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STAGE 6

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Further graphs

Chapter introduction

This chapter contains various loosely related discussions about functions and their graphs.

- Sections 6A–6B deal with some trickier types of inequations, using algebraic and graphical–geometric methods.
- Sections 6C–6E introduce further transformations of the graphs of functions. The graph of the reciprocal of a graphed function is drawn — a procedure that requires a little more discussion of asymptotes. The sum and difference of two graphed functions is drawn. And the compositions of a function with the absolute value function are drawn using reflections in the x-axis and y-axis.
- Section 6F–6G introduce inverse relations and functions graphically, with their corresponding reflections in the diagonal line $y = x$. This is yet another type of transformation of a known graph. Section 6G develops the formal notation for inverse functions.
- The last Section 6H introduces parameters so that equations of functions, and relations in general, can be expressed and graphed in terms of functions x and y of a single parameter.

As always, computer sketching of curves is very useful in demonstrating how the features of a graph are related to the algebraic properties of its equation, and in gaining familiarity with the variety of graphs and their transformations.

Some questions in Sections 6C and 6H use the trigonometry of the general angle and Pythagorean identities, and these questions could be delayed until after trigonometry is reviewed and extended in Chapter 7.

The chapter is conceptually demanding, particularly Section 6D. Readers may prefer to leave Section 6D until later in the year — an appropriate place could be before Chapter 16: Further trigonometry, when the sum $a \sin x + b \cos x$ of two trigonometric graphs needs to be sketched.

6A Solving two particular inequations

Learning intentions

- Solve absolute value inequations of the form $|ax + b| < k$ in three ways.
- Solve inequations with x in the denominator by multiplying through by its square.

This section is devoted to two particular types of inequations:

- Absolute value inequations of the form $|ax + b| < k$.
- Inequations where x is in the denominator, such as $\dfrac{5}{x-4} \geq 1$.

The next Section 6B will solve inequations using a table of test points.

A review of inequations solved so far

So far we have solved linear and quadratic inequations:

- We solved linear inequations like linear equations, except that when multiplying or dividing by a nonzero number, the inequality is reversed.
- We solved quadratic inequations using the graph, and also using test points after finding the zeroes.

Remember that we *solve inequations* such as $x + 2 < 3$, and we *prove inequalities* such as $(x-2)^2 \geq 0$, but that the word 'inequality' is often used for both meanings. Do not be surprised if you are asked to 'solve an inequality'.

Method 1 for solving an absolute value inequation — Sketch the graph

Our task is to solve inequations $|ax + b| < k$, where $a \neq 0$, b, and k are constants (and the inequality sign is $<$, or $\leq$, or $>$, or $\geq$).

Drawing a graph is probably the clearest method of solution. Taking as a standard example the inequation $|2x + 1| > 3$:

1 Solving an absolute value inequation such as $|2x + 1| > 3$ graphically

- Solve the *equation* $|2x + 1| = 3$ using the methods of Section 5E, Box 19.
- Hence draw the graphs of $y = |2x + 1|$ and $y = 3$ on the one set of axes.
- Read the solution off the graph.

Example 1 **Solving an absolute value inequation graphically**

Solve $|2x + 1| > 3$ graphically.

Solution

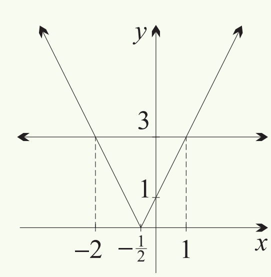
First solve the equation $|2x + 1| = 3$

$$2x + 1 = 3 \quad \text{or} \quad 2x + 1 = -3$$
$$x = 1 \quad \text{or} \quad -2.$$

The graph of $y = |2x + 1|$ is V upwards, with x-intercept $x = -\frac{1}{2}$.

From the sketch of $|2x + 1|$ and $y = 3$ together, the solution of the inequality is

$$x < -2 \quad \text{or} \quad x > 1.$$



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Method 2 for solving an absolute value inequation — Use distance on the number line

We can use distance on the number line to solve the inequality. But we must be careful in dealing with the coefficient of x, particularly when it is negative.

First, we review formulae for absolute value using distance on the number line:

2 Absolute value and distance

- $|x|$ = distance from x to zero on the number line
- $|x - a|$ = distance from x to a on the number line

The easy case, where the coefficient of x is 1

When the coefficient of x is 1, we can immediately use the second formula above.

Example 2 **The coefficient of x is 1**

Solve these inequations on the number line.

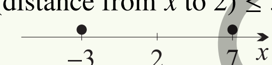
a $|x - 2| \le 5$

b $|x + 3| > 4$

Solution

a $|x - 2| \le 5$

(distance from x to 2) ≤ 5

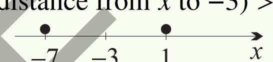


so $-3 \le x \le 7$.

b $|x + 3| > 4$

$|x - (-3)| > 4$

(distance from x to -3) > 4



so $x < -7$ or $x > 1$.

The general case, where the coefficient of x is not 1

There are two initial steps to get it into the form where the coefficient of x is 1.

3 Solving $|ax + b| < k$, for any non-zero value of a

Step 1: Force a to be positive by writing say $|-2x + 3|$ as $|2x - 3|$.
Step 2: Divide through by the now positive number a.
Step 3: Solve using distance, as in the easy case above.

Example 3 The coefficient of x is not 1

Solve these inequations using distance on the number line.

a $|-3x-7|<3$ **b** $|7-\frac{1}{4}x|\geq 3$

Solution

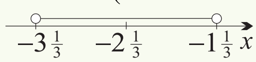
a

$$|-3x-7|<3$$

Rewrite with a positive, $|3x+7|<3$

$\boxed{\div 3}$ $|x-(-2\frac{1}{3})|<1$

(distance from x to $-2\frac{1}{3}$) <1



so $-3\frac{1}{3}<x<-1\frac{1}{3}$.

b

$$|7-\tfrac{1}{4}x|\geq 3$$

Rewrite with a positive, $|\frac{1}{4}x-7|\geq 3$

$\boxed{\times 4}$ $|x-28|\geq 12$

(distance from x to 28) ≥ 12

so $x\leq 16$ or $x\geq 40$.

Method 3 for solving an absolute value inequation — Algebraic approach

We saw in Section 5E how an absolute value equation of the form $|f(x)|=k$ can be solved algebraically by rewriting the equation.

Rewrite an equation $|f(x)|=k$ as $f(x)=k$ or $f(x)=-k$.

A similar approach can be taken to solving an inequation such as $|f(x)|<k$ or $|f(x)|>k$.

4 Solving an absolute value equation or inequation algebraically

- Rewrite an equation $|f(x)|=k$ as $f(x)=k$ or $f(x)=-k$.
- Rewrite an inequation $|f(x)|<k$ as $-k<f(x)<k$.
- Rewrite an inequation $|f(x)|>k$ as $f(x)<-k$ or $f(x)>k$.

Example 4 Solving an absolute value inequation algebraically

a Solve $|9-2x|=5$. **b** Solve $|9-2x|<5$. **c** Solve $|9-2x|\geq 5$.

Solution

a As in Section 5E, the first step in solving the equation $|9-2x|=5$ is

$9-2x=5$ or $9-2x=-5$,

giving two solutions, $x=2$ or $x=7$.

b Using the second dotpoint,

$$|9-2x|<5$$
$$-5<9-2x<5$$

$\boxed{-9}$ $-14<-2x<-4$

$\boxed{\div(-2)}$ $7>x>2$

that is, $2<x<7$.

c Using the third dotpoint,

$$|9-2x|\geq 5$$
$$9-2x\leq -5 \text{ or } 9-2x\geq 5$$

$\boxed{-9}$ $-2x\leq -14$ or $-2x\geq -4$

$\boxed{\div(-2)}$ $x\geq 7$ or $x\leq 2$

that is, $x\leq 2$ or $x\geq 7$.

Inequations that have no solutions, or are inequalities

The absolute value $|f(x)|$ cannot be negative. Thus if k is negative:

- $|f(x)|=k$ and $|f(x)|<k$ have no solutions, and
- $|f(x)|\geq k$ is true for all values of x in the domain of $f(x)$.

Example 5 **Inequations that have no solutions, or are inequalities**

Solve these absolute value inequations algebraically:

a $|x - 7| < -5$

b $|6 - x| \geq -1$

Solution

a The absolute value is never negative, so there are no solutions.

b Absolute value is always at least zero, so every real number is a solution. Thus the inequation is an inequality.

Solving inequations with x in the denominator — multiply through by its square

When we try to solve this inequation, we immediately run into a problem:

$$\frac{5}{x-4} \geq 1.$$

The denominator $x - 4$ is sometimes positive and sometimes negative. Thus if we were to multiply both sides by the denominator $x - 4$, the inequality symbol would reverse sometimes and not other times.

To avoid using cases, the most straightforward approach is to multiply through instead by the *square of the denominator*, which is always positive or zero.

5 Solving an inequation with the variable in the denominator

- *Multiply through by the square of the denominator.*
  - ▷ Be careful to exclude the zeroes of the denominator from the solutions.
  - ▷ Once the fractions have been cleared, there will usually be common factors on both sides. These should *not* be multiplied out, because the factoring will be easier if they are left unexpanded.

An alternative approach using a table of signs is presented in the next section.

Example 6 **Solving an inequation by multiplying through by the square**

Solve $\dfrac{5}{x-4} \geq 1$.

Solution

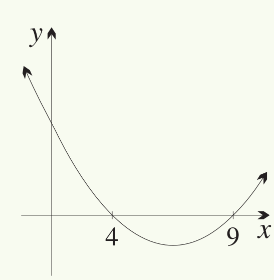
First, $x \neq 4$, because the LHS is undefined when $x = 4$.

The key step is to multiply both sides by $(x - 4)^2$.

$\boxed{\times\ (x-4)^2}$ $\quad 5(x - 4) \geq (x - 4)^2$, and $x \neq 4$,

Don't expand the brackets here! You will only have to re-factor.

$$(x - 4)^2 - 5(x - 4) \leq 0, \text{ and } x \neq 4,$$
$$(x - 4)(x - 4 - 5) \leq 0, \text{ and } x \neq 4,$$
$$(x - 4)(x - 9) \leq 0, \text{ and } x \neq 4.$$



From the diagram, $4 < x \leq 9$.

Notice that $x = 4$ is not a solution of the original inequation because substituting $x = 4$ into the LHS of the original inequation gives $\dfrac{5}{4-4}$, which is undefined.

Exercise 6A

FOUNDATION

1 Solve for x using distance on the number line.

a $|x| = 7$ **b** $|x| = 0$ **c** $|x| \le 2$ **d** $|x| > 5$ **e** $|x| < \frac{1}{4}$ **f** $|x| \ge \frac{3}{2}$

3 Solve:

a $|2x| < 6$ **b** $|5x| \ge -3$ **c** $|3x| \ge 6$
d $|4x| \le 2$ **e** $|12x| < -18$ **f** $|3x| > 5$

3 Multiply both sides by x^2 and hence solve:

a $\frac{1}{x} > 1$ **b** $\frac{3}{x} < 1$ **c** $\frac{1}{x} \ge 2$
d $4 + \frac{3}{x} \ge 0$ **e** $\frac{1}{x} > -1$ **f** $\frac{4}{x} \le -2$

4 Collect like terms where necessary then multiply by x^2 to solve:

a $-\frac{1}{x} > 2$ **b** $-\frac{3}{x} \le 1$ **c** $2 + \frac{1}{x} \ge 1$
d $1 + \frac{2}{x} > 3$ **e** $1 - \frac{1}{x} \le -2$ **f** $3 - \frac{2}{x} < 1$

DEVELOPMENT

5 **a** Using distance on the number line, solve:

i $|x - 1| < 2$ **ii** $|x - 5| \ge 4$ **iii** $|x + 1| > 3$ **iv** $|x + 8| \le 6$

b Next, solve the inequations in parts (a) graphically.

c Finally, solve the inequations in part (a) algebraically.

6 Multiply both sides of each inequation by the square of the denominator and hence solve it. Do not multiply out any common factor.

a $\frac{2}{x+1} \le 1$ **b** $\frac{2}{x-3} > 1$ **c** $\frac{3}{x+4} \ge 2$
d $\frac{5}{2x-3} < 1$ **e** $\frac{2}{3-x} > 1$ **f** $\frac{4}{5-3x} \le -1$

7 In each case, first simplify the inequation, then use multiplication by the square of the denominator to solve for x.

a $1 + \frac{2}{x-3} > -3$ **b** $2 - \frac{1}{x+4} < 3$ **c** $3 + \frac{1}{x-2} \le 2$

8 **a** Solve by any suitable method, and graph the solution on a number line.

i $|x - 2| < 3$ **ii** $|3x - 5| \le 4$ **iii** $|x - 7| \ge 2$
iv $|2x + 1| < 3$ **v** $|6x - 7| > 5$ **vi** $|5x + 4| \ge 6$

b Solve the inequations in parts (ii), (iv), and (vi) by the other two of the three methods that you did *not* use in part (a).

9 First simplify each inequation, then solve for x.

a $|x + 1| + 2 < 3$ **b** $|x - 2| - 1 > 3$ **c** $|x - 3| - 1 \geq 4$

10 Solve for x:

a $\dfrac{x - 1}{x + 2} > -2$ **b** $\dfrac{2 - x}{x - 1} \geq 3$ **c** $\dfrac{x + 1}{x - 1} \leq 2$

d $\dfrac{5x}{2x - 1} \geq 3$ **e** $\dfrac{2x + 5}{x + 3} < 1$ **f** $\dfrac{4x + 7}{x - 2} > -3$

11 **a** Show that the double inequation $2 \leq |x| \leq 6$ has solution $2 \leq x \leq 6$ or $-6 \leq x \leq -2$.

b Similarly solve:

i $2 < |x + 4| < 6$ **ii** $1 \leq |2x - 5| < 4$

12 Say whether each statement is true or false. If it is false, give a counterexample.

a $x^2 > 0$ **b** $x^2 \geq x$ **c** $2^x > 0$ **d** $x \geq \dfrac{1}{x}$

e $2x \geq x$ **f** $x + 2 > x$ **g** $x \geq -x$ **h** $2x - 3 > 2x - 7$

13 Solve these equations and inequations.

a $|4 - 5x| = -2$ **b** $|3x + 2| < -7$ **c** $|5 - x| \geq -6$ **d** $|3x - 5| \leq 0$

CHALLENGE

14 Consider the inequation $\left|x + \dfrac{1}{x}\right| < 2x$.

a Explain why x must be positive. **b** Hence solve the inequation.

15 Solve the inequation $1 + 2x - x^2 \geq \dfrac{2}{x}$.

16 Consider the inequation $|x - a| + |x - b| < c$, where $a < b$.

a If $a \leq x \leq b$, show, using distances on a number line, that there can only be a solution if $b - a < c$.

b If $b < x$, show, using distances on a number line, that $x < \dfrac{a + b + c}{2}$.

c If $x < a$, show, using distances on a number line, that $x > \dfrac{a + b - c}{2}$.

d Hence show that either $\left|x - \dfrac{a + b}{2}\right| < \dfrac{c}{2}$ or there is no solution to the original problem.

e Hence find the solution of $|x + 2| + |x - 6| < 10$.

6B The sign of a function

Learning intentions

- Construct a table of test points dodging around zeroes and discontinuities.
- Use the table of test points to solve various types of inequation.
- Use the table of test points as a first step in curve-sketching.

This section completes the story of a very general method of using test points to analyse the sign of any function, provided only that we can find its zeroes and discontinuities. The pattern of signs produced by these test points can be used to solve inequations, which are the immediate point of this section.

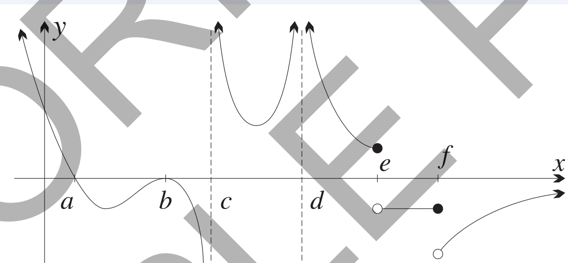
But the pattern also provides the basis for sketching curves of every kind, and will be central particularly to curve sketching using calculus in Year 12.

Where can a function change sign?

Sketching factored cubics in Section 3G, and solving quadratic inequations in Section 4B, used a table of test values dodging around the zeroes to see where the function changed signs. The functions in this chapter, however, may also have breaks or *discontinuities*, and we need to dodge around them as well, as the diagram below demonstrates:

6 Where can a function change sign?

The only places where a function may possibly change sign are zeroes and discontinuities.



- The graph above has discontinuities at $x = c$, $x = d$, $x = e$, and $x = f$, and has zeroes at $x = a$ and $x = b$.
- The function changes sign at the zero $x = a$ and at the discontinuities $x = c$ and $x = e$, and nowhere else.
- Notice that it does not change sign at the zero $x = b$ or at the discontinuities $x = d$ and $x = f$.

The statement in Box 6 goes to the heart of what the real numbers are and what continuity means. In this course, the sketch above is sufficient justification.

A table of signs

As a consequence, we can examine the sign of a function using a table of test values dodging around any zeroes and discontinuities. We add a third row for the sign, so that the table becomes a *table of signs*.

7 Examining the sign of a function

To examine the sign of a function, draw up a a *table of signs* using test values that dodge around all the zeroes and discontinuities.

Finding the zeroes of a function has been a constant concern ever since quadratics were introduced in earlier years. To find discontinuities, we will be using our standard assumption that *all the functions in the course are continuous everywhere in their domains, except where there is an obvious problem.*

Solving polynomial inequations using a table of signs

A simpler form of this table of signs was introduced in Section 3G to sketch cubics. Cubics, and more generally polynomials, do not have any discontinuities, so the test values only need to dodge around their zeroes — if we can find them.

Section 3G's concern was to sketch the function, whereas here the concern is to solve inequations. The graphs are not necessary here, and our sketches will be incomplete, but the sketches allow us to see the whole situation.

Example 7 Sketching a factored polynomial and solving an inequality

a Draw up a table of signs of the function $y = (x - 1)(x - 3)(x - 5)$.

b State where the function is positive and where it is negative.

c Solve the inequation $(x - 1)(x - 3)(x - 5) \le 0$.

d Confirm the answers by sketching what is now known about the graph.

Solution

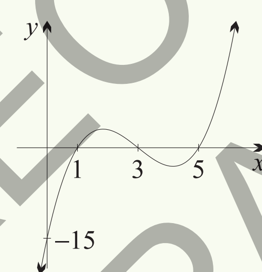
a There are zeroes at 1, 3 and 5, and no discontinuities.

x	0	1	2	3	4	5	6
y	−15	0	3	0	−3	0	15
sign	−	0	+	0	−	0	+

b Hence y is positive for $1 < x < 3$ or $x > 5$, and negative for $x < 1$ or $3 < x < 5$.

c $x \le 1$ or $3 \le x \le 5$.

d



Example 8 Factoring a polynomial, solving an inequality, and sketching the curve

Solve $x^3 + 1 \le x^2 + x$ using a table of signs. Then confirm with a sketch.

Solution

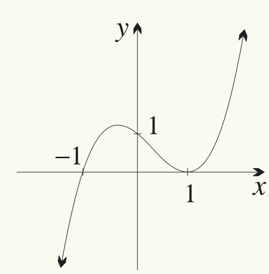
Move all terms to the left, then factor by grouping,

$$x^3 - x^2 - x + 1 \le 0$$
$$x^2(x - 1) - (x - 1) \le 0$$
$$(x^2 - 1)(x - 1) \le 0$$
$$(x + 1)(x - 1)^2 \le 0.$$

The LHS has zeroes at 1 and −1, and no discontinuities.

x	−2	−1	0	1	2
y	−9	0	1	0	3
sign	−	0	+	0	+

Hence $x \le -1$ or $x = 1$.



Solving inequations involving discontinuities

When the function has discontinuities, the method is the same, except that the test values now need to dodge around discontinuities as well as the zeroes.

Again, a sketch is useful for understanding. Because horizontal asymptotes are in Year 12, however, a clue is needed before the graph really tells the story.

Example 9 Examining the sign in a function with discontinuities

a Find the zeroes and discontinuities of $y = \dfrac{x-1}{x-4}$, and examine its sign.

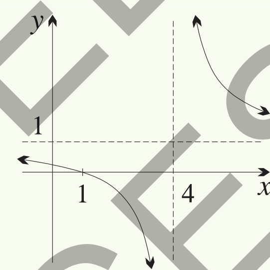
b **i** Write $x - 1 = (x - 4) + 3$, hence find the horizontal asymptote.

ii Identify the vertical discontinuities of the function, and sketch it.

Solution

a There is a zero at $x = 1$, and a discontinuity at $x = 4$.

x	0	1	2	4	5
y	$\frac{1}{4}$	0	$-\frac{1}{2}$	*	4
sign	+	0	−	*	+



Hence y is positive for $x < 1$ or $x > 4$, and negative for $1 < x < 4$.

b **i** $y = \dfrac{(x-4)+3}{x-4}$

$= 1 + \dfrac{3}{x-4}$

Hence $y \to 1$ as $x \to \infty$ and as $x \to -\infty$, so $y = 1$ is a horizontal asymptote in both directions.

ii As $x \to 4^-$, $y \to -\infty$, and as $x \to 4^+$, $y \to \infty$, so $x = 4$ is a vertical asymptote.

Example 10 Examining the sign in a situation that is trivial

Find the zeroes and discontinuities of $y = \dfrac{1}{1+x^2}$, and examine its sign.

Solution

The denominator $1 + x^2$ is always at least 1. Hence the function is defined for all real x, and is always positive.

A table of signs is now unnecessary, but we can draw one up. There are no discontinuities and no zeroes, so one test value $f(0) = 1$ confirms that the function is always positive.

x	0
y	1
sign	+

Comparing the methods of Sections 6A and 6B

We now have two ways to solve an inequation with x in the denominator. For comparison, here is an inequality with x in the denominator solved both ways. First it is solved by multiplying both sides by the square of the denominator. Then it is solved using a table of signs. Compare the two quite different approaches.

Example 11 Constructing a table of test points

Solve $\dfrac{3}{x+2} \le x$ using a table of signs.

Solution

Collecting everything on the left, $\dfrac{3}{x+2} - x \le 0,$

using a common denominator, $\dfrac{3 - x^2 - 2x}{x+2} \le 0,$

and factoring, $\dfrac{(3+x)(1-x)}{x+2} \le 0.$

The LHS has zeroes at $x = -3$ and $x = 1$, and a discontinuity at $x = -2$.

x	-4	-3	$-2\frac{1}{2}$	-2	0	1	2
LHS	$2\frac{1}{2}$	0	$-3\frac{1}{2}$	$*$	$1\frac{1}{2}$	0	$-1\frac{1}{4}$
sign	$+$	0	$-$	$*$	$+$	0	$-$

So the solution is $x \ge 1$ or $-3 \le x < -2$.

(No sketch here — rely just on the table of signs.)

Example 12 Multiplying through by the square of the denominator

Solve $\dfrac{3}{x+2} \le x$ by first multiplying though by the square of the denominator.

Solution

Notice that the common factor $(x + 2)$ is never multiplied out. That would cause a serious problem, because of the effort required to re-factor the expanded cubic.

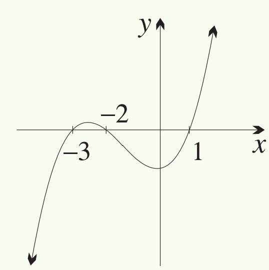
$$\frac{3}{x+2} \le x$$

$\boxed{\times (x+2)^2}$ $\quad 3(x+2) \le x(x+2)^2, \quad$ and $x \ne -2,$

$$x(x+2)^2 - 3(x+2) \ge 0, \quad \text{and } x \ne -2,$$

$$(x+2)(x^2 + 2x - 3) \ge 0, \quad \text{and } x \ne -2,$$

$$(x+2)(x+3)(x-1) \ge 0, \quad \text{and } x \ne -2.$$



The LHS can now be sketched using a table of signs.

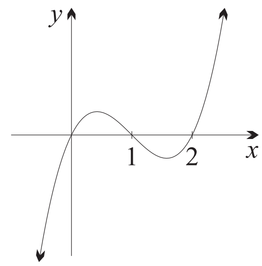
From the graph, $x \ge 1$ or $-3 \le x < -2$.

Exercise 6B FOUNDATION

The purpose of this exercise is to solve inequations using a table of signs. In the case of polynomials, this table also allows the sketch to be drawn, as in Section 3G. A sketch makes the situation clearer, but the sketch is not necessary for obtaining the solution.

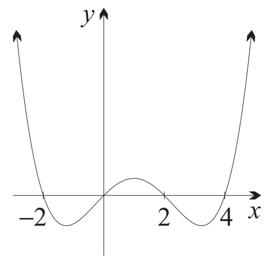
1 Use the given graph of the LHS to help solve each inequation.

a



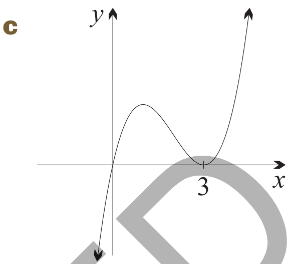
$x(x-1)(x-2) \leq 0$

b



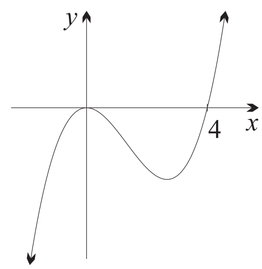
$x(x+2)(x-2)(x-4) < 0$

c



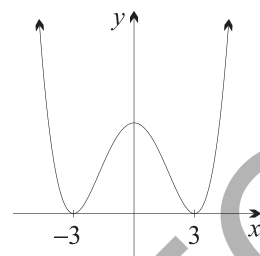
$x(x-3)^2 > 0$

d



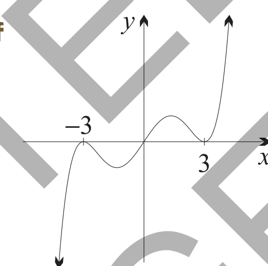
$x^2(x-4) \geq 0$

e



$(x-3)^2(x+3)^2 \leq 0$

f



$x(x-3)^2(x+3)^2 \geq 0$

2 **a** Explain why the zeroes of $y = (x+1)^2(1-x)$ are $x = 1$ and $x = -1$. Then copy and complete the table of signs.

x	-2	-1	0	1	2
y					
sign					

b Use the table of signs to solve $(x+1)^2(1-x) \geq 0$.

c Sketch the graph to confirm the solution in part (b).

3 Use the three steps of the previous question to solve each inequation.

a $(x+1)(x+3) < 0$ **b** $x(x-2)(x-4) \geq 0$ **c** $(x-1)(x+2)^2 \geq 0$

d $x(x-2)(x+2) \leq 0$ **e** $(x-2)x(x+2)(x+4) > 0$ **f** $(x-1)^2(x-3)^2 \leq 0$

4 First factor each polynomial completely, then use the methods of Questions 2 and 3 to sketch its graph. (Hint: take out any common factors first.)

a $f(x) = x^3 - 4x$ **b** $f(x) = x^3 - 5x^2$ **c** $f(x) = x^3 - 4x^2 + 4x$

5 From the graphs in the previous question, or from the tables of signs used to construct them, solve the following inequations. Begin by getting all terms onto the one side.

a $x^3 > 4x$ **b** $x^3 < 5x^2$ **c** $x^3 + 4x \leq 4x^2$

DEVELOPMENT

6 Here is Question 10 from Exercise 6A. This time, collect all terms on the LHS as a simplified single fraction. Find the zeroes and discontinuities to use in a table of signs, and hence solve each inequation.

a $\dfrac{x-1}{x+2} > -2$ **b** $\dfrac{2-x}{x-1} \geq 3$ **c** $\dfrac{x+1}{x-1} \leq 2$

d $\dfrac{5x}{2x-1} \geq 3$ **e** $\dfrac{2x+5}{x+3} < 1$ **f** $\dfrac{4x+7}{x-2} > -3$

7 **a** Find the zeroes and discontinuities of $y = \dfrac{x^2}{x-3}$ and construct a table of signs.

b Hence solve $\dfrac{x^2}{x-3} < 0$.

8 If necessary, collect all terms on the LHS and factor. Find any zeroes and discontinuities, then draw up a table of signs with interlacing values in order to solve the inequation.

a $(x-1)(x-3)(x-5) < 0$ **b** $(x-1)^2(x-3)^2 > 0.$ **c** $\dfrac{x-4}{x+2} \le 0$

d $x^3 > 9x$ **e** $\dfrac{x+3}{x+1} < 0$ **f** $\dfrac{x^2}{x-5} < 0$

g $x^4 \ge 5x^3$ **h** $\dfrac{x^2-4}{x} \ge 0$ **i** $\dfrac{x-2}{x^2+3x} \le 0$

9 **a** Factor each equation completely, and hence find the x-intercepts of the graph. Factor parts (ii) and (iii) by grouping in pairs.

i $y = x^3 - x$ **ii** $y = x^3 - 2x^2 - x + 2$ **iii** $y = x^3 + 2x^2 - 4x - 8$

b For each function in the previous question, examine the sign of the function around each zero, and hence draw a graph of the function.

10 Find all zeroes of these functions, and any values of x where the function is discontinuous. Then analyse the sign of the function by taking test points around these zeroes and discontinuities.

a $f(x) = \dfrac{x}{x-3}$ **b** $f(x) = \dfrac{x-4}{x+2}$ **c** $f(x) = \dfrac{x+3}{x+1}$

11 Multiply through by the square of the denominator, collect all terms on one side and then factor to obtain a factored cubic. Sketch this cubic by examining the intercepts and the sign. Hence solve the original inequation.

a $\dfrac{4}{x+3} \ge x$ **b** $\dfrac{2}{2x+3} < x$ **c** $\dfrac{8}{2x-3} \le 2x-1$

12 Solve the inequations in the previous question by the alternative method of collecting everything on the LHS, finding a common denominator, identifying zeroes and discontinuities, and drawing up a table of signs.

CHALLENGE

13 **a** Prove that $f(x) = 1 + x + x^2$ is positive for all x.

b Prove that $f(x) = 1 + x + x^2 + x^3 + x^4$ is positive for all x. Consider separately the three cases $x \ge 0$, $-1 < x < 0$ and $x \le -1$. Group the five terms into pairs in different ways with the second and third cases.

c Use similar methods to prove that for all integers $n \ge 0$,

$$f(x) = 1 + x + x^2 + \cdots + x^{2n-1} + x^{2n} \text{ is positive for all } x.$$

d Prove that $x = -1$ is the only zero of $f(x) = 1 + x + x^2 + \cdots + x^{2n-1}$, for all positive integers n.

14 Let $f(x) = \dfrac{a}{2}\left(\dfrac{1}{x+a} - \dfrac{1}{x-a}\right)$ with $a > 1$.

a For what values of k does $f(x) = k$ have a solution?

b Solve $f(x) < 2$ using any appropriate method.

c Now choose a suitable value for a and use shifting to solve $\dfrac{1}{x-1} - \dfrac{1}{x-7} < \dfrac{4}{3}$.

6C Sketching reciprocal functions

Learning intentions

- Sketch the reciprocal of a function given the graph of the function.
- Determine its vertical and horizontal asymptotes, and its domain and range.

Section 6C–6E present further ways to transform or combine graphs. In each case, the emphasis will be more on the relationship between the graphs than on the equations of the various functions.

This section sketches the reciprocal function $g(x) = \dfrac{1}{f(x)}$ of a graphed function $f(x)$. Reading from the graph of $y = f(x)$, we develop properties of this reciprocal function $y = g(x)$, and use these properties to sketch $y = g(x)$.

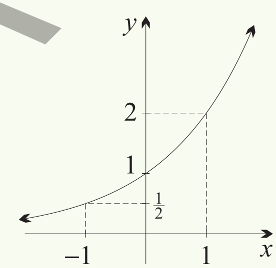
The reciprocal trig functions — cosec, sec, and cot — are developed in Chapter 7.

The sign and horizontal asymptotes

There are many small details involved in sketching reciprocals, so it is better to give several examples, then follow them with some general statements.

Example 13 **Analysing sign and horizontal asymptotes**

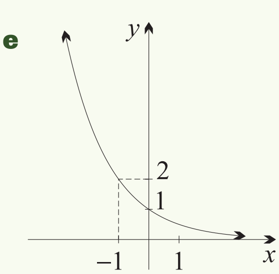
The sketch shows $y = 2^x$. Let $g(x) = 1/f(x)$ be the reciprocal function of $f(x)$. Reasoning from the sketch:



a What sign do the values of $y = g(x)$ have?
b What is the y-intercept of $y = g(x)$?
c What happens to $y = g(x)$ as $x \to \infty$?
d What happens to $y = g(x)$ as $x \to -\infty$?
e Hence sketch $y = g(x)$.
f What are the domain and range of $f(x)$ and $g(x)$?

Solution

a $f(x)$ is always positive, so its reciprocal is always positive.
b $f(0) = 1$, and the reciprocal of 1 is 1, so $g(0) = 1$.
c As $x \to +\infty$, $f(x) \to \infty$, so $g(x) \to 0^+$, so the x-axis is a horizontal asymptote on the right.
d As $x \to -\infty$, $f(x) \to 0^+$, so $g(x) \to +\infty$.
e



f They both have domain all real x, and range $y > 0$.

Note: $g(x) = 2^{-x}$ is, of course, the reflection of $y = f(x)$ in the y-axis. But the intention here is to argue using reciprocals.

8 Graphing the reciprocal — sign and horizontal asymptotes

Let $f(x)$ be a graphed function, and let $g(x) = 1/f(x)$ be its reciprocal function.

- When $f(x)$ is small and positive, $g(x)$ is large and positive.
  - ▷ When $f(x)$ is large and positive, $g(x)$ is small and positive.
- When $f(x) = 1$, $g(x) = 1$, and vice versa.
- If $f(x) \to +\infty$ as $x \to \infty$, then $g(x) \to 0^+$ as $x \to \infty$,
  - ▷ and the x-axis is a horizontal asymptote on the right.
- If $f(x) \to 0^+$ as $x \to \infty$, then $g(x) \to +\infty$ as $x \to \infty$.

The last two dotpoints have equivalent behaviour on the left.
All four dotpoints have equivalent behaviour when $f(x)$ is negative.

Zeroes and vertical asymptotes

Again, an initial example should makes things clear.

Example 14 Analysing zeroes and vertical asymptotes.

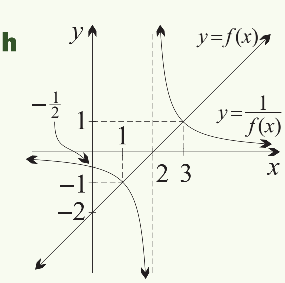
Sketch $f(x) = x - 2$, then with $g(x) = 1/f(x)$, and reasoning from the sketch:

a Where is $g(x)$ undefined?
b Does $y = g(x)$ have any zeroes?
c Where is $y = g(x)$ above and below the x-axis?
d Identify any vertical asymptotes.
e Identify any horizontal asymptotes.
f What is the y-intercept of $y = g(x)$?
g Where does $y = g(x)$ intersect with $y = f(x)$?
h Hence sketch $y = g(x)$ on the same set of axes.
i What are the domain and range of $f(x)$ and $g(x)$?

Solution

a Zero has no reciprocal, so $g(x)$ is undefined at the zero $x = 2$ of $f(x)$.
b The number zero is not the reciprocal of any number, so $g(x)$ is never zero.
c $g(x)$ is positive when $f(x)$ is positive, and negative when $f(x)$ is negative.
d As $x \to 2^+$, $f(x) \to 0^+$, so $g(x) \to +\infty$.
As $x \to 2^-$, $f(x) \to 0^-$, so $g(x) \to -\infty$.
Hence $x = 2$ is a vertical asymptote.
e As $x \to +\infty$, $f(x) \to \infty$, so $g(x) \to 0^+$.
As $x \to -\infty$, $f(x) \to -\infty$, so $g(x) \to 0^-$.
Hence the x-axis is a horizontal asymptote left and right.
f When $x = 0$, $f(0) = -2$, so $g(-2) = -\frac{1}{2}$.
g The only numbers that are their own reciprocal are 1 and -1.
Hence $y = f(x)$ and $y = g(x)$ intersect at $(1, -1)$ and $(3, 1)$.

h



i Domain: For $f(x)$, all real x. For $g(x)$, $x \neq 2$.
Range: For $f(x)$, all real y. For $g(x)$, $y \neq 0$.

9 Graphing the reciprocal — zeroes and vertical asymptotes

Let $f(x)$ be a graphed function, and let $g(x) = 1/f(x)$ be its reciprocal function.

- Zero is the only number that has no reciprocal.
  - ▹ Hence $g(x)$ is undefined at every zero of $f(x)$.
- Zero is the only number that is not a reciprocal of any number.
  - ▹ Hence $g(x)$ is never zero.
- When $f(x)$ has a zero, $y = g(x)$ normally has a vertical asymptote (but check, because there are some rather bizarre exceptions).

Domain and range

These deserve particular attention.

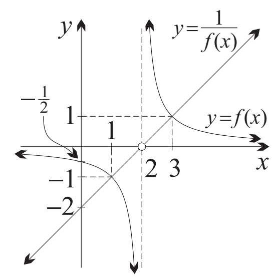
10 Graphing the reciprocal — domain and range

Let $f(x)$ be a graphed function, and let $g(x) = 1/f(x)$ be its reciprocal function.

- The domain of $g(x)$ is the domain of $f(x)$ with all the zeroes of $f(x)$ removed.
- $g(x)$ is never zero, so the range of $g(x)$ does not contain zero.

Two warnings about zero, infinity, and reciprocals

- **Infinity is not a number:** Don't ever be tempted to say that if $f(x)$ has an asymptote at $x = a$, then the function $g(x) = 1/f(x)$ is zero at $x = a$.
  The symbols ∞ and $-\infty$ are part of mathematics, but they are not numbers, and they certainly do not have reciprocals.
- **The reciprocal of the reciprocal may not be the original function:**
  Referring to our previous example, start instead with reciprocal function, that is,
  let $\quad f(x) = \dfrac{1}{x-2}$. Then $f(x)$ is undefined at $x = 2$, so its reciprocal $g(x) = 1/f(x)$ is also undefined at $x = 2$.
  That is, $\quad g(x) = x - 2, \quad$ where $x \neq 2$.
  This is not the original line $y = x - 2$ of the example, because its zero $x = 2$ has now been removed.



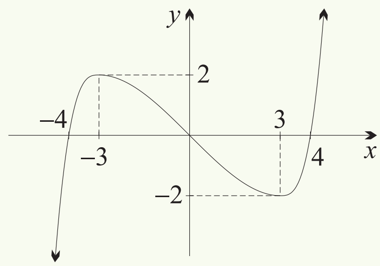
Sketching the reciprocal from the graph, with no equation

In the previous examples, the equation of the function was there for reference. But in the two examples below, only the sketch is given.

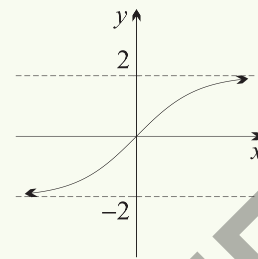
Example 15 **Sketching the reciprocal from a graph**

Given each sketch of $y = f(x)$ below, sketch $g(x) = 1/f(x)$, then state the domain and range of $g(x)$.

a

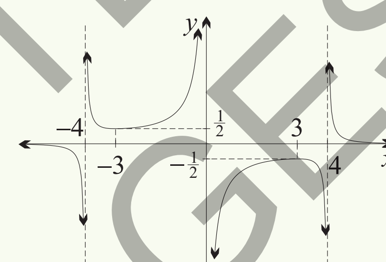


b

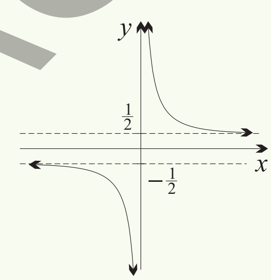


Solution

a The point $(-3, 2)$ is called a *local maximum* because for a small region around $x = -3$, $f(x)$ is greatest at $x = -3$. Similarly, $(3, -2)$ is called a *local minimum*.
Corresponding, $g(x)$ has a local minimum at $(-3, \frac{1}{2})$ and a local maximum at $(3, -\frac{1}{2})$.
The domain of $g(x)$ is $x \neq -4$, 0 or 4 (remove the zeroes), and the range of $g(x)$ is $y \neq 0$.



b As $x \to \infty$, $f(x) \to 2$, so $g(x) \to \frac{1}{2}$, so $y = \frac{1}{2}$ is a horizontal asymptote on the right.
Similarly, as $x \to -\infty$, $f(x) \to -2$, so $g(x) \to -\frac{1}{2}$. Thus $y = -\frac{1}{2}$ is a horizontal asymptote on the left.
Domain of $g(x)$: $x \neq 0$. Range: $y < -\frac{1}{2}$ or $y > \frac{1}{2}$.



Local maxima and minima, and horizontal asymptotes

Part (a) introduced the idea of *local maxima and minima*, and their relationship with the reciprocal function.

Part (b) showed how to deal with horizontal asymptotes other than the x-axis.

Exercise 6C

FOUNDATION

Note: Question 9 and 13 concern the graphs of the three trigonometric functions $\sin\theta$, $\cos\theta$, and $\tan\theta$. The graphs are drawn in the questions, but readers unfamiliar with them may like to delay attempting these questions until after studying Chapter 7.

1 **a** Sketch $y = \dfrac{1}{x - 1}$ after carrying out the following steps.

i State the natural domain, and find the y-intercept.

ii Find any points where $y = 1$ or $y = -1$.

iii Explain why $y = 0$ is a horizontal asymptote.

iv Draw up a table of values and examine the sign.

v Identify any vertical asymptotes, and use the table of signs to write down its behaviour near any vertical asymptotes.

b Repeat for $y = \dfrac{2}{3 - x}$. **c** Repeat for $y = -\dfrac{2}{x + 2}$. **d** Repeat for $y = \dfrac{5}{2x + 5}$.

2 Follow steps (i)–(v) of Question 1(a) to investigate the function $y = \dfrac{2}{(x-1)^2}$ and hence sketch its graph. Show any points where $y = 1$.

3 Investigate the domain, zeroes, sign and asymptotes of the function $y = -\dfrac{1}{(x-2)^2}$ and hence sketch its graph. Show any points where $y = -1$.

4 **a** Let $f(x) = x + 1$.

i Graph $y = f(x)$ showing the intercepts with the axes.

ii Also show the points where $f(x) = 1$ and $f(x) = -1$.

iii Hence on the same number plane sketch $y = \dfrac{1}{f(x)}$.

b Follow similar steps for the function $g(x) = 2 - x$.

DEVELOPMENT

5 Let $y = f(x)$ where $f(x) = \frac{1}{3}(x+1)(x-3)$.

a Show that $y = 1$ at $x = 1 - \sqrt{7}$ and $x = 1 + \sqrt{7}$. Plot these points.

b Complete the graph of $y = f(x)$ showing the vertex, the intercepts with the axes and the points where $f(x) = -1$.

c What is the range of $y = f(x)$?

d Hence sketch $y = \dfrac{1}{f(x)}$ on the same number plane .

e What is the range of $y = \dfrac{1}{f(x)}$?

6 **a** Follow similar steps to question 5 to sketch $y = \frac{1}{4}(4 - x^2)$ and $y = \dfrac{4}{4 - x^2}$.

b What is the range of $y = \frac{1}{4}(4 - x^2)$?

c What is the range of $y = \dfrac{4}{4 - x^2}$?

7 **a** Sketch $y = \frac{1}{2}(x^2 + 1)$ showing the points where $y = 1$.

b What is the minimum value of $\frac{1}{2}(x^2 + 1)$?

c Sketch $y = \dfrac{2}{x^2 + 1}$ on the same number plane.

d Explain why the x-axis is an asymptote to $y = \dfrac{2}{x^2 + 1}$.

e What is the maximum value of $\dfrac{2}{x^2 + 1}$?

8 Let $f(x) = -x^2 + 2x - 3$.

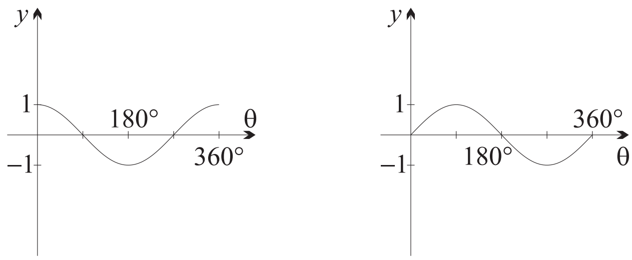
a What is the maximum of $f(x)$?

b Sketch $y = f(x)$ and $y = \dfrac{1}{f(x)}$ on the same number plane.

c Explain why the x-axis is an asymptote for $y = \dfrac{1}{f(x)}$.

d What is the minimum of $\dfrac{1}{f(x)}$?

9 The diagrams show first $y = \cos\theta$ then $y = \sin\theta$ for $0° \le \theta \le 360°$.



a **i** Copy the sketch of $y = \cos\theta$, and add to it the sketch of $y = \dfrac{1}{\cos\theta}$.

ii What are the domain and range of $y = \dfrac{1}{\cos\theta}$ in this interval ?

b **i** Copy the sketch of $y = \sin\theta$, and add to it the sketch of $y = \dfrac{1}{\sin\theta}$.

ii What are the domain and range of $y = \dfrac{1}{\sin\theta}$ in this interval ?

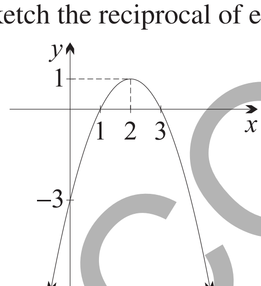
10 Prove that the symmetry of a function is preserved when taking reciprocals. That is, prove that the reciprocal of an even function is an even function, and prove that the reciprocal of an odd function is an odd function.

11 **a** Graph $y = \dfrac{2 + x}{x}$ by first noting that $y = 1 + \dfrac{2}{x}$.

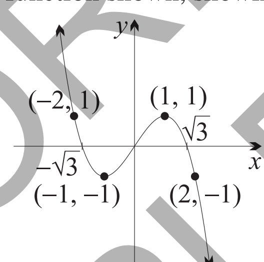
b Hence graph $y = \dfrac{x}{2 + x}$.

12 Sketch the reciprocal of each function shown, showing all significant features.

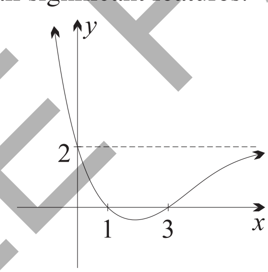
a



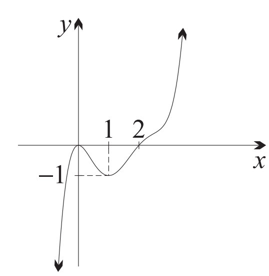
b



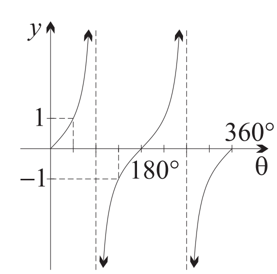
c



d



13 The sketch shows $y = \tan\theta$ for $0° \le \theta \le 360°$.



a What is the domain of $y = \tan\theta$ in this interval?

b Where is $\tan\theta = 0$ in this interval?

c Hence state the domain of $y = \dfrac{1}{\tan\theta}$ in this interval.

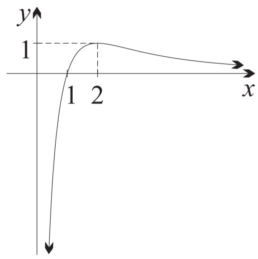
d What is $\lim\limits_{\theta \to 90°} \dfrac{1}{\tan\theta}$?

e Copy the graph of $y = \tan\theta$, and add to it the graph of $y = \dfrac{1}{\tan\theta}$.

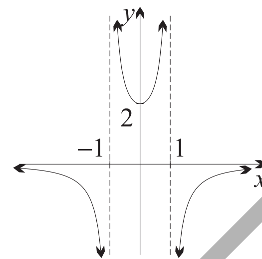
f What is the range of the reciprocal function?

14 In each case the graph of $y = f(x)$ is given. Sketch the graph of $y = \dfrac{1}{f(x)}$, paying careful attention to the domain, any asymptotes, and any relevant limits.

a



b



CHALLENGE

15 The arguments in the solution to Example 15(a) seems to rely on the following assertion about the graphs of a function $y = f(x)$ and its reciprocal function $y = (f(x))^{-1}$:

'When one curve has a local maximum, the other has a local minimum.'

This statement is not strictly true. State the qualification that needs to be made in this statement, and give an example where the qualification is necessary.

16 Let $y = \dfrac{1}{x - 2}$. Write down precisely the equation of the reciprocal function.

6D Sketching sums and differences

Learning intentions

- Sketch the sum and difference of two graphed functions.

The problem addressed in this section is to take the sketches of two functions $f(x)$ and $g(x)$, and working just from those sketches, sketch their sum and difference:

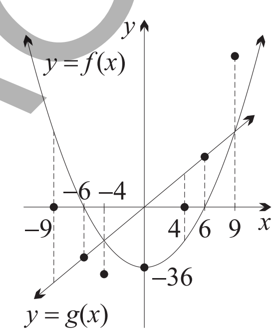
$$s(x) = f(x) + g(x) \qquad \text{and} \qquad d(x) = f(x) - g(x).$$

The pronumerals $s(x)$ and $d(x)$ used here are not any mathematical convention, they are just convenient notation in this section.

The word 'ordinate'

The *ordinate* of a point is the y-coordinate, or more generally, the coordinate on the vertical axis. The operations in this section and the next routinely act on the y-coordinates of points, so this shorter word is useful. The coordinate on the horizontal axis is sometimes referred to as the 'abcissa', plural 'abcissae', but that word is not necessary in the course.

Sketching the sum of two sketched functions

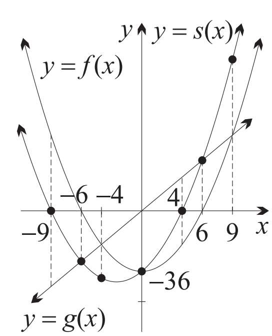


The graphs of two functions $f(x)$ and $g(x)$ are sketched to the right, and we want to sketch the sum $s(x) = f(x) + g(x)$. The equations of the functions will usually not be given, but it is convenient to state them here while developing the method. They are:

$$f(x) = x^2 - 36 \qquad \text{and} \qquad g(x) = 5x,$$

and here is a tables of values:

x	−9	−6	−4	−1	0	1	4	6	9
$f(x)$	45	0	−20	−35	−36	−35	−20	0	45
$g(x)$	−45	−30	−20	−5	0	5	20	30	45
$s(x)$	0	−30	−40	−40	−36	−30	0	30	90



Each ordinate of $s(x)$ is the sum of the ordinates of $f(x)$ and $g(x)$. This is the key idea that leads to everything else. The second sketch adds $y = s(x)$ to the other two graphs.

Now let us consider how we could have sketched $s(x)$ if we had not had the equations.

0 Add the ordinates where possible — the key idea. This has been done in the top diagram.

1 If one curve, say $f(x)$, has a zero at $x = a$, then

$$s(a) = 0 + g(a) = g(a),$$

so $s(x)$ has the same ordinate as $g(x)$ at $x = a$, and the curves meet there.
This happens at both the zeroes $x = 6$ and $x = -6$ of $f(x) = x^2 - 36$, where $s(x)$ meets $g(x)$. It also happens at the zero $x = 0$ of $g(x) = 5x$, where $s(x)$ meets $f(x)$.

2 If the ordinates are opposites at $x = a$, then

$$s(a) = f(a) + g(a) = 0,$$

so $s(x)$ has a zero at $x = a$. This happens at $x = -9$ and at $x = 4$.

3 If the two curves meet at $x = a$, so that their ordinates are equal, then

$$s(a) = f(a) + g(a) = 2f(a) \qquad (\text{or } 2g(a))$$

and the ordinate of $s(x)$ is twice the ordinate of $f(x)$ (or of $g(x)$). This happens at $x = -4$ and at $x = 9$.

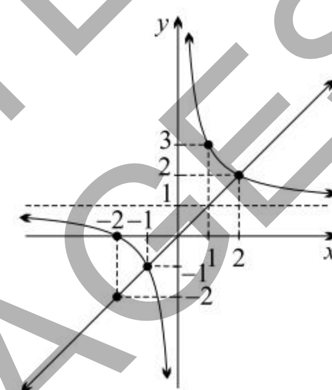
We cannot find from the graphs alone the precise position of the minimum, so in the absence of the equations, this is not required in the sketch. But once we know the equations, the minimum is the vertex of the parabola

$$s(x) = (x^2 - 36) + 5x = x^2 + 5x - 36 = (x + 9)(x - 4),$$

where the average of the zeroes is $x = \frac{1}{2}(4 - 9) = -2\frac{1}{2}$, and $s(-2\frac{1}{2}) = -42\frac{1}{4}$.

An example of adding graphs with asymptotes

Here is an example in which one curve has a horizontal and a vertical asymptote. A succession of steps allow us to sketch the sum $s(x) = f(x) + g(x)$.



1 The curves intersect at $(2, 2)$ and at $(-1, -1)$, so

$$s(2) = f(2) + g(2) = 2 + 2 = 4,$$
$$s(-1) = f(-1) + g(-1) = -1 + (-1) = -2.$$

2 The curve $y = f(x)$ has a zero at $x = -2$, so

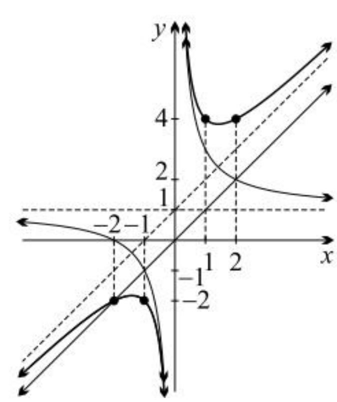
$$s(-2) = f(-2) + g(-2) = 0 + (-2) = -2.$$

3 Add the ordinates at $x = 1$, so

$$s(1) = f(1) + g(1) = 3 + 1 = 4.$$

4 There are no values of x where ordinates are opposites, so $s(x)$ has no zeroes.

5 Because there are no zeroes for $s(x)$, it can only change sign at the asymptote $x = 0$ (next step). Hence $s(x)$ is negative for $x < 0$, and positive for $x > 0$.



Dealing with the horizontal and vertical asymptotes of $f(x)$:

6 Vertically, on both sides of the y-axis:
As $x \to 0^+$, $f(x) \to +\infty$ and $g(x) \to 0$, so $s(x) \to +\infty$.
As $x \to 0^-$, $f(x) \to -\infty$ and $g(x) \to 0$, so $s(x) \to -\infty$.
Thus the y-axis is a vertical asymptote to $y = s(x)$.

7 Horizontally, on the right and the left:
As $x \to \infty$, $f(x) \to 1$ and $g(x) \to +\infty$, so $s(x) \to +\infty$.
As $x \to -\infty$, $f(x) \to 1$ and $g(x) \to -\infty$, so $s(x) \to -\infty$.

Note: There is a final detail that seems beyond the course, but has been added to the diagram for completeness. You will see that a third asymptote, an oblique asymptote, has been drawn. Here is the argument for it. For large values of x, positive or negative, $f(x)$ is almost 1, so $y = s(x)$ is almost the same graph as $y = 1 + g(x)$.
Hence $y = s(x)$ eventually looks like a line parallel to $y = g(x)$.

Having now argued from the graphs alone, we will now reveal the equations of the two functions — they are $f(x) = \dfrac{2}{x} + 1$ and (obviously) $g(x) = x$. Their sum is

$$s(x) = x + 1 + \frac{2}{x} = \frac{x^2 + x + 2}{x}.$$

This has no zeroes because the discriminant $\Delta = 1 - 8 = -7$ is negative, and clearly the graph has a vertical asymptote at $x = 0$.

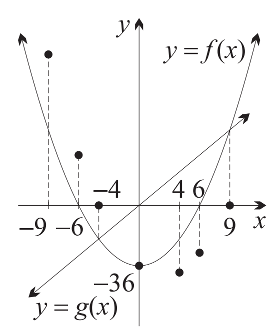
Summary of sketching the sum of two sketched functions

11 Sketching the sum of two sketched functions

- To sketch the sum $s(x) = f(x) + g(x)$ of two sketched functions:
  0 Add the ordinates wherever possible. This is the key idea.
- Some systematic approaches:
  1 If one curve has a zero, then $s(x)$ meets the other curve at that value of x.
  2 If the curves meet, then the ordinate of $s(x)$ is double the ordinate of $f(x)$.
  3 If the ordinates of $f(x)$ and $g(x)$ are opposites, then $s(x)$ has a zero.
  4 The sign of $s(x)$ everywhere will usually be clear now.
- To clarify any vertical or horizontal asymptotic behaviour:
  5 If $f(x) \to \infty$ or $g(x) \to \infty$, then so also does $s(x)$.
  If $f(x) \to -\infty$ or $g(x) \to -\infty$, then so also does $s(x)$.
  If, however, $f(x)$ and $g(x)$ go in opposite directions, we may not be able determine what happens with the sum.

As discussed in Section 5A, a translation of $y = f(x)$ up 5 is $y = f(x) + 5$. This is $f(x) + g(x)$ where $g(x) = 5$ is a constant function, so translations up and down are special cases of the construction in this section.

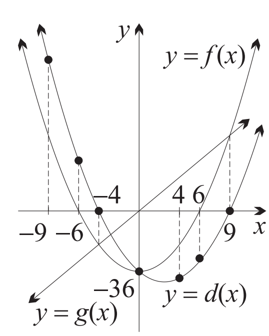
Sketching the difference of two sketched functions



Now let us sketch the difference $d(x) = f(x) - g(x)$ of the same two functions

$$f(x) = x^2 - 36 \qquad \text{and} \qquad g(x) = 5x,$$

but this time we will work from the sketches alone and then confirm the sketch using a table of values. The difference is the sum of $f(x)$ and $-g(x)$, so it could be done by reflecting $g(x)$ in the x-axis and then adding the graphs, but it is easier to sketch it in one step.



0 Subtract the ordinates where possible — the key idea.

1 If $f(x)$ has a zero at $x = a$, then $d(a) = 0 - g(a) = -g(a)$, so the ordinate of $d(x)$ is the opposite of the ordinate of $f(x)$. This happens at $x = -6$ and at $x = 6$.
If $g(x)$ has a zero at $x = a$, then $d(a) = f(a) - 0 = f(a)$, so the ordinate of $d(x)$ is the same as the ordinate of $f(x)$, and the curve $d(x)$ meets the curve $f(x)$. This happens at $x = 0$.

2 If the ordinates of $f(x)$ and $g(x)$ are opposites, then the ordinate of $d(x)$ is double the ordinate of $f(x)$. This happens at $x = -9$ and at $x = 4$.

3 If the two curves meet at $x = a$, then they have equal ordinates there, so $d(x)$ has a zero at $x = a$. This happens at $x = -4$ and at $x = 9$.

Here is a table of values to confirm these arguments:

x	-9	-6	-4	-1	0	1	4	6	9
$f(x)$	45	0	-20	-35	-36	-35	-20	0	45
$g(x)$	-45	-30	-20	-5	0	5	20	30	45
$d(x)$	90	30	0	-30	-36	-40	-40	-30	0

Again, we cannot find from the graphs alone the precise location of the minimum, so it is not needed in the sketch. It is the vertex of the parabola $y = d(x)$. Readers should complete the square for $d(x)$ and show that the vertex is $(2\frac{1}{2}, -42\frac{1}{4})$.

Subtracting graphs with asymptotes

The procedures here are very similar to the previous example of adding graphs with asymptotes, so there is no need for another example — and keep in mind that subtracting graphs means adding the opposite of the second graph.

The structured Question 10 in Exercise 4D below presents the steps in subtracting graphs, and the Enrichment Question 15 deals with the oblique asymptotes.

12 Sketching the difference of two sketched functions

- To sketch the difference $d(x) = f(x) - g(x)$ of two sketched functions.
  0 Subtract the ordinates wherever possible. This is the key idea.
- Some systematic approaches:
  1 If $f(x)$ has a zero at $x = a$, then $d(a) = -g(a)$.
  If $g(x)$ has a zero at $x = a$, then $d(a) = f(a)$.
  2 If the curves meet at $x = a$, then $d(a)$ has a zero there.
  3 If the ordinates are opposites at $x = a$, then $d(a) = 2f(a) = -2g(a)$.
  4 The sign of $d(x)$ everywhere will usually be clear now.
- To clarify any vertical or horizontal asymptotic behaviour:
  5 If $f(x) \to \infty$ or $g(x) \to -\infty$, then $d(x) \to \infty$.
  If $f(x) \to -\infty$ or $g(x) \to \infty$, then $d(x) \to -\infty$.
  If, however, $f(x)$ and $g(x)$ go in the same direction, we may not be able determine what happens with the difference.

Domain and range of the sum and difference

These need attention, but there is no real difficulty involved.

13 Domain and range of the sum and difference

Let $s(x)$ and $d(x)$ be the sum and difference of two functions $f(x)$ and $g(x)$.

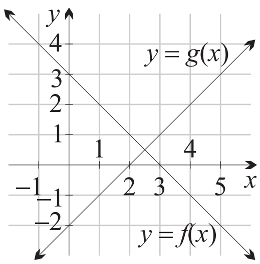
- $s(x)$ and $d(x)$ have domain the intersection of the domains of $f(x)$ and $g(x)$.
- Sort out the ranges of $s(x)$ and $g(x)$ from the graphs, or the equations.

Exercise 6D

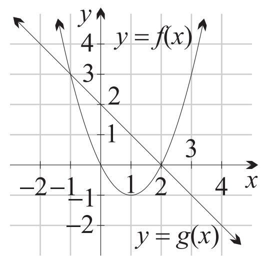
FOUNDATION

1 Each diagram below shows the graph of two functions, $y = f(x)$ and $y = g(x)$. Copy each diagram to your book and draw the graph of $y = f(x) + g(x)$, by adding ordinates. Try to distinguish the original graphs from the graph of the sum — use different colours, or dot the original graphs.

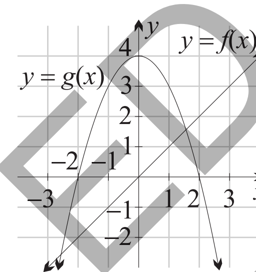
a



b

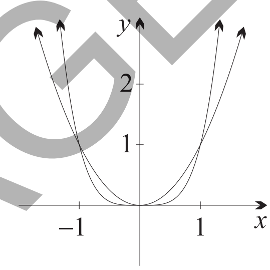


c



2 Copy each diagram in Question 1 to a fresh number plane. Subtract ordinates to sketch the graphs of $y = f(x) - g(x)$.

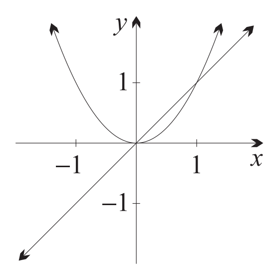
3 The diagram to the right shows the graphs of $y = f(x)$, where $f(x) = x^4$, and of $y = g(x)$, where $g(x) = x^2$.



a Copy the diagram to your book.

b On the same set of axes and in a different colour, sketch $y = f(x) - g(x)$ by subtracting ordinates. Pay careful attention to points where the graphs cross, and to the zeroes of $f(x)$ and $g(x)$.

4 The diagram to the right shows the graphs of $y = f(x)$, where $f(x) = x^2$, and of $y = g(x)$, where $g(x) = x$.



a Copy the diagram to your book.

b On the same set of axes and in a different colour, sketch $y = f(x) + g(x)$ by adding ordinates. Pay careful attention to points where $g(x) = -f(x)$, because at those points $f(x) + g(x) = 0$. Notice also the zeroes of $f(x)$ because $f(x) + g(x) = f(x)$ at those points, and the zeroes of $g(x)$, because $f(x) + g(x) = g(x)$ at those points.

DEVELOPMENT

5 **a** Plot $y = x^3$ and $y = x$ on the same number plane, noting any points of intersection.

b Hence sketch the graph of the difference, $y = x^3 - x$.

6 Sketch $y = x^4$ and $y = x(2 - x)$, then sketch the difference $y = x^4 - x(2 - x)$.

7 When sketching the sums of absolute value graphs in this question, it is helpful to remember that the sum of two linear functions is itself a linear function. Thus the following graphs will be made up of straight-line sections.

a Graph $f(x) = |x + 1|$ and $g(x) = |x - 1|$, then graph:

i $y = f(x) + g(x)$ **ii** $y = f(x) - g(x)$

b Graph $f(x) = |2x|$ and $g(x) = |x - 1|$, then graph:

i $y = f(x) + g(x)$ **ii** $y = f(x) - g(x)$

8 Graphs the functions $y = x^2$ and $y = |x|$ on the same set of axes.

a **i** Explain why the graph of $y = x^2 + |x|$ must lie above both original graphs.

ii Hence sketch the sum $y = x^2 + |x|$. Take care with the shape at $x = 0$.

b **i** Explain why the graph of $y = x^2 - |x|$ is on or below the x-axis for $-1 \le x \le 1$.

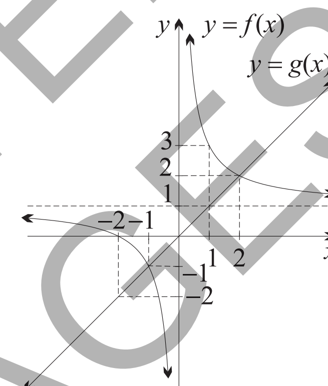
ii Hence sketch the difference $y = x^2 - |x|$. Take care with the shape at $x = 0$.

9 **a** Sketch $y = \sqrt{x}$ and $y = x$, paying careful attention to the domain and to the points where these graphs intersect.

b Hence sketch $y = x - \sqrt{x}$.

10 The graph on the right is a repeat of the one used in the theory to investigate the sum of functions that involves an asymptote. In this question you will investigate their difference

$$d(x) = f(x) - g(x).$$



a The given curves intersect at $x = -1$ and $x = 2$. What feature of the graph of $y = d(x)$ occurs at these values?

b Write down the zero of $f(x)$ and hence evaluate $d(x)$ there.

c Evaluate $d(1)$ by using the ordinates of $f(x)$ and $g(x)$.

d Copy and complete each statement relating to the vertical asymptote of $y = f(x)$.

i As $x \to 0^+$, $f(x) \to \ldots$ and $g(x) \to \ldots$ so $d(x) \to \ldots$

ii As $x \to 0^-$, $f(x) \to \ldots$ and $g(x) \to \ldots$ so $d(x) \to \ldots$

e Copy and complete each statement relating to the horizontal asymptote of $y = f(x)$.

i As $x \to +\infty$, $f(x) \to \ldots$ and $g(x) \to \ldots$ so $d(x) \to \ldots$

ii As $x \to -\infty$, $f(x) \to \ldots$ and $g(x) \to \ldots$ so $d(x) \to \ldots$

f Briefly explain why $d(x)$ approaches $y = 1 - x$ for large values of x.

g Sketch $y = d(x)$ showing all these features.

11 For each pair of functions, first graph $y = f(x)$ and $y = g(x)$ on the same number plane. Then in a different colour graph the sum $y = f(x) + g(x)$.

a $f(x) = \dfrac{1}{x}$ and $g(x) = x$

b $f(x) = \dfrac{1}{x}$ and $g(x) = \dfrac{1}{x+2}$

c $f(x) = 2^x$ and $g(x) = x - 1$

d $f(x) = 2^x$ and $g(x) = 2^{-x}$

12 For each pair of functions in the previous question, sketch the difference $y = f(x) - g(x)$.

13 It may be that if $y = f(x)$ and $y = g(x)$ exhibit even or odd symmetry then we know the symmetry of $s(x) = f(x) + g(x)$ and of $d(x) = f(x) - g(x)$.

a Try to complete the following table, predicting the symmetry of the resulting functions, and test your claims on the examples in this exercise.

	$f(x)$ even, $g(x)$ even	$f(x)$ odd, $g(x)$ odd	$f(x)$ even, $g(x)$ odd
$s(x)$			
$d(x)$			

b Prove formally that if $f(x)$ and $g(x)$ are both odd then $s(x)$ is also odd. That is, prove that $s(-x) = -s(x)$ for all x in its domain.

CHALLENGE

14 Consider the two functions $f(x) = \frac{1}{2}x - 3$ and $g(x) = 1 + \dfrac{1}{x-1}$. The purpose of this question is to sketch the graph $y = f(x) + g(x)$.

a Find the x-coordinates of the points where $f(x) = -g(x)$.

b Plot the line $y = f(x)$ and the hyperbola $y = g(x)$ on the same number plane, and mark the points found in part (a).

c Show that $f(x) + g(x) - (\frac{1}{2}x - 2) \to 0$ as $x \to +\infty$ and as $x \to -\infty$. Add the line $y = \frac{1}{2}x - 3$ to your graph. This is an *oblique asymptote* to the curve $y = f(x) + g(x)$.

d Complete your sketch of the sum $y = f(x) + g(x)$.

15 In the text, the sum $s(x) = f(x) + g(x)$ was drawn for the functions $f(x) = \dfrac{2}{x} + 1$ and $g(x) = x$, and the difference $d(x) = f(x) - g(x)$ was drawn in Question 10. The oblique asymptotes of $s(x)$ and $d(x)$ were also drawn. This question now finds the equations of the oblique asymptotes to the two curves by the method used in the previous question.

a Simplify $s(x) = f(x) + g(x)$, and show that $s(x) - (x + 1) = \dfrac{2}{x}$. Hence explain why $y = x + 1$ is an oblique asymptote to the curve $y = s(x)$.

b Similarly simplify $d(x) = f(x) - g(x)$, and so find its oblique asymptote.

6E Modifying a function using absolute value

Learning intentions

- Sketch $y = |f(x)|$ and $y = f(|x|)$, given the graph of $y = f(x)$.
- Recognise these functions as composites of $f(x)$ with the absolute value function.
- Associate the resulting graphs with reflections in the axes.

The problem in this section is to take some graph $y = f(x)$, and from it sketch

$$y = |f(x)| \qquad \text{and} \qquad y = f(|x|)\,.$$

These operations can be done very simply using reflections in the x-axis and y-axis.

Identifying the composites

Let $g(x) = |x|$ be the absolute value function. Then

$|f(x)| = g(f(x))$ is the composite formed by $f(x)$ followed by $g(x)$, and

$f(|x|) = f(g(x))$ is the composite formed by $g(x)$ followed by $f(x)$.

Thus these two modified functions $y = |f(x)|$ and $y = f(|x|)$ are composites of $f(x)$ and the absolute value function in the two possible orders.

A review of the absolute value function

Our graphs will rely on the algebraic definition of $|x|$ using cases:

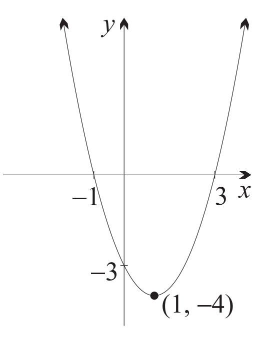
$$|x| = \begin{cases} x, & \text{for } x \geq 0 \quad \text{(for example, } |5| = 5 \text{ and } |0| = 0) \\ -x, & \text{for } x < 0 \quad \text{(for example, } |-5| = 5). \end{cases}$$

Expressed in words:

- The absolute value of a positive number or zero is unchanged.
- The absolute value of a negative number is the opposite, so is positive.

Thus an absolute value can never be negative.

Sketching $y = |f(x)|$ from the sketch of $y = f(x)$

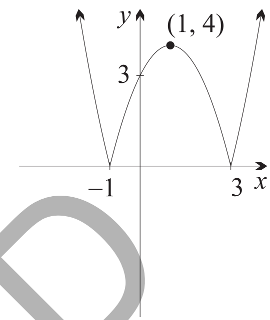


Each absolute value transformation will be illustrated in turn using the graph to the right, which is the parabola

$$f(x) = (x + 1)(x - 3).$$

Graphing $y = |f(x)|$ requires two arguments:

- When the graph is above or on the x-axis, $f(x)$ is positive or zero. Hence $|f(x)| = f(x)$, so the graph is unchanged.



- When the graph is below the x-axis, $f(x)$ is negative. Hence $|f(x)| = -f(x)$ is the opposite of $f(x)$.
  - Thus replace every part of the graph below the x-axis by its reflection back above the x-axis.

A table of values helps to understand the situation:

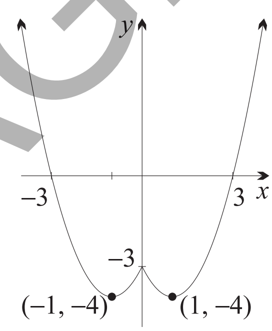
x	−2	−1	0	1	2	3	4
$f(x)$	5	0	−3	−4	−3	0	5
$\lvert f(x)\rvert$	5	0	3	4	3	0	5

14 To sketch $y = |f(x)|$ from the sketch of $y = f(x)$

- Everything above and on the x-axis stays the same.
- Replace everything below the x-axis by its reflection back above the x-axis.

Sketching $y = f(|x|)$ from the sketch of $y = f(x)$

The procedure to graph $y = f(|x|)$ again has two arguments:



- To the right of or on the y-axis, x is positive or zero.
  Hence $|x| = x$, so the graph is unchanged.
- To the left of x-axis, x is is negative, so $|x| = -x$, and $f(|x|) = f(-x)$.
  - Thus replace every part of the graph left of the y-axis by its reflection back across the y-axis.

The table of values illustrating this has a preliminary row for $|x|$:

x	−4	−3	−2	−1	0	1	2	3	4
$\lvert x\rvert$	4	3	2	1	0	1	2	3	4
$f(\lvert x\rvert)$	5	0	−3	−4	−3	−4	−3	0	5

15 To sketch $y = f(|x|)$ from the sketch of $y = f(x)$

- Everything to the right of the y-axis stays the same.
- Replace everything left of the y-axis by its reflection back across the y-axis.

The resulting function is even, that is, it has line symmetry in the y-axis. To illustrate this, the table of values is clearly symmetric about $x = 0$.

Combining these transformations to sketch $y = \left|f(|x|)\right|$

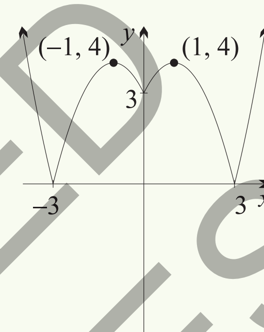
These two absolute value transformation can be combined, as in the two examples below.

Example 16 Combining the two transformations

Earlier in this section, we sketched $f(x) = (x + 1)(x - 3)$ and its transformations $y = |f(x)|$ and $y = f(|x|)$. Use these graphs to sketch $y = \left|f(|x|)\right|$:

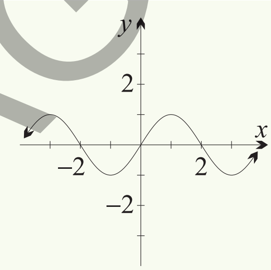
Solution

Start with the earlier sketch of either $|f(x)|$ or $f(|x|)$.



Example 17 Comparing all three methods

Using the graph of $y = f(x)$ sketched to the right, sketch:



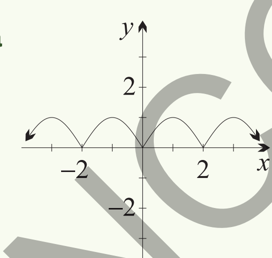
a $y = |f(x)|$

b $y = f(|x|)$

c $y = \left|f(|x|)\right|$

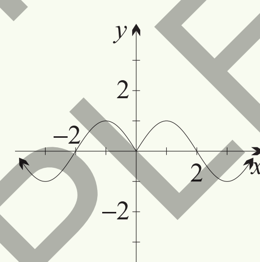
Solution

a



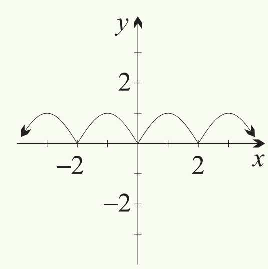
$y = |f(x)|$

b



$y = f(|x|)$

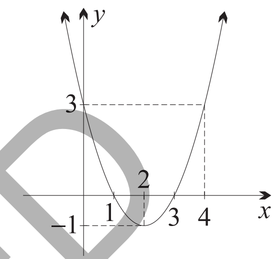
c



$y = \left|f(|x|)\right|$, same as part (a)

Exercise 6E

FOUNDATION



1 The graph of $y = f(x)$ is sketched to the right. To draw each transformation, copy the graph and draw the transformed graph in a different colour on the same axes.

a Replace any part of the graph below the x-axis by its reflection above the x-axis. This will give you the graph of $y = |f(x)|$.

b Graph only the parts of $y = f(x)$ that are to the right of the y-axis. Then add the reflection of these parts in the y-axis. This will give you the graph of $y = f(|x|)$. Notice the symmetry in the y-axis.

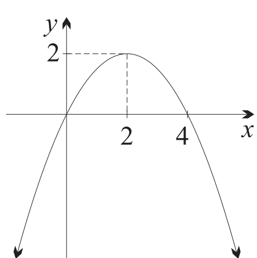
c Using the graph of $y = f(|x|)$, replace any part of the graph below the x-axis by its refection above the x-axis. This gives the graph of $y = \left|f(|x|)\right|$

d Using the graph of $y = |f(x)|$, graph only the parts that are to the right of the y-axis. Then add the reflection of these parts in the y-axis. What do you notice about the result of this and the answer to part (c)?

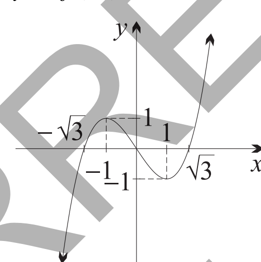
2 In each case, follow the steps of Question 1 and use the given graph of $y = f(x)$ to sketch:

i $y = |f(x)|$ **ii** $y = f(|x|)$ **iii** $y = \left|f(|x|)\right|$

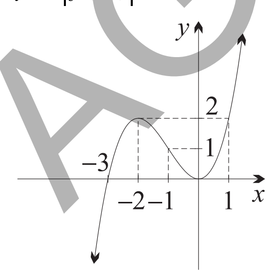
a



b



c



3 Sketch each given function and use it to then sketch:

i $y = |f(x)|$ **ii** $y = f(|x|)$ **iii** $y = \left|f(|x|)\right|$

a $y = x - 1$ **b** $y = 2^{-x} - 1$ **c** $y = \dfrac{1}{1 - x}$

4 **a** Explain why the graphs of $y = 2^x$ and $y = |2^x|$ are the same.

b Write $y = 2^{|x|}$ using cases, then sketch its graph.

DEVELOPMENT

5 **a** Sketch $y = f(x)$ where $f(x) = (x + 1)(x - 2)$, showing all x-intercepts and the vertex.

b Hence sketch:

i $y = |f(x)|$ **ii** $y = f(|x|)$ **iii** $y = \left|f(|x|)\right|$

6 Repeat the steps of the previous question for the function $f(x) = \left|x - 1\right| - 1$.

7 **a** Graph $y = f(x)$, where $f(x) = \dfrac{1}{x - 1} + 1$. Be careful to identify any intercepts with the axes and any asymptotes.

b Hence sketch:

i $y = |f(x)|$ **ii** $y = f(|x|)$ **iii** $y = \left|f(|x|)\right|$

8 **a** Let $f(x) = x(x - 2)$.

i Sketch $y = f(x)$. **ii** Use repeated transformations to sketch $y = |f(|x|)|$.

b Repeat part (a) for the function $f(x) = (x + 1)(3 - x)$.

9 **a** Show that $y = |2 - x|$ is neither even nor odd and graph it.

b Show that $y = \big|2 - |x|\big|$ is even and use part (a) to graph this new function.

c Hence graph $y = 1 - \big|2 - |x|\big|$.

d Finally graph $y = \Big|1 - \big|2 - |x|\big|\Big|$.

10 **a** Let $f(x)$ be any function. Explain why $g(x) = f(|x|)$ is even.

b Let $f(x)$ be an odd function. Show that $h(x) = |f(x)|$ is even.

11 **a** When will the graphs of $y = f(x)$ and $y = |f(x)|$ be the same?

b When will the graphs of $y = f(x)$ and $y = |f(x)|$ be symmetric in the x-axis?

CHALLENGE

12 **a** Read carefully the instruction in Question 1(b) for graphing $y = f(|x|)$. Write a similar instruction for graphing $|y| = f(x)$.

b Test your instruction by graphing $|y| = f(x)$ for each of the functions in Question 2.

13 Sketch $|y| = |x|$.

14 **a** Describe the graph of $y = f(-|x|)$ in terms of the graph of $y = f(x)$.

b What type of symmetry must $y = f(-|x|)$ possess?

15 Let $f(x)$ be any function, and let $g(x) = f(|x|)$ and $h(x) = \frac{1}{2}(f(x) + f(-x))$.

a Prove that both $g(x)$ and $h(x)$ are even functions.

b Are $g(x)$ and $h(x)$ always the same function? If so then prove it, otherwise give a counter-example.

16 **a** Investigate the graphs of the sequence of functions

$$y = |x|, \quad y = \big|1 - |x|\big|, \quad y = \Big|1 - \big|1 - |x|\big|\Big|, \quad \ldots$$

b Show that the 2nd, 4th, 8th, … functions in this sequence can be simplified to

$$y = \big|1 - |x|\big|, \quad y = \Big|1 - \big|2 - |x|\big|\Big|, \quad y = \Bigg|1 - \Big|2 - \big|4 - |x|\big|\Big|\Bigg|, \quad \ldots$$

6F Inverse relations and functions

Learning intentions

- Construct the formula of the inverse of a relation or function.
- Use the horizontal line test to see whether the inverse relation is a function.

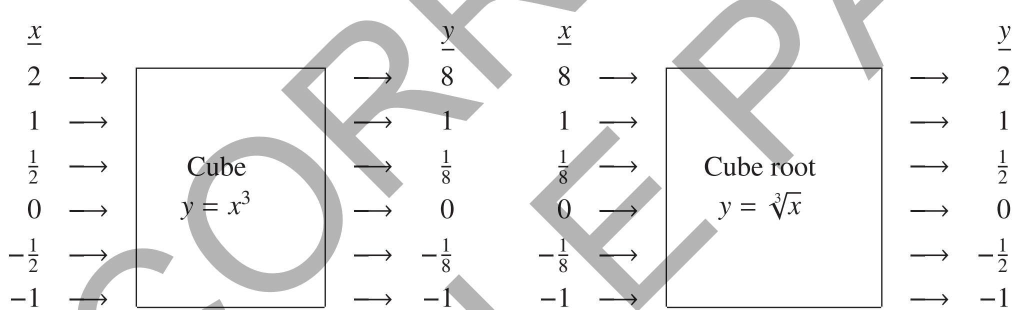
Mathematics is full of inverse processes:

- The inverse of multiplying by 7 is dividing by 7 — and the inverse of dividing by 7 is multiplying by 7.
- The inverse of shifting up 3 is shifting down 3 — and the inverse of shifting down 3 is shifting up 3.
- The inverse of reflecting in the y-axis is reflecting in the y-axis — this process is its own inverse.

This section uses geometric and graphical methods to obtain the inverse of a relation, and to find a simple geometric condition for the inverse to be a function.

Inverse relations

Cube the number 5 and get 125. The inverse process is taking the cube root, which sends 125 back to 5. We can do this with any number, positive, negative or zero, so the cubing function $y = x^3$ has a well-defined *inverse function* $y = \sqrt[3]{x}$ that sends any output back to its original input. When these two function machines are put together in a chain, in either order, the *composition* of the two functions is *the identity function* that maps every number to itself:



The exchanging of input and output can also be seen in the two tables of values, where the two rows are interchanged:

x	2	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	-1	-2
x^3	8	1	$\frac{1}{8}$	0	$-\frac{1}{8}$	-1	-8

x	8	1	$\frac{1}{8}$	0	$-\frac{1}{8}$	-1	-8
$\sqrt[3]{x}$	2	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	-1	-2

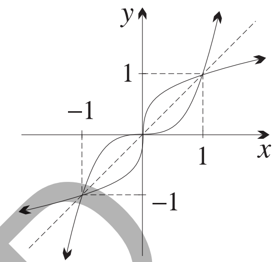
This exchanging of input and output means that the coordinates of each ordered pair are exchanged. Remembering that a relation was defined simply as a set of ordered pairs, we are led to a definition of inverse that can be applied to any relation, whether it is a function or not:

16 Inverse relations

- The *inverse relation* of any relation is obtained by reversing each ordered pair.
- The inverse relation of the inverse relation is the original relation.

The second statement follows from the first because each ordered pair returns to its original state when reversed a second time. For example, the pair $(2, 8)$ in the original becomes the pair $(8, 2)$ in the inverse, and reversed goes back to $(2, 8)$.

Graphing the inverse relation



Reversing an ordered pair means that the original first coordinate is read off the vertical axis, and the original second coordinate is read off the horizontal axis.

Geometrically, this exchanging of the two coordinates can be done by reflecting the point in the diagonal line $y = x$. This can be seen by comparing the graphs of $y = x^3$ and $y = \sqrt[3]{x}$, which are drawn here on the same pair of axes.

17 The graph of the inverse relation

The graph of the inverse relation is obtained by reflecting the original graph in the diagonal line $y = x$.

Domain and range of the inverse relation

Exchanging x- and y-coordinates means the domain and range are exchanged:

18 Domain and range of the inverse relation

- The domain of the inverse is the range of the relation.
- The range of the inverse is the domain of the relation.

Finding the equation and restrictions of the inverse relation

When the coordinates are exchanged, the x-variable becomes the y-variable and the y-variable becomes the x-variable. Hence the method for finding the equation and restrictions of the inverse is:

19 The equation of the inverse relation

- To find the equation and restrictions of the inverse relation, write x for y and y for x every time each variable occurs.
- This process can be applied to any relation whose equation and restrictions are known, whether or not it is a function.

For example, the inverse of the function $y = x^3$ is $x = y^3$. This particular equation can then be solved for y to give $y = \sqrt[3]{x}$, confirming that in this particular case, the inverse relation is again a function.

Example 18 **Graphing function and inverse together**

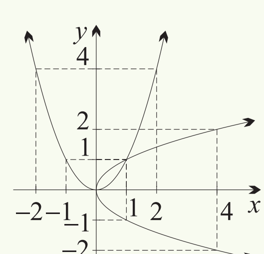
a Write down the inverse relation of the function $y = x^2$.
b Graph both relations on the same number plane, showing the reflection line.
c Write down the domain and range of both relations.
d Is the inverse relation a function?

Solution

a Writing x for y and y for x, the inverse is

$$x = y^2.$$

b



c For the original, domain: all real x, range: $y \geq 0$.
For the inverse, domain: $x \geq 0$, range: all real y.
Notice how the domain and the range have been swapped.

d The inverse is not a function — it fails the vertical line test.

Example 19 **Graphing function and inverse together**

Repeat the previous questions for the function $y = x^3 + 2$.

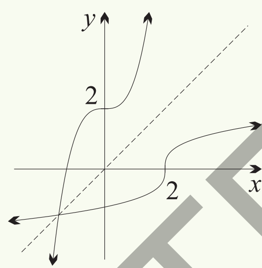
Solution

a Writing x for y and y for x, the inverse is

$$x = y^3 + 2,$$

which is $y = \sqrt[3]{x-2}$.

b



c For both, domain and range are all real numbers.

d The inverse is again a function, by the vertical line test.

Forming the inverse when there are restrictions

When there are any restrictions, then x and y must be swapped in these as well, as in the next example.

Example 20 **Forming the equation of the inverse with restrictions**

Consider the function $y = 2x - 3$, where $x > 1$.

a Write down the equation and restriction of the inverse relation.

b Rewrite the inverse as a function with y as the subject, changing the restriction to a restriction on x.

Solution

a The function is $y = 2x - 3$, where $x > 1$,

so the inverse relation is $x = 2y - 3$, where $y > 1$.

b Solving for y, $y = \frac{1}{2}(x+3)$, where $y > 1$.

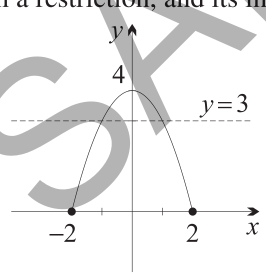
The restriction $y > 1$ is

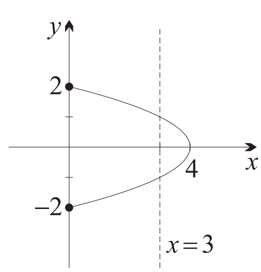
$$\tfrac{1}{2}(x+3) > 1$$
$$x + 3 > 2$$
$$x > -1,$$

so the inverse is the function $y = \frac{1}{2}(x+3)$, where $x > -1$.

Testing graphically whether the inverse relation is a function

In general, the inverse of a relation is not a function. For example, the sketches below show the graphs of another function, with a restriction, and its inverse:





$y = 4 - x^2$, where $-2 \le x \le 2$ $\quad x = 4 - y^2$, where $-2 \le y \le 2$

The first graph is a function, passing the *vertical line test.* But when we read it backwards, the value $y = 3$ gives two answers, $x = 1$ and $x = -1$. Accordingly, when we sketch the inverse relation, we see that it fails the vertical line test, with $x = 3$ crossing the graph twice.

But reflection in $y = x$ exchanges vertical and horizontal lines! Now we can see immediately from the first graph that its inverse is not a function, because it fails the *horizontal line test* — the line $y = 3$ crosses it more than once.

We didn't need to draw the second graph to know that the inverse is not a function. All we need to know is that the first graph fails the horizontal line test.

20 Horizontal line test for whether the inverse is a function

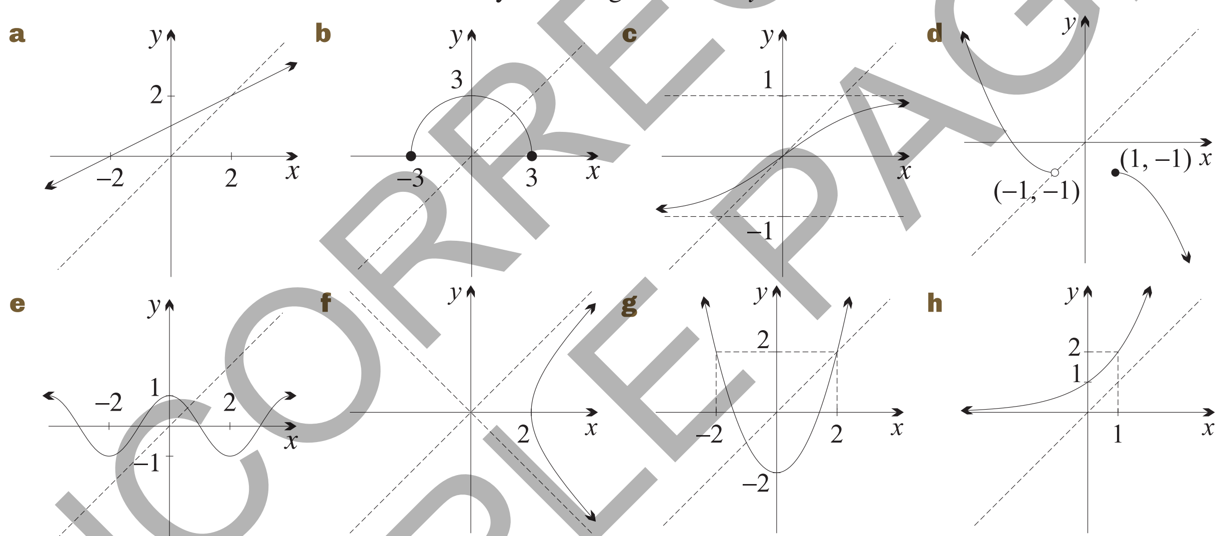
- Geometrically, the inverse relation of a given relation is a function if and only if no horizontal line crosses the original graph more than once.

We could also have done it the slow way — solve the equation $x = 4 - y^2$ of the inverse relation to give $y = \sqrt{4 - x}$ or ≠ $y = -\sqrt{4 - x}$, showing again that for many values of x, there is more than one value of y, so the inverse is not a function.

Exercise 6F

FOUNDATION

1 Draw the inverse relation of each relation by reflecting in the line $y = x$.



2 Use the vertical and horizontal line tests to determine which relations and which inverse relations drawn in Question 1 are also functions.

3 Determine each inverse algebraically by swapping x and y and then making y the subject.

a $y = 3x - 2$ **b** $y = \frac{1}{2}x + 1$ **c** $y = 3 - \frac{1}{2}x$

d $x - y + 1 = 0$ **e** $2x + 5y - 10 = 0$ **f** $y = 2$

4 For each function in the previous question, draw a graph of the function and its inverse on the same number plane to verify the reflection property. Draw a separate number plane for each part.

5 a Find each inverse algebraically by swapping x and y and then making y the subject.

i $y = \frac{1}{x} + 1$ **ii** $y = \frac{1}{x + 1}$ **iii** $y = \frac{x + 2}{x - 2}$ **iv** $y = \frac{3x}{x + 2}$

b For parts (i) and (iv) above, find the domain and range of the function, and the domain and range of the inverse function.

6 Swap x and y and solve for y to find the inverse of each function. What do you notice, and what is the geometric significance of this?

a $y = \dfrac{1}{x}$ **b** $y = \dfrac{2x-2}{x-2}$ **c** $y = \dfrac{-3x-5}{x+3}$ **d** $y = -x$

DEVELOPMENT

7 **a** Graph each relation and its inverse relation, then find the equation of the inverse relation. Which of the four original relations are functions, and which inverse relations are functions?

i $(x-3)^2 + y^2 = 4$ **ii** $(x+1)^2 + (y+1)^2 = 9$

iii $y = x^2 - 4$ **iv** $y = x^2 + 1$

b For parts (i) and (iv) above, find the domain and range of the relation, and the domain and range of the inverse relation.

8 Write down the inverse of each function, solving for y if it is a function. Sketch the function and the inverse on the same graph and observe the symmetry in the line $y = x$.

a $y = x^2$ **b** $y = 2x - x^2$ **c** $y = -\sqrt{x}$ **d** $y = -\sqrt{4 - x^2}$

9 Each function below has a restriction. Write down its inverse relation. Then attempt to solve it for y. If the inverse is a function, rewrite the restriction as a restriction on x. If the inverse is not a function, give a value of x that corresponds to two or more values of y.

a $y = 3x - 10$, where $x < 2$ **b** $y = 13 - 6x$, where $x \geq 3$

c $y = x^3 + 2$, where $x < 3$ **d** $y = x^2 - 3$, where $x \geq -2$

10 **a** Factorise $f(x) = x^2 - 2x - 3$ in order to show that the graph of $y = f(x)$ fails the horizontal line test.

b Let $g(x) = x^2 - 2x - 3$ for $x \geq 1$. Explain why this function has an inverse and find its equation.

11 **a** Show that the inverse function of $y = \dfrac{ax+b}{x+c}$ is $y = \dfrac{b-cx}{x-a}$.

b Hence show that $y = \dfrac{ax+b}{x+c}$ is its own inverse if and only if $a + c = 0$.

CHALLENGE

12 **a** Show that the inverse of $y = \dfrac{2^x + 2^{-x}}{2}$ is not a function.

b Show that the inverse of $y = \dfrac{2^x - 2^{-x}}{2}$ is a function.

13 **a** Show that if the domain of an even function contains a non-zero number, then its inverse is not a function.

b Is the inverse of an odd function always a function? If not, give a counter-example.

14 **a** The polynomial in Question 10(a) has two linear factors and fails the horizontal line test, so its inverse is not a function. Explain why all polynomials with at least two linear factors must automatically fail the horizontal line test.

b Do any polynomials have an inverse which is a function?

6G Inverse function notation

Learning intentions

- Use the alternative definition of inverse function in terms of composition.
- Use the standard notation for the inverse function.
- Restrict a function or relation appropriately so that its inverse is a function.

This section ignores relations, and deals only with functions and inverse functions.

One-to-one functions

A function $f(x)$ must by definition pass the *vertical line test*, meaning that every value of x in the domain corresponds to exactly one y-value. For the inverse of $f(x)$ to be a function also, $f(x)$ must pass the *horizontal line test*, meaning that every value of y in the range corresponds to exactly one x-value.

Thus the condition for the inverse of $f(x)$ to be a function is that $f(x)$ is a *one-to-one correspondence* between the elements of the domain and the elements of the range. Such a function is called simply *one-to-one*.

21 One-to-one functions, and the inverse of a function

- A function $f(x)$ is called *one-to-one*, or a *one-to-one correspondence*, if for every value of y in the range, there is exactly one element x in the domain so that $f(x) = y$.
- The inverse relation of a function $f(x)$ is also a function:
  - ▷ if and only if $f(x)$ passes the horizontal line test, and
  - ▷ if and only if $f(x)$ is one-to-one, and
  - ▷ if and only if the equation of the inverse can be solved uniquely for y.

The composite of a function and its inverse function is the identity

Suppose that $f(x)$ is a one-to-one function with inverse function $g(x)$. Then $g(x)$ is also a one-to-one correspondence, with the same pairing of the domain and range as provided by $f(x)$, but with the corresponding pairs reversed. When we apply $f(x)$ then $g(x)$, it is as if nothing has happened, that is,

$$g(f(x)) = x, \quad \text{for all } x \text{ in the domain of } f(x).$$

And because $g(x)$ sends each number back where it came from, its inverse is $f(x)$, and the other composite $f(g(x))$ is also an identity function,

$$f(g(x)) = x, \quad \text{for all } x \text{ in the domain of } g(x).$$

When we restrict discussion to functions, these two conditions can be taken as an alternative definition of an inverse function:

22 Alternative definition of inverse function using composition

- The function $g(x)$ is the inverse function of a function $f(x)$ if and only if:

  $$g(f(x)) = x, \quad \text{for all } x \text{ in the domain of } f(x), \text{ and}$$
  $$f(g(x)) = x, \quad \text{for all } x \text{ in the domain of } g(x).$$

  That is, if and only if $g(f(x))$ and $f(g(x))$ are both identity functions.
- An *identity function* is a function $I(x)$ such that:

  $$I(x) = x, \ \text{for all } x \text{ in the domain of } I(x).$$

Inverse function notation

Suppose that $f(x)$ is a one-to-one function, that is, its inverse is also a function. Then that inverse function is written as $f^{-1}(x)$. The index -1 used here means 'inverse function' and must not be confused with its more common use for the reciprocal of a number. To return to the original example in the last section,

$$\text{If } f(x) = x^3, \quad \text{then } f^{-1}(x) = \sqrt[3]{x}. \qquad \left(\text{Be careful:} \quad (f(x))^{-1} = \frac{1}{f(x)}\right)$$

As we have seen, the inverse function $f^{-1}(x)$ is also one-to-one, with inverse $f(x)$, and if the function and the inverse function are applied successively in either order, the result is the original number. Using the example above,

$$f^{-1}(f(8)) = \sqrt[3]{8^3} = \sqrt[3]{512} = 8 \qquad \text{and} \qquad f(f^{-1}(8)) = \left(\sqrt[3]{8}\right)^3 = 2^3 = 8.$$

23 Inverse function notation

- If a function $f(x)$ is one-to-one, then its inverse relation is also a one-to-one function, and is written as $f^{-1}(x)$.
- The composite of the function and its inverse, in either order, sends every number for which it is defined back to itself:

$$f^{-1}(f(x)) = x, \quad \text{for all } x \text{ in the domain of } f(x)$$
$$f(f^{-1}(x)) = x, \quad \text{for all } x \text{ in the domain of } f^{-1}(x).$$

- To find the formula for $f^{-1}(x)$ from the formula for $f(x)$:
  - ▷ Convert to $y = \ldots$ notation to generate the inverse relation.
  - ▷ Write y for x and x for y in all the equations and restrictions.
  - ▷ Solve for y, and convert to the notation $f^{-1}(x) = \ldots$.

Examples 21–22 demonstrate the method described in the final dotpoint.

Example 21 Verifying the formal definition of an inverse function

Find the equation of $f^{-1}(x)$ for each function, then verify that

$$f^{-1}(f(x)) = x \qquad \text{and} \qquad f(f^{-1}(x)) = x.$$

a $f(x) = x^3 + 2$

b $f(x) = 6 - 2x$, where $x > 0$

Solution

a Let $y = x^3 + 2$.

Then the inverse has equation $x = y^3 + 2$ (the key step)

and solving for y, $y = \sqrt[3]{x - 2}$.

Hence $f^{-1}(x) = \sqrt[3]{x - 2}$.

Verifying, $f^{-1}(f(x)) = \sqrt[3]{(x^3 + 2) - 2} = \sqrt[3]{x^3} = x$

and $f(f^{-1}(x)) = \left(\sqrt[3]{x - 2}\right)^3 + 2 = (x - 2) + 2 = x$

b Let $y = 6 - 2x$, where $x > 0$.

Then the inverse has equation $x = 6 - 2y$, where $y > 0$ (the key step)

$y = 3 - \frac{1}{2}x$, where $y > 0$.

The restriction $y > 0$ means $3 - \frac{1}{2}x > 0$

$x < 6$,

so $f^{-1}(x) = 3 - \frac{1}{2}x$, where $x < 6$.

Verifying, $f^{-1}(f(x)) = f^{-1}(6 - 2x)$ and $f(f^{-1}(x)) = f(3 - \frac{1}{2}x)$

$= 3 - \frac{1}{2}(6 - 2x)$ $\quad = 6 - 2(3 - \frac{1}{2}x)$

$= 3 - 3 + x$ $\quad = 6 - 6 + x$

$= x$ $\quad = x$

Example 22 Testing whether a function has an inverse function

Find the inverse relation of each function. If the inverse is a function, find an expression for $f^{-1}(x)$, and verify that $f^{-1}(f(x)) = x$ and $f(f^{-1}(x)) = x$.

a $f(x) = \dfrac{1 - x}{1 + x}$ **b** $f(x) = x^2 - 9$

What is surprising about the result of part (a)?

Solution

a Let $y = \dfrac{1 - x}{1 + x}$.

Then the inverse has equation $x = \dfrac{1 - y}{1 + y}$ (the key step)

$\boxed{\times (1 + y)}$ $x + xy = 1 - y$

$y + xy = 1 - x$ (terms in y on one side)

$y(1 + x) = 1 - x$ (now y occurs only once)

$y = \dfrac{1 - x}{1 + x}$.

Hence $f^{-1}(x) = \dfrac{1 - x}{1 + x}$.

Notice that this function $f(x)$ and its inverse $f^{-1}(x)$ are identical, so that if the function $f(x)$ is applied twice, each number is sent back to itself.

Thus $f(f(2)) = f\left(-\frac{1}{3}\right)$ and in general, $f(f(x)) = \dfrac{1 - \frac{1-x}{1+x}}{1 + \frac{1-x}{1+x}}$

$= 1\frac{1}{3} \div \frac{2}{3}$ $\quad = \dfrac{(1 + x) - (1 - x)}{(1 + x) + (1 - x)}$

$= 2$ $\quad = x$

b The function $f(x) = x^2 - 9$ fails the horizontal line test. For example, $f(3) = f(-3) = 0$, which means that the x-axis meets the graph twice. Hence the inverse relation of $f(x)$ is not a function.

Alternatively, the inverse relation is $x = y^2 - 9$, which on solving for y gives

$y = \sqrt{x + 9}$ or $-\sqrt{x + 9}$,

which is not unique, so the inverse relation is not a function.

Restricting the domain so the inverse is a function

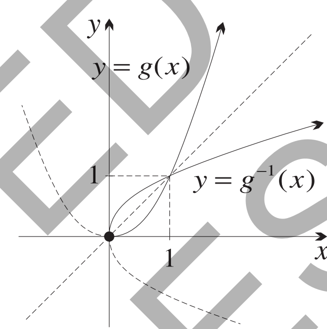
When a function is not one–to-one, that is, its inverse is not a function, it is often convenient to restrict the domain of the function so that this new restricted function has an inverse function.

The clearest example is the squaring function $f(x) = x^2$, whose inverse relation is not a function because, for example, 49 has two square roots, 7 and −7.

If, however, we restrict the domain of $f(x) = x^2$ to $x \geq 0$ and define a new restricted function

$$g(x) = x^2, \quad \text{where } x \geq 0,$$

then the new restricted function $g(x)$ is one-to-one, and thus has an inverse function. This inverse function has equation $g^{-1}(x) = \sqrt{x}$, where as explained earlier, the symbol $\sqrt{\ }$ means 'take the positive square root (or zero)'.



To the right are the graphs of the restricted function and its inverse function, with the unrestricted function and its inverse relation shown dotted.

These ideas will be developed a great deal further in Year 12, when inverse of the tirgonometric functions are developed.

Exercise 6G

FOUNDATION

1 Let $f(x) = 2x - 8$ and $g(x) = \frac{1}{2}x + 4$.

a Verify by substitution that:

i $g(f(5)) = 5$ **ii** $f(g(5)) = 5$ **iii** $g(f(x)) = x$ **iv** $f(g(x)) = x$

b What do you conclude about the functions $f(x)$ and $g(x)$?

2 Each pair of functions $f(x)$ and $g(x)$ are known to be mutual inverses. Show in each case that $f(g(2)) = 2$ and $g(f(2)) = 2$, and that $f(g(x)) = x$ and $g(f(x)) = x$.

a $f(x) = x + 13$ and $g(x) = x - 13$ **b** $f(x) = 7x$ and $g(x) = \frac{1}{7}x$

c $f(x) = 2x + 6$ and $g(x) = \frac{1}{2}(x - 6)$ **d** $f(x) = x^3 - 6$ and $g(x) = \sqrt[3]{x + 6}$

3 **a** Find the inverse function $f^{-1}(x)$ of $f(x) = 2x + 5$. Begin 'Let $y = 2x + 5$', then swap x and y to find the inverse, then solve for y, then write down the equation of $f^{-1}(x)$.

b Check your answer by calculating $f^{-1}(f(x))$ and $f(f^{-1}(x))$.

c Similarly find the inverse functions of each function, and check each answer.

i $f(x) = 4 - 3x$ **ii** $f(x) = x^3 - 2$ **iii** $f(x) = \dfrac{1}{x - 5}$

DEVELOPMENT

4 Explain whether the inverse relation is a function by testing whether it is one-to-one. If it is a function, find $f^{-1}(x)$, specifying its domain. Then verify the two identities $f^{-1}(f(x)) = x$ and $f(f^{-1}(x)) = x$.

a $f(x) = x^2$ **b** $f(x) = \sqrt{x}$ **c** $f(x) = x^4$

d $f(x) = x^3 + 1$ **e** $f(x) = 9 - x^2$ **f** $f(x) = 9 - x^2, x \geq 0$

g $f(x) = 3^{-x^2}$ **h** $f(x) = \dfrac{1 - x}{3 + x}$ **i** $f(x) = x^2, x \leq 0$

j $f(x) = x^2 - 2x, x \geq 1$ **k** $f(x) = x^2 - 2x, x \leq 1$ **l** $f(x) = \dfrac{x + 1}{x - 1}$

5 Let $f(x)$ be the restricted function $f(x) = 3x - 2$, where $1 \le x \le 4$.

a Find the inverse function $f^{-1}(x)$, being careful to add its restriction.

b Show that $f^{-1}(f(x))$ and $f(f^{-1}(x))$ are both identity functions, and find their respective domains. A sketch may make the situation clearer.

6 **a** What is the gradient of the line $y = ax + b$?

b Write down the inverse relation of $y = ax + b$.

c What are the conditions for this inverse relation to be a function?

d When the inverse is a function, solve it for y, find its gradient, and explain why the gradients of the function and its inverse both have the same sign.

e Give an argument using reflection in the line $y = x$ for your answers in part (c).

7 Sketch on separate graphs:

a $y = -x^2$ **b** $y = -x^2$, for $x \ge 0$

Draw the inverse of each on the same graph, then comment on the similarities and differences between parts (a) and (b).

8 The parabola $y = f(x)$, where $f(x) = (x - 3)^2 + 1$, has its vertex at $(3, 1)$. The inverse of $f(x)$ is not a function.

a **i** Janine says that when she applies the restriction $x \ge a$ to $f(x)$ the inverse is a function. What is the least value of a?

ii Find the equation of the inverse in this case.

b **i** Jacob says that when he applies the restriction $x \le b$ to $f(x)$ the inverse is a function. What is the greatest value of b?

ii Find the equation of the inverse in this case.

9 **a** Let $f(x) = x^2 - 2x - 3$ with the restriction $x \ge a$. It is known that the inverse of $f(x)$ is a function. Using the previous question as a guide, find the least value of a and find $f^{-1}(x)$ in that case.

b Do the same for the function $f(x) = 5 - 4x - x^2$ with the restriction $x \le a$.

10 **a** Let $f(x)$ and $g(x)$ be one-to-one functions. That is, both pass the horizontal line test. Let $h(x) = g(f(x))$. Show that the inverse function of $h(x)$ is $h^{-1}(x) = f^{-1}(g^{-1}(x))$.

b **i** Find the inverse function of $h(x) = \dfrac{1}{x-3}$.

ii Express $h(x)$ as the composition of the reciprocal function and a linear function, and hence use part (a) to find its inverse function.

11 Suggest restrictions on the domains of each function to produce a new function whose inverse is also a function (there may be more than one answer). Draw the restricted function and its inverse.

a $y = -\sqrt{4 - x^2}$ **b** $y = \dfrac{1}{x^2}$ **c** $y = x^3 - x$ **d** $y = \sqrt{x^2}$

CHALLENGE

12 **a** Let $f(x) = ax + b$ and $g(x) = \alpha x + \beta$. Find $g(f(x))$, and hence prove that the condition for $f(x)$ and $g(x)$ to be mutually inverse functions is

$$\alpha = a^{-1} \qquad \text{and} \qquad \beta = -a^{-1}b.$$

b Find three linear functions $f(x)$, $g(x)$ and $h(x)$, none of whose graphs pass through the origin, with no two graphs parallel, such that $h\big(g(f(x))\big)$ is the identity function.

13 Does the empty function have an inverse function? If so then what is it?

6H Defining functions and relations parametrically

Learning intentions

- Deal with a function or relation defined parametrically.
- Eliminate the parameter to obtain the Cartesian equation of the curve.

There is an ingenious way of handling curves by making each coordinate a function of a single variable, called a *parameter*. Each point on the curve is then specified by a single number, rather than by a pair of coordinates.

The section uses some trigonometry that is only reviewed in Chapter 7. Angles of any magnitude and the Pythagorean identities are needed. Readers may prefer to delay studying these examples and questions until Chapter 7 is completed.

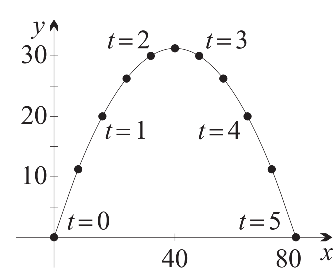
An example of parameters

SDB hits a six at the Sydney Cricket Ground.

- A cameraman in a distant stand, exactly behind the path of the ball, sees the ball rise and fall, with its height y in metres given by $y = -5t^2 + 25t$, where t is time in seconds after the strike.
- A drone filming the shot from high above the Cricket Ground sees the ball move across the ground, with distance x from the batsman given by $x = 16t$.

The cameraman and the drone together have a complete record of the ball's flight. (Ignore parallax here, and regard both observers as 'distant'). From their results, we can draw up a table of values of the (x, y) position of the ball at each time t:

t	0	1	2	$2\frac{1}{2}$	3	4	5
x	0	16	32	40	48	64	80
y	0	20	30	$31\frac{1}{4}$	30	20	0



The resulting (x, y)-graph shows the path of the ball, and each point on the graph can be labelled with the corresponding value of time t.

This variable t is called a *parameter*, and the equations

$$x = 16t, \qquad y = -5t^2 + 25t$$

are called *parametric equations of the curve.*

It is possible to *eliminate the parameter t* from these two equations.

Solving the first equation for t, $\quad t = \frac{x}{16}$,

then substituting into the second, $\quad y = -\frac{5}{256}x^2 + \frac{25}{16}x$,

$$y = \frac{-5x^2 + 400x}{256}$$

and factoring displays the zeroes, $\quad y = \dfrac{5x(80 - x)}{256}$.

24 Parameters

- A curve in the (x, y)-plane may be *parametrically defined*, meaning that x and y are given as functions of a third variable t called a *parameter*.
  - ▷ These two equations for x and y in terms of t are called *parametric equations* of the curve.
- In many situations, the parameter t may be *eliminated* to give a single equation in x and y for the curve.
  - ▷ The single equation in x and y is called the *Cartesian equation* of the curve.

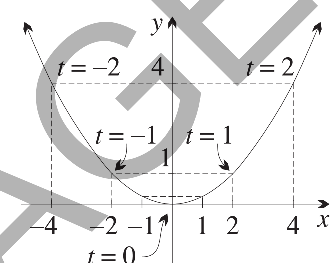
The letter t is often used for the parameter because it stands for 'time'. Other letters are often used, however, particularly θ and φ for an angle, and p and q.

Examples of parametrisation — a parabola

We can reverse the process of eliminating the parameter, and *parametrise* familiar curves. Some straightforward parametrisations are given below of a parabola, a circle, and a rectangular hyperbola.

The parabola $x^2 = 4y$ can be parametrised by the pair of equations

$$x = 2t \qquad \text{and} \qquad y = t^2$$



because elimination of t gives $x^2 = 4y$. The variable point $(2t, t^2)$ now runs along the whole curve as the parameter t takes all the real numbers as its values:

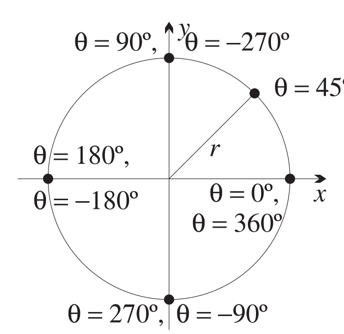
t	-2	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2
x	-4	-2	-1	0	1	2	4
y	4	1	$\frac{1}{4}$	0	$\frac{1}{4}$	1	4

The sketch shows the curve with the seven plotted points labelled by their parameter. The curve can be regarded as a 'bent and stretched number line.'

A parametrisation of the circle

The circle $x^2 + y^2 = r^2$ can be parametrised using trigonometric functions by

$$x = r\cos\theta \qquad \text{and} \qquad y = r\sin\theta.$$



This parametrisation uses the Pythagorean identity, because squaring the two equations and adding them:

$$\begin{aligned} x^2 + y^2 &= r^2\cos^2\theta + r^2\sin^2\theta \\ &= r^2(\cos^2\theta + \sin^2\theta) \\ &= r^2, \quad \text{because } \cos^2\theta + \sin^2\theta = 1. \end{aligned}$$

Notice from the table of values below that with these equations, each parameter corresponds to just one point, but each point corresponds to infinitely many different values of the parameter, all differing by multiples of 360°:

θ	$-360°$	$-270°$	$-180°$	$-90°$	0	$45°$	$90°$	$180°$	$270°$	$360°$
x	r	0	$-r$	0	r	$\frac{1}{2}r\sqrt{2}$	0	$-r$	0	r
y	0	r	0	$-r$	0	$\frac{1}{2}r\sqrt{2}$	r	0	$-r$	0

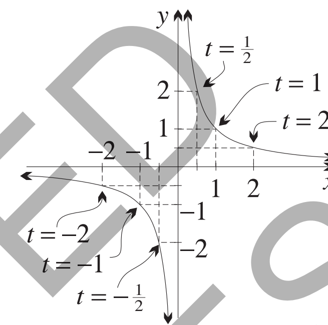
In our previous examples, the map from parameters to points was always one-to-one, but in this parametrisation of the circle, the map is many-to-one.

The circle is a relation, but not a function. The great advantage of parametrising the circle is that it is now described by a pair of *functions*, which in many situations are easier to handle than the original relation.

A parametrisation of the rectangular hyperbola

The rectangular hyperbola $xy = 1$ can be parametrised algebraically by

$$x = t \qquad \text{and} \qquad y = \frac{1}{t}.$$



There is a one-to-one correspondence between the points on the curve and the real numbers, with the one exception that $t = 0$ does not correspond to any point:

t	-2	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2
x	-2	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2
y	$-\frac{1}{2}$	-1	-2	$*$	2	1	$\frac{1}{2}$

Example 23 **Using trig identities to find the Cartesian equaton**

Find the Cartesian equations of the curves defined by the parametric equations:

a $x = 4p,\ y = p^2 + 1$

b $x = \sec\theta,\ y = \sin\theta$

Describe part (a) geometrically.

Solution

a From the first, $p = \frac{1}{4}x$, and substituting into the second,

$$y = \tfrac{1}{16}x^2 + 1,$$

which is a parabola with vertex $(0, 1)$ and concave up.

b Squaring, $x^2 = \sec^2\theta$,

and $y^2 = \sin^2\theta$

$= 1 - \cos^2\theta$,

so $y^2 = 1 - \dfrac{1}{x^2}$

$x^2(1 - y^2) = 1.$

Exercise 6H

FOUNDATION

Note: Some questions in this exercise use trigonometry reviewed in Chapter 7, in particular, angles of any magnitude and the Pythagorean identities:

$$\sin^2\theta + \cos^2\theta = 1 \qquad \tan^2\theta + 1 = \sec^2\theta \qquad 1 + \cot^2\theta = \operatorname{cosec}^2\theta.$$

1 Consider the parametric equations $x = t - 2$ and $y = 2t - 1$.

a Complete the table to the right.

b Explain from the table why the graph is a line.

c From the table, find the y-intercept and the gradient.

d Eliminate t to find the Cartesian equation, and check it from part (c).

t	-2	-1	0	1	2
x					
y					

2 a The parametric equations $x = 2t - 3$ and $y = t + 1$ represent a line.

i Find the points A and B with parameters $t = 0$ and $t = 1$, and hence find the gradient of the line.

ii Find the value of t that makes $x = 0$, and hence find the y-intercept.

iii Check your answers by eliminating t to form a Cartesian equation.

b Repeat the steps in part (a) for these lines:

i $x = 2t - 3$ and $y = 6t - 5$

ii $x = 2t - 3$ and $y = 3t - 2$

3 **a** Complete the table below for the curve $x = 4t$, $y = 2t^2$ and sketch its graph.

t	-6	-4	-2	-1	0	1	2	4	6
x									
y									

b Eliminate the parameter to find the Cartesian equation of the curve.

c The curve is a parabola. What value of t gives the coordinates of the vertex?

4 Repeat the previous question for the curve $x = t$, $y = \frac{1}{2}t^2$.

5 **a** Show that the point $\left(cp, \dfrac{c}{p}\right)$ lies on the hyperbola $xy = c^2$, where c is a constant.

b Complete the table of values below for $x = 2p$, $y = \dfrac{2}{p}$ and sketch the graph.

p	-3	-2	-1	$-\frac{1}{2}$	$-\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{2}$	1	2	3
x											
y											

c Explain what happens as $p \to \infty$, $p \to -\infty$, $p \to 0^+$ and $p \to 0^-$.

6 **a** Show that the point $(a\cos\theta, b\sin\theta)$ lies on the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

b **i** Complete a table of values for the curve $x = 4\cos\theta$, $y = 3\sin\theta$, taking the values $\theta = 0°, 30°, 60°, 90°, 120°, \ldots, 360°$.

ii Sketch the curve and state its Cartesian equation.

DEVELOPMENT

7 Eliminate the parameter and hence find the Cartesian equation of the curve.

a $x = 3 - p$, $y = 2p + 1$

b $x = 1 + 2\tan\theta$, $y = 3\sec\theta - 4$

c $x = p + \dfrac{1}{p}$, $y = p^2 + \dfrac{1}{p^2}$

d $x = \cos\theta + \sin\theta$, $y = \cos\theta - \sin\theta$

8 **a** Show that $x = a + r\cos\theta$ and $y = b + r\sin\theta$ define a circle with centre (a, b) and radius r.

b Hence sketch a graph of the curve $x = 1 + 2\cos\theta$, $y = -3 + 2\sin\theta$.

9 Show by elimination that $x = \dfrac{t^2 - 1}{t^2 + 1}$ and $y = \dfrac{2t}{t^2 + 1}$ almost represent the unit circle $x^2 + y^2 = 1$. What point is missing?

10 [Parameters and Curve Orientation] Let $A = (1, 2)$ and $B = (2, 1)$.

a **i** Show that $x = 1 + t$, $y = 2 - t$, $0 \le t \le 1$ parameterises the line segment AB.

ii Describe how the point $P(1 + t, 2 - t)$ moves as t increases from 0 to 1.

b **i** Show that $x = 2 - u$, $y = 1 + u$, $0 \le u \le 1$ also parameterises the line segment AB.

ii Describe how the point $P(2 - u, 1 + u)$ moves as u increases from 0 to 1.

c The points A and B also lie on the circle with centre $(1, 1)$ and radius 1. Consider the points $P(1 + \cos t, 1 + \sin t)$ and $Q(1 + \sin t, 1 + \cos t)$, where $0° \le t \le 90°$ in both cases. Explain the difference between the curves traced out by the points P and Q.

11 Different parametric representations may result in the same Cartesian equation. The graphical representation, however, may differ due to restrictions in the domain or range.

a Find the Cartesian equation of the curve $x = 2 - t$, $y = t - 1$ and sketch its graph.

b Find the Cartesian equation of the curve $(\sin^2 t, \cos^2 t)$. Explain why $0 \le x \le 1$ and $0 \le y \le 1$ and sketch a graph of the curve.

c Find the Cartesian equation of the curve $x = 4 - t^2$, $y = t^2 - 3$. Explain why $x \le 4$ and $y \ge -3$ and sketch a graph of the curve.

12 A certain graph has parametric equations $x = at + b$ and $y = ct + d$, where the constants a, b, c and d are real numbers.

a Eliminate t from these equations and hence show that they represent a straight line. Express your answer in gradient-intercept form.

b Are there any restrictions that need to be placed on the answer to part (a)?

c How could the answer to part (a) be written in order to avoid these restrictions? Explain your answer.

13 **a** Show that the parametric equations $x = c(\sec\theta - \tan\theta)$ and $y = c(\sec\theta + \tan\theta)$ with $-90^\circ < \theta < 90^\circ$ represents the portion of the hyperbola $xy = c^2$ in the first quadrant.

b What restriction on θ is needed to get the portion in the third quadrant?

14 Find the Cartesian equation of the curve $x = 3 + r\cos\theta$, $y = -2 + r\sin\theta$, and describe it geometrically if:

a r is constant and θ is variable,

b θ is constant and r is variable.

15 [Parameters and Transformations]

a The circle with centre the origin and radius 2 has parametric equations $x = 2\cos\theta$ and $y = 2\sin\theta$ with $-180^\circ < \theta \le 180^\circ$.

i By considering translations, write down the parametric equations when this circle is shifted right 1 and up 3.

ii By considering dilations, write down the parametric equations when the original circle is stretched horizontally by 2 and vertically by $\frac{1}{2}$.

b A certain curve has parametric equations $x = f(t)$ and $y = g(t)$. Ignoring any possible restrictions on t, answer the following.

i Describe the curve generated by $x = f(t) + h$ and $y = g(t) + k$.

ii Describe the curve generated by $x = af(t)$ and $y = b\,g(t)$.

CHALLENGE

16 Show by elimination that $x = \dfrac{2t + 1}{2t^2 + 2t + 1}$ and $y = \dfrac{2t^2 + 2t}{2t^2 + 2t + 1}$ almost represent the unit circle $x^2 + y^2 = 1$. What point is missing?

17 **a** Explain why the parametric equations $x = \cos t$, $y = \sin t$, $z = t$ describe a spiral.

18 **a** Show that the point $(a\sec\theta, b\tan\theta)$ lies on the curve $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$.

b Complete a table of values for the curve $x = 4\sec\theta$, $y = 3\tan\theta$, where $0^\circ \le \theta \le 360^\circ$. What happens when $\theta = 90^\circ$ and $\theta = 270^\circ$?

c Sketch the curve (it has two asymptotes) and state its Cartesian equation.

19 After finding the Cartesian equation, sketch the curve whose parametric equations are

$$x = \tfrac{1}{2}(2^t + 2^{-t}) \qquad \text{and} \qquad y = \tfrac{1}{2}(2^t - 2^{-t})\,.$$

20 A relation is defined parametrically by $x = f(t)$ and $y = g(t)$.

a What transformation of the relation occurs when t is replaced by $-t$ if:

i $f(t)$ and $g(t)$ are both even,

ii $f(t)$ and $g(t)$ are both odd,

iii $f(t)$ is even and $g(t)$ is odd,

iv $f(t)$ is odd and $g(t)$ is even.

b What is the relationship between this relation and the relation defined by $x = g(t)$ and $y = f(t)$?

c Where is the graph of the relation $x = |f(t)|$ and $y = |g(t)|$ located?

d Where is the graph of the relation $x = f(t)$ and $y = f(t)$ located?

Chapter 6 Review

Review activity

- Create your own summary of this chapter on paper or in a digital document.

Chapter 6 Multiple-choice quiz

- This automatically-marked quiz is accessed in the Interactive Textbook. A printable PDF Worksheet version is also available there.

Skills Checklist

Check-list

- Available in the Interactive Textbook, use the checklist to track your understanding of the learning intentions. Printable PDF and word document versions are also available there.

Review

Chapter Review Exercise

Note: Graphing software could be very helpful in this exercise.

1 Solve each inequation.

a $|x| < 3$ **b** $|x + 2| \geq 4$ **c** $|2x - 5| \leq 11$

2 Solve each inequation by multiplying both sides by the square of the denominator.

a $\frac{5}{x} > 1$ **b** $\frac{3}{x - 3} \leq 1$ **c** $\frac{x - 2}{x + 1} \geq 4$

3 Consider the function $f(x) = x(x + 2)(x - 3)$.

a Write down the zeroes of the function, and draw up a table of signs.

b Copy and complete: '$f(x)$ is positive for . . . , and negative for . . . '

c Write down the solution of the inequation $x(x + 2)(x - 3) \leq 0$.

d Sketch the graph of the function to confirm these results.

4 Consider the function $y = (1 - x)(x - 3)^2$.

a Write down the zeroes of the function and draw up a table of signs.

b Hence solve the inequation $(1 - x)(x - 3)^2 \geq 0$.

c Confirm the solution by sketching a graph of the function.

5 Solve each inequation in question 4 using the table-of-signs method. First move everything to the left-hand side, then make the LHS into a single fraction, then identify its zeroes and discontinuities, then draw up a table of signs, then read the solution from the table.

6 Consider the linear function $f(x) = x - 2$.

a Sketch $y = f(x)$, clearly indicating the x- and y-intercepts.

b Also show on your sketch the points where $y = 1$ and $y = -1$.

c Hence sketch the graph of $y = \frac{1}{f(x)}$ on the same number plane.

d Write down the equation of the vertical asymptote of $y = \frac{1}{f(x)}$, then copy and complete the sentence, 'As $x \to 2^-$, $y \to$ . . . , and as $x \to 2^+$, $y \to$ . . . '

7 Consider the quadratic function $f(x) = 3 - x^2$.

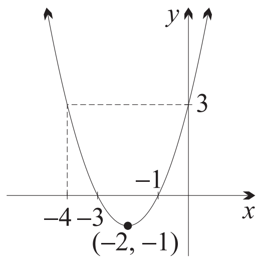
a Sketch $y = f(x)$, clearly indicating the x- and y-intercepts.

b Also show on your sketch the points where $y = 1$ and $y = -1$.

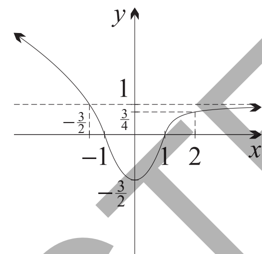
c Hence sketch the graph of $y = \dfrac{1}{f(x)}$ on the same number plane.

8 Sketch the reciprocal of each function graphed below, showing all the important features.

a



b



9 **a** Find the equations of the vertical asymptotes of each function.

i $y = \dfrac{2}{x+1}$ **ii** $y = \dfrac{2x+1}{x-2}$ **iii** $y = \dfrac{4x}{x^2 - 25}$

b In part (iii) above, identify the zeroes and discontinuities and draw up a table of signs. Then describe the behaviour of the curve near each vertical asymptote by copying and completing, 'As $x \to 5^-$, $y \to \dots$, and as $x \to 5^+$, $y \to \dots$' (and similarly for -5).

10 Consider the function $y = \dfrac{2x}{x^2 - 1}$.

a Show that it is an odd function.

b Find the zeroes and discontinuities, and draw up a table of signs.

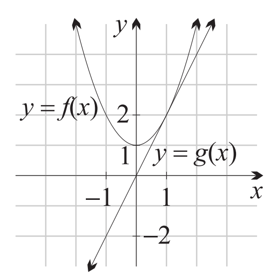
c Identify any vertical and horizontal asymptotes.

d Hence sketch the graph.

11 The graphs of $y = f(x)$ and $y = g(x)$ are sketched to the right. On separate number planes, sketch:

a $y = f(x) + g(x)$

b $y = f(x) - g(x)$



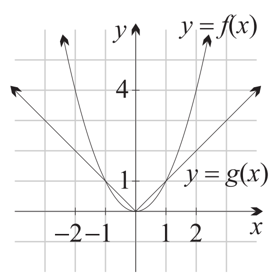
12 The graphs of $y = f(x)$ and $y = g(x)$ are sketched to the right.

a On separate number planes, sketch:

i $y = f(x) + g(x)$

ii $y = f(x) - g(x)$

b What symmetry should be evident in your sketches?

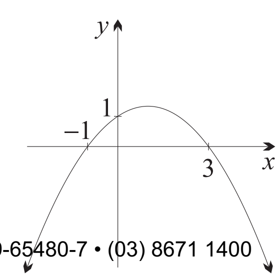


13 The graph of $y = f(x)$ is sketched to the right. On three separate number planes, sketch:

a $y = |f(x)|$

b $y = f(|x|)$

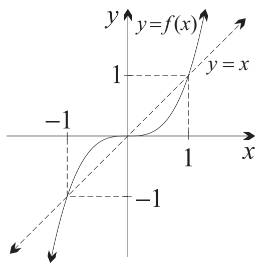
c $y = |f(|x|)|$



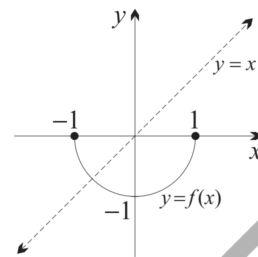
Review

14 Copy each diagram below, then sketch the inverse relation of the function. Also state whether or not the inverse relation is a function.

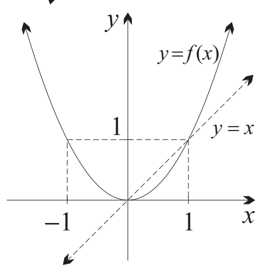
a



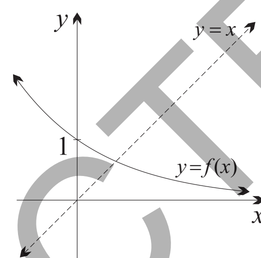
b



c



d



15 Find the equation of the inverse function for each function.

a $y = 5 - 3x$ **b** $y = \dfrac{5}{x-3}$ **c** $y = \dfrac{5x}{x-3}$ **d** $y = x^3 + 5$

16 Find the inverse function $f^{-1}(x)$ of each function, and then confirm algebraically that

$$f(f^{-1}(x)) = x \quad \text{and} \quad f^{-1}(f(x)) = x.$$

a $f(x) = \frac{1}{2}x + 4$ **b** $f(x) = (x+2)^3$ **c** $f(x) = \dfrac{3}{x} - 6$

17 A line is defined parametrically by the equations $x = t + 2$ and $y = 2t + 6$.

a Copy and complete the table below.

t	−5	−4	−3	−2	−1	0	1
x							
y							

b Use the table to sketch the line, marking each point with its t-value.

c Eliminate the parameter t to find the Cartesian equation of the line.

18 A parabola is defined parametrically by the equations $x = \frac{1}{2}t$ and $y = \frac{1}{4}t^2$.

a Copy and complete the table below.

t	−6	−4	−2	−1	0	1	2	4	6
x									
y									

b Use the table to sketch the parabola, marking each point with its t-value.

c Eliminate the parameter t to find the Cartesian equation of the parabola.

19 A curve is defined parametrically by the equations $x = \cos\theta - 1$ and $y = \sin\theta + 1$.

a Use the identity $\cos^2\theta + \sin^2\theta = 1$ to find the Cartesian equation of the curve.

b Describe the curve, and then sketch it.

20 A curve is represented parametrically by the equations $x = 2t$ and $y = \dfrac{1}{t+1}$.

Find the Cartesian equation of the curve and hence sketch it.

11

Polynomials

Chapter introduction

Once the variable x has been introduced into arithmetic, polynomial expressions such as $4x^3 - 7x + 5$ arise naturally. Indeed every expression that can be formed using just the three operations of addition, subtraction and multiplication can be written as a polynomial — simply expand brackets, and collect like terms.

The course has already discussed linear and quadratic functions in detail, and this chapter begins the systematic study of polynomials of higher degree.

Polynomials were mentioned briefly in Section 3G, and some readers will already have familiarity with their graphs, with long division of polynomials, and with the remainder and factor theorems. The later sections in this chapter develop the relationships between the coefficients and the zeroes.

The problem of factoring a given polynomial is a constant theme, and the final Section 11G applies the methods of the chapter to geometric problems about polynomial curves, circles, and rectangular hyperbolas.

11A The language of polynomials

Learning intentions

- Define polynomial, leading term and coefficient, degree, monic, constant.
- Classify zero, linear, quadratic, cubic and quartic polynomials.
- Add, subtract, and multiply polynomials, and observe the resulting degrees.
- Define identically equal polynomials, and use the relationship to the coefficients.

Polynomials are expressions such as the quadratic $x^2 - 5x + 6$ or the quartic $3x^4 - \frac{2}{3}x^3 + 4x + 7$. They have occurred already in the course, but now our language and notation needs to be more precise.

Polynomials and polynomial functions

1 Polynomials

- A *polynomial* is an expression of the form

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0,$$

where $a_0, a_1, \ldots, a_n$ are real numbers, and n is a whole number.
- A *polynomial function* is a function that can be written as a polynomial.

The term a_0 is called the *constant term*. This is the value of the polynomial at $x = 0$, so a_0 is the y-intercept of its graph.

Leading term and degree

The term of highest index with non-zero coefficient is called the *leading term* of the polynomial. Its coefficient is called the *leading coefficient*, and its index is called the *degree*. For example, the polynomial

$$P(x) = -5x^6 - 3x^4 + 2x^3 + x^2 - x + 9$$

has leading term $-5x^6$ and leading coefficient -5, and has degree 6, written as

$$\deg P(x) = 6 .$$

For convenience, the constant term a_0 is regarded in this context as being a term of index 0. This is because a_0 can be written as a_0x^0 provided that $x \neq 0$. But never actually write the term a_0 as a_0x^0, because 0^0 is undefined.

A *monic polynomial* is a polynomial whose leading coefficient is 1. For example, $P(x) = x^3 - 2x^2 - 3x + 4$ is monic. Every non-zero polynomial is a unique multiple of a monic polynomial because $a_n \neq 0$, so

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = a_n\left(x^n + \frac{a_{n-1}}{a_n}x^{n-1} + \cdots + \frac{a_1}{a_n}x + \frac{a_0}{a_n}\right).$$

Some names of polynomials

The zero polynomial, and polynomials of low degree, have standard names.

- The *zero polynomial* $Z(x) = 0$ is a special case. It has a constant term 0. But it has no term with a non-zero coefficient. Hence it has no leading term, no leading coefficient, and most importantly, no degree. Its graph is the x-axis — meaning that every real number is a zero of the zero polynomial.
- A *constant polynomial* is a polynomial whose only term is the constant term:

$$P(x) = 4, \qquad Q(x) = -\tfrac{3}{5}, \qquad R(x) = \pi, \qquad Z(x) = 0.$$

Apart from the zero polynomial, all constant polynomials have degree 0, have no zeroes, and are equal to their leading term and to their leading coefficient.

- A *linear polynomial* is a polynomial of degree 1:

$$P(x) = x - 3, \qquad Q(x) = 4x + 7, \qquad R(x) = -\tfrac{1}{2}x.$$

Warning: *A constant function is a linear function. But a constant polynomial is NOT a linear polynomial, because it does not have degree 1 — it has degree 0 or is the zero polynomial.*

- A polynomial of degree 2 is called a *quadratic polynomial*:

$$P(x) = 3x^2 + 4x - 1, \qquad Q(x) = -\tfrac{1}{2}x - x^2, \qquad R(x) = 9 - x^2.$$

Notice that the coefficient of x^2 must be non-zero for the degree to be 2.

- Polynomials of higher degree are called:

cubics (degree 3), *quartics* (degree 4), *quintics* (degree 5), ...

2 The degree of a polynomial $P(x)$)

- The *leading term* of $P(x)$ is the term of highest index with non-zero coefficient.
  - ▷ The *degree* of $P(x)$ is the index of the leading term.
  - ▷ The *leading coefficient* of $P(x)$ is the coefficient of the leading term.
  - ▷ A *monic polynomial* is a polynomial whose leading coefficient is 1.
- The *zero polynomial* $Z(x) = 0$ has no leading term, and so has no degree.
- A *constant polynomial* $P(x) = a_0$ is a polynomial with only a constant term.
  - ▷ If $a_0 \neq 0$, it has degree 0. But if $a_0 = 0$, it is the zero polynomial $Z(x) = 0$.
- A *linear polynomial* $P(x) = a_1x + a_0$ is a polynomial of degree 1, so $a_1 \neq 0$.
- Polynomials of higher degree are called *quadratic, cubic, quartic, quintic, . . .*

Addition and subtraction

When two polynomials are added or subtracted, the results are again polynomials:

$$(5x^3 - 4x + 3) + (3x^2 - 3x - 2) = 5x^3 + 3x^2 - 7x + 1$$
$$(5x^3 - 4x + 3) - (3x^2 - 3x - 2) = 5x^3 - 3x^2 - x + 5$$

The zero polynomial $Z(x) = 0$ is the *zero for addition*, in the usual sense that $P(x) + Z(x) = P(x)$, for all polynomials $P(x)$. The *opposite polynomial* $-P(x)$ of any polynomial $P(x)$ is obtained by taking the opposite of every coefficient. Then the sum of $P(x)$ and $-P(x)$ is the zero polynomial. For example,

$$(4x^4 - 2x^2 + 3x - 7) + (-4x^4 + 2x^2 - 3x + 7) = 0.$$

The degree of the sum or difference is usually the maximum of the degrees of the two polynomials, as in the two examples above, where the polynomials have degrees 2 and 3 and their sum has degree 3. If, however, the two polynomials have the same degree, the leading terms may cancel out and disappear, for example,

$$(x^2 - 3x + 2) + (9 + 4x - x^2) = x + 11, \quad \text{which has degree 1,}$$

or the two polynomials may be opposites, in which case everything cancels out so that their sum is zero and thus has no degree.

3 Degree of the sum and difference

Let $P(x)$ and $Q(x)$ be non-zero polynomials of degree n and m respectively.

- If $n \neq m$, then $\deg(P(x) + Q(x))$ = maximum of m and n.
- If $n = m$, then $\deg(P(x) + Q(x)) \leq n$ or $P(x) + Q(x) = 0$.

Multiplication

Any two polynomials can be multiplied, giving another polynomial:

$$\begin{aligned}(3x^3 + 2x + 1) \times (x^2 - 1) &= (3x^5 + 2x^3 + x^2) - (3x^3 + 2x + 1)\\ &= 3x^5 - x^3 + x^2 - 2x - 1\end{aligned}$$

The constant polynomial $I(x) = 1$ is the identity for multiplication, in the sense that $P(x) \times I(x) = P(x)$, for all polynomials $P(x)$. Multiplication by the zero polynomial, on the other hand, always gives the zero polynomial.

If two polynomials are non-zero, then the degree of their product is the sum of their degrees, because the leading term of the product is always the product of the two leading terms.

4 Degree of the product

If $P(x)$ and $Q(x)$ are non-zero polynomials, then

$$\deg(P(x) \times Q(x)) = \deg P(x) + \deg Q(x).$$

Identically equal polynomials

We need to be clear what is meant by saying that two polynomials are the same.

5 Identically equal polynomials

- Two polynomials $P(x)$ and $Q(x)$ are called *identically equal* if they are equal for all values of x:

  $P(x) = Q(x)$, for all x, often written as $P(x) \equiv Q(x)$.
- If two polynomials are identically equal, then the corresponding coefficients of the two polynomials are equal.

The second dotpoint needs proof. This is developed in the Enrichment section of Exercise 11B. In the meantime, here is an example that uses the result.

Example 1 Finding coefficients in identically equal polynomials

Find a, b, c, d, and e if $ax^4 + bx^3 + cx^2 + dx + e = (x^2 - 3)^2$ for all x.

Solution

Expanding, $(x^2 - 3)^2 = x^4 - 6x^2 + 9$.
Now comparing coefficients, $a = 1$, $b = 0$, $c = -6$, $d = 0$ and $e = 9$.

Factoring polynomials

The most significant problem of this chapter is the factoring of a given polynomial. For example,

$$x(x + 2)^2(x - 2)^2(x^2 + x + 1) = x^7 + x^6 - 7x^5 - 8x^4 + 8x^3 + 16x^2 + 16x$$

is a routine expansion of a factored polynomial, but it is not at all clear how to move in the other direction from the expanded form back to the factored form.

Polynomial equations

If $P(x)$ is a polynomial, then the equation formed by putting $P(x) = 0$ is a *polynomial equation*. For example, using the polynomial in the previous paragraph, we can form the polynomial equation

$$x^7 + x^6 - 7x^5 - 8x^4 + 8x^3 + 16x^2 + 16x = 0.$$

Solving polynomial equations and factoring polynomial functions are closely related. Using the factoring of the previous paragraph,

$$x(x+2)^2(x-2)^2(x^2+x+1) = 0,$$

so the solutions are $x = 0$, $x = 2$ (double root) and $x = -2$ (double root), where the quadratic factor $x^2 + x + 1$ has no zeroes, because $\Delta = -3$. Thus factoring a polynomial into linear and irreducible quadratic factors solves the corresponding polynomial equation.

The solutions of a polynomial equation are called *roots*, whereas the *zeroes* of a polynomial function are the values of x where the value of the polynomial is zero. The distinction between the two words is not always strictly observed.

Exercise 11A

FOUNDATION

1 State whether or not each expression is a polynomial.

a $3x^2 - 7x$ **b** $\dfrac{1}{x^2} + x$ **c** $\sqrt{x} - 2$

d $3x^{\frac{2}{3}} - 5x + 11$ **e** $\sqrt{3}x^2 + \sqrt{5}x$ **f** $2^x - 1$

g $(x+1)^3$ **h** $\dfrac{7x^{13} + 3x}{4}$ **i** $\log_e x$

j $\frac{4}{3}x^3 - ex^2 + \pi x$ **k** 5 **l** $\dfrac{x-2}{x+1}$

2 For each polynomial, state:

i the degree, **ii** the leading coefficient, **iii** the leading term,
iv the constant term, **v** whether or not the polynomial is monic.

Expand the polynomial first where necessary.

a $4x^3 + 7x^2 - 11$ **b** $10 - 4x - 6x^3$ **c** 2

d x^{12} **e** $x^2(x-2)$ **f** $(x^2 - 3x)(1 - x^3)$

g 0 **h** $x(x^3 - 5x + 1) - x^2(x^2 - 2)$ **i** $6x^7 - 4x^6 - (2x^5 + 1)(5 + 3x^2)$

3 If $P(x) = 5x + 2$ and $Q(x) = x^2 - 3x + 1$, find:

a $P(x) + Q(x)$ **b** $Q(x) + P(x)$ **c** $P(x) - Q(x)$

d $Q(x) - P(x)$ **e** $P(x) \times Q(x)$ **f** $Q(x) \times P(x)$

4 If $P(x) = 5x + 2$, $Q(x) = x^2 - 3x + 1$ and $R(x) = 2x^2 - 3$, show, by simplifying the LHS and RHS separately, that:

a $(P(x) + Q(x)) + R(x) = P(x) + (Q(x) + R(x))$

b $P(x)\,(Q(x) + R(x)) = P(x)Q(x) + P(x)R(x)$

c $(P(x)Q(x))\,R(x) = P(x)\,(Q(x)R(x))$

DEVELOPMENT

5 Factor each polynomial completely, and write down all its zeroes.

a $x^3 - 8x^2 - 20x$ **b** $2x^4 - x^3 - x^2$

c $x^4 - 81$ **d** $x^4 - 5x^2 - 36$

6 For each polynomial, determine:

i the degree, **ii** the leading coefficient, **iii** the constant term.

a $(2x^3 - 3)^3$ **b** $(2x^2 + 1)(3x^3 - 2)(4x^4 + 3)(5x^5 - 4)$

7 **a** The polynomials $P(x)$ and $Q(x)$ have degrees p and q respectively, and $p \neq q$. What is the degree of:

i $P(x)Q(x)$, **ii** $P(x) + Q(x)$?

b What differences would it make if $P(x)$and $Q(x)$ both had the same degree p?

c Give an example of two polynomials, both of degree 2, which have a sum of degree 0.

8 Write down the monic polynomial whose degree, leading coefficient, and constant term are all equal.

9 Find a, b and c, given that the following pairs of polynomials are identically equal.

a $ax^2 + bx + c = 3x^2 - 4x + 1$, for all x.

b $(a - b)x^2 + (2a + b)x = 7x - x^2$, for all x.

c $a(x - 1)^2 + b(x - 1) + c = x^2$, for all x.

d $a(x + 2)^2 + b(x + 3)^2 + c(x + 4)^2 = 2x^2 + 8x + 6$, for all x.

10 **a** Suppose that $P(x) = ax^4 + bx^3 + cx^2 + dx + e$ is even, so that $P(-x) = P(x)$.
Show that $b = d = 0$.

b Suppose that $Q(x) = ax^5 + bx^4 + cx^3 + dx^2 + ex + f$ is odd, so that $Q(-x) = -Q(x)$.
Show that $b = d = f = 0$.

c Give a general statement of the situation in parts (a) and (b).

11 Suppose that $P(x)$, $Q(x)$, $R(x)$ and $S(x)$ are polynomials. Indicate whether the following statements are true or false. Provide a counter-example for any false statements.

a If $P(x)$ is even, then $P'(x)$ is odd. **b** If $Q'(x)$ is even, then $Q(x)$ is odd.

c If $R(x)$ is odd, then $R'(x)$ is even. **d** If $S'(x)$ is odd, then $S(x)$ is even.

CHALLENGE

12 **a** Find a and b so that $x^4 + 1 = (x^2 + ax + 1)(x^2 + bx + 1)$.

b Find a and b so that $x^4 + x^2 + 1 = (x^2 + ax + 1)(x^2 + bx + 1)$.

c Find a and b so that $x^4 - x^2 + 1 = (x^2 + ax + 1)(x^2 + bx + 1)$.

d Show that all the quadratic factors in parts (a), (b) and (c) are irreducible.

13 **a** What is the coefficient of x in the polynomial $B(x) = (1 + x)^n$?

b What is the coefficient of x^n in the polynomial $G(x) = (1 + x + x^2 + \cdots + x^n)^2$?

11B Graphs of polynomial functions

Learning intentions

- Sketch polynomials, using behaviour as $x \to \infty$ and as $x \to -\infty$.
- Define multiple (or repeated) zeroes, and simple (sometimes called single) zeroes.
- Take account of simple and multiple zeroes in sketches.

We have assumed that the graph of a polynomial function is continuous for all values of x. We assume also that it is *smooth*, without sharp corners anywhere.

This section will concentrate on two main concerns.

- How does the graph behave for large positive and negative values of x?
- Given the full factoring of the polynomial, how does the graph behave near its various x-intercepts?

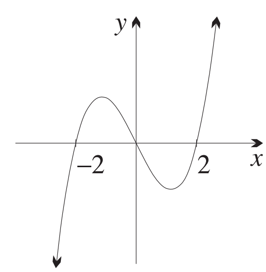
In Year 12, we will pursue further questions about stationary points and inflections that are not also zeroes of the polynomial.

The graphs of polynomial functions

It should be intuitively obvious that for large positive and negative values of x, the behaviour of the curve is governed entirely by the sign of its leading term.

For example, the cubic graph sketched on the right below is

$$P(x) = x^3 - 4x = x(x-2)(x+2).$$



For large positive values of x, the leading term x^3 completely swamps the other term $-4x$. Hence $P(x) \to \infty$ as $x \to \infty$.

Similarly, for large negative values of x, the term $-4x$ is negligible compared with the far bigger negative values of the leading term x^3. Hence $P(x) \to -\infty$ as $x \to -\infty$.

Every polynomial of odd degree has a graph that similarly disappears off diagonally opposite corners. The curve is continuous, so it must be zero somewhere at least once. Here is the general situation.

6 Behaviour of polynomials for large x

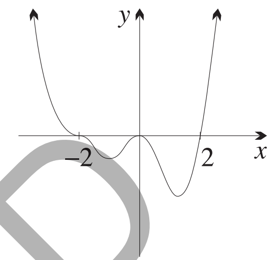
Suppose that $P(x)$ is a polynomial of degree at least 1 with leading term $a_n x^n$.

- As $x \to \infty$, $P(x) \to \infty$ if a_n is positive, and $P(x) \to -\infty$ if a_n is negative.
- As $x \to -\infty$, $P(x)$ behaves the same as when $x \to \infty$ if the degree is even, and $P(x)$ behaves in the opposite way if the degree is odd.
  - ▷ It follows that every polynomial of odd degree has at least one zero.

See the Enrichment section for a formal proof.

Zeroes and sign

If the polynomial can be completely factored, then its zeroes can be quickly read off, and we can construct a table of test values to decide its sign. Here, for example, is the table of test values and the sketch of



$$P(x) = (x+2)^3 x^2 (x-2).$$

x	-3	-2	-1	0	1	2	3
y	45	0	-3	0	-27	0	1125

The function changes sign around $x = -2$ and $x = 2$, where the associated factors $(x+2)^3$ and $(x-2)$ have odd degrees, but not around $x = 0$, where the factor x^2 has even degree.

Because the curve is smooth — without sharp corners — at $x = 0$, we know that the curve will be increasing on the left of $x = 0$, decreasing on the right of $x = 0$, and stationary at $x = 0$. This produces a *turning point* at the origin — the x-axis is a tangent there, and the curve *turns over* smoothly from increasing to stationary to decreasing without crossing the x-axis.

At $x = -2$, our table of values tells us that the curve crosses the x-axis. We shall see in Year 12 that the curve is momentarily flat there, with a *horizontal inflection* on the x-axis at $x = -2$ — the x-axis is a tangent to the curve that actually crosses the curve there. This corresponds to the factor $(x+2)^3$ having odd degree greater than 1. Proving all this requires calculus, but the result is obvious by comparison with the known graph of the very simple polynomial function $y = x^3$ that we first drew in Section 3G.

Multiple zeroes (or repeated zeroes)

Some language is needed here. Take the polynomial $P(x) = (x+2)^3 x^2 (x-2)$.

- The zero $x = -2$ is called a *triple zero*.
- The zero $x = 0$ is called a *double zero* (as with quadratics in Chapter 3).
- The zero $x = 2$ is called a *simple zero*.

The triple zero $x = -2$ and the double zero $x = 0$ are called *multiple* or *repeated zeroes*.

7 Zeroes and their multipliicity

- Suppose that $x - \alpha$ is a factor of a polynomial $P(x)$, and

  $$P(x) = (x-\alpha)^m Q(x), \quad \text{where } Q(x) \text{ is not divisible by } x - \alpha.$$

  Then $x = \alpha$ is called a *zero of multiplicity m*.
- A zero of multiplicity 1 is called a *simple zero*.
- A zero of multiplicity 2 or greater is called a *multiple zero* or a *repeated zero*.
- For low orders, we use the terms *double zero*, *triple zero*, and *quadruple zero*, ...

Note: The term *single zero* is sometimes used in place of *simple zero*, but that term is best avoided because of the constant verbal confusion that it causes.

Behaviour at simple and multiple zeroes

Here is the statement of how a polynomial graph behaves when it crosses the x-axis at a zero, as justified above as far as is possible in Year 11.

8 Multiple roots and the shape of the curve

Suppose that $x = \alpha$ is a zero of a polynomial $P(x)$.

- If $x = \alpha$ is a simple zero (sometimes called a single zero), then the curve crosses the x-axis at $x = \alpha$ at an angle, and is not tangent to the x-axis there.
- If $x = \alpha$ has even multiplicity, then the curve is tangent to the x-axis at $x = \alpha$, and does not cross the x-axis there. The point $(\alpha, 0)$ is a *turning point*.
- If $x = \alpha$ has odd multiplicity at least 3, then the curve is tangent to the x-axis at $x = \alpha$, but crosses the x-axis. The point $(\alpha, 0)$ is a *horizontal inflection*.

Note: When we were dealing only with quadratics in Chapter 3, we often used the terms ‘equal roots’ and ‘distinct roots’. Never use those terms in the context of polynomials, because of the verbal ambiguities that inevitably arise.

Example 2 Sketching factored polynomials near simple and multiple zeroes

In each part, name the zeroes and their multiplicity, and then sketch, showing the behaviour near any x-intercepts:

a $P(x) = (x - 1)^2(x - 2)$

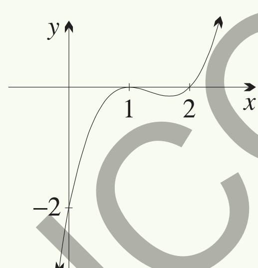
b $Q(x) = x^3(x + 2)^4(x^2 + x + 1)$

c $R(x) = -2(x - 2)^2(x + 1)^5(x - 1)$

Solution

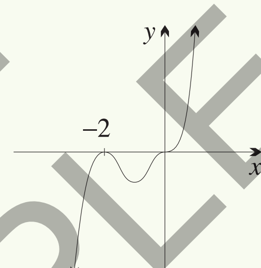
In part (b), $x^2 + x + 1$ has no zeroes, because $\Delta = 1 - 4 < 0$.

a



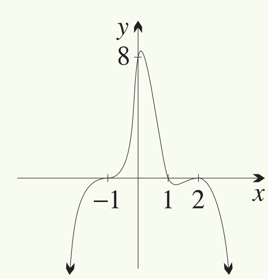
1 is a double zero,
2 is a simple zero.

b



−2 is a double zero,
0 is a triple zero.

c



−1 is a quintuple zero,
1 is a simple zero,
2 is a double zero.

Example 3 Simple and multiple zeroes

Give an example of a polynomial factored into linear factors that:

a has two multiple (or repeated) zeroes, and three simple zeroes (sometimes called ‘single zeroes’).

b has degree 9, and has two triple zeroes and one double zero, and is non-monic.

Solution

a $(x + 5)^2(x - 1)(x - 2)(x - 3)x^7$

b $12(x + 8)^3(x + 7)^3(x + 5)^2(x + 3)$

Exercise 11B

FOUNDATION

1 Write down the zeroes of each polynomial, and classify each zero as a simple zero or a multiple zero.

a $P(x) = (x - 3)(x - 6)^2$ **b** $P(x) = x^3(x + 1)$ **c** $P(x) = 8(x + 5)^4(x - 7)^2(x + 10)$

Note: Readers should be aware that the term 'repeated zero' is often used for 'multiple zero', and that the term 'single zero' is sometimes used for 'simple zero'.

2 Write down the zeroes of each polynomial, and state the multiplicity of each zero.

a $P(x) = (x + 2)(x - 3)^2$ **b** $P(x) = x(x + 7)^3(x - 5)$ **c** $P(x) = (1 - x)^3(3 + x)^5$

3 Write down the roots of each polynomial equation, and identify each root as a simple, double, triple or quadruple root.

a $(x - 2)^2(x + 3) = 0$ **b** $(4 - x)^3(9 + x)^4 = 0$ **c** $x^4(x - 6)^2(x + 8)^3 = 0$

4 Sketch the graphs of these linear polynomials, clearly indicating all intercepts with the axes.

a $P(x) = 2$ **b** $P(x) = x$ **c** $P(x) = x - 4$ **d** $P(x) = 3 - 2x$

5 Sketch the graphs of these quadratic polynomials, clearly indicating all intercepts with the axes.

a $P(x) = x^2$ **b** $P(x) = (x - 1)(x + 3)$ **c** $P(x) = (x - 2)^2$

d $P(x) = 9 - x^2$ **e** $P(x) = 2x^2 + 5x - 3$ **f** $P(x) = 4 + 3x - x^2$

6 Sketch the graphs of these cubic polynomials, clearly indicating all intercepts with the axes.

a $y = x^3$ **b** $y = x^3 + 2$ **c** $y = (x - 4)^3$

d $y = (x - 1)(x + 2)(x - 3)$ **e** $y = x(2x + 1)(x - 5)$ **f** $y = (1 - x)(1 + x)(2 + x)$

g $y = (2x + 1)^2(x - 4)$ **h** $y = x^2(1 - x)$ **i** $y = (2 - x)^2(5 - x)$

7 Sketch the graphs of these quartic polynomials, clearly indicating all intercepts with the axes. Identify all double zeroes, triple zeroes, and quadruple zeroes,

a $F(x) = x^4$ **b** $F(x) = (x + 2)^4$

c $F(x) = x(3x + 2)(x - 3)(x + 2)$ **d** $F(x) = (1 - x)(x + 5)(x - 7)(x + 3)$

e $F(x) = x^2(x + 4)(x - 3)$ **f** $F(x) = (x + 2)^3(x - 5)$

g $F(x) = (2x - 3)^2(x + 1)^2$ **h** $F(x) = (1 - x)^3(x - 3)$

i $F(x) = (2 - x)^2(1 - x^2)$

DEVELOPMENT

8 These polynomials are not factored, but the positions of their zeroes can be found by trial and error. Copy and complete each table of values, then sketch the graph, and state how many zeroes there are, and between which integers they lie.

a $y = x^2 - 3x + 1$

x	−1	0	1	2	3	4
y						

b $y = 1 + 3x - x^3$

x	−2	−1	0	1	2	3
y						

9 Sketch each polynomial function, clearly indicating all intercepts with the axes.

a $P(x) = x(x - 2)^3(x + 1)^2$ **b** $P(x) = (x + 2)^2(3 - x)^3$

c $P(x) = x(2x + 3)^3(1 - x)^4$ **d** $P(x) = (x + 1)(4 - x^2)(x^2 - 3x - 10)$

10 Use the graphs drawn in the previous question to solve these inequations.

a $x(x-2)^3(x+1)^2 > 0$

b $(x+2)^2(3-x)^3 \geq 0$

c $x(2x+3)^3(1-x)^4 \geq 0$

d $(x+1)(4-x^2)(x^2-3x-10) < 0$

11 Factor each polynomial, then sketch it, showing all intercepts with the axes.

a $P(x) = x^4 - 13x^2 + 36$

b $P(x) = 4x^4 - 13x^2 + 9$

c $P(x) = (x^2-5x)^2 - 2(x^2-5x) - 24$

d $P(x) = (x^2-3x+1)^2 - 4(x^2-3x+1) - 5$

CHALLENGE

---

12 Consider the polynomial $P(x) = x^4 - 5x^2 + 4x + 13$.

a Show that $P(x)$ can be expressed in the form $(x^2-a)^2 + (x-b)^2$.

b How many x-intercepts does the graph of $P(x)$ have? Explain your answer.

13 At what points do the graphs of the polynomials $f(x) = (x+1)^n$ and $g(x) = (x+1)^m$ intersect, where $m \neq n$? (Hint: Consider the cases where m and n are odd and even.)

14 **To prove:** *Let $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial of degree at least 1. Then the leading term dominates the behaviour of $P(x)$ for large positive values of x, and for large negative values of x.*

a Write down $\dfrac{P(x)}{x^n}$.

b Take the limit of this expression as $x \to \infty$.

c Draw a conclusion about the behaviour of $P(x)$ as $x \to \infty$.

d Draw a conclusion about the behaviour of $P(x)$ as $x \to -\infty$.

e What conclusion, if any, can you draw about the zeroes of a polynomial of odd degree, and about the zeroes of a polynomial of even degree?

15 **To prove:** *Let $P(x)$ be a polynomial such that $P(x) = 0$ for all x. Then $P(x)$ is the zero polynomial $Z(x) = 0$ that has no term with a non-zero coefficient.*

a Use the previous question to prove that $P(x)$ cannot have degree ≥ 1, and hence is a constant polynomial $P(x) = a_0$.

b Explain why the constant a_0 cannot be non-zero.

16 **To prove:** *If $P(x) = Q(x)$ for all x, then $P(x)$ and $Q(x)$ have the same coefficients.*
Use the previous question, and the polynomial $A(x) = P(x) - Q(x)$, to prove this result.

11C Division of polynomials

Learning intentions

- Apply the division algorithm for polynomials, and compare it with integer division.

Section 11A had examples of adding, subtracting, and multiplying polynomials, operations that are quite straightforward. The division of one polynomial by another, however, is more elaborate.

Division of polynomials

It can happen that the quotient of two polynomials is again a polynomial:

$$\frac{6x^3 + 4x^2 - 9x}{3x} = 2x^2 + \tfrac{4}{3}x - 3 \qquad \text{and} \qquad \frac{x^2 + 4x - 5}{x + 5} = x - 1.$$

But usually, division results in rational functions, not polynomials:

$$\frac{x^4 + 4x^2 - 9}{x^2} = x^2 + 4 - \frac{9}{x^2} \qquad \text{and} \qquad \frac{x + 4}{x + 3} = 1 + \frac{1}{x + 3}.$$

There is a close analogy here between the set **Z** of all integers and the set of all polynomials. In both cases, everything works nicely for addition, subtraction, and multiplication, but the results of division do not usually lie within the set. For example, although $20 \div 5 = 4$ is an integer, the division of two integers usually results in a fraction rather than an integer, as in $23 \div 5 = 4\tfrac{3}{5}$.

In both cases, the best way to handle division is to use remainders.

The division algorithm for integers

```
       16     remainder 5
12 | 197
     12
     --
      77
      72
      --
       5
```

On the right is an example of the well-known long division algorithm for integers, applied here to $197 \div 12$. The number 12 is called the *divisor*, 197 is called the *dividend*, 16 is called the *quotient*, and 5 is called the *remainder*.

The result of the division can be written as $\frac{197}{12} = 16\frac{5}{12}$, but we can avoid fractions completely by writing the result as:

$$197 = 12 \times 16 + 5, \qquad \text{dividend} = \text{divisor} \times \text{quotient} + \text{remainder}.$$

The remainder 5 must be less than 12, otherwise the division process could be continued. Thus the general result for division of integers can be expressed as:

9 Division of integers

- Let p (the *dividend*) and d (the *divisor*) be integers, with $d > 0$. Then there are unique integers q (the *quotient*) and r (the *remainder*) such that

  $$p = dq + r \qquad \text{and} \qquad 0 \le r < d.$$

- When the remainder r is zero, d is a *divisor of* p, and the integer p *factors* as

  $$p = d \times q.$$

The division algorithm for polynomials

The method of dividing one polynomial by another is similar to the method of dividing integers.

10 The method of long division of polynomials

- At each step, divide the leading term of the remainder by the leading term of the divisor. Continue the process for as long as possible.
- Unless otherwise specified, express the final answer in the form

  dividend = divisor × quotient + remainder.

Note on missing terms: In the example below, it is vital to notice that there is no term in x^2 in the dividend. There are two approaches to deal with this:

- Leave a gap for the x^2 column. This is what has been done below.
- If you prefer not to have any gaps, write in the missing term $0x^2$.

Why not try both methods? Both are very satisfactory.

Example 4 Long division of polynomials

Divide $3x^4 - 4x^3 + 4x - 8$ by:

a $x - 2$

b $x^2 - 2$

Give results first in the standard manner, then using rational functions.

Solution

In each part, the steps have been annotated to explain the method.

a

$3x^3 + 2x^2 + 4x + 12$	
$x - 2 \,\big)\, 3x^4 - 4x^3 \qquad + 4x - 8$	(divide x into $3x^4$, giving the $3x^3$ above)
$3x^4 - 6x^3$	(multiply $x - 2$ by $3x^3$ and then subtract)
$2x^3 \qquad + 4x - 8$	(divide x into $2x^3$, giving the $2x^2$ above)
$2x^3 - 4x^2$	(multiply $x - 2$ by $2x^2$ and then subtract)
$4x^2 + 4x - 8$	(divide x into $4x^2$, giving the $4x$ above)
$4x^2 - 8x$	(multiply $x - 2$ by $4x$ and then subtract)
$12x - 8$	(divide x into $12x$, giving the 12 above)
$12x - 24$	(multiply $x - 2$ by 12 and then subtract)
16	(this is the final remainder)

Hence $3x^4 - 4x^3 + 4x - 8 = (x - 2)(3x^3 + 2x^2 + 4x + 12) + 16$,
or, writing the result using rational functions,

$$\frac{3x^4 - 4x^3 + 4x - 8}{x - 2} = 3x^3 + 2x^2 + 4x + 12 + \frac{16}{x - 2}.$$

b

$3x^2 - 4x + 6$	
$x^2 - 2 \,\big)\, 3x^4 - 4x^3 \qquad + 4x - 8$	(divide x^2 into $3x^4$, giving the $3x^2$ above)
$3x^4 \qquad - 6x^2$	(multiply $x^2 - 2$ by $3x^2$ and then subtract)
$-4x^3 + 6x^2 + 4x - 8$	(divide x^2 into $-4x^3$, giving the $-4x$ above)
$-4x^3 \qquad + 8x$	(multiply $x^2 - 2$ by $-4x$ and then subtract)
$6x^2 - 4x - 8$	(divide x^2 into $6x^2$, giving the 6 above)
$6x^2 \qquad - 12$	(multiply $x^2 - 2$ by 6 and then subtract)
$-4x + 4$	(this is the final remainder)

Hence $3x^4 - 4x^3 + 4x - 8 = (x^2 - 2)(3x^2 - 4x + 6) + (-4x + 4)$,

or $\dfrac{3x^4 - 4x^3 + 4x - 8}{x^2 - 2} = 3x^2 - 4x + 6 + \dfrac{-4x + 4}{x^2 - 2}$.

The division theorem

The division process illustrated above can be continued until the remainder is zero or has degree less than the degree of the divisor. Thus the general result for polynomial division is:

11 Division of polynomials

- Suppose that $P(x)$ (the *dividend*) and $D(x)$ (the *divisor*) are polynomials with $D(x) \neq 0$. Then there are unique polynomials $Q(x)$ (the *quotient*) and $R(x)$ (the *remainder*) such that:
  1 $P(x) = D(x)Q(x) + R(x)$,
  2 either $\deg R(x) < \deg D(x)$, or $R(x) = 0$.
- When the remainder $R(x)$ is zero, then $D(x)$ is called a *divisor* of $P(x)$, and the polynomial $P(x)$ *factors* into the product $P(x) = D(x) \times Q(x)$.

For example, in the two long divisions in Example 4 above:

- In part (a), the remainder after division by the degree 1 polynomial $x - 2$ is the polynomial 16 of degree 0.
- In part (b), the remainder after division by the degree 2 polynomial $x^2 - 2$ is the linear polynomial $-4x + 4$ of degree 1.

The uniqueness of the quotient and remainder in the first dotpoint above is proven in the Enrichment section of the following exercise.

Exercise 11C

FOUNDATION

1 Perform each integer division, and write the result in the form $p = dq + r$, where $0 \leq r < d$. For example, $30 \div 7 = 4$, remainder 2, so $30 = 4 \times 7 + 2$.

a $63 \div 5$ **b** $125 \div 8$ **c** $324 \div 11$ **d** $1857 \div 23$

2 Use long division to perform each division. Express each result in the form $P(x) = D(x)\,Q(x) + R(x)$.

a $(x^2 - 4x + 1) \div (x + 1)$ **b** $(x^2 - 6x + 5) \div (x - 5)$

c $(x^3 - x^2 - 17x + 24) \div (x - 4)$ **d** $(2x^3 - 10x^2 + 15x - 14) \div (x - 3)$

e $(4x^3 - 4x^2 + 7x + 14) \div (2x + 1)$ **f** $(x^4 + x^3 - x^2 - 5x - 3) \div (x - 1)$

g $(6x^4 - 5x^3 + 9x^2 - 8x + 2) \div (2x - 1)$ **h** $(10x^4 - x^3 + 3x^2 - 3x - 2) \div (5x + 2)$

3 Express the answers to parts (a)–(d) of the previous question in rational form, that is, as $\dfrac{P(x)}{D(x)} = Q(x) + \dfrac{R(x)}{D(x)}$.

4 Use long division to perform each division. Express each result in the standard form $P(x) = D(x)\,Q(x) + R(x)$.

a $(x^3 + x^2 - 7x + 6) \div (x^2 + 3x - 1)$ **b** $(x^3 - 4x^2 - 2x + 3) \div (x^2 - 5x + 3)$

c $(x^4 - 3x^3 + x^2 - 7x + 3) \div (x^2 - 4x + 2)$ **d** $(2x^5 - 5x^4 + 12x^3 - 10x^2 + 7x + 9) \div (x^2 - x + 2)$

5 **a** If the divisor of a polynomial has degree 3, what are the possible degrees of the remainder?

b On division by $D(x)$, a polynomial has remainder $R(x)$ of degree 2. What are the possible degrees of $D(x)$?

DEVELOPMENT

6 Use long division to perform each division. Take care to ensure that the columns line up correctly. Express each result in the form $P(x) = D(x)\,Q(x) + R(x)$.

a $(x^3 - 5x + 3) \div (x - 2)$

b $(2x^3 + x^2 - 11) \div (x + 1)$

c $(x^3 - 3x^2 + 5x - 4) \div (x^2 + 2)$

d $(2x^4 - 5x^2 + x - 2) \div (x^2 + 3x - 1)$

e $(2x^3 - 3) \div (2x - 4)$

f $(x^5 + 3x^4 - 2x^2 - 3) \div (x^2 + 1)$

Write the answers to parts (c) and (f) above in rational form, that is, in the form $\dfrac{P(x)}{D(x)} = Q(x) + \dfrac{R(x)}{D(x)}$.

7 **a** Use long division to show that $P(x) = x^3 + 2x^2 - 11x - 12$ is divisible by $x - 3$, and hence express $P(x)$ as the product of three linear factors.

b Find the values of x for which $P(x) > 0$.

8 **a** Use long division to show that $F(x) = 2x^4 + 3x^3 - 12x^2 - 7x + 6$ is divisible by $x^2 - x - 2$, and hence express $F(x)$ as the product of four linear factors.

b Find the values of x for which $F(x) \le 0$.

9 **a** Find the quotient and remainder when $x^4 - 2x^3 + x^2 - 5x + 7$ is divided by $x^2 + x - 1$.

b Hence find a and b so that $x^4 - 2x^3 + x^2 + ax + b$ is exactly divisible by $x^2 + x - 1$.

10 **a** Use long division to divide the polynomial $f(x) = x^4 - x^3 + x^2 - x + 1$ by the polynomial $d(x) = x^2 + 4$. Express your answer in the form $f(x) = d(x)q(x) + r(x)$.

b Hence find the values of c and d such that $x^4 - x^3 + x^2 + cx + d$ is divisible by $x^2 + 4$.

11 If $x^4 - 2x^3 - 20x^2 + mx + n$ is exactly divisible by $x^2 - 5x + 2$, find m and n.

12 Suppose that $P(x) = x^4 + x^3 - 5x^2 - 22x + 5$ and $D(x) = x^2 + 3x + 5$.

a Find the polynomials $Q(x)$ and $R(x)$, where $R(x)$ is of lower degree than $D(x)$, so that $P(x) = D(x)\,Q(x) + R(x)$.

b Hence explain why $P(x)$ and $D(x)$ cannot have a common zero.

CHALLENGE

13 Consider the cubic equation $x^3 - kx + (k + 11) = 0$.

a Use long division to show that $k = x^2 + x + 1 + \dfrac{12}{x - 1}$.

b Hence find all the integer values of k for which the equation has at least one *positive* integer solution for x.

14 **To prove:** *Let $P(x)$ and $D(x)$ be polynomials with $D(x) \ne 0$. Then there exists unique polynomials $Q(x)$ and $R(x)$ such that $P(x) = D(x)Q(x) + R(x)$ and either $R(x) = 0$ or $\deg R(x) < \deg D(x)$.*
Existence follows from the long division algorithm, two examples of which have been given. To prove uniqueness, let $Q_1(x)$ and $R_1(x)$ be polynomials such that

$$P(x) = D(x)Q_1(x) + R_1(x), \text{ and either } R_1(x) = 0 \text{ or } \deg R_1(x) < \deg D(x).$$

a Show that $D(x)\big(Q(x) - Q_1(x)\big) = R_1(x) - R(x)$.

b Use the possible degrees of LHS and RHS to prove the result.

11D The remainder and factor theorems

Learning intentions

- Prove and use the remainder theorem to find remainders.
- Prove and use the factor theorem to find zeroes.
- Identify possible zeroes by factoring the constant term.
- Combine the factor theorem and long division to factor a polynomial.

Long division of polynomials is a cumbersome process. It is therefore useful to have two theorems, called the *remainder theorem* and the *factor theorem*, that provide information about the results of a division without the division actually being carried out. In particular, the factor theorem gives a simple test whether a particular linear polynomial is a factor.

The remainder theorem

The remainder theorem is a remarkable result which, in the case of linear divisors, allows the remainder to be found without the long division ever being performed.

12 The remainder theorem

Suppose that $P(x)$ is a polynomial and α is a constant.
Then the remainder after division of $P(x)$ by $x - \alpha$ is $P(\alpha)$.

Proof Because $x - \alpha$ is a polynomial of degree 1, the division theorem tells us that there are unique polynomials $Q(x)$ and $R(x)$ such that

$$P(x) = (x - \alpha)Q(x) + R(x),$$

where either $R(x) = 0$ or $\deg R(x) = 0$.

Hence $R(x)$ is zero or a non-zero constant, which we can write more simply as r,

so that $P(x) = (x - \alpha)Q(x) + r$.

Substituting $x = \alpha$ gives $P(\alpha) = (\alpha - \alpha)Q(\alpha) + r$

and rearranging, $r = P(\alpha)$, as required.

Example 5 **Applying the remainder theorem**

Find the remainder when $3x^4 - 4x^3 + 4x - 8$ is divided by $x - 2$:

a by long division, **b** by the remainder theorem.

Solution

a In Example 4 of Section 11C, performing the division showed that

$$3x^4 - 4x^3 + 4x - 8 = (x - 2)(3x^3 + 2x^2 + 4x + 12) + 16,$$

that is, the remainder is 16.

b Alternatively, substituting $x = 2$ into $P(x)$,

$$\begin{aligned} \text{remainder} &= P(2) \qquad \text{(This is the remainder theorem.)} \\ &= 48 - 32 + 8 - 8 \\ &= 16, \quad \text{as expected.} \end{aligned}$$

Example 6 **Finding coefficients using the remainder theorem**

The polynomial $P(x) = x^4 - 2x^3 + ax + b$ has remainder 3 after division by $x - 1$, and has remainder -5 after division by $x + 1$. Find a and b.

Solution

Applying the remainder theorem for each divisor,

$$P(1) = 3$$
$$1 - 2 + a + b = 3$$
$$a + b = 4. \qquad (1)$$

Also $P(-1) = -5$
$$1 + 2 - a + b = -5$$
$$-a + b = -8. \qquad (2)$$

Adding (1) and (2), $2b = -4$,

and subtracting them, $2a = 12$.

Hence $a = 6$ and $b = -2$.

The factor theorem

The remainder theorem tells us that the number $P(\alpha)$ is the remainder after division by $x - \alpha$. But $x - \alpha$ is a factor if and only if the remainder after division by $x - \alpha$ is zero, so:

13 The factor theorem

Suppose that $P(x)$ is a polynomial and α is a constant.
Then $x - \alpha$ is a factor of $P(x)$ if and only if $P(\alpha) = 0$.

This is a quick and easy way to test whether $x - \alpha$ is a factor of $P(x)$.

Example 7 **Using the factor theorem and long division to factor a polynomial**

a Show that $x - 3$ is a factor of $P(x) = x^3 - 2x^2 + x - 12$, and $x + 1$ is not.
b Then use long division to factor the polynomial completely.

Solution

a $P(3) = 27 - 18 + 3 - 12 = 0$, so $x - 3$ is a factor.
$P(-1) = -1 - 2 - 1 - 12 = -16 \neq 0$, so $x + 1$ is not a factor.

b Long division of $P(x) = x^3 - 2x^2 + x - 12$ by $x - 3$ (which we omit) gives

$$P(x) = (x - 3)(x^2 + x + 4),$$

and because $\Delta = 1 - 16 = -15 < 0$ for the quadratic, this factoring is complete.

Factoring polynomials — the initial approach

The factor theorem gives us the beginnings of an approach to factoring polynomials. This approach will be further refined in the next two sections.

14 Factoring polynomials — the initial approach

- Use trial and error to find an integer zero $x = \alpha$ of $P(x)$.
- Then use long division to factor $P(x)$ in the form $P(x) = (x - \alpha)Q(x)$.

If the coefficients of $P(x)$ are all integers, then all the integer zeroes of $P(x)$ are divisors of the constant term.

Proof We must prove the claim that if the coefficients of $P(x)$ are integers, then every integer zero of $P(x)$ is a divisor of the constant term.

Let $$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0,$$

where the coefficients $a_n, a_{n-1}, \ldots a_1, a_0$ are all integers, and let $x = \alpha$ be an integer zero of $P(x)$.

Substituting into $P(\alpha) = 0$ gives

$$a_n\alpha^n + a_{n-1}\alpha^{n-1} + \cdots + a_1\alpha + a_0 = 0$$

$$\begin{aligned} a_0 &= -a_n\alpha^n - a_{n-1}\alpha^{n-1} - \cdots - a_1\alpha \\ &= \alpha(-a_n\alpha^{n-1} - a_{n-1}\alpha^{n-2} - \cdots - a_1), \end{aligned}$$

so a_0 is an integer multiple of α.

Example 8 Factoring the constant term to find possible zeroes

Factor $P(x) = x^4 + x^3 - 9x^2 + 11x - 4$ completely.

Solution

Because all the coefficients are integers, any integer zero is a divisor of the constant term -4. Thus we test 1, 2, 4, -1, -2 and -4.

$$\begin{aligned} P(1) &= 1 + 1 - 9 + 11 - 4 \\ &= 0, \quad \text{so } x - 1 \text{ is a factor.} \end{aligned}$$

After long division (omitted), $P(x) = (x - 1)(x^3 + 2x^2 - 7x + 4)$.

Let $Q(x) = x^3 + 2x^2 - 7x + 4$, then
$$\begin{aligned} Q(1) &= 1 + 2 - 7 + 4 \\ &= 0, \quad \text{so } x - 1 \text{ is a factor.} \end{aligned}$$

Again after long division (omitted), $P(x) = (x - 1)(x - 1)(x^2 + 3x - 4)$.

Factoring the quadratic, $P(x) = (x - 1)^3(x + 4)$.

Note: In the next two sections, and again in Year 12, we will develop methods that will often allow long division to be avoided.

Exercise 11D

FOUNDATION

1 Without division, find the remainder when $P(x) = x^3 - x^2 + 2x + 1$ is divided by:

a $x - 1$ **b** $x - 3$ **c** $x + 2$
d $x + 1$ **e** $x - 5$ **f** $x + 3$

2 Without division, find which of the following are factors of $F(x) = x^3 + 4x^2 + x - 6$.

a $x - 1$ **b** $x + 1$ **c** $x - 2$

d $x + 2$ **e** $x - 3$ **f** $x + 3$

3 **a** Find k, if $x - 1$ is a factor of $P(x) = x^3 - 3x^2 + kx - 2$.

b Find m, if -2 is a zero of the function $F(x) = x^3 + mx^2 - 3x + 4$.

c When the polynomial $P(x) = 2x^3 - x^2 + px - 1$ is divided by $x - 3$, the remainder is 2. Find p.

d For what value of a is $3x^4 + ax^2 - 2$ divisible by $x + 1$?

4 Factor each polynomial and sketch its graph, indicating all intercepts with the axes. Remember that any integer zeroes are divisors of the constant term.

a $P(x) = x^3 + 2x^2 - 5x - 6$ **b** $P(x) = x^3 + 3x^2 - 25x + 21$

c $P(x) = -x^3 + x^2 + 5x + 3$ **d** $P(x) = x^4 - x^3 - 19x^2 - 11x + 30$

5 Solve these equations by first factoring the LHS.

a $x^3 + 3x^2 - 6x - 8 = 0$ **b** $x^3 - 4x^2 - 3x + 18 = 0$

c $6x^3 - 5x^2 - 12x - 4 = 0$ **d** $2x^4 + 11x^3 + 19x^2 + 8x - 4 = 0$

DEVELOPMENT

6 **a** Show that $P(x) = x^3 - 8x^2 + 9x + 18$ is divisible by both $x - 3$ and $x + 1$.

b By considering the leading term and constant term, express $P(x)$ as a product of three linear factors.

7 **a** Show that $P(x) = 2x^3 - x^2 - 13x - 6$ is divisible by both $x - 3$ and $2x + 1$.

b By considering the leading term and constant term, express $P(x)$ as a product of three linear factors.

8 **a** When the polynomial $P(x) = 2x^3 - x^2 + ax + b$ is divided by $x - 1$ the remainder is 16, and when it is divided by $x + 2$ the remainder is -17. Find a and b.

b The polynomial $P(x)$ is given by $P(x) = x^3 + ax^2 + bx - 18$. Find a and b given that $x + 2$ is a factor of $P(x)$, and -24 is the remainder when $P(x)$ is divided by $x - 1$.

9 Without division, find the remainder when $P(x) = x^3 + 2x^2 - 4x + 5$ is divided by:

a $2x - 1$ **b** $2x + 3$ **c** $3x - 2$

10 **a** If $P(x) = 2x^3 + x^2 - 13x + 6$, evaluate $P\left(\frac{1}{2}\right)$. Hence use long division to express $P(x)$ in fully factored form.

b Given $P(x) = 6x^3 + x^2 - 5x - 2$, evaluate $P\left(-\frac{2}{3}\right)$ then express $P(x)$ in factored form.

11 Is either $x + 1$ or $x - 1$ a factor of $x^n + 1$, where n is a positive integer?
(Hint: Consider the cases where n is even or odd separately.)

12 **a** $P(x)$ is an odd polynomial of degree 3. It has $x + 4$ as a factor, and when it is divided by $x - 3$ the remainder is 21. Find $P(x)$.

b Find p so that $x - p$ is a factor of $4x^3 - (10p - 1)x^2 + (6p^2 - 5)x + 6$.

13 When the polynomial $P(x)$ is divided by $(x - 1)(x + 3)$, the quotient is $Q(x)$ and the remainder is $2x + 5$.

a Write down a division identity based on this information.

b Hence, by evaluating $P(1)$, find the remainder when $P(x)$ is divided by $x - 1$.

c What is the remainder when $P(x)$ is divided by $x + 3$?

14 The polynomial $P(x)$ is divided by $(x-1)(x+2)$. Suppose that the quotient is $Q(x)$ and the remainder is $R(x)$.

a Explain why the general form of $R(x)$ is $ax+b$, where a and b are constants.

b If $P(1)=2$ and $P(-2)=5$, find a and b. (HINT: Use the division identity.)

15 The polynomial $P(x)$ is divided by $(x+4)(x-3)$. If $P(-4)=11$ and $P(3)=-3$, use the same approach as the previous question to find the remainder.

16 **a** When a polynomial is divided by $(2x+1)(x-3)$, the remainder is $3x-1$. What is the remainder when the polynomial is divided by $2x+1$?

b When x^5+3x^3+ax+b is divided by x^2-1, the remainder is $2x-7$. Find a and b.

c When a polynomial $P(x)$ is divided by x^2-5, the remainder is $x+4$. Find the remainder when $P(x)+P(-x)$ is divided by x^2-5. (HINT: Write down the division identity.)

CHALLENGE

17 When the polynomial $P(x)$ is divided by x^2-k^2, the remainder is $ax+b$.

a Show that $a=\frac{1}{2k}\left(P(k)-P(-k)\right)$ and $b=\frac{1}{2}\left(P(k)+P(-k)\right)$.

b Given that $P(x)=8x^5-4x^4+6x^3-11x^2-2x+3$ and $k=\frac{1}{2}$, find a and b, and hence factor $P(x)$ fully.

18 **a** Use the factor theorem to prove that $a+b+c$ is a factor of $a^3+b^3+c^3-3abc$. Then find the other factor. (HINT: Regard it as a polynomial in a.)

b Factor $ab^3-ac^3+bc^3-ba^3+ca^3-cb^3$.

19 **a** If all the coefficients of a monic polynomial are integers, prove that all the rational zeroes are integers. (HINT: Look carefully at the proof under Box 14.)

b If all the coefficients of a polynomial are integers, prove that the denominators of all the rational zeroes (in lowest terms) are divisors of the leading coefficient.

11E Consequences of the factor theorem

Learning intentions

- Develop consequences of the factor theorem to help factor a polynomial.
- Develop bounds on the number of zeroes of a polynomial of a certain degree.
- Develop a test for two polynomials to be identically equal.
- Develop geometric consequences of the factor theorem.

The factor theorem has a number of straightforward but very useful consequences. They are presented here as six successive theorems.

A. Several distinct zeroes

Suppose that several distinct zeroes of a polynomial have been found, probably using test substitutions into the polynomial.

15 Distinct zeroes

Suppose that $\alpha_1, \alpha_2, \dots \alpha_s$ are distinct zeroes of a polynomial $P(x)$. Then $(x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_s)$ is a factor of $P(x)$.

Proof Because α_1 is a zero, $x - \alpha_1$ is a factor, and $P(x) = (x - \alpha_1)P_1(x)$.
Because $P(\alpha_2) = 0$ but $\alpha_2 - \alpha_1 \neq 0$, $P_1(\alpha_2)$ must be zero.
Hence $x - \alpha_2$ is a factor of $P_1(x)$, and $P_1(x) = (x - \alpha_2)P_2(x)$, so $P(x) = (x - \alpha_1)(x - \alpha_1)P_2(x)$.
Continuing similarly for s steps, $(x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_s)$ is a factor of $P(x)$.

B. All distinct zeroes

If n distinct zeroes of a polynomial of degree n can be found, then the factoring is complete, and the polynomial is the product of distinct linear factors.

16 All distinct zeroes

If $\alpha_1, \alpha_2, \dots, \alpha_n$ are n distinct zeroes of a polynomial $P(x)$ of degree n, then

$$P(x) = a(x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_n),$$

where a is the leading coefficient of $P(x)$.

Proof By the previous theorem, $(x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_n)$ is a factor of $P(x)$,
so $P(x) = (x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_n)Q(x)$, for some polynomial $Q(x)$.
But $P(x)$ and $(x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_n)$ both have degree n, so $Q(x)$ is a constant.
Equating coefficients of x^n, the constant $Q(x)$ is the leading coefficient.

Factoring polynomials — finding several zeroes first

If we can find several zeroes of a polynomial, then we have a quadratic or cubic factor, and the long divisions required can be reduced, or even avoided completely.

17 Factoring polynomials — finding several zeroes first

- Use trial and error to find as many integer zeroes of $P(x)$ as possible.
- Using long division, divide $P(x)$ by the product of the known factors.

If the coefficients of $P(x)$ are all integers, then any integer zero of $P(x)$ must be one of the divisors of the constant term.

When this procedure is applied to the polynomial factored in the previous section, one rather than two long divisions is required.

Example 9 Finding several zeroes before long division

Factor $P(x) = x^4 + x^3 - 9x^2 + 11x - 4$ completely (done in Example 8).

Solution

As before, all the coefficients are integers, so any integer zero is a divisor of the constant term −4. That is, we test 1, 2, 4, −1, −2 and −4.

$$P(1) = 1 + 1 - 9 + 11 - 4 = 0, \quad \text{so } x - 1 \text{ is a factor.}$$
$$P(-4) = 256 - 64 - 144 - 44 - 4 = 0, \quad \text{so } x + 4 \text{ is a factor.}$$

After long division by $(x - 1)(x + 4) = x^2 + 3x - 4$ (omitted),

$$P(x) = (x^2 + 3x - 4)(x^2 - 2x + 1).$$

Factoring both quadratics,
$$\begin{aligned} P(x) &= (x - 1)(x + 4) \times (x - 1)^2 \\ &= (x - 1)^3(x + 4). \end{aligned}$$

Note: The methods of the next section will allow this particular factoring to be done with no long divisions.

The next example involves a polynomial that factors into distinct linear factors — nothing more than the factor theorem is required to complete the task.

Example 10 Finding all the zeroes by the factor theorem

Factor $P(x) = x^4 - x^3 - 7x^2 + x + 6$ completely.

Solution

The divisors of the constant term 6 are 1, 2, 3, 6, −1, −2, −3 and −6.

$$P(1) = 1 - 1 - 7 + 1 + 6 = 0, \quad \text{so } x - 1 \text{ is a factor.}$$
$$P(-1) = 1 + 1 - 7 - 1 + 6 = 0, \quad \text{so } x + 1 \text{ is a factor.}$$
$$P(2) = 16 - 8 - 28 + 2 + 6 = -12 \neq 0, \quad \text{so } x - 2 \text{ is not a factor.}$$
$$P(-2) = 16 + 8 - 28 - 2 + 6 = 0, \quad \text{so } x + 2 \text{ is a factor.}$$
$$P(3) = 81 - 27 - 63 + 3 + 6 = 0, \quad \text{so } x - 3 \text{ is a factor.}$$

We now have four distinct zeroes of a polynomial of degree 4.

Hence $P(x) = (x - 1)(x + 1)(x + 2)(x - 3)$. (Notice that $P(x)$ is monic.)

C. The maximum number of zeroes

If a polynomial of degree n had $n + 1$ zeroes, then by the first theorem above (Box 15), it would be divisible by a polynomial of degree $n + 1$, which is impossible.

18 Maximum number of zeroes

A polynomial of degree n has at most n zeroes.

D. A vanishing condition

The previous theorem translates easily into a condition for a polynomial to be the zero polynomial.

19 A vanishing condition

- Suppose that $P(x)$ is a polynomial that has no term of degree more than n, yet is zero for at least $n + 1$ distinct values of x. Then $P(x)$ is the zero polynomial.
- In particular, the only polynomial that is zero for all values of x is the zero polynomial (as was proven in Exercise 11B Enrichment).

Proof Suppose that $P(x)$ had a degree. This degree must be at most n because there is no term of degree more than n. But the degree must also be at least $n + 1$ because there are $n + 1$ distinct zeroes. This is a contradiction, so $P(x)$ has no degree, and is therefore the zero polynomial.

Note: This again highlights the fact that the zero polynomial $Z(x) = 0$ is quite different in nature from all other polynomials. It is the only polynomial with an infinite number of zeroes — in fact every real number is a zero of $Z(x)$. Associated with this is the fact that $x - \alpha$ is a factor of $Z(x)$ for all real values of α, because $Z(x) = (x - \alpha)Z(x)$ (which is trivially true, because both sides are zero for all x). No wonder then that the zero polynomial does not have a degree !

E. A condition for two polynomials to be identically equal

A most important consequence of this last theorem is a condition for two polynomials $P(x)$ and $Q(x)$ to be identically equal.

20 An identically equal condition

- Suppose that $P(x)$ and $Q(x)$ are degree n polynomials that have the same values for at least $n + 1$ values of x.
- Then the polynomials $P(x)$ and $Q(x)$ are identically equal (meaning that they are equal for all values of x), and their coefficients are equal.
- In particular:
  - ▷ A linear polynomial is determined by two values.
  - ▷ A quadratic polynomial is determined by three values.
  - ▷ A cubic polynomial is determined by four values.
  - ▷ A quartic polynomial is determined by five values.

Proof Let $F(x) = P(x) - Q(x)$. Because $F(x)$ is zero whenever $P(x)$ and $Q(x)$ have the same value, it follows that $F(x)$ is zero for at least $n + 1$ values of x, so by the previous theorem, $F(x)$ is the zero polynomial, so $P(x) = Q(x)$ for all values of x.

And we stated before in Box 5 of Section 11A — proven in Exercise 11B Enrichment — that it now follows that the coefficients are equal.

Example 11 **Using the condition that polynomials are identically equal**

Find a, b, c and d, if $x^3 - x = a(x-2)^3 + b(x-2)^2 + c(x-2) + d$ for at least four values of x.

Solution

Because the cubics are equal for four values of x, they are identically equal.

Substituting $x = 2$, $\quad 6 = d$.

Equating coefficients of x^3, $\quad 1 = a$.

Hence the identity is now $\quad x^3 - x = (x-2)^3 + b(x-2)^2 + c(x-2) + 6$.

Substituting $x = 0$, $\quad 0 = -8 + 4b - 2c + 6$

$$2b - c = 1. \qquad (1)$$

Substituting $x = 1$, $\quad 0 = -1 + b - c + 6$

$$b - c = -5. \qquad (2)$$

Solving (1) and (2) simultaneously, $b = 6$ and $c = 11$.

F. Geometric implications of the factor theorem

Here are some of the geometric versions of the factor theorem — they translate the consequences above into the language of coordinate geometry. You will already have seen them in operation when dealing with graphs of quadratics.

21 Geometric implications of the factor theorem

1. The graph of a polynomial function of degree n is completely determined by any $n + 1$ points on the curve.
2. The graphs of two distinct polynomial functions cannot intersect in more points than the maximum of the two degrees.
3. A line cannot intersect the graph of a polynomial of degree n in more than n points.

Example 12 **Examining intersections of curves using the factor theorem**

By factoring the difference $F(x) = P(x) - Q(x)$, describe the intersections between the curves $P(x) = x^4 + 4x^3 + 2$ and $Q(x) = x^4 + 3x^3 + 3x$, and find where $P(x)$ is above $Q(x)$.

Solution

Subtracting, $\quad F(x) = x^3 - 3x + 2$.

Substituting, $\quad F(1) = 1 - 3 + 2 = 0$, $\quad$ so $x - 1$ is a factor.

$F(-2) = -8 + 6 + 2 = 0$, $\quad$ so $x + 2$ is a factor.

After long division by $(x-1)(x+2) = x^2 + x - 2$,

$$F(x) = (x-1)^2(x+2).$$

Hence $y = P(x)$ and $y = Q(x)$ are tangent at $x = 1$, but do not cross there, and also intersect also at $x = -2$, where they cross at an angle — see Box 22 below about this step.

Because $F(x)$ is positive for $-2 < x < 1$ or $x > 1$, and negative for $x < -2$, $P(x)$ is above $Q(x)$ for $-2 < x < 1$ or $x > 1$, and below it for $x < -2$.

Simple and multiple roots when finding intersections of polynomials

When two polynomial graphs cross, as in the worked example above, solving them simultaneously results in a polynomial equation, which may have multiple roots. The behaviour at a multiple root is exactly analogous to the behaviour of a single polynomial at a multiple zero, as described in Box 8 in Section 11B:

22 The behaviour at an intersdection of two polynomial graphs

Suppose that $x =$ å is a root of the polynomial equation obtained when two polynomial graphs $y = P(x)$ and $y = Q(x)$ are solved simultaneously.

- If $x = \alpha$ is a simple root (sometimes called a single root), then the curves cross each other at an angle, and are not tangent to each other there.
- If $x = \alpha$ has even multiplicity, then the curves are tangent to each other, and do not cross.
- If $x = \alpha$ has odd multiplicity at least 3, then the curves are tangent to each other, and cross at the point of intersection.

Again, the proof relies on Year 12 calculus. But this result does follow immediately from the earlier statement in Box 8, if one considers instead the *difference* $F(x) = P(x) - Q(x)$, as was done in the worked example above.

A note for Extension 2 students

The *fundamental theorem of algebra* cannot be proven in the Extension 2 course, but the theorem helps us to understand the importance of complex numbers for polynomials. It tells us that every polynomial equation of degree $n \geq 1$ has exactly n roots, provided first that roots are counted according to their multiplicity, and secondly that complex roots are also counted. For example:

- $x^3 = 0$ has one root $x = 0$, but this root has multiplicity 3.
- $x^3 - 1 = 0$, which factors to $(x - 1)(x^2 + x + 1) = 0$, has root $x = 1$, but also has the two complex roots of $x^2 + x + 1 = 0$.

This means that the graph of a polynomial of degree $n \geq 2$ intersects every line in exactly n points, provided first that points where the line is a tangent are counted according to their multiplicity, and secondly that complex points of intersection are also counted. This theorem provides the fundamental link between the algebra of polynomials and the geometry of their graphs, and allows the degree of a polynomial to be defined algebraically as the highest index, or geometrically as the number of times every line crosses it.

Exercise 11E

FOUNDATION

1 Use the factor theorem to write down in factored form:

a a monic cubic polynomial with zeroes −1, 3 and 4,

b a monic quartic polynomial with zeroes 0, −2, 3 and 1,

c a cubic polynomial with leading coefficient 6 and zeroes at $\frac{1}{3}$, $-\frac{1}{2}$ and 1.

2 **a** Show that 2 and 5 are zeroes of $P(x) = x^4 - 3x^3 - 15x^2 + 19x + 30$.

b Hence explain why $(x - 2)(x - 5)$ is a factor of $P(x)$.

c Divide $P(x)$ by $(x - 2)(x - 5)$ and hence express $P(x)$ as the product of four linear factors.

3 Use trial and error to find as many integer zeroes of $P(x)$ as possible. Use long division to divide $P(x)$ by the product of the known factors and hence express $P(x)$ in factored form.

a $P(x) = 2x^4 - 5x^3 - 5x^2 + 5x + 3$
b $P(x) = 2x^4 - 5x^3 - 5x^2 + 20x - 12$
c $P(x) = 6x^4 - 25x^3 + 17x^2 + 28x - 20$
d $P(x) = 9x^4 - 51x^3 + 85x^2 - 41x + 6$

4 Refer to Box 19 to answer parts (a) and (b).

a The polynomial $(a - 2)x^2 + (1 - 3b)x + (5 - 2c)$ has three zeroes. What are the values of a, b and c?
b The polynomial $(a + 1)x^3 + (b - 3)x^2 + (2c - 1)x + (5 - 4d)$ has four zeroes. What are the values of a, b, c and d?

DEVELOPMENT

5 **a** If $3x^2 - 4x + 7 = a(x + 2)^2 + b(x + 2) + c$ for all x, find a, b and c.
b If $2x^3 - 8x^2 + 3x - 4 = a(x - 1)^3 + b(x - 1)^2 + c(x - 1) + d$ for all x, find a, b, c and d.
c Use similar methods to express $x^3 + 2x^2 - 3x + 1$ as a polynomial in $(x + 1)$.
d If the polynomials $2x^2 + 4x + 4$ and $a(x + 1)^2 + b(x + 2)^2 + c(x + 3)^2$ are equal for three values of x, find a, b and c.

6 **a** A polynomial of degree 3 has a double zero at 2. When $x = 1$ it takes the value 6 and when $x = 3$ it takes the value 8. Find the polynomial.
b Two zeroes of a polynomial of degree 3 are 1 and -3. When $x = 2$ it takes the value -15 and when $x = -1$ it takes the value 36. Find the polynomial.

7 Show that $x^2 - 3x + 2$ is a factor of $P(x) = x^n(2^m - 1) + x^m(1 - 2^n) + (2^n - 2^m)$, where m and n are positive integers.

8 If two polynomials have degrees m and n, where $m > n$, what is the maximum number of intersection points of their graphs?

9 Explain why the graph of a cubic polynomial with three distinct zeroes must have two turning points.

10 The line $y = k$ meets the curve $y = ax^3 + bx^2 + cx + d$ four times. Find the values of a, b, c and d in terms of k.

11 Find, in expanded form, the monic degree five polynomial whose zeroes are 0, -1, 1, $2 - \sqrt{2}$ and $2 + \sqrt{2}$.

12 By factoring the difference $F(x) = P(x) - Q(x)$, describe the intersections between the curves $P(x)$ and $Q(x)$.

a $P(x) = 2x^3 - 4x^2 + 3x + 1$, $Q(x) = x^3 + x^2 - 8$
b $P(x) = x^4 + x^3 + 10x - 4$, $Q(x) = x^4 + 7x^2 - 6x + 8$
c $P(x) = -2x^3 + 3x^2 - 25$, $Q(x) = -3x^3 - x^2 + 11x + 5$
d $P(x) = x^4 - 3x^2 - 2$, $Q(x) = x^3 - 5x$
e $P(x) = x^4 + 4x^3 - x + 5$, $Q(x) = x^3 - 3x^2 - 2x + 5$

13 If a and b are non-zero, and $a + b = 0$, prove that the polynomials $A(x) = x^3 + ax^2 - x + b$ and $B(x) = x^3 + bx^2 - x + a$ have a common factor of degree 2 but are not identical polynomials. What is the common factor?

CHALLENGE

14 **a** Factor $x^n - 1$.

b Suppose that the roots of the equation $x^n - 1 = 0$ are $x = 1, \alpha_1, \alpha_2, \ldots, \alpha_{n-1}$. Show that $(1 - \alpha_1)(1 - \alpha_2)\ldots(1 - \alpha_{n-1}) = n$.

15 Suppose that $P(x) = x^5 + x^2 + 1$ and $Q(x) = x^2 - 2$. If r_1, r_2, r_3, r_4 and r_5 are the five zeroes of $P(x)$, find the value of $Q(r_1) \times Q(r_2) \times Q(r_3) \times Q(r_4) \times Q(r_5)$.

16 Suppose that $P(x)$ is a polynomial of *odd degree* n. It is known that $P(k) = \dfrac{k}{k+1}$ for $k = 0, 1, 2, \ldots, n$.

a Write down the zeroes of the polynomial $(x + 1)\,P(x) - x$.

b Let A be the leading coefficient of the polynomial $(x + 1)\,P(x) - x$. Factor the polynomial, and hence show that

$$A = \frac{1}{1 \times 2 \times 3 \times \cdots \times n \times (n+1)} = \frac{1}{(n+1)!}.$$

c Find $P(n + 1)$.

11F Sums and products of zeroes

Learning intentions

- Develop and use the sum-and-product-of-roots formulae for degrees 2, 3, and 4.
- Use these formulae to help factor polynomials, and for other purposes.

When expanding a monic polynomial with only linear factors, such as

$$P(x) = (x-2)(x-3)(x-5)(x-7) = x^4 - 17x^3 + 101x^2 - 247x + 210,$$

the four zeroes 2, 3, 5 and 7 of the polynomial clearly determine the five coefficients 1, −17, 101, −247 and 210. But what are the formulae for the coefficients?

If we study the coefficient −17 of the term in x^3, and the constant term 210, it does not take long to realise that

$$-17 = -(2+3+5+7) \qquad 210 = +2 \times 3 \times 5 \times 7$$
$$= -(\text{sum of the zeroes}) \qquad = +(\text{product of the zeroes}).$$

It is not at all clear, however, how the coefficients 101 of x^2, and −247 of x, are related to the zeroes. (We invite readers to try working it out before continuing.)

This section answers these questions for quadratic, cubic, and quartic polynomials. Knowing these formulae is a helpful technique when trying to factor a polynomial, and has other more general applications.

The zeroes of a quadratic

Let $P(x) = ax^2 + bx + c$ be a quadratic with zeroes α and β and leading coefficient a. Then $P(x) = a(x-\alpha)(x-\beta)$ by the factor theorem.

Expanding, $\quad P(x) = a(x-\alpha)(x-\beta) = ax^2 - a(\alpha+\beta)x + a\alpha\beta.$

Hence the coefficient of x is $-a$ times the sum of the zeroes, and the constant is $+a$ times the product of the zeroes,

$$b = -a(\alpha+\beta) \qquad \text{and} \qquad c = +a\alpha\beta.$$

This is just what happened with the quartic above, except for multiplication by the leading coefficient a. Solving for the sum and product of the zeroes:

23 Sum and product of zeroes of a quadratic

Let $P(x) = ax^2 + bx + c$ have zeroes α and β. Then

$$\alpha+\beta = -\frac{b}{a} \qquad \text{and} \qquad \alpha\beta = +\frac{c}{a}.$$

Example 13 Using the sum and product of zeroes of a quadratic

a **i** Show by substitution that $x = 5$ is a zero of $P(x) = 3x^2 - 30x + 75$.

ii Use sum-of-zeroes to find the other zero.

iii Use product-of-zeroes to find the other zero.

b Find the sum of the zeroes of $Q(x) = x^2 + 7x - 11$, and hence find its axis of symmetry.

Solution

a **i** $P(5) = 3 \times 25 - 30 \times 5 + 75$

$= 0.$

ii $\alpha + \beta = +\frac{30}{3}$

$\alpha + 5 = 10$

$\alpha = 5$, so 5 is a double zero.

iii $\alpha\beta = \frac{75}{3}$

$\alpha \times 5 = 25$

$\alpha = 5$, so again, 5 is a double zero.

b $\alpha + \beta = -\frac{7}{1}$

$= -7.$

The axis of symmetry is midway between zeroes, so it is $x = -3\frac{1}{2}$

(This calculation is exactly the same as the formula $x = -\dfrac{b}{2a}$ for the axis.)

The zeroes of a cubic

Let $P(x) = ax^3 + bx^2 + cx + d$ be a cubic with zeroes α, β and γ and leading coefficient a. Then $P(x) = a(x - \alpha)(x - \beta)(x - \gamma)$ by the factor theorem.

Expanding, $P(x) = a(x - \alpha)(x - \beta)(x - \gamma)$

$$= ax^3 - a(\alpha + \beta + \gamma)x^2 + a(\alpha\beta + \beta\gamma + \gamma\alpha)x - a\alpha\beta\gamma,$$

so as we saw before with the quartic and the quadratic,

$$b = -a(\alpha + \beta + \gamma) \quad \text{and} \quad d = -a\alpha\beta\gamma,$$

where the negative sign of the product comes from the cube of -1.

But the new phenomenon here is the coefficient of x,

$$c = a(\alpha\beta + \beta\gamma + \gamma\alpha) = +(\text{sum of products of pairs of zeroes}).$$

Solving for the sums and products of the zeroes:

24 Sums and products of zeroes of a cubic

Let $P(x) = ax^3 + bx^2 + cx + d$ have zeroes α, β and γ. Then

$$\alpha + \beta + \gamma = -\frac{b}{a} \quad \text{(sum of the zeroes)}$$

$$\alpha\beta + \beta\gamma + \gamma\alpha = +\frac{c}{a} \quad \text{(sum of products of pairs of zeroes)}$$

$$\alpha\beta\gamma = -\frac{d}{a} \quad \text{(product of the zeroes)}$$

Example 14 **Using the sum and product of zeroes of a cubic**

a Show that -6 is a zero of $P(x) = x^3 - 4x^2 - 39x + 126$.
b Use sum and product of zeroes to find the other two zeroes.
c Check the formula for the sum of products of pairs of zeroes.

Solution

a $P(-6) = -216 - 144 + 234 + 126$
$\quad = 0.$

b $\alpha + \beta - 6 = -\frac{-4}{1}$ and $\alpha\beta \times (-6) = -\frac{126}{1}$
$\alpha + \beta = 10$ $\qquad\qquad \alpha\beta = 21.$
Hence by inspection, α and β are 3 and 7.
(You have been using this 'by inspection' approach for years to solve quadratics.)

c $\alpha\beta + \beta\gamma + \gamma\alpha = 3 \times 7 + 7 \times (-6) + (-6) \times 3$
$\quad = -39$, which is the coefficient of x.

The zeroes of a quartic

Suppose that the four zeroes of the quartic polynomial

$$P(x) = ax^4 + bx^3 + cx^2 + dx + e$$

are α, β, γ and δ. By the factor theorem (see Box 16 in Section 11E), $P(x)$ is a multiple of the product $(x - \alpha)(x - \beta)(x - \gamma)(x - \delta)$:

$$\begin{aligned} P(x) &= a(x - \alpha)(x - \beta)(x - \gamma)(x - \delta) \\ &= ax^4 - a(\alpha + \beta + \gamma + \delta)x^3 + a(\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta)x^2 \\ &\quad - a(\alpha\beta\gamma + \beta\gamma\delta + \gamma\delta\alpha + \delta\alpha\beta)x + a\alpha\beta\gamma\delta. \end{aligned}$$

Equating coefficients of terms in x^3, x^2, x and constants now gives:

25 Zeroes and coefficients of a quartic

$$\alpha + \beta + \gamma + \delta = -\frac{b}{a} \quad \text{(sum of the zeroes)}$$

$$\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = +\frac{c}{a} \quad \text{(sum of products of pairs of zeroes)}$$

$$\alpha\beta\gamma + \beta\gamma\delta + \gamma\delta\alpha + \delta\alpha\beta = -\frac{d}{a} \quad \text{(sum of products of triples of zeroes)}$$

$$\alpha\beta\gamma\delta = +\frac{e}{a} \quad \text{(product of the zeroes)}$$

The second formula gives *the sum of the products of pairs of zeroes*, and the third formula gives *the sum of the products of triples of zeroes*.

The general case

The method is the same for all degrees. Notation is unfortunately a major difficulty here, and the results are better written in words. Suppose that $\alpha_1, \alpha_2, \ldots \alpha_n$ are the n zeroes of the degree n polynomial

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 .$$

26 Zeroes and coefficients of a polynomial

$$\text{sum of the zeroes} = \alpha_1 + \alpha_2 + \cdots + \alpha_n = -\frac{a_{n-1}}{a_n}$$

$$\text{sum of products of pairs of zeroes} = +\frac{a_{n-2}}{a_n}$$

$$\text{sum of products of triples of zeroes} = -\frac{a_{n-3}}{a_n}$$

$$\cdots\cdots$$

$$\text{product of the zeroes} = \alpha_1\alpha_2\cdots\alpha_n = (-1)^n\frac{a_0}{a_n}$$

Notice the alternating signs of the successive results. It is unlikely that anything apart from the first and last formulae would be required.

Example 15 Using the sum and product of zeroes of a cubic

Let α, β and γ be the roots of the cubic equation $x^3 - 3x + 2 = 0$. Use the formulae above to find:

a $\alpha + \beta + \gamma$ **b** $\alpha\beta\gamma$ **c** $\alpha\beta + \beta\gamma + \gamma\alpha$

d $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$ **e** $\alpha^2 + \beta^2 + \gamma^2$ **f** $\alpha^2\beta + \alpha\beta^2 + \beta^2\gamma + \beta\gamma^2 + \gamma^2\alpha + \gamma\alpha^2$

Check the result with the factoring $x^3 - 3x + 2 = (x - 1)^2(x + 2)$ obtained in the solution of Example 12 in Section 11E.

Solution

a $\alpha + \beta + \gamma = \frac{-0}{1} = 0$

b $\alpha\beta\gamma = -\frac{2}{1} = -2$

c $\alpha\beta + \beta\gamma + \gamma\alpha = \frac{-3}{1} = -3$

d $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\beta\gamma + \gamma\alpha + \alpha\beta}{\alpha\beta\gamma}$
$= \frac{-3}{-2} = \frac{3}{2}$

e $(\alpha + \beta + \gamma)^2 = \alpha^2 + \beta^2 + \gamma^2 + 2\alpha\beta + 2\beta\gamma + 2\gamma\alpha$,
so $0^2 = \alpha^2 + \beta^2 + \gamma^2 + 2 \times (-3)$
$\alpha^2 + \beta^2 + \gamma^2 = 6$.

f $\alpha^2\beta + \alpha\beta^2 + \beta^2\gamma + \beta\gamma^2 + \gamma^2\alpha + \gamma\alpha^2$
$= \alpha\beta(\alpha + \beta + \gamma) + \beta\gamma(\beta + \gamma + \alpha)$
$\quad + \gamma\alpha(\gamma + \alpha + \beta) - 3\alpha\beta\gamma$
$= (\alpha\beta + \beta\gamma + \gamma\alpha)(\alpha + \beta + \gamma) - 3\alpha\beta\gamma$
$= (-3) \times 0 - 3 \times (-2)$
$= 6$

Because $x^3 - 3x + 2 = (x - 1)^2(x + 2)$, the actual roots are 1, 1 and −2, hence

a $\alpha + \beta + \gamma = 1 + 1 - 2 = 0$

b $\alpha\beta\gamma = 1 \times 1 \times (-2) = -2$

c $\alpha\beta + \beta\gamma + \gamma\alpha = 1 - 2 - 2 = -3$

d $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = 1 + 1 - \frac{1}{2} = 1\frac{1}{2}$

e $\alpha^2 + \beta^2 + \gamma^2 = 1 + 1 + 4 = 6$

f $\alpha^2\beta + \alpha\beta^2 + \beta^2\gamma + \beta\gamma^2 + \gamma^2\alpha + \gamma\alpha^2$
$= 1 \times 1 + 1 \times 1 + 1 \times (-2)$
$\quad + 1 \times 4 + 4 \times 1 + (-2) \times 1$
$= 6$,

all of which agree with the previous calculations.

Factoring polynomials using the factor theorem and the sum and product of zeroes

Long division can be avoided in many situations by applying the sum and product of zeroes formulae after one or more zeroes have been found. Here are the steps that we have developed so far for factoring polynomials:

27 Factoring polynomials — the steps so far

- Use trial and error to find as many integer zeroes of $P(x)$ as possible.
- Use sum and product of zeroes to find the other zeroes.
- Alternatively, use long division of $P(x)$ by the product of the known factors.

If the coefficients of $P(x)$ are all integers, then any integer zero of $P(x)$ must be one of the divisors of the constant term.

In the next example, we factor a polynomial factored twice already, but this time there is no need for any long division at all.

Example 16 **Using the sum and product of zeroes of a quartic**

Factor $F(x) = x^4 + x^3 - 9x^2 + 11x - 4$ completely.

Solution

As before, $F(1) = 1 + 1 - 9 + 11 - 4 = 0$,

and $F(-4) = 256 - 64 - 144 - 44 - 4 = 0$.

Let the zeroes be 1, −4, α and β.

Then $\alpha + \beta + 1 - 4 = -1$

$\alpha + \beta = 2$. (1)

Also $\alpha\beta \times 1 \times (-4) = -4$

$\alpha\beta = 1$. (2)

From (1) and (2), $\alpha = \beta = 1$, so $F(x) = (x-1)^3(x+4)$.

Example 17 **Using the sum and product of zeroes of a cubic**

Factor completely the cubic $G(x) = x^3 - x^2 - 4$.

Solution

First, $G(2) = 8 - 4 - 4 = 0$.

Let the zeroes be 2, α and β.

Then $2 + \alpha + \beta = 1$

$\alpha + \beta = -1$, (1)

and $2 \times \alpha \times \beta = 4$

$\alpha\beta = 2$. (2)

Substituting (1) into (2), $\alpha(-1 - \alpha) = 2$

$\alpha^2 + \alpha + 2 = 0$

This is an irreducible quadratic, because $\Delta = -7$, so the complete factoring is $G(x) = (x-2)(x^2 + x + 2)$.

Note: This procedure — developing the irreducible quadratic factor from the sum and product of zeroes — is really little easier than the long division it avoids.

Forming identities with the coefficients

If some information can be gained about the roots of a polynomial equation, it may be possible to form an identity with the coefficients of the polynomial.

Example 18 Identities on the coefficients of a cubic

If one zero of the cubic $f(x) = ax^3 + bx^2 + cx + d$ is the opposite of another, prove that $ad = bc$.

Solution

We know that one of the zeroes is the opposite of another, so we can begin with the following sentence, which expresses this fact symbolically:

Let the three zeroes of the cubic be α, $-\alpha$ and β.

Now we can use the formula for the sum of the roots,

$$\alpha + (-\alpha) + \beta = -\frac{b}{a}$$

$$a\beta = -b. \qquad (1)$$

Then using the formula for the product of the roots,

$$\alpha \times (-\alpha) \times \beta = -\frac{d}{a}$$

$$a\alpha^2\beta = d. \qquad (2)$$

There are three products of pairs of roots, namely $-\alpha^2$ and $-\alpha\beta$ and $\beta\alpha$, so using the sum of products of pairs of roots,

$$-\alpha^2 - \alpha\beta + \beta\alpha = \frac{c}{a}$$

$$a\alpha^2 = -c. \qquad (3)$$

Now we must eliminate the roots α and β from equations (1), (2), and (3).

Substituting (3) into (2), $\quad -c\beta = d, \qquad (4)$

and dividing (1) by (4), $\quad -ac = -\frac{b}{d},$

that is, $\quad ad = bc.$

Exercise 11F FOUNDATION

1 If α and β are the roots of the quadratic equation $x^2 - 4x + 2 = 0$, find:

a $\alpha + \beta$ **b** $\alpha\beta$ **c** $\alpha^2\beta + \alpha\beta^2$

d $\frac{1}{\alpha} + \frac{1}{\beta}$ **e** $(\alpha + 2)(\beta + 2)$ **f** $\alpha^2 + \beta^2$

g $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$ **h** $\alpha\beta^3 + \alpha^3\beta$ **i** $\left(\alpha + \frac{1}{\alpha}\right)\left(\beta + \frac{1}{\beta}\right)$

2 If α, β and γ are the roots of the equation $x^3 + 2x^2 - 11x - 12 = 0$, find:

a $\alpha + \beta + \gamma$ **b** $\alpha\beta + \alpha\gamma + \beta\gamma$ **c** $\alpha\beta\gamma$

d $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma}$ **e** $\dfrac{1}{\alpha\beta} + \dfrac{1}{\alpha\gamma} + \dfrac{1}{\beta\gamma}$ **f** $(\alpha + 1)(\beta + 1)(\gamma + 1)$

g $(\alpha\beta)^2\gamma + (\alpha\gamma)^2\beta + (\beta\gamma)^2\alpha$ **h** $\alpha^2 + \beta^2 + \gamma^2$ **i** $(\alpha\beta)^{-2} + (\alpha\gamma)^{-2} + (\beta\gamma)^{-2}$

Now find the roots of the equation $x^3 + 2x^2 - 11x - 12 = 0$ by factoring the LHS. Hence check your answers for the expressions in parts (a)–(i).

3 If α, β, γ and δ are the roots of the equation $x^4 - 5x^3 + 2x^2 - 4x - 3 = 0$, find:

a $\alpha + \beta + \gamma + \delta$ **b** $\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta$

c $\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta$ **d** $\alpha\beta\gamma\delta$

e $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma} + \dfrac{1}{\delta}$ **f** $\dfrac{1}{\alpha\beta} + \dfrac{1}{\alpha\gamma} + \dfrac{1}{\alpha\delta} + \dfrac{1}{\beta\gamma} + \dfrac{1}{\beta\delta} + \dfrac{1}{\gamma\delta}$

g $\dfrac{1}{\alpha\beta\gamma} + \dfrac{1}{\alpha\beta\delta} + \dfrac{1}{\alpha\gamma\delta} + \dfrac{1}{\beta\gamma\delta}$ **h** $\alpha^2 + \beta^2 + \gamma^2 + \delta^2$

4 If α and β are the roots of the equation $2x^2 + 5x - 4 = 0$, find:

a $\alpha + \beta$ **b** $\alpha\beta$ **c** $\alpha^2 + \beta^2$ **d** $|\alpha - \beta|$ (Hint: Square it.)

DEVELOPMENT

5 **a** The polynomial $P(x) = 2x^3 - 5x^2 - 14x + 8$ has zeroes at -2 and 4. Use the sum of the zeroes to find the other zero.

b Suppose that $x - 3$ and $x + 1$ are factors of $P(x) = x^3 - 6x^2 + 5x + 12$. Use the product of the zeroes to find the other factor of $P(x)$.

6 Consider the polynomial $P(x) = x^3 - x^2 - x + 10$.

a Show that -2 is a zero of $P(x)$.

b Suppose that the zeroes of $P(x)$ are -2, α and β. Show that $\alpha + \beta = 3$ and $\alpha\beta = 5$.

c By solving the two equations in part (b) simultaneously, show that $\alpha^2 - 3\alpha + 5 = 0$.

d Hence show that there are no such real numbers α and β.

e Hence state how many times the graph of the cubic crosses the x-axis.

7 Show that $x = 1$ and $x = -2$ are zeroes of $P(x)$, and use the sum and product of zeroes to find the other one or two zeroes. Note any multiple zeroes.

a $P(x) = x^3 - 2x^2 - 5x + 6$ **b** $P(x) = 2x^3 + 3x^2 - 3x - 2$

c $P(x) = x^4 + 3x^3 - 3x^2 - 7x + 6$ **d** $P(x) = 3x^4 - 5x^3 - 10x^2 + 20x - 8$

8 **a** Find the values of a and b for which $x^3 + ax^2 - 10x + b$ is exactly divisible by $x^2 + x - 12$, and then factor the cubic.

b Find the values of a and b for which $x^2 - x - 20$ is a factor of $x^4 + ax^3 - 23x^2 + bx + 60$, and then find all the zeroes.

9 **a** If one of the roots of the equation $x^2 + bx + c = 0$ is twice the other root, show that $2b^2 = 9c$. (Hint: Let the roots be α and 2α.)

b If one of the roots of the equation $x^2 + px + q = 0$ is one more than the other root, show that $p^2 = 4q + 1$. (Hint: Let the roots be α and $\alpha + 1$.)

10 **a** Find the roots of the equation $3x^3 - x^2 - 48x + 16 = 0$ given that the sum of two of them is zero. (Hint: Let the roots be α, $-\alpha$ and β.)

b Find the roots of the equation $2x^3 - 5x^2 - 46x + 24 = 0$ given that the product of two of them is 3. (Hint: Let the roots be α, $\frac{3}{\alpha}$ and β.)

c Find the roots of the equation $x^3 - 27x - 54 = 0$ given that two of them are equal. (Hint: Let the roots be α, α and β.)

d Find the zeroes of the polynomial $P(x) = 2x^3 - 13x^2 + 22x - 8$ given that one of them is the product of the other two. (Hint: Let the zeroes be α, β and $\alpha\beta$.)

11 **a** Two of the roots of the equation $x^3 + 3x^2 - 4x + a = 0$ are opposites. Find the value of a and the three roots.

b Two of the roots of the equation $4x^3 + ax^2 - 47x + 12 = 0$ are reciprocals. Find the value of a and the three roots.

12 Find the zeroes of the polynomial $x^4 - 3x^3 - 8x^2 + 12x + 16$ if they are α, 2α, β and 2β.

13 **a** If one root of the equation $x^3 - bx^2 + cx - d = 0$ is equal to the product of the other two, show that $(c + d)^2 = d(b + 1)^2$.

b The polynomial $P(x) = x^4 + ax^3 + bx^2 + cx + d$ has two zeroes that are opposites and another two zeroes that are reciprocals. Show that:

i $b = 1 + d$ **ii** $c = ad$

14 The cubic equation $x^3 - Ax^2 + 3A = 0$, where $A > 0$, has roots α, β and $\alpha + \beta$.

a Use the sum of the roots to show that $\alpha + \beta = \frac{1}{2}A$.

b Use the sum of the products of pairs of roots to show that $\alpha\beta = -\frac{1}{4}A^2$.

c Show that $A = 2\sqrt{6}$.

15 The polynomial $P(x) = x^3 - Lx^2 + Lx - M$ has zeroes α, $\frac{1}{\alpha}$ and β.

a Show that:

i $\alpha + \frac{1}{\alpha} + \beta = L$ **ii** $1 + \alpha\beta + \frac{\beta}{\alpha} = L$ **iii** $\beta = M$

b Show that either $M = 1$ or $M = L - 1$.

16 Suppose that the polynomial $P(x) = ax^3 + bx^2 + cx + d$ has zeroes α, β and γ.

a Show that $\alpha^2 + \beta^2 + \gamma^2 = \frac{b^2 - 2ac}{a^2}$.

b Hence explain why the polynomial $2x^3 - 3x^2 + 5x - 8$ cannot have three real zeroes.

CHALLENGE

17 If α, β and γ are the roots of the equation $x^3 + 5x - 4 = 0$, evaluate $\alpha^3 + \beta^3 + \gamma^3$. (Hint: $x = \alpha$, $x = \beta$ and $x = \gamma$ satisfy the given equation.)

18 Suppose that the equation $x^2 + bx + c = 0$ has roots α and β.

a Show that b and c are the roots of the equation $x^2 + (\alpha + \beta - \alpha\beta)x - \alpha\beta(\alpha + \beta) = 0$.

b Find the non-zero values of b and c for which the roots of the equation in part (a) are:

i α and β **ii** α^2 and β^2

11G Geometry using polynomial techniques

Learning intentions

- Use the methods of polynomials to solve geometric problems.

This final section adds the methods of the preceding sections, particularly the sum and product of zeroes, to the available techniques for studying the geometry of various curves. The standard technique is to examine the roots of the equation formed in the process of solving two curves simultaneously.

This section is rather demanding, and could all be regarded as Enrichment.

Midpoints and tangents

When two curves intersect, we can form the equation whose solutions are the x- or y-coordinates of points of intersection of the two curves. The midpoint of two points of intersection can then be found using the average of the roots. Tangents can be identified as corresponding to double roots.

The next example could also be done using quadratic equations, but it is a clear example of the use of sum and product of roots.

Example 19 **Dealing with tangents using double roots**

The line $y = 2x$ meets the parabola $y = x^2 - 2x - 8$ at the two points $A(\alpha, 2\alpha)$ and $B(\beta, 2\beta)$.

a Show that α and β are roots of $x^2 - 4x - 8 = 0$, and hence find the coordinates of the midpoint M of AB.

b Use the identity $(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta$ to find the horizontal distance $|\alpha - \beta|$ from A to B. Then use Pythagoras' theorem and the gradient of the line to find the length of AB.

c Find the value of b for which $y = 2x + b$ is a tangent to the parabola, and find the point T of contact.

Solution

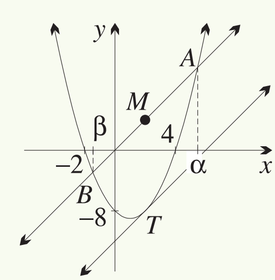
a Solving the line and the parabola simultaneously,

$$x^2 - 2x - 8 = 2x$$
$$x^2 - 4x - 8 = 0.$$

Hence $\alpha + \beta = 4$,

and $\alpha\beta = -8$.

Averaging the roots, M has x-coordinate $x = 2$, and substituting into the line, $M = (2, 4)$.



b We know that $(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta$

$$= 16 + 32$$
$$= 48$$

so $|\alpha - \beta| = 4\sqrt{3}$.

Because the line has gradient 2, the vertical distance is $8\sqrt{3}$,

so using Pythagoras, $AB^2 = (4\sqrt{3})^2 + (8\sqrt{3})^2$

$$= 16 \times 15$$
$$AB = 4\sqrt{15}.$$

c Solving $y = x^2 - 2x - 8$ and $y = 2x + b$ simultaneously,

$$x^2 - 4x - (8 + b) = 0$$

Because the line is a tangent, let the roots be θ and θ.

Then using the sum of roots, $\theta + \theta = 4$,

so $\theta = 2$,

Using the product of roots, $\theta^2 = -8 - b$

and because $\theta = 2$, $b = -12$.

So the line $y = 2x - 12$ is a tangent at $T(2, -8)$.

Geometric problems using sum and product of roots

The sum and product of roots can make some interesting geometric problems quite straightforward.

Example 20 Tangents and the sum and product of roots

A line with gradient m through the point $P(-1, 0)$ on the cubic $y = x^3 - x$ crosses the curve at two further points A and B with x-coordinates α and β respectively.

a Sketch the situation.

b Show that α and β satisfy the cubic equation $x^3 - (m + 1)x - m = 0$.

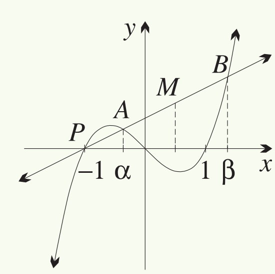
c Show that the midpoint M of AB lies on the vertical line $x = \frac{1}{2}$.

d Find the line through P tangent to the cubic at a point distinct from P.

Solution

a The cubic factors as $y = x(x - 1)(x + 1)$,

so the zeroes are $x = -1$, $x = 0$ and $x = 1$.



b The line through $P(-1, 0)$ has equation

$$y = m(x + 1).$$

Solving the line simultaneously with the cubic,

$$x^3 - x = mx + m$$

$$x^3 - (m + 1)x - m = 0.$$

c The cubic has roots α, β and 1.

Using the sum of roots, $\alpha + \beta + (-1) = 0$ (there is no term in x^2)

so $\alpha + \beta = 1$.

Hence the midpoint M of AB has x-coordinate $\frac{1}{2}(\alpha + \beta) = \frac{1}{2}$, so M lies on the line $x = \frac{1}{2}$.

d The line PM is a tangent at another point on the curve when the points

A, M and B coincide, that is, when $\alpha = \beta$,

and because $\alpha + \beta = 1$, this means that $\alpha = \beta = \frac{1}{2}$.

Using the product of roots, $\alpha \times \beta \times (-1) = m$ (the constant term is $-m$)

$$\tfrac{1}{2} \times \tfrac{1}{2} \times (-1) = m$$

$$m = -\tfrac{1}{4},$$

so the tangent through P is $y = -\frac{1}{4}(x + 1)$.

Exercise 11G

FOUNDATION

Note: Sketches should be drawn in all these questions to make the situation clear.

1 **a** Show that the x-coordinates of the points of intersection of the parabola $y = x^2 - 6x$ and the line $y = 2x - 16$ satisfy the equation $x^2 - 8x + 16 = 0$.

b Solve this equation, and hence show that the line is a tangent to the parabola. Find the point T of contact.

2 **a** Show that the x-coordinates of the points of intersection of the line $y = -2x + b$ and the parabola $y = x^2 - 6x$ satisfy the quadratic equation $x^2 - 4x - b = 0$.

b Suppose that the line is a tangent to the parabola, so that the roots of the quadratic equation are equal. Let these roots be α and α.

i Using the sum of roots, show that $\alpha = 2$.

ii Using the product of roots, show that $\alpha^2 = -b$, and hence find b.

iii Find the equation of the tangent and its point T of contact.

3 The line $y = x + 1$ meets the parabola $y = x^2 - 3x$ at A and B.

a Show that the x-coordinates α and β of A and B satisfy the equation $x^2 - 4x - 1 = 0$.

b Find $\alpha + \beta$, and hence find the coordinates of the midpoint M of AB.

DEVELOPMENT

4 **a** Show that the x-coordinates of the points of intersection of the line $y = 3 - x$ and the cubic $y = x^3 - 5x^2 + 6x$ satisfy the equation $x^3 - 5x^2 + 7x - 3 = 0$.

b Show that $x = 1$ and $x = 3$ are roots of the equation, and use the sum of the roots to find the third root.

c Explain why the line is a tangent to the cubic, then find the point of contact and the other point of intersection.

5 **a** Show that the x-coordinates of the points of intersection of the line $y = mx$ and the cubic $y = x^3 - 5x^2 + 6x$ satisfy the equation $x^3 - 5x^2 + (6 - m)x = 0$.

b Suppose now that the line is a tangent to the cubic at a point other than the origin, so that the roots of the equation are 0, α and α.

i Using the sum of roots, show that $\alpha = \frac{5}{2}$.

ii Using the product of pairs of roots, show that $\alpha^2 = 6 - m$, and hence find m.

iii Find the equation of the tangent and its point T of contact.

6 The line $y = x - 2$ meets the cubic $y = x^3 - 5x^2 + 6x$ at $F(2, 0)$, and also at A and B.

a Show that the x-coordinates α and β of A and B satisfy $x^3 - 5x^2 + 5x + 2 = 0$.

b Find $\alpha + \beta$, and hence find the coordinates of the midpoint M of AB.

c Show that $\alpha\beta = -1$, then use the identity $(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta$ to show that the horizontal distance $|\alpha - \beta|$ between A and B is $\sqrt{13}$.

d Hence use Pythagoras' theorem to find the length AB.

7 A line passes through the point $A(-1, -7)$ on the curve $y = x^3 - 3x^2 + 4x + 1$. Suppose that the line has gradient m and is tangent to the curve at another point P on the curve whose x-coordinate is α.

a Show that the line has equation $y = mx + (m - 7)$.

b Show that the cubic equation whose roots are the x-coordinates of the points of intersection of the line and the curve is $x^3 - 3x^2 + (4 - m)x + (8 - m) = 0$.

c Explain why the roots of this equation are -1, α and α, and hence find the point P and the value of m.

8 The point $P(p, p^3)$ lies on the curve $y = x^3$. A line through P with gradient m intersects the curve again at A and B.

a Find the equation of the line through P.

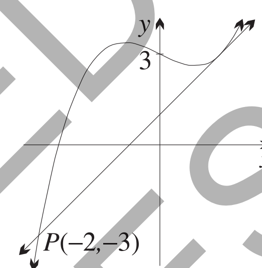
b Show that the x-coordinates of A and B satisfy the equation $x^3 - mx + mp - p^3 = 0$.

c Hence find the x-coordinate of the midpoint M of AB, and show that for fixed p, M always lies on a line that is parallel to the y-axis.

9 **a** The equation $x^3 - (m + 1)x + (6 - 2m) = 0$ has a root at $x = -2$ and a double root at $x = \alpha$. Find α and m.

b Write down the equation of the line ℓ passing through the point $P(-2, -3)$ with gradient m.

c The diagram shows the curve $y = x^3 - x + 3$ and the point $P(-2, -3)$ on the curve. The line ℓ cuts the curve at P, and is tangent to the curve at another point A on the curve. Find the equation of the line ℓ.



10 **a** Use the factor theorem to factor the polynomial $y = x^4 - 4x^3 - 9x^2 + 16x + 20$, given that there are four distinct zeroes, then sketch the curve.

b The line ℓ: $y = mx + b$ touches the quartic $y = x^4 - 4x^3 - 9x^2 + 16x + 20$ at two distinct points A and B. Explain why the x-coordinates α and β of A and B are double roots of the equation $x^4 - 4x^3 - 9x^2 + (16 - m)x + (20 - b) = 0$.

c Use the theory of the sum and product of roots to write down four equations involving α, β, m and b.

d Hence find m and b, and write down the equation of ℓ.

Note: If two curves touch each other at P, then they are tangent to each other at P.

11 **a** Find k and the points of contact if the parabola $y = x^2 - k$ touches the quartic $y = x^4$ at two points.

b Find k and the point T of contact if the parabola $y = x^2 - k$ touches the cubic $y = x^3$.

c Find k and the points of contact if the parabola $y = x^2 - k$ touches the circle $x^2 + y^2 = 1$ at two points.

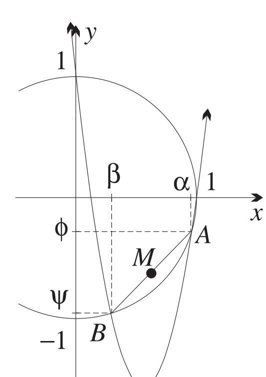
CHALLENGE

12 A circle passing through the origin O is tangent to the hyperbola $xy = 1$ at A, and intersects the hyperbola again at two distinct points B and C. Prove that $OA \perp BC$.

13 The diagram to the right shows the circle $x^2 + y^2 = 1$ and the parabola $y = (\lambda x - 1)(x - 1)$, where λ is a constant. The circle and parabola meet in the four points

$$P(1, 0), \qquad Q(0, 1), \qquad A(\alpha, \varphi), \qquad B(\beta, \psi).$$

The point M is the midpoint of the chord AB.



a Show that the x-coordinates of the points of intersection of the two curves satisfy the equation

$$\lambda^2 x^4 - 2\lambda(1 + \lambda)x^3 + (\lambda^2 + 4\lambda + 2)x^2 - 2(1 + \lambda)x = 0.$$

b Use the formula for the sum of the roots to show that the x-coordinate of M is $\dfrac{\lambda + 2}{2\lambda}$.

c Use a similar method to find the y-coordinate of M, and hence show that M lies on the line through the origin O parallel to PQ.

d For what values of λ is the parabola tangent to the circle in the fourth quadrant?

e For what values of λ are the four points P, Q, A and B distinct, with real numbers as coordinates.

Chapter 11 Review

Review activity

- Create your own summary of this chapter on paper or in a digital document.

Chapter 11 Multiple-choice quiz

- This automatically-marked quiz is accessed in the Interactive Textbook. A printable PDF Worksheet version is also available there.

Skills Checklist

Check-list

- Available in the Interactive Textbook, use the checklist to track your understanding of the learning intentions. Printable PDF and word document versions are also available there.

Review

Chapter Review Exercise

1 Consider the polynomial $P(x) = 2x^3 - 5x^2 - 6x - 11$. State:

a the degree of $P(x)$,
b the leading coefficient of $P(x)$,
c the leading term of $P(x)$,
d the constant term of $P(x)$.

2 The polynomial $P(x)$ has degree 3. Write down the degree of the polynomial:

a $3\,P(x)$
b $(P(x))^3$

3 Find the coefficient of x^2 in the polynomial $P(x) = (x^2 - 3x - 7)(2x^2 + 4x - 9)$.

4 **a** Sketch the graph of the polynomial function $y = (x + 2)^2(x - 1)(x - 3)$, showing all intercepts with the coordinate axes.
b Hence find the values of x for which $(x + 2)^2(x - 1)(x - 3) < 0$.

5 Sketch the graph of the polynomial $P(x) = x^3 - x^5$.

6 Suppose that the polynomial $P(x) = 2x^3 + 7x^2 - 4x + 5$ is divided by $D(x) = x - 3$.
a Find the quotient $Q(x)$ and the remainder $R(x)$.
b Write down a division identity using the information above.

7 Without long division, find the remainder when $P(x) = x^3 - 5x^2 + 1$ is divided by:

a $x - 3$
b $x + 2$

8 **a** Use the factor theorem to show that $x - 2$ is a factor of $P(x) = x^3 - 19x + 30$.
b Hence factor $P(x)$ fully.

9 Find the value of k given that $x + 3$ is a factor of $P(x) = x^3 + 4x^2 + kx - 12$.

10 Find the values of b and c given that $x + 1$ is a factor of $P(x) = x^3 + bx^2 + cx - 7$, and the remainder is -12 when $P(x)$ is divided by $x - 5$.

11 Find the values of h and k given that $x + 2$ is a factor of $Q(x) = (x + h)^2 + k$, and the remainder is 16 when $Q(x)$ is divided by x.

Review

12 The polynomial $P(x)$ is divided by $(x + 1)(x - 2)$. Suppose that the quotient is $Q(x)$ and the remainder is $R(x)$.

a Explain why the general form of $R(x)$ is $ax + b$, where a and b are constants.

b When $P(x)$ is divided by $x + 1$ the remainder is 10, and when $P(x)$ is divided by $x - 2$ the remainder is -8. Find a and b. (HINT: Use the division identity.)

13 Suppose that the polynomial $Q(x) = x^2 - 6x - 4$ has zeroes α and β. Without finding the zeroes, find the value of:

a $\alpha + \beta$ **b** $\alpha\beta$ **c** $\alpha^2\beta + \beta^2\alpha$

d $\frac{1}{\alpha} + \frac{1}{\beta}$ **e** $(\alpha - 3)(\beta - 3)$ **f** $\alpha^2 + \beta^2$

14 If α, β and γ are the roots of the equation $x^3 + 10x^2 + 5x - 20 = 0$, find:

a $\alpha + \beta + \gamma$ **b** $\alpha\beta + \alpha\gamma + \beta\gamma$ **c** $\alpha\beta\gamma$

d $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$ **e** $\frac{1}{\alpha\beta} + \frac{1}{\alpha\gamma} + \frac{1}{\beta\gamma}$ **f** $(\alpha + 2)(\beta + 2)(\gamma + 2)$

g $\alpha^2\beta^2\gamma + \alpha^2\gamma^2\beta + \beta^2\gamma^2\alpha$ **h** $\alpha^2 + \beta^2 + \gamma^2$ **i** $\frac{1}{\alpha^2\beta^2} + \frac{1}{\alpha^2\gamma^2} + \frac{1}{\beta^2\gamma^2}$

15 The equation $x^3 + 5x^2 + cx + d = 0$ has roots -3, 7 and α.

a Use the sum of the roots to find α.

b Use the product of the roots to find d.

c Use the sum of the roots in pairs to find c.

16 The equation $6x^3 - 17x^2 - 5x + 6 = 0$ has roots α, β and γ, where $\alpha\beta = -2$.

a Use the product of the roots to find γ.

b Use the sum of the roots to find the other two roots.

17 One root of the equation $ax^2 + 2bx + c = 0$ is the reciprocal of the square of the other root. Show that $a^3 + c^3 + 2abc = 0$.

18 Solve the equation $9x^3 - 27x^2 + 11x + 7 = 0$ given that the roots are $\alpha - \beta$, α and $\alpha + \beta$.

19 Find the zeroes of the polynomial $P(x) = 8x^3 - 14x^2 + 7x - 1$ given that they are $\frac{\alpha}{\beta}$, α and $\alpha\beta$.

20 The line $y = 9x + 5$ is the tangent to the curve $y = x^3 - 3x^2$ at the point $A(-1, -4)$. The line intersects the curve at another point B. Suppose that the x-coordinate of B is α.

a Write down the cubic equation whose roots are the x-coordinates of A and B.

b Explain why the roots of this equation are -1, -1 and α.

c Hence find the point B.

16

Further trigonometry

Chapter introduction

The first section of this chapter applies trigonometry to three-dimensional problems. Three new issues arise immediately when moving from 2 to 3 dimensions.

- Visualisation is the first new issue here, and a clear diagram displaying all the triangles is required — a task that may involve several attempts.
- The second new issue is to identify all right angles, and to understand what it means for a plane and a line, or two planes, to be perpendicular.
- The third new issue is to understand how to identify more generally the angle between a line and a plane, and between two planes.

Three-dimensional work will continue in the Year 12 study of 2 and 3-dimensional vectors, where these same three issues will arise.

The next two sections develop a series of trig identities involving compound angles and double angles. These are required in Year 12 when the trig functions are differentiated, which, and as always with new functions, will begin with an appeal to geometry. These identities will also be required when sketching functions involving the trig functions, and in modelling various periodic phenomena.

The new identities, together with those developed previously, are applied in Section 16D to the solution of a wide variety of trigonometric equations that arise later when modelling with calculus.

The final Section 16E deals with functions of the form $a \sin x + b \cos x$, which surprisingly turn out to be sine or cosine functions — shifted horizontally, and dilated vertically to give a different amplitude. These results and methods require transformation of trigonometric functions presented in the Year 12 Advanced course, so a preliminary subheading quickly applies the methods of Chapter 5 to the two transformations that are needed.

The methods of this last section are the first step in Fourier analysis, which is the basis of a vast number of wave phenomena in science, engineering and medicine — our ears split a sound wave into individual frequencies, as is clear when we listen to music, and the analysis of gravity waves is at the very edge of physics right now. Joseph Fourier was a French scientist deeply involved with Napoleon Bonaparte and the French Revolution. As well as his ground-breaking work with the analysis of periodic phenomena, he was an expert in Egyptian archaeology.

16A Three-dimensional trigonometry

Learning intentions

- Interpret data given in diagram or written form about a 3D trig problem.
- Apply trigonometry to solve a 3D problem.

Trigonometry is based on triangles, which are two-dimensional objects. When trigonometry is applied to a three-dimensional problem, the diagram must be broken up into a collection of triangles in space, and trigonometry is then applied to each triangle in turn. A carefully drawn diagram is always essential.

Two new ideas about angles are needed — the angle between a line and a plane, and the angle between two planes. Pythagoras' theorem remains fundamental.

Trigonometry and Pythagoras' theorem in three dimensions

Here are the steps in a successful approach to a three-dimensional problem.

1 Trigonometry and Pythagoras' theorem in three dimensions

1. Draw a careful sketch of the situation.
2. Note carefully all the triangles in the figure.
3. Mark, or note, all right angles in these triangles.
4. Always name the triangle you are working with.

Example 1 Using Pythagoras' theorem in a 3D figure

The rectangular prism sketched below has dimensions:

$EF = 5$ cm and $FG = 4$ cm and $CG = 3$ cm.

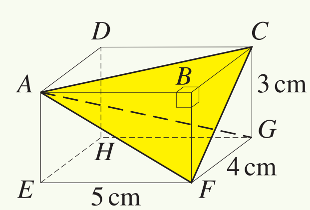
a Use Pythagoras' theorem in $\triangle CFG$ to find the length of the diagonal FC.

b Similarly find the lengths of the diagonals AC and AF.

c Use Pythagoras' theorem in $\triangle ACG$ to find the length of the space diagonal AG.

d Use trigonometry in $\triangle BAF$ to find $\angle BAF$ (nearest minute).

e Use trigonometry in $\triangle GAF$ to find $\angle GAF$ (nearest minute).



Solution

a In $\triangle CFG$, $FC^2 = 3^2 + 4^2$, using Pythagoras,

$FC = 5$ cm.

b In $\triangle ABC$, $AC^2 = 5^2 + 4^2$ (Pythagoras)

$AC = \sqrt{41}$ cm.

In $\triangle ABF$, $AF^2 = 5^2 + 3^2$ (Pythagoras)

$AF = \sqrt{34}$ cm.

c In $\triangle ACG$, the angle $\angle ACG$ is a right angle, and $AC = \sqrt{41}$ and $CG = 3$.

Hence $AG^2 = AC^2 + CG^2$, using Pythagoras,

$= 41 + 3^2$

$AG = \sqrt{50}$

$= 5\sqrt{2}$ cm.

d In $\triangle BAF$, the angle $\angle ABF$ is a right angle, and $AB = 5$ and $BF = 3$.

$$\text{Hence } \tan\angle BAF = \frac{BF}{AB} = \tfrac{3}{5}$$

$$\angle BAF \doteqdot 30^\circ 58'.$$

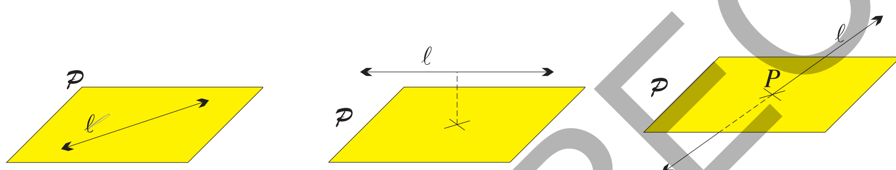
e In $\triangle GAF$, the angle $\angle AFG$ is a right angle, and $AF = \sqrt{34}$ and $FG = 4$.

$$\text{Hence } \tan\angle GAF = \frac{FG}{AF} = \frac{4}{\sqrt{34}}$$

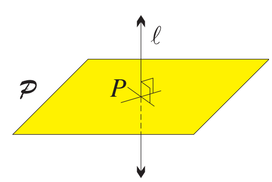
$$\angle GAF \doteqdot 34^\circ 27'.$$

The angle between a line and a plane

In 3D space, a plane $\mathcal{P}$ and a line ℓ can be related in three different ways:

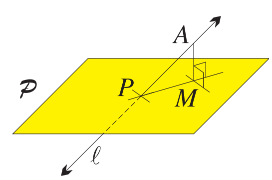


- In the first diagram, the line lies wholly within the plane.
- In the second diagram, the line never meets the plane. We say that the line and the plane are *parallel*.
- In the third diagram, the line intersects the plane in a single point P.



When the line ℓ meets the plane $\mathcal{P}$ in the single point P, it can do so in two distinct ways.

In the upper diagram, the line ℓ is perpendicular to every line in the plane through P. We say that the line is *perpendicular* to the plane.



In the lower diagram, the line ℓ is not perpendicular to $\mathcal{P}$. To construct the angle θ between the line and the plane:

- Choose another point A on the line ℓ.
- Construct the point M in the plane $\mathcal{P}$ so that $AM \perp \mathcal{P}$.

Then $\angle APM$ is the angle between the line and the plane.

Example 2 **Finding the angle between a line and a plane**

Find the angle between a slant edge and the base in a square pyramid of height 8 metres whose base has side length 12 metres (nearest minute).

Solution

Using Pythagoras' theorem in the base $ABCD$,

$$AC^2 = 12^2 + 12^2$$

$$AC = 12\sqrt{2} \text{ metres.}$$

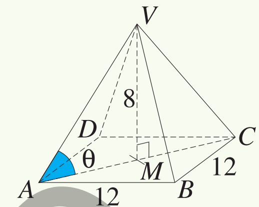
The perpendicular from the vertex V to the base meets the base at the midpoint M of the diagonal AC.

In $\triangle MAV$, $\tan \angle MAV = \dfrac{MV}{MA}$

$= \dfrac{8}{6\sqrt{2}}$

$\angle MAV \doteqdot 43^\circ 19'$,

and this is the angle between the slant edge AV and the base.

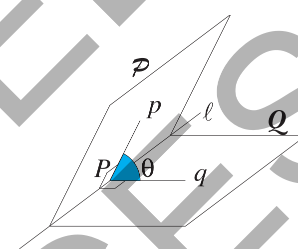


The angle between two planes

In 3D space, two planes that are not parallel intersect in a line ℓ. To construct the angle between the planes:

- Take any point P on this line of intersection.
- Construct the line p through P perpendicular to ℓ lying in the plane $\mathcal{P}$.
- Construct the line q through P perpendicular to ℓ lying in the plane $\mathcal{Q}$.

The angle between the planes $\mathcal{P}$ and $\mathcal{Q}$ is defined to be the angle θ between these two lines p and q.



Example 3 Finding the angle between two planes

In the pyramid of the previous example, find the angle between an oblique face of the pyramid and the base.

Solution

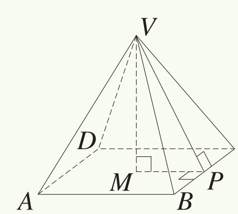
Let P be the midpoint of the edge BC.

Then $VP \perp BC$ and $MP \perp BC$,
so $\angle VPM$ is the angle between the oblique face and the base.

In $\triangle VPM$, $\tan \angle VPM = \dfrac{VM}{PM}$

$= \frac{8}{6}$

so $\angle VPM \doteqdot 53^\circ 8'$.



Three-dimensional problems in which no triangle can be solved

In the following classic problem, there are four triangles forming a tetrahedron, but no triangle can be solved, because no more than two measurements are known in any one of these triangles.

The method is to introduce a pronumeral $h = TF$ for the height, then work around the figure until *four* measurements are known in terms of h in the base triangle — at this point an equation in h can be formed and solved.

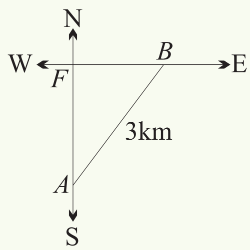
Example 4 Solving a problem in which no triangle can be solved

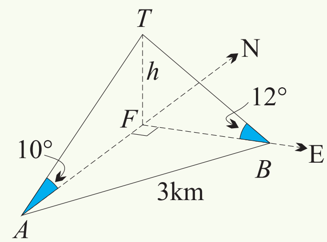
A motorist driving on level ground sees, due north of her, a tower whose angle of elevation is 10°. After driving 3 km further in a straight line, the tower is in the direction due west, with angle of elevation 12°.

a How high is the tower?

b In what direction is she driving?

Solution





Let the tower be TF, and let the motorist be driving from A to B.

a There are four triangles, none of which can be solved.
Let h be the height of the tower.
In $\triangle TAF$, $AF = h\cot 10°$.
In $\triangle TBF$, $BF = h\cot 12°$.
We now have expressions for four measurements in $\triangle ABF$, so we can use Pythagoras' theorem to form an equation in h.
In $\triangle ABF$,

$$AF^2 + BF^2 = AB^2$$
$$h^2\cot^2 10° + h^2\cot^2 12° = 3^2$$
$$h^2(\cot^2 10° + \cot^2 12°) = 9$$
$$h^2 = \frac{9}{\cot^2 10° + \cot^2 12°}$$
$$h \doteqdot 0.407 \text{ km},$$

so the tower is about 407 metres high.

b Let $\theta = \angle FAB$, then in $\triangle AFB$,

$$\sin\theta = \frac{FB}{AB} = \frac{h\cot 12°}{3}$$
$$\theta \doteqdot 64°,$$

so her direction is about N64°E.

The general method of approach

Here is a summary of what has been said about 3D problems:

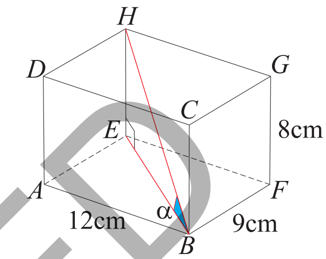
2 Three-dimensional trigonometry

1. Understand angles between a line and a plane, and between two planes.
2. Draw a careful diagram of the situation, marking all right angles.
3. A plan diagram, looking down, is usually a great help.
4. Identify every triangle in the diagram, to see whether it can be solved.
5. If one triangle can be solved, then work from it around the diagram until the problem is solved.
6. If no triangle can be solved, assign a pronumeral to what is to be found, then work around the diagram until an equation in that pronumeral can be formed and solved.

Exercise 16A

FOUNDATION

1 The diagram to the right shows a rectangular prism.

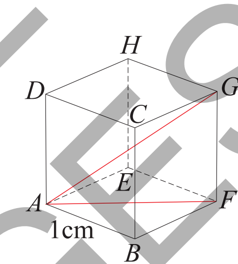


a Use Pythagoras' theorem to find the length of the base diagonal BE.

b Hence find the length of the prism diagonal BH.

c Find, correct to the nearest degree, the angle α that BH makes with the base of the prism.

2 The diagram to the right shows a cube.



a Write down the size of:

i $\angle ABF$ **ii** $\angle AFG$ **iii** $\angle ABG$

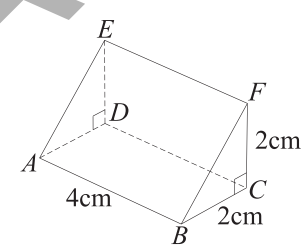
b Use Pythagoras' theorem to find the exact length of:

i AF **ii** AG

c Hence find, correct to the nearest degree:

i $\angle GAF$ **ii** $\angle AGB$

3 The diagram to the right shows a triangular prism.



a Find the exact length of:

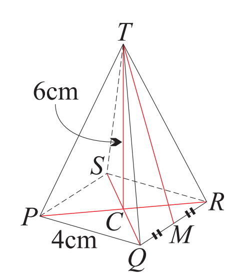
i AC **ii** AF

b What is the size of $\angle ACF$?

c Find $\angle AFC$, correct to the nearest degree.

DEVELOPMENT

4 The diagram to the right shows a square pyramid. The point C is the centre of the base, and TC is perpendicular to the base.



a Write down the size of:

i $\angle CMQ$ **ii** $\angle TCM$ **iii** $\angle TCQ$

b Find the length of:

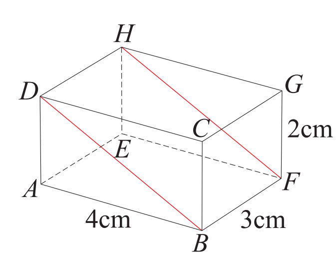
i CM **ii** CQ

c Find, correct to the nearest degree:

i the angle between a side face and the base,

ii the angle between a slant edge and the base.

5 The diagram to the right shows a rectangular prism.



a Write down the size of:

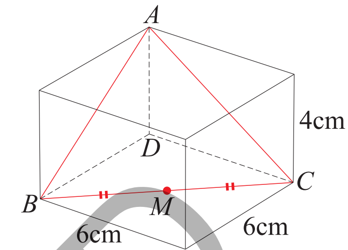
i $\angle ABF$ **ii** $\angle DBF$

b Find, correct to the nearest degree, the angle that the diagonal plane $DBFH$ makes with the base of the prism.

6 The diagram to the right shows a square prism. The plane ABC is inside the prism, and M is the midpoint of the base diagonal BC.

a Find the exact length of MD.

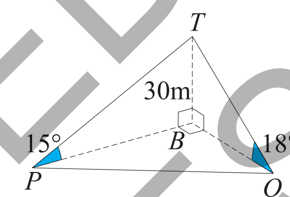
b Hence find, correct to the nearest degree, the angle that the plane ABC makes with the base of the prism.



7 Two landmarks P and Q on level ground are observed from the top T of a vertical tower BT of height 30 m. Landmark P is due south of the tower, while landmark Q is due east of the tower. The angles of elevation of T from P and Q are 15° and 18° respectively.

a Show that $BP = 30 \tan 75°$ and find a similar expression for BQ.

b Find, correct to the nearest metre, the distance between the two landmarks.

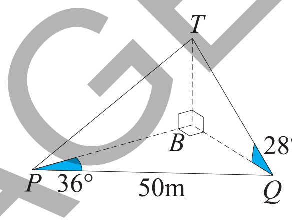


8 A tree BT is due north of an observer at P and due west of an observer at Q. The two observers are 50 m apart and the bearing of Q from P is 36°. The angle of elevation of T from Q is 28°.

a Show that $BQ = 50 \sin 36°$.

b Hence find the height h of the tree correct to the nearest metre.

c Find, correct to the nearest degree, the angle of elevation of T from P.

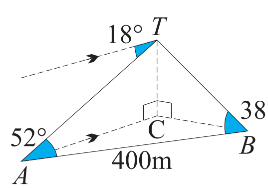


9 Two monuments A and B are 400 m apart on a horizontal plane. The angle of depression of A from the top T of a tall building is 18°. Also, $\angle TAB = 52°$ and $\angle TBA = 38°$.

a Show that $TA = 400 \cos 52°$.

b Find the height h of the building, correct to the nearest metre.

c Find, correct to the nearest degree, the angle of depression of B from T.



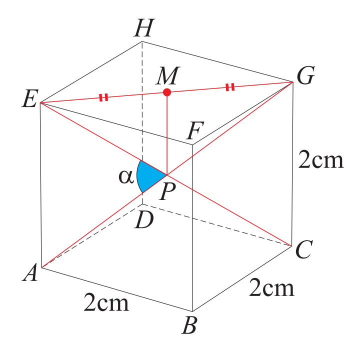
10 The diagram shows a cube of side 2 cm, with diagonals AG and CE intersecting at P. The point M is the midpoint of the face diagonal EG. Let α be the acute angle between the diagonals AG and CE.

a What is the length of PM?

b Find the exact length of EM.

c Write down the exact value of $\tan \angle EPM$.

d Hence find α, correct to the nearest minute.

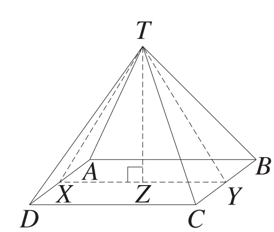


11 The diagram shows a rectangular pyramid. X and Y are the midpoints of AD and BC respectively and T is directly above Z. $TX = 15$ cm, $TY = 20$ cm, $AB = 25$ cm and $BC = 10$ cm.

a Show that $\angle XTY = 90°$.

b Using either similar triangles or Pythagoras' theorem, show that $TZ = 12$ cm.

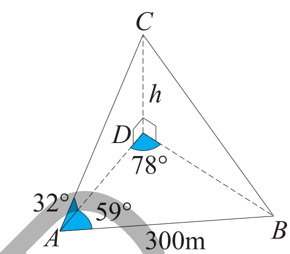
c Hence find, correct to the nearest minute, the angle that the front face DCT makes with the base.



12 Two observers at A and B on horizontal ground are 300 m apart. From A, the angle of elevation of the top C of a tall building DC is 32°. It is also known that $\angle DAB = 59°$ and $\angle ADB = 78°$.

a Show that $AD = \dfrac{300 \sin 43°}{\sin 78°}$.

b Hence find the height of the building, correct to the nearest metre.



13 A balloon B is due north of an observer P and its angle of elevation is 62°. From another observer Q 100 metres from P, the balloon is due west and its angle of elevation is 55°. Let the height of the balloon be h metres and let C be the point on the level ground vertically below B.

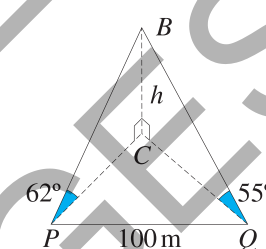
a Show that $PC = h \cot 62°$, and write down a similar expression for QC.

b Explain why $\angle PCQ = 90°$.

c Use Pythagoras' theorem in $\triangle CPQ$ to show that

$$h^2 = \frac{100^2}{\cot^2 62° + \cot^2 55°}.$$

d Hence find h, correct to the nearest metre.



14 From a point P due south of a vertical tower, the angle of elevation of the top of the tower is 20°. From a point Q situated 40 metres from P and due east of the tower, the angle of elevation is 35°. Let h metres be the height of the tower.

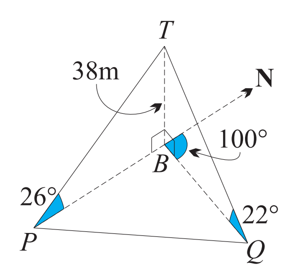
a Draw a diagram to represent the situation.

b Show that $h = \dfrac{40}{\sqrt{\tan^2 70° + \tan^2 55°}}$, and evaluate h, correct to the nearest metre.

15 From two points P and Q on level ground, the angles of elevation of the top T of a 38 m tower are 26° and 22° respectively. Point P is due south of the tower, and the bearing of Q from the tower is 100°T.

a Show that $PB = 38 \tan 64°$, and find a similar expression for QB.

b Use the cosine rule to determine, correct to the nearest metre, the distance between P and Q.



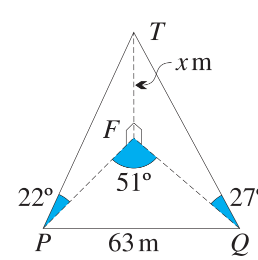
16 In the diagram, TF represents a vertical tower of height x metres standing on level ground. From P and Q at ground level, the angles of elevation of T are 22° and 27° respectively. $PQ = 63$ metres and $\angle PFQ = 51°$.

a Show that $PF = x \cot 22°$ and write down a similar expression for QF.

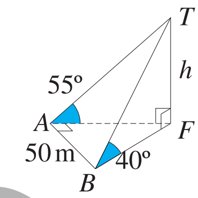
b Use the cosine rule to show that

$$x^2 = \frac{63^2}{\cot^2 22° + \cot^2 27° - 2 \cot 22° \cot 27° \cos 51°}.$$

c Find the value of x to the nearest integer.



17 The diagram shows a tower of height h metres standing on level ground. The angles of elevation of the top T of the tower from two points A and B on the ground nearby are 55° and 40° respectively. The distance AB is 50 metres and the interval AB is perpendicular to the interval AF, where F is the foot of the tower.



a Find AT and BT in terms of h.

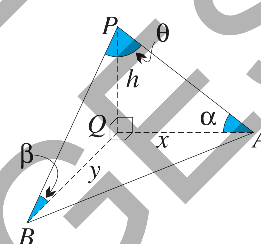
b What is the size of $\angle BAT$?

c Use Pythagoras' theorem in $\triangle BAT$ to show that $h = \dfrac{50 \sin 55° \sin 40°}{\sqrt{\sin^2 55° - \sin^2 40°}}$.

d Hence find the height of the tower, correct to the nearest metre.

CHALLENGE

18 In the diagram of a triangular pyramid, $AQ = x$, $BQ = y$, $PQ = h$, $\angle APB = \theta$, $\angle PAQ = \alpha$ and $\angle PBQ = \beta$. Also, there are three right angles at Q.



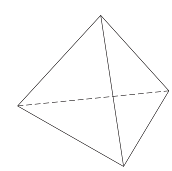
a Show that $x = h \cot \alpha$ and write down a similar expression for y.

b Use Pythagoras' theorem and the cosine rule to show that

$$\cos \theta = \frac{h^2}{\sqrt{(x^2 + h^2)(y^2 + h^2)}}.$$

c Hence show that $\sin \alpha \sin \beta = \cos \theta$.

19 The diagram shows a triangular pyramid, all of whose faces are equilateral triangles — such a solid is called a *regular tetrahedron*. Suppose that the slant edges are inclined at an angle θ to the base. Show that $\cos \theta = \frac{1}{\sqrt{3}}$.



20 A square pyramid has perpendicular height equal to the side length of its base. Show that the angle between a slant edge and a base edge it meets is $\tan^{-1} \sqrt{5}$.

21 Three tourists T_1, T_2 and T_3 at ground level are observing a landmark whose top we shall call L. T_1 is due north of L, T_3 is due east of L, and T_2 is on the line of sight from T_1 to T_3 and between them. The angles of elevation to L from T_1, T_2 and T_3 are 25°, 32° and 36° respectively.

a Show that $\tan \angle LT_1T_2 = \dfrac{\cot 36°}{\cot 25°}$.

b Use the sine rule in $\triangle LT_1T_2$ to find, correct to the nearest minute, the bearing of T_2 from L.

16B Trigonometric functions of compound angles

Learning intentions

- Develop and apply the compound angle trig formulae.

The derivatives of the trigonometric functions will be developed in Year 12. Before that can be done, however, various formulae involving compound angles need to be established, beginning with the expansion of objects such as $\sin(x + h)$, $\tan(x - y)$ and $\cos 2x$. Such trigonometric identities are most important for all sorts of other reasons as well, and must be thoroughly memorised. Their development is the concern of this section and the next.

As with all fundamental results in trigonometry, these formulae require an initial appeal to geometry, in this case circles, Pythagoras' theorem in the form of the distance formula, the cosine rule, and the Pythagorean trigonometric identity.

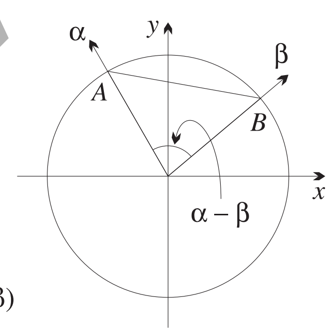
The approach given here begins with the expansion of $\cos(\alpha - \beta)$ and uses that result to derive the other expansions. There are many alternative approaches.

The expansion of $\cos(\alpha - \beta)$

We shall prove that for all real numbers α and β,

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta.$$

Proof Let the rays corresponding to the angles α and β intersect the circle $x^2 + y^2 = r^2$ at the points A and B respectively. Then by the definitions of the trigonometric functions for general angles,



$$A = (r\cos\alpha,\ r\sin\alpha) \quad \text{and} \quad B = (r\cos\beta,\ r\sin\beta).$$

Now we can use the distance formula to find AB^2:

$$\begin{aligned} AB^2 &= r^2(\cos\alpha - \cos\beta)^2 + r^2(\sin\alpha - \sin\beta)^2 \\ &= r^2(\cos^2\alpha + \cos^2\beta + \sin^2\alpha + \sin^2\beta - 2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta) \\ &= 2r^2(1 - \cos\alpha\cos\beta - \sin\alpha\sin\beta), \quad \text{because } \sin^2\theta + \cos^2\theta = 1. \end{aligned}$$

But the angle $\angle AOB$ is $\alpha - \beta$, so by the cosine rule in $\triangle AOB$,

$$\begin{aligned} AB^2 &= r^2 + r^2 - 2r^2\cos(\alpha - \beta) \\ &= 2r^2(1 - \cos(\alpha - \beta)). \end{aligned}$$

Equating these two expressions for AB^2,

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta.$$

Note: It was claimed in the proof that $\angle AOB = \alpha - \beta$. This is not necessarily the case, because it's also possible that $\angle AOB = \beta - \alpha$, or that $\angle AOB$ differs from either of these two values by a multiple of 2π. But the cosine function is even, and it is periodic with period 2π. So it will still follow in every case that $\cos\angle AOB = \cos(\alpha - \beta)$, which is all that is required in the proof.

The six compound-angle formulae

Here are all six compound-angle formulae for the sine, cosine and tangent functions, followed by the remaining five proofs.

3 The compound-angle formulae

A $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

B $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

C $\tan(\alpha + \beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$

D $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

E $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

F $\tan(\alpha - \beta) = \dfrac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$

These formulae are of course true whether the angle is given in degrees or radians.

Proof We proceed from formula E, which has already been proven.

B Replacing β by $-\beta$ in E, which is the expansion of $\cos(\alpha - \beta)$,

$$\begin{aligned}\cos(\alpha + \beta) &= \cos(\alpha - (-\beta))\\ &= \cos\alpha\cos(-\beta) + \sin\alpha\sin(-\beta)\\ &= \cos\alpha\cos\beta - \sin\alpha\sin\beta, \quad \text{because cosine is even and sine is odd.}\end{aligned}$$

A Using the identity $\sin\theta = \cos(\frac{\pi}{2} - \theta)$,

$$\begin{aligned}\sin(\alpha + \beta) &= \cos\left(\tfrac{\pi}{2} - (\alpha + \beta)\right)\\ &= \cos\left((\tfrac{\pi}{2} - \alpha) - \beta\right)\\ &= \cos(\tfrac{\pi}{2} - \alpha)\cos\beta + \sin(\tfrac{\pi}{2} - \alpha)\sin\beta\\ &= \sin\alpha\cos\beta + \cos\alpha\sin\beta.\end{aligned}$$

D Replacing β by $-\beta$, and noting that cosine is even and sine is odd,

$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta.$$

C Because $\tan\theta$ is the ratio of $\sin\theta$ and $\cos\theta$,

$$\begin{aligned}\tan(\alpha + \beta) &= \frac{\sin(\alpha + \beta)}{\cos(\alpha + \beta)}\\ &= \frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta - \sin\alpha\sin\beta}\\ &= \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}, \quad \text{after dividing top and bottom by } \cos\alpha\cos\beta.\end{aligned}$$

F Replacing β by $-\beta$, and noting that the tangent function is odd,

$$\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}.$$

Example 5 Using the compound angle formulae

Express $\sin(x + \frac{2\pi}{3})$ in the form $a\cos x + b\sin x$.

Solution

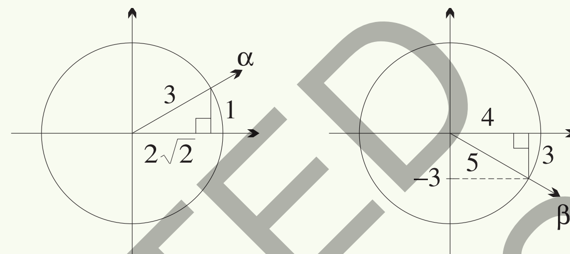
$$\begin{aligned}\sin(x + \tfrac{2\pi}{3}) &= \sin x\cos\tfrac{2\pi}{3} + \cos x\sin\tfrac{2\pi}{3}\\ &= -\tfrac{1}{2}\sin x + \tfrac{1}{2}\sqrt{3}\cos x\end{aligned}$$

Example 6 Using the compound angle formulae

Given that $\sin\alpha = \frac{1}{3}$ and $\cos\beta = \frac{4}{5}$, where α is acute and $-\frac{\pi}{2} < \beta < 0$, find $\sin(\alpha - \beta)$ and $\cos(\alpha + \beta)$.

Solution

First, using the diagrams on the right,
$\cos\alpha = \frac{2}{3}\sqrt{2}$ and $\sin\beta = -\frac{3}{5}$.



So $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

$= \frac{4}{15} + \frac{6}{15}\sqrt{2}$

$= \frac{2}{15}(2 + 3\sqrt{2})$,

and $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$= \frac{8}{15}\sqrt{2} + \frac{3}{15}$

$= \frac{1}{15}(8\sqrt{2} + 3)$.

Further exact values of trigonometric functions

The various compound-angle formulae can be used to find expressions in surds for trigonometric functions of many angles other than ones whose related angles are the standard 30°, 45° and 60°.

Example 7 Finding exact values of angles other than 30°, 45°, and 60°

Find exact values of:

a $\sin 75°$

b $\cos 75°$

Solution

There are many alternative methods.

a $\sin 75° = \sin(30° + 45°)$

$= \sin 30°\cos 45° + \cos 30°\sin 45°$

$= \frac{1}{2} \times \frac{1}{2}\sqrt{2} + \frac{1}{2}\sqrt{3} \times \frac{1}{2}\sqrt{2}$

$= \frac{1}{4}(\sqrt{2} + \sqrt{6})$

b $\cos 75° = \cos(135° - 60°)$

$= \cos 135°\cos 60° + \sin 135°\sin 60°$

$= -\frac{1}{2}\sqrt{2} \times \frac{1}{2} + \frac{1}{2}\sqrt{2} \times \frac{1}{2}\sqrt{3}$

$= \frac{1}{4}(\sqrt{6} - \sqrt{2})$

Exercise 16B

FOUNDATION

1 Use the compound-angle formulae to expand:

a $\sin(x - y)$ **b** $\cos(2A + 3B)$ **c** $\sin(3\alpha + 5\beta)$

d $\cos(\theta - \frac{\varphi}{2})$ **e** $\tan(A + 2B)$ **f** $\tan(3\alpha - 4\beta)$

2 Express as a single trigonometric function:

a $\cos x\cos y - \sin x\sin y$

b $\sin 3\alpha\cos 2\beta + \cos 3\alpha\sin 2\beta$

c $\dfrac{\tan 40° - \tan 20°}{1 + \tan 40°\tan 20°}$

d $\sin 5A\cos 2A - \cos 5A\sin 2A$

e $\cos 70°\cos 20° + \sin 70°\sin 20°$

f $\dfrac{\tan\alpha + \tan 10°}{1 - \tan\alpha\tan 10°}$

3 Use the compound-angle formulae to prove:

a $\sin(90° + A) = \cos A$ **b** $\cos(90° - A) = \sin A$ **c** $\tan(360° - A) = -\tan A$

d $\tan(180° + A) = \tan A$ **e** $\cos(270° - A) = -\sin A$ **f** $\sin(360° - A) = -\sin A$

4 Prove the identities:

a $\sin(A + 45°) = \dfrac{1}{\sqrt{2}}(\sin A + \cos A)$ **b** $2\cos(\theta + \frac{\pi}{3}) = \cos\theta - \sqrt{3}\sin\theta$

c $\tan(\frac{\pi}{4} - A) = \dfrac{1 - \tan A}{1 + \tan A}$ **d** $\sin(A - 30°) = \frac{1}{2}(\sqrt{3}\sin A - \cos A)$

5 **a** If $\tan\alpha = \frac{1}{2}$ and $\tan\beta = \frac{1}{3}$, find $\tan(\alpha + \beta)$.

b If $\cos A = \frac{4}{5}$ and $\sin B = \frac{12}{13}$, where A and B are both acute, find $\sin(A + B)$.

c If $\sin\theta = \frac{2}{3}$ and $\cos\varphi = \frac{1}{4}$, where θ and φ are both acute, find $\cos(\theta - \varphi)$.

6 Prove each identity.

a $\sin(A + B) + \sin(A - B) = 2\sin A\cos B$

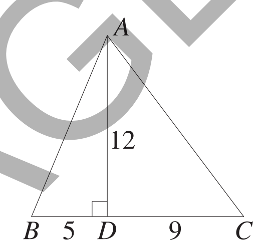
b $\cos(x - y) - \cos(x + y) = 2\sin x\sin y$

c $\sin(x + y) + \cos(x - y) = (\sin x + \cos x)(\sin y + \cos y)$

7 Find from the diagram, using appropriate compound-angle results:

a $\sin\angle BAC$

b $\cos\angle BAC$



DEVELOPMENT

8 **a** By expressing 15° as (45° − 30°), show that:

i $\sin 15° = \dfrac{\sqrt{3} - 1}{2\sqrt{2}}$ **ii** $\cos 15° = \dfrac{\sqrt{3} + 1}{2\sqrt{2}}$ **iii** $\tan 15° = 2 - \sqrt{3}$

b Hence find surd expressions for:

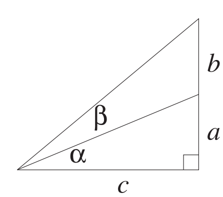
i $\sin 75°$ **ii** $\cos 75°$ **iii** $\tan 75°$

9 In the diagram opposite, it is known that $\tan\beta = \frac{1}{3}$.

a Show that $\tan(\alpha + \beta) = \dfrac{3a + c}{3c - a}$.

b Write down an alternative expression for $\tan(\alpha + \beta)$.

c Hence show that $b = \dfrac{a^2 + c^2}{3c - a}$.



10 If $\sin A = \frac{2}{3}$, where $\frac{\pi}{2} < A < \pi$, and $\tan B = \frac{2}{3}$, where $\pi < B < \frac{3\pi}{2}$, show that $\cos(A + B) = \dfrac{3\sqrt{5} + 4}{3\sqrt{13}}$.

11 Use compound-angle formulae to find the exact value of:

a $\cos 105°$ **b** $\sin\frac{13\pi}{12}$ **c** $\cot 285°$

12 **a** Show that $\sin(A + B)\sin(A - B) = \sin^2 A - \sin^2 B$.

b Hence simplify $\sin^2 75° - \sin^2 15°$.

13 Show that:

a $\sin(\frac{\pi}{4} - \theta)\cos(\frac{\pi}{4} + \theta) + \cos(\frac{\pi}{4} - \theta)\sin(\frac{\pi}{4} + \theta) = 1$

b $\tan 35° + \tan 10° + \tan 35°\tan 10° = 1$

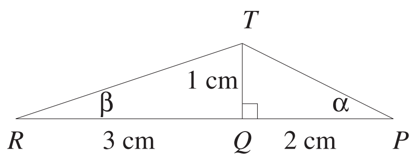
14 Prove each identity:

a $\dfrac{2\sin(x-y)}{\cos(x+y)-\cos(x-y)} = \cot x - \cot y$

b $\dfrac{\sin(\theta+\varphi)}{\cos(\theta-\varphi)} = \dfrac{\tan\theta+\tan\varphi}{1+\tan\theta\tan\varphi}$

c $\dfrac{\sin(\alpha-\beta)}{\sin\alpha\sin\beta} + \dfrac{\sin(\beta-\gamma)}{\sin\beta\sin\gamma} + \dfrac{\sin(\gamma-\alpha)}{\sin\gamma\sin\alpha} = 0$

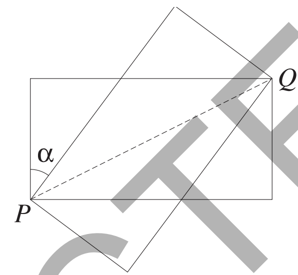
15



a Write down the formula for $\sin(\alpha+\beta)$.

b Hence show that the angles α and β in the diagram above have sum 45°.

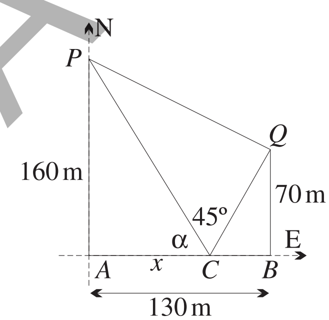
16



The diagram shows two rectangles. Each rectangle is 6 cm long and 3 cm wide, and they share a common diagonal PQ. Show that $\tan\alpha = \frac{3}{4}$.

CHALLENGE

17 In the diagram opposite, P and Q are landmarks that are 160 metres and 70 metres due north of points A and B respectively. Points A and B lie 130 metres apart on a west–east road, and C is a point on the road between A and B so that $\angle PCQ = 45°$. Find AC.
(Let $AC = x$ and $\angle ACP = \alpha$.)



18 **a** Prove the trigonometric identity

$$\tan(A+B+C) = \frac{\tan A + \tan B + \tan C - \tan A\tan B\tan C}{1 - \tan A\tan B - \tan B\tan C - \tan C\tan A}.$$

b Hence show that in any triangle ABC, $\tan A + \tan B + \tan C = \tan A\tan B\tan C$.

16C The double-angle formulae

Learning intentions

- Develop and apply the various double-angle trig formulae.

Substituting $\theta = \alpha = \beta$ into the compound angle formulae gives formulae for double-angle expressions such as $\sin 2\theta$.

Double-angle formulae

In the compound-angle formula $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$, replace both angles α and β by the one angle θ. This gives:

$$\begin{aligned}\sin 2\theta &= \sin(\theta + \theta) = \sin\theta\cos\theta + \cos\theta\sin\theta \\ &= 2\sin\theta\cos\theta.\end{aligned}$$

The same procedure gives expansions of $\cos 2\theta$ and $\tan 2\theta$:

4 The double-angle formulae

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$$

The expansion of $\cos 2\theta$ can then be combined with the Pythagorean identity to give two other forms of the expansion, first by using $\sin^2\theta = 1 - \cos^2\theta$, then by using $\cos^2\theta = 1 - \sin^2\theta$.

5 The $\cos 2\theta$ formulae

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$\cos 2\theta = 2\cos^2\theta - 1$$

$$\cos 2\theta = 1 - 2\sin^2\theta$$

and rearranging the last two identities:

$$\cos^2\theta = \tfrac{1}{2} + \tfrac{1}{2}\cos 2\theta$$

$$\sin^2\theta = \tfrac{1}{2} - \tfrac{1}{2}\cos 2\theta$$

The last two are only rearrangements — perhaps just remember they are there.

Example 8 **Proving further double-angle identities**

Prove that $(\sin x + \cos x)^2 = 1 + \sin 2x$.

Solution

$$\begin{aligned}\text{LHS} &= (\sin x + \cos x)^2 \\ &= \sin^2 x + \cos^2 x + 2\sin x\cos x \\ &= 1 + \sin 2x \\ &= \text{RHS}.\end{aligned}$$

Example 9 **Proving further double-angle identities**

Express $\cot 2x$ in terms of $\cot x$.

Solution

$$\cot 2x = \frac{1}{\tan 2x} = \frac{1 - \tan^2 x}{2\tan x} = \frac{\cot^2 x - 1}{2\cot x}, \quad \text{after division of top and bottom by } \tan^2 x.$$

Further exact values using the double-angle formulae

The double-angle formulae can be used to find exact values of the trigonometric functions at angles such as $22\frac{1}{2}^\circ$.

Example 10 **Finding exact values of further acute angles**

Find the exact value of $\tan 22\frac{1}{2}^\circ$.

Solution

Use the formula $\tan 2\theta = \dfrac{2\tan\theta}{1 - \tan^2\theta}$, with $\theta = 22\frac{1}{2}^\circ$,

which gives, $\tan 45^\circ = \dfrac{2\tan 22\frac{1}{2}^\circ}{1 - \tan^2 22\frac{1}{2}^\circ}$.

Because $\tan 45^\circ = 1$, $1 - \tan^2 22\frac{1}{2}^\circ = 2\tan 22\frac{1}{2}^\circ$

$$\tan^2 22\tfrac{1}{2}^\circ + 2\tan 22\tfrac{1}{2}^\circ - 1 = 0.$$

This quadratic has discriminant $\Delta = 4 + 4 = 4 \times 2$,

so $\tan 22\frac{1}{2}^\circ = \dfrac{-2 + 2\sqrt{2}}{2}$ or $\dfrac{-2 - 2\sqrt{2}}{2}$

$= \sqrt{2} - 1$ (which is positive) or $-\sqrt{2} - 1$ (which is negative).

But $\tan 22\frac{1}{2}^\circ$ is positive because $22\frac{1}{2}^\circ$ is acute, so its value is $\tan 22\frac{1}{2}^\circ = \sqrt{2} - 1$.

Exercise 16C

FOUNDATION

1 Use the double-angle formulae to simplify:

a $2\sin x\cos x$ **b** $\cos^2\theta - \sin^2\theta$ **c** $\dfrac{2\tan\alpha}{1 - \tan^2\alpha}$

d $2\sin 20^\circ\cos 20^\circ$ **e** $2\cos^2 50^\circ - 1$ **f** $\dfrac{2\tan 70^\circ}{1 - \tan^2 70^\circ}$

g $2\sin 3\theta\cos 3\theta$ **h** $1 - 2\sin^2 2A$ **i** $\dfrac{2\tan 4x}{1 - \tan^2 4x}$

2 Prove that:

a $(\cos A - \sin A)(\cos A + \sin A) = \cos 2A$

b $(\sin\alpha - \cos\alpha)^2 = 1 - \sin 2\alpha$

c $\sin 2\theta = 2\sin\theta\sin(\frac{\pi}{2} - \theta)$

d $\dfrac{1}{1-\tan\theta} - \dfrac{1}{1+\tan\theta} = \tan 2\theta$

3 **a** If $\cos\alpha = \frac{4}{5}$, find $\cos 2\alpha$.

b If $\sin x = \frac{2}{3}$, find $\cos 2x$.

c If $\sin\theta = \frac{5}{13}$ and θ is acute, find $\sin 2\theta$.

d If $\tan A = \frac{1}{2}$, find $\tan 2A$.

DEVELOPMENT

4 If $\sin x = \frac{3}{4}$ and $\frac{\pi}{2} < x < \pi$, find the exact value of $\sin 2x$.

5 Prove each identity.

a $\cos^4\alpha - \sin^4\alpha = \cos 2\alpha$

b $\cos 2x + \cos x = (\cos x + 1)(2\cos x - 1)$

c $\dfrac{\sin 2A}{1 - \cos 2A} = \cot A$

d $\cos\theta - \sin\theta\sin 2\theta = \cos\theta\cos 2\theta$

e $\tan(\frac{\pi}{4} + \alpha) - \tan(\frac{\pi}{4} - \alpha) = 2\tan 2\alpha$

f $\dfrac{\sin 2\theta + \sin\theta}{\cos 2\theta + \cos\theta + 1} = \tan\theta$

6 **a** Show that $\dfrac{1 - \cos 2x}{1 + \cos 2x} = \tan^2 x$.

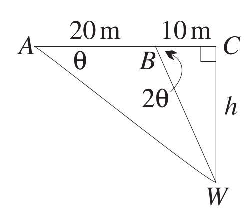
b Hence find the exact value of $\tan\frac{\pi}{8}$.

7 Eliminate θ from each pair of parametric equations.

a $x = 2 + \cos\theta$, $y = \cos 2\theta$

b $x = \tan\theta + 1$, $y = \tan 2\theta$

8 Points A, B, C, and W lie in the same vertical plane. A bird at A observes a worm at W at an angle of depression θ. After flying 20 metres horizontally to B, the angle of depression of the worm is 2θ. If the bird flew another 10 metres horizontally it would be directly above the worm. Let $WC = h$.



a Write $\tan 2\theta$ in terms of $\tan\theta$.

b Use the two right-angled triangles to write two equations in h and θ.

c Use parts (a) and (b) to show that $\dfrac{1}{10} = \dfrac{60}{900 - h^2}$.

d Hence show that $h = 10\sqrt{3}$ metres.

e How could h have been found without trigonometry?

9 In $\triangle ABC$, $\dfrac{b}{c} = \dfrac{4}{3}$ and $B = 2C$. Use the sine rule to show that $\cos C = \dfrac{2}{3}$.

10 **a** By writing 3θ as $2\theta + \theta$ and using compound-angle and double-angle results, prove that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.

b Hence show that $\cos\frac{2\pi}{9}$ is a root of the equation $8x^3 - 6x + 1 = 0$.

11 Use double-angle formulae to show that:

a $2\sin\frac{4\pi}{5}\cos\frac{\pi}{5} = \sin\frac{2\pi}{5}$

b $\cos^2\frac{4\pi}{7} - \sin^2\frac{3\pi}{7} = \cos\frac{6\pi}{7}$

12 Prove each identity.

a $\cot 2\alpha + \tan \alpha = \operatorname{cosec} 2\alpha$

b $\dfrac{\sin 3A}{\sin A} + \dfrac{\cos 3A}{\cos A} = 4\cos 2A$

c $\tan 2x \cot x = 1 + \sec 2x$

d $\dfrac{\sin 2\theta - \cos 2\theta + 1}{\sin 2\theta + \cos 2\theta - 1} = \tan(\theta + \frac{\pi}{4})$

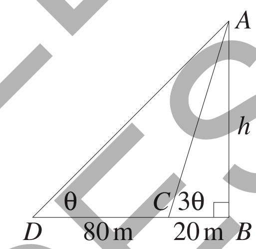
e $\dfrac{1 + \sin 2\alpha}{1 + \cos 2\alpha} = \frac{1}{2}(1 + \tan \alpha)^2$

f $\operatorname{cosec} 4A + \cot 4A = \frac{1}{2}(\cot A - \tan A)$

CHALLENGE

13 **a** Show that $\tan 3\theta = \dfrac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$.

b A tower AB has height h metres. The angle of elevation of the top of the tower at a point C 20 metres from its base is three times the angle of elevation at a point D 80 metres further away from its base. Use the identity in part (a) to show that $h = \frac{100}{\sqrt{7}}$ metres.



14 **a** Write down the exact value of $\cos 45°$.

b Hence show that:

i $\cos 22\frac{1}{2}° = \frac{1}{2}\sqrt{2 + \sqrt{2}}$

ii $\cos 11\frac{1}{4}° = \frac{1}{2}\sqrt{2 + \sqrt{2 + \sqrt{2}}}$

15 **a** Show that $\sqrt{8 - 4\sqrt{3}} = \sqrt{6} - \sqrt{2}$.

b Show that $\tan 165° = \sqrt{3} - 2$.

c Hence show that $\tan 82\frac{1}{2}° = \sqrt{6} + \sqrt{3} + \sqrt{2} + 2$.

16D Trigonometric equations

Learning intentions

- Apply the compound angle formulae to the solution of trigonometric equations.
- Decide which method or methods to use for particular trigonometric equations.

Trigonometric equations occur whenever trigonometric functions and their graphs are being analysed, and careful study of them is essential. They were addressed in degrees in Chapter 7, and in radians in Chapter 13. The identities of the last two sections now allow further equations to be solved.

Solving a trig equations using factoring

As we have seen, factoring is fundamental everywhere in our course. The difference of squares is the immediately obvious approach to the next worked example.

Example 11 **Solving a trig equation by factoring.**

Solve $\sin^2 x - 4\cos^2 x = 0$, for $-\pi \le x \le \pi$, by factoring (three decimal places).

Solution

The equation is $\sin^2 x - 4\cos^2 x = 0.$

Using difference of squares, $(\sin x + 2\cos x)(\sin x - 2\cos x) = 0$

$\div \cos^2 x$ $\quad (\tan x + 2)(\tan x - 2) = 0.$

Hence $\tan x = 2$ or $\tan x = -2.$

The related angle is 1.107148... (leave in the calculator)
so $x \doteqdot -2.034,\ -1.107,\ 1.107,$ or $2.034.$

Equations requiring algebraic substitutions or factoring

This is a review of the method as presented earlier.

Example 12 **Solving trig equations using substitution or factoring**

Solve $2\cos x = 1 + \sec x$, for $0 \le x \le 2\pi$:

a using a substitution,

b using identities and factoring.

Solution

a Let $u = \cos x$.

Then $2u = 1 + \dfrac{1}{u}$

$2u^2 - u - 1 = 0$

$(2u + 1)(u - 1) = 0$

$u = 1$ or $u = -\frac{1}{2}$,

so, $\cos x = 1$ or $\cos x = -\frac{1}{2}$.
Hence $x = 0,\ 2\pi,\ \frac{2\pi}{3}$, or $\frac{4\pi}{3}$.

b $2\cos x = 1 + \sec x$

$\times \cos x$ $\quad 2\cos^2 x = \cos x + 1$

$2\cos^2 x - \cos x - 1 = 0$

$(2\cos x + 1)(\cos x - 1) = 0$

$\cos x = 1$ or $\cos x = -\frac{1}{2}$

Hence $x = 0,\ 2\pi,\ \frac{2\pi}{3}$, or $\frac{4\pi}{3}$.

Equations with more than one trigonometric function, but the same angle

This is where trigonometric identities come into play.

6 Equations with more than one trigonometric function

Trigonometric identities can usually be used to produce an equation in only one trigonometric function.

Example 13 Using trig identities to solve a trig equation

Solve the equation $2\tan\theta = \sec\theta$, for $0^\circ \le \theta \le 360^\circ$:

a using the ratio identities,

b by squaring both sides.

Solution

a
$$2\tan\theta = \sec\theta$$
$$\frac{2\sin\theta}{\cos\theta} = \frac{1}{\cos\theta}$$
$$\sin\theta = \tfrac{1}{2}$$
$$\theta = 30^\circ \text{ or } 150^\circ$$

b Squaring,
$$4\tan^2\theta = \sec^2\theta$$
$$4\sec^2\theta - 4 = \sec^2\theta$$
$$\sec^2\theta = \tfrac{4}{3}$$
$$\cos\theta = \tfrac{1}{2}\sqrt{3} \text{ or } -\tfrac{1}{2}\sqrt{3}$$
$$\theta = 30^\circ, 150^\circ, 210^\circ, \text{ or } 330^\circ.$$
Checking each solution, $\theta = 30^\circ$ or 150°.

Warning — The dangers of squaring an equation

Method (b) in the previous example is intended as a warning — avoid squaring an equation if possible, because squaring may introduce extra solutions.

If an equation has to be squared, each solution must be checked in the original equation to see whether it is a solution or not. Here are two very simple equations, one algebraic and one trigonometric, where the effect of squaring is obvious.

a Suppose that $\sqrt{x} = -5$.
Squaring, $x = 25$.
But $\sqrt{25} = 5$, not -5.
In fact, there are no solutions.

b Suppose that $\cos x = 1$.
Squaring,
$$\cos^2 x = 1$$
$$1 - \sin^2 x = 1$$
$$\sin^2 x = 0$$
$$\sin x = 0.$$
$$x = n\pi, \quad \text{where } n \in \mathbf{Z}.$$
But $\cos n\pi = -1$, for every odd multiple of π.

Equations involving different angles

When different angles are involved in the same trigonometric equation, the usual approach is to use compound-angle identities to change all the trigonometric functions to functions of the one angle.

7 Equations involving different angles

Use compound-angle identities to change all the trigonometric functions to functions of the one angle.

Frequently such an equation can be solved by more than one method.

Example 14 Using the double-angle formulae

Use the tan 2θ formula to solve $\tan 4x = -\tan 2x$, for $0 \le x \le \frac{\pi}{2}$.

Solution

$$\tan 4x = -\tan 2x$$
$$\frac{2\tan 2x}{1-\tan^2 2x} = -\tan 2x$$
$$2\tan 2x = -\tan 2x + \tan^3 2x$$
$$\tan^3 2x - 3\tan 2x = 0$$
$$\tan 2x(\tan^2 2x - 3) = 0$$
$$\tan 2x = 0, \text{ or } \sqrt{3}, \text{ or } -\sqrt{3}.$$

The restriction on $2x$ is $0 \le 2x \le \pi$, so the solutions are

$$2x = 0, \text{ or } \pi, \text{ or } \tfrac{\pi}{3}, \text{ or } \tfrac{2\pi}{3}$$
$$x = 0, \text{ or } \tfrac{\pi}{2}, \text{ or } \tfrac{\pi}{6}, \text{ or } \tfrac{\pi}{3}.$$

Equations involving different angles and functions

The six trig functions are all very closely related. The best approach is usually:

8 Approaching trigonometric equations

- First, try to get all the angles the same.
- Then try to get all the trigonometric functions the same.

Example 15 Solving equations involving different angles and functions

Solve $\cos 2x = 4\sin^2 x - 14\cos^2 x$, for $0 \le x \le 2\pi$:

a by changing all the angles to x,

b by changing all the angles to $2x$.

Solution

a
$$\cos 2x = 4\sin^2 x - 14\cos^2 x$$
$$\cos^2 x - \sin^2 x = 4\sin^2 x - 14\cos^2 x$$
$$15\cos^2 x = 5\sin^2 x$$
$$\tan x = \sqrt{3} \text{ or } -\sqrt{3}$$
$$x = \tfrac{\pi}{3}, \tfrac{2\pi}{3}, \tfrac{4\pi}{3}, \text{ or } \tfrac{5\pi}{3}$$

b
$$\cos 2x = 4\sin^2 x - 14\cos^2 x$$
$$\cos 2x = 4(\tfrac{1}{2} - \tfrac{1}{2}\cos 2x) - 14(\tfrac{1}{2} + \tfrac{1}{2}\cos 2x)$$
$$10\cos 2x = -5$$
$$\cos 2x = -\tfrac{1}{2}$$
$$2x = \tfrac{2\pi}{3}, \tfrac{4\pi}{3}, \tfrac{8\pi}{3}, \text{ or } \tfrac{10\pi}{3}$$
$$x = \tfrac{\pi}{3}, \tfrac{2\pi}{3}, \tfrac{4\pi}{3}, \text{ or } \tfrac{5\pi}{3}$$

Homogeneous equations

Equations homogeneous in $\sin x$ and $\cos x$ are an important special case.

9 Homogeneous equations

- An equation is called *homogeneous in* $\sin x$ *and* $\cos x$ if the sum of the indices of $\sin x$ and $\cos x$ in each term is the same (and the sum is called the *degree*).
- To solve an equation homogeneous in $\sin x$ and $\cos x$, divide through by a power of $\cos x$ to produce an equation in $\tan x$ alone.

The expansions of $\sin 2x$ and $\cos 2x$ are homogeneous of degree 2 in $\sin x$ and $\cos x$. Also, $1 = \sin^2 x + \cos^2 x$ can be regarded as being homogeneous of degree 2.

Example 16 **Solve homogeneous trig equations**

Solve $\sin 2x + \cos 2x = \sin^2 x + 1$, for $0 \le x \le 2\pi$.

Solution

Expanding, $2\sin x\cos x + (\cos^2 x - \sin^2 x) = \sin^2 x + (\sin^2 x + \cos^2 x)$

$3\sin^2 x - 2\sin x\cos x = 0$ (this is homogeneous in $\sin x$ and $\cos x$.)

$\boxed{\div \cos^2 x}$ $3\tan^2 x - 2\tan x = 0$

$\tan x(3\tan x - 2) = 0$

$\tan x = 0$ or $\tan x = \frac{2}{3}$.

Hence $x = 0$ or π or 2π, or $x \doteqdot 0.588$ or 3.730.

A trigonometric equation can often be solved in more than one way

Worked Example 11 above was a classic difference-of-squares equation. But it is also a classic for a homogeneous equation approach.

Example 17

Trigonometric equations often have alternative approaches Solve $\sin^2 x - 4\cos^2 x = 0$, for $-\pi \le x \le \pi$, as a homogeneous equation.

Solution

Dividing through by $\cos^2 x$ gives $\tan^2 x - 4 = 0$.

The rest is obvious, and proceeds along the same lines as worked Example 11.

The Equations $\sin x = \sin\alpha$, $\cos x = \cos\alpha$, and $\tan x = \tan\alpha$

It is possible to solve these equations very neatly and quickly, provided that you keep your eye on the All Stations To Central quadrants diagram.

- One solution is α.
- We know from the ASTC quadrants diagram that there are solutions in one of the other three quadrants — find one example of these solutions.
- Add integer multiples of 360° (or of 2π if working in radians) to these two solutions, to find all the solutions satisfying the given restriction on x.
- But if α is a boundary angle, just read the solutions off the graph

Example 18 **Solving equations of the form $\sin x = \sin\alpha$**

a Solve $\sin x = \sin 25°$, for $-180° \le x \le 180°$.
b Solve $\tan x = \tan 0.2$, for $0 \le x \le 2\pi$.
c Solve $\cos x = \cos 160°$, for $-360° \le x \le 360°$.
d Solve $\sin x = \sin\frac{3\pi}{2}$, for $-2\pi \le x \le 2\pi$.

Solution

a $\sin x$ is positive in quadrants 1 and 2, so the two solutions are 25° and 155°.

b By the ASTC diagram, $\tan x > 0$ in quadrants 1 and 3.
Hence the two solutions in $0 \le x \le 2\pi$ are 0.2, and $0.2 + \pi$.
(Alternatively, the graph of $y = \tan x$ has period π, and each branch between two asymptotes is increasing at every point.)

c The angle 160° has related angle 20°, and $\cos x < 0$ in quadrants 2 and 3, so the two solutions in $0° \le x \le 360°$ are 160° and 200°.
Hence the four solutions in $-360° \le x \le 360°$ are $-200°$, $-160°$, 160°, and 200°.

d This is a boundary angle — from the graph, the two solutions are $\frac{3\pi}{2}$ and $-\frac{\pi}{2}$.

Exercise 16D

FOUNDATION

1 Consider the equation $\sin 2x - \cos x = 0$.
- **a** By using a double-angle formula and then factoring, show that $\cos x = 0$ or $\sin x = \frac{1}{2}$.
- **b** Hence solve the equation for $0 \le x \le 2\pi$.

2 Consider the equation $\cos 2x - \cos x = 0$.
- **a** By using a double-angle formula and then factoring, show that $\cos x = 1$ or $-\frac{1}{2}$.
- **b** Hence solve the equation for $0 \le x \le 2\pi$.

3 Consider the equation $\sin(x + \frac{\pi}{4}) = 2\cos(x - \frac{\pi}{4})$.
- **a** Use compound-angle formulae to show that $\tan x = -1$.
- **b** Hence solve the equation for $0 \le x \le 2\pi$.

DEVELOPMENT

4 Use compound-angle formulae to solve, for $0 \le \theta \le 2\pi$.
- **a** $\sin(\theta + \frac{\pi}{6}) = 2\sin(\theta - \frac{\pi}{6})$
- **b** $\cos(\theta - \frac{\pi}{6}) = 2\cos(\theta + \frac{\pi}{6})$
- **c** $\cos 4\theta \cos\theta + \sin 4\theta \sin\theta = \frac{1}{2}$
- **d** $\cos 3\theta = \cos 2\theta \cos\theta$

(Hint: In part (d), write $\cos 3\theta$ as $\cos(2\theta + \theta)$.)

5 Use double-angle formulae to solve, for $0 \le x \le 2\pi$.
- **a** $\sin 2x = \sin x$
- **b** $\sin 2x + \sqrt{3}\cos x = 0$
- **c** $3\sin x + \cos 2x = 2$
- **d** $\cos 2x + 3\cos x + 2 = 0$
- **e** $\tan 2x + \tan x = 0$
- **f** $\sin 2x = \tan x$

6 Solve, for $0° \le \theta \le 360°$, giving solutions correct to the nearest minute where necessary.
- **a** $2\sin 2\theta + \cos\theta = 0$
- **b** $2\cos^2\theta + \cos 2\theta = 0$
- **c** $2\cos 2\theta + 4\cos\theta = 1$
- **d** $8\sin^2\theta\cos^2\theta = 1$
- **e** $3\cos 2\theta + \sin\theta = 1$
- **f** $\cos 2\theta = 3\cos^2\theta - 2\sin^2\theta$
- **g** $10\cos\theta + 13\cos\frac{1}{2}\theta = 5$
- **h** $\tan\theta = 3\tan\frac{1}{2}\theta$

7 Consider the equation $\tan(\frac{\pi}{4} + \theta) = 3\tan(\frac{\pi}{4} - \theta)$.
- **a** Show that $\tan^2\theta - 4\tan\theta + 1 = 0$.
- **b** Hence use the quadratic formula to solve the equation for $0 \le \theta \le \pi$.

8 Given the equation $2\cos x - 1 = 2\cos 2x$:

a Show that $\cos x = \frac{1}{4}(1 + \sqrt{5})$ or $\cos x = \frac{1}{4}(1 - \sqrt{5})$.

b Hence solve the equation for $0 \le x \le 2\pi$.

9 **a** Show that $\sin(\alpha + \beta)\sin(\alpha - \beta) = \sin^2\alpha - \sin^2\beta$.

b Hence solve the equation $\sin^2 3\theta - \sin^2\theta = \sin 2\theta$, for $0 \le \theta \le \pi$.

10 **a** Show that $\sin 3x = 3\sin x - 4\sin^3 x$.

b Hence solve the equation $\sin 3x + \sin 2x = \sin x$, for $0 \le x \le 2\pi$.

11 **a** Given the equation $\sin(\theta + \frac{\pi}{6}) = \cos(\theta - \frac{\pi}{4})$, show that $\tan\theta = \sqrt{6} - \sqrt{3} - \sqrt{2} + 2$.

b Hence solve the equation for $0 \le \theta \le 2\pi$.

12 Solve, for $0 \le \theta \le 2\pi$, giving solutions correct to two decimal places where necessary.

a $\cos^2 2\theta = \sin^2\theta$

b $\cos 2\theta + 3 = 3\sin 2\theta$ [HINT: Write 3 as $3(\cos^2\theta + \sin^2\theta)$.]

CHALLENGE

13 **a** Use the product-to-sum identity $2\cos A\cos B = \cos(A + B) + \cos(A - B)$ to prove the sum-to-product identity $\cos P + \cos Q = 2\cos\left(\frac{P + Q}{2}\right)\cos\left(\frac{P - Q}{2}\right)$.

b Hence solve the equation $\cos 4x + \cos x = 0$, for $0 \le x \le \pi$.

14 **a** Show that $\cos 3x = 4\cos^3 x - 3\cos x$.

b By substituting $x = 2\cos\theta$, show that the equation $x^3 - 3x - 1 = 0$ has roots $x = 2\cos 20°$, $-2\sin 10°$ or $-2\cos 40°$.

c Use a similar approach to find, correct to three decimal places, the three real roots of the equation $x^3 - 12x = 8\sqrt{3}$.

15 **a** If $t = \tan x$, show that $\tan 4x = \dfrac{4t(1 - t^2)}{1 - 6t^2 + t^4}$.

b If $\tan 4x \tan x = 1$, show that $5t^4 - 10t^2 + 1 = 0$.

c Show that $\sin A\sin B = \frac{1}{2}(\cos(A - B) - \cos(A + B))$ and that $\cos A\cos B = \frac{1}{2}(\cos(A - B) + \cos(A + B))$.

d Hence show that $\frac{\pi}{10}$ and $\frac{3\pi}{10}$ both satisfy $\tan 4x \tan x = 1$.

e Hence write down, in trigonometric form, the four real roots of the polynomial equation $5x^4 - 10x^2 + 1 = 0$.

16 Consider the equation $\cos 2x = \sin 3x$, for $0° \le x \le 360°$.

a Solve the equation over the restricted domain by writing $\sin 3x$ as $\cos(90° - 3x)$.

b Alternatively, use the identity $\sin 3x = 3\sin x - 4\sin^3 x$ and the factorisation $4u^3 - 2u^2 - 3u + 1 = (u - 1)(4u^2 + 2u - 1)$ to show that $\sin x = 1, \dfrac{-1 + \sqrt{5}}{4}$ or $\dfrac{-1 - \sqrt{5}}{4}$.

c Hence determine the exact value of $\sin 18°$.

16E The sum of sine and cosine functions

Learning intentions

- Express $a \sin x + b \cos x$ in the form $R \sin(x + \alpha)$ or related forms.
- Understand the result as a dilated and shifted sine or cosine function.
- Solve trig equations of the form $a \sin x + b \cos x = c$ and related inequations.

This section analyses what happens when the sine and cosine curves are added, and, more generally, when multiples of the two curves are added. The surprising result is that $y = a \sin x + b \cos x$ is still a sine or cosine wave, whatever the values of a and b are, but shifted sideways and stretched vertically.

These forms for $a \sin x + b \cos x$ also give a systematic method of solving any equation of the form $a \cos x + b \sin x = c$.

Preliminary notes on vertical dilations and horizontal shifts of $y = \sin x$ and $y = \cos x$

The boxed results below follow immediately from the notes on transformations in Chapter 5. They also establish the amplitude of vertically dilated sine or cosine functions.

10 Vertical dilations and horizontal shifts of $y = \sin x$ and $y = \cos x$

- When shifted right α:

$$y = \sin x \longrightarrow y = \sin(x - \alpha) \quad \text{and} \quad y = \cos x \longrightarrow y = \cos(x - \alpha),$$

  and the resulting wave functions still have *amplitude 1*.
- When dilated vertically with factor $R > 0$:

$$y = \sin x \longrightarrow y = R \sin x \quad \text{and} \quad y = \cos x \longrightarrow y = R \cos x,$$

  and the resulting wave functions now have *amplitude R*.
- When shifted right α and dilated vertically with factor $R > 0$ (in either order):

$$y = \sin x \longrightarrow y = R \sin(x - \alpha) \quad \text{and} \quad y = \cos x \longrightarrow y = \cos(x - \alpha),$$

  and the resulting wave functions have *amplitude R*.

Example 19 Sketching vertical dilations and horizontal shifts of $\sin x$ and $\cos x$

Sketch, over $-360° \le x \le 360°$, on single pairs of axes:

a $y = \sin x$ and $y = 2 \sin x$

b $y = \cos x$ and $y = 2 \cos x$

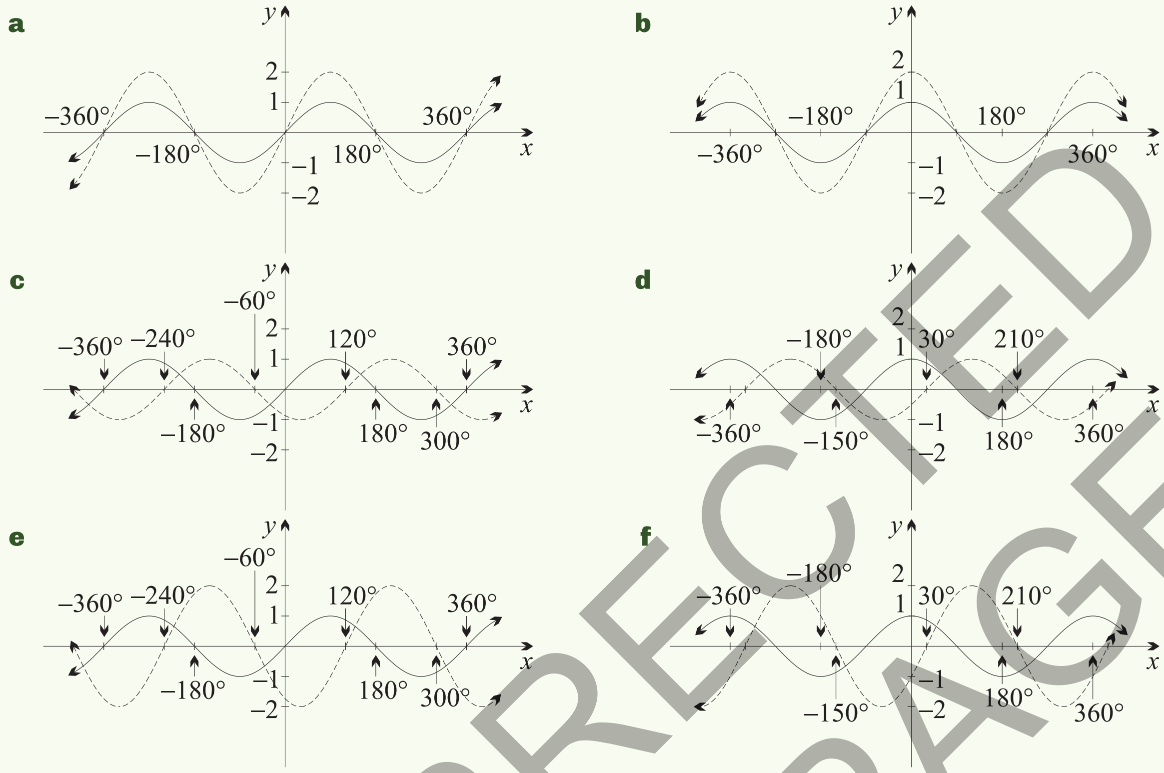
c $y = \sin x$ and $y = \sin(x - 120°)$

d $y = \cos x$ and $y = \cos(x - 120°)$

e $y = \sin x$ and $y = 2 \sin(x - 120°)$

f $y = \cos x$ and $y = 2 \cos(x - 120°)$

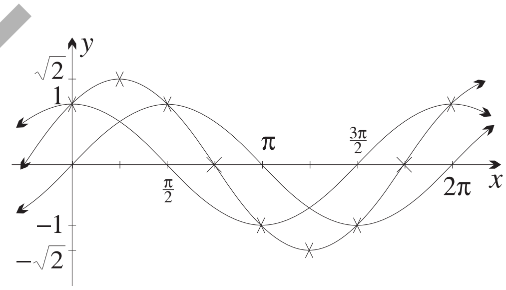
Solution



Sketching $y = \sin x + \cos x$ by graphical methods

We can now return to the sum of $y = \sin x$ and $y = \cos x$. Chapter 6 prepared the ground for this by sketching the sum of two given sketched graphs.

The diagram to the right shows the two graphs of $y = \sin x$ and $y = \cos x$. From these two graphs, the sum function $y = \sin x + \cos x$ has been drawn on the same diagram — the crosses represent obvious points to mark on the graph of the sum.



- The new graph obviously has the same period 2π as $y = \sin x$ and $y = \cos x$. It looks like a wave, and within $[0, 2\pi]$ there are zeroes at the two values $x = \frac{3\pi}{4}$ and $x = \frac{7\pi}{4}$ where $\sin x$ and $\cos x$ take opposite values.
- The new amplitude is bigger than 1. The value at $x = \frac{\pi}{4}$ is $\frac{1}{2}\sqrt{2} + \frac{1}{2}\sqrt{2} = \sqrt{2}$, so if the maximum occurs there, as seems likely, the amplitude is $\sqrt{2}$.
- This suggests that the resulting sum function is $y = \sqrt{2}\sin(x + \frac{\pi}{4})$, because it is the stretched sine curve shifted left $\frac{\pi}{4}$. We can check this by expansion:

$$\begin{aligned}\sqrt{2}\sin(x + \tfrac{\pi}{4}) &= \sqrt{2}\left(\sin x \cos \tfrac{\pi}{4} + \cos x \sin \tfrac{\pi}{4}\right)\\ &= \sin x + \cos x, \quad \text{because } \cos \tfrac{\pi}{4} = \sin \tfrac{\pi}{4} = \frac{1}{\sqrt{2}}.\end{aligned}$$

  That is exactly what we expected from the sketches of the graphs.

The general algebraic approach — the auxiliary angle

It is true in general that any function of the form $f(x) = a\sin x + b\cos x$ can be written as a single wave function. There are four possible standard forms in which the wave can be written, and the procedure is done by expanding the standard form and equating coefficients of $\sin x$ and $\cos x$.

11 Auxiliary-angle method

- Any function of the form $f(x) = a\sin x + b\cos x$, where a and b are constants (not both zero), can be written in any one of the four forms:

  $y = R\sin(x - \alpha)$ $\quad y = R\cos(x - \alpha)$

  $y = R\sin(x + \alpha)$ $\quad y = R\cos(x + \alpha)$

  where $R > 0$ and $0° \le \alpha < 360°$. The constant $R = \sqrt{a^2 + b^2}$ is the same for all forms, but the *auxiliary angle* α depends on which form is chosen.
- To begin the process, expand the standard form and equate coefficients of $\sin x$ and $\cos x$.
- Be careful to identify the quadrant in which the auxiliary angle α lies.

The next example continues with the example given at the start of the section, and shows the systematic algorithm used to obtain the required form.

Example 20 Using the auxiliary angle to rewrite a sum of sine and cosine

Express $y = \sin x + \cos x$ in the two forms:

a $R\sin(x + \alpha)$, **b** $R\cos(x + \alpha)$,

where, in each case, $R > 0$ and $0 \le \alpha < 2\pi$. Then sketch the curve, showing all intercepts and maxima and minima in the interval $0 \le x \le 2\pi$.

Solution

a Expanding, $R\sin(x + \alpha) = R\sin x\cos\alpha + R\cos x\sin\alpha$,

so for all x, $\sin x + \cos x = R\sin x\cos\alpha + R\cos x\sin\alpha$.

Equating coefficients of $\sin x$, $R\cos\alpha = 1$, (1)

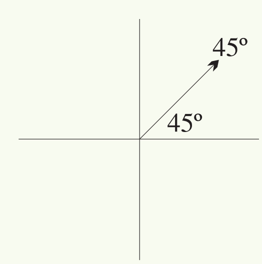
equating coefficients of $\cos x$, $R\sin\alpha = 1$. (2)

Squaring and adding, $R^2 = 2$,

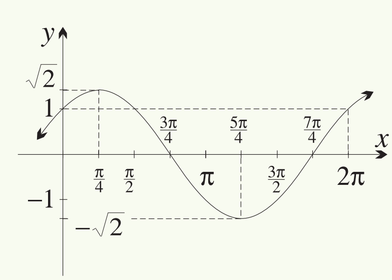
and because $R > 0$, $R = \sqrt{2}$.

From (1), $\cos\alpha = \dfrac{1}{\sqrt{2}}$, (1A)

and from (2), $\sin\alpha = \dfrac{1}{\sqrt{2}}$, (2A)



so the related angle α is in the 1st quadrant, with related angle $\frac{\pi}{4}$.
Hence $\alpha = \frac{\pi}{4}$, and $\sin x + \cos x = \sqrt{2}\sin(x + \frac{\pi}{4})$.
The graph is $y = \sin x$ shifted left by $\frac{\pi}{4}$, and stretched vertically by a factor of $\sqrt{2}$. Thus the x-intercepts are $x = \frac{3\pi}{4}$ and $x = \frac{7\pi}{4}$, there is a maximum of $\sqrt{2}$ when $x = \frac{\pi}{4}$, and a minimum of $-\sqrt{2}$ when $x = \frac{5\pi}{4}$.



b Expanding, $R\cos(x+\alpha) = R\cos x\cos\alpha - R\sin x\sin\alpha$,
so for all x, $\sin x + \cos x = R\cos x\cos\alpha - R\sin x\sin\alpha$.
Equating coefficients of $\cos x$, $R\cos\alpha = 1$, (1)
equating coefficients of $\sin x$, $R\sin\alpha = -1$. (2)
Squaring and adding, $R^2 = 2$,
and because $R > 0$, $R = \sqrt{2}$.
From (1), $\cos\alpha = \dfrac{1}{\sqrt{2}}$, (1A)
and from (2), $\sin\alpha = -\dfrac{1}{\sqrt{2}}$, (2A)
so the auxiliary angle α is in the 4th quadrant, with related angle $\frac{\pi}{4}$.
Hence $\alpha = \frac{7\pi}{4}$, so $\sin x + \cos x = \sqrt{2}\cos(x + \frac{7\pi}{4})$.
The graph above could equally well be obtained from this.
It is $y = \cos x$ shifted left by $\frac{7\pi}{4}$ and stretched vertically by a factor of $\sqrt{2}$.
(Alternatively, shift right by $\frac{\pi}{4}$, because the period is 2π.)

Approximating the auxiliary angle

Unless special angles are involved, the auxiliary angle α will need to be approximated on the calculator. Degrees or radian measure may be used, but the next example uses degrees to make the working a little more intuitive.

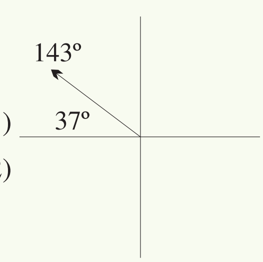
Example 21 Approximating the auxiliary angle and sketching the curve

a Express $y = 3\sin x - 4\cos x$ in the form $y = R\cos(x-\alpha)$, where $R > 0$ and $0° \le \alpha < 360°$, giving α correct to the nearest degree.

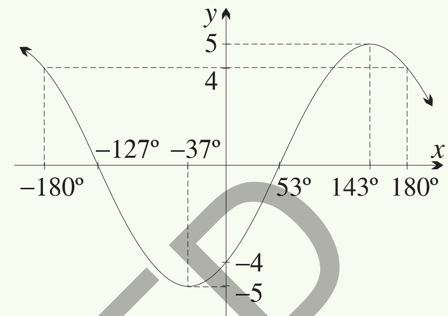
b Sketch the curve, showing, correct to the nearest degree, all intercepts and turning points in the interval $-180° \le x \le 180°$.

Solution

a Expanding, $R\cos(x-\alpha) = R\cos x\cos\alpha + R\sin x\sin\alpha$,
so for all x, $3\sin x - 4\cos x = R\cos x\cos\alpha + R\sin x\sin\alpha$.
Equating coefficients of $\cos x$, $R\cos\alpha = -4$, (1)
equating coefficients of $\sin x$, $R\sin\alpha = 3$. (2)
Squaring and adding, $R^2 = 25$,
and because $R > 0$, $R = 5$.
From (1), $\cos\alpha = -\frac{4}{5}$, (1A)
and from (2), $\sin\alpha = \frac{3}{5}$, (2A)
so α is in the 2nd quadrant, with related angle about 37°.
Hence the auxiliary angle is $\alpha \doteqdot 143°$, and $3\sin x - 4\cos x = 5\cos(x-\alpha)$,



b The graph is $y = \cos x$ shifted right by $\alpha \doteqdot 143°$ and stretched vertically by a factor of 5.
Thus the x-intercepts are $x \doteqdot 53°$ and $x \doteqdot -127°$, there is a maximum of 5 when $x \doteqdot 143°$, and a minimum of -5 when $x \doteqdot -37°$.
(Find the y-intercept -4 by substituting $x = 0$ into the original equation.)



A note on the calculator and its approximations for the auxiliary angle

In worked Example 21, the exact value of α is $\alpha = 180° - \sin^{-1}\frac{3}{5}$, because α is in the 2nd quadrant. The calculator holds this value correct to many decimal places.

When there are subsequent calculations to do, as in worked Example 22 below, this value should be stored in memory and used whenever the auxiliary angle is required. Re-entry of an approximation may lead to rounding errors.

Solving equations of the form $a \sin x + b \cos x = c$, and associated inequations

Once the LHS is in one of the four standard forms, the solutions to this equation are easily obtained. Worked Example 22 continues from worked Example 21.

Note: It is always important to keep track of restrictions on the compound angle.

Example 22 Using the auxiliary angle to solve equations and inequations

a Using the previous worked example, and its calculator approximation, solve $3 \sin x - 4 \cos x = -2$, for $-180° \le x \le 180°$, correct to the nearest degree.

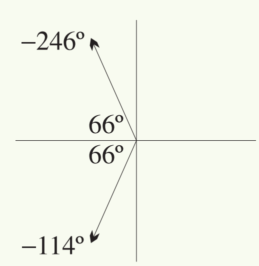
b Hence use the graph to solve $3 \sin x - 4 \cos x \le -2$, for $-180° \le x \le 180°$.

Solution

a We know that $3 \sin x - 4 \cos x = 5 \cos(x - \alpha)$, where $\alpha \doteqdot 143°$,

so the equation is $5 \cos(x - \alpha) = -2$,

$$\cos(x - \alpha) = -\tfrac{2}{5}.$$



Thus $x - \alpha$ is in quadrant 2 or 3, with related angle about 66°.

The restriction on x is $-180° \le x \le 180°$

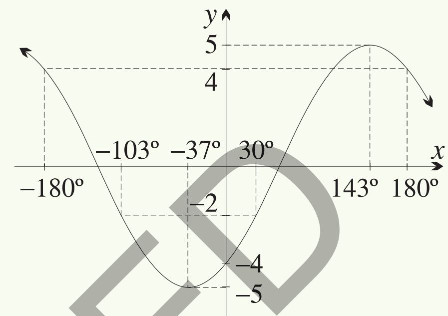
that is, about $-323° \le x - \alpha \le 37°$,

so $x - \alpha \doteqdot -114°$ or $-246°$

$$x \doteqdot 30° \text{ or } -103°.$$

Be careful to use the calculator's memory here, for the angle α, and for the angle 66° above. Never re-enter approximations of those angles.

b The graph to the right shows the previously drawn graph of $y = 3\sin x - 4\cos x$ with the horizontal line $y = -2$ added. This roughly verifies the answers in part (a).
It also shows that the solution of the inequation $3\sin x - 4\cos x \le -2$ is about $-103° \le x \le 30°$.



12 The auxiliary-angle method for equations of the form $a\sin x + b\cos x = c$

- Get the LHS into one of the four forms

$$R\sin(x+\alpha) \quad \text{or} \quad R\sin(x-\alpha) \quad \text{or} \quad R\cos(x+\alpha) \quad \text{or} \quad R\cos(x-\alpha).$$

- Solve the resulting equation, keeping approximations in the calculator memory.
- Use the graph to solve any associated inequation.

Exercise 16E

FOUNDATION

1 **a** What transformation moves $y = \cos x$ to $y = 4\cos x$? Draw both curves on one set of axes, and state the new amplitude.
b What transformation moves $y = \sin x$ to $y = \sin(x - \frac{\pi}{4})$? Draw both curves on one set of axes.

2 Find R and α exactly, if $R > 0$ and $0 \le \alpha < 2\pi$, and:
a $R\sin\alpha = \sqrt{3}$ and $R\cos\alpha = 1$,
b $R\sin\alpha = 3$ and $R\cos\alpha = 3$.

3 Find R (exactly) and α (correct to the nearest minute), if $R > 0$ and $0° \le \alpha < 360°$, and:
a $R\sin\alpha = 5$ and $R\cos\alpha = 12$,
b $R\cos\alpha = 2$ and $R\sin\alpha = 4$.

4 **a** If $\cos x - \sin x = A\cos(x + \alpha)$, show that $A\cos\alpha = 1$ and $A\sin\alpha = 1$.
b Find the positive value of A by squaring and adding.
c Find α, if $0 \le \alpha < 2\pi$.
d State the maximum and minimum values of $\cos x - \sin x$, and the first positive values of x for which they occur.
e Solve the equation $\cos x - \sin x = -1$, for $0 \le x \le 2\pi$.
f Write down the amplitude and period of $\cos x - \sin x$. Hence sketch $y = \cos x - \sin x$, for $0 \le x \le 2\pi$. Indicate on your sketch the line $y = -1$ and the solutions to the equation in part (e).

5 **a** If $\sqrt{3}\cos x - \sin x = B\cos(x + \theta)$, show that $B\cos\theta = \sqrt{3}$ and $B\sin\theta = 1$.
b Find B, if $B > 0$, by squaring and adding.
c Find θ, if $0 \le \theta < 2\pi$.
d State the greatest and least possible values of $\sqrt{3}\cos x - \sin x$, and the values of x closest to $x = 0$ for which they occur.
e Solve the equation $\sqrt{3}\cos x - \sin x = 1$, for $0 \le x \le 2\pi$.
f Sketch $y = \sqrt{3}\cos x - \sin x$, for $0 \le x \le 2\pi$. On the same diagram, sketch the line $y = 1$. Indicate on your diagram the solutions to the equation in part (e).

6 Let $4\sin x - 3\cos x = A\sin(x - \alpha)$, where $A > 0$ and $0° \le \alpha < 360°$.

a Show that $A\cos\alpha = 4$ and $A\sin\alpha = 3$.

b Show that $A = 5$ and $\alpha = \tan^{-1}\frac{3}{4}$.

c Hence solve the equation $4\sin x - 3\cos x = 5$, for $0° \le x \le 360°$. Give the solution(s) correct to the nearest minute.

7 Consider the equation $2\cos x + \sin x = 1$.

a Let $2\cos x + \sin x = B\cos(x - \theta)$, where $B > 0$ and $0° \le \theta < 360°$. Show that $B = \sqrt{5}$ and $\theta = \tan^{-1}\frac{1}{2}$.

b Hence find, correct to the nearest minute where necessary, the solutions of the equation, for $0° \le x \le 360°$.

8 Let $\cos x - 3\sin x = D\cos(x + \varphi)$, where $D > 0$ and $0° \le \varphi < 360°$.

a Show that $D = \sqrt{10}$ and $\varphi = \tan^{-1} 3$.

b Hence solve $\cos x - 3\sin x = 3$, for $0° \le x \le 360°$. Give the solutions correct to the nearest minute where necessary.

9 Consider the equation $\sqrt{5}\sin x + 2\cos x = -2$.

a Transform the LHS into the form $C\sin(x + \alpha)$, where $C > 0$ and $0° \le \alpha < 360°$.

b Find, correct to the nearest minute where necessary, the solutions of the equation, for $0° \le x \le 360°$.

DEVELOPMENT

10 Solve each equation, for $0° \le x \le 360°$, by transforming the LHS into a single-term sine or cosine function. Give solutions correct to the nearest minute.

a $3\sin x + 5\cos x = 4$ **b** $6\sin x - 5\cos x = 7$

c $7\cos x - 2\sin x = 5$ **d** $9\cos x + 7\sin x = 3$

11 Find A and α exactly, if $A > 0$ and $0 \le \alpha < 2\pi$, and:

a $A\sin\alpha = 1$ and $A\cos\alpha = -\sqrt{3}$, **b** $A\cos\alpha = -5$ and $A\sin\alpha = -5$.

12 **a** **i** Express $\sqrt{3}\cos x + \sin x$ in the form $A\cos(x + \theta)$, where $A > 0$ and $0 < \theta < 2\pi$.

ii Hence solve $\sqrt{3}\cos x + \sin x = 1$, for $0 \le x < 2\pi$.

b **i** Express $\cos x - \sin x$ in the form $B\sin(x + \alpha)$, where $B > 0$ and $0 < \alpha < 2\pi$.

ii Hence solve $\cos x - \sin x = 1$, for $0 \le x < 2\pi$.

c **i** Express $\sin x - \sqrt{3}\cos x$ in the form $C\sin(x + \beta)$, where $C > 0$ and $0 < \beta < 2\pi$.

ii Hence solve $\sin x - \sqrt{3}\cos x = -1$, for $0 \le x < 2\pi$.

d **i** Express $-\cos x - \sin x$ in the form $D\cos(x - \varphi)$, where $D > 0$ and $0 < \varphi < 2\pi$.

ii Hence solve $-\cos x - \sin x = 1$, for $0 \le x < 2\pi$.

13 Solve, for $0° \le x \le 360°$, giving solutions correct to the nearest minute:

a $2\sec x - 2\tan x = 5$ **b** $2\operatorname{cosec} x + 5\cot x = 3$

14 **a** Given the equation $\sin\theta + \cos\theta = \cos 2\theta$, show that $\tan\theta = -1$ or $\cos\theta - \sin\theta = 1$.

b Hence solve $\sin\theta + \cos\theta = \cos 2\theta$, for $0 \le \theta < 2\pi$.

15 Solve, for $0 \le x \le 2\pi$:

a $\sin x - \cos x = \sqrt{1.5}$ **b** $\sqrt{3}\sin 2x - \cos 2x = 2$ **c** $\sin 4x + \cos 4x = 1$

16 **a** Show that $(\sqrt{3}+1)\cos 2x + (\sqrt{3}-1)\sin 2x = 2\sqrt{2}\cos(2x - \frac{\pi}{12})$.

b Hence solve $(\sqrt{3}+1)\cos 2x + (\sqrt{3}-1)\sin 2x = 2$, for $-\pi < x \le \pi$.

17 The current I in amperes in a circuit after t seconds is given by $I = 2\sin(t - \frac{\pi}{3}) - \cos(t + \frac{\pi}{2})$. Find the maximum current, and the earliest time that it occurs correct to the nearest tenth of a second.

18 **a** **i** Show that $\sin x - \cos x = \sqrt{2}\sin(x - \frac{\pi}{4})$.

ii Hence sketch the graph of $y = \sin x - \cos x$, for $0 \le x \le 2\pi$.

iii Use your sketch to determine the values of x in the domain $0 \le x \le 2\pi$ for which $\sin x - \cos x > 1$.

b Use a similar approach to part (a) to solve, for $0 \le x \le 2\pi$:

i $\sin x + \sqrt{3}\cos x \le 1$

ii $\sin x - \sqrt{3}\cos x < -1$

iii $\left|\sqrt{3}\sin x + \cos x\right| < 1$

iv $\cos x - \sin x \ge \frac{1}{2}\sqrt{2}$

CHALLENGE

19 Consider the function $y = 4\sin(3x + \frac{\pi}{6}) + 2\sin 3x$.

a Show that the maximum value of the function is $2\sqrt{5 + 2\sqrt{3}}$.

b Find, correct to one decimal place, the first positive value of x for which the maximum value occurs.

20 **a** Show that if $\cos(x - \alpha) = \cos\beta$, then $\tan x = \tan(\alpha + \beta)$ or $\tan x = \tan(\alpha - \beta)$.

Hint: If $\cos A = \cos B$, then $A = 2n\pi + B$ or $A = 2n\pi - B$, where n is an integer.

b Show that $2\cos x + 11\sin x = 5\sqrt{5}\cos\left(x - \tan^{-1}\frac{11}{2}\right)$.

c Consider the equation $2\cos x + 11\sin x = 10$, for $0 \le x < 2\pi$.

i By writing the equation in the form $\cos(x - \alpha) = \cos\beta$ and using part (a), show that $\tan x = \frac{4}{3}$ or $-\frac{24}{7}$.

ii Deduce that the equation has roots $\tan^{-1}\frac{4}{3}$ and $\pi - \tan^{-1}\frac{24}{7}$.

iii Prove that one of the roots is twice the other.

Chapter 16 Review

Review activity

- Create your own summary of this chapter on paper or in a digital document.

Chapter 16 Multiple-choice quiz

- This automatically-marked quiz is accessed in the Interactive Textbook. A printable PDF Worksheet version is also available there.

Skills Checklist

Check-list

- Available in the Interactive Textbook, use the checklist to track your understanding of the learning intentions. Printable PDF and word document versions are also available there.

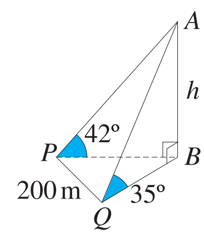
Chapter Review Exercise

1 From two points P and Q on horizontal ground, the angles of elevation of the top T of a 10 m monument are 16° and 13° respectively. It is also known that $\angle PBQ = 70°$, where B is the base of the monument.

a Show that $PB = 10 \tan 74°$, and write down a similar expression for QB.

b Hence determine, correct to the nearest metre, the distance between P and Q.

2 The points P, Q and B lie in a horizontal plane. From P, which is due west of B, the angle of elevation of the top of a tower AB of height h metres is 42°. From Q, which is on a bearing of 196° from the tower, the angle of elevation of the top of the tower is 35°. The distance PQ is 200 metres.



a Show that $PB = h \cot 42°$, and write down a similar expression for QB.

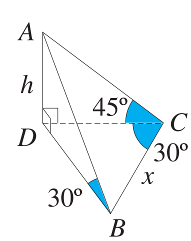
b Explain why $\angle PBQ = 74°$.

c Use the cosine rule to show that

$$h^2 = \frac{200^2}{\cot^2 42° + \cot^2 35° - 2\cot 35° \cot 42° \cos 74°}.$$

d Hence find the height of the tower, correct to the nearest metre.

3 A triangular pyramid $ABCD$ has base BCD and perpendicular height AD.



a Find BD and CD in terms of h.

b Use the cosine rule to show that $2h^2 = x^2 - \sqrt{3}hx$.

c Let $u = \frac{h}{x}$. Write the result of the previous part as a quadratic equation in u, and hence show that

$$\frac{h}{x} = \frac{\sqrt{11} - \sqrt{3}}{4}.$$

4 Simplify, using the compound-angle results:

a $\cos 3\theta \cos \theta + \sin 3\theta \sin \theta$

b $\sin 50° \cos 10° - \cos 50° \sin 10°$

c $\dfrac{\tan 41° + \tan 9°}{1 - \tan 41° \tan 9°}$

d $\cos 15° \cos 55° - \sin 15° \sin 55°$

e $\sin 4\alpha \cos 2\alpha + \cos 4\alpha \sin 2\alpha$

f $\dfrac{1 + \tan 2\theta \tan \theta}{\tan 2\theta - \tan \theta}$

5 Simplify, using the double-angle results:

a $2\sin 2\theta \cos 2\theta$ **b** $\cos^2 \frac{1}{2}x - \sin^2 \frac{1}{2}x$ **c** $2\cos^2 3\alpha - 1$

d $\dfrac{2\tan 35°}{1 - \tan^2 35°}$ **e** $1 - 2\sin^2 25°$ **f** $\dfrac{2\tan 4x}{1 - \tan^2 4x}$

6 Given that the angles A and B are acute, and that $\sin A = \frac{3}{5}$ and $\cos B = \frac{5}{13}$, find:

a $\cos A$ **b** $\cos 2A$ **c** $\cos(A + B)$

d $\sin 2B$ **e** $\tan 2A$ **f** $\tan(B - A)$

7 **a** By writing 75° as 45° + 30°, show that:

i $\sin 75° = \dfrac{\sqrt{3} + 1}{2\sqrt{2}}$ **ii** $\cos 75° = \dfrac{\sqrt{3} - 1}{2\sqrt{2}}$

b Hence show that:

i $\sin 75° \cos 75° = \frac{1}{4}$ **ii** $\sin 75° - \cos 75° = \sin 45°$

iii $\sin^2 75° - \cos^2 75° = \sin 60°$ **iv** $\sin^2 75° + \cos^2 75° = 1$

8 Use the compound-angle and double-angle results to find the exact value of:

a $2\sin 15° \cos 15°$ **b** $\cos 35° \cos 5° + \sin 35° \sin 5°$

c $\dfrac{\tan 110° + \tan 25°}{1 - \tan 110° \tan 25°}$ **d** $1 - 2\sin^2 \frac{\pi}{8}$

e $\cos \frac{\pi}{12} \sin \frac{\pi}{12}$ **f** $\sin \frac{8\pi}{9} \cos \frac{2\pi}{9} - \cos \frac{8\pi}{9} \sin \frac{2\pi}{9}$

9 Prove each identity.

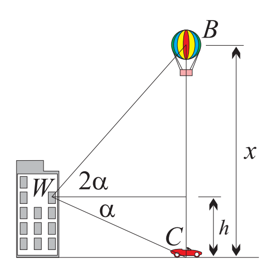
a $(\sin\alpha - \cos\alpha)^2 = 1 - \sin 2\alpha$ **b** $\cos A - \sin 2A \sin A = \cos A \cos 2A$

c $\sin 2\theta(\tan\theta + \cot\theta) = 2$ **d** $\cot\alpha \sin 2\alpha - \cos 2\alpha = 1$

e $\dfrac{\sin 2x}{1 + \cos 2x} = \tan x$ **f** $\dfrac{1 - \tan^2\theta}{1 + \tan^2\theta} = \cos 2\theta$

g $\dfrac{1}{1 - \tan A} - \dfrac{1}{1 + \tan A} = \tan 2A$ **h** $\tan 2A(\cot A - \tan A) = 2$, (provided $\cot A \neq \tan A$)

10 An office-worker is looking out a window W of a building standing on level ground. From W, a car C has an angle of depression α, while a balloon B directly above the car has an angle of elevation 2α. The height of the balloon above the car is x, and the height of the window above the ground is h.



a Show that $\dfrac{\tan\alpha}{h} = \dfrac{\tan 2\alpha}{x - h}$.

b Hence show that $\dfrac{h}{x} = \dfrac{1 - \tan^2\alpha}{3 - \tan^2\alpha}$.

11 Solve each equation for $0 \leq x \leq 2\pi$.

a $\sin 2x + \cos x = 0$ **b** $\cos 2x = \sin x$ **c** $2\cos 2x + 8\cos x + 5 = 0$

d $\cos 2x + 5\sin x + 2 = 0$ **e** $2\sin(x - \frac{\pi}{6}) = \cos(x - \frac{\pi}{3})$ **f** $\tan 2x = 3\tan x$

12 **a** Use compound and double-angle formulae to prove that $\cos 3x = 4\cos^3 x - 3\cos x$.

b Hence solve $\cos 3x + \sin 2x + \cos x = 0$, for $0 \leq x \leq 2\pi$.

13 **a** Express $\sin x - \cos x$ in the form $R\sin(x - \alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$.

b Hence solve $\sin x - \cos x = \sqrt{2}$, for $0 \leq x \leq 2\pi$.

14 **a** Express $\sqrt{3}\cos x + \sin x$ in the form $A\cos(x - \theta)$, where $A > 0$ and $0 < \theta < \frac{\pi}{2}$.

b Hence solve $\sqrt{3}\cos x + \sin x = -1$, for $0 \leq x \leq 2\pi$.

15 **a** Express $2\sin x + \sqrt{5}\cos x$ in the form $R\sin(x + \alpha)$, where $R > 0$ and α is acute.

b Hence solve $2\sin x + \sqrt{5}\cos x = 3$, for $0° \leq x \leq 360°$, writing the solution in degrees correct to one decimal place.

16 **a** Express $3\cos x - 2\sin x$ in the form $A\cos(x + \theta)$, where $A > 0$ and θ is acute.

b Hence solve $3\cos x - 2\sin x = 1$, for $0° \leq x \leq 360°$, writing the solutions correct to the nearest minute.

17

Combinatorics

Chapter introduction

Arithmetic begins with counting, and throughout all branches and applications of mathematics, counting continues to be important.

- How many NSW number plates are possible with the current pattern:

  Letter + Letter + Letter + Digit + Digit + Letter ?

- How many possible 8-bit bytes have three ones and five zeroes?
- How many ways can four chickens be chosen from a flock of 20 chickens? This chapter introduces and develops some of the standard methods of counting.

After introducing *factorial notation*, such as $5! = 5 \times 4 \times 3 \times 2 \times 1$, three successive basic counting methods are addressed:

- Counting ordered selections allowing repetition. This is done using powers.
- Counting ordered selections without repetition, called *permutations*, This requires the numbers ${}^{n}\mathrm{P}_{r}$, which are evaluated using factorials.
- Counting unordered selections, which are just subsets, and are also called *combinations*. This requires the numbers ${}^{n}\mathrm{C}_{r}$, also evaluated using factorials.

Counting arrangements in a circle or around a round table is also addressed.

Counting is particularly important in probability. What is the probability of being dealt two aces and two jacks in a Bridge hand of 13 cards? The counting methods developed here are applied to probability to count sample and event spaces.

The final Chapter 18 examines binomial expansions $(x + y)^n$. It turns out that the basic counting methods here, and particularly the numbers ${}^{n}\mathrm{C}_{r}$, are essential for understanding these expansions.

A note on the order of the chapters: Some basic set theory is obviously helpful in combinatorics. In particular, the standard counting rule:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

often makes a complicated situation much clearer, and complementary sets occur routinely. Readers who have not yet studied the Advanced Chapter 14: Probability are advised to work at the very least through the introductory set theory in Section 14A before embarking on this chapter. Far better would be to work through the whole of Chapter 14, because probability is central to Section 17F.

17A Factorial notation

Learning intentions

- Define and evaluate factorials.
- Manipulate factorials by unrolling and/or cancelling.

Products such as $5 \times 4 \times 3 \times 2 \times 1 = 120$, called *factorials*. They occur throughout mathematics, but are particularly important in combinatorics. This chapter introduces factorials and some of their elementary arithmetic.

The need for factorial notation

How many ways can five people form a queue at a bus stop? Solutions of such problems will be formalised in the next section as the *multiplication principle*, but this particular problem can be solved straightforwardly, and the boxes below help to visualise the solution. The result below is a *factorial:*

- Choose the person at the head of the queue in five ways.
- Four people remain, so the person in second place can be chosen in four ways.
- Three people remain, so the person in third place can be chosen in three ways.
- Two people remain, so the person in fourth place can be chosen in two ways.
- And there is now only one way to choose the person in the last place.

1st place	2nd place	3rd place	4th place	5th place
5	4	3	2	1

Number of possible queues $= 5 \times 4 \times 3 \times 2 \times 1 = 120$.

This product is called ‘5 factorial’, with symbol $5! = 5 \times 4 \times 3 \times 2 \times 1$.

Many apparently unrelated situations in mathematics also generate factorials. For example, what is the fifth derivative of $y = x^5$?

$$\begin{aligned} y' &= 5x^4 \\ y'' &= 5 \times 4 \times x^3 \\ y''' &= 5 \times 4 \times 3 \times x^2 \\ y^{(4)} &= 5 \times 4 \times 3 \times 2 \times x \\ y^{(5)} &= 5 \times 4 \times 3 \times 2 \times 1 = 120. \end{aligned}$$

The definition of n factorial

Here is the general definition of $n!$, called ‘n *factorial*’, stated two ways:

1 The definition of n factorial

- For each whole number $n \ge 1$, the number $n!$ (n *factorial*) is the product of all whole numbers from n down to 1:

$$n! = n \times (n-1) \times (n-2) \times \cdots \times 3 \times 2 \times 1,$$

  and define $0! = 1$.
- The better definition, however, is *recursive* — first define $0! = 1$, and then for each successive value of n, say exactly how to define $n!$ in terms of $(n-1)!$.

$$\begin{cases} 0! = 1, \\ n! = n \times (n-1)!, \quad \text{for all whole numbers } n \ge 1. \end{cases}$$

The recursive definition has three advantages. It avoids the dots $\cdots$ in the first definition, the value $0! = 1$ is built more firmly into the definition, and as we shall soon see, it gives a better insight into how to manipulate factorial notation

The fact that $0!$ is defined to be 1 needs some explanation.

- An *empty product* is regarded as being 1, because if nothing has yet been multiplied, the register remains at 1 where it was originally set in preparation for performing multiplication.
- In a similar way, an *empty sum* is 0, because if nothing has yet been added, the register remains at 0 where it was originally set in preparation for performing addition.

Using the recursive definition that defines each factorial in terms of its predecessor:

$0! = 1$	$4! = 4 \times 3! = 24$	$8! = 8 \times 7! = 40\,320$
$1! = 1 \times 0! = 1$	$5! = 5 \times 4! = 120$	$9! = 9 \times 8! = 362\,880$
$2! = 2 \times 1! = 2$	$6! = 6 \times 5! = 720$	$10! = 10 \times 9! = 3\,628\,800$
$3! = 3 \times 2! = 6$	$7! = 7 \times 6! = 5040$	$11! = 11 \times 10! = 39\,916\,800$

and so on, increasing very quickly indeed. If your calculator has a factorial button labelled x! or n!, use it straight away to see that at least the *calculator* believes that $0! = 1$. Notice also the error message if n is not a whole number, because the domain of the function $n!$ is the whole numbers $0, 1, 2, \ldots$.

Unrolling factorials

The recursive definition of $n!$ given and used above is the key idea in many calculations. Successive applications of the definition can be thought of as *unrolling* the factorial further and further:

$$\begin{aligned} 8! &= 8 \times 7! && \text{(unrolling once)} \\ &= 8 \times 7 \times 6! && \text{(unrolling twice)} \\ &= 8 \times 7 \times 6 \times 5! && \text{(unrolling three times)} \end{aligned}$$

and so on. This idea is vital when there are fractions involved.

Example 1 **Simplifying a factorial expression by unrolling**

Simplify each expression using unrolling techniques:

a $\dfrac{10!}{7!}$ **b** $\dfrac{(n+2)!}{(n-1)!}$ **c** $\dfrac{n!}{(n-r)!}$

Solution

a $$\begin{aligned} \frac{10!}{7!} &= \frac{10 \times 9 \times 8 \times 7!}{7!} \\ &= 10 \times 9 \times 8 \\ &= 720 \end{aligned}$$

b $$\begin{aligned} \frac{(n+2)!}{(n-1)!} &= \frac{(n+2)(n+1)n(n-1)!}{(n-1)!} \\ &= (n+2)(n+1)n \end{aligned}$$

c $$\begin{aligned} \frac{n!}{(n-r)!} &= \frac{n(n-1)(n-2)\cdots(n-r+1)(n-r)!}{(n-r)!} \\ &= \underbrace{n(n-1)(n-2)\cdots(n-r+1)}_{r \text{ factors}} \end{aligned}$$

Example 2 Simplifying a factorial expression using a common denominator

Simplify each expression using a common denominator.

a $\dfrac{1}{8!} - \dfrac{1}{10!}$

b $\dfrac{1}{n!} + \dfrac{1}{(n+1)!}$

Solution

a $\dfrac{1}{8!} - \dfrac{1}{10!} = \dfrac{10 \times 9}{10!} - \dfrac{1}{10!}$

$= \dfrac{89}{10!}$

b $\dfrac{1}{n!} + \dfrac{1}{(n+1)!} = \dfrac{n+1}{(n+1)!} + \dfrac{1}{(n+1)!}$

$= \dfrac{n+2}{(n+1)!}$

2 Calculating by unrolling factorials

- A factorial can be unrolled one step at a time:

$$7! = 7 \times 6! = 7 \times 6 \times 5! = 7 \times 6 \times 5 \times 4! = \cdots$$

- Cancel fractions with factorials by unrolling until the factorials cancel:

$$\frac{7!}{4!} \div \frac{7 \times 6 \times 5 \times 4!}{4!} = 7 \times 6 \times 5 = 210.$$

Example 3 Examining the prime factors in a factorial

a Without using any device or the internet, find how many final zeroes there are when 25! is evaluated.

b Can this be checked on a calculator or other device?

Solution

a Everything depends on the fact that $10 = 5 \times 2$, and 5 and 2 are primes.
There are plenty of factors of 2 in 25! , so the number of final zeroes is the number of factors of 5 in the product. There are factors of 5 in 5, 10, 15, 20 and 25 (which is 5^2), making 6 altogether, so 25! has 6 final zeroes.

b It depends on the device or webpage. Without some reasoning, it will need to give the answer correct to 26 significant figures!

Exercise 17A

FOUNDATION

1 Use arguments similar to that used at the very start of this section to express the following using factorial notation.

a The number of ways to form a queue with 8 people.

b The number of ways to arrange 4 distinct books on a bookself.

c The number of ways 3 particular people can be placed first, second and third in a competition, assuming that each is placed in one of these three positions.

d Sam drops his portable keyboard and all 101 keys fall off. How many ways can he put the keys back on the keyboard, assuming that they are all interchangable?

e When the teacher calls on students in alphabetical order, Andrzej Zywiec is upset that he is always called last. In a class of 20, how many ways could the roll be called, if the alphabetical ordering restriction were dropped?

2 Use the definition of $n!$ and methods of unrolling factorials to evaluate each expression. Do not use a calculator.

a $3!$ **b** $5!$ **c** $1!$ **d** $\dfrac{15!}{14!}$

e $\dfrac{10!}{8! \times 2!}$ **f** $\dfrac{15!}{13! \times 3!}$ **g** $\dfrac{12!}{3! \times 9!}$ **h** $\dfrac{8!}{4! \times 4!}$

3 Use the factorial button $\boxed{n!}$ on your calculator to evaluate each expression.

a $7!$ **b** $10!$ **c** $0!$ **d** $\dfrac{9!}{4!}$

e $\dfrac{8!}{3!}$ **f** $\dfrac{10!}{5! \times 3! \times 2!}$ **g** $\dfrac{15!}{3! \times 5! \times 9!}$ **h** $\dfrac{12!}{2! \times 3! \times 4! \times 5!}$

4 If $f(x) = x^6$, find expressions for:

a $f'(x)$ **b** $f''(x)$ **c** $f'''(x)$ **d** $f''''(x)$ **e** $f^{(5)}(x)$ **f** $f^{(6)}(x)$ **g** $f^{(7)}(x)$

5 Simplify by unrolling factorials appropriately:

a $\dfrac{n!}{(n-1)!}$ **b** $n \times (n-1)!$ **c** $\dfrac{n(n-1)!}{n!}$ **d** $\dfrac{(n+1)!}{(n-1)!}$

e $\dfrac{(n+2)!}{n!}$ **f** $\dfrac{(n-2)!}{n!}$ **g** $\dfrac{(n-2)!\,(n-1)!}{n!\,(n-3)!}$ **h** $\dfrac{n!\,(n-1)!}{(n+1)!}$

6 Simplify by taking out a common factor:

a $8! - 7!$ **b** $(n+1)! - n!$ **c** $8! + 6!$

d $(n+1)! + (n-1)!$ **e** $9! + 8! + 7!$ **f** $(n+1)! + n! + (n-1)!$

DEVELOPMENT

7 Write each expression as a single fraction.

a $\dfrac{1}{n!} + \dfrac{1}{(n-1)!}$ **b** $\dfrac{1}{n!} - \dfrac{1}{(n+1)!}$ **c** $\dfrac{1}{(n+1)!} - \dfrac{1}{(n-1)!}$

8 **a** If $f(x) = x^n$, find:

i $f'(x)$ **ii** $f''(x)$ **iii** $f^{(n)}(x)$ **iv** $f^{(k)}(x)$, where $k \le n$.

b If $f(x) = \dfrac{1}{x}$, find:

i $f'(x)$ **ii** $f''(x)$ **iii** $f^{(5)}(x)$ **iv** $f^{(n)}(x)$.

9 **a** Show that $k \times k! = (k+1)! - k!$

b Hence by considering each individual term as a difference of two terms, sum the series $1 \times 1! + 2 \times 2! + 3 \times 3! + \cdots + n \times n!$

10 **a** Find the largest power of:

i 2, **ii** 10, that is a divisor of $10!$

b Find the largest power of:

i 2, **ii** 5, **iii** 7, **iv** 13, that is a divisor of $100!$

11 [A relationship between higher derivatives of polynomials and factorials]

a If $f(x) = 11x^3 + 7x^2 + 5x + 3$, show that:

i $f(0) = 3 \times 0!$ **ii** $f'(0) = 5 \times 1!$ **iii** $f''(0) = 7 \times 2!$

iv $f'''(0) = 11 \times 3!$ **v** $f^{(k)}(0) = 0$, for all $k \geq 4$.

Hence explain why $f(x)$ can be written $f(x) = \dfrac{f(0)}{0!} + \dfrac{f'(0)\,x}{1!} + \dfrac{f''(0)\,x^2}{2!} + \dfrac{f'''(0)\,x^3}{3!}$.

b Show that if $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ is any polynomial, then

$$f^{(k)}(0) = \begin{cases} a_k\, k!\,, & \text{for } k = 0, 1, 2, \ldots, n, \\ 0, & \text{for } k > n, \end{cases}$$

and hence explain why $f(x) = \dfrac{f(0)}{0!} + \dfrac{f'(0)\,x}{1!} + \dfrac{f''(0)\,x^2}{2!} + \dfrac{f'''(0)\,x^3}{3!} + \cdots$.

Because $f(x)$ is a polynomial, this *power series* has only finitely many terms. There are ways of generalising this to some other functions that are not polynomials.

12 **a** Evaluate $\dfrac{k}{(k+1)!}$, for $k = 1, 2, 3, 4$ and 5.

b Evaluate $\dfrac{1}{(1+1)!} + \dfrac{2}{(2+1)!} + \cdots + \dfrac{n}{(n+1)!}$, for $n = 1, 2, 3, 4$ and 5.

c Write the expression in part (b) as S_n, to indicate that it is a sum whose value depends on n. That is,

$$S_n = \frac{1}{(1+1)!} + \frac{2}{(2+1)!} + \cdots + \frac{n}{(n+1)!}.$$

Make a reasonable guess about the value of S_n. Hence find $\lim_{n\to\infty} S_n$.

d Prove that $\dfrac{k}{(k+1)!} = \dfrac{1}{k!} - \dfrac{1}{(k+1)!}$, and hence produce a proof of part (c) using a collapsing sum.

CHALLENGE

13 Express using factorial notation:

a $30 \times 28 \times 26 \times \cdots \times 2$ **b** $29 \times 27 \times 25 \times \cdots \times 1$ **c** $\dfrac{30 \times 28 \times 26 \times \cdots \times 2}{29 \times 27 \times 25 \times \cdots \times 1}$

14 [Stirling's formula] The following formula is too difficult to prove at this stage, but it is most important because it provides a continuous function that approximates $n!$ for integer values of n:

$$n! \doteqdot \sqrt{2\pi}\, n^{n+\frac{1}{2}}\, e^{-n}, \quad \text{in the sense that the percentage error} \to 0 \text{ as } n \to \infty.$$

Show that the formula has an error of approximately 2.73% for 3! and 0.83% for 10! Find the percentage error for 60!

17B Ordered selections with and without repetition

Learning intentions

- Develop the multiplication principle of counting, using boxes to illustrate.
- Count ordered arrangements allowing repetition.
- Count ordered arrangements without repetition.
- Understand permutations, use the notation ${}^nP_r$, and evaluate ${}^nP_r$ using factorials.
- Develop the counting principle — deal with the difficulties first.

Two questions dominate discussion of counting in the next four sections.

- Are the selections we are counting ordered or unordered?
- If they are ordered, is repetition allowed or not?

These questions generate the three situations described in the chapter introduction.

Sections 17B–17D develop the theory of counting ordered selections, with and without repetition, then Section 17E deals with unordered selections.

A general counting principle — the multiplication principle

This morning, Stefan chose a shirt to wear from the 12 that he owns. Then without worrying about a match, he chose a pair of trousers from the 6 pairs he owns. How many possible outfits could he have walked out of the house wearing?

The answer is $12 \times 6 = 72$. For each choice of trousers, he has 12 choices of shirt, making 12 possible outfits. But there were 6 choices of trousers, so

$$\text{number of outfits} = 12 + 12 + 12 + 12 + 12 + 12 = 12 \times 6\,.$$

The multiplication principle may be applied to more than two successive choices. Stefan also chose a pair of shoes from his 10 pairs, and a tie from his extraordinary collection of 100 wonderful ties. Taking these choices into account as well,

$$\text{number of outfits} = 12 \times 6 \times 10 \times 100 = 72\,000\,.$$

3 A general counting principle — the multiplication principle

- Suppose that a selection is to be made in r stages. Suppose that the first stage can be chosen in n_1 ways, the second in n_2 ways, . . . , the rth in n_r ways. Then

$$\text{number of ways of choosing the complete selection} = n_1 \times n_2 \times \cdots \times n_r$$

- The successive choices can be visualised in a box diagram, as described below,

Using boxes to visualise a multiplication principle question

An ordered selection can usually be regarded as a sequence of choices made one after the other. A box diagram is an efficient setting-out here, and keeps track of these successive choices. In the situation above with Stefan's outfits:

shirt	trousers	shoes	tie
12	6	10	100

Number of outfits $= 12 \times 6 \times 10 \times 100 = 72\,000$.

Example 4 Using the multiplication principle

How many five-letter words can be formed in which the second and fourth letters are vowels and the other three letters are consonants?

Note: Unless otherwise indicated, always take the letter 'y' as a consonant.

Solution

There are 5 vowels and $26 - 5 = 21$ consonants. We can select each letter in order:

1st letter	2nd letter	3rd letter	4th letter	5th letter
21	5	21	5	21

Hence number of words $= 21 \times 5 \times 21 \times 5 \times 21 = 231\,525$.

Ordered selections allowing repetition

This is the first situation listed in the chapter introduction. Suppose that r-letter words are formed from n distinct letters, where any letter can be used any number of times. Then each successive letter in the word can be chosen in n ways:

1st letter	2nd letter	3rd letter	...	rth letter
n	n	n	...	n

giving n^r distinct words altogether. The result is thus a simple power:

4 Ordered selections allowing repetition

If r letters are chosen successively from n distinct letters, allowing repetition of letters, and placed in order, then:

number of arrangements $= n^r$.

Example 5 Counting ordered selections allowing repetition

a How many six-digit numbers can be formed entirely from odd digits?

b How many of these numbers contain at least one seven?

Solution

a There are five odd digits, so the number of such numbers is 5^6.

b We first count the number of these six-digit numbers not containing 7. Such numbers are formed from the digits 1, 3, 5 and 9, so there are 4^6 of them. Subtracting this from the answer to part (a), number of numbers $= 5^6 - 4^6 = 11\,529$.

Ordered selections without repetition

This is the second of the three situations listed in the chapter introduction. Counting ordered selections without repetition typically involves factorials, because as each stage is completed, the number of possible objects diminishes by 1.

Note: The phrases 'with (or allowing) replacement' and 'with (or allowing) repetition' are used almost interchangeably. Similarly, the phrases 'without replacement' and 'without repetition' are also used almost interchangeably.

Example 6 **Counting ordered selections without repetition**

In how many ways can a class of 16 select a 4-person committee consisting of a president, a vice-president, a treasurer, and a secretary?

Select in order the president, the vice-president, the treasurer and the secretary.

president	vice-president	treasurer	secretary
16	15	14	13

Hence there are $16 \times 15 \times 14 \times 13 = \dfrac{16!}{12!}$ possible committees (that is, 43 680).

The language of permutations

A *permutation* is an arrangement of objects chosen from a certain finite set without repetition (that is, without replacement). For example, the words ABC, CED, EAB and DBC are some of the many 3-letter permutations taken from the 5-member set {A, B, C, D, E}.

The symbol ${}^n\text{P}_r$ is used to denote the number of permutations of r letters chosen without repetition from a set of n distinct letters. The previous example is easily generalised to show that there are $\dfrac{n!}{(n-r)!}$ such permutations, so this becomes the formula for ${}^n\text{P}_r$:

1st letter	2nd letter	3rd letter	4th letter	$\cdots$	rth letter
n	$n-1$	$n-2$	$n-3$	$\ldots$	$n-r+1$

Hence
$$\begin{aligned} {}^n\text{P}_r &= n(n-1)(n-2)(n-3)\cdots(n-r+1) \\ &= \frac{n(n-1)(n-2)(n-3)\times\cdots\times 2\times 1}{(n-r)(n-r-1)\times\cdots\times 2\times 1} \\ &= \frac{n!}{(n-r)!}\,. \end{aligned}$$

5 Permutations — ordered selections without repetition

- A *permutation* is an arrangement of objects chosen from a certain finite set without repetition (that is, without replacement).
- We often imagine permutations as *words* formed by choosing, without repetition, letters from a set of distinct letters.
- The number of r-letter permutations chosen from a set of n distinct letters, is
$$ {}^n\text{P}_r = \frac{n!}{(n-r)!}\,. $$
- The number of permutations of all n distinct letters is
$$ {}^n\text{P}_n = \frac{n!}{0!} = n!\,. $$

The term ‘distinct objects’ is often necessary in this situation.

Scientific calculators have a button labelled ${}^n\text{P}_r$ that will find values of ${}^n\text{P}_r$. For low values of n and r, the answers are exact, but for higher values they are only approximations. Make a practice of evaluating these number by hand when it is reasonable to do so, because such calculations greatly help the intuition.

Example 7 **Counting permutations taken from a 5-member set.**

a How many 3-letter words can be formed without repetition from the 5-member set {A, B, C, D, E}?
b How many 5-letter words without repetition can be formed from the set?

Solution

a By Box 5, number of permutations $= \dfrac{5!}{(5-3)!} = \dfrac{5!}{2!} = 5 \times 4 \times 3 = 60$.
b By Box 5, number of permutations $= 5! = 120$.

The permutations of a set

A *permutation of a set* is an arrangement of all the members of the set in some order. Box 5 established that the number of such permutation is ${}^{n}\mathrm{P}_{n} = n!$.

Example 8 **Permutations of all members and of all but one member**

a How many 8-digit numbers, and how many 9-digit numbers, can be formed from the 9 non-zero digits if no repetition is allowed?
b Comment on the two answers in part (a).

Solution

a Number of 8-digit numbers $= {}^{9}\mathrm{P}_{8} = \dfrac{9!}{1!} = 9!$

Number of 9-digit numbers $= {}^{9}\mathrm{P}_{9} = \dfrac{9!}{0!} = 9!$

b The results are the same, because every 8-digit number can be extended in one and only one way to a 9-digit number, that is, by adding the unused digit.

Example 9 **Using the calculator for very large numbers**

In how many ways can 20 people and 40 people form a queue? Answer correct to two significant figures.

Solution

With 20 people, number of ways $= 20! \doteqdot 2.4 \times 10^{18}$.

With 40 people, number of ways $= 40! \doteqdot 8.2 \times 10^{47}$.

A general counting principle — deal with the restrictions first

Many problems have restrictions in the way things can be arranged. These restrictions should be dealt with first. It is also important to keep in mind that the order of the boxes usually represents the order the choices are made in, not the ordering of the objects, and that they can be used in surprisingly flexible ways.

6 A general counting principle — deal with the difficulties first

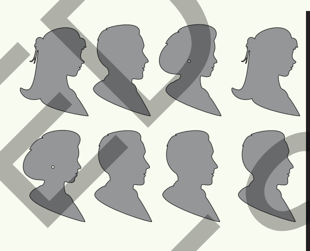
- When counting ordered selections, deal with any restrictions first.
- Usually place the boxes in the order in which the selections are made.

Example 10 Dealing with the difficulties first

Eight people form two queues, each with four people. Albert will only stand in the left-hand queue, Beth only in the right-hand queue, and Charles and Diana insist on standing in the same queue. In how many ways can the two queues be formed?

Solution

Place Albert in any of the 4 possible positions in the left-hand queue,
Then place Beth in any of the 4 positions in the right-hand queue.
Place Charles in any of the remaining 6 positions.
Place Diana in one of the 2 remaining positions in the same queue.
There remain 4 unfilled positions, which can be filled in 4! ways:



Albert	Beth	Charles	Diana	last four positions
4	4	6	2	4!

Hence number of ways $= 4 \times 4 \times 6 \times 2 \times 4!$
$= 4608.$

Exercise 17B

FOUNDATION

1 List all the permutations of the letters of the word DOG. How many are there?

2 List all the permutations of the letters EFGHI, beginning with F, taken three at a time.

3 Find how many arrangements of the letters of the word FRIEND are possible if the letters are taken:
a four at a time, **b** six at a time.

4 Find how many four-digit numbers can be formed using the digits 5, 6, 7, 8 and 9 if:
a no digit is to be repeated, **b** any of the digits can occur more than once.

5 How many three-digit numbers can be formed using the digits 2, 3, 4, 5 and 6 if no digit can be repeated? How many of these are greater than 400?

6 In how many ways can seven people be seated in a row of seven different chairs?

7 The symbol ${}^{n}\mathrm{P}_{r}$ is the number of ways of arranging r objects selected without replacement from n objects. Use this expression, and the ${}^{n}\mathrm{P}_{r}$ button on your calculator, to answer the following questions.
a From a group of 10, three people line up to buy tickets. How many ways can this happen?
b Five cards are each labelled uniquely with one of the digits 1, 2, 3, 4, 5. Three of the five cards are placed down in a row. How many ways can the cards be arranged?
c One hundred people each buy one ticket in a lottery. How many ways can the first three places be awarded?

8 The expression n^r is the number of ways of arranging r objects selected with replacement from n objects. Use this expression to answer the following questions.

a How many three-digit numbers can be written down from the digits 1 to 9? The digits need not be distinct.

b One hundred people each buy one ticket in a lottery. Tickets are selected one by one at random from a barrel and then replaced between selections. How many ways can the first three places be awarded?

c A computer sends a string of ten binary digits, that is, each symbol can only be 0 or 1. How many such ten-digit strings are possible?

9 Eight runners are participating in a 400-metre race.

a In how many ways can they finish?

b In how many ways can the gold, silver and bronze medals be awarded?

10 **a** If you toss a coin and roll a die, how many outcomes are possible?

b If you toss two coins and roll three dice, how many outcomes are possible?

11 A woman has four hats, three blouses, five skirts, two handbags and six pairs of shoes. In how many ways can she be attired, assuming that she wears one of each item?

12 Jack has six different football cards and Meg has another eight different football cards. In how many ways can one of Jack's cards be exchanged for one of Meg's cards?

DEVELOPMENT

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13 In Sydney, landline phone numbers, ignoring the area code, consist of eight digits. Let us consider those starting with the digit 9.

a How many such phone numbers are possible?

b How many of these end in an odd number?

c How many consist of odd digits only?

d How many are there that do not contain a zero, and in which the consecutive digits alternate between odd and even?

14 Users of automatic teller machines are required to enter a four-digit PIN (personal identification number). Find how many PINs:

a are possible,

b consist of four distinct digits,

c consist of odd digits only,

d start and end with the same digit.

15 **a** If repetitions are not allowed, how many four-digit numbers can be formed from the digits 1, 2, . . . 8, 9?

b How many of these end in 1?

c How many of these are even?

d How many are divisible by 5?

e How many are greater than 7000?

16 Repeat the previous question if repetitions are allowed.

17 In Tasmania, a car licence plate consists of two letters followed by four digits. Find how many of these are possible:

a if there are no restrictions,

b if there is no repetition of letters or digits,

c if the second letter is X and the third digit is 3,

d if the letters are D and Q and the digits are 3, 6, 7 and 9.

18 **a** In how many ways can the letters of the word NUMBER be arranged?
b How many begin with N?
c How many begin with N and end with U?
d In how many is the N somewhere to the left of the U?

19 In how many ways can a boat crew of eight women be arranged if three of the women can only row on the bow side and two others can only row on the stroke side?

20 A motor bike can carry three people: the driver, one passenger behind the driver and one in the sidecar. If among five people, only two can drive, in how many ways can the bike be filled?

21 **a** How many five-digit numbers can be formed without repetition from the digits 2, 3, 4, 5 and 6?
b How many of these numbers are greater than 56 432?
c How many of these numbers are less than 56 432?

22 **a** Integers are formed from the digits 2, 3, 4 and 5, with repetitions not allowed.
i How many such numbers are there? **ii** How many of them are even?
b Repeat the two parts to this question if repetitions are allowed.

23 **a** How many five-digit numbers can be formed from the digits 0, 1, 2, 3 and 4 if repetitions are not allowed?
b How many of these are odd?
c How many are divisible by 5?

24 **a** If ${}^{8}P_r = 336$, find the value of r.
b If $7 \times {}^{2n}P_n = 4 \times {}^{2n+1}P_n$, find n.
c Using the result ${}^{n}P_r = \dfrac{n!}{(n-r)!}$, prove that:
i ${}^{n+1}P_r = {}^{n}P_r + r \times {}^{n}P_{r-1}$
ii ${}^{n}P_r = {}^{n-2}P_r + 2r \times {}^{n-2}P_{r-1} + r(r-1) \times {}^{n-2}P_{r-2}$

CHALLENGE

25 Recall that a whole number is divisible by 3 if and only if the sum of the digits is divisible by 3.
a How many 5-digit whole numbers are divisible by 3?
b How many 5-digit whole numbers with no 0s are divisible by 3?
c How many 5-digit whole numbers with no 0s are divisible by 2?
d How many 5-digit whole numbers with no 0s are divisible by 6?

17C Ordered selections — three more principles

Learning intentions

- Develop the grouping principle — order the groups, then order each group.
- Develop the complementary principle — deal with the complementary situation.
- Develop the cases principle — split the problem up into cases.

Section 17B developed the multiplication principle, and the principle of dealing with the difficulties first. This section develops three further principles.

A general counting principle — grouping

In some ordering problems, particular members must be grouped together. This produces a *compound ordering*, in which the various groups must first be ordered, and then the individuals ordered within each group.

7 A general counting principle — use grouping in counting

- First order the groups, then order the individuals within each group.
- A group may consist of a single individual.

Example 11 Using grouping to count arrangements

Four boys and four girls form a queue at the bus stop. One couple want to stand together. The other three girls want to stand together, but the other three boys don't care where they stand. How many acceptable ways are there of forming the queue?

Solution

There are five groups — the couple, the group of three girls, and the three groups each consisting of one individual boy. First order the five groups, then order the couple, then order the three girls.

order the 5 groups	order the couple	order the 3 girls
5!	2!	3!

Hence number of ways $= 5! \times 2! \times 3! = 1440$.

A general counting principle — deal with the complementary situation

Many probability questions have already been solved in Chapter 14 using complementary events. The same principle applies to counting — count the unacceptable orderings, then subtract them from the total number of orderings. The word 'not' may or may not be there to prompt you.

Example 12 Using the complementary situation in counting

How many 7-letter words can be formed from the letters A, B, C, D, E, F, G if the two vowels must be separated by at least one consonant?

Solution

Number of orderings without restriction = 7!

Number of orderings with A and E together = 2! × 6!

(Order the A and E and glue them together, then order the 6 groups.)

Hence number of orderings with A and E apart = 7! − 2 × 6!

= 3600 .

8 A general counting principle — deal with the complementary situation

Count the unacceptable orderings, then subtract to count the acceptable orderings.

A general counting principle — use cases

Sometimes the fiddly conditions of a problem mean that one set of boxes is not enough to solve it, and separate cases need to be considered.

In this situation, the cases should not overlap, or if they do, the number in the overlap needs to be subtracted.

9 A general counting principle — use cases

- The cases should not overlap.
- If the cases do overlap, the overlap needs to be subtracted.

This same principle was expressed in Section 14A on set theory, by the formula in Box 4 of that chapter:

$$|A \cup B| = |A| + |B| - |A \cap B| .$$

Example 13 Using cases in a difficult counting problem

a How many whole numbers less than 1000 are odd and greater than 500, or multiples of 5 and less than 200?
b How many whole numbers less than 1000 are odd and greater than 200, or multiples of 5 and less than 500?

Solution

a There are 250 odd numbers between 500 and 1000.
There are 40 multiples of 5 less than 200 (including zero).
The two cases do not overlap, so

Total = 250 + 40 = 290.

b There are 400 odd numbers between 200 and 1000.
There are 100 multiples of 5 less than 500 (including zero).
But the two cases overlap this time — there are 30 odd numbers from 200 to 500 that are multiples of 5.
Hence

Total = 400 + 100 − 30 = 470.

Exercise 17C

FOUNDATION

1 How many rearrangements are there of the letters of each word, if the vowels must be together?

a BOARDS **b** RIO **c** QUIT **d** TROUNCE

2 How many arrangements of the word MATHS are possible, if:

a the T and H must remain together?

b the TH must remain together and in this order?

3 How many ways can Andrew, Becky, Courtney, Dion and Ellie sit in a row, if Andrew and Becky sit together and Dion and Ellie sit together?

4 In how many ways can three different mathematics books, six different science books and four different English books be placed on a shelf, if the books relating to each subject are to be kept together?

5 A class is asked to determine how many ways the letters of the word SOLAR can be be arranged, if the first two positions cannot both be vowels.

a Jack decides to use cases (the word either starts with a vowel or it does not). Use Jack's method to answer the question.

b Jill decides to consider the complementary situation (both first positions are vowels). Use Jill's method to answer the question.

6 A family of two parents and two children are going on a car trip. Only the parents can drive. If the father drives, then the mother will sit in the back seat where she feels safer. The father always sits in a front seat of the car. How many arrangements are possible, if two sit in the front and two in the back?

7 How many numbers can be written down using each digit of 789 at most once, if the result must be at least 80?

DEVELOPMENT

8 Find how many arrangements of the letters of the word UNIFORM are possible:

a if the vowels must occupy the first, middle and last positions,

b if the word must start with U and end with M,

c if all the consonants must be in a group at the end of the word,

d if the M is somewhere to the right of the U.

9 Find how many arrangements of the letters of the word BEHAVING:

a end in NG, **b** begin with three vowels, **c** have three vowels occurring together.

10 A Maths test is to consist of six questions. In how many ways can it be arranged so that:

a the shortest question is first and the longest question is last,

b the shortest and longest questions are next to one another?

11 In Morse code, letters are formed by a sequence of dashes and dots. How many different letters is it possible to represent if a maximum of ten symbols are used?

12 Four boys and four girls are to sit in a row. Find how many ways this can be done if:

a the boys and girls alternate, **b** the boys and girls sit in distinct groups.

13 Five-letter words are formed without repetition from the letters of PHYSICAL.

a How many consist only of consonants?
b How many begin with P and end with S?
c How many begin with a vowel?
d How many contain the letter Y?
e How many have the two vowels occurring next to one another?
f How many have the letter A immediately following the letter L?

14 **a** How many seven-letter words can be formed without repetition from the letters of the word INCLUDE?
b How many of these do not begin with I?
c How many end in L?
d How many have the vowels and consonants alternating?
e How many have the C immediately following the D?
f How many have the letters N and D separated by exactly two letters?
g How many have the letters N and D separated by more than two letters?

15 Repeat parts (a)–(d) of the previous question if repetition is allowed.

16 **a** In how many ways can ten people be arranged in a line:
i without restriction,
ii if one particular person must sit at either end,
iii if two particular people must sit next to one another,
iv if neither of two particular people can sit on either end of the row?
b In how many ways can n people be placed in a row of n chairs:
i if one particular person must be on either end of the row,
ii if two particular people must sit next to one another,
iii if two of them are not permitted to sit at either end?

17 Five boys and four girls form a queue at the cinema. There are two brothers who want to stand together, the remaining three boys wish to stand together, and the four girls don't mind where they stand. In how many ways can the queue be formed?

18 Eight people are to form two queues of four. In how many ways can this be done if:
a there are no restrictions,
b Jim will only stand in the left-hand queue,
c Sean and Liam must stand in the same queue?

19 There are eight swimmers in a race. In how many ways can they finish if there are no dead heats and the swimmer in Lane 2 finishes:
a immediately after the swimmer in Lane 5,
b any time after the swimmer in Lane 5?

20 Five backpackers arrive in a city where there are five youth hostels.

a How many different accommodation arrangements are there if there are no restrictions on where the backpackers stay?

b How many different accommodation arrangements are there if each backpacker stays at a different youth hostel?

c Suppose that two of the backpackers are brother and sister and wish to stay in the same youth hostel. How many different accommodation arrangements are there if the other three can go to any of the other youth hostels?

21 Numbers less than 4000 are formed from the digits 1, 3, 5, 8 and 9, without repetition.

a How many such numbers are there?

b How many of them are odd?

c How many of them are divisible by 5?

d How many of them are divisible by 3?

CHALLENGE

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22 [Derangements] A *derangement* of n distinct letters is a permutation of them so that no letter appears in its original position. For example, DABC is a derangement of ABCD, but DACB is not. Denote the number of derangements of n letters by $D(n)$.

a By listing all the derangements of A, AB, ABC and ABCD, find the values of $D(1)$, $D(2)$, $D(3)$ and $D(4)$.

b Suppose that we have formed a derangement of the five letters ABCDE. Let the last letter in the derangement be X, and exchange X with E — this puts E back to its original position. Either X is now also in its original position so that three letters are away from their original positions, or X is not in its original position so that four letters are away from their original positions. Hence explain why

$$D(5) = 4 \times D(4) + 4 \times D(3)\,.$$

c Use this formula to evaluate $D(5)$. Then apply the corresponding arguments and formulae to evaluate $D(6)$, $D(7)$ and $D(8)$.

17D Ordered selections with identical elements

Learning intentions

- Count rearrangements of a word whose letters are not all distinct.
- Use cases to count more complicated situations.
- Develop the special cases of words with only two distinct letters.

So far, our permutations have been drawing letters from a word whose letters are all distinct. This section deals with rearrangements of *all the letters of a word whose letters are not all distinct.*

Counting with identical elements

Finding the number of different words formed using all the letters of the word 'PRESSES' is complicated by the fact that there are three Ss and two Es. If the seven letters were all different, we would conclude that

$$\text{number of ways} = 7!$$

But we have *overcounted by a factor of* $2! = 2$, because the Es can be interchanged without changing the word. We have also *overcounted by a factor of* $3! = 6$, because the three Ss can be permuted amongst themselves in $3!$ ways without changing the word. Taking account of both overcountings:

$$\text{number of ways} = \frac{7!}{2! \times 3!} = 420.$$

This method is easily generalised. Continuing in the language of 'words':

10 Counting with identical elements

Suppose that a word of n letters has r_1 alike of one type, r_2 alike of another type, ..., r_k alike of a final type. Then the number of distinct words that can be formed by rearranging the letters is

$$\text{number of words} = \frac{n!}{r_1! \times r_2! \times \cdots \times r_k!}.$$

If $r_i = 1$, that is, if there is only one letter of the ith type, then $r_i! = 1$, so it doesn't matter whether we include it in the formula or not.

Example 14 **Rearranging objects that are not all distinct**

Three identical wine glasses and five identical tumblers are to be arranged in a row across the front of a cupboard.

a In how many ways can this be done (counting only the patterns)?

b How does this change if one of the wine glasses becomes clearly chipped, and two tumblers break and are replaced by two identical tumblers different from the other three?

Solution

a There are 8 glasses, 3 alike of one type and 2 alike of another,

so number of ways $= \dfrac{8!}{3! \times 5!} = 56.$

b There are now 2 alike of one type, 1 of another, 3 of another, 2 of another,

so number of ways $= \dfrac{8!}{2! \times 1! \times 3! \times 2!} = 1680$ (the $1!$ can be omitted).

Using cases for counting

As mentioned in the previous section, many counting problems are too complicated to be analysed completely by a single box diagram or a single method. In such situations, the use of cases is unavoidable. Attention should be given, however, to minimising the number of different cases that need to be considered.

Example 15 A difficult counting problem requiring cases

How many 6-letter words can be formed by using the 7 letters of the word 'PRESSES'?

Solution

We omit in turn each of the four letters P, R, E and S.
This leaves six letters, which we must then arrange in order.

a If an S is omitted, there are then 2 Es and 2 Ss,
so number of words $= \dfrac{6!}{2! \times 2!} = 180.$

b If an E is omitted, there are then 3 Ss,
so number of words $= \dfrac{6!}{3!} = 120.$

c If P or R is omitted (2 cases), there are then 2 Es and 3 Ss,
so number of words $= \dfrac{6!}{3! \times 2!} \times 2$ (doubling for the two cases)
$= 120.$

Hence total number of words $= 180 + 120 + 120 = 420.$

Counting words consisting of just two different letters

This situation involves two counting formulae that will be vital in the next section.

These two formulae are special cases of methods that have already been developed in Sections 17B and this section, and are best introduced with an example.

Example 16 Counting words consisting of just two different letters

An opinion poll asks 8 independent questions, each to be answered Y or N.

a How many possible answer sheets are there?

b How many possible answer sheets have 5 Ys and 3 Ns?

Solution

Answering the poll generates a word such as YNNYYNYY made up of Ys and Ns.

a By Box 4 in Section 17B, the number of words is 2^8. Or using boxes:

1st	2nd	3rd	4th	5th	6th	7th	8th
2	2	2	2	2	2	2	2

b By Box 10 above, the number of words is $\dfrac{8!}{5! \times 3!}$.

Both formulae for the general case now follow easily from this example.

11 Rearrangements of words containing only two different letters

- The number of n-letter words consisting only of Ys and Ns is:

$$\text{number of words} = 2^n \,.$$

- The number of n-letter words consisting of r Ys and $n - r$ Ns is:

$$\text{number of words} = \frac{n!}{r! \times (n-r)!} \,.$$

After the next section, we will be able to use the concise symbol ${}^nC_r$ for this expression $\dfrac{n!}{r! \times (n-r)!}$.

Example 17 Arrangements of objects of just two types

a In a queue of ten adults, how many patterns of men and women are possible?

b Six women and four men form a queue at the bus stop. How many patterns of men and women are possible?

Solution

a By Box 11 Dotpoint 1, the number of possible patterns is $2^{10} = 1024$.

b By Box 11 Dotpoint 2, the number of possible patterns is $\dfrac{10!}{6! \times 4!} = 210$.

Exercise 17D

FOUNDATION

1 Find the number of permutations of the following words if all the letters are used.

a BOB **b** ALAN **c** SNEEZE
d TASMANIA **e** BEGINNER **f** FOOTBALLS
g EQUILATERAL **h** COMMITTEE **i** WOOLLOOMOOLOO

2 The six digits 1, 1, 1, 2, 2, 3 are used to form a six-digit number. How many numbers can be formed?

3 Six coins are lined up on a table. Find how many patterns are possible if there are:

a five tails and one head, **b** four heads and two tails, **c** three tails and three heads.

4 Eight balls, identical except for colour, are arranged in a line. Find how many different arrangements are possible if:

a all balls are of a different colour,

b there are seven red balls and one white ball,

c there are six red balls, one white ball and one black ball,

d there are three red balls, three white balls and two black balls.

5 Five identical green chairs and three identical red chairs are arranged in a row. Find how many arrangements are possible:

a if there are no restrictions, **b** if there must be a green chair on either end.

DEVELOPMENT

6 A motorist travels through eight sets of traffic lights, each of which is red or green. He is forced to stop at three sets of lights.

a In how many ways could this happen?

b What other number of red lights would give an identical answer to part (a)?

7 In how many ways can the letters of the word SOCKS be arranged in a line:

a without restriction,

b so that the two Ss are together,

c so that the two Ss are separated by at least one other letter,

d so that the K is somewhere to the left of the C?

8 a Find the number of arrangements of the letters in SLOOPS if:

i there are no restrictions,

ii the two Os are together,

iii the two Os are to be separated,

iv the Os are together and the Ss are together.

b In how many arrangements of the letters in TATTOO are the two Os separated?

9 In how many ways can the letters of the word DECISIONS be arranged:

a without restriction,

b so that the vowels and consonants alternate,

c so that the vowels come first followed by the consonants,

d so that the N is somewhere to the right of the D?

10 In how many ways can the letters of the word PROPORTIONALITY be arranged so that the vowels and consonants still occupy the same places?

11 A form has ten questions in order, each of which requires the answer ‘Yes’ or ‘No’. Find the number of ways the form can be filled in:

a without restriction,

b if the first and last answers are ‘Yes’,

c if two are ‘Yes’ and eight are ‘No’,

d if five are ‘Yes’ and five are ‘No’,

e if more than seven answers are ‘Yes’,

f if an odd number of answers are ‘Yes’,

g if exactly three answers are ‘Yes’, and they occur together,

h if the first and last answers are ‘Yes’ and exactly four more are ‘Yes’.

12 Containers are identified by a row of coloured dots on their lids. The colours used are yellow, green and purple. In any arrangement, there are to be no more than three yellow dots, no more than two green dots and no more than one purple dot.

a If six dots are used, what is the number of possible codes?

b What is the number of different codes possible if only five dots are used?

13 a How many five-letter words can be formed by using the letters of the word STRESS?

b How many five-letter words can be formed by using the letters of the word BANANA?

14 Find how many arrangements of the letters of the word TRANSITION are possible if:

a there are no restrictions,

b the Is are together,

c the Is are together, and so are the Ns, and so are the Ts,

d the Ns occupy the end positions,

e an N occupies the first but not the last position,

f the letter N is not at either end,

g the vowels are together.

15 Ten coloured marbles are placed in a row.

a If they are all of different colours, how many arrangements are possible?

b What is the minimum number of colours needed to guarantee at least 10 000 different patterns? (This will need a guess-and-check approach.)

CHALLENGE

16 Find how many arrangements there are of the letters of the word GUMTREE if:

a there are no restrictions,

b the Es are together,

c the Es are separated by:

i one, **ii** two, **iii** three, **iv** four, **v** five letters,

d the G is somewhere between the two Es,

e the M is somewhere to the left of both Es and the U is somewhere between them,

f the G is somewhere to the left of the U and the M is somewhere to the right of the U.

17 If the letters of the word GUMTREE and the letters of the word KOALA are combined and arranged into a single twelve-letter word, in how many of these arrangements do the letters of KOALA appear in their correct order, but not necessarily together?

18 Bob is about to hang his eight shirts in the wardrobe. He has four different styles of shirt, with two identical shirts in each style. How many different arrangements are possible if no two identical shirts are next to one another?

19 [Derangements and the number e] A *derangement* was defined in question 22 of Exercise 14C as a permutation that leaves no letter unmoved, and $D(n)$ was defined as the number of derangements of n letters. Another approach to finding a formula for $D(n)$ uses the *inclusion–exclusion principle* introduced in the Enrichment section of Exercise 12C. Consider, for example, the derangements of ABCD.

a How many permutations are there of ABCD?

b How many of these leave in its original position:

i A, **ii** B, **iii** C, **iv** D?

Each of these numbers will need to be subtracted from the answer to part (a).

c In doing part (b), however, we have subtracted some of the permutations twice. For example, some of them would leave both A and B unmoved. Thus we need to add back the number of permutations that leave any two particular letters unmoved. How many of these are there?

d Now you will need to subtract the number of permutations that leave three letters unmoved, and so on. Hence find an expression for $D(4)$.

e Rearrange your expression into the form $D(4) = 4!\left(\frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\right)$.

f Explain how this can be generalised to $D(n) = n!\left(\frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!}\right)$.

g We shall prove in an Enrichment question of the Year 12 volume that the sequence in the brackets of part (f) above converges to $1/e$ as $n \to \infty$. Give both a combinatorial and a probabilistic interpretation of this result in terms of the ratio of the numbers of permutations and derangements of n letters.

17E Counting unordered selections

Learning intentions

- Count all subsets, and all r-member subsets, of an n-member set.
- Use the notation ${}^n\mathrm{C}_r$, and prove that ${}^n\mathrm{C}_0 + {}^n\mathrm{C}_1 + {}^n\mathrm{C}_2 + \cdots + {}^n\mathrm{C}_n = 2^n$.
- Develop the formula ${}^n\mathrm{C}_{n-r} = {}^n\mathrm{C}_r$, and explain it using complements.

This section turns attention away from the ordered selections of the previous three sections towards the counting of unordered selections. An unordered selection of distinct objects chosen from a certain set is just *a subset of the set.* Thus given an n-member set S, there are two basic questions to ask:

- How many subsets does S have altogether?
- How many r-member subsets does S have, for $r = 0, 1, \ldots, n$?

An example of the result

Here is an example to illustrate the situation. Let S be the five-member set

$$S = \{ \mathrm{A, B, C, D, E} \}.$$

Here is the list of all the subsets of S, arranged by the number of members:

1	0-member subsets:	$\emptyset$ (the empty set)
5	1-member subsets:	{ A }, { B }, { C }, { D }, { E }
10	2-member subsets:	{ A, B }, { A, C }, { A, D }, { A, E }, { B, C }, { B, D }, { B, E }, { C, D }, { C, E }, { D, E }
10	3-member subsets:	{ A, B, C }, { A, B, D }, { A, B, E }, { A, C, D }, { A, C, E }, { A, D, E }, { B, C, D }, { B, C, E }, { B, D, E }, { C, D, E }
5	4-member subsets:	{ A, B, C, D }, { A, B, C, E }, { A, B, D, E }, { A, C, D, E }, { B, C, D, E }
1	5-member subset:	{ A, B, C, D, E }

making $1 + 5 + 10 + 10 + 5 + 1 = 32 = 2^5$ subsets altogether.

How many subsets does S have altogether?

Choosing a subset of S requires looking at each member of S in turn and deciding whether to include it in the subset or not. Let us write the code Y for 'Yes' if it is included, and N for 'No' if it is not included. Using this code, each subset of S corresponds to a five-letter word made up of Ys and Ns. For example,

$$\mathrm{YYNYN} \longleftrightarrow \{ \mathrm{A, B, D} \} \quad \text{and} \quad \mathrm{NNNNY} \longleftrightarrow \{ \mathrm{E} \}.$$

Hence the total number of subsets of S is the number of five-letter words made up of Ys and Ns. By Box 11 of Section 17D, or using the box to the right to visualise the choices, this is 2^5, so

A	B	C	D	E
2	2	2	2	2

$$\text{Number of subsets of } S = 2^5 = 32, \text{ as demonstrated above.}$$

This same argument applies to sets of any finite size, so the general result is:

$$\text{Number of subsets of an } n\text{-member set} = 2^n.$$

How many three-member subsets does S have?

Using the same code of Ys and Ns, the number of 3-member subsets of S is the number of 5-letter words with 3 Ys and 2Ns. Thus by Box 11, second dotpoint:

$$\text{Number of 3-member subsets of } S = \frac{5!}{3! \times 2!} = 10, \text{ as demonstrated above.}$$

Again, this argument applies to sets of any finite size, so the general result is:

$$\text{Number of r-member subsets of an } n\text{-member set} = \frac{n!}{r! \times (n-r)!}.$$

The notation ${}^n\mathrm{C}_r$

The notation ${}^n\mathrm{C}_r$ is a convenient shorthand, and means the number of r-member subsets of an n-member set. Thus we have the formula ${}^n\mathrm{C}_r = \dfrac{n!}{r! \times (n-r)!}$.

12 Counting the subsets of a set

Let S be an n-member set, and let $0 \le r \le n$.

- The symbol ${}^n\mathrm{C}_r$ denotes the number of r-member subsets of an n-member set.
- Total number of all the subsets of $S = 2^n$.
- Number of r-member subsets of $S = {}^n\mathrm{C}_r = \dfrac{n!}{r!\,(n-r)!}$.
- Adding up the subsets with $0, 1, 2, \dots n$ members gives the formula:

  $${}^n\mathrm{C}_0 + {}^n\mathrm{C}_1 + {}^n\mathrm{C}_2 + \cdots + {}^n\mathrm{C}_n = 2^n,$$

  because LHS and RHS are both counting the total number of subsets of S.

The symbol ${}^n\mathrm{C}_r$ is spoken as '*n choose r*', and has an alternative notation $\binom{n}{r}$.

The notations ${}^n\mathrm{P}_r$ and ${}^n\mathrm{C}_r$ are intended to be similar. The letter C stands for 'combination' — an old term for 'unordered selection' — just as the letter P stands for 'permutation'. By a convenient, but false, etymology, C also stands for 'Choose', hence the more recent convention of saying 'n choose r' for ${}^n\mathrm{C}_r$.

Scientific calculators have a button labelled ${}^n\mathrm{C}_r$ that will find values of ${}^n\mathrm{C}_r$. The answers are exact for low values of n and r, but as with ${}^n\mathrm{P}_r$, they are only approximations for higher values. Again, make a practice of evaluating these numbers by hand when reasonable — such calculations greatly help the intuition.

A proof moving from ordered selections to unordered selections

There is another standard way to prove the formula for ${}^n\mathrm{C}_r$. In Section 17B we saw that the number of three-letter words formed, without repetition, from the set $S = \{A, B, C, D, E\}$ of five letters is ${}^5\mathrm{P}_3 = 5 \times 4 \times 3 = 60$. When we turn to unordered selections, however, there are six distinct words that all correspond, for example, to the three-member subset { B, C, E }:

$$\text{BCE, BEC, CBE, CEB, EBC, ECB} \longleftrightarrow \{\,B, C, E\,\}$$

The reason for this is that there are ${}^3\mathrm{P}_3 = 3 \times 2 \times 1 = 6$ ways of ordering the subset { B, C, E }. Thus the correspondence between three-letter words and three-member subsets is many-to-one, with a six-fold overcounting. Hence the number of three-member subsets is $60 \div 6 = 10$, as illustrated above.

In general, ${}^n\mathrm{P}_r = \dfrac{n!}{(n-r)!}$ words of r letters can be formed without repetition from the members of an n-member set S. But every r-member subset can be ordered in ${}^r\mathrm{P}_r = r!$ ways, so the correspondence between the ordered selections and the unordered selections is many-to-one, with overcounting by a factor of $r!$.

Hence the number of (unordered) r-member subsets of the set S is

$$^{n}\mathrm{P}_r \div {}^{r}\mathrm{P}_r = \frac{n!}{(n-r)!} \div r! = \frac{n!}{r! \times (n-r)!}.$$

Calculations of $^{n}\mathrm{C}_r$

Here are some examples of using the formula to calculate $^{n}\mathrm{C}_r$ for some values of n and r.

Example 18 Calculations involving the formula for $^{n}\mathrm{C}_r$

a Use the formula for $^{n}\mathrm{C}_r$ to evaluate:

i $^{8}\mathrm{C}_5$ **ii** $^{n}\mathrm{C}_3$

b Find $^{16}\mathrm{C}_5$, leaving your answer factored into primes.

c Evaluate the terms on the LHS to prove $^{5}\mathrm{C}_0 + {}^{5}\mathrm{C}_1 + {}^{5}\mathrm{C}_2 + {}^{5}\mathrm{C}_3 + {}^{5}\mathrm{C}_4 + {}^{5}\mathrm{C}_5 = 2^5$.

Solution

a **i** $^{8}\mathrm{C}_5 = \dfrac{8!}{3! \times 5!} = \dfrac{8 \times 7 \times 6 \times 5!}{3 \times 2 \times 1 \times 5!} = 56$

ii $^{n}\mathrm{C}_3 = \dfrac{n!}{3! \times (n-3)!} = \dfrac{n(n-1)(n-2)(n-3)!}{3! \times (n-3)!} = \dfrac{n(n-1)(n-2)}{6}$

b $^{16}\mathrm{C}_5 = \dfrac{16 \times 15 \times 14 \times 13 \times 12}{1 \times 2 \times 3 \times 4 \times 5}$

$= 2 \times 14 \times 13 \times 12$

$= 2^4 \times 3 \times 7 \times 13$ (Check this on the calculator.)

c LHS $= \dfrac{5!}{0! \times 5!} + \dfrac{5!}{1! \times 4!} + \dfrac{5!}{2! \times 3!} + \dfrac{5!}{3! \times 2!} + \dfrac{5!}{4! \times 1!} + \dfrac{5!}{5! \times 0!}$

$= 1 + 5 + 10 + 10 + 5 + 1$

$= 32 =$ RHS, as shown on the first page of this section.

A natural (or canonical) correspondence — $^{n}\mathrm{C}_r = {}^{n}\mathrm{C}_{n-r}$

Suppose that two people are to be chosen from five to make afternoon tea. This task can be done in two ways:

- Choose the two people out of five who will *make the tea*.
- Choose the three people out of five who will *not make the tea*.

Thus for every choice of two people out of five, the remaining three people is a corresponding choice of three people out of five. This confirms that $^{5}\mathrm{C}_2 = {}^{5}\mathrm{C}_3$.

But it also gives a one-to-one correspondence between the two-member subsets and the three-member subsets of a five-member set:

$\{A, B\} \longleftrightarrow \{C, D, E\}$ $\{B, D\} \longleftrightarrow \{A, C, E\}$

$\{A, C\} \longleftrightarrow \{B, D, E\}$ $\{B, E\} \longleftrightarrow \{A, C, D\}$

$\{A, D\} \longleftrightarrow \{B, C, E\}$ $\{C, D\} \longleftrightarrow \{A, B, E\}$

$\{A, E\} \longleftrightarrow \{B, C, D\}$ $\{C, E\} \longleftrightarrow \{A, B, D\}$

$\{B, C\} \longleftrightarrow \{A, D, E\}$ $\{D, E\} \longleftrightarrow \{A, B, C\}$

In this *natural* or *canonical correspondence* between the 2-member subsets and the 3-member subsets, every subset T is paired with its complement $\overline{T}$:

$$T \longleftrightarrow \overline{T}$$

13 A canonical correspondence

Let n and r be whole numbers with $0 \le r \le n$, and let S be an n-member set.

- ${}^nC_r = {}^nC_{n-r}$
- The r-member subsets of S are *naturally* or *canonically* paired up with the $(n - r)$-member subsets of S by pairing each subset with its complement.

The correspondence is by no means restricted to mathematics. In normal language, a situation can often be described just as well by saying what it is not as by saying what it is — we are all familiar with 'days when no rain fell' or 'unforgivable actions' or 'invisible enemies'.

Example 19 **Illustrating the formulae for ${}^nC_r$ involving complements**

Write ${}^7C_2$ and ${}^7C_5$ in factorial notation, showing that they are equal.

Solution

$${}^7C_2 = \frac{7!}{2! \times 5!} \qquad {}^7C_5 = \frac{7!}{5! \times 2!}$$

Using ${}^nC_r$ in counting problems

Here are a number of examples of counting problems. As always, words such as 'at least', 'at most', 'not' and 'excluding' should always be regarded as warnings that the problem may best be solved by considering the complementary situation.

Example 20 **Using ${}^nC_r$ in counting problems**

Ten people meet to play doubles tennis.

a In how many ways can four people be selected from this group to play the first game? (Ignore the subsequent organisation into pairs.)

b How many of these ways will include Maria and exclude Alex?

c If there are four women and six men, in how many ways can two men and two women be chosen for this game?

d Again with four women and six men, in how many ways will women be in the majority?

Solution

a Number of ways $= {}^{10}C_4 = 210$.

b Because Maria is included, three further people must be chosen, and because Alex is excluded, there are now eight people to choose these three from.
Hence number of ways $= {}^8C_3 = 56$.

c Number of ways of choosing the two women $= {}^4C_2 = 6$,
number of ways of choosing the two men $= {}^6C_2 = 15$,
so number of ways of choosing all four $= 15 \times 6 = 90$.

d Number of ways with one man and three women $= {}^6C_1 \times {}^4C_3 = 6 \times 4 = 24$,
number of ways with four women $= 1$,
so number of ways with a majority of women $= 24 + 1 = 25$.

Example 21 Counting using ${}^nC_r$ with complications

Let $S = \{2, 4, 6, 8, 10, 12\}$ be the set of the first six positive even numbers.

a How many subsets of S contain at least two numbers?

b How many subsets with at least two numbers do not contain 8?

c How many subsets with at least two numbers do not contain 8 but do contain 10?

Solution

a Number of 1-member and 0-member subsets $= {}^6C_1 + {}^6C_0 = 6 + 1 = 7$,

so number with at least 2 members $= 2^6 - 7 = 57$.

b We consider now the 5-member set $T = \{2, 4, 6, 10, 12\}$.

Number of 1-member and 0-member subsets $= {}^5C_1 + {}^5C_0 = 6$,

so number with at least 2 members $= 2^5 - 6 = 26$.

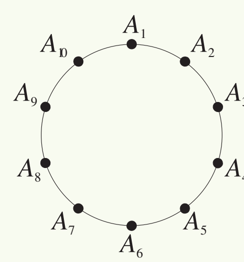
c Because 10 has already been chosen, we need to choose a subset, with at least one member, from the four-member set $U = \{2, 4, 6, 12\}$.

Number of 0-member subsets $= 1$ (the empty set),

so number of such subsets with at least one member $= 2^4 - 1 = 15$.

Example 22 A harder geometric question involving ${}^nC_r$

Ten points $A_1, A_2, \dots, A_{10}$ are arranged equally spaced around a circle.



a How many triangles can be drawn with these points as vertices?

b How many pairs of such triangles can be drawn, if the vertices of the two triangles are distinct?

c In how many such pairs will the triangles:

i not overlap, **ii** overlap?

Solution

a To form a triangle, we must choose 3 points out of 10,

so number of triangles $= {}^{10}C_3 = 120$.

b To form two triangles, first choose 6 points out of 10, which can be done in ${}^{10}C_6 = 210$ ways.

Take any one of those 6 points, then choose the other 2 points in its triangle, which can be done in ${}^5C_2 = 10$ ways.

The second triangle is then drawn using the other three points.

Hence number of pairs of triangles $= 210 \times 10 = 2100$.

c To form two non-overlapping triangles, we first choose 6 points out of 10, which again can be done in ${}^{10}C_6 = 210$ ways.

These 6 points can be made into two non-overlapping triangles in 3 ways, by arranging the 6 points in cyclic order, and choosing 3 adjacent points.

i Hence number of non-overlapping pairs $= 210 \times 3 = 630$.

ii By subtraction, number of overlapping pairs $= 2100 - 630 = 1470$.

Exercise 17E

FOUNDATION

1 Two people are chosen from a group of five people called P, Q, R, S and T. List all possible combinations, and find how many there are.

2 Find in how many ways you can form a group of:

a two people from a choice of seven,
b three people from a choice of seven,
c two people from a choice of six,
d five people from a choice of nine.

3 a Find how many possible combinations there are if, from a group of ten people:

i two people are chosen,
ii eight people are chosen.

b Why are the answers identical?

4 From a party of twelve men and eight women, find how many groups can be chosen consisting of:

a five men and three women,
b four women and four men.

5 Four numbers are to be selected from the set of the first eight positive integers. Find how many possible combinations there are if:

a there are no restrictions,
b there are two odd numbers and two even numbers,
c there is exactly one odd number,
d all the numbers must be even,
e there is at least one odd number.

6 Four balls are simultaneously drawn from a bag containing three identical green balls and six identical blue balls. Find how many ways there are of drawing the four balls if:

a the balls may be of any colour,
b there are exactly two green balls,
c there are at least two green balls,
d there are more blue balls than green balls.

7 A committee of five is to be chosen from six men and eight women. Find how many committees are possible if:

a there are no restrictions,
b all members are to be female,
c all members are to be male,
d there are exactly two men,
e there are four women and one man,
f there is a majority of women,
g a particular man must be included,
h a particular man must not be included.

DEVELOPMENT

8 a What is the number of combinations of the letters of the word EQUATION taken four at a time (without repetition)?

b How many of the four-letter combinations contain four vowels?

c How many of the four-letter combinations contain the letter Q?

9 A team of seven netballers is to be chosen from a squad of twelve players A, B, C, D, E, F, G, H, I, J, K and L. In how many ways can they be chosen:

a with no restrictions,
b if the captain C is to be included,
c if J and K are both to be excluded,
d if A is included but H is not,
e if one of F and L is to be included and the other excluded?

10 **a** Consider the digits 9, 8, 7, 6, 5, 4, 3, 2, 1 and 0. Find how many five-digit numbers are possible if the digits are to be in:

  i descending order, **ii** ascending order.

b Why do these two questions involve unordered selections?

11 Twelve people arrive at a restaurant. There is one table for six, one table for four and one table for two. In how many ways can they be assigned to a table?

12 Twenty students, ten male and ten female, are to travel from school to the sports ground. Eight of them go in a minibus, six of them in cars, four of them on bikes and two walk.

a In how many ways can they be distributed for the trip?

b In how many ways can they be distributed if none of the boys walk?

13 Ten points $P_1, P_2, \ldots, P_{10}$ are chosen in a plane, no three of the points being collinear.

a How many lines can be drawn through pairs of the points?

b How many triangles can be drawn using the given points as vertices?

c How many of these triangles have P_1 as one of their vertices?

d How many of these triangles have P_1 and P_2 as vertices?

14 Ten points are chosen in a plane. Five of the points are collinear, but no other set of three of the points is collinear.

a How many sets of three points can be selected from the five that are collinear?

b How many triangles can be formed using three of the ten points as vertices?

15 From a standard deck of 52 playing cards, find how many five-card hands can be dealt:

a consisting of black cards only, **b** consisting of diamonds only,

c containing all four kings, **d** consisting of three diamonds and two clubs,

e consisting of three twos and another pair, **f** consisting of one pair and three of a kind.

16 **a** In how many ways can a group of six people be divided into:

  i two unequal groups (neither group being empty), **ii** two equal groups?

b Repeat part (a) for four people.

c Repeat part (a) for eight people.

17 Find how many diagonals there are in:

a a quadrilateral, **b** a pentagon, **c** a decagon, **d** a polygon with n sides.

18 Twelve points are arranged in order around a circle.

a How many triangles can be drawn with these points as vertices?

b In how many pairs of such triangles are the vertices of the two triangles distinct?

c In how many such pairs will the triangles:

  i not overlap, **ii** overlap?

19 Let $S = \{1, 3, 5, 7, 9, 11, 13, 15, 17, 19\}$ be the set of the first ten positive odd integers.

a How many subsets does S have?

b How many subsets of S contain at least three numbers?

c How many subsets with at least three numbers do not contain 7?

d How many subsets with at least three numbers do not contain 7 but do contain 19?

20 In how many ways can two numbers be selected from $1, 2, \ldots, 8, 9$ so that their sum is:

a even, **b** odd, **c** divisible by 3, **d** divisible by 5, **e** divisible by 6?

21 There are ten basketballers in a team. Find in how many ways:

a the starting five can be chosen, **b** they can be split into two teams of five.

22 Nine players are to be divided into two teams of four and one umpire.

a In how many ways can the teams be formed?

b If two particular people cannot be on the same team, how many different combinations are possible?

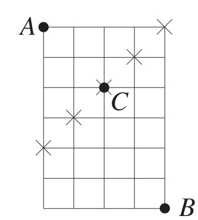
23 By considering their prime factorisations, find the number of positive divisors of:

a $2^3 \times 3^2$ **b** 1 000 000 **c** 315 000 **d** $2^a \times 5^b \times 13^c$

24 **a** The six faces of a number of identical cubes are painted in six distinct colours. How many different cubes can be formed?

b A die fits perfectly into a cubical box. How many ways are there of putting the die into the box?

25 The diagram shows a 6×4 grid. The aim is to walk from the point A in the top left-hand corner to the point B in the bottom right-hand corner by walking along the black lines either downwards or to the right. A single move is defined as walking along one side of a single small square, thus it takes you ten moves to get from A to B.



a Find how many different routes are possible:

i without restriction,

ii if you must pass through C,

iii if you cannot move along the top line of the grid,

iv if you cannot move along the second row from the top of the grid.

b Notice that every route must pass through one and only one of the five crossed points. Hence prove that ${}^{10}C_4 = {}^4C_0 \times {}^6C_4 + {}^4C_1 \times {}^6C_3 + {}^4C_2 \times {}^6C_2 + {}^4C_3 \times {}^6C_1 + {}^4C_4 \times {}^6C_0$.

c Draw another suitable diagonal and, using a method similar to that in part (b), prove that ${}^{10}C_4 = {}^5C_0 \times {}^5C_4 + {}^5C_1 \times {}^5C_3 + {}^5C_2 \times {}^5C_2 + {}^5C_3 \times {}^5C_1 + {}^5C_4 \times {}^5C_0$.

d Draw up a similar 6×6 grid, then using the same idea as that used in parts (b) and (c), prove that ${}^{12}C_6 = ({}^6C_0)^2 + ({}^6C_1)^2 + ({}^6C_2)^2 + ({}^6C_3)^2 + ({}^6C_4)^2 + ({}^6C_5)^2 + ({}^6C_6)^2$.

CHALLENGE

26 A piece of art receives an integer mark from zero to 100 for each of the categories design, technique and originality. In how many ways is it possible to score a total mark of 200?

27 How many different combinations are there of three different integers between one and thirty inclusive such that their sum is divisible by three?

28 **a** How many doubles tennis games are possible, given a group of four players?

b In how many ways can two games of doubles tennis be arranged, given a group of eight players?

c Six men and six women are to play in three games of doubles tennis. Find how many ways the pairings can be arranged if:

i there are no restrictions,

ii each game is to be a game of mixed doubles.

29 Referring to Question 18 of Exercise 17D, Bob's interest in shirts has matured recently — he now has $2n$ shirts, with two identical shirts in each of n distinct styles. He still wants to hang his shirts in the wardrobe so that no two identical shirts are next to one another.

a Show that the number of allowed arrangements is

$$ {}^{n}C_0\,(2n)!\,2^{-n} - {}^{n}C_1\,(2n-1)!\,2^{1-n} + {}^{n}C_2\,(2n-2)!\,2^{2-n} + \cdots + (-1)^n\,{}^{n}C_n\,n!\,2^0 .$$

b Evaluate this number for $n = 1, 2, 3, 4, 5$ and 6, and find the corresponding ratio of the total number of arrangements to the number of allowed arrangements.

30 Computing is based on *binary strings*, which are sequences of 0s and 1s, such as 00101 and 1001110. The problem in this question is to find the number N of binary strings that contain exactly a 0s and at most b 1s, where a and b are whole numbers. We will count such strings in two different ways, and hence prove an interesting identity.

a Using ${}^{n}C_r$ notation, find how many binary strings with exactly a 0s contain:

i no 1s

ii one 1

iii two 1s

iv three 1s

v r 1s.

b Hence prove that $N = {}^{a}C_0 + {}^{a+1}C_1 + {}^{a+2}C_2 + \cdots + {}^{a+b}C_b$.

c Let S be a binary string consisting of a 0s and r 1s, where r is at most b. Extend S to a longer binary string $S01\ldots1$ by adding 0 on the right, and then adding $b-r$ 1s. How many 0s and 1s are there in the resulting string, and what is its total length?

d Describe the inverse process by which a string of $(a+1)$ 0s and b 1s can be converted to a string of a 0s and at most b 1s, and show that the two processes are one-to-one correspondences.

e Hence prove that $N = {}^{a+b+1}C_b$.

f Show that ${}^{a}C_0 + {}^{a+1}C_1 + {}^{a+2}C_2 + \cdots + {}^{a+b}C_b = {}^{a+b+1}C_b$.

17F Using counting in probability

Learning intentions

- Apply counting procedures to finding sample and event spaces in probability.

The purpose of this section is to apply the counting procedures of the last four sections to questions about probability. In these more complicated questions, counting procedures are required for counting both sample space and event space.

Counting the sample space and the event space

As always in this topic, the two questions that need to be asked are:

- Are the selections we are counting ordered or unordered?
- If they are ordered, is repetition allowed or not?

If the questions can be done with ordered selections or with unordered selections, it is usually easier to use unordered selections because the numbers are smaller.

Example 23 Calculate probability using unordered and ordered selections

Three cards are dealt from a pack of 52.

a Find the probability that one club and two hearts are dealt, in any order.

b Find the probability that one club and two hearts are dealt in that order.

Solution

a Let the sample space be the set of all unordered selections of 3 cards from 52,

so number of unordered hands $= {}^{52}C_3 = \dfrac{52 \times 51 \times 50}{3 \times 2 \times 1}$.

We can now choose the hand by choosing 1 club from 13 in ${}^{13}C_1 = 13$ ways, and choosing the 2 hearts from 13 in ${}^{13}C_2 = 78$ ways, so the hand can be chosen in 13×78 ways.

Hence $P(\text{1 club and 2 hearts}) = 13 \times 78 \times \dfrac{3 \times 2 \times 1}{52 \times 51 \times 50} = \dfrac{39}{850}$.

b Let the sample space be the set of all ordered selections of 3 cards from 52,

so number of ordered hands $= {}^{52}P_3 = 52 \times 51 \times 50$,

and number of such hands in the order $\clubsuit\heartsuit\heartsuit = 13 \times 13 \times 12$.

Hence $P(\clubsuit\heartsuit\heartsuit) = \dfrac{13 \times 13 \times 12}{52 \times 51 \times 50} = \dfrac{13}{850}$.

Note: The answer to part (b) must be $\frac{1}{3}$ of the answer to part (a), because in a hand with one club and two hearts, the club can be any one of three positions. This indicates that it would be quite reasonable to do part (a) using ordered selections, and to do part (b) using unordered selections, although the methods chosen above are more natural to the way in which each question was worded.

Problems requiring a variety of methods

The sample spaces in the next two examples are easily found, but a variety of methods is needed to establish the sizes of the various event spaces.

Example 24 Finding probability using a variety of methods

A five-digit number is chosen at random. Find the probability:

a that it is at least 60 000,

b that it consists only of even digits,

c that the digits are distinct,

d that the digits are distinct and in increasing order.

Solution

The first digit of a five-digit number cannot be zero, giving nine choices, but the other digits can be any one of the ten digits.

Hence the number of five-digit numbers $= 9 \times 10 \times 10 \times 10 \times 10 = 90\,000$.

a To be at least 60 000, the first digit must be 6, 7, 8 or 9,
so the number of favourable numbers is $4 \times 10 \times 10 \times 10 \times 10 = 40\,000$.
Hence $P(\text{at least } 60\,000) = \frac{4}{9}$.

b If all the digits are even, there are four choices for the first digit (it cannot be zero) and five choices for each of the other four.
Hence number of such numbers $= 4 \times 5 \times 5 \times 5 \times 5 = 2500$,

and $P(\text{all digits are even}) = \dfrac{2500}{90\,000} = \dfrac{1}{36}$.

c This is counting without replacement:

1st digit	2nd digit	3rd digit	4th digit	5th digit
9	9	8	7	6

so number of such numbers $= 9 \times 9 \times 8 \times 7 \times 6$

and $P(\text{digits are distinct}) = \dfrac{9 \times 9 \times 8 \times 7 \times 6}{90\,000} = \dfrac{189}{625}$.

d Every *unordered* five-member subset of the set of nine non-zero digits can be arranged in exactly one way into a five-digit number with the digits in increasing order. (Note that the digit zero cannot be used, because a number can't begin with the digit zero.)

Hence number of such numbers

$= \text{number of unordered 5-member subsets of } \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

$= {}^9C_5$

$= 126$,

so $P(\text{digits are distinct and in increasing order}) = \dfrac{126}{90\,000} = \dfrac{7}{5000}$.

Example 25 A harder example using a variety of methods

Continue with Example 24, where a five-digit number is chosen at random, giving a sample space of size 90 000. Find the probability that

a the number contains at least one 4,

b the number contains at least one 4 and at least one 5,

c the number contains exactly three 7s,

d the number contains at least three 7s.

Solution

a This can be approached using the complementary event:

$$\text{number of five-digit numbers without a } 4 = 8 \times 9 \times 9 \times 9 \times 9 = 52\,488,$$

so $$\text{number of five-digit numbers with a } 4 = 90\,000 - 52\,488 = 37\,512.$$

Hence $$P(\text{at least one } 4) = \frac{37\,512}{90\,000} = \frac{521}{1250}.$$

b This is best approached using the counting rule $|A \cup B| = |A| + |B| - |A \cap B|$.

Let A be the set of five-digit numbers without a 4, and let B be the set of five-digit numbers without a 5.

Then $A \cap B$ is the set of five-digit number with no 4 **and** no 5, and $A \cup B$ is the set of five-digit number with no 4 **or** no 5.

From line 2 of the solution to part (a), $|A| = 52\,488$,

and similarly, $|B| = 52\,488$,

then proceeding again as in part (a), $|A \cap B| = 7 \times 8 \times 8 \times 8 \times 8$

$$= 8\,672.$$

Now we apply the counting rule,

$$\begin{aligned} |A \cup B| &= |A| + |B| - |A \cap B| \\ &= 52\,488 + 52\,488 - 28\,672 \\ &= 76\,304, \end{aligned}$$

so there are 76 304 five-digit numbers with no 4 **or** no 5.

We want the set of all five-digit numbers with at least one 4 **and** at least one 5,

which is the complementary set $\overline{A \cup B}$, and

$$\begin{aligned} \left|\overline{A \cup B}\right| &= 90\,000 - 76\,304 \\ &= 13\,696. \end{aligned}$$

Hence

$$\begin{aligned} P(\text{at least one 4 and at least one 5}) &= \frac{13\,696}{90\,000}, \\ &= \frac{856}{5625}. \end{aligned}$$

c Counting the number of five-digit numbers with exactly three 7s requires cases. First we count the five-digit strings with exactly three 7s, by first placing the three 7s, and then choosing the first and second non-7 digits:

position of the three 7s	choose first non-7	choose second non-7
${}^5C_3 = 10$	9	9

giving 810 such strings. Secondly, we must subtract the number of five-digit strings with exactly three 7s and beginning with zero:

position of the three 7s	choose the other non-7
${}^4C_3 = 4$	9

giving 36 such strings. Hence there are 810 − 36 = 774 such numbers, and

$$P(\text{number has exactly three 7s}) = \frac{774}{90\,000} = \frac{43}{5000}.$$

d The number 77 777 is the only five-digit number with five 7s.

Any five-digit number with exactly four 7s has one of the five forms

$$*7\,777, \qquad 7*777, \qquad 77*77, \qquad 77\,7*7, \qquad 77\,77*,$$

where the $*$ in $*7\,777$ is a non-zero digit. There are eight numbers of the first form, and nine of the other four forms, giving 44 numbers altogether.

Hence the number with at least three 7s = 774 + 44 + 1 = 819,

and $P(\text{at least three 7s}) = \dfrac{819}{90\,000} = \dfrac{91}{10\,000}$.

Exercise 17F

FOUNDATION

1 A committee of three is to be selected from the nine members in a club.

a How many different committees can be formed?

b If there are five men in the club, what is the probability that the selected committee consists entirely of males?

2 The integers from 1 to 10 inclusive are written on ten separate pieces of paper. Four pieces of paper are drawn at random. Find the probability that:

a the four numbers drawn are 1, 2, 3 and 6,

b the number 9 is one of the numbers drawn,

c the number 8 is not drawn,

d the number 7 is drawn but the number 1 is not.

3 A bag contains three red, seven yellow and five blue balls. If three balls are drawn from the bag simultaneously, find the probability that:

a all three balls are yellow,

b all the balls are of the same colour,

c there are two red balls and one blue ball,

d all the balls are of different colours.

4 A sports committee of five members is to be chosen from eight AFL footballers and seven soccer players. Find the probability that the committee will contain:

a only AFL footballers,

b only soccer players,

c three soccer players and two AFL footballers,

d at least one soccer player,

e at most one soccer player,

f Ian, a particular soccer player.

5 From a standard pack of 52 cards, three are selected at random. Find the probability that:

a they are the jack of spades, the two of clubs and the seven of diamonds,

b all three are aces,

c they are all diamonds,

d they are all of the same suit,

e they are all picture cards,

f two are red and one is black,

g one is a seven, one is an eight and one is a nine,

h two are sevens and one is a six,

i exactly one is a diamond,

j at least two of them are diamonds.

6 Repeat the previous question if the cards are selected from the pack one at a time, and each card is replaced before the next one is drawn.

7 Three boys and three girls are to sit in a row. Find the probability that:

a the boys and girls alternate,

b the boys sit together and the girls sit together,

c two specific girls sit next to one another.

8 A family of five are seated in a row at the cinema. Find the probability that:

a the parents sit on the end and the three children are in the middle,

b the parents sit next to one another.

9 Six people, of whom Patrick and Jessica are two, arrange themselves in a row. Find the probability that:

a Patrick and Jessica occupy the end positions,

b Patrick and Jessica are not next to each other.

DEVELOPMENT

10 The letters of PROMISE are arranged randomly in a row. Find the probability that:

a the word starts with R and ends with S,

b the letters P and R are next to one another,

c the letters P and R are separated by at least three letters,

d the vowels and the consonants alternate,

e the vowels are together.

11 The digits 3, 3, 4, 4, 4 and 5 are placed in a row to form a six-digit number. If one of these numbers is selected at random, find the probability that:

a it is even,

b it ends in 5,

c the 4s occur together,

d the number starts with 5 and then the 4s and 3s alternate,

e the 3s are separated by at least one other number.

12 The letters of the word PRINTER are arranged in a row. Find the probability that:

a the word starts with the letter E,

b the letters I and P are next to one another,

c there are three letters between N and T,

d there are at least three letters between N and T.

13 The letters of KETTLE are arranged randomly in a row. Find the probability that:

a the two letters E are together,

b the two letters E are not together,

c the two letters E are together and the two letters T are together,

d the Es and Ts are together in one group.

14 The letters of ENTERTAINMENT are arranged in a row. Find the probability that:

a the letters E are together,

b two Es are together and one is apart,

c all the letters E are apart,

d the word starts and ends with E.

15 A tank contains 20 tagged fish and 80 untagged fish. On each day, four fish are selected at random, and after noting whether they are tagged or untagged, they are returned to the tank. Answer the following questions, correct to three significant figures.

a What is the probability of selecting no tagged fish on a given day?

b What is the probability of selecting at least one tagged fish on a given day?

c Calculate the probability of selecting no tagged fish on every day for a week.

d What is the probability of selecting no tagged fish on exactly three of the seven days during the week?

16 A bag contains seven white and five black discs. Three discs are chosen from the bag. Find the probability that all three discs are black, if the discs are chosen:

a without replacement,
b with replacement,
c so that after each draw the disc is replaced with one of the opposite colour.

17 Six people are to be divided into two groups, each with at least one person. Find the probability that:

a there will be three in each group,
b there will be two in one group and four in the other,
c there will be one group of five and an individual.

18 A three-digit number is formed from the digits 3, 4, 5, 6 and 7 (no repetitions allowed). Find the probability that:

a the number is 473,
b the number is odd,
c the number is divisible by 5,
d the number is divisible by 3,
e the number starts with 4 and ends with 7,
f the number contains the digit 3,
g the number contains the digits 3 and 5,
h the number contains the digit 3 or 5,
i all digits in the number are odd,
j the number is greater than 500.

19 The digits 1, 2, 3 and 4 are used to form numbers that may have 1, 2, 3 or 4 digits in them. If one of the numbers is selected at random, find the probability that:

a it has three digits,
b it is even,
c it is greater than 200,
d it is odd and greater than 3000,
e it is divisible by 3.

20 **a** A senate committee of five members is to be selected from six Labor and five Liberal senators. What is the probability that Labor will have a majority on the committee?

b The senate committee is to be selected from N Labor and five Liberal senators. Use trial and error to find the minimum value of N, given that the probability of Labor having a majority on the committee is greater than $\frac{3}{4}$.

21 Four basketball teams A, B, C and D each consist of ten players, and in each team, the players are numbered 1, 2, … 9, 10. Five players are to be selected at random from the four teams. Find the probability that of the five players selected:

a three are numbered 4 and two are numbered 9,
b at least four are from the same team.

22 A poker hand of five cards is dealt from a standard pack of 52. Find the probability of obtaining:

a one pair,
b two pairs,
c three of a kind,
d four of a kind,
e a full house (one pair and three of a kind),
f a straight (five cards in sequence regardless of suit, ace high or low),
g a flush (five cards of the same suit),
h a royal flush (ten, jack, queen, king and ace in a single suit).

23 Four adults are standing in a room that has five exits. Each adult is equally likely to leave the room through any one of the five exits.

- **a** What is the probability that all four adults leave the room via the same exit?
- **b** What is the probability that three particular adults use the same exit and the fourth adult uses a different exit?
- **c** What is the probability that any three of the four adults come out the same exit, and the remaining adult comes out a different exit?
- **d** What is the probability that no more than two adults come out any one exit?

24 **a** Five diners in a restaurant choose randomly from a menu featuring five main courses. Find the probability that exactly one of the main courses is not chosen by any of the diners.

- **b** Repeat the question if there are n diners and a choice of n main courses.

25 [The birthday problem]

- **a** Assuming a 365-day year, find the probability that in a group of three people there will be at least one birthday in common. Answer correct to two significant figures.
- **b** If there are n people in the group, find an expression for the probability of at least one common birthday.
- **c** By choosing a number of values of n, plot a graph of the probability of at least one common birthday against n for $n \leq 50$.
- **d** How many people need to be in the group before the probability exceeds 0.5?
- **e** How many people need to be in the group before the probability exceeds 90%?

CHALLENGE

---

26 During the seven games of the football season, Max and Bert must each miss three consecutive games. The games to be missed by each player are randomly and independently selected.

- **a** What is the probability that they both have the first game off together?
- **b** What is the probability that the second game is the first one where Max and Bert are both missing?
- **c** What is the probability that Max and Bert miss at least one of the same games?

27 Eight players make the quarter-finals at Wimbledon. The winner of each of the quarter-finals plays a semi-final to see who enters the final.

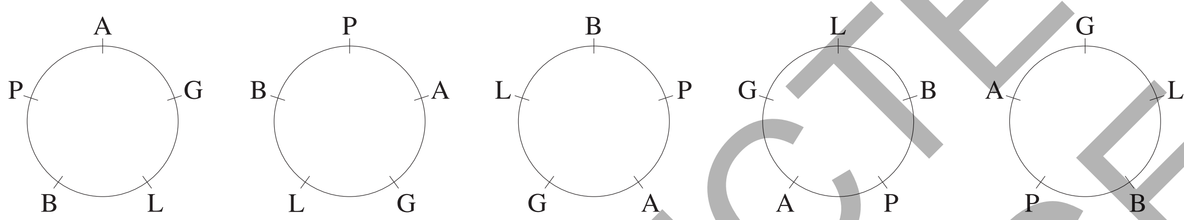
- **a** Assuming that all eight players are equally likely to win a match, show that the probability that any two particular players will play each other is $\frac{1}{4}$.
- **b** What is the probability that two particular people will play each other if the tournament starts with 16 players?
- **c** What is the probability that two particular players will meet in a similar knockout tournament if 2^n players enter?

17G Arrangements in a circle

Learning intentions

- Develop procedures to count arrangements in a circle.
- Count arrangements in a circle to calculate probabilities.

Arrangements in a circle, or around a round table, are complicated because two arrangements are regarded as equivalent if one can be rotated to produce the other. For example, all the five round-table seatings below of King Arthur, Queen Guinevere, Sir Lancelot, Sir Bors and Sir Percival are to be regarded as the same:



The basic algorithm

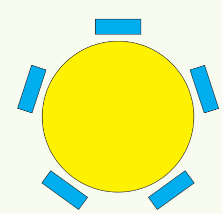
The most straightforward way of counting arrangements in a circle is to seat the people in order, dealing with the restrictions first as always, but reckoning that there is essentially only one way to seat the first person who sits down, because until that time, all the seats are identical.

14 Counting Arrangements in a Circle

There is essentially only one way to seat the first person, because until then, all the seats are identical.

Example 26 Counting arrangements in a circle

King Arthur, Queen Guinevere, Sir Lancelot, Sir Bors and Sir Percival sit around a round table. Find in how many ways this can be done:



a without restriction,

b if Queen Guinevere sits at King Arthur's leftt hand,

c if Queen Guinevere sits between Sir Lancelot and Sir Bors,

d if King Arthur and Sir Lancelot do not sit together.

Solution

a

Seat Arthur	Seat Guinevere	Seat Lancelot	Seat Bors	Seat Percival
1	4	3	2	1

Number of ways = 24.

b

Seat Arthur	Seat Guinevere	Seat Lancelot	Seat Bors	Seat Percival
1	1	3	2	1

Number of ways = 6.

c

Seat Guinevere	Seat Lancelot	Seat Bors	Seat Arthur	Seat Percival
1	2	1	2	1

Number of ways = 4.

d

Seat Arthur	Seat Lancelot	Seat Guinevere	Seat Bors	Seat Percival
1	2	3	2	1

Number of ways = 12.

Arranging groups around a circle

When arranging groups around a circle, the principle is the same as the principle for compound orderings established in Section 17C.

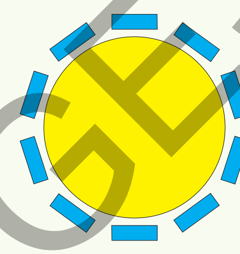
15 Arranging groups around a circle

- First choose an order for each group.
- Then arrange the groups around the circle, reckoning that there is essentially only one way to place the first group.

Example 27 Using grouping to count arrangements in a circle

Five boys and five girls are to sit around a table. Find in how many ways this can be done:

a without restriction,
b if the boys and girls alternate,
c if there are five couples, all of whom sit together,
d if the boys sit together and the girls sit together,
e if four couples sit together, but Walter and Maude do not.



Solution

a

1st	2nd	3rd	4th	5th	6th	7th	8th	9th	10th
1	9	8	7	6	5	4	3	2	1

Number of ways = 9! = 362 880.

b

1st boy	2nd boy	3rd boy	4th boy	5th boy	1st girl	2nd girl	3rd girl	4th girl	5th girl
1	4	3	2	1	5	4	3	2	1

Number of ways = 5! × 4! = 2880.

c Each couple can be ordered in 2 ways, giving 2^5 orderings of the five couples. Then seat the five couples around the table:

1st couple	2nd couple	3rd couple	4th couple	5th couple
1	4	3	2	1

Number of ways = $2^5 \times 4! = 768$.

d The boys can be ordered in 5! ways, and the girls in 5! ways also. Then seat the two groups around the table:

group of boys	group of girls
1	1

Number of ways = 5! × 5! × 1 = 14 400.

e Order each of the four couples in 2 ways, giving 16 orderings of the couples. There are now four couples and two individuals to seat around the table, with the restriction that Maude does not sit next to Walter:

Walter	Maude	1st couple	2nd couple	3rd couple	4th couple
1	3	4	3	2	1

Number of ways = $2^4 \times 3 \times 4! = 1152$.

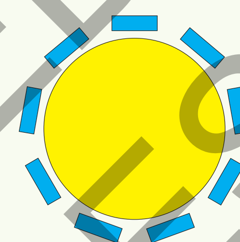
Probability in arrangements around a circle

As always, counting allows probability problems to be solved by counting the sample space and the event space.

Example 28 Finding probabilities with arrangements in a circle

Three Tasmanians, three New Zealanders, and three Queenslanders are seated at random around a round table. What is the probability that the three groups are seated together?

Solution



Using the same boxes as before, there are $1 \times 8!$ possible orderings. To find the number of favourable orderings, first order each group in $3! = 6$ ways, then order the three groups around the table in $1 \times 2 \times 1 = 2$ ways, so the total number of favourable orderings is $6 \times 6 \times 6 \times 2$.

$$\text{Hence } P(\text{groups are together}) = \frac{6 \times 6 \times 6 \times 2}{8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1} = \frac{3}{280}.$$

Exercise 17G

FOUNDATION

1 **a** In how many ways can five people be arranged:
  i in a line, **ii** in a circle?

b In how many ways can ten people be arranged:
  i in a line, **ii** in a circle?

2 Eight people are arranged in:
  a a straight line, **b** a circle.

In how many ways can they be arranged so that two particular people sit together?

3 Bob, Betty, Ben, Brad and Belinda are to be seated at a round table. In how many ways can this be done:
  a if there are no restrictions,
  b if Betty sits on Bob's right-hand side,
  c if Brad is to sit between Bob and Ben,
  d if Belinda and Betty sit apart,
  e if Ben and Belinda sit apart, but Betty sits next to Bob?

4 Four boys and four girls are arranged in a circle. In how many ways can this be done:
  a if there are no restrictions,
  b if the boys and the girls alternate,
  c if the boys and girls are in distinct groups,
  d if a particular boy and girl wish to sit next to one another,
  e if two particular boys do not wish to sit next to one another,
  f if one particular boy wants to sit between two particular girls?

DEVELOPMENT

5 The letters A, E, I, P, Q and R are arranged in a circle. Find the probability that:

a the vowels are together,
b A is opposite R,
c the vowels and consonants alternate,
d at least two vowels are next to one another.

6 In how many ways can the integers 1, 2, 3, 4, 5, 6, 7, 8 be placed in a circle if:

a there are no restrictions,
b all the even numbers are together,
c the odd and even numbers alternate,
d at least three odd numbers are together,
e the numbers 1 and 7 are adjacent,
f the numbers 3 and 4 are separated?

7 A committee of three women and seven men is to be seated randomly at a round table.

a What is the probability that the three females will sit together?
b The committee elects a president and a vice-president. What is the probability that they are sitting opposite one another?

8 Find how many arrangements of n people around a circle are possible if:

a there are no restrictions,
b two particular people must sit together,
c two particular people sit apart,
d three particular people sit together.

9 Twelve marbles are to be placed in a circle. In how many ways can this be done if:

a all the marbles are of different colours,
b there are eight red, three blue and one green marble?

10 There are two round tables, one oak and one mahogany, each with five seats. In how many ways may a group of ten people be seated?

11 A sports committee consisting of four rowers, three basketballers and two cricketers sits at a circular table.

a How many different arrangements of the committee are possible if the rowers and basketballers both sit together in groups, but no rower sits next to a basketballer?
b One rower and one cricketer are related. If the conditions in (a) apply, what is the probability that these two members of the committee will sit next to one another?

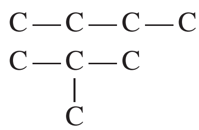
CHALLENGE

12 A group of n men and $n + 1$ women sit around a circular table. What is the probability that no two men sit next to one another?

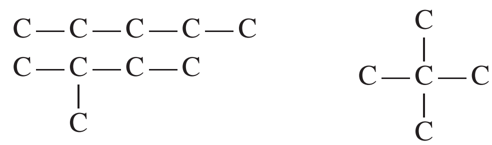
13 **a** Consider a necklace of six differently coloured beads. Because the necklace can be turned over, clockwise and anti-clockwise arrangements of the beads do not yield different orders. Hence find how many different arrangements there are of the six beads on the necklace.
b In how many ways can ten different keys be placed on a key ring?
c In how many ways can one yellow, two red and four green beads be placed on a bracelet if the beads are identical apart from colour? (This will require a listing of patterns to see if they are identical when turned over.)

Isomers — a possible computing project

Because of branching, it is difficult to count the number of possible hydrocarbons with say 20 carbon atoms. For example, for butane C_4H_{10} there are two carbon chains, and for pentane C_5H_{12} there are three carbon chains:



butane



pentane

This is a hard problem. If you can write computer programmes, you could write your own programme to investigate the number of possible carbon chains for hydrocarbons with increasing numbers of carbon atoms. Perhaps your programme can display diagrams of all the chains, as with butane and pentane above.

Chapter 17 Review

Review activity

- Create your own summary of this chapter on paper or in a digital document.

Chapter 17 Multiple-choice quiz

- This automatically-marked quiz is accessed in the Interactive Textbook. A printable PDF Worksheet version is also available there.

Skills Checklist

Check-list

- Available in the Interactive Textbook, use the checklist to track your understanding of the learning intentions. Printable PDF and word document versions are also available there.

Review

Chapter Review Exercise

1 How many ways can 8 people line up in a queue?

2 By unrolling the factorial (see Box 2 in Section 14A) simplify:

a $\frac{9!}{7!}$ b $\frac{(n+1)!}{(n-1)!}$ c $(k+1)! - k!$

3 Use your calculator to evaluate:

a ${}^{12}C_7$ b ${}^{10}C_3 \div {}^{6}C_3$ c $\binom{20}{17}$

4 How many four-letter words, with no repeated letters, can be formed from the letters of JACKSON?

5 A number plate in a certain country consists of 3 letters A–Z followed by 4 digits 0–9. How many number plates are possible?

6 Find how many arrangements of the letters of the word FOUNDER are possible:

a if the vowels and consonants must alternate,
b if the word must start with N and end with D,
c if all the consonants must be in a group at the end of the word,
d if the R is somewhere to the right of the U.

7 Five boys and five girls are to sit in a row. Find how many ways this can be done if:

a there are no restrictions,
b the boys and girls sit in distinct groups,
c a particular boy and girl must sit together.

8 How many ways can the letters of REPORTER be arranged?

9 How many words of three or four letters may be formed using the letters of SAMUEL?

10 A quiz consists of ten questions, each taking the answer Yes or No. How many ways is it possible to get 6 correct and 4 wrong answers?

11 A committee of seven is to be chosen from six men and ten women. Find how many committees are possible if:

a there are no restrictions,
b all members are to be female,
c all members are to be male,
d there are to be exactly two men,
e there are to be four women and three men,
f there is to be a majority of women,
g a particular man must be included,
h a particular man must not be included,
i Mustafa refuses to be on a committee with Ying Yue.

12 Eight people arrive at a restaurant. Find how many ways can they be assigned to:

a a large table for five and a smaller table for three,
b two quite different tables for four,
c two indistinguishable tables for four.

13 A committee of six is formed at random from four men and three women. What is the probability that it will have more men than women?

14 From a standard pack of 52 cards, three are selected at random. Find the probability that:

a they are the queen of spades, the three of clubs and the nine of hearts,
b all three are kings,
c they are all clubs,
d they are all of the same suit,
e one is red and two are black,
f one is a three, one is a five, and one is an eight,
g two are fives and one is a seven,
h at least two of them are spades.

15 Three boys and three girls are arranged in a circle. In how many ways can this be done:

a if there are no restrictions,
b if the boys and the girls alternate,
c if the boys and girls are in distinct groups,
d if a particular boy and girl wish to sit next to one another,
e if two particular boys do not wish to sit next to one another,
f if a particular boy wants to sit opposite a particular girl?

18

The binomial theorem and Pascal's triangle

Chapter introduction

In Chapter 11 we discussed the factoring of a polynomial into irreducible factors, so that it could be written in a form such as

$$P(x) = (x - 4)^2(x + 1)^3(x^2 + x + 1).$$

In this chapter we will now study in more detail the individual binomial power factors such as $(x - 4)^2$ and $(x + 1)^3$ that appear in such a factoring, and their expansions. For example, we have already seen that

$$(x + 1)^3 = x^3 + 3x^2 + 3x + 1.$$

The coefficients in the general expansion of $(x + y)^n$ will be investigated through the patterns they form when they are written down in *Pascal's triangle*. It turns out that basic counting methods of the previous chapter are essential for understanding these expansions.

Pascal's triangle displays in a clear visual form the interrelationships between the counting methods of Chapter 17 and the binomial expansions of this chapter. There are a great number of symmetries and other patterns. Sections 18E and 18F investigate these patterns using various combinatoric and algebraic methods.

Binomial expansions will be used in the Year 12 topic of binomial distributions

18A Binomial expansions and Pascal's triangle

Learning intentions

- Investigate the binomial expansions $(1 + x)^n$, for low whole numbers n.
- Use these expansions to construct the first few rows of Pascal's triangle.
- Notice and explain patterns, such as the addition property, in Pascal's triangle.
- Use Pascal's triangle to solve problems involving a binomial expansion.

A *binomial* is a polynomial with two terms, such as $1 + x$ or $3x^4 - \frac{1}{2}x^2$.

A *binomial expansion* is the expansion of a power of a binomial, for example

$$(1 + x)^3 = 1 + 3x + 3x^2 + x^3 .$$

This first section is restricted to the expansion of $(1 + x)^n$ and to the various techniques arising from such expansions. The techniques are based on Pascal's triangle and its three most basic properties.

Some expansions of $(1 + x)^n$

Here are the expansions of $(1 + x)^n$ for low values of n. The calculations have been carried out using two rows so that like terms can be written above each other in columns. In this way, the process by which the coefficients build up can be followed better.

Examine the last expansion in particular, and work out why the boxed $\boxed{4}$ is the sum of the boxed $\boxed{1}$ and $\boxed{3}$.

$$\begin{aligned}
(1 + x)^0 &= 1 \\
(1 + x)^1 &= 1 + x \\
(1 + x)^2 &= 1(1 + x) + x(1 + x) \\
&= 1 + x \\
&\quad + x + x^2 \\
&= 1 + 2x + x^2
\end{aligned}$$

$$\begin{aligned}
(1 + x)^3 &= 1(1 + x)^2 + x(1 + x)^2 \\
&= 1 + 2x + x^2 \\
&\quad + x + 2x^2 + x^3 \\
&= 1 + 3x + 3x^2 + x^3 \\
(1 + x)^4 &= 1(1 + x)^3 + x(1 + x)^3 \\
&= 1 + 3x + 3x^2 + \boxed{1}\,x^3 \\
&\quad + x + 3x^2 + \boxed{3}\,x^3 + x^4 \\
&= 1 + 4x + 6x^2 + \boxed{4}\,x^3 + x^4
\end{aligned}$$

Notice how the expansion of $(1 + x)^2$ has 3 terms, that of $(1 + x)^3$ has 4 terms, and so on. In general, the expansion of $(1 + x)^n$ has $n + 1$ terms, from the constant term in $x^0 = 1$ to the term in x^n. Be careful — this is inclusive counting — there are $n + 1$ numbers from 0 to n inclusive.

Note: The power 0^0 is undefined, and as a consequence, there should be continual qualifications to avoid it. But as is customary in this topic, we will omit them all, leaving the reader to interpret situations where a problem may arise.

Pascal's triangle and the addition property

When the coefficients in the expansions of $(1 + x)^n$ are arranged in a table, the result is known as *Pascal's triangle*. The table below contains the first five rows of the triangle, copied from the expansions above, plus the next four rows, obtained by continuing these calculations up to $(1 + x)^8$.

Coefficient of:

n	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8
0	1								
1	1	1							
2	1	2	1						
3	1	3	[3]	[1]					
4	1	4	6	[4]	1				
5	1	5	10	10	5	1			
6	1	6	15	20	15	6	1		
7	1	7	21	35	35	21	7	1	
8	1	8	28	56	70	56	28	8	1

Three basic properties of this triangle should quickly become obvious. They will be used in this section, and proven formally later.

1 Three basic properties of Pascal's triangle

1 Each row starts and ends with 1.

2 Each row is reversible. That is, the rows are symmetric.

3 [The addition property] Every number in the triangle, apart from the 1s, is the sum of the number directly above, and the number above and to the left.

The first two properties should be reasonably obvious after looking at the expansions at the start of the section. The third property, called the *addition property*, however, needs attention. Three numbers in Pascal's triangle above have been boxed as an example of this — notice how

$3 + 1 = 4.$

The binomial expansions on the last page were written with the columns aligned, and with these particular coefficients boxed, to make this property stand out.

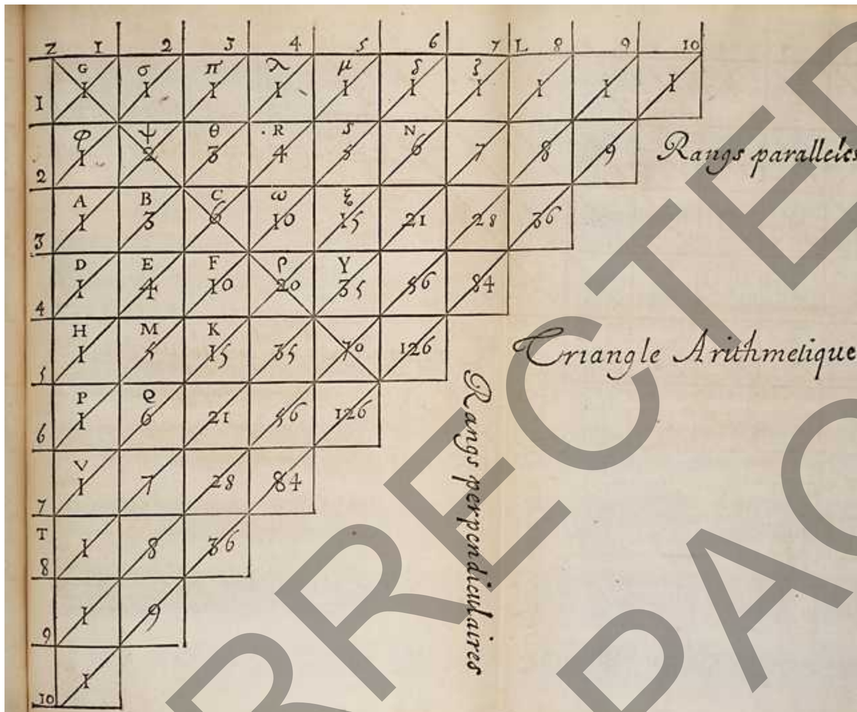
- The sum $3 + 1 = 4$ arises because the coefficient of x^3 in the expansion of $(1 + x)^4$ is the sum of the coefficients of x^3 and x^2 in the expansion of $(1 + x)^3$.

Pascal's triangle can be constructed using these rules, and the first question in Exercise 18A asks for the first thirteen rows to be calculated.

Pascal's triangle is often drawn in the layout to the right below, like an equilateral triangle. This layout displays the left-right symmetry well, but it is less useful for our purposes because we want the columns to line up with the powers of x. Nevertheless, you should be able to work from both layouts of the triangle.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1

Pascal did not discover the triangle, although he did write an important treatise on it in 1653. It was known to the ancient Indian mathematician Pingala in the 2nd century BC, and later to mediaeval Persian and Chinese mathematicians, but the first known occurrence in Europe is in a 1527 book on business calculations by Petrus Adrianus. Below is Pascal's version of the triangle from his treatise, with yet another layout.



Using Pascal's triangle

Examples 1–5 illustrate various calculations involving the coefficients of $(1 + x)^n$ for low values of n.

Example 1 **Using Pascal's triangle to write out expansions**

Use Pascal's triangle to write out the expansions of:

a $(1 + 2a)^6$ **b** $(1 - x)^4$ **c** $(1 - \frac{2}{3}x)^5$

Solution

a $(1 + 2a)^6 = 1 + 6(2a) + 15(2a)^2 + 20(2a)^3 + 15(2a)^4 + 6(2a)^5 + (2a)^6$

$= 1 + 12a + 60a^2 + 160a^3 + 240a^4 + 192a^5 + 64a^6$

b Be particularly careful with the alternating signs in this expansion.

$(1 - x)^4 = 1 + 4(-x) + 6(-x)^2 + 4(-x)^3 + (-x)^4$

$= 1 - 4x + 6x^2 - 4x^3 + x^4$

c The signs also alternate in this expansion.

$(1 - \frac{2}{3}x)^5 = 1 + 5(-\frac{2}{3}x) + 10(-\frac{2}{3}x)^2 + 10(-\frac{2}{3}x)^3 + 5(-\frac{2}{3}x)^4 + (-\frac{2}{3}x)^5$

$= 1 - \frac{10}{3}x + \frac{40}{9}x^2 - \frac{80}{27}x^3 + \frac{80}{81}x^4 - \frac{32}{243}x^5$

Example 2 Using Pascal's triangle to approximate powers

By expanding the first few terms of $(1 + 0.02)^8$, find an approximation of 1.02^8 correct to five decimal places.

Solution

$$(1 + 0.02)^8 = 1 + 8 \times 0.02 + 28 \times (0.02)^2 + 56 \times (0.02)^3 + 70 \times (0.02)^4 + \cdots$$
$$= 1 + 0.16 + 0.0112 + 0.000\,448 + 0.000\,011\,20 + \cdots$$
$$\doteqdot 1.171\,66.$$

The remaining four terms are too small to affect the fifth decimal place.

Example 3 Examining a product of binomials powers

a Write out the expansion of $\left(1 + \frac{5}{x}\right)^2$, then write out the first four terms in the expansion of $(1 - x)^8$.

b Hence find, in the expansion of $\left(1 + \frac{5}{x}\right)^2 (1 - x)^8$:

i the term independent of x,

ii the term in x.

Solution

a $\left(1 + \frac{5}{x}\right)^2 = 1 + 10x^{-1} + 25x^{-2}$

$(1 - x)^8 = 1 - 8x + 28x^2 - 56x^3 + \cdots$

b Hence in the expansion of $\left(1 + \frac{5}{x}\right)^2 (1 - x)^8$:

i constant term $= 1 \times 1 + (10x^{-1}) \times (-8x) + (25x^{-2}) \times (28x^2)$
$= 1 - 80 + 700$
$= 621.$

ii term in $x = 1 \times (-8x) + (10x^{-1}) \times (28x^2) + (25x^{-2}) \times (-56x^3)$
$= -8x + 280x - 1400x$
$= -1128x.$

Example 4 Finding pronumerals given a condition on a binomial expansion

a Write down the terms in x^4 and x^3 in the expansion of $(1 + 2kx)^6$.

b Find k if these terms in x^4 and x^3 have coefficients in the ratio $2 : 3$.

Solution

a $(1 + 2kx)^6 = \cdots + 15(2kx)^2 + 20(2kx)^3 + 15(2kx)^4 + \cdots$
$= \cdots + 60k^2x^2 + 160k^3x^3 + 240k^4x^4 + \cdots$

(Alternatively, just write down the two terms.)

b Put $\dfrac{240k^4}{160k^3} = \dfrac{2}{3}$.

Then $\frac{3}{2}k = \frac{2}{3}$

$k = \frac{4}{9}$.

Example 5 A harder example — expanding a power of a trinomial

Expand $(1 + x + x^2)^4$ using Pascal's triangle, by writing $1 + x + x^2 = 1 + (x + x^2)$, and writing $x + x^2 = x(1 + x)$.

Solution

$$\begin{aligned}(1 + x + x^2)^4 &= \big(1 + (x(1 + x))\big)^4 \\ &= 1 + 4x(1 + x) + 6x^2(1 + x)^2 + 4x^3(1 + x)^3 + x^4(1 + x)^4 \\ &= 1 + 4x(1 + x) + 6x^2(1 + 2x + x^2) + 4x^3(1 + 3x + 3x^2 + x^3) \\ &\quad + x^4(1 + 4x + 6x^2 + 4x^3 + x^4) \\ &= 1 + (4x + 4x^2) + (6x^2 + 12x^3 + 6x^4) + (4x^3 + 12x^4 + 12x^5 + 4x^6) \\ &\quad + (x^4 + 4x^5 + 6x^6 + 4x^7 + x^8) \\ &= 1 + 4x + 10x^2 + 16x^3 + 19x^4 + 16x^5 + 10x^6 + 4x^7 + x^8\end{aligned}$$

Exercise 18A

FOUNDATION

Note: Question 1 constructs the rows of Pascal's triangle up to $n = 12$. This is 'your copy' of Pascal's triangle — keep it in a prominent place for constant use in this chapter. Your calculator may give you particular values, but it does not display the patterns.

1 Complete all the rows of Pascal's triangle for $n = 0, 1, 2, 3, \ldots, 12$.

2 Using Pascal's triangle to provide the binomial coefficients, give the expansions of:

a $(1 + x)^6$ **b** $(1 - x)^6$ **c** $(1 + x)^9$ **d** $(1 - x)^9$

e $(1 + c)^5$ **f** $(1 + 2y)^4$ **g** $\left(1 + \dfrac{x}{3}\right)^7$ **h** $(1 - 3z)^3$

i $\left(1 - \dfrac{1}{x}\right)^8$ **j** $\left(1 + \dfrac{2}{x}\right)^5$ **k** $\left(1 + \dfrac{y}{x}\right)^5$ **l** $\left(1 + \dfrac{3x}{y}\right)^4$

3 Continue the calculations of the expansions of $(1 + x)^n$ at the beginning of this section, expanding $(1 + x)^5$ and $(1 + x)^6$ in the same manner. Keep your work in columns, so that the addition property of Pascal's triangle is clear.

4 Find the specified term in each expansion.

a For $(1 + x)^{11}$:
- **i** find the term in x^2,
- **ii** find the term in x^8.

b For $(1 - x)^7$:
- **i** find the term in x^3,
- **ii** find the term in x^5.

c For $(1 + 2x)^6$:
- **i** find the term in x^4,
- **ii** find the term in x^5.

d For $\left(1 - \dfrac{3}{x}\right)^4$:
- **i** find the term in x^{-1},
- **ii** find the term in x^{-2}.

5 Expand $(1+x)^9$ and $(1+x)^{10}$, and show that the sum of the coefficients in the second expansion is twice the sum of the coefficients in the first expansion.

DEVELOPMENT

6 Without expanding, simplify:

a $1+3(x-1)+3(x-1)^2+(x-1)^3$

b $1-6(x+1)+15(x+1)^2-20(x+1)^3+15(x+1)^4-6(x+1)^5+(x+1)^6$

7 Find the coefficient of x^4 in the expansion of $(1-x)^4+(1-x)^5+(1-x)^6$.

8 Find integers a and b such that:

a $(1+\sqrt{3})^5=a+b\sqrt{3}$

b $(1-\sqrt{5})^3=a+b\sqrt{5}$

9 Verify by direct expansion, and by taking out the common factor, that:

a $(1+x)^4-(1+x)^3=x(1+x)^3$

b $(1+x)^7-(1+x)^6=x(1+x)^6$

10 Do not use a calculator in this question.

a Expand the first few terms of $(1+x)^6$, hence evaluate 1.003^6 to five decimal places.

b Similarly, expand $(1-4x)^5$, and hence evaluate 0.96^5 to five decimal places.

11 **a** **i** Expand $(1+x)^4$ as far as the term in x^2.

ii Hence find the coefficient of x^2 in the expansion of $(1-5x)(1+x)^4$.

b **i** Expand $(1+2x)^5$ as far as the term in x^3.

ii Hence find the coefficient of x^3 in the expansion of $(2-3x)(1+2x)^5$.

c **i** Expand $(1-3x)^4$ as far as the term in x^3.

ii Hence find the coefficient of x^3 in the expansion of $(2+x)^2(1-3x)^4$.

12 **a** When $(1+2x)^5$ is expanded in increasing powers of x, the third and fourth terms in the expansion are equal. Find the value of x.

b When $(1+x)^7$ is expanded in increasing powers of x, the sixth term is the average of the fifth and seventh terms in the expansion. Find the value of x.

13 Find the coefficient of:

a x^3 in $(3-4x)(1+x)^4$

b x in $(1+3x+x^2)(1-x)^3$

c x^4 in $(5-2x^3)(1+2x)^5$

d x^0 in $\left(1-\frac{x}{3}\right)^3\left(1+\frac{2}{x}\right)^2$

14 Determine the value of the term independent of x in the expansion of:

a $(1+2x)^4\left(1-\frac{1}{x^2}\right)^6$

b $\left(1-\frac{x}{3}\right)^5\left(1+\frac{2}{x}\right)^3$

15 **a** In the expansion of $(1 + x)^6$:

i find the term in x^2,

ii find the term in x^3,

iii find the ratio of the term in x^2 to the term in x^3,

iv find the values of (i), (ii) and (iii) when $x = 3$.

b In the expansion of $\left(1 + \dfrac{2}{3x}\right)^7$:

i find the term in x^{-5},

ii find the term in x^{-6},

iii find the ratio of the term in x^{-5} to the term in x^{-6},

iv find the values of (i), (ii) and (iii) when $x = 2$.

16 **a** Find the coefficients of x^4 and x^5 in the expansion of $(1 + kx)^8$. Hence find k if these coefficients are in the ratio 1 : 4.

b Find the coefficients of x^5 and x^6 in the expansion of $(1 - \frac{3}{4}kx)^9$. Hence find k if these coefficients are equal.

17 [Patterns in Pascal's triangle] Check the following results using the triangle you constructed in Question 1. (Some of these will be proven later.)

a The sum of the numbers in the row beginning $1, n, \ldots$ is equal to 2^n.

b If the second member of a row is a prime number, all the numbers in that row excluding the 1s are divisible by it.

c [The hockey-stick pattern] Start at the top 1 of any column of Pascal's triangle, and go vertically down any number of rows. Then the sum of these numbers is the number in the next row down and to the right. For example, if you start at the 1 in the x^2 column, and add downwards:

$$1 + 3 + 6 + 10 + 15 + 21 = 56, \quad \text{which is one row down in the } x^3 \text{ column.}$$

d [The powers of 11] If a row is made into a single number by using each element as a digit of the number, the number is a power of 11 (except that after the row 1, 4, 6, 4, 1, the pattern gets confused by carrying).

e Find the diagonal and the column containing the triangular numbers, and show that adding adjacent pairs gives the square numbers.

CHALLENGE

18 By writing $(1 + x + 3x^2)^6$ as $(1 + A)^6$, where $A = x + 3x^2$, expand $(1 + x + 3x^2)^6$ as far as the term in x^3. Hence evaluate $(1.0103)^6$ to four decimal places.

19 [The Pascal pyramid] By considering the expansion of $(1 + x + y)^n$, where $0 \le n \le 4$, calculate the first five layers of the Pascal pyramid.

18B Binomial expansions with several variables

Learning intentions

- Understand that a polynomial, and a binomial, can have more than one variable.
- Understand that a binomial expansion $(x + y)^n$ is homogeneous in x and y.
- Understand that the coefficients of $(x + y)^n$ also come from Pascal's triangle.
- Solve problems with general binomial expansions.

The more general case of the expansion of $(x + y)^n$ is similar. Because x and y are both variables, however, the symmetries of the expansion will be more obvious.

A minor point about language — we are now widening the term *polynomial* to include terms made up of powers of any number of variables. Thus $5xy^4$ is a monomial, $x + y$ is a binomial, and $x^2 + 2xy + y^2$ and $x + y + zt$ are trinomials.

Some expansions of $(x + y)^n$

Here are the expansions of $(x + y)^n$ for low values of n. Again, the calculations have been carried out with like terms written in the same column so that the addition property is clear — see the boxed coefficients.

$$\begin{aligned}
(x+y)^0 &= 1\\
(x+y)^1 &= x+y\\
(x+y)^2 &= x(x+y)+y(x+y)\\
&= x^2 + xy\\
&\quad + xy + y^2\\
&= x^2 + 2xy + y^2
\end{aligned}$$

$$\begin{aligned}
(x+y)^3 &= x(x+y)^2 + y(x+y)^2\\
&= x^3 + 2x^2y + xy^2\\
&\quad + x^2y + 2xy^2 + y^3\\
&= x^3 + 3x^2y + 3xy^2 + y^3\\
(x+y)^4 &= x(x+y)^3 + y(x+y)^3\\
&= x^4 + 3x^3y + 3x^2y^2 + \boxed{1}\,xy^3\\
&\quad + x^3y + 3x^2y^2 + \boxed{3}\,xy^3 + y^4\\
&= x^4 + 4x^3y + 6x^2y^2 + \boxed{4}\,xy^3 + y^4
\end{aligned}$$

First, the coefficients in the expansions are exactly the same as before, so as before they can be taken from Pascal's triangle. This is easily proven by substituting $y = 1$ — this turns the expansion of $(x + y)^n$ into the expansion of $(x + 1)^n$.

Secondly, the pattern for the indices of x and y is straightforward.

- The expansion of $(x + y)^3$ has four terms, and in each term the indices of x and y are whole numbers adding to 3.
- Similarly the expansion of $(x + y)^4$ has five terms, and in each term the indices of x and y are whole numbers adding to 4.

More generally, in each successive expansion, beginning with $(x + y)^0 = 1$, the terms of the previous expansion are multiplied first by x, and then by y, so the sum of the two indices goes up by 1.

2 The expansion of $(x + y)^n$ is homogeneous of degree n in x and y together

- The expansion of $(x + y)^n$ has $n + 1$ terms, and in each term the indices of x and y are whole numbers adding to n.
  - ▷ The expansion of $(x + y)^n$ is called *homogeneous of degree n in x and y together*, because in each term, the sum of the indices of x and y is n.
- The coefficients in the expansion of $(x + y)^n$ are the same as the coefficients in the expansion of $(1 + x)^n$. Proof: Substitute $y = 1$.

Using the general expansion

The general expansion of $(x + y)^n$ is applied similarly to the expansion of $(1 + x)^n$.

Example 6 Expanding a general binomial

Use Pascal's triangle to write out the expansions of:

a $(2 - 3x)^4$

b $(5x + \frac{1}{5}a)^5$

Solution

a
$$\begin{aligned}(2 - 3x)^4 &= 2^4 + 4 \times 2^3 \times (-3x) + 6 \times 2^2 \times (-3x)^2 \\ &\quad + 4 \times 2 \times (-3x)^3 + (-3x)^4 \\ &= 16 - 96x + 216x^2 - 216x^3 + 81x^4\end{aligned}$$

b
$$\begin{aligned}(5x + \tfrac{1}{5}a)^5 &= (5x)^5 + 5 \times (5x)^4 \times \tfrac{1}{5}a + 10 \times (5x)^3 \times (\tfrac{1}{5}a)^2 \\ &\quad + 10 \times (5x)^2 \times (\tfrac{1}{5}a)^3 + 5 \times (5x) \times (\tfrac{1}{5}a)^4 + (\tfrac{1}{5}a)^5 \\ &= 3125x^5 + 625ax^4 + 50a^2x^3 + 2a^3x^2 + \tfrac{1}{25}a^4x + \tfrac{1}{3125}a^5\end{aligned}$$

Example 7 Finding particular terms in a binomial expansion

Use Pascal's triangle to write out the expansion of $(2x + x^{-2})^6$, leaving the terms unsimplified. Hence find:

a the term independent of x,

b the term in x^{-3}.

Solution

$$\begin{aligned}(2x + x^{-2})^6 &= (2x)^6 + 6 \times (2x)^5 \times (x^{-2}) + 15 \times (2x)^4 \times (x^{-2})^2 \\ &\quad + 20 \times (2x)^3 \times (x^{-2})^3 + 15 \times (2x)^2 \times (x^{-2})^4 + 6 \times (2x) \times (x^{-2})^5 + (x^{-2})^6\end{aligned}$$

a Constant term

$$\begin{aligned}&= 15 \times (2x)^4 \times (x^{-2})^2 \\ &= 15 \times 2^4 \times x^4 \times x^{-4} \\ &= 240\end{aligned}$$

b Term in x^{-3}

$$\begin{aligned}&= 20 \times (2x)^3 \times (x^{-2})^3 \\ &= 20 \times 2^3 \times x^3 \times x^{-6} \\ &= 160x^{-3}\end{aligned}$$

Example 8 Finding a term in a binomial expansion

Expand $(2-3x)^7$ as far as the term in x^2, and hence find the term in x^2 in the expansion of $(5+x)(2-3x)^7$.

Solution

$$(2-3x)^7 = 2^7 + 7\times 2^6\times(-3x) + 21\times 2^5\times(-3x)^2 + \cdots$$
$$= 128 - 1344\,x + 6048\,x^2 - \cdots.$$

Hence the term in x^2 in the expansion of $(5+x)(2-3x)^7$

$$= 5\times 6048\,x^2 - x\times 1344\,x$$
$$= 28\,896\,x^2.$$

Exercise 18B

FOUNDATION

1 Use Pascal's triangle to expand:

a $(x+y)^4$ **b** $(x-y)^4$ **c** $(r-s)^6$ **d** $(p+q)^{10}$
e $(a-b)^9$ **f** $(2x+y)^5$ **g** $(p-2q)^7$ **h** $(3x+2y)^4$
i $(a-\frac{1}{2}b)^3$ **j** $(\frac{1}{2}r+\frac{1}{3}s)^5$ **k** $\left(x+\dfrac{1}{x}\right)^6$

2 Use Pascal's triangle to expand:

a $(1+x^2)^4$ **b** $(1-3x^2)^3$ **c** $(x^2+2y^3)^6$
d $\left(x-\dfrac{1}{x}\right)^9$ **e** $(\sqrt{x}+\sqrt{y}\,)^7$ **f** $\left(\dfrac{2}{x}+3x^2\right)^5$

3 Simplify without expanding the brackets:

a $y^5+5y^4(x-y)+10y^3(x-y)^2+10y^2(x-y)^3+5y(x-y)^4+(x-y)^5$
b $a^4-4a^3(a-b)+6a^2(a-b)^2-4a(a-b)^3+(a-b)^4$
c $x^3+3x^2(2y-x)+3x(2y-x)^2+(2y-x)^3$
d $(x+y)^6-6(x+y)^5(x-y)+15(x+y)^4(x-y)^2-20(x+y)^3(x-y)^3+15(x+y)^2(x-y)^4$
$-6(x+y)(x-y)^5+(x-y)^6$

4 **a** **i** Expand $(4+x)^5$ as far as the term in x^3.
ii Hence find the coefficient of x^3 in the expansion of $(3-x)(4+x)^5$.
b **i** Expand $(1-2x)^6$ as far as the term in x^4.
ii Hence find the coefficient of x^4 in the expansion of $(1-3x)(1-2x)^6$.
c **i** Expand $(3-y)^7$ as far as the term in y^4.
ii Hence find the coefficient of y^4 in the expansion of $(1-y)^2(3-y)^7$.

DEVELOPMENT

5 **a** Expand and simplify $(x+y)^6+(x-y)^6$.
b Hence (and without a calculator) prove that

$$5^6+5^5\times3^3+5^3\times3^5+3^6=2^5(2^{12}+1).$$

6 Find the coefficient of:

a x^3 in $(2-5x)(x^2-3)^4$ **b** x^5 in $(x^2-3x+11)(4+x^3)^3$
c x^0 in $(3-2x)^2\left(x+\dfrac{2}{x}\right)^5$ **d** x^9 in $(x+2)^3(x-2)^7$

7 **a** **i** Use Pascal's triangle to expand $(x+h)^3$.

ii If $f(x) = x^3$, simplify $f(x+h) - f(x)$.

iii Hence use the definition $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$ to differentiate x^3.

b Similarly, differentiate x^5 from first principles.

8 **a** Show that $(3+\sqrt{5})^6 + (3-\sqrt{5})^6 = 20\,608$.

b Show that $(2+\sqrt{7})^4 + (2-\sqrt{7})^4$ is rational.

c If $(\sqrt{6}+\sqrt{3})^3 - (\sqrt{6}-\sqrt{3})^3 = a\sqrt{3}$, where a is an integer, find the value of a.

9 Show that $\dfrac{1}{(\sqrt{3}-1)^4} + \dfrac{1}{(\sqrt{3}+1)^4} = \dfrac{(\sqrt{3}+1)^4 + (\sqrt{3}-1)^4}{(\sqrt{3}-1)^4(\sqrt{3}+1)^4}$ by putting the LHS over a common denominator. Then simplify the expression using Pascal's triangle.

10 Expand $(x+2y)^5$ and hence evaluate:

a $(1.02)^5$ correct to to five decimal places,

b $(0.98)^5$ correct to to five decimal places.

11 **a** Expand:

i $\left(x + \dfrac{1}{x}\right)^3$ **ii** $\left(x + \dfrac{1}{x}\right)^5$ **iii** $\left(x + \dfrac{1}{x}\right)^7$

b Hence, if $x + \dfrac{1}{x} = 2$, evaluate:

i $x^3 + \dfrac{1}{x^3}$ **ii** $x^5 + \dfrac{1}{x^5}$ **iii** $x^7 + \dfrac{1}{x^7}$

12 Find the coefficients of x and x^{-3} in the expansion of $\left(3x - \dfrac{a}{x}\right)^5$. Hence find the values of a if these coefficients are in the ratio 2 : 1.

13 The coefficients of the terms in a^3 and a^{-3} in the expansion of $\left(ma + \dfrac{n}{a^2}\right)^6$ are equal, where m and n are non-zero real numbers. Prove that $m^2 : n^2 = 10 : 3$.

CHALLENGE

14 **a** Expand $\left(x + \dfrac{1}{x}\right)^6$.

b If $U = x + \dfrac{1}{x}$, express $x^6 + \dfrac{1}{x^6}$ in the form $U^6 + AU^4 + BU^2 + C$. State the values of A, B and C.

15 **a** By starting with $((x+y)+z)^3$, expand $(x+y+z)^3$.

b Find the term independent of x in the expansion of $(x + 1 + x^{-1})^4$.

16 [The Sierpinski triangle fractal]

a Draw an equilateral triangle of side length 1 unit on a piece of white paper. Join the midpoints of the sides of this triangle to form a smaller triangle. Colour it black. Repeat this process on all white triangles that remain. What do you notice?

b Draw up Pascal's triangle in the shape of an equilateral triangle, then colour all the even numbers black and leave the odd numbers white. What do you notice? This pattern will be more evident if you take at least the first 16 rows — perhaps use a computer program to generate 100 rows of Pascal's triangle.

18C The binomial theorem

Learning intentions

- Prove the binomial theorem using combinatorics.
- Apply the binomial theorem to problems with binomial expansions.
- Apply the binomial theorem to problems with binomial expansions.

The reader has probably realised already that the entries in Pascal's triangle are exactly the numbers ${}^nC_r$ that arose in the previous chapter when counting r-member subsets of an n-member set. Once this has been proven below, we can write the general binomial expansion out explicitly as

$$(x+y)^n = {}^nC_0\, x^n + {}^nC_1\, x^{n-1}y + {}^nC_2\, x^{n-2}y^2 + \cdots + {}^nC_n\, y^n,$$

or using the alternative notation $\binom{n}{r}$ for 'n choose r',

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n}y^n,$$

This result is known as the *binomial theorem*. It gives a striking connection between the counting methods of the previous chapter and the algebraic structures in this chapter, showing yet again the unity between algebra and arithmetic.

Proving the binomial theorem

We have already established that

$$(x+y)^n = *x^n + *x^{n-1}y + *x^{n-2}y^2 + \cdots + *y^n,$$

where $*$ denotes the different coefficients. What remains to be proven is that the coefficient of $x^{n-r}y^r$ is ${}^nC_r$. As usual, the most straightforward way to establish this is by looking at how things work in a special case. For example, let us establish why the coefficient of x^2y^2 in the expansion of $(x+y)^4$ is ${}^4C_2 = 6$.

Expanding $(x+y)^4 = (x+y)(x+y)(x+y)(x+y)$ without collecting like terms, or even applying the commutative law of multiplication, gives $2^4 = 16$ terms:

$(x+y)^4 = xxxx$ (the term with 4 x s)

$+ xxxy + xxyx + xyxx + yxxx$ (the terms with 3 x s and 1 y)

$+ xxyy + xyxy + yxxy + xyyx + yxyx + yyxx$ (2 x s and 2 y s)

$+ xyyy + yxyy + yyxy + yyyx$ (the terms with 1 x and 3 y s)

$+ yyyy$ (the term with 4 y s)

which then becomes $x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$ after collecting like terms.

The middle row above shows that the coefficient of x^2y^2 is 6 because there are six terms with two x s and two y s. Why are there six terms? The reason is:

- these six terms are all the words formed with 4 letters, 2 alike of one kind, and 2 of another.

We showed in the last chapter that the number of such words is $\dfrac{4!}{2! \times 2!} = 6$.

More generally, when we expand $(x+y)^n$, the coefficient of x^ry^{n-r} is the number of words formed with n letters, r alike of one kind and $n-r$ alike of another.

Now we can bring in the theory of the last chapter, where we used the symbol ${}^nC_r$ for the number of r-member subsets of an n-member set.

Otherwise expressed, this is the number of unordered selections of r objects chosen from n objects. We showed there that

$$^n\mathrm{C}_r = \frac{n!}{r! \times (n-r)!},$$

and we also showed that this was equal to the number of n-letter words with r letters alike of one kind and $n-r$ letters alike of another. This means that the coefficient of $x^r y^{n-r}$ can be written as $^n\mathrm{C}_r$, giving a concise form of the expansion:

3 The binomial theorem, and a formula for the entries in Pascal's triangle

- For all whole numbers n,

$$(x+y)^n = {}^n\mathrm{C}_0\, x^n + {}^n\mathrm{C}_1\, x^{n-1}y + {}^n\mathrm{C}_2\, x^{n-2}y^2 + \cdots + {}^n\mathrm{C}_n\, y^n,$$

where $^n\mathrm{C}_r = \dfrac{n!}{r!\,(n-r)!}$ = number of r-member subsets of an n-member set, for $r = 0, 1, \ldots, n$.
- This gives a notation and a formula for each binomial expansion coefficient.
- We now have two interpretations of Pascal's triangle:
  - ▷ It is a table of coefficients in the binomial expansion.
  - ▷ It is a table of the number of r-member subsets of an n-member set.

We can also write $^n\mathrm{C}_r = \dfrac{n \times (n-1) \times \cdots \times (n-r+1)}{1 \times 2 \times \cdots \times r}$.

Remember too that the notations $^n\mathrm{C}_r$ and $\binom{n}{r}$ mean exactly the same thing. But don't mix the two notations up in the same problem because it looks dreadful.

Examples of the binomial theorem

We calculated $^n\mathrm{C}_r$ in Section 17E. These examples use the formula for $^n\mathrm{C}_r$ to calculate coefficients in binomial expansions.

Example 9 **Using the binomial theorem to calculate terms and coefficients**

Use the binomial theorem to calculate:

a the coefficient of x^8 in $(1+x)^{12}$,

b the term in x^3y^4 in $(x+y)^7$,

c the coefficient of A^5B^5 in $(2A-3B)^{10}$ (factored into primes),

d The term in $a^{n-4}b^4$ in $(a-2b)^n$.

Solution

a

$$\begin{aligned}(1+x)^{12} &= \cdots + {}^{12}\mathrm{C}_8 \times 1^4 \times x^8 + \cdots, \\ \text{so coefficient of } x^8 &= \frac{12!}{8! \times 4!} \\ &= \frac{12 \times 11 \times 10 \times 9}{4 \times 3 \times 2 \times 1} \\ &= 495.\end{aligned}$$

b
$$(x+y)^7 = \cdots + \binom{7}{4} \times x^3 \times y^4 + \cdots \text{ (alternative notation for } {}^n\mathrm{C}_r\text{)},$$
$$\begin{aligned}\text{so term in } x^3y^4 &= \frac{7!}{4! \times 3!}\, x^3y^4 \\ &= \frac{7\times 6\times 5}{3\times 2\times 1}\, x^3y^4 \\ &= 35\, x^3y^4.\end{aligned}$$

c
$$(2A-3B)^{10} = \cdots + {}^{10}\mathrm{C}_5 \times (2A)^5 \times (-3B)^5 + \cdots,$$
$$\begin{aligned}\text{so coefficient of } A^5B^5 &= -\frac{10!}{5! \times 5!} \times 2^5 \times 3^5 \\ &= -\frac{10\times 9\times 8\times 7\times 6}{5\times 4\times 3\times 2\times 1} \times 2^5 \times 3^5 \\ &= -9 \times 4 \times 7 \times 2^5 \times 3^5 \\ &= -2^7 \times 3^7 \times 7.\end{aligned}$$

d
$$(a-2b)^n = \cdots + {}^n\mathrm{C}_4 \times a^{n-4} \times (-2b)^4 + \cdots,$$
$$\begin{aligned}\text{so term in } a^{n-4}b^4 &= \frac{n!}{4! \times (n-4)!} \times 2^4 \times a^{n-4}b^4 \\ &= \frac{n\times(n-1)\times(n-2)\times(n-3)}{4\times 3\times 2\times 1} \times 2^4 \times a^{n-4}b^4 \\ &= \frac{2n(n-1)(n-2)(n-3)}{3}\, a^{n-4}b^4.\end{aligned}$$

Some important values of ${}^n\mathrm{C}_r$, and the symmetry of the rows of Pascal's triangle

The formulae for ${}^n\mathrm{C}_r$ for $n = 0$, 1 and 2 are important enough to be memorised:

$$\begin{aligned}{}^n\mathrm{C}_0 &= \frac{n!}{0!\; n!} \\ &= 1\end{aligned} \qquad \begin{aligned}{}^n\mathrm{C}_1 &= \frac{n!}{1!\; (n-1)!} \\ &= n\end{aligned} \qquad \begin{aligned}{}^n\mathrm{C}_2 &= \frac{n!}{2!\; (n-2)!} \\ &= \tfrac{1}{2}n(n-1)\end{aligned}$$

We saw in Chapter 17 that ${}^n\mathrm{C}_r = {}^n\mathrm{C}_{n-r}$ for $r = 0, 1, 2, \ldots, n$. It follows therefore that ${}^n\mathrm{C}_n = 1$, ${}^n\mathrm{C}_{n-1} = n$ and ${}^n\mathrm{C}_{n-2} = \frac{1}{2}n(n-1)$.

4 Some important values of ${}^n\mathrm{C}_r$, and the row symmetry of Pascal's triangle

- For all whole numbers n,
$$\begin{aligned}{}^n\mathrm{C}_0 = {}^n\mathrm{C}_n &= 1,\\ {}^n\mathrm{C}_1 = {}^n\mathrm{C}_{n-1} &= n,\\ {}^n\mathrm{C}_2 = {}^n\mathrm{C}_{n-2} &= \tfrac{1}{2}n(n-1).\end{aligned}$$
  In particular, all the rows of Pascal's triangle begin and end with 1.
- For all whole numbers n, and for $r = 0, 1, 2, \ldots, n$,
$${}^n\mathrm{C}_r = {}^n\mathrm{C}_{n-r}.$$
  That is, the rows are reversible.

Check all this on your copy of Pascal's triangle.

Example 10 Binomial expansion problems on finding a pronumerals

Find the value of n if:

a ${}^nC_2 = 55$

b ${}^nC_2 + {}^nC_1 + {}^nC_0 = 29$

Solution

We know that ${}^nC_0 = 1$ and ${}^nC_1 = n$ and ${}^nC_2 = \frac{1}{2}n(n-1)$.

a
$$\frac{1}{2}n(n-1) = 55$$
$$n^2 - n - 110 = 0$$
$$(n-11)(n+10) = 0$$
Because $n \geq 0$, $n = 11$.

b
$${}^nC_2 + {}^nC_1 + {}^nC_0 = 29$$
$$\frac{1}{2}n(n-1) + n + 1 = 29$$
$$n^2 - n + 2n + 2 = 58$$
$$n^2 + n - 56 = 0$$
$$(n-7)(n+8) = 0$$
Because $n \geq 0$, $n = 7$.

Exercise 18C

FOUNDATION

1 Use the result ${}^nC_r = \dfrac{n!}{r!\,(n-r)!}$ to evaluate the following. Do not use a calculator — you will need to unroll the factorial symbol. Check your answers against Pascal's triangle.

a ${}^4C_3$ **b** ${}^6C_3$ **c** ${}^9C_1$ **d** ${}^7C_3$

2 Repeat the previous question for these binomial coefficients. Remember the alternative notation $\binom{n}{r} = {}^nC_r$.

a $\binom{5}{2}$ **b** $\binom{8}{8}$ **c** $\binom{11}{10}$ **d** $\binom{10}{4}$

3 Use your calculator button ${}^nC_r$ to evaluate:

a ${}^{15}C_{10}$ **b** $\binom{13}{5}$ **c** ${}^{12}C_7$

d $\binom{12}{3} \div \binom{5}{2}$ **e** $\dfrac{{}^{15}C_8}{{}^6C_4}$ **f** $\dfrac{{}^{19}C_6}{{}^7C_5}$

4 **a** Expand $(1+x)^4$, and hence write down the values of ${}^4C_0$, ${}^4C_1$, ${}^4C_2$, ${}^4C_3$ and ${}^4C_4$.

b Hence find:

i ${}^4C_0 + {}^4C_1 + {}^4C_2 + {}^4C_3 + {}^4C_4$

ii ${}^4C_0 - {}^4C_1 + {}^4C_2 - {}^4C_3 + {}^4C_4$

5 Use the values of ${}^nC_r$ from Pascal's triangle to find:

a ${}^6C_0 + {}^6C_2 + {}^6C_4 + {}^6C_6$

b ${}^6C_1 + {}^6C_3 + {}^6C_5$

c ${}^2C_2 + {}^3C_2 + {}^4C_2 + {}^5C_2$

d $({}^5C_0)^2 + ({}^5C_1)^2 + ({}^5C_2)^2 + ({}^5C_3)^2 + ({}^5C_4)^2 + ({}^5C_5)^2$

6 **a** Evaluate:

i $\binom{8}{3}$ and $\binom{8}{5}$,

ii $\binom{7}{4}$ and $\binom{7}{3}$.

b If $\binom{n}{3} = \binom{n}{2}$, find the value of n.

c If $\binom{12}{4} = \binom{12}{n}$, find the value of n.

7 **a** By evaluating the LHS and RHS, verify the following result for $n = 8$ and $r = 3$:

$$\binom{n}{r-1} + \binom{n}{r} = \binom{n+1}{r}$$

b Use this identity to solve each equation for n.

i $\binom{5}{3} + \binom{5}{4} = \binom{n}{4}$ **ii** $\binom{n}{7} + \binom{n}{8} = \binom{11}{8}$

8 Find the specified term or coefficient in each expansion.

a For $(1 + x)^{18}$, find the term in x^5.

b For $(1 + 6x)^{12}$, find the term in x^9.

c For $(1 - 7x)^8$, find the term in x^3.

d For $(1 + 3x)^7$, find the coefficient of x^2.

9 Find the specified terms in each expansion.

a For $(2 + x)^7$, find the term in x^2.

b For $(x + \frac{1}{2}y)^{14}$, find the term in x^9y^5.

c For $(\frac{1}{2}x - 3y^2)^{11}$, find the term in $x^{10}y^2$.

d For $(a - b^{\frac{1}{2}})^{20}$, find the term in a^2b^9.

10 **a** Find $x \neq 0$ if the terms in x^{10} and x^{11} in the expansion of $(5 + 2x)^{15}$ are equal.

b Find $x \neq 0$ if the terms in x^{13} and x^{14} in the expansion of $(2 - 3x)^{17}$ are equal.

DEVELOPMENT

11 **a** In the expansion of $(1 + x)^{16}$, find the ratio of the term in x^{13} to the term in x^{11}.

b Find the ratio of the coefficients of x^{14} and x^5 in the expansion of $(1 + x)^{20}$.

c In the expansion of $(2 + x)^{18}$, find the ratio of the coefficients of x^{10} and x^{16}.

12 **a** Use the binomial theorem to obtain formulae for:

i ${}^nC_0$ **ii** ${}^nC_1$ **iii** ${}^nC_2$ **iv** ${}^nC_3$

b Hence solve each equation for n.

i ${}^9C_2 - {}^nC_1 = {}^6C_3$ **ii** ${}^nC_2 = 36$ **iii** ${}^nC_2 + {}^6C_2 = {}^7C_2$

iv ${}^nC_2 + {}^nC_1 = 22 - {}^nC_0$ **v** ${}^nC_1 + {}^nC_2 = {}^5C_2$ **vi** ${}^nC_3 + {}^nC_2 = 8\,{}^nC_1$

c Use the formula for ${}^nC_2$ to show that ${}^nC_2 + {}^{n+1}C_2 = n^2$, and verify the result on the third column of Pascal's triangle.

13 The expression $(1 + ax)^n$ is expanded in increasing powers of x. Find the values of a and n if the first three terms are:

a $1 + 28x + 364x^2 + \cdots$ **b** $1 - \frac{10}{3}x + 5x^2 - \cdots$

14 **a** In the expansion of $(2 + 3x)^n$, the coefficients of x^5 and x^6 are in the ratio 4 : 9. Find the value of n.

b In the expansion of $(1 + 3x)^n$, the coefficients of x^8 and x^{10} are in the ratio 1 : 2. Find the value of n.

15 In the expansion of $\left(x + \frac{1}{x}\right)^{40}\left(x - \frac{1}{x}\right)^{40}$, find the term independent of x. Give your answer in the form ${}^nC_r$, and also correct to four significant figures.

16 [Divisibility problems]

a Use the binomial theorem to show that $7^n + 2$ is divisible by 3, where n is a positive integer. (Hint: Write $7 = 6 + 1$.)

b Use the binomial theorem to show that $5^n + 3$ is divisible by 4, where n is a positive integer.

c Suppose that b, c and n are positive integers, and $a = b + c$. Use the binomial expansion of $(b + c)^n$ to show that $a^n - b^{n-1}(b + cn)$ is divisible by c^2. Hence show that $5^{42} - 2^{48}$ is divisible by 9.

17 **a** Use the binomial theorem to expand $(x + h)^n$.

b Hence use the definition $f'(x) = \lim_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$ to differentiate x^n from first principles.

CHALLENGE

18 Use the formula ${}^nC_r = \dfrac{n!}{r!(n-r)!}$ to prove that if the second member of a row is a prime number, all the numbers in that row, excluding the 1s, are divisible by it.

19 [These geometrical results should be related to the numbers in Pascal's triangle.]

a Place three points on the circumference of a circle. How many line segments and triangles can be formed using these three points?

b Place four points on the circumference of a circle. How many segments, triangles and quadrilaterals can be formed using these four points?

c What happens if five points are placed on the circle?

d How many pentagons could you form if you placed seven points on the circumference of a circle?

20 [A more general form of the binomial theorem]

a Show that the binomial expansion can be written as

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots.$$

b In this form, it can be shown that the expansion is true for negative or fractional values of n, provided that the RHS is regarded as the limit of an infinite sum of powers of x. This is called the *power series expansion* of $(1 + x)^n$. Assuming this, generate the binomial expansions of:

i $\dfrac{1}{1-x}$ **ii** $\dfrac{1}{(1-x)^2}$ **iii** $\dfrac{1}{(1+x)^2}$ **iv** $\sqrt{1+x}$

c Verify, using your expansions in parts (b)(i) and (ii), that $\dfrac{d}{dx}\left(\dfrac{1}{1-x}\right) = \dfrac{1}{(1-x)^2}$. (This assumes that a power series can be differentiated term-by-term.)

18D Using the general term

Learning intentions

- Identify and use the general term of a binomial expansion.

This section involves writing down and using the general term of a binomial expansion. This makes it easier to identify particular terms, and to compare two expansions, both of which are required in the final two sections.

The general term of a binomial expansion

The *general term* of a binomial expansion is an algebraic expression that is a function of r, so that every term is obtained by substituting the appropriate value of r. For example, the general term of the standard binomial expansion

$$(x+y)^n = {}^nC_0\,x^n + {}^nC_1\,x^{n-1}y + {}^nC_2\,x^{n-2}y^2 + \cdots + {}^nC_n\,y^n$$

is ${}^nC_r x^{n-r}y^r$, or we can choose to use ${}^nC_r x^r y^{n-r}$.

5 The general term of a binomial expansion

$$(x+y)^n = {}^nC_0\,x^n + {}^nC_1\,x^{n-1}y + {}^nC_2\,x^{n-2}y^2 + \cdots + {}^nC_n\,y^n$$

has general term ${}^nC_r x^{n-r}y^r$, or we can choose to use ${}^nC_r x^r y^{n-r}$.

Example 11 Finding and using the general term

a Find the general term in the expansion of $(x^2+3x)^7$.

b In the answers to these three parts, give each coefficient factored into primes:

i Find the term in x^{12}. **ii** Find the term in x^{10}. **iii** Find the term in x^6.

Solution

a General term
$$\begin{aligned} &= {}^7C_r \times (x^2)^{7-r} \times (3x)^r \\ &= {}^7C_r \times x^{14-2r} \times 3^r \times x^r \\ &= {}^7C_r \times 3^r \times x^{14-r}. \end{aligned}$$

b **i** To obtain the term in x^{12}, put $14 - r = 12$

$r = 2.$

Hence the term in x^{12}
$$\begin{aligned} &= {}^7C_2 \times 3^2 \times x^{12} \\ &= \frac{7\times 6}{1\times 2} \times 3^2 \times x^{12} \\ &= 3^3 \times 7 \times x^{12}. \end{aligned}$$

ii To obtain the term in x^{10}, put $14 - r = 10$

$r = 4.$

Hence the term in x^{10}
$$\begin{aligned} &= {}^7C_4 \times 3^4 \times x^{10} \\ &= \frac{7\times 6\times 5}{1\times 2\times 3} \times 3^4 \times x^{10} \\ &= 3^4 \times 5 \times 7 \times x^{10}. \end{aligned}$$

iii To find the term in x^6, put $14 - r = 6$

$r = 8.$

This is impossible, because evaluating ${}^7C_r$ requires $0 \le r \le 7$, so there is no such term (or alternatively, the term in x^6 is zero).

Example 12 Finding and using the general term

a Find the general term in the expansion of $(2x^2 - x^{-1})^9$.
b Find the term in x^{12}, giving the coefficient factored into primes.
c Find the constant term.
d Find the term of lowest index (with non-zero coefficient).

Solution

a General term $= {}^9C_r \times (2x^2)^{9-r} \times (-x^{-1})^r$
$= (-1)^r \times {}^9C_r \times 2^{9-r} \times x^{18-2r} \times x^{-r}$
$= (-1)^r \times {}^9C_r \times 2^{9-r} \times x^{18-3r}$.

b To obtain the term in x^{12}, put $18 - 3r = 12$
$r = 2$.

Hence the term in $x^{12} = (-1)^2 \times {}^9C_2 \times 2^7 \times x^{12}$
$= \dfrac{9 \times 8}{1 \times 2} \times 2^7 \times x^{12}$
$= 2^9 \times 3^2 \times x^{12}$.

(It is often best to leave large numbers factored into primes, as in parts (b) and (c).)

c To obtain the constant term, put $18 - 3r = 0$
$r = 6$.

Hence constant term $= (-1)^6 \times {}^9C_6 \times 2^3 \times x^0$
$= +\dfrac{9 \times 8 \times 7}{1 \times 2 \times 3} \times 2^3$
$= 2^5 \times 3 \times 7$

d The lowest possible index is when $r = 9$ (look at ${}^9C_r$), and this term is
$(-1)^9 \times {}^9C_9 \times 2^0 \times x^{18-27} = -x^{-9}$,
which is obvious by simply looking at the original expansion.

Exercise 18D

FOUNDATION

1 Write down the general term for each binomial expansion.

a $(1 + x)^{13}$ **b** $(1 + 2x)^7$ **c** $(5 + 7x)^{12}$
d $(2x - y)^9$ **e** $(x + 2x^{-1})^5$ **f** $\left(6x - \dfrac{2}{x}\right)^8$

2 Consider the expansion $\left(x^2 + \dfrac{1}{x}\right)^9$.

a Show that each term in the expansion of $\left(x^2 + \dfrac{1}{x}\right)^9$ can be written as ${}^9C_i\, x^{18-3i}$.
b Hence find the coefficients of:
i x^3 **ii** x^{-3} **iii** x^0

3 In the expansion of $(2x^3 + 3x^{-2})^{10}$, the general term is ${}^{10}C_k \left(2x^3\right)^{10-k} \left(3x^{-2}\right)^k$.

a Show that this general term can be written as ${}^{10}C_k\, 2^{10-k}\, 3^k\, x^{30-5k}$.

b Hence find the coefficients of these terms, giving your answers factored into primes:

i x^{10} **ii** x^{-5} **iii** x^0

4 **a** Show that the general term in the expansion of $\left(\frac{x}{2} - \frac{5}{x}\right)^{15}$ can be written as

$${}^{15}C_j\,(-1)^j\,5^j\,2^{j-15}\,x^{15-2j}.$$

b Hence find, without simplifying, the coefficients of:

i x^{11} **ii** x **iii** x^{-5}

DEVELOPMENT

5 Find the term independent of x in each expansion.

a $\left(x + \frac{3}{x}\right)^8$ **b** $\left(2x^3 - \frac{1}{x}\right)^{12}$ **c** $(5x^4 - 2x^{-1})^{10}$ **d** $(ax^{-2} + \frac{1}{2}x)^6$

6 Find the coefficient of the power of x specified in each expansion.

a x^{15} in $\left(x^3 - \frac{2}{x}\right)^9$ **b** x^{-14} in $(x - 3x^{-4})^{11}$

c the constant term in $\left(\frac{1}{2x^3} + x\right)^{20}$ **d** x^7 in $\left(5x^2 + \frac{1}{2x}\right)^8$

e x^{-1} in $(x^{-1} + \frac{2}{7}x)^5$

7 Determine the coefficients of the specified term in each expansion.

a For $(3 + x)(1 - x)^{15}$, find the coefficient of the term in x^4.

b For $(2 - 5x + x^2)(1 + x)^{11}$, find the coefficient of the term in x^9.

c For $(x - 3)(x + 2)^{15}$, find the coefficient of the term in x^{12}.

d For $(1 - 2x - 4x^2)\left(1 - \frac{3}{x}\right)^9$, find the coefficient of the term in x^0.

8 **a** Find the coefficient of x in the expansion of $\left(x + \frac{1}{x}\right)^5 \left(x - \frac{1}{x}\right)^4$.

b Find the coefficient of x^2 in the expansion of $\left(x - \frac{1}{x}\right)^9 \left(x + \frac{1}{x}\right)^5$.

c Find the coefficient of y^{-3} in the expansion of $\left(y + \frac{1}{y}\right)^{10} \left(y - \frac{1}{y}\right)^7$.

9 **a** In the expansion of $(2 + ax + bx^2)(1 + x)^{13}$, the coefficients of x^0, x^1 and x^2 are all equal to 2. Find the values of a and b.

b In the expansion of $(1 + x)^n$, the coefficient of x^5 is 1287. Find the value of n by trial and error, and hence find the coefficient of x^{10}.

CHALLENGE

10 Show that there will always be a term independent of x in the expansion of $\left(x^p + \dfrac{1}{x^{2p}}\right)^{3n}$, where n is a positive integer, and find that term.

11 **a** Write down the term in x^r in the expansion of $(a - bx)^{12}$.

b In the expansion of $(1 + x)(a - bx)^{12}$, the coefficient of x^8 is zero. Find the value of the ratio $\dfrac{a}{b}$ in simplest form.

12 **a** By writing it as $\left((1 - x) + x^2\right)^4$, expand $(1 - x + x^2)^4$ in ascending powers of x as far as the term containing x^4.

b In the expansion of $(1 + 3x + ax^2)^n$, where n is a positive integer, the coefficient of x^2 is 0. Find, in terms of n:

i the value of a,

ii the coefficient of x^3.

18E Identities in Pascal's triangle

Learning intentions

- Prove a Pascal's triangle identity by substituting into a binomial expansion.
- Prove a Pascal's triangle identity by equating coefficients in a binomial expansion.

There are a great number of patterns in Pascal's triangle. Some are quite straightforward to recognise and to prove, others are more complicated. They are very important in all applications of the binomial theorem, and in particular, some will be important in Year 12 binomial distributions.

Note: Here is where you absolutely need to refer constantly to the Pascal triangle up to $n = 12$ that you calculated in the first question of Exercise 18A. This section and the next are all about searching for patterns in the triangle.

Identities in Pascal's triangle — their identification and their proofs

Each pattern in Pascal's triangle is described by an identity on the binomial coefficients ${}^nC_r$. These identities sometimes have a rather forbidding appearance, and it is important to take the time to interpret each identity as some sort of pattern in Pascal's triangle. Use small values of n such as $n = 3$, $n = 4$, and $n = 5$ to work out what the identity is saying.

Here again is the first part of Pascal's triangle formed by the numbers ${}^nC_r$. Each identity that is obtained should be interpreted as a pattern in the triangle and verified there, either before or after the proof is completed.

$n\backslash r$	0	1	2	3	4	5	6	7	8	9	10
0	1										
1	1	1									
2	1	2	1								
3	1	3	3	1							
4	1	4	6	4	1						
5	1	5	10	10	5	1					
6	1	6	15	20	15	6	1				
7	1	7	21	35	35	21	7	1			
8	1	8	28	56	70	56	28	8	1		
9	1	9	36	84	126	126	84	36	9	1	
10	1	10	45	120	210	252	210	120	45	10	1

There are a number of approaches to proving identities, and many patterns can be proven by two or more of these approaches. This section covers two approaches (plus a third in the Enrichment section) — these are all based on the binomial expansion. The final section covers another three, based more on combinatorics.

A method of proof — Substituting a value into a binomial expansion

Substitute into some form of the binomial expansion

$$(x+y)^n = {}^n\mathrm{C}_0\, x^n y^0 + {}^n\mathrm{C}_1\, x^{n-1} y^1 + {}^n\mathrm{C}_2\, x^{n-2} y^2 + \cdots + {}^n\mathrm{C}_n\, x^0 y^n\,.$$

Example 13 Proving identities by substituting into a binomial expansion

Obtain identities by substituting into the basic form of the binomial expansion as written out above:

a $x = 1$ and $y = 1$ **b** $x = 1$ and $y = -1$ **c** $x = 1$ and $y = 2$

Then explain what pattern each identity describes in selected rows of Pascal's triangle triangle.

Solution

a Substituting $x = 1$ and $y = 1$, and swapping sides, gives

$${}^n\mathrm{C}_0 + {}^n\mathrm{C}_1 + {}^n\mathrm{C}_2 + \cdots + {}^n\mathrm{C}_n = 2^n,$$

which was proven in Section 17E by combinatorics. In Pascal's triangle, this means that the sum of every row is 2^n. For example, $1 + 5 + 10 + 10 + 5 + 1 = 32 = 2^5$.

b Substituting $x = 1$ and $y = -1$, and swapping sides, gives

$${}^n\mathrm{C}_0 - {}^n\mathrm{C}_1 + {}^n\mathrm{C}_2 - \cdots + (-1)^n\,{}^n\mathrm{C}_n = 0.$$

This means that the alternating sum of every row is zero.

For odd n, this is trivial: $1 - 5 + 10 - 10 + 5 - 1 = 0,$

but for even n, $1 - 6 + 15 - 20 + 15 - 6 + 1 = 0.$

c Substituting $x = 1$ and $y = 2$, and swapping sides, gives

$$1 \times {}^n\mathrm{C}_0 + 2 \times {}^n\mathrm{C}_1 + 2^2 \times {}^n\mathrm{C}_2 + \cdots + 2^n \times {}^n\mathrm{C}_n = 3^n.$$

Taking as an example the row 1, 4, 6, 4, 1,

$$1 \times 1 + 2 \times 4 + 4 \times 6 + 8 \times 4 + 16 \times 1 = 81 = 3^4.$$

A method of proof — Equating coefficients

This method involves taking two equal expansions and equating coefficients.

The following identity is quite dramatic, because it shows how squaring the terms relates each row to the row twice as far down the triangle.

Example 14 Proving identities by equating coefficients

a Taking $\left(x + \dfrac{1}{x}\right)^n \left(x + \dfrac{1}{x}\right)^n = \left(x + \dfrac{1}{x}\right)^{2n}$ and expanding and equating constants, prove that

$$({}^n\mathrm{C}_0)^2 + ({}^n\mathrm{C}_1)^2 + ({}^n\mathrm{C}_2)^2 + \cdots + ({}^n\mathrm{C}_n)^2 = {}^{2n}\mathrm{C}_n\,.$$

b Then interpret the identity in Pascal's triangle.

Solution

a On the RHS, the constant term is ${}^{2n}C_n \times x^n \times x^{-n} = {}^{2n}C_n$. (1)
On the LHS, the first factor $(x + x^{-1})^n$ can be expanded as

$$\begin{aligned}(x + x^{-1})^n &= {}^nC_0\, x^n + {}^nC_1\, x^{n-1}(x^{-1})^1 + {}^nC_2\, x^{n-2}(x^{-1})^2 + \cdots + {}^nC_n\,(x^{-1})^n \\ &= {}^nC_0\, x^n + {}^nC_1\, x^{n-2} + {}^nC_2\, x^{n-4} + \cdots + {}^nC_n\, x^{-n},\end{aligned}$$

so the constant terms in the product on the LHS arise from products such as

$${}^nC_3\, x^3 \times {}^nC_{n-3}\, x^{-3} = {}^nC_3 \times {}^nC_{n-3}.$$

Thus the constant term on the LHS is the sum of the products

$${}^nC_0 \times {}^nC_n + {}^nC_1 \times {}^nC_{n-1} + {}^nC_2 \times {}^nC_{n-2} + \cdots + {}^nC_n \times {}^nC_0,$$

and because ${}^nC_{n-k} = {}^nC_k$, by the symmetry of the row,
the constant term on the LHS is $({}^nC_0)^2 + ({}^nC_1)^2 + ({}^nC_2)^2 + \cdots + ({}^nC_n)^2$. (2)
Equating the two constant terms at (1) and (2) gives the required identity:

$$({}^nC_0)^2 + ({}^nC_1)^2 + ({}^nC_2)^2 + \cdots + ({}^nC_n)^2 = {}^{2n}C_n.$$

b This means that if the entries of any row are squared and added, the sum is the middle entry in the row twice as far down. Look at the Pascal triangle on the first page of this section, and pick out the rows indexed by 3 and 6.

3	1	3	3	1			
6	1	6	15	20	15	6	1

The sum of the squares of the row indexed by 3 is $1^2 + 3^2 + 3^2 + 1^2 = 20$, which is the middle term ${}^6C_3$ of the row indexed by 6.
Adding instead the squares of the entries in the row indexed by 6,

$$1^2 + 6^2 + 15^2 + 20^2 + 15^2 + 6^2 + 1 = 924.$$

which is ${}^{12}C_6$, hopefully calculated in the very first question of Exercise 18A.

Exercise 18E FOUNDATION

1 [Substitution] Begin with the binomial expansion for $n = 4$,

$$(1 + x)^4 = {}^4C_0 + {}^4C_1\, x + {}^4C_2\, x^2 + {}^4C_3\, x^3 + {}^4C_4\, x^4, \qquad (*)$$

and use it to prove the following results. Explain each result in terms of the row 1 4 6 4 1 indexed by $n = 4$ in Pascal's triangle.

a By substituting $x = 1$, show that ${}^4C_0 + {}^4C_1 + {}^4C_2 + {}^4C_3 + {}^4C_4 = 2^4$.
b By substituting $x = -1$, show that ${}^4C_0 + {}^4C_2 + {}^4C_4 = {}^4C_1 + {}^4C_3$.
c Hence, by using the result of part (a), show that ${}^4C_0 + {}^4C_2 + {}^4C_4 = 2^3$.

2 [Substitution] Use the general binomial expansion,

$$(1+x)^n = {}^nC_0 + {}^nC_1\,x + \cdots + {}^nC_n\,x^n \qquad (*)$$

to prove the following results. Explain each result in terms of the row 1 5 10 10 5 1 indexed by $n = 5$ in Pascal's triangle.

a By substituting $x = 1$, show that ${}^nC_0 + {}^nC_1 + \cdots + {}^nC_n = 2^n$.

b By substituting $x = -1$, show that ${}^nC_0 + {}^nC_2 + {}^nC_4 + \cdots = {}^nC_1 + {}^nC_3 + {}^nC_5 + \cdots$

c Hence, by using the result of part (a), show that ${}^nC_0 + {}^nC_2 + {}^nC_4 + \cdots = 2^{n-1}$.

3 [Substitution] This question follows the same steps as Question 2, and uses the expansion

$$(1+x)^{2n} = {}^{2n}C_0 + {}^{2n}C_1\,x + \cdots + {}^{2n}C_{2n}\,x^{2n}. \qquad (*)$$

a Show that ${}^{2n}C_0 + {}^{2n}C_1 + \cdots + {}^{2n}C_{2n} = 2^{2n}$.

b Show that ${}^{2n}C_1 + {}^{2n}C_3 + {}^{2n}C_5 + \cdots + {}^{2n}C_{2n-1} = 2^{2n-1}$.

c Check both results in Pascal's triangle, using $n = 3$ and $n = 4$.

4 [Comparing coefficients]

a Use the binomial expansion of $(1+x)^n$ to write an expansion of $(1+x)(1+x)^n$.

b By equating coefficients on both sides in the identity $(1+x)(1+x)^n = (1+x)^{n+1}$ deduce an identity involving:

i ${}^nC_0, {}^nC_1, {}^{n+1}C_1$.

ii ${}^nC_1, {}^nC_2, {}^{n+1}C_2$.

iii ${}^nC_k, {}^nC_{k+1}, {}^{n+1}C_{k+1}$

c What identity have you written down?

DEVELOPMENT

5 [Comparing coefficients]

a Show that ${}^nC_k = {}^nC_{n-k}$.

b **i** By comparing coefficients of x^{10} on both sides of $(1+x)^{10}(1+x)^{10} = (1+x)^{20}$, write down an identity involving

$${}^{10}C_0\,{}^{10}C_{10} + {}^{10}C_1\,{}^{10}C_9 + {}^{10}C_2\,{}^{10}C_8 + \cdots + {}^{10}C_{10}\,{}^{10}C_0$$

ii Write down the value of $({}^{10}C_0)^2 + ({}^{10}C_1)^2 + \cdots + ({}^{10}C_{10})^2$

c By comparing coefficients of x^n on both sides of the identity $(1+x)^n(1+x)^n = (1+x)^{2n}$, show that $({}^nC_0)^2 + ({}^nC_1)^2 + \cdots + ({}^nC_n)^2 = {}^{2n}C_n$.

d Check this identity on the Pascal triangle by adding the squares of the rows indexed by $n = 1, 2, 3, 4, 5$ and 6.

6 **a** By comparing coefficients of x^{n+1} on both sides of $(1+x)^n(1+x)^n = (1+x)^{2n}$, show that

$${}^nC_0 \times {}^nC_1 + {}^nC_1 \times {}^nC_2 + {}^nC_2 \times {}^nC_3 + \cdots + {}^nC_{n-1} \times {}^nC_n = \frac{(2n)!}{(n-1)!\,(n+1)!}.$$

b Check this identity on the rows indexed by $n = 3, 4, 5$ and 6 of the Pascal triangle.

7 [Comparing coefficients]

a By equating coefficients of x^4 in the expansion of $(1+x)^4(1-x)^4 = \left(1-x^2\right)^4$, prove that $\left({}^4C_0\right)^2 - \left({}^4C_1\right)^2 + \left({}^4C_2\right)^2 - \left({}^4C_3\right)^2 + \left({}^4C_4\right)^2 = {}^4C_2$.

b Generalise this result, and prove it, by considering the expansion of

$$(1+x)^{2n}(1-x)^{2n} = \left(1-x^2\right)^{2n}.$$

c Check your identity on the Pascal triangle, for $n = 4$, 5 and 6.

8 [Comparing coefficients]

a By expanding both sides of the identity $(1+x)^{n+4} = (1+x)^n(1+x)^4$, show that

$$\binom{n+4}{r} = \binom{n}{r} + 4\binom{n}{r-1} + 6\binom{n}{r-2} + 4\binom{n}{r-3} + \binom{n}{r-4},$$

and state the necessary restriction on r.

b Check this identity in Pascal's triangle, using $n = r = 5$ and using $n = 6$ and $r = 4$.

9 [Substitution] Use the expansion of $(1+x)^{2n}$, to prove that

$${}^{2n}C_0 + {}^{2n}C_1 + \cdots + {}^{2n}C_n = 2^{2n-1} + \frac{(2n)!}{2(n!)^2}.$$

10 [Comparing coefficients]

a Find the coefficient of x^{n+r} in the expansion of $(1+x)^{3n}$.

b By writing $(1+x)^{3n}$ as $(1+x)^n(1+x)^{2n}$, prove that for $0 < r \le n$,

$$\binom{n}{0}\binom{2n}{r} + \binom{n}{1}\binom{2n}{r+1} + \binom{n}{2}\binom{2n}{r+2} + \cdots + \binom{n}{n}\binom{2n}{r+n} = \binom{3n}{n+r}.$$

c Check this identity in Pascal's triangle, using $n = 4$ and $r = 3$.

11 [Comparing coefficients]

a Show that $x^n(1+x)^n\left(1+\dfrac{1}{x}\right)^n = (1+x)^{2n}$.

b Hence prove that $1 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}$.

CHALLENGE

The next question follows the substitution method of Question 1–2, but applies calculus to the binomial expansion first. If you haven't yet done differentiation, you will need to skip this question for the moment. This approach of differentiation-then-substitution is not in the course, but it presents no problems once you have done differentiation. We have included it here in Enrichment because some of the resulting identities are needed in Year 12 binomial distributions.

12 [Substitution and differentiation] Use the general binomial expansion,

$$(1+x)^n = {}^nC_0 + {}^nC_1\, x + \cdots + {}^nC_n\, x^n \qquad (*)$$

to prove the following results. Explain each result in terms of the row 1 5 10 10 5 1 indexed by $n = 5$ in Pascal's triangle.

a Differentiate both sides of the binomial expansion $(*)$ above.

b By substituting $x = 1$, show that $1 \times {}^nC_1 + 2 \times {}^nC_2 + \cdots + n \times {}^nC_n = n2^{n-1}$.

c By substituting $x = -1$, show that $1 \times {}^nC_1 - 2 \times {}^nC_2 + \cdots + (-1)^{n-1} n \times {}^nC_n = 0$.

13 **a** Use the results of the previous question to show that

$$2 \times {}^nC_1 + 3 \times {}^nC_2 + \cdots + (n+1) \times {}^nC_n = 2^{n-1}(n+2)\,.$$

b Check this identity in Pascal's triangle, using $n = 4$, 5 and 6.

14 **a** By considering the identity $(1 - x^2)^n = (1 + x)^n(1 - x)^n$, show that for n even

$$\binom{n}{0}^2 - \binom{n}{1}^2 + \binom{n}{2}^2 - \cdots + (-1)^n\binom{n}{n}^2 = \frac{(-1)^{\frac{n}{2}}(n+2)(n+4)\cdots(2n)}{2 \times 4 \times \cdots \times n}$$

$$= \frac{(-1)^{\frac{n}{2}}\, n!}{(\frac{n}{2})^2}$$

b What is the corresponding result if n is odd?

15 If $(1 + x)^n = c_0 + c_1x + c_2x^2 + \cdots + c_nx^n$, show that

$$c_0c_2 + c_1c_3 + c_2c_4 + \cdots + c_{n-2}c_n = \frac{(2n)!}{(n+2)!\,(n-2)!}\,.$$

18F Further identities in Pascal's triangle

Learning intentions

- Prove a Pascal's triangle identity using combinatorics associated with $^n\mathrm{C}_r$.
- Prove a Pascal's triangle identity using the factorial expression of $^n\mathrm{C}_r$.
- Prove a Pascal's triangle identity using previously proven identities.

This final section proves Pascal's triangle identities using three methods that are closer to the combinatorics and factorials of Chapter 17.

The reader is thus invited to have a double view of binomial expansions. On the one hand, they are algebraic expansions, with an appearance complicated by the many terms in the expansion. On the other hand, they are very close to the combinatorial situation presented in Chapter 17, because each coefficient $^n\mathrm{C}_r$ is the number of r-member subsets of an n-member set.

A method of proof — Using combinatorics

The following example displays the combinatorics approach well, but it can also be done by equating coefficients. This allows a comparison of methods.

Example 15 Proving identities using combinatorics

a I have 4 twenty-cent pieces and 6 five-cent pieces in my pocket. How many subsets of 3 coins are there in my pocket?

  i Ignoring the two types of coins, write the result as $^n\mathrm{C}_r$.

  ii Count separately the subsets with 0, 1, 2 and 3 five-cent coins.

  iii What Pascal's triangle identity have you proven?

b Prove the identity by equating appropriate coefficients in

$$(1 + x)^4(1 + x)^6 = (1 + x)^{10}.$$

c Use the values from Pascal's triangle to verify the identity.

d Identify visually all together the entries in the Pascal Triangle, identify the pattern, and try to reproduce it elsewhere on the triangle.

e Suggest a generalisation of the identity.

Solution

a **i** Number of subsets = $^{10}\mathrm{C}_3$.

  ii Each 3-coin subset has 3 twenty-cents and 0 five-cents, or 2 twenty-cents and 1 five -cent, or 1 twenty-cent and 2 five-cents, or 0 twenty-cents and 3 five-cents. Hence

$$\text{Number of subsets} = {}^4\mathrm{C}_3 \times {}^6\mathrm{C}_0 + {}^4\mathrm{C}_2 \times {}^6\mathrm{C}_1 + {}^4\mathrm{C}_1 \times {}^6\mathrm{C}_2 + {}^4\mathrm{C}_0 \times {}^6\mathrm{C}_3.$$

  iii $^4\mathrm{C}_3 \times {}^6\mathrm{C}_0 + {}^4\mathrm{C}_2 \times {}^6\mathrm{C}_1 + {}^4\mathrm{C}_1 \times {}^6\mathrm{C}_2 + {}^4\mathrm{C}_0 \times {}^6\mathrm{C}_3 = {}^{10}\mathrm{C}_3$.

b The term in x^3 on the right is $^{10}\mathrm{C}_3 x^3$.

The term in x^3 on the left is the sum of the four products:

$$^4\mathrm{C}_3 \times {}^6\mathrm{C}_0 x^3 + {}^4\mathrm{C}_2 \times {}^6\mathrm{C}_1 x^3 + {}^4\mathrm{C}_1 \times {}^6\mathrm{C}_2 x^3 + {}^4\mathrm{C}_0 \times {}^6\mathrm{C}_3 x^3.$$

This proves the same identity.

c Substituting into the identity from Pascal's triangle (your copy or Section 18E),

$$\text{LHS} = 4 \times 1 + 6 \times 6 + 4 \times 15 + 1 \times 20 = 4 + 36 + 60 + 20 = 120 = \text{RHS}.$$

d Visualisation is left to the reader.

e Here is one of many possible generalisations. For $n \geq m \geq 3$,

$${}^{m}\mathrm{C}_3 \times {}^{n}\mathrm{C}_0 + {}^{m}\mathrm{C}_2 \times {}^{n}\mathrm{C}_1 + {}^{m}\mathrm{C}_1 \times {}^{n}\mathrm{C}_2 + {}^{m}\mathrm{C}_0 \times {}^{n}\mathrm{C}_3 = {}^{m+n}\mathrm{C}_3.$$

Test it out on the Pascal triangle for $n = 5$ and $m = 4$.
Then perhaps prove it by analogy with parts (a) or (b).

A method of proof — Writing ${}^{n}\mathrm{C}_r$ in terms of factorials

This method is not particularly easy to follow, but the following example is the best-known application of the method. Again, the result has other proofs, and it is interesting to compare them.

Example 16 Proving identities by writing ${}^{n}\mathrm{C}_r$ in terms of factorials

a By writing LHS and RHS in terms of factorials, prove the *addition property*:

$${}^{n}\mathrm{C}_r = {}^{n-1}\mathrm{C}_{r-1} + {}^{n-1}\mathrm{C}_r, \quad \text{for } n \geq 1 \text{ and } 1 \leq r \leq n.$$

b Give a combinatorial proof of the addition property.

c Give a proof comparing coefficients.

Solution

a
$$\begin{aligned}
\text{RHS} &= \frac{(n-1)!}{(r-1)! \times (n-r)!} + \frac{(n-1)!}{r! \times (n-r-1)!} \\
&= \frac{(n-1)!}{(r-1)! \times (n-r-1)!} \times \left(\frac{1}{1 \times (n-r)} + \frac{1}{r \times 1} \right) \\
&= \frac{(n-1)!}{(r-1)! \times (n-r-1)!} \times \left(\frac{r}{r \times (n-r)} + \frac{n-r}{r \times (n-r)} \right) \\
&= \frac{(n-1)!}{(r-1)! \times (n-r-1)!} \times \frac{n}{r \times (n-r)} \\
&= \frac{n!}{r! \times (n-r)!} \\
&= \text{LHS}
\end{aligned}$$

b Represent ${}^{n}\mathrm{C}_r$ as the number of r-person subsets of a crowd of n people.
There is bloke called Geoffrey in the crowd,
and every r-person subset either contains Geoffrey, or does not contain Geoffrey.
Number of r-member subsets that contain Geoffrey $= {}^{n-1}\mathrm{C}_{r-1}$, (1)

because we need to choose another $r - 1$ people out of $n - 1$ people.

Number of r-member subsets not containing Geoffrey $= {}^{n-1}\mathrm{C}_r$, (2)

because we need to choose all r people out of $n - 1$ people.

Adding the counts in (1) and (2) gives ${}^{n}\mathrm{C}_r = {}^{n-1}\mathrm{C}_{r-1} + {}^{n-1}\mathrm{C}_r$.

c Consider the terms in x^n in the expansion of $(1 + x)^n = (1 + x)^{n-1} \times (1 + x)$.

$$\text{Term in } x^r \text{ on the LHS} = {}^{n}C_r\, x^r.$$

$$\begin{aligned}\text{Term in } x^r \text{ on the RHS} &= {}^{n-1}C_r\, x^r \times 1 \; + \; {}^{n-1}C_{r-1}\, x^{r-1} \times x.\\ &= ({}^{n-1}C_r + {}^{n-1}C_{r-1})x^r.\end{aligned}$$

Equating the coefficients of x^r, $\quad {}^{n}C_r = {}^{n-1}r - 1 + {}^{n-1}C_r$.

A method of proof — Using previously proven identities

Occasionally a previously proven identity can be used in a proof.

Example 17 Proving identities by using previously proven identities

a Use the addition property to prove that

$${}^{n+1}C_2 - {}^{n}C_2 = n, \quad \text{for all } n \geq 2.$$

b Hence show that ${}^{n+1}C_2 = 1 + 2 + 3 + \cdots + n$, for all whole numbers $n \geq 1$.

c Interpret part (b) on the Pascal triangle.

Solution

a By the addition property,

$$\begin{aligned}{}^{n+1}C_2 &= {}^{n}C_2 + {}^{n}C_1\\ {}^{n+1}C_2 - {}^{n}C_2 &= {}^{n}C_1\\ &= \frac{n!}{(n-1)! \times 1!}\\ &= n.\end{aligned}$$

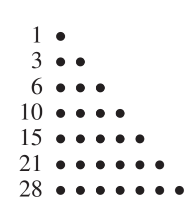
b This proof uses *recursion*, which will become *mathematical induction* in Year 12.
First, we can rewrite the result to part (a) as ${}^{n+1}C_2 = n + {}^{n}C_2$.
Then applying this formula repeatedly,

$$\begin{aligned}{}^{n+1}C_2 &= n + {}^{n}C_2\\ &= n + (n-1) + {}^{n-1}C_2\\ &= n + (n-1) + (n-2) + {}^{n-2}C_2\\ &= n + (n-1) + (n-2) + \cdots + 2 + {}^{2}C_2 \qquad \text{(after many more steps)}\\ &= n + (n-1) + (n-2) + \cdots + 2 + 1.\end{aligned}$$

c Look at the third column of the Pascal triangle (your copy or Section 18E):

$1 = 1, \quad 3 = 1 + 2, \quad 6 = 1 + 2 + 3, \quad 10 = 1 + 2 + 3 + 4, \quad 15 = 1 + 2 + 3 + 4 + 5, \ldots$

Note: The numbers 1, 3, 6, 10, 15, ... are called the *triangular numbers* because they are the successive total numbers of dots in 1, 2, 3, 4, 5, ... rows of a triangular array. We will meet them again with sequences and series in Year 12.



Remarks on the various methods of proof

The reader has probably gained some impressions about these various methods.

- Substitution is very simple, but does not develop many new ideas.
- Equating coefficients and combinatoric approaches are both excellent methods, and generate interesting new identities.
- Expanding factorials can be very fiddly.
- Using previously proven results may be useful in some situations, as in all mathematics.

6 Some methods of proving identities in Pascal's triangle

- Substitute values into a binomial expansion.
- Equate coefficients in two equal binomial expansions.
- Use combinatoric arguments.
- Expand the factorials in the expressions for ${}^{n}C_{r}$.
- Use previously proven identities.

- Differentiation-then-substitution (see Exercise 18E Enrichment) is not in the course, but it is straightforward, and generates many interesting identities, some of which will be needed in Year 12 binomial distributions.

Exercise 18F

FOUNDATION

For the next three questions, you should use the definition of the symbol ${}^{n}C_{r}$ as the number of ways of choosing an r-member subset from an n-member set.

1 [Combinatorics] Consider the set $S = \{A, B, C, D, E\}$.

a Consider the subset {A,B}. What letters have been omitted from the subset that are in the whole set?

b List all the subsets of {A, B, C, D, E} containing 2 letters.

c Along side each of the sets in (a) write down the set that does NOT contain those two letters.

d Use the combinatorial definition of ${}^{n}C_{r}$ and parts (a)-(c) to explain why ${}^{5}C_{2} = {}^{5}C_{3}$.

e Explain why ${}^{n}C_{r} = {}^{n}C_{n-r}$ for any whole numbers n and r with $r \le n$.

2 [Combinatorics] Consider the set $S = \{A, B, C, D\}$.

a Write down all the subsets of S, then explain why there are exactly 2^4 of them.

b Hence explain why $\binom{4}{0} + \binom{4}{1} + \binom{4}{2} + \binom{4}{3} + \binom{4}{4} = 2^4$.

c Explain why $\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n$, for any whole number n.

3 [A combinatorics proof of the addition property] Consider the sets

$$S = \{\ A, B, C, D, E\ \} \quad \text{and} \quad U = \{\ A, B, C, D\ \}.$$

a **i** Write down all the 3-letter subsets of S that do not not contain E.

ii Explain why this is a list of all the 3-letter subsets of U.

b **i** Write down all the 3-letter subsets of S that contain E.

ii Explain how to pair them up with all the 2-letter subsets of U.

c Hence explain why ${}^{4}C_{2} + {}^{4}C_{3} = {}^{5}C_{3}$.

d Explain why ${}^{n}C_{r-1} + {}^{n}C_{r} = {}^{n+1}C_{r}$, for any whole numbers n and r with $1 \le r \le n$.

4 [The addition property and the formula for ${}^nC_r$] In this question, you will prove that if a, b, c and d are any four consecutive terms in any row of Pascal's triangle, then

$$\frac{a}{a+b} + \frac{c}{c+d} = \frac{2b}{b+c}.$$

a Consider the row 1, 7, 21, 35, 35, 21, 7, 1 indexed by $n = 7$. Show that the identity holds for each sequence a, b, c, d of four consecutive terms from this row.

b Choose four consecutive terms from any other row and show that the identity holds.

c Prove the identity by letting $a = {}^nC_{r-1}$, $b = {}^nC_r$, $c = {}^nC_{r+1}$ and $d = {}^nC_{r+2}$. You will need to use the addition property, then the formula ${}^nC_r = \dfrac{n!}{r!(n-r)!}$.

5 [Two combinatorics proofs] We have seen in question 2(a) that substituting $x = -1$ and $x = 1$ into the binomial expansion proves

$${}^nC_0 + {}^nC_1 + {}^nC_2 + {}^nC_3 + \cdots = 2^n. \qquad (1)$$

$${}^nC_0 - {}^nC_1 + {}^nC_2 - {}^nC_3 + \cdots = 0, \quad \text{for } n \ge 1. \qquad (2)$$

Here are combinatorics proofs of these results.

a Let S be an n-member set, and interpret each ${}^nC_r$ as the number of r-member subsets of S. Hence prove the first identity (1).

b To prove the second identity (2), choose a fixed element A in the set S. Pair up each subset U not containing A with the unique subset $U \cup \{A\}$ containing A.

i Explain why the procedure arranges all the subsets of S uniquely into pairs.

ii Explain why one member of each pair has an even number of members, and the other has an odd number of members.

iii Hence prove that ${}^nC_0 + {}^nC_2 + \cdots = {}^nC_1 + {}^nC_3 + \cdots$.

6 [The formula for ${}^nC_r$] This question involves the formula ${}^nC_r = \dfrac{n!}{r!(n-r)!}$.

a Prove that $\dfrac{n!}{r!\,(n-r)!} + \dfrac{n!}{(r-1)!\,(n-r+1)!} = \dfrac{(n+1)!}{r!\,(n-r+1)!}$.

b What result have you proven?

DEVELOPMENT

7 [An inequality proven using the formula for ${}^nC_r$]

a Prove that ${}^{2n}C_r < {}^{2n}C_{r+1}$, for all whole numbers n and r with $r < n$.

b Prove that ${}^{2n+1}C_r < {}^{2n+1}C_{r+1}$, for all whole numbers n and r with $r < n$.

8 [The hockey-stick identity] Look at the column indexed by $r = 2$ in Pascal's triangle:

$$^2C_2 + {}^3C_2 + {}^4C_2 + {}^5C_2 + {}^6C_2 = 1 + 3 + 6 + 10 + 15 = 35 = {}^7C_3\,. \qquad (*)$$

The general form of this well-known *hockey-stick identity* is

$$^rC_r + {}^{r+1}C_r + {}^{r+2}C_r + \cdots + {}^nC_r = {}^{n+1}C_{r+1} \qquad (**)$$

where n and r are whole numbers with $0 \le r \le n$. Here are proofs of this identity using the addition property, and using a combinatorial proof.

a [A proof using the addition property]

i To prove the particular result $(*)$, start with the right-hand side and write

$$^7C_3 = {}^6C_3 + {}^6C_2$$

then keep expanding the term with $r = 3$. To complete the proof, you will need to use the fact that $^3C_3 = {}^2C_2 = 1$.

ii Generalise this to a proof of the identity $(**)$.

b [A combinatorics proof]

i The number $^7C_3$ is the number of 3-member subsets of a 7-member set S. To choose a 3-member subset U of $S = \{1, 2, 3, 4, 5, 6, 7\}$, make the choice in the following way:

- First choose from S the greatest number k that will be in the subset U. This must be one of the numbers 3, 4, 5, 6 or 7, because U is to have three members, so it will have two numbers smaller than k.
- Then choose the remaining two numbers in U from the $k - 1$ possible numbers $1, 2, \ldots, k - 1$.

Explain why this method of choosing the subset yields the identity $(*)$.

ii Generalise this to a proof of the identity $(**)$.

9 [A combinatorial approach to the sum of squares in a row] In this question we consider binary words consisting only of the letters A and B.

a Consider a binary word consisting of $a + b$ letters, with A occuring a times and B occuring b times. Show that there are $^{a+b}C_a = {}^{a+b}C_b$ permutations of such a word.

b How many possible permutations are there of a binary word with $2n$ letters, if A and B both occur n times?

c A word with $2n$ letters may be split down the middle into two words of n letters. Consider the example where two As fall in the first n-letter word.

i How many arrangements are there of the first n-letter binary word with two As?

ii How many arrangements are there of the second n-letter binary word with $n - 2$ of the As and 2 Bs?

iii How many arrangements are there of a ten-letter binary word with two As in the first half and two Bs in the second half?

d Hence prove that

$$({}^nC_0)^2 + ({}^nC_1)^2 + ({}^nC_2)^2 + ({}^nC_3)^2 + \cdots + ({}^nC_n)^2 = {}^{2n}C_n\,.$$

10 **a** When the entries of the row 1, 5, 10, 10, 5, 1 indexed by $n = 5$ in Pascal's triangle are multiplied by 0, 1, 2, 3, 4, 5 respectively, the results are 0, 5, 20, 30, 20, 5. Ignoring the zero, this is five times the row 1, 4, 6, 4, 1. Formulate this result algebraically, for $n = 5$ and then for generally n, and prove it using the binomial theorem.

b When the entries of the row 1, 5, 10, 10, 5, 1 are divided by 1, 2, 3, 4, 5 and 6 respectively, the result is 1, $2\frac{1}{2}$, $3\frac{1}{3}$, $2\frac{1}{2}$, 1, $\frac{1}{6}$. If you add $\frac{1}{6}$ at the start, this is $\frac{1}{6}$th of the row 1, 6, 15, 20, 15, 6, 1. Formulate this result algebraically, for $n = 5$ and then for general n, and prove it using the binomial theorem.

CHALLENGE

11 **a** Find the value of $\dfrac{{}^nC_r}{{}^nC_{r-1}}$.

b Evaluate $\dfrac{{}^nC_1}{{}^nC_0} + 2\dfrac{{}^nC_2}{{}^nC_1} + 3\dfrac{{}^nC_3}{{}^nC_2} + \cdots + n\dfrac{{}^nC_n}{{}^nC_{n-1}}$.

c Prove the following identity, and verify it using the row indexed by $n = 4$:

$$({}^nC_0 + {}^nC_1)({}^nC_1 + {}^nC_2)\cdots({}^nC_{n-1} + {}^nC_n) = {}^nC_0\,{}^nC_1\,{}^nC_2\cdots{}^nC_n \times \frac{(n+1)^n}{n!}.$$

12 **a** **i** If $r > p + 1$, show that ${}^rC_p = {}^{r+1}C_{p+1} - {}^rC_{p+1}$.

ii Hence deduce that for $n > p$, ${}^pC_p + {}^{p+1}C_p + {}^{p+2}C_p + \cdots + {}^nC_p = {}^{n+1}C_{p+1}$.

iii What is the significance of this result in the Pascal's triangle?

Chapter 18 Review

Review activity

- Create your own summary of this chapter on paper or in a digital document.

Chapter 18 Multiple-choice quiz

- This automatically-marked quiz is accessed in the Interactive Textbook. A printable PDF Worksheet version is also available there.

Skills Checklist

Check-list

- Available in the Interactive Textbook, use the checklist to track your understanding of the learning intentions. Printable PDF and word document versions are also available there.

Review

Chapter Review Exercise

1 Write out the first five rows of Pascal's triangle.

2 Use your answer to the previous question to expand:

a $(1 + x)^5$ **b** $(1 + 2x)^5$ **c** $(1 - 3x)^3$ **d** $(1 - xy)^4$

3 **a** Expand $(1 + 7x)^5$ as far as the term in x^2.

b Hence find the coefficient of x^2 in the expansion of $(1 - 5x)(1 + 7x)^5$.

4 **a** Expand $(1 + x)^7$.

b Hence find the first decimal place of 1.02^7.

5 Expand:

a $(3 + 2x)^4$ **b** $(5 - x)^3$ **c** $(2x + 4y)^5$ **d** $(x - \frac{1}{x})^4$

6 Use the result ${}^nC_r = \dfrac{n!}{r!\,(n-r)!}$ to evaluate each binomial coefficient. Do not use a calculator — you will need to unroll the factorial symbol. Check your answers against the copy of Pascal's triangle that you developed in Exercise 15A.

a ${}^5C_3$ **b** ${}^7C_4$ **c** ${}^8C_5$ **d** ${}^{120}C_{60} \div {}^{119}C_{60}$

7 Use your calculator to evaluate:

a ${}^{10}C_8$ **b** $\binom{12}{7}$ **c** ${}^9C_6$

d $\binom{10}{3} \div \binom{3}{2}$ **e** $\dfrac{{}^{10}C_5}{{}^6C_2}$ **f** $\dfrac{{}^{16}C_6}{{}^{10}C_5}$

8 Use your understanding of the patterns in Pascal's triangle to simplify:

a ${}^nC_0$ **b** ${}^9C_4 - {}^9C_5$

c ${}^5C_0 + {}^5C_1 + {}^5C_2 + {}^5C_3 + {}^5C_4 + {}^5C_5$ **d** ${}^{2n+1}C_n - {}^{2n}C_n - {}^{2n}C_{n-1}$

9 Solve ${}^nC_2 = 28$.

10 Find the coefficient of x^8 in the expansion of $(1 - 3x)(1 + 5x)^{14}$.

11 **a** Write out the first few terms in the expansion of $(1 + x)^n$.

b By an appropriate substitution, prove that:

$$\binom{n}{0} + 2 \times \binom{n}{1} + 4 \times \binom{n}{2} + 8 \times \binom{n}{3} + \cdots = 3^n$$

c Verify this result for the row indexed by $n = 4$ in Pascal's triangle.

12 **a** Prove that ${}^nC_0 + {}^nC_1 + {}^nC_2 + {}^nC_3 + \cdots = 2^n$.

b What is the significance of this result for Pascal's triangle?

c How may this result be interpreted as a sum of combinations?

13 **a** Show that the general term in the expansion $(2x + \frac{1}{x^2})^{18}$ can be written:

$${}^{18}C_r 2^{18-r} x^{18-3r}$$

b Hence find in this expansion:

i the term independent of x

ii the coefficient of x^9

iii the term in x^9

14 In this question we shall prove the identity

$${}^nC_r + 2 \times {}^nC_{r+1} + {}^nC_{r+2} = {}^{n+2}C_{r+2} \qquad (*)$$

a Check this identity in the case $n = 4$ and $r = 0$.

b Prove $(*)$ using the addition property ${}^nC_r + {}^nC_{r+1} = {}^{n+1}C_{r+1}$.

c Prove $(*)$ by equating coefficients of x^{r+2} in the identity $(1 + x)^2(1 + x)^n = (1 + x)^{n+2}$.

d A committee of 8 is to be formed from 12 people.

i How many committees of 8 can be formed?

The group of 12 includes Brad and Janet.

ii How many contain neither Brad nor Janet?

iii How many contain one of either Brad or Janet?

iv How many contain both?

e Use part (d) to prove $(*)$ in the case $n = 10$ and $r = 6$. Generalise your argument.