $-\pi, -\frac{3\pi}{4}, -\frac{2\pi}{3}, -\frac{\pi}{3}, -\frac{\pi}{4}, 0, \frac{\pi}{4}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{3\pi}{4}, \pi$

d $-\pi, -\frac{5\pi}{6}, -\frac{\pi}{2}, -\frac{\pi}{6}, 0, \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \pi$

12 a $\frac{\pi}{6}, \frac{5\pi}{6}$

b $0, \frac{\pi}{6}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{5\pi}{6}, \pi$

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c $0, \frac{\pi}{12}, \frac{\pi}{3}, \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{2\pi}{3}, \frac{11\pi}{12}, \pi$

d $\frac{\pi}{10}, \frac{\pi}{6}, \frac{3\pi}{10}, \frac{\pi}{2}, \frac{7\pi}{10}, \frac{5\pi}{6}, \frac{9\pi}{10}$

17 $\frac{1-\cos(100x)}{2\sin(x)}$

Chapter 11 review

Short-response questions

Technology-free

1 a 5, 1 **b** 4, −2 **c** 4, −4 **d** 2, 0 **e** $1, \frac{1}{3}$

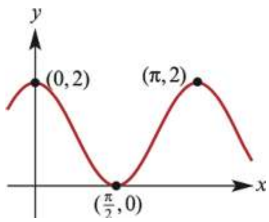
2 a $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$ **b** $\frac{3\pi}{8}, \frac{7\pi}{8}, \frac{11\pi}{8}, \frac{15\pi}{8}$

c $\frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$ **d** $\frac{\pi}{8}, \frac{7\pi}{8}, \frac{9\pi}{8}, \frac{15\pi}{8}$

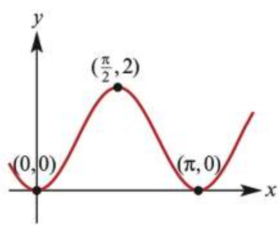
4 a $\frac{140}{221}$ **b** $\frac{171}{221}$ **c** $\frac{171}{140}$

5 $\frac{1}{2}$

6 a $y = 2\cos^2 x$



b $y = 2\sin^2 x$



7 a 1 **b** 0

9 a $\frac{-1}{9}$ **b** $\frac{-4\sqrt{5}}{9}$ **c** $\frac{8\sqrt{5}}{81}$

11 $2-\sqrt{3}$

12 a $0, \frac{\pi}{2}, 2\pi$ **b** $\frac{7\pi}{6}, \frac{11\pi}{6}$ **c** $0, \pi, 2\pi$

d $\frac{\pi}{2}, \frac{3\pi}{2}$ **e** $\frac{\pi}{6}, \frac{\pi}{3}, \frac{7\pi}{6}, \frac{4\pi}{3}$

f $\frac{7\pi}{12}, \frac{3\pi}{4}, \frac{19\pi}{12}, \frac{7\pi}{4}$

13 $\frac{56}{65}$

14 a $\sqrt{85}\cos(\theta-\alpha)$ where $\alpha = \cos^{-1}\left(\frac{2}{\sqrt{85}}\right)$

b i $\sqrt{85}$ **ii** $\frac{2}{\sqrt{85}}$

iii $\theta = \cos^{-1}\left(\frac{2}{\sqrt{85}}\right) + \cos^{-1}\left(\frac{1}{\sqrt{85}}\right)$

15 a $0, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \pi$ **b** $0, \frac{\pi}{3}, \pi$

17 b $-\frac{2}{3}$ or $\frac{1}{2}$

18 $\frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6}$

19 $\frac{\pi}{12}, \frac{5\pi}{12}$

Technology-active

20 b $P = 10\sqrt{5}\cos(\theta-\alpha)$ where $\alpha = \cos^{-1}\left(\frac{2}{\sqrt{5}}\right)$;

$\theta = 70.88°$

c $k = 25$ **d** $\theta = 45°$

21 a $AD = \cos\theta + 2\sin\theta$

b $AD = \sqrt{5}\cos(\theta-\alpha)$ where

$\alpha = \cos^{-1}\left(\frac{1}{\sqrt{5}}\right) \approx 63°$

c Max length of AD is $\sqrt{5}$ m when $\theta = 63°$

d $\theta = 79.38°$

22 b ii $a = 1,\ b = 1$

c $\frac{1+\sqrt{2}-\sqrt{3}}{1+\sqrt{3}+\sqrt{6}} = \frac{2\sqrt{2}-\sqrt{3}-1}{\sqrt{3}-1}$

$= \sqrt{6}+\sqrt{2}-\sqrt{3}-2$

23 a ii $2\cos\left(\frac{\pi}{5}\right)$

b iii $4\cos^2\left(\frac{\pi}{5}\right) - 2\cos\left(\frac{\pi}{5}\right) - 1 = 0$

iv $\frac{1+\sqrt{5}}{4}$

Multiple-choice questions

1 A	**2** A	**3** B	**4** B
5 C	**6** D	**7** D	**8** A
9 D	**10** D	**11** C	**12** A

Chapter 12

Exercise 12A

1

	Re(z)	Im(z)
a	2	3
c	$\frac{1}{2}$	$-\frac{3}{2}$
e	0	3

	Re(z)	Im(z)
b	4	5
d	−4	0
f	$\sqrt{2}$	$-2\sqrt{2}$

2 a $a = 2,\ b = -2$

b $a = 3,\ b = 2$ or $a = 2,\ b = 3$

c $a = 5,\ b = 0$ **d** $a = \frac{2}{3},\ b = -\frac{1}{3}$

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3 **a** $6-8i$ **b** $6-i$ **c** $-6-2i$
d $7-3\sqrt{2}i$ **e** $-2-3i$ **f** $4+2i$
g $6-4i$ **h** $-4+6i$ **i** $-1+11i$
j -1

4 **a** $4i$ **b** $6i$ **c** $\sqrt{2}i$
d $-i$ **e** -1 **f** 1
g -2 **h** -12 **i** -4

5 **a** $1+2i$ **b** $-3+4i$
c $-\sqrt{2}-2i$ **d** $-\sqrt{6}-3i$

6 -1

Exercise 12B

1 **a** $15+8i$ **b** $-8i$ **c** $-2+16i$
d $2i$ **e** 5 **f** $-4+19i$

2 **a** -2 **b** 7

3 **a** $2+5i$ **b** $-1-3i$
c $\sqrt{5}+2i$ **d** $5i$

4 **a** 25 **b** 2 **c** 13 **d** 5

5 **a** $2+i$ **b** $-3-2i$ **c** $-4+7i$
d $-4-7i$ **e** $-4-7i$ **f** $-1+i$
g $-1-i$ **h** $-1-i$

6 **a** $5\sqrt{2}$ **b** $5\sqrt{2}$ **c** 5 **d** $\sqrt{130}$

7 **a** $2+4i$ **b** 20 **c** 4
d $8-16i$ **e** $-8i$ **f** 8
g $\frac{1}{10}(1+2i)$ **h** $-4-2i$

8 $a=\frac{1}{29},\ b=-\frac{17}{29}$

9 **a** $\frac{7}{17}-\frac{6}{17}i$ **b** i **c** $\frac{7}{2}-\frac{1}{2}i$
d $-\frac{1}{2}-\frac{1}{2}i$ **e** $\frac{2}{13}+\frac{3}{13}i$ **f** $\frac{3}{20}+\frac{1}{20}i$

10 $a=\frac{5}{2},\ b=-\frac{3}{2}$

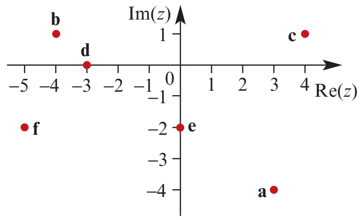
11 **a** $-\frac{42}{5}(1-2i)$ **b** $-\frac{1}{2}(1-i)$
c $\frac{1}{17}(4+i)$ **d** $\frac{1}{130}(6+43i)$
e $2-2i$

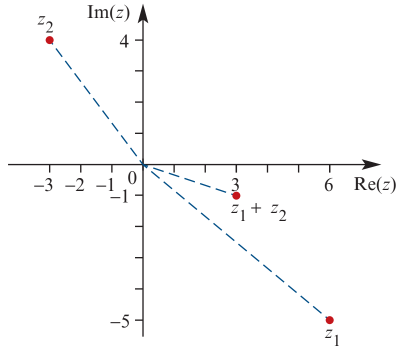
12 $a=-2, b=-3$ or $a=2, b=3$

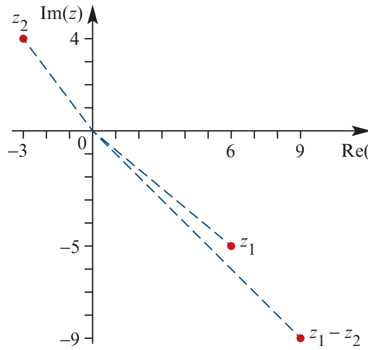
13 $a=2$ or $a=\frac{9}{2}$

Exercise 12C

1 $A=3+i,\quad B=2i,\quad C=-3-4i$
$D=2-2i,\quad E=-3,\quad F=-1-i$

2


3 **a** $z_1+z_2=3-i$


b $z_1-z_2=9-9i$


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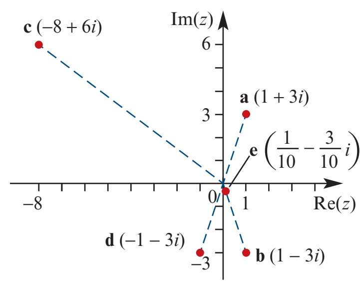

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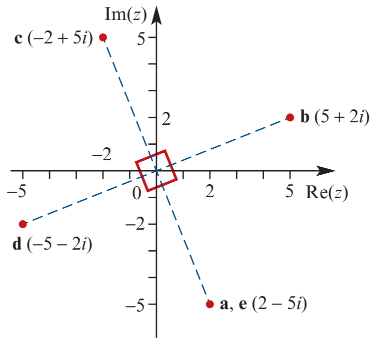


Exercise 12D

1 **a** $\pm i$ **b** $\pm 3i$ **c** $\pm 4i$ **d** $\pm\frac{5}{2}i$ **e** $\pm\sqrt{2}i$
f $\pm 2i$ **g** $\pm 5i$ **h** $\pm\frac{1}{2}i$ **i** $\pm\frac{3}{4}i$ **j** $\pm\sqrt{3}i$
k $\pm\sqrt{5}i$ **l** $-1\pm i$ **m** $2\pm\sqrt{5}i$
n $-3\pm\sqrt{3}i$ **o** $2\pm 2i$

2 **a** $-1 \pm \sqrt{2}i$ **b** $2 \pm i$ **c** $-3 \pm \sqrt{3}i$
d $2 \pm i$ **e** $\frac{1}{3}(-1 \pm \sqrt{2}i)$ **f** $\frac{1}{2}(-1 \pm i)$

3 **a** $\frac{1}{2}(-3 \pm \sqrt{3}i)$ **b** $2 \pm i$ **c** $-3 \pm \sqrt{3}i$
d $2 \pm 2i$ **e** $\frac{1}{3}(-1 \pm \sqrt{2}i)$ **f** $\frac{1}{4}(\sqrt{2} \pm \sqrt{6}i)$

4 **a** $b = -2,\ c = 2$ **b** $b = 4,\ c = 29$

5 **a** $\pm 2i$ **b** $\pm 3i$ **c** $\pm\sqrt{5}i$ **d** $2 \pm 4i$
e $-1 \pm 7i$ **f** $1 \pm \sqrt{2}i$ **g** $\frac{1}{2}(-3 \pm \sqrt{3}i)$
h $\frac{1}{4}(-5 \pm \sqrt{7}i)$ **i** $\frac{1}{6}(1 \pm \sqrt{23}i)$ **j** $1 \pm 2i$
k $\frac{1}{2}(3 \pm \sqrt{11}i)$ **l** $3 \pm \sqrt{5}i$

7 $c = \frac{2}{3}$

Exercise 12E

1 $2, -2 \pm i$ **2** $-1, -1 \pm i$
3 $3, 3 \pm 2i$ **4** $2, 1 \pm \sqrt{2}i$
5 $3, \pm 2i$ **6** $-2, 1, \pm 3i$
7 **a** $1, \pm i$ **b** $-1, 1 \pm 2i$
c $-2, 1 \pm i$ **d** $3, -3 \pm \sqrt{3}i$

8 $a = -5,\ b = 8,\ c = -6$

9 $c = 12$

10 $b = -9, c = 28, d = -30$

12 **a** 3 **b** 5

13 **a** $z^4 + 5z^2 + 4$
b $(z - i)(z + i)(z - 1)(z + 2)$

14 **a and b** two non-real solutions of multiplicity 1 and one real solution of multiplicity 1. For example,
$P(z) = z(z - i)(z + i) = z^3 + z$
one real solution of multiplicity 1 and one real solution of multiplicity 2. For example,
$P(z) = z^2(z - 1) = z^3 - z^2$
one real solution of multiplicity 3. For example,
$P(z) = z^3$
three real solutions of multiplicity 1. For example,
$P(z) = z(z - 1)(z + 1) = z^3 - z$

15 $1 + 2i, 1 - 2i, -3, \frac{4}{3}$

17 **b** $b = -6, c = 21, d = -26$

Exercise 12F

1 **a** $2\operatorname{cis}\left(\frac{\pi}{3}\right)$ **b** $\sqrt{2}\operatorname{cis}\left(-\frac{\pi}{4}\right)$
c $4\operatorname{cis}\left(\frac{5\pi}{6}\right)$ **d** $4\sqrt{2}\operatorname{cis}\left(-\frac{3\pi}{4}\right)$
e $24\operatorname{cis}\left(-\frac{\pi}{3}\right)$ **f** $\frac{1}{\sqrt{2}}\operatorname{cis}\left(\frac{3\pi}{4}\right)$

2 **a** $3i$ **b** $\frac{1}{\sqrt{2}}(1 + \sqrt{3}i) = \frac{\sqrt{2}}{2}(1 + \sqrt{3}i)$
c $\sqrt{3} + i$ **d** $-\frac{5}{\sqrt{2}}(1 - i) = -\frac{5\sqrt{2}}{2}(1 - i)$
e $-6(\sqrt{3} - i)$ **f** $3(1 - i)$
g $-\frac{5}{2}(1 + \sqrt{3}i)$ **h** $-\frac{5}{2}(1 + \sqrt{3}i)$

3 **a** $3\sqrt{2}(1 + i)$ **b** $6(1 + \sqrt{3}i)$
c $-\frac{5}{2}(1 - \sqrt{3}i)$ **d** $18(1 + \sqrt{3}i)$
e $-18(1 + \sqrt{3}i)$ **f** $\sqrt{3}(1 + i)$
g $\sqrt{3} + i$ **h** -4
i $-4(1 - \sqrt{3}i)$ **j** $-\frac{5}{2}$

4 **a** $\left(\frac{5 - 2\sqrt{3}}{2}, \frac{5\sqrt{3} + 2}{2}\right)$
b $\left(\frac{5\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$
c $(x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta)$

6 $\operatorname{cis}\theta$

7 **c** $\sin\theta = \frac{1}{2i}(\operatorname{cis}\theta - \operatorname{cis}(-\theta))$

Exercise 12G

1 **a** $2\operatorname{cis}\left(\frac{\pi}{2}\right) = 2i$ **b** $2\sqrt{2}\operatorname{cis}\left(\frac{3\pi}{4}\right) = -2 + 2i$
c $4\operatorname{cis}\pi = -4$ **d** $4\sqrt{2}\operatorname{cis}\left(-\frac{3\pi}{4}\right) = -4 - 4i$
e $16\operatorname{cis}0 = 16$

2 **a** $4\operatorname{cis}\left(\frac{2\pi}{3}\right) = -2 + 2\sqrt{3}i$
b $8\operatorname{cis}\pi = -8$
c $16\operatorname{cis}\left(-\frac{2\pi}{3}\right) = -8 - 8\sqrt{3}i$
d $32\operatorname{cis}\left(-\frac{\pi}{3}\right) = 16 - 16\sqrt{3}i$
e $64\operatorname{cis}0 = 64$ **f** $512\operatorname{cis}\pi = -512$
g $4096\operatorname{cis}0 = 4096$

3 **a** $4\operatorname{cis}\left(-\frac{\pi}{3}\right) = 2 - 2\sqrt{3}i$
b $27\operatorname{cis}0 = 27$
c $81\operatorname{cis}\left(-\frac{2\pi}{3}\right) = -\frac{81}{2}(1 + \sqrt{3}i)$
d $16\operatorname{cis}\pi = -16$

4 **a** $8\operatorname{cis}\left(\frac{\pi}{3}\right)$ **b** $\frac{8}{27}\operatorname{cis}\left(\frac{\pi}{8}\right)$
c $27\operatorname{cis}\left(\frac{5\pi}{6}\right)$ **d** $-32i$
e -216 **f** $1024\operatorname{cis}\left(-\frac{\pi}{12}\right)$
g $\frac{27}{4}\operatorname{cis}\left(-\frac{\pi}{20}\right)$

5 **a** $64\operatorname{cis}0 = 64$ **b** $\frac{\sqrt{2}}{8}\operatorname{cis}\left(-\frac{3\pi}{4}\right)$

c $128\,\text{cis}\left(-\frac{2\pi}{3}\right)$

d $\frac{\sqrt{3}}{72}\,\text{cis}\left(-\frac{\pi}{2}\right) = -\frac{\sqrt{3}}{72}i$

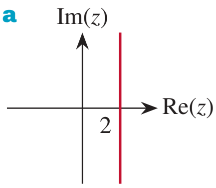
e $\sqrt{2}\,\text{cis}\left(-\frac{\pi}{4}\right)$ **f** $\frac{64\sqrt{3}}{3}\,\text{cis}\left(\frac{3\pi}{4}\right)$

g $\frac{\sqrt{2}}{2}\,\text{cis}\left(\frac{\pi}{2}\right) = \frac{\sqrt{2}}{2}i$ **h** $\frac{1}{4}\,\text{cis}\left(-\frac{2\pi}{15}\right)$

i $8\sqrt{2}\,\text{cis}\left(\frac{11\pi}{12}\right)$

Exercise 12H

1 a 5 **b** $\sqrt{2}$ **c** 13 **d** $2\sqrt{2}$ **e** 13 **f** $\sqrt{3}$

2 a


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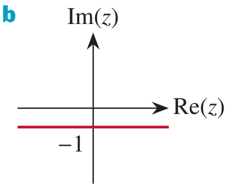

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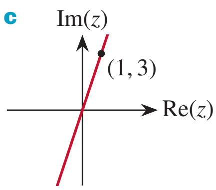

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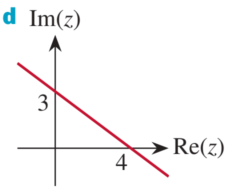

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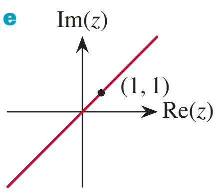

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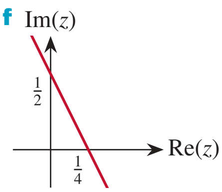

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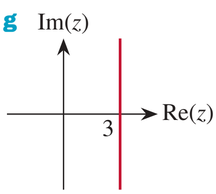

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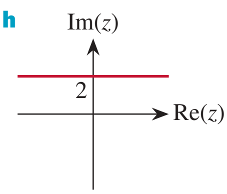

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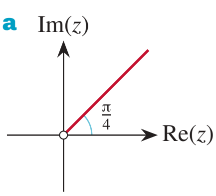

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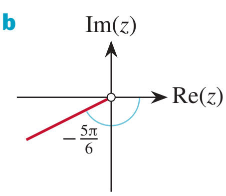

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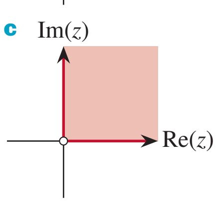

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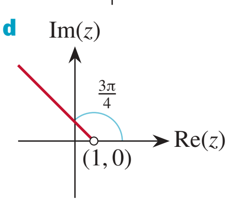

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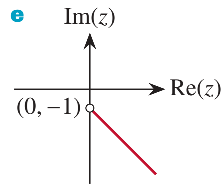

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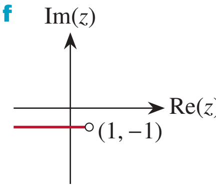

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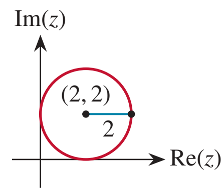

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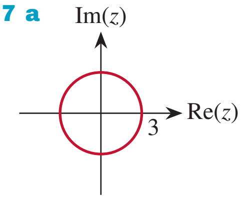

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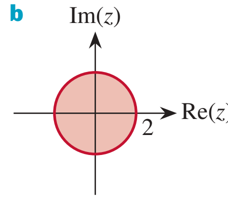

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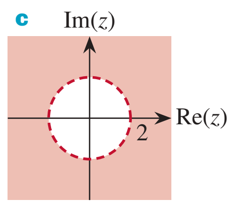

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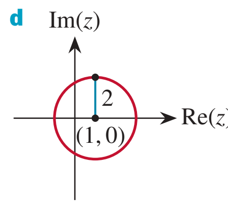

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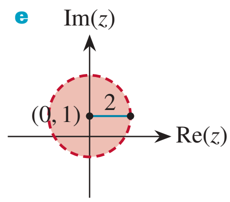

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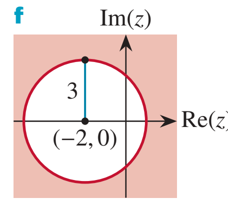

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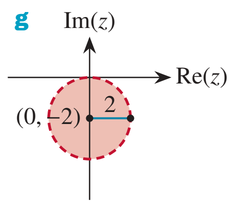

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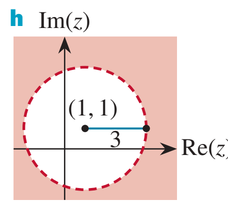

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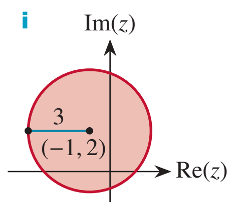

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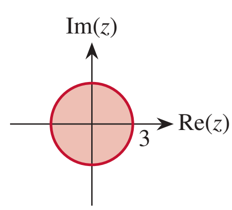

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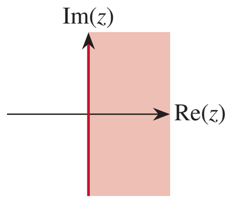

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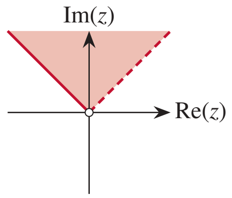

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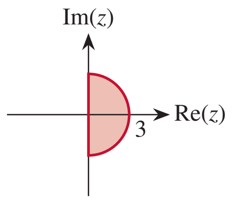

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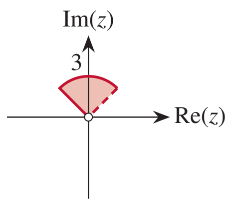

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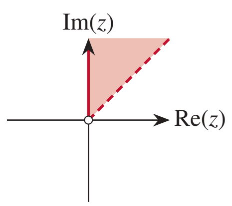

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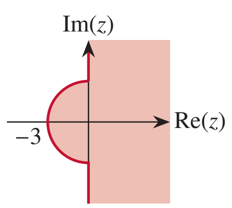

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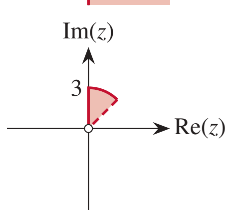
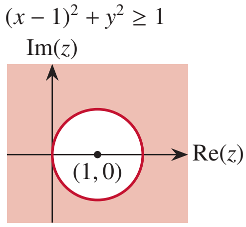


9 a $(x-1)^2 + y^2 \geq 1$

b



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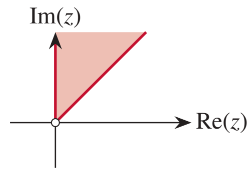

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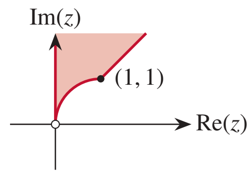
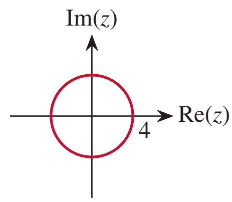


10 Centre $(0, 0)$, radius 1

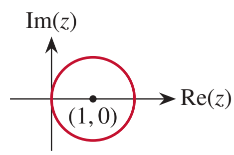
11 $x = \frac{1}{2}(y^2 - 2y)$

12 b



13 Centre $(-9, 0)$, radius 3

14 a



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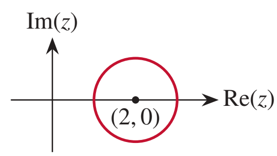

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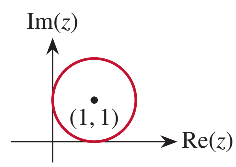

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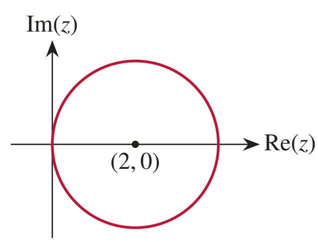

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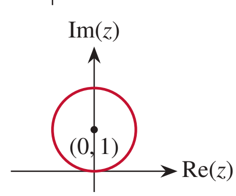

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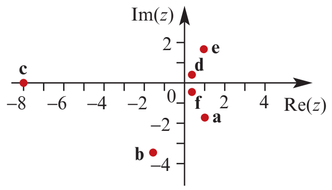
Chapter 12 review

Short-response questions

Technology-free

1 **a** $(2m + 3p) + (2n + 3q)i$ **b** $p - qi$
c $(mp + nq) + (np - mq)i$
d $\dfrac{(mp + nq) + (np - mq)i}{p^2 + q^2}$ **e** $2m$
f $(m^2 - n^2 - p^2 + q^2) + (2mn - 2pq)i$
g $\dfrac{m - ni}{m^2 + n^2}$ **h** $\dfrac{(mp + nq) + (mq - np)i}{m^2 + n^2}$
i $\dfrac{3((mp + nq) + (np - mq)i)}{p^2 + q^2}$

2

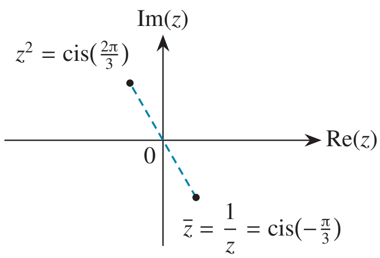


a $1 - \sqrt{3}i$ **b** $-2 - 2\sqrt{3}i$
c -8 **d** $\dfrac{1}{4}(1 + \sqrt{3}i)$
e $1 + \sqrt{3}i$ **f** $\dfrac{1}{4}(1 - \sqrt{3}i)$

3 **a** $\sqrt{2}\,\text{cis}\left(\dfrac{\pi}{4}\right)$ **b** $2\,\text{cis}\left(-\dfrac{\pi}{3}\right)$
c $\sqrt{13}\,\text{cis}\left(\tan^{-1}\left(\dfrac{\sqrt{3}}{6}\right)\right)$ **d** $6\,\text{cis}\left(\dfrac{\pi}{4}\right)$
e $6\,\text{cis}\left(-\dfrac{3\pi}{4}\right)$ **f** $2\,\text{cis}\left(-\dfrac{\pi}{6}\right)$

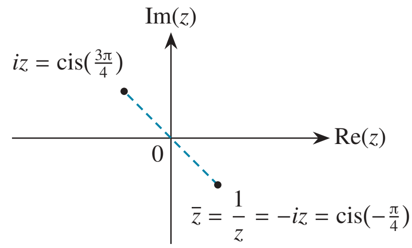
4 **a** $-1 - \sqrt{3}i$ **b** $\dfrac{3\sqrt{2}}{2} + \dfrac{3\sqrt{2}}{2}i$
c $-\dfrac{3\sqrt{2}}{2} + \dfrac{3\sqrt{2}}{2}i$ **d** $\dfrac{3\sqrt{2}}{2} + \dfrac{3\sqrt{2}}{2}i$
e $-\dfrac{3\sqrt{3}}{2} - \dfrac{3}{2}i$ **f** $1 - i$

5



a $z^2 = \text{cis}\left(\dfrac{2\pi}{3}\right)$ **b** $\bar{z} = \text{cis}\left(-\dfrac{\pi}{3}\right)$
c $\dfrac{1}{z} = \text{cis}\left(-\dfrac{\pi}{3}\right)$ **d** $\text{cis}\left(\dfrac{2\pi}{3}\right)$

6



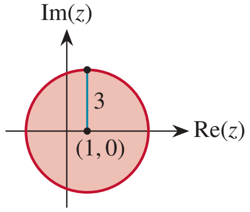
a $iz = \text{cis}\left(\dfrac{3\pi}{4}\right)$ **b** $\bar{z} = \text{cis}\left(-\dfrac{\pi}{4}\right)$
c $\dfrac{1}{z} = \text{cis}\left(-\dfrac{\pi}{4}\right)$ **d** $-iz = \text{cis}\left(-\dfrac{\pi}{4}\right)$

7 **a** $\pm 2i$ **b** $\pm\sqrt{3}i$ **c** $-2 \pm i$
d $\dfrac{1}{4}(3 \pm \sqrt{23}i)$

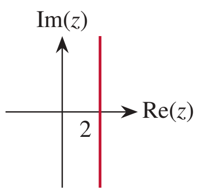
8 $2, \pm 2i$

9 **b** $\dfrac{11}{12}, \pm i$ **c** $n = \pm 1$

10 **a**



b



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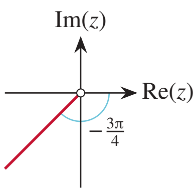

11 $-64(\sqrt{3} + i)$

12 $z = \sqrt{2}\,\text{cis}\left(\dfrac{\pi}{4}\right), w = \sqrt{2}\,\text{cis}\left(-\dfrac{\pi}{4}\right)$,
$\dfrac{1}{2}(1 + i)$

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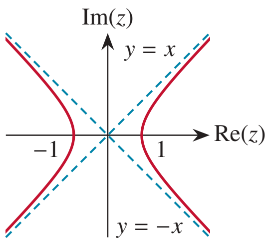
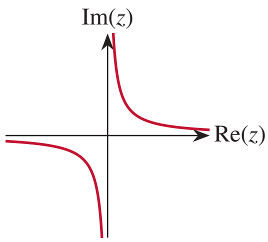
13 **a** $z^2 = (x^2 - y^2) + (2xy)i$
b



c



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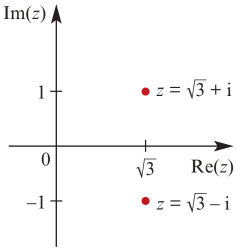
14 a $z = \sqrt{3} + i$ or $z = \sqrt{3} - i$

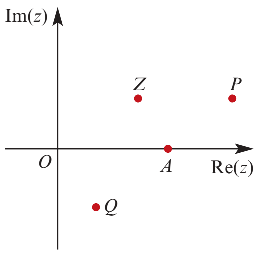
b i



ii $x^2 + y^2 = 4$ **iii** $a = 2$

15 a i $6\sqrt{2}$ **ii** 6

16 a



b $\sqrt{2} + 1$

19 a $|z + 1| = \sqrt{2 + 2\cos\theta} = 2\cos\left(\frac{\theta}{2}\right)$, $\operatorname{Arg}(z + 1) = \frac{\theta}{2}$

b $|z - 1| = \sqrt{2 - 2\cos\theta} = 2\sin\left(\frac{\theta}{2}\right)$, $\operatorname{Arg}(z - 1) = \frac{\pi + \theta}{2}$

c $\left|\frac{z-1}{z+1}\right| = \tan\left(\frac{\theta}{2}\right)$, $\operatorname{Arg}\left(\frac{z-1}{z+1}\right) = \frac{\pi}{2}$

20 a $\Delta = b^2 - 4ac$

b $b^2 < 4ac$

c i $-\frac{b}{a}, \frac{\sqrt{ac}}{a}$ **ii** $\frac{b^2}{2ac} - 1$

21 a $z_1 = \frac{1}{2}(-1 + \sqrt{3}i)$, $z_2 = \frac{1}{2}(-1 - \sqrt{3}i)$

c $|z_1| = 1$, $\operatorname{Arg}(z_1) = \frac{2\pi}{3}$; $|z_2| = 1$, $\operatorname{Arg}(z_2) = -\frac{2\pi}{3}$

d $\frac{\sqrt{3}}{4}$

Multiple-choice questions

1 C **2** D **3** C **4** D **5** D
6 D **7** D **8** D **9** B **10** D
11 C **12** B **13** D **14** D **15** C
16 B **17** C **18** C **19** D

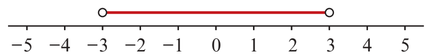
Chapter 13

Exercise 13A

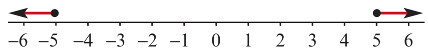
1 a 8 **b** 8 **c** 2 **d** −2
e −2 **f** 4

2 a 3, −1 **b** $\frac{7}{2}, -\frac{1}{2}$ **c** $\frac{12}{5}, -\frac{6}{5}$ **d** 12, −6
e −1, 7 **f** $\frac{4}{3}, -4$ **g** $-\frac{2}{5}, -4$

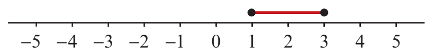
3 a $(-3, 3)$



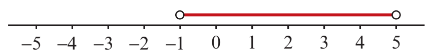
b $(-\infty, -5] \cup [5, \infty)$



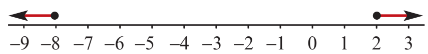
c $[1, 3]$



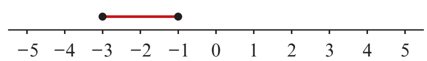
d $(-1, 5)$



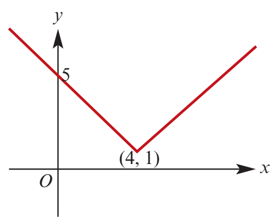
e $(-\infty, -8] \cup [2, \infty)$



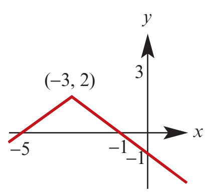
f $[-3, -1]$



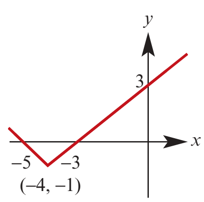
4 a Range $[1, \infty)$



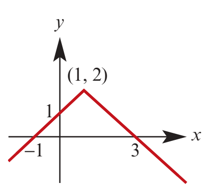
b Range $(-\infty, 2]$



c Range $[-1, \infty)$



d Range $(-\infty, 2]$



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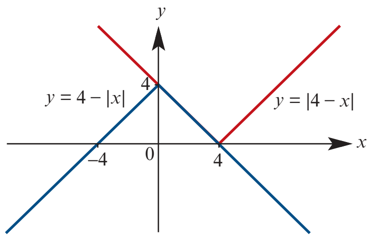
5 a $-5 \le x \le 5$ **b** $x \le -2$ or $x \ge 2$

c $1 \le x \le 2$ **d** $-\dfrac{1}{5} < x < 1$

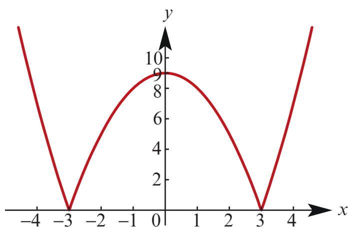
e $x \le -4$ or $x \ge 10$ **f** $1 \le x \le 3$

6 a

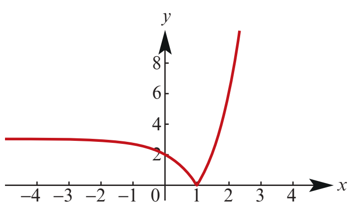


b $0 \le x \le 4$

7 a



b



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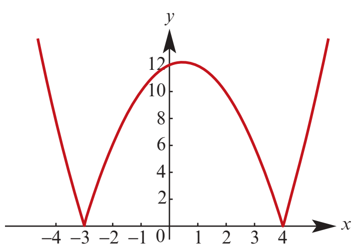
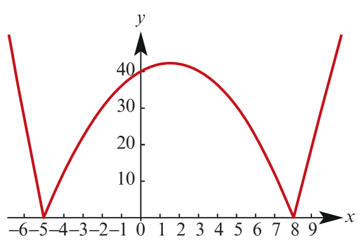
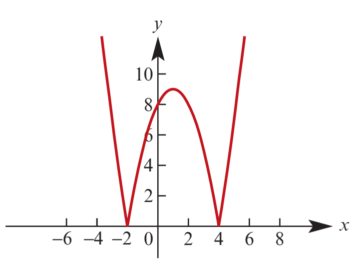


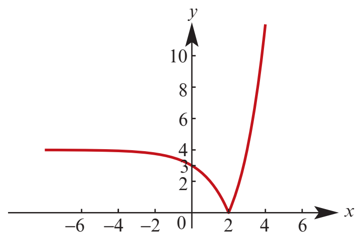
d



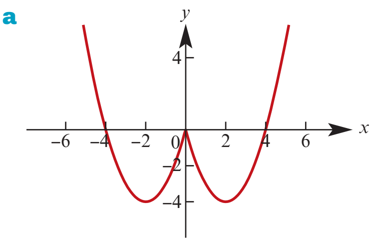
e



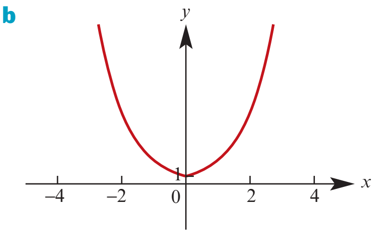
f



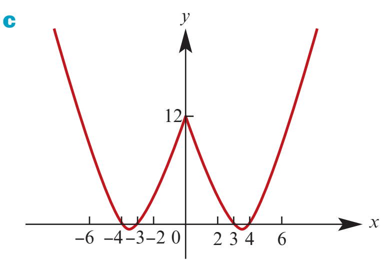
8 a



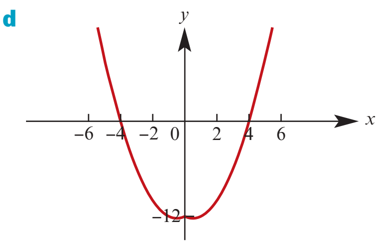
b



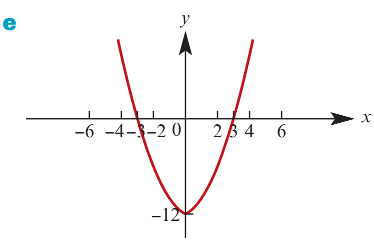
c



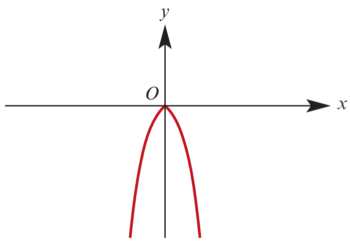
d



e



f



9 a $\dfrac{2-\sqrt{6}}{2}, \dfrac{2+\sqrt{6}}{2}, \dfrac{2-\sqrt{2}}{2}, \dfrac{2+\sqrt{2}}{2}$

b $1-\sqrt{2}, 1+\sqrt{2}, 1$ **c** $-2, 4$

d $3-\sqrt{17}, 3+\sqrt{17}, 2, 4$ **e** $-2, 8$

f $3-3\sqrt{2}, 3+3\sqrt{2}, 3$

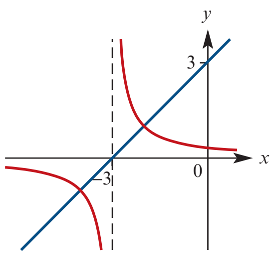
10 a $x \le -2$ **b** $x = -9$ or $x = 11$

c $x = -\dfrac{5}{4}$ or $x = \dfrac{15}{4}$

11 $a = 1, \; b = 1$

Exercise 13B

1 a



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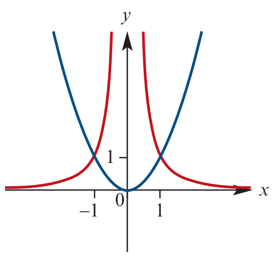
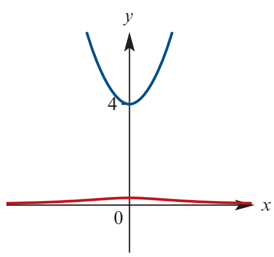
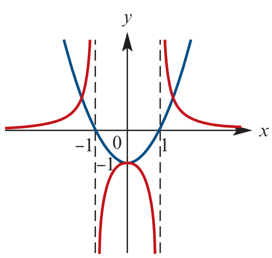


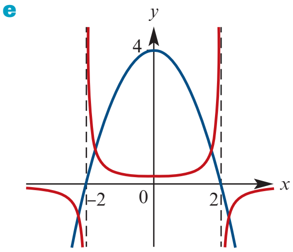
c



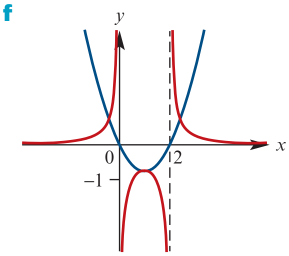
d



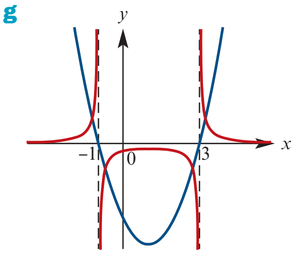
e



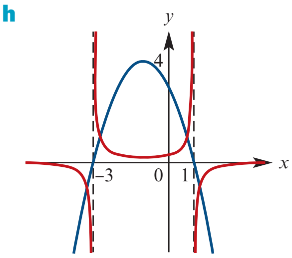
f



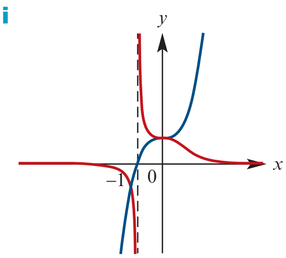
g



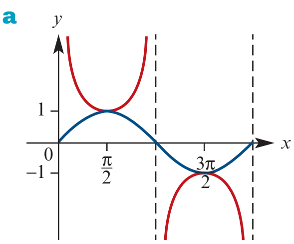
h



i



2 a



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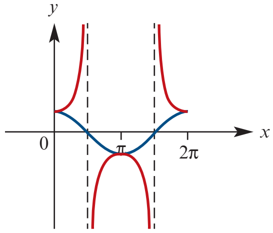
b



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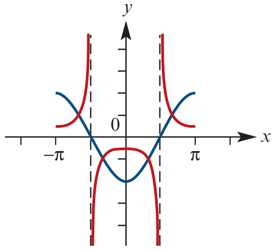

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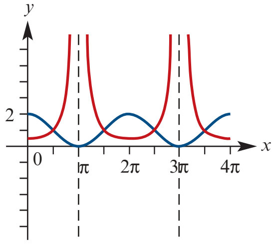

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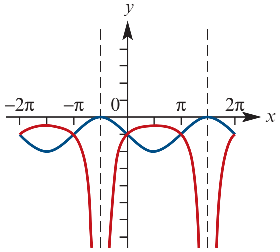

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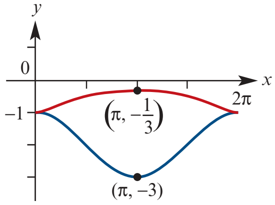

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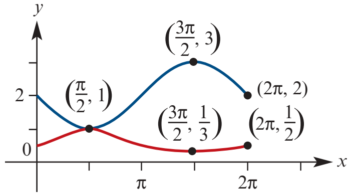

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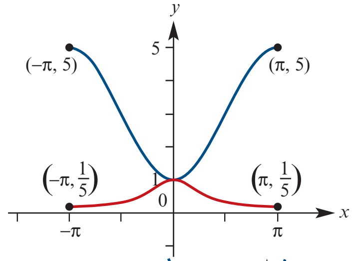
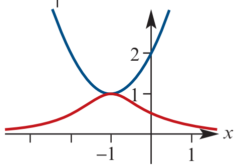
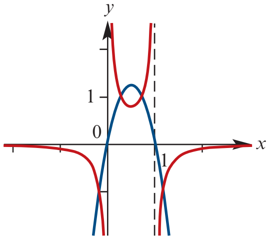


3 a $(-1, 1)$ **b**

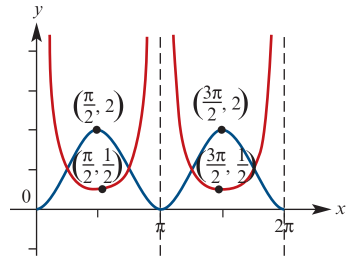


4 a

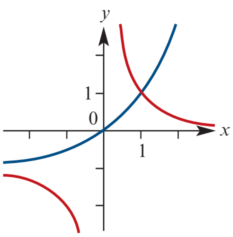


b $\left(\frac{5 \pm 3\sqrt{5}}{10}, -1\right), \left(\frac{5 \pm \sqrt{5}}{10}, 1\right)$

5



6



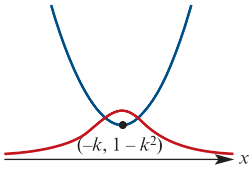
7 a $f(x) = (x + k)^2 + 1 - k^2$

b **i** $-1 < k < 1$

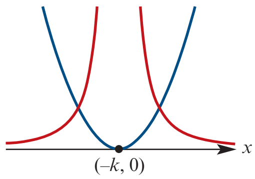
ii $k = \pm 1$

iii $k > 1$ or $k < -1$

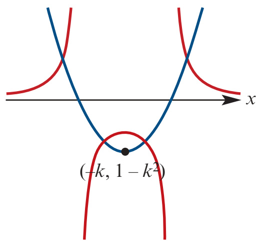
c **i**



ii



iii



8 $b = -1$ and $c = -6$

9 $b = -\frac{17}{4}$ and $c = \frac{15}{4}$.

10 $c = \frac{3}{2}$

11 $h = \frac{1}{4}$

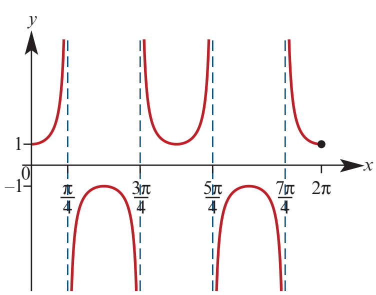
12 $c \in (-\infty, 0) \cup (0, \frac{1}{4})$

13 a $(0, \frac{1}{4}]$. **b** $(-\infty, -\frac{1}{4}] \cup (0, \infty)$

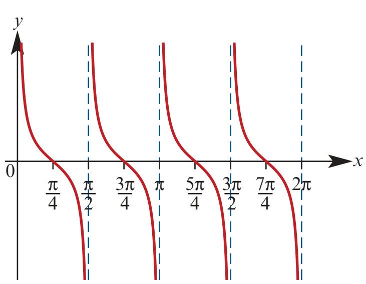
c $\left[-\frac{16}{63}, -\frac{1}{4}\right)$

Exercise 13C

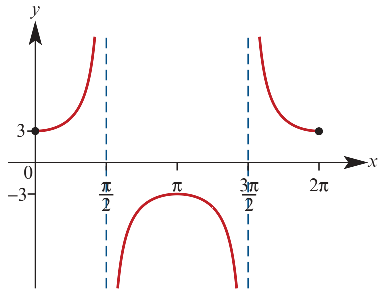
1 a



b



c



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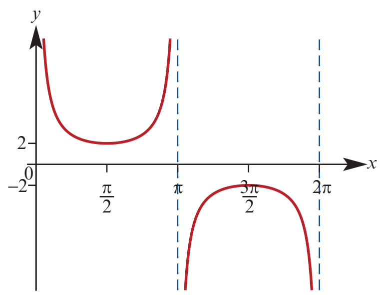

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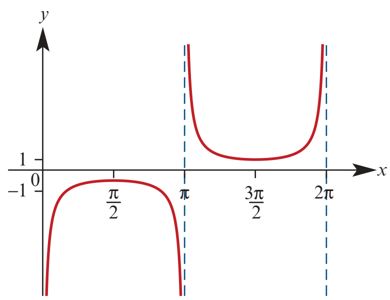

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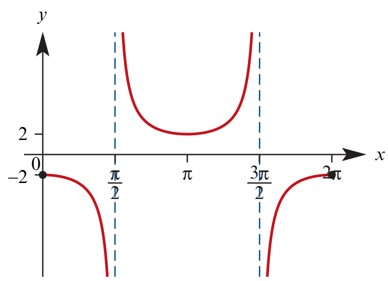
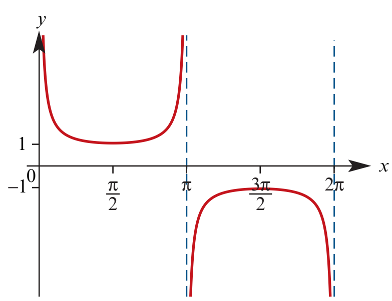


2 a



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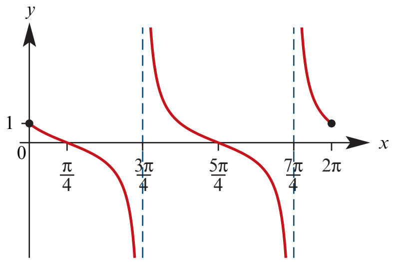
b



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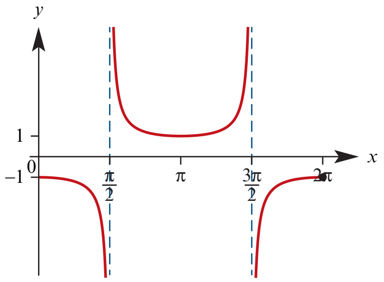

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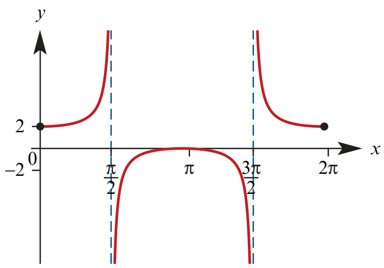

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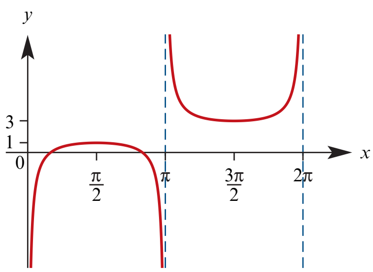

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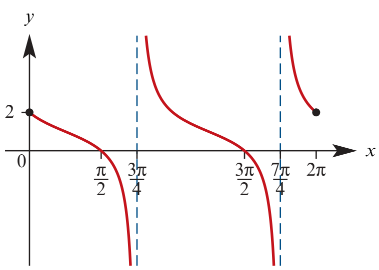
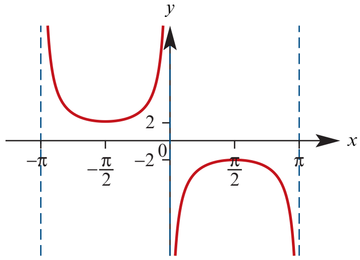
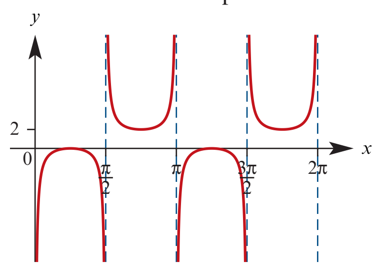


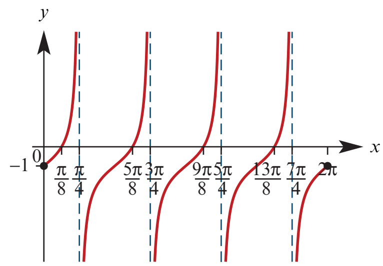
3
- Reflection in the x-axis
- Dilation of factor 2 from the x-axis
- Translation $\frac{\pi}{2}$ units to the right



4
- Reflection in the y-axis
- Dilation of factor $\frac{1}{2}$ from the y-axis
- Translation 1 unit up



5
- Reflection in the x-axis
- Dilation of factor $\frac{1}{2}$ from the y-axis
- Translation $\frac{\pi}{4}$ units to the right and 1 unit down



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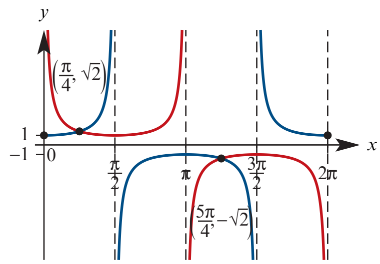

Chapter 13 review

Short-response questions

Technology-free

1 **a** 9 **b** $\frac{1}{400}$ **c** 4
d 4 **e** $\pi - 3$ **f** $4 - \pi$

2 $x = 0$ or $x = 2$ or $x = 4$

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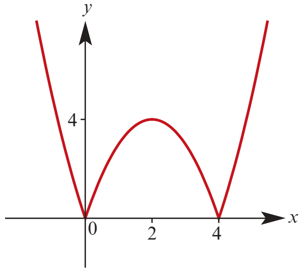
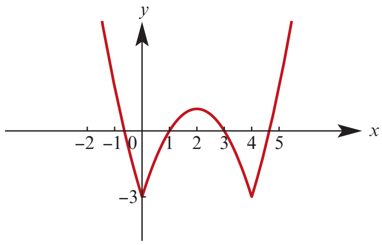
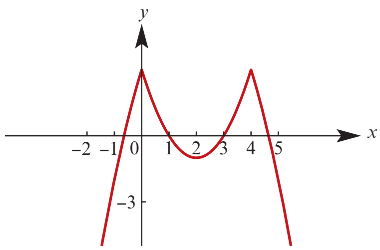
3 a Range $[0, \infty)$



b Range $[-3, \infty)$



c Range $(-\infty, 3]$

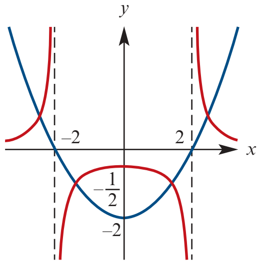


4 a $n = \pm 2$ or $n = \pm 4$

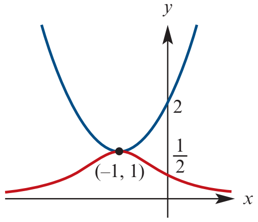
b i $x = \pm 1$ **ii** $x \leq 0$

c $x < -1$ or $x > 1$

5 a



b



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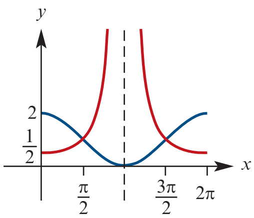

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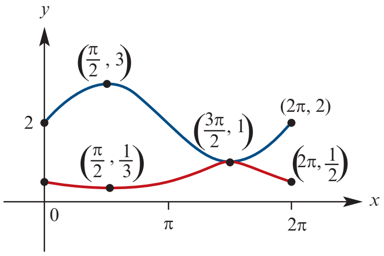
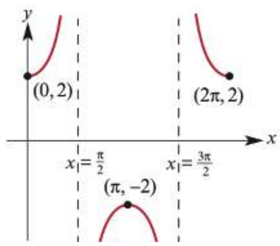


6 a



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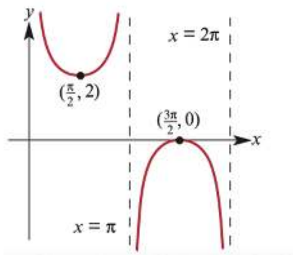
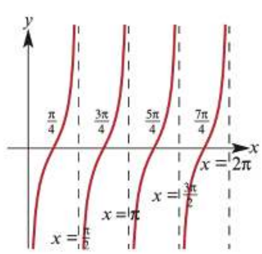


c



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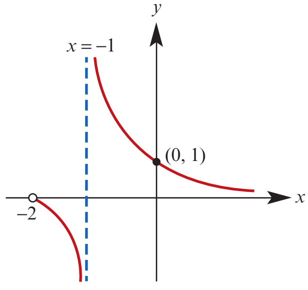

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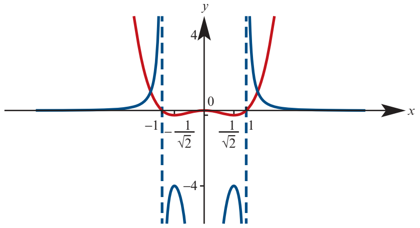
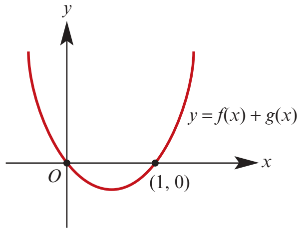


Technology-active

9 a i



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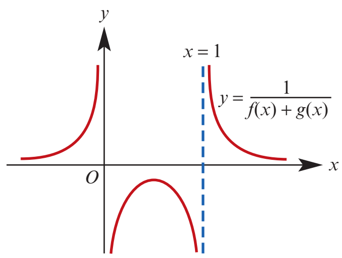

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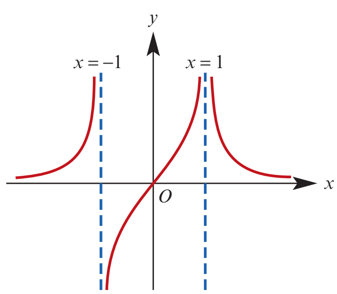


b $f(x) = x^2 - 1,\ \ g(x) = (x - 1)^2$

c i $f(x) + g(x) = 2x^2 - 2x$

ii $\dfrac{1}{f(x) + g(x)} = \dfrac{1}{2x^2 - 2x}$

iii $\dfrac{1}{f(x)} + \dfrac{1}{g(x)} = \dfrac{2x}{(x-1)^2(x+1)}$

10 a $-\dfrac{2}{m}$

b $0 < m < 1$

c i $y = -\dfrac{1}{m}x + 2$

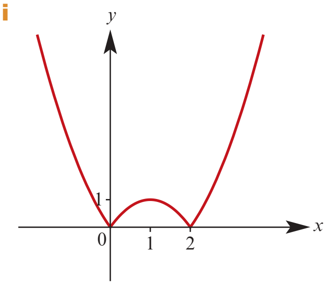
ii $\left(\dfrac{4m}{1 - m^2}, \dfrac{2(m^2 + 1)}{m^2 - 1}\right)$

iii The line ℓ is parallel to the graph of $y = f(x)$ for $x < -2$

iv $m = 3$

d $x = -\dfrac{1}{m}$ or $x = -\dfrac{3}{m}$

11 a i

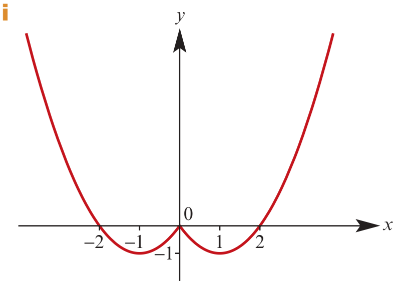


ii $(0, 0), (a, 0)$

iii $\left(\dfrac{a}{2}, \dfrac{a^2}{4}\right)$

iv $a = 3$ or $a = -5$

b i



ii $(0, 0), (a, 0), (-a, 0)$

iii $\left(\dfrac{a}{2}, -\dfrac{a^2}{4}\right), \left(-\dfrac{a}{2}, -\dfrac{a^2}{4}\right)$

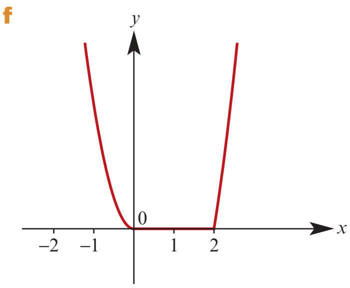
iv $a = -3$

c $x = 0$ or $x \geq a$

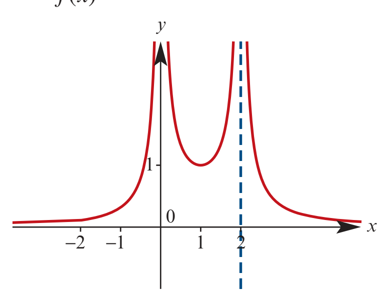
d $x \geq 0$

e $h(x) = |x^2 - ax| + |x|^2 - a|x|$

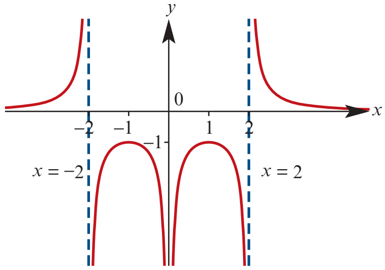
f



g $y = \dfrac{1}{f(x)}$



$y = \dfrac{1}{g(x)}$



Multiple-choice questions

1 D **2** D **3** B **4** C **5** D **6** D
7 D **8** A **9** B **10** B **11** D **12** A
13 D **14** D **15** A **16** A **17** D **18** C
19 D **20** D

Chapter 14

Exercise 14A

1 a i Obtuse **ii** Straight
iii Acute **iv** Right
b i $\angle HFB$ **ii** $\angle BFE$
iii $\angle HFG$ **iv** $\angle BFE$
c i $\angle CBD, \angle BFE, \angle ABF, \angle HFG$
ii $\angle CBA, \angle BFH, \angle DBF, \angle EFG$

2 a $a = 65°,\ b = 65°$
b $x = 40°,\ y = 130°$
c $a = 60°,\ b = 70°,\ c = 50°,\ d = 60°,$ $e = 50°,\ f = 130°$
d $\alpha = 60°,\ \beta = 120°$
e $\alpha = 90°,\ \beta = 93°$
f $\alpha = 108°,\ \beta = 90°,\ \theta = 108°$

4 a $\angle B = \angle D = 180° - \alpha$ **b** $\angle C = \alpha$

9 a $\theta = 107°$ **b** $\theta = 55°$

Exercise 14B

1 a Yes **b** Yes **c** Yes **d** No

2 a Scalene **b** Isosceles **c** Equilateral

3 Must be greater than 10 cm and less than 30 cm

4 a 6, 6.5, 7 **b** No

6 a $\theta = 46°$, straight angle;
$\beta = 70°$, supplementary to $\angle EBC$;
$\gamma = 64°$, alternate angles ($\angle CBD$);
$\alpha = 46°$, corresponding angles ($\angle EBD$)
b $\gamma = 80°$, angle sum of triangle;
$\beta = 80°$, vertically opposite (γ);
$\theta = 100°$, supplementary to β;
$\alpha = 40°$, alternate angles ($\angle BAD$)
c $\alpha = 130°$, supplementary to $\angle ADC$;
$\beta = 65°$, co-interior angles $\angle CDA$;
$\gamma = 65°$, co-interior angles $\angle CDA$
d $\alpha = 60°$, equilateral triangle
e $\alpha = 60°$, straight angle;
$\beta = 60°$, angle sum of triangle
f $a = 55°$, straight angle;
$b = 55°$, corresponding angles (a);
$g = 45°$, vertically opposite;
$c = 80°$, angle sum of triangle;
$e = 100°$, straight angle;
$f = 80°$, corresponding angles (c)
g $m = 68°$, corresponding angles;
$n = 60°$, angle sum of triangle;
$p = 52°$, straight angle;
$q = 60°$, alternate angles (n);
$r = 68°$, alternate angles (m)

7 a Sum = 720°; Angles = 120°
b Sum = 1800°; Angles = 150°
c Sum = 3240°; Angles = 162°

8 a Together they form 10 straight angles
b 360°

10 10

Exercise 14C

1 a A and C (SAS)
b All of them (AAS)
c A and B (SSS)

2 a $\triangle ABC \equiv \triangle CDA$ (SSS)
b $\triangle CBA \equiv \triangle CDE$ (SAS)
c $\triangle CAD \equiv \triangle CAB$ (SAS)
d $\triangle ADC \equiv \triangle CBA$ (RHS)
e $\triangle DAB \equiv \triangle DCB$ (SSS)
f $\triangle DAB \equiv \triangle DBC$ (SAS)

6 a $a = b = c = d = 60°$

7 a $a = 108°,\ b = 36°,\ c = 72°,\ d = 36°,$ $e = 36°,\ f = 36°$

Exercise 14D

1 16.58 m **2** 41 m
3 13.9 cm **4** 18.38 cm
5 a 50 cm^2 **b** 32 cm^2
6 $2\sqrt{5}$ cm **7** 2 cm^2
8 $2\sqrt{6} \approx 4.9$ cm **9** **b, d**
12 13.86 cm **14** $XY = 2.8$ cm
15 $x = 1.375,\ y = 2.67$ **16** $3\sqrt{2}$ cm

Exercise 14E

1 a AAA, 11.25 cm **b** AAA, $11\frac{2}{3}$ cm
c AAA, 3 cm **d** AAA, 7.5 cm

2 a AAA, 6 cm **b** AAA, $1\frac{1}{3}$ cm

Exercise 14F

See solutions supplement

Exercise 14G

1 a i $\overrightarrow{AB} = \boldsymbol{c}$ **ii** $\overrightarrow{OB} = \boldsymbol{a} + \boldsymbol{c}$ **iii** $\overrightarrow{AC} = \boldsymbol{c} - \boldsymbol{a}$
b $|\boldsymbol{c}|^2 - |\boldsymbol{a}|^2$

2 a $\overrightarrow{AC} = -\boldsymbol{a} + \boldsymbol{c}$, $\overrightarrow{BC} = \boldsymbol{a} + \boldsymbol{c}$ **b** $|\boldsymbol{c}|^2 - |\boldsymbol{a}|^2$

9 a $\overrightarrow{EX} = -\frac{2}{3}\boldsymbol{a} + \frac{8}{3}\boldsymbol{b}$, $\overrightarrow{MX} = -\frac{1}{6}\boldsymbol{a} + \frac{2}{3}\boldsymbol{b}$

10 a i $\frac{4}{5}\boldsymbol{p}$ **ii** $\frac{1}{5}\boldsymbol{p}$ **iii** $-\boldsymbol{p}$ **iv** $\frac{1}{5}(\boldsymbol{q} - \boldsymbol{p})$ **v** $\frac{1}{5}\boldsymbol{q}$
b RS and OQ are parallel
c $ORSQ$ is a trapezium
d 120 cm^2

11 a i $\frac{1}{3}\boldsymbol{a} + \frac{2}{3}\boldsymbol{b}$ **ii** $\frac{k}{7}\boldsymbol{a} + \frac{6}{7}\boldsymbol{b}$
b i 3 **ii** $\frac{7}{2}$

12 a i $\overrightarrow{OD} = 2\boldsymbol{i} - \frac{1}{2}\boldsymbol{j}$, $\overrightarrow{OE} = \frac{15}{4}\boldsymbol{i} + \frac{9}{4}\boldsymbol{j}$
ii $\frac{\sqrt{170}}{4}$
b i $p\left(\frac{15}{4}\boldsymbol{i} + \frac{9}{4}\boldsymbol{j}\right)$
ii $(q + 2)\boldsymbol{i} + \left(4q - \frac{1}{2}\right)\boldsymbol{j}$
c $p = \frac{2}{3}$, $q = \frac{1}{2}$

13 a $\boldsymbol{r} + \boldsymbol{t}$ **b** $\frac{1}{2}(\boldsymbol{s} + \boldsymbol{t})$

Chapter 14 review

Short-response questions

1 a $\overrightarrow{OM} = \frac{1}{2}(\boldsymbol{a} - \boldsymbol{c})$, $\overrightarrow{ON} = \frac{1}{2}\boldsymbol{a} + \frac{3}{2}\boldsymbol{c}$
b $\overrightarrow{MN} = 2\boldsymbol{c}$

4 $AF : FE = 2 : 1$

5 a $\overrightarrow{AL} = \overrightarrow{AB} + \frac{1}{2}\overrightarrow{AD}$, $\overrightarrow{AM} = \frac{1}{2}\overrightarrow{AB} + \overrightarrow{AD}$

7 a Rectangle **b** 16 cm

11 $\sqrt{34}$ cm

12 a $x = 7$ cm, $y = 7$ cm, $\alpha = 45°$, $\beta = 40°$
b $\alpha = 125°$, $\beta = 27.5°$
c $\theta = 52°$, $\alpha = 52°$, $\beta = 65°$, $\gamma = 63°$

14 b i 20 cm **ii** 10 cm
c $XP : PY = 2 : 1$, $PQ : YZ = 2 : 3$

15 $\frac{210}{23}$ m **16** $\frac{15}{8}$

17 12.25 **18** 12

19 a $\frac{1}{3}$ **b** $\frac{1}{3}$ **c** $\frac{2}{3}$ **d** $\frac{2}{3}$

21 a $\triangle EBC$ **c** $\frac{h}{q} = \frac{x}{x + y}$ **e** $\frac{20}{9}$

22 $x = 8$ or $x = 11$

23 a $\triangle BDR$ and $\triangle CDS$; $\triangle BDT$ and $\triangle BCS$; $\triangle RSB$ and $\triangle DST$
b $\frac{z}{y} = \frac{p}{p + q}$

Multiple-choice questions

1 C **2** D **3** A **4** D **5** C
6 C **7** A **8** B **9** C **10** C
11 D **12** D **13** C **14** B **15** B
16 B **17** A **18** A **19** D **20** B

Chapter 15

Exercise 15A

1 a $x = 100$, $y = 50$
b $x = 126$, $y = 252$, $z = 54$
c $y = 145$, $z = 290$
d $x = 180$, $y = 90$
e $x = 45$, $y = 90$, $z = 270$

2 a $x = 68$, $y = 121$ **b** $x = 112$, $y = 87$
c $x = 50$, $y = 110$

3 110°, 110°, 140°

4 $\angle ABC = 98°$, $\angle BCD = 132°$, $\angle CDE = 117°$, $\angle DEA = 110°$, $\angle EAB = 83°$

7 60° or 120°

8 $\angle P = 78°$, $\angle Q = 74°$, $\angle R = 102°$, $\angle S = 106°$

Exercise 15B

1 a $x = 73$, $y = 81$ **b** $x = 57$, $q = 57$
c $x = 53$, $y = 74$, $z = 53$
d $x = 60$, $y = 60$, $z = 20$, $w = 100$
e $w = 54$, $x = 54$, $y = 72$, $z = 54$

2 a 40° **b** 40° **c** 80°

3 32° and 148°

5 $\angle ACB = 40°$, $\angle ABC = 70°$, $\angle BAT = 40°$

Chapter 15 review

Short-response questions

Technology-free

1 $\angle MCN = 18°$

2 a $x = 110$, $y = 70$ **b** $x = 35$, $y = 35$
c $x = 47$, $y = 53$, $z = 100$
d $x = 40$, $y = 40$, $z = 70$

6 a $x = 66$ **b** $x = 116$ **c** $x = 66$, $y = 114$

12 b 24 cm^2

Multiple-choice questions

1 B **2** A **3** D **4** A **5** B
6 A **7** A **8** A

Chapter 16

Exercise 16A

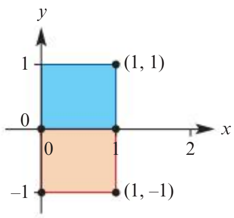
1 a $(-2, 6)$ **b** $(-8, 22)$
c $(26, 2)$ **d** $(-4, -2)$

2 a $(3, 2)$ **b** $(-4, 9)$
c $(8, 3)$ **d** $(7, 11)$

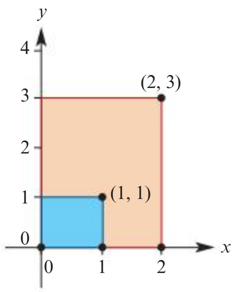
3 a $\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$ **b** $\begin{bmatrix} 11 & -3 \\ 3 & -8 \end{bmatrix}$
c $\begin{bmatrix} 2 & 0 \\ 1 & -3 \end{bmatrix}$ **d** $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

4 Unit square is blue; image is red

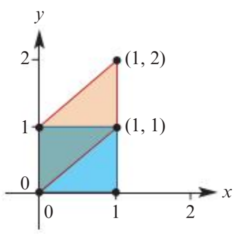
a



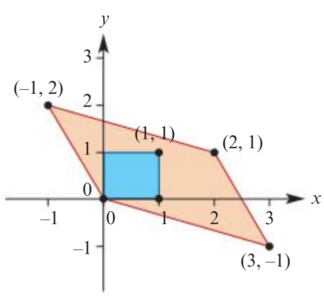
b



c



d



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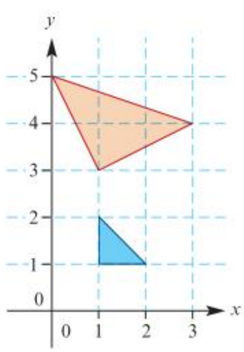


6 $\begin{bmatrix} 3 & 5 \\ 4 & 6 \end{bmatrix}\begin{bmatrix} -2 \\ 4 \end{bmatrix} = \begin{bmatrix} 14 \\ 16 \end{bmatrix}$

7 $\begin{bmatrix} -3 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$

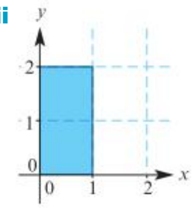
8 a $\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$ or $\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$

b $\begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix}$ or $\begin{bmatrix} -2 & 1 \\ 1 & -1 \end{bmatrix}$

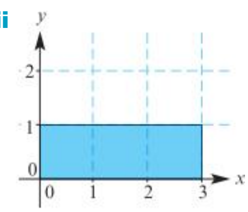
c $\begin{bmatrix} 1 & -2 \\ -1 & -3 \end{bmatrix}$ or $\begin{bmatrix} -2 & 1 \\ -3 & -1 \end{bmatrix}$

Exercise 16B

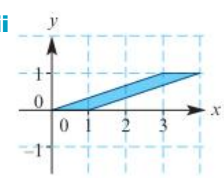
1 a i $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ **ii**



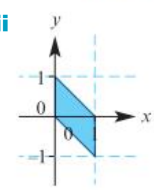
b i $\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$ **ii**



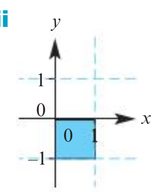
c i $\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$ **ii**



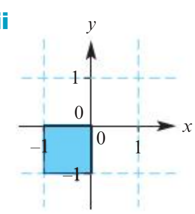
d i $\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$ **ii**



e i $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ **ii**

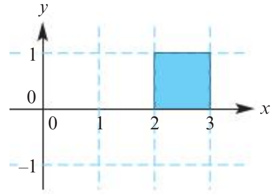


f i $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ **ii**



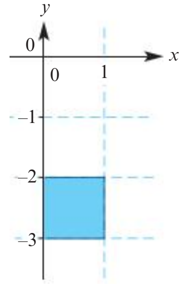
2 a i $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} x+2 \\ y \end{bmatrix}$

ii



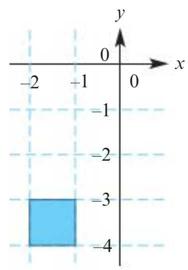
b i $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ -3 \end{bmatrix} = \begin{bmatrix} x \\ y-3 \end{bmatrix}$

ii



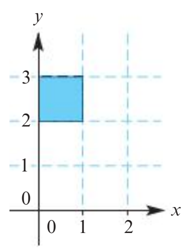
c i $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -2 \\ -4 \end{bmatrix} = \begin{bmatrix} x-2 \\ y-4 \end{bmatrix}$

ii



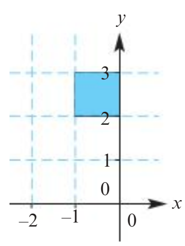
d i $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} x \\ y+2 \end{bmatrix}$

ii



e i $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} x-1 \\ y+2 \end{bmatrix}$

ii



Exercise 16C

1 a $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ **b** $\begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$

c $\begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$ **d** $\begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$

2 a $(-3, 2)$ **b** $\left(\frac{5\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

3 a $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ **b** $\begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$

c $\begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$ **d** $\begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix}$

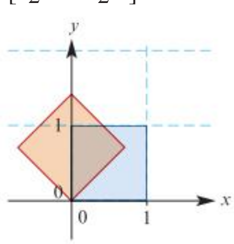
4 a $\begin{bmatrix} -\frac{4}{5} & \frac{3}{5} \\ \frac{3}{5} & \frac{4}{5} \end{bmatrix}$ **b** $\begin{bmatrix} -\frac{12}{13} & \frac{5}{13} \\ \frac{5}{13} & \frac{12}{13} \end{bmatrix}$

c $\begin{bmatrix} \frac{5}{13} & \frac{12}{13} \\ \frac{12}{13} & -\frac{5}{13} \end{bmatrix}$ **d** $\begin{bmatrix} -\frac{4}{5} & -\frac{3}{5} \\ -\frac{3}{5} & \frac{4}{5} \end{bmatrix}$

5 a $\begin{bmatrix} \frac{1-m^2}{m^2+1} & \frac{2m}{m^2+1} \\ \frac{2m}{m^2+1} & \frac{m^2-1}{m^2+1} \end{bmatrix}$ **b** $\left(\frac{-23}{37}, \frac{47}{37}\right)$

6 a $\begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$

b



c $\sqrt{2} - 1$

7 a $C\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right), B\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

b Equilateral

c $y = -\sqrt{3}x,\ y = 0,\ y = \sqrt{3}x$

Exercise 16D

1 $\begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}$ **2** $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

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3 a $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

b $\begin{bmatrix} \cos 180^\circ & -\sin 180^\circ \\ \sin 180^\circ & \cos 180^\circ \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

4 a $\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$ **b** $\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$ **c** No

5 a $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ **b** $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ **c** Yes

6 a $(x, y) \to (-x - 3, y + 5)$

b $(x, y) \to (-x + 3, y + 5)$ **c** Yes

7 a $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ **b** $\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

c $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ **d** $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

8 a $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

b $\begin{bmatrix} \cos 90^\circ & -\sin 90^\circ \\ \sin 90^\circ & \cos 90^\circ \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

9 $\theta = 180^\circ k$, where $k \in \mathbb{Z}$

10 a 2θ

b $\begin{bmatrix} \cos^2\theta - \sin^2\theta & -2\sin\theta\cos\theta \\ 2\sin\theta\cos\theta & \cos^2\theta - \sin^2\theta \end{bmatrix}$

c $\cos(2\theta) = \cos^2\theta - \sin^2\theta$

$\sin(2\theta) = 2\sin\theta\cos\theta$

11 a $x' = y + 1$ **b** $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$y' = x + 2$

12 a $\begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$ **b** $\begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$

c $\begin{bmatrix} \frac{\sqrt{2}+\sqrt{6}}{4} & \frac{\sqrt{2}-\sqrt{6}}{4} \\ \frac{\sqrt{6}-\sqrt{2}}{4} & \frac{\sqrt{6}+\sqrt{2}}{2} \end{bmatrix}$

d $\cos 15^\circ = \dfrac{\sqrt{2}+\sqrt{6}}{4}$, $\sin 15^\circ = \dfrac{\sqrt{6}-\sqrt{2}}{4}$

13 $\begin{bmatrix} \cos(2\theta - 2\varphi) & -\sin(2\theta - 2\varphi) \\ \sin(2\theta - 2\varphi) & \cos(2\theta - 2\varphi) \end{bmatrix}$,
rotation matrix for angle $2\theta - 2\varphi$

Exercise 16E

1 a $\begin{bmatrix} 1 & -1 \\ -3 & 4 \end{bmatrix}$ **b** $\begin{bmatrix} \frac{2}{7} & \frac{1}{14} \\ \frac{1}{7} & -\frac{3}{14} \end{bmatrix}$

c $\begin{bmatrix} \frac{2}{3} & -\frac{1}{2} \\ \frac{1}{3} & 0 \end{bmatrix}$ **d** $\begin{bmatrix} \frac{5}{7} & -\frac{3}{7} \\ \frac{4}{7} & -\frac{1}{7} \end{bmatrix}$

2 a $(x, y) \to (x - 2y, 2x - 5y)$

b $(x, y) \to (y, -x + y)$

3 a $(-1, 1)$ **b** $(-\frac{1}{2}, 1)$

4 $\begin{bmatrix} -4 & 3 \\ -1 & 1 \end{bmatrix}$

5 $(0, 0)$, $(-1, -2)$, $(1, 1)$, $(0, -1)$

6 a $\mathbf{A} = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$ **b** $\mathbf{A}^{-1} = \begin{bmatrix} \frac{1}{k} & 0 \\ 0 & 1 \end{bmatrix}$

7 a $\mathbf{A} = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$ **b** $\mathbf{A}^{-1} = \begin{bmatrix} 1 & -k \\ 0 & 1 \end{bmatrix}$

8 a $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

b Reflecting twice in the same axis will return any point (x, y) to its original position

9 a $\begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$

b Reflecting twice in the same line will return any point (x, y) to its original position

Exercise 16F

1 a $y = -3x - 1$ **b** $y = \dfrac{3x}{2} + 1$ **c** $y = \dfrac{9x}{2} + 3$

d $y = 3x - 1$ **e** $y = -9x + 3$ **f** $y = \dfrac{-x - 1}{3}$

g $y = \dfrac{x - 1}{3}$

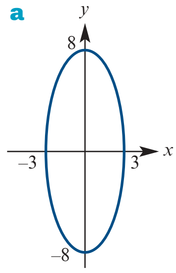
2 a $y = 6 - \dfrac{9x}{2}$ **b** $y = \dfrac{x + 2}{3}$

c $y = \dfrac{2 - 3x}{7}$ **d** $y = \dfrac{5x - 2}{12}$

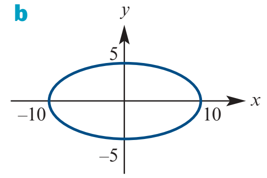
3 $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

4 $\begin{bmatrix} -3 & 0 \\ 0 & 6 \end{bmatrix}$

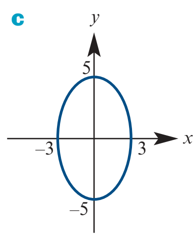
5 a



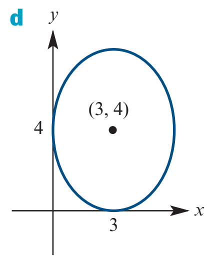
b



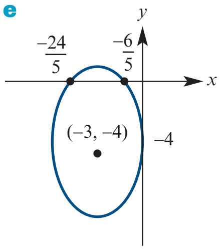
c



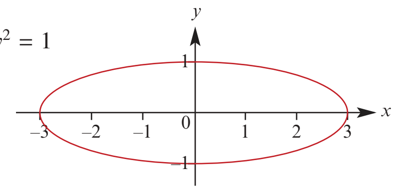
d



e



6 $\frac{x^2}{3^2} + y^2 = 1$

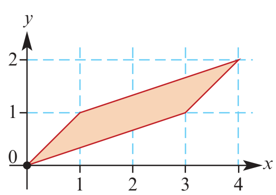


7 $y = -(x+1)^2 - 1$

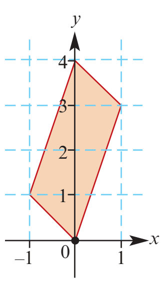
8 $y = (x-1)^2 - 3$

Exercise 16G

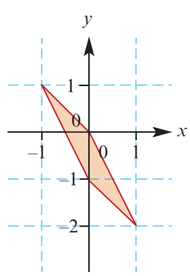
1 a Area = 2



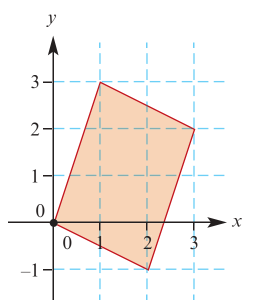
b Area = 4



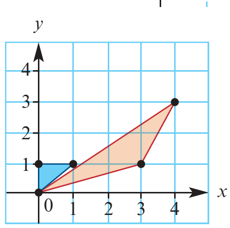
c Area = 1



d Area = 7

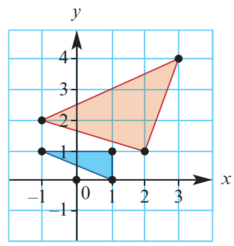


2 a



b Original area = $\frac{1}{2}$; image area = $\frac{5}{2}$

3 a



b Original area = 1; image area = 5

4 $m = \pm 2$

5 $m = -1, 2$

6 a i $\det\begin{bmatrix}1 & k\\0 & 1\end{bmatrix} = 1$

ii $\det\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix} = 1$

iii $\det\begin{bmatrix}\cos(2\theta) & \sin(2\theta)\\ \sin(2\theta) & -\cos(2\theta)\end{bmatrix} = -1$

b i Dilation of factor k from the y-axis and dilation of factor $\frac{1}{k}$ from the x-axis

ii Determinant of matrix is 1

7 b $x = -1$

8 $m > 2$ or $m < 1$

9 $\begin{bmatrix}1 & \pm\frac{\sqrt{3}}{2}\\ 0 & \pm\frac{1}{2}\end{bmatrix}$

10 a $\begin{bmatrix}a & b\\c & d\end{bmatrix}$

Exercise 16H

1 $\begin{bmatrix}x'\\y'\end{bmatrix} = \begin{bmatrix}y\\-x+4\end{bmatrix}$

2 $\begin{bmatrix}x'\\y'\end{bmatrix} = \begin{bmatrix}-x-2\\-y+2\end{bmatrix}$

3 a $\begin{bmatrix}x'\\y'\end{bmatrix} = \begin{bmatrix}y+1\\x-1\end{bmatrix}$ **b** $\begin{bmatrix}x'\\y'\end{bmatrix} = \begin{bmatrix}-y-1\\-x-1\end{bmatrix}$

c $\begin{bmatrix}x'\\y'\end{bmatrix} = \begin{bmatrix}x\\-y+2\end{bmatrix}$ **d** $\begin{bmatrix}x'\\y'\end{bmatrix} = \begin{bmatrix}-x-4\\y\end{bmatrix}$

4 a $\mathbf{A} = \begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}$ **b** $\mathbf{B} = \begin{bmatrix}1 & 0\\0 & k\end{bmatrix}$

c $\mathbf{C} = \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$

d $\mathbf{CBA} =$

$$\begin{bmatrix}\cos^2\theta + k\sin^2\theta & \cos\theta\sin\theta - k\sin\theta\cos\theta\\ \cos\theta\sin\theta - k\sin\theta\cos\theta & \sin^2\theta + k\cos^2\theta\end{bmatrix}$$

5 $\begin{bmatrix}\cos^2\theta & \cos\theta\sin\theta\\ \cos\theta\sin\theta & \sin^2\theta\end{bmatrix}$

6 $\begin{bmatrix}x'\\y'\end{bmatrix} = \begin{bmatrix}x+1\\y-1\end{bmatrix}$

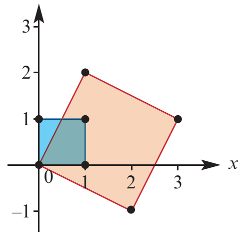
Chapter 16 review

Short-response questions

Technology-free

1 a $(7, 4)$ **b** $\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$

c Area = 5



d $(x, y) \to \left(\frac{2}{5}x - \frac{1}{5}y, \frac{1}{5}x + \frac{2}{5}y\right)$

2 a $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ **b** $\begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$ **c** $\begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}$

d $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ **e** $\begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$ **f** $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

3 a $\begin{bmatrix} -\frac{4}{5} & \frac{3}{5} \\ \frac{3}{5} & \frac{4}{5} \end{bmatrix}$ **b** $\left(\frac{4}{5}, \frac{22}{5}\right)$

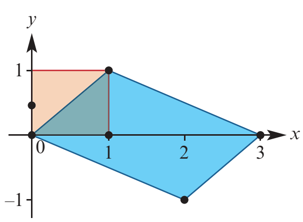
4 a $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ **b** $\begin{bmatrix} 0 & -1 \\ 2 & 0 \end{bmatrix}$ **c** $\begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$

5 a $(x, y) \to (x - 3, -y + 4)$

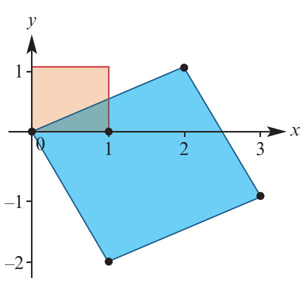
b $(x, y) \to (x - 3, -y - 4)$

6 a $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$ **b** $\mathbf{A}^{-1} = \begin{bmatrix} 1 & 0 \\ -k & 1 \end{bmatrix}$

7 a Area of image = 3 square units

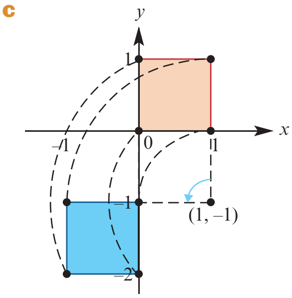


b Area of image = 5 square units



8 a $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -y \\ x - 2 \end{bmatrix}$ **b** $(1, 0)$

c



Technology-active

9 a $\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$ **b** $\begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$

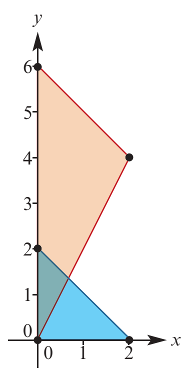
c Product of these two matrices:

$$\begin{bmatrix} \frac{-1 + \sqrt{3}}{2\sqrt{2}} & -\frac{1 + \sqrt{3}}{2\sqrt{2}} \\ \frac{1 + \sqrt{3}}{2\sqrt{2}} & \frac{-1 + \sqrt{3}}{2\sqrt{2}} \end{bmatrix}$$

d $\cos 75^\circ = \frac{-1 + \sqrt{3}}{2\sqrt{2}} = \frac{-\sqrt{2} + \sqrt{6}}{4}$

$\sin 75^\circ = \frac{1 + \sqrt{3}}{2\sqrt{2}} = \frac{\sqrt{2} + \sqrt{6}}{4}$

10 a



b Original area = 2 square units;
Image area = 6 square units

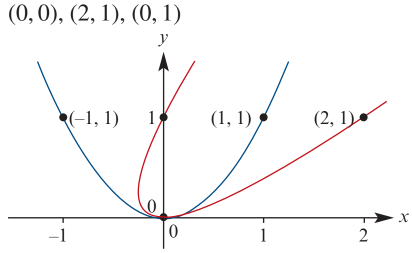
c 8π cubic units

11 a $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

b Shear of factor 1 parallel to the x-axis

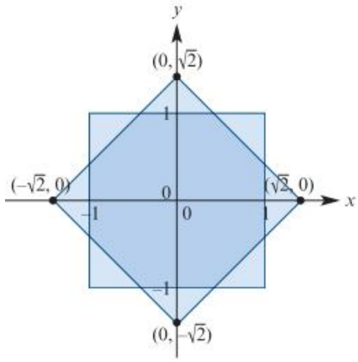
c $(0, 0), (2, 1), (0, 1)$

d



12 a $(0, \sqrt{2}), (\sqrt{2}, 0), (0, -\sqrt{2}), (-\sqrt{2}, 0)$

b



c $13 - 8\sqrt{2}$ square units

13 b **i** The composition of two rotations is a rotation

ii The composition of two reflections is a rotation

iii The composition of a reflection followed by a rotation is a reflection

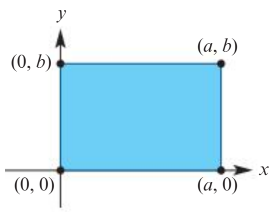
iv The composition of a rotation followed by a reflection is a reflection

c $\begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$

14 a $\begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{bmatrix}$ **b** $A'(-1, -3)$ **c** $2\sqrt{10}$

d Isosceles **f** $2\sqrt{10}$

15 a



b $O(0, 0), A(a\cos\theta, a\sin\theta),$
$B(-b\sin\theta, b\cos\theta),$
$C(a\cos\theta - b\sin\theta, a\sin\theta + b\cos\theta)$

16 a $y = \frac{1}{m} - \frac{x}{m}$; $(1, 0), \left(\frac{1 - m^2}{1 + m^2}, \frac{2m}{1 + m^2}\right)$

b $y = 1 - \frac{x}{m}$; $(0, 1), \left(\frac{2m}{1 + m^2}, \frac{m^2 - 1}{1 + m^2}\right)$

c $\begin{bmatrix} \frac{1 - m^2}{1 + m^2} & \frac{2m}{1 + m^2} \\ \frac{2m}{1 + m^2} & \frac{m^2 - 1}{1 + m^2} \end{bmatrix}$

Multiple-choice questions

1 B **2** D **3** A **4** D **5** C
6 A **7** D **8** D **9** D **10** D
11 B **12** D **13** D **14** A **15** D
16 C **17** B **18** D

Chapter 17

Short-response questions

1 a $4 - 3i$ **b** $-2 + 7i$ **c** $-1 + 9i$ **d** $13 + i$
e $13 - i$ **f** $-\frac{7}{34} + \frac{11}{34}i$ **g** $\frac{1}{5} + \frac{2}{5}i$

2 $z = 1 + 3i$ or $z = 1 - 3i$

3 Range $= \left(-\infty, -\frac{1}{4}\right]$

4 a $6\operatorname{cis}\left(\frac{\pi}{4}\right)$ **b** $6\operatorname{cis}\left(-\frac{\pi}{3}\right)$ **c** $10\operatorname{cis}\left(-\frac{\pi}{6}\right)$

5 $\frac{\pi}{18}, \frac{11\pi}{18}, \frac{13\pi}{18}$

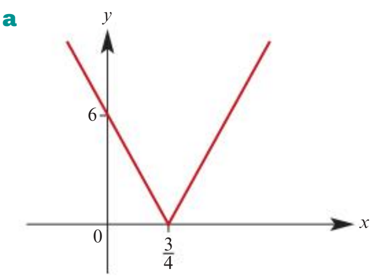
6 $z = -7 + i$

7 a $\pi, \frac{2\pi}{3}$ **b** $\frac{\pi}{2}$

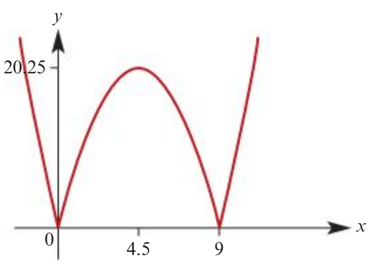
c $\pi - \cos^{-1}\left(\frac{1}{5}\right), \pi + \cos^{-1}\left(\frac{1}{5}\right)$

8 $f(x) = \frac{\sqrt{5}}{2}\sin(x + \alpha)$ where $\alpha = \tan^{-1}\left(\frac{1}{2}\right)$;
Maximum $= \frac{\sqrt{5}}{2}$, Minimum $= \frac{1}{2}$

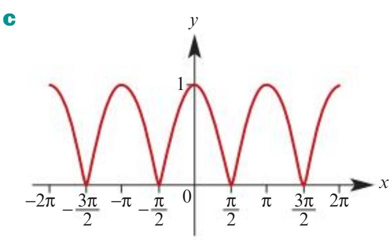
9 a



b

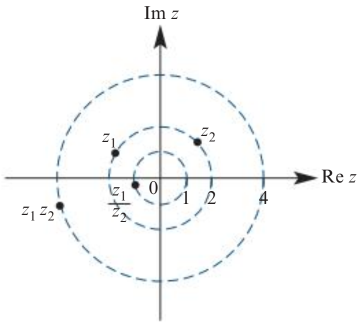


c



10 a $2\operatorname{cis}\left(\frac{\pi}{3}\right), \sqrt{2}\operatorname{cis}\left(-\frac{\pi}{4}\right)$ **b** $-\sqrt{3} + i$

11 a



b $z_1z_2 = 4\operatorname{cis}\left(-\frac{11\pi}{12}\right)$, $\frac{z_1}{z_2} = \operatorname{cis}\left(\frac{7\pi}{12}\right)$

12 a $\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$ **b** $\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}$ **c** $\begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$

d $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ **e** $\begin{bmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{bmatrix}$ **f** $\begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$

13 a $\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$ **b** $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

14 5

15 a $\frac{5}{4}$ **b** $\frac{4}{3}$ **c** $-\frac{\sqrt{3}}{3}$ **d** $\frac{2\sqrt{3}}{3}$

16 $\pm\frac{\sqrt{6}}{3}$

17 a 6 **b** $4i$ **c** 13 **d** 10
e 36 **f** -16 **g** $24i$ **h** $24i$

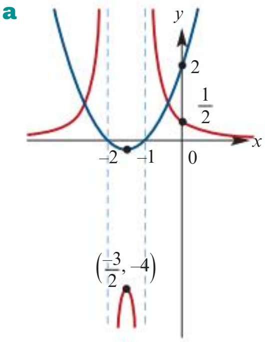
18 a 90° **b** 54° **c** 80° **d** 220°
e $x = 96°$, $y = 70°$ **f** 46°

19 a 40° **b** 140° **c** 50°

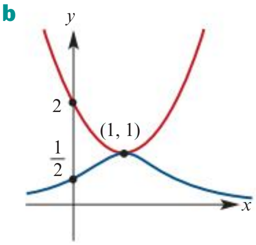
20 a $3 - 5i$ **b** $-1 + i$ **c** $-4 - 7i$
d $\frac{8 - i}{13}$ **e** $2 + i$ **f** $\frac{-2 + i}{5}$
g $-2 - i$ **h** $\frac{8 + i}{5}$ **i** $\frac{13 - i}{34}$
j $3 - i$ **k** $\frac{-1 - 3i}{2}$ **l** $-3 - 4i$

21 a $(z - 7i)(z + 7i)$
b $(z - 1 - 3i)(z - 1 + 3i)$
c $9\left(z - \frac{1}{3} - \frac{2}{3}i\right)\left(z - \frac{1}{3} + \frac{2}{3}i\right)$
d $4\left(z + \frac{3}{2} - i\right)\left(z + \frac{3}{2} + i\right)$

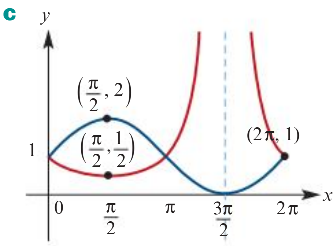
22 a



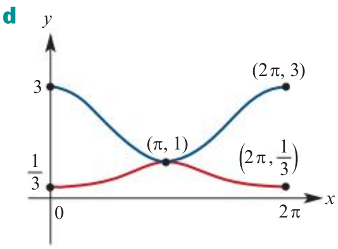
b



c



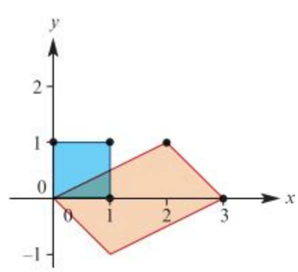
d



23 a $-\frac{4}{3}, 6$ **b** $-\frac{20}{3}, \frac{16}{3}$ **c** $-3, 7$
d 0, 5 **e** $-2, 8$ **f** $\frac{1 \pm \sqrt{21}}{2}, \frac{1 \pm \sqrt{13}}{2}$

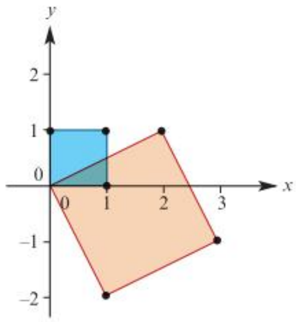
24 a $(x, y) \to (-x + 2, y - 1)$
b $(x, y) \to (-x - 2, y - 1)$

25 a Area = 3

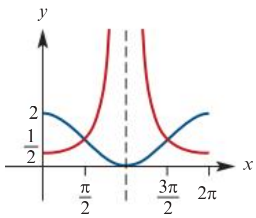


 ISBN 978-1-009-53737-7 Cambridge University Press

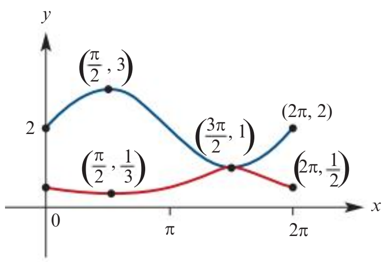
b Area = 5



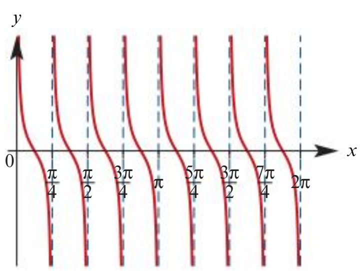
26 a



b



27 a



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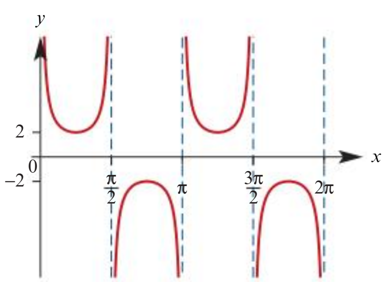

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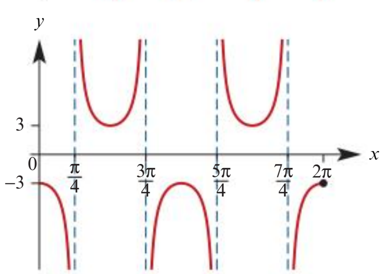
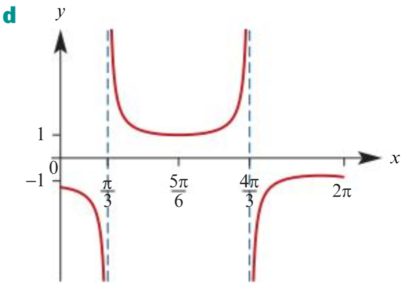
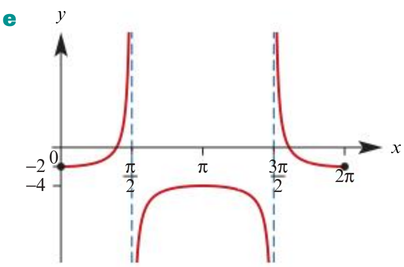


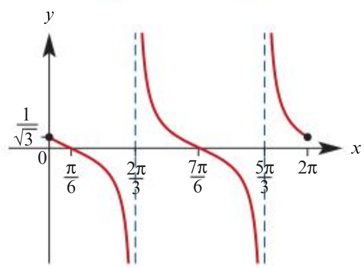
d



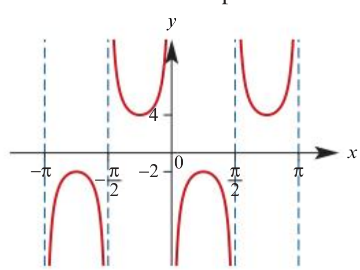
e



f

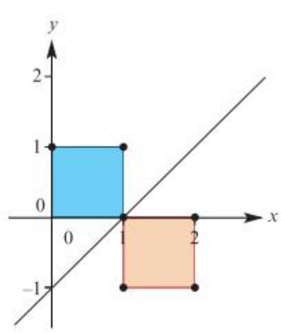


28
- Reflection in the x-axis
- Dilation of factor 3 from the x-axis
- Dilation of factor $\frac{1}{2}$ from the y-axis
- Translation 1 unit up



29 a $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} y+1 \\ x-1 \end{bmatrix}$ **b** $(0, 0) \to (1, -1)$

c

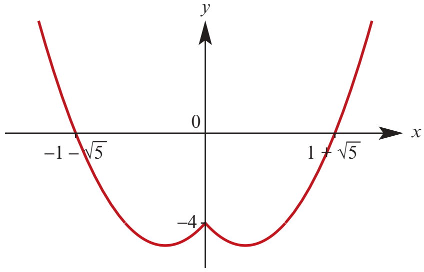


30 a 38° **b** 52° **c** 68°

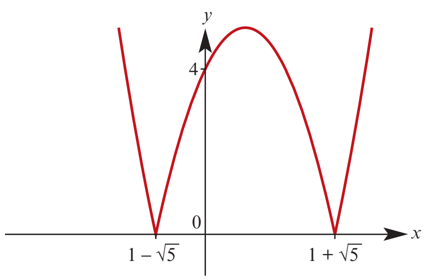
31 a $2 + i, \ -2 - i$

b $z = -1 - i$ or $z = -i$

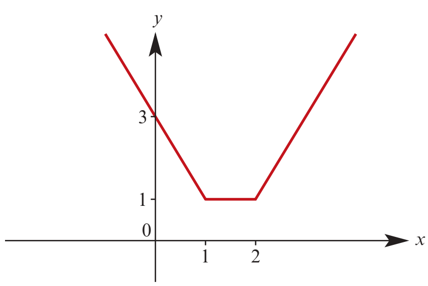
32 a



b



c

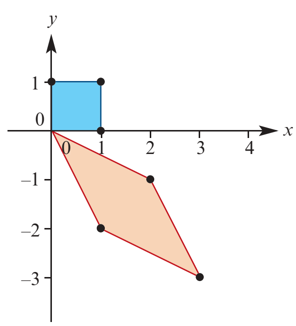


34 $x = \frac{3\pi}{4}$ or $x = \frac{7\pi}{4}$

Technology-active

38 a $(7, -8)$ **b** $\begin{bmatrix} 2 & 1 \\ -1 & -2 \end{bmatrix}$

c Area = 3



d $(x, y) \to \left(\frac{2}{3}x + \frac{1}{3}y, -\frac{1}{3}x - \frac{2}{3}y\right)$

39 a $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ **b** $\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$ **c** $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$

d $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ **e** $\begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$

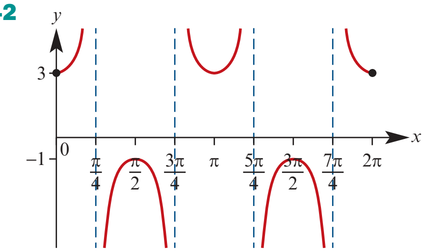
f $\begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$ **g** $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ **h** $\begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$

40 a $\frac{1}{2}(\sin(4x) - \sin(2x))$

b $\theta = \frac{(2n+1)\pi}{2}$ or $\theta = \frac{(2n+1)\pi}{4}, n \in \mathbb{Z}$

41 $\mathbb{R} \setminus (0, 6)$

42



43 $x = \frac{7\pi}{12}$ or $x = \frac{23\pi}{12}$

44 a $\begin{bmatrix} -\frac{15}{17} & \frac{8}{17} \\ \frac{8}{17} & \frac{15}{17} \end{bmatrix}$ **b** $\left(\frac{2}{17}, \frac{76}{17}\right)$

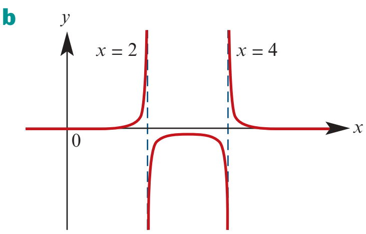
45 $z = \frac{\sqrt{7}+1}{6}(1 + i)$ or $z = \frac{1-\sqrt{7}}{6}(1 + i)$

46 a i $0, 2$ **ii** $-10, 4$ **iii** $\frac{1}{2}$

b $\left[-\frac{7}{2}, \infty\right)$

47 a $4(x-3)^2 - 4$

b



Local maximum at $\left(3, -\frac{1}{4}\right)$

48 $a = 6$

49 $z = -5 + 2i$ or $z = 5 - 2i$

50 $-\frac{a}{4} < x < \frac{5a}{2}$

51 a $x + y = 240°$ **b** $x = 28°$

52 a
- Dilation of factor p from the x-axis
- Translation of h units in the x-direction and k units in the y-direction

b $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & p \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} h \\ k \end{bmatrix}$

c $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{q} \end{bmatrix} \left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -m \\ -n \end{bmatrix} \right)$

d $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{p}{q} \end{bmatrix} \left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -m \\ -n \end{bmatrix} \right) + \begin{bmatrix} h \\ k \end{bmatrix}$

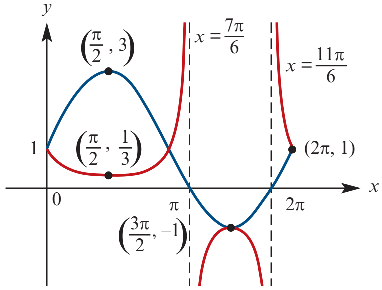
53 a i $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ **ii** $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$

iii $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

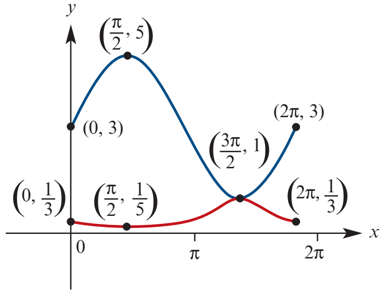
b $\begin{bmatrix} 1 - k\sin\theta\cos\theta & k\cos^2\theta \\ -k\sin^2\theta & 1 + k\sin\theta\cos\theta \end{bmatrix}$

c A shear of factor k parallel to the line $y = x\tan\theta$. (The positive direction is given by the vector $[1, \theta]$ and the positive side of the line is given by the vector $[1, \frac{\pi}{2} + \theta]$.)

54 a i

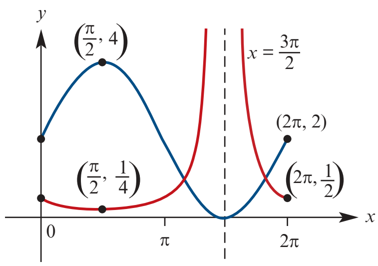


ii



b $k = 2$

c



55 $\frac{\pi}{4}, \frac{\pi}{6}, \frac{5\pi}{6}$

56 $z = 1,\ z = \frac{1}{2}(-1 + \sqrt{3}i)$ or $z = \frac{1}{2}(-1 - \sqrt{3}i)$

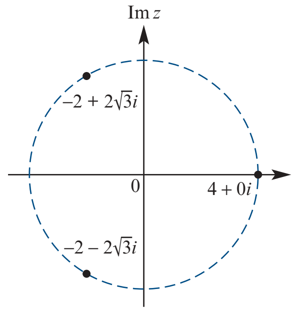
57 a $a = 5,\ b = 3$ **b** $z = 4i$

60 b $(\tan x - \sec x)^2 = \frac{1 - \sin x}{1 + \sin x}$

62 a $-2 + 2\sqrt{3}i,\ -2 - 2\sqrt{3}i$

b $4\operatorname{cis}\left(\frac{2\pi}{3}\right),\ 4\operatorname{cis}\left(-\frac{2\pi}{3}\right)$ **c** $64,\ 64$

d



e $4\sqrt{3},\ 4\sqrt{3}$

63 b $(-6, 2, 6), (-6, -2, 6), (-8, 3, 3), (-3, -3, 8)$

64 d i $\sqrt{26}$ **ii** $\sqrt{13}$ **iii** $2\sqrt{2}$ **iv** $2\sqrt{5}$

e i $\frac{\sqrt{10}}{5}(-1 - 3i)$ **ii** $\frac{\sqrt{10}}{5}(1 + 3i)$

iii $\frac{\sqrt{2}}{2}(1 + i)$

65 a $z^2 = \frac{1}{2}(-1 + \sqrt{3}i),\ z^3 = -1,$

$z^4 = \frac{1}{2}(-1 - \sqrt{3}i),\ z^5 = \frac{1}{2}(1 - \sqrt{3}i),$

$z^6 = 1$

66 a i 3 m **ii** 5 m **iii** 3 m **iv** 3 m

b $x = 4 - \cos(2t)$

c i 4 m from O **ii** π s **iii** 1 m

d $t = \frac{\pi}{2}$ s

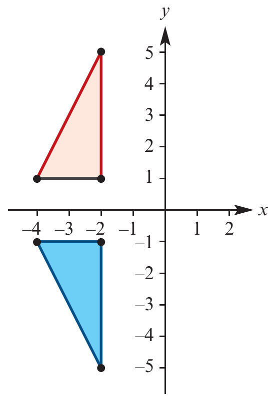
e $t = \frac{\pi}{6}$ s

67 b $\angle BCA = x°,\ \angle BOA = 2x°,\ \angle TAB = x°,$
$\angle TBA = x°$

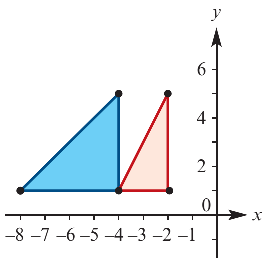
70 $z = \frac{1}{\sqrt[3]{16}}\left(\sqrt[3]{1 + \sqrt{3}i} + \sqrt[3]{1 - \sqrt{3}i}\right)$

72 a $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x + 6 \\ y + 3 \end{bmatrix}$

b



c



d $y = 2(x+3)^2 + 2$ **e** $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x+3 \\ -2y+4 \end{bmatrix}$

73 a $(4, 1)$

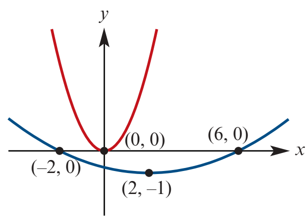
b **i** Rectangle with vertices $A'(0, 0)$, $B'(0, 1)$, $C'(4, 1)$, $E'(4, 0)$

ii 1 **iii** 4 **iv** k

c $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 4x \\ y \end{bmatrix}$

d **i** $y = \dfrac{1}{16}x^2$ **ii** $y = \dfrac{1}{16}(x-2)^2 - 1$

iii



e $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x+2 \\ \frac{1}{5}(y+3) \end{bmatrix}$

74 b **i** $x^2 + (y-1)^2 = 1$

ii $\left(x + \dfrac{4}{5}\right)^2 + \left(y - \dfrac{3}{5}\right)^2 = 1$

c $(0, 0), \left(-\dfrac{4}{5}, \dfrac{8}{5}\right)$

75 a $(-3, 11)$ **b** $\dfrac{1}{10}\begin{bmatrix} 3 & -1 \\ -2 & 4 \end{bmatrix}$

c $a = 2,\ b = 3$ **d** $(5a, 5a)$

e $\lambda = 2,\ b = -2a;\ \lambda = 5,\ b = a$

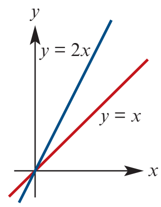
76 a $\begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$ **b** $\begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$

c $a = \sqrt{2},\ b = 0$ **d** $c = \dfrac{3\sqrt{2}}{2},\ d = \dfrac{\sqrt{2}}{2}$

e **i** $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2}x' + \frac{\sqrt{2}}{2}y' \\ -\frac{\sqrt{2}}{2}x' + \frac{\sqrt{2}}{2}y' \end{bmatrix}$

ii $\sqrt{2}(y-x) = (x+y)^2$

77 a

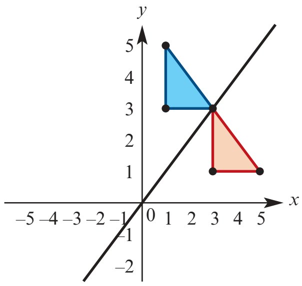


b $a = 2,\ b = \dfrac{\pi}{4}$

c $\begin{bmatrix} \frac{3}{\sqrt{10}} & -\frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \end{bmatrix}$

78 a **i** $(3, 1)$ **ii** $A'(3, 1)$, $B'(5, 1)$, $C'(3, 3)$

iii



b **ii** $(-1, -1), (2, 2)$

iv $(-1, -1), (2, 2)$,
$\left(\frac{1}{2}(-1+\sqrt{5}), \frac{1}{2}(-1-\sqrt{5})\right)$,
$\left(\frac{1}{2}(-1-\sqrt{5}), \frac{1}{2}(-1+\sqrt{5})\right)$

Multiple-choice questions

1 A **2** B **3** C **4** C **5** D **6** B
7 C **8** B **9** D **10** C **11** B **12** D
13 D **14** C **15** D **16** D **17** A **18** C
19 C **20** A **21** C **22** D **23** C **24** A
25 B **26** C **27** D **28** C **29** A **30** D
31 B **32** D **33** C **34** D **35** D **36** B
37 B **38** C **39** B **40** D **41** D **42** C
43 B **44** B

Problem-solving and modelling

See solutions supplement

Appendix A

Exercise A1

1 a 4π **b** 3π **c** $-\dfrac{5\pi}{2}$ **d** $\dfrac{\pi}{12}$

e $-\dfrac{\pi}{18}$ **f** $-\dfrac{7\pi}{4}$

2 a 225° **b** −120° **c** 105° **d** −330°

e 260° **f** −165°

3 a 0 **b** −1 **c** 1 **d** −1

4 a −1 **b** −1 **c** 1 **d** 1

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Exercise A2

1 a $-\frac{1}{\sqrt{2}}$ **b** $-\frac{1}{\sqrt{2}}$ **c** 1 **d** 1 **e** $\frac{1}{\sqrt{2}}$

f $\frac{1}{\sqrt{2}}$ **g** 0 **h** $\frac{\sqrt{3}}{2}$ **i** 0 **j** 0

k 1 **l** 0 **m** $-\frac{1}{2}$ **n** -1 **o** -1

2 a $\frac{1}{\sqrt{2}}$ **b** $\frac{1}{2}$ **c** $\frac{\sqrt{3}}{2}$ **d** $-\frac{1}{2}$

e $\frac{1}{\sqrt{2}}$ **f** $\frac{\sqrt{3}}{2}$

3 a 0.6 **b** 0.6 **c** 0.3 **d** −0.3
e −0.3 **f** 0.6 **g** −0.6 **h** −0.3

4 $\cos x = -\frac{\sqrt{3}}{2}$, $\tan x = -\frac{1}{\sqrt{3}}$

5 $\sin x = -\frac{\sqrt{51}}{10}$, $\tan x = \frac{\sqrt{51}}{7}$

6 $\cos x = -\frac{\sqrt{3}}{2}$, $\tan x = \frac{1}{\sqrt{3}}$

7 $\cos x = \frac{\sqrt{91}}{10}$, $\tan x = -\frac{3\sqrt{91}}{91}$

Exercise A3

1 $2\pi - a,\ 2\pi - b,\ 2\pi - c,\ 2\pi - d$

2 a $\frac{4\pi}{3}, \frac{5\pi}{3}$ **b** $\frac{2\pi}{3}, \frac{5\pi}{6}, \frac{5\pi}{3}, \frac{11\pi}{6}$

c $\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$ **d** $\frac{5\pi}{6}, \frac{3\pi}{2}$

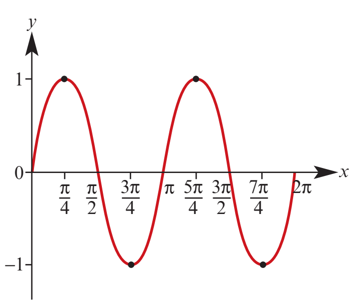
e $0, \frac{\pi}{3}, \pi, \frac{4\pi}{3}, 2\pi$ **f** $\frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{2}, \frac{5\pi}{3}$

3 a $-\frac{5\pi}{6}, -\frac{\pi}{6}$ **b** $0, -\frac{2\pi}{3}, -\frac{\pi}{3}, \frac{\pi}{3}, \frac{2\pi}{3}, -\pi, \pi$

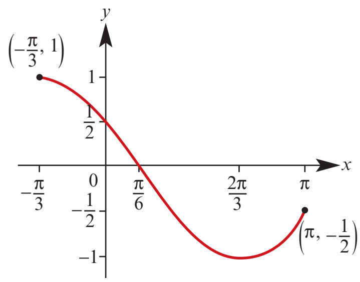
c 0 **d** $0, -\frac{2\pi}{3}$ **e** $-\frac{5\pi}{6}, -\frac{\pi}{2}, \frac{\pi}{6}, \frac{\pi}{2}$

Exercise A4

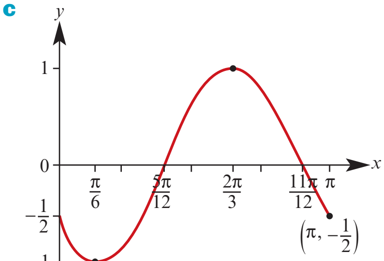
1 a



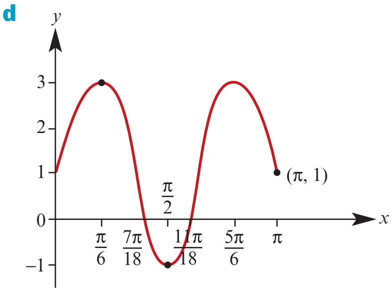
b



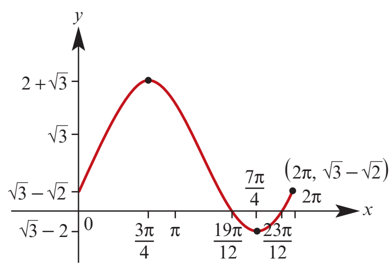
c



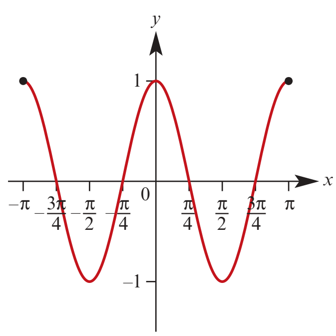
d



e



f



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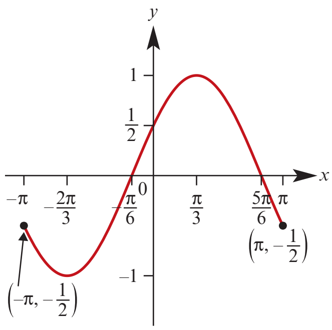
2 a



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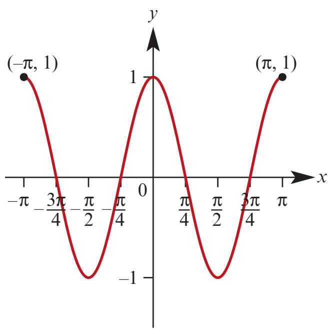

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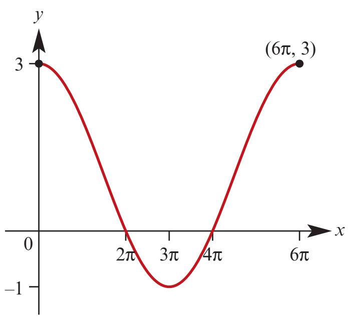
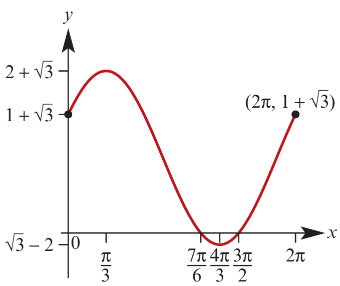
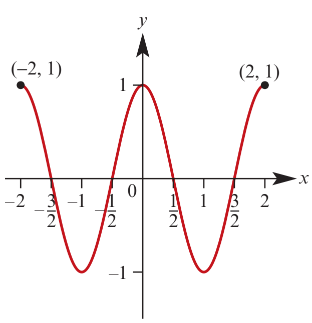


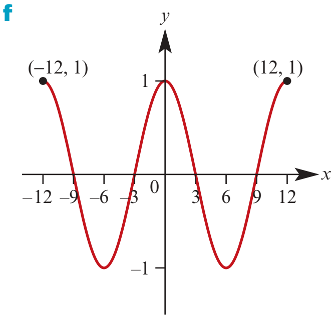
d



e



f



Exercise A5

1 a 1 **b** $\sqrt{3}$ **c** $\dfrac{1}{\sqrt{3}}$

2 a $\sqrt{3}$ **b** $\dfrac{1}{\sqrt{3}}$ **c** -1

3 a $\dfrac{-\sqrt{17}}{17}$ **b** $\dfrac{-4\sqrt{17}}{17}$ **c** $\dfrac{-1}{4}$ **d** $\dfrac{-1}{4}$

4 a $\dfrac{\sqrt{21}}{7}$ **b** $\dfrac{-2\sqrt{7}}{7}$ **c** $\dfrac{\sqrt{3}}{2}$ **d** $\dfrac{-\sqrt{3}}{2}$

5 a $\dfrac{3\pi}{4}, \dfrac{7\pi}{4}$ **b** $\dfrac{\pi}{3}, \dfrac{4\pi}{3}$ **c** $\dfrac{\pi}{6}, \dfrac{7\pi}{6}$

d $-\dfrac{7\pi}{8}, -\dfrac{3\pi}{8}, \dfrac{\pi}{8}, \dfrac{5\pi}{8}$ **e** $-\dfrac{5\pi}{6}, -\dfrac{\pi}{3}, \dfrac{\pi}{6}, \dfrac{2\pi}{3}$

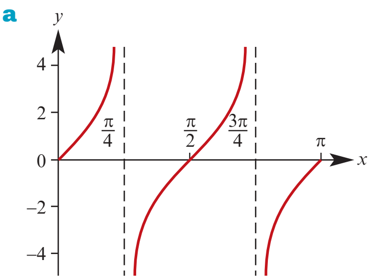
f $-\dfrac{7\pi}{12}, -\dfrac{\pi}{12}, \dfrac{5\pi}{12}, \dfrac{11\pi}{12}$

6 a $\dfrac{3\pi}{8}, \dfrac{7\pi}{8}, \dfrac{11\pi}{8}, \dfrac{15\pi}{8}$

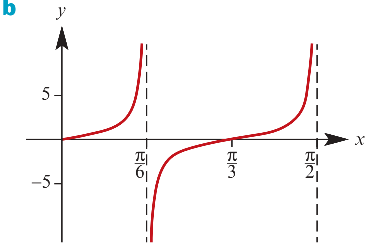
b $-\dfrac{7\pi}{8}, -\dfrac{3\pi}{8}, \dfrac{\pi}{8}, \dfrac{5\pi}{8}$

c $-\dfrac{13\pi}{18}, -\dfrac{7\pi}{18}, -\dfrac{\pi}{18}, \dfrac{5\pi}{18}, \dfrac{11\pi}{18}, \dfrac{17\pi}{18}$ **d** $-\dfrac{\pi}{6}$

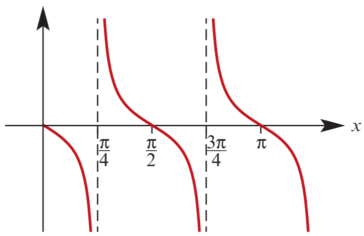
7 a



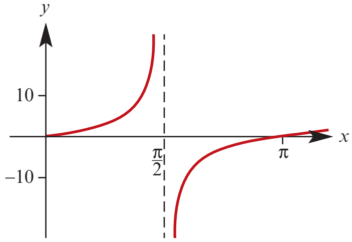
b



c



d



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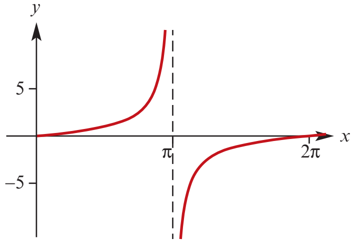

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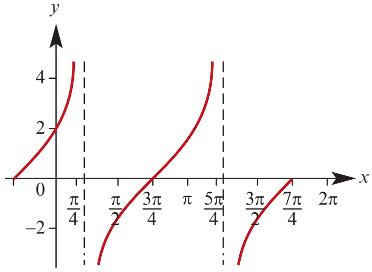
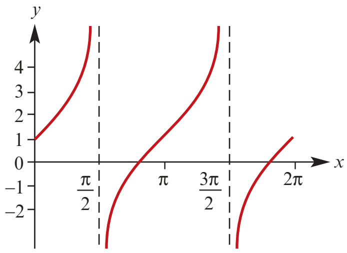
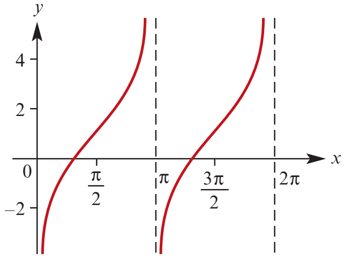


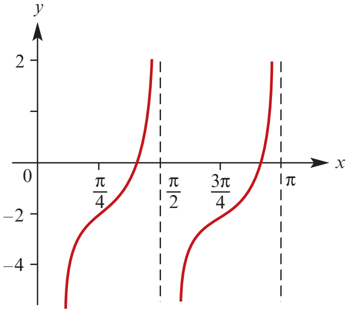
g



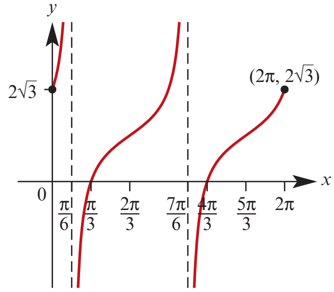
h



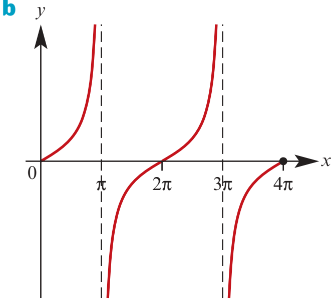
i



8 a



b



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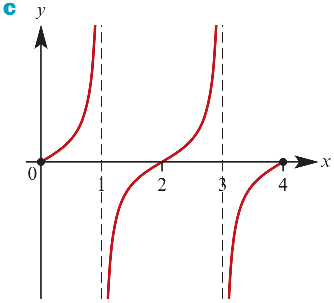


Exercise A6

1 a i 2π **ii** 4π **iii** -4π

b i $\frac{4\pi}{3}, \frac{8\pi}{3}$ **ii** $\frac{14\pi}{3}, \frac{10\pi}{3}$ **iii** $\frac{-14\pi}{3}, \frac{-10\pi}{3}$

2 a $x = 2n\pi \pm \frac{\pi}{6}, n \in \mathbb{Z}$

b $x = \frac{2n\pi}{3} + \frac{\pi}{9}$ or $x = \frac{2n\pi}{3} + \frac{2\pi}{9}, n \in \mathbb{Z}$

c $x = n\pi + \frac{\pi}{3}, n \in \mathbb{Z}$

3 a $\frac{\pi}{6}, \frac{5\pi}{6}$ **b** $\frac{\pi}{12}, \frac{11\pi}{12}$ **c** $\frac{\pi}{3}, \frac{5\pi}{6}$

4 $\frac{-11\pi}{6}, \frac{-7\pi}{6}, \frac{\pi}{6}, \frac{5\pi}{6}$

5 $\frac{-\pi}{3}, \frac{\pi}{3}, \frac{5\pi}{3}$

6 **a** $x = n\pi - \frac{\pi}{6}$ or $x = n\pi - \frac{\pi}{2}, n \in \mathbb{Z}$

b $x = \frac{n\pi}{2} - \frac{\pi}{12}, n \in \mathbb{Z}$

c $x = 2n\pi + \frac{5\pi}{6}$ or $x = 2n\pi - \frac{\pi}{2}, n \in \mathbb{Z}$

7 $x = \frac{(4n-1)\pi}{4}$ or $x = n\pi, n \in \mathbb{Z}$;

$x = \frac{-5\pi}{4}, -\pi, -\frac{\pi}{4}, 0, \frac{3\pi}{4}, \pi, \frac{7\pi}{4}$

8 $x = \frac{n\pi}{3}, n \in \mathbb{Z}$; $\quad x = -\pi, \frac{-2\pi}{3}, \frac{-\pi}{3}, 0$

9 $x = \frac{6n-1}{12}$ or $x = \frac{3n+2}{6}, n \in \mathbb{Z}$;

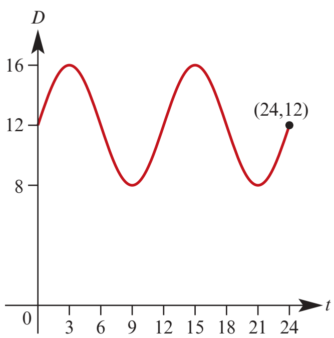
$x = \frac{-2}{3}, \frac{-7}{12}, \frac{-1}{6}, \frac{-1}{12}, \frac{1}{3}, \frac{5}{12}, \frac{5}{6}, \frac{11}{12}$

Exercise A7

1 **a** **i** 0.00 hours **ii** 24.00 hours

b 14 February ($t = 1.48$), 27 October ($t = 9.86$)

2 **a**



b $0 \le t \le 6, \ 12 \le t \le 18$

c 15.9 m

3 **a** 7 metres **b** 1 metre

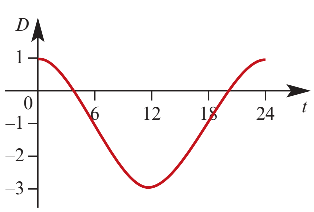
c $t = \frac{1}{4}$ or $t = \frac{5}{4}$ **d** $t = \frac{3}{4}$ or $t = \frac{7}{4}$

e Particle oscillates between $x = 1$ and $x = 7$

4 **a** 19.5°C

b $D = -1 + 2\cos\left(\frac{\pi t}{12}\right)$

c



d $4 < t < 20$

5 **a** 2 a.m.

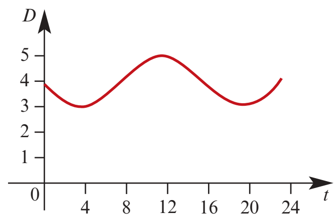
b 8 a.m. and 8 p.m.

6 **a** **i** $\frac{3}{2}$ **ii** 12 **iii** $d(t) = \frac{7}{2} - \frac{3}{2}\cos\left(\frac{\pi}{6}t\right)$

iv 1.5 m

b Between 9 p.m. and 3 a.m. and between 9 a.m. and 3 p.m. each day

7 **a**



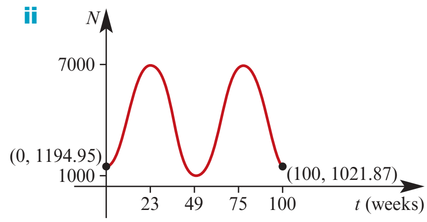
b The boat can enter at 8 a.m. and must leave by 4 p.m.

c The boat can enter at 6:40 a.m. and must leave by 5:20 p.m.

8 **a** **i** 52 weeks **ii** 3000 **iii** [1000, 7000]

b **i** $N(0) = 1194.95, \ N(100) = 1021.87$

ii



c **i** $t = 23, 75$ **ii** $t = 49$

d $14\frac{1}{3} < t < 31\frac{2}{3}, \ 66\frac{1}{3} < t < 83\frac{2}{3}$

e $d = 25\,000, \ a = 15\,000, \ b = 10, \ c = 5$

Appendix A review

Short-response questions

Technology-free

1 **a** $\frac{13\pi}{6}$ **b** $\frac{14\pi}{3}$ **c** $\frac{37\pi}{6}$ **d** $\frac{71\pi}{12}$ **e** $\frac{11\pi}{12}$

f $\frac{5\pi}{2}$ **g** $\frac{7\pi}{3}$ **h** $\frac{13\pi}{6}$ **i** $\frac{2\pi}{9}$

2 **a** 330° **b** 765° **c** 405° **d** 105°

e 1530° **f** −495° **g** −225° **h** −585°

i 1035°

3 **a** $\frac{1}{\sqrt{2}}$ **b** $-\frac{1}{\sqrt{2}}$ **c** −1 **d** 0 **e** $\frac{\sqrt{3}}{2}$

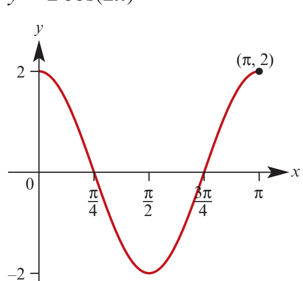
f −1 **g** $-\sqrt{3}$ **h** 1

4 **a** $4, 4\pi$ **b** $5, \frac{\pi}{3}$ **c** $\frac{1}{3}, \frac{\pi}{2}$ **d** $2, \frac{2\pi}{5}$

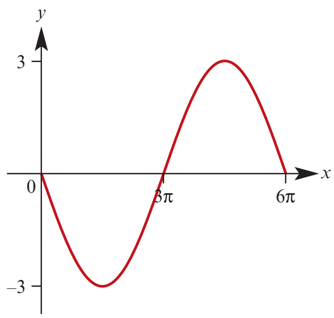
e 7, 8 **f** $\frac{2}{3}, 3$

5 **a** 5, 1 **b** 9, −1

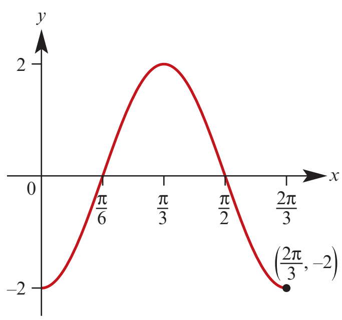
6 **a** $y = 2\cos(2x)$



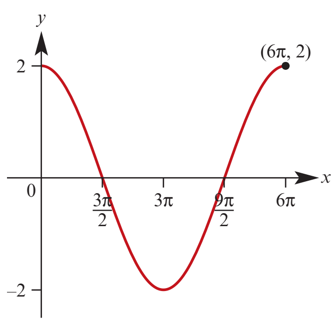
b $y = -3\sin\left(\frac{x}{3}\right)$



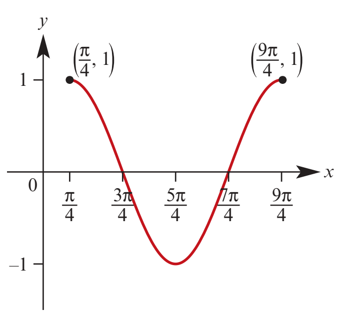
c $y = -2\cos(3x)$



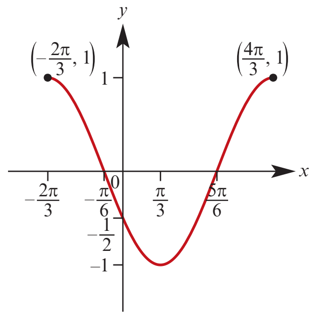
d $y = 2\cos\left(\frac{x}{3}\right)$



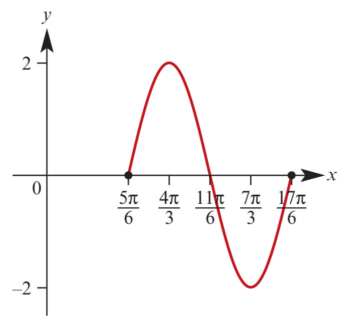
e $y = \cos\left(x - \frac{\pi}{4}\right)$



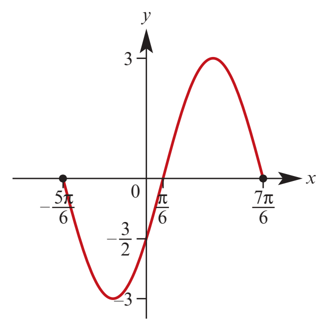
f $y = \cos\left(x + \frac{2\pi}{3}\right)$



g $y = 2\sin\left(x - \frac{5\pi}{6}\right)$



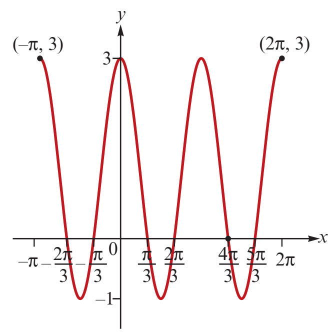
h $h = -3\sin\left(x + \frac{5\pi}{6}\right)$



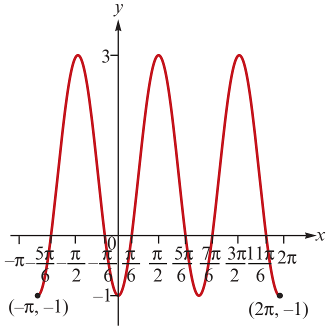
7 **a** $-\frac{5\pi}{6}, \frac{5\pi}{6}$ **b** $-\frac{7\pi}{12}, -\frac{5\pi}{12}, \frac{5\pi}{12}, \frac{7\pi}{12}$

c $\pi, \frac{5\pi}{3}$ **d** $\frac{2\pi}{3}$ **e** $\pi, \frac{5\pi}{3}$

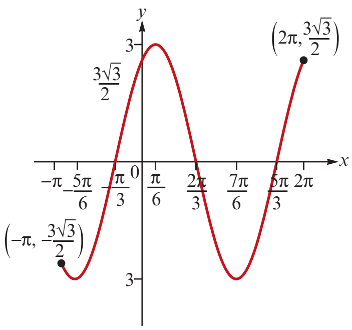
8 **a** $f(x) = 2\cos(2x) + 1$



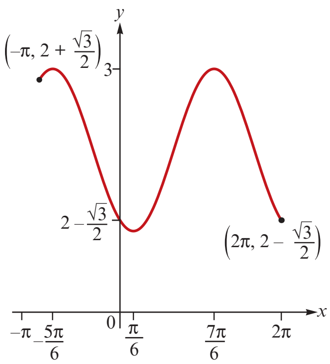
b $f(x) = 1 - 2\cos(2x)$



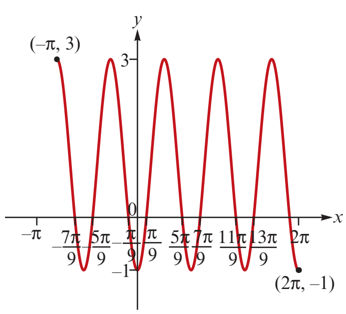
c $f(x) = 3\sin\left(x + \frac{\pi}{3}\right)$



d $f(x) = 2 - \sin\left(x + \frac{\pi}{3}\right)$



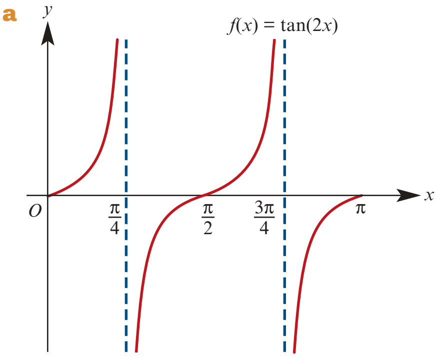
e $f(x) = 1 - 2\cos(3x)$



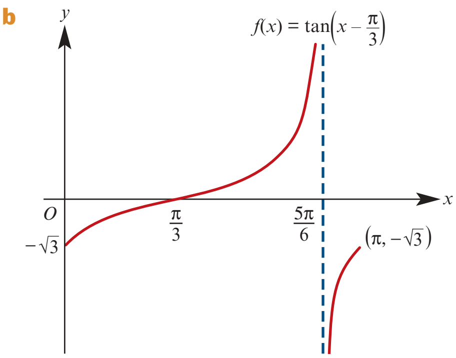
9 a $\frac{2\pi}{3}, \frac{5\pi}{3}$ **b** $\frac{\pi}{9}, \frac{4\pi}{9}, \frac{7\pi}{9}, \frac{10\pi}{9}, \frac{13\pi}{9}, \frac{16\pi}{9}$

c $\frac{3\pi}{2}$ **d** $\frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}, \frac{13\pi}{8}$

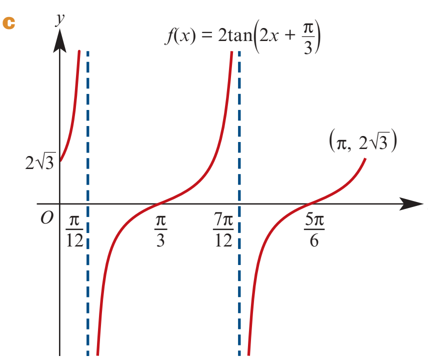
10 a



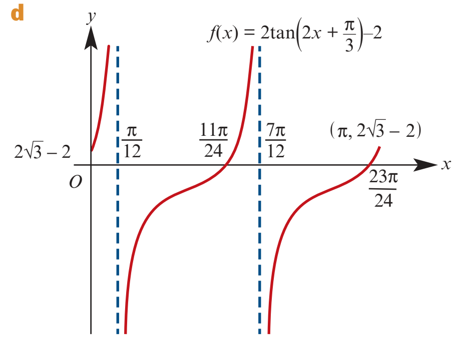
b



c



d



11 a $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$

b $\frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}$

c $\frac{\pi}{18}, \frac{11\pi}{18}, \frac{13\pi}{18}, \frac{23\pi}{18}, \frac{25\pi}{18}, \frac{35\pi}{18}$

d $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

12 0°, 60°, 180°, 300°, 360°

13 a $x = n\pi - \frac{\pi}{4}, n \in \mathbb{Z}$

b $x = \frac{2n\pi}{3}, n \in \mathbb{Z}$

c $x = n\pi - \frac{\pi}{4}, n \in \mathbb{Z}$

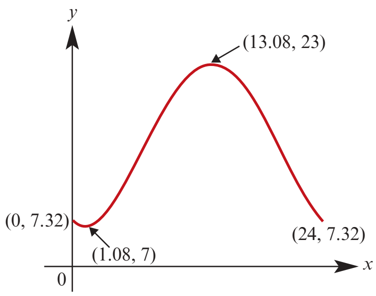
Technology-active

14 a 7.3°C

b Min = 7°C; max = 23°C

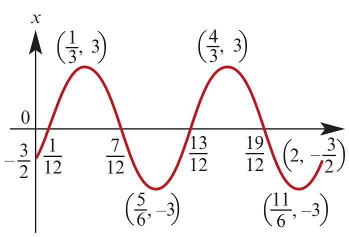
c Between 9:40 a.m. and 4:30 p.m.

d



15 a $a = \frac{\pi}{6}$

b



c 3 m **d** $\frac{5}{6}$ s **e** 1 s **f** $\frac{1}{4}$ s

g i 24 m **ii** 30 m

16 a i $a = 13.4$ **ii** $b = 2$ **iii** $k = 12$

b 3 a.m., 9 a.m., 3 p.m., 9 p.m.

c $2 < t < 10,\ 14 < t < 22$

17 a $p = 6,\ q = 4.2$

b 3 a.m., 3 p.m.

c 6 m

d 7 a.m., 11 a.m., 7 p.m., 11 p.m.

e 8 hours

18 a i $-1 < k < 1$ or $1 < k < 3$

ii $k = -1$ or $k = 3$

iii $k < -1$ or $k > 3$

b A translation of 1 unit in the negative direction of the y-axis, followed by a dilation of factor $\frac{1}{2}$ from the x-axis and a dilation of factor 3 from the y-axis

c i $h = \frac{\pi}{2}, \frac{7\pi}{6}$ **ii** $h = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$

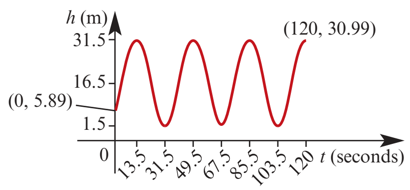
19 a For N: Max = 7000, occurs in April ($t = 4$); Min = 1000, occurs in October ($t = 10$).
For M: Max = 8500, occurs in June ($t = 6$); Min = 2500, occurs at end of January and November ($t = 1$ and $t = 11$)

b $t = 4.31$ (May), population is 6961; $t = 0.24$ (January), population is 2836

c 14 556 in June ($t = 5.19$)

d $t = 7.49$ (August)

20 a



b 5.89 m **c** 27.51 s **d** 6 times

e 20 times **f** 4.21 m **g** 13.9 m

Multiple-choice questions

1 A **2** D **3** D **4** D **5** D
6 D **7** C **8** D **9** B **10** A
11 D **12** A **13** A

Appendix B

Exercise B1

1 $(x-1)^2 + (y+2)^2 = 4^2$

2 $(x+4)^2 + (y-3)^2 = 5^2$

3 a $y = -x$

4 a $y = \frac{x}{2} + \frac{3}{4}$

5 $(0, 3)$ or $(3, 0)$

6 $\left(\frac{9}{10}, \frac{3}{10}\right)$

7 $(6, 8)$ or $\left(\frac{72}{17}, \frac{154}{17}\right)$

8 a $2y - x = 1$ **b** $x + y = 2$ **c** $P(1, 1)$

d $(x-1)^2 + (y-1)^2 = 5^2$

9 $y = 2x + 1, \quad x \neq 2$

10 $y = 6$

11 a The rays $x = 0,\ y \geq 0$ and $y = 0,\ x \geq 0$

b $\left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{2}$

12 $(x-4)^2 + y^2 = 4$

13 The lines $y = 1$ and $y = 5$

14 3

Exercise B2

1 $y = \frac{x^2}{12}$ **2** $y = -\frac{x^2}{12} - 1$ **3** $x = \frac{y^2}{12} - 1$

4 a $x = \frac{y^2}{4c}$ **b** $\left(\frac{1}{12}, 0\right)$

 ISBN 978-1-009-53737-7

5 a $y = \dfrac{1}{2b - 2c}(x^2 - 2ax + a^2 + b^2 - c^2)$

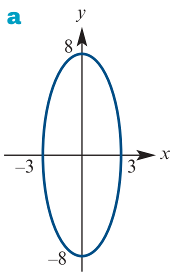
b $y = -\dfrac{1}{2}(x^2 - 2x - 4)$

6 $y = -1$ or $y = 19$

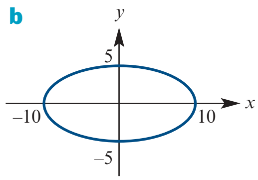
7 $(2, 1 + \sqrt{3})$ or $(2, 1 - \sqrt{3})$

Exercise B3

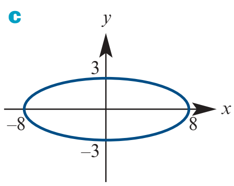
1 a



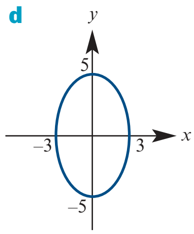
b



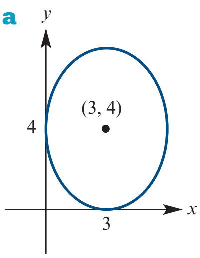
c



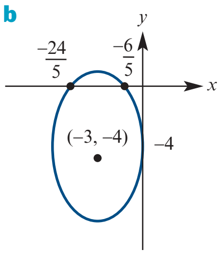
d



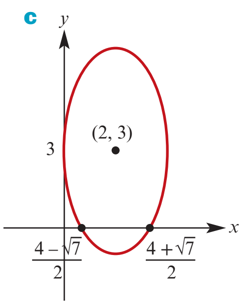
2 a



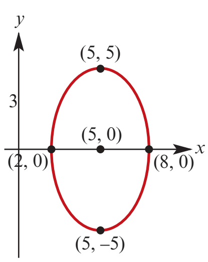
b



c



d



3 a $\dfrac{x^2}{25} + \dfrac{y^2}{16} = 1$ **b** $\dfrac{(x-2)^2}{9} + \dfrac{y^2}{4} = 1$

c $\dfrac{(x+1)^2}{4} + (y-1)^2 = 1$

4 $\dfrac{x^2}{4} + \dfrac{y^2}{3} = 1$

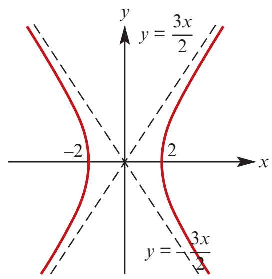
5 $\dfrac{x^2}{5} + \dfrac{y^2}{9} = 1$

6 $\dfrac{(x-4)^2}{16} + \dfrac{y^2}{12} = 1$

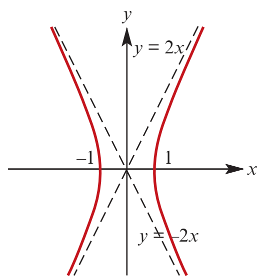
7 $\dfrac{x^2}{25} + \dfrac{y^2}{9} = 1$

Exercise B4

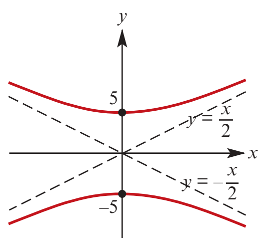
1 a



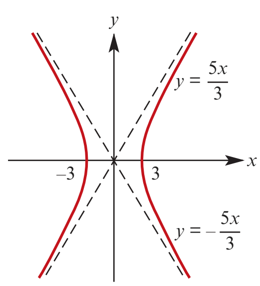
b



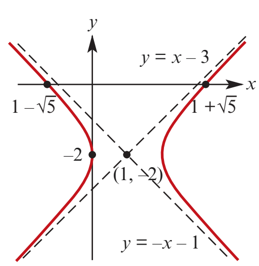
c



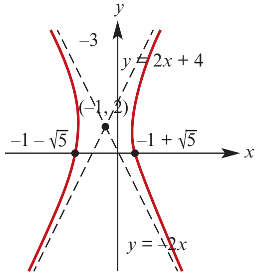
d



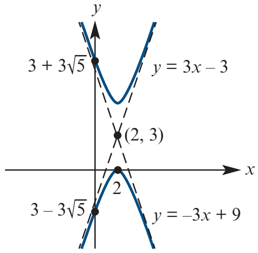
2 a



b



c



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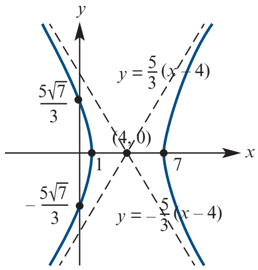
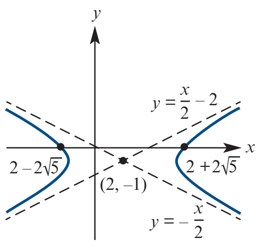
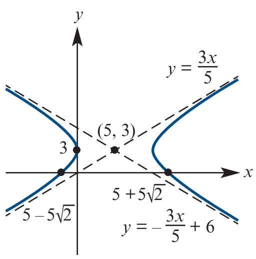


e



f



3 $\dfrac{x^2}{9} - \dfrac{y^2}{7} = 1$

4 $5x^2 - 4y^2 = 20$

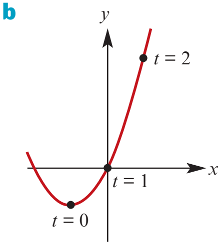
5 $\dfrac{(x+3)^2}{16} - \dfrac{y^2}{48} = 1$

6 $\dfrac{(y+5)^2}{4} - \dfrac{x^2}{12} = 1$

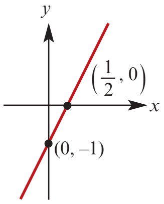
Exercise B5

1 a $y = x^2 + 2x$

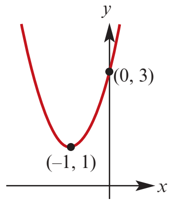
b



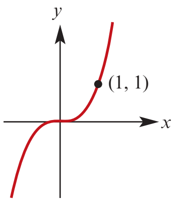
2 a $y = 2x - 1$



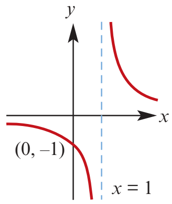
b $y = 2(x+1)^2 + 1$



c $y = x^3$



d $y = \dfrac{1}{x-1}$



3 a $x^2 + y^2 = 2^2$

b $\dfrac{(x+1)^2}{3^2} + \dfrac{(y-2)^2}{2^2} = 1$

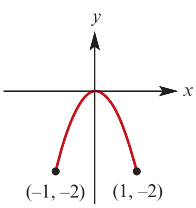
c $x = 3\cos t - 3$ and $y = 3\sin t + 2$ (other answers are possible)

d $x = 3\cos t - 2$ and $y = 2\sin t + 1$ (other answers are possible)

4 $x = t$ and $y = 3t + 1$ (other answers are possible)

5 a $y = -2x^2$ where $-1 \le x \le 1$

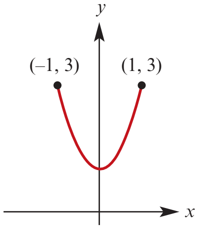
b



6 $\left(-\dfrac{3}{5}, -\dfrac{4}{5}\right), \left(\dfrac{3}{5}, \dfrac{4}{5}\right)$

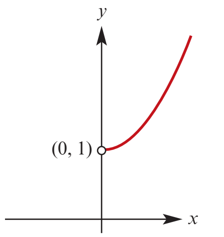
7 a $y = 2x^2 + 1$ **b** $-1 \le x \le 1$ **c** $1 \le y \le 3$

d

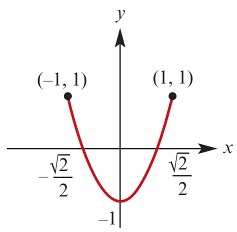


8 a $y = x^2 + 1$ **b** $x > 0$ **c** $y > 1$

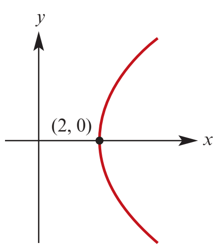
d



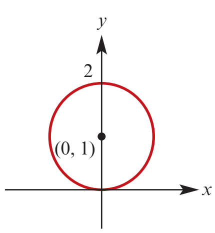
9 $y = -1 + 2x^2$ where $-1 \le x \le 1$



10 b



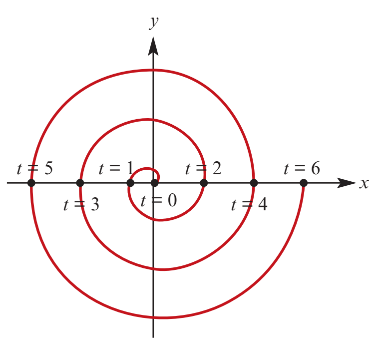
11 a



c $x = \dfrac{2t}{t^2 + 1}$

$y = \dfrac{2}{t^2 + 1}$

12



Appendix B review

Short-response questions

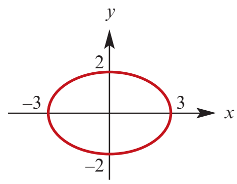
Technology-free

1 $y = \dfrac{x}{3}$

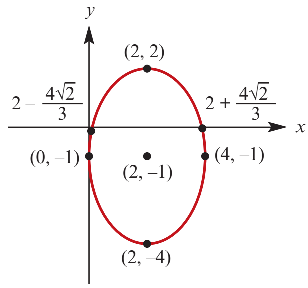
2 $(x - 3)^2 + (y - 2)^2 = 6^2$

3 $C(-2, 4)$, $r = \sqrt{20}$

4 a

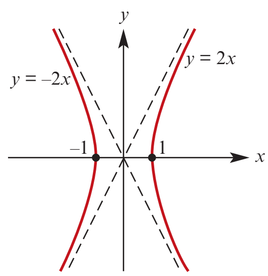


b

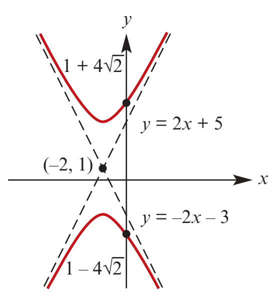


5 $C(-2, 0)$; Intercepts $(0, 0)$, $(-4, 0)$

6 a



b



7 $\dfrac{(x - 2)^2}{4} - \dfrac{(y - 5)^2}{12} = 1$

8 a $y = 4 - 2x$ **b** $x^2 + y^2 = 2^2$

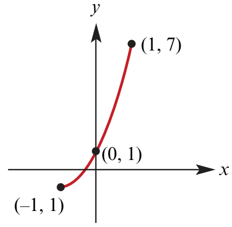
c $\dfrac{(x - 1)^2}{3^2} + \dfrac{(y + 1)^2}{5^2} = 1$

d $y = 1 - 3x^2$ where $-1 \le x \le 1$

9 a $y = 2(x + 1)^2 - 1$ **b** $-1 \le x \le 1$

c $-1 \le y \le 7$

d



Technology-active

10 a $y = 2x - \frac{9}{2}$

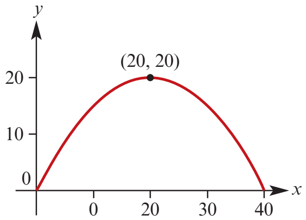
b $(x - 8)^2 + (y + 1)^2 = 20$

11 a $y = \frac{x^2}{12} + 1$ **b** $\frac{x^2}{12} + \frac{(y-6)^2}{16} = 1$

c $\frac{(y+4)^2}{16} - \frac{x^2}{48} = 1$

12 a $y = \frac{1}{20}x(40 - x)$

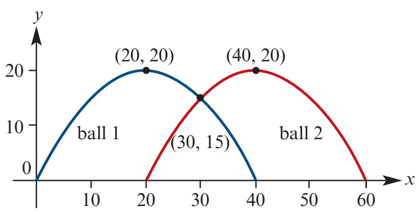
b



c 20 metres

d $y = -\frac{1}{20}(x - 20)(x - 60)$

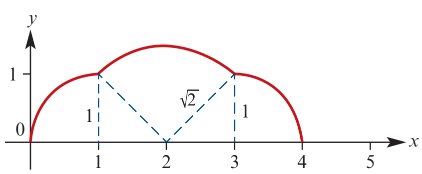
e



f (30, 15)

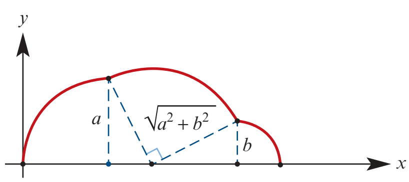
g Yes (same position at same time)

14 a



b Distance $= \frac{\pi}{2}(2 + \sqrt{2})$

c



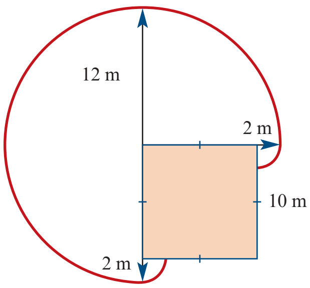
Distance $= \frac{\pi}{2}(a + \sqrt{a^2 + b^2} + b)$

d Area $= \frac{\pi}{2}(a^2 + b^2) + ab$

15 c $x = t$ and $y = -t + 3$

e $k > \frac{8}{\sqrt{5}}$ or $k < -\frac{8}{\sqrt{5}}$

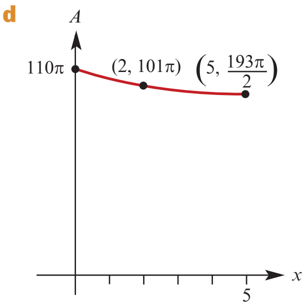
16 a



b 110π m^2

c $A(x) = \begin{cases} \frac{3\pi x^2}{4} - 6\pi x + 110\pi, & 0 \le x \le 2 \\ \frac{\pi x^2}{2} - 5\pi x + 109\pi, & 2 < x \le 5 \end{cases}$

d



e i $x = 0$ **ii** $x = 5$

Multiple-choice questions

1 A	**2** D	**3** C	**4** D	**5** C
6 C	**7** D	**8** D	**9** D	**10** B
11 C	**12** B			