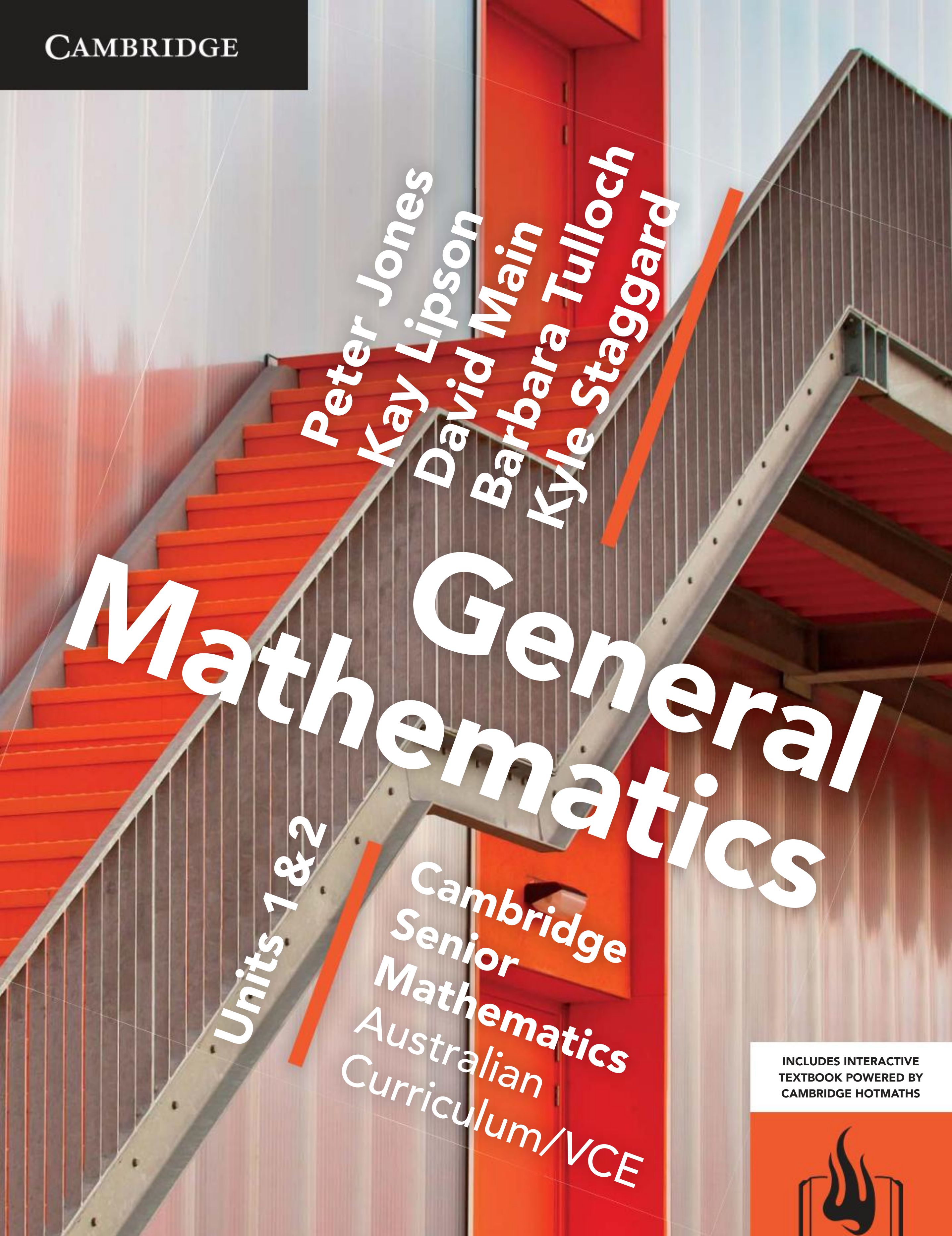

CAMBRIDGE

Peter Jones
Kay Lipson
David Main
Barbara Tulloch
Kyle Staggard

General Mathematics

Units 1&2

Cambridge Senior Mathematics
Australian Curriculum/VCE

INCLUDES INTERACTIVE TEXTBOOK POWERED BY CAMBRIDGE HOTMATHS

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Introduction

General Mathematics Units 1&2 has been specifically written for students studying General Mathematics as a complete preparation for the new 2016 Further Mathematics curriculum.

This book accurately reflects the content and pedagogical intent of the new General Mathematics curriculum. Chapters are carefully ordered to ensure that students' knowledge and skills follow a logical progression from Unit 1 to Unit 2. It also ensures that students have a sound preparation for studying Further Mathematics the following year.

Two major changes in the structure of the new General Mathematics curriculum have strongly influenced the writing of *General Mathematics Units 1&2*. These changes are the inclusion of 'Recursion and financial modelling' as a core area of study and the reduction in the number of applications modules to be studied from three to two.

Chapter 8 *Number patterns and recursion* introduces recursion relations through the generation and analysis of arithmetic and geometric sequences, as well as their practical applications. This introductory material is followed by a section on the use of recursion to model growth and decay in financial contexts (8H), which provides a direct pathway to the *Recursion and financial modelling* topic in Further Mathematics.

Of the remaining chapters *Computation and practical arithmetic* is new, while all other chapters have been updated and rewritten to meet the needs of the new curriculum and provide clear pathways into the compulsory data analysis topic and application modules in Further Mathematics (see the chart on the following page). Many new modelling, applications and problem-solving tasks have been added.

As with the predecessor to this book, *Essential Standard General Mathematics*, all chapters provide carefully graded exercise sets to help students develop the key skills and knowledge specified in the General Mathematics Study Design. In addition, each chapter has a Review section including multiple-choice, short-answer and extended-response questions to help students consolidate their learning. An extensive Glossary of terms is also provided to ensure that students can quickly access the definitions of key mathematical or statistical terms.

The TI-Nspire calculator examples and instructions, including the appendices, have been completed by Russell Brown, and those for the Casio ClassPad have been completed by Maria Schaffner.

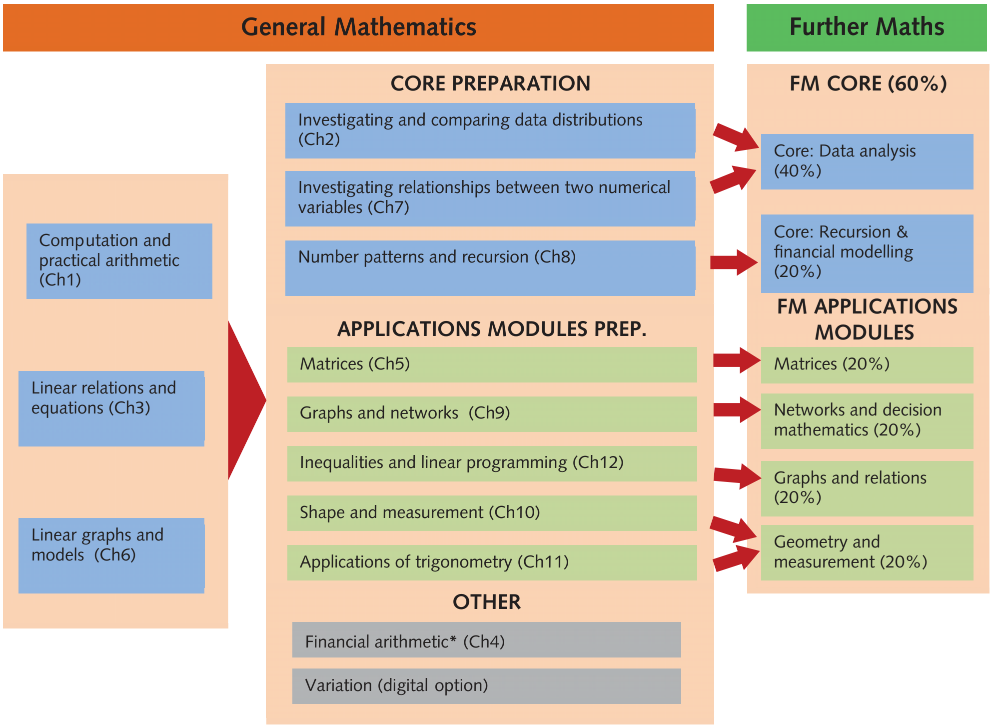
The integration of the features of the textbook and the new digital components of the package, powered by Cambridge HOTmaths, are illustrated in the following pages.

About Cambridge HOTmaths

Cambridge HOTmaths is a comprehensive, award-winning mathematics learning system – an interactive online maths learning, teaching and assessment resource for students and teachers, for individuals or whole classes, for school and at home. Its digital engine or platform is used to host and power the interactive textbook and the Online Teaching Suite, and selected topics from HOTmaths' own Years 9 and 10 course areas are available for revision of prior knowledge. All this is included in the price of the textbook.

Links between General Mathematics topics and Further Mathematics

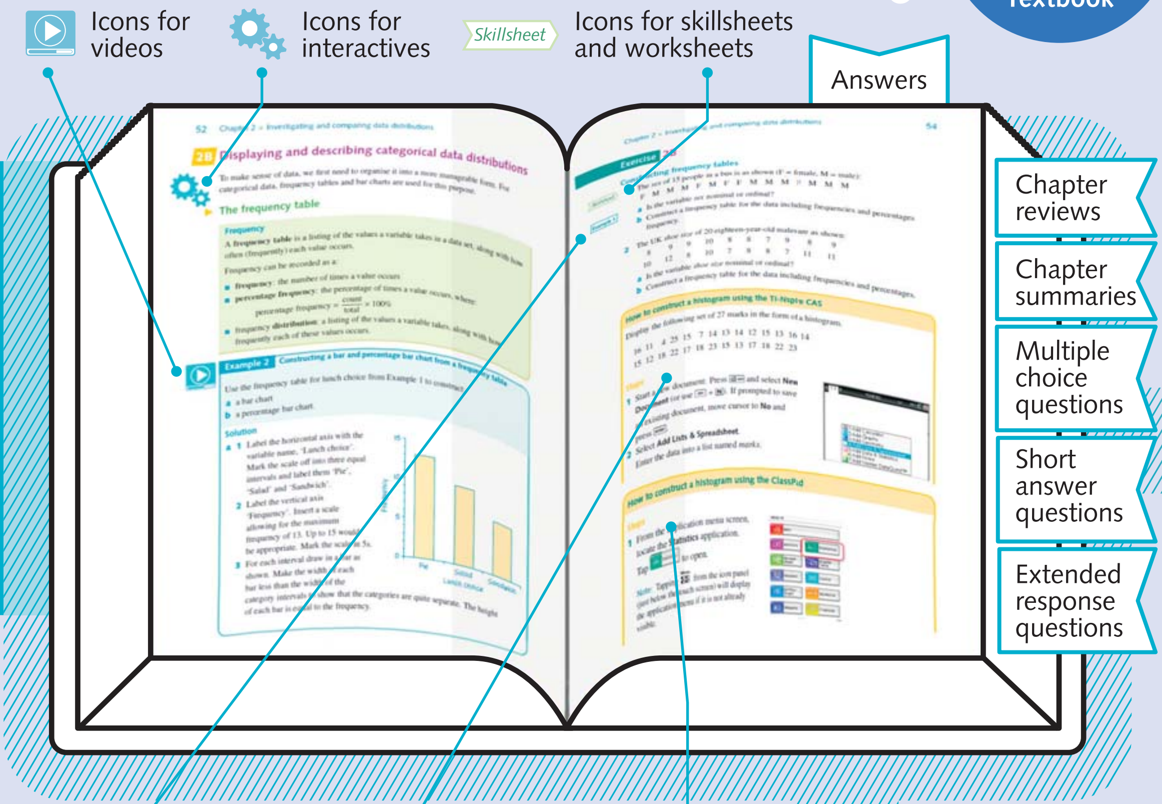
This chart outlines how the topics in General Mathematics prepares students for Further Mathematics.



**Note that Chapter 4* Financial arithmetic *has a consumer arithmetic focus, a knowledge of which is not required to prepare students for the financial modelling topic in Further Mathematics. The prerequisite financial knowledge for the latter topic is now located in section 8H of General Mathematics Chapter 8* Number patterns and recursion

An overview of the Cambridge complete teacher and learning resource

For more detail, see the guide in the online Interactive Textbook



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Chapter summaries

Multiple choice questions

Short answer questions

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TI-Nspire OS4.0 examples

Casio ClassPad II examples

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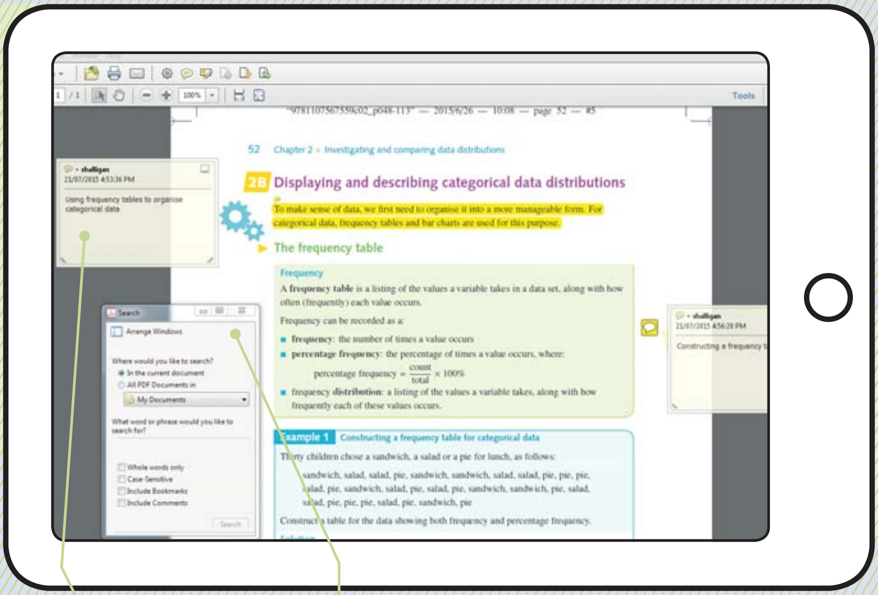

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52 Chapter 2 ▪ Investigating and comparing data distributions

2B Displaying and describing categorical data distributions

To make sense of data, we first need to organise it into a more manageable form. For categorical data, frequency tables and bar charts are used for this purpose.

The frequency table

Frequency

A **frequency table** is a listing of the values a variable takes in a data set, along with how often (frequently) each value occurs.

Frequency can be recorded as a:

- **frequency**: the number of times a value occurs
- **percentage frequency**: the percentage of times a value occurs, where:

$$\text{percentage frequency} = \frac{\text{count}}{\text{total}} \times 100\%$$

- frequency **distribution**: a listing of the values a variable takes, along with how frequently each of these values occurs.

Example 1 Constructing a frequency table for categorical data

Thirty children chose a sandwich, a salad or a pie for lunch, as follows:

sandwich, salad, salad, pie, sandwich, sandwich, salad, salad, pie, pie, pie, salad, pie, sandwich, salad, pie, salad, pie, sandwich, sandwich, pie, salad, salad, pie, pie, pie, salad, pie, sandwich, pie

Construct a table for the data showing both frequency and percentage frequency.

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Using frequency tables to organise categorical data

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Constructing a frequency t

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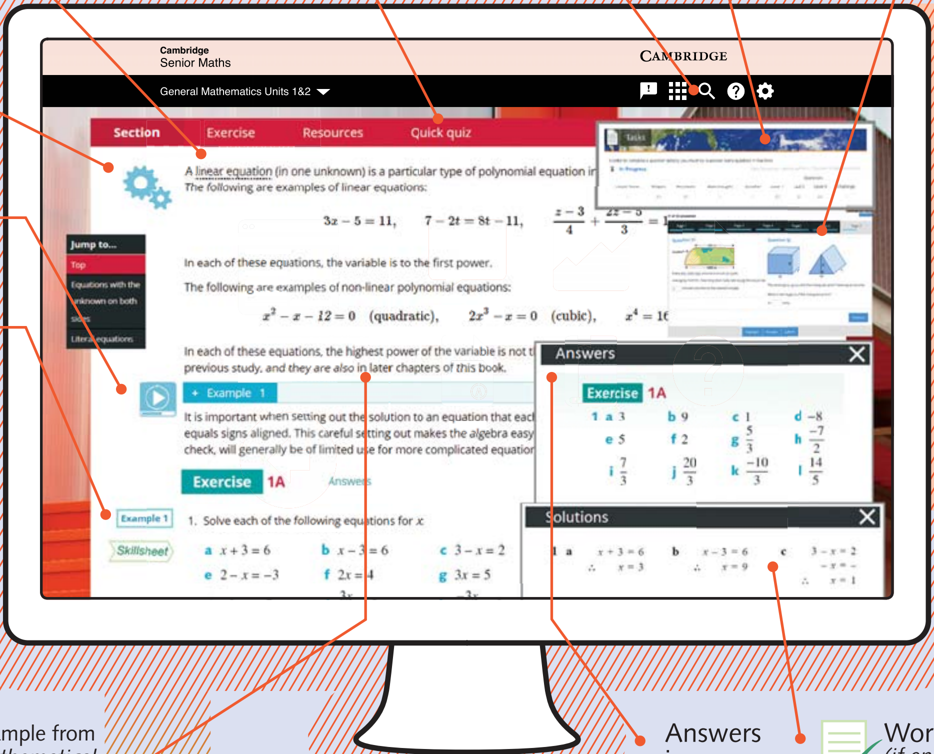

A linear equation (in one unknown) is a particular type of polynomial equation i
The following are examples of linear equations:

$$3z-5=11, \qquad 7-2t=8t-11, \qquad \frac{z-3}{4}+\frac{2z-5}{3}=1$$

In each of these equations, the variable is to the first power.

The following are examples of non-linear polynomial equations:

$$x^2-x-12=0 \quad \text{(quadratic)}, \qquad 2x^3-x=0 \quad \text{(cubic)}, \qquad x^4=16$$

In each of these equations, the highest power of the variable is not t
previous study, and they are also in later chapters of this book.

Example 1

It is important when setting out the solution to an equation that eac
equals signs aligned. This careful setting out makes the algebra easy
check, will generally be of limited use for more complicated equation

Exercise 1A

1. Solve each of the following equations for x

a $x+3=6$ b $x-3=6$ c $3-x=2$

e $2-x=-3$ f $2x=4$ g $3x=5$

Answers

Exercise 1A

1 a 3 b 9 c 1 d −8

e 5 f 2 g $\frac{5}{3}$ h $\frac{-7}{2}$

i $\frac{7}{3}$ j $\frac{20}{3}$ k $\frac{-10}{3}$ l $\frac{14}{5}$

Solutions

1 a $x+3=6$ $\therefore x=3$ b $x-3=6$ $\therefore x=9$ c $3-x=2$ $-x=-$ $x=1$

Example from *Mathematical Methods Units 1 & 2 Interactive Textbook*

Answers in pop-ups

Worked solutions *(if enabled by teacher)*

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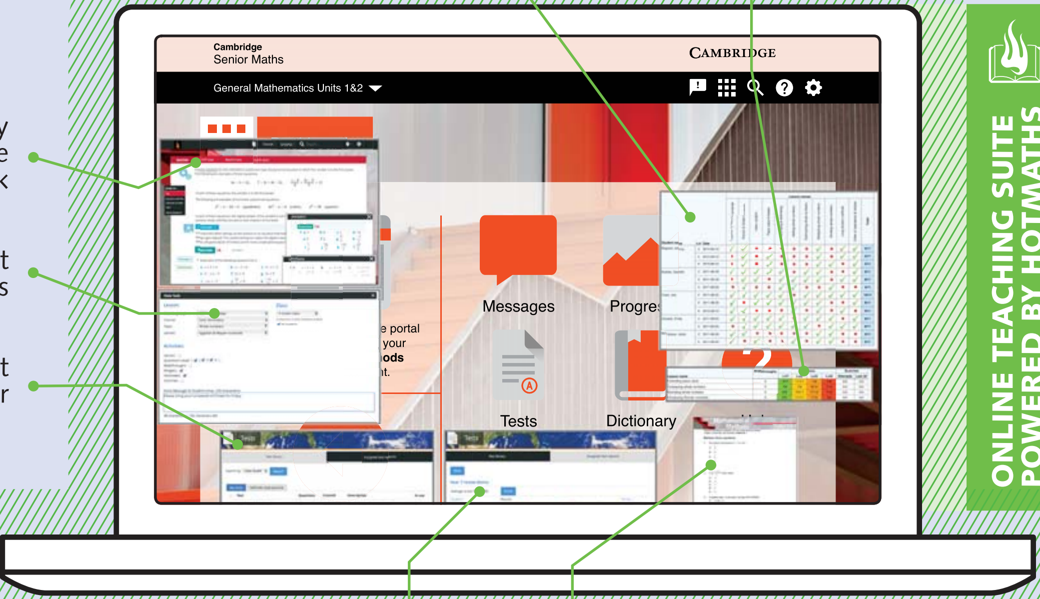

Student results

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1

Computation and practical arithmetic

- How do we use a variety of mathematical operations in the correct order?
- How do we add, subtract, multiply and divide directed numbers?
- How do we find powers and roots of numbers?
- How do we round numbers to specific place values?
- How do we write numbers in standard form?
- What are and how do we use significant figures?
- How do we convert units of measurements?
- How do we express ratios in their simplest form?
- How do we solve practical problems involving ratios, percentages and the unitary method?
- How do we use and interpret log scales that represent quantities that range over multiple orders of magnitude?

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Introduction

This chapter revises basic methods of computation used in general mathematics. It will allow you to carry out the necessary numerical calculations for solving problems. We will begin with the fundamentals.

1A Order of operations

Adding, subtracting, multiplying, dividing and squaring are some examples of operations that are used in mathematics. When carrying out a sequence of arithmetic operations, it is necessary to observe a definite sequence of rules. These rules, defining the order of operations, have been devised and standardised to avoid confusion.

Order of operation

The rules are to:

- always complete the operations in brackets first
- then carry out the division and multiplication operations (in order, from left to right)
- then carry out the addition and subtraction operations (in order, from left to right).

These rules can also be remembered by using **BODMAS**.

B	**B**rackets come first
O	If a fraction **O**f a number is required or **O**rders (powers, square roots), you complete that next
DM	**D**ivision and **M**ultiplication, working left to right across the page
AS	**A**ddition and **S**ubtraction, working left to right across the page

A calculator, with *algebraic logic*, will carry out calculations in the correct order of operations. However, particular care must be taken with brackets.

Pronumeral

A number or **pronumeral** (letter) placed in front of a bracket means that you multiply everything in the bracket by that number or pronumeral.

4(8) means $4 \times 8 = 32$

$5(x - 9) = 5x - 45$

$a(3a + 6) = 3a^2 + 6a$

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Example 1 Using correct order of operation

Evaluate the following.

a $3 + 6 \times 8$ **b** $(3 + 6) \times 8$ **c** $8 \div 2 - 2$ **d** $23 - (8 - 5)$

e $(4)\,3 - 2$ **f** $3 + 5(x - 1)$ **g** $(3 \times 8.5 - 4) - (4.1 + 5.4 \div 2)$

Solution

a $$\begin{aligned} 3 + 6 \times 8 &= 3 + 48 \\ &= 51 \end{aligned}$$

b $$\begin{aligned} (3 + 6) \times 8 &= 9 \times 8 \\ &= 72 \end{aligned}$$

c $$\begin{aligned} 8 \div 2 - 2 &= 4 - 2 \\ &= 2 \end{aligned}$$

d $$\begin{aligned} 23 - (8 - 5) &= 23 - 3 \\ &= 20 \end{aligned}$$

e $$\begin{aligned} (4)3 - 2 &= 12 - 2 \\ &= 10 \end{aligned}$$

f $$\begin{aligned} 3 + 5(x - 1) &= 3 + 5x - 5 \\ &= 5x - 2 \end{aligned}$$

g $$\begin{aligned} (3 \times 8.5 - 4) - (4.1 + 5.4 \div 2) &= (25.5 - 4) - (4.1 + 2.7) \\ &= 21.5 - 6.8 \\ &= 14.7 \end{aligned}$$

Exercise 1A

Example 1a–d

1 Evaluate the following, without using a calculator.

a $5 + 4 \times 8$ **b** $4 \times 3 - 7$

c $7 \times 6 - 4 + 4 \times 3$ **d** $15 \div 3 + 2$

e $3 + 12.6 \div 3$ **f** $4 \times (8 + 4)$

g $15 - 9 \div 2 + 4 \times (10 - 4)$ **h** $(3.7 + 5.3) \div 2$

i $8.6 - 3 \times 2 - 6 \div 3$ **j** $(3 \times 4 - 3) \div (2 - 3 \times 4)$

Example 1e

2 Use your calculator to find the answers to the following.

a $(8.23 - 4.5) + (3.6 + 5.2)$ **b** $(17 - 8.7) - (73 - 37.7)$

c $(6.2 + 33.17) \times (6.9 - 6.1)$ **d** $(3.2 + 0.5 \div 2.5) \div (8.6 - 1.3 \times 4)$

Example 1f–g

3 Evaluate the following.

a $9(3)$ **b** $2(x - 7)$ **c** $10(5 - y)$ **d** $w(8 - 2)$

e $k(k + 8)$ **f** $27(2) - 3(8)$ **g** $(5 - 3)x + 7(2)$ **h** $3(5) \times 2 - 8$

i $3(x + 1) - 8$ **j** $4 - 2(x + 3)$

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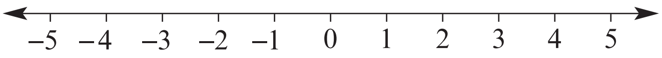
1B Directed numbers

Positive and negative numbers are *directed numbers* and can be shown on a number line.

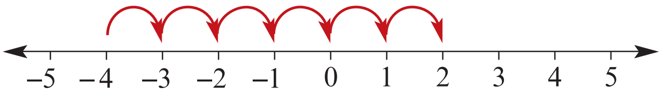
▶ Addition and subtraction

It is often useful to use a number line when adding directed numbers.



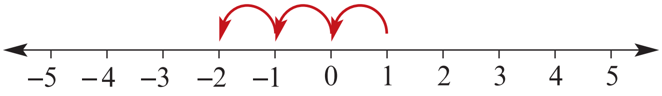
Adding a positive number means that you move to the right.

Example: $-4 + 6 = 2$



Adding a negative number means that you move to the left.

Example: $1 + (-3) = -2$



When subtracting directed numbers, you add its opposite.

Example: $-2 - 3$ is the same as $-2 + (-3) = -5$

Example: $7 - (-9) = 7 + 9 = 16$

▶ Multiplication and division

Multiplying or dividing two numbers with the *same* sign gives a *positive* value.

Multiplying or dividing two numbers with *different* signs gives a *negative* value.

Multiplication and division with directed numbers

$+ \times + = +$ $+ \times - = -$
$- \times - = +$ $- \times + = -$
$+ \div + = +$ $+ \div - = -$
$- \div - = +$ $- \div + = -$

Example 2 Using directed numbers

Evaluate the following.

a $6 - 13$ **b** $(-5) - 11$ **c** $9 - (-7)$ **d** $(-10) - (-9)$

e 5×-3 **f** $(-8) \times (-7)$ **g** $(-16) \div 4$ **h** $(-60) \div (-5)$

i $(-100) \div (-4) \div (-5)$ **j** $(-3)^2$

Solution

a $6 - 13 = 6 + (-13) = -7$

b $(-5) - 11 = (-5) + (-11)$
$= -16$

c $9 - (-7) = 9 + 7$
$= 16$

d $(-10) - (-9) = (-10) + 9$
$= -1$

e $5 \times -3 = -15$

f $(-8) \times (-7) = 56$

g $(-16) \div 4 = -4$

h $(-60) \div (-5) = 12$

i $(-100) \div (-4) \div (-5) = 25 \div (-5)$
$= -5$

j $(-3)^2 = (-3) \times (-3)$
$= 9$

Exercise 1B

Example 2a–d

1 Without using a calculator, find the answers to the following.

a $6 - 7$ **b** $-10 + 6$ **c** $-13 + (-3)$ **d** $-7 + 10$

e $-7 - 19$ **f** $(-18) - 7$ **g** $(-9) - 3$ **h** $4 - (-18)$

i $18 - (-4)$ **j** $15 - (-17)$ **k** $16 - (-12)$ **l** $(-3) - (-13)$

m $(-12) - (-6)$ **n** $(-21) - (-8)$

Example 2e–j

2 Without using a calculator, find the answers to the following.

a $(-6) \times 2$ **b** $(-6)(-4)$ **c** $(-10) \div (-4)$ **d** $15 \div (-3)$

e $(5 + 2) \times 6 - 6$ **f** $-(-4) \times -3$ **g** $-7(-2 + 3)$ **h** $-4(-7 - (2)(4))$

i $-(3 - 2)$ **j** $-6 \times (-5 \times 2)$ **k** $-6(-4 + 3)$ **l** $-(-12 - 9) - 2$

m $-4 - 3$ **n** $-(-4 - 7(-6))$ **o** $(-5)(-5) + (-3)(-3)$

p $8^2 + 4(0.5)(8)(6)$

1C Powers and roots

▶ Squares and square roots

When a number is multiplied by itself, we call this the *square* of the number.

$$4 \times 4 = 4^2 = 16$$

- 16 is called the *square* of 4 (or 4 squared).
- 4 is called the *square root* of 16.
- The square root of 16 can be written as $\sqrt{16} = 4$. ($\sqrt{}$ is the square root symbol)

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Cubes and cube roots

When a number is squared and then multiplied by itself again, we call this the *cube* of the number.

$$4 \times 4 \times 4 = 4^3 = 64$$

- 64 is called the *cube* of 4 (or 4 cubed).
- 4 is called the *cube root* of 64.
- The cube root of 64 can be written as $\sqrt[3]{64} = 4$. ($\sqrt[3]{\ }$ is the cube root symbol)

Other powers

When a number is multiplied by itself a number of times, the values obtained are called *powers* of the original number.

For example, $4 \times 4 \times 4 \times 4 \times 4 = 1024 = 4^5$, which is read as '4 to the *power* of 5'.

- 4 is the fifth root of 1024.
- $\sqrt[5]{1024}$ means the fifth root of 1024.
- Another way of writing $\sqrt{16}$ is $16^{\frac{1}{2}}$, which is read as '16 to the half'.
- Likewise, $8^{\frac{1}{3}}$, read as '8 to the third', means $\sqrt[3]{8} = 2$.
- Powers and roots of numbers can be evaluated on the calculator by using the ^ button.

Example 3 Finding the power or root of a number using a calculator

a Find 8^3.

b Find $8^{\frac{1}{3}}$.

Solution

a

```
8^3
                    512
```

b

```
8^(1/3)
                      2
```

Exercise 1C

Example 3

1 Find the value of the following.

a 10^4 **b** 7^3 **c** $\sqrt{25}$ **d** $\sqrt[3]{8}$

e 2^6 **f** 12^4 **g** $9^{\frac{1}{2}}$ **h** $169^{\frac{1}{2}}$

i $1\,000\,000^{\frac{1}{2}}$ **j** $64^{\frac{1}{3}}$ **k** $32^{\frac{1}{5}}$

2 Find the value of the following.

a $\sqrt{10^2 + 24^2}$ **b** $\sqrt{39^2 - 36^2}$ **c** $\sqrt{12^2 + 35^2}$

d $\sqrt{(4+2)^2 - 11}$ **e** $10(3+5) - (\sqrt{9} - 2)$ **f** $\sqrt{(3+2)^2 - (5-2)^2}$

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1D Approximations, decimal places and significant figures

Approximations occur when we are not able to give exact numerical values in mathematics. Some numbers are too long (e.g. 0.573 128 9 or 107 000 000 000) to work with and they are rounded to make calculations easier. Calculators are powerful tools and have made many tasks easier that previously took a considerable amount of time. Nevertheless, it is still important to understand the processes of rounding and estimation.

Some questions do not require an exact answer and a stated degree of accuracy is often sufficient. Some questions may only need an answer rounded to the nearest tenth, hundredth etc. Other questions may ask for an answer correct to two decimal places or to three significant figures.

▶ Rules for rounding

Rules for rounding

1. Look at the value of the digit to the right of the specified digit.
2. If the value is 5, 6, 7, 8 or 9, *round the digit up*.
3. If the value is 0, 1, 2, 3 or 4, *leave the digit unchanged*.

Example 4 Rounding to the nearest thousand

Round 34 867 to the nearest thousand.

Solution

1. Look at the first digit after the thousands. It is an 8.

   ⇓
   34 867

2. As it is 5 or more, increase the digit to its left by one. So the 4 becomes a 5. The digits to the right all become zero. Write your answer.

   35 000

Note: 34 867 is closer to 35 000 than 34 000

▶ Scientific notation (standard form)

When we work with very large or very small numbers, we often use *scientific notation*, also called *standard form*.

To write a number in scientific notation we express it as a number between 1 and 10 multiplied by a power of 10.

Scientific notation

Large numbers

$$\overbrace{249000000000} = 2.49 \times 100\,000\,000\,000$$
$$= 2.49 \times 10^{11}$$

The decimal point needs to be moved 11 places to the right to obtain the basic numeral.

Multiplying by $10^{\text{positive power}}$ gives the effect of moving the decimal point to the right to make the number larger.

Small numbers

$$0.\overbrace{000000002} = 2.0 \div 1\,000\,000\,000$$
$$= 2.0 \times 10^{-9}$$

The decimal point needs to be moved 9 places to the left to obtain the basic numeral.

Multiplying by $10^{\text{negative power}}$ gives the effect of moving the decimal point to the left to make the number smaller.

Example 5 Writing a number in scientific notation

Write the following numbers in scientific notation.

a 7 800 000 **b** 0.000 000 5

Solution

a **1** Write 7 800 000 as a number between 1 and 10 (7.8) and decide what to multiply it by to make 7 800 000.

$7\,800\,000 = 7.8 \times 1\,000\,000$

$7\,\overbrace{8\,0\,0\,0\,0\,0}^{\text{6 places}}$

2 Count the number of places the decimal point needs to move and whether it is to the left or right.

Decimal point needs to move 6 places to the right from 7.8 to make 7 800 000.

3 Write your answer.

$7\,800\,000 = 7.8 \times 10^6$

b **1** Write 0.000 000 5 as a number between 1 and 10 (5.0) and decide what to divide it by to make 0.000 000 5

$0.000\,000\,5 = 5.0 \div 10\,000\,000$

$0.\overbrace{0\,0\,0\,0\,0\,0\,5}^{\text{7 places}}$

2 Count the number of places the decimal point needs to move and whether it is to the left or right.

Decimal point needs to move 7 places to the left from 5.0 to make 0.000 000 5

3 Write your answer.

$0.000\,000\,5 = 5.0 \times 10^{-7}$

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Example 6 Writing a scientific notation number as a basic numeral

Write the following scientific notation numbers as basic numerals.

a 3.576×10^7 **b** 7.9×10^{-5}

Solution

a 1 Multiplying 3.576 by 10^7 means that the decimal point needs to be moved 7 places to the right.

3.576×10^7

$\overset{\text{7 places}}{3.5\,6\,7\,0\,0\,0\,0} \times 10^7$

2 Move the decimal place 7 places to the right and write your answer. Zeroes will need to be added as placeholders.

$= 35\,760\,000$

b 1 Multiplying 7.9 by 10^{-5} means that the decimal point needs to be moved 5 places to the left.

7.9×10^{-5}

$\overset{\text{5 places}}{0.0\,0\,0\,0\,7\,9} \times 10^{-5}$

2 Move the decimal place 5 places to the left and write your answer.

$= 0.000\,079$

Significant figures

The first non-zero digit, reading from left to right in a number, is the first *significant figure*. It is easy to think of significant figures as all non-zero figures, except where the zero is between non-zero figures. The number of significant figures is shown in red below.

For example:

Number	Significant figures	Explanation
596.36	5	All numbers provide useful information.
5000	1	We do not know anything for certain about the hundreds, tens or units places. The zeroes may be just placeholders or they may have been rounded off to give this value.
0.0057	2	Only the 5 and 7 tell us something. The other zeroes are placeholders.
0.00570	3	The last zero tells us that the measurement was made accurate to the last digit.
8.508	4	Any zeroes between significant digits are significant.
0.00906	3	Any zeroes between significant digits are significant.
560.0	4	The zero in the tenths place means that the measurement was made accurate to the tenths place. The first zero is between significant digits and is therefore significant.

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Rules for significant figures

1. All non-zero digits are significant.
2. All zeroes between significant digits are significant.
3. After a decimal point, all zeroes to the right of non-zero digits are significant.

Example 7 Rounding to a certain number of significant figures

Round 93.738 095 to:

a two significant figures **b** one significant figure **c** five significant figures

Solution

a **1** Count the significant figures in 93.738 095

There are eight significant figures.

2 For two significant figures, start counting two non-zero numbers from the left.

93.738 095

3 The next number (7) is 5 or more so we increase the previous number (3) by one (making it 4). Write your answer.

= 94 (two significant figures)

b **1** For one significant figure, count one non-zero number from the left.

93.738 095

2 The next number (3) is less than 5 so we leave the previous number (9) as it is and replace the 3 with 0 to make only one significant figure. Write your answer.

= 90 (one significant figure)

c **1** For five significant figures, start counting five non-zero numbers from the left.

93.738 095

2 The next number (0) is less than 5 so do not change the previous number (8). Write your answer.

= 93.738 (five significant figures)

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Example 8 Rounding to a certain number of significant figures

Round 0.006 473 5 to:

a four significant figures **b** three significant figures **c** one significant figure

Solution

a	**1**	Count the significant figures.	There are five significant figures.
	2	Count four non-zero numbers starting from the left.	0.006 473 5
	3	The next number (5) is 5 or more. Increase the previous number (3) by one (4). Write your answer.	= 0.006 474 (four significant figures)
b	**1**	For three significant figures, count three non-zero numbers from the left.	0.006 473 5
	2	The next number (3) is less than 5 so we leave the previous number (7) as it is. Write your answer.	= 0.006 47 (three significant figures)
c	**1**	For one significant figure, count one non-zero number from the left.	0.006 473 5
	2	The next number (4) is less than 5 so do not change the previous number (6). Write your answer.	= 0.006 (one significant figure)

▶ Decimal places

23.798 is a decimal number with three digits after the decimal point. The first digit (7) after the decimal point is the first (or one) decimal place. Depending on the required accuracy we round to one decimal place, two decimal places, etc.

Example 9 Rounding correct to a number of decimal places

Round 94.738 295 to:

a two decimal places **b** one decimal place **c** five decimal places

Solution

a	**1**	For two decimal places, count two places after the decimal point and look at the next digit (8).	94.738 295
	2	As 8 is 5 or more, increase the digit to the left of 8 by one. (3 becomes 4) Write your answer.	= 94.74 (to two decimal places)

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b **1** For one decimal place, count one place after the decimal point and look at the next digit (3). 94.738 295

2 As 3 is less than 5, the digit to the left of 3 remains unchanged. Write your answer. = 94.7 (to one decimal place)

c **1** For five decimal places, count five places after the decimal point and look at the next digit (5). 94.738 295

2 As the next digit (5) is 5 or more, the digit to the left of 5 needs to be increased by one. As this is a 9, the next higher number is 10, so the previous digit also needs to change to the next higher number. Write your answer. = 94.738 30 (to five decimal places)

Exercise 1D

Example 4

1 Round off to the nearest whole number.

a 87.15 **b** 605.99 **c** 2.5 **d** 33.63

Example 4

2 Round off to the nearest hundred.

a 6827 **b** 46 770 **c** 79 999 **d** 313.4

Example 6

3 Write these scientific notation numbers as basic numerals.

a 5.3467×10^4 **b** 3.8×10^6 **c** 7.89×10^5 **d** 9.21×10^{-3}

e 1.03×10^{-7} **f** 2.907×10^6 **g** 3.8×10^{-12} **h** 2.1×10^{10}

Example 5

4 Write these numbers in scientific notation.

a 792 000 **b** 14 600 000 **c** 500 000 000 000 **d** 0.000 009 8

e 0.145 697 **f** 0.000 000 000 06 **g** 2 679 886 **h** 0.0087

5 Express the following approximate numbers, using scientific notation.

a The mass of the Earth is 6 000 000 000 000 000 000 000 000 kg.

b The circumference of the Earth is 40 000 000 m.

c The diameter of an atom is 0.000 000 000 1 m.

d The radius of the Earth's orbit around the Sun is 150 000 000 km.

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Example 7, 8

6 For each of the following numbers, state the number of significant figures.

a 89 156 **b** 608 765 **c** 900 000 000 000 **d** 0.709

e 0.10 **f** 0.006 **g** 450 000 **h** 0.008 007

7 Write the following correct to the number of significant figures indicated in each of the brackets.

a 4.8976 (2) **b** 0.078 74 (3)

c 1506.892 (5) **d** 5.523 (1)

8 Calculate the following and give your answer correct to the number of significant figures indicated in each of the brackets.

a $4.3968 \times 0.000\,743\,8$ (2) **b** $0.611\,35 \div 4.1119$ (5)

c $3.4572 \div 0.0109$ (3) **d** $50\,042 \times 0.0067$ (3)

Example 9

9 Use a calculator to find answers to the following. Give each answer correct to the number of decimal places indicated in the brackets.

a 3.185×0.49 (2) **b** $0.064 \div 2.536$ (3)

c 0.474×0.0693 (2) **d** $12.943 \div 6.876$ (4)

e $0.006\,749 \div 0.000\,382$ (3) **f** $38.374\,306 \times 0.007\,493$ (4)

10 Calculate the following, correct to two decimal places.

a $\sqrt{7^2 + 14^2}$ **b** $\sqrt{3.9^2 + 2.6^2}$ **c** $\sqrt{48.71^2 - 29^2}$

1E Conversion of units

The modern metric system in Australia is defined by the International System of Units (SI), which is a system of measuring and has three main units.

The three main SI units of measurement

m the *metre* for length

kg the *kilogram* for mass

s the *second* for time

Larger and smaller units are based on these by the addition of a prefix. When solving problems, we need to ensure that the units we use are the same. We may also need to convert our answer into specified units.

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Conversion of units

To convert units remember to:

- use multiplication (×) when you convert from a larger unit to a smaller unit
- use division (÷) when you convert from a smaller unit to a larger unit.

The common units used for measuring *length* are kilometres (km), metres (m), centimetres (cm) and millimetres (mm). The following chart is useful when converting units of length, and can be adapted to other metric units.

$$\begin{tikzcd}
\textit{Length} & \text{km} \arrow[r, bend left, "\times 1000"] & \text{m} \arrow[l, bend left, "\div 1000"] \arrow[r, bend left, "\times 100"] & \text{cm} \arrow[l, bend left, "\div 100"] \arrow[r, bend left, "\times 10"] & \text{mm} \arrow[l, bend left, "\div 10"] \\
\textit{Area} & \text{km}^2 \arrow[r, bend left, "\times 1000^2"] & \text{m}^2 \arrow[l, bend left, "\div 1000^2"] \arrow[r, bend left, "\times 100^2"] & \text{cm}^2 \arrow[l, bend left, "\div 100^2"] \arrow[r, bend left, "\times 10^2"] & \text{mm}^2 \arrow[l, bend left, "\div 10^2"] \\
\textit{Volume} & \text{km}^3 \arrow[r, bend left, "\times 1000^3"] & \text{m}^3 \arrow[l, bend left, "\div 1000^3"] \arrow[r, bend left, "\times 100^3"] & \text{cm}^3 \arrow[l, bend left, "\div 100^3"] \arrow[r, bend left, "\times 10^3"] & \text{mm}^3 \arrow[l, bend left, "\div 10^3"]
\end{tikzcd}$$

The common units for measuring *liquids* are kilolitres (kL), litres (L) and millilitres (mL).

1 kilolitre = 1000 litres
1 litre = 1000 millilitres

The common units for measuring *mass* are tonnes (t), kilograms (kg), grams (g) and milligrams (mg).

1 tonne = 1000 kilograms
1 kilogram = 1000 grams
1 gram = 1000 milligrams

Note: Strictly speaking the litre and tonne are not included in the SI, but are commonly used with SI units.

The following prefixes are useful to remember.

Prefix	Symbol	Definition	Decimal
micro	μ	millionth	0.000 001
milli	m	thousandth	0.001
centi	c	hundredth	0.01
deci	d	tenth	0.1
kilo	k	thousand	1000
mega	M	million	1 000 000
giga	G	billion	1 000 000 000

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Example 10 Converting between units

Convert these measurements into the units given in the brackets.

a 5.2 km (m) **b** 339 cm^2 (m^2) **c** 9.75 cm^3 (mm^3)

Solution

a As there are 1000 metres in a kilometre and we are converting from kilometres (km) to a smaller unit (m), we need to multiply 5.2 by 1000.

$$5.2 \times 1000$$
$$= 5200 \text{ m}$$

b As there are 100^2 square centimetres in a square metre and we are converting from square centimetres (cm^2) to a larger unit (m^2), we need to divide 339 by 100^2.

$$339 \div 100^2$$
$$= 0.039 \text{ m}^2$$

c As there are 10^3 cubic millimetres in a cubic centimetre and we are converting from cubic centimetres (cm^3) to a smaller unit (mm^3), we need to multiply 9.75 by 10^3.

$$9.75 \times 10^3$$
$$= 9750 \text{ mm}^3$$

Sometimes a measurement conversion requires more than one step.

Example 11 Converting between units requiring more than one step

Convert these measurements into the units given in the brackets.

a 40 000 cm (km) **b** 0.000 22 km^2 (cm^2) **c** 0.08m^3 (mm^3)

Solution

a As there are 100 centimetres in a metre and 1000 metres in a kilometre and we are converting from centimetres (cm) to a larger unit (km), we need to divide 40 000 by (100 × 1000) = 100 000.

$$40\,000 \div 100\,000$$
$$= 0.4 \text{ km}$$

b As there are 100^2 square centimetres in a square metre and 1000^2 square metres in a square kilometre and we are converting from square kilometres (km^2) to a smaller unit (cm^2), we need to multiply 0.000 22 by ($100^2 \times 1000^2$).

$$0.000\,22 \times 100^2 \times 1000^2$$
$$= 2\,200\,000 \text{ cm}^2$$

c As there are 10^3 cubic millimetres in a cubic centimetre and 100^3 cubic centimetres in a cubic metre and we are converting from cubic metres (m^3) to a smaller unit (mm^3), we need to multiply 0.08 by ($10^3 \times 100^3$).

$$0.08 \times 10^3 \times 100^3$$
$$= 80\,000\,000 \text{ mm}^3$$

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Exercise 1E

Example 10

1 Convert the following measurements into the units given in brackets.

a 5.7 m (cm) **b** 1.587 km (m) **c** 8 cm (mm) **d** 670 cm (m)
e 0.0046 km (cm) **f** 289 mm^2 (cm^2) **g** 5.2 m^2 (cm^2) **h** 0.08 km^2 (m^2)
i 3700 mm^2 (cm^2) **j** 6 m^2 (mm^2) **k** 500 mL (L) **l** 0.7 kg (g)
m 2.3 kg (mg) **n** 567 000 mL (kL) **o** 793 400 mg (g) **p** 75.5 kg (mg)
q 0.5 L (mL)

Example 11

2 Convert the following measurements into the units indicated in brackets and give your answer in standard form.

a 5 tonne (kg) **b** 6000 mg (kg) **c** 27 100 km^2 (m^2) **d** 33 m^3 (cm^3)
e 487 m^2 (km^2) **f** 28 mL (L) **g** 6 km (cm) **h** 1125 mL (kL)
i 50 000 m^3 (km^3) **j** 340 000 mm^3 (m^3)

3 Find the total sum of these measurements. Express your answer in the units given in brackets.

a 14 cm, 18 mm (mm) **b** 589 km, 169 m (km)
c 3.4 m, 17 cm, 76 mm (cm) **d** 300 mm^2, 10.5 cm^2 (cm^2)

4 A wall in a house is 7860 mm long. How many metres is this?

5 A truck weighs 3 tonne. How heavy is this in kilograms?

6 An Olympic swimming pool holds approximately 2.25 megalitres of water. How many litres is this?

7 Baking paper is sold on a roll 30 cm wide and 10 m long. How many baking trays of width 30 cm and length 32 cm could be covered with one roll of baking paper?

1F Percentages

Per cent is an abbreviation of the Latin words *per centum*, which mean ‘by the hundred’.

A **percentage** is a rate or a proportion expressed as part of one hundred. The symbol used to indicate percentage is %. Percentages can be expressed as common fractions or as decimals.

For example: 17% (17 per cent) means 17 parts out of every 100.

$$17\% = \frac{17}{100} = 0.17$$

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Conversions

1 To convert a fraction or a decimal to a percentage, multiply by 100.

2 To convert a percentage to a decimal or a fraction, divide by 100.

Example 12 Converting fractions to percentages

Express $\frac{36}{90}$ as a percentage.

Solution

Method 1 (by hand)

1 Multiply the fraction $\frac{36}{90}$ by 100.

2 Evaluate and write your answer.

Note: The above calculation can be performed on the ClassPad calculator.

$$\frac{36}{90} \times 100$$
$$= 40\%$$

Method 2 (using CAS)

1 Enter 36 ÷ 90 on calculator.

2 Press % sign and EXE (Casio) or ENTER (Ti-Nspire).

3 Write your answer.

36/90%

40

Expressed as a percentage, $\frac{36}{90}$ is 40%.

Example 13 Converting a decimal to a percentage

Express 0.75 as a percentage.

Solution

1 Multiply 0.75 by 100.

2 Evaluate and write your answer.

$$0.75 \times 100$$
$$= 75\%$$

Example 14 Converting a percentage to a fraction

Express 62% as a common fraction.

Solution

1 As 62% means 62 out of 100, this can be written as a fraction $\frac{62}{100}$.

$$62\% = \frac{62}{100}$$

2 Simplify the fraction by dividing both the numerator and the denominator by 2.

$$= \frac{62 \div 2}{100 \div 2}$$
$$= \frac{31}{50}$$

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Example 15 **Converting a percentage to a decimal**

Express 72% as a decimal.

Solution

1 Write 72% as a fraction over 100 and express this as a decimal.

$$\frac{72}{100} = 0.72$$

Finding a percentage of a quantity

To find a percentage *of* a number or a quantity, remember that in mathematics *of* means *multiply*.

Example 16 **Finding a percentage of a quantity**

Find 15% of $140.

Solution

Method 1

1 Write out problem and rewrite 15% as a fraction out of 100.

$$15\% \text{ of } 140$$
$$= \frac{15}{100} \text{ of } 140$$

2 Change *of* to *multiply*.

$$= \frac{15}{100} \times 140$$

3 Perform the calculation and write your answer.

$$= 21$$

Note: The above calculation can be performed on the CAS calculator.

Method 2 (using CAS)

1 Enter 15%140 on calculator.

2 Press EXE (Casio) or ENTER (Ti-Nspire).

```
15%140
                21
```

3 Write your answer.

21

Comparing two quantities

One quantity or number may be expressed as a percentage of another quantity or number (both quantities must always be in the same units). Divide the quantity by what you are comparing it with and then multiply by 100 to convert it to a percentage.

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Example 17 Expressing a quantity as a percentage of another quantity

There are 18 girls in a class of 25 students. What percentage of the class are girls?

Solution

1 Work out the fraction of girls in the class.

$\text{Girls} = \frac{18}{25}$

2 Convert the fraction to a percentage by multiplying by 100.

$\frac{18}{25} \times 100$

3 Evaluate and write your answer.

$= 72$

72% of the class are girls.

Example 18 Expressing a quantity as a percentage of another quantity with different units

Express 76 mm as a percentage of 40 cm.

Solution

1 First convert 40 centimetres to millimetres by multiplying by 10, as there are 10 millimetres in 1 centimetre.

$40 \text{ cm} = 40 \times 10$

$= 400 \text{ mm}$

2 Write 76 millimetres as a fraction of 400 millimetres.

$\frac{76}{400}$

3 Multiply by 100 to convert to a percentage.

$\frac{76}{400} \times 100$

4 Evaluate and write your answer.

$= 19\%$

Exercise 1F

Example 12, 13

1 Express the following as percentages.

a $\frac{1}{4}$ **b** $\frac{2}{5}$ **c** $\frac{3}{20}$ **d** $\frac{7}{10}$

e 0.19 **f** 0.79 **g** 2.15 **h** 39.57

i 0.073 **j** 1

Example 14, 15

2 Express the following as:

i common fractions, in their lowest terms

ii decimals.

a 25% **b** 50% **c** 75% **d** 68%

e 5.75% **f** 27.2% **g** 0.45% **h** 0.03%

i 0.0065% **j** 100%

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Example 16 **3** Find the following, correct to three significant figures.

a 15% of \$760
b 22% of \$500
c 17% of 150 m
d $13\frac{1}{2}\%$ of \$10 000
e 2% of 79.34 cm
f 19.6% of 13.46
g 0.46% of 35 €
h 15.9% of \$28 740
i 22.4% of \$346 900
j 1.98% of \$1 000 000

Example 17 **4** From a class, 28 out of 35 students wanted to take part in a project. What percentage of the class wanted to take part?

5 A farmer lost 450 sheep out of a flock of 1200 during a drought. What percentage of the flock were lost?

6 In a laboratory test on 360 light globes, 16 globes were found to be defective. What percentage were satisfactory, correct to one decimal place?

7 After three rounds of a competition, a basketball team had scored 300 points and 360 points had been scored against them. Express the points scored by the team as a percentage of the points scored against them. Give your answer correct to two decimal places.

8 In a school of 624 students, 125 are in year 10. What percentage of the students are in year 10? Give your answer to the nearest whole number.

Example 18 **9** Express 75 cm as a percentage of 2 m.

10 In a population of $3\frac{1}{4}$ million people, 2 115 000 are under the age of 16. Calculate the percentage, to two decimal places, of the population who are under the age of 16.

11 The cost of producing a chocolate bar that sells for \$1.50 is 60c. Calculate the profit made on a bar of chocolate as a percentage of the production cost of a bar of chocolate.

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1G Percentage increase and decrease

When increasing or decreasing a quantity by a given percentage, the percentage increase or decrease is always calculated as a percentage of the *original* quantity.

Example 19 Calculating the new price following a percentage increase

Sally's daily wage of \$175 is increased by 15%. Calculate her new daily wage.

Solution

Method 1

1 First find 15% of \$175 by rewriting 15% as a fraction out of 100 and changing *of* to multiply (or use a calculator).

15% of 175

$$= \frac{15}{100} \times 175$$

2 Perform the calculation and write your answer.

$= 26.25$

3 As \$175 is to be increased by 15%, add \$26.25 to the original amount of \$175.

$175 + 26.25$

$= 201.25$

4 Write your answer in a sentence.

Sally's new daily wage is \$201.25.

Method 2

1 An increase of 15% means that the new amount will be the original amount (in other words, 100%) plus an extra 15%.

Find 115% of 175.

115% of 175

$$= \frac{115}{100} \times 175$$

$= 201.25$

2 Perform the calculation.

3 Write your answer in a sentence.

Sally's new daily wage is \$201.25.

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Example 20 Calculating the new amount following a percentage decrease

A primary school's fun run distance of 2.75 km is decreased by 20% for students in years 2 to 4. Find the new distance.

Solution

Method 1

1 First find 20% of 2.75 by writing 20% as a fraction out of 100 and changing *of* to multiply (or use a calculator).

20% of 2.75

$$= \frac{20}{100} \times 2.75$$

2 Evaluate and write your answer.

$$= 0.55$$

3 As 2.75 km is to be decreased by 20%, subtract 0.55 km from the original 2.75 km.

$$2.75 - 0.55$$

$$= 2.2$$

4 Write your answer in a sentence.

The new distance is 2.2 km.

Method 2

1 A decrease of 20% means that the new amount will be the original amount (100%) minus 20%. Find 80% of 2.75.

80% of 2.75

$$= \frac{80}{100} \times 2.75$$

$$= 2.2$$

2 Perform the calculation.

3 Write your answer in a sentence.

The new distance is 2.2 km.

Example 21 Calculating a new price with a percentage discount

If a shop offers a discount of 15% on items in a sale, what would be the sale price of a pair of jeans originally priced at $95?

Solution

Method 1

1 Find 15% of 95.

$$15\% \text{ of } 95 = \frac{15}{100} \times 95$$

$$= 14.25$$

2 As jeans are discounted by 15%, this is a decrease, so we need to subtract the discounted price of $14.25 from the original price of $95.

$$95 - 14.25$$

$$= 80.75$$

3 Write your answer in a sentence.

The sale price would be $80.75

Method 2

1 A discount of 15% means that the new amount is 85% of 95.

85% of 95

$$= \frac{85}{100} \times 95$$

2 Perform the calculation.

$$= 80.75$$

3 Write your answer in a sentence.

The sale price would be $80.75

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Finding a percentage change

If we are given the original price and the new price of an item, we can find the percentage change. To find percentage change, we compare the change (increase or decrease) with the original number.

Percentage change

$$\text{Percentage change} = \frac{\text{change}}{\text{original}} \times 100$$

Thus:

$$\text{Percentage discount} = \frac{\text{discount}}{\text{original}} \times 100$$

$$\text{Percentage increase} = \frac{\text{increase}}{\text{original}} \times 100$$

Example 22 Calculating a percentage increase

A university increased its total size at the beginning of an academic year by 3000 students. If the previous number of students was 35 000, by what percentage, correct to two decimal places, did the student population increase?

Solution

1 To find the percentage increase, use the formula:

$$\text{Percentage increase} = \frac{\text{increase}}{\text{original}} \times 100$$

Substitute increase as 3000 and original as 35 000.

$$\text{Percentage increase} = \frac{\text{increase}}{\text{original}} \times 100$$

$$= \frac{3000}{35\,000} \times 100$$

2 Evaluate.

$$= 8.5714\ldots$$

3 Write your answer correct to two decimal places.

Student population increased by 8.57%.

Example 23 Calculating the percentage discount

Calculate the percentage discount obtained when a calculator with a normal price of \$38 is sold for \$32 to the nearest whole per cent.

Solution

1 Find the amount of discount given by subtracting the new price, \$32, from the original price \$38.

$$\text{Discount} = \$38 - \$32$$
$$= \$6$$

2 To find the percentage discount, use formula:

$$\text{Percentage discount} = \frac{\text{discount}}{\text{original}} \times 100$$

Substitute discount as 6 and original as 38 and evaluate.

$$\text{Percentage discount} = \frac{\text{discount}}{\text{original}} \times 100$$
$$= \frac{6}{38} \times 100$$
$$= 15.7895\ldots$$

3 Write your answer to the nearest whole per cent.

The percentage discount is 16%.

Exercise 1G

Example 21

1 A jewellery store has a promotion of 20% discount on all watches.

a How much discount will you get on a watch marked \$185?

b What is the sale price of the watch?

2 A store gave different savings discounts on a range of items in a sale. Copy and complete the following table.

	Normal price	% Discount	Saving	Sale price
a	\$89.99	5		
b	\$189.00	10		
c	\$499.00	15		
d	\$249.00	20		
e	\$79.95	22.5		
f	\$22.95	25		
g	\$599.00	27.5		
h	\$63.50	30		
i	\$1000.00	33		

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3 In a particular shop the employees are given a $12\frac{1}{2}\%$ discount on any items they purchase. Calculate the actual price an employee would pay for each of the following:

a \$486 laptop
b \$799 HD LED television
c \$260 iPod
d \$750 digital camera
e \$246 digital video recorder

4 A clothing store offers 6% discount for cash sales. A customer who paid cash purchased the following items:

One pair of jeans \$95.95
A leather belt at \$29.95
Two jumpers at \$45 each

Calculate:

a the total saving
b the actual amount paid for the goods.

5 Which results in the larger sum of money, increasing \$50 by 10% or decreasing \$60 by 8%?

Example 19 **6** The production of a particular model of car is increased from 14 000 by 6% over a 12-month period. What is the new production figure?

7 If a new car is sold for \$23 960 and three years later it is valued at \$18 700, calculate the percentage depreciation, correct to two decimal places.

Example 22 **8** A leading tyre manufacturer claims that a new tyre will average 12% more life than a previous tyre. The owner of a taxi fleet finds that the previous tyre averaged 24 000 km before replacement. How many kilometres should the new tyre average?

Example 23 **9** Calculate the percentage discount for each of the following, to the nearest whole number.

	Normal price	Selling price	% Discount
a	\$60.00	\$52.00	
b	\$250.00	\$185.00	
c	\$5000.00	\$4700.00	
d	\$3.80	\$2.90	
e	\$29.75	\$24.50	
f	\$12.95	\$10.00	

10 A second-hand car advertised for sale at \$13 990 was sold for \$13 000. Calculate, correct to two decimal places, the percentage discount obtained by the purchaser.

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11 A sport shop advertised the following items in their end-of-year sale. Calculate the percentage discount for each of the items to the nearest whole number.

		Normal price	Selling price	% Discount
a	Shoes	\$79.99	\$65.00	
b	12 pack of golf balls	\$29.99	\$19.99	
c	Exercise bike	\$1099.00	\$599.00	
d	Basket ball	\$49.99	\$39.99	
e	Sports socks	\$14.95	\$10.00	
f	Hockey stick	\$299.00	\$250.00	

12 Find the percentage increase that has been applied in each of the following:

a a book that is increased from \$20 to \$25

b an airfare that is increased from \$300 to \$420

c accommodation costs that are increased from \$540 to \$580.50.

1H Ratio and proportion

Ratios are used to numerically compare the values of two or more quantities.

A *ratio* can be written as **a : b** (read as 'a to b'). It can also be written as a fraction $\frac{a}{b}$.

The order of the numbers or numerals in a ratio is important. a : b is *not* the same as b : a

Example 24 Expressing quantities as a ratio

In a year 10 class of 26 students there are 14 girls and 12 boys. Express the number of girls to boys as a ratio.

Solution

As there are 14 girls and 12 boys, the ratio of girls to boys is 14 : 12.

Note: This could also be written as a fraction $\frac{14}{12}$.

Example 25 Expressing more than two quantities as a ratio

A survey of the same group of 26 students showed that 10 students walked to school, 11 came by public transport, and 5 were driven by their parents. Express as a ratio the number of students who walked to school to the number of students who came by public transport to the number of students who were driven to school.

Solution

The order of the numbers in a ratio is important.

10 students walked, 11 used public transport and 5 were driven so the ratio is 10 : 11 : 5.

Example 26 Expressing quantities as a ratio

A cordial bottle has instructions to mix 1 part cordial with 4 parts water. Express this as a ratio.

Solution

The ratio of cordial to water is 1 : 4. This could also be written as $\frac{1}{4}$.

Note: The reverse ratio of water *to* cordial is 4 : 1, which could also be written as $\frac{4}{1}$.

Exercise 1H

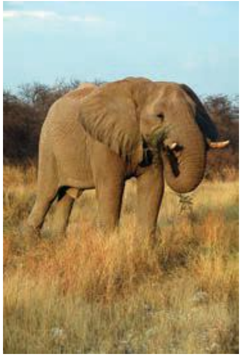
Example 24 **1** A survey of a group of 50 year 11 students in a school showed that 35 of them have a part-time job and 15 do not. Express the number of students having a part-time job to those who do not as a ratio.

Example 25 **2** The table below shows the average life expectancy of some animals.

Animal	Life expectancy
Chimpanzee	40 years
Elephant	70 years
Horse	40 years
Kangaroo	9 years
Tortoise	120 years
Mouse	4 years
Whale	80 years

Find the ratios between the life expectancies of the following animals.

- **a** Whale to horse
- **b** Elephant to kangaroo
- **c** Whale to tortoise
- **d** Chimpanzee to mouse
- **e** Horse to mouse to whale



1I Expressing ratios in their simplest form

Ratios can be simplified by dividing through by a common factor or by multiplying each term as required.

Example 27 Simplifying ratios

Simplify the follow ratios.

a $15:20$ **b** $0.4:1.7$ **c** $\frac{3}{4}:\frac{5}{3}$

Solution

a **1** Divide both 15 and 20 by 5.

2 Evaluate and write your answer.

$$15:20$$
$$=\frac{15}{5}:\frac{20}{5}$$
$$=3:4$$

b **1** Multiply both 0.4 and 1.7 by 10 to give whole numbers.

2 Evaluate and write your answer.

$$0.4:1.7$$
$$=0.4\times 10:1.7\times 10$$
$$=4:17$$

c Method 1

1 Multiply both fractions by 4.

$$\frac{3}{4}\times 4:\frac{5}{3}\times 4$$
$$=3:\frac{20}{3}$$

2 Multiply both sides of the equation by 3.

3 Write your answer.

$$=3\times 3:\frac{20}{3}\times 3$$
$$=9:20$$

Method 2

1 Multiply both $\frac{3}{4}$ and $\frac{5}{3}$ by the lowest common multiple (LCM) of 3 and 4, which is 12, to eliminate fractions.

2 Evaluate and write your answer.

$$\frac{3}{4}:\frac{5}{3}$$
$$=\frac{3}{4}\times 12:\frac{5}{3}\times 12$$
$$=9:20$$

In each of the above examples, the ratios are equivalent and the information is unchanged. For example, the ratio:

12 : 8 is equivalent to the ratio 24 : 16 (multiply both 12 and 8 by 2)

and

12 : 8 is also equivalent to the ratio 3 : 2 (divide both 12 and 8 by 4)

Ratios

1 When ratios are written in terms of the smallest possible whole numbers, they are expressed in their *simplest form*.

2 The order of the figures in a ratio is important. 3 : 5 is *not* the same as 5 : 3.

3 Both parts of a ratio must be expressed in the same unit of measurement.

Example 28 Simplifying ratios with different units

Express 15 cm to 3 m as a ratio in its simplest form.

Solution

1 Write down the ratio.

$$15 \text{ cm} : 3 \text{ m}$$

2 Convert 3 m to cm, by multiplying 3 m by 100, so that both parts of the ratio will be in the same units.

$$15 \text{ cm} : 3 \times 100 \text{ cm}$$
$$= 15 \text{ cm} : 300 \text{ cm}$$

3 Simplify the ratio by dividing both 15 and 300 by 15.

$$= 15 : 300$$
$$= \frac{15}{15} : \frac{300}{15}$$

4 Write your answer.

$$= 1 : 20$$

Example 29 Finding missing values in a ratio

Find the missing value for the equivalent ratios 3 : 7 = ☐ : 28.

Solution

1 Let the unknown value be *x* and write the ratios as fractions.

2 Solve for *x*.

$$3 : 7 = x : 28$$
$$\frac{3}{7} = \frac{x}{28}$$

Method 1 (by hand)

1 Multiply both sides of equation by 28.

$$\frac{3}{7} \times 28 = \frac{x}{28} \times 28$$

2 Evaluate and write your answer.

$$x = 12$$
$$3 : 7 = 12 : 28$$

Method 2 (using CAS)

Use the solve function.

$$\text{solve}\left(\frac{3}{7} = \frac{x}{28}, x\right)$$
$$x = 12$$

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Exercise 1I

Example 27

1 Express the following ratios in their simplest forms.

a 12 : 15 **b** 10 : 45 **c** 22 : 55 : 33 **d** 1.3 : 3.9

e 2.7 : 0.9 **f** $\frac{5}{3} : \frac{1}{4}$ **g** 18 : 8

Example 28

2 Express the following ratios in their simplest form after making sure that each quantity is expressed in the same units.

a 60 L to 25 L **b** \$2.50 to \$50 **c** 75 cm to 2 m

d 5 kg to 600 g **e** 15 mm to 50 cm to 3 m **f** 1 km to 1 m to 1 cm

g 5.6 g to 91 g **h** \$30 to \$6 to \$1.20 to 60c

Example 29

3 For each of the following equivalent ratios find the missing value.

a $1 : 4 = \boxed{} : 20$ **b** $15 : 8 = 135 : \boxed{}$ **c** $600 : 5 = \boxed{} : 1$

d $2 : 5 = 2000 : \boxed{}$ **e** $3 : 7 = \boxed{} : 56$

4 Which of the following statements are true and which are false? For those that are false, suggest a correct replacement statement, if possible.

a The ratio 4 : 3 is the same as 3 : 4.

b The ratio 3 : 4 is equivalent to 20 : 15.

c 9 : 45 is equivalent to 1 : 5.

d The ratio 60 to 12 is equivalent to 15 to 3, which is the same as 4 to 1.

e If the ratio of a father's age to his daughter's age is 7 : 1, then the girl is 7 years old when her father is 56.

f If my weekly allowance is $\frac{5}{8}$ of that of my friend, then the ratio of my monthly allowance to the allowance of my friend is 20 : 32.

5 The following recipe is for Anzac biscuits.

Anzac biscuits (makes 25)

100 grams rolled oats

175 grams plain all-purpose flour, sifted

125 grams butter

2 tablespoons golden syrup

60 grams desiccated coconut

125 grams soft brown sugar

3 tablespoons boiling water

1 teaspoon bicarbonate of soda

a What is the unsimplified ratio of rolled oats : coconut : flour : brown sugar : butter?

b Simplify the ratio from part **a**.

c You want to adapt the recipe to make 75 biscuits. What quantity of each ingredient do you need?

1J Dividing quantities in given ratios

Example 30 Dividing quantities in given ratios

Calculate the number of students in each class if 60 students are divided into classes in the following ratios.

a $1 : 3$ **b** $5 : 1$ **c** $1 : 2 : 7$

Solution

a	**1**	Add up the total number of parts. (Remember that a $1 : 3$ ratio means that there is 1 part for every 3 parts).	The total number of parts is $1 + 3 = 4$.
	2	Divide the number of students (60) by the number of parts (4) to give the number of students in one group.	$60 \div 4 = 15$ One group of students will have $1 \times 15 = 15$ students.
	3	Work out how many students in the other group by multiplying the number of parts (3) by the number of students in one group (15).	The other group will have $3 \times 15 = 45$ students.
	4	Check this gives a total of 60 students and write your answer.	$15 + 45 = 60$ The two groups will have 15 and 45 students.
b	**1**	Add up the total number of parts. (Remember that a $1 : 5$ ratio means that there is 1 part for every 5 parts).	The total number of parts is $1 + 5 = 6$.
	2	Divide the number of students (60) by the number of parts (6) to give the number of students in one group.	$60 \div 6 = 10$ One group of students will have $1 \times 10 = 10$ students.
	3	Work out how many students in the other group by multiplying the number of parts (5) by the number of students in one group (10).	The other group will have $5 \times 10 = 50$ students.
	4	Check this gives a total of 60 students and write your answer.	$10 + 50 = 60$. The two groups will have 10 and 50 students.

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c	**1** To divide 60 students into classes in the ratio 1 : 2 : 7, first add up the total number of parts.	The total number of parts is 1 + 2 + 7 = 10.
	2 Divide the number of students (60) by the number of parts (10) to give the number of students in one group.	60 ÷ 10 = 6 One group of students will have 1 × 6 = 6 students.
	3 Work out how many students in the other two groups by multiplying the number of parts (2) and (7) by the number of students in one group (6).	The other groups will have 2 × 6 = 12 students and 7 × 6 = 42 students.
	4 Check that this gives 60 students and write your answer.	6 + 12 + 42 = 60 The three groups will have 6, 12 and 42 students.

Exercise 1J

Example 30a,b

1 If a 40 m length of rope is cut in the following ratios, what will be the lengths of the individual pieces of rope?

a 4 : 1 **b** 1 : 7 **c** 60 : 20
d 8 : 8

Example 30c

2 If a sum of $500 were shared among a group of people in the following ratios, how much would each person receive?

a 6 : 4 **b** 1 : 4 : 5 **c** 1 : 8 : 1 **d** 8 : 9 : 8
e 10 : 5 : 4 : 1

3 A basket contains bananas, mangos and pineapples in the ratio 10 : 1 : 4. If there are 20 pineapples in the basket, calculate:

a the number of bananas
b the number of mangos
c the total amount of fruit in the basket.

4 7.5 litres of cordial is required for a children's party. If the ratio of cordial to water is 1 : 4:

a how many litres of cordial is required?
b how many litres of water is required?

5 The scale on a map is 1 : 20 000 (in cm). If the measured distance on the map between two historical markers is 15 centimetres, what is the actual distance between the two markers in kilometres?

1K Unitary method

Ratios can be used to calculate unit prices, i.e. the price of one item. This method is known as the *unitary method* and can be used to solve a range of ratio problems.

Example 31 Using the unitary method

If 24 golf balls cost \$86.40, how much do 7 golf balls cost?

Solution

1 Find the cost of 1 golf ball by dividing \$86.40 (the total cost) by 24 (the number of golf balls).	\$86.40 ÷ 24 = \$3.60 \$3.60 × 7 = \$25.20
2 Multiply the cost of one golf ball (\$3.60) by 7. Write your answer.	7 golf balls cost \$25.20

Exercise 1K

Example 31

1 Use the unitary method to answer the following questions.

- **a** If 12 cakes cost \$14.40, how much do 13 cakes cost?
- **b** If a clock gains 20 seconds in 5 days, how much does the clock gain in three weeks?
- **c** If 17 textbooks cost \$501.50, how much would 30 textbooks cost?
- **d** If an athlete can run 4.5 kilometres in 18 minutes, how far could she run in 40 minutes at the same pace?

2 If one tin of red paint is mixed with four tins of yellow paint, it produces five tins of orange paint. How many tins of the red and yellow paint would be needed to make 35 tins of the same shade of orange paint?

3 If a train travels 165 kilometres in 1 hour 50 minutes at a constant speed, calculate how far it could travel in:

- **a** 3 hours
- **b** $2\frac{1}{2}$ hours
- **c** 20 minutes
- **d** 70 minutes
- **e** 3 hours and 40 minutes
- **f** $\frac{3}{4}$ hour

4 Ice creams are sold in two different sizes. A 35 g cone costs \$1.25 and a 73 g cone costs \$2.00. Which is the better buy?

5 A shop sells 2 L containers of Brand A milk for \$2.99, 1 L of Brand B milk for \$1.95 and 600 mL of Brand C milk for \$1.42. Calculate the best buy.

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6 You need 6 large eggs to bake 2 chocolate cakes. How many eggs will you need to bake 17 chocolate cakes?

7 A car uses 45 litres of petrol to travel 495 kilometres. Under the same driving conditions calculate:

- **a** how far the car could travel on 50 litres of petrol
- **b** how much petrol the car would use to travel 187 kilometres.

1L Logarithms

Consider the numbers:

0.01, 0.1, 1, 10, 100, 1000, 10 000, 100 000, 1 000 000

Such numbers can be written more compactly as:

$10^{-2}, 10^{-1}, 10^0, 10^2, 10^3, 10^4, 10^5, 10^6$

In fact, if we make it clear we are only talking about powers of 10, we can merely write down the powers:

$-2, -1, 0, 1, 2, 3, 4, 5, 6$

These powers are called the **logarithms** of the numbers or *logs* for short. When we use logarithms to write numbers as powers of 10, we say we are working with logarithms to the base 10.

Powers of 10

$10^7 = 10\,000\,000$
$10^6 = 1\,000\,000$
$10^5 = 100\,000$
$10^4 = 10\,000$
$10^3 = 1000$
$10^2 = 100$
$10^1 = 10$
$10^0 = 1$
$10^{-1} = 0.1$
$10^{-2} = 0.01$
$10^{-3} = 0.001$

Knowing the powers of 10 is important when using logarithms to the base 10.

Example 32 Evaluating a logarithm

Write the number 100 as a power or 10 and then write down its logarithm.

Solution

1 Write 100 as a power of 10. $100 = 10^2$

2 Write down the logarithm.

$$\log(100) = \log(10^2)$$
$$= 2$$

Example 33 Evaluating a logarithm giving a negative value

Write the number 0.001 as a power of 10 and then write down its logarithm.

Solution

1 Write 0.001 as a power of 10. $0.001 = 10^{-3}$

2 Write down the logarithm.

$$\log(0.001) = \log(10^{-3})$$
$$= -3$$

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Example 34 Using a CAS calculator to find logs

Find the log of 45, correct to one decimal place.

Solution

1 Open a calculator screen, type log(45) and press ENTER (Ti-Nspire) or EXE (Casio).

$\log_{10}(45)$ 1.65321

2 Write the answer correct to one decimal place.

$\log(45) = 1.65\ldots$

$= 1.6$ to one decimal place

Example 35 Using a CAS calculator to evaluate a number if log is known

Find the number whose log is 3.1876, correct to one decimal place.

Solution

1 If the log of a number is 3.1876, then the number is $10^{3.1876}$.

$10^{3.1876}$ 1540.3

2 Enter the expression and press ENTER (Ti-Nspire) or EXE (Casio).

$10^{3.1876} = 1540.281\ldots$

3 Write the answer correct to one decimal place.

$= 1540.3$ to one decimal place

Exercise 1L

Example 32, 33

1 Write the number as a power of 10 and then write down its logarithm.

a 1000 **b** 1 000 000 **c** 0.0001 **d** 10 000 000

e 1 **f** 10 **g** 0.000 000 001

Example 34

2 Use your calculator to evaluate, correct to three decimal places.

a log(300) **b** log(5946) **c** log(10 390) **d** log(0.0047)

e log(0.6) **f** log(0.089) **g** log(7.25)

Determining numbers from logs

Example 35

3 Find the numbers, correct to two decimal places, with logs of:

a 2.5 **b** −1.5

c 0.5 **d** 0

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1M Order of magnitude

Increasing an object by an order of magnitude of 1 means that the object is ten times larger.

An increase of order of magnitude	Increase in size
1	10^1 = 10 times larger
2	10^2 = 100 times larger
3	10^3 = 1000 times larger
6	10^6 = 1 000 000 times larger

Decreasing an object by an order of magnitude 1 means that the object is ten times smaller.

An decrease of order of magnitude	Decrease in size
1	$10^{-1} = 0.1 = \frac{1}{10}$ smaller
2	$10^{-2} = 0.01 = \frac{1}{100}$ smaller
3	$10^{-3} = 0.001 = \frac{1}{1000}$ smaller
6	$10^{-6} = 0.000\,001 = \frac{1}{1\,000\,000}$ smaller

An increase of order of magnitude

In general, an *increase* of n orders of magnitude is the equivalent of multiplying a quantity by 10^n.

A decrease of order of magnitude

In general, a *decrease* of n orders of magnitude is the equivalent of dividing a quantity by 10^n or multiplying a quantity by 10^{-n}.

It is easy to see the order of magnitude of various numbers when they are written in standard form (e.g. 200 in standard form is $2 \times 100 = 2 \times 10^2$).

Example 36 Finding the order of magnitude of a number written in standard form

What is the order of magnitude of 1200?

Solution

1 Write 1200 in standard form.

$1200 = 1.2 \times 10^3$

2 Look at the power of 10 to find the order of magnitude. Write your answer.

The power of 10 is 3 so the order of magnitude of 1200 is 3.

Note: The order of magnitude of 1.2 is 0.

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Exercise 1M

Example 36

1 What is the order of magnitude of the following numbers?

a 46 000 **b** 559 **c** 3 000 000 000

d 4.21×10^{12} **e** 600 000 000 000

2 A city has two TAFE colleges with 4000 students each. What is the order of magnitude of the total number of school students in the city?

3 At the football stadium, 35 000 people attend a football match each week. What is the order of magnitude of the number of people who attend 8 weeks of games?

4 A builder buys 9 boxes containing 1000 screws to build a deck.

a What is the order of magnitude of the total number of screws?

Once the deck is completed, the number of screws left is 90.

b What is the order of magnitude of the number of screws that are left?

1N Logarithmic scales

Some numbers in science are very large or very small.

		Scientific notation
Distance: Earth to Sun	150 000 000 km	1.5×10^8 km
Distance: Earth to moon	384 000 km	3.84×10^5 km
Mass: hydrogen atom	0.000 000 000 000 000 000 000 000 001 673 kg	1.673×10^{-27} kg
Wavelength: yellow light	0.000 000 55 m	5.5×10^{-7} m

Logarithmic scale graphs are useful when plotting a range of very small to very large numbers. Converting values to a logarithmic scale can make it easier to read and interpret values.

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Example 37 Converting values to logarithms in order to sketch a graph

Plot the heartbeat/minute of mammals against their body weight.

Animal	Body weight (g)	Heartbeat/minute
Shrew	2.5	0.40
Chick	50	400
Rabbit	1000	205
Monkey	5000	190
Tree kangaroo	8000	192
Giraffe	900 000	65
Elephant	5 000 000	30
Blue whale	170 000 000	16

Note: To plot the heartbeat/minute of mammals against their body weight, we will be starting from a very small weight value of 50 grams for a chick to 170 tonne = 170 000 kilograms = 170 000 000 grams for a blue whale.

Plotting the body weight values on a horizontal axis is difficult because of the large range of values for the body weight of mammals.

However, if the body weight values are written more compactly as logarithms (powers) of 10, then these logarithms can be placed on a logarithmic scale graph. For example, we have seen that $\log_{10} 100 = 2$. This can also be expressed as $\log(100) = 2$.

Note: $\log_{10} x$ is often written as $\log(x)$.

Solution

1 Convert mammals' body weight to logarithms.

Weight of chick is 50 grams.

Find logarithm (log) of 50

50 is between 10 and 100

$\log(10) = \log(10^1) = 1$

$\log(100) = \log(10^2) = 2$

So log(50) is between 1 and 2.

Use calculator to find log(50).

$\log(50) = 1.70$ (correct to two decimal places).

Weight of tree kangaroo is 8000 grams.

Find log(8000).

8000 is between 1000 and 10 000.

$\log(1000) = \log(10^3) = 3$

$\log(8000) = 3.90$ (correct to two decimal places).

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$\log(10\,000) = \log(10^4) = 4$

So log(8000) must be between 3 and 4.

Use calculator to find log(8000).

Weight of giraffe is 900 000 grams.

Find log(900 000).

900 000 is between 100 000 and 1 000 000.

$\log(100\,000) = \log(10^5) = 5$

$\log(1\,000\,000) = \log(10^6) = 6$

So log(900 000) is between 5 and 6.

Use a calculator to log(900 000)

log(900 000) = 5.95 (correct to two decimal places).

Weight of blue whale 170 000 000 grams (or 1.7×10^8).

Use calculator to find log(170 000 000)

log(170 000 000) = 8.23 (correct to two decimal places).

2 Use a calculator to find the logarithms (logs) of the body weight of the different mammals. Record your results.

Animal	Body weight (g)	Log (weight)
Shrew	2.5	0.40
Chick	50	1.70
Rabbit	1000	3.00
Monkey	5000	3.70
Tree kangaroo	8000	3.90
Giraffe	900 000	5.95
Elephant	5 000 000	6.70
Blue whale	170 000 000	8.23

3 Plot the logarithms of the animals' body weights on the horizontal axis of the graph and the heart rate on the vertical axis.

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On a log scale:

- In moving from 1 to 2 we are actually increasing by a factor of 10.
- In moving from 2 to 3 we are increasing by a factor of 10.
- To go from 2 to 5 we will have multiplied by a factor of $10 \times 10 \times 10 = 1000$.

For example:

- The weight of the elephant is represented by the logarithm of 6.70 and the weight of the monkey is represented by the logarithm of 3.70. The difference between these logarithms is $6.70 - 3.70 = 3$, which means that the elephant is $10^3 = 1000$ times heavier than the monkey.

Or, in a more complex situation:

- The weight of the rabbit is represented by the logarithm of 3 and the weight of the giraffe is represented by the logarithm of 5.95. The difference between these logarithms is $5.95 - 3 = 2.95$ and represents $10^{2.95} = 891.25$ indicating that the giraffe is 891.25 times heavier than a rabbit.

Other real-life examples that use a logarithmic scale

The earthquake magnitude scale

The strength of an earthquake is measured by the Moment Magnitude Scale (MMS), which takes the logarithm of the energy emitted by the quake. It is a modern modification of the earlier Richter Scale. The picture opposite shows the impact of the 2011 earthquake on the Christchurch Cathedral.

The numbers $1, 2, 3, 4, 5, 6, \ldots$ on the MMS indicate an intensity that is ten times stronger than the previous number. For example:

- A magnitude 5 earthquake is 10 times stronger than a magnitude 4 earthquake.
- A magnitude 6 earthquake is $10 \times 10 = 100$ times stronger than a magnitude 4 earthquake.
- A magnitude 7 earthquake is $10 \times 10 \times 10 = 1000$ times stronger than a magnitude 4 earthquake.

Decibels (the loudness of sound)

When using the decibel scale to measure the loudness of sound, the least audible sound is assigned 0.

Thus:

- A sound 10 ($= 10^1$) times louder than 0 is assigned a decibel value of 10.
- A sound 100 ($= 10^2$) times louder than 0 is assigned a decibel value of 20.
- A sound 1000 ($= 10^3$) times louder than 0 is assigned a decibel value of 30.
- A change in power by a factor of 10 corresponds to a 10 dB change in level.

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Example 38 Measuring the strength of an earthquake

The 2011 Tokyo earthquake was magnitude 9.0 on the MMS. To the nearest hundred, how much more intense was this compared to the 2011 Christchurch earthquake which was magnitude 6.3?

Solution

1	Remembering that 9.0 and 6.3 are logarithmic values, subtract 6.3 from 9.0	$9.0 - 6.3 = 2.7$
2	As this is a log value, evaluate $10^{2.7}$.	$10^{2.7} = 501.187\ldots$
3	Round to the nearest hundred and write your answer.	The Tokyo earthquake was 500 times stronger than the Christchurch earthquake.

Example 39 Calculating the intensity of sound

The sound of a normal conversation is 60 decibels and the sound from sitting at the front row of a rock concert is 110 decibels. How much louder is the sound of the rock concert to the sound of normal conversation?

Solution

1	First find out the difference in decibels by subtracting 60 from 110.	$110 - 60 = 50$
2	Each increase of 10 decibels corresponds to 10 times the loudness. Divide 50 by 10 = 5, which corresponds to 5 lots of 10 times the loudness $10 \times 10 \times 10 \times 10 \times 10 = 10^5$.	$50 \div 10 = 5$ There are 5 lots of 10 decibels, which means $10 \times 10 \times 10 \times 10 \times 10 = 10^5$
3	Evaluate 10^5.	$10^5 = 100\,000$
4	Write your answer.	The sound at the front row of the rock concert is 100 000 times louder than normal conversation.

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Exercise 1N

Example 38

1 How many times stronger is a magnitude 7 earthquake than a magnitude 5 earthquake?

2 A magnitude 7.4 was recorded in the Solomon Islands in April 2014. Earlier that month, a magnitude 7.7 earthquake was recorded near the coast of Northern Chile. How much stronger than the Solomon Islands earthquake was the Chilean earthquake? Give your answer to the nearest whole number.

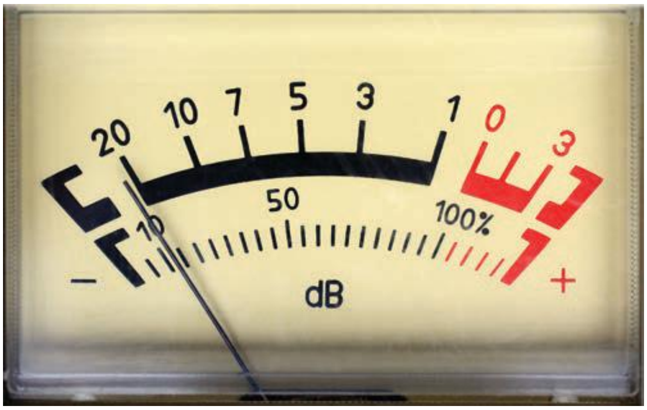
3 How much stronger is a magnitude 6.7 earthquake compared to one of 6.2? Give your answer correct to two decimal places.

Example 37

4 Use the logarithmic values for the animals' weights in Example 37 to find how much heavier than a shrew is a tree kangaroo, to the nearest thousand.

Example 39

5 If the sound of a normal conversation is 60 decibels, and the sound of a train going through a tunnel is 90 decibels, how much louder is the sound of the train than a conversation?



6 The sound of a vacuum cleaner is 80 decibels and someone whispering is 20 decibels. How much softer is the sound of someone whispering than the sound of a vacuum cleaner?

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Key ideas and chapter summary

Order of operation	The order of operations is important. Remember BODMAS or BOMDAS **B**rackets come first **O**f or **O**rders (powers, square roots) **D**ivision and **M**ultiplication come next, working from left to right then **A**ddition and **S**ubtraction, working from left to right
Directed numbers	Multiplying or dividing two numbers with the **same** sign gives a **positive** value. Multiplying or dividing two numbers with **different** signs gives a **negative** value.
Scientific notation	To write a number in scientific notation express it as a number between 1 and 10 multiplied by a power of 10.
Rounding	5.417 rounded to two decimal places is 5.42 (number after the 1 is 7 so round up).
Significant figures	All non-zero digits are significant. All zeroes between significant digits are significant. After a decimal point, all zeroes to the right of non-zero digits are significant.

Conversion of measurements

$$\begin{tikzcd}
\textit{Length} & \text{km} \arrow[r, bend left, "\times 1000"] & \text{m} \arrow[l, bend left, "\div 1000"] \arrow[r, bend left, "\times 100"] & \text{cm} \arrow[l, bend left, "\div 100"] \arrow[r, bend left, "\times 10"] & \text{mm} \arrow[l, bend left, "\div 10"] \\
\textit{Area} & \text{km}^2 \arrow[r, bend left, "\times 1000^2"] & \text{m}^2 \arrow[l, bend left, "\div 1000^2"] \arrow[r, bend left, "\times 100^2"] & \text{cm}^2 \arrow[l, bend left, "\div 100^2"] \arrow[r, bend left, "\times 10^2"] & \text{mm}^2 \arrow[l, bend left, "\div 10^2"] \\
\textit{Volume} & \text{km}^3 \arrow[r, bend left, "\times 1000^3"] & \text{m}^3 \arrow[l, bend left, "\div 1000^3"] \arrow[r, bend left, "\times 100^3"] & \text{cm}^3 \arrow[l, bend left, "\div 100^3"] \arrow[r, bend left, "\times 10^3"] & \text{mm}^3 \arrow[l, bend left, "\div 10^3"]
\end{tikzcd}$$

1 kilolitre = 1000 litres
1 litre = 1000 millilitres
1 tonne = 1000 kilograms
1 kilogram = 1000 grams
1 gram = 1000 milligrams

Percentages

To convert a fraction or a decimal to a percentage, multiply by 100.
To convert a percentage to a decimal or a fraction, divide by 100.

$$\text{Percentage change} = \frac{\text{change}}{\text{original quantity or price}} \times 100$$

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Ratios	The order of the figures in a ratio is important. 4 : 3 is not the same as 3 : 4. Ratios can be simplified. Eg. 6 : 2 = 3 : 1
Logarithmic scales	A logarithmic scale is often used to plot very large and/or very small numbers on a linear scale.

Skills check

Having completed this chapter you should be able to:

- use a variety of mathematical operations in the correct order
- add, subtract, multiply and divide directed numbers
- find powers and roots of numbers
- round numbers to specific place values
- write numbers in scientific notation (standard form)
- understand and use significant figures
- convert units of measurements
- express ratios in their simplest form
- solve practical problems involving ratios, percentages and the unitary method
- use and interpret log scales when used to represent quantities that range over multiple orders of magnitude.

Multiple-choice questions

1 Evaluate $4 + 7 \times 3$.

A 33 **B** 30 **C** 19 **D** 14 **E** 25

2 Evaluate $3 + (6 \div 3) - 2$.

A 3 **B** 6 **C** 1 **D** 9 **E** 8

3 $(8.7 - 4.9) \times (5.4 + 2.8)$ is equal to:

A 23.32 **B** 31.16 **C** -14.96 **D** 12.0 **E** -31.48

4 Evaluate $(-3) \times 4 \times 5$.

A 60 **B** 6 **C** -60 **D** 27 **E** 3

5 Evaluate $(-2) + 8$.

A 10 **B** 6 **C** -10 **D** -6 **E** 28

6 Evaluate $(-2) - (-3)$.

A -5 **B** 5 **C** 1 **D** -1 **E** 6

7 Evaluate 5 − (−9)

A −4 **B** 59 **C** 44 **D** −14 **E** 14

8 3.895 rounded to two decimal places is:

A 3.8 **B** 3.99 **C** 4.0 **D** 3.90 **E** 3.89

9 4679 rounded to the nearest hundred is:

A 5000 **B** 4600 **C** 4700 **D** 4670 **E** 4680

10 5.21×10^5 is the same as:

A 52 100 000 **B** 521 000 **C** 52 105 **D** 0.000 052 1 **E** 260.50

11 0.0048 written in scientific notation is:

A 48×10^{-4} **B** 48×10^{-3} **C** 4.8×10^{3} **D** 4.8×10^{-3} **E** 4.8×10^{-4}

12 28 037.2 rounded to two significant figures is:

A 28 000 **B** 20 000.2 **C** 20 007 **D** 7.2 **E** 28 000.2

13 0.030 69 rounded to two significant figures is:

A 0.03 **B** 0.000 69 **C** 0.0307 **D** 0.031 **E** 0.0306

14 5.1 m^2 is the same as:

A 510 cm^2 **B** 0.0051 km^2 **C** 51 000 cm^2 **D** 5100 mm^2 **E** 51 cm^2

15 56% as a fraction in its simplest form is:

A 0.56 **B** $\frac{56}{100}$ **C** $\frac{0.56}{100}$ **D** $\frac{5.6}{100}$ **E** $\frac{28}{50}$

16 15% of $1600 is equal to:

A $24 **B** $150 **C** $240 **D** $1840 **E** $24 000

17 An item with a cost price of $450 is marked up by 30%. Its selling price is:

A $585 **B** $135 **C** $480 **D** $1350 **E** $463.50

18 A box contains 5 green marbles, 7 blue marbles and 3 yellow marbles. The ratio of blue marbles to total marbles is:

A 7 : 5 : 3 **B** 7 : 8 **C** 7 : 15 **D** 5 : 7 : 3 **E** 5 : 7 : 3 : 15

19 $750 is divided in the ratio 1 : 3 : 2. The smallest share is:

A $250 **B** $125 **C** $375 **D** $750 **E** $150

20 In simplest ratio form the ratio of 450 grams to 3 kilograms is:

A 3 : 20 **B** 450 : 3 **C** 9 : 60 **D** 150 : 1 **E** 15 : 100

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Short-answer questions

1 Evaluate the following.

a $3 + 2 \times 4$ **b** $25 \div (10 - 5) + 5$ **c** $14 - 21 \div 3$

d $(12 + 12) \div 12 + 12$ **e** $27 \div 3 \times 5 + 4$ **f** $4 \times (-2) + 3$

g $\dfrac{10 - 8}{2}$ **h** $\dfrac{4(3 + 12)}{2}$ **i** $\dfrac{-5 + 9}{2}$

2 Calculate the following and give your answer correct to two decimal places where appropriate.

a 5^3 **b** $\sqrt{64} - 5$ **c** $9^{\frac{1}{2}} + 9^{\frac{1}{2}}$ **d** $\sqrt{8}$

e $\sqrt{25 - 9}$ **f** $\sqrt{25} - 9$ **g** $\dfrac{6^3}{(10 \div 2)^2}$ **h** $\sqrt{6^2 + 10^2}$

3 Write each of the following in scientific notation.

a 2945 **b** 0.057 **c** 369 000 **d** 850.9

4 Write the basic numeral for each of the following.

a 7.5×10^3 **b** 1.07×10^{-3} **c** 4.56×10^{-1}

5 Write the following correct to the number of significant figures indicated in the brackets.

a 8.916 (2) **b** 0.0589 (2) **c** 809 (1)

6 Write the following correct to the number of decimal places indicated in the brackets.

a 7.145 (2) **b** 598.241 (1) **c** 4.0789 (3)

7 Convert the following measurements into the units given in brackets.

a 7.07 cm (mm) **b** 2170 m (km) **c** 0.1 m^2 (cm^2)

d 2.5 km^2 (m^2) **e** 0.0005 m^2 (cm^2) **f** 0.000 53 cm^3 (mm^3)

g 5.8 kg (mg) **h** 0.07 L (mL)

8 Express the following percentages as decimals.

a 75% **b** 40% **c** 27.5%

9 Express the following percentages as fractions, in their lowest terms.

a 10% **b** 20% **c** 22%

10 Evaluate the following.

a 30% of 80 **b** 15% of \$70 **c** $12\frac{1}{2}$% of \$106

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11 A new LED smart television was valued at \$1038. During a sale it was discounted by 5%.

a What was the amount of discount?

b What was the sale price?

12 Tom's weekly wage of \$750 is increased by 15%. What is his new weekly wage?

13 A 15-year-old girl working at a local bakery is paid \$12.50 per hour. Her pay will increase to \$15 per hour when she turns 16. What will be the percentage increase to her pay (to the nearest per cent)?

14 A leather jacket is reduced from \$516 to \$278. Calculate the percentage discount (to the nearest per cent).

15 After dieting for three months, Melissa who weighed 78 kg lost 4 kg and Jody's weight dropped from 68 kg to 65 kg. Calculate the percentage weight loss, correct to two decimal places, for each girl.

16 True or false?

a The ratio 3 : 2 is the same as 2 : 3

b 1 : 5 = 3 : 12

c 20 cm : 1 m is written as 20 : 1 in simplest form

d 3 : 4 = 9 : 12

17 If a sum of \$800 were to be shared among a group of people in the following ratios, how much would each person receive?

a 4 : 6 **b** 1 : 4 **c** 2 : 3 : 5 **d** 2 : 2 : 4

18 A recipe for pizza dough requires 3 parts wholemeal flour for each 4 parts of plain flour. How many cups of wholemeal flour are needed if 24 cups of plain flour are used?

19 The scale on a map is 1 : 1000. Find the actual distance (in metres) between two markers if the distance between the two markers on a map is:

a 2.7 cm **b** 140 mm

20 If 5 kilograms of mincemeat costs \$50, how much does 2 kilograms of mincemeat cost?

21 A truck uses 12 litres of petrol to travel 86 kilometres. How far will it travel on 42 litres of petrol?

22 A earthquake measured 6 on the MMS. How many times stronger is a magnitude 6 earthquake compared to a magnitude 3 earthquake?

2

Investigating and comparing data distributions

- What are categorical and numerical data?
- What is a bar chart and when is it used?
- What is a histogram and when is it used?
- What is are dot and stem plots and when are they used?
- What are the mean, median, range, interquartile range and standard deviation?
- What are the properties of these summary statistics and when is each used?
- How do we construct and interpret boxplots?

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Introduction

In this information age we increasingly have to interpret data. This data may be presented in charts, diagrams or graphs, or it may simply be lists of words or numbers. There may be a lot of relevant information embodied in the data, but the story it has to tell will not always be immediately obvious. Various statistical procedures are available which will help us extract the relevant information from data sets. In this chapter, we will look at some of the techniques that are used when the data are collected from a single variable and that can help us to answer real world questions.

2A Types of data

Consider the following situation. In completing a survey, students are asked to:

- indicate their sex by circling an ‘F’ for female or an ‘M’ for male on the form
- indicate their preferred coffee cup size when buying takeaway coffee as ‘small’, ‘medium’ or ‘large’
- write down the number of brothers they have
- measure and write their hand span in centimetres.

The information collected from four students is displayed in the table below.

Sex	Coffee size	Number of brothers	Hand span (in cm)
M	Large	0	23.6
F	Small	2	19.6
F	Small	1	20.2
M	Large	1	24.0

Since the answers to each of the questions in the survey will vary from student to student, each question defines a different **variable** namely: *sex*, *coffee size*, *number of brothers* and *hand span*. The values we collect about each of these variables are called **data**.

The data in the table fall into two broad types: *categorical* or *numerical*.

Categorical data

The data arising from the students responses to the first and second questions in the survey are called **categorical data** because the data values can be used to place the person into one of several groups or categories. However, the properties of the data generated by these two questions differ slightly.

- The question asking students to use an ‘M’ or ‘F’ to indicate their sex will prompt a response of either M or F. This identifies the respondent as either male or female but tells us no more. We call this **nominal data** because it simply *names* or *nominates*.
- However, the question with responses ‘small’, ‘medium’ and ‘large’ that indicates the students’ preferred coffee size tells us two things. Firstly, it names the coffee size, but secondly it enables us to order the students according to their preferred coffee sizes. We call this **ordinal data** because it enables us to both name and order their responses.

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Numerical data

The data arising from the responses to the third and fourth questions in the survey are called **numerical data** because they have values for which arithmetic operations such as adding and averaging make sense. However, the properties of the data generated by these questions differs slightly.

- The question asking students to write down the number of brothers they have will prompt whole number responses like 0, 1, 2, . . .
  Because the data can only take particular numerical values it is called **discrete data**. Discrete data arises in situations where counting is involved. For this reason, discrete data is sometimes called count data.
- In response to the hand span question, students who wrote 24 cm could have an actual hand span of anywhere between 23.5 and 24.4 cm, depending on the accuracy of the measurement and how the student rounded their answer. This is called **continuous data**, because the variable we are measuring, in this case, *hand span*, can take any numerical value within a specified range.
  Continuous data are often generated when measurement is involved. For this reason, continuous data is sometimes called measurement data.

Types of variables

Categorical variables

Variables that generate categorical data are called **categorical variables** or, if we need a finer distinction, **nominal** or **ordinal** variables. For example, *sex* is a nominal variable, while *coffee size* is an ordinal variable.

Numerical variables

Variables that generate numerical data are called numerical variables or, if we need a finer distinction, discrete or continuous variables. For example, *number of brothers* is a **discrete variable**, while *hand span* is a continuous variable.

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Exercise 2A

Classifying data

1 Classify the categorical data arising from people answering the following questions as either nominal or ordinal.

- **a** What is your favourite football team?
- **b** How often do you exercise? Choose one of ‘never’, ‘once a month’, ‘once a week’, ‘every day’.
- **c** Indicate how strongly you agree with ‘alcohol is the major cause of accidents’ by selecting one of ‘strongly agree’, ‘agree’, ‘disagree’, ‘strongly disagree’.
- **d** What language will you study next year, ‘French’, ‘Chinese’, ‘Spanish’ or ‘none’?

2 Classify the data generated in each of the following as categorical or numerical.

- **a** Kindergarten pupils bring along their favourite toys, and they are grouped together under the headings ‘dolls’, ‘soft toys’, ‘games’, ‘cars’ and ‘other’.
- **b** The number of students on each of 20 school buses are counted.
- **c** A group of people each write down their favourite colour.
- **d** Each student in a class is weighed in kilograms.
- **e** Students are weighed and then classified as ‘light’, ‘average’ or ‘heavy’.
- **f** People rate their enthusiasm for a certain rock group as ‘low’, ‘medium’ or ‘high’.

3 Classify the data generated in each of the following situations as nominal, ordinal or numerical (discrete or continuous).

- **a** The different brand names of instant soup sold by a supermarket are recorded.
- **b** A group of people are asked to indicate their attitude to capital punishment by selecting a number from 1 to 5, where 1 = strongly disagree, 2 = disagree, 3 = undecided, 4 = agree and 5 = strongly agree.
- **c** The number of computers per household was recorded during a census.

Classifying variables

4 Classify the numerical variables identified below (in italics) as discrete or continuous.

- **a** The *number of pages* in a book
- **b** The *price* paid to fill the tank of a car with petrol
- **c** The *volume* of petrol (in litres) used to fill the tank of a car
- **d** The *time* between the arrival of successive customers at an ATM
- **e** The *number of people* at a football match

2B Displaying and describing categorical data distributions

To make sense of data, we first need to organise it into a more manageable form. For categorical data, frequency tables and bar charts are used for this purpose.

▶ The frequency table

Frequency

A **frequency table** is a listing of the values a variable takes in a data set, along with how often (frequently) each value occurs.

Frequency can be recorded as a:

- **frequency**: the number of times a value occurs
- **percentage frequency**: the percentage of times a value occurs, where:

$$\text{percentage frequency} = \frac{\text{count}}{\text{total}} \times 100\%$$

- frequency **distribution**: a listing of the values a variable takes, along with how frequently each of these values occurs.

Example 1 Constructing a frequency table for categorical data

Thirty children chose a sandwich, a salad or a pie for lunch, as follows:

sandwich, salad, salad, pie, sandwich, sandwich, salad, salad, pie, pie, pie, salad, pie, sandwich, salad, pie, salad, pie, sandwich, sandwich, pie, salad, salad, pie, pie, pie, salad, pie, sandwich, pie

Construct a table for the data showing both frequency and percentage frequency.

Solution

	Frequency	
Lunch choice	Number	%
Sandwich	7	23.3
Salad	10	33.3
Pie	13	43.3
Total	30	99.9

1. Set up a table as shown. The variable *lunch choice* has three categories: ‘sandwich’, ‘salad’ and ‘pie’.
2. Count the number of children choosing a sandwich, a salad or a pie. Record in the ‘Number’ column.
3. Add the frequencies to find the total number.
4. Convert the frequencies into percentages and record in the ‘%’ column. For example, percentage frequency for pies equals $\frac{13}{30} \times 100\% = 43.3\%$
5. Total the percentages and record. Note that the percentages add up to 99.9%, not 100%, because of rounding.

▶ Bar charts

When there are a lot of data, a frequency table can be used to summarise the information, but we generally find that a graphical display is also useful. When the data are categorical, the appropriate display is a **bar chart**.

Bar charts

In a bar chart:

- frequency or percentage frequency is shown on the vertical axis
- the variable being displayed is plotted on the horizontal axis
- the height of the bar (column) gives the frequency (or percentage)
- the bars are drawn with gaps to indicate that each value is a separate category
- there is one bar for each category.

Example 2 Constructing a bar and percentage bar chart from a frequency table

Use the frequency table for lunch choice from Example 1 to construct:

a a bar chart

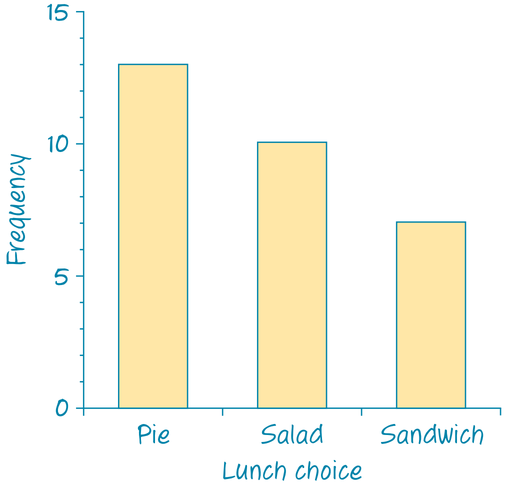
b a percentage bar chart.

Solution

a **1** Label the horizontal axis with the variable name, 'Lunch choice'. Mark the scale off into three equal intervals and label them 'Pie', 'Salad' and 'Sandwich'.

2 Label the vertical axis 'Frequency'. Insert a scale allowing for the maximum frequency of 13. Up to 15 would be appropriate. Mark the scale in 5s.

3 For each interval draw in a bar as shown. Make the width of each bar less than the width of the category intervals to show that the categories are quite separate. The height of each bar is equal to the frequency.



Note: For nominal variables it is common, but not necessary, to list categories in decreasing order by frequency. This makes later interpretation easier.

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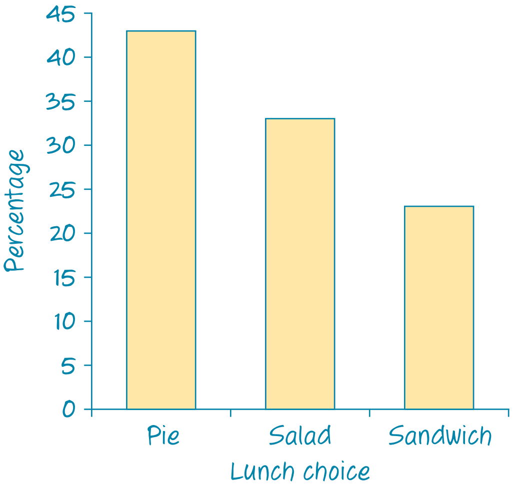
b To construct a percentage bar chart of the lunch choice data, follow the same procedure as above but label the vertical axis ‘Percentage’. Insert a scale allowing for a maximum percentage frequency up to 45%. Mark the vertical scale in intervals of 5%.
The height of each bar is equal to the percentage.



The mode or modal category

One of the features of a data set that is quickly revealed with a bar chart is the **mode** or **modal category**. This is the most frequently occurring category. In a bar chart, this is given by the category with the tallest bar. For the bar chart in Example 2, the modal category is clearly ‘pie’. That is, the most frequent or popular lunch choice was a pie.

When is the mode useful?

The mode is most useful when a single value or category in the frequency table occurs more often (frequently) than the others. Modes are of particular importance in popularity polls, answering questions like ‘Which is the most frequently watched TV station between the hours of 6 p.m. and 8 p.m.?’ or ‘When is a supermarket in peak demand?’

Exercise 2B

Skillsheet

Constructing frequency tables

1 The *sex* of 15 people in a bus is as shown (F = female, M = male):

F M M M F M F F M M M F M M M

Example 1

a Is the variable *sex* nominal or ordinal?

b Construct a frequency table for the data including frequencies and percentages frequency.

2 The UK *shoe size* of 20 eighteen-year-old males are as shown:

8	9	9	10	8	8	7	9	8	9
10	12	8	10	7	8	8	7	11	11

a Is the variable *shoe size* nominal or ordinal?

b Construct a frequency table for the data including frequencies and percentages.

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Analysing frequency tables and constructing bar charts

Example 2

3 The table below shows the frequency distribution of the favourite type of fast food (*food type*) of a group of students.

a Complete the table.

b Is the variable *food type* nominal or ordinal?

c How many students preferred Chinese food?

d What percentage of students chose chicken as their favourite fast food?

e What was the favourite type of fast food for these students?

f Construct a bar chart of the frequencies (number).

Food type	Frequency	
	Number	%
Hamburgers	23	33.3
Chicken	7	10.1
Fish and chips	6	
Chinese	7	10.1
Pizza	18	
Other	8	11.6
Total		99.9

4 The following responses were received to a question regarding the return of capital punishment.

a Complete the table.

b Is the data used to generate this table nominal or ordinal?

c How many people said 'Strongly agree'?

d What percentage of people said 'Strongly disagree'?

e What was the most frequent response?

f Construct a frequencies bar chart.

Capital punishment	Frequency	
	Number	%
Strongly agree	21	8.2
Agree	11	4.3
Don't know	42	
Disagree		
Strongly disagree	129	50.4
Total	256	100.0

5 A bookseller noted the types of books purchased during a particular day, with the following results.

a Complete the table.

b Is the variable *type of book* nominal or ordinal?

c How many books purchased were classified as 'Fiction'?

d What percentage of books were classified as 'Children'?

e How many books were purchased in total?

f Construct a bar chart of the percentage frequencies (%).

Type of book	Frequency	
	Number	%
Children	53	22.8
Fiction	89	
Cooking	42	18.1
Travel	15	
Other	33	14.2
Total	232	

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6 A survey of secondary school students' preferred ways of spending their leisure time at home gave the following results.

a How many students were surveyed?

b Is the variable *leisure activity* nominal or ordinal?

c What percentage of students said that they preferred to spend their leisure time phoning a friend?

d What was the most popular way of spending their leisure time for these students?

e Construct a bar chart of the percentage frequencies (%).

	Frequency	
Leisure activity	Number	%
Watch TV	84	42
Read	26	13
Listen to music	46	23
Watch a movie	24	12
Phone friends	8	4
Other	12	6
Total	200	100

2C Interpreting and describing frequency tables and bar charts

As part of this subject, you will be expected to complete a statistical investigation. Under these circumstances, constructing a frequency table or a bar chart is not an end in itself. It is merely a means to an end. The end is being able to understand something about the variables you are investigating that you didn't know before.

To complete the investigation, you will need to communicate this finding to others. To do this, you will need to know how to describe and interpret any patterns you observe in the context of your data investigation in a written report that is both systematic and concise. The purpose of this section is to help you develop such skills.

Some guidelines for describing the distribution of a categorical variable and communicating your findings

- Briefly summarise the context in which the data were collected including the number of people (or things) involved in the study.
- If there is a clear modal category, make sure that it is mentioned.
- Include relevant counts or percentages in the report.
- If there are a lot of categories, it is not necessary to mention every category.
- Either counts or percentages can be used to describe the distribution.

These guidelines are illustrated in the following examples.

Example 3 Using a frequency table to describe the outcome of an investigation involving a categorical variable

A group of 30 children were offered a choice of a sandwich, a salad or a pie for lunch and their responses collected and summarised in the frequency table opposite.

Lunch choice	Frequency
Sandwich	7
Salad	10
Pie	13
Total	30

Use the frequency table to report on the relative popularity of the three lunch choices quoting appropriate frequencies to support your conclusions.

Solution

Report

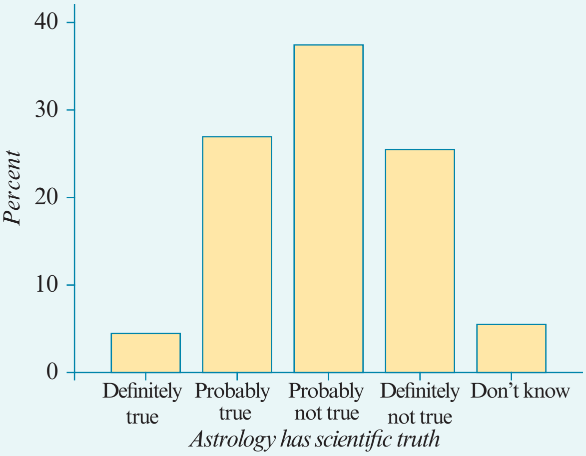
A group of 30 children were offered a choice of a sandwich, a salad or a pie for lunch. The most popular lunch choice was pie, chosen by 13 of the children. Ten of the children chose a salad. The least popular option was sandwich, chosen by only 7 of the children.

Example 4 Using a frequency table and a percentage bar chart to describe the outcome of an investigation involving a categorical variable

A sample of 200 people were asked to comment on the statement ‘Astrology has scientific truth’ by selecting one of the options ‘definitely true’, ‘probably true’, ‘probably not true’, definitely not true’ or ‘don’t know’.

The data are summarised in the following frequency table and bar chart. Note that the categories in the frequency table can be ordered in a definite order because the data are ordinal.

Astrology has scientific truth	Frequency	
	Number	%
Definitely true	9	4.5
Probably true	54	27.0
Probably not true	75	37.5
Definitely not true	51	25.5
Don’t know	11	5.5
Total	200	100.0



Write a report summarising the findings of this investigation quoting appropriate percentages to support your conclusion.

Solution

Report

Two hundred people were asked to respond to the statement 'Astrology has scientific truth'.

The majority of respondents did not agree, with 37.5% responding that they believed that this statement was probably not true, and another 25.5% declaring that the statement was definitely not true. Over one quarter (27%) of the respondents thought that the statement was probably true, while only 4.5% of subjects thought that the statement was definitely true.

Exercise 2C

Interpreting and describing frequency tables and bar charts

Example 3

1 A group of 69 students were asked to nominate their preferred type of fast food. The results are summarised in the percentage frequency table opposite. Use the information in the table to complete the report below by filling in the blanks.

Fast food type	%
Hamburgers	33.3
Chicken	10.1
Fish and chips	8.7
Chinese	10.1
Pizza	26.1
Other	11.6
Total	99.9

Report

A group of ______ students were asked their favourite type of fast food. The most popular response was ______ (33.3%), followed by pizza (______). The rest of the group were almost evenly split between chicken, fish and chips, Chinese and other, all around 10%.

2 Two hundred and fifty-six people were asked whether they agreed that there should be a return to capital punishment in their state. Their responses are summarised in the table opposite. Use the information in the table to complete the report below by filling in the blanks.

Capital punishment	%
Strongly agree	8.2
Agree	4.3
Don't know	16.4
Disagree	20.7
Strongly disagree	50.4
Total	100.0

Report

A group of 256 people were asked whether they agreed that there should be a return to capital punishment in their state. The majority of these people ______ (50.4%), followed by ______ who disagreed. Levels of support for return to capital punishment were quite low, with only 4.3% agreeing and 8.2% strongly agreeing. The remaining ______ said that they didn't know.

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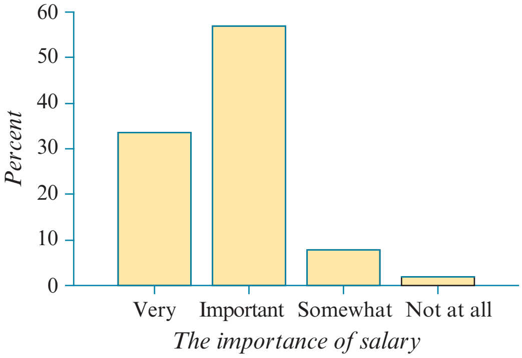
3 A group of 200 students were asked how they prefer to spend their leisure time. The results are summarised in the frequency table below.
Use the information in the table to write a brief report of the results of this investigation.

Leisure activity	%
Internet and digital games	42
Read	13
Listen to music	23
Watch TV or go to movies	12
Phone friends	4
Other	6
Total	100

Example 4 **4** A group of 579 employees from a large company were asked to rate the importance of salary in determining how they felt about their job. Their responses are shown in the following frequency table and bar chart.

Importance of salary	%
Very important	33.5
Important	56.8
Somewhat important	7.8
Not at all important	1.9
Total	100.0



Write a report describing how these employees rated the importance of salary in determining how they felt about their job.

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2D Displaying and describing numerical data

Frequency tables can also be used to organise numerical data. For discrete numerical data, the process is the same as that for categorical data, as shown in the following example.

▶ Discrete data

Example 5 Constructing a frequency table for discrete numerical data

The number of brothers and sisters (siblings) reported by each of the 30 students in year 11 are as follows:

2 3 4 0 3 2 3 0 4 1 0 0 1 2 3
0 2 1 1 4 5 3 2 5 6 1 1 1 0 2

Construct a frequency table for these data.

Solution

1. Find the maximum and the minimum values in the data set. Here the minimum is 0 and the maximum is 6.
2. Construct a table as shown, including all the values between the minimum and the maximum.
3. Count the number of 0s, 1s, 2s, etc. in the data set. For example, there are seven 1s. Record these values in the number column.
4. Add the frequencies to find the total.
5. Convert the frequencies to percentages, and record in the per cent (%) column.

   For example, percentage of 1s equals $\frac{7}{30} \times 100 = 23.3\%$.
6. Total the percentages and record.

Number of siblings	Frequency	
	Number	%
0	6	20.0
1	7	23.3
2	6	20.0
3	5	16.7
4	3	10.0
5	2	6.7
6	1	3.3
Total	30	100.0

▶ Grouping data

Some variables can only take on a limited range of values, for example, the variable *number of children in a family*. For these variables, it makes sense to list each of these values individually when forming a frequency distribution.

In other cases, the variable can take on a large range of values: for example, the variable *age* might take values from 0 to 100 or even more. Listing all possible ages would be tedious and would produce a large and unwieldy table. To solve this problem we **group the data** into a small number of convenient intervals.

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These grouping intervals should be chosen according to the following principles:

- Every data value should be in an interval.
- The intervals should not overlap.
- There should be no gaps between the intervals.

The choice of intervals can vary but there are some guidelines.

- A division, which results in about 5 to 15 groups, is preferred.
- Choose an interval width that is easy for the reader to interpret such as 10 units, 100 units or 1000 units (depending on the data).
- By convention, the beginning of the interval is given the appropriate exact value, rather than the end. As a result, intervals of 0–49, 50–99, 100–149 would be preferred over the intervals 1–50, 51–100, 101–150 etc.

Grouped discrete data

Example 6 Constructing a frequency table for a discrete numerical variable

A group of 20 people were asked to record how many cups of coffee they drank in a particular week, with the following results:

2	0	9	10	23	25	0	0	34	32
5	0	17	14	3	6	0	33	23	0

Construct a table of these data showing both frequency (count) and percentage frequency.

Solution

1 The minimum number of cups of coffee drunk is 0 and the maximum is 34. Intervals beginning at 0 and ending at 34 would ensure that all the data are included. Interval width of 5 will mean that there are 7 intervals. Note that the endpoints are within the interval, so that the interval 0 – 4 includes 5 values: 0, 1, 2, 3, 4.

2 Set up the table as shown.

3 Count the number of data values in each interval to complete the number column.

	Frequency	
Cups of coffee	Number	%
0–4	8	40
5–9	3	15
10–14	2	10
15–19	1	5
20–24	2	10
25–29	1	5
30–34	3	15
Total	20	100

4 Convert the frequencies into percentages and record in the per cent (%) column. For example, for the interval 5–9: % frequency $= \frac{3}{20} \times 100 = 15\%$

5 Total the percentages and record.

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Grouped continuous data

Example 7 Constructing a frequency table for a continuous numerical variable

The following are the heights of the 41 players in a basketball club, in centimetres.

178.1 185.6 173.3 193.4 183.1
184.6 202.4 170.9 183.3 180.3
185.8 189.1 178.6 194.7 185.3
191.1 189.7 191.1 180.4 180.0
193.8 196.3 189.6 183.9 177.7
178.9 193.0 188.3 189.5 182.0
183.6 184.5 188.7 192.4 203.7
180.1 170.5 179.3 184.1 183.8
174.7

Construct a frequency table of these data.

Solution

1. Find the minimum and maximum heights, which are 170.5 cm and 203.7 cm. A minimum value of 170 and a maximum of 204.9 will ensure that all the data are included.
2. Interval width of 5 cm will mean that there are 7 intervals from 170 to 204.9, which is within the guidelines of 5–15 intervals.
3. Set up the table as shown. All values of the variable that are from 170 to 174.9 have been included in the first interval. The second interval includes values from 175 to 179.9, and so on for the rest of the table.

	Frequency	
Heights	Number	%
170–174.9	4	9.8
175–179.9	5	12.2
180–184.9	13	31.7
185–189.9	9	22.0
190–194.9	7	17.1
195–199.9	1	2.4
200–204.9	2	4.9
Total	41	100.1

4. The number of data values in each interval is then counted to complete the number column of the table.
5. Convert the frequencies into percentages and record in the per cent (%) column.
   For example, for the interval 175.0–179.9: % frequency $= \frac{5}{41} \times 100 = 12.2\%$
6. Total the percentages and record.

The interval that has the highest frequency is called the **modal interval**. Here the modal interval is 180.0–184.9, as 13 players (31.7%) have heights that fall into this interval.

Histograms

As with categorical data, we would like to construct a visual display of a frequency table for numerical data. The graphical display of a frequency table for a numerical variable is called a **histogram**. A histogram looks similar to a bar chart but, because the data is numerical, there is a natural order to the plot and the bar widths depend on the data values.

Histograms

In a histogram:

- frequency (number or percentage) is shown on the vertical axis
- the values of the variable being displayed are plotted on the horizontal axis
- each column corresponds to a data value, or a data interval if the data is grouped; alternatively, for ungrouped discrete data, the actual data value is located at the middle of the column
- the height of the column gives the frequency (number or percentage).

Example 8 Constructing a histogram for ungrouped discrete data

Construct a histogram for the data in the frequency table.

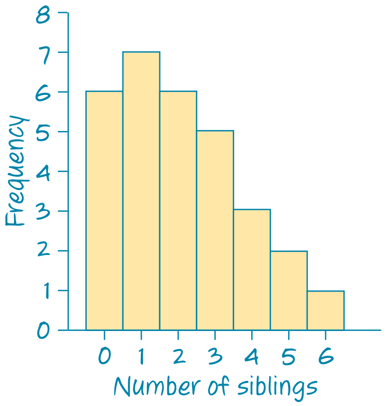
Siblings	Frequency
0	6
1	7
2	6
3	5
4	3
5	2
6	1
Total	30

Solution

1 Label the horizontal axis with the variable name ‘Number of siblings’. Mark in the scale in units, so that it includes all possible values.

2 Label the vertical axis ‘Frequency’. Insert a scale allowing for the maximum frequency of 7. Up to 8 would be appropriate. Mark the scale in units.

3 For each value for the variable draw in a column. The data is discrete, so make the width of each column 1, starting and ending halfway between data values. For example, the column representing 2 siblings starts at 1.5 and ends at 2.5. The height of each column is equal to the frequency.



Example 9 Constructing a histogram for continuous data

Construct a histogram for the data in the frequency table.

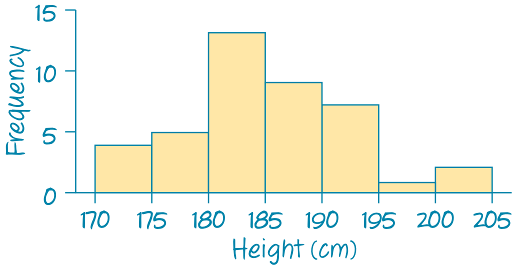
Height (cm)	Frequency
170.0–174.9	4
175.0–179.9	5
180.0–184.9	13
185.0–189.9	9
190.0–194.9	7
195.0–199.9	1
200.0–204.9	2
Total	41

Solution

1 Label the horizontal axis with the variable name ‘Height’. Mark in the scale using the beginning of each interval as the scale points; that is, 170, 175, . . .

2 Label the vertical axis ‘Frequency’. Insert a scale allowing for the maximum frequency of 13. Up to 15 would be appropriate. Mark the scale in units.

3 For each interval draw in a column. Each column starts at the beginning of the interval and finishes at the beginning of the next interval. Make the height of each column equal to the frequency.



Constructing a histogram using a CAS calculator

It is relatively quick to construct a histogram from a frequency table. However, if you only have the data (as you mostly do), it is a very slow process because you have to construct the frequency table first. Fortunately, a CAS calculator will do this for us.

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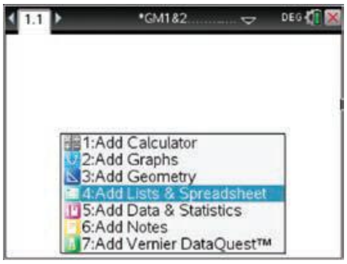
How to construct a histogram using the TI-Nspire CAS

Display the following set of 27 marks in the form of a histogram.

16 11 4 25 15 7 14 13 14 12 15 13 16 14
15 12 18 22 17 18 23 15 13 17 18 22 23

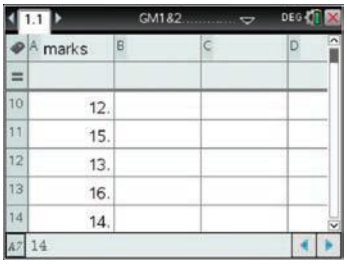
Steps

1 Start a new document: Press ⌂on and select **New Document** (or use ctrl + N). If prompted to save an existing document, move cursor to **No** and press enter.



2 Select **Add Lists & Spreadsheet**.
Enter the data into a list named *marks*.

- **a** Move the cursor to the name space of column A (or any other column) and type in *marks* as the list name. Press enter.
- **b** Move the cursor down to row 1, type in the first data value and press enter. Continue until all the data has been entered. Press enter after each entry.

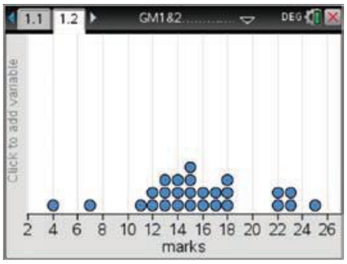


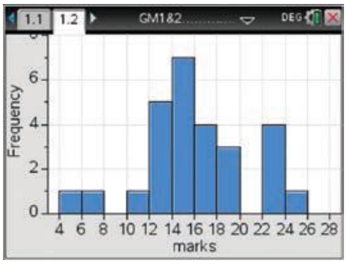
	A marks	B	C	D
=				
10	12.			
11	15.			
12	13.			
13	16.			
14	14.			

A7 14

3 Statistical graphing is done through the **Data & Statistics** application.
Press ctrl + I and select **Add Data & Statistics** (or press ⌂on, arrow to , and press enter).
Note: A random display of dots will appear – this is to indicate that data are available for plotting. It is not a statistical plot.

- **a** Press tab to show the list of variables that are available. Select the variable **marks**. Press enter to paste the variable **marks** to that axis.
- **b** A dot plot is displayed as the default plot. To change the plot to a histogram, press menu**>Plot Type>Histogram** and then press enter or 'click' (press). Your screen should now look like that shown opposite. This histogram has a column (or bin) width of 2 and a starting point of 3.





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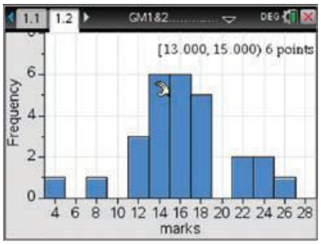
4 Data analysis



a Move cursor onto any column. A ☝ will appear and the column data will be displayed as shown opposite.

b To view other column data values move the cursor to another column.

Note: If you click on a column it will be selected. To deselect any previously selected columns move the cursor to the open area and press ☝.

Hint: If you accidentally move a column or data point, press `ctrl` + `esc` to undo the move.

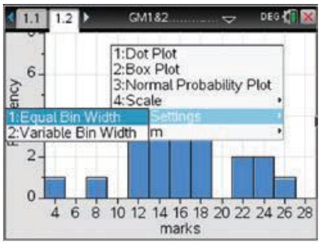
5 Change the histogram column (bin) width to 4 and the starting point to 2.

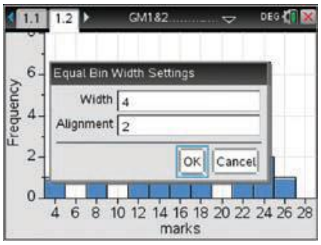
a Press `ctrl` + `menu` to get the contextual menu as shown (below left).

Hint: Pressing `ctrl` + `menu` with the cursor on the histogram gives you access to a contextual menu that enables you to do things that relate only to histograms.

b Select **Bin Settings**.

c In the settings menu (below right) change the **Width** to **4** and the **Starting Point (Alignment)** to **2** as shown. Press `enter`.



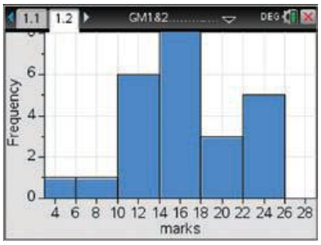


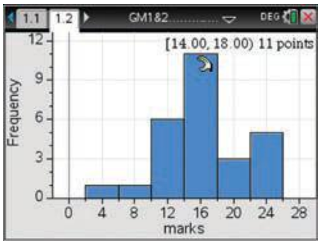
Equal Bin Width Settings

Width 4

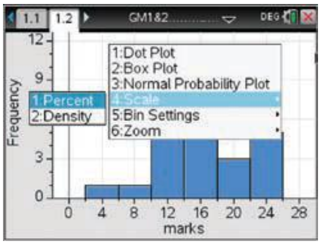
Alignment 2

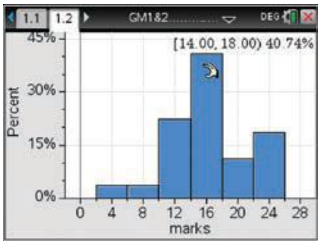
d A new histogram is displayed with column width of 4 and a starting point of 2 but it no longer fits the viewing window (below left). To solve this problem press `ctrl` + `menu` **>Zoom>Zoom-Data** and `enter` to obtain the histogram as shown below right.





6 To change the frequency axis to a percentage axis, press `ctrl` + `menu`**>Scale>Percent** and then press `enter`.





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How to construct a histogram using the ClassPad

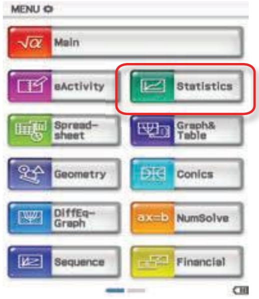
Display the following set of 27 marks in the form of a histogram.

16 11 4 25 15 7 14 13 14 12 15 13 16 14
15 12 18 22 17 18 23 15 13 17 18 22 23

Steps

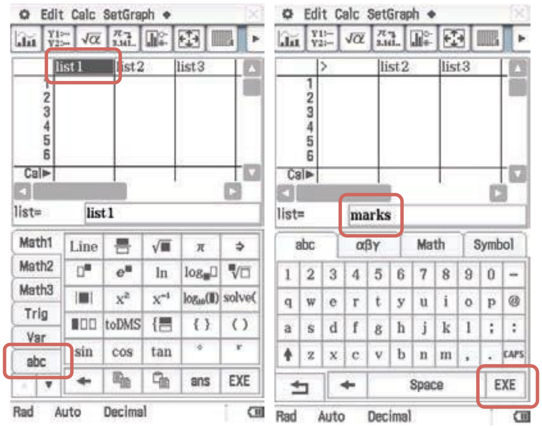
1 From the application menu screen, locate the **Statistics** application.
Tap Statistics to open.

Note: Tapping Menu from the icon panel (just below the touch screen) will display the application menu if it is not already visible.

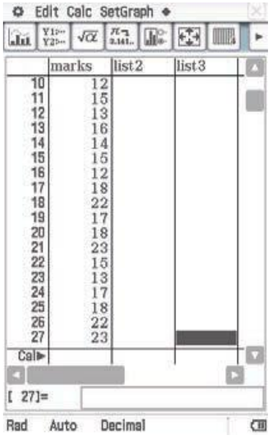


2 Enter the data into a list named **marks**.

- **a** Highlight the heading of the first list by tapping.
- **b** Press Keyboard and tap abc.
- **c** Type **marks** and press EXE.
- **d** Starting in row 1, type in each data value. Press EXE or ▼ to move down the list.



Your screen should be like the one shown at right.



	marks
10	12
11	15
12	13
13	16
14	14
15	15
16	12
17	18
18	22
19	17
20	18
21	23
22	15
23	13
24	17
25	18
26	22
27	23

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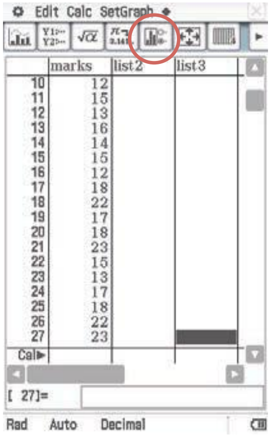
3 To plot a statistical graph:

a Tap ▦ at the top of the screen. This opens the **Set StatGraphs** dialog box.

b Complete the dialog box. For:

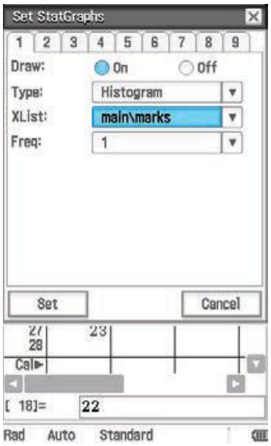
- **Draw:** select **On**
- **Type:** select **Histogram (▼)**
- **XList:** select **main\marks (▼)**
- **Freq:** leave as **1**.

c Tap Set to confirm your selections.



	marks	list2	list3
10	12		
11	15		
12	13		
13	16		
14	14		
15	15		
16	12		
17	18		
18	22		
19	17		
20	18		
21	23		
22	15		
23	13		
24	17		
25	18		
26	22		
27	23		

[27]=



Set StatGraphs

Draw: ● On ○ Off

Type: Histogram

XList: main\marks

Freq: 1

[18]= 22

Note: To make sure only this graph is drawn, select **SetGraph** from the menu bar at the top and confirm there is a tick only beside **StatGraph1** and no other box.

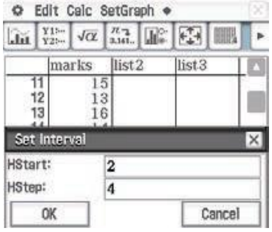
4 To plot the graph:

a Tap 📊 in the toolbar.

b Complete the **Set Interval** dialog box as given below. For:

- **HStart:** type in **2**
- **HStep:** type in **4**.

c Tap **OK**.



	marks	list2	list3
11	15		
12	13		
13	16		

Set Interval

HStart: 2

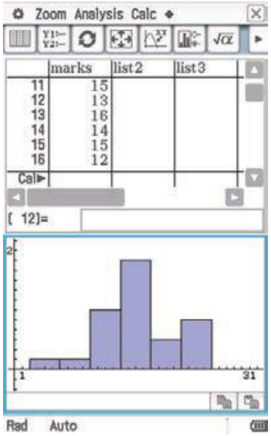
HStep: 4

5 The screen is split in two. Tapping Resize from the icon panel will allow the graph to fill the entire screen. Tap Resize to return to half-screen size.

6 Tapping ▦ places a marker on the first column of the histogram and tells us that:

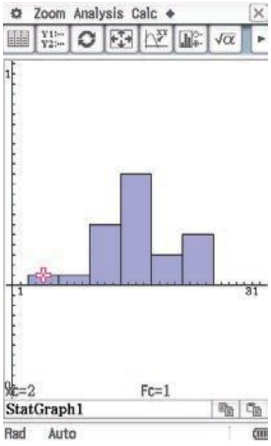
- the first interval begins at 2 ($x_c = 2$)
- for this interval, the frequency is 1 ($F_c = 1$).

To find the frequencies and starting points of the other intervals, use the arrow (▶) to move from interval to interval.



	marks	list2	list3
11	15		
12	13		
13	16		
14	14		
15	15		
16	12		

[12]=



xc=2 Fc=1

StatGraph1

Exercise 2D

Constructing frequency tables for numerical data

Example 5

1 The number of magazines purchased in a month by 15 different people was as follows:

0 5 3 0 1 0 2 4 3 1 0 2 1 2 1

Construct a frequency table for the data, including both the frequency and percentage frequency.

Example 6

2 The amount of money carried by 20 students is as follows:

\$4.55	\$1.45	\$16.70	\$0.60	\$5.00	\$12.30	\$3.45	\$23.60	\$6.90	\$4.35
\$0.35	\$2.90	\$1.70	\$3.50	\$8.30	\$3.50	\$2.20	\$4.30	\$0.00	\$11.50

Construct a frequency table for the data, including both the number and percentage in each category. Use intervals of \$5, starting at \$0.

Analysing frequency tables and constructing histograms

Example 7

3 A group of 28 students were asked to draw a line that they estimated to be the same length as a 30 cm ruler. The results are shown in the frequency table below.

Length of line (cm)	Frequency	
	Number	%
28.0–28.9	1	3.6
29.0–29.9	2	7.1
30.0–30.9	8	28.6
31.0–31.9	9	32.1
32.0–32.9	7	25.0
33.0–33.9	1	3.6
Total	28	100.0

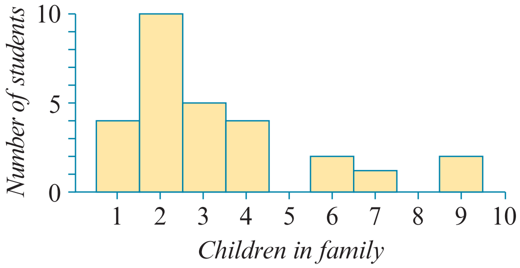
a How many students drew a line with a length:
  i from 29.0 to 29.9 cm?
  ii of less than 30 cm?
  iii of 32 cm or more?

b What percentage of students drew a line with a length:
  i from 31.0 to 31.9 cm?
  ii of less than 31 cm?
  iii of 30 cm or more?

c Use the table to construct a histogram using the counts.

Interpreting histograms

4 The number of children in the family for each student in a class is shown in the histogram.



a How many students are the only child in a family?

b What is the most common number of children in a family?

c How many students come from families with 6 or more children?

d How many students are there in the class?

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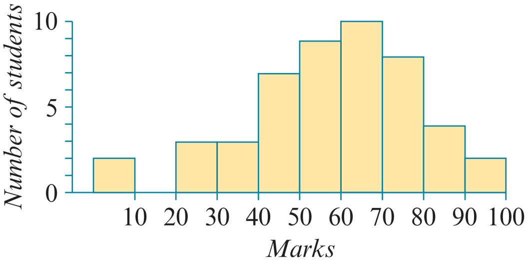
5 The following histogram gives the scores on a general knowledge quiz for a class of year 11 students.



a How many students scored from 10 to 19 marks?

b How many students attempted the quiz?

c What is the modal interval?

d If a mark of 50 or more is designated as a pass, how many students passed the quiz?

Constructing histograms using a CAS calculator and their analysis

Example 8 **6** A student purchased 21 new textbooks from a schoolbook supplier with the following prices (in dollars):

41.65	34.95	32.80	27.95	32.50	53.99	63.99	17.80	13.50	18.99	42.98
38.50	59.95	13.20	18.90	57.15	24.55	21.95	77.60	65.99	14.50	

a Use a CAS calculator to construct a histogram with column width 10 and starting point 10. Name the variable *price*.

b For this histogram:

- **i** what is the range of the third interval?
- **ii** what is the 'frequency' for the third interval?
- **iii** what is the modal interval?

7 The maximum temperatures for several capital cities around the world on a particular day, in degrees Celsius, were:

17	26	36	32	17	12	32	2	16	15	18	25
30	23	33	33	17	23	28	36	45	17	19	37

a Use a CAS calculator to construct a histogram with column width 2 and starting point 0. Name the variable *maxtemp*.

b For this histogram:

- **i** what is the starting point of the second column?
- **ii** what is the 'frequency' for this interval?

c Use the window menu to redraw the histogram with a column width of 5 and a starting point of 0.

d For this histogram:

- **i** how many cities had maximum temperatures from 20°C to 25°C?
- **ii** what is the modal interval?

2E Characteristics of distributions of numerical data: shape, location and spread

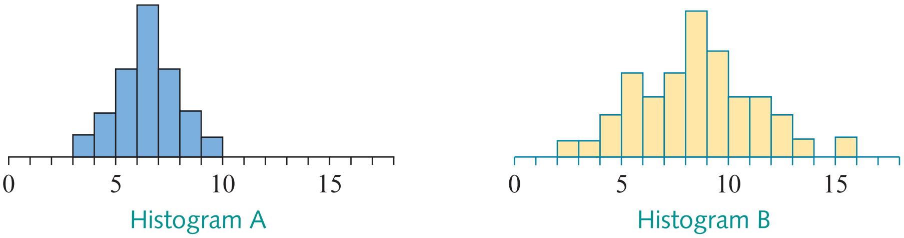
Distributions of numerical data are characterised by their shape and special features such as centre and spread.

Shape of a distribution

Symmetry and skew

A distribution is said to be **symmetric** if it forms a mirror image of itself when folded in the 'middle' along a vertical axis. Otherwise, the distribution is **skewed**.

Histogram A is symmetric, while Histogram B shows a distribution that is approximately symmetric.



Histogram A

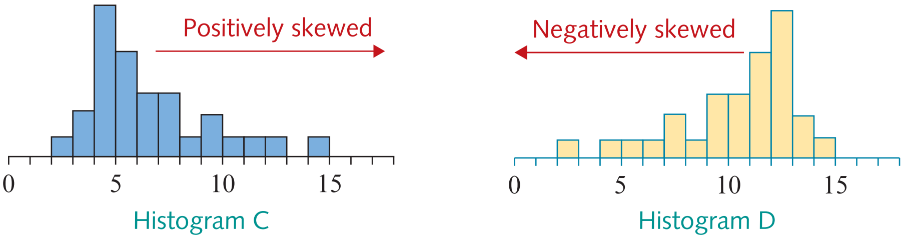
Histogram B

Positive and negative skew

A histogram may be positively or negatively skewed.

- It is **positively skewed** if it has a short tail to the left and a long tail pointing to the right (because of the many values towards the positive end of the distribution).
- It is **negatively skewed** if it has a short tail to the right and a long tail pointing to the left (because of the many values towards the negative end of the distribution).

Histogram C is an example of a positively skewed distribution, and Histogram D is an example of a negatively skewed distribution.



Histogram C

Histogram D

Knowing whether a distribution is skewed or symmetric is important, as this gives considerable information concerning the choice of appropriate summary statistics, as will be seen in the next section.

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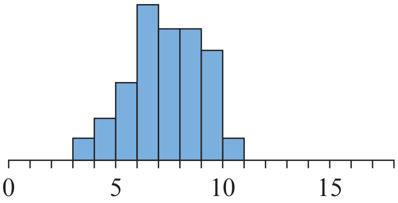
▶ Location and spread

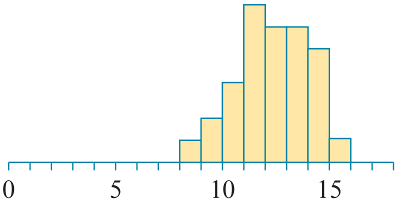
Comparing location

Two distributions are said to differ in **location** if the values of the data in one distribution are generally larger than the values of the data in the other distribution.

Consider, for example, the following histograms, shown on the same scale. Histogram F is identical in shape and width to Histogram E but moved horizontally several units to the right, indicating that these distributions differ in location.



Histogram E

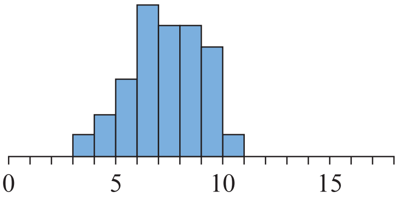


Histogram F

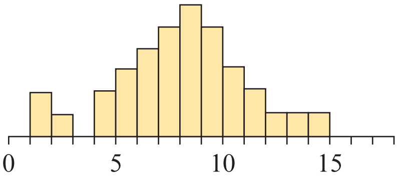
Comparing spread

Two distributions are said to differ in **spread** if the values of the data in one distribution tend to be more variable (spread out) than the values of the data in the other distribution.

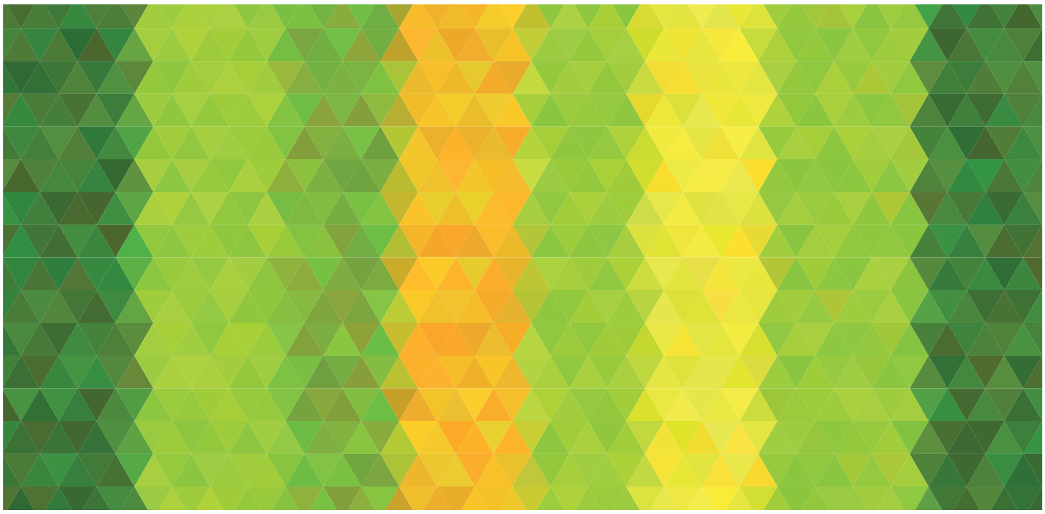
Histograms G and H illustrate the difference in spread. While both are centred at about the same place, Histogram H is more spread out.



Histogram G



Histogram H



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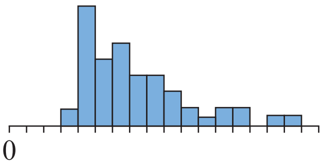
Exercise 2E

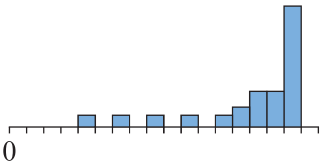
Describing shape using histograms

1 Describe the shape of each of the following histograms.

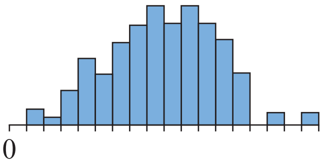
a



b

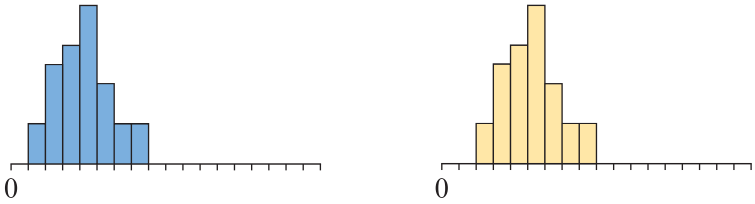


c

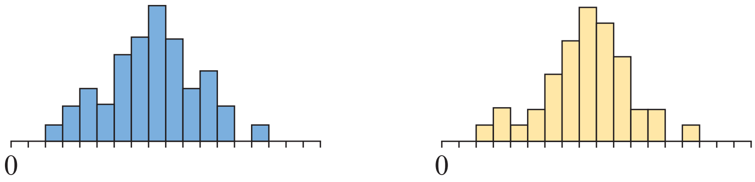


2 Do the following pairs of distributions differ in spread, location, both or neither? Assume that each pair of histograms is drawn on the same scale.

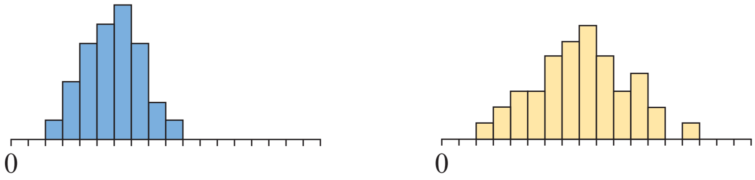
a



b



c



2F Dot plots and stem-and-leaf plots

▶ Dot plots

The simplest display of numerical data is a **dot plot**.

Dot plot

A dot plot consists of a number line with each data point marked by a dot. When several data points have the same value, the points are stacked on top of each other.

Dot plots are a great way to display fairly small data sets where the data takes a limited number of values.

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Example 10 Constructing a dot plot

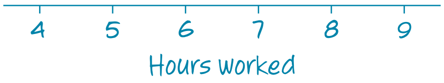
The number of hours worked by each of 10 students in their part-time jobs is as follows:

6 9 5 8 6 4 6 7 6 5

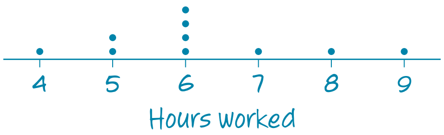
Construct a dot plot of these data.

Solution

1 Draw in a number line, scaled to include all data values. Label the line with the variable being displayed.



2 Plot each data value by marking in a dot above the corresponding value on the number as shown.



Stem-and-leaf plots

The **stem-and-leaf plot** or **stem plot** is another plot used for small data sets.

Example 11 Constructing a stem plot

The following is a set of marks obtained by a group of students on a test:

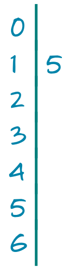
15 2 24 30 25 19 24 33 41 60 42 35 35
28 28 19 19 28 25 20 36 38 43 45 39

Display the data in the form of an ordered stem-and-leaf plot.

Solution

1 The data set has values in the units, tens, twenties, thirties, forties, fifties and sixties. Thus, appropriate stems are 0, 1, 2, 3, 4, 5 and 6. Write these down in ascending order, followed by a vertical line.

2 Now attach the leaves. The first data value is 15. The stem is 1 and the leaf is 5. Opposite the 1 in the stem, write the number 5, as shown.



The second data value is 2. The stem is 0 and the leaf is 2. Opposite the 0 in the stem, write the number 2, as shown.

0	2
1	5
2	
3	
4	
5	
6	

Continue systematically working through the data, following the same procedure, until all points have been plotted. You will then have the *unordered* stem plot, as shown.

0	2
1	5 9 9 9
2	4 5 4 8 8 8 5 0
3	0 3 5 5 6 8 9
4	1 2 3 5
5	
6	0

3 Ordering the leaves in increasing value as they move away from the stem gives the *ordered* stem plot, as shown. Write the name of the variable being displayed (*Marks*) at the top of the plot and add a key (1|5 means 15 marks).

Marks 1 | 5 means 15 marks

0	2
1	5 9 9 9
2	0 4 4 5 5 8 8 8
3	0 3 5 5 6 8 9
4	1 2 3 5
5	
6	0

It can be seen from this plot that the distribution is approximately symmetric, with one test score, 60, which seems to stand out from the rest. When a value sits away from the main body of the data, it is called an **outlier**.

▶ Choosing between plots

We now have three different plots that can be used to display numerical data: the histogram, the dot plot and the stem plot. They all allow us to make judgements concerning the important features of the distribution of the data, so how would we decide which one to use?

While there are no hard and fast rules, the following guidelines are often used.

Plot	Used best when	How usually constructed
Dot plot	small data sets (say $n < 30$) discrete data	by hand or with technology when constructing histograms as well
Stem plot	small data sets (say $n < 50$)	by hand
Histogram	large data sets (say $n > 30$)	with technology

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Exercise 2F

Constructing and analysing dot plots

Example 10

1 The number of children in each of 15 families is as follows:

0 7 2 2 2 4 1 3 3 2 2 2 0 0 1

a Construct a dot plot of the number of children.

b What is the mode of this distribution?

2 A group of 20 people were asked how many times in the last week they had shopped at a particular supermarket. Their responses were as follows:

0 1 1 0 0 6 0 1 2 2

3 4 0 0 1 1 2 3 2 0

a Construct a dot plot of this data.

b How many people did not shop at the supermarket in the last week?

3 The number of goals scored in an AFL game by each player on one team is as follows:

0 0 0 0 0 0 0 0 0 0 0

0 0 0 1 1 1 1 2 2 3 6

a Construct a dot plot of the number of goals scored.

b What is the mode of this distribution?

c What is the shape of the distribution of goals scored?

4 In a study of the service offered at her café, Amanda counted the number of people waiting in the queue every 5 minutes from 12 noon until 1 p.m.:

Time	12:00	12:05	12:10	12:15	12:20	12:25	12:30	12:35	12:40	12:45	12:50	12:55	1:00
Number	0	2	4	4	7	8	6	5	0	1	2	1	1

a Construct a dot plot of the number of people waiting in the queue.

b When does the peak demand at the café seem to be?

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Constructing and analysing stem plots

Example 11 **5** The marks obtained by a group of students on an English examination are as follows:

92	65	35	89	79	32	38	46	26	43	83	79
50	28	84	97	69	39	93	75	58	49	44	59
78	64	23	17	35	94	83	23	66	46	61	52

a Construct a stem plot of the marks.
b How many students obtained 50 or more marks?
c What was the lowest mark?

6 The stem plot on the right shows the ages, in years, of all the people attending a meeting.

a How many people attended the meeting?
b What is the shape of the distribution of ages?
c How many of these people were less than 43 years old?

Age (years)   1 | 2 represents 12 years

0 | 2 7
2 | 1 4 5 5 7 8 9
3 | 0 3 4 4 5 7 8 9
4 | 0 1 2 2 3 3 4 5 7 8 8 8
5 | 2 4 5 6 7 9
6 | 3 3 3 8
7 | 0

7 An investigator recorded the amount of time for which 24 similar batteries lasted in a toy. Her results (in hours) were:

26	40	30	24	27	31	21	27	20	30	33	22
4	26	17	19	46	34	37	28	25	31	41	33

a Make a stem plot of these times.
b How many of the batteries lasted for more than 30 hours?

8 The amount of time (in minutes) that a class of students spent on homework on one particular night were:

10	27	46	63	20	33	15	21	16	14	15
39	70	19	37	56	20	28	23	0	29	10

a Make a stem plot of these times.
b How many students spent more than 60 minutes on homework?
c What is the shape of the distribution?

9 The prices of a selection of shoes at a discount outlet are as follows:

\$49	\$75	\$68	\$79	\$75	\$39	\$35	\$52	\$149	\$84
\$36	\$95	\$28	\$25	\$78	\$45	\$46	\$76	\$82	

a Construct a stem plot of this data.
b What is the shape of the distribution?

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2G Summarising data

A statistic is any number computed from data. Certain special statistics are called **summary statistics**, because they numerically summarise important features of the data set. Of course, whenever any set of data is summarised into just one or two numbers, much information is lost. However, if a summary statistic is well chosen, it may reveal important information hidden in the data set.

For a single data distribution, the most commonly used summary statistics are either measures of centre or measures of spread.

Measures of centre

The mean

The most commonly used measure of the centre of a distribution of a numerical variable is the **mean**. The mean is calculated by summing the data values and then dividing by their number. The mean of a set of data is what many people call the ‘average’.

The mean

$$\text{mean} = \frac{\text{sum of data values}}{\text{total number of data values}}$$

For example, consider the set of data: 1, 5, 2, 4

$$\text{Mean} = \frac{1+5+2+4}{4} = \frac{12}{4} = 3$$

Some notation

Because the rule for the mean is relatively simple, it is easy to write in words. However, later you will meet other rules for calculating statistical quantities that are extremely complicated and hard to write out in words. To overcome this problem, we use a shorthand notation that enables complex statistical formulas to be written out in a compact form.

In this notation we use:

- the Greek capital letter sigma, $\sum$, as a shorthand way of writing ‘sum of’
- a lower case x to represent a data value
- a lower case x with a bar, $\bar{x}$ (pronounced ‘x bar’), to represent the mean of the data values
- n to represent the total number of data values.

The rule for calculating the mean then becomes: $\bar{x} = \dfrac{\sum x}{n}$

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Example 12 Calculating the mean

The following data set shows the number of premierships won by each of the current AFL teams, until the end of 2014. Find the mean of the number of premierships won.

Team	Premierships won
Carlton	16
Essendon	16
Collingwood	15
Melbourne	12
Hawthorn	12
Brisbane Lions	11
Richmond	10
Geelong	9
Sydney	5
Kangaroos	4
West Coast	3
Adelaide	2
Port Adelaide	1
Western Bulldogs	1
St Kilda	1
Fremantle	0
Gold Coast	0
GWS	0

Solution

1 Write down the formula and the value of *n*.

$$\bar{x} = \frac{\sum x}{n} \qquad n = 18$$

2 Substitute into the formula and evaluate.

$$\bar{x} = \frac{16 + 16 + 15 + ::: + 1 + 1 + 0 + 0 + 0}{18}$$

3 We do not expect the mean to be a whole number, so give your answer to one decimal place.

$$= \frac{118}{18}$$

$$= 6.6$$

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The median

Another useful measure of the centre of a distribution of a numerical variable is the middle value, or **median**. To find the value of the median, all the observations are listed in order and the middle one is the median.

For example, the median of the following data set is 6, as there are five observations on either side of this value when the data are listed in order.

median = 6
↓
2 3 4 5 5 6 7 7 8 8 11

When there is an even number of data values, the median is defined as the midpoint of the two middle values. For example, the median of the following data set is 6.5, as there are six observations on either side of this value when the data are listed in order.

median = 6.5
↓
2 3 4 5 5 6 7 7 8 8 11 11

Returning to the premiership data. As the data are already given in order, it only remains to determine the middle observation.

Since there are 18 entries in the table there is no actual middle observation, so the median is chosen as the value halfway between the two middle observations, in this case the ninth and tenth (5 and 4).

$$\text{median} = \frac{1}{2}(5 + 4) = 4.5$$

The interpretation here is that, of the teams in the AFL, half (or 50%) have won the premiership 5 or more times and half (or 50%) have won the premiership 4 or less times.

The following rule is useful for locating the median in a larger data set stem plot.

Determining the median

To compute the median of a distribution:

- arrange all the observations in ascending order according to size
- if n, the number of observations, is odd, then the median is the $\frac{n+1}{2}$th observation from the end of the list
- if n, the number of observations, is even, then the median is found by averaging the two middle observations in the list. That is, to find the median the $\frac{n}{2}$ and the $\left(\frac{n}{2} + 1\right)$th observations are added together and divided by 2.

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Example 13 Determining the median

Find the median age of 23 people whose ages are displayed in the ordered stem plot below.

Age (years)   1 | 2 represents 12 years

0 | 2 5
2 | 1 4 5 8
3 | 0 3 4 6
4 | 0 1 2 5 7
5 | 2 4 5 8
6 | 3 5 9 9

Solution

As the data are already given in order, it only remains to determine the middle observation.

1 Write down the number of observations.

$n = 23$

2 The median is located at the $\frac{n+1}{2}$th position.

median is at the $\frac{23+1}{2} = 12$th position

Thus the median age is 41 years.

Note: We can check to see whether we are correct by counting the number of data values either side of the median. They should be equal.

Comparing the mean and median

In Example 12 we found that the mean number of premierships won by the 18 AFL clubs was $\bar{x} = 6.5$. By contrast, in Example 13, we found that the median number of premierships won was 4.5.

These two values are quite different and the interesting question is: Why are they different, and which is the better measure of centre in this situation?

To help us answer this question, consider a stem plot of these data values.

Premierships won

0 | 0 0 0 1 1 1 2 3 4
0 | 5 9
1 | 0 1 1 2
1 | 5 6 6

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From the stem-and-leaf plot it can be seen that the distribution is positively skewed. This example illustrates a property of the mean. When the distribution is skewed or if there are one or two very extreme values, then the value of the mean may be far from the centre. The median is not so affected by unusual observations and always gives the middle value.

▶ Measures of spread

A measure of spread is calculated in order to judge the *variability* of a data set. That is, are most of the values clustered together, or are they rather spread out?

The range

The simplest measure of spread can be determined by considering the difference between the smallest and the largest observations. This is called the **range**.

The range

The range (R) is the simplest measure of spread of a distribution.

The range is the difference between the largest and smallest values in the data set.

$$R = \text{largest data value} - \text{smallest data value}$$

Example 14 Finding the range

Consider the marks, for two different tasks, awarded to a group of students:

Task A

2	6	9	10	11	12	13	22	23	24	26	26	27	33	34
35	38	38	39	42	46	47	47	52	52	56	56	59	91	94

Task B

11	16	19	21	23	28	31	31	33	38	41	49	52	53	54
56	59	63	65	68	71	72	73	75	78	78	78	86	88	91

Find the range of each of these distributions.

Solution

For task A the minimum mark is 2 and the maximum mark is 94.

Range for Task A $= 94 - 2 = 92$

For Task B, the minimum mark is 11 and the maximum mark is 91.

Range for Task B $= 91 - 11 = 80$

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The range for Task A is greater than the range for Task B. Is the range a useful summary statistic for comparing the spread of the two distributions? To help make this decision, consider the stem plots of the data sets:

Task A

0	2 6 9
1	0 1 2 3
2	2 3 4 6 6 7
3	3 4 5 8 8 9
4	2 6 7 7
5	2 2 6 6 9
6	
7	
8	
9	1 4

Task B

0	
1	1 6 9
2	1 3 8
3	1 1 3 8
4	1 9
5	2 3 4 6 9
6	3 5 8
7	1 2 3 5 8 8 8
8	6 8
9	1

From the stem-and-leaf plots of the data it appears that the spread of marks for the two tasks is not really described by the range. It is clear that the marks for Task A are more concentrated than the marks for Task B, except for the two unusual values for Task A.

Another measure of spread is needed, one which is not so influenced by these extreme values. The statistic we use for this task is the **interquartile range**.

The interquartile range

The interquartile range (IQR) gives the spread of the middle 50% of data values.

Determining the interquartile range

To find the interquartile range of a distribution:

- arrange all observations in order according to size
- divide the observations into two equal-sized groups, and if n is odd, omit the median from both groups
- locate Q_1, the *first quartile*, which is the median of the lower half of the observations, and Q_3, the *third quartile*, which is the median of the upper half of the observations.

The interquartile range IQR is then: $\text{IQR} = Q_3 - Q_1$

Definitions of the **quartiles** of a distribution sometimes differ slightly from the one given here. Using different definitions may result in slight differences in the values obtained, but these will be minimal and should not be considered a difficulty.

Example 15 Finding the interquartile range (IQR)

Find the interquartile ranges for Tasks A and B in Example 14 and compare.

Solution

1 There are 30 values in total. This means that there are fifteen values in the lower 'half', and fifteen in the upper 'half'. The median of the lower half (Q_1) is the 8th value.	Task A Lower half: 2 6 9 10 11 12 13 22 23 24 26 26 27 33 34 $Q_1 = 22$
2 The median of the upper half (Q_3) is the 8th value.	Upper half: 35 38 38 39 42 46 47 47 52 52 56 56 59 91 94 $Q_3 = 47$
3 Determine the IQR.	$IQR = Q_3 - Q_1 = 47 - 22 = 25$
4 Repeat the process for Task B.	Task B $Q_1 = 31$ $Q_3 = 73$ $IQR = Q_3 - Q_1 = 73 - 31 = 42$
5 Compare the IQR for Task A to the IQR for Task B.	The IQR shows the variability of Task A marks is smaller than the variability of Task B marks.

The interquartile range describes the range of the middle 50% of the observations. It measures the spread of the data distribution around the median (M). Since the upper 25% and the lower 25% of the observations are discarded, the interquartile range is generally not affected by outliers in the data set, which makes it a reliable measure of spread.

The standard deviation

The **standard deviation (s)**, measures the spread of a data distribution about the mean ($\bar{x}$).

The standard deviation

The standard deviation is defined to be:

$$s = \sqrt{\frac{\sum(x - \bar{x})^2}{n - 1}}$$

where n is the number of data values (sample size) and $\bar{x}$ is the mean.

Although it is not easy to see from the formula, the standard deviation is an average of the squared deviations of each data value from the mean. We work with the *squared* deviations because the sum of the deviations around the mean will always be zero. For technical reasons we average by dividing by $n - 1$, not n. In practice this is not a problem, as dividing by $n - 1$ compared to n generally makes very little difference to the final value.

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Normally, you will use your calculator to determine the value of a standard deviation. However, to understand what is involved when your calculator is doing the calculation, you should know how to calculate the standard deviation from the formula.

Example 16 Calculating the standard deviation

Use the formula:

$$s = \sqrt{\frac{\sum(x - \bar{x})^2}{n - 1}}$$

to calculate the standard deviation of the data set: 2, 3, 4.

Solution

1 To calculate s, it is convenient to set up a table with columns for:
x the data values
$(x - \bar{x})$ the deviations from the mean
$(x - \bar{x})^2$ the squared deviations.

	x	$(x - \bar{x})$	$(x - \bar{x})^2$
	2	−1	1
	3	0	0
	4	1	1
Sum	9	0	2

2 First find the mean and then complete the table as shown.

$$\bar{x} = \frac{\sum x}{n} = \frac{2 + 3 + 4}{3} = \frac{9}{3} = 3$$

3 Substitute the required values into the formula and evaluate.

$$s = \sqrt{\frac{\sum(x - \bar{x})^2}{n - 1}} = \sqrt{\frac{2}{3 - 1}} = 1$$

Using a CAS calculator to calculate summary statistics

As you can see, calculating the various summary statistics you have encountered in this section is sometimes rather complicated and generally time consuming. Fortunately, it is no longer necessary to carry out these computations by hand, except in the simplest cases.

How to calculate measures of centre and spread using the TI-Nspire CAS

The table shows the monthly rainfall figures for a year in Melbourne.

Month	J	F	M	A	M	J	J	A	S	O	N	D
Rainfall (mm)	48	57	52	57	58	49	49	50	59	67	60	59

Determine the mean and standard deviation, median and interquartile range, and range.

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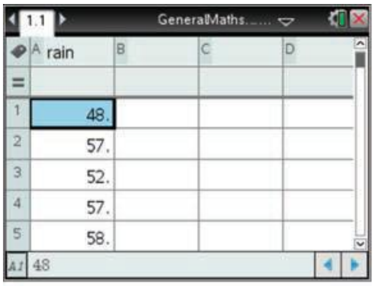
Steps

1 Start a new document: Press ⌂on and select **New Document** (or press ctrl + N).

2 Select **Add Lists & Spreadsheet**.
Enter the data into a list named *rain* as shown.
Statistical calculations can be done in the **Lists & Spreadsheet** application or the **Calculator** application.



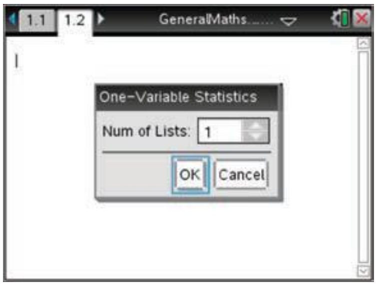
	A rain	B	C	D
=				
1	48.			
2	57.			
3	52.			
4	57.			
5	58.			

A1 48

3 Press ctrl + I and select **Add Calculator** (or press ⌂on and arrow to ⊠ and press enter).

a Press menu>**Statistics>Stat Calculations>One-Variable Statistics**, enter.

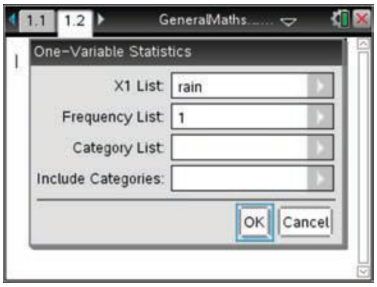
b Press the tab key to highlight OK and enter.



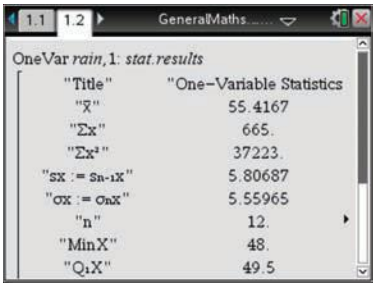
One-Variable Statistics
Num of Lists: 1

c Use the ▶ arrow and enter to paste in the list name **rain**. Press esc to exit the popup screen and generate statistical results screen below.

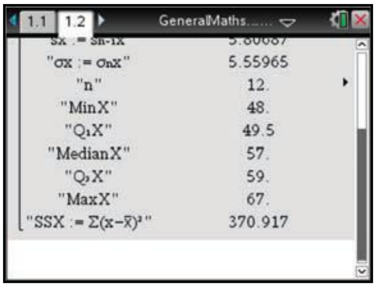
Notes: 1 The sample standard deviation is **sx**.
2 Use the ▲ arrows to scroll through the results screen to see the full range of statistics calculated.



One-Variable Statistics
X1 List: rain
Frequency List: 1
Category List:
Include Categories:



```
OneVar rain,1: stat.results
      "Title"       "One-Variable Statistics
       "x̄"           55.4167
       "Σx"          665.
       "Σx²"         37223.
   "sx := sn-1x"     5.80687
   "σx := σnx"       5.55965
       "n"           12.
      "MinX"         48.
      "Q1X"          49.5
```



```
   "σx := σnx"       5.55965
       "n"           12.
      "MinX"         48.
      "Q1X"          49.5
    "MedianX"        57.
      "Q3X"          59.
      "MaxX"         67.
 "SSX := Σ(x-x̄)²"    370.917
```

4 Write the answers correct to one decimal place.

$\bar{x} = 55.4, S = 5.8$

$M = 57$

$IQR = Q_3 - Q_1 = 59 - 49.5 = 9.5$

$R = max - min = 67 - 48 = 19$

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How to calculate measures of centre and spread using the ClassPad

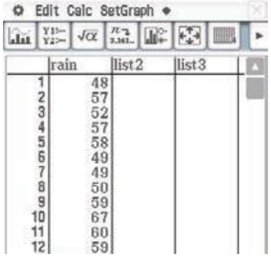
The table shows the monthly rainfall figures for a year in Melbourne.

Month	J	F	M	A	M	J	J	A	S	O	N	D
Rainfall (mm)	48	57	52	57	58	49	49	50	59	67	60	59

Determine the mean and standard deviation, median and interquartile range, and range.

Steps

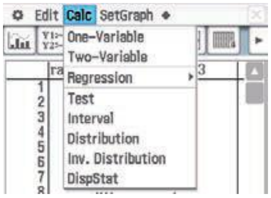
1 Open the **Statistics** application and enter the data into the column labelled **rain**.



```
 1  48
 2  57
 3  52
 4  57
 5  58
 6  49
 7  49
 8  50
 9  59
10  67
11  60
12  59
```

2 To calculate the mean, median, standard deviation, and quartiles:

- Select **Calc** from the menu bar.
- Tap **One-Variable** and open the **Set Calculation** dialog box.



3 Complete the dialog box. For:

- **XList:** select **main \ rain** (▼)
- **Freq:** leave as **1**.

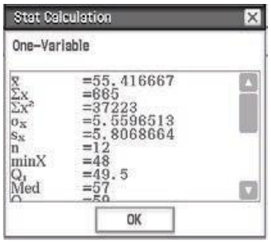
XList: main\rain

Freq: 1

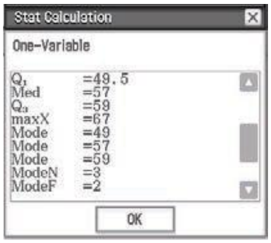
4 Tap **OK** to confirm your selections.

Notes: 1 The sample standard deviation is given by s_x.

2 Use the ▲▼ side-bar arrows to scroll through the results screen to obtain values for additional statistics if required.



```
x̄      =55.416667
Σx     =665
Σx²    =37223
σx     =5.5596513
sx     =5.8068664
n      =12
minX   =48
Q₁     =49.5
Med    =57
```



```
Q₁     =49.5
Med    =57
Q₃     =59
maxX   =67
Mode   =49
Mode   =57
Mode   =59
ModeN  =3
ModeF  =2
```

5 Write the answers correct to one decimal place.

$\bar{x} = 55.4, s = 5.8$

$M = 57$

$IQR = Q_3 - Q_1 = 59 - 49.5 = 9.5$

$R = \max - \min = 67 - 48 = 19$

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Exercise 2G

Calculating the mean, median and IQR without a calculator

Example 12

1 Find, without using a calculator, the mean and median for each of these data sets.

a 2 5 7 2 9

b 4 11 3 5 6 1

c 15 25 10 20 5

d 101 105 98 96 97 109

e 1.2 1.9 2.3 3.4 7.8 0.2

Example 14

Example 15

2 Find, without using a calculator, the median and IQR and range of each of these ordered data sets.

a 2 2 5 7 9 11 12 16 23

b 1 3 3 5 6 7 9 11 12 12

c 21 23 24 25 27 27 29 31 32 33

d 101 101 105 106 107 107 108 109

e 0.2 0.9 1.0 1.1 1.2 1.2 1.3 1.9 2.1 2.2 2.9

Example 13

3 Without a calculator, determine the median and the IQR for the data displayed in the following stem plots.

a Monthly rainfall (mm)

4	8 9 9
5	0 2 7 7 8 9 9
6	0 7

b Battery time (hours)

0	4
1	7 9
2	0 1 2 4 5 6 6 7 7 8
3	0 0 1 1 3 3 4 7
4	0 1 6

Using a calculator to determine summary statistics

4 The following table gives the area, in hectares, of each of the suburbs of a city:

3.6	2.1	4.2	2.3	3.4	40.3	11.3	19.4	28.4	27.6	7.4	3.2	9.0

a Find the mean and the median areas.

b Which is a better measure of centre for this data set? Explain your answer.

5 The prices, in dollars, of apartments sold in a particular suburb during one month were:

$387 500	$329 500	$293 400	$600 000	$318 000	$368 000	$750 000
$333 500	$335 500	$340 000	$386 000	$340 000	$404 000	$322 000

a Find the mean and the median of the prices.

b Which is a better measure of centre of this data set? Explain your answer.

Example 16

6 A manufacturer advertised that a can of soft drink contains 375 mL of liquid. A sample of 16 cans yielded the following contents:

357 375 366 360 371 363 351 369
358 382 367 372 360 375 356 371

Find the mean and standard deviation, median and IQR, and range for the volume of drink in the cans. Give answers correct to one decimal place.

7 The serum cholesterol levels for a sample of 20 people are:

231 159 203 304 248 238 209 193 225 244
190 192 209 161 206 224 276 196 189 199

Find the mean and standard deviation, median and IQR, and range of the serum cholesterol levels. Give answers correct to one decimal place.

8 Twenty babies were born at a local hospital on one weekend. Their birth weights are given in the stem plot.

Birth weight (kg) 3 | 6 represents 3.6 kg
2 | 1 5 7 9 9
3 | 1 3 3 4 4 5 6 7 7 9
4 | 1 2 2 3 5

Find the mean and standard deviation, median and IQR, and range of the birth weights.

9 The results of a student's chemistry experiment were as follows:

7.3 8.3 5.9 7.4 6.2 7.4 5.8 6.0

a **i** Find the mean and the median of the results.

ii Find the IQR and the standard deviation of the results.

b Unfortunately, when the student was transcribing his results into his chemistry book, he made a small error and wrote:

7.3 8.3 5.9 7.4 6.2 7.4 5.8 60

i Find the mean and the median of these results.

ii Find the interquartile range and the standard deviation of these results.

c Describe the effect the error had on the summary statistics in parts **a** and **b**.

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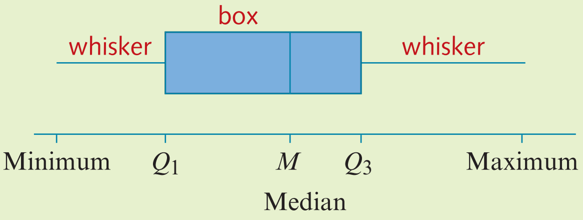
2H Boxplots

Knowing the median and quartiles of a distribution means that quite a lot is known about the central region of the data set. If something is known about the tails of the distribution as well, then a good picture of the whole data set can be obtained. This can be achieved by knowing the **maximum** and **minimum** values of the data.

When we list the *median*, the *quartiles* and the *maximum* and *minimum* values of a data sets, we have what is known as a **five-number summary**. Its pictorial (graphical) representation is called a **boxplot** or a box-and-whisker plot.

Boxplots



- A boxplot is a graphical representation of a five-number summary.
- A box is used to represent the middle 50% of scores.
- The median is shown by a vertical line drawn within the box.
- Lines (whiskers) extend out from the lower and upper ends of the box to the smallest and largest data values of the data set respectively.

Example 17 Constructing a boxplot from a five-number summary

The following are the monthly rainfall figures for a year in Melbourne.

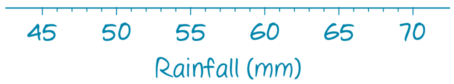
Month	J	F	M	A	M	J	J	A	S	O	N	D
Rainfall (mm)	48	57	52	57	58	49	49	50	59	67	60	59

Construct a boxplot to display this data, given the five-number summary:

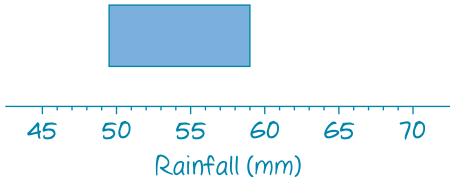
Min = 48, $Q_1 = 49.5$, $M = 57$, $Q_3 = 59$, Max = 67

Solution

1 Draw in a labelled and scaled number line that covers the full range of values.



2 Draw in a box starting at $Q_1 = 49.5$ and ending at $Q_3 = 59$.



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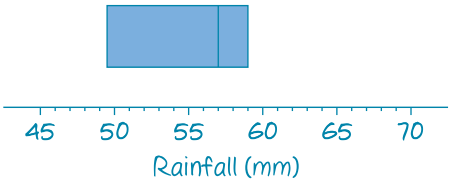
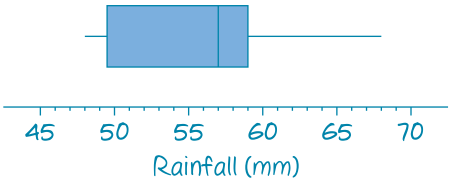
3 Mark in the median value with a vertical line segment at $M = 57$.



4 Draw in the whiskers, lines joining the midpoint of the ends of the box, to the minimum and maximum values, 48 and 67, respectively.



Boxplots with outliers

An extension of the boxplot can also be used to identify possible outliers in a data set.

Outlier

An *outlier* is a data value that appears to be rather different from other observations.

Sometimes it is difficult to decide whether or not an observation is an outlier. For example, a boxplot might have one extremely long whisker. How might we explain this?

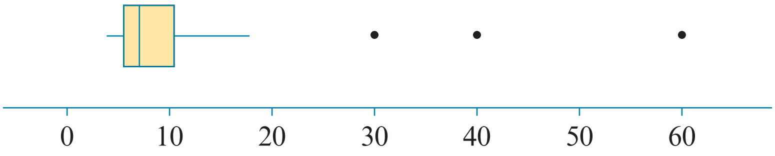
- One explanation is that the data distribution is extremely skewed with lots of data values in its tail.
- Another explanation is that the long whisker hides one or more outliers.

By modifying the boxplots, we can decide which explanation is most likely.

Designating outliers

Any data point in a distribution that lies more than 1.5 interquartile ranges above the third quartile or more than 1.5 interquartile ranges below the first quartile could be an outlier.

These data values are plotted individually in the boxplot, and the whisker now ends at the largest or smallest data value that is not outside these limits. An example of a boxplot displaying outliers is shown below.



Upper and lower fences

When constructing a boxplot to display outliers, we must first determine the location of what we call the *upper and lower fences*. These are imaginary lines drawn one and a half the interquartile range (or box widths) above and below the ends of the box (see over page). Data values outside these fences are classified as possible outliers and plotted separately.

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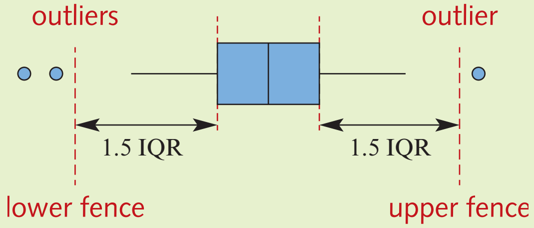
Using a boxplot to display possible outliers

In a boxplot, possible outliers are defined as those values that are:

- greater than $Q_3 + 1.5 \times \text{IQR}$ (upper fence)
- less than $Q_1 - 1.5 \times \text{IQR}$ (lower fence).



When drawing a boxplot, any observation identified as an outlier is indicated by a dot. The whiskers then end at the smallest and largest values that are not classified as outliers.

Example 18 Constructing a boxplot showing outliers

The number of hours that each of 33 students spent on a school project is shown below.

2	3	4	9	9	13	19	24	27	35	36
37	40	48	56	59	71	76	86	90	92	97
102	102	108	111	146	147	147	166	181	226	264

Construct a boxplot for this data set that can be used to identify possible outliers.

Solution

1 From the ordered list, state the minimum and maximum values. Find the median, the $\frac{1}{2}(33 + 1)$th = 17th value.

min. = 2, max. = 264,
median = 71

2 Determine Q_1 and Q_3. There are 33 values, so Q_1 is halfway between the 8th and 9th values and Q_3 is halfway between the 25th and the 26th values.

first quartile, $Q_1 = \dfrac{24 + 27}{2} = 25.5$

third quartile, $Q_3 = \dfrac{108 + 111}{2} = 109.5$

3 Determine the IQR.

$\text{IQR} = Q_3 - Q_1 = 109.5 - 25.5 = 84$

4 Determine the upper and lower fences.

$$\begin{aligned}\text{Lower fence} &= Q_1 - 1.5 \times \text{IQR}\\ &= 25.5 - 1.5 \times 84\\ &= -100.5\end{aligned}$$

$$\begin{aligned}\text{Upper fence} &= Q_3 + 1.5 \times \text{IQR}\\ &= 109.5 + 1.5 \times 84\\ &= 235.5\end{aligned}$$

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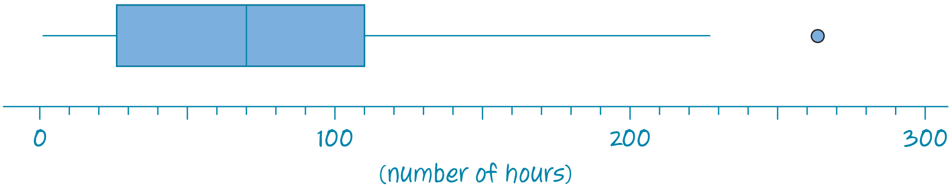
5 Locate any values outside the fences, and the values that lie just inside the limits (the whiskers will extend to these values).

There is one outlier 264.
The largest value that is not an outlier is 226.

6 The boxplot can now be constructed as shown below. The circle denotes the outlier.



There is one possible outlier, the student who spent 264 hours on the project.

It is clearly very time-consuming to construct boxplots displaying outliers by hand. Fortunately, your CAS calculator will do it for you automatically as we will see below.

How to construct a boxplot using the TI-Nspire CAS

The number of hours that each of 33 students spent on a school project is shown below.

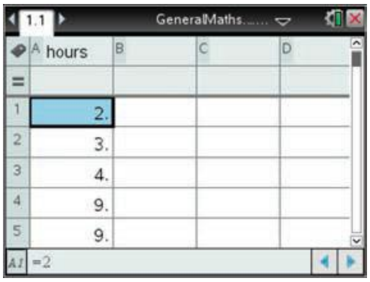
2	3	4	9	9	13	19	24	27	35	36
37	40	48	56	59	71	76	86	90	92	97
102	102	108	111	146	147	147	166	181	226	264

Construct a boxplot for this data set that can be used to identify possible outliers.

Steps

1 Press ⌂on and select **New Document** (or use ctrl + N).

2 Select **Add Lists & Spreadsheet**.
Enter the data into a list called **hours** as shown.

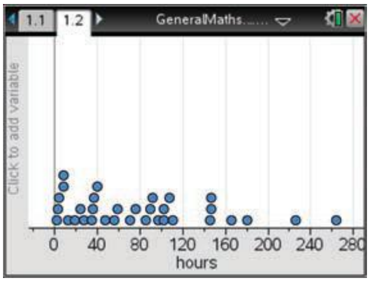


	A hours	B	C	D
=				
1	2.			
2	3.			
3	4.			
4	9.			
5	9.			

A1 =2

3 Statistical graphing is done through the **Data & Statistics** application. Press ctrl + I and select **Add Data & Statistics** (or press ⌂on, arrow to , and press enter).

Note: A random display of dots will appear – this is to indicate list data is available for plotting. It is not a statistical plot.



a Press tab to show the list of variables. Select the variable **hours**. Press enter to paste the variable **hours** to that axis. A dot plot is displayed as the default plot.

b To change the plot to a boxplot press menu>**Plot Type>BoxPlot**, then enter or click' (press). Outliers are indicated by a dot(s).

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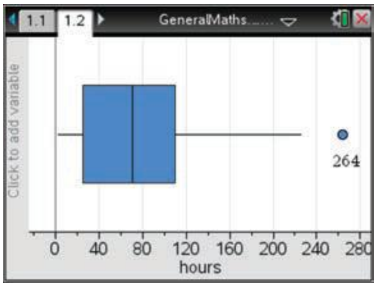
4 Data Analysis

Move the cursor over the plot to display the key values (or use menu>**Analyze>Graph Trace**).

Starting at the far left of the plot, we see that the:

- minimum value is 2: **minX** = **2**
- first quartile is 25.5: $\mathbf{Q_1}$ = **25.5**
- median is 71: **Median** = **71**
- third quartile is 109.5: $\mathbf{Q_3}$ = **109.5**
- maximum value is 264: **maxX** = **264**. It is also an outlier.



How to construct a boxplot using the ClassPad

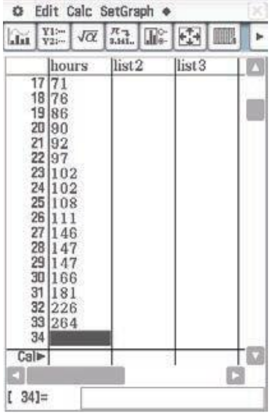
The number of hours that each of 33 students spent on a school project is shown below.

2	3	4	9	9	13	19	24	27	35	36
37	40	48	56	59	71	76	86	90	92	97
102	102	108	111	146	147	147	166	181	226	264

Construct a box plot for this data set that can be used to identify possible outliers.

Steps

1 Open the **Statistics** application and enter the data into a column labelled **hours**.



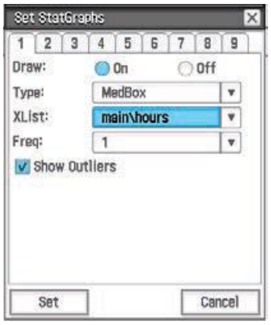
	hours	list2	list3
17	71		
18	76		
19	86		
20	90		
21	92		
22	97		
23	102		
24	102		
25	108		
26	111		
27	146		
28	147		
29	147		
30	166		
31	181		
32	226		
33	264		
34			

2 Open the **Set StatGraphs** dialog box by tapping 📊 in the toolbar. Complete the dialog box as shown, right. For:

- **Draw:** select **On**
- **Type:** select **MedBox** (▼)
- **XList:** select **main\hours** (▼)
- **Freq:** leave as **1**.

Tap the **Show Outliers** box.

Tap Set to exit.



Draw: ● On ○ Off
Type: MedBox
XList: main\hours
Freq: 1
☑ Show Outliers

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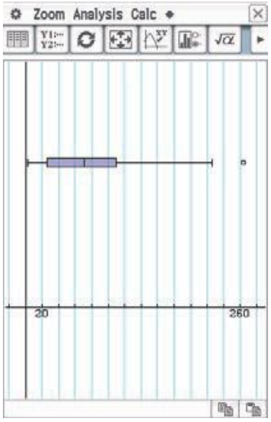
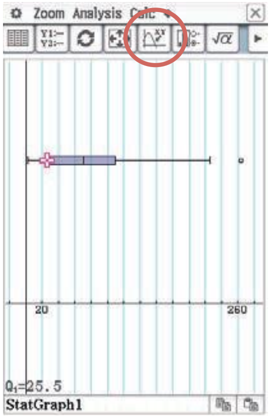


3 Tap to plot the boxplot.

4 Tap Resize to obtain a full-screen display.



5 Key values can be read from the boxplot by tapping .
Use the arrows (◀ and ▶) to move from point to point on the boxplot.
Starting at the far left of the plot, we see that the:

- minimum value is 2 (**minX** = **2**)
- first quartile is 25.5 (**Q_1** = **25.5**)
- median is 71 (**Median** = **71**)
- third quartile is 109.5 (**Q_3** = **109.5**)
- maximum value is 264 (**maxX** = **264**). It is also an outlier.

Exercise 2H

Constructing a boxplot from a five-number summary

Example 17

1 The heights (in centimetres) of a class of girls are:

160	165	123	143	154	180	133	123	157	157
135	140	140	150	154	159	149	167	176	163
154	167	168	132	145	143	157	156		

The five-number summary for this data is:

Min = 123, Q_1 = 141.5, M = 154, Q_3 = 161.5, Max = 180

Use this five-number summary to construct a boxplot (there are no outliers).

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2 The data shows how many weeks each of the singles in the Top 41 has been in the charts, in a particular week.

24	11	5	7	4	15	13	4	12	14	3	12	4	4
3	10	17	8	6	2	18	15	5	6	9	14	4	5
14	12	16	11	6	7	12	4	16	2	8	10	1	

The five-number summary for this data is:

Min = 1, $Q_1 = 4$, $M = 8$, $Q_3 = 13.5$, Max = 24

Use this five-number summary to construct a boxplot (there are no outliers).

Example 18

3 The amount of pocket money paid per week to a sample of year 8 students is:

\$5.00	\$10.00	\$12.00	\$8.00	\$7.50	\$12.00	\$15.00
\$10.00	\$10.00	\$0.00	\$5.00	\$10.00	\$20.00	\$15.00
\$26.00	\$13.50	\$15.00	\$5.00	\$15.00	\$25.00	\$16.00

The five-number summary for this data is:

Min = 0, $Q_1 = 7.75$, $M = 12$, $Q_3 = 15$, Max = 26

Use this five number summary to construct a boxplot (there is one outlier).

Constructing boxplots from raw data

4 The length of time, in years, that employees have been employed by a company is:

5	1	20	8	6	9	13	15	4	2
15	14	13	4	16	18	26	6	8	2
6	7	20	2	1	1	5	8		

Use a CAS calculator to construct the boxplot.

5 The times (in seconds) that 35 children took to tie up a shoelace are:

8	6	18	39	7	10	5	8	6	14	11	10
8	35	6	6	14	15	6	7	6	5	8	11
8	15	8	8	7	8	8	6	29	5	7	

Use a CAS calculator to construct the boxplot.

6 A researcher is interested in the number of books people borrow from a library. She selected a sample of 38 people and recorded the number of books each person had borrowed in the previous year. Here are her results:

7	28	0	2	38	18	0	0	4	0	0	5	13
2	13	1	1	14	1	8	27	0	52	4	11	0
0	12	28	15	10	1	0	2	0	1	11	0	

a Use a CAS calculator to construct a boxplot of the data.

b Use the boxplot to identify any possible outliers and write down their values.

7 The following table gives the prices for units sold in a particular suburb in one month (in thousands of dollars):

356	366	375	389	432
445	450	450	495	510
549	552	579	585	590
595	625	725	760	880
940	950	1017	1180	1625

a Use a CAS calculator to construct a boxplot of the data.

b Use the boxplot to identify any possible outliers and write down their values.

8 The time taken, in seconds, for a group of children to complete a puzzle is:

8	6	18	39	7	10	5	8	6	14	11	5
10	8	60	6	6	14	15	6	7	6	5	7
8	11	8	15	8	8	7	8	8	6	29	

a Use a CAS calculator to construct a boxplot of the data.

b Use the boxplot to identify any possible outliers and write down their values.

9 The percentage of people using the internet in 23 countries is given in the table:

Country	Internet users (%)
Afghanistan	5.45
Argentina	55.80
Australia	79.00
Brazil	48.56
Bulgaria	51.90
China	42.30
Colombia	48.98
Greece	55.07
Hong Kong SAR, China	72.90
Iceland	96.21
India	12.58

Country	Internet users (%)
Italy	55.83
Malaysia	65.80
Morocco	55.42
New Zealand	82.00
Saudi Arabia	54.00
Singapore	72.00
Slovenia	68.35
South Africa	41.00
United Kingdom	87.48
United States	79.30
Venezuela	49.05
Vietnam	39.49

a Use a CAS calculator to construct a boxplot of the data.

b Use the boxplot to identify any possible outliers and write down their values.

2I Comparing the distribution of a numerical variable across two or more groups

It makes sense to compare the distributions of data sets when they are concerned with the *same* numerical variable, say *height* measured for different groups of people, for example, a basketball team and a gymnastic team.

For example, it would be useful to compare the distributions for each of the following:

- the maximum daily temperatures in Melbourne in March and the maximum daily temperatures in Sydney in March
- the test scores for a group of students who had not had a revision class and the test scores for a group of students who had a revision class.

In each of these examples, we can actually identify *two variables*. One is a *numerical variable* and the other is a *categorical variable*.

For example:

- The variable maximum daily *temperature* is numerical while the variable *city*, which takes the values 'Melbourne' or 'Sydney', is categorical.
- The variable *test score* is numerical while the variable *attended a revision class*, which takes the values 'yes' or 'no', is categorical.

Thus, when we compare two data sets in this section, we will be actually investigating the relationship between two variables: a numerical variable and a categorical variable.

The outcome of these investigations will be a brief written report that compares the distribution of the numerical variable across two or more groups defined as categorical variables. The starting point for these investigations will be, as always, a graphical display of the data. To this end you will meet and learn to interpret two new graphical displays: the **back-to-back stem plot** and the **parallel boxplot**.

▶ Comparing distributions using back-to-back stem plots

A back-to-back stem plot differs from the stem plots you have met in the past in that it has a single stem with two sets of leaves, one for each of the two groups being compared.

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Example 19 Comparing distributions using back-to-back stem plots

The following back-to-back stem plot displays the distributions of life expectancies for males (in years) in several countries in the years 1970 and 2010.

In this situation, *life expectancy* is the numerical variable. *Year*, which takes the values 1970 and 2010, is the categorical variable.

Male life expectancy (in years)

1970		2010
8	3	
	4	
9 4 2	5	8
9 9 9 8 7 7 1 0	6	9
4	7	1 2 4 4 6 8 9 9
	8	0 0 0

5|8 = 58 years

M = 67 years M = 76 years
IQR = 12.5 years IQR = 8 years

Use the back-to-back stem plot and the summary statistics provided to compare these distributions in terms of centre and spread and draw an appropriate conclusion.

Solution

1 Centre: Use the medians to compare centres.	The median life expectancy of males in 2010 (M = 76 years) was nine years higher than in 1970 (M = 67 years).
2 Spread: Use the IQRs to compare spreads.	The spread of life expectancies of males in 2010 (IQR = 12.5 years) was different to the spread in 1970 (IQR = 8).
3 Conclusion: Use the above observations to write a general conclusion.	In conclusion, the median life expectancy for these countries has increased over the last 40 years, and the variability in life expectancy between countries has decreased.

▶ Comparing distributions using parallel boxplots

Back-to-back stem plots can be used to compare the distribution of a numerical variable across two groups when the data sets are small. Parallel boxplots can also be used to compare distributions. Unlike back-to-back stem plots, boxplots can also be used when there are more than two groups.

By drawing boxplots on the same axis, both the centre and spread for the distributions are readily identified and can be compared visually.

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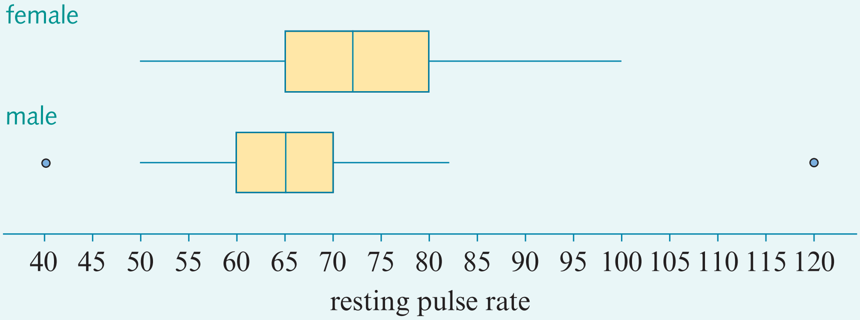
When comparing distributions of a numerical variable across two or more groups using parallel boxplots, the report should address the key features of:

- centre (the median)
- spread (the IQR)
- possible outliers.

Example 20 Comparing distributions across two groups using parallel boxplots

The following parallel boxplots display the distribution of pulse rates (in beats/minute) for a group of female students and a group of male students.

Use the information in the boxplots to write a report comparing these distributions in terms of centre, spread and outliers in the context of the data.



Solution

1 Centre: Compare the medians. Estimate values of these medians from the plot (the vertical lines in the boxes).	The median pulse rate for females ($M = 72$ beats/minute) is higher than that for males ($M = 65$ beats/minute).
2 Spread: Compare the spread of the two distributions using IQRs (the widths of the boxes).	The spread of pulse rates for females ($IQR = 15$) is higher than for males ($IQR = 10$).
3 Outliers: Locate on any outliers and describe.	There are no female outliers. The males with pulse rates of 40 and 120 were outliers.
4 Conclusion: Use the above observations to write a general conclusion.	In conclusion, the median pulse rate for females was higher than for males and female pulse rates were generally more variable than male pulse rates.

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Exercise 2I

Comparing groups using back-to-back stem plots

Example 19

1 The stem plot displays the age distribution of ten females and ten males admitted to a regional hospital on the same day.

a Calculate the median and the IQR for admission ages of the females and males in this sample.

b Write a report comparing these distributions in terms of centre and spread in the context of the data.

Females		Males
9	0	
5 0	1	3 6
7	2	1 4 5 6 7
7 1	3	4
3 0	4	0 7
0	5	
	6	
9	7	

4 | 0 = 40 years

2 The stem plot opposite displays the mark distribution of students from two different mathematics classes (Class A and Class B) who sat the test. The test was marked out of 100.

a How many students in each class scored less than 50%?

b Determine the median and the IQR for the marks obtained by the students in each class.

c Write a report comparing these distributions in terms of centre and spread in the context of the data.

Marks %

Class B		Class A
3 2	1	9
	2	2
	3	9
	4	5 7 8
	5	5 8
9	6	5 8
6 4 3 3 2 2 1 0 0	7	1 6 7 9 9
8 8 4 4 3 2 1 1 0 0	8	0 1 2 2 5 5 9
8 1	9	1 9

7 | 1 = 71

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3 The following table shows the number of nights spent away from home in the past year by a group of 21 Australian tourists and by a group of 21 Japanese tourists:

Australian

3	14	15	3	6	17	2
7	4	8	23	5	7	21
9	11	11	33	4	5	3

Japanese

14	3	14	7	22	5	15
26	28	12	22	29	23	17
32	5	9	23	6	44	19

a Construct a back-to-back stem-and-leaf plot of these data sets.

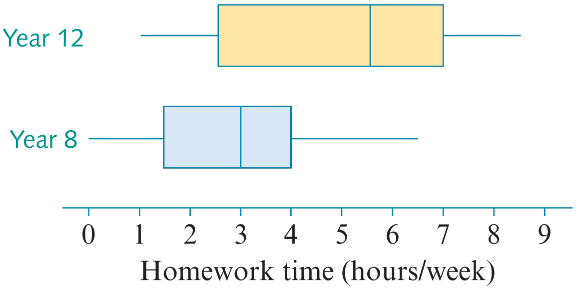
b Determine the median and IQR for the two distributions.

c Write a report comparing the distributions of the number of nights spent away by Australian and Japanese tourists in terms of centre and spread.

Comparing groups using parallel boxplots

Example 20

4 The boxplots below display the distributions of homework time (in hours/week) of a sample of year 8 and a sample of year 12 students.



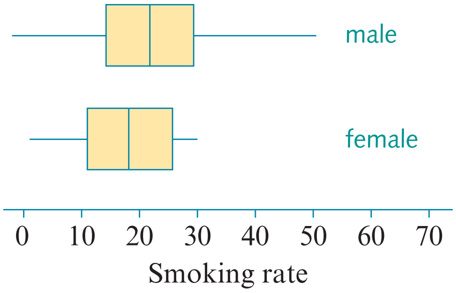
a Estimate the median and IQRs from the boxplots.

b Use these medians and IQRs to write a report comparing these distributions in terms of centre and spread in the context of the data.

5 The boxplots below display the distribution of smoking rates (%) of males and females from several countries.

a Estimate the median and IQRs from the boxplots.

b Use the information in the boxplots to write a report comparing these distributions in terms of centre and spread in the context of the data.



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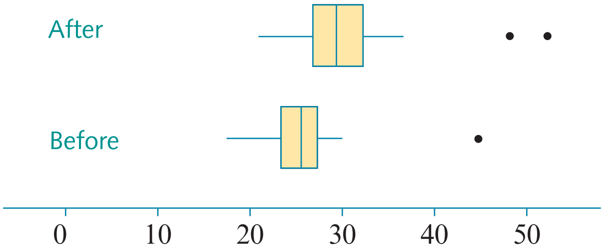
6 The boxplots below display the distributions of the number of sit-ups a person can do in one minute, both before and after a fitness course.



a Estimate the median, IQRs and the values of any outliers from the boxplots.

b Use these medians and IQRs to write a report comparing these distributions in terms of centre and spread in the context of the data.

7 To test the effect of alcohol on coordination twenty randomly selected participants were timed to complete a task with both 0% blood alcohol and 0.05% blood alcohol. The times taken (in seconds) are shown in the accompanying table.

0% blood alcohol									
38	36	35	35	43	46	42	47	40	48
35	34	40	44	30	25	39	31	29	44

0.05% blood alcohol									
39	32	35	39	36	34	41	64	44	38
43	42	46	46	50	32	32	41	40	50

a Draw boxplots for each of the sets of scores on the same scale.

b Use the information in the boxplots to write a report comparing the distributions of the times taken to complete a task with 0% blood alcohol and 0.05% blood alcohol in terms of centre (medians), spread (IQRs) and outliers.

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2J Statistical investigation

Exercise 2J

1 To investigate the age of parents at the birth of their first child, a hospital recorded the ages of the mothers and fathers for the first 40 babies born in the hospital for each of the years 1970, 1990 and 2010.

The data is given below:

1970 Mother									
23	22	33	19	19	26	20	15	26	17
18	31	24	20	29	28	25	45	28	22
1970 Father									
29	15	39	29	22	35	32	26	37	29
25	31	20	34	28	22	33	25	34	46
1990 Mother									
28	14	38	28	21	34	31	25	36	28
24	30	19	33	27	21	32	24	33	45
1990 Father									
31	27	46	31	26	28	30	27	43	37
39	22	27	35	31	29	32	27	38	35
2010 Mother									
30	26	45	32	25	27	29	26	42	36
38	21	26	34	37	28	28	37	37	34
2010 Father									
37	31	39	36	21	34	34	23	17	37
23	33	31	32	24	39	45	30	35	34

Use appropriate displays and summary statistics to answer the following questions:

- **a** How do the ages of the mothers compare to the ages of fathers in each time period?
- **b** How have the ages of the mothers changed over the three time periods?
- **c** How have the ages of the fathers changed over the three time periods?
- **d** Has the relationship between mothers' ages and fathers' ages changed over time?

In each case write a brief report to summarise your findings.

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Key ideas and chapter summary

Types of data	Data can be classified as **numerical** or **categorical**.
Frequency table	A **frequency table** is a listing of the values that a variable takes in a data set, along with how often (frequently) each value occurs. **Frequency** can be recorded as the number of times a value occurs or a **percentage**, the percentage of times a value occurs.
Categorical data	**Categorical data** arises when classifying or naming some quality or attribute. When the categories are naming the groups, the data is called **nominal**. When there is an inherent order in the categories, the data is called **ordinal**.
Bar chart	A **bar chart** is used to display the frequency distribution of a categorical variable.
Mode, modal category/class	The **mode** (or modal category) is the value of a variable (or the category) that occurs most frequently. The **modal interval**, for **grouped data,** is the interval that occurs most frequently.
Numerical data	**Numerical data** arises from measuring or counting some quantity. **Discrete** numerical data can only take particular values, usually whole numbers, and often arises from counting. **Continuous** numerical data describes numerical data that can take any value, sometimes in an interval, and often arises from measuring.
Histogram	A **histogram** is used to display the frequency distribution of a numerical variable: suitable for medium to large-sized data sets.
Stem plot	A **stem plot** is a visual display of a numerical data set, an alternative display to the histogram: suitable for small to medium-sized data sets. Leading digits are shown as the stem and the final digit as the leaf.
Dot plot	A **dot plot** consists of a number line with each data point marked by a dot. Suitable for small to medium sized data sets.
Describing the distribution of a numerical variable	The **distribution of a numerical variable** can be described in terms of **shape** (**symmetric** or **skewed**: positive or negative), **centre** (the midpoint of the distribution) and **spread**.
Summary statistics	**Summary statistics** are used to give numerical values to special features of a data distribution such as centre and spread.
Mean	The **mean** ($\bar{x}$) is a summary statistic that can be used to locate the centre of a symmetric distribution.

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Range	The **range** (R) is the difference between the smallest and the largest data values. It is the simplest measure of spread. range = largest value − smallest value
Standard deviation	The **standard deviation** (s) is a summary statistic that measures the spread of the data values around the mean.
Median	The **median** (M) is a summary statistic that can be used to locate the centre of a distribution. It is the midpoint of a distribution, so that 50% of the data values are less than this value and 50% are more. If the distribution is clearly skewed or there are outliers, the median is preferred to the mean as a measure of centre.
Quartiles	**Quartiles** are summary statistics that divide an ordered data set into four equal groups.
Interquartile range	The **interquartile range (IQR)** gives the spread of the middle 50% of data values in an ordered data set. If the distribution is highly skewed or there are outliers, the IQR is preferred to the standard deviation as a measure of spread.
Five-number summary	The median, the first quartile, the third quartile, along with the minimum and the maximum values in a data set, are known as a **five-number summary**.
Outliers	**Outliers** are data values that appear to stand out from the rest of the data set.
Boxplot	A **boxplot** is a visual display of a five-number summary with adjustments made to display outliers separately when they are present.

Skills check

Having completed this chapter you should be able to:

- differentiate between nominal, ordinal, discrete and continuous data
- interpret the information contained in a frequency table
- identify the mode from a frequency table and interpret it
- construct a bar chart or histogram from a frequency table
- construct a histogram from raw data using a graphics calculator
- construct a dot plot and stem-and-leaf plot from raw data
- recognise symmetric, positively skewed and negatively skewed distributions
- identify potential outliers in a distribution from its histogram or stem plot

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- locate the median and quartiles of a data set and hence calculate the IQR
- produce a five-number summary from a set of data
- construct a boxplot from a five-number summary
- construct a boxplot from raw data using a graphics calculator
- use a boxplot to identify key features of a data set such as centre and spread
- use the information in a back-to-back stem plot or a boxplot to describe and compare distributions
- calculate the mean and standard deviation of a data set
- understand the difference between the mean and the median as measures of centre and be able to identify situations where it is more appropriate to use the median
- write a short paragraph comparing distributions in terms of centre, spread and outliers.

Multiple-choice questions

1 In a survey, a number of people were asked to indicate how much they exercised by selecting one of the options 'never', 'seldom', 'sometimes' or 'regularly'. The resulting variable was named *level of exercise*. The type of data generated is:

A variable **B** numerical **C** nominal **D** ordinal **E** metric

2 For which of the following variables is a bar chart an appropriate display?

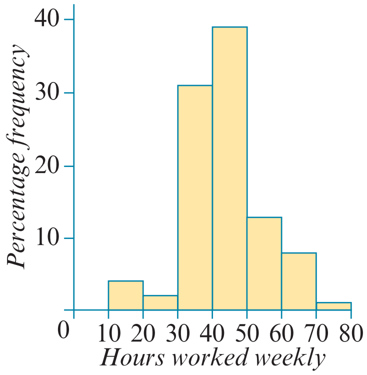
A Weight (kg)
B Age (years)
C Distance between towns (km)
D Hair colour
E Reaction time (seconds)

3 For which of the following variables is a histogram an appropriate display?

A Hair colour
B Sex (male, female)
C Distances between towns on a long road trip (km)
D Postcode
E Weight (under weight, average, over weight)

The following information relates to Questions 4 to 7

The number of hours worked per week by employees in a large company is shown in the following percentage frequency histogram.



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4 The percentage of employees who work from 20 to less than 30 hours per week is closest to:

A 1% **B** 2% **C** 6% **D** 10% **E** 33%

5 The percentage of employees who worked *less* than 30 hours per week is closest to:

A 2% **B** 3% **C** 4% **D** 6% **E** 30%

6 The modal interval for hours worked is:

A 10 to less than 20 **B** 20 to less than 30 **C** 30 to less than 40
D 40 to less than 50 **E** 50 to less than 60

7 The median number of hours worked is in the interval:

A 10 to less than 20 **B** 20 to less than 30 **C** 30 to less than 40
D 40 to less than 50 **E** 50 to less than 60

The following information relates to Questions 8 to 11

A group of 18 employees of a company were asked to record the number of meetings they had attended in the last month.

1 1 2 3 4 5 5 6 7 9 10 12 14 14 16 22 23 44

8 The range of meetings is:

A 22 **B** 23 **C** 24 **D** 43 **E** 44

9 The median number of meetings is:

A 6 **B** 7 **C** 7.5 **D** 8 **E** 9

10 The mean number of meetings is closest to:

A 7 **B** 8 **C** 9 **D** 10 **E** 11

11 The interquartile range (IQR) of the number of meetings is:

A 0 **B** 4 **C** 9.5 **D** 10 **E** 14

12 The heights of six basketball players (in cm) are:

178.1 185.6 173.3 193.4 183.1 193.0

The mean and standard deviation are closest to:

A mean = 184.4; standard deviation = 8.0
B mean = 184.4; standard deviation = 7.3
C mean = 182.5; standard deviation = 7.3
D mean = 182.5; standard deviation = 8.0
E mean = 183.1; standard deviation = 7.3

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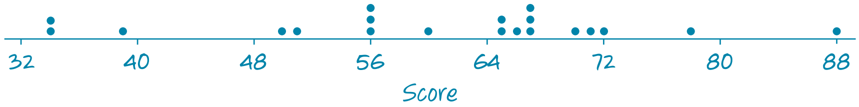
The following information relates to Questions 13 and 14

The dot plot below gives the examination scores in mathematics for a group of 20 students.



13 The number of students who scored 56 on the examination is:

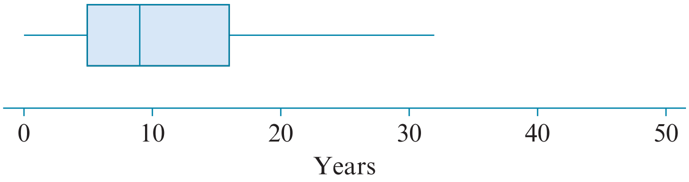
A 1 **B** 2 **C** 3 **D** 4 **E** 5

14 The percentage of students who scored between 40 and 80 on the exam is closest to:

A 60% **B** 70% **C** 80% **D** 90% **E** 100%

The following information relates to Questions 15 to 18

The number of years for which a sample of people have lived at their current address is summarised in the boxplot.



15 The range is closest to:

A 10 **B** 15 **C** 20

D 25 **E** 30

16 The median number of *years lived at this address* is closest to:

A 5 **B** 9 **C** 12 **D** 15 **E** 47

17 The interquartile range of the number of *years lived at this address* is closest to:

A 5 **B** 10 **C** 15 **D** 20 **E** 45

18 The percentage who have lived at this address for more than 15 years is closest to:

A 10% **B** 25% **C** 50% **D** 60% **E** 75%

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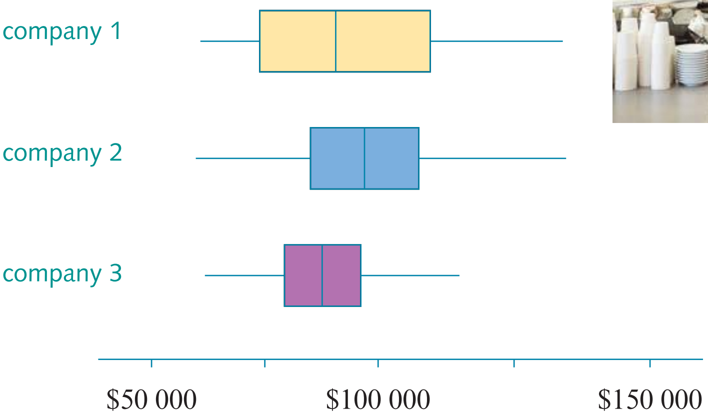
The following information relates to Questions 19 to 21

The amount paid per annum to the employees of each of three large companies is shown in the boxplots.



19 The company with the lowest median wage is:

A company 1 **B** company 2 **C** company 3
D company 1 and company 2 **E** company 2 and company 3

20 The company with the largest general spread (IQR) in wages is:

A company 1 **B** company 2 **C** company 3
D company 1 and company 2 **E** company 2 and company 3

21 Which of the following statements is *not* true?

A All workers in company 3 earned less than \$125 000 per year.
B More than half of workers in company 2 earned less than \$100 000 per year.
C 75% of workers in company 2 earned less than the median wage in company 3.
D More than half of the workers in company 1 earned more than the median wage in company 3.
E More than 25% of the workers in company 1 earned more than the median wage at company 2.

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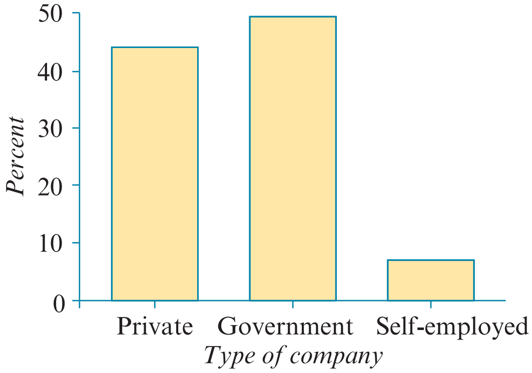
Short-answer questions

1 Classify the data that arises from the following situations as nominal, ordinal, discrete or continuous.

a The number of phone calls a hotel receptionist receives each day.

b Interest in politics on a scale from 1 to 5, where 1 = very interested, 2 = quite interested, 3 = somewhat interested, 4 = not very interested, and 5 = uninterested.

2 The following bar chart shows the percentage of working people in a certain town who are employed in private companies, work for the government or are self-employed.



a Is the data categorical or numerical?

b Approximately what percentage of the people are self-employed?

3 A researcher asked a group of people to record how many cigarettes they had smoked on a particular day. Here are her results:

0	0	9	10	23	25	0	0	34	32	0	0	30	0	4
5	0	17	14	3	6	0	33	23	0	32	13	21	22	6

Using class intervals of width 5, construct a histogram of this data.

4 A teacher recorded the time taken (in minutes) by each of a class of students to complete a test:

56	57	47	68	52	51	43	22	59	51	39
54	52	69	72	65	45	44	55	56	49	50

a Make a dot plot of this data.

b Make a stem-and-leaf plot of these times.

c Use this stem plot to find the median and quartiles for the time taken.

5 The monthly phone bills, in dollars, for a group of people are given below:

285	185	210	215	320	680	280
265	300	210	270	190	245	315

Find the mean and standard deviation, the median and the IQR, and the range of the monthly phone bills.

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6 Geoff decided to record the time (in minutes) it takes him to complete his mail round each working day for four weeks. His data is recorded below:

170 189 201 183 168 182 161 166 167 173 182 167 188 211
164 176 161 187 180 201 147 188 186 176 174 193 185 183

Find the mean and standard deviation of his mail round times.

7 A group of students was asked to record the number of SMS messages that they sent in one 24-hour period. The following five-number summary was obtained from the data set.

$$\text{Min} = 0, \quad Q_1 = 3, \quad M = 5, \quad Q_3 = 12, \quad \text{Max} = 24$$

Use the summary to construct a boxplot of this data.

8 The following data gives the number of students absent from a large secondary college on each of 36 randomly chosen school days:

7 22 12 15 21 16 23 23 17 23 8 16
7 3 21 30 13 2 7 12 18 14 14 0
15 16 13 21 10 16 11 4 3 0 31 44

a Construct a boxplot of this data.
b What was the median number of students absent each day during this period?
c On what percentage of days were more than 20 students absent?

Extended-response questions

1 The divorce rates (in percentages) of 19 countries are:

27 18 14 25 28 6 32 44 53 0
26 8 14 5 15 32 6 19 9

a Is the data categorical or numerical?
b Construct an ordered stem plot of divorce rates by hand.
c Construct a dot plot of divorce rates by hand.
d What shape is the distribution of divorce rates?
e What percentage of the 19 countries have divorce rates greater than 30%?
f Calculate the mean and median of the distribution of divorce rates.
g Use your calculator to construct a histogram of the data with class intervals of width 10.
  i What is the shape of the histogram?
  ii How many of the 19 countries have divorce rates from 10% to less than 20%?

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2 Metro has decided to improve its service on the Lilydale line. Trains were timed on the run from Lilydale to Flinders Street, and their times recorded over a period of six weeks at the same time each day.
The journey times are shown below (in minutes):

60	61	70	72	68	80	76	65	69	79	82
90	59	86	70	77	64	57	65	60	68	60
63	67	74	78	65	68	82	89	75	62	64
58	64	69	59	62	63	89	74	60		

a Use your CAS calculator to construct a histogram of the times taken for the journey from Lilydale to Flinders Street.

  i On how many days did the trip take 65–69 minutes?

  ii What shape is the histogram?

  iii What percentage of trains took less than 65 minutes to reach Flinders Street?

b Use your calculator to determine the following summary statistics for the *time* taken (correct to two decimal places):

$$\bar{x},\ s,\ \text{Min},\ Q_1,\ M, Q_3,\ \text{Max}$$

c Use the summary statistics to complete the following report.

  i The mean time taken from Lilydale to Flinders Street was ☐ minutes.

  ii 50% of the trains took more than ☐ minutes to travel from Lilydale to Flinders Street.

  iii The range of travelling times was ☐ minutes, while the interquartile range was ☐ minutes.

  iv 25% of trains took more than ☐ minutes to travel to Flinders Street.

  v The standard deviation of travelling times was ☐ minutes.

d Summary statistics for the year before Metro took over the Lilydale line from Connex are:

$$\text{Min} = 55, \quad Q_1 = 65, \quad M = 70, \quad Q_3 = 89, \quad \text{Max} = 99$$

Construct boxplots for the last year Connex ran the line and for the data from Metro on the same plot.

e Use the information from the boxplots to write a report comparing the distribution of travelling times for the two transport corporations in terms of centre (medians) and spread (IQRs).

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Chapter 3

3

Linear relations and equations

- How do we use a formula?
- How do we create a table of values?
- How do we use a graphics calculator to create a table of values?
- How do we solve linear equations?
- What is a literal equation?
- How do we solve literal equations?
- How do we develop a formula?
- How do we transpose a formula?
- How do we find the intersection of two linear graphs?
- What are simultaneous equations?
- How do we solve simultaneous equations?
- How can we use simultaneous equations to solve practical problems?

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Introduction

Linear relations and equations connect two or more variables such that they yield a straight line when graphed. They have many applications in technology, science and business.

3A Substitution of values into a formula

A **formula** is a mathematical relationship connecting two or more variables.

For example:

- $C = 45t + 150$ is a formula for relating the cost, C dollars, of hiring a plumber for t hours. C and t are the variables.
- $P = 4L$ is a formula for finding the perimeter of a square, where P is the perimeter and L is the side length of the square. P and L are the variables.

By substituting all known variables into a formula, we are able to find the value of an unknown variable.

Example 1 Using a formula

The cost of hiring a windsurfer is given by the rule:

$$C = 40t + 10$$

where C is the cost in dollars and t is the time in hours. How much will it cost to hire a windsurfer for 2 hours?

Solution

1 Write the formula.	$C = 40t + 10$
2 To determine the cost of hiring a windsurfer for 2 hours, substitute $t = 2$ into the formula. Note: 40(2) means 40×2	$C = 40(2) + 10$
3 Evaluate.	$C = 90$
4 Write your answer.	It will cost \$90 to hire a windsurfer for 2 hours.

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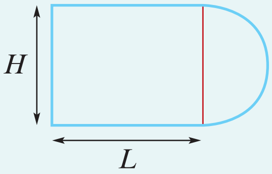
Example 2 Using a formula

The perimeter of the shape shown can be given by the formula:

$$P = 2L + H\left(1 + \frac{\pi}{2}\right)$$



In this formula, L is the length of the rectangle and H is the height. Find the perimeter correct to one decimal place, if $L = 16.1$ cm and $H = 3.2$ cm.

Solution

1 Write the formula. $P = 2L + H\left(1 + \frac{\pi}{2}\right)$

2 Substitute values for L and H into the formula. $P = 2 \times 16.1 + 3.2\left(1 + \frac{\pi}{2}\right)$

3 Evaluate. $P = 40.4$ (correct to one decimal place)

4 Give your answer with correct units. The perimeter of the shape is 40.4 cm.

Exercise 3A

Example 1

1 The cost of hiring a dance hall is given by the rule:

$$C = 50t + 1200$$

where C is the total cost in dollars and t is the number of hours for which the hall is hired.

Find the cost of hiring the hall for:

a 4 hours **b** 6 hours **c** 4.5 hours

2 The distance, d km, travelled by a car in t hours at an average speed of v km/h is given by the formula:

$$d = v \times t$$

Find the distance travelled by a car travelling at a speed of 95 km/h for 4 hours.

3 Taxi fares are calculated using the formula:

$$F = 1.3K + 4$$

where K is the distance travelled in kilometres and F is the cost of the fare in dollars. Find the costs of the following trips.

a 5 km **b** 8 km **c** 20 km

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Example 2

4 The circumference, C, of a circle with radius, r, is given by:

$$C = 2\pi r$$

Find correct to two decimal places, the circumferences of the circles with the following radii.

a A stained glass window with $r = 25$ cm

b An earring with $r = 3$ mm

c A DVD of $r = 5.4$ cm

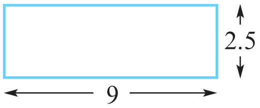
d A circular garden bed with $r = 7.2$ m

5 If the perimeter of a rectangle is given by $P = 2(L + W)$, find the value of P for the following rectangles.

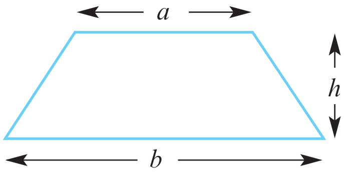
a $L = 3$ and $W = 4$

b $L = 15$ and $W = 8$

c



6 The area of a trapezium as shown is $A = \frac{1}{2}h(a + b)$. Find A if:



a $h = 1, a = 3, b = 5$

b $h = 5, a = 2.5, b = 3.2$

c $h = 2.7, a = 4.1, b = 8.3$

7 The formula used to convert temperature from degrees Fahrenheit to degrees Celsius is:

$$C = \frac{5}{9}(F - 32)$$

Use this formula to convert the following temperatures to degrees Celsius. Give your answers correct to one decimal place.

a 50°F

b 0°F

c 212°F

d 92°F

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8 The formula for calculating simple interest is:

$$I = \frac{PRT}{100}$$

where P is the principal (amount invested or borrowed), R is the interest rate per annum and T is the time (in years). In the following questions, give your answers to the nearest cent (correct to two decimal places).

a Frank borrows \$5000 at 12% for 4 years. How much interest will he pay?

b Chris borrows \$1500 at 6% for 2 years. How much interest will he pay?

c Jane invests \$2500 at 5% for 3 years. How much interest will she earn?

d Henry invests \$8500 for 3 years with an interest rate of 7.9%. How much interest will he earn?

9 In Australian football, a goal, G, is worth 6 points and a behind, B, is worth 1 point. The total number of points, is given by:

$$\text{total number of points} = 6 \times \text{number of goals} + \text{number of behinds}$$

a Find the number of points if:

- **i** 2 goals and 3 behinds are kicked
- **ii** 5 goals and 7 behinds are kicked
- **iii** 8 goals and 20 behinds are kicked

b In a match, Redteam scores 4 goals and 2 behinds and Greenteam scores 3 goals and 10 behinds. Which team wins the match?

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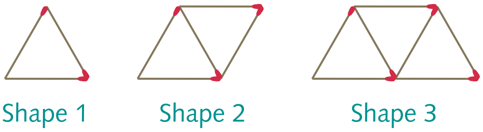
10 The number of matchsticks used for each shape below follows the pattern 3, 5, 7, . . .



The rule for finding the number of matches used in this sequence is:

number of matches = $a + (n - 1)d$

where a is the number of matches in the first shape ($a = 3$), d is the number of extra matches used for each shape ($d = 2$) and n is the shape number.
Find the number of matches in the:

a 6th shape **b** 11th shape **c** 50th shape

11 Suggested cooking times for roasting x kilograms of meat are given in the following table.

Meat type	Minutes/kilogram
Chicken (well done)	45 min/kg + 20 mins
Lamb (medium)	55 min/kg + 25mins
Lamb (well done)	65 min/kg + 30 mins
Beef (medium)	55 min/kg + 20 mins
Beef (well done)	65 min/kg + 30 mins

a How long, to the nearest minute, will it take to cook:

- **i** a 2 kg chicken?
- **ii** 2.25 kg beef (well done)?
- **iii** a piece of lamb weighing 2.4 kg (well done)?
- **iv** 2.5 kg beef (medium)?

b At what time should you put a 2 kg leg of lamb into the oven to have served medium at 7:30 p.m.?

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3B Constructing a table of values

We can use a formula to construct a *table of values*. This can be done by substitution (by hand) or using your TI-Nspire or ClassPad.

Example 3 Constructing a table of values

The formula for converting degrees Celsius to degrees Fahrenheit is given by:

$$F = \frac{9}{5}C + 32$$

Use this formula to construct a table of values for F using values of C in intervals of 10 between $C = 0$ and $C = 100$.

Solution

Draw up a table of values for $F = \frac{9}{5}C + 32$, and then substitute values of $C = 0, 10, 20, 30, \ldots$ into the formula to find F.

If $C = 0, F = \frac{9}{5}(0) + 32$

If $C = 10, F = \frac{9}{5}(10) + 32 = 50$

and so on.

The table would then look as follows:

C	0	10	20	30	40	50	60	70	80	90	100
F	32	50	68	86	104	122	140	158	176	194	212

How to construct a table of values using the TI-Nspire CAS

The formula for converting degrees Celsius to degrees Fahrenheit is given by:

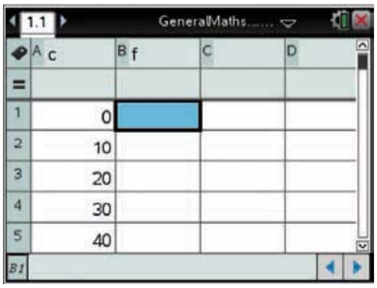
$$F = \frac{9}{5}C + 32$$

Use this formula to construct a table of values for F using values of C in intervals of 10 between $C = 0$ and $C = 100$.

Steps

1 Start a new document: **Press** ctrl + N

2 Select **Add Lists & Spreadsheet**. **Name** the lists c (for Celsius) and f (for Fahrenheit).
Enter the data 0–100 in intervals of 10 into a list named c, as shown.



	A c	B f	C	D
=				
1	0			
2	10			
3	20			
4	30			
5	40			

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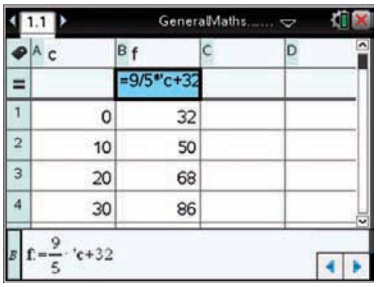
3 Place cursor in the grey formula cell in column B (i.e. list f) and type in: **= 9 ÷ 5 × c + 32**

Hint: If you typed in c you will need to select **Variable Reference** when prompted. This prompt occurs because c can also be a column name. Alternatively, pressing the var key and selecting **c** from the variable list will avoid this issue.

Press enter to display the values given.

Use the ▼ arrow to move down through the table.



	A c	B f	C	D
=		=9/5*c+32		
1	0	32		
2	10	50		
3	20	68		
4	30	86		

B $f:=\frac{9}{5}\cdot c+32$

How to construct a table of values using the ClassPad

The formula for converting degrees Celsius to degrees Fahrenheit is given by:

$$F = \frac{9}{5}C + 32$$

Use this formula to construct a table of values for F using values of C in intervals of 10 between $C = 0$ and $C = 100$.

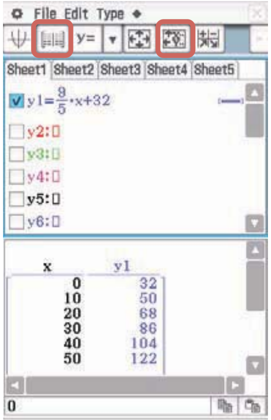
Steps

1 Enter the data into your calculator using the **Graph & Table** application. From the application menu screen, locate the built-in **Graph & Table** application, Graph& Table. Tap to open. Tapping Menu from the icon panel (just below the touch screen) will display the application menu if it is not already visible.

2 **a** Adjacent to **y1=** type in the formula $\frac{9}{5}x + 32$. Then press EXE.

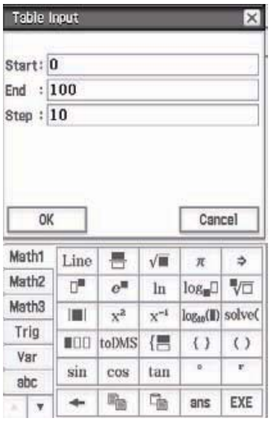
b Tap the **Table Input** icon to set the table entries as shown and tap OK.

c Tap the icon to display the required table of values. Scrolling down will show more values in the table.



☑ $y1=\frac{9}{5}\cdot x+32$
☐ y2:□
☐ y3:□
☐ y4:□
☐ y5:□
☐ y6:□

x	y1
0	32
10	50
20	68
30	86
40	104
50	122



Start: 0
End : 100
Step : 10

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Exercise 3B

1 A football club wishes to purchase pies at a cost of \$2.15 each. If C is the cost (\$) and x is the number of pies, complete the table showing the amount of money needed to purchase from 40 to 50 pies.

x	40	41	42	43	44	45	46	47	48	49	50
C(\$)	86	88.15	90.3								

2 The circumference of a circle is given by:

$$C = 2\pi r$$

where r is the radius. Complete the table of values to show the circumferences of circles with radii from 0 to 1 cm in intervals of 0.1 cm. Give your answers correct to three decimal places.

r (cm)	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
C (cm)	0	0.628	1.257	1.885							

3 A phone bill is calculated using the formula:

$$C = 40 + 0.18n$$

where C is the total cost and n represents the number of calls made. Complete the table of values to show the cost for 50, 60, 70, . . . 130 calls.

n	50	60	70	80	90	100	110	115	120	125	130
C(\$)	49	50.80	52.60								

4 The amount of energy (E) in kilojoules expended by an adult male of mass (M) at rest, can be estimated using the formula:

$$E = 110 + 9M$$

Complete the table of values in intervals of 5 kg for males of mass 60–120 kg to show the corresponding values of E.

M (kg)	60	65	70	75	80	85	90	95	100	105	110	115	120
E (kJ)	650	695											

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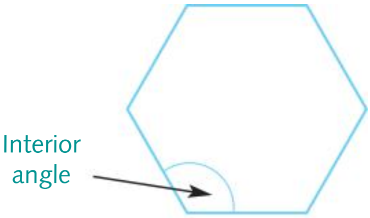
5 The sum, S, of the interior angles of a polygon with n sides is given by the formula:

$$S = 90(2n - 4)$$

Construct a table of values showing the sum of the interior angles of polygons with 3 to 10 sides.



n	3	4	5					
S	180°	360°						

Example 3 **6** A car salesman's weekly wage, E dollars, is given by the formula:

$$E = 60n + 680$$

where n is the number of cars sold.

a Construct a table of values to show how much his weekly wage will be if he sells from 0 to 10 cars.

b Using your table of values, if the salesman earns \$1040 in a week, how many cars did he sell?

7 Anita has \$10 000 that she wishes to invest at a rate of 4.5% per annum. She wants to know how much interest she will earn after 1, 2, 3, ...10 years. Using the formula:

$$I = \frac{PRT}{100}$$

where P is the principal and R is the interest rate (%), construct a table of values with a calculator to show how much interest, I, she will have after $T = 1, 2, \ldots 10$ years.

8 The formula for finding the amount, A, accumulated at compound interest is given by:

$$A = P \times \left(1 + \frac{r}{100}\right)^t$$

where P is the principal, r is the annual interest rate (%) and t is the time in years. Construct a table of values showing the amount accumulated when \$5000 is invested at a rate of 5.5% over 5, 10, 15, 20 and 25 years. Give your answers to the nearest dollar.

3C Solving linear equations with one unknown

Practical applications of mathematics often involve the need to be able to solve **linear equations**. An *equation* is a mathematical statement that says that two things are equal. For example, these are all equations:

$x - 3 = 5$ $\quad$ $2w - 5 = 17$ $\quad$ $3m = 24$

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Linear equations come in many different forms in mathematics but are easy to recognise because the powers on the unknown values are always 1. For example:

- $m - 4 = 8$ is a linear equation, with unknown value m
- $3x = 18$ is a linear equation, with unknown value x
- $4y - 3 = 17$ is a linear equation, with unknown value y
- $a + b = 0$ is a linear equation, with unknown values a and b
- $x^2 + 3 = 9$ is *not* a linear equation (the power of x is 2 not 1), with unknown value x
- $c = 16 - d^2$ is *not* a linear equation (the power of d is 2), with unknowns c and d.

The process of finding the unknown value is called *solving the equation*. When solving an equation, *opposite* (or *inverse*) operations are used so that the unknown value to be solved is the only term remaining on one side of the equation. Opposite operations are indicated in the table below.

Operation	+	−	×	÷	x^2 (power of 2, square)	$\sqrt{x}$ (square root)
Opposite operation	−	+	÷	×	$\sqrt{x}$ (square root)	x^2 (power of 2, square)

Remember: The equation must remain *balanced*. To balance an equation add or subtract the *same* number on *both* sides of the equation or multiply or divide *both* sides of the equation by the *same* number.

Example 4 Solving a linear equation

Solve the equation $x + 6 = 10$.

Solution

Method 1: By inspection

Write the equation.

$$x + 6 = 10$$

What needs to be added to 6 to make 10?

The answer is 4.

$$\therefore x = 4$$

Method 2: Inverse operations

This method requires the equation to be 'undone', leaving the unknown value by itself on one side of the equation.

1 Write the equation.

$$x + 6 = 10$$

2 Subtract 6 from both sides of the equation. This is the opposite process to adding 6.

$$x + 6 - 6 = 10 - 6$$
$$\therefore x = 4$$

3 Check your answer by substituting the found value for x into the original equation. If each side gives the same value, the solution is correct.

$$\begin{aligned} \text{LHS} &= x + 6 \\ &= 4 + 6 \\ &= 10 \\ &= \text{RHS} \end{aligned}$$

∴ Solution is correct.

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Example 5 Solving a linear equation by hand

Solve the equation $3y = 18$.

Solution

1 Write the equation.

$$3y = 18$$

2 The opposite process of multiplying by 3 is dividing by 3. Divide both sides of the equation by 3.

$$\frac{3y}{3} = \frac{18}{3}$$

$$\therefore y = 6$$

3 Check that the solution is correct by substituting $y = 6$ into the original equation.

$$\begin{aligned} \text{LHS} &= 3y \\ &= 3 \times 6 \\ &= 18 \\ &= \text{RHS} \end{aligned}$$

$\therefore$ Solution is correct.

Example 6 Solving a linear equation by hand

Solve the equation $4(x - 3) = 24$.

Solution

Method 1

1 Write the equation.

$$4(x - 3) = 24$$

2 Expand the brackets.

$$4x - 12 = 24$$

3 Add 12 to both sides of the equation.

$$4x - 12 + 12 = 24 + 12$$

4 Divide by 4.

$$4x = 36$$

$$\frac{4x}{4} = \frac{36}{4}$$

$$\therefore x = 9$$

5 Check that the solution is correct by substituting $x = 9$ into the original equation (see 4 below).

Method 2

1 Write the equation.

$$4(x - 3) = 24$$

2 Divide by 4.

$$\frac{4(x - 3)}{4} = \frac{24}{4}$$

$$x - 3 = 6$$

3 Add 3.

$$x - 3 + 3 = 6 + 3$$

$$\therefore x = 9$$

4 Check that the solution is correct by substituting $x = 9$ into the orginal equation.

$$\begin{aligned} \text{LHS} &= 4(x - 3) \\ &= 4(9 - 3) \\ &= 4 \times 6 \\ &= 24 \\ &= \text{RHS} \end{aligned}$$

$\therefore$ Solution is correct.

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Example 7 Solving a linear equation using CAS

Solve the equation $-4 - 5b = 8$.

Solution

1 Use the **solve(** command on your CAS calculator to solve for b as shown opposite.

Note: Set the mode of your calculator to Approximate (TI-Nspire) or Decimal (ClassPad) before using **solve(**.

solve $(-4 - 5b = 8, b)$ $\quad b = -2.4$

2 Check that the solution is correct by substituting $x = -2.4$ into the original equation.

$$\begin{aligned} \text{LHS} &= -4 - 5b \\ &= -4 - 5 \times -2.4 \\ &= -4 + 12 \\ &= 8 = \text{RHS} \end{aligned}$$

$\therefore$ Solution is correct.

Exercise 3C

Example 4

1 Solve the following linear equations.

a $x + 6 = 15$ **b** $y + 11 = 26$ **c** $t + 5 = 10$ **d** $m - 5 = 1$

e $g - 3 = 3$ **f** $f - 7 = 12$ **g** $f + 5 = 2$ **h** $v + 7 = 2$

i $x + 11 = 10$ **j** $g - 3 = -2$ **k** $b - 10 = -5$ **l** $m - 5 = -7$

m $2 + y = 8$ **n** $6 + e = 9$ **o** $7 + h = 2$ **p** $3 + a = -1$

q $4 + t = -6$ **r** $8 + s = -3$ **s** $9 - k = 2$ **t** $5 - n = 1$

u $3 - a = -5$ **v** $10 - b = -11$

Example 5

2 Solve the following linear equations.

a $5x = 15$ **b** $3g = 27$ **c** $9n = 36$ **d** $2x = -16$

e $6j = -24$ **f** $4m = 28$ **g** $2f = 11$ **h** $2x = 7$

i $3y = 15$ **j** $3s = -9$ **k** $-5b = 25$ **l** $4d = -18$

m $\frac{r}{3} = 4$ **n** $\frac{q}{5} = 6$ **o** $\frac{x}{8} = 6$ **p** $\frac{t}{-2} = 6$

q $\frac{h}{-8} = -5$ **r** $\frac{m}{-3} = -7$ **s** $\frac{14}{a} = 7$ **t** $\frac{24}{f} = -12$

u $2a + 15 = 27$ **v** $\frac{y}{4} - 10 = 0$ **w** $13 = 3r - 11$ **x** $\frac{x + 1}{3} = 2$

y $\frac{3m}{4} = 6$ **z** $\frac{2x - 1}{3} = 4$

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Example 6

3 Solve the following linear equations.

a $2(y-1)=6$ **b** $8(x-4)=56$ **c** $3(g+2)=12$

d $3(4x-5)=21$ **e** $8(2x+1)=16$ **f** $3(5m-2)=12$

g $\dfrac{2(a-3)}{5}=6$ **h** $\dfrac{4(r+2)}{6}=10$

4 Solve these equations by firstly collecting all like terms.

a $2x=x+5$ **b** $2a+1=a+4$ **c** $4b-10=2b+8$

d $7-5y=3y-17$ **e** $3(x+5)-4=x+11$ **f** $6(c+2)=2(c-2)$

g $2f+3=2-3(f+3)$ **h** $5(1-3y)-2(10-y)=-10y$

Example 7

5 Solve the following linear equations using CAS. Give answers correct to one decimal place where appropriate.

a $3a+5=11$ **b** $4b+3=27$ **c** $2w+5=9$

d $7c-2=12$ **e** $3y-5=16$ **f** $4f-1=7$

g $3+2h=13$ **h** $2+3k=6$ **i** $-4(g-4)=-18$

j $\dfrac{2(s-6)}{7}=4$ **k** $\dfrac{5(t+1)}{2}=8$ **l** $\dfrac{-4(y-5)}{5}=2.4$

m $2(x-3)+4(x+7)=10$ **n** $5(g+4)-6(g-7)=25$ **o** $5(p+4)=25+(7-p)$

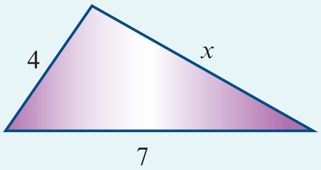
3D Developing a formula: setting up linear equations in one unknown

In many practical problems, we often need to set up a linear equation before finding the solution to a problem. Some practical examples are given below showing how a linear equation is set up and then solved.

Example 8 Setting up a linear equation

Find an equation for the perimeter of the triangle shown.

Note: Perimeter is the distance around the outside of a shape.



Solution

1 Choose a variable to represent the perimeter.

Let P be the perimeter.

2 Add up all sides of the triangle and let them equal the perimeter, P.

$P = 4 + 7 + x$

3 Write your answer.

$P = 11 + x$

The required equation is

$P = 11 + x$

Example 9 Setting up and solving a linear equation

If 11 is added to a certain number, the result is 25. Find the number.

Solution

1 Choose a variable to represent the number. Let n be the number.

2 Using the information, write an equation.

$n + 11 = 25$

3 Solve the equation by subtracting 11 from both sides of the equation.

$n + 11 - 11 = 25 - 11$

$\therefore n = 14$

4 Write your answer. The required number is 14.

Example 10 Setting up and solving a linear equation

At a recent show, Chris spent $100 on 8 showbags, each costing the same price.

a Using x as the cost of one showbag, write an equation showing the cost of 8 showbags.

b Use the equation to find the cost of one showbag.

Solution

a 1 Write the cost of one showbag using the variable given. Let x be the cost of one showbag.

2 Use the information to write an equation. Remember: $8 \times x = 8x$

$8x = 100$

b 1 Write the equation.

$8x = 100$

2 Solve the equation by dividing both sides of the equation by 8.

$$\frac{8x}{8} = \frac{100}{8}$$

$\therefore x = 12.5$

3 Write your answer. The cost of one showbag is \$12.50.

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Example 11 Setting up and solving a linear equation

A car rental company has a fixed charge of $110 plus $84 per day for the hire of a car. The Brown family have budgeted $650 for the hire of a car during their family holiday. For how many days can they hire a car?

Solution

1 Choose a variable (d) for the number of days that the car is hired for. Use the information to write an equation.

Let d be the number of days that the car is hired for.

$$110 + 84d = 650$$

2 Solve the equation.
First, subtract 110 from both sides of the equation.
Then divide both sides of the equation by 84.

$$110 + 84d - 110 = 650 - 110$$
$$84d = 540$$
$$\frac{84d}{84} = \frac{540}{84}$$
$$\therefore d = 6.428$$

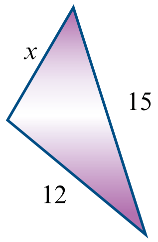
3 Write your answer in terms of complete days.

Car hire works on a daily rate so 6.428 days is not an option. We therefore round down to 6 days to ensure that the Brown family stays within their budget of $650.
The Brown family could hire a car for 6 days.

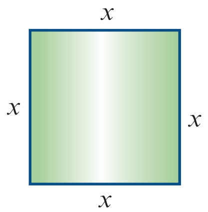
Exercise 3D

Example 8 **1** Find an expression for the perimeter, P, of each of the following shapes.

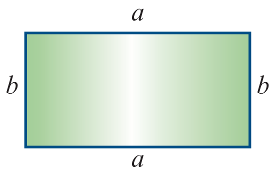
a



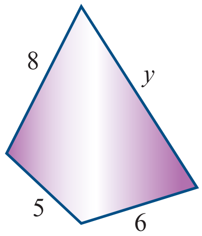
b



c

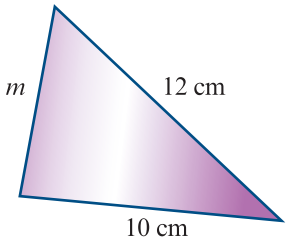


d



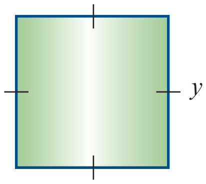
2 **a** Write an expression for the perimeter of the triangle shown.

b If the perimeter, P, of the triangle is 30 cm, solve the equation to find the value of m.



3 **a** Write an equation for the perimeter of the square shown.

b If the perimeter is 52 cm, what is the length of one side?



Example 9 **4** Seven is added to a number and the result is 15.

a Write an equation using n to represent the number.

b Solve the equation for n.

5 Five is added to twice a number and the result is 17. What is the number?

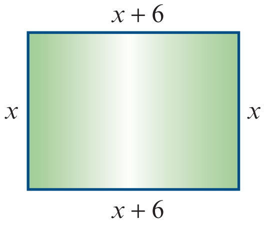
6 When a number is doubled and 15 is subtracted, the result is 103. Find the number.

7 The perimeter of a rectangle is 84 cm. The length of the rectangle is 6 cm longer than the width, as shown in the diagram.

a Write an expression for the perimeter, P, of the rectangle.

b Find the value of x.

c Find the lengths of the sides of the rectangle.



Example 10, 11 **8** Year 11 students want to run a social. The cost of hiring a band is $820 and they are selling tickets for $15 per person. The profit, P, is found by subtracting the band hire cost from the money raised from selling tickets. The students would like to make a profit of $350. Use the information to write an equation, and then solve the equation to find how many tickets they need to sell.

9 The cost for printing cards at the Stamping Printing Company is $60 plus $2.50 per card. Kate paid $122.50 to print invitations for her party. How many invitations were printed?

10 A raffle prize of $1000 is divided between Anne and Barry so that Anne receives 3 times as much as Barry. How much does each receive?

11 Bruce cycles x kilometres, then walks half as far as he cycles. If the total distance covered is 45 km, find the value of x.

12 Amy and Ben live 17.2 km apart. They cycle to meet each other. Ben travels at 12 km/h and Amy travels at 10 km/h.

a How long (in minutes) until they meet each other?

b What distance have they both travelled?

3E Solving literal equations

A **literal equation** is an equation whose **solution** will be expressed in terms of another variable rather than a number.

For example:

- $2x + 3 = 13$ is a linear equation in one unknown, whose solution for x is the number 5
- $x + 2y = 6$ is a *literal* equation in two unknowns, whose solution for x is $6 - 2y$. It is called a literal equation because its solution for x is expressed in terms of another variable.

Note: While the following exercises can be completed with a CAS calculator, you should check that you are able to complete them by hand.

Example 12 Solving a literal equation

If $x + 5y = 9$, find an expression for x.

Solution

Method 1 (by hand)

Strategy: Rearrange the equation so that x becomes the subject.

1 Write the original equation.

$$x + 5y = 9$$

2 Subtract $5y$ from both sides of the equation to give your answer.

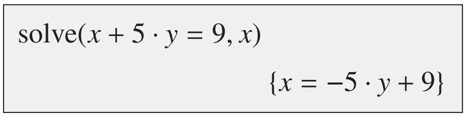
$$x + 5y - 5y = 9 - 5y$$

$$\therefore x = 9 - 5y$$

Method 2 (using CAS)

Use the **solve(** command as shown opposite.

Note: $x = -5y + 9$ is the same as $x = 9 - 5y$



solve($x + 5 \cdot y = 9, x$)

$\{x = -5 \cdot y + 9\}$

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Example 13 Solving a literal equation

If $2a - 4b = 8$, find an expression for a.

Solution

Strategy: Rearrange the equation so that a becomes the subject.

1 Write the original equation.

$$2a - 4b = 8$$

2 Add $4b$ to both sides of the equation.

$$2a - 4b + 4b = 8 + 4b$$

$$2a = 8 + 4b$$

3 Divide both sides of the equation by 2 to give your answer.

$$\therefore a = 4 + 2b$$

Note: All terms must be divided by 2

Example 14 Solving a literal equation

If $3x + 2y = 4$, find:

a an expression for y

b an expression for x.

Solution

Method 1 (by hand)

Strategy: Rearrange the equation so that y, and then x, becomes the subject.

a an expression for y

1 Write the original equation.

$$3x + 2y = 4$$

2 Subtract $3x$ from both sides of the equation.

$$3x + 2y - 3x = 4 - 3x$$

$$2y = 4 - 3x$$

3 Divide both sides of the equation by 2 to give your answer.

$$\therefore y = 2 - \tfrac{3}{2}x$$

b an expression for x

1 Write the original equation.

$$3x + 2y = 4$$

2 Subtract $2y$ from both sides of the equation.

$$3x + 2y - 2y = 4 - 2y$$

$$3x = 4 - 2y$$

3 Divide both sides of the equation by 3 to give your answer.

$$x = \frac{4}{3} - \frac{2}{3}y$$

Method 2 (using CAS)

Use the **solve(** command as shown.

$\text{solve}(3 \cdot x + 2 \cdot y = 4, y)$

$$\left\{y = \frac{-3 \cdot x}{2} + 2\right\}$$

$\text{solve}(3 \cdot x + 2 \cdot y = 4, x)$

$$\left\{x = \frac{-2 \cdot y}{3} + \frac{4}{3}\right\}$$

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Example 15 Solving a literal equation

If $y = mx + c$, find an expression for m.

Solution

Method 1 (by hand)

Strategy: Rearrange the equation so that m becomes the subject.

1 Write the original equation.

2 Subtract c from both sides of the equation.

$$y = mx + c$$

$$y - c = mx + c - c$$

3 Divide both sides of the equation by x.

$$y - c = mx$$

$$\frac{y-c}{x} = \frac{mx}{x}$$

4 Rewrite the equation with the required variable, m, on the left-hand side.

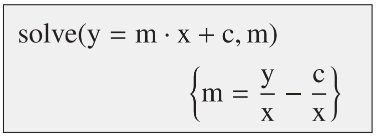
$$\frac{y-c}{x} = m$$

$$\therefore m = \frac{y-c}{x}$$

Method 2 (using CAS)

Use the **solve(** command as shown opposite.

Note: $m = \frac{y}{x} - \frac{c}{x}$ is the same as $m = \frac{y-c}{x}$



solve(y = m · x + c, m)

$$\left\{m = \frac{y}{x} - \frac{c}{x}\right\}$$

Exercise 3E

Skillsheet

Example 12, 13

1 Find an expression for x.

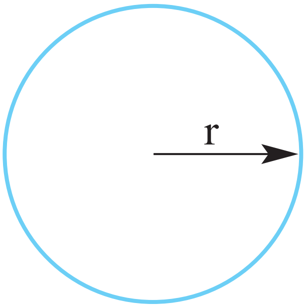
a $x + 7y = 22$
b $x - 4y = 11$
c $x + 6y = 5$
d $5y = x - 12$
e $2y = x + 5$
f $4y = 8 - x$
g $4y = 2x + 6$
h $3x - 5 = 2y$
i $5y + 2x = 10$
j $4x + 12y = -24$
k $2x - 3y = 12$
l $7y - 5x = 40$
m $y = ax + 2$
n $mx + y = 9$
o $4y = 2mx - 6$
p $ny + mx = d$
q $y - ax + b = 0$
r $sx - ty + a = 0$

Example 14

2 The formula for the circumference, C, of a circle is given by:

$$C = 2\pi r$$

Find an expression for r.



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3 The sum, S, of the interior angles of a polygon with number of sides n is given by:

$$S = 180n - 360$$

a Find an expression for n.

b Calculate the number of sides, n, of a polygon with interior angle sum, $S = 900°$.

Example 15

4 The velocity, v, of an object is described by the rule:

$$v = u + at$$

where u is the initial velocity, a is the acceleration and t is the time in seconds.

a Find an expression for t. **b** Find t *if* $v = 30, u = 12$ and $a = 5$.

5 The formula for the area of a triangle is:

$$A = \frac{1}{2}bh$$

where b is the base and h is the height. Find an expression for h in terms of b and A.

6 The formula for converting temperature in Celsius to temperature in Fahrenheit is given by:

$$F = \frac{9}{5}C + 32$$

Find an expression for C.

7 The formula for simple interest is given by:

$$I = \frac{PRT}{100}$$

where P is the principal, R is the interest rate per annum (%) and T is the time in years.

a Rearrange the formula to make T the subject.

b For how many years does \$5000 need to be invested at a rate of 4% to obtain interest of at least \$1000?

8 The formula for the surface area of a cylinder is given by if $S = 2\pi r(r + h)$.

a Make h the subject of the formula.

b Find h if $S = 1221$ cm^2 and $r = 10.5$ cm (correct to one decimal place).

9 The equation of a straight line is given by $y = mx + c$ where m is the gradient, c is the y-intercept and (x, y) are the coordinates of a point on the line.

a Rearrange the equation to make c the subject.

b Find the y-intercept, c, if the gradient, m, is -2 and x is 4 and y is 5.

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3F Developing a formula: setting up linear equations in two unknowns

It is often necessary to develop formulas so that problems can be solved. Constructing a formula is similar to developing an equation from a description.

Example 16 Setting up and solving a linear equation in two unknowns

Sausage rolls cost \$1.30 each and party pies cost 75 cents each.

a Construct a formula for finding the cost, C dollars, of buying x sausage rolls and y party pies.

b Find the cost of 12 sausage rolls and 24 party pies.

Solution

a **1** Work out a formula using x.

One sausage roll costs \$1.30.

Two sausage rolls cost $2 \times \$1.30 = \$2.60$.

Three sausage rolls cost $3 \times \$1.30 = \$3.90$ *etc*.

Write a formula using x.

x sausage rolls cost

$x \times 1.30 = 1.3x$

2 Work out a formula using y.

One party pie costs \$0.75.

Two party pies cost $2 \times \$0.75 = \$1.50$

Three party pies cost $3 \times \$0.75 = \$2.25$ *etc*.

Write a formula using y.

y party pies cost

$y \times 0.75 = 0.75y$

3 Combine to get a formula for total cost, C.

$C = 1.3x + 0.75y$

b **1** Write the formula for C.

$C = 1.3x + 0.75y$

2 Substitute $x = 12$ and $y = 24$ into the formula.

$C = 1.3 \times 12 + 0.75 \times 24$

3 Evaluate.

$C = 33.6$

4 Give your answer in dollars and cents.

The total cost for 12 sausage rolls and 24 party pies is \$33.60.

Exercise 3F

Skillsheet

Example 16

1 Balloons cost 50 cents each and streamers costs 20 cents each.

a Construct a formula for the cost, C, of x balloons and y streamers.

b Find the cost of 25 balloons and 20 streamers.

2 Tickets to a concert cost \$40 for adults and \$25 for children.

a Construct a formula for the total amount, C, paid by x adults and y children.

b How much money altogether was paid by 150 adults and 315 children?

3 At the football canteen, chocolate bars cost \$1.60 and muesli bars cost \$1.40.

a Construct a formula to show the total money, C, made by selling x chocolate bars and y muesli bars.

b How much money would be made if 55 chocolate bars and 38 muesli bars were sold?

4 At the bread shop, custard tarts cost \$1.75 and iced doughnuts \$0.70 cents.

a Construct a formula to show the total cost, C, if x custard tarts and y iced doughnuts are purchased.

b On Monday morning, Mary bought 25 custard tarts and 12 iced doughnuts. How much did it cost her?

5 At the beach café, Marion takes orders for coffee and milkshakes. A cup of coffee costs \$3.50 and a milkshake costs \$5.00.

a Let x = number of coffees ordered and y = number of milkshakes ordered. Using x (coffee) and y (milkshakes) write a formula showing the cost, C, of the number of coffee and milkshakes ordered.

b Marion took orders for 52 cups of coffee and 26 milkshakes. How much money did this make?

6 Joe sells budgerigars for \$30 and parrots for \$60.

a Write a formula showing the money, C, made by selling x budgerigars and y parrots.

b Joe sold 28 parrots and 60 budgerigars. How much money did he make?

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7 James has been saving fifty-cent and twenty-cent pieces.

a If James has x fifty-cent pieces and y twenty-cent pieces, write a formula to show the number, N, of coins that James has.

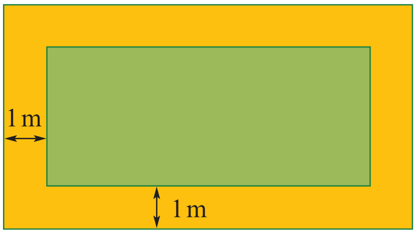
b Write a formula to show the value, V dollars, of James' collection.

c When James counts his coins, he has 45 fifty-cent pieces and 77 twenty-cent pieces. How much money does he have in total?

8 A tennis coach buys four cans of tennis balls and empties them into a large container. The container already has twelve balls in it. Altogether, there are now 32 tennis balls. How many tennis balls were in each can?

9 Maria is five years older than George. The sum of their ages is 37. What are their ages?

10 A rectangular lawn is twice as long as it is wide. The lawn has a path 1 metre wide around it. The length of the perimeter of the outside of the path is 48 metres. What is the width of the lawn? Give your answer correct to the nearest centimetre.



3G Setting up and solving simple non-linear equations (optional topic)

Not all equations that are solved in mathematics are linear equations. Some equations are non-linear. A **non-linear equation** has variables with powers other than one.

For example:

- $y = x^2 + 2$ is a non-linear equation with two unknowns, x and y
- $d^2 = 25$ is a non-linear equation with one unknown, d
- $6m^3 = 48$ is a non-linear equation with one unknown, m.

Some examples of non-linear formulas are:

- $A = \pi r^2$ is the formula for finding the area of a circle, A. The variable r, for the radius, has a power of 2.
- $V = \pi r^2 h$ is the formula for the volume of a cylinder, V. It includes the variables V (volume), h (height), and r (radius), which has a power of 2.

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Example 17 Solving a non-linear equation

Solve the equation $x^2 = 81$.

Solution

Method 1 (by hand)

1 Write the equation.

$$x^2 = 81$$

2 Take the square root of both sides of the equation.
(The opposite process of squaring a number is to take the square root.)

$$\sqrt{x^2} = \sqrt{81}$$
$$\therefore x = \pm 9$$

Note: Both the positive and negative answers should be given, as $-9 \times -9 = 81$ and $9 \times 9 = 81$.

Method 2 (using CAS)

Use the **solve(** command as shown opposite.

solve($x^2 = 81,\ x$) $x = -9$ or $x = 9$

Note: Set the mode of your calculator to Approximate (TI-Nspire) or Decimal (ClassPad) before using **solve(**.

Example 18 Solving a non-linear equation

Solve the equation $a^3 = -512$.

Solution

Method 1 (by hand)

1 Write the equation.

$$a^3 = -512$$

2 Take the cube root of both sides of the equation. (The opposite process of cubing a number is to take the cube root.)

$$\sqrt[3]{a^3} = \sqrt[3]{-512}$$
$$\therefore a = -8$$

Note: $\sqrt[3]{-152} = (-512)^{\frac{1}{3}}$

$(-8) \times (-8) \times (-8) = -512$

but $8 \times 8 \times 8 = 512$

Method 2 (using CAS)

Use the **solve(** command as shown opposite.

solve($a^3 = -512, a$) $a = -8$

Example 19 Solving a non-linear equation

The surface area, S, of a sphere of radius, r, is given by the equation $S = 4\pi r^2$. Find the radius of a sphere with a surface area of 600 cm^2.

Solution
(using CAS)

1 Write the equation. $S = 4\pi r^2$

2 Substitute $S = 600$ into the equation. $600 = 4\pi r^2$

3 Use the **solve(** command as shown opposite to solve the equation.

solve$(600 = 4\pi r^2, r)$

$r = -6.90988$ or $r = 6.90988$

4 Noting that the radius must have a positive value, write the value of r correct to two decimal places.

Correct to two decimal places, the radius of the sphere is 6.91 cm.

Example 20 Using a non-linear formula

A circular garden is to be planted with a radius of 4 metres. What is the area of the garden?

Give your answer correct to two decimal places.

Solution

1 Write the formula. $A = \pi r^2$

2 Substitute $r = 4$ into the formula. $A = \pi(4)^2$

3 Evaluate. $A = 50.2654\ldots$

4 Give your answer correct to two decimal places and with correct units.

The area of the garden is 50.27 cm^2, correct to two decimal places.

Example 21 Using a non-linear formula

The intensity I of light at a depth of d cm in a coloured liquid is given by $I = \dfrac{4800}{d^2}$ for values of d greater than 5 cm. Find the intensity of light at a depth of 10 cm.

Solution

1 Write the formula. $I = \dfrac{4800}{d^2}$

2 Substitute $d = 10$ into the formula. $I = \dfrac{4800}{10^2}$

3 Evaluate. $I = 48$

4 Write your answer.

The intensity is 48.

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Exercise 3G

1 Evaluate the following.

a 4^2 **b** $(-9)^2$ **c** 7^2 **d** 3^3

e 2^3 **f** 6^3 **g** $(-5)^3$ **h** 4^4

i $(-10)^4$

Example 17, 18

2 Solve the following non-linear equations correct to two decimal places.

a $a^2 = 12$ **b** $b^2 = 72$ **c** $c^2 = 568$

d $d^3 = 76$ **e** $e^3 = 300$ **f** $f^3 = -759$

3 Solve the following non-linear equations correct to two decimal places.

a $3x^2 = 24$ **b** $5y^2 = 25$ **c** $2a^2 = 55$ **d** $6f^2 = 33$

e $4h^2 = 48$ **f** $11c^2 = 75$ **g** $x^3 = 81$ **h** $r^3 = 18$

i $y^3 = 96$ **j** $2r^3 = 50$ **k** $4m^3 = 76$ **l** $8b^3 = 21$

m $2p^2 - 1 = 8$ **n** $3q^3 + 5 = 101$ **o** $2(r^2 + 8) = 64$ **p** $2(3y^2 - 1) = 52$

q $\dfrac{f^3 - 2}{2} = 31$ **r** $5(2n^2 + 7) = 110$

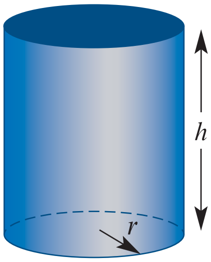
Example 19, 20

Example 21

4 The volume of a cylinder is given by:

$$V = \pi r^2 h$$

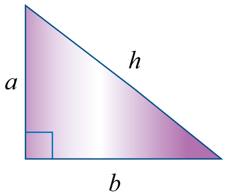
where h is the height and r is the radius of the base. Find, correct to two decimal places, the value of h when $V = 450$ cm^2 and $r = 10$ cm.



5 Pythagoras' theorem states that, for any right-angled triangle, the hypotenuse, h, is given by:

$$h^2 = a^2 + b^2$$

where a and b are the other two sides of the triangle.
Find a, correct to one decimal place, when $h = 17.5$ cm and $b = 7.8$ cm.



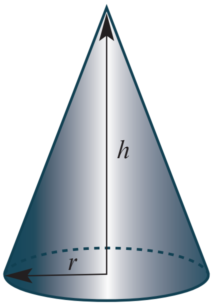
6 The volume, V, of a cone is given by:

$$V = \frac{1}{3}\pi r^2 h$$

where r is the radius and h is the height of the cone.

a Find, to the nearest centimetre, the radius if the height of the cone is 15 cm and the volume is 392.7 cm^3.

b Find, to the nearest centimetre, the height if the radius of the cone is 7.5 cm and the volume is 562.8 cm^3.



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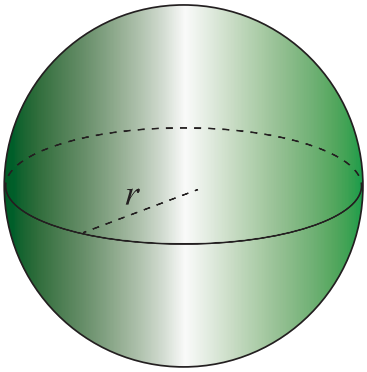
7 The volume of a sphere is 147 cm^3. If the formula for the volume of a sphere is:

$$V = \frac{4}{3}\pi r^3$$

find the radius, r, correct to two decimal places.



3H Transposition of formulas

Sometimes we wish to find the value of a variable (also called the pronumeral) that is not the subject of the formula. We thus rearrange the formula to make another variable the subject. This is called *transposing* the formula. We can transpose linear and non-linear formulas. The rules for transposing formulas are similar to those for solving equations.

Note: The need to transform formulas is not as important nowadays with the use of a CAS calculator as the SOLVE function can be used as seen earlier in this chapter. Nevertheless it is a useful skill.

Example 22 Transposing a linear formula

Transpose the formula $F = \frac{9}{5}C + 32$ to make C the subject. Use the rearranged formula to calculate the value of C when $F = 212$.

Solution

1 Write down the formula.

$$F = \frac{9}{5}C + 32$$

2 Subtract 32.

$$F - 32 = \frac{9}{5}C + 32 - 32$$

$$F - 32 = \frac{9}{5}C$$

3 Multiply by 5.

$$5(F - 32) = \frac{9}{5} \times 5C$$

$$5(F - 32) = 9C$$

4 Divide by 9.

$$\frac{5(F - 32)}{9} = \frac{9C}{9}$$

5 Write out answer with C as subject.

$$C = \frac{5(F - 32)}{9}$$

6 Substitute $F = 212$ into formula.

$$C = \frac{5(212 - 32)}{9}$$

7 Evaluate.

$$C = 100$$

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Exercise 3H

Example 22

1 Transpose the following formulas to make the variable (pronumeral) shown in brackets the subject.

a $A = 2\pi rh$ (r) **b** $I = \dfrac{PRT}{100}$ (R) **c** $V = \pi r^2 h$ (h)

d $V = \pi r^2 h$ (r) **e** $s = ut + \dfrac{1}{2}at^2$ (a)

2 The velocity, v m/s, of an object is described by the rule $v = 2t + 5$ where t is the time for a journey in seconds.

a Transpose the equation to make t the subject.

b Find, correct to two decimal places, the value of t when $v = 14.9$ m/s.

3 The volume of a cone is given by the rule $V = \dfrac{1}{3}\pi r^2 h$.

a Transpose the rule to make r the subject.

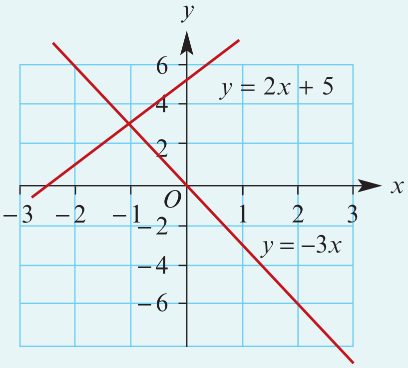
b If the volume, V, is 100 cm^3 and the height of the cone, h, is 10 cm, find the length of the radius, r, correct to two decimal places.

3I Finding the point of intersection of two linear graphs

Two straight lines will always intersect unless they are parallel. The point at which two straight lines intersect can be found by sketching the two graphs on the one set of axes and then reading off the coordinates at the point of intersection. When we find the *point of intersection*, we are said to be **solving the equations simultaneously**.

Example 23 Finding the point of intersection of two linear graphs

The graphs of $y = 2x + 5$ and $y = -3x$ are shown.
Write their point of intersection.



Solution

From the graph it can be seen that the point of intersection is (−1, 3).

Note: A CAS calculator can also be used to find the point of intersection.

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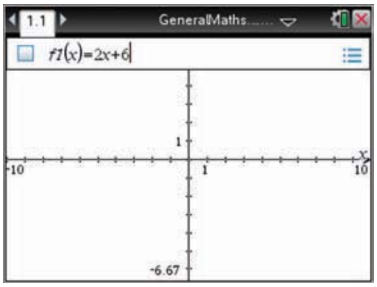
How to find the point of intersection of two linear graphs using the TI-Nspire CAS

Use a graphics calculator to find the point of intersection of the simultaneous equations $y = 2x + 6$ and $y = -2x + 3$.

Steps

1 Start a new document (ctrl + N) and select **Add Graphs**.

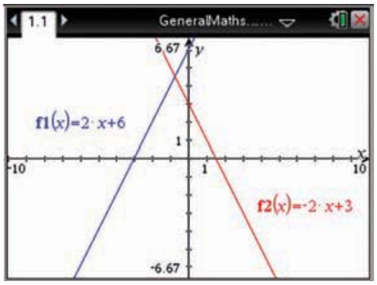
2 Type in the first equation as shown. Note that $f1(x)$ represents the y. Press ▼ and the edit line will change to $f2(x)$ and the first graph will be plotted. Type in the second equation and press enter to plot the second graph.



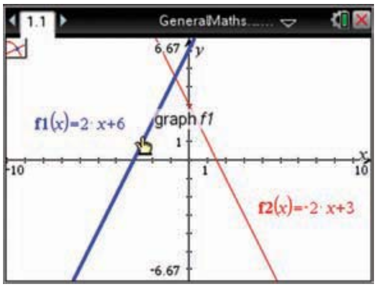
Hint: If the entry line is not visible press tab.

Hint: To see all entered equations move the cursor onto the ☰ and press del.

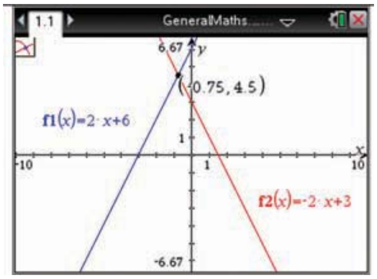
Note: To change window settings, press menu**>Window/Zoom>Window Settings** and change to suit. Press enter when finished.



3 To find the point of intersection, press menu**>Geometry>Points & Lines>Intersection Point(s)**.
Move the cursor to one of the graphs until it flashes, press , then move to the other graph and press . The solution will appear.
Alternatively, use menu**>Analyze Graph>Intersection**.



4 Press enter to display the solution on the screen. The coordinates of the point of intersections are $x = -0.75$ and $y = 4.5$.



Note: you can also find the intersection point using menu**>Analyze Graph>Intersection**.

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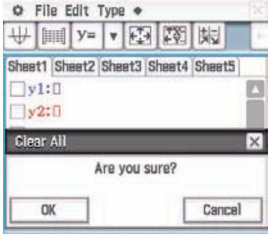
How to find the point of intersection of two linear graphs using the ClassPad

Use a graphics calculator to find the point of intersection of the simultaneous equations $y = 2x + 6$ and $y = -2x + 3$.

Steps

1 Open the built-in **Graphs and Tables** application. Tapping Menu from the icon panel (just below the touch screen) will display the Application menu if it is not already visible.

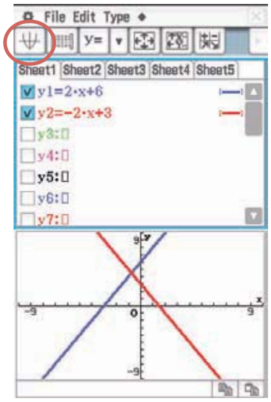


Are you sure?

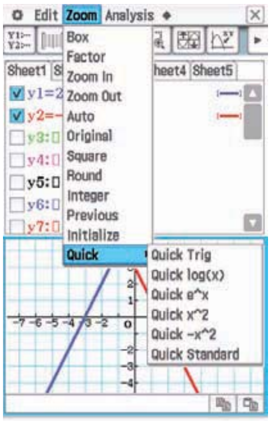
2 If there are any equations from previous questions go to **Edit Clear** all and tap **OK**.

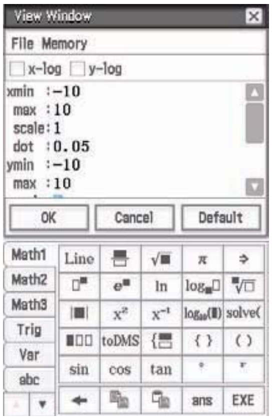
3 Enter the equations into the graph editor window. Tick the boxes. Tap the icon to plot the graphs.

4 To adjust the graph window, tap **Zoom, Quick Standard**. Alternatively, tap the icon and complete the **View Window** dialog box.



☑ y1=2·x+6
☑ y2=−2·x+3

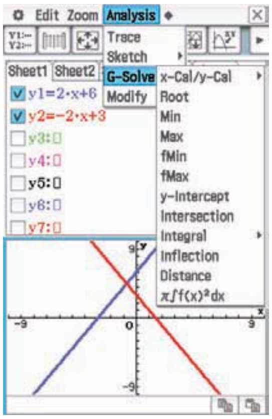


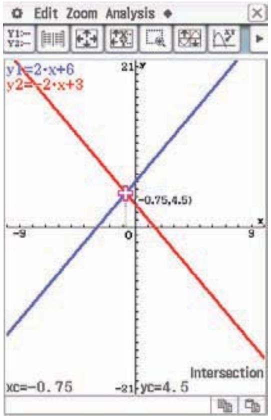


xmin :−10
max :10
scale:1
dot :0.05
ymin :−10
max :10

5 Solve by finding the point of intersection. Select **Analysis** from the menu bar, then **G-solve**, then **Intersect**. Tap Resize to view graph only.

The required solution is displayed on the screen: $x = -0.75$ and $y = 4.5$.





xc=−0.75 yc=4.5

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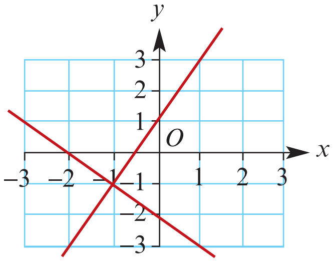
Exercise 3I

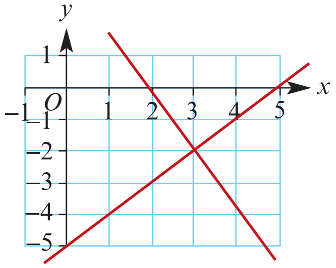
Example 23

1 State the point of intersection for each of these pairs of straight lines.

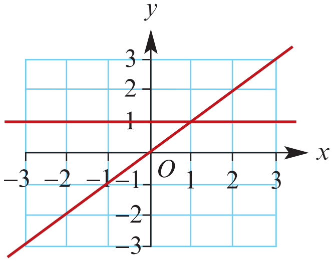
a



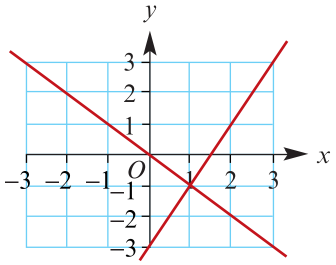
b



c



d



2 Using a CAS calculator, find the point of intersection of each of these pairs of lines.

a $y = x - 6$ and $y = -2x$

b $y = x + 5$ and $y = -x - 1$

c $y = 3x - 2$ and $y = 4 - x$

d $y = x - 1$ and $y = 2x - 3$

e $y = 2x + 6$ and $y = 6 + x$

f $x - y = 5$ and $y = 2$

g $x + 2y = 6$ and $y = 3 - x$

h $2x + y = 7$ and $y - 3x = 2$

i $3x + 2y = -4$ and $y = x - 3$

j $y = 4x - 3$ and $y = 3x + 4$

k $y = x - 12$ and $y = 2x - 4$

l $y + x = 7$ and $2y + 5x = 5$

m $y = 2x + 3$ and $y = 2x - 7$

3J Solving simultaneous linear equations algebraically

When solving simultaneous equations algebraically, there are two methods that can be used: *substitution* or *elimination*. Both methods will be demonstrated here.

▶ Method 1: Substitution

When solving simultaneous equations by substitution, the process is to substitute one variable from one equation into the other equation.

Substitution is useful if one equation is in the form $y = mx + c$, where m and c are constants.

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Example 24 Solving simultaneous equations by substitution

Solve the pair of simultaneous equations $y = 5 - 2x$ and $3x - 2y = 4$.

Solution

1 Number the two equations as (1) and (2).

$$y = 5 - 2x \qquad (1)$$
$$3x - 2y = 4 \qquad (2)$$

2 Substitute the y-value from equation (1) into equation (2).

substitute (1) into (2)

$$3x - 2(5 - 2x) = 4$$

3 Expand the brackets and then collect like terms.

$$3x - 10 + 4x = 4$$
$$7x - 10 = 4$$

4 Solve for x. Add 10 to both sides of the equation.

Divide both sides of the equation by 7.

$$7x - 10 + 10 = 4 + 10$$
$$7x = 14$$
$$\frac{7x}{7} = \frac{14}{7}$$
$$\therefore x = 2$$

5 To find y, substitute $x = 2$ into equation (1).

Substitute $x = 2$ into (1).

$$y = 5 - 2(2)$$
$$y = 5 - 4$$
$$\therefore y = 1$$

6 Check by substituting $x = 2$ and $y = 1$ into equation (2).

$$\text{LHS} = 3(2) - 2(1)$$
$$= 6 - 2 = 4 = \text{RHS}$$

7 Write your solution.

$$x = 2, y = 1$$

Example 25 Solving simultaneous equations by substitution

Solve the pair of simultaneous equations $y = x + 5$ and $y = -3x + 9$.

Solution

1 Number the two equations as (1) and (2).

$$y = x + 5 \qquad (1)$$
$$y = -3x + 9 \qquad (2)$$

2 Both equations are expressions for y, so they can be made equal to each other.

$$x + 5 = -3x + 9$$

3 Solve for x.

Add $3x$ to both sides of the equation.

$$x + 5 + 3x = -3x + 9 + 3x$$
$$4x + 5 = 9$$

Subtract 5 from both sides of the equation.

$$4x + 5 - 5 = 9 - 5$$
$$4x = 4$$

Divide both sides of the equation by 4.

$$\frac{4x}{4} = \frac{4}{4}$$
$$\therefore x = 1$$

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4 Find y by substituting $x = 1$ into either equation (1) or equation (2).	Substitute $x = 1$ into (1). $y = 1 + 5$ $\therefore y = 6$
5 Check by substituting $x = 1$ and $y = 6$ into the other equation.	$LHS = 6$ $RHS = -3(1) + 9$ $= -3(1) + 9$ $= -3 + 9$ $= 6$
6 Write your answer.	$x = 1, y = 6$

Method 2: Elimination

When solving simultaneous equations by elimination, one of the unknown variables is eliminated by the process of adding or subtracting multiples of the two equations.

Example 26 Solving simultaneous equations by elimination

Solve the pair of simultaneous equations $x + y = 3$ and $2x - y = -9$.

Solution

1 Number the two equations. On inspection, it can be seen that if the two equations are added, the variable y will be eliminated as the y-coefficients have equal but opposite signed values.	$x + y = 3 \quad (1)$ $2x - y = 9 \quad (2)$
2 Add equations (1) and (2).	$(1) + (2): \quad 3x = 12$
3 Solve for x. Divide both sides of the equation by 3.	$\frac{3x}{3} = \frac{12}{3}$ $\therefore x = 4$
4 Substitute $x = 4$ into equation (1) to find the corresponding y value.	Substitute $x = 4$ into (1). $4 + y = 3$
5 Solve for y. Subtract 4 from both sides of the equation.	$4 + y - 4 = 3 - 4$ $\therefore y = -1$
6 Check by substituting $x = 4$ and $y = -1$ into equation (2).	$LHS = 2(4) - (-1)$ $= 8 + 1 = 9 = RHS$
7 Write your answer.	$x = 4, y = -1$

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Example 27 Solving simultaneous equations by elimination

Solve the pair of simultaneous equations $3x + 2y = 2.3$ and $8x - 3y = 2.8$.

Solution

1 Label the two equations (1) and (2).

$$\begin{aligned} 3x + 2y &= 2.3 \quad (1) \\ 8x - 3y &= 2.8 \quad (2) \end{aligned}$$

2 Multiply equation (1) by 3 and equation (2) by 2 to give $6y$ in both equations.

Remember: Each term in equation (1) must be multiplied by 3 and each term in equation (2) by 2.

$$\begin{aligned} (1) \times 3 \qquad 9x + 6y &= 6.9 \quad (3) \\ (2) \times 2 \qquad 16x - 6y &= 5.6 \quad (4) \end{aligned}$$

3 Add equation (4) to equation (3) to eliminate $6y$.

$$(3) + (4) \qquad 25x = 12.5$$

4 Solve for x. Divide both sides of the equation by 25.

$$\frac{25x}{25} = \frac{12.5}{25}$$

$$x = 0.5$$

5 To find y, substitute $x = 0.5$ into equation (1).

$$3(0.5) + 2y = 2.3$$

$$1.5 + 2y = 2.3$$

6 Solve for y. Subtract 1.5 from both sides of the equation.

$$1.5 + 2y - 1.5 = 2.3 - 1.5$$

$$2y = 0.8$$

7 Divide both sides of the equation by 2.

$$\frac{2y}{2} = \frac{0.8}{2}$$

$$y = 0.4$$

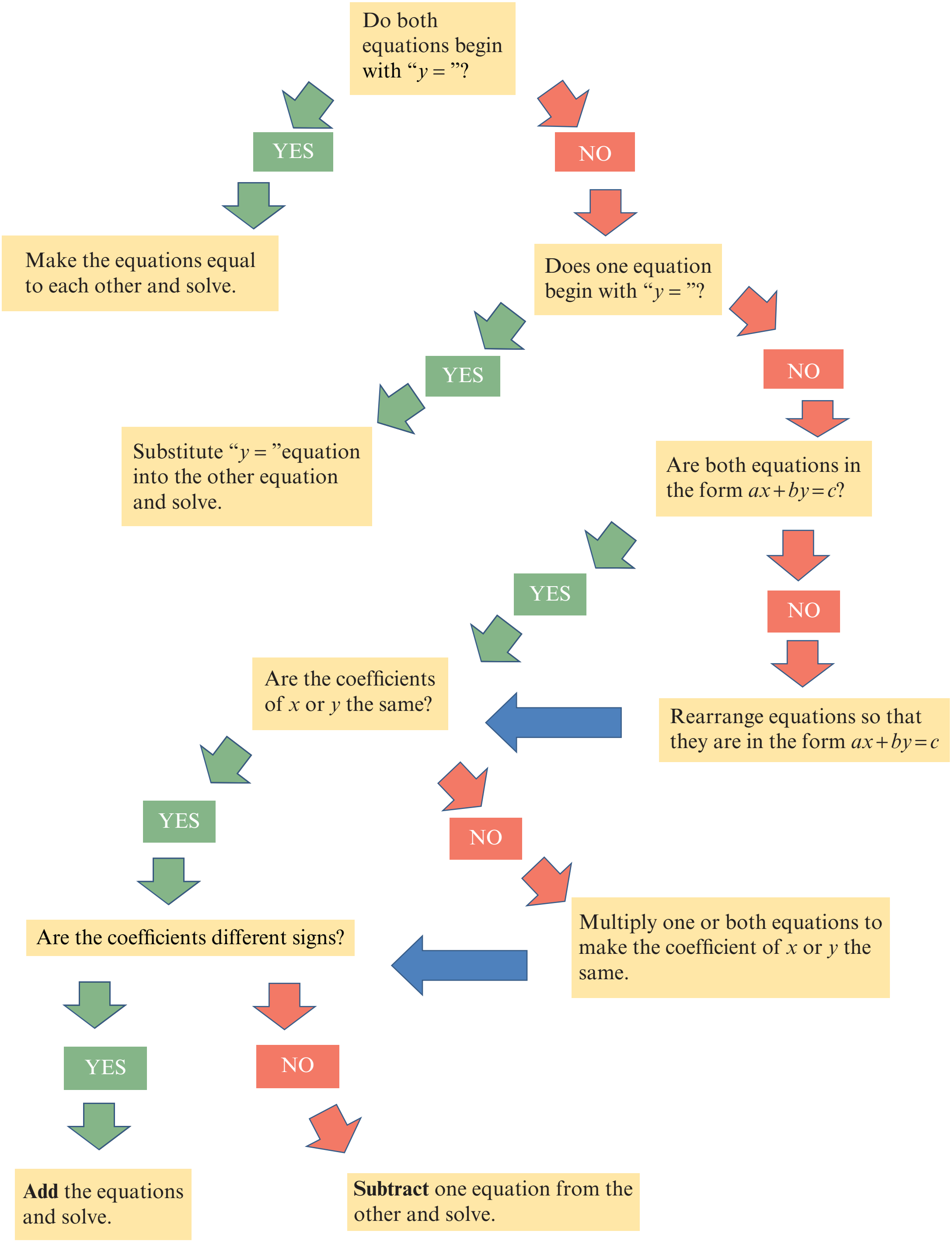
8 Check by substituting $x = 0.5$ and $y = 0.4$ into equation (1).

$$\begin{aligned} LHS &= 3(0.5) + 2(0.4) \\ &= 2.3 = RHS \end{aligned}$$

9 Write your answer.

$$x = 0.5, y = 0.4$$

▶ Deciding whether to use the substitution or elimination method



Exercise 3J

Example 24–27

1 Solve the following pairs of simultaneous equations by any algebraic method (elimination or substitution).

a $y = x - 1$
$3x + 2y = 8$

b $y = x + 3$
$6x + y = 17$

c $x + 3y = 15$
$y - x = 1$

d $x + y = 10$
$x - y = 8$

e $2x + 3y = 12$
$4x - 3y = 6$

f $3x + 5y = 8$
$x - 2y = -1$

g $2x + y = 11$
$3x - y = 9$

h $2x + 3y = 15$
$6x - y = 11$

i $3p + 5q = 17$
$4p + 5q = 16$

j $4a + 3b = 7$
$6a - 3b = -27$

k $3f + 5g = -11$
$-3f - 2g = 8$

l $4x - 3y = 6$
$5y - 2x = 4$

2 Solve the following pairs of simultaneous equations by any suitable method.

a $y = 6 - x$
$2x + y = 8$

b $3x + 2y = 0$
$-3x - y = 3$

c $3x + y = 4$
$y = 2 - 4x$

d $3x + 5y = 9$
$y = 3$

e $2x + 3y = 5$
$y = 7 - 2x$

f $4x + 3y = -28$
$5x - 6y = -35$

3K Solving simultaneous linear equations using a CAS calculator

How to solve a pair of simultaneous linear equations algebraically using the TI-Nspire CAS

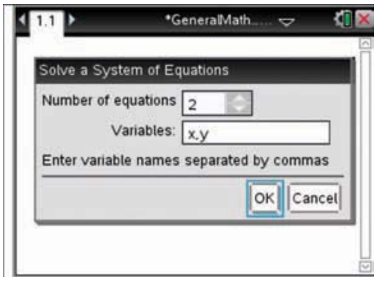
Solve the following pair of simultaneous equations:

$$24x + 12y = 36$$
$$45x + 30y = 90$$

Steps

1 Start a new document and select **Add Calculator**.

2 Press menu>**Algebra>Solve System of Equations>Solve System of Equations** and complete the pop-up screen as shown (the default settings are for two equations with variables x & y).
A simultaneous equation template will be pasted to the screen.



Solve a System of Equations
Number of equations: 2
Variables: x,y
Enter variable names separated by commas

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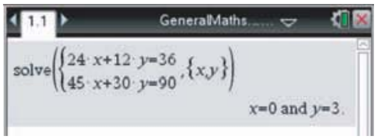
3 Enter the equations as shown into the template. Use the tab key to move between entry boxes.

4 Press enter to display the solution, $x = 0$ and $y = 3$.



$\text{solve}\left(\begin{cases}24\cdot x+12\cdot y=36\\45\cdot x+30\cdot y=90\end{cases},\{x,y\}\right)$

$x=0$ and $y=3$

5 Write your answer.

$x = 0, y = 3$

How to solve a pair of simultaneous linear equations algebraically using the ClassPad

Solve the following pair of simultaneous equations:

$$24x + 12y = 36$$
$$45x + 30y = 90$$

Steps

1 Open the built-in **Main** application √α.

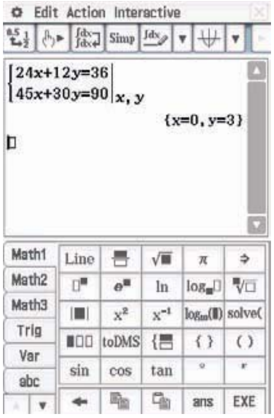
  a Press Keyboard on the front of the calculator to display the built-in keyboard.

  b Tap the simultaneous equations icon: {▤

  c Enter the information

$$\begin{cases}24x + 12y = 36\\45x + 30y = 90\end{cases}\Bigg|_{x,\, y}$$

2 Press EXE to display the solution, $x = 0$ and $y = 3$.



$\begin{cases}24x+12y=36\\45x+30y=90\end{cases}\Big|_{x,\,y}$

$\{x=0, y=3\}$

3 Write your answer.

$x = 0, y = 3$

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Exercise 3K

1 Solve the following simultaneous equations using a CAS calculator.

a $2x + 5y = 3$
$x + y = 3$

b $3x + 2y = 5.5$
$2x - y = -1$

c $3x - 8y = 13$
$-2x - 3y = 8$

d $2h - d = 3$
$8h - 7d = 18$

e $2p - 5k = 11$
$5p + 3k = 12$

f $5t + 4s = 16$
$2t + 5s = 12$

g $2m - n = 1$
$2n + m = 8$

h $15x - 4y = 6$
$-2y + 9x = 5$

i $2a - 4b = -12$
$2b + 3a - 2 = -2$

j $3y = 2x - 1$
$3x = 2y + 1$

k $2.9x - 0.6y = 4.8$
$4.8x + 3.1y = 5.6$

3L Practical applications of simultaneous equations

Simultaneous equations can be used to solve problems in real situations. It is important to define the unknown quantities with appropriate variables before setting up the equations.

Example 28 Using simultaneous equations to solve a practical problem

Tickets for a movie cost \$19.50 for adults and \$14.50 for children. Two hundred tickets were sold giving a total of \$3265. How many children's tickets were sold?

Solution

Strategy: Using the information given, set up a pair of simultaneous equations to solve.

1 Choose appropriate variables to represent the cost of an adult ticket and the cost of a child ticket.

Let a be the number of adults' tickets sold and c be the number of children's tickets sold.

2 Write two equations using the information given in the question. Label the equations as (1) and (2).

Note: The total number of adult and children's tickets is 200, which means that $a + c = 200$.

$$a + c = 200 \qquad (1)$$
$$19.5a + 14.5c = 3265 \qquad (2)$$

3 Rearrange equation (1) to make a the subject.

$$a = 200 - c \qquad (3)$$

4 Substitute a from (3) into equation (2).

$$19.5(200 - c) + 14.5c = 3265$$

5 Expand the brackets and then collect like terms.

$$3900 - 19.5c + 14.5c = 3265$$
$$3900 - 5c = 3265$$

6 Solve for c. Subtract 3900 from both sides of the equation.

Divide both sides of the equation by −5.

$$3900 - 5c - 3900 = 3265 - 3900$$
$$-5c = -635$$
$$\frac{-5c}{-5} = \frac{-635}{-5}$$
$$\therefore c = 127$$

7 To solve for a, substitute $c = 127$ into equation (1).

$$a + 127 = 200 \quad (1)$$

8 Subtract 127 from both sides.

$$a + 127 - 127 = 200 - 127$$
$$a = 73$$

9 Check by substituting, $a = 73$ and $c = 127$ into equation (2).

$$127 + 73 = 200$$

10 Write your solution.

127 children's tickets were sold.

Example 29 Using simultaneous equations to solve a practical problem

The perimeter of a rectangle is 48 cm. If the length of the rectangle is three times the width, determine its dimensions.

Solution

Strategy: Using the information given, set up a pair of simultaneous equations to solve.

1 Choose appropriate variables to represent the dimensions of width and length.

Note: If the length, l, of a rectangle is three times its width, w, then this can be written as $l = 3w$

Let W = width
L = length

2 Write two equations from the information given in the question. Label the equations as (1) and (2).

Remember: The perimeter of a rectangle is the distance around the outside and can be found using $2w + 2l$.

$$2W + 2L = 48 \quad (1)$$
$$L = 3W \quad (2)$$

3 Solve the simultaneous equations by substituting equation (2) in equation (1).

Substitute $L = 3W$ into (1).
$$2W + 2(3W) = 48$$

4 Expand the brackets.

$$2W + 6W = 48$$

5 Collect like terms.

$$8W = 48$$

6 Solve for w. Divide both sides by 8.

$$\frac{8W}{8} = \frac{48}{8}$$
$$\therefore W = 6$$

7 Find the corresponding value for l by substituting $w = 6$ into equation (2).

Substitute $W = 6$ into (2).
$$L = 3(6)$$
$$\therefore L = 18$$

8 Give your answer in the correct units.

The dimensions of the rectangle are width 6 cm and length 18 cm.

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Example 30 Using simultaneous equations to solve a practical problem

Mark buys 3 roses and 2 gardenias for \$15.50. Peter buys 5 roses and 3 gardenias for \$24.50. How much did each type of flower cost?

Solution

Strategy: Using the information given, set up a pair of simultaneous equations to solve.

1 Choose appropriate variables to represent the cost of roses and the cost of gardenias.

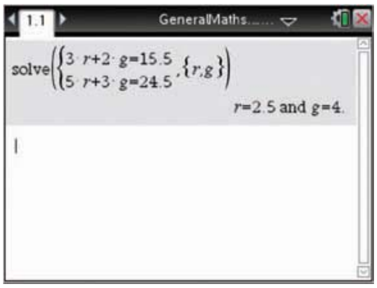
Let r be the cost of a rose and g be the cost of a gardenia.

2 Write equations using the information given in the question. Label the equations (1) and (2).

$3r + 2g = 15.5 \quad (1)$

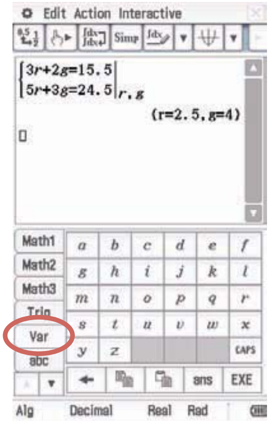
$5r + 3g = 24.5 \quad (2)$

3 Use your CAS calculator to solve the two simultaneous equations.



$\text{solve}\left(\begin{cases}3\cdot r+2\cdot g=15.5\\5\cdot r+3\cdot g=24.5\end{cases},\{r,g\}\right)$

$r=2.5$ and $g=4.$



$\begin{cases}3r+2g=15.5\\5r+3g=24.5\end{cases}\Big|_{r,g}$

$\{r=2.5, g=4\}$

4 Write down the solutions.

$r = 2.50$ and $g = 4$

5 Check by substituting $r = 2.5$ and $g = 4$ into equation (2).

$\text{LHS} = 5(25) + 3(4)$

$= 12.5 + 12 = 24.5 = \text{RHS}$

6 Write your answer with the correct units.

Roses cost \$2.50 each and gardenias cost \$4 each.

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Exercise 3L

Example 28–30

1 Jessica bought 5 textas and 6 pencils for \$12.75, and Tom bought 7 textas and 3 pencils for \$13.80.

a Using t for texta and p for pencil, find a pair of simultaneous equations to solve.

b How much did one pencil and one texta cost each?

2 Peter buys 50 litres of petrol and 5 litres of motor oil for \$109. His brother Anthony buys 75 litres of petrol and 5 litres of motor oil for \$146.

a Using p for petrol and m for motor oil, write down a pair of simultaneous equations to solve.

b How much do a litre of petrol and a litre of motor oil cost each?

3 Six oranges and ten bananas cost \$7.10. Three oranges and eight bananas cost \$4.60.

a Write down a pair of simultaneous equations to solve.

b Find the cost each of an orange and a banana.

4 The weight of a box of nails and a box of screws is 2.5 kg. Four boxes of nails and a box of screws weigh 7 kg. Determine the weight of each.

5 An enclosure at a wildlife sanctuary contains wombats and emus. If the number of heads totals 28 and the number of legs totals 88, determine the number of each species present.

6 The perimeter of a rectangle is 36 cm. If the length of the rectangle is twice its width, determine its dimensions.

7 The sum of two numbers x and y is 52. The difference between the two numbers is 8. Find the values of x and y.

8 The sum of two numbers is 35 and their difference is 19. Find the numbers.

9 Bruce is 4 years older than Michelle. If their combined age is 70, determine their individual ages.

10 A boy is 6 years older than his sister. In three years time he will be twice her age. What are their present ages?

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11 A chocolate thickshake costs \$2 more than a fruit smoothie. Jack pays \$27 for 3 chocolate thickshakes and 4 fruit smoothies. How much do a chocolate thickshake and a fruit smoothie cost each?

12 In 4 years time a mother will be three times as old as her son. Four years ago she was five times as old as her son. Find their present ages.

13 The registration fees for a mathematics competition are \$1.20 for students aged 8–12 years and \$2 for students 13 years and over. One hundred and twenty-five students have already registered and an amount of \$188.40 has been collected in fees. How many students between the ages of 8 and 12 have registered for the competition?

14 A computer company produces 2 laptop models: standard and deluxe. The standard laptop requires 3 hours to manufacture and 2 hours to assemble. The deluxe model requires $5\frac{1}{2}$ hours to manufacture and $1\frac{1}{2}$ hours to assemble. The company allows 250 hours for manufacturing and 80 hours for assembly over a limited period. How many of each model can be made in the time available?

15 A chemical manufacturer wishes to obtain 700 litres of a 24% acid solution by mixing a 40% solution with a 15% solution. How many litres of each solution should be used?

16 In a hockey club there are 5% more boys than there are girls. If there is a total of 246 members in the club, what is the number of boys and the number of girls?

17 The owner of a service station sells unleaded petrol for \$1.42 and diesel fuel for \$1.54. In five days he sold a total of 10 000 litres and made \$14 495. How many litres of each type of petrol did he sell? Give your answer to the nearest litre.

18 James had \$30 000 to invest. He chose to invest part of it at 5% and the other part at 8%. Overall he earned \$2100 in interest. How much did he invest at each rate?

19 The perimeter of a rectangle is 120 metres. The length is one and a half times the width. Calculate the width and length.

20 Three classes, A, B and C, in a school are such that class A has two thirds the number of students that class B has and class C has five sixths the number of students in class B. If the total number of all pupils in the three classes is 105, how many are there in each class?

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3M Problem solving and modelling

Exercise 3M

1 As part of its urban renewal strategy, Camtown council makes $\frac{1}{4}$ hectare of land available for building middle-income homes. The project manager decides to build 10 houses on blocks of varying sizes.

There are five small blocks, three medium-sized blocks and two large blocks. The medium-sized blocks are 100 m^2 larger than the small blocks and the large blocks are 200 m^2 larger than the medium-sized ones.

What are the sizes of the blocks?

Note: 1 hectare = 10 000 m^2

2 A shop sells fruit in two types of gift boxes: standard and deluxe. Each standard box contains 1 kg of peaches and 2 kg of apples and each deluxe box contains 2 kg of peaches and 1.5 kg of apples. On one particular day the shop sold 12 kg of peaches and 14 kg of apples in gift boxes. How many containers of each kind of box were sold on the day?

3 A person's total body water is dependent on their gender, age, height and weight. Total body water (TBW) for males and females can be found by the following formulas:

$$\underset{\text{(litres)}}{\mathit{Male\ TBW}} = 2.447 - 0.09516 \times \underset{\text{(years)}}{\mathit{age}} + 0.1074 \times \underset{\text{(cm)}}{\mathit{height}} + 0.3362 \times \underset{\text{(kg)}}{\mathit{weight}}$$

$$\underset{\text{(litres)}}{\mathit{Female\ TBW}} = -2.097 + 0.1069 \times \underset{\text{(cm)}}{\mathit{height}} + 0.2466 \times \underset{\text{(kg)}}{\mathit{weight}}$$

a What is the TBW for a female of height 175 cm and weight 62 kg? Give your answer correct to two decimal places.

b Calculate the TBW, correct to two decimal places, for a 45-year-old male of height 184 cm and weight 87 kg.

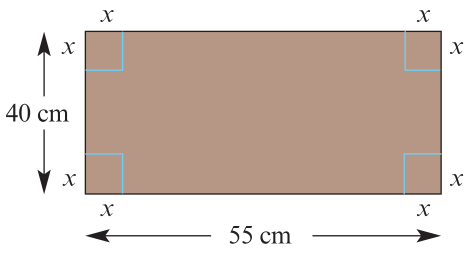
c A healthy 27-year-old female has a TBW of 32 and weighs 62 kg. What is her height, to the nearest cm?

d What would be the TBW, correct to two decimal places, for a 78-year-old man of height 174 cm and weight 80 kg?

e Over a period of a week, the 78-year-old man's weight rapidly increases to 95 kg due to fluid retention. What is his new TBW, correct to two decimal places?

f Construct a table showing the TBW for a 22-year-old male of height 185 cm with weights in increments of 5 kg from 60–120 kg. Give you answers correct to two decimal places.

4 A cardboard storage box is made by cutting equal squares of side length x, from a piece of cardboard measuring 55 cm by 40 cm.



a The volume, V, of the box is given by:

$$V = length \times width \times height$$

Find an expression for the volume in terms of x.

b If the volume of the box is to be 7000 cm^3, find two possible values for the height of the box, correct to two decimal places.

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Key ideas and chapter summary

AS

Formula	A **formula** is a mathematical relationship connecting two or more variables.
Linear equation	A **linear equation** is an equation whose unknown values are always to the power of 1.
Non-linear equation	A **non-linear equation** is one whose unknown values are *not* all to the power of 1.
Literal equation	A **literal equation** is an equation whose solution will be expressed in terms of other unknown values rather than numbers. Example: The solution for x of the equation $x + 3y = 7$ is $x = 7 - 3y$.
Simultaneous equations	Two straight lines will always intersect, unless they are parallel. At the point of intersection the two lines will have the same coordinates. When we find the point of intersection, we are solving the equations simultaneously. **Simultaneous equations** can be solved graphically, algebraically or by using a CAS calculator. Example: $3x + 2y = 6$ $4x - 5y = 12$ are a pair of simultaneous equations.

Skills check

Having completed the current chapter you should be able to:

- substitute values in linear relations and formulas
- construct tables of values from given formulas
- solve linear equations
- use linear equations to solve practical problems
- solve literal equations
- develop formulas from descriptions
- transpose formulas
- solve simultaneous equations graphically, algebraically and with a CAS calculator.

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Multiple-choice questions

1 If $a = 4$, then $3a + 5 =$

A 39
B 12
C 17
D 27

2 If $b = 1$, then $2b - 9 =$

A -11
B -7
C 12
D 21

3 If $C = 50t + 14$ and $t = 8$, then $C =$

A 522
B 1100
C 72
D 414

4 If $P = 2L + 2W$, then the value of P when $L = 6$ and $W = 2$ is:

A 48
B 16
C 12
D 30

5 If $x = -2, y = 3$ and $z = 7$, then $\dfrac{z - x}{y} =$

A 3
B $\dfrac{5}{3}$
C $-\dfrac{5}{3}$
D -3

6 If $a = 2, b = 5, c = 6$ and $d = 10$, then $bd - ac =$

A 38
B 24
C 7
D 484

7 The area of a circle is given by $A = \pi r^2$. If $r = 6$ cm, then the area of the circle is:

A 18.84 cm^2
B 37.70 cm^2
C 35 531 cm^2
D 113.10 cm^2

8 The solution to $4x = 24$ is:

A $x = 2$
B $x = 6$
C $x = 20$
D $x = 96$

9 The solution to $\dfrac{x}{3} = -8$ is:

A $\dfrac{8}{3}$
B 24
C $-\dfrac{8}{3}$
D -24

10 The solution to $2v + 5 = 11$ is:

A $v = 8$
B $v = 3$
C $v = 6$
D $v = 16$

11 The solution to $4x^2 = 64$ is:

A $x = \pm 16$
B $x = \pm 2$
C $x = \pm 4$
D $x = \pm 8$

12 The solution to $3k - 5 = -14$ is:

A $k = 115.67$
B $k = 19$
C $k = 3$
D $k = -3$

13 The cost of hiring a car for a day is \$60 plus 0.25c per kilometre. Michelle travels 750 kilometres. Her total cost is:

A \$810
B \$187.50
C \$247.50
D \$188.10

14 Given $v = u + at$ and $v = 11.6$ when $u = 6.5$ and $a = 3.7$, the value of t correct to two decimal places is:

A 1.378 **B** 9.84 **C** 1.37 **D** 1.38

15 The area of a trapezium is given by $A = \frac{(a+b)h}{2}$. An expression for h is:

A $\frac{2A}{a+b}$ **B** $\frac{A-2}{a+b}$ **C** $\frac{A-(a+b)}{2}$ **D** $2A - (a+b)$

16 The solution to $5(x^2 + 1) = 50$ is:

A $x = \pm 7$ **B** $x = \pm\sqrt{11}$ **C** $x = \pm 3$ **D** $x = \pm\sqrt{\frac{49}{5}}$

17 The solution to the pair of simultaneous equations

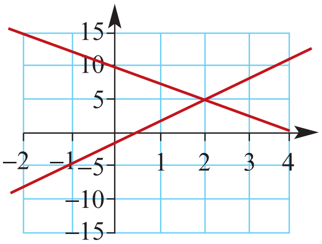
$y = 5x$

$y = 2x + 6$ is:

A $(-2, 0)$ **B** $(-1, -5)$ **C** $(3, 0)$ **D** $(2, 10)$

18 The point of intersection of the lines shown in the diagram is:

A $(5, 2)$ **B** $(0, 0)$ **C** $(0, 9)$ **D** $(2, 5)$



19 The solution to the pair of simultaneous equations

$2x + 3y = -6$

$x + 3y = 0$ is:

A $(-6, -2)$ **B** $(6,2)$ **C** $(-2, 6)$ **D** $(-6, 2)$

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Short-answer questions

1 Solve the following equations for x.

a $x + 5 = 15$ **b** $x - 7 = 4$ **c** $16 + x = 24$

d $9 - x = 3$ **e** $2x + 8 = 10$ **f** $3x - 4 = 17$

g $x + 4 = -2$ **h** $3 - x = -8$ **i** $6x + 8 = 26$

j $3x - 4 = 5$ **k** $\frac{x}{5} = 3$ **l** $\frac{x}{-2} = 12$

2 If $P = 2l + 2b$, find P if:

a $l = 12$ and $b = 8$ **b** $l = 40$ and $b = 25$.

3 If $A = \frac{1}{2}bh$, find A if:

a $b = 6$ and $h = 10$ **b** $b = 12$ and $h = 9$.

4 The formula for finding the circumference of a circle is given by $C = 2\pi r$, where r is the radius. Find the circumference of a circle with radius 15 cm, correct to two decimal places.

5 For the equation $y = 33x - 56$, construct a table of values for values of x in intervals of 5 from -20 to 25.

a For what value of x is $y = 274$?

b When $y = -221$, what value is x?

6 Solve the following non-linear equations.

a $a^2 = 49$ **b** $b^2 = 8836$ **c** $2c^2 = 128$ **d** $d^3 = 27$

e $e^3 = -64$ **f** $3f^3 = 81$ **g** $g^4 = 16$ **h** $h^5 = 3125$

7 I think of a number, double it and add 4. If the result is 6, what is the original number?

8 Four less than three times a number is 11. What is the number?

9 Find the point of intersection of the following pairs of lines.

a $y = x + 2$ and $y = 6 - 3x$ **b** $y = x - 3$ and $2x - y = 7$

c $x + y = 6$ and $2x - y = 9$

10 Solve the following pairs of simultaneous equations.

a $y = 5x - 2$ and $2x + y = 12$ **b** $x + 2y = 8$ and $3x - 2y = 4$

c $2p - q = 12$ and $p + q = 3$ **d** $3p + 5q = 25$ and $2p - q = 8$

e $3p + 2q = 8$ and $p - 2q = 0$

Extended-response questions

1 The cost, C, of hiring a boat is given by $C = 8h + 25$ where h represents hours.

a What is the cost if the boat is hired for 4 hours?

b For how many hours was the boat hired if the cost was \$81?

2 A phone bill is calculated using the formula $C = 25 + 0.50n$, where n is the number of calls made.

a Complete the table of values below for values of n from 60 to 160.

n	60	70	80	90	100	110	120	130	140	150	160
C											

b What is the cost of making 160 phone calls?

3 An electrician charges \$80 up front and \$45 for each hour, h, that he works.

a Write a linear equation for the total charge, C, of any job.

b How much would a 3-hour job cost?

4 Two families went to the theatre. The first family bought tickets for 3 adults and 5 children and paid \$73.50. The second family bought tickets for 2 adults and 3 children and paid \$46.50.

a Write down two simultaneous equations that could be used to solve the problem.

b What was the cost of an adult's ticket?

c What was the cost of a child's ticket?

5 The perimeter of a rectangle is 10 times the width. The length is 9 metres more than the width. Find the width of the rectangle.

6 A secondary school offers three languages: French, Indonesian and Japanese. In year 9, there are 105 students studying one of these languages. The Indonesian class has two-thirds the number of students that the French class has and the Japanese class has five-sixths the number of students of the French class. How many students study each language?

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Chapter 4

4

Financial arithmetic

- How do we determine the new price when discounts or increases are applied?
- How do we determine the percentage discount or increase applied, given the old and new prices?
- How do we determine the old price, given the new price and the percentage discount or increase?
- What do we mean by simple interest, and how is it calculated?
- What do we mean by compound interest, and how is it calculated?
- What is hire purchase, and how is the interest rate determined?

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Introduction

There is no doubt that an understanding of financial arithmetic will be the most useful life skill that you will develop in mathematics. Without this knowledge you could end up spending a lot of money unnecessarily.

4A Percentages and applications

Note: If you need help with percentages, the skills are covered in Chapter 1, page 16.

Discounts and markups

Suppose an item is discounted, or marked *down,* by 10%. The amount of the discount and the new price are:

$$\begin{aligned}\text{discount} &= 10\% \text{ of original price} \\ &= 0.10 \times \text{original price}\end{aligned} \quad \textit{and} \quad \begin{aligned}\text{new price} &= 100\% \text{ of old price} - 10\% \text{ of old price} \\ &= 90\% \text{ of old price} \\ &= 0.90 \times \text{old price}\end{aligned}$$

Applying discounts

In general, if $r\%$ discount is applied:

$$\text{discount} = \frac{r}{100} \times \text{original price} \qquad \begin{aligned}\text{new price} &= \text{original price} - \text{discount} \\ &= \frac{(100 - r)}{100} \times \text{original price}\end{aligned}$$

Example 1 Calculating the discount and the new price

a How much is saved if a 10% discount is offered on an item marked $50.00?

b What is the new discounted price of this item?

Solution

a Evaluate the discount.

$$\text{Discount} = \frac{10}{100} \times \$50 = \$5.00$$

b Evaluate the new price by either:

- subtracting the discount from the original price, or

$$\begin{aligned}\text{New price} &= \text{original price} - \text{discount} \\ &= \$50.00 - \$5.00 = \$45.00\end{aligned}$$

or

- calculating 90% of the original price.

$$\text{New price} = \frac{90}{100} \times \$50 = \$45.00$$

Sometimes, prices are increased or marked *up*.
If a price is increased by 10%:

$$\begin{aligned}\text{increase} &= 10\% \text{ of original price} \\ &= 0.10 \times \text{original price}\end{aligned} \quad \textit{and} \quad \begin{aligned}\text{new price} &= 100\% \text{ of old price} + 10\% \text{ of old price} \\ &= 110\% \text{ of old price} \\ &= 1.10 \times \text{old price}\end{aligned}$$

Applying markups

In general, if $r\%$ increase is applied:

$$\text{increase} = \frac{r}{100} \times \text{original price} \qquad \begin{aligned}\text{new price} &= \text{original price} + \text{increase} \\ &= \frac{(100 + r)}{100} \times \text{original price}\end{aligned}$$

Example 2 Calculating the increase and the new price

a How much is added if a 10% increase is applied to an item marked \$50?

b What is the new increased price of this item?

Solution

a Evaluate the increase.

$$\text{Increase} = \frac{10}{100} \times 50 = \$5.00$$

b Evaluate the new price by either:

- adding the increase to the original price, or
- calculating 110% of the original price.

$$\begin{aligned}\text{New price} &= \text{original price} + \text{increase} \\ &= \$50.00 + 5.00 = \$55.00\end{aligned}$$

or

$$\text{New price} = \frac{110}{100} \times 50 = \$55.00$$

▶ Calculating the percentage change

Given the original and new price of an item, we can work out the **percentage change**.

Calculating percentage discount or increase

$$\text{Percentage discount} = \frac{\text{discount}}{\text{original price}} \times \frac{100}{1}\%$$

$$\text{Percentage increase} = \frac{\text{increase}}{\text{original price}} \times \frac{100}{1}\%$$

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Example 3 Calculating the percentage discount or increase

a The price of an item was reduced from \$50 to \$45. What percentage discount was applied?

b The price of an item was increased from \$50 to \$55. What percentage increase was applied?

Solution

a **1** Determine the amount of the discount.

$$\text{discount} = \text{original price} - \text{new price}$$
$$= 50.00 - 45.00 = \$5.00$$

2 Express this amount as a percentage of the original price.

$$\text{percentage discount} = \frac{5.00}{50.00} \times \frac{100}{1}$$
$$= 10\%$$

b **1** Determine the amount of the increase.

$$\text{increase} = \text{new price} - \text{original price}$$
$$= 55.00 - 50.00 = \$5.00$$

2 Express this amount as a percentage of the original price.

$$\text{percentage increase} = \frac{5.00}{50.00} \times \frac{100}{1}$$
$$= 10\%$$

Calculating the original price

Sometimes we are given the new price and the percentage increase or decrease ($r\%$), and asked to determine the original price. Since we know that:

- for a discount: $\text{new price} = \dfrac{(100 - r)}{100} \times \text{original price}$
- for an increase: $\text{new price} = \dfrac{(100 + r)}{100} \times \text{original price}$

we can rearrange these formulas to give rules for determining the original price as follows.

Calculating the original price

When $r\%$ discount has been applied: $\text{original price} = \text{new price} \times \dfrac{100}{(100 - r)}$

When $r\%$ increase has been applied: $\text{original price} = \text{new price} \times \dfrac{100}{(100 + r)}$

Example 4 Calculating the original price

Suppose that Cate has a \$50 gift voucher from her favourite shop.

a If the store has a '10% off' sale, what is the original value of the goods she can now purchase? Give answer correct to the nearest cent.

b If the store raises its prices by 10%, what is the original value of the goods she can now purchase? Give answer correct to the nearest cent.

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Solution

a Substitute new price = 50 and $r = 10$ into the formula for a $r\%$ discount.

$$\text{Original price} = 50 \times \frac{100}{90} = \$55.555\ldots$$
$$= \$55.56 \text{ to nearest cent}$$

b Substitute new price = 50 and $r = 10$ into the formula for a $r\%$ increase.

$$\text{Original price} = 50 \times \frac{100}{110} = \$45.454\ldots$$
$$= \$45.45 \text{ to nearest cent}$$

Goods and services tax (GST)

The **goods and services tax (GST)** is a tax of 10% that is added to the price of most goods (such as cars) and services (such as insurance). We can consider this a special case of the previous rules, where $r = 10$.

Consider the cost of an item after GST is added – this is the same as finding the new price when there has been a 10% increase in the cost of the item. Thus:

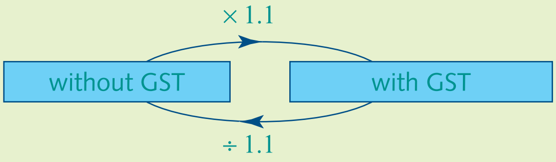
$$\text{cost with GST} = \text{cost without GST} \times \frac{110}{100} = \text{cost without GST} \times 1.1$$

Similarly, finding the cost of an item before GST was added is the same as finding the original cost when a 10% increase has been applied. Thus:

$$\text{Cost without GST} = \text{cost with GST} \times \frac{100}{110} = \frac{\text{cost with GST}}{1.1}$$

Finding the cost with and without GST

- Cost with GST = cost without GST × 1.1

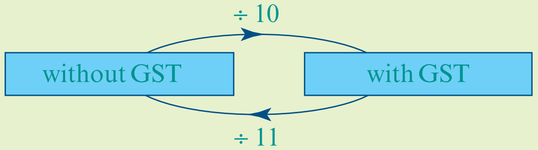


- Cost without GST = $\dfrac{\text{cost with GST}}{1.1}$

We can also directly calculate the actual amount of GST from either the cost without GST or the cost with GST.

Finding the amount of GST

- Amount of GST $= \dfrac{\text{cost without GST}}{10}$



- Amount of GST $= \dfrac{\text{cost with GST}}{11}$

Example 5 Calculating GST

a If the cost of electricity supplied in one quarter is \$288.50, how much GST will be added to the bill?

b If the selling price of a washing machine is \$990:

  i what is the price without GST?

  ii how much of this is GST?

Solution

a Substitute \$288.50 into the rule for GST from cost without GST.

$GST = 288.50 \div 10 = \$28.85$

b **i** Substitute \$990 into the rule for cost with GST.

$$\begin{aligned}\text{Cost without GST} &= 990 \div 1.1\\ &= \$900\end{aligned}$$

ii We can either determine the amount of the GST by subtraction or by direct substitution into the formula.

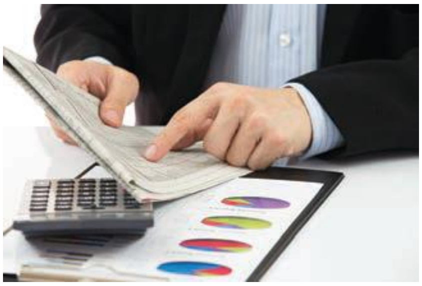
$GST = 990 - 900 = 90$

or

$GST = 990 \div 11 = \$90$

▶ Shares and dividends

Investors often choose to invest their money in shares. A **share** is a unit of ownership in a company. All shares are equal in value, and each share entitles the person who owns it to an equal claim on the company's profits.



For example:

- if there are 100 shares in a company and you own 20, then you own 20% of the shares in a company
- if the company makes a profit of \$100 000 in one year, then you are entitled to 20% of that profit (which would be \$20 000).

Example 6 Calculating profit from shares

There are 500 shares in the Kanz Construction Company. Richard owns 25 shares.

a What percentage of the company does Richard own?

b The company declares an annual profit of \$780 000. How much profit is Richard entitled to?

Solution

a We need to convert 25 out of 500 into a percentage.

Percentage ownership

$$= \frac{25}{500} \times \frac{100}{1}$$

$$= 5\%$$

b Richard is entitled to 5% of the profit.

$$\text{Profit} = 780\,000 \times \frac{5}{100}$$

$$= \$39\,000$$

Investors are not only interested in the amount of profit they are entitled to, they also want to interpret this profit in light of the amount that they have invested in shares. One measure of this is the **price-to-earnings ratio** of the shares.

Price-to-earnings ratio

$$\text{Price-to-earnings ratio} = \frac{\text{current share price}}{\text{profit per share}}$$

The lower the price-to-earnings ratio the less you are investing for each dollar of profit, which is better for the investor.

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Example 7 Price-to-earnings ratio

Suppose shares in Company A have a market value of \$20, and a 12-month profit per share of \$1.85 is declared, while shares in Company B have a market value of \$50, and a profit per share of \$3.30 is declared for the same 12-month period.

a What is the price-to-earnings ratio for each company for that time period? Give answers correct to one decimal place.

b Which shares have been a better investment?

Solution

a Substitute in the formula above.

A: price-to-earnings ratio

$$= \frac{20}{1.85} = 10.8$$

B: price-to-earnings ratio

$$= \frac{50}{3.30}$$

$$= 15.15\ldots = 15.2 \text{ to one d.p.}$$

b Compare the ratios.

Company A: price-to-earnings ratio lower.

In practice, companies do not share all of their profits with shareholders, but they do pay **dividends**. Dividends can be specified in one of two ways:

- as the number of dollars each share receives
- as a percentage of the current price of the shares, called the dividend yield.

Percentage dividend yield

$$\text{Dividend yield} = \frac{\text{dividend}}{\text{current share price}} \times \frac{100}{1}\%$$

Example 8 Dividends

Miller has 3000 shares in Alphabet Childcare Centres. The current market price of the shares is \$3.50 each and the company has recently paid a dividend of 40 cents per share to each shareholder.

a How much does Miller receive in dividends in total?

b What is the percentage dividend yield for this share? Give answer correct to one decimal place.

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Solution

a Total dividend = number of shares × dividend per share

Total dividend $= 3000 \times 0.40$

$= \$1200$

b Use the percentage dividend rule by substituting \$0.40 for the share dividend and \$3.50 for the share price.

$$\text{Dividend yield} = \frac{0.40}{3.50} \times \frac{100}{1}\%$$

$= 11.4\%$ to one d.p.

Exercise 4A

Review of percentages

1 Calculate the following as percentages. Give answers correct to one decimal place.

a \$200 of \$410 **b** \$6 of \$24.60 **c** \$1.50 of \$13.50

d \$24 of \$260 **e** 30c of 90c **f** 50c of \$2

2 Calculate the amount of the following percentage increases and decreases. Give answers to the nearest cent.

a 10% increase on \$26 000 **b** 5% increase on \$4000

c 12.5% increase on \$1600 **d** 15% increase on \$12

e 10% decrease on \$18 650 **f** 2% decrease on \$1 000 000

Discounts, mark-ups and mark-downs

Example 1

3 Calculate the amount of the discount for the following, to the nearest cent.

a 24% discount on \$360 **b** 72% discount on \$250

c 6% discount on \$9.60 **d** 9% discount on \$812

Example 2

4 Calculate the new increased price for each of the following.

a \$260 marked up by 12% **b** \$580 marked up by 8%

c \$42.50 marked up by 60% **d** \$5400 marked up by 17%

5 Calculate the new discounted price for each of the following.

a \$2050 discounted by 9% **b** \$11.60 discounted by 4%

c \$154 discounted by 82% **d** \$10 600 discounted by 3%

e \$980 discounted by 13.5% **f** \$2860 discounted by 8%

6 **a** The price of an item was reduced from \$25 to \$19. What percentage discount was applied?

b The price of an item was increased from \$25 to \$30. What percentage increase was applied?

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Example 4

7 Find the original prices of the items that have been marked down as follows.

a Marked down by 10%, now priced $54.00

b Marked down by 25%, now priced $37.50

c Marked down by 30%, now priced $50.00

d Marked down by 12.5%, now priced $77.00

8 Find the original prices of the items that have been marked up as follows.

a Marked up by 20%, now priced $15.96

b Marked up by 12.5%, now priced $70.00

c Marked up by 5%, now priced $109.73

d Marked up by 2.5%, now priced $5118.75

9 Mikki has a card that entitles her to a 7.5% discount at the store where she works. How much will she pay for boots marked at $230?

10 The price per litre of petrol is $1.80 on Friday. When Rafik goes to fill up his car, he finds that the price has increased by 2.3%. If his car holds 50 L of petrol, how much will he pay to fill the tank?

GST calculations

Example 5

11 Find the GST payable on each of the following (give your answer correct to the nearest cent).

a A gas bill of $121.30

b A telephone bill of $67.55

c A television set costing $985.50

d Gardening services of $395

12 The following prices are without GST. Find the price after GST has been added to the following.

a A dress worth $139

b A bedroom suite worth $2678

c A home video system worth $9850

d Painting services of $1395

13 If a computer is advertised for $2399 including GST, how much would the computer have cost without GST?

14 What is the amount of the GST that has been added if the price of a car is advertised as $39 990 including GST?

15 The telephone bill is $318.97 after GST is added.

a What was the price before GST was added?

b How much GST must be paid?

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Shares and dividends

Example 6 **16** Nick owns 500 of the 100 000 shares available in the Lucky Insurance Company.

- **a** What percentage of the company does Nick own?
- **b** If the company declares an annual profit of $2 500 000, how much profit is Nick entitled to?

Example 7 **17** Suppose shares in Company A have a market value of $42.50, and the company makes a 12-month profit of $4.85 per share, while shares in Company B have a market value of $8, and they make a profit per share of $0.80 for the same 12-month period.

- **a** What is the price-to-earnings ratio for each company for that time period?
- **b** Which shares have been a better investment?

18 Michael has $5000 to invest in shares. He has decided to invest in either the Alpha Oil Company or Omega Mining.

- **a** The price-to-earnings ratio for the Alpha Oil Company is 10. If the share price is $5.00, what is the profit per share?
- **b** The price-to-earnings ratio for Omega Mining is also 10. If the share price is $10.00, what is the profit per share?
- **c** Michael decides to spend his money equally between the two share investments. How many shares in each company does he buy?
- **d** Suppose that, in the next 12 months, the share price of:
  - Alpha Oil is expected to increase by 10% while the price-to-earnings ratio is expected to remain at 10
  - Omega Mining is expected to increase by 8% while the price-to-earnings ratio is expected to reduce to 8.

  What is the expected gain to Michael from these changes?

Example 8 **19** Suppose Taj has 500 shares in Bunyip Plumbing Supplies. If the current market price of the shares is $4.60 each, and the company declares a dividend of 50 cents per share.

- **a** How much does Taj receive in dividends in total?
- **b** What is the percentage dividend for this share?

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4B Simple interest

When you borrow money, you have to pay for the use of that money. When you invest money, someone else will pay you for the use of your money. The amount you pay when you borrow or the amount you are paid when you invest is called **interest**. There are many different ways of calculating interest. The simplest of all is called, rather obviously, **simple interest**. Simple interest is a fixed percentage of the amount invested or borrowed and is *calculated on the original amount*.

Suppose we invest \$1000 in a bank account that pays simple interest at the rate of 5% per annum. This means that, for each year we leave the money in the account, interest of 5% of the original amount will be paid to us.

In this instance, the amount of interest paid to us is 5% of \$1000 or $\$1000 \times \frac{5}{100} = \$50$

If the money is left in the account for several years, the interest will be paid yearly.

To calculate simple interest we need to know:

- the initial investment, called the **principal**
- the **interest rate**, usually as % per annum (p.a.)
- the length of time the money is invested.

Example 9 Calculating simple interest from first principles

How much interest will be earned if investing \$1000 at 5% p.a. simple interest for 3 years?

Solution

1 Calculate the interest for the first year.

$$\text{Interest} = 1000 \times \frac{5}{100} = \$50$$

2 Calculate the interest for the second year.

$$\text{Interest} = 1000 \times \frac{5}{100} = \$50$$

3 Calculate the interest for the third year.

$$\text{Interest} = 1000 \times \frac{5}{100} = \$50$$

4 Calculate the total interest.

$$\begin{aligned}\text{Interest for 3 years} &= 50 + 50 + 50\\ &= \$150\end{aligned}$$

The same rules apply when simple interest is applied to a loan rather than an investment.

The simple interest formula

Since the amount of interest in a simple interest investment is the same each year, we can apply a general rule.

$$\text{interest} = \frac{\text{amount invested or borrowed} \times \text{interest rate (per annum)} \times \text{length of time (in years)}}{100}$$

This rule gives rise to the following formula.

Simple interest formula

To calculate the simple interest earned or owed:

$$I = \frac{P \times r \times t}{100} = \frac{Prt}{100}$$

where I = the total interest earned or paid in dollars

P = is the principal (the initial amount borrowed or invested) in dollars

r = is the percentage interest rate per annum

t = the time in years of the loan or investment.

Example 10 Calculating simple interest for periods other than one year

Calculate the amount of simple interest that will be paid on an investment of \$5000 at 10% simple interest per annum for 3 years and 6 months.

Solution

Apply the formula with P = \$5000, r = 10% and t = 3.5 (since 3 years and 6 months is equal to 3.5 years).

$$I = \frac{Prt}{100}$$

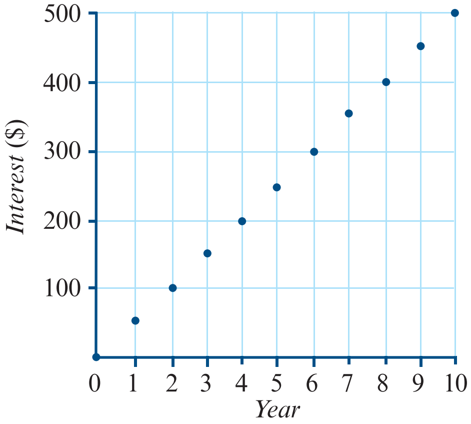
$$= 5000 \times \frac{10}{100} \times 3.5$$

$$= \$1750$$

The graph below shows the total amount of interest earned after 1, 2, 3, 4, . . . years, when \$1000 is invested at 5% per annum simple interest for a period of years.

As we would expect from the simple interest rule, the graph is linear.

The slope of a line which could be drawn through these points is equal to the amount of interest added each year, in this case \$50.



A CAS calculator enables us to investigate the growth in simple interest with time using both the tables and graphing facilities of the calculator.

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Using the TI-Nspire CAS to explore a simple interest investment

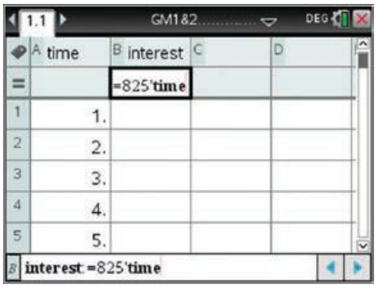
$10 000 is invested at a simple interest rate of 8.25% per annum for a ten-year period. Plot the growth in interest earned over this period.

Steps

1 Find a rule for the interest earned after t years for a simple interest investment when $P = \$10\,000$ and $r = 8.25\%$.

$$I = \frac{Prt}{100} = \frac{10\,000 \times 8.25 \times t}{100}$$
$$= 825t$$

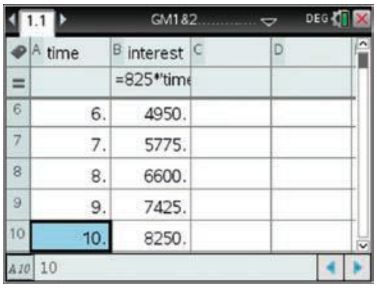
2 Start a new document (ctrl+N) and select **Add Lists & Spreadsheet**.
Name the lists **time** (to represent time in years) and **interest**.
Enter the data 1–10 into the list named **time** as shown.
Note: you can also use the sequence command to do this.



	A time	B interest	C	D
=		=825·time		
1	1.			
2	2.			
3	3.			
4	4.			
5	5.			

B interest:=825·time

3 Place the cursor in the grey formula cell in the list named **interest** and type **=825 ×time**.
Note: you can also use the h key and paste time from the variable list.
Press enter to display the values.
By scrolling down the table (use ▼) we can see interest of $8250 will be earned after 10 years.



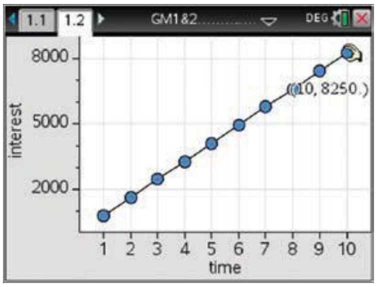
	A time	B interest	C	D
=		=825*time		
6	6.	4950.		
7	7.	5775.		
8	8.	6600.		
9	9.	7425.		
10	10.	8250.		

A10 10

4 Press ctrl+I and select **Data & Statistics** and plot the graph as shown.

a To connect the data points. Move the cursor to the graphing area and press ctrl+menu. Select **Connect Data Points**.

b To display a value:
Move the cursor over the data points or use menu>**Analyze>Graph Trace** and the horizontal arrow keys to move from point to point.



From the plot we can see that the graph of the amount of simple interest earned is linear. The slope of the graph is equal to the interest added each year.

Note: you can also graph this example in the **Graphs** application.

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Using the ClassPad to explore the growth of the interest earned in a simple interest investment

$10 000 is invested at a simple interest rate of 8.25% per annum for a ten year period. Plot the growth in interest earned over this period.

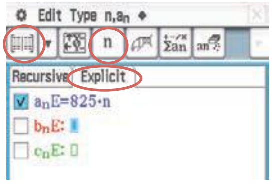
Steps

1 Find a rule for the interest earned after t years for a simple interest investment when $P = \$10\,000$ and $r = 8.25\%$.

$$I = \frac{Prt}{100} = \frac{10\,000 \times 8.25 \times t}{100}$$
$$= 825t$$

2 Enter the rule in the sequence.

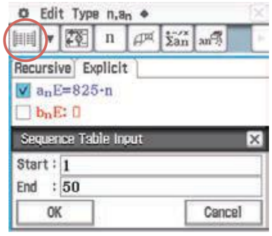
- Open the **Sequence** application.
- Select the **Explicit** tab.
- Move the cursor to the box opposite $\boldsymbol{a_n}$**E:**
- Type ***825n*** using the n in the toolbar for n years.
- Press EXE to confirm your entry indicated by a tick in the square to the left of $\boldsymbol{a_n}$**E:**



☑ aₙE=825·n
☐ bₙE:
☐ cₙE:

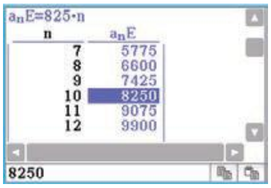
3 To display the terms of the sequence in a table.

- Tap the icon.
- Tap the **Sequence TableInput** icon in the toolbar.
- Adjust the **Start** and **End** values if required.
- Scroll down the table to find the interest amount $8250 earned after 10 years.



☑ aₙE=825·n
☐ bₙE:

Start : 1
End : 50



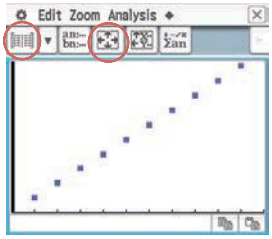
aₙE=825·n

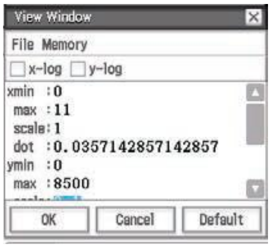
n	aₙE
7	5775
8	6600
9	7425
10	8250
11	9075
12	9900

8250

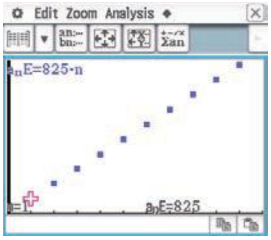
4 To graph the sequence of simple interest values.

- Select the **Sequence Grapher** icon from the toolbar.
- Select the **View Window** icon from the toolbar.
- Set the values as shown and Tap **OK** to confirm your settings (leave dot settings as they are).
- Select **Analysis** and then **Trace** to place a cursor on the values.
- Use the cursor key to view other points.





xmin : 0
max : 11
scale : 1
dot : 0.0357142857142857
ymin : 0
max : 8500



aₙE=825·n
aₙE=825

From the plot we can see that the graph of the amount of simple interest earned is linear. The slope of the graph is equal to the interest paid each year.

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Calculating the amount of a simple interest loan or investment

To determine the total value or amount of a **simple interest** loan or investment the total interest used is added to the initial amount borrowed or invested (the principal).

Total value of a simple interest loan

Total amount after t years (A) = principal (P) + interest (I)

$$\text{or } A = P + I$$

Example 11 Calculating the total amount owed on a simple interest loan

Find the total amount owed on a simple interest loan of $16 000 at 8% per annum after 2 years.

Solution

1 Apply the formula $\frac{Prt}{100}$ with $P = \$16\,000$, $r = 8\%$ and $t = 2$ to find the total interest accrued.

$$I = \frac{Prt}{100}$$

$$= 16\,000 \times \frac{8}{100} \times 2 = \$2560$$

2 Find the total amount owed by adding the interest to the principal.

$$A = P + I$$

$$= 16\,000 + 2560 = \$18\,560$$

Interest paid to bank accounts

One very useful application of simple interest is in the calculation of the interest earned on a bank account. When we keep money in the bank, interest is paid. The amount of interest paid depends on:

- the rate of interest the bank is paying
- the amount on which the interest is calculated.

Generally, banks will pay interest on the **minimum monthly balance**, which is the lowest amount the account contains in each calendar month. When this principle is used, we will assume that all months are of equal length, as illustrated in the next example.

Example 12 Calculating interest paid to a bank account

The table shows the entries in Tom's bank account.

Date	Transaction	Debit	Credit	Total
30 June	Pay		400.00	400.00
3 July	Cash	50.00		350.00
15 July	Cash		100.00	450.00
1 August				450.00

If the bank pays interest at a rate of 3% per annum on the minimum monthly balance, find the interest payable for the month of July correct to the nearest cent.

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Solution

1 Determine the minimum monthly balance for July.

The minimum balance in the account for July was \$350.00.

2 Determine the interest payable on \$350.00

$$I = \frac{Prt}{100}$$

$$= 350 \times \frac{3}{100} \times \frac{1}{12} = 0.875$$

$$= \$0.88 \text{ or } 88 \text{ cents}$$

Exercise 4B

Calculating simple interest

Example 10

1 Calculate the amount of interest earned from each of the following simple interest investments. Give answers correct to the nearest cent.

	Principal	Interest rate	Time
a	\$400	5%	4 years
b	\$750	8%	5 years
c	\$1000	7.5%	8 years
d	\$1250	10.25%	3 years
e	\$2400	12.75%	15 years
f	\$865	15%	2.5 years
g	\$599	10%	6 months
h	\$85.50	22.5%	9 months
i	\$15 000	33.3%	1.25 years

Exploring the growth of interest in a simple interest loan or investment

2 **a** A loan of \$900 is taken out at a simple interest rate of 16.5% per annum.
Use your CAS calculator to construct a graph of the simple interest owed against time (in years) for the next 10 years.

b Use the table of values to determine the amount of interest owed after 5 years.

3 **a** Ben decides to invest his savings of \$1850 from his holiday job for five years at 13.25% per annum simple interest.
Use your graphics calculator to construct a graph of the simple interest earned against time (in years) for the next 10 years.

b Use the table of values to determine the amount of interest owed after 4 years.

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Simple interest loans and investments

Example 11

4 Calculate the total amount to be repaid for each following simple interest loans. Give answers correct to the nearest cent.

	Principal	Interest rate	Time
a	\$500	5%	4 years
b	\$780	6.5%	3 years
c	\$1200	7.25%	6 months
d	\$2250	10.75%	8 months
e	\$2400	12%	18 months

5 A simple interest loan of \$20 000 is taken out for 5 years. Calculate:

a the simple interest owed after 5 years if the rate of interest is 12% per annum

b the total amount to be repaid after 5 years.

6 A sum of \$10 000 was invested in a fixed term account for 3 years paying a simple interest rate of 6.5% per annum.
Calculate:

a the total amount of interest earned after 3 years

b the total amount of the investment at the end of 3 years.

7 A loan of \$1200 is taken out at a simple interest rate of 14.5% per annum.
How much is owed, in total, after 3 months?

8 A company invests \$1 000 000 in the short-term money market at 11% per annum simple interest. How much interest is earned by this investment in 30 days? Give your answer to the nearest cent.

9 A building society offers the following interest rates for its cash management accounts.

	Interest rate (per annum) on term (months)				
Balance	1–<3	3–<6	6–<12	12–<24	24–<36
\$20 000–\$49 999	2.85%	3.35%	3.85%	4.35%	4.85%
\$50 000–\$99 999	3.00%	3.50%	4.00%	4.50%	5.00%
\$100 000–\$199 999	3.40%	3.90%	4.40%	4.90%	5.40%
\$200 000 and over	4.00%	4.50%	5.00%	5.50%	6.00%

Using this table, find the simple interest earned by each of the following investments. Give your answers to the nearest cent.

a \$25 000 for 2 months

b \$125 000 for 6 months

c \$37 750 for 18 months

d \$200 000 for 2 years

e \$74 386 for 8 months

f \$145 000 for 23 months

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Interest paid into bank accounts

Example 12 **10** An account at a bank is paid interest of 4% per annum on the minimum monthly balance, credited to the account at the beginning of the next month.

Date	Transaction	Debit	Credit	Balance
1 October				5000.00
7 October	Cash	1000.00		4000.00
31 October	Cash		500.00	

a What was the balance of the account at the end of October?

b How much interest was paid for the month?

11 The minimum monthly balances for three consecutive months are:

$240.00 $350.50 $478.95

How much interest is earned over the three-month period if it is calculated on the minimum monthly balance at a rate of 3.5% per annum?

12 The bank statement below shows transactions for a savings account that earns simple interest at a rate of 4.5% per annum on the minimum monthly balance.

Date	Transaction	Debit	Credit	Balance
1 March				500.00
15 March	Cash		250.00	750.00
31 March	Cash		250.00	1000.00
1 April				1000.00

How much interest was earned in March?

13 The bank statement below shows transactions over a three-month period for a savings account that earns simple interest at a rate of 3.75% per annum on the minimum monthly balance.

Date	Transaction	Debit	Credit	Balance
1 March				650.72
8 April	Cash		250.00	900.72
21 May	Cash		250.00	1150.72
1 June				1150.72

a What were the minimum monthly balances in March, April and May?

b How much was earned over this three-month period?

4C Rearranging the simple interest formula

The simple interest formula can be rearranged to find any one of the variables as long as the other three are known.

▶ Calculating the interest rate

Interest rate

To find the annual interest rate, $r\%$, given the values of P, I and t:

$$r = \frac{100I}{Pt}$$

where P is the principal, I is the amount of interest accrued in t years.

Example 13 Calculating the interest rate

Find the rate of simple interest if:

a a principal of \$8000 increases to \$11 040 in 4 years

b a principal of \$5000 increases to \$5500 in 9 months.

Solution

a **1** Find the amount of interest earned on the investment.

Interest:

$$I = 11\,040 - 8000$$
$$= \$3040$$

2 Apply the formula $r = \frac{100I}{Pt}$ with $P = \$8000$, $I = \$3040$ and $t = 4$.

$$r = \frac{100I}{Pt} = \frac{100 \times 3040}{8000 \times 4}$$
$$= 9.5\%$$

Interest rate is 9.5% per annum

b **1** Find the amount of interest earned on the investment.

Interest:

$$I = 5500 - 5000$$
$$= \$500$$

2 Apply the same formula with $P = \$5500$, $I = \$500$ and $t = \frac{9}{12} = 0.75$ years

$$r = \frac{100I}{Pt} = \frac{100 \times 500}{5000 \times 0.75}$$

$= 13.3\%$ to one decimal place

Interest rate is 13.3% per annum

Note: You need to convert the time in months to years by substituting in the formula.

▶ Calculating the time period

Time period

To find the number of years or term of an investment, t years, given P, I and r:

$$t = \frac{100I}{Pr}$$

where P is the principal, I is the amount of interest and $r\%$ is the annual interest rate.

Example 14 Calculating the time period of a loan or investment

Find the length of time it would take for \$5000 invested at an interest rate of 12% per annum to:

a earn \$1800 interest

b earn \$404 interest.

Give answer in days to the nearest day.

Solution

a Apply the formula $t = \frac{100I}{Pr}$ with $P = \$5000$, $I = \$1800$ and $r = 12$.

$$t = \frac{100I}{Pr} = \frac{100 \times 1800}{5000 \times 12}$$
$$= 3 \text{ years}$$

b Apply the same formula with $P = \$5000$, $I = \$404$ and $r = 12$ assuming that there are 365 days in a year.

$$t = \frac{100I}{Pr} = \frac{100 \times 404}{5000 \times 12}$$
$$= 0.673\ldots \text{ years}$$
$$= 365 \times 0.673$$
$$= 245.766\ldots \text{ days}$$
$$= 246 \text{ days (to the nearest day)}$$

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Calculating the principal

Calculating the principal

- To find the value of the principal, P, given the values of I, r and t use the formula:

$$P = \frac{100I}{rt}$$

where I is the amount of interest accrued, r% is the annual interest rate and t is the time in years.

- To find the value of the principal, P, given the values of A, r and t:

$$P = \frac{A}{\left(1 + \frac{rt}{100}\right)}$$

where A is the amount of the investment or loan, r% is the annual interest rate and t is the time in years.

Example 15 Calculating the principal of a loan or investment

a Find the amount that should be invested in order to earn \$1500 interest over 3 years at an annual interest rate of 5%.

b Find the amount that should be invested at an annual interest rate of 5% if you require the value of the investment to be \$15 600 in 4 years time.

Solution

a Since we are given the value of the interest, I, use the formula $P = \frac{100I}{rt}$ with $I = \$1500$, $r = 5$ and $t = 3$ years.

$$P = \frac{100I}{rt} = \frac{100 \times 1500}{5 \times 3}$$
$$= \$10\,000$$

b Here we are *not* given the value of the interest, I, but the value of the total investment, A.

Use the formula $P = \frac{A}{\left(1 + \frac{rt}{100}\right)}$

with $A = \$15\,600$, $r = 5$ and $t = 4$.

$$P = \frac{A}{\left(1 + \frac{rt}{100}\right)}$$
$$= \frac{15\,600}{\left(1 + \frac{5 \times 4}{100}\right)}$$
$$= \frac{15\,600}{1.2}$$
$$= \$13\,000$$

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Exercise 4C

Skillsheet

Simple interest: calculating interest rate

Example 13

1 Find the annual interest rate if a simple interest investment of $5000 amounts to $6500 in 2.5 years.

2 Find the annual interest rate if a simple interest investment of $500 amounts to $550 in 8 months.

Simple interest: calculating time

Example 14

3 Calculate the time taken for $2000 to earn $975 at 7.5% simple interest.

4 Calculate the time in days for $760 to earn $35 at 4.75% simple interest.

Simple interest: calculating principle

Example 15

5 Calculate the principal that earns $514.25 in 10 years at 4.25% simple interest.

6 Calculate the principal that earns $780 in 100 days at 6.25% per annum simple interest.

Simple interest: mixed problems

7 Calculate the answers to complete the following table.

Principal	Rate	Time	Simple interest	Total investment
$600	6%	5 years	**a**	**b**
$880	6.5%	**c**	$171.60	**d**
$1290	**e**	6 months	$45.15	**f**
g	10%	4 months	$150.00	**h**
$3600	**i**	200 days	$98.63	**j**
$980	7.5%	**k**	**l**	$1200.50
m	7.25%	6 months	$52.50	**n**

Applications

8 If Geoff invests $30 000 at 10% per annum simple interest until he has $42 000, for how many years will he need to invest the money?

9 Josh decides to put $5000 into an investment account that pays 5.0% per annum simple interest. If he leaves the money there until it doubles, how long will this take, to the nearest month?

10 A personal loan of $15 000 over a 3-year period costs $500 per month to repay.

a How much money will be repaid in total?

b How much of the money repaid is interest?

4D Compound interest

We have seen that simple interest is calculated *only* on the original amount borrowed or invested. A more common form of interest, known as **compound interest**, calculates the interest on the original amount plus any interest accrued to that time.

Calculating compound interest

Consider, for example, \$250 invested at 10% per annum, where the interest is added to the account each year.

After 1 year:

$$\begin{aligned}\text{interest} &= \text{amount invested} \times \text{interest rate} \times \text{time}\\ &= \$250 \times 10\% \times 1\\ &= 250 \times \frac{10}{100} \times 1\\ &= \$25\end{aligned}$$

so that after one year, the amount of money in the account is:

$$\begin{aligned}\text{amount} &= \text{amount at the start of year} + \text{interest earned}\\ &= \$250 + \$25\\ &= \$275\end{aligned}$$

After 2 years:

$$\text{interest} = \$275 \times 10\% \times 1 = \$27.50$$

so that after two years, the amount of money in the account is:

$$\$275 + \$27.50 = \$302.50$$

After 3 years:

$$\text{interest} = \$302.50 \times 10\% \times 1 = \$30.25$$

so that after three years, the amount of money in the account is:

$$\$302.50 + \$30.25 = \$332.75$$

And so on.

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If we tabulate this information, we will see that using compound interest, the amount of interest owed or paid increases each year.

After	Amount invested	Interest earned	Total amount of investment
1 year	\$250	\$25	\$(250 + 25) = \$275
2 years	\$275	\$27.50	\$(275 + 27.50) = \$302.50
3 years	\$302.50	\$30.25	\$(302.50 + 30.25) = \$332.75
and so on			

Calculating compound interest in this way can be very tedious. However, there is a pattern to the calculations that enables us to develop a formula.

Start by recalling that the multiplying factor to increase a quantity by 10% is $\left(1 + \frac{10}{100}\right) = 1.1$

Using this factor we have the value of the investment, A, is:

$$
\begin{aligned}
A &= \$250 \times 1.1 = \$275 && \text{(after 1 year)}\\
A &= \$250 \times 1.1 \times 1.1 &&\\
&= \$250 \times (1.1)^2 = \$302.50 && \text{(after 2 years)}\\
A &= \$250 \times 1.1 \times 1.1 \times 1.1 &&\\
&= \$250 \times (1.1)^3 = \$332.75 && \text{(after 3 years)}
\end{aligned}
$$

and so on until, the value of the investment:

$$A = \$250 \times (1.1)^n \qquad \text{(after } n \text{ years)}$$

Thus the value of the investment after 10 years would be: $A = \$250 \times 1.1^{10} = \$648.44$

Following this pattern, we can write down a general formula for calculating the amount of a compound investment after a given amount of time.

The compound interest formula

In general, the amount, A, of a compound interest investment is given by:

$$A = P\left(1 + \frac{r}{100}\right)^t$$

where P is the initial amount invested (the principal), r% the annual interest rate and t is time in years.

Note: This formula can also be used to determine the amount of debt accrued by a compound interest loan.

To find the *total amount of interest* earned subtract the initial investment from the final amount.

Determining the interest earned

Interest earned (I) = value of the investment (A) − initial amount invested (P)

$$I = A - P$$

Example 16 Calculating the amount of the investment and interest

a Determine, to the nearest dollar, the amount of money accumulated after 3 years if \$2000 is invested at an interest rate of 8% per annum, compounded annually.

b Determine the total amount of interest earned.

Solution

a Substitute $P = \$2000, t = 3, r = 8$ into the formula giving the amount of the investment.

$$A = P \times \left(1 + \frac{r}{100}\right)^t = 2000 \times \left(1 + \frac{8}{100}\right)^3$$
$$= \$2519 \text{ to the nearest dollar}$$

b Subtract the principal from this amount to determine the interest earned.

$$I = A - P = 2519 - 2000$$
$$= \$519$$

The formulas for compound interest can also be applied when money is borrowed, as shown in the following example.

Example 17 Calculating the amount of the debt and interest owed

a Determine, to the nearest dollar, the amount of money owed after 2 years if \$10 000 is borrowed at an interest rate of 10% per annum, compounded annually.

b Determine the amount of interest owed.

Solution

a Substitute $P = \$10\,000, t = 2, r = 10$ into the formula giving the amount of the debt.

$$A = P \times \left(1 + \frac{r}{100}\right)^t = 10\,000 \times \left(1 + \frac{10}{100}\right)^2$$
$$= \$12\,100$$

b Subtract the principal from this amount to determine the interest owed.

$$I = A - P$$
$$= 12\,100 - 10\,000$$
$$= \$2100$$

Another way to determine compound interest is to enter the appropriate formula into a CAS calculator, and examine the interest earned using the calculator's tables and graphing facilities.

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Using the TI-Nspire CAS to investigate compound interest problems

a Set up a table to enable the amount of money accumulated after t years if \$2000 is invested at an interest rate of 8% per annum compounding.

b Use the table to determine the value of the loan after three years and the amount of interest earned.

c Plot the growth in the amount of money in the investment for ten years and note the shape of the graph.

Steps

1 Substitute P = \$2000 and r = 8 into the formula for compound interest.

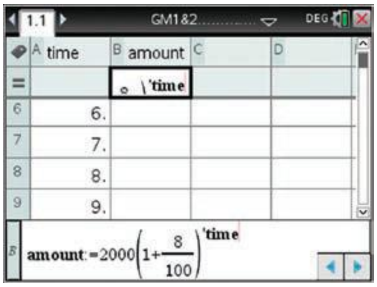
$$A = 2000 \times \left(1 + \frac{8}{100}\right)^{t}$$

2 Start a new document (ctrl + N) and select **Add Lists & Spreadsheet**.

Name the lists **time** (to represent time in years) and **amount**.

Enter the data 1–10 into the list named **time** as shown.

Note: you can also use the sequence command to do this.



	A time	B amount	C	D
=		∘ \'time		
6	6.			
7	7.			
8	8.			
9	9.			

B amount:=2000$\left(1+\frac{8}{100}\right)^{\text{'time}}$

3 Place the cursor in the grey formula cell in the list named **amount** and type in:

= **2000** ×(**1** + **8** ÷ **100**)^ **time**

Note: you can also use the var key and paste **time** from the variable list.

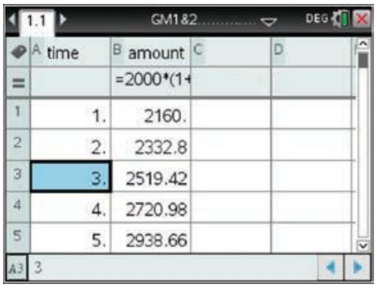
Press enter to display the values as shown.

By scrolling down the table we can see that:

a the amount of money accumulated after 3 years is \$2519.42

b interest earned = \$2519.42 – \$2000

= \$519.42



	A time	B amount	C	D
=		=2000*(1+		
1	1.	2160.		
2	2.	2332.8		
3	3.	2519.42		
4	4.	2720.98		
5	5.	2938.66		

A3 3

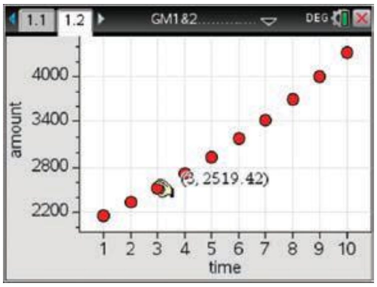
4 Press ctrl + I and select **Add Data & Statistics** and plot the graph as shown.

Note:

a To connect the data points: Move the cursor to the graphing area and press ctrl + menu. Select **Connect Data Points**.

b To display a value: Move the cursor over the data points or use b **>Analyze>Graph Trace**.

c You can use ctrl + menu and select **Zoom>Window Settings** and set the Ymin to 0 if you prefer.



From the plot we see that, for compound interest, the graph of the amount of money accumulated curves upwards with time.

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Using the ClassPad to investigate compound interest problems

a Set up a table to enable the amount of money accumulated after t years if \$2000 is invested at an interest rate of 8% per annum compounding.

b Use the table to determine the value of the loan after three years and interest earned.

c Plot the growth of the investment for ten years and note the shape of the graph.

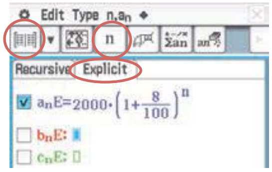
Steps

1 Substitute P = \$2000 and r = 8 into the formula for compound interest.

$$A = 2000 \times \left(1 + \frac{8}{100}\right)^t$$

2 To form a table of values.

- Open the **Sequence** application.
- Select the **Explicit** tab.
- Move the cursor to the box opposite $\mathbf{a_n E}$::
- Type **2000** × (**1** + **8**/**100**)^n. Use the **n** is found in the toolbar $\boxed{\text{n}}$.
- Press $\boxed{\text{EXE}}$ to confirm your entry, indicated by a tick in the square to the left of $\mathbf{a_n E}$:



☑ $a_nE=2000\cdot\left(1+\frac{8}{100}\right)^n$

☐ b_nE:

☐ c_nE:

3 To view a table of values.

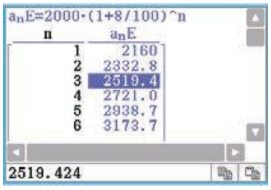
- Tap from the toolbar.
- Scroll down the table to see that the:

a amount of money accumulated after 3 years is \$2519.42

b interest earned

= \$2519.42 – \$2000

= \$519.42



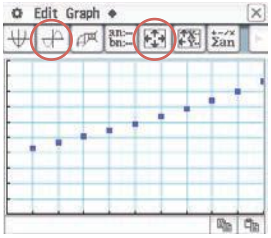
anE=2000·(1+8/100)^n

n	anE
1	2160
2	2332.8
3	2519.4
4	2721.0
5	2938.7
6	3173.7

2519.424

4 To graph the sequence of compound interest values.

- Select the **Sequence Grapher** icon.
- Select the **View Window** icon.
- Set the values as shown. Use a y scale of 500.
- Tap **OK** to confirm your settings.



From the plot we see that, for compound interest, the graph of amount of money accumulated curves upwards with time.

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▶ Comparing simple and compound interest

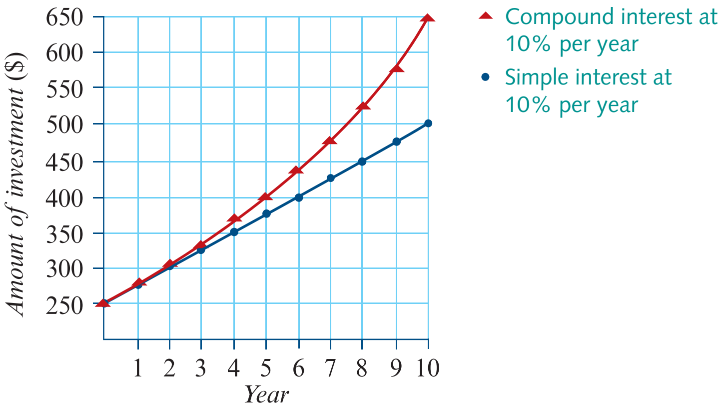
Earlier we saw that the growth in the value of simple interest investments and loans was linear. By contrast, as we have just seen, the growth in compound investments or loans was non-linear following a curve that became increasingly steep.

The difference in the growth pattern becomes clear if we compare a **simple interest** investment with a **compound interest** investment with the same principle ($250) and rate of interest (12% per annum). The results are displayed in the table below.

	Amount of investment ($)	
Year (*n*)	10% simple interest	10% compound interest
0	250	250.00
1	275	275.00
2	300	302.50
3	325	332.75
4	350	366.03
5	375	402.63
6	400	442.89
7	425	487.18
8	450	535.90
9	475	589.49
10	500	648.44

From the table, we can see that after the first month the value of the compound interest investment is higher and this advantage increases over time.

The difference between the two investment strategies is even clearer when graphed.



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Exercise 4D

Skillsheet Note: In the following exercises, give all answers correct to the nearest cent.

Compound interest investments

Example 16

1 An amount of \$3500 is invested at 5% compound interest per annum for 5 years. Determine:

a the final value of this investment **b** the total amount of interest earned.

2 An amount of \$7000 is invested at 8% compound interest per annum for 4 years. Determine:

a the final value of this investment **b** the total amount of interest earned.

3 Calculate the difference between the simple interest and the compound interest on an investment of \$3000 at 7.9% per annum over 5 years.

Compound interest loans

Example 17

4 A person borrows \$1250 at 7.5% compound interest per annum for 3 years. Determine:

a the total amount of money owed after 3 years

b the amount of interest owed.

5 A person borrows \$1000 at 6.0% compound interest per annum for 5 years. Determine:

a the total amount of money owed after 5 years

b the amount of interest owed.

6 Calculate the difference between the simple interest and the compound interest on a loan of \$2000 at 7% per annum over 5 years.

Exploring compound interest loans and investments with a CAS calculator

7 \$850 is borrowed at 13.25% per annum compound interest for 8 years.

a Construct a table to display the total amount owed after t years for up to 8 years.

b How much is owed in total after 5 years and how much of that is interest?

c Plot the growth in the amount of money in the investment for 8 years and note the shape of the graph.

8 Peter invests \$3000 at 5.65% per annum compound interest for 10 years.

a Construct a table to display the total amount owed after t years for up to 5 years.

b How much is owed in total after 4 years and how much of that is interest?

c Plot the growth in the amount of money in the investment for 10 years and note the shape of the graph.

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4E Time payment agreements

When you go shopping there are generally two options for payment. The first option is to pay at the time using *cash* or a *debit card*, a card that directly debits money from your bank account. Of course, this option is only possible if you have enough money with you, or in the bank, at the time of purchase.

The second option is to purchase on **credit**. In a financial context, credit means taking delivery of a good or service without paying for it at the time but with a commitment to pay for it later. These are sometimes called *time payment agreements*. Common forms of time payment agreements include **hire-purchase** agreements, **personal loans** and *credit cards*.

The cost of credit includes interest and possibly other fees and charges. It is really important to calculate what this total cost is before purchasing on credit so we can understand the real cost of the purchase.

▶ Hire purchase

In a hire-purchase payment, the buyer will take the item they have purchased and then pay it off by making regular payments of an agreed amount. When all the payments have been made, the buyer owns the item. In some agreements, a deposit is paid at the beginning of the agreement and this reduces the amount of the payments. We are interested in calculating the interest rate being charged in these contracts because it is not always stated explicitly.

For a hire-purchase agreement we do can this in two ways. Firstly, by calculating the flat rate of interest charged and, secondly, by calculating the effective rate of interest.

▶ Flat rate of interest

If we calculate the total interest paid as a proportion of the original debt and express this as an annual rate this is called the **flat rate of interest**.

The flat rate of interest is exactly the same as the simple rate of interest but is often called by this name in time payment agreements.

Flat rate of interest r_f

The flat rate of interest is calculated as a percentage of the *original amount* owed.

The annual flat rate of interest rate, r_f, is given by:

$$r_f = \frac{100I}{Pt}$$

where I = total interest paid

P = principal owing after the deposit has been deducted

t = the time in years

The following example illustrates the calculation of the flat rate interest for a typical hire-purchase agreement.

Example 18 Calculating the flat rate of interest for a hire-purchase agreement

Josh buys a sound system costing \$1400. He pays a deposit of \$500. The remaining \$900 he owes must be repaid by making six monthly payments of \$160.

Solution

a To calculate the flat rate of interest we must first work out how much interest Josh will pay.

Interest = total paid − purchase price

$$\begin{aligned}\text{Total paid} &= \text{deposit} + \text{repayments}\\ &= 500 + 6 \times 160\\ &= \$1460\\ \text{Interest paid} &= 1460 - 1400\\ &= \$60\end{aligned}$$

b To calculate the flat rate of interest charged, apply the formula $r_f = \dfrac{100I}{Pt}$.

- P = \$900 (since only \$900 is owed after the deposit is paid)
- I = \$60 (calculated in **a**)
- t = 0.5 (since the \$900 owed is repaid in 6 months).

$$r_f = \frac{100I}{Pt} = \frac{100 \times 60}{900 \times 0.5}$$

$= 13.3\%$ to one decimal place

Thus, the flat rate of interest for this hire-purchase agreement is 13.3% p.a.

In the example above, we see that under this hire-purchase agreement, Josh is paying a flat rate of interest of 13.3%. However, in reality, Josh is actually paying a much higher effective rate of interest. This is because he is making regular repayments throughout the course of the agreement. As a result, for a lot of the time he actually owes quite a lot less than the original \$900 he borrowed.

▶ Effective interest rate

One way of determining the **effective interest rate** is to work out the *average amount* owed over the period of the loan. This value can then be used as the **principal (*P*)** in the interest calculation.

Returning to Josh's situation, if no interest needed to be paid, Josh would only pay \$150 each month to pay off the \$900 he owes in six months $\left(\dfrac{900}{6} = \$150\right)$.

Since Josh pays \$160 each month, this amount can be considered as paying \$150 off the principal. The additional \$10 is the amount he is charged in interest.

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So, how much does Josh actually owe at different times through the hire-purchase agreement? To see this we have constructed the following table.

Month	Principal owed (\$)	Principal paid (\$)	Interest paid (\$)	Total payment (\$)
1	900	150	10	160
2	750	150	10	160
3	600	150	10	160
4	450	150	10	160
5	300	150	10	160
6	150	150	10	160
Total	3150	900	60	960

From the table, we can see the amount that Josh owes (the principal of his loan) decreases by $150 each month.

To find the *average* principal owed by Josh over the period of the hire-purchase agreement, we proceed as follows:

$$\text{average principal owed} = \frac{\text{total amount of principal owed}}{\text{number of months}} = \frac{3150}{6} = \$525$$

We can then use this amount to calculate the effective interest rate:

$$\begin{aligned}\text{effective interest rate} &= \frac{100I}{\text{average amount owed} \times t}\\ &= \frac{100 \times 60}{525 \times 0.5}\\ &= 22.9\% \text{ to 1 d.p.}\end{aligned}$$

Not surprisingly, the *effective* interest rate (22.9%) is very much higher than the *flat rate* of interest (13.3%) that we calculated earlier.

Fortunately, we do not have to make a table to calculate the average amount of interest owed for each new hire-purchase problem because there is rule that will do this for us.

The rule is:

$$\text{average amount owed} = \left(\frac{n+1}{2n}\right)P$$

where n is the number of payments and P is the **principal** (the total amount owed).

Using this rule we arrive at the following for calculating the effective annual interest rate.

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Effective annual interest rate (r_e)

The effective rate of interest is calculated as a percentage of the average amount owed.

The annual effective rate of interest rate, r_e, is given by:

$$r_e = \frac{100I}{Pt} \times \frac{2n}{n+1}$$

or, noting that $\frac{100I}{Pt} = r_f$

$$r_e = r_f \times \frac{2n}{n+1}$$

where r_f is the annual flat rate of interest and n is the number of payments made.

The second version of the formula is probably the most commonly used, because the flat rate of interest is generally calculated first.

We can further simplify the above rule for calculating the effective interest rate by noting that, for large values of n, the expression for $\frac{n}{n+1}$ is approximately equal to one.

This gives a very quick way of estimating the effective interest rate per annum from the flat interest rate per annum as follows.

A quick way for estimating the effective interest rate

For a large number of payments (n):

$$r_e \approx r_f \times 2$$

That is, the *effective rate* of interest is approximately twice the *flat rate* of interest.

Example 19 Determining the effective interest rate from the flat interest rate

Repayments on Tom's hire-purchase agreement are based on a flat rate interest of 15% p.a., and he is to make 60 repayments.

a Calculate the value of the effective interest rate.

b Check your answer by using the approximate rule.

Solution

a Substitute $r_f = 15$ and $n = 60$ in the formula for effective interest rate and evaluate.

$$r_e = r_f \times \frac{2n}{n+1} = 15 \times \frac{2 \times 60}{60+1}$$
$$= 15 \times \frac{120}{61} = 29.5\%$$

b Use the approximate rule.

$$r_e \approx r_f \times 2 = 15 \times 2 = 30\%$$

Note: Since n is large in this example, we expect the answers to **a** and **b** to be similar.

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Exercise 4E-1

Calculating the real cost of an item purchased under hire-purchase

Example 18

1 The cash price of a tennis racquet is $330. To buy it through a hire-purchase agreement requires a deposit of $30 and 12 equal monthly instalments of $28.
Calculate:

a the total cost of buying the racquet by hire-purchase

b the extra cost of buying by hire-purchase.

2 A bicycle is on sale price for $300. It can be bought through hire-purchase with a deposit of $60 and 10% interest on the outstanding balance, to be repaid in 10 monthly instalments.
Calculate:

a the amount of each monthly instalment

b the total cost of buying the bicycle by hire-purchase.

3 A hire-purchase agreement offers gym equipment, with a marked price of $897, for $87 deposit and $46.80 a month payable over 2 years.
Calculate:

a the total hire-purchase price

b the amount of interest charged.

Calculating an effective rate from a flat rate of interest

Example 19

4 The flat rate of interest charged for a hire-purchase agreement is 10% p.a. The hire-purchase agreement is to be paid out by making 24 monthly payments. Calculate the effective interest rate.

5 The flat rate of interest charged for a hire-purchase agreement is 12% p.a. The hire-purchase agreement is to be paid out by making 52 weekly payments. Calculate the effective interest rate.

Calculating flat and effective rates of interest for a hire-purchase agreement

6 Equipment, which normally costs $750, can be bought through a hire-purchase agreement with a $200 deposit and $26.40 a month for 30 months. Calculate:

a the amount of interest being charged

b the flat rate of interest per annum

c the effective rate of interest per annum.

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7 A customer bought a new car for \$36 000 on a hire-purchase agreement. A deposit of \$4000 was paid. The remaining amount owed is to be paid by making 18 monthly payments of \$2100.

Calculate:

a the total amount paid for the car

b the flat interest rate per annum calculated on the amount owed

c the effective rate of interest per annum.

▶ Personal loans

Another form of financing offered is a **personal loan**. A personal loan involves borrowing a sum of money for personal use from a financial institution such as a bank. The money can be used for purchases such as a car, holiday or computer.

A fixed interest rate personal loan has an interest rate that stays the same for the full loan term. It generally has the following characteristics:

- Regular repayments are made (weekly, monthly, etc.).
- Interest is calculated on the unpaid balance of the loan when each payment is due.
- Repayment amounts are calculated taking into account the total amount of the loan, the loan term, bank fees and the annual interest rate.
- The repayments ensure that the loan is repaid in full within the agreed term of the loan. Unlike hire-purchase agreements, personal loans do not involve a deposit.

Calculating the flat rate and effective rate of interest of a personal loan is one way to determine its true cost.

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Example 20 Analysing a personal loan

To buy a new car Samar takes out a personal loan of \$30 000 from his bank, for which he is required to make fortnightly repayments of \$409 for 4 years. Assume that there are 26 fortnightly in a year.

a How much does it cost to pay out the loan in full?

b How much interest does he pay on this loan?

c What is the equivalent flat rate of interest?

d What is the equivalent effective rate of interest?

Solution

a Determine the total amount repaid.

Total paid = number of payments × payment amount

$$\text{Total paid} = 4 \times 26 \times \$409 = \$42\,536$$

It will cost Samar \$ 42 536 to pay off the loan.

b Determine the amount of interest paid.

Interest paid = total amount repaid − amount owed

$$\text{Interest} = \$42\,536 - \$30\,000 = \$12\,536$$

Samar will pay \$12 536 in interest.

c Use the formula $r_f = \dfrac{100I}{Pt}$ and the values of I, P and t to find the flat rate of interest.

$$r_f = \frac{100I}{Pt} = \frac{100 \times 12\,536}{30\,000 \times 4} = 10.44\ldots = 10.4\% \text{ to one decimal place}$$

The annual flat rate of interest is 10.4%.

d Use the formula $r_e = r_f \times \dfrac{2n}{(n+1)}$ and the values of I, P and t and n to find the effective rate of interest.

$$r_e = 10.4 \times \frac{2 \times 104}{(104 + 1)} = 10.4 \times \frac{2 \times 104}{105} = 20.601\ldots = 20.6\% \text{ to one decimal place}$$

The annual effective rate of interest is 20.6%.

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Exercise 4E-2

Analysing personal loans

Example 20

1 To buy some new furniture Meg takes out a personal loan of \$15 000 from a credit union bank, for which she is required to make monthly repayments of \$720 for 2 years.

- **a** How much does it cost to pay out the loan in full?
- **b** How much interest does she pay on this loan?
- **c** What is the equivalent flat rate of interest?
- **d** What is the equivalent effective rate of interest?

2 To buy some new furniture Ross takes out a personal loan of \$25 000 for which he has to make monthly repayments of \$1500 for 18 months.

- **a** How much does it cost to pay out the loan in full?
- **b** How much interest does he pay on this loan?
- **c** What is the equivalent flat rate of interest?
- **d** What is the equivalent effective rate of interest?

3 To buy a new caravan a personal loan of \$40 000 is obtained. The loan is to be paid off by making *fortnightly* payments of \$632 for 3 years.

- **a** How much does it cost to pay out the loan in full? (Assume 26 fortnights in a year)
- **b** How much interest does he pay on this loan?
- **c** What is the equivalent flat rate of interest?
- **d** What is the equivalent effective rate of interest?

► Credit cards

Calculating credit card debt is an application of **compound interest** where interest is calculated daily.

Calculating credit card debt

If a credit card debt of \$$P$ accumulates at the rate of r% per annum, compounding daily, then the amount of debt accumulated after n days is given by:

$$A = P\left(1 + \frac{r/365}{100}\right)^n = P\left(1 + \frac{r}{36\,500}\right)^n$$

and the amount of interest payable after n days is given by:

$$I = A - P$$

Note: To determine the daily interest rate the annual interest rate r is divided by 365.

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Example 21 Calculating credit card interest

Determine how much interest is payable on a credit card debt of $5630 at an interest rate of 17.8% per annum for 27 days.

Solution

1 Calculate the value of debt after 27 days using the rule:

$$A = P\left(1 + \frac{r}{36\,500}\right)^n$$

$P = \$5630$, $r = 17.8\%$ and $n = 27$

$$A = P\left(1 + \frac{r}{36\,500}\right)^n$$

$$= 5630\left(1 + \frac{17.8}{36\,500}\right)^{27}$$

$= \$5704.60$ to the nearest cent

2 The amount of interest payable is obtained by subtracting the original debt P from the value of the debt after 27 days.

$$I = A - P$$

$$= \$5704.60 - \$5630.00$$

$$= \$74.60$$

Taking into account interest-free periods

Most credit cards offer a maximum interest-free period, which means that if you pay for your purchase within that time you won't pay any interest.

Usually the credit card has a statement period, which runs for about 30 days. After the statement closes, there are an additional number of days, usually 15 to 25, to pay the full balance before an interest rate applies. The actual number of interest-free days varies depending on when you make your purchase and the number of days remaining in your statement period. For example, suppose your statement begins on 1 June and ends on 30 June. Your bank gives you another 25 days after 30 June to pay your bill in full before you are charged any interest on the items you bought in June.

Thus:

- if you make a purchase on 10 June, you will have 20 days remaining until your statement period closes plus 25 days to make your full payment. This means that you have 45 interest-free days before you will be charged interest.
- if you make a purchase on 28 June, you only have 2 days remaining until your statement period closes. Adding that to the 25 days available to make the payment and you only have 27 days before you will be charged interest.

Not all statement periods will start at the beginning of the month, this will vary from person to person. However, the statement periods will be the same for all individuals with a particular credit card.

These principles are illustrated in the next example.

Example 22 Calculating credit card interest with an interest-free period

Janelle pays for her holiday in Bali using her credit card. Her bank offers a 30-day statement period plus a further 25 days interest free. After that time, the bank charges interest at a rate of 20% per annum compounding daily.

The cost of the holiday is $1500 and Janelle makes the purchase on 16 August, which is day 10 of her statement period. She intends to pay off the credit card on 1 November. At this date, how much will she need to pay back? (Assume no interest is payable on the last day).

Solution

1 Determine the number of interest free days.

Since Janelle is in day 10 of her statement period she has 20 + 25 days = 45 interest free days.

2 Determine the number of days for which interest is payable. Since the purchase was made on 16 August, start counting from 17 August.

Number of days:

17 August–30 August = 14 days

1 September–30 September = 30 days

1 October–31 October = 31 days

Total days = 14 + 30 + 31 = 75

Total interest payable days = 75 − 45
= 30

3 Calculate the amount payable using the rule $A = P\left(1 + \frac{r}{36\,500}\right)^n$.

$P = \$1500, r = 20\%, n = 30$

$$A = 1500\left(1 + \frac{20}{36\,500}\right)^{30}$$

$= \$1524.85$ to the nearest cent

Janelle will need to pay back $1524.85.

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Exercise 4E-3

Calculating credit card interest

Example 21 **1** Determine the amount of interest payable on the following credit card debts.

- **a** \$2000 at an interest rate of 18.9% per annum for 52 days
- **b** \$785 at an interest rate of 24% per annum for 200 days
- **c** \$12 000 at an interest rate of 22.5% per annum for 60 days
- **d** \$837 at an interest rate of 21.7% per annum for 90 days

Example 22 **2** Matt has two credit cards, each with different borrowing terms.

- Credit card A charges 22% p.a. interest and offers up to 60 days interest free.
- Credit card B charges 19% p.a. but only offers 40 days interest free.

He wishes to buy an item costing \$2000 on his credit card, which he will purchase at the beginning of the statement period whichever card he uses, so as have the maximum interest free days. Which credit card should he use:

- **a** if he is going to pay off the card 30 days after purchase?
- **b** if he is going to pay off the card 60 days after purchase?
- **c** if he is going to pay off the card 90 days after purchase?
- **d** if he is going to pay off the card 240 days after purchase?

3 Joe buys a skateboard costing \$830 on his credit card. He buys it on the first day of his statement period so he has the maximum number of interest-free days, which is 55. After that time, the bank charges interest at a rate of 24% per annum compounding daily. How much will he owe on his credit card:

- **a** at the end of 20 weeks?
- **b** at the end of 40 weeks?

4 Brett spends \$3000 on some home theatre equipment on his credit card. His bank offers a 30-day statement period and then a further 30 days interest free. After that time, the bank charges interest at a rate of 23.5% per annum compounding daily. Brett makes the purchase on 2 April, which is day 2 of his statement period. He intends to pay off the credit card on 1 July. At this date, how much will he need to pay back? (Assume no interest is payable on the last day).

5 Sacha buys a computer costing \$1470 on her credit card. Her bank offers a 30-day statement period and then a further 10 days interest free. After that time, the bank charges interest at a rate of 18.5% per annum compounding daily.
Sacha makes the purchase on 10 January, which is day 12 of her statement period. She intends to pay off the credit card on 1 March. At this date, how much will she need to pay back? (Assume no interest is payable on the last day).

4F Inflation

Effect of inflation on prices

Inflation is a term that describes the continuous upward movement in the general level of prices. This has the effect of steadily *reducing* the **purchasing power** of your money; that is, what you can actually buy with your money.

In the early 1970s, inflation rates were very high, up to around 16% and 17%. Inflation in Australia has been relatively low in recent years.

- Since 1970, inflation has averaged 6.8% per year.
- Since 1990, it has averaged 2.1% per year.

Example 23 Determining the effect of inflation on prices over a short period of time

Suppose that inflation is recorded as 2.7% in 2012 and 3.5% in 2013 and that a loaf of bread costs \$2.20 at the end of 2011. If the price of bread increases with inflation, what will be the price of the loaf at the end of 2013?

Solution

1 Determine the price of the loaf of bread at the end of 2012 after a 2.7% increase.

$$\text{Increase on the 2012 price} = 2.20 \times \frac{2.7}{100} = 0.06$$

2 Calculate the price at the end of 2012.

$$\text{Price (2012)} = 2.20 + 0.06 = \$2.26$$

3 Determine the price of the loaf of bread at the end of 2013 after a further 3.5% increase.

$$\text{Increase in price (2013)} = 2.26 \times \frac{3.5}{100} = 0.08$$

4 Calculate the price at the end of 2013.

$$\text{Price (2013)} = 2.26 + 0.08 = \$2.34$$

While the difference in price seen in Example 23 does not seem a lot, you will be aware from earlier compound interest examples that even if inflation holds steady at a low 2.1% per year for 20 years, prices will still increase a lot, as the following example shows.

Example 24 Determining the effect of inflation on prices over a long period

Suppose that a one-litre carton of milk costs \$1.70 today.

a What will be the price of the one-litre carton of milk in 20 years time if the average annual inflation rate is 2.1%?

b What will be the price of the one-litre carton of milk in 20 years time if the average annual inflation rate is 6.8%?

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Solution

a **1** This is the equivalent of investing \$1.70 at 2.1% interest compounding annually, so we can use the compound interest formula.

$$A = P \times \left(1 + \frac{r}{100}\right)^t$$

2 Substitute $P = \$1.70, t = 20$ and $r = 2.1$ in the formula to find the price in 20 years.

$$\text{Price} = 1.70 \times \left(1 + \frac{2.1}{100}\right)^{20}$$
$$= \$2.58 \text{ to the nearest cent}$$

b Substitute $P = 1.70, t = 20$ and $r = 6.8$ in the formula and evaluate.

$$\text{Price} = 1.70 \times \left(1 + \frac{6.8}{100}\right)^{20}$$
$$= \$6.34 \text{ to the nearest cent}$$

Effect of inflation on the purchasing power of money

Another way of looking at the effect of inflation on our money is to consider what a sum of money today would buy in the future. That is, to convert projected dollar numbers back into present-day values so you can think in today's money values.

Suppose, for example, that you put \$100 in a box under the bed and leave it there for 10 years. When you go back to the box, there is still \$100, but, what could you buy with this amount in 10 years time? To find out we need to 'deflate' this amount back to current-day purchasing power dollars.

We can do this using the compound interest formula.

Suppose there has been an average inflation rate of 4% over the 10-year period. Substituting $A = 100$, $r = 4$ and $t = 10$ gives:

$$100 = P \times \left(1 + \frac{4}{100}\right)^{10} = P \times (1 + 0.04)^{10}$$

Rearranging this equation, or using your CAS calculator to solve it, gives

$$P = \frac{100}{(1 + 0.04)^{10}}$$
$$= \$67.56 \text{ to the nearest cent}$$

That is, the money that was worth \$100 when it was put away has a purchasing power of only \$67.58 after 10 years if the inflation rate has averaged 4% per annum.

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Example 25 Investigating purchasing power

If savings of \$100 000 are hidden in a mattress in 2016, what is the purchasing power of this amount in 8 years time if the average inflation rate over this period is 3.7%? Give your answer to the nearest dollar.

Solution

1 Write the compound interest formula with P (the purchasing power, which is unknown), $A = 100\,000$ (current value), $r = 3.7$ and $t = 8$.

$$A = P \times \left(1 + \frac{r}{100}\right)^t$$

$$100\,000 = P \times \left(1 + \frac{3.7}{100}\right)^8$$

2 Use your CAS calculator to solve this equation for P and write your answer.

The purchasing power of \$100 000 in 8 years is \$74 777, to the nearest dollar.

Exercise 4F

Effect of inflation on prices

Example 23

1 Suppose that inflation is recorded as 2.7% in 2017 and 3.5% in 2018, and that a magazine costs \$3.50 at the end of 2016. Assume that the price increases with inflation.

a What will be the price of the magazine at the end of 2017?

b What will be the price of the magazine at the end of 2018?

2 Suppose that Henry receives a salary increase at the end of each year equal to the rate of inflation for that year. Inflation is recorded as 3.2% in 2017 and 5.3% in 2018, and Henry's weekly salary is \$825 at the end of 2016.

a What will Henry's salary be at the end of 2017?

b What will Henry's salary be at the end of 2018?

Example 24

3 Suppose that the cost of petrol per litre is \$1.80 today.

a What will be the price of petrol per litre in 20 years time if the average annual inflation rate is 1.9%?

b What will be the price of petrol per litre in 20 years time if the average annual inflation rate is 7.1%?

4 A house is sold at auction for \$500 000. If the price of the house increases with the inflation rate, what will be the price of the house in 12 years time:

a if the average inflation rate over the 12-year period is 2.6%?

b if the average inflation rate over the 12-year period is 6.9%?

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Effect of inflation on purchasing power

Example 25

5 If savings of $200 000 are hidden in a mattress today, what is the purchasing power of that money in 10 years time:

- **a** if the average inflation rate over the 10-year period is 3%?
- **b** if the average inflation rate over the 10-year period is 13%?

6 If Jo puts $1000 cash in her safe, what is its purchasing power in 20 years time:

- **a** if the average inflation rate over the 20-year period is 2.6%?
- **b** if the average inflation rate over the 20-year period is 6.9%?
- **c** if the average inflation rate over the 20-year period is 14.3%?

4G Financial investigation: buying a car

Exercise 4G

One of the first big purchases you are likely to make is buying your first car. How much can you afford to pay for that car, and how will you go about financing the purchase?

In this investigation you are going to use available resources to determine the best strategy. You will need to investigate each of the following themes.

1 What can you afford?
Assuming that you will need to finance the car, what can you afford to repay each week or fortnight? You will need to consider your likely salary, as well as your other living costs. Some of the major banks will give advice regarding this amount and include ‘affordability’ calculators on their websites.

2 How should you finance the car?
Compare some different forms of finance (such as variable interest personal loans, fixed interest personal loans, credit cards), some different financial institutions, as well as some of the financing options offered directly by the car dealerships to determine your best finance option.

3 What car should you buy?
Cars often depreciate in value very quickly, especially if they are purchased new. In the worst-case scenario, you can end up owing more money on a car than its current market value! Compare the depreciation of two or three different brands of car when purchased new, and at various stages over the period of time for which you have decided to finance it. How much will the car be worth when it is finally paid for?

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Key ideas and chapter summary

Percentage increase or decrease	**Percentage increase or decrease** is the amount of the increase or decrease of value or quantity expressed as a percentage of the original value or quantity.
Simple interest	**Simple interest** is paid on an investment or loan on the basis of the original amount invested or borrowed called the principal (P). The amount of simple interest is constant from year to year.
GST	**GST** (goods and services tax) is a 10% tax that is added to most purchases.
Shares	A **share** is a unit of ownership of a company.
Price-to-earnings ratio	**Price-to-earnings ratio** is a measure of the profit of a company, given by the current share price/profit per share. A lower value of the price-to-earnings ratio may indicate a better investment.
Dividend yield	A **dividend** paid by company expressed as a percentage of the share price.
Minimum monthly balance	The lowest amount an account contains in each calendar month is its **minimum monthly balance**.
Compound interest	Under **compound interest**, the interest paid on a loan or investment is credited or debited to the account at the end of each period. The interest earned for the next period is based on the principal plus previous interest earned. The amount of interest earned increases each year.
Hire-purchase agreement	Under a **hire-purchase agreement**, the purchaser hires an item from the vendor and makes periodic payments at an agreed rate of interest. At the end of a hire-purchase agreement, the item belongs to the purchaser.
Flat rate of interest	The **flat rate of interest** (r_f) is given by $r_f = \dfrac{100I}{Pt}$ where P is the principal owing after any deposit has been deducted, I is the interest and t is the duration in years.
Effective interest rate	For time payment arrangements, **the effective rate of interest** (r_e) is given by: $r_e = r_f \times \dfrac{2n}{n+1}$ where n = total number of payments.
Purchasing power	**Purchasing power** describes what you can actually buy with your money.
Credit	**Credit** is an advance of money from a financial institution, such as a bank, that does not have to be paid back immediately but which attracts interest after an interest-free period.

Personal loan	A **personal loan** is a sum of money borrowed from a financial institution such as a bank for buying an item for personal use. Regular repayments are required to repay the loan.
Inflation	**Inflation** is the continuous upward movement of the economy that increases prices over time or, conversely, decreases the spending power of money over time.

Skills check

Having completed this chapter you should be able to:

- calculate the amount of the discount and the new price when an $r\%$ discount is applied
- calculate the amount of the increase and the new price when an $r\%$ increase is applied
- calculate the percentage discount or increase given the old and new prices
- calculate the original price given the new price and the percentage discount or increase
- determine the new price of a good or service after the GST has been added
- determine the original cost of a good or service given the inclusive of GST
- determine the price-to-earnings ratio of shares
- determine dividend yield of shares
- use the formula for simple interest to find the value of any one of the variables I, P, r or t when the values of the other three are known
- determine the interest payable on a bank account, paid on the minimum monthly balance
- calculate the amount of an investment after simple interest has been added
- plot the value of simple interest (I) against time (t) to show a linear relationship
- use the formula for compound interest to solve problems involving investments and loans
- plot the value of compound interest (I) against time (t) to show a non-linear relationship
- determine the annual flat and effective rate of interest for a hire-purchase agreement or a personal loan
- determine the interest payable when an item is purchased using a credit card
- determine the new price of an item after a period of inflation
- determine the effect of inflation on the purchasing power of money.

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Multiple-choice questions

1 The amount saved if a 10% discount is offered on an item marked \$120 is:

A \$20 **B** \$12 **C** \$1.20 **D** \$10.90 **E** \$10

2 If a 20% discount is offered on an item marked \$30, the new discounted price of the item is:

A \$10 **B** \$24 **C** \$6 **D** \$25 **E** \$28

3 If a 15% increase is applied to an item marked \$60, the new price of the item is:

A \$69 **B** \$9 **C** \$75 **D** \$67 **E** \$70

4 If GST is applied to an item marked \$121.50, the new price of the item is:

A \$131.50 **B** \$111.50 **C** \$12.15 **D** \$133.65 **E** \$110.45

5 The telephone bill including GST is \$318.97. The price before GST was:

A \$289.97 **B** \$31.90 **C** \$350.87 **D** \$29.00 **E** \$328.97

6 Shares in Company A have a market value of \$22.50. If the company makes a 12-month profit of \$2.85 per share, the price-to-earnings ratio for that time period is:

A 12.67 **B** 0.13 **C** 7.89 **D** \$25.35 **E** \$19.65

7 How much interest is earned if \$2000 is invested for 1 year at a simple interest rate of 4% per annum?

A \$2080 **B** \$160 **C** \$8 **D** \$800 **E** \$80

8 The total value of an investment of \$1000 after 3 years if simple interest is paid at the rate of 5.5% per annum is:

A \$55 **B** \$1055 **C** \$1165 **D** \$1174.24 **E** \$3165

9 What is the interest rate, per annum, if a deposit of \$1500 earns interest of \$50 over a period of 6 months?

A 0.56% **B** 5.45% **C** 5.55% **D** 6.45% **E** 6.67%

10 \$2400 is invested at a rate of 4.25% compound interest, paid annually. The value of this investment after 6 years is closest to:

A \$3080 **B** \$3074 **C** \$3012 **D** \$680 **E** \$674

11 \$3600 is invested at a rate of 5% p.a. compounding annually. The value of the investment after 4 years is given by:

A $3600(1 + (5/100)^4)$ **B** $3600(1 + 5/100)^4$

C $3600(1 + 4 \times 0.05)$ **D** $3600 + 4 \times 3600 \times 0.05$

E $3600(5/100)^4$

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12 The value of $8500 compounding annually for 5 years at 6% p.a. is closest to:

A $2550 **B** $10 731 **C** $11 050 **D** $11 375 **E** $11 700

13 Sue has a credit card debt of $3000. She has 35 interest-free days left, but she will not be able to pay the amount for 90 days. If the interest rate is 18.9% per annum, the amount she will need to repay is closest to:

A $3055 **B** $3087 **C** $3143 **D** $87 **E** $55

The following information relates to Questions 14 to 17

To renovate her kitchen Annie take out a personal loan of $20 000 from the bank, for which she is required to make monthly repayments of $685 for 3 years.

14 The total cost of paying off the loan is:

A $2055 **B** $4660 **C** $20 000 **D** $24 660 **E** $44 660

15 The total interest paid is:

A $2055 **B** $4660 **C** $20 000 **D** $24 660 **E** $44 660

16 The flat rate of interest for this loan is closest to:

A 23.30% **B** 7.8% **C** 18.9% **D** 6.3% **E** 27.0%

17 The effective rate of interest for this loan is closest to:

A 45.3% **B** 15.2% **C** 36.8% **D** 12.3% **E** 52.5%

18 The price of a newspaper is $2 today. If the price increases with inflation, what will be the price of the newspaper in 10 years time if the average annual inflation rate is 1.8%?

A $2.39 **B** $2.10 **C** $2.18 **D** $20 **E** $5.57

The following information relates to Questions 19 to 22

Janet buys a car costing $23 000. She pays a $5000 deposit and then makes payments of $440 per month for the next 5 years.

19 How many payments does Janet make under this arrangement?

A 5 **B** 12 **C** 20 **D** 40 **E** 60

20 How much interest does Janet pay under this arrangement?

A $3400 **B** $8400 **C** $3120 **D** $1600 **E** $1250

21 What is the annual flat rate of interest?

A 36.5% **B** 18.4% **C** 14.6% **D** 9.3% **E** 7.3%

22 What is the annual effective rate of interest?

A 46.7% **B** 36.5% **C** 18.4% **D** 14.6% **E** 9.3%

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Short-answer questions

1 Rabbit Easter Eggs were reduced from \$2.99 to \$2.37 because they were not selling quickly. Bilby Easter Eggs were discounted from \$4.79 to \$3.83.

a Which type of Easter egg had the larger percentage reduction?

b Calculate the difference in the percentage rates.

2 After Christmas, all stock in JDs was discounted by 20%. The sale price of a pair of cross-trainers was \$110. Calculate the original marked price.

3 If the selling price of a computer is \$1990:

a what is the price without GST? **b** how much of this is GST?

4 How much additional interest is earned if \$8000 is invested for 7 years at 6.5% when interest is compounded annually, as compared with simple interest paid at the same rate?

5 Zara buys a leather jacket costing \$450 on her credit card. She buys it on the last day of her statement period, so she has the minimum number of interest free days, which is 25. After that time, the bank charges interest at a rate of 22.6% per annum compounding daily. How much will she owe on her credit card:

a in 30 days? **b** in 90 days?

6 A television set, which normally costs \$880, can be bought through hire purchase with a \$200 deposit and a payment of \$30 a month for 30 months. Calculate:

a the amount of interest being charged

b the flat rate of interest

c the effective rate of interest.

Extended-response questions

1 **a** The wholesale price of a digital camera is \$350. The maximum profit that a retailer is allowed to make when selling this particular camera is 75% of the wholesale price. Calculate the maximum retail price of the camera.

b Suppose that the wholesale price of the camera increases at 5% per annum simple interest for the next 5 years.

  i What is the new wholesale price of the camera?

  ii By how much will the wholesale price have increased at the end of 5 years?

  iii What is the new retail price of the camera (with 75% profit)?

  iv What percentage increase is this in the retail price determined in part **a**?

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2 Suppose that you have $30 000 to invest, and there are two alternative plans for investment:

Plan A offers 5.3% per annum simple interest.

Plan B offers 5.0% per annum compound interest, compounding annually.

- **a** Use your graphics calculator to construct a graph of the interest earned under Plan A against time.
- **b** On the same axes, use your graphics calculator to construct a graph of the interest earned under Plan B against time.
- **c** Which of the plans would you choose, A or B, if the investment is for:
  - **i** 3 years?
  - **ii** 6 years?

3 The Smiths bought a new car priced at $34 800 and paid a deposit of $5000 cash. They borrowed the balance of the purchase price. They then agreed to repay the loan plus interest in equal monthly payments of $850 over 4 years.

- **a** Calculate the total amount of interest to be repaid over the term of this loan.
- **b** Calculate, correct to one decimal place, the annual simple interest rate charged on this loan.

4 The phone that Emily wants to buy usually costs $400 but is on sale for $350.

- **a** What percentage discount does this amount to?
- **b** Emily considers entering into a hire-purchase agreement to buy the phone where she pays no deposit and 24 monthly payments of $22.50.
  - **i** How much interest would Emily pay under this agreement on the purchase price of $350?
  - **ii** What is the annual flat rate of interest that this represents? Express your answer as a percentage correct to one decimal place.
  - **iii** What is the effective annual rate of interest that this represents? Express your answer as a percentage correct to one decimal place.
- **c** Another option available to Emily is to use her credit card, which attracts an interest rate of 20% per annum compounding daily. How much interest would Emily pay on the purchase price of $350 if she makes no payments for 2 years, assuming that she has 60 interest free days?
- **d** Would you recommend her to purchase the phone using the hire-purchase agreement or use her credit card?

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5

Matrices

- What is a matrix?
- How is the order of a matrix defined?
- How are the positions of the elements of a matrix specified?
- What are the rules for adding and subtracting matrices?
- How do we multiply a matrix by a scalar?
- What is the method for multiplying a matrix by another matrix?
- What are the properties of the identity and inverse matrices?
- How can your graphics calculator be used to do matrix operations?
- How can communications be represented by matrices?

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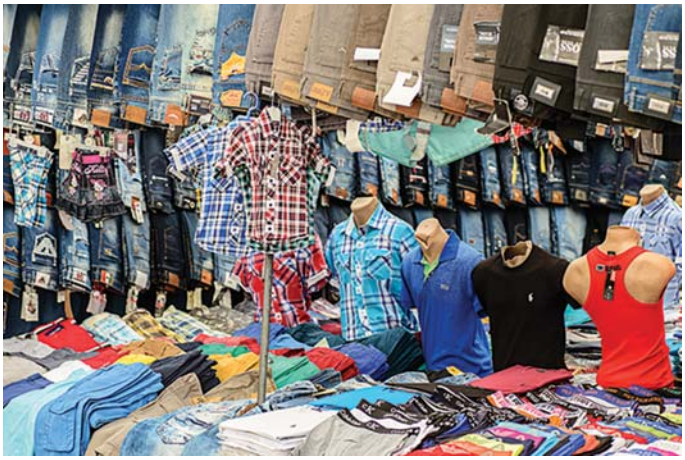
Introduction

A **matrix** (plural matrices) is a rectangular group of numbers set out in rows and columns. Matrices can be used to store information, solve sets of simultaneous equations, find optimal solutions in business, analyse networks, transform shapes in geometry, encode information and devise the best strategies in game theory. We will explore some of these applications while learning the basic theory of matrices.

5A The basics of a matrix

A market stall operates on Friday and Saturday. Sales could be recorded using matrix A.



Matrix A :

$$A = \begin{array}{r} \\ \textit{Friday} \\ \textit{Saturday} \\ \\ \end{array} \begin{array}{ccc} \textit{Shirts} & \textit{Jeans} & \textit{Belts} \\ \end{array}$$

$$A = \begin{array}{rcccl} & \textit{Shirts} & \textit{Jeans} & \textit{Belts} & \\ \textit{Friday} & 6 & 8 & 4 & \textit{row } 1 \\ \textit{Saturday} & 3 & 7 & 1 & \textit{row } 2 \\ & \textit{column } 1 & \textit{column } 2 & \textit{column } 3 & \end{array}$$

Rows	Columns
Friday sales are listed in row 1.	The number of shirts sold is listed in column 1.
Saturday sales are listed in row 2.	The number of pairs of jeans sold is listed in column 2.
	The number of belts sold is listed in column 3.

We can read the following information from the matrix:

- on Friday 8 pairs of jeans were sold
- on Saturday 1 belt was sold
- the total number of items sold on Friday was 6 + 8 + 4 = 18
- the total number of belts sold was 4 + 1 = 5.

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Order of a matrix

The **order** (or size) **of a matrix** is written as: number of rows × number of columns

$$\begin{array}{r} \text{row } 1 \\ \text{row } 2 \end{array} \begin{bmatrix} 6 & 8 & 4 \\ 3 & 7 & 1 \end{bmatrix} \qquad \begin{array}{c} \begin{bmatrix} 6 & 8 & 4 \\ 3 & 7 & 1 \end{bmatrix} \\ \begin{array}{ccc} \text{column } 1 & \text{column } 2 & \text{column } 3 \end{array} \end{array}$$

Think: ‘rows in a cinema’

Think: ‘columns of the Parthenon’

The order of matrix A in the market stall example above is 2×3; that is, 2 rows × 3 columns. It is called a ‘two by three’ matrix.

In writing down the order of a matrix, the number of rows is always given first, then the number of columns. *Rows first*, then *columns*.

Remember: When you walk into a cinema, you go to your *row first*.

Matrices are usually named using capital letters such as A, B, O.

Elements of a matrix

The numbers within a matrix are called its **elements**.

Locating an element in a matrix

a_{ij} is the element in *row i*, *column j*.

For example, in the matrix:

$$A = \begin{bmatrix} 6 & 8 & 4 \\ 3 & 7 & 1 \end{bmatrix}$$

- element a_{13} is in row 1, column 3 and its value is 4
- element a_{22} is in row 2, column 2 and its value is 7.

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Example 1 Interpreting the elements of a matrix

Matrix B shows the number of boys and girls in years 10 to 12 at a particular school.

$$B = \begin{array}{r} \\ \text{Year 10} \\ \text{Year 11} \\ \text{Year 12} \end{array} \begin{array}{c} \begin{array}{cc} \textit{Boys} & \textit{Girls} \end{array} \\ \begin{bmatrix} 57 & 63 \\ 48 & 54 \\ 39 & 45 \end{bmatrix} \end{array}$$

a Give the order of matrix B.

b What information is given by the element b_{12}?

c Which element gives the number of girls in year 12?

d How many boys in total?

e How many students in year 11?

Solution

a Count the rows, count the columns.
Remember: Order is rows × columns.

The order of matrix B is 3×2.

b The element b_{12} is in row 1 and column 2. This is where the year 10 row meets the girls column.

There are 63 girls in year 10.

c Year 12 is row 3. Girls are column 2.

The number of year 12 girls is given by b_{32}.

d The sum of the boys column gives the total number of boys.

The total number of boys is 144.

e The sum of the year 11 row gives the total number of students in year 11.

There are 102 students in year 11.

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Row matrices

A **row matrix** has a *single row* of elements.

In matrix A, the Friday sales from the market stall can be represented by a 1 × 3 *row* matrix.

$$A = \begin{array}{c} \\ \textit{Friday} \\ \textit{Saturday} \end{array} \begin{array}{c} \begin{array}{ccc} \textit{Shirts} & \textit{Jeans} & \textit{Belts} \end{array} \\ \begin{bmatrix} 6 & 8 & 4 \\ 3 & 7 & 1 \end{bmatrix} \end{array} \qquad \begin{array}{cccc} & \textit{Shirts} & \textit{Jeans} & \textit{Belts} \\ \textit{Friday} & [\,6 & 8 & 4\,] \end{array}$$

Column matrices

A **column matrix** has a *single column* of elements.

In matrix A, the sales of jeans from the market stall can be represented by a 2 × 1 *column* matrix.

$$\begin{array}{cc} & \textit{Jeans} \\ \textit{Friday} & 8 \\ \textit{Saturday} & 7 \end{array}$$

Although they appear to be very simple, row and column matrices have useful properties that will be explored in this chapter.

Square matrices

In **square matrices** the number of *rows* equals the number of *columns*.

Here are three examples.

$$[9] \qquad \begin{bmatrix} 5 & 4 \\ 4 & 2 \end{bmatrix} \qquad \begin{bmatrix} 0 & 4 & 3 \\ 8 & 1 & 6 \\ 2 & 0 & 7 \end{bmatrix}$$

1 × 1 2 × 2 3 × 3

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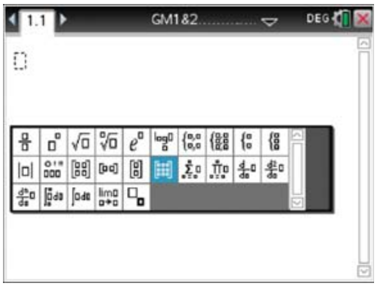
How to enter a matrix using the TI-Nspire CAS

Enter the matrix $B = \begin{bmatrix} 57 & 63 \\ 48 & 51 \\ 39 & 45 \end{bmatrix}$ into the TI-Nspire CAS. Display the element $b_{2\,1}$.

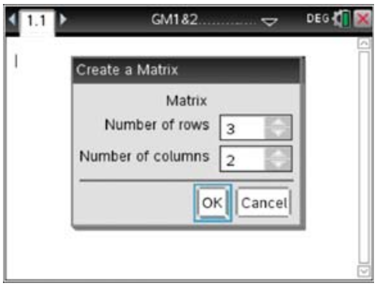
Steps

1 Press ⌂ on **>New Document>Add Calculator**.



2 Press ⊞ and use the cursor ▼ ▶ arrows to highlight the matrix template shown. Press enter.

Note: Math Templates can also be accessed by pressing ctrl + menu **>Math Templates**.



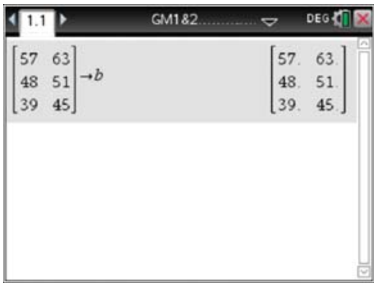
Create a Matrix

Matrix

Number of rows 3

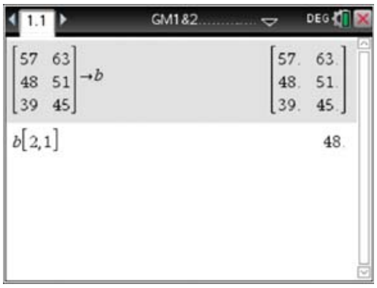
Number of columns 2

3 Press ◀ then ▲ or ▼ to select the **Number of rows** required (number of rows in this example is 3). Press tab to move to the next entry and repeat for the **Number of columns** (the number of columns in this example is 2). Press tab to highlight **OK** and press enter.



$\begin{bmatrix} 57 & 63 \\ 48 & 51 \\ 39 & 45 \end{bmatrix} \to b$ $\begin{bmatrix} 57. & 63. \\ 48. & 51. \\ 39. & 45. \end{bmatrix}$

4 Type in the values into the matrix template. Use tab or the arrow keys to move to the required position in the matrix to enter each value. When the matrix has been completed press tab or ▶ to move outside the matrix and press ctrl + var followed by B. This will store the matrix as the variable **b**. Press enter.



$\begin{bmatrix} 57 & 63 \\ 48 & 51 \\ 39 & 45 \end{bmatrix} \to b$ $\begin{bmatrix} 57. & 63. \\ 48. & 51. \\ 39. & 45. \end{bmatrix}$

$b[2,1]$ 48.

5 When you type B (or b) in the graphics calculator, it will paste in the matrix $\begin{bmatrix} 57 & 63 \\ 48 & 51 \\ 39 & 45 \end{bmatrix}$.

6 To display element $b_{2\,1}$ (the element in position Row 2, Column 1), type in **b[2,1]** and press enter.

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How to enter a matrix using the ClassPad

Enter the matrix $B = \begin{bmatrix} 57 & 63 \\ 48 & 51 \\ 39 & 45 \end{bmatrix}$ into the ClassPad calculator. Display the element b_{21}.

Steps

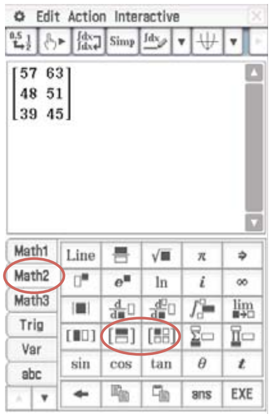
1 Open the soft **Keyboard** in the **Main** application $\sqrt{\alpha}$.

2 Select the **Math2** keyboard.

3 Tap the 2 × 2 matrix ▦ followed by the 2 × 1 matrix ▤ icon. This will add a third row and create a 3 × 2 matrix.

4 Enter the values of $\begin{bmatrix} 57 & 63 \\ 48 & 51 \\ 39 & 45 \end{bmatrix}$.

Note: Tap in each new position to enter the new value or use the cursor key ▭ on the hard keyboard to navigate to a new position.

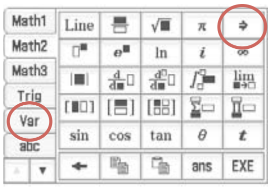


$\begin{bmatrix} 57 & 63 \\ 48 & 51 \\ 39 & 45 \end{bmatrix}$

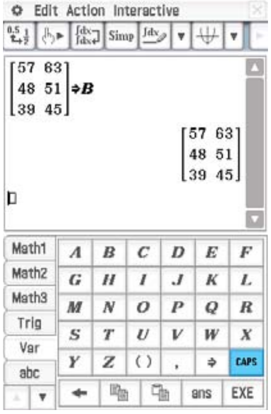
5 To assign the matrix the variable name B.

a Move the cursor to the very right-hand side of the matrix.

b From the keyboard, tap the variable assignment key ⇒, followed by the var, then CAPS (for uppercase letters) and ***B***. Press **EXE** to confirm your choice.



Note: Until it is reassigned, ***B*** will represent the matrix as defined above.



$\begin{bmatrix} 57 & 63 \\ 48 & 51 \\ 39 & 45 \end{bmatrix} \Rightarrow B$

$\begin{bmatrix} 57 & 63 \\ 48 & 51 \\ 39 & 45 \end{bmatrix}$

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Exercise 5A

Example 1

1 Matrix C is shown on the right.

$$C = \begin{bmatrix} 2 & 4 & 16 & 7 \\ 6 & 8 & 9 & 3 \\ 5 & 6 & 10 & 1 \end{bmatrix}$$

a Write down the order of the matrix C.

b State the value of:

i c_{13} **ii** c_{24} **iii** c_{31}

c Find the sum of the elements in row 3.

d Find the sum of the elements in column 2.

2 For each of the following matrices:

i state the order **ii** find the values of the required elements.

a $A = \begin{bmatrix} 5 & 6 & 8 \\ 4 & 7 & 9 \end{bmatrix}$ Find a_{12} and a_{22}

b $B = \begin{bmatrix} 6 & 8 & 2 \end{bmatrix}$ Find b_{13} and b_{11}

c $C = \begin{bmatrix} 4 & 5 \\ 3 & 1 \\ 8 & -4 \end{bmatrix}$ Find c_{32} and c_{12}

d $D = \begin{bmatrix} 8 \\ 6 \\ 9 \end{bmatrix}$ Find d_{31} and d_{11}

e $E = \begin{bmatrix} 10 & 12 \\ 15 & 13 \end{bmatrix}$ Find e_{21} and e_{12}

f $F = \begin{bmatrix} 8 & 11 & 2 & 6 \\ 4 & 1 & 5 & 7 \\ 6 & 14 & 17 & 20 \end{bmatrix}$ Find f_{34} and f_{23}

3 Name which of the matrices in Question **2** are:

a row matrices **b** column matrices **c** square matrices.

4 For matrix D, give the values of the following elements.

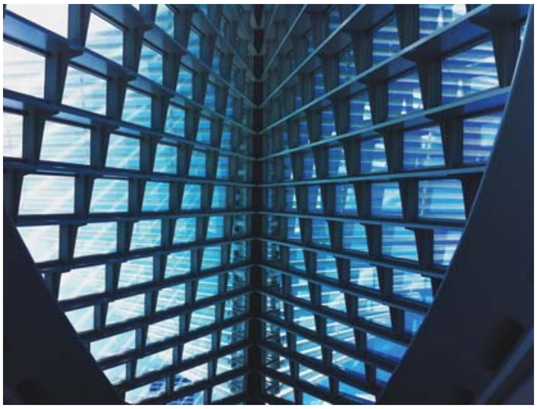
$$D = \begin{bmatrix} 3 & 4 & 6 & 11 & 2 \\ 5 & 1 & 9 & 10 & 4 \\ 8 & 7 & 2 & 0 & 1 \\ 6 & 8 & 5 & 8 & 2 \end{bmatrix}$$

a d_{23} **b** d_{45}

c d_{11} **d** d_{24}

e d_{42}

5 From Question **4**, enter the matrix D into your graphics calculator. Use it to check your answers to Question **4**.



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6 Some students were asked which of four sports they preferred to play and the results were entered in the following matrix.

$$S = \begin{array}{r} \\ \mathit{Year\ 10} \\ \mathit{Year\ 11} \\ \mathit{Year\ 12} \end{array}\begin{array}{c} \begin{array}{cccc} \mathit{Tennis} & \mathit{Basketball} & \mathit{Football} & \mathit{Hockey} \end{array} \\ \begin{bmatrix} 19 & 18 & 31 & 14 \\ 16 & 32 & 22 & 12 \\ 21 & 25 & 5 & 7 \end{bmatrix} \end{array}$$

a How many year 11 students preferred basketball?

b Write down the order of matrix S.

c What information is given by s_{23}?

7 $A = \begin{bmatrix} 2 & 7 & -1 \\ 0 & 5 & 3 \\ -3 & 4 & 8 \\ 7 & 6 & -6 \end{bmatrix}$ $B = \begin{bmatrix} 4 \\ -5 \end{bmatrix}$ $C = \begin{bmatrix} 8 & -2 \end{bmatrix}$ $D = \begin{bmatrix} 4 & -3 & 0 & 1 & 9 \\ 6 & 11 & 2 & 7 & 5 \end{bmatrix}$

a Write down the order of each matrix A, B, C and D.

b Identify the elements: $a_{32}, b_{21}, c_{11}$and d_{24} of matrices A, B, C and D respectively.

8 Matrix F shows the number of hectares of land used for different purposes on two farms, X and Y.

Row 1 represents Farm X and row 2 represents Farm Y. Columns 1, 2 and 3 show the amount of land used for wheat, cattle and sheep (W, C, S) respectively, in hectares.

$$F = \begin{array}{c} \begin{array}{ccc} W & C & S \end{array} \\ \begin{bmatrix} 150 & 300 & 75 \\ 200 & 0 & 350 \end{bmatrix} \end{array}\begin{array}{l} \\ X \\ Y \end{array}$$

a How many hectares are used on:

- **i** Farm X for sheep?
- **ii** Farm X for cattle?
- **iii** Farm Y for wheat?

b Calculate the total number of hectares used on both farms for wheat.

c Write down the information that is given by:

i f_{22} **ii** f_{13} **iii** f_{11}

d Which element of matrix F gives the number of hectares used:

- **i** on Farm Y for sheep?
- **ii** on Farm X for cattle?
- **iii** on Farm Y for wheat?

e State the order of matrix F.

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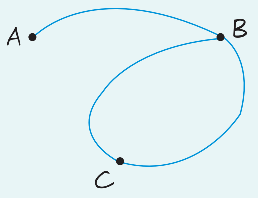
5B Using matrices to model (represent) practical situations

A **network** diagram of points (vertices) joined by lines (edges). It can be used to show connections or relationships. The information in network diagrams can be recorded in a matrix and used to solve related problems.

Example 2 Using a matrix to represent connections

The network diagram drawn shows the ways to travel between three towns, A, B and C.



a Use a matrix to represent the connections. Each element should describe the number of ways to travel *directly* from one town to another.

b What information is given by the sum of the second column of the matrix?

Solution

a As there are three towns, A, B and C, use a 3×3 matrix to show the direct connections.

There are 0 roads directly connecting any town to itself. So enter 0 where column A crosses row A, and so on.

$$\begin{array}{cc} \begin{matrix} A & B & C \end{matrix} & \\ \begin{bmatrix} 0 & & \\ & 0 & \\ & & 0 \end{bmatrix} & \begin{matrix} A \\ B \\ C \end{matrix} \end{array}$$

If there were a road directly connecting town A to itself, it would be a loop from A back to A.

There is one road directly connecting B to A (or A to B). So enter 1 where column B crosses row A and where column A crosses row B.

$$\begin{array}{cc} \begin{matrix} A & B & C \end{matrix} & \\ \begin{bmatrix} 0 & 1 & \\ 1 & 0 & \\ & & 0 \end{bmatrix} & \begin{matrix} A \\ B \\ C \end{matrix} \end{array}$$

There are no direct roads between C and A. So enter 0 where column C crosses row A and where column A crosses row C.

$$\begin{array}{cc} \begin{matrix} A & B & C \end{matrix} & \\ \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & \\ 0 & & 0 \end{bmatrix} & \begin{matrix} A \\ B \\ C \end{matrix} \end{array}$$

There are 2 roads between C and B. Enter 2 where column C crosses row B and where column B crosses row C.

$$\begin{array}{cc} \begin{matrix} A & B & C \end{matrix} & \\ \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix} & \begin{matrix} A \\ B \\ C \end{matrix} \end{array}$$

b The second column shows the number of roads directly connected to town B.

The sum of the second column is the total number of roads directly connected to town B.

$1 + 0 + 2 = 3$

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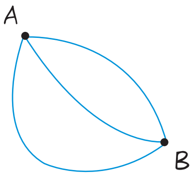
Exercise 5B

Example 2

1 The road network show roads connecting towns.

a In each case use a matrix to record the number of ways of travelling *directly* from one town to another.

i



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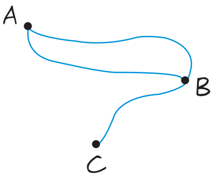
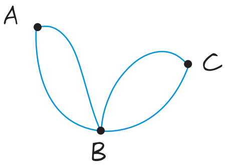


iii



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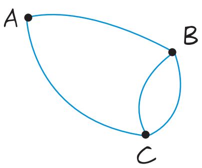

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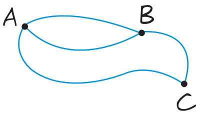

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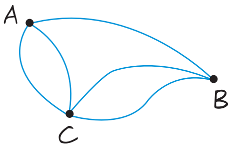


b What does the sum of the second column of each matrix represent?

2 The matrices record the number of ways of going directly from one town to another.

a In each case draw graphs to show the direct connections between towns A, B and C.

i

$$\begin{array}{cccc} A & B & C & \\ \left[\begin{matrix} 0 \\ 1 \\ 1 \end{matrix}\right. & \begin{matrix} 1 \\ 0 \\ 0 \end{matrix} & \left.\begin{matrix} 1 \\ 0 \\ 0 \end{matrix}\right] & \begin{matrix} A \\ B \\ C \end{matrix} \end{array}$$

ii

$$\begin{array}{cccc} A & B & C & \\ \left[\begin{matrix} 0 \\ 1 \\ 1 \end{matrix}\right. & \begin{matrix} 1 \\ 0 \\ 1 \end{matrix} & \left.\begin{matrix} 1 \\ 1 \\ 0 \end{matrix}\right] & \begin{matrix} A \\ B \\ C \end{matrix} \end{array}$$

iii

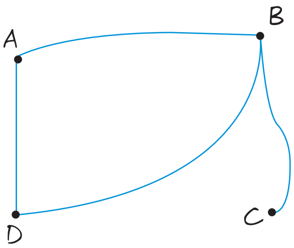
$$\begin{array}{cccc} A & B & C & \\ \left[\begin{matrix} 0 \\ 1 \\ 2 \end{matrix}\right. & \begin{matrix} 1 \\ 0 \\ 0 \end{matrix} & \left.\begin{matrix} 2 \\ 0 \\ 0 \end{matrix}\right] & \begin{matrix} A \\ B \\ C \end{matrix} \end{array}$$

iv

$$\begin{array}{cccc} A & B & C & \\ \left[\begin{matrix} 0 \\ 2 \\ 2 \end{matrix}\right. & \begin{matrix} 2 \\ 0 \\ 0 \end{matrix} & \left.\begin{matrix} 2 \\ 0 \\ 0 \end{matrix}\right] & \begin{matrix} A \\ B \\ C \end{matrix} \end{array}$$

b State the information that is given by the sum of the first column in the matrices of part **a**.

3 The network diagram opposite has lines showing which people from the four people A, B, C and D have met.



a Represent the graph using a matrix. Use 0 when two people have *not* met and 1 when they have met.

b How can the matrix be used to tell who has met the most people?

c Who has met the most people?

d Who has met the least number of people?

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5C Adding and subtracting matrices

Rules for adding and subtracting matrices

1. Matrices are added by adding the elements that are in the same positions.
2. Matrices are subtracted by subtracting the elements that are in the same positions.
3. **Matrix addition and subtraction** can only be done if the two matrices have the *same order*.

Example 3 Adding and subtracting of matrices

Complete the following addition and subtraction of matrices.

a $\begin{bmatrix} 2 & 4 \\ 5 & 1 \end{bmatrix} + \begin{bmatrix} 9 & 8 \\ 9 & -1 \end{bmatrix}$

b $\begin{bmatrix} 7 & 3 \\ 2 & 8 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 4 & 2 \\ -1 & 9 \\ 3 & 7 \end{bmatrix}$

Solution

a **1** Write the addition.

$$\begin{bmatrix} 2 & 4 \\ 5 & 1 \end{bmatrix} + \begin{bmatrix} 9 & 8 \\ 9 & -1 \end{bmatrix}$$

2 Add the elements that are in the same positions.

$$= \begin{bmatrix} 2+9 & 4+8 \\ 5+9 & 1+(-1) \end{bmatrix}$$

3 Evaluate each element.

$$= \begin{bmatrix} 11 & 12 \\ 14 & 0 \end{bmatrix}$$

b **1** Write the subtraction.

$$\begin{bmatrix} 7 & 3 \\ 2 & 8 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 4 & 2 \\ -1 & 9 \\ 3 & 7 \end{bmatrix}$$

2 Subtract the elements that are in the same positions.

$$= \begin{bmatrix} 7-4 & 3-2 \\ 2-(-1) & 8-9 \\ 1-3 & 0-7 \end{bmatrix}$$

3 Evaluate each element.

$$= \begin{bmatrix} 3 & 1 \\ 3 & -1 \\ -2 & -7 \end{bmatrix}$$

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▶ The zero matrix, 0

In a **zero matrix** every element is zero.

The following are examples of zero matrices.

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Just as in arithmetic with ordinary numbers, adding or subtracting a zero matrix does not make any change to the original matrix. For example:

$$\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} \qquad \begin{bmatrix} 5 \\ 6 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

Also, subtracting any matrix from itself gives a zero matrix. For example:

$$\begin{bmatrix} 9 & 4 & 8 \end{bmatrix} - \begin{bmatrix} 9 & 4 & 8 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$$

Exercise 5C

Skillsheet

Example 3

1 **a** $\begin{bmatrix} 4 & 3 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 5 & 7 \\ 6 & 1 \end{bmatrix}$ **b** $\begin{bmatrix} 8 & 6 \\ 9 & 3 \end{bmatrix} + \begin{bmatrix} -1 & 2 \\ 4 & 0 \end{bmatrix}$ **c** $\begin{bmatrix} 3 & 5 \\ 7 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

d $\begin{bmatrix} 9 \\ 8 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ **e** $\begin{bmatrix} 8 & 6 \\ 2 & 9 \end{bmatrix} - \begin{bmatrix} 5 & 2 \\ 1 & 3 \end{bmatrix}$ **f** $\begin{bmatrix} 7 & 4 \\ 5 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 6 \\ 2 & -8 \end{bmatrix}$

g $\begin{bmatrix} 4 & 2 \end{bmatrix} + \begin{bmatrix} 8 & 5 \end{bmatrix}$ **h** $\begin{bmatrix} 7 & -5 \end{bmatrix} - \begin{bmatrix} 7 & -5 \end{bmatrix}$ **i** $\begin{bmatrix} 4 & -3 \end{bmatrix} + \begin{bmatrix} -4 & 3 \end{bmatrix}$

j $\begin{bmatrix} 4 & -3 & 2 & -1 \end{bmatrix} - \begin{bmatrix} 6 & -5 & -1 & 8 \end{bmatrix}$

2 Using the matrices given:

$$A = \begin{bmatrix} 3 & -2 \\ 2 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 5 & 7 \\ 1 & 3 \end{bmatrix} \quad C = \begin{bmatrix} 6 & 2 \\ 1 & 0 \\ 3 & -8 \end{bmatrix} \quad D = \begin{bmatrix} -3 & 5 \\ 4 & -2 \\ 1 & 7 \end{bmatrix} \quad E = \begin{bmatrix} 8 \\ 3 \end{bmatrix}$$

find, where possible:

a $A + B$ **b** $B + A$ **c** $A - B$ **d** $B - A$

e $B + E$ **f** $C + D$ **g** $B + C$ **h** $D - C$

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3 Two people shared the work of a telephone poll surveying voting intentions. The results for each person's survey are given in matrix form.

Sample 1:

$$\begin{array}{c} \\ \textit{Men} \\ \textit{Women} \end{array} \begin{array}{c} \begin{array}{cccc} \textit{Liberal} & \textit{Labor} & \textit{Democrat} & \textit{Green} \end{array} \\ \begin{bmatrix} 19 & 21 & 7 & 3 \\ 18 & 17 & 11 & 4 \end{bmatrix} \end{array}$$

Sample 2:

$$\begin{array}{c} \\ \textit{Men} \\ \textit{Women} \end{array} \begin{array}{c} \begin{array}{cccc} \textit{Liberal} & \textit{Labor} & \textit{Democrat} & \textit{Green} \end{array} \\ \begin{bmatrix} 24 & 21 & 3 & 2 \\ 19 & 20 & 6 & 5 \end{bmatrix} \end{array}$$

Write a matrix showing the overall result of the survey.

4 The weights and heights of four people were recorded and then checked again one year later.

2004 results:

$$\begin{array}{c} \\ \textit{Weight (kg)} \\ \textit{Height (cm)} \end{array} \begin{array}{c} \begin{array}{cccc} \textit{Aida} & \textit{Bianca} & \textit{Chloe} & \textit{Donna} \end{array} \\ \begin{bmatrix} 32 & 44 & 59 & 56 \\ 145 & 155 & 160 & 164 \end{bmatrix} \end{array}$$

2005 results:

$$\begin{array}{c} \\ \textit{Weight (kg)} \\ \textit{Height (cm)} \end{array} \begin{array}{c} \begin{array}{cccc} \textit{Aida} & \textit{Bianca} & \textit{Chloe} & \textit{Donna} \end{array} \\ \begin{bmatrix} 38 & 52 & 57 & 63 \\ 150 & 163 & 167 & 170 \end{bmatrix} \end{array}$$

a Write the matrix that gives the changes in each person's weight and height after one year.

b Who gained the most weight?

c Which person had the greatest height increase?

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5D Scalar multiplication

A *scalar* is just a number. Multiplying a matrix by a number is called **scalar multiplication**.

Multiplying a matrix by a scalar

Scalar multiplication is the process of multiplying a matrix by a number (a scalar).

In scalar multiplication each element is multiplied by that scalar (number).

The following is an example of scalar multiplication of a matrix.

$$5\begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 5\times 1 & 5\times 2 \\ 5\times 2 & 5\times 0 \end{bmatrix} = \begin{bmatrix} 5 & 10 \\ 10 & 0 \end{bmatrix}$$

Example 4 Scalar multiplication

If $A = \begin{bmatrix} 5 & 1 \\ -3 & 0 \end{bmatrix}$, find $3A$.

Solution

1 If $A = \begin{bmatrix} 5 & 1 \\ -3 & 0 \end{bmatrix}$, then $3A = 3\begin{bmatrix} 5 & 1 \\ -3 & 0 \end{bmatrix}$.

$$3A = 3\begin{bmatrix} 5 & 1 \\ -3 & 0 \end{bmatrix}$$

2 Multiply each number in the matrix by 3.

$$= \begin{bmatrix} 3\times 5 & 3\times 1 \\ 3\times -3 & 3\times 0 \end{bmatrix}$$

3 Evaluate each element.

$$= \begin{bmatrix} 15 & 3 \\ -9 & 0 \end{bmatrix}$$

Scalar multiplication has many practical applications. It is particularly useful in scaling up the elements of a matrix, for example, add the GST to the cost of the prices of all items in a shop by multiplying a matrix of prices by 1.1.

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Example 5 Application of scalar multiplication

A gymnasium has the enrolments in courses shown in this matrix.

$$\begin{array}{c} \\ \textit{Men} \\ \textit{Women} \end{array} \begin{array}{c} \begin{array}{ccc} \textit{Body building} & \textit{Aerobics} & \textit{Fitness} \end{array} \\ \begin{bmatrix} 70 & 20 & 80 \\ 10 & 50 & 60 \end{bmatrix} \end{array}$$

The manager wishes to double the enrolments in each course. Show this in a matrix.

Solution

1 Each element in the matrix is multiplied by 2.

$$2 \times \begin{bmatrix} 70 & 20 & 80 \\ 10 & 50 & 60 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \times 70 & 2 \times 20 & 2 \times 80 \\ 2 \times 10 & 2 \times 50 & 2 \times 60 \end{bmatrix}$$

2 Evaluate each element.

$$= \begin{array}{c} \\ \text{Men} \\ \text{Women} \end{array} \begin{array}{c} \begin{array}{ccc} \text{Body building} & \text{Aerobics} & \text{Fitness} \end{array} \\ \begin{bmatrix} 140 & 40 & 160 \\ 20 & 100 & 120 \end{bmatrix} \end{array}$$

Scalar multiplication can also be used in conjunction with addition and subtraction of matrices.

Example 6 Scalar multiplication and subtraction of matrices

If $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, find the matrix equal to $2A - 3B$.

Solution

1 Write $2A - 3B$ in expanded matrix form.

$$2A - 3B = 2 \times \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} - 3 \times \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

2 Multiply the elements in A by 2 and the elements in B by 3.

$$= \begin{bmatrix} 2 & 2 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix}$$

3 Subtract the elements in corresponding positions.

$$= \begin{bmatrix} 2 - 0 & 2 - 3 \\ 0 - 3 & 2 - 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$$

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How to add, subtract and scalar multiply matrices using the TI-Nspire CAS

If $A = \begin{bmatrix} 2 & -3 \\ 5 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 6 \\ 1 & -2 \end{bmatrix}$, find:

a $A + B$ **b** $A - B$ **c** $9A$ **d** $15A - 11B$

Steps

1 Press ⌂on**>New Document>Add Calculator**.

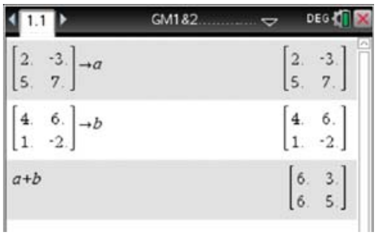
2 Enter the matrices A and B into your calculator.

Note: refer to page 220 if you are unsure how to enter a matrix into your calculator.

$$\begin{bmatrix} 2. & -3. \\ 5. & 7. \end{bmatrix} \to a \qquad \begin{bmatrix} 2. & -3. \\ 5. & 7. \end{bmatrix}$$

$$\begin{bmatrix} 4. & 6. \\ 1. & -2. \end{bmatrix} \to b \qquad \begin{bmatrix} 4. & 6. \\ 1. & -2. \end{bmatrix}$$

a To calculate $A + B$, type $A + B$ and then press enter to evaluate.



$$\begin{bmatrix} 2. & -3. \\ 5. & 7. \end{bmatrix} \to a \qquad \begin{bmatrix} 2. & -3. \\ 5. & 7. \end{bmatrix}$$

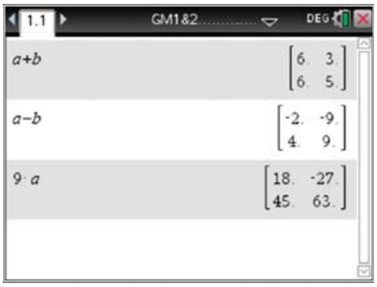
$$\begin{bmatrix} 4. & 6. \\ 1. & -2. \end{bmatrix} \to b \qquad \begin{bmatrix} 4. & 6. \\ 1. & -2. \end{bmatrix}$$

$$a+b \qquad \begin{bmatrix} 6. & 3. \\ 6. & 5. \end{bmatrix}$$

$$A + B = \begin{bmatrix} 6 & 3 \\ 6 & 5 \end{bmatrix}$$

b To calculate $A - B$, type A – B and then press enter to evaluate.

c To calculate $9A$, type $9A$ and then press enter to evaluate.



$$a+b \qquad \begin{bmatrix} 6. & 3. \\ 6. & 5. \end{bmatrix}$$

$$a-b \qquad \begin{bmatrix} -2. & -9. \\ 4. & 9. \end{bmatrix}$$

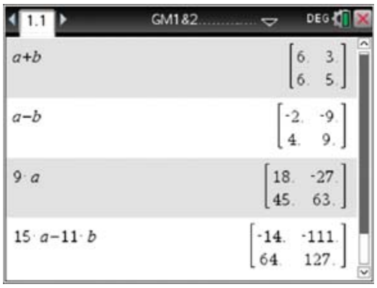
$$9 \cdot a \qquad \begin{bmatrix} 18. & -27. \\ 45. & 63. \end{bmatrix}$$

$$A - B = \begin{bmatrix} -2 & -9 \\ 4 & 9 \end{bmatrix}$$

$$9A = \begin{bmatrix} 18 & -27 \\ 45 & 63 \end{bmatrix}$$

d To calculate $15A - 11B$, type $15A - 11B$ and then press enter to evaluate.

$$15A - 11B = \begin{bmatrix} -14 & -111 \\ 64 & 127 \end{bmatrix}$$



$$a+b \qquad \begin{bmatrix} 6. & 3. \\ 6. & 5. \end{bmatrix}$$

$$a-b \qquad \begin{bmatrix} -2. & -9. \\ 4. & 9. \end{bmatrix}$$

$$9 \cdot a \qquad \begin{bmatrix} 18. & -27. \\ 45. & 63. \end{bmatrix}$$

$$15 \cdot a-11 \cdot b \qquad \begin{bmatrix} -14. & -111. \\ 64. & 127. \end{bmatrix}$$

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How to add, subtract and scalar multiply matrices using the ClassPad

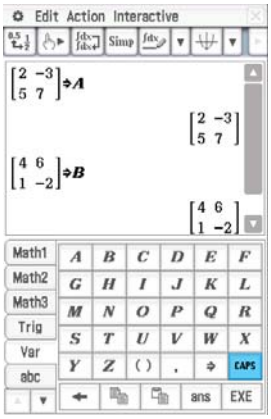
If $A = \begin{bmatrix}2 & -3\\5 & 7\end{bmatrix}$ and $B = \begin{bmatrix}4 & 6\\1 & -2\end{bmatrix}$, find:

a $A + B$ **b** $A - B$ **c** $9A$ **d** $15A - 11B$

Steps

1 Enter the matrices A and B into your calculator.

Note: Refer to page 221 if you are unsure how to enter a matrix into your calculator.



$\begin{bmatrix}2 & -3\\5 & 7\end{bmatrix}\Rightarrow A$

$\begin{bmatrix}2 & -3\\5 & 7\end{bmatrix}$

$\begin{bmatrix}4 & 6\\1 & -2\end{bmatrix}\Rightarrow B$

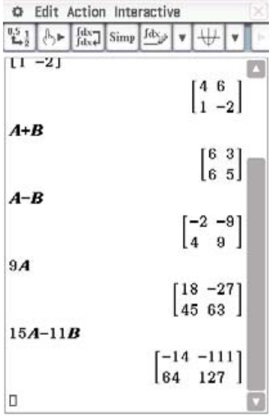
$\begin{bmatrix}4 & 6\\1 & -2\end{bmatrix}$

a To calculate $A + B$, type $\boldsymbol{A} + \boldsymbol{B}$ and then press EXE to evaluate.

b To calculate $A - B$, type $\boldsymbol{A} - \boldsymbol{B}$ and then press EXE to evaluate.

c To calculate $9A$, type **9*A*** and then press EXE to evaluate.

d To calculate $15A - 11B$, type **15*A* − 11*B*** and then press EXE to evaluate.



$\begin{bmatrix}4 & 6\\1 & -2\end{bmatrix}$

$A+B$

$\begin{bmatrix}6 & 3\\6 & 5\end{bmatrix}$

$A-B$

$\begin{bmatrix}-2 & -9\\4 & 9\end{bmatrix}$

$9A$

$\begin{bmatrix}18 & -27\\45 & 63\end{bmatrix}$

$15A-11B$

$\begin{bmatrix}-14 & -111\\64 & 127\end{bmatrix}$

$$A + B = \begin{bmatrix}6 & 3\\6 & 5\end{bmatrix}$$

$$A - B = \begin{bmatrix}-2 & -9\\4 & 9\end{bmatrix}$$

$$9A = \begin{bmatrix}18 & -27\\45 & 63\end{bmatrix}$$

$$15A - 11B = \begin{bmatrix}-14 & -111\\64 & 127\end{bmatrix}$$

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Exercise 5D

Example 4

1 Calculate the values of the following.

a $2\begin{bmatrix}7 & -1\\4 & 9\end{bmatrix}$ **b** $5\begin{bmatrix}0 & -2\\5 & 7\end{bmatrix}$ **c** $-4\begin{bmatrix}16 & -3\\1.5 & 3.5\end{bmatrix}$ **d** $1.5\begin{bmatrix}1.5 & 0\\-2 & 5\end{bmatrix}$

e $3\begin{bmatrix}6 & 7\end{bmatrix}$ **f** $6\begin{bmatrix}-2\\5\end{bmatrix}$ **g** $\frac{1}{2}\begin{bmatrix}4 & 6 & 0\\0 & 3 & 1\end{bmatrix}$ **h** $-1\begin{bmatrix}3 & 6 & -8\end{bmatrix}$

Example 6

2 Given the matrices:

$$A = \begin{bmatrix}3 & -4\\2 & 5\end{bmatrix} \quad B = \begin{bmatrix}7 & 6\\1 & -4\end{bmatrix} \quad C = \begin{bmatrix}-3 & 4\\-2 & -5\end{bmatrix} \quad O = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$$

find the matrix required for:

a $3A$ **b** $2B + 4C$ **c** $5A - 2B$ **d** $2O$ **e** $3B + O$

3 Enter the matrices $A = \begin{bmatrix}3 & -1\\4 & 2\end{bmatrix}$ and $B = \begin{bmatrix}-2 & 1\\0 & 5\end{bmatrix}$ into your graphics calculator and use them to evaluate:

a $17A - 14B$ **b** $29B - 21A$ **c** $9A + 7B$ **d** $3(5A - 4B)$

4 For the matrices:

$$A = \begin{bmatrix}1\\1\\0\end{bmatrix} \quad B = \begin{bmatrix}0\\1\\1\end{bmatrix} \quad C = \begin{bmatrix}0 & 1 & 1 & 1\end{bmatrix} \quad D = \begin{bmatrix}0 & 0 & 1 & 1\end{bmatrix}$$

find the matrix for:

a $3A + 4B$ **b** $5C - 2D$ **c** $2(3A + 4B)$ **d** $3(5C - 2D)$

Example 5

5 The expenses arising from costs and wages for each section of three stores, A, B and C, are shown in the Costs matrix. The Sales matrix shows the money from the sale of goods in each section of the three stores. Figures represent the nearest million dollars.

Costs:

$$\begin{array}{c}\\ A\\ B\\ C\end{array}\begin{array}{ccc}\textit{Clothing} & \textit{Furniture} & \textit{Electronics}\\ \end{array}$$

$$\begin{array}{cccc} & \textit{Clothing} & \textit{Furniture} & \textit{Electronics}\\ A & 12 & 10 & 15\\ B & 11 & 8 & 17\\ C & 15 & 14 & 7\end{array}$$

Sales:

$$\begin{array}{cccc} & \textit{Clothing} & \textit{Furniture} & \textit{Electronics}\\ A & 18 & 12 & 24\\ B & 16 & 9 & 26\\ C & 19 & 13 & 12\end{array}$$

a Write a matrix showing the profits in each section of each store.

b If 30% tax must be paid on profits, show the amount of tax that must be paid by each section of each store. No tax needs to be paid for a section that has made a loss.

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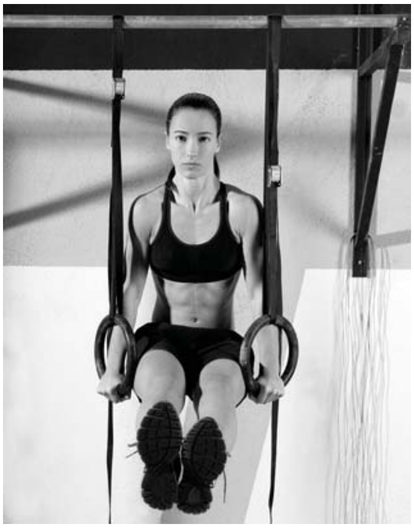
6 Zoe competed in the gymnastics rings and parallel bars events in a three-day gymnastics tournament. A win was recorded as 1 and a loss as 0. The three column matrices show the results for Saturday, Sunday and Monday.

$$\begin{array}{lccc} & \textit{Sat} & \textit{Sun} & \textit{Mon} \\ \textit{Gymnastics rings} & \left[\begin{matrix}1\\0\end{matrix}\right] & \left[\begin{matrix}1\\1\end{matrix}\right] & \left[\begin{matrix}1\\1\end{matrix}\right] \\ \textit{Parallel bars} & & & \end{array}$$

a Give a 2×1 column matrix which records her total wins for each of the two types of events.

b Zoe received \$50 for each win. Give a 2×1 matrix which records her total prize money for each of the two types of events.



5E Matrix multiplication

Matrix multiplication is the multiplication of a matrix by another matrix. Not to be confused with the scalar multiplication, which is the multiplication of a matrix by a number.

The matrix multiplication of two matrices A and B can be written as $A \times B$ or just AB. Although it is called multiplication and the symbol × may be used, matrix multiplication is not the simple multiplication of numbers but a routine involving the sum of pairs of numbers that have been multiplied.

For example, the method of matrix multiplication can be demonstrated by using a practical example. The numbers of CDs and DVDs sold by Fatima and Gaia are recorded in matrix N. The selling prices of the CDs and DVDs are shown in matrix P.

$$N = \begin{array}{l} \\ \textit{Fatima} \\ \textit{Gaia} \end{array} \begin{array}{cc} \textit{CDs} & \textit{DVDs} \\ \left[\begin{matrix} 7 \\ 5 \end{matrix}\right. & \left.\begin{matrix} 4 \\ 6 \end{matrix}\right] \end{array} \qquad P = \begin{array}{l} \\ \textit{CDs} \\ \textit{DVDs} \end{array} \begin{array}{c} \$ \\ \left[\begin{matrix} 20 \\ 30 \end{matrix}\right] \end{array}$$

We want to make a matrix, S, that shows the value of the sales made by each person.

Fatima sold: **7** CDs at **\$20** + **4** DVDs at **\$30**.
Gaia sold: **5** CDs at **\$20** + **6** DVDs at **\$30**.

$$S = \begin{array}{l} \\ \textit{Fatima} \\ \textit{Gaia} \end{array} \begin{array}{c} \$ \\ \left[\begin{matrix} 7 \times 20 + 4 \times 30 \\ 5 \times 20 + 4 \times 30 \end{matrix}\right] \end{array}$$

The steps used in this example follow the routine for the matrix multiplication of $N \times P$.

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As we move across the *first row* of matrix N we move down the *column* of matrix P, adding the products of the pairs of numbers as we go.

$$N \times P$$

$$\begin{bmatrix} 7 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 20 \\ 30 \end{bmatrix} = \begin{bmatrix} 7 \times 20 + 4 \times 30 \\ \square \quad \square \end{bmatrix}$$

Then we move across the *second row* of matrix N and down the *column* of matrix P, adding the products of the pairs of numbers as we go.

$$\begin{bmatrix} 7 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 20 \\ 30 \end{bmatrix} = \begin{bmatrix} 7 \times 20 + 4 \times 30 \\ 5 \times 20 + 6 \times 30 \end{bmatrix}$$

$$= \begin{bmatrix} 140 + 120 \\ 100 + 180 \end{bmatrix}$$

$$\begin{array}{cc} \$ & \\ = \begin{bmatrix} 260 \\ 280 \end{bmatrix} & \begin{array}{l} \textit{Fatima} \\ \textit{Gaia} \end{array} \end{array}$$

Rules for matrix multiplication

Because of the way the products are formed, the number of columns in the first matrix must equal the number of rows in the second matrix. Otherwise, we say that matrix multiplication is not defined, meaning it is not possible.

Matrix multiplication

For matrix multiplication to be defined:

Think of the orders as two railway carriages that must be the same where they meet.

order of 1st matrix order of 2nd matrix

$$\underbrace{m \times n \qquad n}_{\text{must be the same}} \times p$$

In our example of the CD and DVD sales:

order of 1st matrix order of 2nd matrix

$$2 \times \underbrace{2 \qquad 2}_{\text{the same}} \times 1$$

Notice that the outside numbers give the order of the product matrix: the matrix made by multiplying the two matrices. In our case, the answer is a 2 × 1 matrix.

Order of the product matrix

The order of the product matrix is given by:

Think: when the 'railway carriages' meet, the result has an order given by the end numbers.

order of 1st matrix order of 2nd matrix

$$\underbrace{m \times n \qquad n \times p}_{\text{order of answer}}$$

$$m \times p$$

We will check that these two important rules hold in the examples that follow.

Methods of matrix multiplication

Some people like to think of the matrix multiplication of $A \times B$ using a *run and dive* description.

Matrix multiplication of $A \times B$

The *run and dive* description of matrix multiplication is to add the products of the pairs made as you:

- *run* along the first row of A and *dive* down the first column of B
- repeat running along the first row of A and diving down the next column of B until all columns of B have been used
- now start running along the next row of A and repeat diving down each column of B, entering your results in a new row
- repeat this routine until all rows of A have been used.

This procedure can be very tedious and error prone, so we will only do simple cases by hand so that you understand the process. Then a graphics calculator will be used to do matrix multiplication.

Example 7 Matrix multiplication

For the following matrices: $A = \begin{bmatrix} 5 & 2 \\ 4 & 6 \\ 1 & 3 \end{bmatrix}$ $B = \begin{bmatrix} 8 \\ 9 \end{bmatrix}$ $C = \begin{bmatrix} 2 & 4 & 7 \end{bmatrix}$ $D = \begin{bmatrix} 8 \\ 6 \\ 5 \end{bmatrix}$

- decide whether the matrix multiplication in each question below is defined
- if matrix multiplication is defined, give the order of the answer matrix and then do the matrix multiplication.

a AB **b** BA **c** CD

Solution

a AB

1 Write the order of each matrix.

$$\begin{array}{cc} A & B \\ 3 \times 2 & 2 \times 1 \end{array}$$

2 The inside numbers are the same.

Matrix multiplication is defined for $A \times B$.

3 The outside numbers give the order of $A \times B$.

The order of the product AB is 3×1.

4 Move across the first row of A and down the column of B, adding the products of the pairs.

$$\begin{bmatrix} 5 & 2 \\ 4 & 6 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 8 \\ 9 \end{bmatrix} = \begin{bmatrix} 5 \times 8 + 2 \times 9 \\ \\ \\ \end{bmatrix}$$

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5 Move across the second row of *A* and down the column of *B*, adding the products of the pairs.

$$\begin{bmatrix} 5 & 2 \\ 4 & 6 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 8 \\ 9 \end{bmatrix} = \begin{bmatrix} 5 \times 8 + 2 \times 9 \\ 4 \times 8 + 6 \times 9 \end{bmatrix}$$

6 Move across the third row of *A* and down the column of *B*, adding the products of the pairs.

$$\begin{bmatrix} 5 & 2 \\ 4 & 6 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 8 \\ 9 \end{bmatrix} = \begin{bmatrix} 5 \times 8 + 2 \times 9 \\ 4 \times 8 + 6 \times 9 \\ 1 \times 8 + 3 \times 9 \end{bmatrix}$$

7 Tidy up by doing some arithmetic.

$$= \begin{bmatrix} 40 + 18 \\ 32 + 54 \\ 8 + 27 \end{bmatrix}$$

8 Write your answer.

$$\text{So } A \times B = \begin{bmatrix} 58 \\ 86 \\ 35 \end{bmatrix}$$

b *BA*

1 Write the order of each matrix.

$$\begin{array}{cc} B & A \\ 2 \times 1 & 3 \times 2 \end{array}$$

2 Are the inside numbers the same? No.

Multiplication is not defined for $B \times A$.

c *CD*

1 Write the order of each matrix.

$$\begin{array}{cc} C & D \\ 1 \times 3 & 3 \times 1 \end{array}$$

2 Are the inside numbers the same? Yes.

Multiplication is defined for $C \times D$.

3 The outside numbers give the order of $C \times D$.

The order of the product *CD* is 1×1.

4 Move across the row of *C* and down the column of *D*, adding the products of the pairs.

$$\begin{bmatrix} 2 & 4 & 7 \end{bmatrix}\begin{bmatrix} 8 \\ 6 \\ 5 \end{bmatrix} = \begin{bmatrix} 2 \times 8 + 4 \times 6 + 7 \times 5 \end{bmatrix}$$

5 Tidy up by doing some arithmetic.

$$= [16 + 24 + 35]$$

6 Write your answer.

$$\text{So } C \times D = [75]$$

In the previous example, $AB \neq BA$. Usually, when we reverse (*commute*) the order of the matrices in matrix multiplication, we get a different answer. This differs from ordinary arithmetic, where multiplication gives the same answer when the terms are commuted, for example, $3 \times 4 = 4 \times 3$.

Matrix multiplication

In general, matrix multiplication is not commutative. That is: $AB \neq BA$

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How to multiply two matrices using the TI-Nspire CAS

If $C = \begin{bmatrix} 11 & 5 \\ 10 & 9 \end{bmatrix}$ and $D = \begin{bmatrix} 6 \\ 1 \end{bmatrix}$, find the matrix CD.

Steps

1 ⌂on >**New Document>Add Calculator**.

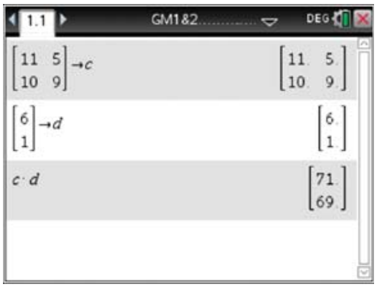
2 Enter the matrices C and D into your calculator.

3 To calculate matrix CD, type in c × d. Press enter to evaluate.

Note: you must put a multiplication sign between the c and d.

Check: C has dimension 2 × 2 and B has dimension 2 × 1. So, matrix CD should be a 2 × 1 matrix, as it is.

4 Write your answer.



$\begin{bmatrix} 11 & 5 \\ 10 & 9 \end{bmatrix} \to c$ $\qquad \begin{bmatrix} 11. & 5. \\ 10. & 9. \end{bmatrix}$

$\begin{bmatrix} 6 \\ 1 \end{bmatrix} \to d$ $\qquad \begin{bmatrix} 6. \\ 1. \end{bmatrix}$

$c \cdot d$ $\qquad \begin{bmatrix} 71. \\ 69. \end{bmatrix}$

$$CD = \begin{bmatrix} 71 \\ 69 \end{bmatrix}$$

How to multiply two matrices using the ClassPad

Find $A \times B$: $A = \begin{bmatrix} 6 & 2 & 5 \\ 4 & 8 & 1 \end{bmatrix}$ $\quad B = \begin{bmatrix} 7 & 4 & 1 & 8 \\ -2 & 0 & 3 & -5 \\ 9 & 2 & 0 & -1 \end{bmatrix}$

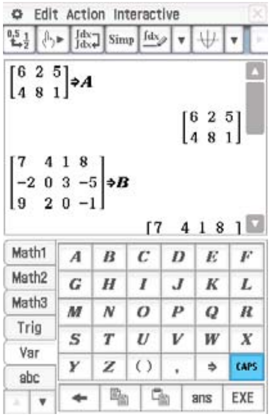
Steps

1 Enter the matrices A and B into your calculator.

Note: Refer to page 221 if you are unsure how to enter a matrix into your calculator.

2 To calculate $A \times B$, type $\boldsymbol{A} \times \boldsymbol{B}$ or $\boldsymbol{AB}$ and then press EXE to evaluate.

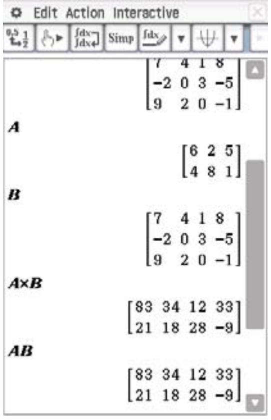
3 *Check:* A has dimensions 2×3 and B has dimensions 3×4. So, matrix AB should be a 2×4 matrix, which it is.



$\begin{bmatrix} 6 & 2 & 5 \\ 4 & 8 & 1 \end{bmatrix} \Rightarrow A$

$\begin{bmatrix} 6 & 2 & 5 \\ 4 & 8 & 1 \end{bmatrix}$

$\begin{bmatrix} 7 & 4 & 1 & 8 \\ -2 & 0 & 3 & -5 \\ 9 & 2 & 0 & -1 \end{bmatrix} \Rightarrow B$



$\begin{bmatrix} 7 & 4 & 1 & 8 \\ -2 & 0 & 3 & -5 \\ 9 & 2 & 0 & -1 \end{bmatrix}$

A

$\begin{bmatrix} 6 & 2 & 5 \\ 4 & 8 & 1 \end{bmatrix}$

B

$\begin{bmatrix} 7 & 4 & 1 & 8 \\ -2 & 0 & 3 & -5 \\ 9 & 2 & 0 & -1 \end{bmatrix}$

$A \times B$

$\begin{bmatrix} 83 & 34 & 12 & 33 \\ 21 & 18 & 28 & -9 \end{bmatrix}$

AB

$\begin{bmatrix} 83 & 34 & 12 & 33 \\ 21 & 18 & 28 & -9 \end{bmatrix}$

4 Write your answer.

$$AB = \begin{bmatrix} 83 & 34 & 12 & 33 \\ 21 & 18 & 28 & -9 \end{bmatrix}$$

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Exercise 5E

Example 7

1 For the following matrices: $A = \begin{bmatrix} 6 & 2 \\ 3 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$ $C = \begin{bmatrix} 1 & 3 \\ 0 & 8 \\ 2 & -5 \end{bmatrix}$

- decide whether the matrix multiplication in each question below is defined
- if matrix multiplication is defined, give the order of the answer matrix and then do the matrix multiplication.

a AB **b** BA **c** CB **d** BC

e AA **f** BB **g** AC **h** CA

2 Write the orders of each pair of matrices and decide if matrix multiplication is defined. If matrix multiplication is defined, find the answer.

a $\begin{bmatrix} 3 & 4 \end{bmatrix}\begin{bmatrix} 6 \\ 5 \end{bmatrix}$ **b** $\begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ **c** $\begin{bmatrix} 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

d $\begin{bmatrix} 8 & -2 & -1 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ **e** $\begin{bmatrix} 1 & 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$ **f** $\begin{bmatrix} 1 & 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

3 For the matrices: $A = \begin{bmatrix} 2 & 3 \end{bmatrix}$ $B = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$ find:

a **i** $3A$ **ii** $5A$ **iii** $8A$ **iv** $3A + 5A$

b **i** $6B$ **ii** $6B + B$ **iii** $7B$

c **i** $2A$ **ii** $3B$ **iii** AB **iv** $2A \times 3B$ **v** $6AB$

4 Do these matrix multiplications without using a graphics calculator.

a $\begin{bmatrix} 4 & 2 \\ 1 & 6 \end{bmatrix}\begin{bmatrix} 3 \\ 5 \end{bmatrix}$ **b** $\begin{bmatrix} 8 & 4 \\ 5 & 9 \end{bmatrix}\begin{bmatrix} 7 \\ 2 \end{bmatrix}$ **c** $\begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 4 & 2 \\ 6 & 7 \end{bmatrix}$

d $\begin{bmatrix} 3 & 5 \\ 1 & 8 \end{bmatrix}\begin{bmatrix} 2 & 4 \\ -2 & -3 \end{bmatrix}$ **e** $\begin{bmatrix} 8 & 3 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ **f** $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 9 & 6 \\ 7 & 8 \end{bmatrix}$

g $\begin{bmatrix} 2 & 5 \\ 4 & 1 \\ 0 & 7 \end{bmatrix}\begin{bmatrix} 8 \\ 3 \end{bmatrix}$ **h** $\begin{bmatrix} 7 & 4 \\ 1 & 0 \\ 6 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ **i** $\begin{bmatrix} 5 & 8 \end{bmatrix}\begin{bmatrix} 7 \\ 6 \end{bmatrix}$

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j $\begin{bmatrix} 3 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 9 \end{bmatrix}$ **k** $\begin{bmatrix} 4 & 1 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 6 \\ 1 \end{bmatrix}$ **l** $\begin{bmatrix} 1 & 0 & 3 \end{bmatrix}\begin{bmatrix} 4 \\ 5 \\ 0 \end{bmatrix}$

m $\begin{bmatrix} 6 & 2 \end{bmatrix}\begin{bmatrix} 4 \\ 3 \end{bmatrix}$ **n** $\begin{bmatrix} 3 & 2 & 1 \end{bmatrix}\begin{bmatrix} 6 \\ 5 \\ 8 \end{bmatrix}$ **o** $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}$

5 Use your graphics calculator to do the matrix multiplications in Question **4**.

6 $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$

a Find AB. **b** Find BA. **c** Does $AB = BA$?

7 Use these matrices to find the required products.

$$C = \begin{bmatrix} 9 & 8 \\ 7 & 6 \end{bmatrix} \quad D = \begin{bmatrix} 8 & 6 \\ 4 & 2 \end{bmatrix} \quad E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad F = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

a CD **b** CE **c** CF **d** DE **e** DF

8 Perform the following matrix multiplications using your graphics calculator.

a $\begin{bmatrix} 6 & 8 & 12 \\ 14 & 17 & 11 \end{bmatrix}\begin{bmatrix} 26 & 9 & 21 & 6 \\ 8 & -7 & -4 & 9 \\ 13 & 10 & 5 & 26 \end{bmatrix}$ **b** $\begin{bmatrix} 15 & 9 & 23 & 72 \end{bmatrix}\begin{bmatrix} -6 \\ 22 \\ -8 \\ 19 \end{bmatrix}$

c $\begin{bmatrix} 16 \\ 10 \\ 24 \\ -18 \end{bmatrix}\begin{bmatrix} -31 & 47 & 61 & -14 \end{bmatrix}$ **d** $\begin{bmatrix} 8 & -7 & 9 \\ 6 & 11 & 14 \\ 3 & 21 & -5 \end{bmatrix}\begin{bmatrix} 8 & -19 & 24 \\ 33 & 16 & 19 \\ 4 & 0 & 13 \end{bmatrix}$

9 Noting that $A^2 = A \times A, A^3 = A \times A \times A$, etc., calculate:

i A^2 **ii** A^3 **iii** A^4

for each of the following matrices.

a $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ **b** $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ **c** $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

d $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ **e** $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

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5F Applications of matrices

Data represented in matrix form can be multiplied to produce new useful information.

Example 8 Business application of matrices

Fatima and Gaia's store has a special sales promotion. One free cinema ticket is given with each DVD purchased. Two cinema tickets are given with the purchase of each computer game.

The number of DVDs and games sold by Fatima and Gaia are given in matrix S.

The selling price of a DVD and a game, together with the number of free tickets is given by matrix P.

$$S = \begin{array}{c} \\ \mathit{Fatima} \\ \mathit{Gaia} \end{array} \begin{array}{c} \begin{array}{cc} \mathit{DVDs} & \mathit{Games} \end{array} \\ \begin{bmatrix} 7 & 4 \\ 5 & 6 \end{bmatrix} \end{array} \qquad P = \begin{array}{c} \\ \mathit{DVDs} \\ \mathit{Games} \end{array} \begin{array}{c} \begin{array}{cc} \$ & \mathit{Tickets} \end{array} \\ \begin{bmatrix} 20 & 1 \\ 30 & 2 \end{bmatrix} \end{array}$$

From the matrix product $S \times P$ and interpret.

Solution

1 Complete the matrix multiplication, $S \times P$.

$$\begin{array}{c} \\ \text{Fatima} \\ \text{Gaia} \end{array} \begin{array}{c} \begin{array}{cc} \text{DVDs} & \text{Games} \end{array} \\ \begin{bmatrix} 7 & 4 \\ 5 & 6 \end{bmatrix} \end{array} \times \begin{array}{c} \\ \text{DVDs} \\ \text{Games} \end{array} \begin{array}{c} \begin{array}{cc} \$ & \text{Tickets} \end{array} \\ \begin{bmatrix} 20 & 1 \\ 30 & 2 \end{bmatrix} \end{array}$$

$$= \begin{array}{c} \\ \text{Fatima} \\ \text{Gaia} \end{array} \begin{array}{c} \begin{array}{cc} \$ & \text{Tickets} \end{array} \\ \begin{bmatrix} 7 \times 20 + 4 \times 30 & 7 \times 1 + 4 \times 2 \\ 5 \times 20 + 6 \times 30 & 5 \times 1 + 6 \times 2 \end{bmatrix} \end{array}$$

$$= \begin{array}{c} \\ \text{Fatima} \\ \text{Gaia} \end{array} \begin{array}{c} \begin{array}{cc} \$ & \text{Tickets} \end{array} \\ \begin{bmatrix} 260 & 15 \\ 280 & 17 \end{bmatrix} \end{array}$$

2 Interpret the matrix.

Fatima had sales of \$260 and gave out 15 tickets.

Gaia had sales of \$280 and gave out 17 tickets.

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Properties of row and column matrices

Row and column matrices provide efficient ways of extracting information from data stored in large matrices. Matrices of a convenient size will be used to explore some of the surprising and useful properties of row and column matrices.

Example 9 Using row and column matrices to extract information

Three rangers completed their monthly park surveys of feral animal sightings in the matrix S.

$$S = \begin{array}{r|cccc|} & cats & dogs & foxes & rabbits \\ Aaron & 27 & 9 & 34 & 59 \\ Barra & 18 & 15 & 10 & 89 \\ Chloe & 35 & 6 & 46 & 29 \end{array} \qquad A = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

a Evaluate $S \times B$.

b What information about matrix S is given in the product $S \times B$?

c Evaluate $A \times S$.

d What information about matrix S is given in the product $A \times S$?

Solution

a Matrix multiplication of a 3×4 and a 4×1 matrix produces a 3×1 matrix.

$$S \times B = \begin{bmatrix} 27 & 9 & 34 & 59 \\ 18 & 15 & 10 & 89 \\ 35 & 6 & 46 & 29 \end{bmatrix} \times \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

b Look at the second last step in the working of $S \times B$.

$$= \begin{bmatrix} 27 + 9 + 34 + 59 \\ 18 + 15 + 10 + 89 \\ 35 + 6 + 46 + 29 \end{bmatrix} = \begin{bmatrix} 129 \\ 132 \\ 116 \end{bmatrix}$$

Each row of SB gives the sum of the rows in S. Namely, the total sightings made by each ranger.

c Matrix multiplication of a 1×3 and a 3×4 matrix produces a 1×4 matrix.

$$A \times S = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \times \begin{bmatrix} 27 & 9 & 34 & 59 \\ 18 & 15 & 10 & 89 \\ 35 & 6 & 46 & 29 \end{bmatrix}$$

d In the second last step of part **c**, we see that each element is the sum of the sightings for each type of animal.

$$A \times S = \begin{bmatrix} 27 + 18 + 35 & 9 + 15 + 6 & 34 + 10 + 46 & 59 + 89 + 29 \end{bmatrix}$$
$$= \begin{bmatrix} 80 & 30 & 90 & 177 \end{bmatrix}$$

Each column of AS gives the sum of the columns in S, which gives the sum of the sightings of each type of animal.

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Exercise 5F

General applications

Example 8

1 One matrix below shows the number of milkshakes and sandwiches that Helen had for lunch. The number of kilojoules (kJ) present in each food is given in the other matrix.

$$\begin{array}{cc} & \begin{array}{cc} Milkshakes & Sandwiches \end{array} \\ Helen & \begin{bmatrix} 2 & 3 \end{bmatrix} \end{array} \qquad \begin{array}{cc} & kJ \\ Milkshakes & \left[\begin{array}{c} 1400 \\ 1000 \end{array}\right. \\ Sandwiches & \end{array}$$

$$\begin{array}{c|c} & kJ \\ \hline Milkshakes & 1400 \\ Sandwiches & 1000 \end{array}$$

Use a matrix product to calculate how many kilojoules Helen had for lunch.

2 The first matrix shows the number of cars and bicycles owned by two families. The second matrix records the wheels and seats for cars and bicycles.

$$\begin{array}{ccc} & Cars & Bicycles \\ Smith & 2 & 3 \\ Jones & 1 & 4 \end{array} \qquad \begin{array}{ccc} & Wheels & Seats \\ Car & 4 & 5 \\ Bicycle & 2 & 1 \end{array}$$

Use a matrix product to find a matrix that gives the numbers of wheels and seats owned by each family.

3 Eve played a game of darts. The parts of the dartboard that she hit during one game are recorded in matrix H. The bull's eye is a small area in the centre of the dartboard. The points scored for hitting different regions of the dartboard are shown in matrix P.

$$H = \begin{array}{cccc} & Bull's\ eye & Inner\ region & Outer\ region \\ Hits & 2 & 13 & 5 \end{array}$$

$$P = \begin{array}{cc} Points & \\ 20 & Bull's\ eye \\ 5 & Inner\ region \\ 1 & Outer\ region \end{array}$$

Use matrix multiplication to find a matrix giving her score for the game.

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Business applications

4 On a Saturday morning Michael's café sold 18 quiches, 12 soups and 64 coffees. A quiche costs \$5, soup costs \$8 and a coffee costs \$3.

a Use a row matrix to record the number of each type of item sold.

b Write the costs of each item in a column matrix.

c Use matrix multiplication of the matrices from parts **a** and **b** to find the total value of the mornings sales.

5 Han's stall at the football made the sales shown in the table.

Tubs of chips	Pasties	Pies	Sausage rolls
90	84	112	73

The selling prices were: chips \$4, pastie \$5, pie \$5 and a sausage roll \$3.

a Record the numbers of each product sold in a row matrix.

b Write the selling prices in a column matrix.

c Find the total value of the sales by using matrix multiplication of the row and column matrices found in parts **a** and **b**.

6 Supermarkets sell eggs in boxes of 12, apples in packets of 8 and yoghurt tubs in sets of 4. This is represented by matrix A.

$$A = \begin{array}{c} \\ Items\ per\ packet \end{array} \begin{array}{c} \begin{array}{ccc} Eggs & Apples & Yoghurt \end{array} \\ \begin{bmatrix} 12 & 8 & 4 \end{bmatrix} \end{array}$$

The cost for each type of packet is given by matrix B.

$$B = \begin{array}{c} \\ Eggs \\ Apples \\ Yoghurt \end{array} \begin{array}{c} \$ \\ \begin{bmatrix} 7 \\ 4 \\ 3 \end{bmatrix} \end{array}$$

The sales of each type of packet are given by matrix C as a column matrix and by matrix D as a row matrix.

$$C = \begin{array}{c} \\ Eggs \\ Apples \\ Yoghurt \end{array} \begin{array}{c} Packets \\ \begin{bmatrix} 100 \\ 50 \\ 30 \end{bmatrix} \end{array} \qquad D = \begin{array}{c} \\ Packets \end{array} \begin{array}{c} \begin{array}{ccc} Eggs & Apples & Yoghurt \end{array} \\ \begin{bmatrix} 100 & 50 & 30 \end{bmatrix} \end{array}$$

Choose the appropriate matrices and use matrix multiplication to find:

a the total number of items sold (counting each egg, apple or yoghurt tub as an item)

b the total value of all sales.

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Using row and column matrices to extract information from a matrix

Example 9

7 The number of study hours completed by three students over four days is shown in matrix H.

$$H = \begin{array}{c} \\ Issie \\ Jack \\ Kaiya \end{array} \begin{array}{c} \begin{array}{cccc} Mon & Tues & Wed & Thur \end{array} \\ \left[\begin{array}{cccc} 2 & 3 & 2 & 3 \\ 1 & 4 & 0 & 2 \\ 3 & 4 & 3 & 2 \end{array}\right] \end{array}$$

Using matrix multiplication with a suitable row or column matrix:

a produce a matrix showing the total study hours for each student

b hence, find a matrix with the average hours of study for each student

c obtain a matrix with the total number of hours studied on each night of the week

d hence, find a matrix with the average number of hours studied each night.

8 Matrix R records four students' results in five tests.

$$R = \begin{array}{c} \\ Ellie \\ Felix \\ George \\ Hannah \end{array} \begin{array}{c} \begin{array}{ccccc} T1 & T2 & T3 & T4 & T5 \end{array} \\ \left[\begin{array}{ccccc} 87 & 91 & 94 & 86 & 88 \\ 93 & 76 & 89 & 62 & 95 \\ 73 & 61 & 58 & 54 & 83 \\ 66 & 79 & 83 & 90 & 91 \end{array}\right] \end{array}$$

Choose an appropriate row or column matrix and use matrix multiplication to:

a obtain a matrix with the sum of each student's results

b hence, give a matrix with each student's average test score

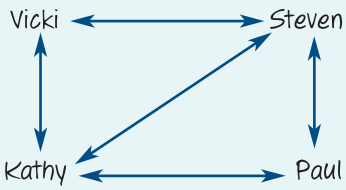
c derive a matrix with the sum of the scores for each test

d hence, give a matrix with the average score on each test.

5G Communications and connections

Social networks, communication pathways and connections can be represented and analysed using matrix techniques.

Example 10 Applying matrices to social networks



The diagram shows the communications within a group of friends. In this diagram:

- a double-headed arrow connecting two names indicates that those two people communicate with each other.
- if there is no arrow directly connecting two people, they do not communicate.

a These links are called one-step connections because there is just one direct step in making contact with the other person. Record the social links in a matrix N, using the first letter of each name to label the columns and rows. Explain how the matrix should be read.

b Explain why there is a symmetry about the leading diagonal of the matrix.

c What information is given by the sum of a column or row?

d N^2 gives the number of two-step communications between people. Namely, how many ways one person can communicate with someone via another person. Find the matrix N^2, the square of matrix N.

e Use the matrix N^2 to find the number of two-step ways Kathy can communicate with Steven and write the connections.

f In the N^2 matrix there is a 3 where S column meets the S row. This indicates that there are three two-step communications Steven can have with himself.
Explain how this can be given a sensible interpretation.

Solution

a

$$N = \begin{array}{cc} \begin{array}{cccc} V & S & K & P \end{array} & \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} & \begin{array}{c} V \\ S \\ K \\ P \end{array} \end{array}$$

For example, reading from the column S down to the row K a 1 indicates that Steven communicates with Kathy. The number 0 is used where there is no communication.

b The symmetry occurs because the communication is two way. For example, Vicki communicates with Steven and Steven communicates with Vicki.

c The sum of a column or row gives the total number of people that a given person can communicate with.
For example, Kathy can communicate with: $1 + 1 + 0 + 1 = 3$ people.

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d

$$N^2 = \begin{array}{c} \\ \\ \\ \\ \\ \end{array}\begin{array}{cccc|c} V & S & K & P & \\ 2 & 1 & 1 & 2 & V \\ 1 & 3 & 2 & 1 & S \\ 1 & 2 & 3 & 1 & K \\ 2 & 1 & 1 & 2 & P \end{array}$$

e Reading down the K column to the S row the 2 indicates there are 2 two-step communications between Kathy and Steven.

These can be found in the arrows diagram.

Kathy → Vicki → Steven

Kathy → Paul → Steven

f There are three ways Steven can communicate with himself via another person.

Steven → Vicki → Steven

Steven → Kathy → Steven

Steven → Paul → Steven

For example, using the first case above, Steven might ring Vicki and ask her to ring him back later to remind him of an appointment.

The matrix N^3 would give the three-step communications between people. The number of ways of communicating with someone via two people.

The matrix methods of investigating communications can be applied to friendships, travel between towns and other types of two-way connections.

Exercise 5G

Example 10

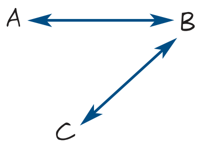
1 Assume that communications are a two-way process. So if *A* communicates with *B* then *B* communicates with *A*. The letters represent the names of people. Find the error in this communications matrix.

$$\begin{array}{cccc|c} A & B & C & D & \\ 0 & 1 & 1 & 1 & A \\ 1 & 0 & 1 & 0 & B \\ 1 & 1 & 0 & 1 & C \\ 0 & 0 & 1 & 0 & D \end{array}$$

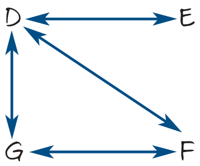
Where column *A* meets row *B* the number 1 indicates that *A* communicates with *B*. The number 0 is used to show there is no direct communication between two people.

2 Write the matrix for each communications diagram. Use the number 1 when direct communication between two people exists and 0 for no direct communication.

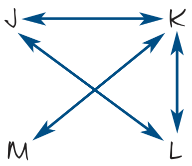
a



b



c



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3 Road connections between towns are recorded in the matrices below. The letters represent towns. Where column A meets row C, the number 1 indicates that there is a road directly connecting town A to town C. The number 0 is used to show when there is no road directly connecting two towns.

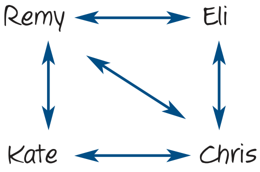
Draw a diagram corresponding to each matrix showing the roads connecting the towns.

a
$$\begin{array}{cccc} A & B & C & \\ 0 & 0 & 1 & A \\ 0 & 0 & 1 & B \\ 1 & 1 & 0 & C \end{array}$$

b
$$\begin{array}{ccccc} P & Q & R & S & \\ 0 & 1 & 0 & 1 & P \\ 1 & 0 & 1 & 0 & Q \\ 0 & 1 & 0 & 1 & R \\ 1 & 0 & 1 & 0 & S \end{array}$$

c
$$\begin{array}{ccccc} T & U & V & W & \\ 0 & 0 & 1 & 1 & T \\ 0 & 0 & 1 & 0 & U \\ 1 & 1 & 0 & 1 & V \\ 1 & 0 & 1 & 0 & W \end{array}$$

4 Communication connections between Chris, Eli, Kate and Remy are shown in the diagram.



a Write a matrix Q to represent the connections. Label the columns and rows in alphabetical order using the first letter of each name. Enter 1 to indicate that two people communicate directly or 0 if they do not.

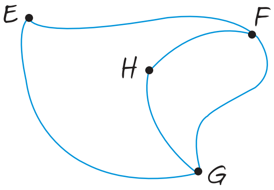
b What information is given by the sum of column R?

c **i** Find Q^2.

ii Using the matrix Q^2, find the total number of ways that Eli can communicate with a person via another person.

iii Write the chain of connections for each way that Eli can communicate to a person via another person.

5 Roads connecting the towns Easton, Fields, Hillsville and Gorges are shown in the diagram. The first letter of each town is used.



a Use a matrix R to represent the road connections. Label the columns and rows in alphabetical order using the first letter of town's name. Write 1 when two towns are directly connected by a road and write 0 if they are not connected.

b What does the sum of column F reveal about the town Fields?

c **i** Find R^2.

ii How many ways are there to travel from Fields to a town via another town? Include ways of starting and ending at Fields.

iii List the possible ways of part **ii**.

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5H Identity and inverse matrices

Identity matrix

In ordinary arithmetic, the number 1 is called the *multiplicative identity element*. When 1 multiplies a number, the answer is always *identical* to the original number. Is there a matrix that can multiply any matrix and give an answer identical to the original matrix? Consider the following example.

Example 11 The identity matrix

$$A = \begin{bmatrix} 5 & 2 \\ 8 & 3 \end{bmatrix} \text{ and } I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

a Find AI.

b Find IA.

Solution

a AI

1 Write the order of each matrix. The inside numbers are the same, so matrix multiplication is defined. The outside numbers tell us that the answer is a 2 × 2 matrix.

2 Do the matrix multiplication by hand or using your calculator.

$$\begin{array}{cc} A & I \\ 2\times 2 & 2\times 2 \end{array}$$

$$A \times I = \begin{bmatrix} 5 & 2 \\ 8 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5\times 1 + 2\times 0 & 5\times 0 + 2\times 1 \\ 8\times 1 + 3\times 0 & 8\times 0 + 3\times 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 2 \\ 8 & 3 \end{bmatrix}$$

b IA

1 Write the order of each matrix. The inside numbers are the same, so matrix multiplication is defined. The outside numbers tell us that the answer is a 2 × 2 matrix.

2 Do the matrix multiplication by hand or using your calculator.

$$\begin{array}{cc} I & A \\ 2\times 2 & 2\times 2 \end{array}$$

$$I \times A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 5 & 2 \\ 8 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1\times 5 + 0\times 8 & 1\times 2 + 0\times 3 \\ 0\times 5 + 1\times 8 & 0\times 2 + 1\times 3 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 2 \\ 8 & 3 \end{bmatrix}$$

Identity matrix for 2 × 2 matrices

Identity matrix for 2 × 2 matrices

The identity matrix for any 2×2 matrix A is $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ where $AI = A = IA$

The **identity matrix**, I, also has the special property that it is *commutative* in matrix multiplication. When I is one of the matrices in the multiplication, the answer is the same when the order of the matrices is commuted (reversed).

In Example 10: $AI = IA = \begin{bmatrix} 5 & 2 \\ 8 & 3 \end{bmatrix}$

Remember that matrix multiplication is not usually commutative.

Only square matrices have identity matrices. The *identity matrix for any square matrix* is a square matrix of the same order with 1s along the *leading diagonal* (from the top left to the bottom right) and 0s in all the other positions.

$$[1] \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

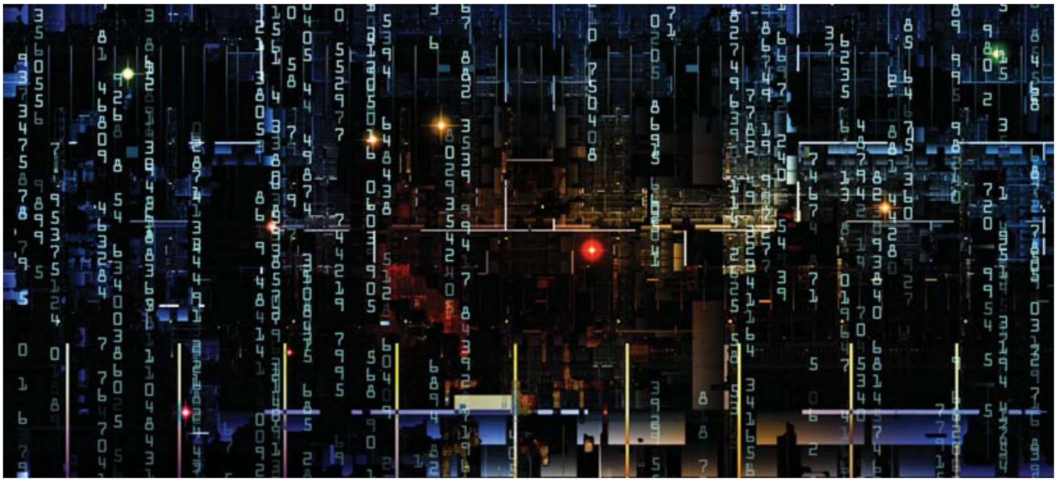
▶ The inverse of a matrix and its evaluation

Written as A^{-1}, the **inverse of matrix** A is a matrix that multiplies A to make the identity matrix I.

$$A \times A^{-1} = I = A^{-1} \times A$$

Only square matrices have inverses.

Finding the inverse of a matrix is best done using your CAS calculator.



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How to find the inverse of a matrix using a CAS calculator

Find the inverse of the matrix $A = \begin{bmatrix} 8 & 4 \\ 3 & 2 \end{bmatrix}$.

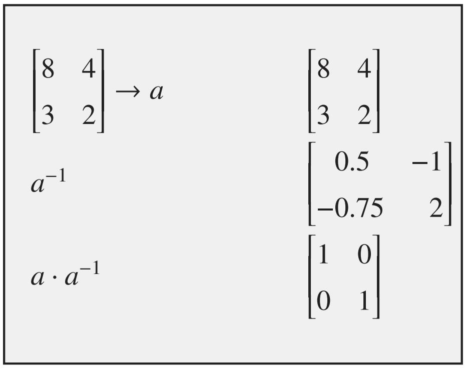
Steps

1 Enter the matrix A into your calculator.

2 Type in $A\boxed{\wedge}-1$ and evaluate.

3 Form the product AA^{-1}. It should give you the identity matrix $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Note: Not all matrices have inverses. If you ask your calculator to calculate the inverse of such a matrix, it will give you an error message.



$\begin{bmatrix} 8 & 4 \\ 3 & 2 \end{bmatrix} \rightarrow a$ $\quad\begin{bmatrix} 8 & 4 \\ 3 & 2 \end{bmatrix}$

a^{-1} $\quad\begin{bmatrix} 0.5 & -1 \\ -0.75 & 2 \end{bmatrix}$

$a \cdot a^{-1}$ $\quad\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

4 Write your answer.

The inverse of A is

$$A^{-1} = \begin{bmatrix} 0.5 & -1 \\ -0.75 & 2 \end{bmatrix}$$

Exercise 5H

Example 11

1 Which of these matrices is the 2×2 identity matrix

A $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ **B** $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ **C** $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ **D** $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ **E** $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$

2 Use your graphics calculator to find the inverse of each matrix. Check by showing that $AA^{-1} = I$ for each matrix.

a $\begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}$ **b** $\begin{bmatrix} 9 & 4 \\ 4 & 2 \end{bmatrix}$ **c** $\begin{bmatrix} 2 & 9 \\ 1 & 4 \end{bmatrix}$ **d** $\begin{bmatrix} 4 & 7 \\ 2 & 3 \end{bmatrix}$

e $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ **f** $\begin{bmatrix} 3 & 2 & 1 \\ 0 & 1 & 4 \\ 2 & 0 & 2 \end{bmatrix}$ **g** $\begin{bmatrix} 1 & 0 & -5 \\ 0 & 3 & 8 \\ 1 & 2 & 0 \end{bmatrix}$ **h** $\begin{bmatrix} 0 & 1 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 1 \end{bmatrix}$

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5I Encoding and decoding information

History has many accounts of the vital role that codes have played in protecting sensitive information used in wars and conspiracies. Mary, Queen of Scots, sent encoded messages from prison to Catholic supporters who planned to overthrow the Protestant Queen Elizabeth. Elizabeth was reluctant to execute her cousin Mary without direct evidence linking her with the plot. The charges were laid by the Principal Secretary, Sir Francis Walsingham. Unfortunately for Mary, Walsingham was also England's spymaster. He used an expert to break the code, and Mary was executed.

In the past, commonly used codes replaced each letter of the alphabet with a randomly chosen number or symbol. The intended recipient could use a list of the changes to change each number back to a letter. The weakness in this type of code is that, in the English language, *E* is the most frequently occurring letter, followed by *T* and then *A*. A table of frequencies for letters can be used to replace numbers occurring with about the same frequency and hence to break the code.

Matrices can be used to encode words so that each letter does *not* have the same number throughout the coded message. This makes the message extremely difficult to decode without knowing the secret encoding matrix. Mary may have survived if she had read the following example. We will use this simple table with the letters numbered in order, but use a matrix to encode the final message.

A	B	C	D	E	F	G	H	I	J	K	L	M	N
1	2	3	4	5	6	7	8	9	10	11	12	13	14

O	P	Q	R	S	T	U	V	W	X	Y	Z	space
15	16	17	18	19	20	21	22	23	24	25	26	27

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Example 12 Using matrices to encode and decode messages

Encode the message 'Meet me' then show how to decode the encoded message. Use the encoding matrix C where:

$$C = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$$

Solution

Encoding the message

1 Write the letters in a 2×4 matrix, M.

$$M = \begin{bmatrix} M & E & E & T \\ & M & E & \end{bmatrix}$$

2 Use the table above to replace each letter with its number.

Tip: Do all of the following steps using your graphics calculator.

$$M = \begin{bmatrix} 13 & 5 & 5 & 20 \\ 27 & 13 & 5 & 27 \end{bmatrix}$$

3 Multiply the message matrix M by the secret encoding matrix C.

Notice that the three Es now have three different numbers in this encoded message matrix, CM.

$$CM = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 13 & 5 & 5 & 20 \\ 27 & 13 & 5 & 27 \end{bmatrix}$$

$$= \begin{bmatrix} 107 & 49 & 25 & 121 \\ 67 & 31 & 15 & 74 \end{bmatrix}$$

Decoding the message

1 Work out the *inverse* of the secret encoding matrix. Use a calculator.

The inverse of $\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ is $C^{-1} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$.

2 Multiply the encoded message matrix CM by the inverse of the encoding matrix, C^{-1}, to return to the message matrix M.

$$C^{-1} \times CM = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 107 & 49 & 25 & 121 \\ 67 & 31 & 15 & 74 \end{bmatrix}$$

$$M = \begin{bmatrix} 13 & 5 & 5 & 20 \\ 27 & 13 & 5 & 27 \end{bmatrix}$$

3 Use the table above to replace each number with its letter.

$$M = \begin{bmatrix} M & E & E & T \\ & M & E & \end{bmatrix}$$

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A 2×2 matrix can be used to encode any information written as a matrix with two rows. Credit card numbers consisting of 16 digits can be written into a 2×8 matrix and encoded using a 2×2 matrix.

Tip: Choose a 2×2 encoding matrix with small positive numbers so that the numbers in the encoded message do not get too large.

A	**B**	**C**	**D**	**E**	**F**	**G**	**H**	**I**	**J**	**K**	**L**	**M**	**N**
1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**

O	**P**	**Q**	**R**	**S**	**T**	**U**	**V**	**W**	**X**	**Y**	**Z**	**space**
15	**16**	**17**	**18**	**19**	**20**	**21**	**22**	**23**	**24**	**25**	**26**	**27**

Exercise 5I

Example 12

1 Use the table for swapping letters with numbers and matrix C to *encode* each of the following messages.

$$C = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

a $\begin{bmatrix} F & B & I & \\ K & N & O & W \end{bmatrix}$ **b** $\begin{bmatrix} M & A & P & \\ L & O & S & T \end{bmatrix}$

c $\begin{bmatrix} A & P & E & \\ F & A & C & E \end{bmatrix}$ **d** $\begin{bmatrix} F & I & N & D \\ & T & O & M \end{bmatrix}$

e $\begin{bmatrix} N & O & & G & U & A & R & D \\ & T & O & N & I & G & H & T \end{bmatrix}$ **f** $\begin{bmatrix} M & E & E & T & & A & N & N \\ A & T & & J & O & H & N & S \end{bmatrix}$

2 Use the letters to numbers table and the *inverse* of matrix S to *decode* the following messages.

$$S = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$$

a $\begin{bmatrix} 10 & 27 & 33 & 62 \\ 19 & 45 & 53 & 97 \end{bmatrix}$ **b** $\begin{bmatrix} 28 & 41 & 52 & 59 \\ 47 & 62 & 85 & 91 \end{bmatrix}$

c $\begin{bmatrix} 20 & 39 & 63 & 59 \\ 34 & 66 & 101 & 91 \end{bmatrix}$ **d** $\begin{bmatrix} 37 & 65 & 29 & 79 \\ 56 & 109 & 44 & 131 \end{bmatrix}$

e $\begin{bmatrix} 22 & 45 & 48 & 67 & 73 & 15 & 36 & 59 \\ 40 & 75 & 82 & 114 & 119 & 23 & 57 & 91 \end{bmatrix}$ **f** $\begin{bmatrix} 26 & 51 & 30 & 35 & 58 & 23 & 39 & 58 \\ 46 & 84 & 55 & 66 & 89 & 37 & 59 & 89 \end{bmatrix}$

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3 Use the encoding matrix B to do the following.

$$B = \begin{bmatrix}1 & 1\\ 1 & 2\end{bmatrix}$$

a Encode the credit card number written into this 2×8 matrix:

$$\begin{bmatrix}3 & 1 & 4 & 7 & 2 & 3 & 8 & 1\\ 6 & 0 & 5 & 8 & 9 & 3 & 0 & 7\end{bmatrix}$$

b Decode the encoded credit card number received in this matrix:

$$\begin{bmatrix}7 & 10 & 8 & 13 & 12 & 0 & 12 & 12\\ 8 & 19 & 9 & 19 & 20 & 0 & 16 & 21\end{bmatrix}$$

4 Make up an encoded message to fit within a 2×8 matrix. Give a classmate the 2×2 encoding matrix and see whether they can decode it.

5J Solving simultaneous equations using matrices

How to solve simultaneous equations using a CAS calculator

Solve to find x and y:

$$5x + 2y = 21$$
$$7x + 3y = 29$$

Steps

1 The two simultaneous equations can be represented by the matrix equation shown.

$$\begin{bmatrix}5x + 2y\\ 7x + 3y\end{bmatrix} = \begin{bmatrix}21\\ 29\end{bmatrix}$$

2 The left-hand side of the matrix equation in step **1** can be written as the product of two matrices.

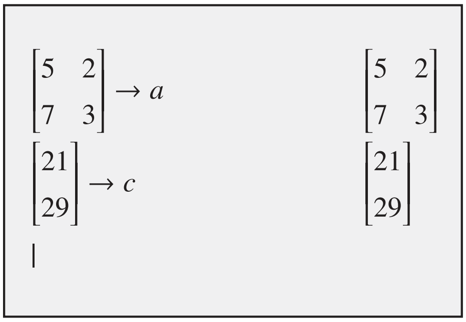
$$\begin{bmatrix}5 & 2\\ 7 & 3\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}21\\ 29\end{bmatrix}$$

3 Name the matrices as shown. Matrix X contains the solutions to the simultaneous equations.

$$A \times X = C$$

4 Enter matrix A and matrix C.

(If you need help, see 'How to enter a matrix into your graphics calculator' (pages 220, 221).)



$\begin{bmatrix}5 & 2\\ 7 & 3\end{bmatrix} \to a$ $\qquad\begin{bmatrix}5 & 2\\ 7 & 3\end{bmatrix}$

$\begin{bmatrix}21\\ 29\end{bmatrix} \to c$ $\qquad\begin{bmatrix}21\\ 29\end{bmatrix}$

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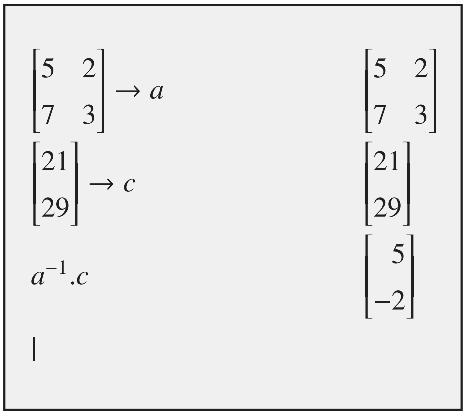
5 We want to find the values of matrix *B*. $B = A^{-1} \times C$

Since

$$\begin{aligned} A \times X &= C \\ A^{-1} \times A \times X &= A^{-1} \times C \\ I \times X &= A^{-1} \times C \\ X &= A^{-1} \times C \end{aligned}$$

6 Enter the matrix product $A^{-1} \times C$.

Note: Order is critical here: $X = A^{-1}C$, *not* CA^{-1}.



$\begin{bmatrix} 5 & 2 \\ 7 & 3 \end{bmatrix} \to a$ $\begin{bmatrix} 5 & 2 \\ 7 & 3 \end{bmatrix}$

$\begin{bmatrix} 21 \\ 29 \end{bmatrix} \to c$ $\begin{bmatrix} 21 \\ 29 \end{bmatrix}$

$a^{-1}.c$ $\begin{bmatrix} 5 \\ -2 \end{bmatrix}$

7 Write matrix *X*. $X = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$

8 Write the solutions to the equations. So $x = 5$ and $y = -2$.

Exercise 5J

Use matrix methods on your graphics calculator to solve the following simultaneous equations.

1 $3x + 2y = 12$
$5x + y = 13$

2 $4x + 3y = 10$
$x + 2y = 5$

3 $4x - 3y = 10$
$3x + y = 1$

4 $8x + 3y = 50$
$5x + 2y = 32$

5 $6x + 7y = 68$
$4x + 5y = 46$

6 $6x - 5y = -27$
$7x + 4y = -2$

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5K Extended application and problem solving tasks

Exercise 5K

Skillsheet **1** Scalar multiplication occurs when a number multiplies a matrix.

For a 2×2 matrix it has the general form:

$$k\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}$$

Suggest why the matrix $\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$ below is called a *scalar matrix*.

Give examples to support your view.

Information can be stored in matrices with hundreds of columns and rows. Using conveniently sized matrices, we will explore methods that can be used for extracting information from huge matrix data banks.

2 The mobile phone bills of Anna, Boyd and Charlie for the four quarters of 2014 are recorded in the matrix P. We will investigate the effect of multiplying by matrix E.

$$P = \begin{array}{r} \\ Anna \\ Boyd \\ Charlie \end{array} \begin{array}{c} \begin{array}{cccc} Q1 & Q2 & Q3 & Q4 \end{array} \\ \begin{bmatrix} 47 & 43 & 52 & 61 \\ 56 & 50 & 64 & 49 \\ 39 & 41 & 44 & 51 \end{bmatrix} \end{array} \qquad E = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

a Find $P \times E$ and comment on the result of that matrix multiplication.

b State the matrix F needed to extract the second quarter, $Q2$, costs.

A matrix is needed so it can multiply matrix P to extract the four quarterly costs on Charlie's phone bill.

c What will be the order of the matrix that displays Charlie's quarterly costs?

d State the order of a matrix G that when it multiplies a 3×4, gives a 1×4 matrix as the result? Should the matrix G pre-multiply or post-multiply matrix P? Pre-multiply means it multiplies at the front (left) of matrix P. While post-multiply means the matrix multiplies when written after (to the right of) matrix P.

e Suggest a suitable matrix G that will multiply matrix P and produce a matrix of Charlie's quarterly phone bills. Check that it works.

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Key ideas and chapter summary

Matrix	A **matrix** is a rectangular array of numbers set out in rows and columns within square brackets. The rows are horizontal; the columns are vertical.
Order of a matrix	The **order (size) of a matrix** is: number of rows × number of columns. The number of rows is always given first.
Elements of a matrix	The **elements of a matrix** are the numbers within it. The position of an element is given by its row and column in the matrix. Element $a_{i\,j}$ is in row i and column j. Row is always given first.
Connections	A matrix can be used to record various types of connections, such as social communications and roads directly connecting towns.
Equal matrices	Two matrices are equal when they have the same numbers in the same positions. To do this they need to have the same order (shape).
Adding matrices	Matrices of the same order can be *added* by adding numbers in the same positions.
Subtracting matrices	Matrices of the same order can be *subtracted* by subtracting numbers in the same positions.
Zero matrix, *O*	A **zero matrix** is any matrix with zeroes in every position.
Scalar multiplication	**Scalar multiplication** is the multiplication of a matrix by *a number*.
Matrix multiplication	The process of multiplying a matrix by a matrix.
Identity matrix, *I*	An *identity matrix* I behaves like the number 1 in arithmetic. Any matrix multiplied by I remains unchanged. For 2×2 matrices, $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ where $AI = A = IA$ Matrix multiplication by the identity matrix is commutative.
Inverse matrix, A^{-1}	When any matrix A is multiplied by its *inverse matrix* A^{-1}, the answer is I, the identity matrix. That is: $A \times A^{-1} = I$

Encoding and decoding

Any information put into a matrix M with 2 rows can be *encoded* by multiplying M by a 2×2 encoding matrix C to form matrix CM.
To *decode* the information, multiply matrix CM by the inverse matrix C^{-1}, so that $C^{-1} \times CM$ gives matrix M.

Solving simultaneous equations

Two **simultaneous equations**, for example:

$$5x + 2y = 21$$

$$7x + 3y = 29$$

can be written in matrix form as

$$\begin{bmatrix} 5 & 2 \\ 7 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 21 \\ 29 \end{bmatrix}$$

$$A \times X = C$$

The equations can be solved (for x and y) by finding the values in matrix X, as

$$X = \begin{bmatrix} x \\ y \end{bmatrix} = A^{-1} \times C$$

Order is critical: $X = A^{-1}C$, *not* CA^{-1}.
This is best done using a graphics calculator.

Skills check

Having completed this chapter you should be able to:

- state the order of a given matrix
- describe the location of an element within a matrix
- decide whether two matrices are equal
- add and subtract matrices
- perform scalar multiplication of a matrix
- identify a zero matrix
- decide whether it is possible to do matrix multiplication with two given matrices
- give the order of the matrix resulting from matrix multiplication
- perform matrix multiplication
- state the identity matrix for a $n \times n$ matrix and know its properties
- find the inverse of a matrix if it exists
- represent and solve communications
- encode and decode information using matrices
- use matrix methods to solve simultaneous equations.

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Multiple-choice questions

Use the matrices F, G, H in Questions 1 and 2

$$F = \begin{bmatrix} 4 & 8 & 6 \\ 5 & 1 & 7 \end{bmatrix} \quad G = \begin{bmatrix} 9 \\ 2 \\ 0 \end{bmatrix} \quad H = \begin{bmatrix} 4 & 2 \\ 5 & 3 \end{bmatrix}$$

1 The order of matrix F is:

A 6 **B** 2×3 **C** 3×2 **D** $2 + 3$ **E** $3 + 2$

2 The element f_{21} is:

A 3 **B** 2 **C** 8 **D** 1 **E** 5

3 Three students were asked the number of electronic devices their family owned. The results are shown in the matrix.

$$\begin{array}{rc} & \begin{array}{ccc} \mathit{TVs} & \mathit{VCRs} & \mathit{PCs} \end{array} \\ \begin{array}{r} \mathit{Caroline} \\ \mathit{Delia} \\ \mathit{Emir} \end{array} & \begin{bmatrix} 4 & 3 & 2 \\ 1 & 0 & 5 \\ 2 & 1 & 3 \end{bmatrix} \end{array}$$

The number of PCs owned by Emir's family is:

A 1 **B** 2 **C** 3 **D** 4 **E** 5

4 The matrix gives the numbers of roads directly connecting one town to another. The total number of roads directly connecting town E to other towns is:

$$\begin{array}{cl} \begin{array}{ccc} D & E & F \end{array} & \\ \begin{bmatrix} 0 & 2 & 1 \\ 2 & 0 & 3 \\ 1 & 3 & 0 \end{bmatrix} & \begin{array}{l} D \\ E \\ F \end{array} \end{array}$$

A 0 **B** 2 **C** 3 **D** 5 **E** 12

5 For these two matrices to be equal, the required value of x is:

$$\begin{bmatrix} 4 & 3x \\ 7 & 9 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 7 & 9 \end{bmatrix}$$

A 2 **B** 3 **C** 4 **D** 6 **E** 18

Use matrices M and N in Questions 6 to 10

$$M = \begin{bmatrix}7 & 6\\4 & 3\end{bmatrix} \quad N = \begin{bmatrix}5 & -2\\1 & 0\end{bmatrix}$$

6 The matrix $M + N$ is:

A $\begin{bmatrix}12 & 8\\5 & 3\end{bmatrix}$
B $\begin{bmatrix}12 & 4\\4 & 3\end{bmatrix}$
C $\begin{bmatrix}12 & 4\\4 & 0\end{bmatrix}$
D $\begin{bmatrix}12 & 4\\5 & 3\end{bmatrix}$
E $\begin{bmatrix}12 & 4\\5 & 0\end{bmatrix}$

7 The matrix $M - N$ is:

A $\begin{bmatrix}2 & 8\\3 & 0\end{bmatrix}$
B $\begin{bmatrix}2 & 4\\3 & 0\end{bmatrix}$
C $\begin{bmatrix}2 & 4\\3 & 3\end{bmatrix}$
D $\begin{bmatrix}2 & 8\\3 & 3\end{bmatrix}$
E $\begin{bmatrix}2 & 8\\4 & 3\end{bmatrix}$

8 The matrix $N - N$ is:

A 0
B $\begin{bmatrix}0\end{bmatrix}$
C $\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$
D $\begin{bmatrix}5 & -2\\1 & 0\end{bmatrix}$
E $\begin{bmatrix}-5 & 2\\-1 & 0\end{bmatrix}$

9 The matrix $2N$ is:

A $\begin{bmatrix}10 & -4\\2 & 0\end{bmatrix}$
B $\begin{bmatrix}7 & 0\\3 & 2\end{bmatrix}$
C $\begin{bmatrix}10 & -4\\1 & 0\end{bmatrix}$
D $\begin{bmatrix}10 & -2\\2 & 0\end{bmatrix}$
E $\begin{bmatrix}10 & -4\\2 & 2\end{bmatrix}$

10 The matrix $2M + N$ is:

A $\begin{bmatrix}14 & 10\\7 & 5\end{bmatrix}$
B $\begin{bmatrix}14 & 6\\7 & 5\end{bmatrix}$
C $\begin{bmatrix}24 & 8\\10 & 6\end{bmatrix}$
D $\begin{bmatrix}19 & 14\\9 & 6\end{bmatrix}$
E $\begin{bmatrix}19 & 10\\9 & 6\end{bmatrix}$

Use the matrices P, Q, R, S in Questions 11 to 14

$$P\begin{bmatrix}5 & 4 & 1\\7 & 6 & 8\end{bmatrix} \quad Q = \begin{bmatrix}2\\6\\0\end{bmatrix} \quad R = \begin{bmatrix}4 & 7\end{bmatrix} \quad S = \begin{bmatrix}1 & 3\\0 & 4\end{bmatrix}$$

11 Matrix multiplication is not defined for:

A PQ
B SS
C SP
D PS
E RS

12 The order of matrix QR is:

A 1×1
B 3×2
C 2×3
D 6
E 5

13 Which of the following matrix multiplications gives a 1×3 matrix?

A QQ
B RQ
C PR
D QR
E RP

14 The matrix multiplication PQ gives the matrix:

A $\begin{bmatrix}34\\50\end{bmatrix}$
B $\begin{bmatrix}10 & 24 & 0\\14 & 36 & 0\end{bmatrix}$
C $\begin{bmatrix}10 & 14\\24 & 0\\36 & 0\end{bmatrix}$
D $\begin{bmatrix}5 & 4 & 1\\7 & 6 & 8\\2 & 6 & 0\end{bmatrix}$
E $\begin{bmatrix}34 & 50\end{bmatrix}$

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15 The identity matrix for 2 × 2 matrices is:

A $\begin{bmatrix} 1 \end{bmatrix}$ **B** $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ **C** $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ **D** $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ **E** $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

16 The inverse of matrix $A = \begin{bmatrix} 4 & 5 \\ 2 & 3 \end{bmatrix}$ is:

A $\begin{bmatrix} -4 & -5 \\ -2 & -3 \end{bmatrix}$ **B** $\begin{bmatrix} 5 & 4 \\ 3 & 2 \end{bmatrix}$ **C** $\frac{1}{2}\begin{bmatrix} 3 & -2 \\ -5 & 4 \end{bmatrix}$ **D** $\begin{bmatrix} 3 & -5 \\ -2 & 4 \end{bmatrix}$ **E** $\frac{1}{2}\begin{bmatrix} 3 & -5 \\ -2 & 4 \end{bmatrix}$

Short-answer questions

Use matrix A in Questions 1 to 4

$$A = \begin{bmatrix} 4 & 2 & 1 & 0 \\ 3 & 4 & 7 & 9 \end{bmatrix}$$

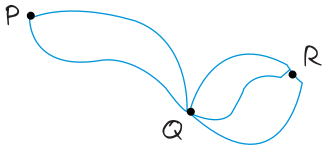
1 State the order of matrix A.

2 Identify the element $a_{1\,3}$.

3 If $C = [5 \quad 6]$, find CA.

4 If the order of a matrix B was 4 × 1, what would be the order of the matrix resulting from AB?

5 Roads are shown joining towns P, Q and R. Use a matrix to record the number roads directly connecting one town to another town.



6 Use the matrices below

$$A = \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 5 \\ 7 & 6 \end{bmatrix} \quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

to find:

a $3A$ **b** $A + B$ **c** $B - A$ **d** $2A + B$

e $A - A$ **f** AB **g** BA **h** A^{-1}

i A^2 **j** AI **k** AA^{-1}

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Extended-response questions

1 Farms A and B have their livestock numbers recorded in the matrix shown.

$$\begin{array}{cc} & \begin{array}{ccc} \textit{Cattle} & \textit{Pigs} & \textit{Sheep} \end{array} \\ \begin{array}{r} \textit{Farm } A \\ \textit{Farm } B \end{array} & \begin{bmatrix} 420 & 50 & 100 \\ 300 & 40 & 220 \end{bmatrix} \end{array}$$

a How many pigs are on Farm B?

b What is the total number of sheep on both farms?

c Which farm has the largest total number of livestock?

2 A bakery recorded the sales for Shop A and Shop B of cakes, pies and rolls in a Sales matrix, S. The prices were recorded in the Prices matrix, P.

$$S = \begin{array}{cc} & \begin{array}{ccc} \textit{Cakes} & \textit{Pies} & \textit{Rolls} \end{array} \\ \begin{array}{r} A \\ B \end{array} & \begin{bmatrix} 12 & 25 & 18 \\ 15 & 21 & 16 \end{bmatrix} \end{array} \qquad P = \begin{array}{cc} & \$ \\ \begin{array}{r} \textit{Cakes} \\ \textit{Pies} \\ \textit{Rolls} \end{array} & \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \end{array}$$

a How many pies were sold by Shop B?

b What is the selling price of pies?

c Calculate the matrix product SP.

d What information is contained in matrix SP?

e Which shop had the largest income from its sales? How much were its takings?

3 Patsy and Geoff decided to participate in a charity fun run.

a Patsy plans to walk for 4 hours and jog for 1 hour. Geoff plans to walk for 3 hours and jog for 2 hours. Write out matrix A, filling in the missing information.

$$A = \begin{array}{cc} & \begin{array}{cc} \textit{Hours walking} & \textit{Hours jogging} \end{array} \\ \begin{array}{r} \textit{Patsy} \\ \textit{Geoff} \end{array} & \begin{bmatrix} \square & \square \\ \square & \square \end{bmatrix} \end{array}$$

b Walking raises \$2 per hour and consumes 1500 kJ/h (kilojoules per hour).
Jogging raises \$3 per hour and consumes 2500 kJ/h.
Write out matrix B, filling in the missing information.

$$B = \begin{array}{cc} & \begin{array}{cc} \$ & kJ \end{array} \\ \begin{array}{r} \textit{Walking} \\ \textit{Jogging} \end{array} & \begin{bmatrix} \square & \square \\ \square & \square \end{bmatrix} \end{array}$$

c Use matrix multiplication to find a matrix that shows the money raised and the kilojoules consumed by each person.

4 **a** The 16 digits of a credit card are recorded in matrix *D*.

$$D = \begin{bmatrix} 1 & 0 & 2 & 1 & 3 & 2 & 1 & 1 \\ 2 & 1 & 4 & 2 & 2 & 1 & 0 & 3 \end{bmatrix} \quad A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

Use matrix *A* to encode the credit card numbers. Give the matrix with the encoded credit card numbers.

b Matrix *E* consists of 16 credit card numbers that have been encoded using matrix *A*. Decode matrix *E* to the matrix with the original credit card numbers.

$$E = \begin{bmatrix} 5 & 6 & 3 & 9 & 2 & 7 & 4 & 4 \\ 3 & 3 & 2 & 5 & 2 & 5 & 2 & 3 \end{bmatrix}$$

5 We are told that 2 apples and 3 bananas cost \$6. This can be represented by the equation $2x + 3y = 6$ where x represents the cost of an apple and y the cost of a banana.

a Write an equation for: 6 apples and 5 bananas cost \$14.

b With your equation and the equation given, use matrix methods on your graphics calculator to find the cost of an apple and the cost of a banana.

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6

Linear graphs and models

- What is a linear graph?
- How do we determine the slope of a straight-line graph?
- How do find the equation of a straight line from its graph?
- How do we sketch a straight-line graph from its equation?
- How do we use straight-line graphs to model practical situations?

Introduction

Many everyday situations can be described and investigated using a linear graph and its equation. Examples include the depreciating value of a newly purchased car, and the short-term growth of a newly planted tree. In this chapter, you will revise the properties of linear graphs and their equations and apply these ideas to modelling linear growth and decay in the real world.

6A Drawing straight-line graphs

▶ Plotting straight-line graphs

Relations defined by equations such as:

$$y = 1 + 2x \qquad y = 3x - 2 \qquad y = 10 - 5x \qquad y = 6x$$

are called *linear* relations because they generate *straight-line graphs*.

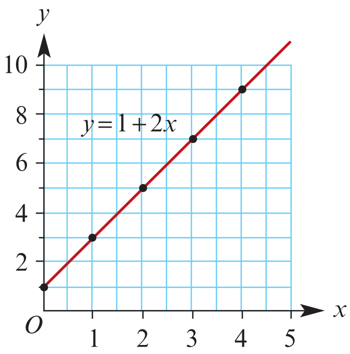
For example, consider the relation $y = 1 + 2x$. To plot this graph, we can form a table.

x	0	1	2	3	4
y	1	3	5	7	9

We can then plot the values from the table on a set of axes, as shown opposite.

The points appear to lie on a straight line.

A ruler can then be used to draw in this straight line to give the graph of $y = 1 + 2x$.



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Example 1 Constructing a graph from a table of values

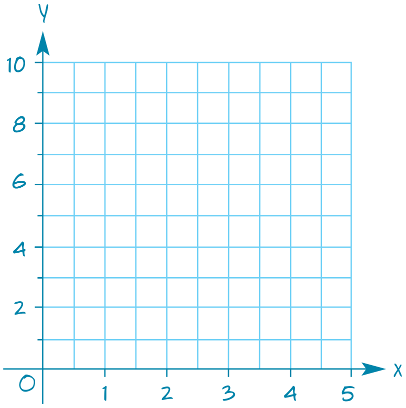
Plot the graph of $y = 8 - 2x$ by forming a table of values of y using $x = 0, 1, 2, 3, 4$.

Solution

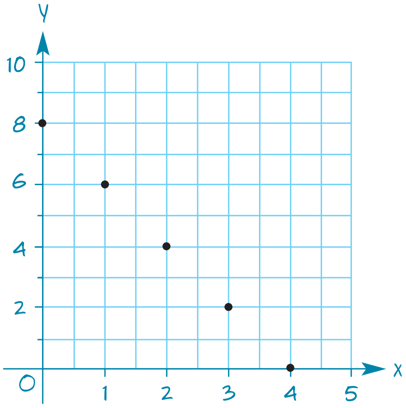
1 Set up table of values.
When $x = 0, y = 8 - 2 \times 0 = 8$.
When $x = 1, y = 8 - 2 \times 1 = 6$, and so on.

x	0	1	2	3	4
y	8	6	4	2	0

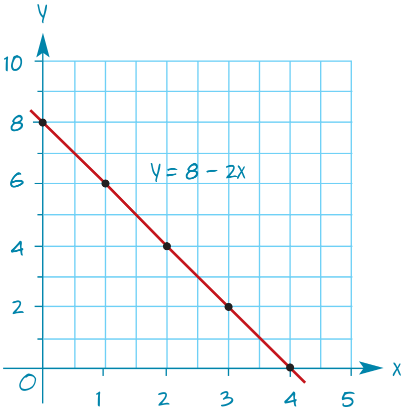
2 Draw, label and scale a set of axes to cover all values.



3 Plot the values in the table on the graph by marking with a dot (•). The first point is (0, 8). The second point is (1, 6), and so on.



4 The points appear to lie on a straight line. Use a ruler to draw in the straight line. Label the line $y = 8 - 2x$.



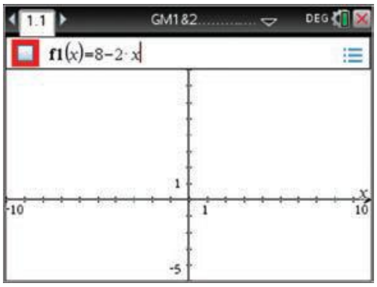
A graphics calculator can also be used to draw straight-line graphs, although it can take some fiddling around with scaling to get the exact graph you want. However, one advantage of using a graphics calculator is that, when drawing the graph, it automatically generates a table of values for you.

How to draw a straight-line graph and show a table of values using the TI-Nspire

Use a graphics calculator to draw the graph $y = 8 - 2x$ and show a table of values.

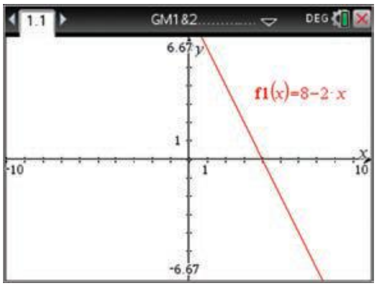
Steps

1 Start a new document (ctrl + N) and select **Add Graphs**.

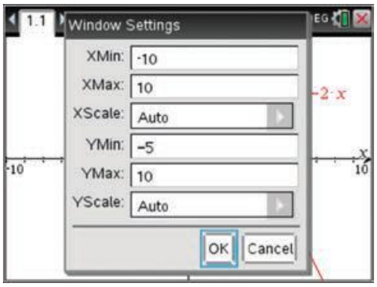


2 Type in the equation as shown. Note that f1(x) represents the y. Press enter to obtain the graph below.

Hint: If the function entry line is not visible, press tab.



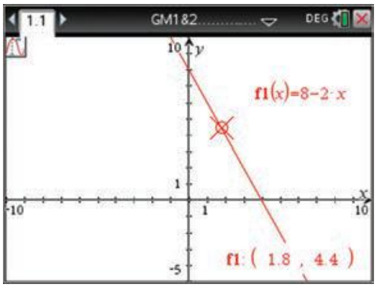
3 Change the window setting to see the key features of the graph. Use menu>**Window/Zoom**>**Window Settings** and edit as shown. Use the tab key to move between the entry lines. Press enter when finished editing the settings. The re-scaled graph is shown below.


Window Settings

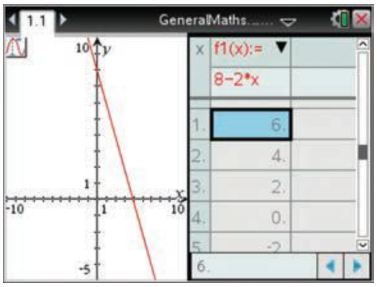
XMin: -10
XMax: 10
XScale: Auto
YMin: -5
YMax: 10
YScale: Auto

4 To show values on the graph, use menu>**Trace**>**Graph Trace** and then use the ◄► arrows to move along the graph.

Note: Press esc to exit the **GraphTrace** tool.



5 To show a table of values, press ctrl + T. Use the ▲▼ arrows to scroll through the values in the table.



x	f1(x):=
	8−2*x
1.	6.
2.	4.
3.	2.
4.	0.
5.	-2.
6.	

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How to draw a straight-line graph and show a table of values using the ClassPad

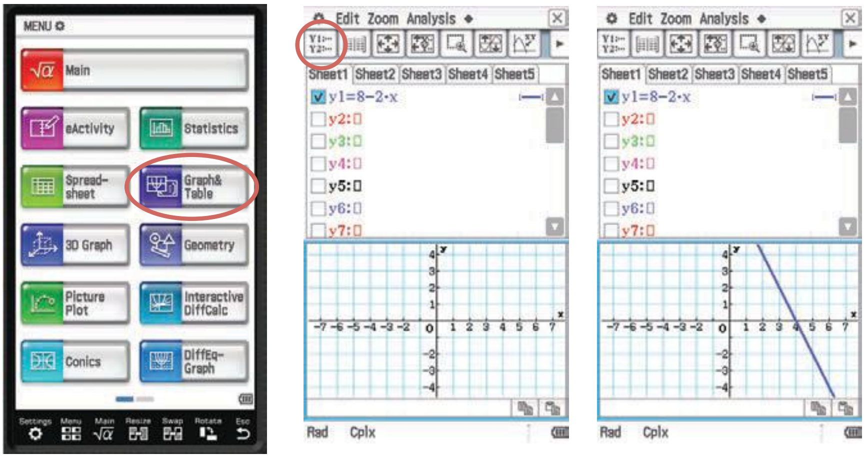
Use a graphics calculator to draw the graph of $y = 8 - 2x$ and show a table of values.

Steps

1 Open the **Graphs and Table** application.

2 Enter the equation into the graph editor window by typing $\mathbf{8 - 2x}$ and press **EXE**.

3 Tap the icon to plot the graph.

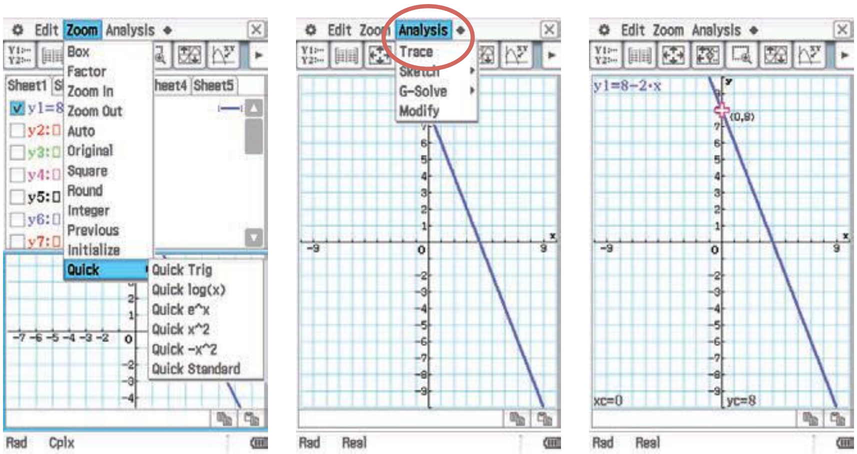


4 To adjust the graph screen go to **Zoom > Quick > Quick Standard**.

5 Tap resize **Resize** from the toolbar to increases the size of the graph window.

6 Select **Analysis>Trace** to place a cursor on the graph. The coordinates of the point will be displayed at the location of the cursor. E.g. (0, 8).

7 Use the cursor key to move the cursor along the line.



8 Tap the icon from the toolbar to display a table of values.

9 Tap the icon from the toolbar to open the **Table Input** dialog box. The values displayed in the table can be adjusted by changing the values in this window.

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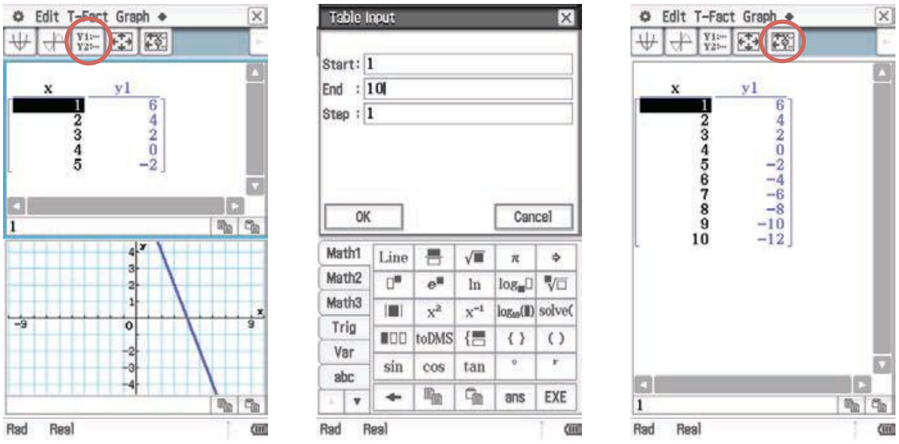


x	y1
1	6
2	4
3	2
4	0
5	−2

Table Input

Start: 1

End : 10

Step : 1

x	y1
1	6
2	4
3	2
4	0
5	−2
6	−4
7	−6
8	−8
9	−10
10	−12

Exercise 6A

Plotting by hand

Example 1

1 Plot the graph of the linear equations below by first forming a table of values of y for $x = 0, 1, 2, 3, 4$.

a $y = 1 + 2x$ **b** $y = 2 + x$ **c** $y = 10 - x$ **d** $y = 9 - 2x$

Using a calculator

2 Use your CAS calculator to plot a graph for the window given and generate a table of values.

a $y = 4 + x$
$-5 \le x \le 5$
$-1 \le y \le 10$

b $y = 2 + 3x$
$-1 \le x \le 5$
$-1 \le y \le 20$

c $y = 10 + 5x$
$-1 \le x \le 5$
$-1 \le y \le 40$

d $y = 5x$
$-5 \le x \le 5$
$-25 \le y \le 25$

e $y = -5x$
$-5 \le x \le 5$
$-25 \le y \le 25$

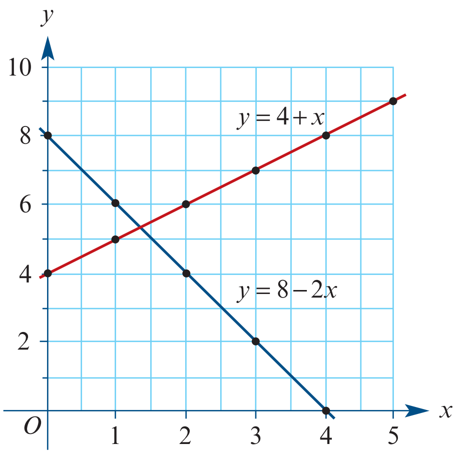
f $y = 100 - 5x$
$-1 \le x \le 25$
$-25 \le y \le 125$

Conceptual understanding

3 Two straight-line graphs, $y = 4 + x$ and $y = 8 - 2x$, are plotted as shown opposite.

a Reading from the graph of $y = 4 + x$, determine the missing coordinates: (0, ?), (2, ?), (?, 7), (?, 9).

b Reading from the graph of $y = 8 - 2x$, determine the missing coordinates: (0, ?), (1, ?), (?, 4), (?, 2).



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6B Determining the slope of a straight line

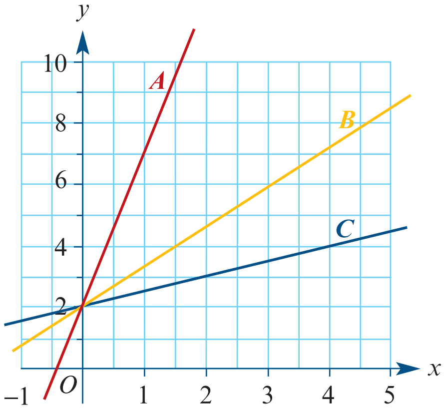
Positive and negative slopes

One thing that makes one straight-line graph look different from another is its steepness or **slope**. Another name for slope is **gradient**[1].

For example, the three straight lines on the graph opposite all cut the y-axis at $y = 2$, but they have quite different slopes.

Line A has the steepest slope while Line C has the gentlest slope. Line B has a slope somewhere in between.

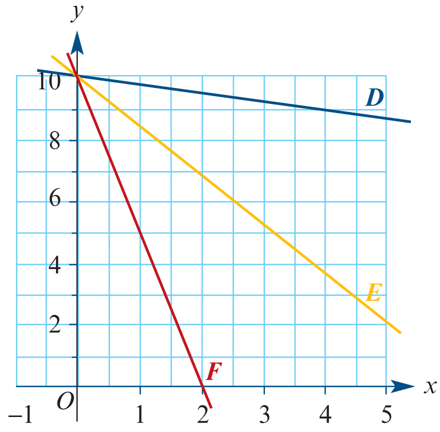
In all cases, the lines have **positive slopes**; that is, they rise from left to right.



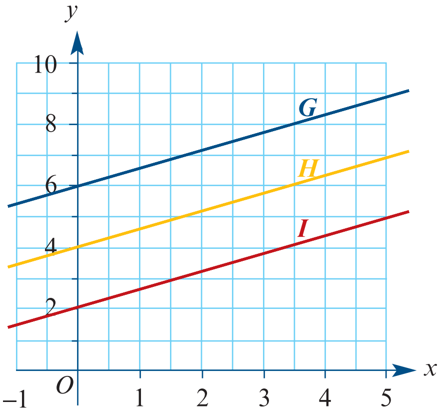
Similarly, the three straight lines on the graph opposite all cut the y-axis at $y = 10$, but they have quite different slopes.

In this case, Line D has the gentlest slope while Line F has the steepest slope. Line E has a slope somewhere in between.

In all cases, the lines have **negative slopes**; that is, they fall from left to right.



By contrast, the three straight lines G, H, I on the graph opposite cut the y-axis at different point, but they all have the *same* slope.



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[1] Note: For linear graphs, the terms slope and gradient mean the same thing. However, when dealing with practical applications of linear graphs and, most particularly, in statistics applications, the word ‘slope’ is preferred. For this reason, and to be consistent with the VCE General Mathematics and Further Mathematics curricula, this is the term used throughout this book.

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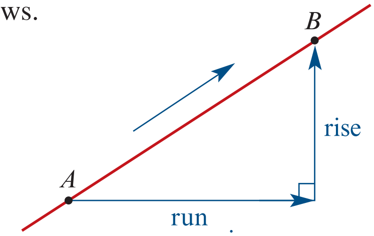
▶ Determining the slope

When talking about the **slope of a straight line**, we want to be able to do more than say that it has a gentle positive slope. We would like to be able to give the slope a value that reflects this fact. We do this by defining the slope of a line as follows.

First, two points A and B on the line are chosen.
As we go from A to B along the line, we move:

- up by a distance called the **rise**
- and across by a distance called the **run**.

The slope is found by dividing the rise by the run.



$$\text{slope} = \frac{\text{rise}}{\text{run}}$$

Example 2 Finding the slope of a line from a graph: positive slope

Find the slope of the line through the points (1, 4) and (4, 8).

Solution

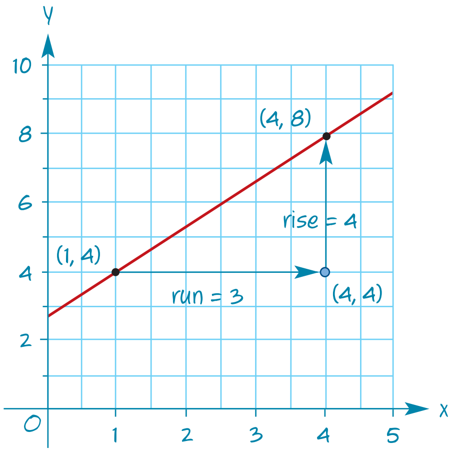
$$\text{slope} = \frac{\text{rise}}{\text{run}}$$

$$\text{rise} = 8 - 4 = 4$$

$$\text{run} = 4 - 1 = 3$$

$$\therefore\ \text{slope} = \frac{4}{3} = 1.33 \text{ (to 2d.p.)}$$

Note: To find the 'rise', look at the y-coordinates.
To find the 'run', look at the x-coordinates.



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Example 3 Finding the slope of a line from a graph: negative slope

Find the slope of the line through the points (0, 10) and (4, 2).

Solution

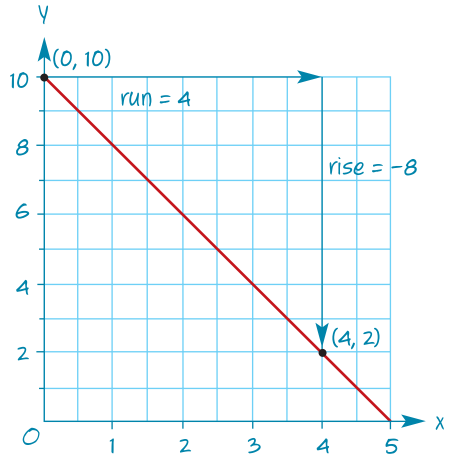
$$\text{slope} = \frac{\text{rise}}{\text{run}}$$

$$\text{rise} = 2 - 10 = -8$$

$$\text{run} = 4 - 0 = 4$$

$$\therefore\ \text{slope} = \frac{-8}{4} = -2$$

Note: In this example, we have a negative ‘rise’ which represents a ‘fall’.



A formula for finding the slope of a line

While the ‘rise/run’ method for finding the slope of a line will always work, some people prefer to use a formula for calculating the slope. The formula is derived as follows.

Label the coordinates of point A: (x_1, y_1).

Label the coordinates of point B: (x_2, y_2).

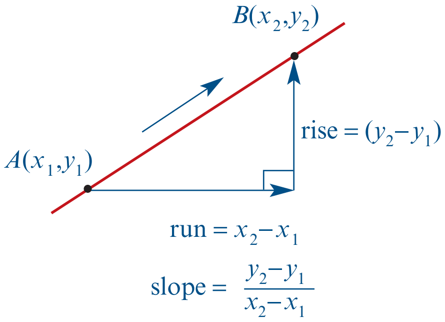
By definition: $\quad \text{slope} = \dfrac{\text{rise}}{\text{run}}$.

From the diagram:

$$\text{rise} = y_2 - y_1$$

$$\text{run} = x_2 - x_1$$

By substitution: $\quad \text{slope} = \dfrac{y_2 - y_1}{x_2 - x_1}$



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Example 4 Finding the slope of a line using the formula for the slope

Find the slope of the line through the points (1, 7) and (4, 2) using the formula for the slope of a line. Give your answer correct to two decimal places.

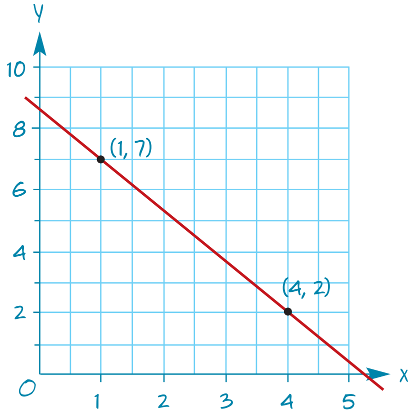
Solution

$$\text{slope} = \frac{y_2 - y_1}{x_2 - x_1}$$

Let $(x_1, y_1) = (1, 7)$ and $(x_2, y_2) = (4, 2)$.

$$\text{slope} = \frac{2 - 7}{4 - 1}$$

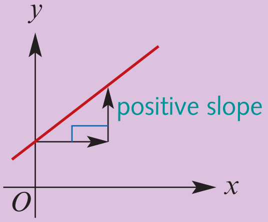
$$= -1.67 \text{ (to 2d.p.)}$$



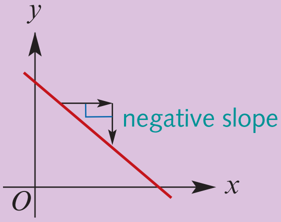
Note: To use this formula it does not matter which point you call (x_1, y_1) and which point you call (x_2, y_2), the rule still works.

Section summary

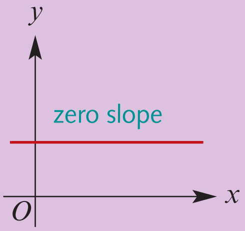
A straight-line graph that rises from left to right is said to have a **positive slope** (positive rise).



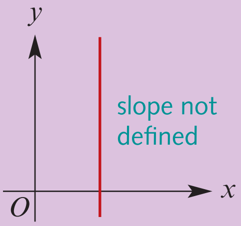
A straight-line graph that falls from left to right is said to have a **negative slope** (negative rise).



A straight-line graph that is horizontal has **zero slope** (‘rise’ = 0).



The slope is **not defined** for a straight-line graph that is vertical.



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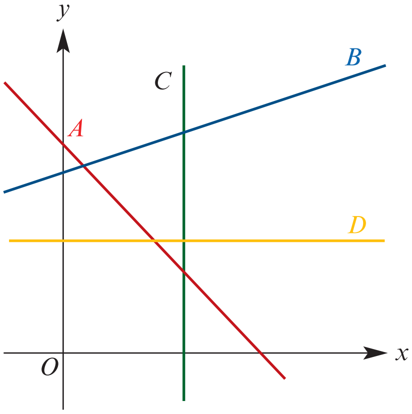
Exercise 6B

Basic ideas

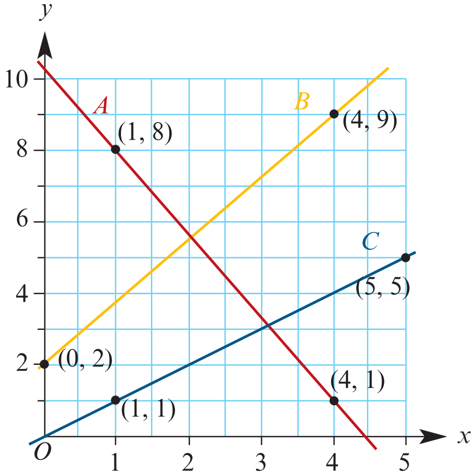
1 Without calculation, identify the slope of each of the straight-line graphs A, B, C and D as: positive, negative, zero, or not defined.



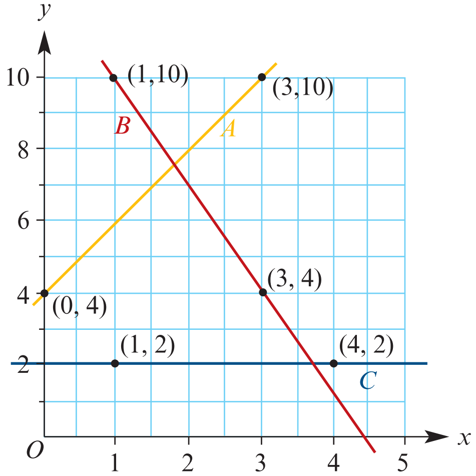
Calculating slopes of lines

Example 2, 3

2 Find the slope of each of the lines (A, B, C) shown on the graph below.



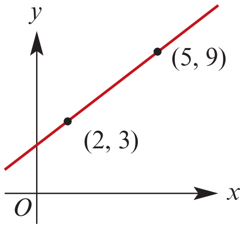
3 Find the slope of each of the lines (A, B, C) shown on the graph below.



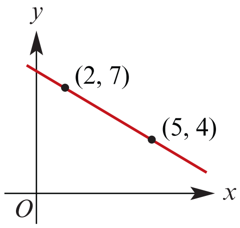
Example 4

4 Find the slope of each of the lines shown.

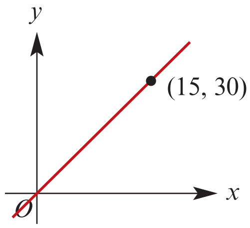
a



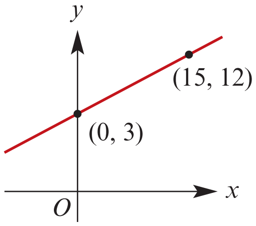
b



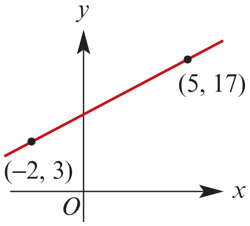
c



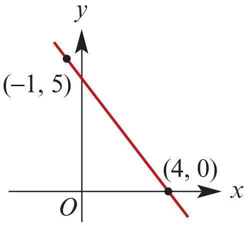
d



e



f



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6C The intercept–slope form of the equation of a straight line

▶ Determining the intercept and slope of a straight-line graph from its equation

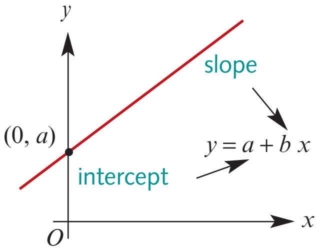
When we write the equation of a straight line in the form:[2]

$$y = a + bx$$

we are using what is called the **intercept–slope form of the equation** of a straight line.

We call $y = a + bx$ the intercept-slope form of the equation of a straight line because:

- a = the **y-intercept** of the graph
- b = the slope of the graph.



The intercept–slope form of the equation of a straight line is useful in modelling relationships in many practical situations. It is also the form used in **bivariate** (two-variable) statistics.

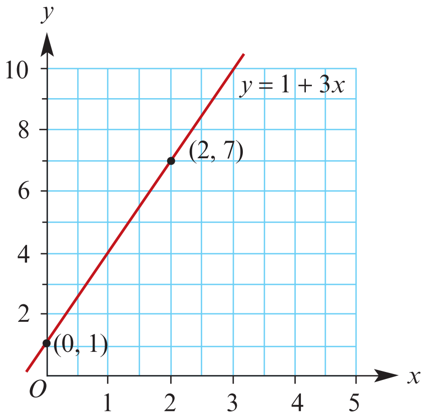
An example of the equation of a straight line written in intercept–slope form is $y = 1 + 3x$.

Its graph is shown opposite.

From the graph we see that the:

$$y\text{-intercept} = 1$$

$$\text{slope} = \frac{7-1}{2-0} = \frac{6}{2} = 3$$



That is:

- the y-intercept corresponds to the (*constant*) term in the equation (intercept = 1)
- the slope is given by the *coefficient of x* in the equation (slope = 3).

---

[2] Note: You may be used to writing the equation of straight line as $y = mx+c$. However, when we are using a straight-line graph to model (represent) real world phenomena, we tend to reverse the order of the terms and use 'a' for the intercept and 'b' for the slope (rather than 'c' and 'm') and write the equation as $y = a + bx$. This is particularly true in performing statistical computations where your calculator will use 'a' for slope and 'b' for the y-intercept. You will see this later in the book, so it is worth making the change now. This is also the language used in the VCE Further Mathematics written examinations.

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Intercept–slope form of the equation of a straight line

If the equation of a straight line is in the intercept–slope form:

$$y = a + bx$$

then: a = the y-intercept of the graph (where the graph cuts the y-axis)

b = the slope of the graph

Example 5 Finding the intercept and slope of a line from its equation

Write down the y-intercept and slope of each of the straight-line graphs defined by the following equations.

a $y = -6 + 9x$ **b** $y = 10 - 5x$ **c** $y = -2x$ **d** $y - 4x = 5$

Solution

For each equation:

1 Write the equation. If it is not in intercept–slope form, rearrange the equation.

2 Write down the y-intercept and slope. When the equation is in intercept–slope form, the value of:

a = the y-intercept (the constant term)

b = the slope (the coefficient of x)

a $y = -6 + 9x$
y-intercept = −6, slope = 9

b $y = 10 - 5x$
y-intercept = 10, slope = −5

c $y = -2x$ or $y = 0 - 2x$
y-intercept = 0, slope = −2

d $y - 4x = 5$ or $y = 5 + 4x$
y-intercept = 5, slope = 4

Example 6 Writing down the equation of a straight line given its *y*-intercept and slope

Write down the equations of the straight lines with the following y-intercepts and slopes.

a y-intercept = 9, slope = 6 **b** y-intercept = 2, slope = −5

c y-intercept = −3, slope = 2

Solution

The equation of a straight line is $y = a + bx$. In this equation, a = y-intercept and b = slope. Form an equation by inserting the given values of the y-intercept and the slope for a and b in the standard equation $y = a + bx$.

a y-intercept = 9, slope = 6
equation : $y = 9 + 6x$

b y-intercept = 2, slope = −5
equation : $y = 2 + (-5)x$
or $y = 2 - 5x$

c y-intercept = −3, slope = 2
equation : $y = -3 + 2x$
or $y = 2x - 3$

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▶ Sketching straight-line graphs

Because only two points are needed to draw a straight line, all we need to do is find two points on the graph and then draw a line passing through these two points. When the equation of a straight line is written in intercept–slope form, one point on the graph is immediately available: the *y*-intercept. A second point can then be quickly calculated by substituting a suitable value of *x* into the equation.

When we draw a graph in this manner, we call it a *sketch graph*.

Example 7 Sketching a straight-line graph from its equation

Sketch the graph of $y = 8 + 2x$.

Solution

1 Write the equation of the line.

$y = 8 + 2x$

2 As the equation is in intercept–slope form, the *y*-intercept is given by the constant term. Write it down.

$y\text{-intercept} = 8$

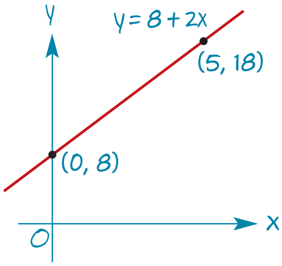
3 Find a second point on the graph. Choose an *x*-value (not 0) that makes the calculation easy: $x = 5$ would be suitable.

When $x = 5$, $y = 8 + 2(5) = 18$

$\therefore$ (5, 18) is a point on the line.

4 To sketch the graph:

- draw a set of labelled axes
- mark in the two points with coordinates
- draw a straight line through the points
- label the line with its equation.



Exercise 6C

Finding slope and intercept of a straight-line graph from its equation

Example 5

1 Write down the *y*-intercepts and slopes of the straight lines with the following equations.

a $y = 5 + 2x$
b $y = 6 - 3x$
c $y = 15 - 5x$
d $y + 3x = 10$
e $y = 3x$
f $4y + 8x = -20$
g $x = y - 4$
h $x = 2y - 6$
i $2x - y = 5$
j $y - 5x = 10$
k $2.5x + 2.5y = 25$
l $y - 2x = 0$
m $y + 3x - 6 = 0$
n $10x - 5y = 20$
o $4x - 5y - 8 = 7$
p $2y - 8 = 2(3x - 6)$

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Finding the equation of a straight-line graph given its y-intercept and slope

Example 6

2 Write down the equation of a line that has:

a y-intercept = 2, slope = 5
b y-intercept = 5, slope = 10
c y-intercept = −2, slope = 4
d y-intercept = 12, slope = −3
e y-intercept = −2, slope = −5
f y-intercept = 1.8, slope = −0.4
g y-intercept = 2.9, slope = −2
h y-intercept = −1.5, slope = −0.5

Sketching straight-line graphs from their equation

Example 7

3 Sketch the graphs of the straight lines with the following equations, clearly showing the y-intercepts and the coordinates of one other point.

a $y = 5 + 2x$
b $y = 5 + 5x$
c $y = 20 - 2x$
d $y = -10 + 10x$
e $y = 4x$
f $y = 16 - 2x$

6D Finding the equation of a straight-line graph from its intercept and slope

We have learned how to construct a straight-line graph from its equation. We can also determine the equation from a graph. In particular, if the graph shows the y-intercept, it is a relatively straightforward procedure.

Finding the equation of a straight-line graph from its intercept and slope

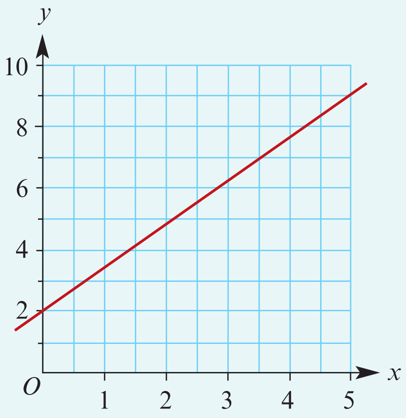
To find the equation of a straight line in intercept–slope form ($y = a + bx$) from its graph:

1. identify the y-intercept (a)
2. use two points on the graph to find the slope (b)
3. substitute these two values into the standard equation $y = a + bx$.

Note: This method *only works* when the graph *scale includes* $x = 0$.

Example 8 Finding the equation of a line: intercept–slope method

Determine the equation of the straight-line graph shown opposite.



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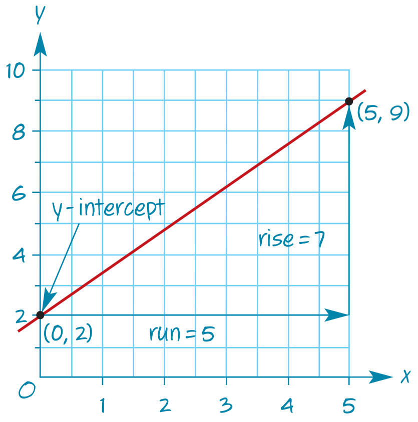
Solution

1. Write the general equation of a line in intercept–slope form.
2. Read the y-intercept from the graph.
3. Find the slope using two well-defined points on the line, for example, (0, 2) and (5, 9).
4. Substitute the values of a and b into the equation.
5. Write your answer.

$y = a + bx$



y-intercept $= 2$ so $a = 2$

$$\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{7}{5} = 1.4 \text{ so } b = 1.4$$

$y = 2 + 1.4x$

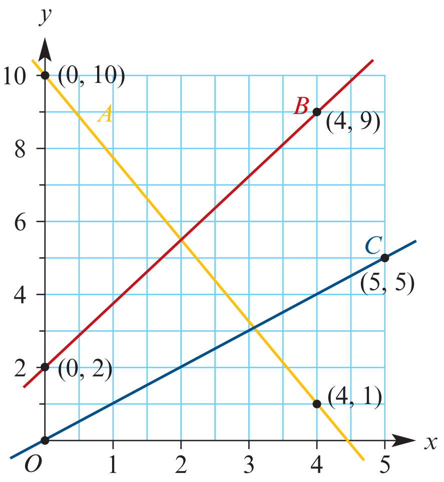
$y = 2 + 1.4x$ is the equation of the line.

Exercise 6D

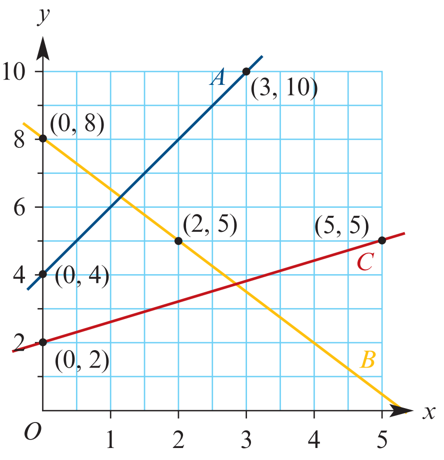
Skillsheet Finding the equation a line from its graph using the intercept–slope method

Example 8

1 Find the equation of each of the lines (A, B, C) shown on the graph below.



2 Find the equations of each of the lines (A, B, C) shown on the graph below.



6E Finding the equation of a straight-line graph using two points on the graph

Unfortunately, not all straight-line graphs show the y-intercept. When this happens, we have to use the two-point method for finding the equation of the line.

Finding the equation of a straight-line graph using two points

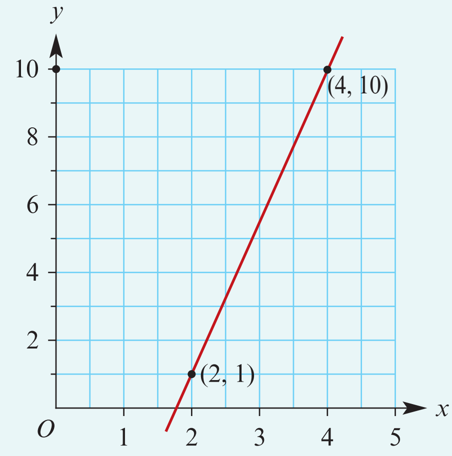
The general equation of a straight-line graph is $y = a + bx$.

1. Use the coordinates of the two points to determine the slope (b).
2. Substitute this value for the slope into the equation. There is now only one unknown, a.
3. Substitute the coordinates of one of the two points on the line into this new equation and solve for the unknown (a).
4. Substitute the values of a and b into the general equation $y = a + bx$ to obtain the equation of the straight line.

Note: This method *works* in all circumstances.

Example 9 Finding the equation of a straight-line using two points on the graph

Find the equation of the line that passes through the points (2, 1) and (4, 10).



Solution

1. Write down the general equation of a straight-line graph.

$y = a + bx$

2. Use the coordinates of the two points on the line to find the slope (b).

$$\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{10 - 1}{4 - 2} = \frac{9}{2} = 4.5$$

so $b = 4.5$

3. Substitute the value of b into the general equation.

$y = a + 4.5x$

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4 To find the value of a, substitute the coordinates of one of the points on the line (either will do) and solve for a.	Using the point (2, 1): $1 = a + 4.5 \times 2$ or $1 = a + 9$ or $a = -8$
5 Substitute the values of a and b into the general equation $y = a + bx$ to find the equation of the line.	Thus, the equation of the line is: $y = -8 + 4.5x$

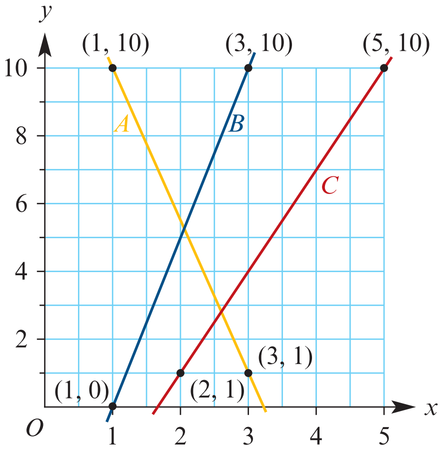
Exercise 6E

Skillsheet

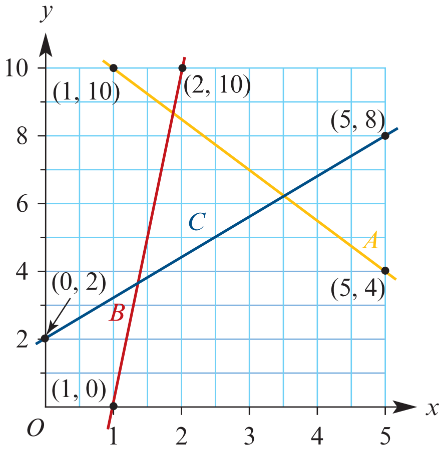
Finding the equation of a line given any two points on its graph

Example 9

1 Find the equation of each of the lines (A, B, C) on the graph below. Write your answers in the form $y = a + bx$.



2 Find the equations of each of the lines (A, B, C) on the graph below. Write your answers in the form $y = a + bx$.



6F Finding the equation of a straight-line graph from two points using a CAS calculator

While the intercept–slope method of finding the equation of a line from its graph is relatively quick and easy to apply, using the two-point method to find the equation of a line can be time consuming. An alternative to using either of these methods is to use the line-fitting facility of your CAS calculator. You will meet this method again when you study the topic ‘Investigating relationships between two numerical variables’ later in the year.

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How to find the equation of a line from two points using the TI-Nspire CAS

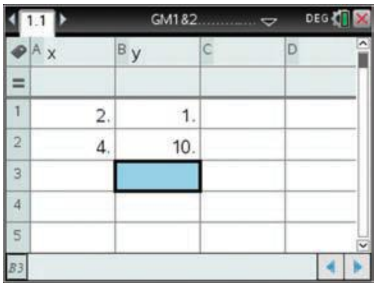
Find the equation of the line that passes through the two points (2, 1) and (4, 10).

Steps

1 Write the coordinates of the two points. Label one point *A*, the other *B*.

2 Start a new document (ctrl + N) and select **Add Lists & Spreadsheet**.

Enter the coordinate values into lists named ***x*** and ***y***.

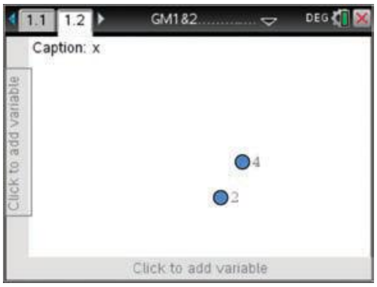


	A x	B y	C	D
=				
1	2.	1.		
2	4.	10.		
3				
4				
5				

3 Plot the two points on a scatterplot. Press ctrl + N and select **Add Data & Statistics**.

(or press on and arrow to and press enter)

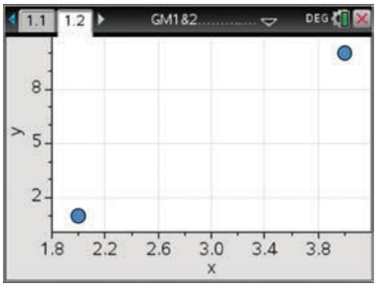
Note: A random display of dots will appear – this is to indicate list data is available for plotting. It is not a statistical plot.



4 To construct a scatterplot:

a Press tab and select the variable ***x*** from the list. Press enter to paste the variable ***x*** to the ***x***-axis.

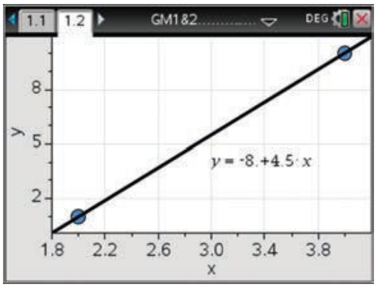
b Press tab again and select the variable ***y*** from the list. Press enter to paste the variable ***y*** to the ***y***-axis axis to generate the required scatter plot.



5 Use the **Regression** command to draw a line through the two points and determine its equation.

Press menu**>Analyze>Regression>Show Linear (a+bx)** and enter to complete the task.

Correct to one decimal place, the equation of the line is: $y = -8.0 + 4.5x$.



6 Write your answer.

The equation of the line is $y = -8 + 4.5x$.

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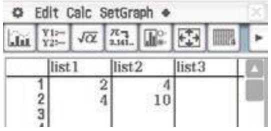
How to find the equation of a line from two points using the ClassPad

Find the equation of the line that passes through the two points (2, 4) and (4, 10).

Steps

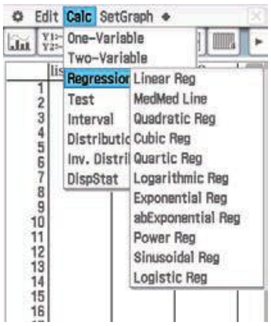
1 Open the **Statistics** application and enter the coordinate values into the lists as shown.



	list1	list2	list3
1	2	4	
2	4	10	
3			

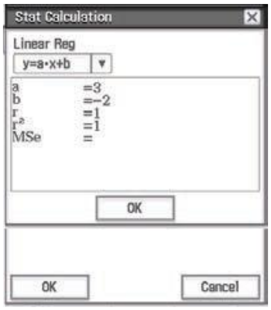
2 To find the equation of the line $y = ax + b$ that passes through the two points:

- Select **Calc** from the menu bar
- Select **Regression** and **Linear Reg**
- Ensure that the **Set Calculation** dialog box is set as shown
- Press OK.

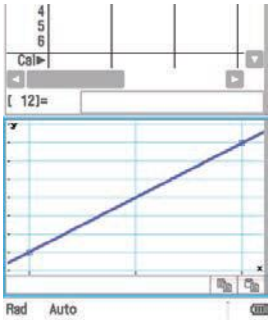


Linear Reg
XList: list1
YList: list2
Freq: 1
Copy Formula: Off
Copy Residual: Off

3 The results are given in a **Stat Calculation** dialog box.



Linear Reg
y=a·x+b
a =3
b =−2
r =1
r² =1
MSe =



The equation of the line is $y = 3x - 2$.

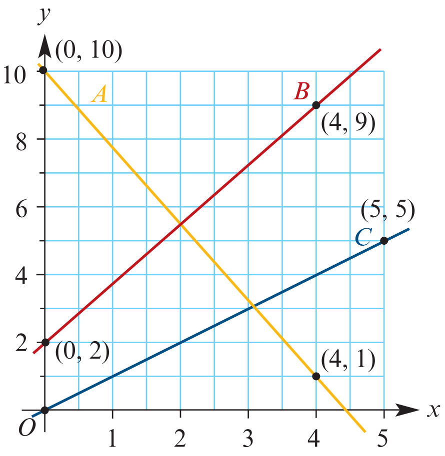
Note: Tapping **OK** will automatically display the graph window with the line drawn through the two points. This confirms that the line passes through the two points.

Exercise 6F

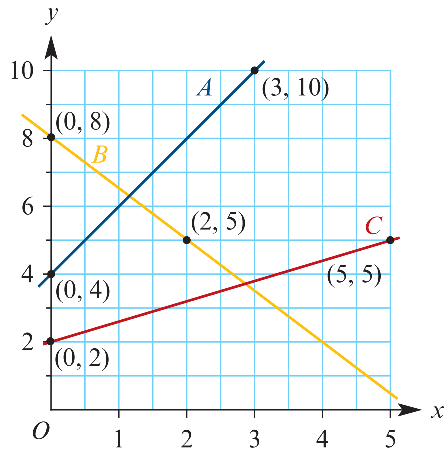
Using a CAS calculator to find the equation of a line from two points

Note: This exercise repeats Exercises 6D and 6E, but this time using a graphics calculator.

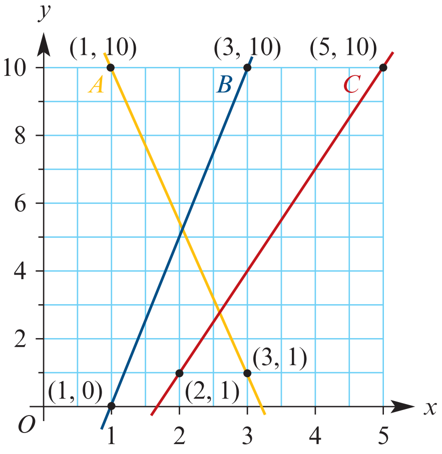
1 Use a graphics calculator to find the equation of each of the lines (*A, B, C*) shown on the graph below. Write your answers in the form $y = a + bx$.



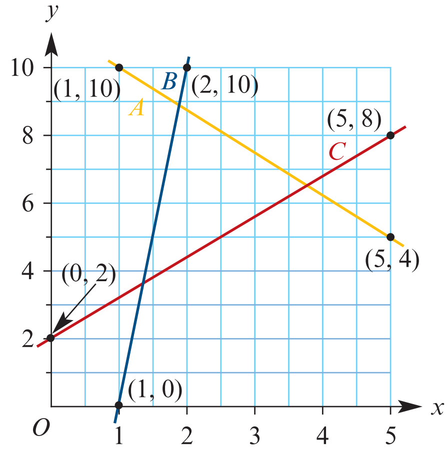
2 Use a graphics calculator to find the equation of each of the lines (*A*, *B*, *C*) shown on the graph below. Write your answers in the form $y = a + bx$.



3 Use a graphics calculator to find the equation of each of the lines (*A*, *B*, *C*) shown on the graph below. Write your answers in the form $y = a + bx$.



4 Use a graphics calculator to find the equation of each of the lines (*A*, *B*, *C*) shown on the graph below. Write your answers in the form $y = a + bx$.



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6G Linear modelling

Many real life relationships between two variables can be described mathematically by linear (straight-line) equations. This is called *linear modelling*.

These linear models can be then used to solve problems such as finding the time taken to fill a partially filled swimming pool with water, estimating the depreciating value of a car over time or describing the growth of a plant over time.

▶ Modelling plant growth with a linear equation

Some plants, such as tomato plants, grow remarkably quickly.

When first planted, the height of this plant was 5 cm.

The plant then grows at a constant rate of 6 cm per week for the next 10 weeks.

From this information, we can now construct a mathematical model that can be used to chart the growth of the plant over the following weeks and predict its height at any time during the first 10 weeks after planting.

Constructing a linear model

Let h be the height of the plant (in cm).

Let t be the time (in weeks) after it was planted.

For a linear growth model we can write:

$$h = a + bt$$

where:

- a is the initial height of the plant; in this case, 5 cm (in graphical terms, the y-intercept)
- b is the constant rate at which the plant's height increases each week; in this case, 6 cm per week (in graphical terms, the slope of the line).

Thus we can write: $h = 5 + 6t \quad \text{for } 0 \le t \le 10$

The graph for this model is plotted opposite.

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Three important features of the linear model $h = 5 + 6t$ for $0 \le t \le 10$ should be noted:

- The *h-intercept* gives the height of the plant at the start; that is, its height when $t = 0$. The plant was 5 cm tall when it was first planted.
- The *slope* of the graph gives the growth rate of the plant. The plant grows at a rate of 6 cm per week; that is, each week the height of the plant increases by 6 cm.
- The graph is only plotted for $0 \le t \le 10$. This is because the model is only valid for the time when the plant is growing at the constant rate of 6 cm a week.

Note: The expression $0 \le t \le 10$ is included to indicate the range of number of weeks for which the model is valid. In more formal language this would be called the domain of the model.

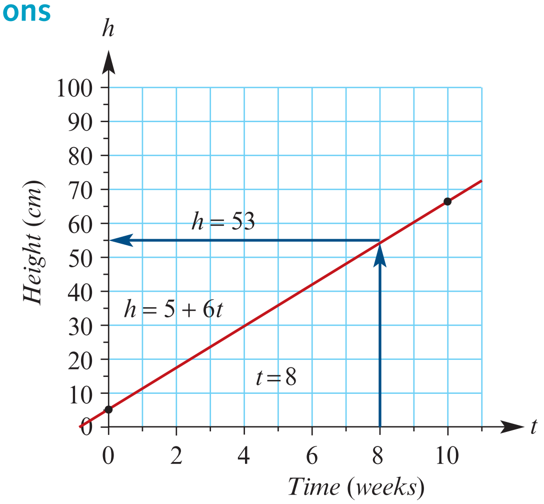
Using a linear model to make predictions

To use the mathematical model to make predictions, we simply substitute a value of t into the model and evaluate.

For example, after eight weeks growth ($t = 8$), the model predicts the height of the plant to be

$$h = 5 + 6(8) = 53 \text{ cm}$$

This value could also be read directly from the graph, as shown opposite.



Exercise 6G-1

Constructing and analysing linear models

1 A tree is 910 cm tall when first measured. For the next five years its height increases at a constant rate of 16 cm per year.
Let h be the height of the tree (in cm).
Let t the time in years after the tree was first measured.

- **a** Write down a linear model in terms of h and t to represent this situation.
- **b** Sketch the graph showing the coordinates of the intercept and its end point.
- **c** Use the model to predict the height of the tree 4.5 years after it was first measured.

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2 An empty 20 L cylindrical beer keg is to be filled with beer at a constant rate of 5 litres per minute.
Let V be the volume of beer in the keg after t minutes.

- **a** The beer keg is filled in 4 minutes, write down a linear model in terms of V and t to represent this situation.
- **b** Sketch the graph showing the coordinates of the intercept and its end point.
- **c** Use the model to predict the volume of beer in the keg after 3.2 minutes.

3 A home waste removal service charges \$80 to come to your property. It then charges \$120 for each cubic metre of waste it removes. The maximum amount of waste that can be removed in one visit is 8 cubic metres.
Let c be the total charge for removing w cubic metres of waste.

- **a** Write down a linear model in terms of c and w to represent this situation.
- **b** Sketch the graph showing the coordinates of the intercept and its end point.
- **c** Use the model to predict the cost of removing 5 cubic metres of waste.

4 A motorist fills the tank of her car with unleaded petrol, which costs \$1.57 per litre. Her tank can hold a maximum of 60 litres of petrol. When she started filling her tank, there was already 7 litres in her tank.
Let c be the cost of adding v litres of petrol to the tank.

- **a** Write down a linear model in terms of c and v to represent this situation.
- **b** Sketch the graph of showing the coordinates of the intercept and its end point.
- **c** Use the model to predict the cost of filling the tank of her car with petrol.

5 A business buys a new photocopier for \$25 000. It plans to depreciate its value by \$4375 per year for five years, at which time it will be sold.
Let V be the value of the photocopier after t years.

- **a** Write down a linear model in terms of V and t to represent this situation.
- **b** Sketch the graph showing the coordinates of the intercept and its end point.
- **c** Use the model to predict the depreciated value of the photocopier after 2.6 years.

6 A swimming pool when full contains 10 000 litres of water. Due to a leak, it loses on average 200 litres of water per day.
Let V be the volume of water remaining in the pool after t days.

- **a** The pool continues to leak. How long will it take to empty the pool?
- **b** Write down a linear model in terms of V and t to represent this situation.
- **c** Sketch the graph showing the coordinates of the intercept and its end point.
- **d** Use the model to predict the volume of water left in the pool after 30 days.

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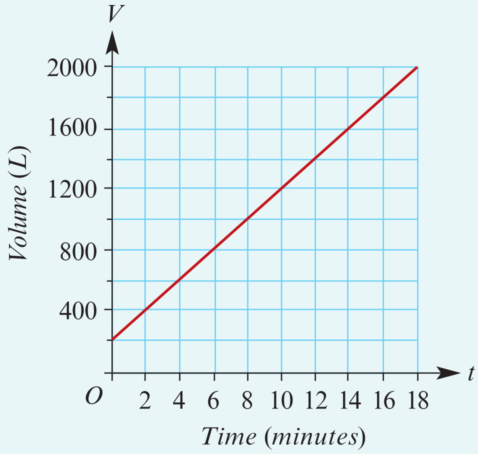
► Interpreting and analysing the graphs of linear models

Example 10 Graphs of linear models with a positive slope

Water is pumped into a partially full tank. The graph gives the volume of water V (in litres) after t minutes.



a How much water is in the tank at the start ($t = 0$)?

b How much water is in the tank after 10 minutes ($t = 10$)?

c The tank holds 2000 L. How long does it take to fill?

d Find the equation of the line in terms of V and t.

e Use the equation to calculate the volume of water in the tank after 15 minutes.

f At what rate is the water pumped into the tank; that is, how many litres are pumped into the tank each minute?

Solution

a Read from the graph (when $t = 0$, $V = 200$). **a** 200 L

b Read from the graph (when $t = 10$, $V = 1200$). **b** 1200 L

c Read from the graph (when $V = 2000$, $t = 18$). **c** 18 minutes

d The equation of the line is $V = a + bt$.

a is the V-intercept. Read from the graph.
b is the slope. Calculate using two points on the graph, say (0, 200) and (18, 2000).

Note: You can use your calculator to find the equation of the line if you wish.

d $V = a + bt$

$a = 200$

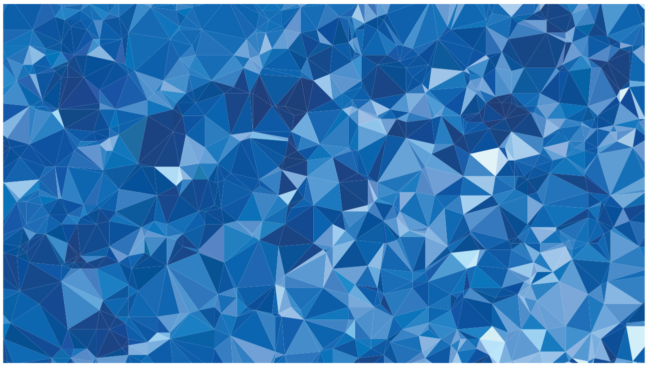
$$b = \text{slope} = \frac{\text{rise}}{\text{run}} = \frac{2000 - 200}{18 - 0}$$

$$= 100$$

$$\therefore\ V = 200 + 100t \ (t \geq 0)$$

e Substitute $t = 15$ into the equation. Evaluate. **e** $V = 200 + 100(15) = 1700$ L

f The rate at which water is pumped into the tank is given by the slope of the graph, 100 (from **d**). **f** 100 L/min



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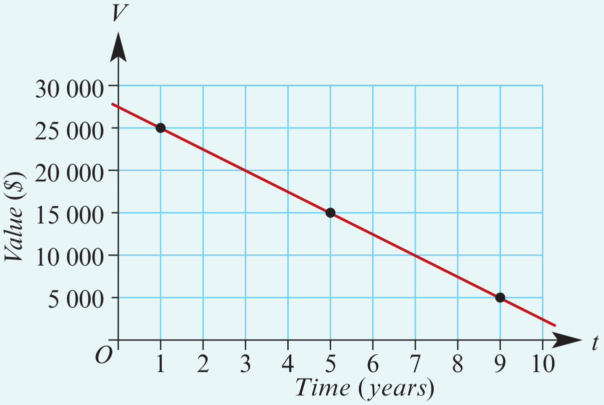
Example 11 Graphs of linear models with a negative slope



The value of new cars depreciates with time. The graph shows how the value V (in dollars) of a new car depreciates with time t (in years).

a What was the value of the car when it was new?

b What was the value of the car when it was 5 years old?

c Find the equation of the line in terms of V and t.

d At what rate does the value of the car depreciate with time; that is, by how much does its value decrease each year?

e When does the equation predict the car will have no (zero) value?

Solution

a Read from the graph (when $t = 0, V = 27\,500$).

a \$27 500

b Read from the graph (when $t = 5, V = 15\,000$).

b \$15 000

c The equation of the line is $V = a + bt$.

- a is the V-intercept. Read from the graph.
- b is the slope. Calculate using two points on the graph, say (1, 25 000) and (9, 5000).

Note: You can use your calculator to find the equation of the line if you wish.

c

$$V = a + bt$$

$$a = 27\,500$$

$$b = \text{slope} = \frac{25\,000 - 5000}{1 - 9}$$

$$= -2500$$

$$\therefore\ V = 27\,500 - 2500t \quad \text{for} \quad t \geq 0$$

d The slope of the line is –2500, so the car depreciates in value by \$2500 per year.

d \$2500 per year

e Substitute into the equation and solve for t.

e

$$0 = 27\,500 - 2500t$$

$$2500t = 27\,500$$

$$\therefore t = \frac{27\,500}{2500} = 11 \text{ years}$$

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Exercise 6G-2

Interpreting the graphs of linear models in their context

Example 10, 11

1 A phone company charges a monthly service fee plus the cost of calls. The graph opposite gives the total monthly charge, C dollars, for making n calls. This includes the service fee.

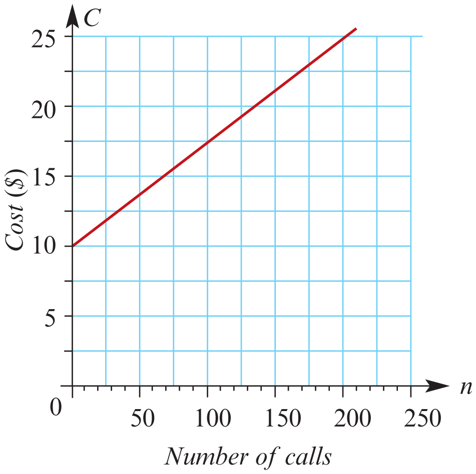
a How much is the monthly service fee ($n = 0$)?

b How much does the company charge if you make 100 calls a month?

c Find the equation of the line in terms of C and n.

d Use the equation to calculate the cost of making 300 calls in a month.

e How much does the company charge per call?



2 The graph opposite shows the volume of saline solution, V mL, remaining in the reservoir of a saline drip after t minutes.

a How much saline solution was in the reservoir at the start?

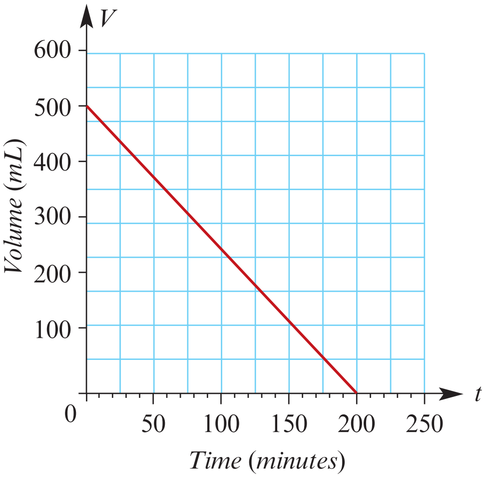
b How much saline solution remains in the reservoir after 40 minutes? Read the result from the graph.

c How long does it take for the reservoir to empty?

d Find the equation of the line in terms of V and t.

e Use the equation to calculate the amount of saline solution in the reservoir after 115 minutes.

f At what rate (in mL/minute) is the saline solution flowing out of the drip?



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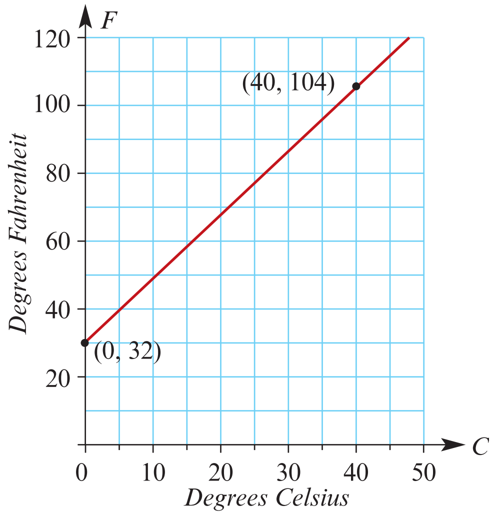
3 The graph opposite can be used to convert temperatures in degrees Celsius (C) to temperatures in degrees Fahrenheit (F).



a Find the equation of the line in terms of F and C.

b Use the equation to predict the temperature in degrees Fahrenheit when the temperature in degrees Celsius is:

i 50°C **ii** 150°C

iii −40°C

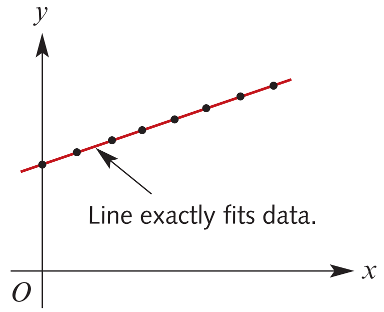
c Complete the following sentence by filling in the box.

When the temperature in Celsius increases by 1 degree, the temperature in Fahrenheit increases by ☐ degrees.

▶ Line of best fit

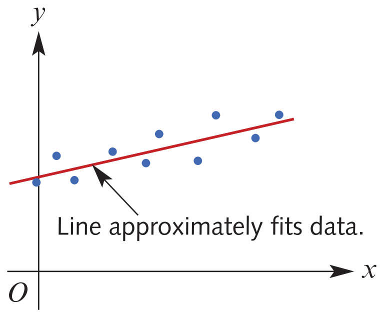
For all of the linear models we have dealt with so far, the slope for the model was either given or calculated from a straight-line graph. Linear models can also be created from data.

Real life relationships between variables are rarely perfectly linear, as shown in the plot opposite.



The data in the plot opposite do not lie exactly on the line, but the values do follow a linear trend. See the plot opposite.

Being able to model this linear trend is a good start to understanding any underlying relationship between the variables.



When a straight line does not exactly fit a set of data, we draw in a **line of best fit**. There are many ways of doing this. The simplest is to use a ruler to draw a line on the graph that seems to balance out the points around the line. This is called **fitting a line 'by eye'**.

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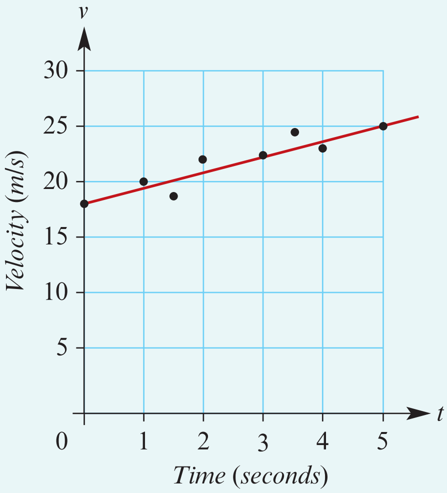
Example 12 Using a line of best fit to model a linear trend in data

A straight line has been fitted to a set of data that recorded the velocity v (in m/s) of an accelerating car at time t seconds.



a What is the car's velocity when $t = 0$?

b The slope of the line can be used to find the car's average acceleration (in m/s^2). What is the value of the slope?

c Write down the equation of the line in terms of v and t.

d Use the equation to predict the car's velocity after:

  i 2.5 seconds

  ii 7 seconds.

e When will the car's velocity be 30 m/s?

Solution

a Read from the graph (when $t = 0$, $v = 18$).

a $v = 18$ m/s

b Calculate the slope by using two points on the graph, say (0, 18) and (5, 25).

b $\text{slope} = \dfrac{\text{rise}}{\text{run}} = \dfrac{25 - 18}{5 - 0} = 1.4$

$\therefore$ Average acceleration $= 1.4$ m/s^2

c The equation of the line is $v = a + bt$.

- a is the y-intercept. Read from the graph.
- b is the slope, already calculated.

c
$$v = a + bt$$
$$a = 18$$
$$b = 1.4$$
$$\therefore v = 18 + 1.4t$$

d Substitute:

  i $t = 2.5$ in the equation of the straight line and evaluate

  ii $t = 7$ in the equation of the straight line and evaluate.

d **i** $v = 18 + 1.4 \times 2.5 = 21.5$

$\therefore$ velocity after 2.5 seconds $= 21.50$ m/s

ii $v = 18 + 1.4 \times 7 = 27.8$

$\therefore$ velocity after 7 seconds $= 27.8$ m/s

e Substitute $v = 30$ into the equation and solve for t.

e
$$30 = 18.5 + 1.4\,t$$
$$30 - 18 = 1.4\,t$$
$$12 = 1.4\,t$$
$$t = \frac{12}{1.4} = 8.6$$

The car will have a velocity of 30 m/s after 8.6 seconds (to 1 d.p.).

Extrapolation and interpolation

Not all predictions we make with a line of best fit are equally reliable. The most reliable are made when we use the line to make a prediction that lies *within* the range of data used to construct the line. When we do this, we say we are *interpolating*.

For example, when we used the equation of the line of best fit to predict the velocity of the car after 2.5 seconds we were interpolating.

Interpolation

Using a model to make a prediction within the range of data used to construct the model is called **interpolation**.

Predictions are far less reliable when we use the line to make a prediction that lies *outside* the range of data used to construct the line. When we do this, we say we are *extrapolating*.

For example, when we used the equation of the line of best fit to predict the velocity of the car after 7 seconds or when we were using the model to predict the time the car would take to reach a speed of 30 m/s, we were extrapolating; that is, we were going beyond the data.

Extrapolation

Using a model to make a prediction beyond the range of data used to construct the model is called **extrapolation**.

Extrapolation is a less reliable process than interpolation because we are working beyond the range of data used to construct the model and we do not know if the model still applies.

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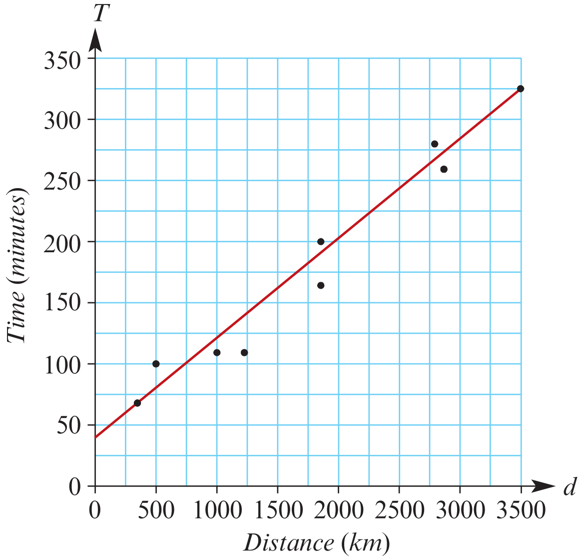
Exercise 6G-3

Determining the equation of a line of best fit from its graph

Example 12

1 A straight line has been fitted by eye to a plot of travelling time T (in minutes) against distance travelled d (in km), for nine plane trips between nine different cities.



a Determine the equation of the line in terms of T and d. Give the values of the intercept and slope.

b Use the line to predict the travelling time between two cities:

i 2500 km apart

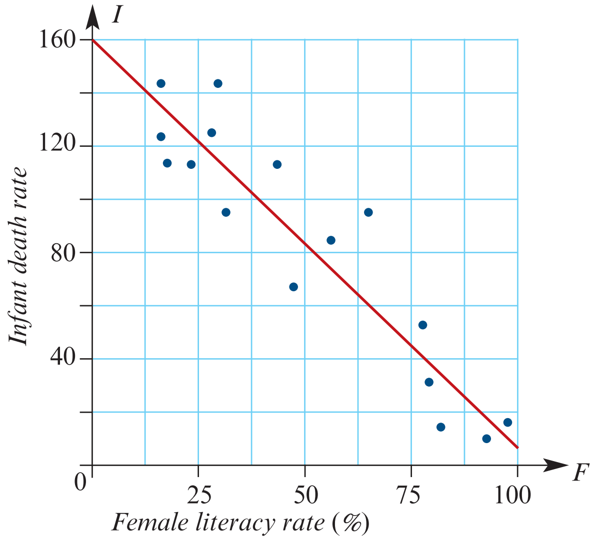
ii 4500 km apart.

c When making the predictions in **b**, are you interpolating or extrapolating? Explain your answer.

d Complete the following sentence by filling in the box.

When the distance travelled increased by 100 km, the travelling time increased by ☐ minutes.

2 A straight line has been fitted by eye to a plot of infant death rate I (per 100 000 people) against female literacy rate F (%) for a number of countries.



a Determine the equation of the line in terms of I and F.

b Predict the infant death rate where the female literacy level is:

i 40% **ii** 60% **iii** 95%

c When making the predictions in **b**, are you interpolating or extrapolating? Explain.

d Complete the following sentence by filling in the box.

When the female literacy rate increases by 1%, the infant death rate decreases by ☐ per 100 000.

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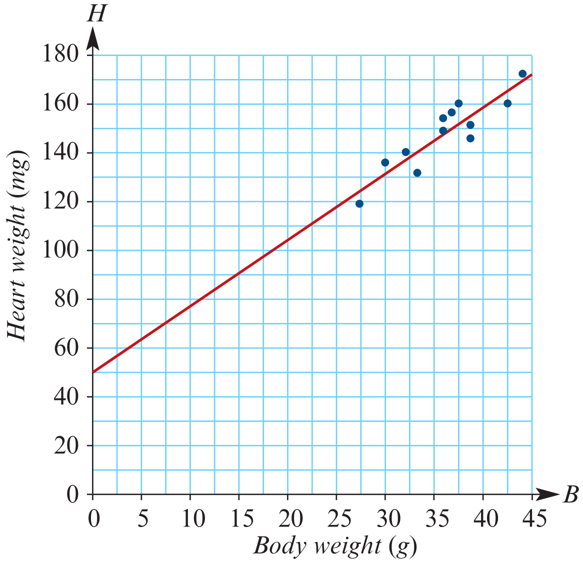
3 A straight line has been fitted by eye to a plot of heart weight H (in mg) against body weight B (in g) for 12 laboratory mice.



a Determine the equation of the line in terms of H and B. Give the values of the intercept and slope.

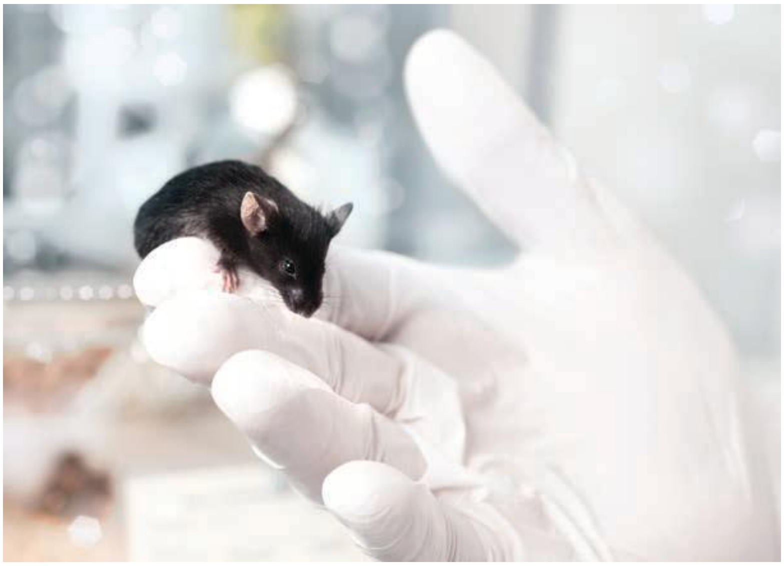
b Use the equation to predict the heart weight of a mouse with a body weight of :

i 40 g **ii** 20 g

c When making the predictions in **b**, are you interpolating or extrapolating? Explain your answer.

d Complete the following sentence by filling in the box.

When the body weight increases by 1g, heart weight increases by ☐.



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▶ Piecewise linear graphs

Sometimes a situation requires two linear graphs to obtain a suitable model. The graphs we use to model such situations are called **piecewise linear graphs**.

Example 13 Constructing a piecewise linear graph model

The amount, C dollars, charged to supply and deliver x m^3 of crushed rock is given by the equations:

$$C = 50 + 40x \qquad (0 \le x < 3)$$

$$C = 80 + 30x \qquad (3 \le x < 8)$$

a Use the appropriate equation to determine the cost to supply and deliver the following amounts of crushed rock.

i 2.5 m^3 **ii** 3 m^3 **iii** 6 m^3

b Use the equations to construct a piecewise linear graph for $0 \le x \le 8$.

Solution

a **1** Write the equations.

2 Then, in each case:

- choose the appropriate equation.
- substitute the value of x and evaluate.
- write down your answer.

a $C = 50 + 40x \ (0 \le x < 3)$

$C = 80 + 30x \ (3 \le x \le 8)$

i When $x = 2.5$

$C = 50 + 40(2.5) = 150$

Cost for 2.5 m^3 of crushed rock is \$150.

ii When $x = 3$

$C = 80 + 30(3) = 170$

Cost for 3 m^3 of crushed rock is \$170.

iii When $x = 6$,

$C = 80 + 30(6) = 260$

Cost for 6 m^{3}of crushed rock is \$260.

b The graph has two line segments.

1 Determine the coordinates of the end points of both lines.

2 Draw a set of labelled axes and mark in the points with their coordinates.

3 Join up the end points of each line segment with a straight line.

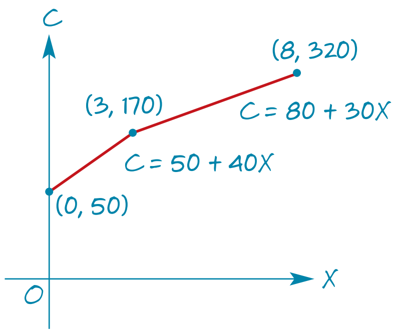
4 Label each line segment with its equation.

b $x = 0 : C = 50 + 40(0) = 50$

$x = 3 : C = 50 + 40(3) = 170$

$x = 3 : C = 80 + 30(3) = 170$

$x = 8 : C = 80 + 30(8) = 320$



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Exercise 6G-4

Using a piecewise linear graph to model and analyse practical situations

Example 13

1 An empty tank is being filled from a mountain spring. For the first 30 minutes, the equation giving the volume, V, of water in the tank (in litres) at time t minutes is:

$$V = 15t \quad (0 \leq t \leq 30)$$

After 30 minutes, the flow from the spring slows down. For the next 70 minutes, the equation giving the volume of water in the tank at time, t, as given by the equation:

$$V = 150 + 10t \quad (30 < t \leq 100)$$

a Use the appropriate equation to determine the volume of water in the tank after:

i 20 minutes **ii** 30 minutes **iii** 60 minutes **iv** 100 minutes.

b Use the equations to construct a piecewise linear graph for $0 \leq t \leq 100$.

2 For the first 25 seconds of the journey of a train between stations, the speed, S, of the train (in metres/second) after t seconds is given by:

$$S = 0.8t \quad (0 \leq t \leq 25)$$

For the next 180 seconds, the train travels at a constant speed of 20 metres/second as given by the equation:

$$S = 20 \quad (25 < t \leq 205)$$

Finally, after travelling for 205 seconds, the driver applies the brakes and the train comes to rest after a further 25 seconds as given by the equation:

$$S = 184 - 0.8t \quad (205 < t \leq 230)$$

a Use the appropriate equation to determine the speed of the train after:

i 10 seconds **ii** 60 seconds **iii** 180 seconds **iv** 210 seconds.

b Use the equations to construct a piecewise linear graph for $0 \leq t \leq 230$.

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Key ideas and chapter summary

Slope of a straight-line graph

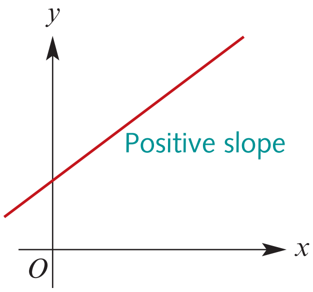
slope of a straight-line graph is defined to be:

$$\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}$$

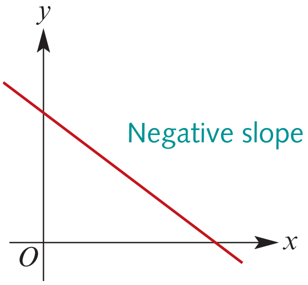
where (x_1, y_1) and (x_2, y_2) are two points on the line.

Positive and negative slope

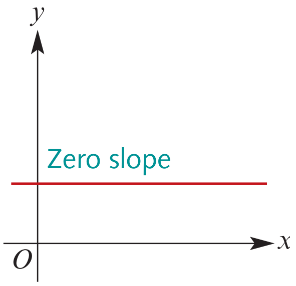
If the line rises to the right, the slope is **positive**.



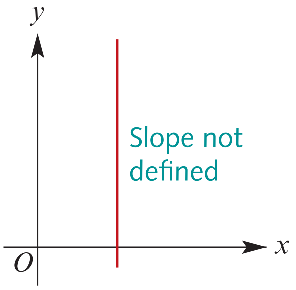
If the line falls to the right, the slope is **negative**.



If the line is horizontal, the slope is **zero**.



If the line is vertical, the slope is **undefined**.



Equation of straight-line graph: the intercept–slope form

The equation of a straight line can take several forms.

The **intercept–slope form** is:

$$y = a + bx$$

where a is the **y-intercept** and b is the **slope** of the line.

Linear model

A **linear model** has a linear equation or relation of the form:

$$y = a + bx \quad \text{where} \quad c \le x \le d$$

where a, b, c and d are constants.

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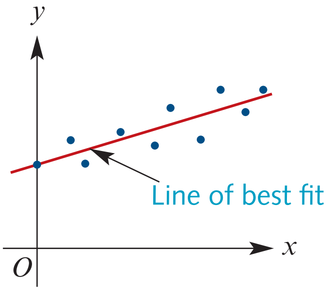
Line of best fit	A **line of best fit** is used to model the linear relationship between two variables. This is needed when the data values do not lie exactly on a straight line. 
Extrapolation	Using a line of best fit to make a prediction beyond the range of data used to construct the model is called **extrapolation**.
Interpolation	Using a line of best fit to make a prediction within the range of data used to construct the line is called **interpolation**.
Piecewise linear graphs	**Piecewise linear graphs** are used in practical situations where more than one linear equation is needed to model the relationship between two variables.

Skills check

Having completed this chapter you should be able to:

- recognise a linear equation written in intercept–slope form
- determine the intercept and slope of a straight-line graph from its equation
- determine the slope of a straight line from its graph
- determine the *y*-intercept of a straight line from its graph (if shown)
- determine the equation of a straight line, given its graph
- construct a linear model to represent a practical situation using a linear equation or a straight-line graph
- interpret the slope and the intercept of a straight-line graph in terms of its context and use the equation to make predictions being aware of the dangers of extrapolation
- analyse and interpret a line of best fit
- construct a piecewise linear graph used to model a practical situation.

Multiple-choice questions

1 The equation of a straight line is $y = 4 + 3x$. When $x = 2$, y is:

A 2 **B** 3 **C** 4 **D** 6 **E** 10

2 The equation of a straight line is $y = 5 + 4x$. The y-intercept is:

A 2 **B** 3 **C** 4 **D** 5 **E** 20

3 The equation of a straight line is $y = 10 - 3x$. The slope is:

A −3 **B** 0 **C** 3 **D** 7 **E** 10

4 The equation of a straight line is $y - 2x = 3$. The slope is:

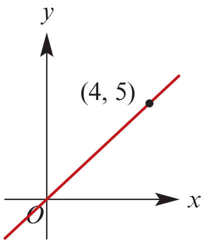
A −3 **B** −2 **C** 0 **D** 2 **E** 3

5 The slope of the line passing through the points (5, 8) and (9, 5) is:

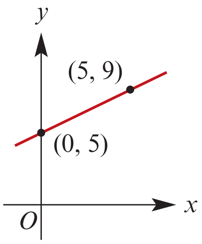
A −1.3 **B** −1 **C** −0.75 **D** 0.75 **E** 1.3

6 The graph of $y = 4 + 5x$ is:

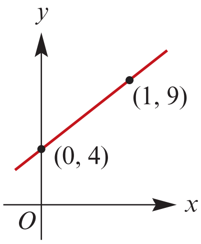
A



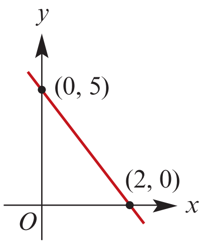
B



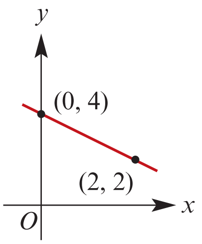
C



D

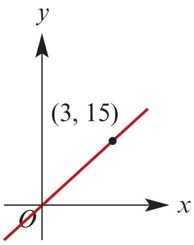


E

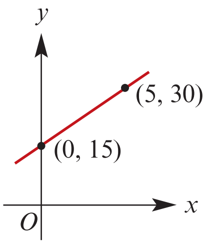


7 The graph of $y = 15 - 3x$ is:

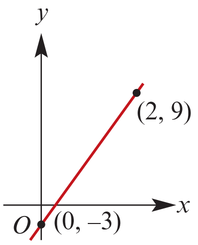
A



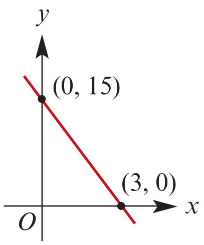
B



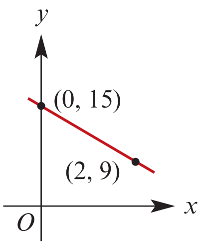
C



D



E



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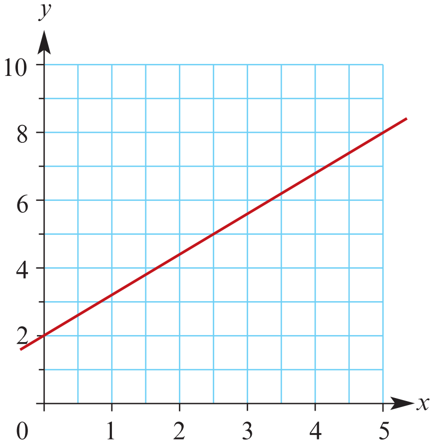
Questions 8 and 9 relate to the following graph



8 The y-intercept is:

A −2

B 0

C 2

D 5

E 8

9 The slope is:

A 1.6

B 1.2

C 2

D 5

E 8

10 The slope of the line in the graph shown opposite is:

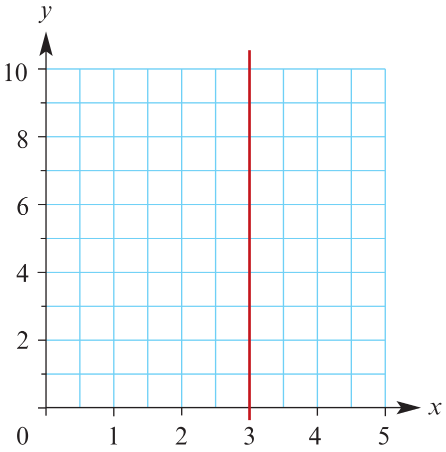
A negative

B zero

C positive

D three

E not defined



11 The equation of the graph shown opposite is:

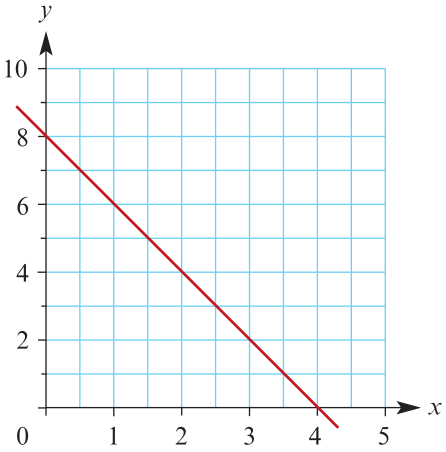
A $y = -2 + 8x$

B $y = 4 - 2x$

C $y = 8 - 2x$

D $y = 4 + 2x$

E $y = 8 + 2x$



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12 The equation of the graph shown opposite is:

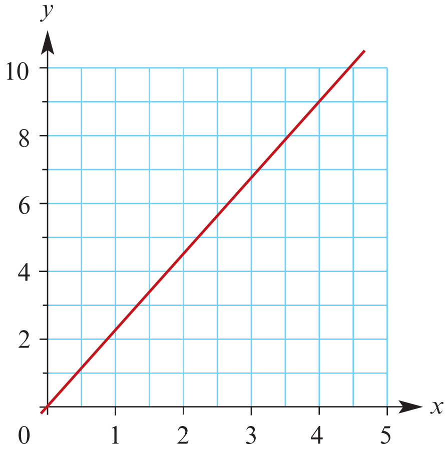
A $y = -2.25x$

B $y = 2.25.x$

C $y = -9x$

D $y = 9x$

E $y = 1 + 2x$



13 Which of the following points lies on the line $y = -5 + 10x$?

A (1, –5) **B** (1, 5) **C** (1, 15) **D** (2, 20) **E** (2, 23)

14 The equation of the graph shown opposite is:

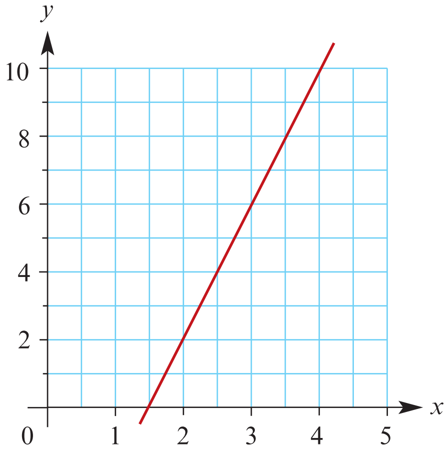
A $y = -6 + 4x$

B $y = -8 + 4x$

C $y = -4 - 4x$

D $y = 2.5 - 4x$

E $y = -4 + 2.5x$



15 The graph opposite shows the height of a small sapling, h, as it increases with time, t. Its growth rate is closest to:

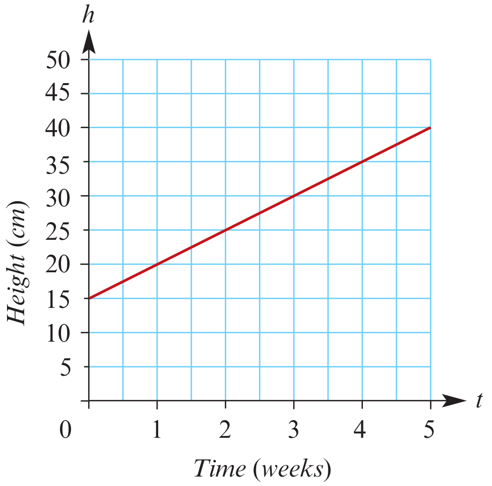
A 1 cm/week

B 3 cm/week

C 5 cm/week

D 8 cm/week

E 15 cm/week



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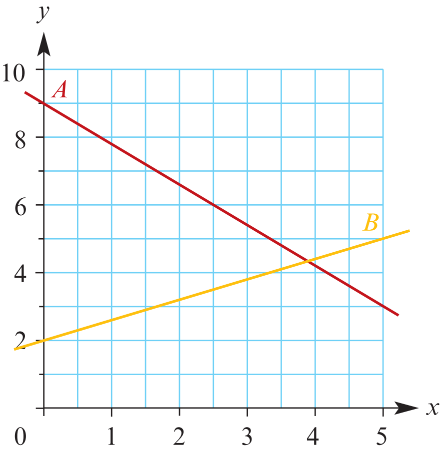
Short-answer questions

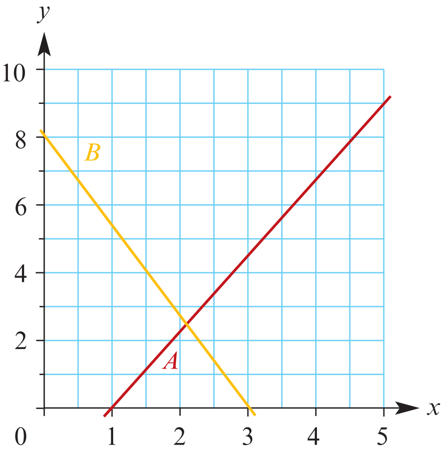
1 Plot the graphs of these linear relations by hand.

a $y = 2 + 5x$ **b** $y = 12 - x$ **c** $y = -2 + 4x$

2 Find the slope of each of the lines A and B shown on the graph below.



3 Find the slope of each of the lines A and B shown on the graph below.



4 A linear model for the amount C, in dollars, charged to deliver w cubic metres of builder's sand is given by $c = 95 + 110w$ for $0 \leq w \leq 7$

a Use the model to determine the total cost of delivering 6 cubic metres of sand.

b When the initial cost of $95 is paid, what is the cost for each additional cubic metres of builder's sand?

Extended-response questions

1 A new piece of machinery is purchased by a business for $300 000. Its value is then depreciated each month using the graph opposite.

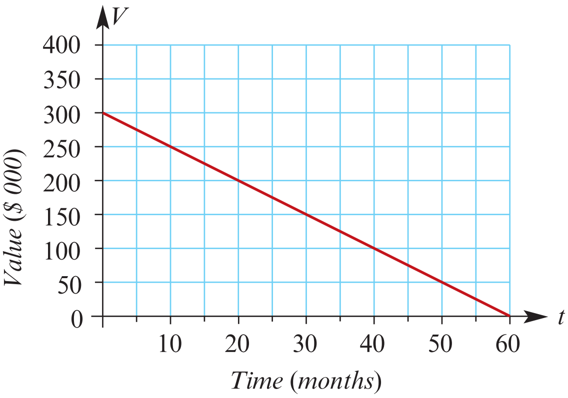
a What is the value of the machine after 20 months?

b When does the line predict that the machine has no value?

c Find the equation of the line in terms of value V and time t.

d Use the equation to predict the value of the machine after 3 years.

e By how much does the machine depreciate in value each month?



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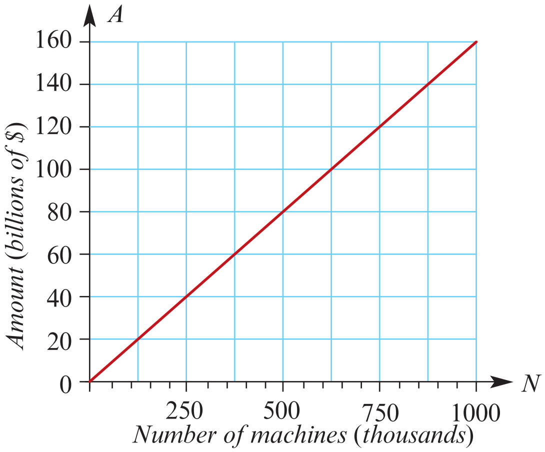
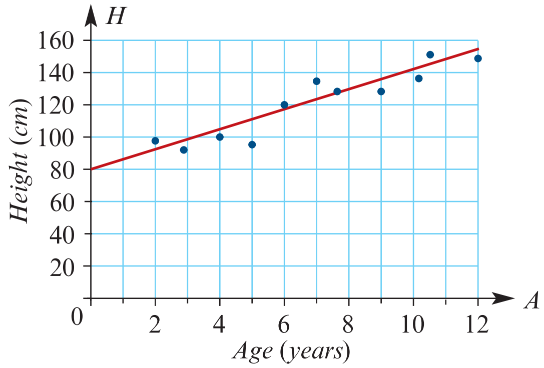
2 The amount of money transacted through ATMs has increased with the number of ATMs available. The graph charts this increase.



- **a** What was the amount of money transacted through ATMs when there were 500 000 machines?
- **b** Find the equation of the line in terms of amount of money transacted, A, and number of ATMs, N. (Leave A in billions and N in thousands).
- **c** Use the equation to predict the amount transacted when there were 600 000 machines.
- **d** If the same rule applies, how much money is predicted to be transacted through ATM machines when there are 1 500 000 machines?
- **e** By how much does the amount of money transacted through ATMs increase with each 1000 extra ATMs?

3 The heights, H, of a number of children are shown plotted against age, A. Also shown is a line of best fit.



- **a** Find the equation of the line of best fit in terms of H and A.
- **b** Use the equation to predict the height of a child aged 3.
- **c** Complete the following sentence by filling in the box. The equation of the line of best fit tells us that, each year, children's heights increase by ☐ cm.

4 To conserve water one charging system increases the amount people pay as the amount of water used increases. The charging system is modelled by:

$$C = 5 + 0.4x \ (0 \leq x < 30) \qquad C = -31 + 1.6x \ (x \geq 30)$$

C is the charge in dollars and x is the amount of water used in kilolitres (kL).

- **a** Use the appropriate equation to determine the charge for using:
  - **i** 20 kL
  - **ii** 30 kL
  - **iii** 50 kL
- **b** How much does a kilolitre of water cost when you use:
  - **i** less than 30 kL?
  - **ii** more than 30 kL?
- **c** Use the equations to construct a segmented graph for $0 \leq x \leq 50$.

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7

Investigating relationships between two numerical variables

- What is a scatterplot, how is it constructed and what does it tell us?
- What is Pearson's correlation coefficient, how is it calculated and what does it tell us?
- How do we fit a straight line to a scatterplot by eye?
- How do we fit a straight line to a scatterplot using the least squares method?
- How do we interpret the intercept and slope of a line fitted to a scatterplot?
- How do we use a line fitted to a scatterplot to make predictions?
- What is the difference between interpolation and extrapolation?

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Introduction

Much of the analysis that is carried out in statistics is not concerned with a single variable. Rather it is concerned with the association (relationship) between two or more variables. Is the new treatment for a cold more effective than the old treatment? Are females more likely to believe in astrology than males? Do younger people spend more time using social media than older people? These are all questions concerned with understanding the relationships between two variables. To determine how to go about answering these questions requires the role of the variables to be clearly identified.

7A Response and explanatory variables

When we analyse **bivariate data**, we try to answer questions such as: ‘Is there a relationship between these two variables?’ and more specifically ‘Does knowing the value of one of the variables tell us anything about the value of the other variable?’

For example, let us take as our two variables the *mark* a student obtained on a test and the amount of *time* they spent studying for that test. It seems reasonable that the more time one spends studying, the better mark you will achieve. That is, the amount of *time* spent studying may help to *explain* the *mark* obtained. For this reason, in statistics we call this the **explanatory variable (EV)**. And, since the *mark* may go up or down in response to amount of *time* spent studying, we call *mark* the **response variable (RV)**. In general, the value of explanatory variable for a case is thought to partially explain the value of the response variable for that individual.

Response and explanatory variables

When investigating associations (relationships between two variables), the explanatory variable (EV) is the variable we expect to explain or predict the value of the response variable (RV).

Note: The explanatory variable is sometimes called the independent variable (IV) and the response variable the dependent variable (DV).

Identifying response and explanatory variables

It is important to be able to identify the explanatory and response variables before starting to explore the relationship between two numerical variables. Consider the following examples.

Example 1 Identifying the response and explanatory variables

We wish to investigate the question ‘Do older people sleep less?’ The variables here are *age* and *time spent sleeping*. Which is the response variable (RV) and which is the explanatory variable (EV)?

Solution

When looking to see if the length of time people spent sleeping is explained by their age, *age* is the EV and *time spent sleeping* is the RV.

EV: age
RV: time spent sleeping

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Example 2 Identifying the response and explanatory variables

We wish to investigate the relationship between kilojoule consumption and weight loss. The variables in the investigation are *kilojoule consumption* and *weight loss.* Which is the response variable (RV) and which is the explanatory variable (EV)?

Solution

Since we are looking to see if the weight loss can be explained by the amount people eat, *kilojoule consumption* is the EV and *weight loss* is the RV.

EV: kilojoule consumption
RV: weight loss

Example 3 Identifying the response and explanatory variables

Can we predict people's height from their wrist circumference? The variables in this investigation are *height* and *wrist circumference*. Which is the response variable (RV) and which is the explanatory variable (EV)?

Solution

Since we wish to predict height from wrist circumference, *wrist circumference* is the EV. *Height* is then the RV.

EV: wrist circumference
RV: height

It is important to note that, in Example 3, we could have asked the question the other way around, that is 'Can we predict people's wrist circumference from their height?' In that case *height* would be the EV and *wrist circumference* would be the RV. The way we ask our statistical question is an important factor when there is no obvious EV and RV.

Exercise 7A

Identifying explanatory and response variables

1 Identify the EV and RV in each of the following situations. The variable names are italicised.

Example 1–3

- **a** We wish to investigate the relationship between the *age* of a certain type of tree and its *diameter*. We want to be able to predict the diameter of a tree from its age.
- **b** A study is to be made of the relationship between *weight loss* and the number of *weeks* a person is on a diet.
- **c** Data is collected to investigate the relationship between *age* of a second hand textbook and its selling *price*.
- **d** The relationship between the number of *hours* a gas heating system is used and the *amount* of gas used is to be investigated.
- **e** A study is to be made of the relationship between the number of *runs* a cricketer scores and the number of *balls bowled* to them.

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7B Scatterplots and their construction

The first step in investigating an association between two numerical variables is to construct a visual display of the data, which we call a **scatterplot**.

Constructing a scatterplot manually

We will illustrate the process by constructing a scatterplot of the marks students obtained on an examination (the RV) and the times they spent studying for the examination (the EV).

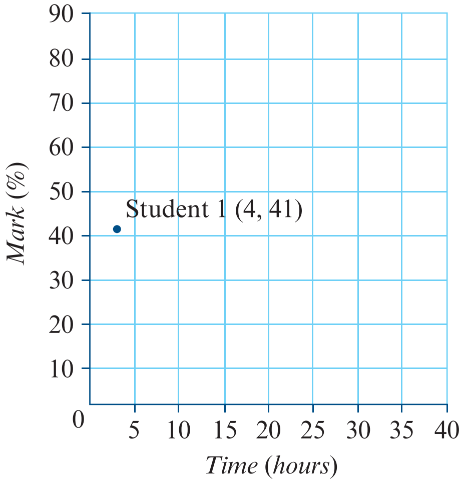
Student	1	2	3	4	5	6	7	8	9	10
Time (hours)	4	36	23	19	1	11	18	13	18	8
Mark (%)	41	87	67	62	23	52	61	43	65	52

In a scatterplot, each point represents a single case, in this instance a student.

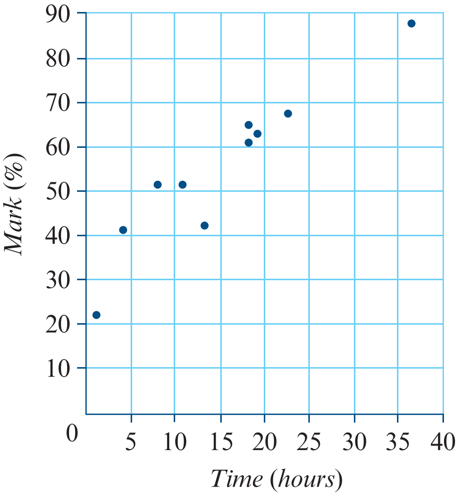
When constructing a scatterplot, it is conventional to use the *vertical* or *y-axis* for the response variable (RV) and the *horizontal* or *x-axis* for the explanatory variable (EV).

- The horizontal or *x*-coordinate of the point represents the time spent studying.
- The vertical or *y*-coordinate of the point represents the mark obtained.

The scatterplot opposite shows the point for Student 1, who studied 4 hours for the examination and obtained a mark of 41.



To complete the scatterplot the points for each remaining student, are plotted as shown.



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Example 4 The elements of a scatterplot

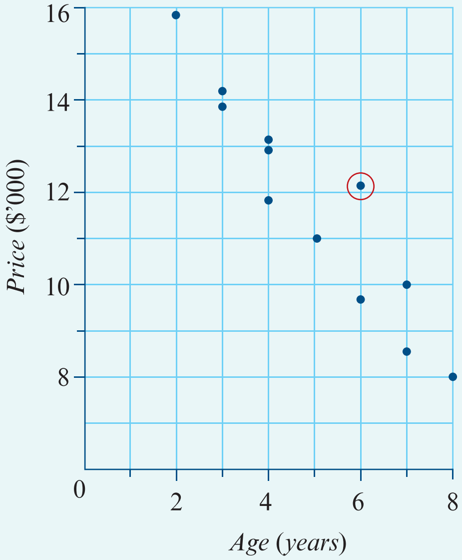
The scatterplot shown has been constructed from data collected to investigate the relationship between the *price* of a second hand car and its *age*.

Use the scatterplot to answer the following questions.

1 Which is the explanatory variable and which is the response variable?

2 How many cars are in the data set?

3 How old is the car circled? What is its price?



Solution

1 The EV will be on the horizontal axis and the RV on the vertical axis.	EV: Age RV: Price
2 The number of cars is equal to the number of points on the scatterplot.	12 Cars
3 The x-coordinate of the point will be the cars age and the y-coordinate its price.	The car is 6 years old, and its price is \$12 000.

▶ Using a graphics calculator to construct a scatterplot

While you need to understand the principles of constructing a scatterplot, and maybe need to construct one by hand for a few points, in practice you will use a graphics calculator to complete this task.

How to construct a scatterplot using the TI-Nspire CAS

The data below shows the marks that 10 students obtained on an examination and the time they spent studying for the examination.

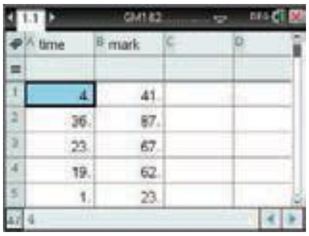
Time (hours)	4	36	23	19	1	11	18	13	18	8
Mark (%)	41	87	67	62	23	52	61	43	65	52

Use a calculator to construct a scatterplot. Use *time* as the explanatory variable.

Steps

1 Start a new document (ctrl + N) and select **Add Lists & Spreadsheet**.

Enter the data into lists named **time** and **mark**.



	A time	B mark	C	D
=				
1	4	41		
2	36.	87.		
3	23.	67.		
4	19.	62.		
5	1.	23.		

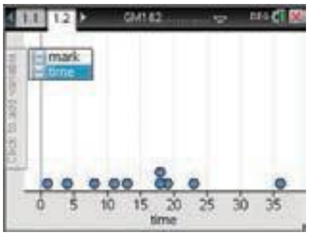
2 Statistical graphing is done through the **Data & Statistics** application.

Press ctrl + I and select **Add Data & Statistics** (or press ⌂ on and arrow to and press enter).

Note: A random display of dots will appear – this is to indicate list data is available for plotting. It is not a statistical plot.

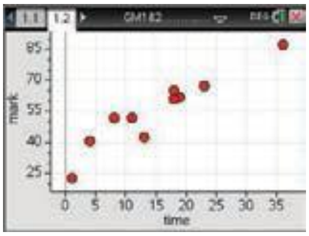
3 To construct a scatterplot.

a Press tab and select the variable **time** from the list. Press enter to paste the variable **time** to the x-axis.



b Press tab again and select the variable **mark** from the list. Press enter to paste the variable **mark** to the y-axis axis to generate the required scatterplot. The plot is automatically scaled.

Note: To change colour, move cursor over the plot and press ctrl + menu**>Color>Fill Color.**



How to construct a scatterplot using the ClassPad

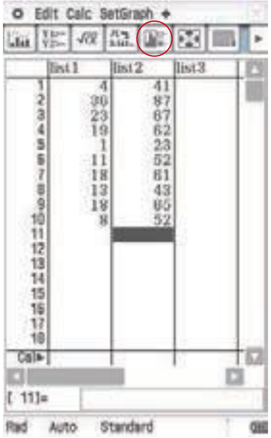
The data below give the marks that students obtained on an examination and the times they spent studying for the examination.

Time (hours)	4	36	23	19	1	11	18	13	18	8
Mark (%)	41	87	67	62	23	52	61	43	65	52

Use a calculator to construct a scatterplot. Use *time* as the explanatory variable.

Steps

1. Open the **Statistics** application .
2. Enter the values into lists with **time** in list1 and **mark** in list2.
3. Tap to open the **Set StatGraphs** dialog box.
4. Complete the dialog box as shown and Tap SET.



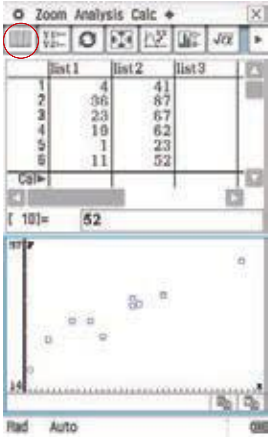
	list1	list2	list3
1	4	41	
2	36	87	
3	23	67	
4	19	62	
5	1	23	
6	11	52	
7	18	61	
8	13	43	
9	18	65	
10	8	52	

[11]=

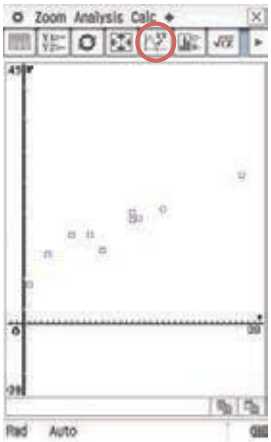
Draw: ● On ○ Off
Type: Scatter
XList: list1
YList: list2
Freq: 1
Mark: square

[10]= 52

5. Tap to plot a scaled graph in the lower-half of the screen.
6. Tap to give a full-screen sized graph. Tap to return to a half-screen.
7. Tap to place a marker on the first data point (**x_c = 4, y_c = 41**).
8. Use the horizontal cursor arrow to move from point to point.



[10]= 52



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Exercise 7B

Scatterplots: basic principles

1 For which of the following pairs of variables would it be appropriate to construct a scatterplot to investigate a possible assosiation?

A Car *colour* (blue, green, black, . . .) and its *size* (small, medium, large)

B A food's *taste* (sweet, sour, bitter) and its *sugar content* (in grams)

C The *weights* (in kg/cm) of 12 oranges and *lengths* (in centimetres) of nine bananas

D The *time* people spend exercising each day and their *resting pulse rate*

E The *arm spans* (in centimetres) and *gender* (male, female) of a group of students

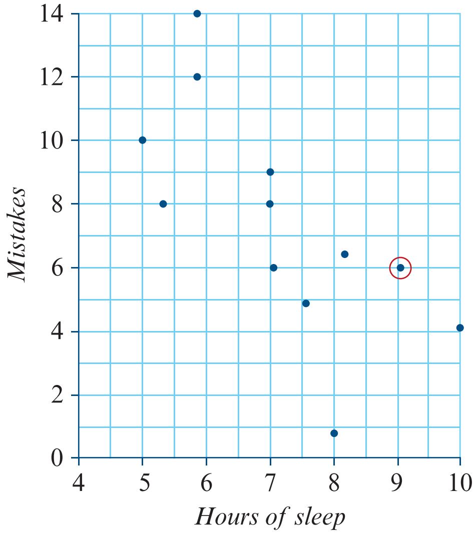
Example 4

2 The scatterplot shown has been constructed from data collected to investigate the relationship between the amount of sleep a person has the night before a test and the number of mistakes they make on the test. Use the scatterplot to answer the following questions.

a Which is the EV and which is the RV?

b How many people are in the data set?

c How much sleep had the individual circled had, and how many mistakes did they make?



Constructing scatterplots

3 The table below shows the heights and weights of eight people.

Height (cm)	190	183	176	178	185	165	185	163
Weight (kg)	77	73	70	65	65	65	74	54

Use your calculator to construct a scatterplot with the variable *height* as the explanatory variable and the variable *weight* as the response variable.

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4 The table below shows the ages of 11 couples when they got married.

Age of wife	26	29	27	21	23	31	27	20	22	17	22
Age of husband	29	43	33	22	27	36	26	25	26	21	24

Use your calculator to construct a scatterplot with the variable *wife* (age of wife) as the explanatory variable and the variable *husband* (age of husband) as the response variable.

5 The table below shows the numbers of seats and airspeeds (in km/h) of eight aircraft.

Airspeed	830	797	774	736	757	765	760	718
Number	405	296	288	258	240	193	188	148

Use your calculator to construct a scatterplot with the variable *number* as the explanatory variable and the variable *airspeed* as the response variable.

6 The table below shows the response times of 10 patients (in minutes) given a pain relief drug and the drug dosages (in milligrams).

a Which variable is the explanatory variable?

b Use your calculator to construct an appropriate scatterplot.

Drug dosage	0.5	1.2	4.0	5.3	2.6	3.7	5.1	1.7	0.3	0.6
Response time	65	35	15	10	22	16	10	18	70	50

7 The table below shows the numbers of people in a cinema at 5-minute intervals after the advertisements started.

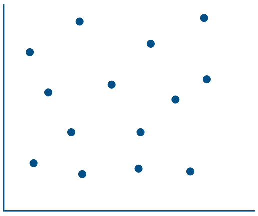
Number in cinema	87	102	118	123	135	137
Time	0	5	10	15	20	25

a Which is the explanatory variable?

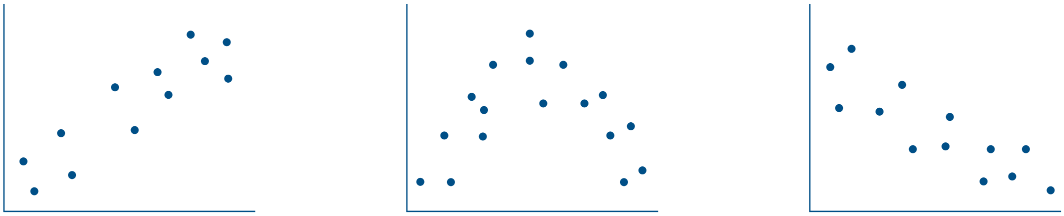
b Use your calculator to construct an appropriate scatterplot.

7C How to interpret a scatterplot

What features do we look for in a scatterplot to help us identify and describe any associations present? First we look to see if there is a *clear pattern* in the scatterplot. In the example opposite, there is *no clear pattern* in the points. The points are *randomly scattered* across the plot, so we conclude that there is *no association*.



For the three examples below, there is a *clear* (but different) *pattern* in each set of points, so we conclude that there is an *association* in each case.



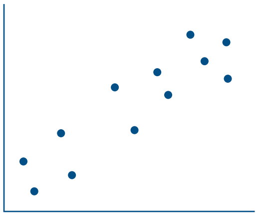
Having found a clear pattern, we need to be able to describe these associations clearly, as they are obviously quite different. There are three things we look for in the pattern of points:

- **direction**
- **form**
- **strength.**

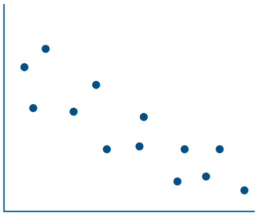
Direction of an association

We begin by looking at the overall pattern in the scatterplot.

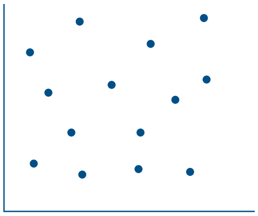
- If the points in the scatterplot tend to move up the plot as we go from left to right, then we say there is a **positive association** between the variables. That is, the values of the explanatory variable and the response variable tend to increase together.



- If the points in the scatterplot tend to move down the plot as we go from left to right, then we say there is a **negative association** between the variables. That is, as the values of the explanatory variable increase, the values of the response variable tend to decrease.



- If there is *no pattern* in the scatterplot; that is, the points just seem to randomly scatter across the plot, then we say there is **no association** between the variables.



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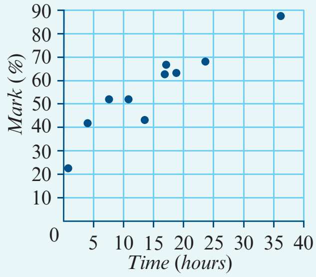
Example 5 Direction

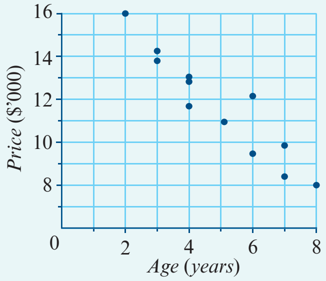
Classify each of the following scatterplots as exhibiting positive, negative or no association. Where there is an association, describe the direction of the association in the scatterplot and what it means in terms of the variables involved.

a



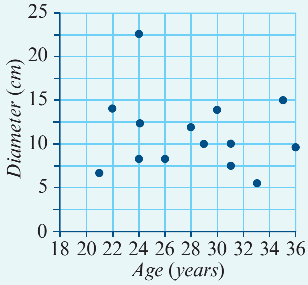
mark: mark on an exam
time: time spent studying

b



price: price of a car
age: age of car

c



diameter: calf diameter
age: age of person

Solution

a There is a *clear pattern* in the scatterplot. The points move *upwards* from left to right.

The direction of the association is positive. Students who spend more time studying for the exam tended to get higher marks.

b There is a *clear pattern* in the scatterplot. The points move *downwards* from left to right.

The direction of the association is negative. The price of a car tends to decrease with age.

c There is no clear pattern in the scatterplot.

There is no association between calf diameter and age.

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In general terms, we can interpret the *direction of an association* as follows.

Direction of an association

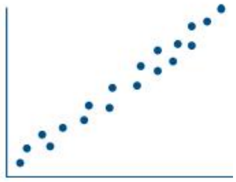
- Two variables have a *positive association* when the value of the response variable tends to increase as the value of the explanatory variable increases.
- Two variables have a *negative association* when the value of the response variable tends to decrease as the value of the explanatory variable increases.
- Two variables have *no association* when there is no consistent change in the value of the response variable when the values of the explanatory variable increase.

Form of an association

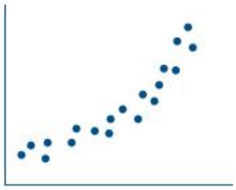
The next thing that interests us in an association is its general form: Do the points in a scatterplot tend to follow a linear pattern or a curved pattern?

For example:

- the association shown in the scatterplot opposite is *linear*. We can imagine the points in the scatterplot to be scattered around some *straight line.*



- the association shown in the scatterplot opposite is *non-linear*. We can imagine the points in the scatterplot to be scattered around a *curved line* rather than a straight line.



In general terms, we can describe the *form of an association* as follows.

Form

A scatterplot is said to have a *linear form* when the points tend to follow a straight line.
A scatterplot is said to have a *non-linear form* when the points tend to follow a curved line.

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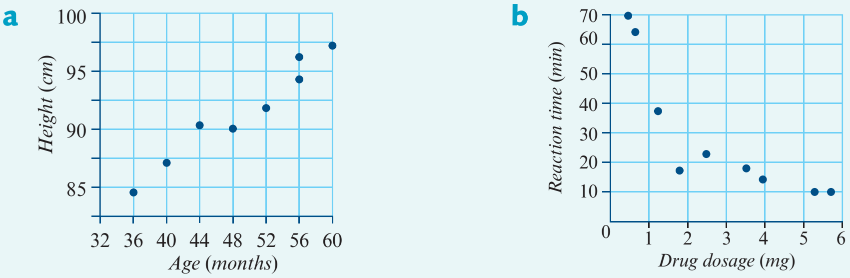
Example 6 Form of an association

Classify the *form* of the association in each of the following scatterplots as linear or non-linear.



Solution

a There is a *clear straight-line pattern*. The association is linear.

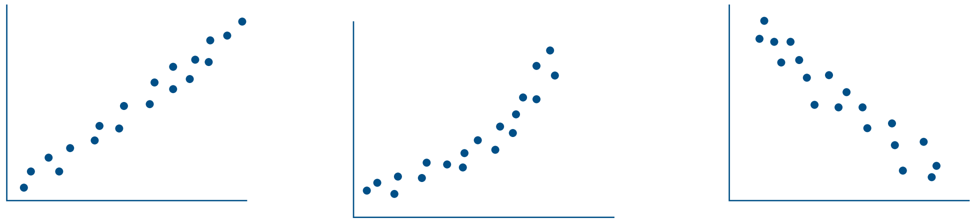
b There is a *clear curved pattern*. The association is non-linear.

Strength of an association

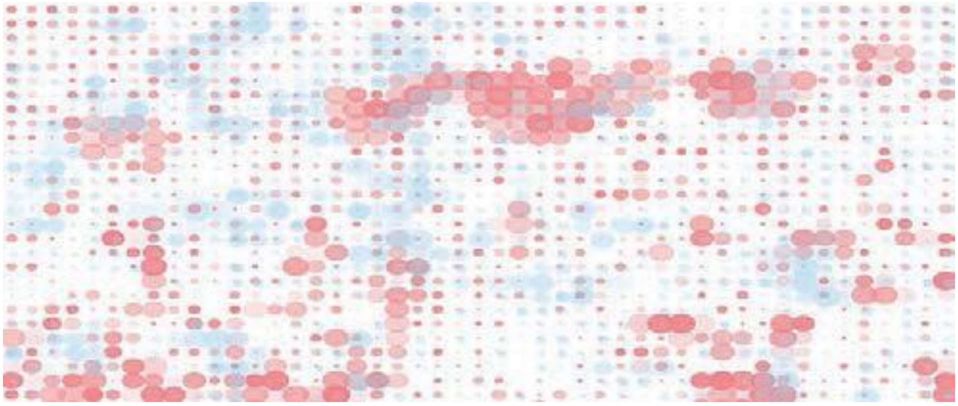
The **strength of an association** is a measure of how much scatter there is in the scatterplot.

Strong association

When there is a *strong association* between the variables, the points will tend to follow a single stream. A pattern is clearly seen. There is only a small amount of scatter in the plot.



Strong positive association Strong positive association Strong negative association



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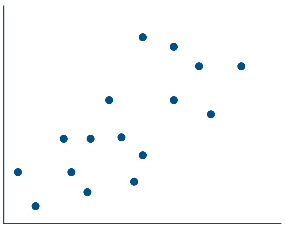
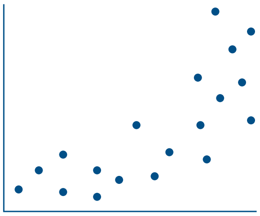
Moderate association

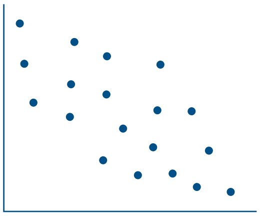
As the amount of scatter in the plot increases, the pattern becomes less clear. This indicates that the association is less strong. In the examples below, we might say that there is a *moderate association* between the variables.



Moderate positive association



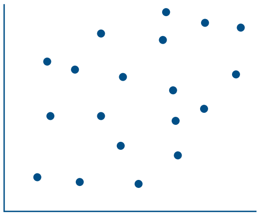
Moderate positive association



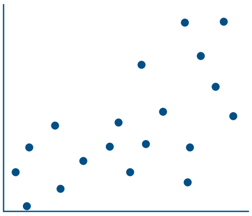
Moderate negative association

Weak association

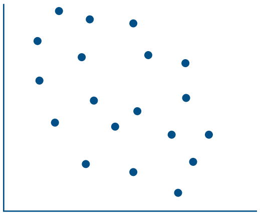
As the amount of scatter increases further, the pattern becomes even less clear. This indicates that any association between the variables is weak. The scatterplots below are examples of *weak association* between the variables.



Weak positive association



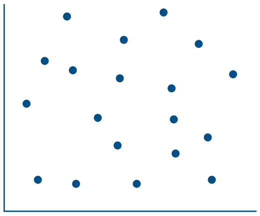
Weak positive association



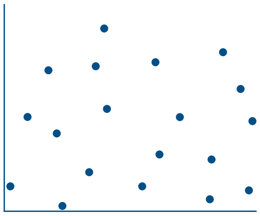
Weak negative association

No association

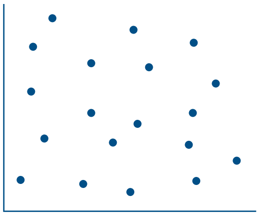
Finally, when all we have is scatter, as seen in the scatterplots below, no pattern can be seen. In this situation we say that there is **no association** between the variables.



No association



No association



No association

The scatterplots on the previous page and above should help you to get a feel for the strength of an association from the amount of scatter in a scatterplot. At the moment, you only need be able to estimate the strength of an association as strong, moderate, weak or none, by comparing it with the standard scatterplots given above. Later in this chapter, you will learn about a statistic, the **correlation coefficient**, which can be used to give a value to the strength of linear association.

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Exercise 7C

Identifying and describing associations

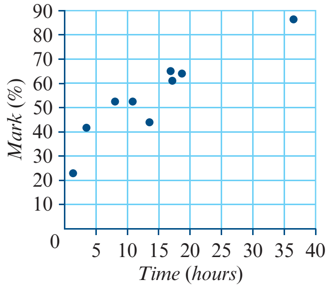
1 For each of the following pairs of variables, indicate whether you expect an association to exist and, if so, whether you would expect the association to be positive or negative.

- **a** *fitness level* and *amount* of daily exercise
- **b** *foot length* and *height*
- **c** *comfort level* and *temperature* above 30°C
- **d** *foot length* and *intelligence*
- **e** *time* taken to get to school and *distance* travelled
- **f** *number of pages* in a book and its *price*

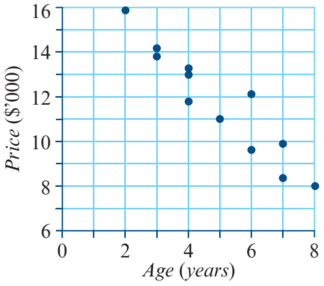
Example 5, 6

2 The variables in each of the following scatterplots are associated. In each case classifying the association according to its strength (strong/medium/weak) using the scatterplots on previous pages to assess, form (linear/non-linear) and direction (positive/negative). Also note the presence of outliers (if any).

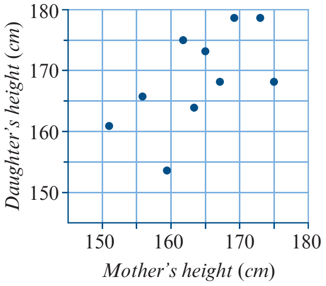
a



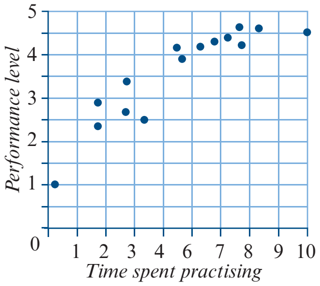
b



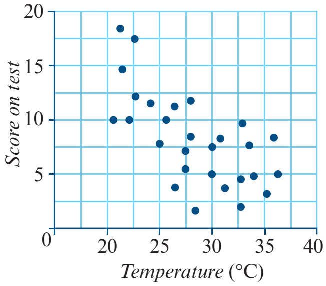
c



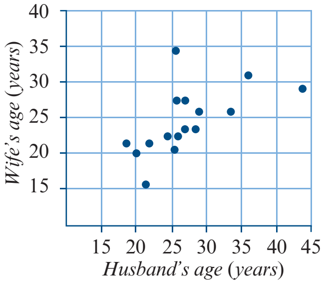
d



e



f



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7D Pearson's correlation coefficient (r)

When an association is linear, the most commonly used measure of strength of the relationship is Pearson's correlation coefficient, r. It gives a numerical measure of the degree to which the points in the scatterplot tend to cluster around a straight line.

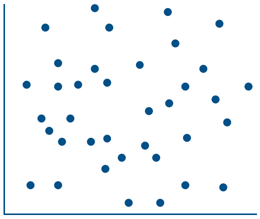
There are two key assumptions when using Pearson's correlation coefficient, r. These are:

- the data is numerical
- the association is linear.

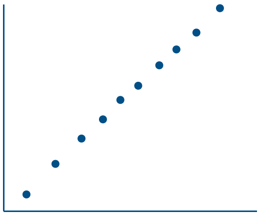
Properties of Pearson's correlation coefficient

Pearson's correlation coefficient has the following properties:

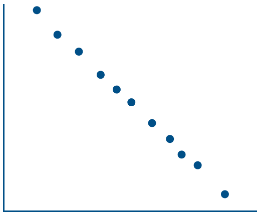
- no linear association, $r = 0$.
- a perfect positive linear association, $r = +1$.
- a perfect negative linear association, $r = -1$.



$r = 0$

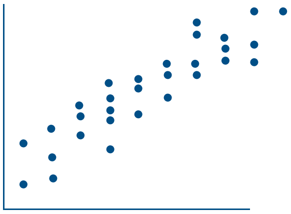


$r = +1$

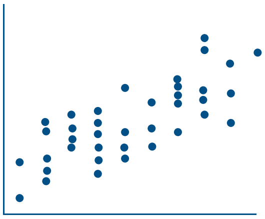


$r = -1$

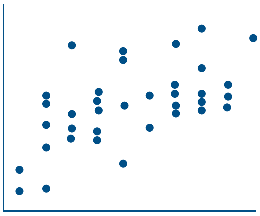
In practice, the value of r will be somewhere between $+1$ and -1 and rarely exactly zero as shown in the selection of scatterplots below.



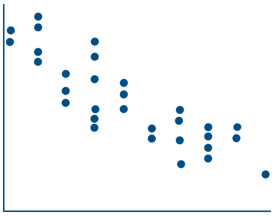
$r = 0.915$



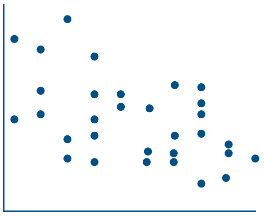
$r = 0.767$



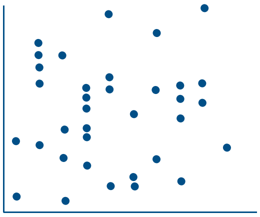
$r = 0.551$



$r = -0.874$



$r = -0.501$



$r = 0.150$

These scatterplots illustrate an important point – the stronger the association, the larger the magnitude of Pearson's correlation coefficient.

Pearson's correlation coefficient

The Pearson's correlation coefficient, r:

- measures the *strength* of *a linear relationship***,** with larger values indicating stronger relationships
- has a value between –1 and +1
- is positive if the direction of the linear relationship is positive
- is negative if the direction of the linear relationship is negative.

Guidelines for classifying the strength of a linear relationship using the correlation coefficient

Pearson's correlation coefficient, r, can be used to classify the strength of a linear association as follows:

strong positive linear association r between 0.75 and 0.99
moderate positive linear association r between 0.5 and 0.74
weak positive linear association r between 0.25 and 0.49
no linear association r between −0.24 and 0.24
weak negative linear association r between −0.25 and −0.49
moderate negative linear association r between −0.5 and −0.74
strong negative linear association r between −0.725 and −0.99

Correlation and causation

The existence of even a strong association between two variables is not, in itself, sufficient to imply that altering one variable *causes* a change in the other. It only implies that this *may* be the explanation. Alternative explanations to consider are that:

- it may be that both the measured variables are affected by a third and different variable, which is a common response.
- it could be that the apparent effect of one variable is caused by a totally different variable, thereby confounding the results.
- finally it may just be pure chance that the data gathered for the two variables seems to be associated, which is coincidence.

Correlations must be interpreted with care, as we will see from the following examples.

Common response

The issue

If data about the variable crime rates and unemployment in a range of cities were gathered, a high correlation would be found. Can it be inferred that high unemployment causes high crime rates?

A possible non-causal explanation

The explanation could be that both of these variables are dependent on another variable such as *level of education,* which may be related to both *unemployment* and *crime rates*. If those with lower levels of education are more likely to be unemployed, and also more likely to be involved in crime, then these variables may vary together, without one necessarily being the direct cause of the variation in the other. The source of the correlation could be entirely due to a common response to a third variable.

Confounding

The issue

Suppose we were to find a high correlation between the smoking rate and heart disease across a group of countries. Can we conclude that smoking causes heart disease?

A possible non-causal explanation

One possible explanation is that people who smoke are also prone to neglect other lifestyle factors such as exercise and diet. It could well be that people who smoke also tend not to exercise regularly, and it is the lack of exercise which causes heart disease. In this case, the effect of smoking on heart disease is confounded (mixed up) with these other factors (lack of exercise and poor diet), which are also known to be associated with heart disease. This makes it impossible to decide the actual cause or causes of a smoker's heart diseases.

Coincidence

The issue

It turns out that there is a strong correlation ($r = 0.95$) between cheese consumption and the number of people who died becoming tangled in their bed sheets. (Source: http://cambridge.edu.au/redirect/?id=5920)

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Can we conclude that eating cheese causes one to sleep so restlessly that it increases the likelihood of death by bed sheet?

A possible non-causal explanation

A better explanation is that this correlation is purely coincidence.

To help you understand the issues associated with correlation, and causality, you should watch the video 'The question of causation', at: http://cambridge.edu.au/redirect/?id=5921

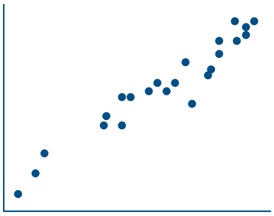
Exercise 7D

The correlation coefficient: assumptions and estimation

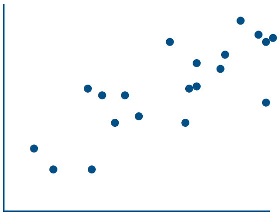
1 What are the two key assumptions justifying the use of Pearson's correlation coefficient to quantify the strength of the association between two variables?

2 Estimate the value of the correlation coefficient, r, using the plots on page 321 as a guide.

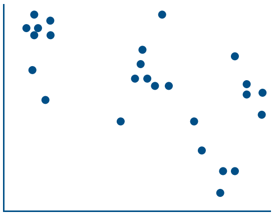
a



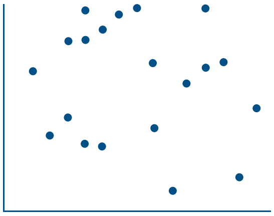
b



c



d



3 Use the guidelines on page 322 to classify the strength of a linear relationship for which Pearson's correlation coefficient is calculated to be:

a $r = 0.20$ **b** $r = -0.30$ **c** $r = -0.85$ **d** $r = 0.33$

e $r = 0.95$ **f** $r = -0.74$ **g** $r = 0.65$ **h** $r = -0.24$

i $r = -0.48$ **j** $r = 0.29$ **k** $r = 1$ **l** $r = -1$

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Correlation and causality

4 If the heights and the scores obtained on a test of mathematical ability by a group of primary school students in prep to year 6 were recorded, a strong correlation would be found. Can it be inferred from this that taller people are better at mathematics? Give a possible non-causal explanation.

5 There is a strong positive correlation between the number of bars and the number of school teachers in cities around the world. Can we conclude from this that school teachers spend a lot of time in bars? Give a possible non-causal explanation.

6 There is a strong negative correlation between birth rate and life expectancy in a country. Can we conclude that decreasing the birth rate in a country will help increase the life expectancy of its citizens? What confounding variable(s) could equally explain this correlation?

7E Determining the value of Pearson's correlation coefficient, *r*

As discussed in the previous section, when a relationship is linear, the most commonly used measure of strength of the association is Pearson's correlation coefficient.

The symbol we use to represent this correlation coefficient is r.
The formula for calculating r is:

$$r = \frac{1}{n-1}\sum\left(\frac{x-\bar{x}}{s_x}\right)\left(\frac{y-\bar{y}}{s_y}\right)$$

In this formula, $\bar{x}$ and s_x are the mean and standard deviation of the x scores and $\bar{y}$ and s_y are the mean and standard deviation of the y scores.

After the mean and standard deviation, Pearson's product-moment correlation coefficient is one of the most frequently computed descriptive statistics. It is a powerful tool, but it is also easily misused. The presence of a linear relationship should always be confirmed with a scatterplot before Pearson's correlation coefficient is calculated. And, like the mean and the standard deviation, Pearson's correlation coefficient can be very sensitive to the presence of outliers, particularly for small data sets.

Pearson's correlation coefficient, r, is rather tedious to calculate by hand and is usually evaluated with the aid of technology.

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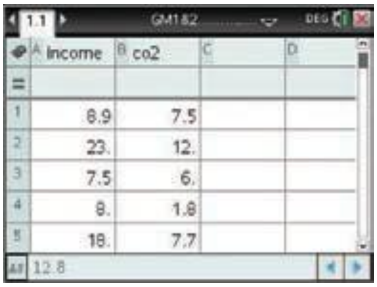
How to calculate Pearson's correlation coefficient, *r*, using the TI-Nspire CAS

The following data shows the per capita income (in \$000) and the carbon dioxide emissions (in tonnes) of 11 countries.
Determine the value of Pearson's correlation coefficient, *r*, for these data.

Income (\$ 000)	8.9	23.0	7.5	8.0	18.0	16.7	5.2	12.8	19.1	16.4	21.7
CO_2 (tonnes)	7.5	12.0	6.0	1.8	7.7	5.7	3.8	5.7	11.0	9.7	9.9

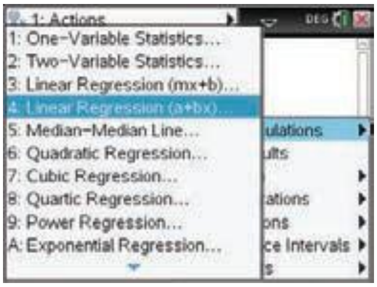
Steps

1. Start a new document by pressing ctrl + N.
2. Select **Add Lists & Spreadsheet**.
   Enter the data into lists named income and co2.
3. Statistical calculations can be done in the Calculator application. Press ctrl + I and select **Calculator**.



	A income	B co2	C	D
=				
1	8.9	7.5		
2	23.	12.		
3	7.5	6.		
4	8.	1.8		
5	18.	7.7		

4. Press menu **>Statistics>Stat Calculations>Linear Regression (a + bx)** to generate the screen opposite.

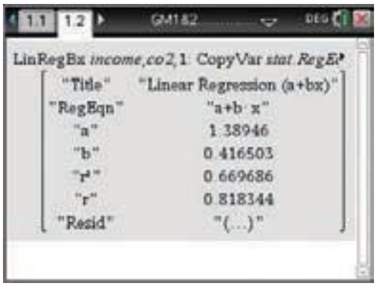


5. Press menu to generate the pop-up screen as shown. To select the variable for the X List entry use ▶ and enter to select and paste in the list name **income**. Press tab to move to the Y List entry, use ▶ ▼ and enter to select and paste in the list name **co2**.

Linear Regression (a+bx)

X List: income
Y List: co2
Save RegEqn to: f1
Frequency List: 1
Category List:
Include Categories:

6. Press enter to exit the pop-up screen and generate the results shown in the screen opposite. The value of the correlation coefficient is $r = 0.818344\ldots$ or 0.818, correct to three decimal places.



```
LinRegBx income,co2,1: CopyVar stat.RegE
 "Title"      "Linear Regression (a+bx)"
 "RegEqn"     "a+b·x"
 "a"          1.38946
 "b"          0.416503
 "r²"         0.669686
 "r"          0.818344
 "Resid"      "{...}"
```

The value of the correlation coefficient is $r = 0.818$, to three decimal places.

How to calculate Pearson's correlation coefficient, *r*, using the ClassPad

The following data show the per capita income (in $000) and the carbon dioxide emissions (in tonnes) of 11 countries.

Determine the value of Pearson's correlation coefficient, *r*, for the given data.

Income ($ 000)	8.9	23.0	7.5	8.0	18.0	16.7	5.2	12.8	19.1	16.4	21.7
CO_2 (tonnes)	7.5	12.0	6.0	1.8	7.7	5.7	3.8	5.7	11.0	9.7	9.9

Steps

1. Open the **Statistics** application Statistics.
2. Enter the data into the columns.
   - **Income** in List1
   - **CO_2** in List2
3. Select **Calc>Regression>Linear Reg** from the menu bar.

```
     list1   list2
1     8.9     7.5
2      23      12
3     7.5       6
4       8     1.8
5      18     7.7
6    16.7     5.7
7     5.2     3.8
8    12.8     5.7
9    19.1      11
10   16.4     9.7
11   21.7     9.9
```

4. Press EXE.

   This opens the **Set Calculation** dialog box as shown to the right.
5. Tap **OK** to confirm your selections.

Linear Reg

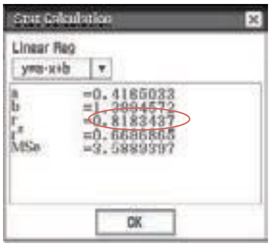
XList: list1

YList: list2

Freq: 1

Copy Formula: Off

Copy Residual: Off



```
Linear Reg
y=a·x+b
a      =0.4165033
b      =1.2894572
r      =0.8183437
r²     =0.6696865
MSe    =3.5889397
```

The value of the correlation coefficient is $r = 0.818344\ldots$ or 0.818, correct to three decimal places.

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Exercise 7E

Calculating the correlation coefficient

1 The table below shows the *weight* (in kg) and blood *glucose* level (in mg/100 mL) of eight adults.

Weight	82.1	70.1	76.6	82.1	83.9	73.2	66.0	77.5
Glucose	101	89	98	100	108	104	94	89

Use your calculator to determine the value of Pearson's correlation coefficient for this data set. Write your answer correct to three decimal places.

2 The table below shows the scores a group of nine students obtained on two class tests, *Test 1* and *Test 2*, as part of their school based assessment.

Test 1	33	45	27	42	50	38	17	35	29
Test 2	43	46	36	34	48	34	29	41	28

Use your calculator to determine the value of Pearson's correlation coefficient for this data set. Write your answer correct to three decimal places.

3 The table below shows the carbohydrate content (*carbs*) and the fat content (*fat*) in 100 g of nine breakfast cereals.

Carbs (g)	88.7	67.0	77.5	61.7	86.8	32.4	72.4	77.1	86.5
Fat (g)	0.3	1.3	2.8	7.6	1.2	5.7	9.4	10.0	0.7

Use your calculator to determine the value of Pearson's correlation coefficient for this data set. Write your answer correct to three decimal places.

Identifying and describing associations using a scatterplot and the correlation coefficient

4 In a survey of nine problem gamblers, the respondents were asked the *amount* (in dollars) they had spent on gambling and the *number* of hours that they had spent gambling in the past week. This data collected is recorded in the table below.

Hours	10	11	12	15	20	21	25	35	40
Amount	500	530	300	750	1000	1200	2000	2300	5000

a The aim is to predict the amount of money spent on gambling from the time spent gambling. Which is the explanatory variable and which is the response variable?

b Construct a scatterplot of these data.

c Determine the value of the correlation coefficient, r, to three decimal places.

d Describe the association between the variables *amount* and *hours* in terms of strength, direction and form.

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5 The following data was recorded through the National Health Survey:

Region	Percentage with eye disease	
	Male (%)	Female (%)
Australia	40.7	49.1
Other Oceania countries	46.1	66.2
United Kingdom	74.5	75.0
Other North-West Europe	71.2	71.5
Southern & Eastern Europe	71.6	74.6
North Africa & the Middle East	52.2	57.5
South-East Asia	47.7	54.8
All other countries	56.0	62.0

a Which is the explanatory variable and which is the response variable?

b Construct a scatterplot of these data, with percentage of males on the horizontal axis and percentage of females on the vertical axis.

c Determine the value of the correlation coefficient, r, to three decimal places.

d Describe the association between the male and female eye disease percentages for these countries in terms of strength, direction and form and outliers (if any).

7F Using the least squares line to model a linear association

Once we identify a linear association between two numerical variables, we can go one step further by fitting a linear model to the data and find its equation. This model gives us a better understanding of the nature of the relationship between the two variables. We can also use the model to make predictions based on this linear model.

The process of modelling an association with a straight line is known as **linear regression** and the resulting line is often called the **regression line**.

In Chapter 6, Linear models and graphs, you learned to fit a linear model to a scatterplot by eye. This method is somewhat problematic because everyone will get a different answer. A better approach is to fit the line according to an appropriate mathematical strategy known as the **least squares method**.

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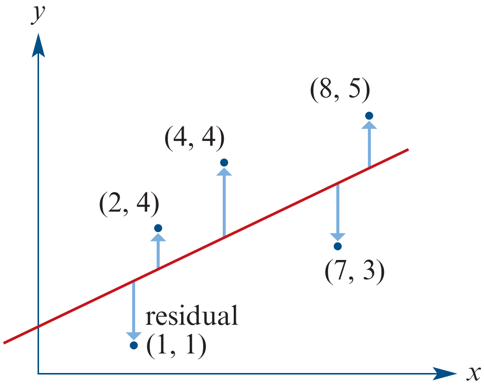
The least squares regression line (the theory)

Residuals (errors of prediction)

We wish to fit a line to a scatterplot that best fits the data in some way. We know that, unless all of the points in the scatterplot lie exactly on a straight line, no line will perfectly fit the data. There will always be errors.



The blue arrows show the errors that would be made if we used the line shown to predict the line the y-coordinates of each of the data points. In statistics, errors of prediction are often called residuals.

Because we will eventually want to use our line to make predictions, one way of fitting the line is to find the line that has the smallest value for the sum of the residuals. However, because some errors will be positive and some will be negative these errors will tend to cancel out. We have met this problem before when calculating standard deviation, and we solve it the same way, by squaring the residuals to make them positive. The least squares line is then found by finding the values of the intercept and slope for the line that minimises the sum of these squared residuals. The exact solution for these values can be found mathematically, using the techniques of calculus. Although the mathematics is beyond that required for General Mathematics, we will make use of these results, which are summarised below.

The equation of the least squares regression line

The equation of the least squares regression line is given by $y = a + bx$, where:

the *slope* (b) is given by: $b = \dfrac{rs_y}{s_x}$

and

the *intercept* (a) is then given by : $a = \bar{y} - b\bar{x}$

Here:

- r is the correlation coefficient
- s_x and s_y are the standard deviations of x and y
- $\bar{x}$ and $\bar{y}$ are the mean values of x and y.

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▶ The least squares regression line (the practice)

The assumptions made in using the least squares method to model a linear association are the same as those for Pearson's correlation coefficient. These are:

- the variables are numerical
- the association is linear.

Your CAS calculator can be used to fit a least squares regression line to a scatterplot and find its equation. In this subject, you are not required to perform this calculation by hand.

How to determine and graph the least squares regression line using the TI-Nspire CAS

The following data show the per capita income (*income*) and the carbon dioxide emissions (CO_2) of 11 countries.

Income ($ 000)	8.9	23.0	7.5	8.0	18.0	16.7	5.2	12.8	19.1	16.4	21.7
CO_2*(tonnes)*	7.5	12.0	6.0	1.8	7.7	5.7	3.8	5.7	11.0	9.7	9.9

a Construct a scatterplot to display these data with *income* as the explanatory variable (EV) .

b Fit a least squares regression line to a scatterplot and determine its equation.

c Write the equation of the regression line in terms of the variables *income* and CO_2 with the coefficients given correct to three significant figures.

d Determine and write down the value of the correlation coefficient, r, correct to three significant figures.

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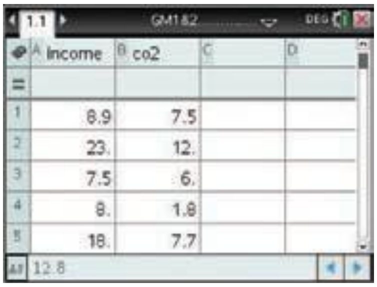
Steps

1 Start a new document by pressing ctrl + N.

2 Select **Add Lists & Spreadsheet**. Enter the data into lists named **income** and **co2**.



	A income	B co2	C	D
=				
1	8.9	7.5		
2	23.	12.		
3	7.5	6.		
4	8.	1.8		
5	18.	7.7		

A1 12.8

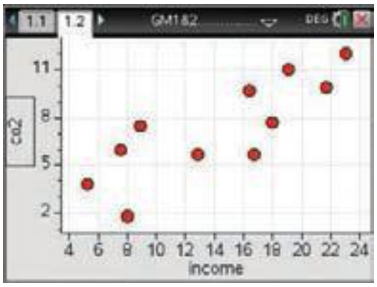
3 Identify the explanatory variable (EV) and the response variable (RV).

EV: **income**

RV: **co2**

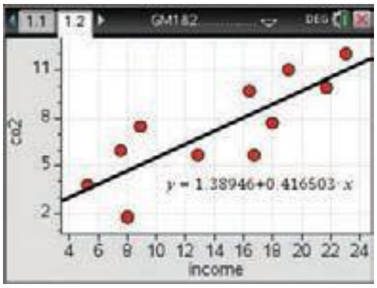
Note: In saying that we want to predict **co2** from **income** we are implying that income is the EV.

4 Press ctrl + I, select **Data & Statistics** and construct a scatterplot with the **income** (EV) on the horizontal (or x-) axis and **co2** (RV) on the vertical (or y-) axis.



5 Press menu **>Analyze>Regression>Show Linear (a+bx)** to display the least squares regression line on the scatterplot. Simultaneously, the equation of the regression line is shown written using the variables y and x:

$$y = 1.389\ldots + 0.416\ldots \text{ or}$$
$$y = 1.39 + 0.417x \text{ to 3 sig. figures}$$



$y = 1.38946+0.416503 \cdot x$

Note: The calculator assumes that the variable on the x-axis is the EV.

6 Write down the equation of the least squares regression line in terms of the variables **income** and **co2**. Write the coefficients correct to three significant figures.

$CO_2 = 1.39 + 0.417 \times \text{income}$

7 **a** Press ctrl + I and select **Calculator** to open the **Calculator** application.

b Now press var, locate then select **stat.r** and press enter to display the value of r.

```
stat.r                                   0.818344
```

8 Write down the value of the correlation coefficient correct to three significant figures.

$r = 0.818$

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How to determine and graph the least squares regression line using the ClassPad

The following data show the per capita income (in $000) and the carbon dioxide emissions (in tonnes) of 11 countries.

Income ($ 000)	8.9	23.0	7.5	8.0	18.0	16.7	5.2	12.8	19.1	16.4	21.7
CO_2 (tonnes)	7.5	12.0	6.0	1.8	7.7	5.7	3.8	5.7	11.0	9.7	9.9

a Determine and graph the equation of the least squares regression line that will enable CO_2 emissions to be predicted from income.

b Write the equation in terms of the variables *income* and co_2 with the coefficients given correct to three significant figures.

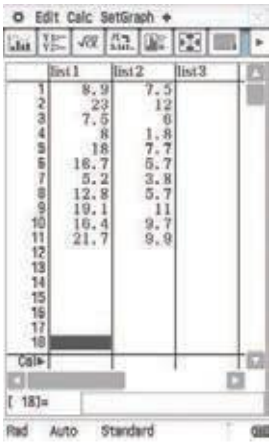
c Determine and write down the value of the correlation coefficient, r, to three significant figures.

Steps

1 Open the **Statistics** application.

2 Enter the data into columns:

- ***Income*** in **List1**
- ***CO_2*** in **List2**

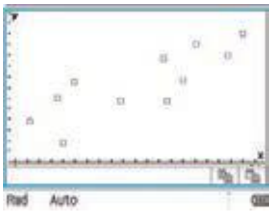


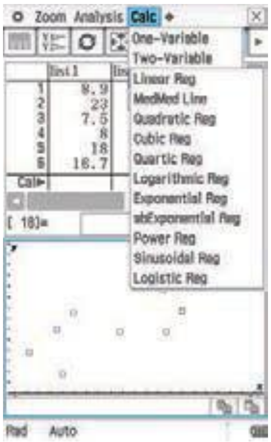
	list1	list2	list3
1	8.9	7.5	
2	23	12	
3	7.5	6	
4	8	1.8	
5	18	7.7	
6	16.7	5.7	
7	5.2	3.8	
8	12.8	5.7	
9	19.1	11	
10	16.4	9.7	
11	21.7	9.9	
12			
13			
14			
15			
16			
17			
18			

Draw: ● On ○ Off
Type: Scatter
XList: list1
YList: list2
Freq: 1
Mark: square

3 Tap to open the **Set StatGraphs** dialog box and complete as shown.
Tap Set to confirm your selections.

4 Tap in the toolbar at the top of the screen to plot the scatterplot in the bottom half of the screen.





	list1
1	8.9
2	23
3	7.5
4	8
5	18
6	16.7

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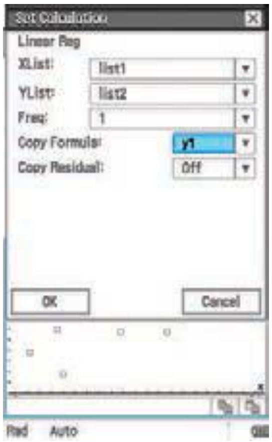
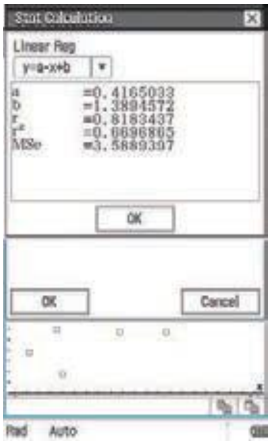
5 To calculate the equation of the least squares regression line.

- Tap **Calc>Regression>Linreg** from the menu bar.
- Complete the **Set Calculations** dialog box as shown.
- Tap **OK** to confirm your selections in the **Set Calculations** dialog box.
- This generates the regression results in **Stat Calculation** shown opposite.



Linear Reg
XList: list1
YList: list2
Freq: 1
Copy Formula: y1
Copy Residual: Off



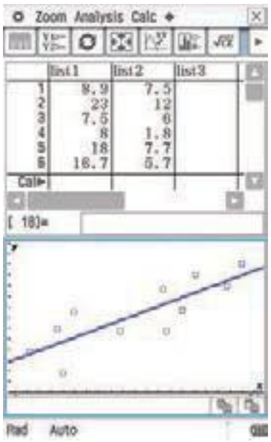
```
Linear Reg
y=a·x+b
a      =0.4165033
b      =1.3894572
r      =0.8183437
r²     =0.6696865
MSe    =3.5889397
```

6 Tapping **OK** a second time automatically plots and displays the regression line on the plot.

7 Write down the equation of the least squares line in terms of the variables *income* and CO_2 and the value of the correlation coefficient to three decimal places.

$CO_2 = 1.39 + 0.417 \times \text{income}$

$r = 0.818$



	list1	list2	list3
1	8.9	7.5	
2	23	12	
3	7.5	6	
4	8	1.8	
5	18	7.7	
6	16.7	5.7	

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Exercise 7F

Skillsheet

Fitting a least squares line and calculating the correlation coefficient

Note: In fitting lines to data, the slope and intercept are calculated accurate to a given number of *significant figures*, as required in the General Mathematics and Further Mathematics Study Design. Chapter 1 has a section on significant figures if you need to revise the basic ideas, page 9.

1 The table below shows the weight (in kg) and blood glucose level (in mg/100 mL) of eight adults.

Weight (kg)	82.1	70.1	76.6	82.1	83.9	73.2	66.0	77.5
Glucose (mg/100 mL)	101	89	98	100	108	104	94	89

a Construct a scatterplot to display these data with *weight* as the EV.

b Fit a least squares regression line to the scatterplot and determine its equation.

c Write the equation of the regression line in terms of the variables *glucose* and *weight* with the coefficients given correct to three significant figures.

d Determine the correlation coefficient to three significant figures.

2 The table below shows the scores a group of nine students obtained on two class tests, Test 1 and Test 2, as part of their school based assessment.

Test 1	33	45	27	42	50	38	17	35	29
Test 2	43	46	36	34	48	34	29	41	28

a Construct a scatterplot to display these data with *test 1* as the EV.

b Fit a least squares regression line to the scatterplot and determine its equation.

c Write the equation of the regression line in terms of the variables *test 2* and *test 1* with the coefficients given correct to three significant figures.

d Determine the correlation coefficient to three significant figures.

3 The table below shows the carbohydrate content (*carbs*) and the fat content (*fat*) in 100 g of nine breakfast cereals.

Carbs	88.7	67.0	77.5	61.7	86.8	32.4	72.4	77.1	86.5
Fat	0.3	1.3	2.8	7.6	1.2	5.7	9.4	10.0	0.7

a Construct a scatterplot to display these data with *carbs* as the EV.

b Fit a least squares regression line to the scatterplot and determine its equation.

c Write the equation of the regression line in terms of the variables *fat* and *carbs* with the coefficients given correct to three significant figures.

d Determine the correlation coefficient to three significant figures.

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4 The table below shows the *age* and *height* of six young girls.

Age (months)	36	40	44	52	56	60
Height (cm)	84	87	90	92	94	96

a Construct a scatterplot to display these data with *age* as the EV.

b Fit a least squares regression line to the scatterplot and determine its equation.

c Write the equation of the regression line in terms of the variables *height* and *age* with the coefficients given correct to three significant figures.

d Determine the correlation coefficient to three significant figures.

5 The following table gives the *shoe size* and *weight* in kilograms of 10 adult males.

Shoe size	9.5	10.0	10.5	10.5	11	9.0	8.5	9.5	7.5	8
Weight (kg)	64	85	70	80	82	73	70	66	55	70

a Construct a scatterplot to display these data with *shoe size* as the EV.

b Fit a least squares regression line to the scatterplot and determine its equation.

c Write the equation of the regression line in terms of the variables *weight* and *shoe size* with the coefficients given correct to three significant figures.

d Determine the correlation coefficient to three significant figures.

7G Using a regression line to make predictions: interpolation and extrapolation

The aim of linear regression is to model the relationship between two numerical variables by using the equation of a straight line. This equation can then be used to make predictions.

The data below shows the times that 10 students spent studying for an exam and the marks they subsequently obtained.

Time (hours)	4	36	23	19	1	11	18	13	18	8
Mark (%)	41	87	67	62	23	52	61	43	65	52

If we fitted a linear model to this data using the least squares method we would have an equation close to:

$$mark = 30.8 + 1.62 \times time$$

Using this equation, and rounding off to the nearest whole number, we would predict that a student who spent:

- 0 hours studying would obtain a mark of 31% $(\text{mark} = 30.8 + 1.62 \times 0 = 31\%)$
- 8 hours studying would obtain a mark of 44% $(\text{mark} = 30.8 + 1.62 \times 8 = 44\%)$
- 12 hours studying would obtain a mark of 50% $(\text{mark} = 30.8 + 1.62 \times 12 = 50\%)$
- 30 hours studying would obtain a mark of 79% $(\text{mark} = 30.8 + 1.62 \times 30 = 79\%)$
- 80 hours studying would obtain a mark of 160% $(\text{mark} = 30.8 + 1.62 \times 80 = 160\%)$

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This last result, 160%, points to one of the limitations of substituting into a regression equation without thinking carefully. Using this regression equation, we predict that a student who studies for 80 hours will obtain a mark of more than 100%: impossible. Something is wrong!

The problem is that we are using the regression equation to make predictions well outside the range of values used to calculate this equation. The maximum time any student spent studying for this exam was 36 hours; yet, we are using the equation we calculated to try to predict the exam mark for someone who studies for 80 hours. Without knowing that the model works equally well for someone who spends 80 hours studying, which we don't, we are venturing into unknown territory and can have little faith in our predictions.

As a general rule, a regression equation only applies to the range of data values used to determine the equation. Thus, we are reasonably safe using the line to make predictions that lie roughly within this data range, say from 1 to 36 hours. The process of making a prediction within the range of data used to derive the regression equation is called **interpolation** and we can have some faith in these predictions.

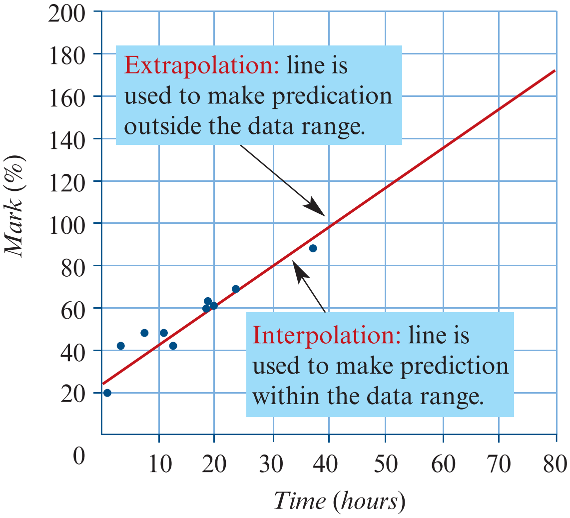
However, we must be extremely careful about how much faith we put into predictions made outside the data range. Making predictions outside the data range is called **extrapolation**.

Interpolation and extrapolation

Predicting *within* the range of data is called **interpolation**.

Predicting *outside* the range of data is called **extrapolation**.

For example, if we use the regression line to predict the examination mark for 30 hours of studying time, we would be interpolating. However, if we use the regression line to predict the examination mark for 50 hours of studying time, we would be extrapolating. Extrapolation is a less reliable process than interpolation because we are going beyond the original data.



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Example 7 Using a regression line to make predictions

The equation relating the weights and heights of a group of students whose heights ranged from 163 cm to 190 cm is:

$$weight = -40 + 0.6 \times height$$

Use this equation to predict the weight (in kg) of a student with the following heights. Are you interpolating or extrapolating?

a 170 cm **b** 65 cm

Solution

a Substitute 170 into the equation and evaluate.

The weight of a person of height 170 cm is predicted to be:

$weight = -40 + 0.6 \times 170 = 62$ kg

Interpolating: predicting within range of data.

b Substitute 65 into the equation and evaluate.

The weight of a person of height 65 cm is predicted to be:

$weight = -40 + 0.6 \times 65 = -1$ kg

which is not possible.

Extrapolating: we are predicting well outside the range of data.

Exercise 7G

Using a regression line to make predictions: interpolation and extrapolation

1 Complete the following sentences.

Using a regression line to make a prediction:

a within the range of data is called ☐.

b outside the range of data is called ☐.

Example 7

2 For children between the ages of 36 and 60 months, the equation relating their *height* (in cm) to their *age* (in months) is:

$$height = 72 + 0.4 \times age$$

Use this equation to predict the height (to the nearest cm) of a child with the following age. Are you interpolating or extrapolating?

a 40 months old **b** 55 months old **c** 70 months old

3 For shoe sizes between 6 and 12, the equation relating a person's *weight* (in kg) to *shoe size* is:

$$weight = 48.1 + 2.2 \times shoe\ size$$

Use this equation to predict the weight (to the nearest kg) of a person with the following shoe size. Are you interpolating or extrapolating?

a 5 **b** 8 **c** 11

4 When preparing between 25 and 100 *meals*, a cafeteria's *cost* (in dollars) is given by the equation:

$$cost = 175 + 5.8 \times meals$$

Use this equation to predict the cost (to the nearest dollar) of preparing the following meals. Are you interpolating or extrapolating?

a no meals **b** 60 meals **c** 89 meals

5 For women of heights from 150 cm to 180 cm, the equation relating a *daughter's height* (in cm) to her *mother's height* (in cm) is:

$$daughter\text{'s } height = 18.3 + 0.91 \times mother\text{'s } height$$

Use this equation to predict (to the nearest cm) the adult height of a woman whose mother is the following heights. Are you interpolating or extrapolating?

a 168 cm tall **b** 196 cm tall **c** 155 cm tall

7H Interpreting the slope and intercept of a regression line

Whatever method is used to determine the equation of a straight line, the intercept and slope of the line can be interpreted in terms of the variables in the question. This interpretation gives us further insights into the relationship between the variables.

Interpreting the slope and intercept of a regression line

For the regression line $y = a + bx$:

- the *slope* (b) predicts the change in the *response variable* (y) for each one-unit increase or decrease in the *explanatory variable* (x).
- the *intercept* (a) predicts the value of the *response variable* (y) when the *explanatory variable* (x) equals 0.

Note: The interpretation of the y-intercept in a data context can be problematic when $x = 0$ is not within the range of observed x values.

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Example 8 Interpreting the slope and intercept of a regression line

A regression line is used to model the association between the *time* 10 students spent studying for an examination and their *mark*. The equation of the regression line is:

$$mark = 30.8 + 1.62 \times time$$

a **i** Write down the value of the intercept.

ii Interpret the intercept in this context of these variables.

b **i** Write down the value of the slope.

ii Interpret the slope in the context these variables.

Solution

a **i** In a linear equation of the form $y = a + bx$, a is the intercept. Here $y = mark$ and $x = time$ so $a = 30.8$.

intercept = 30.8

ii The intercept ($a = 30.8$) predicts the average *mark* (y) when the study *time* (x) equals 0.

The intercept predicts that students who spend no time studying for the examination will obtain a mark of 30.8.

b **i** In a linear equation of the form $y = a + bx$, b is the slope. Here $y = mark$ and $x = time$ so $b = 1.62$.

slope = 1.62

ii The slope ($b = 1.62$) predicts the change in the *mark* (y) associated with a one unit increase in the variable *time* (x).

The slope predicts an increase of 1.62 marks for each extra hour of study.

Exercise 7H

Interpreting intercept and slope and using them to make predictions

Example 8

1 The equation $price = 37\,650 - 4200 \times age$ can be used to predict the *price* of a used car (in dollars) from its *age* (in years).

a **i** For this regression equation, write down the value of the intercept.

ii Interpret the intercept in this context of the variables in the equation.

b **i** For this regression equation, write down the value of the slope.

ii Interpret the slope in the context of the variables in the equation.

c Use the equation to predict the average price of a 5-year-old second-hand car.

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2 The following regression equation can be used to predict the flavour rating of yoghurt from its percentage fat content.

$$flavour\ rating = 40 + 2.0 \times calories$$

a **i** For this regression equation, write down the value of the intercept.
  ii Interpret the intercept in this context of the variables in the equation.

b **i** For this regression equation, write down the value of the slope.
  ii Interpret the slope in the context of the variables in the equation.

c Use the equation to predict the flavour rating of a yoghurt with 30% fat content. Give your answer to the nearest whole number.

3 Students sit for two exams two weeks apart. The following regression equation can be used to predict the students' mark on exam 2 from the mark they obtained on exam 1.

$$mark\ on\ exam\ 2 = 15.7 + 0.650 \times mark\ on\ exam\ 1$$

a **i** For this regression equation, write down the value of the intercept.
  ii Interpret the intercept in this context of the variables in the equation.

b **i** For this regression equation, write down the value of the slope.
  ii Interpret the slope in the context of the variables in the equation.

c Use the equation to predict the mark on exam 2 for a student who obtains a mark of 20 on exam 1. Give answer to the nearest mark.

4 It has been suggested that the blood *glucose* level (in mg/100 mL) of adults can be predicted from their *weight* (in kg).

$$glucose = 51 + 0.62 \times weight$$

a **i** For this regression equation, write down the value of the intercept.
  ii Interpret the intercept in the context of the variables in the equation.

b **i** For this regression equation, write down the value of the slope.
  ii Interpret the slope in the context of the variables in the equation.

c Use the equation to predict the blood glucose level of a person who weighs 75 kg. Give your answer correct to one decimal place.

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7I Statistical investigation

The table gives details of Australian Test Cricket Captains, based on all tests from 1930 up to and including the South Africa test in March 2014.

Name	Years	Played	Won	Name	Years	Played	Won
W M Woodfull	1930–34	25	14	B C Booth	1965–66	2	0
V Y Richardson	1935–36	5	4	W M Lawry	1967–71	25	9
D G Bradman	1936–48	24	15	B N Jarman	1968	1	0
W A Brown	1945–46	1	1	I M Chappell	1970–75	30	15
A L Hassett	1949–53	24	14	G S Chappell	1975–83	48	21
A R Morris	1951–55	2	0	G N Yallop	1978–79	7	1
I W Johnson	1954–57	17	7	K J Hughes	1978–85	28	4
R R Lindwall	1956–57	1	0	A R Border	1984–94	93	32
I D Craig	1957–58	5	3	M A Taylor	1994–1999	50	26
R Benaud	1958–64	28	12	S R Waugh	1999–2004	57	41
R N Harvey	1961	1	1	R Ponting	2004–2010	77	48
R B Simpson	1963–78	39	12	M Clarke	2010–	37	19

Exercise 7I

1 **a** Construct a stem plot and a boxplot for the number of tests played by the captains.

b Describe the distribution of the number of tests captained in terms of shape, centre, spread and outliers quoting the values of appropriate statistics.

c Who is Australia's longest serving captain on the basis of this data?

2 **a** Construct a scatterplot matches won (*won*) against the matches played (*played*) by each captain. *Played* is the EV. Estimate the value of correlation coefficient r.

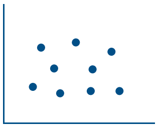
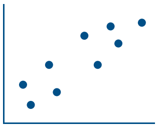
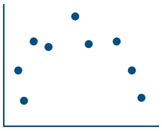
b Describe the association between matches played and matches won in terms of strength, direction, form and outliers if any.

c Fit a least squares regression line to the data and write its equation in terms of the variables *played* and *won*.

d Write down the slope of the least squares regression line and interpret it in terms of the variables *played* and *won*.

e Determine the correlation coefficient, r, and compare to your earlier estimate.

Key ideas and chapter summary

Explanatory and response variables	The **explanatory variable** is used to explain or predict the value of the **response variable**.
Scatterplot	A two-dimensional data plot where each point represents the value of two related variables in a bivariate data set. In a scatterplot, the **response variable (RV)** is plotted on the *vertical* axis and the **explanatory variable (EV)** on the *horizontal* axis. A scatterplot is used to help identify and describe the relationship between two **numerical** variables.
Identifying associations (relationships) between two numerical variables	A random cluster of points (no clear pattern) indicates that there is **no association** between the variables.  A *clear pattern* in the scatterplot indicates that there is an **association between the variables**   
Describing associations in scatterplots	Associations are described in terms of: ■ **direction** (positive or negative) and outliers ■ **form** (linear or non-linear) ■ **strength** (strong, moderate, weak or none)
Pearson's correlation coefficient (*r*)	**Pearson's correlation coefficient** (r) is a statistic that measures the direction and strength of a linear relationship between a pair of variables.
Linear regression	A straight line can be used to model a linear association between two numerical variables. The relationship can then be described by a rule of the form $y = a + bx$ In this equation: ■ y is the **response variable** ■ x is the **explanatory variable** ■ a is the **y-intercept** ■ b is the **slope of the line**.
The least squares method	The **least squares method** for fitting a line to a scatterplot minimises the sum of the squares of the residuals. It works best with no outliers.

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Making predictions	The **regression line** $y = a + bx$ enables the value of y to be predicted for a given value of x by substitution into the equation.
Interpolation and extrapolation	Predicting *within* the range of data is called **interpolation**. Predicting *outside* the range of data is called **extrapolation**.
Correlation and causation	An association (correlation) between two variables does not automatically imply that the observed relationship between the variables is **causal**. Alternative non-causal explanations for the association include, a common response to a third variable, a confounding variable or simply coincidence.

Skills check

Having completed this chapter you should be able to:

- use a scatterplot to describe an observed association between two numerical variables in terms of direction, form and strength, and the meaning of the association within the context of the data
- estimate the value of the correlation coefficient, r, from a scatterplot and calculate its value from data using technology
- recognise that an association (correlation) between two variables does not automatically imply that the observed relationship between the variables is causal and provide alternative non-causal explanations for the association based on common response, a confounding variable or coincidence.
- identify the explanatory variable and use the equation of the least squares line fitted to the data to model an observed linear association
- use the model to make predictions, being aware of the dangers of extrapolation
- interpret the slope and intercept of the model in the context of data.

Multiple-choice questions

1 For which one of the following pairs of variables would it be appropriate to construct a scatterplot?

A *eye colour* (blue, green, brown, other) and *hair colour* (black, brown, blonde, other)
B *test score* and *sex* (male, female)
C *political party preference* (Labor, Liberal, Other) and *age* in years
D *age* in years and *blood pressure* in mmHg
E *height* in cm and *sex* (male, female)

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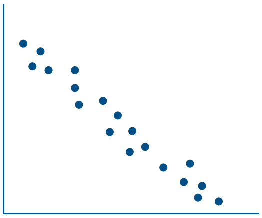
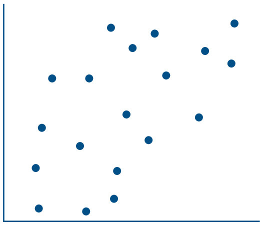
2 For the scatterplot shown, the association between the variables is best described as:



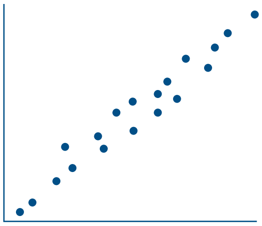
A weak linear negative
B strong linear negative
C no association
D weak linear positive
E strong linear positive

3 For the scatterplot shown, the association between the variables is best described as:



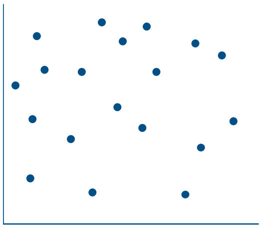
A weak linear negative
B weak non-linear negative
C no association
D weak linear positive
E strong non-linear positive

4 For the scatterplot shown, the association between the variables is best described as:



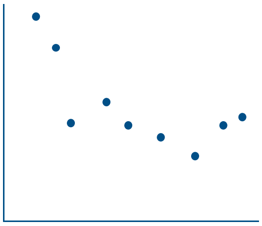
A weak linear positive
B strong linear positive
C no association
D moderate linear positive
E strong non-linear positive

5 For the scatterplot shown, the association between the variables is best described as:



A weak non-linear negative
B strong linear negative
C no association
D weak non-linear positive
E weak linear positive

6 For the scatterplot shown, the relationship between the variables is best described as:



A weak negative linear
B strong negative linear
C no association
D weak negative non-linear
E strong negative non-linear

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7 The association between birth weight and infant mortality rate is negative. Given this information, it can be concluded that:

A birth weight and infant mortality rate are not related.

B infant mortality tends to increase as birth weight increases.

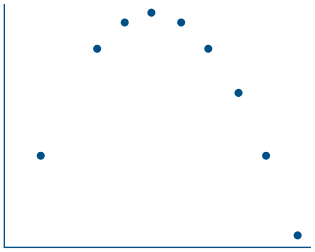
C infant mortality tends to decrease as birth weight decrease.

D infant mortality tends to decrease as birth weight increases.

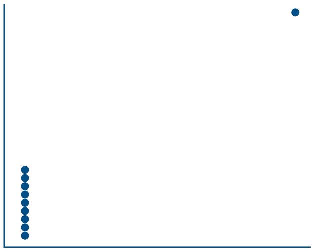
E the values of infant mortality are in general less than the corresponding value of birth weight.

8 For which of the following scatterplots plots would it make sense to calculate the correlation coefficient (r) to indicate the strength of the association between the variables?

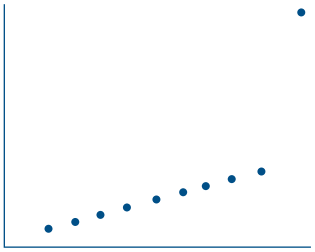
A



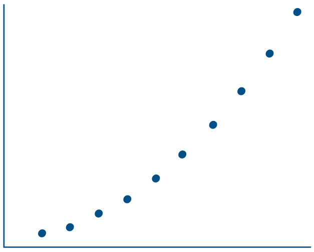
B



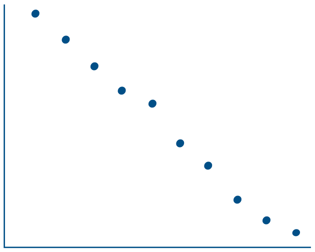
C



D



E



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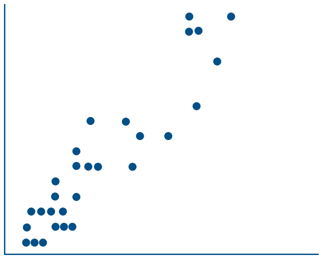
9 For the scatterplot shown, the value of the Pearson's correlation coefficient, r, is closest to:

A 0.28 **B** 0.41 **C** 0.63

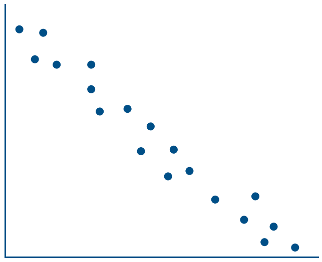
D 0.86 **E** 0.99



10 For the scatterplot shown, the value of the Pearson's correlation coefficient, r, is closest to:

A −0.90 **B** −0.64 **C** −0.23

D 0.64 **E** 0.90



11 A correlation coefficient of $r = -0.32$ would classify a linear relationship as:

A weak positive **B** weak negative **C** moderately positive

D close to zero **E** moderately weak

The following information relates to Questions 12 and 13

The weekly *income* and weekly *expenditure* on food for a group of 10 university students is given in the following table.

Income ($/week)	150	250	300	600	300	380	950	450	850	1000
Expenditure ($/week)	40	60	70	120	130	150	200	260	460	600

12 The value of the Pearson correlation coefficient, r, for these data is closest to:

A 0.2 **B** 0.4 **C** 0.6 **D** 0.7 **E** 0.8

13 The least squares regression line that enables weekly *expenditure* (in dollars) on food to be predicted from weekly *income* (in dollars) is closest to:

A $expenditure\ on\ food = 0.482 + 42.864 \times income$

B $expenditure\ on\ food = 0.482 - 42.864 \times income$

C $expenditure\ on\ food = -42.864 + 0.482 \times income$

D $expenditure\ on\ food = 239.868 + 1.355 \times income$

E $expenditure = 1.355 + 239.868 \times income$

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The following information relates to Questions 14 and 15

The equation of a regression line that enables weekly *amount* spent on entertainment (in dollars) to be predicted from weekly *income* is given by:

$$amount = 40 + 0.10 \times income$$

14 Using this equation the amount spent on entertainment by an individual with a weekly income of \$600 is predicted to be:

A \$40 **B** \$46 **C** \$100 **D** \$240 **E** \$24 060

15 From the equation of the regression line it can be concluded that, on average:

A the weekly *amount* spent on entertainment increases by 40 cents a week for each extra dollar of weekly income.

B the weekly *amount* spent on entertainment increases by 10 cents a week for each extra dollar of weekly income.

C the weekly *income* increases by 10 cents for each dollar increase in the amount spent on entertainment each week.

D \$40 is spent on entertainment each week.

E \$40.10 is spent on entertainment each week.

Short-answer questions

1 The following table gives the *number* of times the ball was inside the team's 50 metre line in an AFL football game and the team's final score (in points) in that game.

Number	64	57	34	61	51	52	53	51	64	55	58	71
Score (points)	90	134	76	92	93	45	120	66	105	108	88	133

a Which variable is the RV? **b** Construct a scatterplot of *score* against *number*.

c Use the scatterplot to describe the association in terms of strength and direction.

2 The *distance* travelled to work and the *time* taken for ten company employees are given in the following table. *Distance* is the response variable.

a determine the value of the Pearson correlation coefficient, *r*, for this set of data

b determine the equation of least squares line for this data and write the equation in terms of the variables *distance* and *time*.

Distance (km)	*Time (min)*
12	15
50	75
40	50
25	50
45	80
20	50
10	10
3	5
10	10
30	35

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3 The regression equation:

$$\textit{taste score} = -22 + 7.3 \times \textit{magnesium content}$$

can be used to predict the *taste score* of a country town's drinking water from its *magnesium content* (in mg/litre).

a Which variable is the explanatory variable?

b Write down and interpret the slope of the regression line.

c Use the regression line to predict the taste score of a country town's drinking water whose magnesium content is 16 milligrams/litre, correct to one decimal place.

4 The *time* taken to complete a task and the number of *errors* on the task were recorded for a sample of 10 primary school children.

Time (s)	22.6	21.7	21.7	21.3	19.3	17.6	17.0	14.6	14.0	8.8
Errors	2	3	3	4	5	5	7	7	9	9

a Determine the equation of the least squares line that fits this data with *errors* as the response variable.

b Determine the value of Pearson's correlation coefficient to two decimal places.

Extended-response questions

1 A marketing firm wanted to investigate the relationship between the number of times a song was played on the radio (*played*) and the number of downloads sold the following week (*weekly sales*).

The following data was collected for a random sample of ten songs.

Played	47	34	40	34	33	50	28	53	25	46
Weekly sales	3950	2500	3700	2800	2900	3750	2300	4400	2200	3400

a Which is the explanatory variable and which is the response variable?

b Construct a scatterplot of this data.

c Determine the value of the Pearson correlation coefficient, r, for this data.

d Describe the relationship between *weekly sales* and *played* in terms of direction, strength and form and outliers (if any).

e Determine the equation for the least squares regression line and write it down in terms of the variables *weekly sales* and *played*.

f Interpret the slope and intercept of the regression line in the context of the problem.

g Use your equation to predict the number of downloads of a song when it was played on the radio 100 times in the previous week.

h In making this prediction, are you interpolating or extrapolating?

2 To test the effect of driving instruction on driving skill, 10 randomly selected learner drivers were given a *score* on a driving skills test. The number of *hours* of instruction for each learner was also recorded. The results are displayed in the table below.

Hours	19	2	5	9	16	4	19	26	14	8
Score	32	12	17	19	23	16	28	36	30	23

a Which is the explanatory variable and which is the response variable?

b Construct a scatterplot of these data.

c Determine the correlation coefficient, r, give answer correct to four decimal places.

d Describe the association between *score* and *hours* in terms of direction, strength and form and outliers (if any).

e Determine the equation for the least squares regression line and write it down in terms of the variables *score* and *hours*. Give coefficients correct to three significant figures.

f Interpret the slope and the intercept (if appropriate) of the regression line.

g Predict the score after 10 hours of instruction to the nearest point.

3 To investigate the association between marks on an assignment and the final examination mark, the following data was collected.

Assignment mark	80	77	71	78	65	80	68	64	50	66
Exam mark	83	83	79	75	68	84	71	69	66	58

a A scatterplot shows that there is a strong positive linear relationship between the *assignment mark* and the final *exam marks*. The correlation coefficient is $r = 0.76$. Given this information, a student wrote: 'Good final exam marks are the result of good assignment marks'. Comment on this statement.

b Determine the equation of the least square regression line and write it down in terms of the variables final *exam mark* and *assignment mark* . Write the coefficients correct to two significant figures.

c Interpret the intercept and slope of the least squares regression line in terms of the variables in the study.

d Use your regression equation to predict the final exam mark for a student who scored 50 on the assignment. Give your answer correct to the nearest mark.

e How reliable is the prediction made in part **d**?

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8

Number patterns and recursion

- What is a sequence?
- How do we make an arithmetic sequence?
- How can we generate a sequence recursively using a graphics calculator?
- What is the rule used to find the *n*th term of an arithmetic sequence?
- How is a geometric sequence different from an arithmetic sequence?
- What is the rule for the *n*th term of a geometric sequence?
- How does a recursive relation help us to generate the terms of a sequence recursively?
- What is a Fibonacci sequence?
- How can we use sequences to model growth and decay?

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Introduction

We will be investigating sequences that can be generated by a rule. Some sequences make each new term by adding a constant amount. Others multiply each term by a fixed number to make the next term. The family of Fibonacci sequences have their own method of generating sequences of numbers that occur in situations as diverse as the number of petals on flowers to the analysis of trends in the stock market.

Finding the rule that fits the known values of a sequence will enable us to predict the values that follow.

Sequences have applications in puzzles, patterns found in the natural world, finance, and many forms of growth and decay.

8A Number patterns

A **sequence** is a list of numbers in a particular order. The numbers or items in a sequence are called the **terms** of the sequence. They may be generated randomly or by a particular rule.

▶ Randomly generated sequences

Recording the numbers obtained while tossing a die would give a randomly generated sequence, such as:

$$3, 1, 2, 2, 6, 4, 3, \ldots$$

Because there is no pattern in the sequence there is no way of predicting the next term.

Consequently random sequences are of no relevance to this chapter.

▶ Rule based sequences

Writing down odd numbers starting at 1 would result in a sequence generated by a rule:

$$1, 3, 5, 7, 9, 11, 13, \ldots$$

There is a rule that allows us to state the next term in the sequence.

'add 2 to the current odd number'

For example, to find the term after 13, just add 2 to 13, to get $13 + 2 = 15$.

The group of three dots (. . .) at the end of the sequence is called an *ellipsis*. An ellipsis is used to show that the sequence continues.

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In this chapter we will look at sequences that can be generated by a rule.

Example 1 Looking for a rule for a sequence of numbers

Look for a pattern or rule in each sequence and find the next number.

a 2, 8, 14, 20, . . . **b** 5, 15, 45, 135, . . . **c** 7, 4, 1, −2, . . .

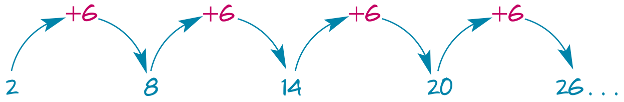
d 1, 4, 9, 16, . . . **e** 1, 1, 2, 3, 5, . . .

Solution

a 2, 8, 14, 20, . . .

1 Add 6 to make each new term.

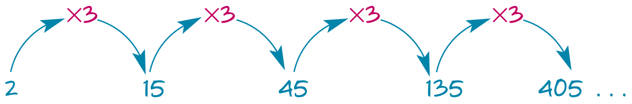
2 Add 6 to 20 to make the next term, 26.



b 5, 15, 45, 135, . . .

1 Multiply by 3 to make each new term.

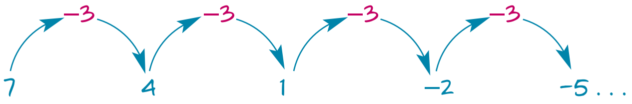
2 Multiply 135 by 3 to make the next term, 405.



c 7, 4, 1, −2, . . .

1 Subtract 3 each time to make the next term.

2 Subtract 3 from −2 to get the next term, −5.



d 1, 4, 9, 16, . . .

1 Each term is the square of its position in the sequence.

2 The fifth term is $5^2 = 25$.

1st,	2nd,	3rd,	4th,	5th,	
1^2,	2^2,	3^2,	4^2,	5^2,	
1,	4,	9,	16,	25,	. . .

e 1, 1, 2, 3, 5, . . .

1 Add the previous two terms to make each new term.

2 The next term is 3 + 5 = 8.

Note: This is called a **Fibonacci sequence**.

		1 + 1	1 + 2	2 + 3	3 + 5	
1,	1,	2,	3,	5,	8,	. . .

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▶ Naming the terms in a sequence

The symbols t_1, t_2, t_3, are used as labels or names for the first, second and third terms in the sequence. In the labels t_1, t_2, t_3 the numbers 1, 2, 3 are called *subscripts*[1].

The subscripts tell us the position of each term in the sequence. So t_{10} is just a name for the tenth term in the sequence.

Example 2 Naming terms in a sequence

For the sequence: 2, 8, 14, 20, 26, 32, . . . state the values of:

a t_1 **b** t_4 **c** t_6

Solution

1 Write the name for each term under its value in the sequence.

2,	8,	14,	20,	26,	32,
t_1	t_2	t_3	t_4	t_5	t_6

2 Read the value of each required term.

$t_1 = 2, \qquad t_4 = 20 \qquad t_6 = 32$

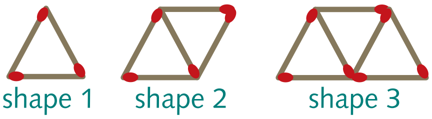
Exercise 8A

Looking for and applying patterns

Example 1 **1** Find the next term in each sequence.

- **a** 3, 7, 11, 15, . . .
- **b** 10, 9, 8, 7, . . .
- **c** 4, 8, 16, 32, . . .
- **d** January, February, . . .
- **e** 1, 2, 1, 2, . . .
- **f** 48, 24, 12, 6, . . .
- **g** 31, 24, 17, 10, . . .
- **h** 1, 8, 27, 64, . . .
- **i** a, c, e, g, . . .
- **j** ♣, ♦, ♥, ♠, . . .
- **k** 1, 3, 9, 27, . . .
- **l** ⇒, ⇐, ⇒, ⇐, . . .
- **m** 2, 6, 18, 54, . . .
- **n** ↑, →, ↓, ←, . . .
- **o** M, T, W, T, . . .
- **p** O, T, T, F, . . .
- **q** A, E, F, H, . . .

2 **a** Draw the next shape in this sequence of matchstick shapes:



b Write the sequence for the number of matchsticks used in each shape shown.

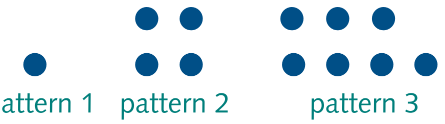
c Find the number of matchsticks needed to make shape 4 and shape 5.

d Give the first number and the rule for making each subsequent number in the sequence.

---

[1] Do not confuse the subscripts, which give the position of each term, with the powers we see in say, x^2, which means x multiplied by x. Also notice that t_3 is just the name for the term in the third position. It is not saying that the value of the term is 3. In the Example 2, $t_3 = 14$.

3 **a** Draw the next pattern in this sequence of dots:



b Complete the table.

Pattern number	1st	2nd	3rd	4th	5th
Number of dots	1	4			

c Give the first number of dots and the rule for making each subsequent number of dots in the sequence.

4 Describe how terms are generated in each number sequence and give the next two terms.

a 5, 8, 11, 14, . . .
b 19, 28, 37, 46, . . .
c 38, 34, 30, 26, . . .
d 66, 58, 50, 42, . . .
e 3, 6, 12, 24, . . .
f 4, 12, 36, 108, . . .
g 128, 64, 32, 16, . . .
h 3, –6, 12, –24, . . .
i 1, 2, 3, 5, . . .

Subscript notation

Example 2

5 Find the required terms from the sequence: 6, 11, 16, 21, . . .

a t_1
b t_3
c t_2
d t_4
e t_5
f t_6

6 For each sequence state the value of the named terms:

i t_1
ii t_4
iii t_5

a 6, 10, 14, 18, . . .
b 2, 8, 32, 128, . . .
c 29, 22, 15, 8, . . .
d 96, 48, 24, 12, . . .

8B Arithmetic sequences

As we have seen earlier, in some cases the rule that connects the values of the terms in a sequence is something like 'each new term is made by adding or subtracting a fixed amount to or from the current term'.

Examples include:

Sequence	Rule
2, 7, 12, . . .	'to find the next term in the sequence, add 5 to the current term'
200, 300, 400, . . .	'to find the next term in the sequence, add 100 to the current term'
50, 45, 40, . . .	'to find the next term in the sequence, subtract 5 from the current term'
100, 99, 98, . . .	'to find the next term in the sequence, subtract 1 from the current term'

Sequences that are generated by adding or subtracting a fixed amount to the previous term are called **arithmetic sequences**.

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▶ The common difference

The fixed amount we add or subtract to form an arithmetic sequence recursively is called the **common difference**. The symbol d is often used to represent the common difference.

If the sequence is *known* to be arithmetic, the common difference can be calculated by simply subtracting any pair of successive terms.

Common difference, *d*

In an *arithmetic sequence*, the fixed number added to (or subtracted from) each term to make the next term is called the *common difference, d,* where:

$$\begin{aligned} d &= \text{any term} - \text{its previous term} \\ &= t_2 - t_1 \\ &= t_3 - t_2 \\ &= t_4 - t_3 \end{aligned}$$

and so on.

For example, the common difference for the arithmetic sequence, 30, 25, 20, . . . is:

$$d = t_2 - t_1 = 25 - 30 = -5$$

Often it is not necessary to calculate the common difference in this formal way. It may be easy to see what amount has been repeatedly added (or subtracted) to make each new term.

Example 3 Finding the common difference in an arithmetic sequence

Find the common difference for the following arithmetic sequences and use it to find the 4th term in the sequence:

a 2, 5, 8, . . . **b** 25, 23, 21, . . .

Solution

1 Because we know the sequence is arithmetic, all we need to do is find the difference in value between terms 1 and 2.

2 To find the 4th term, add the common difference to the 3rd term.

a $d = t_2 - t_1 = 5 - 2 = 3$

$t_4 = t_3 + d = 8 + 3 = 11$

b $d = t_2 - t_1 = 25 - 23 = -2$

$t_4 = t_3 + d = 21 + (-2) = 19$

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Identifying arithmetic sequences

If a sequence is arithmetic, the difference between successive terms will be constant. We can use this idea to see whether or not a sequence is arithmetic.

Example 4 Identifying an arithmetic sequence

Which of the following sequences is arithmetic?

a 21, 28, 35, 42, . . .

b 2, 6, 18, 54, . . .

Solution

a 21, 28, 35, 42, . . .

1 Determine whether the difference between successive terms is constant.

Differences:
$28 - 21 = 7$
$35 - 28 = 7$
$42 - 35 = 7$

2 Write your conclusion.

As the differences between successive terms are constant, the sequence is arithmetic.

b 2, 6, 18, 54, . . .

1 Determine whether the difference between successive terms is constant.

Differences:
$6 - 2 = 4$
$18 - 6 = 12$
$54 - 18 = 36$

2 Write your conclusion.

As the differences between successive terms are not constant, the sequence is not arithmetic.

Using repeated addition on a CAS calculator to generate a sequence

As we have seen, a recursive rule based on repeated addition, such as ‘to find the next term, add 6’, is a quick and easy way of generating the next few terms of a sequence. However, it becomes tedious to do by hand if we want to find, say, the next 20 terms.

Fortunately, your CAS calculator can semi-automate the process.

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Using TI-Nspire CAS to generate the terms of an arithmetic sequence recursively

Generate the first six terms of the arithmetic sequence: 2, 7, 12, . . .

Steps

```
2                    2.
Ans+5
```

1 Press ⌂on>**New Document> Add Calculator**.
2 Enter the value of the first term **2**. Press enter
The calculator stores the value 2 as Answer (you cannot see this yet).

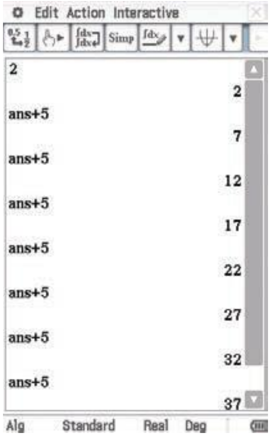
```
2                    2.
2.+5                 7.
7.+5                12.
12.+5               17.
17.+5               22.
22.+5               27.
```

3 The **common difference** for the sequence is 5. So, type in +**5**.
4 Press enter The second term in the sequence, 7, is generated.
5 Pressing enter again generates the next term, 12. Keep pressing enter until the desired number of terms is generated.
6 Write down the terms. *The first 6 terms of the sequence are: 2, 7, 12, 17, 22, 27.*

Using Class Pad to generate the terms of an arithmetic sequence recursively

Generate the first six terms of the arithmetic sequence: 2, 7, 12, . . .

Steps



```
2
                     2
ans+5
                     7
ans+5
                    12
ans+5
                    17
ans+5
                    22
ans+5
                    27
ans+5
                    32
ans+5
                    37
```

1 Tap Main √α to open the Main application.
2 Starting with a clean screen, enter the value of the first term, **2**. Press EXE. The calculator stores the value 2 as **ans**. (You can’t see this yet.)
3 The common difference for this sequence is 5. So, type **+ 5**. Then press EXE. The second term in the sequence, **7**, is displayed.
4 Pressing EXE again generates the next term, **12**. Keep pressing EXE until the required number of terms is generated.
5 Write down the terms. *The first 6 terms of the sequence are: 2, 7, 12, 17, 22, 27.*

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▶ Graphs of arithmetic sequences

If we plot the values of the terms of an arithmetic sequence (t_n) against their number (n) or position in the sequence, we will find that the points lie on a straight line. We could anticipate this because, as we progress through the sequence, the value of successive terms increases by the same amount (the common difference, d).

The advantages of graphing a sequence are that the straight line required for an arithmetic sequence is immediately obvious and any exceptions would stand out very clearly. An upward slope indicates regular growth and a downward slope reveals decay at a constant rate.

Example 5 Graphing the terms of an increasing arithmetic sequence ($d > 0$)

The sequence 4, 7, 10, . . . is arithmetic with common difference $d = 3$.

a Construct a table showing the term number (n) and its value (t_n) for the first four terms in the sequence.

b Use the table to plot the graph.

c Describe the graph.

Solution

a Show the term numbers and values of the first four terms in a table.

1 Write the term numbers in top row of the table.

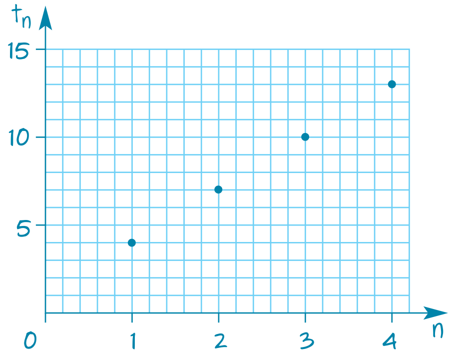
2 Write the values of the terms in the bottom row.

Term number, n	1	2	3	4
Term value, t_n	4	7	10	13

b Use the table to plot the graph.

1 Use the horizontal axis, n, the term numbers.
Use the vertical axis for the value of each term, t_n.

2 Plot each point from the table.



c Describe the graph.

1 Are the points along a straight line or a curve?

2 Is the line of the points rising (positive slope) or falling (negative slope)?

The points of an arithmetic sequence with $d = 3$ lie along a rising straight line.

The line has a positive slope.

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Example 6 Graphing the terms of a decreasing arithmetic sequence ($d < 0$)

The sequence 9, 7, 5, . . . is arithmetic with common difference $d = -2$

a Construct a table showing the term number (n) and its value (t_n) for the first four terms in the sequence.

b Use the table to plot the graph.

c Describe the graph.

Solution

a Show the term numbers and values of the first four terms in a table.

1 Write the term numbers in the top row of the table.

2 Write the values of the terms in the bottom row.

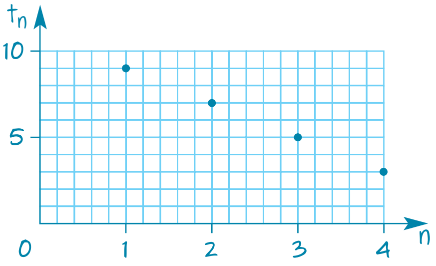
Term number, n	1	2	3	4
Term value, t_n	9	7	5	3

b Use the table to plot the graph.

1 Use the horizontal axis, n, for the term numbers.

Use the vertical axis for the value of each term, t_n.

2 Plot each point from the table.



c Describe the graph.

1 Are the points along a straight line or a curve?

2 Is the line of the points rising (positive slope) or falling (negative slope)?

The points of an arithmetic sequence with $d = -2$ lie along a falling straight line.

The line has a negative slope.

Graphs of arithmetic sequences are:

- points along a line with positive slope, when a constant amount is added ($d > 0$)
- points along a line with negative slope, when a constant amount is subtracted ($d < 0$).

A line with *positive* slope rises from left to right. A *negative* slope falls from left to right.

Exercise 8B

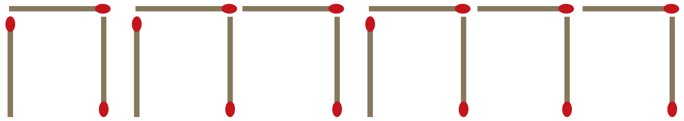
Basic ideas of arithmetic sequences

Example 4

1 Find out which of the sequences below is arithmetic. Give the common difference for each sequence that is arithmetic.

a 8, 11, 14, 17, . . .
b 7, 15, 22, 30, . . .
c 11, 7, 3, −1, . . .
d 12, 9, 6, 3, . . .
e 16, 8, 4, 2, . . .
f 1, 1, 1, 1, . . .

2 The numbers of matches used to construct the succession of shapes shown follow an arithmetic sequence.



a Write the sequence for the number of matches used in each structure.

b Find the common difference for this sequence.

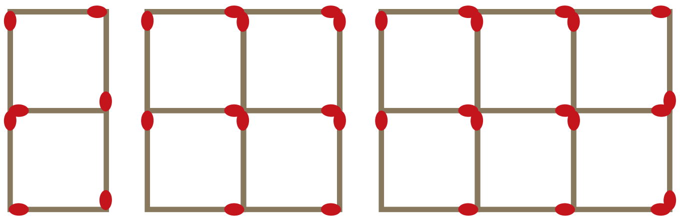
c How many matches would be used in the next two structures?

3 Matchsticks have been used to make the shapes below:

a Give the sequence for the number of matchsticks used in each shape.

b Is the sequence for the number of matchsticks used arithmetic? If it is arithmetic, give the common difference.

c How many matchsticks would be used in each of the next two shapes?



Finding common differences and generating new terms

4 For each of these arithmetic sequences, find the common difference and the 5th term.

a 5, 11, 17, 23, . . .
b 17, 13, 9, 5, . . .
c 11, 15, 19, 23, . . .
d 8, 4, 0, −4, . . .
e 35, 30, 25, 20, . . .
f 1.5, 2, 2.5, 3, . . .

5 Give the next two terms in each of these arithmetic sequences.

a 17, 23, 29, 35, . . .
b 14, 11, 8, 5, . . .
c 2, 1.5, 1.0, 0.5, . . .
d 27, 35, 43, 51, . . .
e 33, 21, 9, −3, . . .
f 0.8, 1.1, 1.4, 1.7, . . .

6 Find the missing terms in the following arithmetic sequences.

a 8, 13, 18, 23, □, □, . . .
b 14, 8, 2, −4, □, □, . . .
c 6, 15, □, □, 42, . . .
d 23, 18, □, □, 3, −2, . . .
e 3, □, □, 27, 35, 43, . . .
f □, □, 29, 37, 45, 53, . . .
g □, □, 7, −4,−15, −26, . . .
h 36, □, 22, □, 8, 1, . . .
i 15, □, 31, □, 47, □, . . .

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Generating the terms of an arithmetic sequence using a CAS calculator

7 Use your CAS calculator or generate the first five terms of the following arithmetic sequences.

a 3, 8, . . . **b** 16, 9, . . . **c** 1.6, 3.9, . . . **d** 8.7, 5.6, . . . **e** 293, 226, . . .

8 Using your CAS calculator:

a generate the first six terms of the arithmetic sequence 1, 6, . . . and write down t_6.

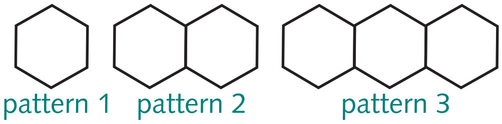
b generate the first 12 terms of the arithmetic sequence 45, 43, . . . and write down t_{12}.

c generate the first 10 terms of the arithmetic sequence 15, 14, . . . and write down t_{10}.

d generate the first 15 terms of the arithmetic sequence 0, 3, . . . and write down t_{15}.

Using a CAS calculator to solve practical problems

9 The number of sticks used to make the hexagonal patterns opposite form the arithmetic sequence: 6, 11, 16, . . .



a Write the common difference for this sequence.

b Using your CAS calculator, determine the number of matches needed to form:

i pattern 6 **ii** pattern 10

10 After one week of business Fumbles Restaurant had 320 wine glasses.
After two weeks, they only had 305 wine glasses. On average 15 glasses are broken each week.
Use your CAS calculator, to determine how many weeks it takes at that breakage rate for there to be only 200 glasses left?

11 Elizabeth stored 100 songs on her music storage device in the first month. In each month that followed she stored seven more songs.
Using your CAS calculator:

a determine the number of songs she had stored after each of the first 4 months

b determine the number of songs she had stored by the end of the year.

Graphing arithmetic sequences

Example 5, 6

12 For the following arithmetic sequences:

a 3, 5, 7, . . . **b** 11, 8, 5, . . .

i write down the next term

ii show the term positions and values of the first four terms in a table

iii use the table to plot a graph

iv describe the graph.

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8C Arithmetic sequence applications

Finding the *n*th term in an arithmetic sequence

Repeated addition can be used to make each new term in an arithmetic sequence. However, this would be a very tedious process if we wanted to know t_{50}, the term in the 50th position. We would have to make every term until we got to the 50th term. A CAS calculator can help, but a lot of careful counting would be required.

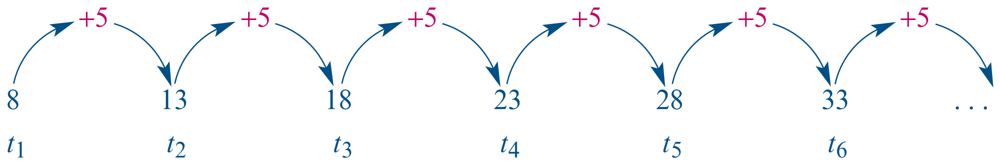
We need a rule that will help us calculate any term t_n using n, its position number, together with the values of the first term and the common difference.

Consider the arithmetic sequence: 8, 13, 18, 23, 28, 33, . . .

The common difference for this sequence is $d = 5$ $(13 - 8 = 5)$.

Thus, this sequence can be formed recursively using repeated addition using the rule 'to find the next term, add 5'.

This is illustrated pictorially in the diagram below.



Using the information from the diagram we can write recursively:

$$t_1 = 8$$

$$t_2 = t_1 + 5 = 8 + 5 = 13$$

$$t_3 = t_2 + 5 = (t_1 + 5) + 5 = t_1 + 2 \times 5 = 8 + 10 = 18$$

$$t_4 = t_3 + 5 = (t_2 + 5) + 5 = t_2 + 2 \times 5 = (t_1 + 5) + 2 \times 5 = t_1 + 3 \times 5 = 8 + 3 \times 5 = 23$$

focussing on the red coloured expressions a rule is emerging:

'to calculate the *n*th term in the sequence write down the first term, 8, then add $(n - 1) \times 5$s'. For example, using the rule we have:

$$t_5 = 8 + (5 - 1) \times 5 = 8 + 4 \times 5 = 28$$

and

$$t_6 = 8 + (6 - 1) \times 5 = 8 + 5 \times 5 = 33$$

(which checks out with the sequence we were given).

What we do not know is the value of the 10th term. However, we now have a quick way of working it out using our new rule as follows:

$$t_{10} = 8 + (10 - 1) \times 5 = 8 + 9 \times 5 = 53$$

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We are now in a position to write a general rule for finding the *n*th term of an arithmetic sequence.

Rule for finding the *n*th term of an arithmetic sequence

In an *arithmetic sequence*, the rule for the term in the *n*th position is:

$$t_n = a + (n - 1)d$$

where $a = t_1$ = first term

d = common difference

n = position number of the term

Example 7 Finding the *n*th term of an arithmetic sequence

Find t_{30}, the 30th term in the arithmetic sequence: 21, 18, 15, 12, . . .

Strategy: To use $t_n = a + (n - 1)d$ we need to know a, d and n.

Solution

1 The first term is 21.

$a = 21$

2 The term t_{30} is in the 30th position.

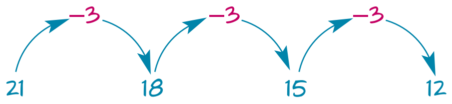
$n = 30$

3 Find the difference in the first two terms.

$d = t_2 - t_1$

$= 18 - 21 = -3$

4 Check that this difference generates the sequence.



5 Substitute the values of a, d and n into $t_n = a + (n - 1)d$.

$t_n = a + (n - 1)d$

$t_{30} = 21 + (30 - 1)(-3)$

6 Evaluate.

$= 21 + 29(-3)$

$= -66$

7 Write your answer.

The 30th term, t_{30}, is -66.

The rule for the *n*th term of an arithmetic sequence gives us a quick way of finding the required term in practical applications.

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▶ Applications

Example 8 Application of an arithmetic sequence

The hire of a car costs \$180 for the first day and \$150 for each extra day.

a How much would it cost to hire the car for 7 days?

b Find a rule for the cost of hiring the car for n days.

Solution

Strategy: The cost of hiring the car for a given number of days follows an arithmetic sequence. The cost for the first day, \$180, is the first term. The common difference is \$150 because this is the amount added to find the cost for each extra day.

a **1** Identify values for a, n and d. — $a = 180, n = 7, d = 150$

2 Substitute the values for a, n and d into $t_n = a + (n - 1)d$. — $t_n = a + (n - 1)d$

$t_7 = 180 + (7 - 1)(150)$

3 Evaluate. — $= 180 + (6)(150)$

$= 1080$

4 Write your answer. — It would cost \$1080 to hire the car for 7 days.

b Substitute $a = 180$ and $d = 150$ into $t_n = a + (n - 1)d$. — $t_n = a + (n - 1)d$

$t_n = 180 + (n - 1)(150)$

Note: This rule is saying that it costs \$180 for the first day and the additional $(n - 1)$ days each cost \$150.

Exercise 8C

Notation: working out *a* and *d*

1 Give the value of a and d in each of the following arithmetic sequences.

a 7, 11, 15, 19, . . .
b 8, 5, 2, −1, . . .
c 14, 23, 32, 41, . . .
d 62, 35, 8, −19, . . .
e −9, −4, 1, 6, . . .
f −13, −19, −25, −31, . . .

Finding the *n*th term

Example 7

2 Find the required term in each of these arithmetic sequences.

a 18, 21, 24, 27, . . . Find t_{35}.
b −14, −6, 2, 10, . . . Find t_{41}.
c 27, 14, 1, −12, . . . Find t_{37}.
d 16, 31, 46, 61, . . . Find t_{29}.
e −19, −23, −27, −31, . . . Find t_{26}.
f 0.8, 1.5, 2.2, 2.9, . . . Find t_{36}.
g 82, 68, 54, 40, . . . Find t_{21}.
h 9.4, 8.8, 8.2, 7.6, . . . Find t_{29}.

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3 Find the 40th term in an arithmetic sequence that starts at 11 and has a common difference of 8.

4 The first term in an arithmetic sequence is 27 and the common difference is 19. Find the 100th term, t_{100}.

5 A sequence started at 100 and had 7 subtracted each time to make new terms. Find the 20th term, t_{20}.

Applications

Example 8

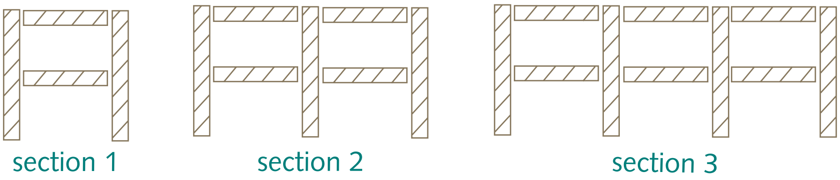
6 A canoe hire shop charges $15 for the first hour and $12 for each extra hour. How much would it cost to hire a canoe for 10 hours?

7 At the end of its first year after planting, a tree was 2.50 m high. It grew 0.75 m in each following year. How high was it 18 years after it was planted?

8 Bronwen swam her first race of 50 metres in 68.4 seconds. She hopes to reduce her time by 0.3 seconds each time she races. Give her times for the first four races if she succeeds.

9 Tristan had $250 on the first day of his holidays. If he spent $23 on each of the following days, how much did he have left after the 10th day of his holidays?

10 A single section of fencing is made from four logs. Two sections use seven logs. Examples of one, two and three-section fences are shown below.



a How many logs are needed for a fence with three sections?

b How many logs would be needed to build a fence with 20 sections?

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8D Using a recurrence relation to generate and analyse an arithmetic sequence

Consider the arithmetic sequence below:

$$10, 15, 20, \ldots$$

We can continue to generate the terms of this sequence by recognising that it uses the rule:

'to find the next term, add 5 to the current term and keep repeating the process'.

A **recurrence relation** is a way of expressing this rule in a precise mathematical language.

$$10, 15, 20, \ldots$$

is

$$t_1 = 10, \qquad t_{n+1} = t_n + 5$$

The rule tells us that the first term is 10, and each subsequent term is equal to the current term plus 5.

Understanding this, we can now proceed to generate the sequence term-by-term:

$$t_1 = 10$$
$$t_2 = t_1 + 5 = 15$$
$$t_3 = t_2 + 5 = 20$$
$$t_4 = t_3 + 5 = 25$$
$$t_5 = t_4 + 5 = 30 \text{ and so on.}$$

The recurrence relation for generating an arithmetic sequence

The recurrence relation:

$$t_1 = a, \qquad t_{n+1} = t_n + d$$

can be used to generate an arithmetic sequence with first term $t_1 = a$ and common difference d.

The *nth term* of a sequence generated by this recurrence relation is given by:

$$t_n = a + (n - 1)d$$

While in theory, we can continue using the recursive process to calculate the value of any term in the sequence above, this can become very tedious. Instead, we can use the rule for calculating the *n*th term of an arithmetic sequence to do the job.

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Example 9 Using a recurrence relation to generate an arithmetic sequence

a Generate and graph the first five terms of the sequence defined by the recurrence relation:

$$t_1 = 24, \quad t_{n+1} = t_n - 2$$

b Use a rule to calculate the value of the 33rd term in the sequence.

Solution

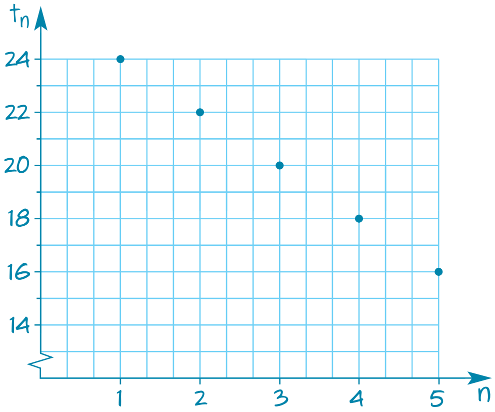
a **1** Write down the recurrence relation. $t_1 = 24, t_{n+1} = t_n - 2$

2 Write down the first term. $t_1 = 24$

3 Use the rule, which translates into 'to find the next term subtract two from the previous term', to generate the first five terms in the sequence.

$t_2 = t_1 - 2 = 24 - 2 = 22$
$t_3 = t_2 - 2 = 22 - 2 = 20$
$t_4 = t_3 - 2 = 20 - 2 = 18$
$t_5 = t_4 - 2 = 18 - 2 = 16$

4 To graph the terms, plot t_n against n.



b The rule for calculating the nth term of an arithmetic sequence is $t_n = a + (n - 1)d$
To find the 33rd term in this sequence use a = 24, $d = -2$ and $n = 33$.

$t_n = a + (n - 1)d$
$a = 24, d = -2, n = 33$
$t_{33} = 24 + (33 - 1)(-2) = -40$

▶ Notation

In the following example, a recurrence relation is used to generate the perimeters of a sequence of shapes. When the quantity being analysed represents some physical quantity, in this case the perimeter of a shape, it is common the use a letter other than the generic symbol t to name the terms in the sequence. In this example, the symbol P is used (P for perimeter).

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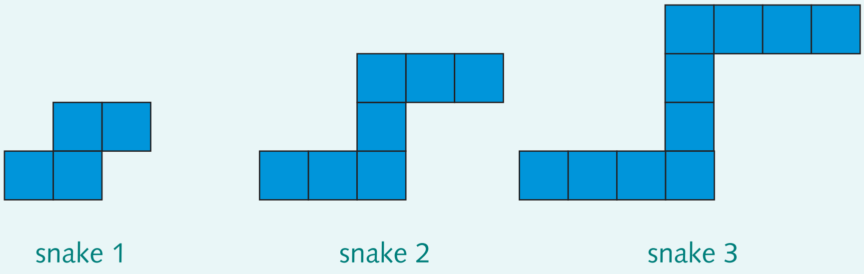
Example 10 Application of arithmetic sequences using a recurrence relation and the rule for the *n*th term

The Snake shapes below are made using blocks, each with a side length of 1 unit.



The perimeter of each Snake shape can be found by counting the sides of the blocks around the outside of the shape. The perimeter of each new snake built is 6 units longer than the previous snake.

Let P_n be the perimeter of the Snake shape n.

A recurrence relation that models this situation is:

$$P_1 = 10, \qquad P_{n+1} = P_n + 6$$

a **i** What does $P_1 = 10$ tell us?

ii What does the rule in the recurrence relation tell us?

b Use the recurrence relation to generate the perimeters of the first four snakes in this sequence and use these perimeters to construct a table showing the Snake number (n) and its perimeter (P_n).

c Use the table to plot the Snake perimeters against shape number and comment on the form of the graph.

d Use the rule for the *n*th term for this sequence to predict the perimeter of the 50th snake, in this sequence.

Solution

a **i** P_1 represents the perimeter of Snake 1.

ii The rule is $P_{n+1} = P_n + 6$.

i P_1 tells us that the perimeter of Snake 1 is ten units.

ii The rule tells us that the perimeter of each new snake in the sequence of snakes is 6 units longer than the perimeter of the previous snake.

b Starting with $P_1 = 10$, generate the next four perimeters using the recurrence rule.

$P_1 = 10$

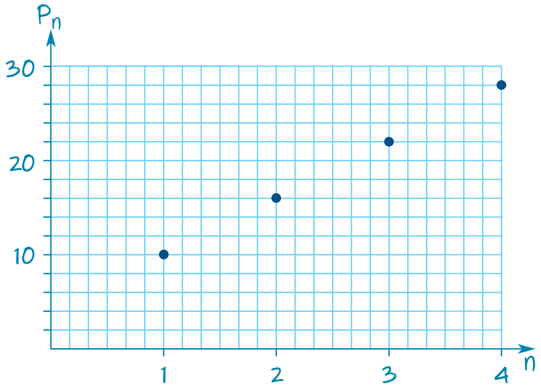
$P_2 = P_1 + 6 = 10 + 6 = 16$

$P_3 = P_2 + 6 = 16 + 6 = 22$

$P_4 = P_3 + 6 = 22 + 6 = 28$

c Construct a table using the perimeters. Use the table to plot a graph with n on the horizontal axis and P_n on the vertical axis.

Snake number	1	2	3	4
Perimeter	10	16	22	28



d Using the symbol P_n, rather than t_n, the rule for the nth term is $P_n = a + (n - 1)d$ Substitute $a = 10$, $n = 50$, $d = 6$ to obtain P_{50}.

The plot is linear, indicating linear growth.
$P_{50} = 10 + (50 - 1)6 = 304$
The perimeter of Snake 50 is 304 units.

Exercise 8D

Skillsheet

Using a recurrence relation to generate and graph the terms of an arithmetic sequence

Example 9

1 **a** Generate and graph the first five terms of the sequence defined by the recurrence relation: $t_1 = 15, t_{n+1} = t_n + 5$ where $n \geq 1$.

b Calculate the value of the 45th term in the sequence.

2 **a** Generate and graph the first five terms of the sequence defined by the recurrence relation: $t_1 = 60, t_{n+1} = t_n - 5$ where $n \geq 1$.

b Calculate the value of the 10th term in the sequence.

3 **a** Generate and graph the first five terms of the sequence defined by the recurrence relation: $t_1 = 15, t_{n+1} = t_n + 35$ where $n \geq 1$.

b Calculate the value of the 15th term in the sequence.

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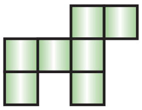
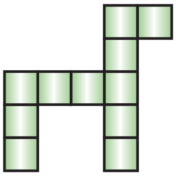
Applications

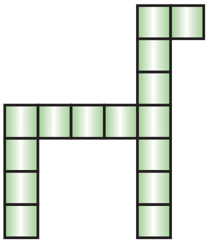
Example 10 **4** The Llama shapes have been made using blocks.



Llama 1



Llama 2



Llama 3

Let B_n be the number of blocks used to make the nth Llama shape.

The number of blocks used to make each Llama shape is generated by the recurrence relation:

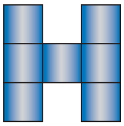
$$B_1 = 7, \qquad B_{n+1} = B_n + 4$$

a Count and record the number of blocks used to make the first, second and third Llama shapes.

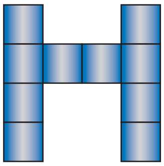
b Use the recurrence relation for B_n to generate the first five terms of the sequence of perimeters for these shapes.

c Use a rule to calculate the number of blocks needed to make the Llama 10 shape.

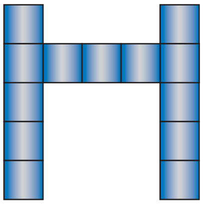
5 The BBQ shapes have been made using blocks, each with a side length of 1 unit.



BBQ 1



BBQ 2



BBQ 3

The perimeter of each BBQ shape can be found by counting the sides of the blocks around the outside of the shape.

Let P_n be the perimeter of the nth BBQ shape.

The perimeters for this sequence of BBQ shapes is generated by the recurrence relation:

$$P_1 = 16, \qquad P_{n+1} = P_n + 6$$

a Count and record the perimeters of the first, second and third BBQ shapes.

b Use the recurrence relation for P_n to generate the first four terms of this sequence of perimeters.

c Draw the fourth BBQ shape, find its perimeter and check if the recurrence relation correctly predicted the perimeter.

d Use the rule for the nth term for this sequence to predict the perimeter of the 10th BBQ shape.

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8E Geometric sequences

▶ The common ratio, *r*

In a **geometric sequence**, each new term is made by multiplying the previous term by a fixed number called the **common ratio**, ***r***. This repeating or recurring process is another example of a sequence generated by *recursion*.

In the sequence:

$$\begin{tikzcd} 2, \arrow[r, bend left, "\times 3"] & 6, \arrow[r, bend left, "\times 3"] & 18, \arrow[r, bend left, "\times 3"] & 54, \arrow[r, bend left, "\times 3"] & \ldots \\ t_1 & t_2 & t_3 & t_4 & \end{tikzcd}$$

each new term is made by multiplying the previous term by 3. The common ratio is 3.

In the sequence:

$$\begin{tikzcd} t_1 \arrow[r, bend left, "\times \frac{1}{2}"] & t_2 \arrow[r, bend left, "\times \frac{1}{2}"] & t_3 \arrow[r, bend left, "\times \frac{1}{2}"] & t_4 \arrow[r, bend left, "\times \frac{1}{2}"] & \ldots \end{tikzcd}$$

each new term is made by halving the previous term. In this sequence we are multiplying each term by $\times\frac{1}{2}$, which is equivalent to dividing by 2. The common ratio is $\frac{1}{2}$.

New terms in a geometric sequence $t_1, t_2, t_3, t_4, \ldots$ are made by multiplying the previous term by the common ratio, r.

$$\begin{tikzcd} t_1 \arrow[r, bend left, "\times r"] & t_2 \arrow[r, bend left, "\times r"] & t_3 \arrow[r, bend left, "\times r"] & t_4 \arrow[r, bend left, "\times r"] & \ldots \end{tikzcd}$$

So $t_1 \times r = t_2$ and $t_2 \times r = t_3$ and so on.

$$r = \frac{t_2}{t_1} \qquad r = \frac{t_3}{t_2}$$

Common ratio, *r*

In a *geometric sequence*, the *common ratio, r,* is found by dividing the next term by the current term.

$$\text{Common ratio } r = \frac{\text{any term}}{\text{the previous term}} = \frac{t_2}{t_1} = \frac{t_3}{t_2} = \frac{t_4}{t_3} = \ldots$$

As preparation for our study of growth and decay we will be using $r > 0$.

Example 11 Finding the common ratio *r*

Find the common ratio in each of the following geometric sequences.

a 3, 12, 48, 192, . . .

b 16, 8, 4, 2, . . .

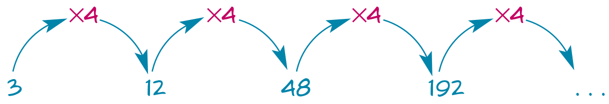
Solution

a 3, 12, 48, 192, . . .

1 The common ratio is equal to any term divided by its previous term.

Common ratio, $r = \frac{t_2}{t_1} = \frac{12}{3} = 4$

2 Check that multiplying by 4 makes each new term.



3 Write your answer.

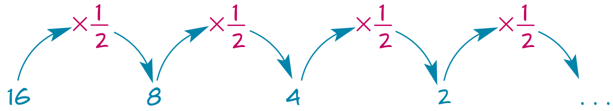
The common ratio is 4.

b 16, 8, 4, 2, . . .

1 Find the common ratio, *r*.

Common ratio, $r = \frac{t_1}{t_2} = \frac{8}{16} = \frac{1}{2}$

2 Check that multiplying by $\frac{1}{2}$ makes each new term.



3 Write your answer.

The common ratio is $\frac{1}{2}$.

▶ Identifying geometric sequences

To identify a sequence as geometric, it is necessary to find a common ratio between successive terms.

Example 12 Identifying a geometric sequence

Which of the following is a geometric sequence?

a 2, 10, 50, 250, . . .

b 3, 6, 18, 36, . . .

Solution

a 2, 10, 50, 250, . . .

1 Find the ratio (multiplier) between successive terms.

$$r = \frac{t_2}{t_1} \qquad r = \frac{t_3}{t_2} \qquad r = \frac{t_4}{t_3}$$

$$r = \frac{10}{2} \qquad r = \frac{50}{10} \qquad r = \frac{250}{50}$$

$$r = 5 \qquad r = 5 \qquad r = 5$$

2 Check that the ratios are the same.

The common ratio is 5.

3 Write your conclusion.

The sequence is geometric.

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b 3, 6, 18, 36, . . .

1 Find the ratios of successive terms.

$$r = \frac{t_2}{t_1} \qquad r = \frac{t_3}{t_2} \qquad r = \frac{t_4}{t_3}$$

$$r = \frac{6}{3} \qquad r = \frac{18}{6} \qquad r = \frac{36}{18}$$

$$r = 2 \qquad r = 3 \qquad r = 2$$

2 Are the ratios the same? The ratios are not the same.

3 Write your conclusion. The sequence is not geometric.

Using repeated multiplication on a CAS calculator to generate a geometric sequence

As we have seen, using a recursive rule based on repeated multiplication, such as ‘to find the next term, multiply by 2’, is a quick and easy way of generating the next few terms of an geometric sequence. It would be tedious to find the next 50 terms.

Fortunately, your CAS calculator can semi-automate the process of performing multiple repeated multiplications and do this very quickly.

How to use recursion to generate the terms of a geometric sequence with the TI-Nspire CAS

Generate the first six terms of the geometric sequence: 1, 3, 9, 27, . . .

Steps

1 Press ⌂ on **>New Document> Add Calculator**.

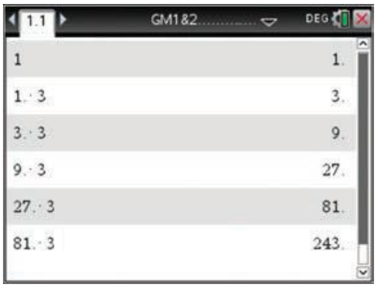
2 Enter the value of the first term **1**. Press enter
The calculator stores the value 1 as Answer (you cannot see this yet).

3 The common ratio for the sequence is 3. So, type in **×3**.

```
1                    1.
Ans·3
```

4 Press enter The second term in the sequence, 3, is generated.

5 Pressing enter again generates the next term, 9. Keep pressing enter until the desired number of terms is generated.



```
1                    1.
1.·3                 3.
3.·3                 9.
9.·3                27.
27.·3               81.
81.·3              243.
```

6 Write down the first six terms of the sequence.

The first six terms of the sequence are:
1, 3, 9, 27, 81, 241

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How to use recursion to generate the terms of a geometric sequence with the ClassPad

Generate the first six terms of the geometric sequence: 1, 3, 9, 27, . . .

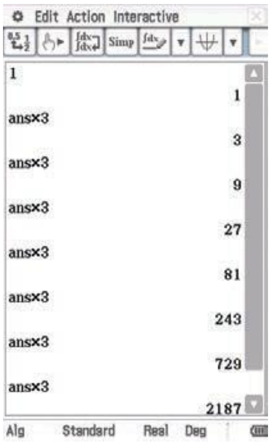
Steps

1 Tap √α to open the **Main** application.

2 Starting with a clean screen, enter the value of the first term, **1**. Press **EXE**.
The calculator stores the value 1 as **ans**wer. (You can't see this yet.)

3 The common ratio for this sequence is 3. So, type **× 3**. Then press **EXE**. The second term in the sequence (i.e. **3**) is displayed.

4 Pressing **EXE** again generates the next term, **9**. Keep pressing **EXE** until the required number of terms is generated.



```
1
                 1
ans×3
                 3
ans×3
                 9
ans×3
                27
ans×3
                81
ans×3
               243
ans×3
               729
ans×3
              2187
```

5 Write down the first six terms of the sequence. The first six terms of the sequence are: 1, 3, 9, 27, 81, 241

▶ Graphs of geometric sequences

In contrast with the straight-line graph of an arithmetic sequence, the values of a geometric sequence lie along a curve. Graphing the values of a sequence is a valuable tool for understanding applications involving growth and decay.

The graph of a geometric sequence clearly displays a curve of increasing values associated with growth or decreasing values indicating decay. As we will see, this depends on the value of the common ratio, r.

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Example 13 Graphing an increasing geometric sequence ($r > 1$)

Consider the geometric sequence: 2, 6, 18, . . .

a Find the next term.

b Show the positions and values of the first four terms in a table.

c Use the table to plot the graph.

d Describe the graph.

Solution

a **1** Find the common ratio using $r = \frac{t_2}{t_1}$.

2 Check that this ratio makes the given terms.

3 Multiply 18 by 3 to make the next term, 54.

4 Write your answer.

Common ratio, $r = \frac{t_2}{t_1} = \frac{6}{2} = 3$

$$2 \xrightarrow{\times 3} 6 \xrightarrow{\times 3} 18$$

The next term is 54.

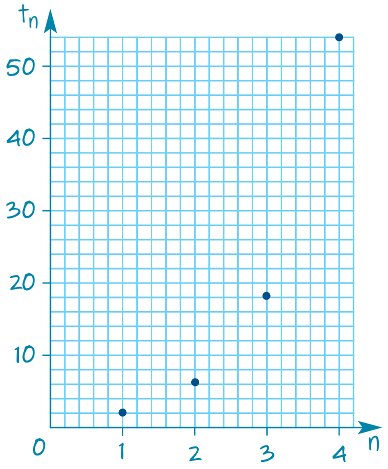
b **1** Number the positions along the top row of the table.

2 Write the terms in the bottom row.

Position, n	1	2	3	4
Term, t_n	2	6	18	54

c **1** Use the horizontal axis, n, for the position of each term.
Use the vertical axis, t_n, for the value of each term.

2 Plot each point from the table.



d Describe the pattern revealed by the graph.

The values lie along a curve and they are increasing.

Example 14 Graphing a decreasing geometric sequence ($0 < r < 1$)

Consider the geometric sequence: 32, 16, 8, . . .

a Find the next term.

b Show the positions and values of the first four terms in a table.

c Use the table to plot the graph.

d Describe the graph.

Solution

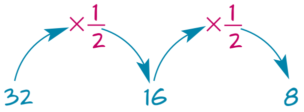
a **1** Find the common ratio using $r = \frac{t_2}{t_1}$.

2 Check that this ratio makes the given terms.

3 Multiply 8 by $\times\frac{1}{2}$ to make the next term, 4.

4 Write your answer.

Common ratio, $r = \frac{t_2}{t_1} = \frac{16}{32} = \frac{1}{2}$



The next term is 4.

b **1** Number the positions along the top row of the table.

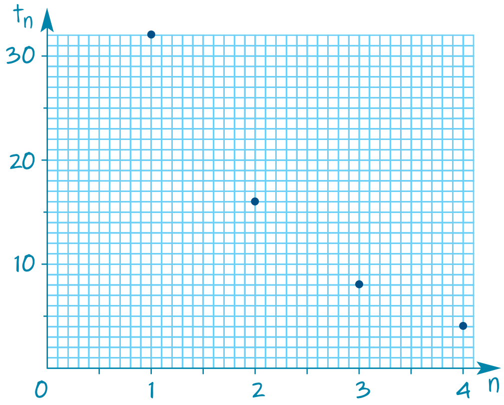
2 Write the terms in the bottom row.

Position, n	1	2	3	4
Term, t_n	32	16	8	4

c **1** Use the horizontal axis, n, for the position of each term.

Use the vertical axis, t_n, for the value of each term.

2 Plot each point from the table.



d Describe the pattern revealed by the graph.

The graph is a curve with values decreasing and approaching zero.

Graphs of geometric sequences (for r positive)

Graphs of geometric sequences for $r > 0$ are:

- *increasing* when r is greater than 1, $r > 1$
- *decreasing* towards zero when r is less than 1, $0 < r < 1$

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Exercise 8E

Showing a sequence is geometric

1 Find out which of the following sequences are geometric. Give the common ratio for each sequence that is geometric.

a 4, 8, 16, 32, . . .
b 1, 3, 9, 27, . . .
c 5, 10, 15, 20, . . .
d 5, 15, 45, 135, . . .
e 24, 12, 6, 3, . . .
f 3, 6, 12, 18, . . .
g 4, 8, 12, 16, . . .
h 27, 9, 3, 1, . . .
i 2, 4, 8, 16, . . .

Finding common ratios and generating new terms

Example 11, 12

2 Find the common ratio for each of the following geometric sequences.

a 3, 6, 12, 24, . . .
b 64, 16, 4, 1, . . .
c 6, 30, 150, 750, . . .
d 2, 8, 32, 128, . . .
e 32, 16, 8, 4, . . .
f 2, 12, 72, 432, . . .
g 10, 100, 1000, 10 000, . . .
h 3, 21, 147, 1029, . . .
i 7, 56, 448, 3584, . . .

3 Find the missing terms in each of these geometric sequences.

a 7, 14, 28, □, □, . . .
b 3, 15, 75, □, □, . . .
c 4, 12, □, □, 324, . . .
d □, □, 20, 40, 80, . . .
e 2, □, 32, 128, □, . . .
f 3, □, 27, □, 243, 729, . . .

Generating the terms of an geometric sequence using a CAS calculator

4 Use your graphics calculator to generate each sequence and find t_6, the sixth term.

a 7, 35, 175, . . .
b 3, 18, 108, . . .
c 96, 48, 24, . . .
d 4, 28, 196, . . .
e 160, 80, 40, . . .
f 11, 99, 891, . . .

Graphing geometric sequences

Example 13, 14

5 Consider each of the geometric sequences below.

i Find the next term.
ii Show the first four terms in a table.
iii Use the table to plot a graph.
iv Describe the graph.

a 3, 6, 12, . . .
b 8, 4, 2, . . .

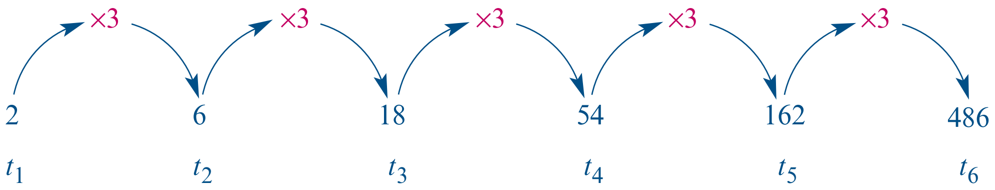
8F Geometric sequence applications

Finding the *n*th term in a geometric sequence

Repeated multiplication can be used to calculate each new term in a geometric sequence. However, to find the 50th term we would first have to calculate all the previous terms.

In this section we will derive a rule to find any term t_n by just using its position number, n, the common ratio, r, and the first term, a, in the sequence.

Consider the geometric sequence: 2, 6, 18, 54, 162, 486, . . .



Using the information from the diagram:

$$t_1 = 2$$

$$t_2 = t_1 \times 3 = 2 \times 3 = 6$$

$$t_3 = t_2 \times 3 = (t_1 \times 3) \times 3 = t_1 \times 3^2 = 2 \times 9 = 18$$

$$t_4 = t_3 \times 3 = (t_2 \times 3) \times 3 = t_2 \times 3^2 = (t_1 \times 3) \times 3^2 = t_1 \times 3^3 = 2 \times 27 = 54$$

The expressions in red show how each term can be found using the first term t_1 and the common ratio, r, (in this example $r = 3$). The rule could be stated as:

'to calculate the *n*th term in the sequence, write down the first term and multiply it by the common ratio to the power of $(n - 1)$'.

For example, using the rule to find t_5, the fifth term:

$$t_5 = 2 \times 3^{5-1} = 2 \times 3^4 = 162$$

and

$$t_6 = 2 \times 3^{6-1} = 2 \times 3^5 = 486$$

Notice that these are the values given in the sequence.

Rule for finding the *n*th term of a geometric sequence

In a geometric sequence, the rule for the term in the *n*th position is:

$$t_n = ar^{n-1}$$

where $a = t_1$ = first term, r = common ratio and n = position number of the term.

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Example 15 Finding the *n*th term of a geometric sequence

Find t_{15}, the 15th term in the geometric sequence: 3, 6, 12, 24, . . .

Strategy: To use $t_n = ar^{n-1}$, we need to know a, r and n.

Solution

1 The first term is 3.

$a = 3$

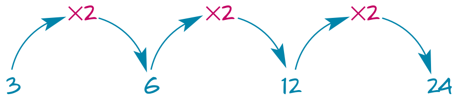
2 The term t_{15} is in the 15th position.

$n = 15$

3 Find the ratio of the first two terms.

$r = \frac{t_2}{t_1} = \frac{6}{3} = 2$

4 Check that this ratio generates the sequence.



5 Substitute the values of a, r and n into $t_n = ar^{n-1}$.

$t_n = ar^{n-1}$

6 Use a calculator to find t_{15}.

$$t_{15} = 3 \times 2^{15-1}$$
$$= 3 \times 2^{14}$$
$$= 49\,152$$

7 Write your answer.

The 15th term, t_{15}, is 49 152.

▶ Percentage change

Let's investigate the sequence generated when each term is 10% less than the previous term. This sequence would arise if someone had \$100 and decided that on each new day they would spend 10% of whatever money they had left.

They start with \$100.

$t_1 = 100$

On the second day they spent 10% of \$100, which is \$10, leaving \$90.
The second term is \$90.

$$t_2 = 100 - 10\% \text{ of } 100$$
$$= 100 - 10$$
$$= 90$$

On the third day they spent 10% of \$90, which is \$9, leaving \$81.

The third term is \$81.

$$t_3 = 90 - 10\% \text{ of } 90$$
$$= 90 - 9$$
$$= 81$$

The sequence is 100, 90, 81, . . .

To find the ratio, use $r = \frac{t_2}{t_1}$

$$r = \frac{t_2}{t_1} = \frac{90}{100} = 0.90$$

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We can make more terms in the sequence by multiplying by 0.90 to make each new term.

$$100 \xrightarrow{\times 0.90} 90 \xrightarrow{\times 0.90} 81 \xrightarrow{\times 0.90} 72.90 \xrightarrow{\times 0.90} 65.61 \ldots$$

So on the fifth day the person had \$65.61 left.

Note: A 10% reduction means that there will only be 90% left. In other words, each new term will be 90% of the current term. So $r = 90\% = 0.90$

In general, an $P\%$ reduction means that each new term will be $(100\% - P\%)$ or $1 - \dfrac{P}{100}$ of the current term.

Rule for finding the common ratio *r*, given the percentage change

For an $P\%$ *reduction* when making each new term, use $r = 1 - \dfrac{P}{100}$

For an $P\%$ *increase* when making each new term, use $r = 1 + \dfrac{P}{100}$

Example 16 Calculating the common ratio from percentage change

State, correct to two decimal places, the first four terms in each of these geometric sequences for the changes given.

a Starts at 200 and each new term is 4% *less* than the previous term.

b Starts at 500 and each new term is 12% *more* than the previous term.

Solution

a **1** The first term is 200.

$a = 200$

2 Use $r = 1 - \dfrac{P}{100}$ with $P = 4$ *or* as each new term is 4% *less* than the previous term, the new term is 96% of the previous term.
So $r = 96\% = 0.96$.

$r = 1 - \dfrac{4}{100} = 0.96$
or
$r = 96\% = 0.96$

3 Starting at 200, multiply by 0.96 to make each new term.

$$200 \xrightarrow{\times 0.96} 192 \xrightarrow{\times 0.96} 184.32 \xrightarrow{\times 0.96} 176.947$$

4 Write your answer correct to two decimal places.

The first four terms are: 200, 192, 184.32, 176.95.

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b	**1** The first term is 500.	$a = 500$
	2 Use $r = 1 + \dfrac{P}{100}$ with $P = 12$ *or* as each new term is 12% *more* than the previous term, the new term is 112% of the previous term. So $r = 112\% = 1.12$.	$r = 1 + \dfrac{12}{100} = 1.12$ or $r = 112\% = 1.12$
	3 Starting at 500, multiply by 1.12 to make each new term	$500 \xrightarrow{\times 1.12} 560 \xrightarrow{\times 1.12} 627.2 \xrightarrow{\times 1.12} 702.464$
	4 Write your answer correct to two decimal places.	The first four terms are: 500, 560, 627.2, 702.46

Percentage change is often central to the study of growth and decay in business and the environment.

▶ Applications

Example 17 Application of a geometric sequence

As a park ranger, Megan has been working on a project to increase the number of rare native orchids in Wilsons Promontory National Park.

At the start of the project, a survey found 200 of the orchids in the park. It is assumed from similar projects that the number of orchids will increase by about 18% each year.

a State the first term a, and the common ratio r, for the geometric sequence for the number of orchids each year.

b Find a rule for the number of orchids at the start of the nth year.

c How many orchids are predicted in 10 years time?

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Solution

a **1** The number of orchids starts at 200.	$a = 200$
2 Use $r = 1 + \frac{P}{100}$ with $P = 18$ *or* an 18% increase means each year there are 118% of the previous year. So $r = 1.18$	$r = 1 + \frac{P}{100}$ Put $P = 18$. $= 1 + \frac{18}{100}$ $= 1.18$
b Substitute $a = 200$ and $r = 1.18$ into $t_n = ar^{n-1}$.	$t_n = ar^{n-1}$ $t_n = 200 \times 1.18^{n-1}$ The number of orchids t_n, at the start of the nth year is given by $t_n = 200 \times 1.18^{n-1}$ $t_n = 200 \times 1.18^{n-1}$
c **1** Substitute $n = 10$ into the rule for t_n.	$t_{10} = 200 \times 1.18^{10-1}$
2 Evaluate.	$= 200 \times 1.18^{9}$ ≈ 887
3 Write your answer.	There will be about 887 orchids in 10 years time.

Exercise 8F

Notation: working out *a* and *r*

1 Give the value of the first term, a, and common ratio, r, in each of the following geometric sequences.

a 12, 24, 48, . . .
b 6, 18, 54, . . .
c 2, 8, 32, . . .
d 56, 28, 14, . . .
e 36, 12, 4, . . .
f 8, 56, 392, . . .
g 1, 10, 100, . . .
h 100, 10, 1, . . .
i 17, 221, 2873, . . .

Finding the *n*th term

Example 15

2 Find the tenth term, t_{10}, in each of these geometric sequences.

a 4, 12, 36, . . .
b 3, 6, 12, . . .
c 2, 6, 18, . . .
d 1, 4, 16, . . .
e 10, 30, 90, . . .
f 512, 256, 128, . . .

3 State the values of the first term, a, and the common ratio, r, for the geometric sequences with the rules given.

a $t_n = 3 \times 4^{n-1}$
b $t_n = 5 \times 2^{n-1}$
c $t_n = 8 \times 7^{n-1}$
d $t_n = 200 \times 1.10^{n-1}$
e $t_n = 600 \times 0.90^{n-1}$
f $t_n = 3 \times 2^{n-1}$

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4 Find the required term in each of these geometric sequences.

a	64, 32, 16, . . .	Find t_7.
b	9, 18, 36, . . .	Find t_8.
c	1, 2, 4, . . .	Find t_{20}.
d	1, 3, 9, . . .	Find t_{13}.
e	729, 243, 81, . . .	Find t_8.
f	100 000,10 000,1000, . . .	Find t_{10}.

5 Find the 10th term, t_{10}, in the geometric sequence that starts at 6 and has a common ratio of 2.

6 The first term in a geometric sequence is 200 and the common ratio is 1.10. Find the 30th term, t_{30}, correct to two decimal places.

7 A sequence starts at 1000 and each term is multiplied by 0.90 to make the next term. Find the 50th term, t_{50}, correct to two decimal places.

8 Give the rule for t_n for each of these geometric sequences.

a 9, 18, 36, 72, . . .
b 54, 18, 6, 2, . . .
c 4, 20, 100, 500, . . .
d 6, 42, 294, 2058, . . .
e 5, 20, 80, 320, . . .
f 8, 24, 72, 216, . . .

Applications of geometric sequences

Example 17

9 The zoom feature on a photocopy machine was set to 300%, making the new image 3 times larger than the original. A small photo, only 1 cm wide, of a person's face was photocopied. The new image was then photocopied to make a further enlargement, and the process was repeated.

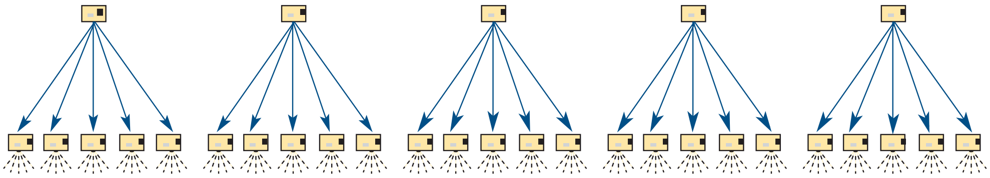
Start with the original width and then list the widths of the next three images produced.

10 A gambler's strategy is to place a first bet of \$50 and to double the bet each time he loses. His first three bets are all losses.

a Write out the sequence of the bets he has made.

b What is the total of all his losses?

11 On the first day of the month, Dorian posted five letters to friends. Each was told to post 5 letters to their friends, with instructions to repeat the process.



Assume that each letter posted is delivered the next day, and that each recipient posts their five letters immediately.

a Write the sequence for the number of letters posted on each of the first 4 days.

b What was the total number of letters posted over all of the 4 days?

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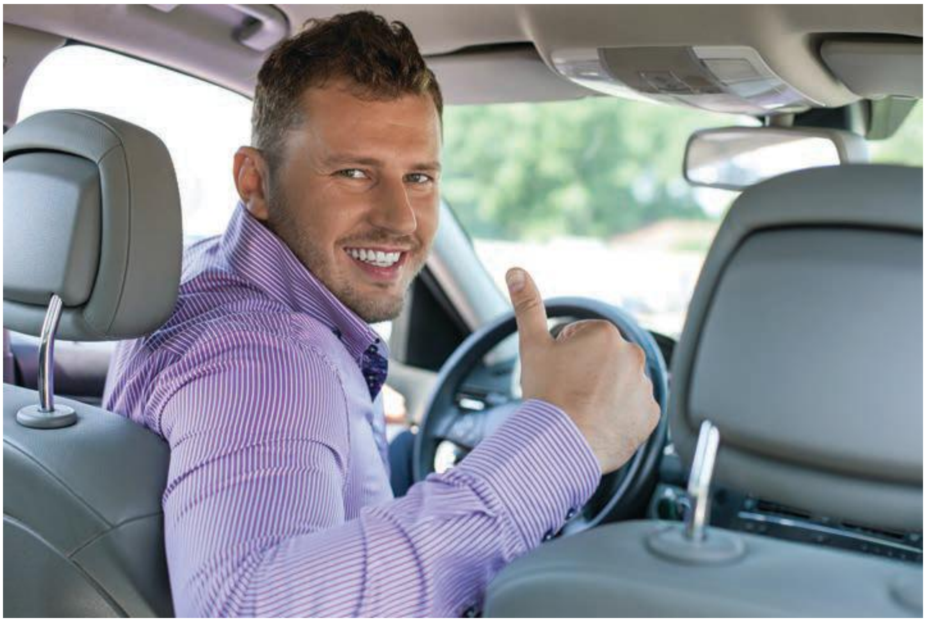
Percentage change and applications

Example 16 **12** State, correct to two decimal places, the first four terms in each of these geometric sequences for the changes given.

a Starts at 100 and decreases by 5%
b Starts at 100 and increases by 20%
c Starts at 5000 and increases by 3%
d Starts at 7000 and decreases by 4%

13 Tom purchased his car for \$30 000 and expects it to depreciate in value by 15% each year. What will be the value of his car after 12 years?



14 Kathy invested \$60 000 so that it grows by 8% each year. What will her investment be worth at the start of the seventh year?

15 The population of 2000 in a small country town is expected to decrease by 10% each year.

a State the first term, a, and the common ratio, r, for the geometric sequence of the population.
b Give the rule for the population at the start of each year.
c What will be the population at the start of the tenth year?
d After how many years will the population be less than 500?

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8G Using a recurrence relation to generate and analyse a geometric sequence

Consider the geometric sequence below:

$2, 6, 18, \ldots$

We can continue to generate the terms of this sequence by recognising that it uses the rule:

'to find the next term multiply the current term by 3 and keep repeating the process'.

A *recurrence relation* is a way of expressing this rule in a precise mathematical language.

The recurrence relation that generates that sequence $2, 6, 18, \ldots$ is $t_1 = 2, \quad t_{n+1} = 3t_n$

The rule tells us that:
'the first term is 2, and each subsequent term is equal to the current term multiplied by 3'.

Understanding this, we proceed to generate the sequence term-by-term as follows:

$$t_1 = 2$$
$$t_2 = 3t_1 = 6$$
$$t_3 = 3t_2 = 18$$
$$t_4 = 3t_3 = 54$$
$t_5 = 3t_4 = 162$ and so on.

The recurrence relation for generating a geometric sequence

The recurrence relation:

$$t_1 = a, \qquad t_{n+1} = rt_n$$

can be used to generate an geometric sequence with first term $t_1 = a$ and common ratio r.

The nth term of a sequence generated by this recurrence relation is given by:

$$t_n = ar^{n-1}$$

While in theory, we can continue using the recursive process to calculate the value of any term in the sequence above, this can become very tedious. Instead, we can use the rule for calculating nth term of an geometric sequence to do the job.

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Example 18 Using a recurrence relation to generate a geometric sequence

a Generate and graph the first five terms of the sequence defined by the recurrence relation:

$$t_1 = 5, \quad t_{n+1} = 2t_n$$

b Use a rule to calculate the value of the 10th term in the sequence.

Solution

a **1** Write down the recurrence relation. $t_1 = 5, \quad t_{n+1} = 2t_n$

2 Write down the first term. $t_1 = 5$

3 Use the rule, which translates into 'to find the next term multiply the previous term by 2' to generate the first five terms in the sequence.

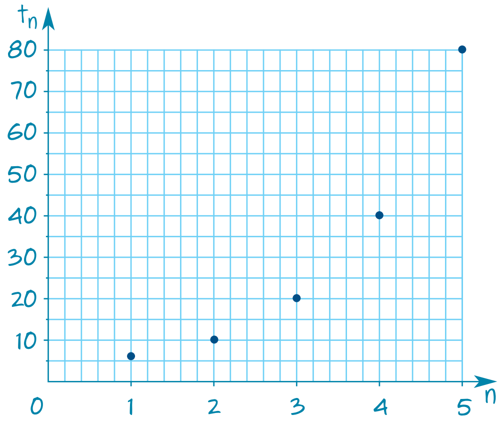
$t_2 = 2t_1 = 2 \times 5 = 10$

$t_3 = 2t_2 = 2 \times 10 = 20$

$t_4 = 2t_3 = 2 \times 20 = 40$

$t_5 = 2t_4 = 2 \times 40 = 80$

4 To graph the terms, plot t_n against n for $1 \leq n \leq 5$.



b The rule for calculating the nth term of an arithmetic sequence is $t_n = ar^{n-1}$. To find the 10th term in this sequence use $a = 5$, $r = 2$ and $n = 10$.

$t_n = ar^{n-1}$

$t_{10} = 5 \times 2^{(10-1)} = 5 \times 2^9 = 2560$

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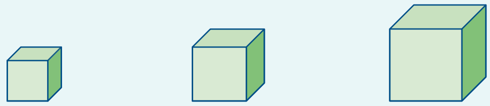
Example 19 Application of geometric sequences using a recurrence relation and the rule for the *n*th term



The volume of cube 1 is 8 cm^3.

The volume of each successive cube is 1.5 times the volume of the previous cube.

Let V_n be the volume (in cm^3) of the *n*th cube in this sequence of cubes.

A recurrence relation that can be used to generate the volumes of this sequence of cubes is:

$$V_1 = 8 \text{ cm}^3, \qquad V_{n+1} = 1.5V_n$$

a Use the recurrence relation to generate the volumes of the first four cubes in this sequence and use these volumes to construct a table showing the cube number (n) and its volume (V_n).

b Use the table to plot the volume of the cube against cube number and comment on the form of the graph.

c Use the rule for the *n*th term for this sequence to predict the volume of the 10th cube in this sequence.

Solution

a Starting with $V_1 = 8$, generate the next three volumes using the recurrence rule:

$$V_{n+1} = 1.5 \times V_n$$

$V_{n+1} = 1.5 \times V_n$
$V_1 = 8$
$V_2 = 1.5 \times V_1 = 1.5 \times 8 = 12$
$V_3 = 1.5 \times V_2 = 1.5 \times 12 = 18$
$V_4 = 1.5 \times V_3 = 1.5 \times 18 = 27$

Construct a table using these volumes.

Cube number	1	2	3	4
Volume	8	12	18	27

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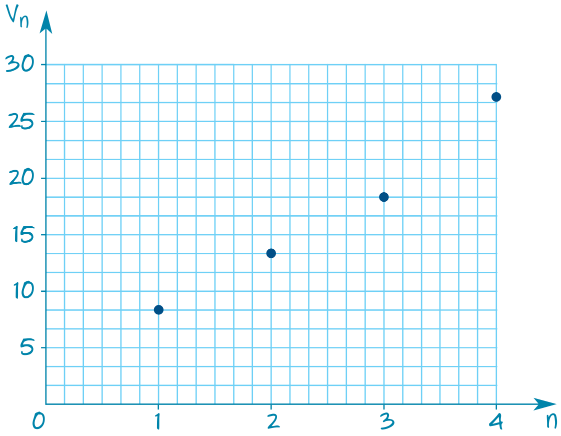
b Use the table to plot a graph with n on the horizontal axis and V_n on the vertical axis.



The plot is non-linear and curves upwards.

c Using the symbol V_n rather than t_n, the rule for the nth term is $V_n = ar^{n-1}$.
Substitute $a = 8, n = 10, r = 1.5$ to obtain V_{10}.

$V_{10} = 8 \times 1.5^{(10-1)} = 308$ cm^3 to the nearest cm^3.
The volume of cube 10 is 308 cm^3.

Exercise 8G

Using a recurrence relation to generate and graph the terms of a geometric sequence

Example 18

1 **a** Generate and graph the first five terms of the geometric sequence defined by the recurrence relation: $t_1 = 1000, \ t_{n+1} = 1.1t_n$.

b Calculate the value of the 13th term in the sequence correct to two decimal places.

2 **a** Generate and graph the first five terms of the geometric sequence defined by the recurrence relation: $t_1 = 256, \ t_{n+1} = 0.5t_n$.

b Calculate the value of the 10th term in the sequence.

3 **a** Generate and graph the first five terms of the geometric sequence defined by the recurrence relation: $t_1 = 10\,000, \ t_{n+1} = 1.25t_n$. Give values to the nearest whole number.

b Calculate the value of the 25th term in the sequence.

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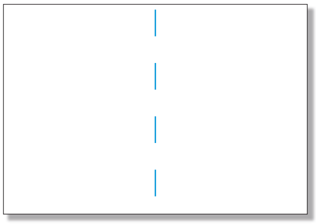
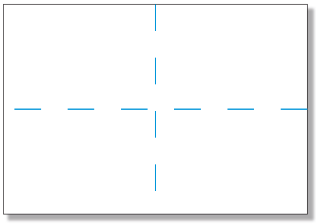
Application

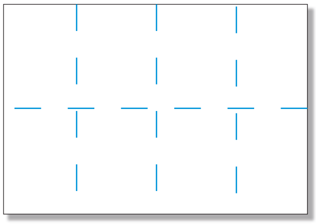
Example 19 **4** A sheet of paper is in the shape of a rectangle. When the sheet is folded once and opened, 2 rectangles are formed either side of the crease. When a sheet is folded twice and opened, 4 rectangles are created, and so on.



1 Fold



2 Folds



3 Folds

Let R_n be the number of rectangles created by n folds.

The sequence for the number of rectangles created is generated by the recurrence relation:

$$R_1 = 2, \qquad R_{n+1} = 2R_n$$

- **a** Use the recurrence relation for R_n to generate the first four terms of the sequence.
- **b** Count and record the number of rectangles created by the first, second and third folds. Then fold a sheet of A4 paper four times to check that the recurrence relation correctly predicted the number of rectangles created.
- **c** Use the rule for the nth term to calculate the number of rectangles after 10 folds.
- **d** Using your calculator, generate the terms of the sequence to check your answer to **c**.

8H Using recurrence relations to model growth and decay

▶ Linear growth and decay

Linear growth and **decay** is commonly found in the world around us. Linear growth or decay in a sequence occurs when a quantity increases or decreases by the same amount at regular intervals. Everyday examples include the payment of simple interest or the depreciation of the value of a new car by a constant amount each year. This sort of depreciation is commonly called flat-rate or straight line depreciation.

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Example 20 Using a recurrence relation to model linear growth: simple interest

The following recurrence relation can be used to model a simple interest investment of \$2000 paying interest at the rate of 7.5% per annum.

$$V_0 = 2000, \qquad V_{n+1} = V_n + 150$$

In the recurrence relation, V_n is the value of the investment after n years.

a Use the recurrence relation to find the value of the investment after 1, 2 and 3 years.

b When will the investment reach \$2750 in value?

Solution

a 1 Write down the recurrence relation. The recurrence relation, tells you that: ‘to find the next value, add 150 to the current value’.

$V_0 = 2000, \qquad V_{n+1} = V_n + 150$

2 With $V_0 = 2000$ as the starting point, use the recurrence relation to generate the terms V_1, V_2, V_3.

$V_0 = 2000$

$V_1 = V_0 + 150 = 2000 + 150 = \$2150$

$V_2 = V_1 + 150 = 2150 + 150 = \$2300$

$V_3 = V_2 + 150 = 2300 + 150 = \$2450$

This can also be done on your CAS calculator as shown opposite.

```
2000                  2000
2000 + 150            2150
2150 + 150            2300
2300 + 150            2450
```

b This question is best aswered using your CAS calculator to generate the values of successive terms and counting the number of steps (years) until the value of the invesment is \$2750.

```
2000                  2000
2000 + 150            2150
2150 + 150            2300
2300 + 150            2450
2450 + 150            2600
2600 + 150            2750
```

Write your conclusion.

The investment will have a value of \$2750 after 5 years.

Note: Count the number of times 150 has been added, *not* the number of terms on the calculator screen.

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Example 21 Using a recurrence relation to model linear decay: flat rate depreciation

The following recurrence relation can be used to model the flat rate depreciation of a car purchased for \$18 500 depreciating at a flat rate of 10% per year.

$$V_0 = 18\,500, \quad V_{n+1} = V_n - 1850$$

In the recurrence relation, V_n is the value of the car after n years.

a Use the recurrence relation to find the value of the car after 1, 2 and 3 years.

b How long will it take for the car's value to depreciate to zero?

Solution

a 1 Write down the recurrence relation. The recurrence relation, tells you that: 'to find the next value subtract 1850 from the current value'.

$V_0 = 18\,500, \quad V_{n+1} = V_n - 1850$

2 With $V_0 = 18\,500$ as the starting point, use the recurrence relation to generate the terms V_1, V_2, V_3.

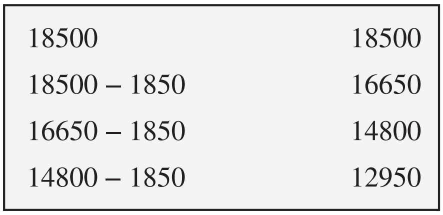
$V_0 = 18\,500$

$V_1 = V_0 - 1850 = 18\,500 - 1850 = \$16\,650$

$V_2 = V_1 - 1850 = 16\,650 - 1850 = \$14\,800$

$V_3 = V_2 - 1850 = 14\,800 - 1850 = \$12\,950$

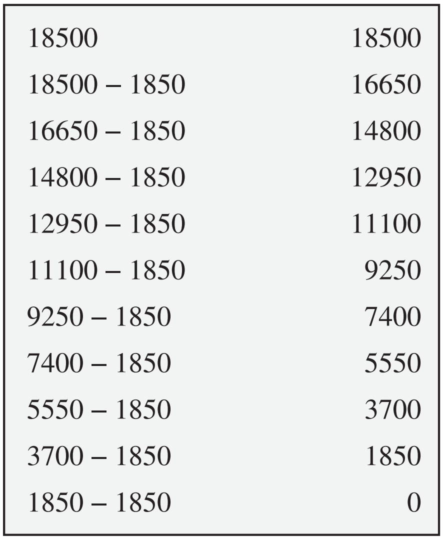
This can also be done on your CAS calculator.



```
18500                 18500
18500 – 1850          16650
16650 – 1850          14800
14800 – 1850          12950
```

b This question is best answered using your calculator and counting the number of steps (years) until the depreciated value of the car is \$0.

Write your conclusion.



```
18500                 18500
18500 – 1850          16650
16650 – 1850          14800
14800 – 1850          12950
12950 – 1850          11100
11100 – 1850           9250
9250 – 1850            7400
7400 – 1850            5550
5550 – 1850            3700
3700 – 1850            1850
1850 – 1850               0
```

The car will have no value after 10 years

Note: Count the number of times 1850 has been subtracted, *not* the number of terms on the calculator screen.

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▶ A rule for the *n*th term in a sequence modelling linear growth or decay

While we can generate as many terms as we like in a sequence using a recurrence relation, for linear growth and decay, it is possible to derive a rule for calculating any term in the sequence directly.

For example, if we invest \$1000 in a simple interest investment paying 5% interest per annum, the amount of your investment will increase by the same amount, \$50, each year.

Letting V_n be the value of the investment after n years, we can write:

$$\begin{aligned} V_0 &= 1000 && \text{(the amount we started with)} \\ V_1 &= 1000 + 1 \times 50 && \text{(after 1 year's interest has been added)} \\ V_2 &= 1000 + 2 \times 50 && \text{(after 2 year's interest has been added)} \\ V_3 &= 1000 + 3 \times 50 && \text{(after 3 year's interest has been added)} \end{aligned}$$

and so on.

More generally, after n year's interest has been added,

$$V_n = 1000 + n \times 50$$

With this rule, we can now predict the value of the nth term in the sequence without having to generate all of the other terms first.

For example, using this rule, the value of the investment after 20 years would be:

$$V_{20} = 1000 + 20 \times 50 = \$2000.$$

This rule can be readily generalised to apply to any situation involving linear growth or decay as follows:

A rule for the *n*th term in a sequence used to model linear growth or decay

Let V_n be the value of the nth term of the sequence used to model linear growth or decay.

For *growth*, the value of the nth term in this sequence generated by the recurrence relation:

$$V_0 = \text{starting or initial value}, \quad V_{n+1} = V_n + D$$

is given by:

$$V_n = V_0 + n \times D$$

For *decay*, the value of the nth term in this sequence generated by the recurrence relation:

$$V_0 = \text{starting or initial value}, \quad V_{n+1} = V_n - D$$

is given by:

$$V_n = V_0 - n \times D$$

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Example 22 Using a rule for determining the *n*th term for linear growth or decay

a The following recurrence relation can be used to model a simple interest investment of $4000 paying interest at the rate of 6.5% per year.

$$V_0 = 4000, \quad V_{n+1} = V_n + 260$$

i How much interest is added to the investment each year?

ii Use a rule to find the value of the investment after 15 years.

iii Use a rule to find when the value of the investment first exceeds $10 000.

Solution

a i This value can be read directly from the recurrence relation or calculated by finding 6.5% of $4000 = 0.065 × 4000 = $260

$260

ii Because it is linear growth, use the rule: $V_n = V_0 + n \times D$
Here $V_0 = 4000$, $n = 15$ and $D = 260$.

$$V_n = 4000 + 15 \times 260$$
$$= \$7900$$

iii Substitute $V_n = 10000$, $V_0 = 4000$ and $D = 260$ into the rule: $V_n = V_0 + n \times D$, and solve for n.

$$10\,000 = 4000 + n \times 260$$
$$6000 = n \times 260$$
$$n = \frac{6000}{260}$$
$$= 23.07\ldots \text{ years}$$

Write your conclusion.

Note: Because the interest is only paid into the account after a whole number of years, any decimal answer will need to be *rounded up* to the next whole number.

The value of the investment will first exceed $10 000 after 24 years.

▶ Geometric growth and decay

Geometric growth and decay is also commonly found in the world around us. Geometric growth or decay in a sequence occurs when the quantity being modelled increases or decreases by the same percentage at regular intervals. Everyday examples include the payment of compound interest or the depreciation of the value of a new car by a constant percentage of its depreciated value each year. This sort of depreciation is commonly called reducing balance depreciation.

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Example 23 Using a recurrence relation to model geometric growth: a compound interest investment

The following recurrence relation can be used to model a compound interest investment of \$1000 paying interest at the rate of 8% per annum.

$$V_0 = 1000, \quad V_{n+1} = 1.08V_n$$

In the recurrence relation, V_n is the value of the investment after n years.

a Use the recurrence relation to find the value of the investment after 1, 2 and 3 years.

b When will the investment first exceed \$1500 in value?

Solution

a **1** Write down the recurrence relation. The recurrence rule tells you that: 'to find the next value multiply the current value by 1.08'.

$V_0 = 1000, \quad V_{n+1} = 1.08V_n$

2 With $V_0 = 1000$ as the starting point, use the recurrence relation to generate the terms V_1, V_2, V_3.

$V_0 = 1000$

$V_1 = 1.08V_0 = 1.08 \times 1000 = \$1080$

$V_2 = 1.08V_1 = 1.08 \times 1080 = \$1066.40$

$V_3 = 1.08V_2 = 1.08 \times 1066.40$

$= \$1259.71\ldots$

This can also be done with your CAS calculator.

```
1000                     1000
1000 · 1.08              1080
1080 · 1.08            1166.4
1166.4 · 1.08         1259.71
```

b This is best done using your CAS calculator and counting the number of steps (years) until the value of the investment first exceeds \$1500.

```
1000                          1000
1000 · 1.08                   1080
1080 · 1.08                 1166.4
1166.4 · 1.08              1259.71
1259.712 · 1.08            1360.49
1360.48896 · 1.08          1469.33
1469.3280768 · 1.08        1586.87
```

Write your conclusion.

The investment will first exceed \$1500 after 6 years.

Note: Count the number of times the value of the investment is increased by 8%.

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Example 24 Using a recurrence relation to model geometric decay: reducing balance depreciation

A car is purchased for \$18 500. The following recurrence relation can be used to model the car's value as it depreciates by 10% of its value each year.

$$V_0 = 18\ 500, \quad V_{n+1} = 0.9V_n$$

In the recurrence relation, V_n is the value of the car after n years.

a Use the recurrence relation to find the value of the car after 1, 2 and 3 years.

b When will the value of the car first be worth less than \$10 000?

Solution

a **1** Write down the recurrence relation.

$V_0 = 18\ 500, \quad V_{n+1} = 0.90V_n$

2 The recurrence relation, tells you that: 'to find the next value multiply the current value by 0.90'. With $V_0 = 18\ 500$ as the starting point, use the recurrence relation to generate the terms V_1, V_2, V_3.

$V_0 = 18\ 500$

$V_1 = 0.9V_0 = 0.90 \times 18\ 500 = \$16\ 650$

$V_2 = 0.9V_1 = 0.90 \times 16\ 650 = \$14\ 985$

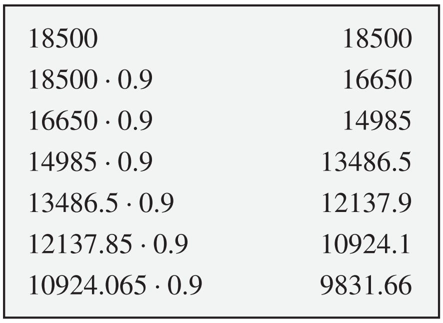
$V_3 = 0.9V_2 = 0.90 \times 14\ 985$

$= \$13\ 486.50$

This can also be done with your calculator.

```
18500                 18500
18500 · 0.9           16650
16650 · 0.9           14985
14985 · 0.9         13486.5
```

b This is best done using you calculator and counting the number of times the car's value has been decreased by 10% until the value of the car is first less than \$10 000.



```
18500                 18500
18500 · 0.9           16650
16650 · 0.9           14985
14985 · 0.9         13486.5
13486.5 · 0.9       12137.9
12137.85 · 0.9      10924.1
10924.065 · 0.9     9831.66
```

Write your conclusion.

The value of the car is first less than \$10 000 after 6 years.

▶ A rule for *n*th term in a sequence representing geometric growth or decay

While we can generate as many terms as we like in a sequence using a recurrence relation, it is possible to derive a rule for calculating any term in the sequence directly. We will derive this rule for a specific situation, but the rule is easily generalised to include any situation involving geometric growth.

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For example, if we invest \$1000 in a compound interest investment paying 8% interest per annum, the amount of your investment will increase by the same percentage 8%, each year.

Letting V_n be the value of the investment after n years, we can write:

$$V_0 = 1000 \qquad \text{(the amount we started with)}$$

$$V_1 = 1.08 \times V_o \qquad \text{(after 1 year)}$$

$$V_2 = 1.08 \times V_1 = 1.08 \times 1.08 \times V_o = 1.08^2 V_o \qquad \text{(after 2 years)}$$

$$V_3 = 1.08 \times V_2 = 1.08 \times (1.08 \times 1.08 \times V_o) = 1.08^3 V_o \qquad \text{(after 3 years)}$$

or more generally, after n years

$$V_n = 1.08^n \times V_o$$

With this rule, we can now predict the value of the nth term in the sequence without having to generate all of the other terms first.

For example, if the \$1000 was invested for 20 years then, using this rule, the value of the investment would be:

$$V_{20} = 1.08^{20} \times 1000 = \$4661 \qquad \text{to the nearest dollar.}$$

This rule can be readily generalised to apply to any situation involving geometric growth and decay as follows.

A rule for the *n*th term in a sequence used to model geometric growth or decay

Let V_n be the value of the nth term of sequence used to model geometric growth or decay.

The value of the nth term in this sequence is given by:

$$V_n = R^n V_0$$

where V_0 = starting or initial value.

For *growth*, R is greater than 1. For *decay*, R is between 0 and 1.

Example 25 Using a rule for determining the *n*th term for geometric growth or decay

a The following recurrence relation can be used to model a compound interest investment of \$1000 paying interest at the rate of 10% per year.

$$V_0 = 1000, \quad V_{n+1} = 1.1V_n$$

Use a rule to find the value of the investment after 15 years correct to the nearest dollar.

b The following recurrence relation can be used to model the reducing balance depreciation of a car purchased for \$18 500 where the value of the car depreciates in value at the rate of 10% per year.

$$V_0 = 18\,500, \quad V_{n+1} = 0.9V_n$$

Use a rule to find the value of the investment after 12 years correct to the nearest dollar.

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Solution

a Use the rule $V_n = R^n V_0$ where $V_0 = 1000$, $n = 15$ and $R = 1.1$

$V_n = 1.1^{15} \times 1000$
$= \$4177.24\ldots$
$= \$4177$ (nearest dollar)

b Use the rule $V_n = R^n V_0$ where $V_0 = 18\,500$, $n = 12$ and $R = 0.90$

$V_n = 0.9^{12} \times 18\,500$
$= \$5224.90\ldots$
$= \$5225$ (nearest dollar)

Exercise 8H

Linear growth: Using a recurrence relation to analyse a simple interest investment

Example 20

1 The following recurrence relation can be used to model a simple interest investment of \$10 000 paying interest at the rate of 4.5% per year.

$$V_0 = 10\,000, \quad V_{n+1} = V_n + 450$$

In the recurrence relation, V_n is the value of the investment after n years.

a Use the recurrence relation to find the value of the investment after 1, 2 and 3 years.

b When will the investment reach \$14 500 in value?

2 The following recurrence relation can be used to model a simple interest investment.

$$V_0 = 8000, \quad V_{n+1} = V_n + 400$$

In the recurrence relation, V_n is the value in dollars of the investment after n years.

a Use the recurrence relation to find the value of the investment after 1, 2 and 3 years.

b When will the investment reach \$12 000 in value?

c How much was invested at the start?

d What was the interest rate?

Linear decay: Using a recurrence relation to analyse flat rate depreciation

Example 21

3 A tractor cost \$90 000 when new. Its value depreciates at a flat rate of 10% or \$9000 per year.

Let V_n be the value (in dollars) of the tractor after n years.

A recurrence relation that models the depreciating value of this tractor over time is:

$$V_0 = 90\,000, \quad V_{n+1} = V_n - 9000$$

a Use the recurrence relation to find the value of the tractor after three years.

b The tractor will be sold after 5 years. How much will it be worth then?

c If the tractor continues to depreciate at the same rate for the rest of its life, how many years will it take to have zero value?

d A different model of tractor costs \$95 000 and depreciates at a flat rate of 12% of its original value per year. Write down a recurrence relation to model this situation.

4 Let V_n be the value (in dollars) of a computer after n years.

A recurrence relation that models the depreciating value of this computer over time is:

$$V_0 = 2400, \quad V_{n+1} = V_n - 300$$

- **a** What was the value of the computer when it was new?
- **b** By how much (in dollars) was the computer depreciated each year?
- **c** What was the percentage flat rate of depreciation?
- **d** After how many years will the value of the computer be \$600?
- **e** When will the computer devaluate to half of its new price?

A rule for the *n*th term in a sequence modelling linear growth or decay

Example 22

5 **a** The following recurrence relation can be used to model a simple interest investment of \$32 000 paying interest at the rate of 2.5% per year.

$$V_0 = 32\,000, \quad V_{n+1} = V_n + 800$$

- **i** How much interest is added to the investment each year?
- **ii** Use a rule to find the value of the investment after 15 years.
- **iii** Use a rule to find when the value of the investment reaches \$40 000.

b The following recurrence relation can be used to model the flat rate depreciation of a motorbike purchased for \$4000 depreciating at a flat rate of 12.5% per year.

$$V_0 = 4000, \quad V_{n+1} = V_n - 500$$

- **i** How much does the motorbike depreciate in value each year?
- **ii** Use a rule to find the value of the motor bike after 4 years.
- **iii** Use a rule to find when the depreciated value of the bike is zero.

Geometric growth: Using a recurrence relation to analyse a compound interest investment

Example 23

6 The following recurrence relation can be used to model a compound interest investment of \$10 000 paying interest at the rate of 4.5% per year.

$$V_0 = 10\,000, \quad V_{n+1} = 1.045V_n$$

In the recurrence relation, V_n is the value of the investment after n years.

- **a** Use the recurrence relation to find the value of the investment after 1, 2 and 3 years.
- **b** When will the value of the investment first exceed \$12 000 in value?

7 The following recurrence relation can be used to model a compound interest investment.

$$V_0 = 8000, \quad V_{n+1} = 1.075V_n$$

In the recurrence relation, V_n is the value in dollars of the investment after n years.

- **a** How much was invested at the start?
- **b** Use the recurrence relation to determine when the value of the investment first exceeds \$10 000.
- **c** What was the interest rate?

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Geometric decay: Using a recurrence relation to analyse reducing balance depreciation

Example 24

8 A tractor cost \$90 000 when new. Its value depreciates at a reducing balance rate of 15% per year.

Let V_n be the value (in dollars) of the tractor after n years.

A recurrence relation that model the depreciating value of this car over time is:

$$V_0 = 90\,000, \quad V_{n+1} = 0.85V_n$$

a Use the recurrence relation to find by hand the value of the tractor after two years.

b The tractor will be sold after 10 years. How much will it be worth then?

9 Let V_n be the value (in dollars) of a refrigerator after n years.

A recurrence relation that models the depreciating value of this refrigerator over time is:

$$V_0 = 1200, \quad V_{n+1} = 0.56V_n$$

a What was the value of the refrigerator when it was new?

b After how many years will the value of the refrigerator first be less than \$200?

c When will the refrigerator devaluate to less than half of its new price?

d What was the percentage rate of depreciation?

A rule for the *n*th term in a sequence modelling geometric growth or decay

Example 25

10 **a** The following recurrence relation can be used to model a compound interest investment of \$30 000 paying interest compounding at the rate of 3.5% per year.

$$V_0 = 30\,000, \quad V_{n+1} = 1.035V_n$$

Use a rule to find the value of the investment after 15 years.

b The following recurrence relation can be used to model the reducing balance depreciation of a motor bike purchased for \$4000 depreciating at the rate of 17.5% per year.

$$V_0 = 4000, \quad V_{n+1} = 0.825V_n$$

Use a rule to find the value of the motor bike after 4 years.

8I The Fibonacci sequence

The sequence

$$1, \quad 1, \quad 2, \quad 3, \quad 5, \quad 8, \quad 13, \ldots$$

is known as the *Fibonacci sequence* and its numbers often occur in nature. It is named after the Italian merchant and mathematician Fibonacci (Leonardo di Pisa, 1170–1250). This sequence has the property that, *after* the first two terms, each successive term is the sum of the preceding two terms:

$$t_1 = 1$$

$$t_2 = 1$$

$$t_3 = 2 = t_1 + t_2$$

$$t_4 = 3 = t_2 + t_3$$

$$t_5 = 5 = t_3 + t_4$$

$$t_6 = 8 = t_4 + t_5 \text{ and so on.}$$

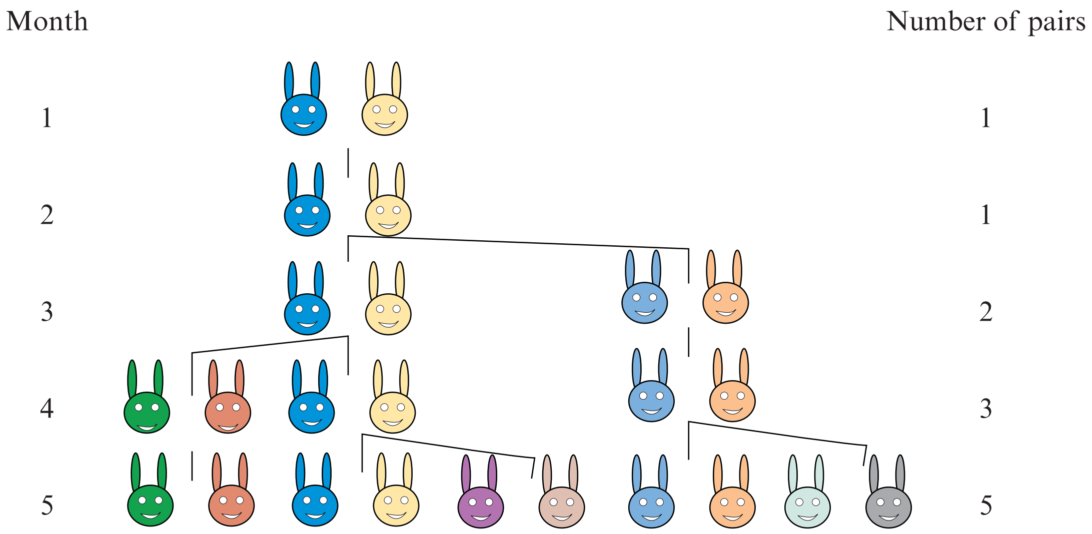
Fibonacci proposed this sequence as a way of modelling the growth in the number of rabbits produced from a single pair of breeding rabbits.

The breeding rabbit problem

The situation he set out to model was as follows. We start out with a newly born pair of rabbits (one male, one female).

- At the start of the 2nd month, they mate.
- At the start of the 3rd month, the female produces a new pair of rabbits (one male, one female).
- At the start of the 4th month, the older female produces a new pair of rabbits (one male, one female) and the new pair mate, and so on.

Assuming the same breeding pattern continues and no rabbit dies, the problem is to predict the number of pairs of rabbits in successive generations. The situation can be represented pictorially as shown below. As we can see, the number of pairs of rabbits follows the Fibonacci sequence: 1, 1, 2, 3, 5 . . .



Fibonacci numbers occur surprisingly often in nature. A pinecone has 8 spirals in one direction and 13 spirals twisting in the opposite direction.

Fibonacci numbers are also found when counting the segments in the spirals of pineapples and the outer petals of artichokes.

The number of petals on many varieties of flowers are often Fibonacci numbers.

The Fibonacci sequence has also turned out to have a number of interesting mathematical properties, some of which we will investigate now.

Example 26 Generating terms in a Fibonacci sequence from adjacent terms

For the Fibonacci sequence, $t_{10} = 55$, and $t_{11} = 89$, find the values of:

a t_{12} **b** $t_8 + t_9$ **c** t_{13}

Solution

Strategy: The key fact in answering all of these questions is that in a Fibonacci sequence, ‘after the first two terms, each new term is the sum of the preceding two terms’.

a Find t_{12}.

$$t_{12} = t_{10} + t_{11}$$
$$= 55 + 89 = 144$$

b Find $t_8 + t_9$.

$$t_8 + t_9 = t_{10} = 55$$

c Find t_{13}.
Use $t_{12} = 144$, from part **a**.

$$t_{13} = t_{11} + t_{12}$$
$$= 89 + 144 = 233$$

▶ The recurrence relation for a Fibonacci sequence

Like most regular sequences, the Fibonacci sequence can be generated using a recurrence relation. Listing the terms of the sequence we have:

$$t_1 = 1$$
$$t_2 = 1$$
$$t_3 = t_1 + t_2 \ (= 1 + 1 = 2)$$
$$t_4 = t_2 + t_3 \ (= 1 + 2 = 3)$$
$$t_5 = t_3 + t_4 \ (= 2 + 3 = 5)$$
$$\vdots$$
$$t_n = t_{n-2} + t_{n-1}$$
$$\vdots$$

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Having an expression for the *n*th term in the sequence, we can now write down the recurrence relation.

Recurrence relation for the Fibonacci sequence

$t_1 = 1$ and $t_2 = 1$ $\quad t_n = t_{n-2} + t_{n-1}$

Note: This recurrence relation differs from those you have met earlier. All the recurrence relations you have dealt with so far are *first-order* recurrence relations. That is, they link terms that are only *one step* apart in the sequence. The recurrence relation that generates the Fibonacci sequence is an example of a *second-order* recurrence relation. It links terms that are *two steps* apart in the sequence.

While the recurrence relation defining the Fibonacci sequence gives a rule for determining values of terms, simply remembering that 'after the first two terms, each new term is the sum of the preceding two terms' will help you answer many questions.

There is a family of Fibonacci type sequences in which each new term, after the first two terms, is found by adding the two previous terms. However, the sequences start with different pair of numbers for the first two terms. For example, the *Lucas sequence* starts with $t_1 = 1$ and $t_2 = 3$.

Example 27 Generating terms in a Lucas sequence using a recurrence relation

Generate the first five terms in the Lucas sequence using the recurrence relation:

$t_1 = 1$ and $t_2 = 3$ $\quad t_n = t_{n-2} + t_{n-1}$

Solution

Use $n = 3$.

$$t_3 = t_1 + t_2$$
$$= 1 + 3 = 4$$

Use $n = 4$.

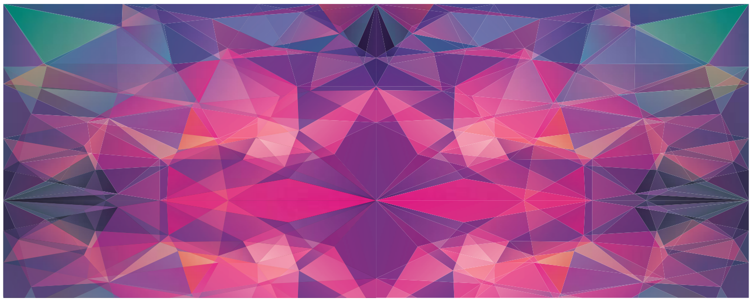
$$t_4 = t_2 + t_3$$
$$= 3 + 4 = 7$$

Use $n = 5$.

$$t_4 = t_3 + t_4$$
$$= 4 + 7 = 11$$

Write your answer.

The first five terms of a Lucas sequence are:
1, 3, 4, 7, 11



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Fibonacci magic

Display the first twenty or more values of the Fibonacci sequence.

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584, 4181, . . .

1 Ask your friend to point to the number in the sequence.
2 Suppose they pointed to 233.
3 You say that all the numbers in the Fibonacci sequence from 1 to 233 add to 609.
4 Ask them to check on a calculator that you are correct.
5 The trick is to look at the term in the sequence two steps further along from 233.
6 It is 610. Just subtract 1 from 610 to get 609.
7 Let’s try a simple example that is easy to check.
8 Suppose your friend pointed to 8.
9 The number two steps along from 8 is 21.
10 So the sum of the sequence up to 8 should be 20.
11 Check.

Exercise 8I

1 Is the difference equation defining the Fibonacci sequence a first-order or second-order difference equation?

Example 26

2 For the Fibonacci sequence, $t_7 = 13$ and $t_8 = 21$. Find the values of:

a t_9
b $t_5 + t_6$
c $t_6 + t_7$
d t_{10}

3 A sequence has the rule: $t_n = t_{n-2} + t_{n-1}$. If $t_9 = 76$ and $t_{10} = 123$, find:

a t_{11}
b $t_7 + t_8$
c $t_8 + t_9$
d t_{12}

4 The Fibonacci sequence can be defined by the recurrence relation:

$$t_1 = 1 \text{ and } t_2 = 1 \quad t_n = t_{n-2} + t_{n-1}$$

a Use the recurrence relation to generate the first 10 terms by hand.
b Graph the first 10 terms.

Example 27

5 The Lucas sequence can be defined by the recurrence relation:

$$t_1 = 1 \text{ and } t_2 = 3 \quad t_n = t_{n-2} + t_{n-1}$$

Use the recurrence relation to generate the first 10 terms by hand.

6 A sequence has the recurrence relation:

$$t_1 = 3 \text{ and } t_2 = 5 \quad t_n = t_{n-2} + t_{n-1}$$

a Use the recurrence relation to find the first 5 terms by hand.
b Would the sequence be different after the first two terms if $t_1 = 5$ and $t_2 = 3$?

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Key ideas and chapter summary

Sequence

A **sequence** is a list of numbers in a particular order.

Arithmetic sequence

In an **arithmetic sequence**, each new term is made by adding a fixed number, called the common difference, d, to the previous term. Example: 3, 5, 7, 9, . . . is made by adding 2 to each term. The common difference, d, is found by taking any term and subtracting its previous term, e.g. $t_2 - t_1$.

In our example above, $d = 5 - 3 = 2$.

Recurrence relation for an arithmetic sequence

A **recurrence relation for an arithmetic sequence** has the form

$$t_1 = a, \quad t_{n+1} = t_n + d$$

where d = common difference and a = first term.

In our example: $t_1 = 3, \quad t_{n+1} = t_n + 2$

Rule for finding t_n, the nth term in an arithmetic sequence:

$$t_n = a + (n - 1)d$$

To find t_{10} in our example: put $n = 10$, $a = 3$, $d = 2$

$$\begin{aligned} t_{10} &= 3 + (10 - 1) \times 2 \\ &= 3 + (9) \times 2 = 21 \end{aligned}$$

The graph of an arithmetic sequence:

- values lie along a straight line
- increasing values when $d > 0$ (positive slope)
- decreasing values when $d < 0$ (negative slope)

Linear growth and decay

An arithmetic sequence can be used to model **linear growth** ($d > 0$) or **linear decay** ($d < 0$).

Geometric sequence

In a **geometric sequence**, each term is made by multiplying the previous term by a fixed number, called the common ratio, r.

Example: 5, 20, 80, 320, . . . is made by multiplying each term by 4.

The common ratio, r, is found by dividing any term by its previous term, e.g. $\frac{t_2}{t_1}$.

In our example: $r = \frac{20}{5} = 4$

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Recurrence relation for a geometric sequence

Recurrence relation for a geometric sequence:

$$t_1 = a, \quad t_{n+1} = r \times t_n$$

where r = common ratio and a = first term.

In our example: $t_1 = 5, \quad t_{n+1} = 4 \times t_n$

Rule for finding t_n, the nth term, in a geometric sequence:

$$t_n = ar^{n-1}$$

where a = first term and r = common ratio.

To find t_7 in our example: put $n = 7$, $a = 5$, $r = 4$

$$\begin{aligned} t_7 &= 5 \times (4)^{7-1} \\ &= 5 \times (4)^6 = 20\,480 \end{aligned}$$

The graph of a geometric sequence:

- values increase when $r > 1$
- values decrease towards zero when $0 < r < 1$

Percentage change

Recurring **percentage change** generates geometric growth.

Recurrence relation

A **recurrence relation** gives the information needed to make each new term in a sequence using the previous term(s). A more general form is:

$$t_1 = a, \qquad t_{n+1} = r \times t_n + d$$

starting value rule

Example: $t_{n+1} = 5t_n + 2$, $t_1 = 4$ tells us to multiply each term by 5 then add 2 to make each new term. Start at 4. 4, 22, 112, 562, . . .

Fibonacci sequence

In a **Fibonacci sequence** each new term is made by adding the two previous terms. The first two terms, 1, 1, are given. Other sequences, such as the Lucas sequence, have a different pair of starting values.

Recurrence relation for a Fibonacci sequence

The **recurrence relation for an Fibonacci sequence** is,

$$t_1 = 1, t_2 = 1, \quad t_n = t_{n-2} + t_{n-1}$$

Skills check

Having completed the current chapter you should be able to:

- decide whether a sequence is arithmetic, geometric or neither
- give the first term, *a*, and find the common difference, *d*, of an arithmetic sequence
- give the first term, *a*, and find the common ratio, *r*, of a geometric sequence
- find the *n*th term of an arithmetic or geometric sequence when given a few terms in the sequence
- find the required terms of a Fibonacci sequence
- generate a sequence using its recurrence relation
- generate the terms of a sequence using a graphics calculator
- calculate the common ratio, *r*, required for a given percentage change
- solve application problems involving sequences.

Multiple-choice questions

1 In the sequence: 1, 4, 7, 10, 13, . . . the 4th term is:

A 4 **B** 10 **C** 3 **D** 1 **E** 2

2 Which of the following is an arithmetic sequence?

A 2, 4, 8, . . . **B** 2, 6, 18, . . . **C** 2, 4, 6, . . . **D** 2, 3, 5, . . . **E** 2, 4, 7, . . .

3 In the sequence: 27, 19, 11, 3, . . . the value of the common difference, *d*, is:

A 8 **B** 3 **C** −8 **D** −24 **E** −5

4 In the sequence: 63, 56, 49, 42, . . . the 15th term is:

A −35 **B** 161 **C** 15 **D** 168 **E** −42

5 Using the recurrence relation $t_1 = 5, t_{n+1} = t_n + 6$, the 7th term would be:

A 11 **B** 41 **C** 18 **D** 47 **E** 13

6 The sum of the first 5 terms of the sequence: 7, 11, 15, . . . is:

A 19 **B** 23 **C** 75 **D** 240 **E** 1792

7 The common ratio, *r*, of the sequence: 17, 221, 2873, . . . is:

A 13 **B** 17 **C** 21 **D** 204 **E** 221

8 Which of the following sequences is geometric?

A 2, 6, 10, 14, . . . **B** 2, 6, 12, 24, . . . **C** 54, 18, 6, 2, . . .
D 54, 27, 9, 3, . . . **E** 1, 3, 9, 18, . . .

9 The recurrence relation for the sequence: 3, 6, 12, 24, . . . is:

A $t_1 = 3, t_{n+1} = t_n + 3$ **B** $t_1 = 3, t_{n+1} = 3t_n$

C $t_1 = 3, t_{n+1} = 2t_n$ **D** $t_1 = 2, t_{n+1} = 3t_n$

E $t_1 = 2, t_{n+1} = 3t_n + 6$

10 The 20th term, t_{20}, of the sequence: 3, 6, 12, 24, . . . is:

A 177 **B** 168 **C** 3 145 728

D 786 432 **E** 1 572 864

11 A 7% increase is made by using a common ratio of:

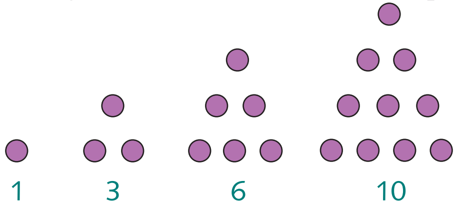
A 7 **B** 0.07 **C** 1.7 **D** 1.07 **E** 107

12 Which of the following sequences indicates exponential growth?

A 3, 6, 9, 12, . . . **B** 3, 6, 12, 24, . . . **C** 24, 12, 6, 3, . . .

D 100, 200, 300, 400, . . . **E** 400, 300, 200, 100, . . .

13 Triangular numbers are based on the pattern:



The first four triangular numbers are 1, 3, 6, 10. The sixth triangular number is:

A 21 **B** 18 **C** 15 **D** 36 **E** 14

Short-answer questions

1 Find t_{20}, the 20th term in the sequence: 7, 11, 15, 19, . . .

2 Microwave cooking instructions for heating muffins is to heat one muffin for 45 seconds and to allow another 30 seconds for each extra muffin. State the heating times for 1, 2, 3 and 4 muffins.

3 Find t_{10}, the 10th term in the sequence: 3, 6, 12, . . .

4 A basketball was dropped from a height of 48 metres. After each bounce it reached only half of the previous height. Starting with the height of 48 metres, list the next three heights that it reached after each bounce.

5 Find t_5, the 5th term, of the sequence generated by the recurrence relation, $t_1 = 7$, $t_{n+1} = 2t_n$.

6 The terms of a sequence start at 1000 and each new term is 8% less than the previous term. Find the 6th term, correct to two decimal places.

7 Match the descriptions given in **a** to **d** with the graphs of the sequences shown.

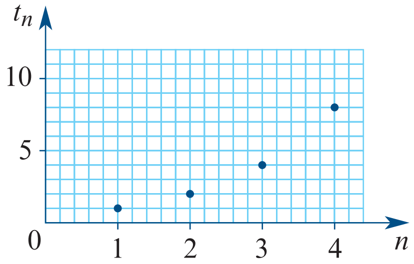
a Arithmetic with $d = 2$

b Arithmetic with $d = -2$

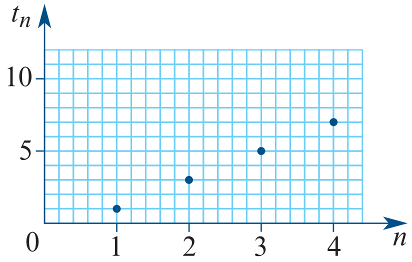
c Geometric with $r = 2$

d Geometric with $r = \frac{1}{2}$

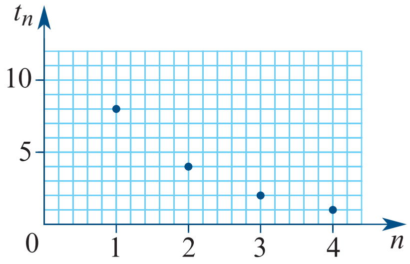
A



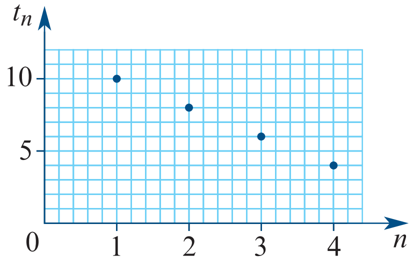
B



C



D



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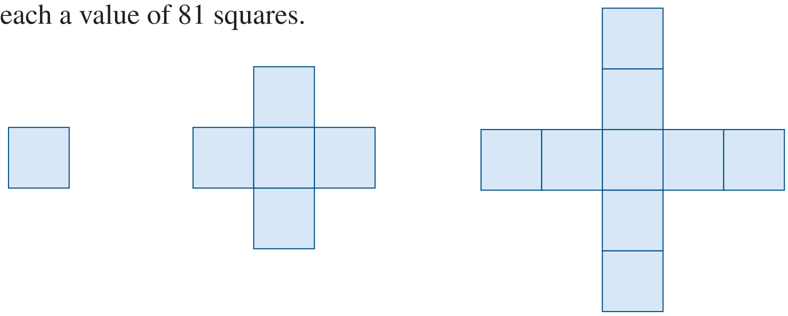
Extended-response questions

1 **a** How many squares are in the tenth shape in the sequence below?

b What is the number of the shape with 81 squares? Hint: Use your calculator to generate the terms of the sequence and count the terms starting at shape 1 until you reach a value of 81 squares.



2 The number of bacteria in a colony doubles each day. On the first day there were 300 bacteria.

a State the value of the first term, a, and the common ratio, r.

b How many bacteria will there be on the eighth day?

c On which day will the number of bacteria be 614 400? Hint: Use your calculator to generate the terms of the sequence and count the terms starting at day 1 until you reach a value of 614 400 bacteria.

3 Margaret started working for a company on an annual salary of \$65 000 with a guaranteed increase of 9% each year.

a Give the value of the first term, a, and the common ratio, r, for the sequence of her salaries each year.

b Find Margaret's salary for the twenty-fifth year working with the company.

c In which year will her salary be \$100 010.56? Hint: Use your calculator to generate the terms of the sequence and count the terms starting at year 1 until you reach a value of \$100 010.56.

4 A snail, starting from the bottom of a drainpipe, climbs 486 centimetres during the first day, 324 centimetres the next day and 216 centimetres the following day. Assuming that this pattern continues, answer the following.

a Find the common ratio for the sequence of distances travelled.

b How far will the snail travel on the fourth day?

c What will be the total distance climbed after 4 days?

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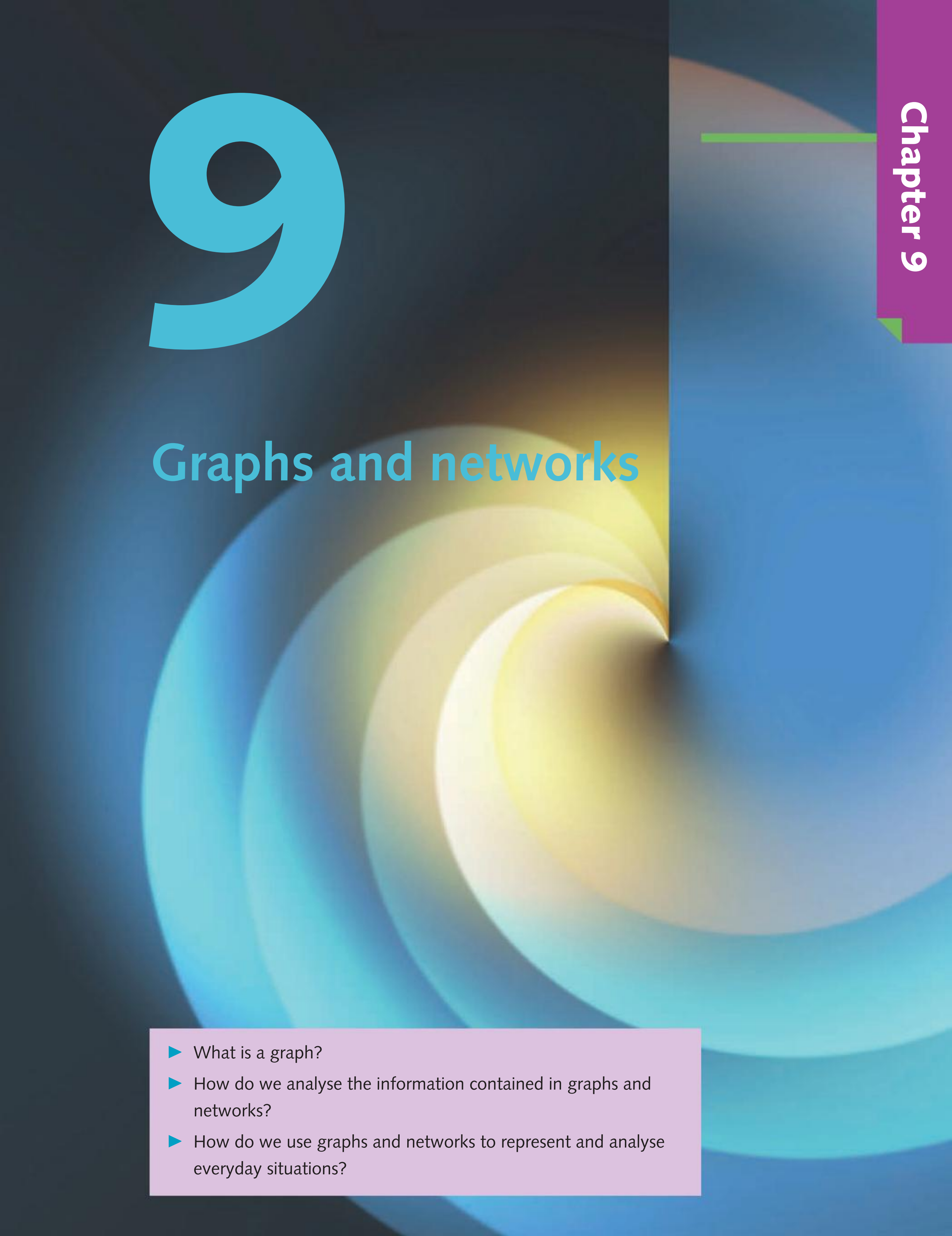


9

Graphs and networks

- What is a graph?
- How do we analyse the information contained in graphs and networks?
- How do we use graphs and networks to represent and analyse everyday situations?

Introduction

This chapter covers mathematical graphs (as opposed to statistical ones) and shows how they can model networks, maps and many kinds of organisational charts and diagrams.

9A Graph theory basics

▶ The Königsberg bridge problem

The problem that began the scientific study of graphs and networks is known as the *Königsberg bridge problem*. The problem began as follows:

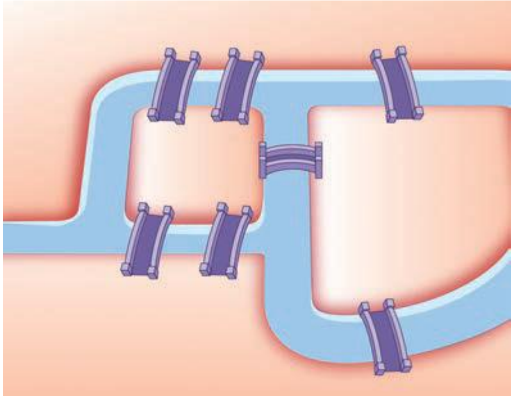
The centre of the old German city of Königsberg was located on an island in the middle of the Pregel River. The island was connected to the banks of the river and to another island by five bridges. Two other bridges connected the second island to the banks of the river, as shown below.

A view of Königsberg as it was in Euler's day.

The problem simplified

A simplified view of the situation is shown in the drawing opposite.

Can a continuous walk be planned so that all bridges are crossed only once?



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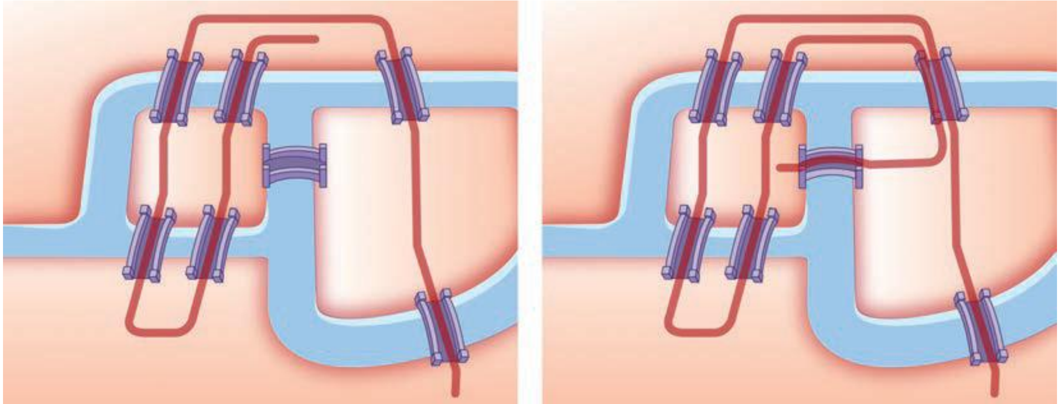
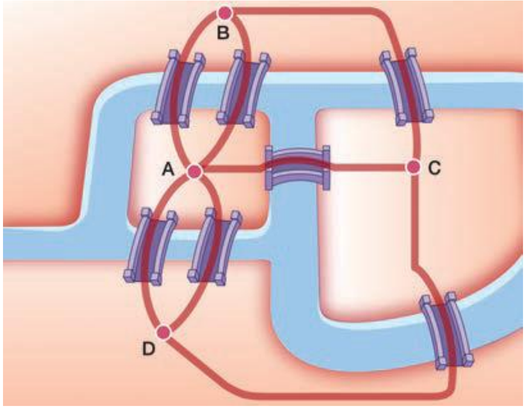
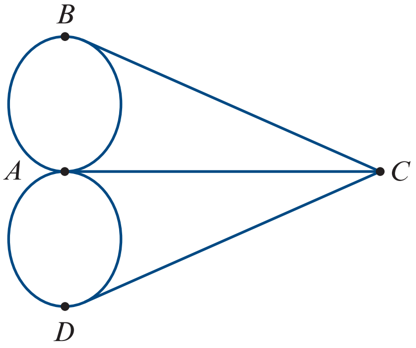
Whenever someone tried to walk the route, they either ended up missing a bridge or crossing one of the bridges more than once. Two such walks are marked on the diagrams that follow. See if you can trace out a walk on the diagram above that crosses every bridge, but only once.



Enter the mathematician

The Königsberg bridge problem was well-known in 18th century Europe and attracted the attention of the Swiss mathematician Euler (pronounced 'Oil-er'). He started analysing the problem by drawing a simplified diagram to represent the situation, as shown below. We now call this this type of simplified diagram a *graph*.





Euler's diagram

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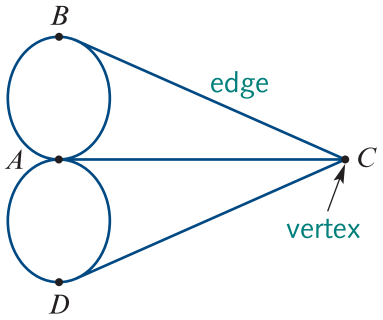
The elements of a graph

A graph is made up of dots and lines joining these dots.

- The *dots* in a graph are called *vertices* (plural of **vertex**). These vertices represent the riverbanks and the islands.
- The *lines* in a graph are called **edges**. These edges represent the bridges.

Euler found that the solution of the Königsberg bridge problem was connected to the *degree* of each vertex.

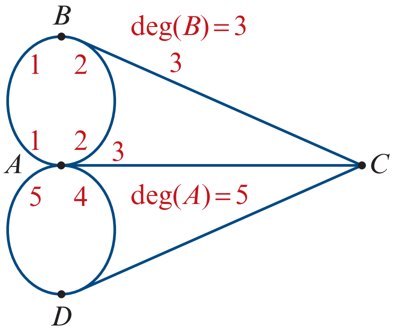


Degree of a vertex

The **degree of a vertex** is the number of edges attached to the vertex.

For the Königsberg bridge graph:

- the degree of vertex A is 5 (because five edges are attached to vertex A): write this as $\deg(A) = 5$
- vertices B, C and D are all of degree 3 (three edges touch each of these vertices): write this as $\deg(B) = 3$, etc.



The degree of a vertex may be even or odd.

- The *degree* of a vertex will be *even* if there are an even number of edges (2, 4, 6, . . .) attached to the vertex.
- The *degree* of a vertex will be *odd* if there are an odd number of edges (1, 3, 5, . . .) attached to the vertex.

All four vertices in the Königsberg bridge graph have an odd degree.

Euler's discovery

Euler was able to prove that a graph with *all odd* vertices cannot be traced or drawn without lifting the pencil or going over the same edge more than once. The problem was solved. The seven bridges of Königsberg could *not* be crossed in a single walk without either missing a bridge or crossing one bridge more than once.

With this analysis, a new area of mathematics was developed which has many practical applications in today's world. These include: analysing friendship networks, scheduling airline flights, designing electrical circuits, planning large-scale building projects, and many more. This relatively new area of mathematics is now called *graph theory*, some aspects of which we will explore in this chapter.

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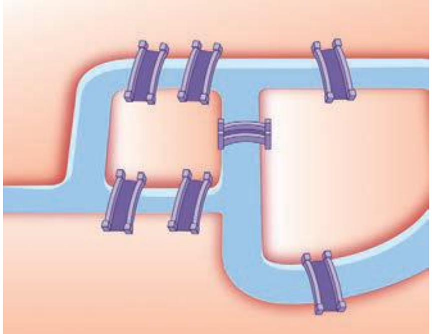
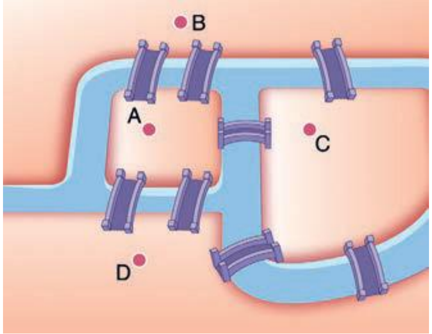
Exercise 9A



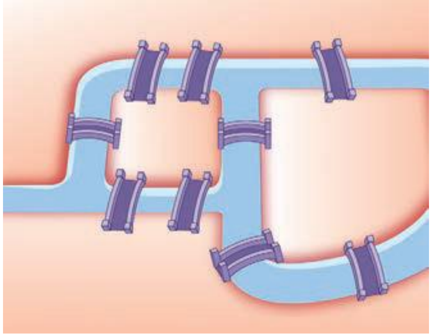
1 We now know that it is impossible to trace out a continuous walk that crosses each of the Königsberg bridges only once. If you don't believe this, try it for yourself on the diagram opposite.

In this exercise, you will investigate how things would change if the number of bridges is changed.



a The picture opposite shows a situation in which an eighth bridge has been added.

- **i** With a pencil, or the tip of your finger, see whether you can trace out a continuous walk that crosses each of the bridges only once. Such a walk exists.
- **ii** Construct a graph to represent this new situation with eight bridges. Labelled dots have been placed on the picture to help you draw your graph.
- **iii** Your graph should have only two odd vertices. Check to see.
- **iv** As you will learn later, when the graph has only two odd vertices, you can only complete the task if you start at the places represented by the odd vertices. You will then finish at the place represented by the other. Check to see.



b A ninth bridge has been added as shown opposite.

- **i** With a pencil, or the tip of your finger, see whether you can trace out a continuous walk that crosses each of the bridges only once. Such a walk exists.
- **ii** Construct a graph to represent this situation.
- **iii** Your graph should *not* have any odd vertices; that is, they should all be even. Check to see.
- **iv** As you will learn later, when the graph has only even vertices, you can start your walk from any island or any river bank and still complete the task. Check to see.

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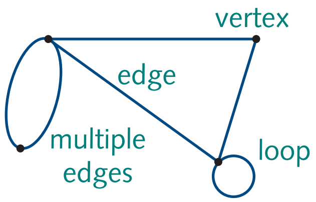
9B What is a graph?

It is now time to introduce some definitions that we need to know in order to work with graphs. This chapter uses a specialised mathematical sense of the word ‘graph’ as opposed to its everyday use in statistics, science and other contexts.

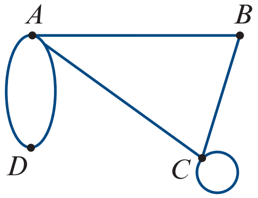
▶ Graph elements: definitions

- A **graph** is a diagram that consists of a set of points called *vertices* that are joined by a set of lines called **edges**. Each edge joins two vertices.
- A **loop** is an edge in a graph that joins a **vertex** in a graph to itself.
- Two or more edges that connect the same vertices are called **multiple edges**.



- The **degree of a vertex** is the number of edges attached to the vertex. The degree of a vertex is denoted deg(V). For example, in the graph below, deg(A) = 4, deg(B) = 2, deg(C) = 4 and deg(D) = 2.

  Note: A loop contributes two degrees to a vertex because a loop is attached to its vertex at both ends.



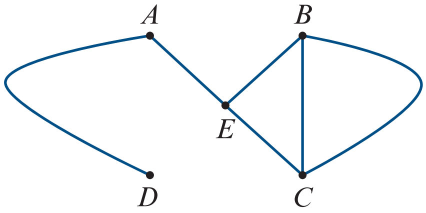
- In any graph, the *sum of degrees* of the vertices is equal to *twice the number of edges*.

  Note: A *loop* contributes *one edge* to the graph.

Exercise 9B

1 For each graph shown, complete the associated statements by filling in the boxes.

a



i The graph has ☐ vertices.
ii The graph has ☐ edges.
iii The graph has ☐ loops.
iv deg(A) = ☐
v deg(E) = ☐
vi The graph has ☐ odd vertices.
vii The graph has ☐ even vertices.

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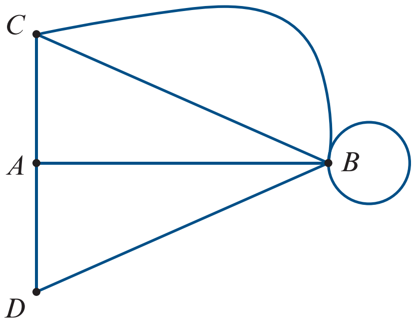
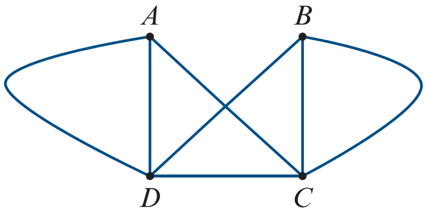
b



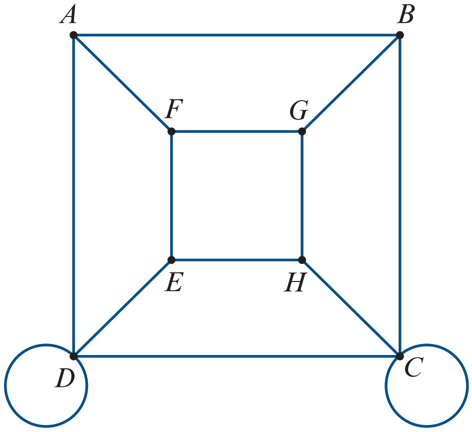
i The graph has ☐ vertices.
ii The graph has ☐ edges.
iii The graph has ☐ loops.
iv deg(B) = ☐
v deg(D) = ☐
vi The graph has ☐ odd vertices.
vii The graph has ☐ even vertices.

c



i The graph has ☐ vertices.
ii The graph has ☐ edges.
iii The graph has ☐ loops.
iv deg(A) = ☐
v deg(C) = ☐
vi The graph has ☐ odd vertices.
vii The graph has ☐ even vertices.

d



i The graph has ☐ vertices.
ii The graph has ☐ edges.
iii The graph has ☐ loops.
iv deg(C) = ☐
v deg(F) = ☐
vi The graph has ☐ odd vertices.
vii The graph has ☐ even vertices.

2 What is the sum of the degrees of the vertices of a graph with:

a five edges? **b** three edges? **c** one edge?

In each case draw an example of the graph and check your answer.

3 Why do you think that the sum of the vertex degrees of a graph will always equal twice the number of edges?

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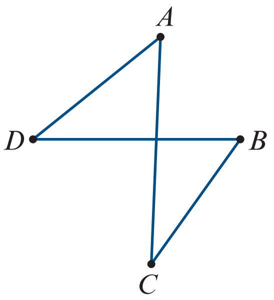
4 Consider the graph opposite.
A loop is added at vertex A:

a how will this change the degree of vertex A?

b how many edges are added to the graph?



5 Play a game of Sprouts with a classmate or a friend. The game of *Sprouts* is played between two people and involves drawing a graph.

Rules for the game of Sprouts

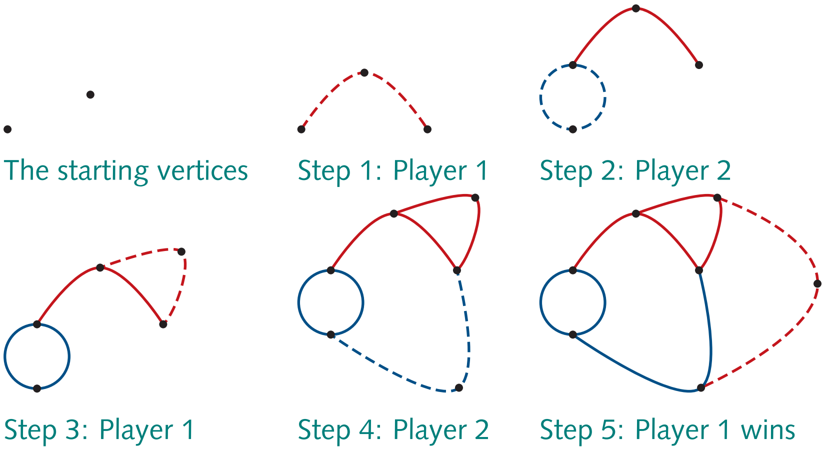
Two or more points are drawn on a piece of paper. These are graph vertices. Players then take turns adding edges according to the following rules:

1. Each edge must join two vertices or itself.
2. Every time a new edge is drawn, a new vertex must be added on the edge.
3. Edges cannot cross nor pass through a vertex.
4. No vertex may have a degree greater than 3.
5. The last player able to add a new edge wins.

For more information see: http://cambridge.edu.au/redirect/?id=5922.

A sample game of Sprouts is played out below.

Sample game of Sprouts



Player 1 wins because Player 2 cannot draw in a new edge without creating a vertex of degree greater than 3.

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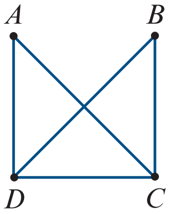
9C Isomorphic, connected graphs and adjacency matrices

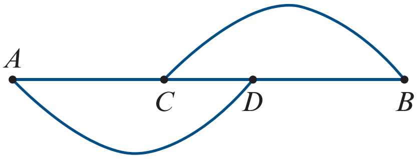
Isomorphic graphs

Different looking graphs can contain the same information. When this happens, we say that these graphs are equivalent or **isomorphic**.

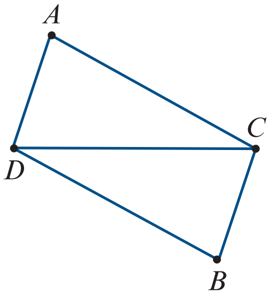
For example, the three graphs below look quite different but, in graphical terms, they are equivalent.



Graph 1



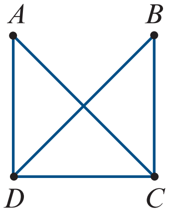
Graph 2



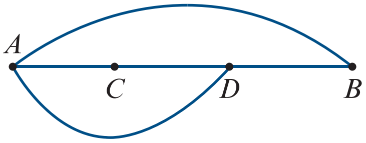
Graph 3

This is because they contain the same information. Each graph has the same number of edges (5) and vertices (4), corresponding vertices have the same degree and the edges join the vertices in the same way (*A* to *C*, *A* to *D*, *B* to *C*, *B* to *D*, and *D* to *C*).

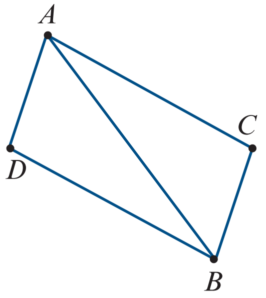
However, the three graphs below, although having the same numbers of edges and vertices, are not isomorphic. This is because corresponding vertices do *not* have the same degree and the edges do *not* connect the same vertices.



Graph 1



Graph 2



Graph 3

Isomorphic graphs

Two graphs are said to be isomorphic (equivalent) if:

- they have the same numbers of edges and vertices
- corresponding vertices have the same degree and the edges connect the same vertices.

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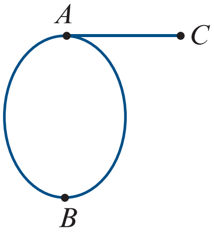
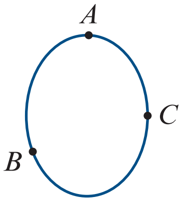
Exercise 9C-1

1 In each of the following sets of three graphs, two of the graphs are isomorphic. In each case, identify the isomorphic graphs.

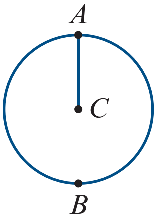
a



Graph 1

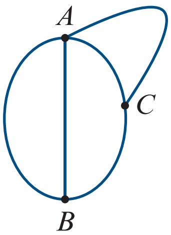


Graph 2

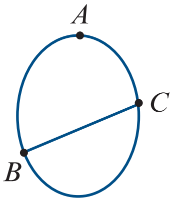


Graph 3

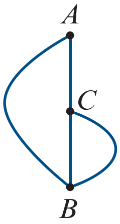
b



Graph 1

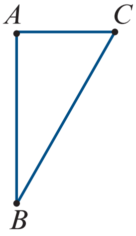


Graph 2

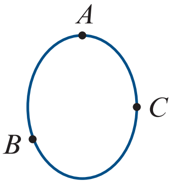


Graph 3

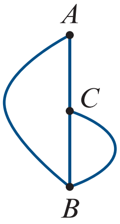
c



Graph 1

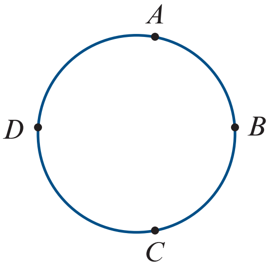


Graph 2

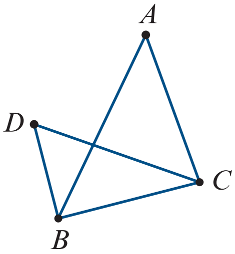


Graph 3

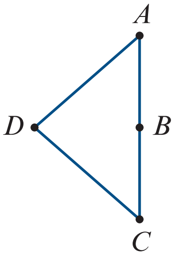
d



Graph 1

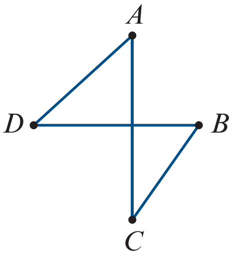


Graph 2

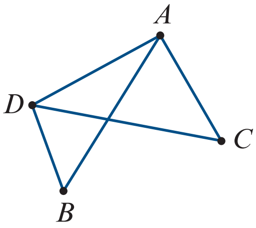


Graph 3

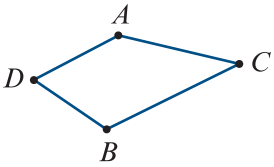
e



Graph 1



Graph 2



Graph 3

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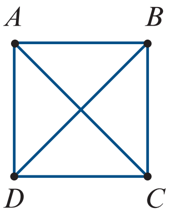
Connected graphs and bridges

Connected graphs

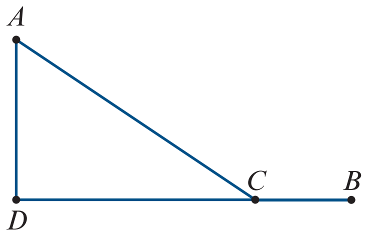
So far, all the graphs we have encountered have been *connected*. In a connected graph, every vertex is connected to every other vertex either directly or via another vertex. That is, every vertex in the graph can be reached from every other vertex in the graph.

The three graphs shown below are all connected.



Graph 1

Graph 2

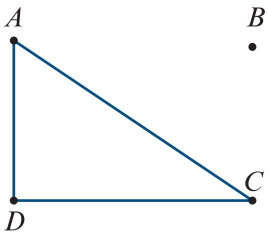


Graph 3

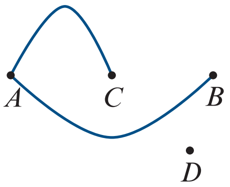
The graphs are connected because, starting at any vertex, say *A*, you can always find a path along the edges of the graph to take you to every other vertex. For example:

- In Graph 1 you can get directly from vertex *A* to vertex *B* by travelling along edge *AB*. A similar statement can be made about all the other vertices.
- In Graph 2 you can get from vertex *A* to vertex *B* indirectly via vertices *C* and *D*. All other vertices are accessible from vertex *A* in a similar manner.
- In Graph 3 you can get from vertex *A* to vertex *B* indirectly by travelling via vertex *C*, or via vertices *D* and *C*. All other vertices are directly accessible from vertex *A*.

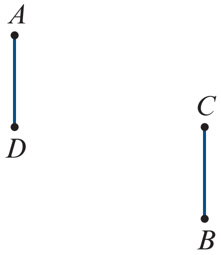
However, the three graphs below are *not connected*, because there is not a path along the edges that connects vertex *A* (for example) to every other vertex in the graph.



Graph 1



Graph 2



Graph 3

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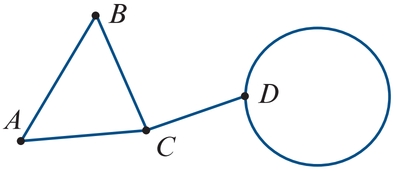
Bridges

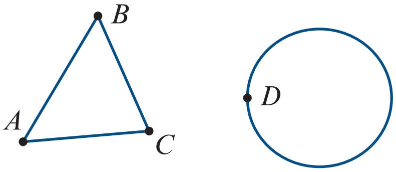
Connected graphs have applications in a range of problems such as planning airline routes, communication systems and computer networks, where a single missing connection can lead to an inoperable system. Such critical connections are called bridges.

A **bridge** is an edge in a connected graph that, if removed, leaves the graph disconnected.

In Graph 1 below, edge *CD* is a bridge because removing *CD* from the graph leaves it disconnected, see Graph 2.



Graph 1



Graph 2

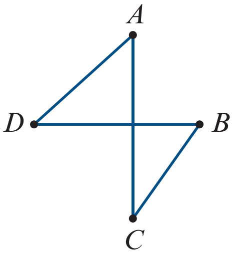
Connected graphs and bridges

- A graph is *connected* if every vertex in the graph is accessible from every other vertex in the graph along a path formed by the edges of the graph.
- A *bridge* is a single edge in a connected graph that if removed leaves the graph disconnected. A graph can have more than one bridge.

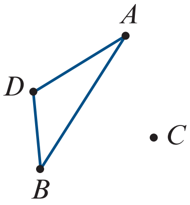
Exercise 9C-2

1 Which of the following graphs are connected?

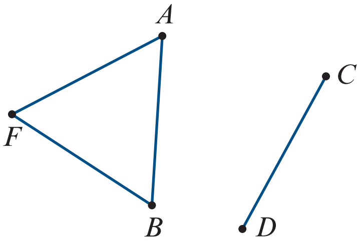
A



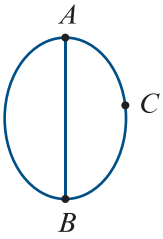
B



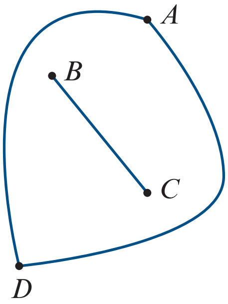
C



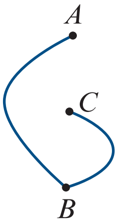
D



E



F



2 Draw a connected graph with:

a three vertices and three edges

b three vertices and five edges

c four vertices and six edges

d five vertices and five edges

3 Draw a graph that is *not* connected with:

a three vertices and two edges

b four vertices and three edges

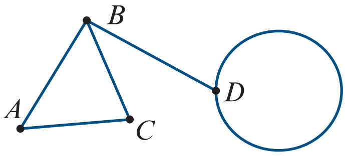
c four vertices and four edges

d five vertices and three edges

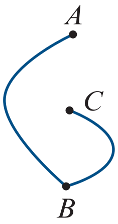
4 What is the smallest number of edges that can form a connected graph with four vertices?

5 Identify the bridge (or bridges) in the graphs below.

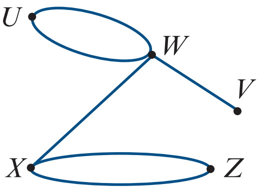
a



b



c



6 Draw a graph with four vertices in which every edge is a bridge.

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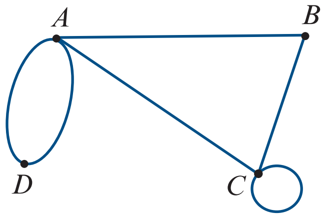
Adjacency matrices

Matrices are a compact way of communicating the information in a graph. Adjacency matrices are useful when the information in a graph needs to be entered into a computer.

There are various types of matrices that can be used to represent the information in a graph. We will only consider one type, the adjacency matrix.

An **adjacency matrix** is a square matrix that uses a zero or an integer to record the number of edges connecting each pair of vertices in the graph.

For example:



$$\begin{array}{c|cccc|} & A & B & C & D \\ A & 0 & 1 & 1 & 2 \\ B & 1 & 0 & 1 & 0 \\ C & 1 & 1 & 1 & 0 \\ D & 2 & 0 & 0 & 0 \end{array}$$

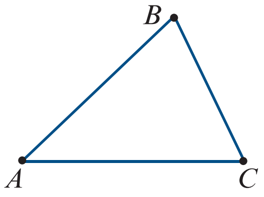
On the left we have a graph with four vertices *A*, *B*, *C*, *D*.
On the right we have a 4 × 4 adjacency matrix (four rows and four columns).
The rows and columns are labelled *A* to *D* as shown to match the vertices in the graph.
The numbers in the matrix refer to the number of edges joining the corresponding vertices.
For example, in this matrix:

- the '0' in row *A*, column *A* indicates that no edges connect vertex *A* to vertex *A*.
- the '1' in row *B*, column *A* indicates that one edge connects vertex *A* to vertex *B*.
- the '2' in row *D*, column *A* indicates that two edges connect vertex *A* to vertex *D*.
- the '1' in row *C*, column *C* indicates that one edge connects vertex *C* to vertex *C* (a loop) and so on until the matrix is complete.

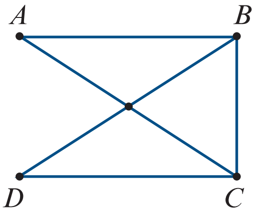
Exercise 9C-3

1 Construct an adjacency matrix for each of the following graphs.

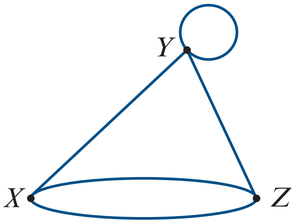
a



b



c



2 Construct a graph for each of the following adjacency matrices.

a
$$\begin{array}{c|ccc|} & A & B & C \\ A & 0 & 2 & 1 \\ B & 2 & 0 & 1 \\ C & 1 & 1 & 0 \end{array}$$

b
$$\begin{array}{c|cccc|} & A & B & C & D \\ A & 0 & 1 & 0 & 1 \\ B & 1 & 0 & 1 & 1 \\ C & 0 & 1 & 0 & 1 \\ D & 1 & 1 & 1 & 0 \end{array}$$

c
$$\begin{array}{c|ccc|} & X & Y & Z \\ X & 0 & 1 & 1 \\ Y & 1 & 1 & 1 \\ Z & 1 & 1 & 0 \end{array}$$

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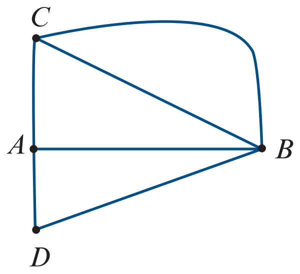
9D Planar graphs and euler's formula

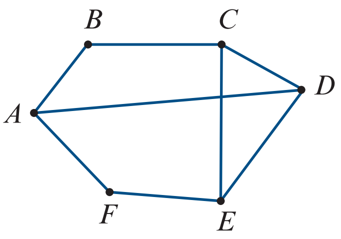
▶ Planar graphs

A **planar graph** can be drawn on a plane (page surface) so that no edges intersect (cross), except at the vertices.

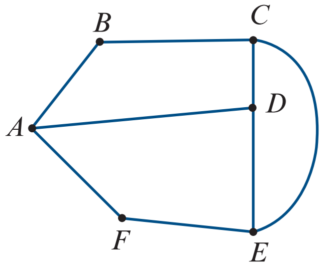


A planar graph: no intersecting edges

Some graphs do not initially appear to be planar; for example, Graph 1 shown below left. However, Graph 2 (below right) is equivalent (isomorphic) to Graph 1. Graph 2 is clearly planar.



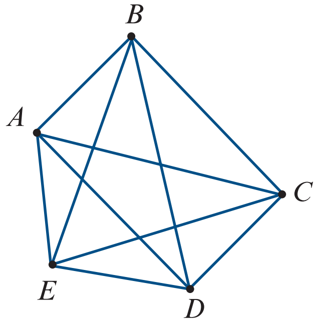
Graph 1: non-planar graph as drawn



Graph 2: planar form of Graph 1

Not all graphs are planar.

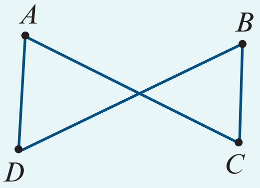
For example, the graph opposite cannot be redrawn in an equivalent planar form, no matter how hard you try.



Non-planar graph

Example 1 Redrawing a graph in planar form

Redraw the graph shown opposite in a planar form.



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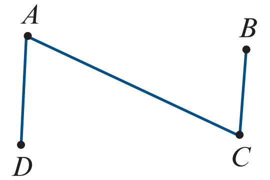
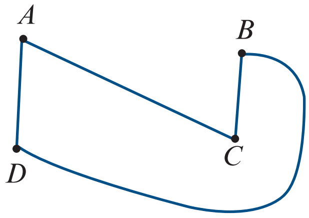
Solution (there are others)

1 Redraw the graph with edge DB removed.

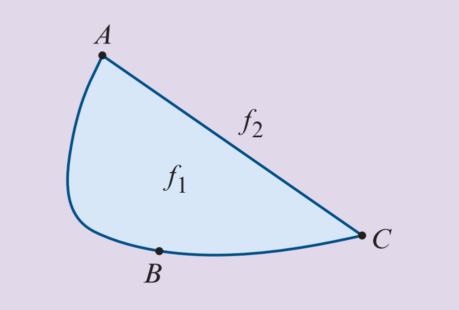
Note: We have removed edge DB because it intersects edge AC.



2 Replace edge DB as a curved line that avoids intersecting with the other three edges. The graph is now in an equivalent planar form: no edges intersect, except at vertices.



Faces of a graph

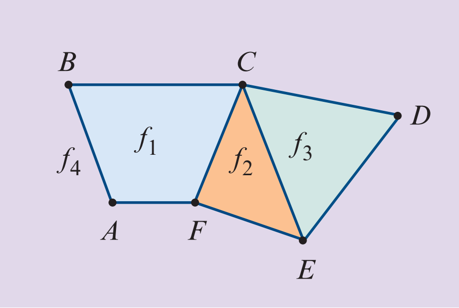


The graph opposite can be regarded as dividing the paper it is drawn on into two regions.

In the language of graphs, these regions are called the **faces** of the graph.

One face, f_1, is bounded by the graph.

The other face, f_2, is the region surrounding the graph. This 'outside' face is infinite.



The graph opposite divides the paper into four regions, so we say that it has four faces: f_1, f_2, f_3 and f_4. Here f_4 is an infinite face.

Euler's formula

Euler discovered that, for connected planar graphs, there is a relationship between the *number of vertices*, v, the *number of edges*, e, and the *number of faces*, f. This relationship can be expressed in words as:

number of vertices − number of edges + number of faces = 2

or in symbols as:

$$v - e + f = 2$$

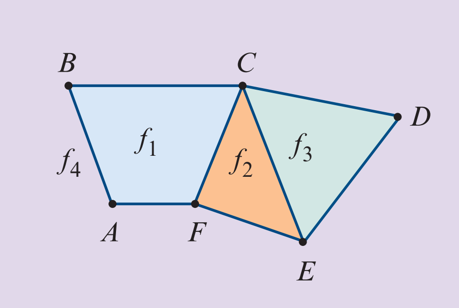
This is known as **euler's formula**.

For example, for the graph opposite:

$$v = 6, \ e = 8 \text{ and } f = 4.$$

So $\quad v - e + f = 6 - 8 + 4 = 2$

confirming euler's formula.



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Euler's formula

For a connected planar graph:

number of vertices − number of edges + number of faces = 2

or

$$v - e + f = 2$$

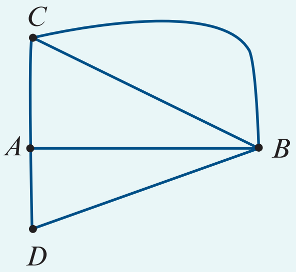
where v = number vertices, e = number of edges and f = number of faces.

Example 2 Verifying euler's formula

Consider the connected planar graph shown.

a Write down the number of vertices, v, the number of edges, e, and the number of faces, f.

b Verify euler's formula.



Solution

a **1** There are four vertices: A, B, C, D, so $v = 4$.

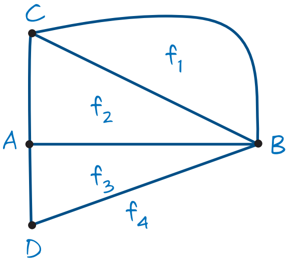
Number of vertices: $v = 4$

2 There are six edges: AB, AC, AD, BC ($\times 2$) and BD, so $e = 6$.

Number of edges: $e = 6$

3 There are four faces, so $f = 4$.

Tip: Mark the faces on the diagram. Do not forget the infinite face f_4 that surrounds the graph.



Number of faces: $f = 4$

b **1** Write down euler's formula.

Euler's formula: $v - e + f = 2$

2 Substitute the values of v, e, and f. Evaluate.

$v - e + f = 4 - 6 + 4 = 2$

3 Write your conclusion.

$\therefore$ euler's formula is verified.

Example 3 Using euler's formula to find the number of faces of a graph

A connected planar graph has four vertices and five edges. Find the number of faces.

Solution

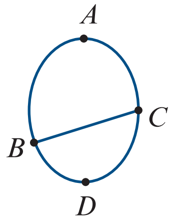
1 Write down v and e.	$v = 4, e = 5$
2 Write down euler's formula.	Euler's formula: $v - e + f = 2$
3 Substitute the values of v and e.	$4 - 5 + f = 2$ $-1 + f = 2$
4 Solve for f.	$f = 3$
5 Write your answer.	The graph has three faces.

Exercise 9D

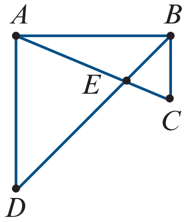
Example 1

1 Which of the following graphs are drawn in planar form?

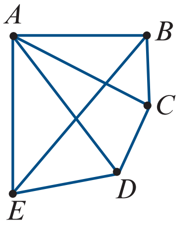
A



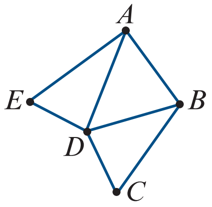
B



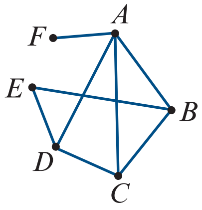
C



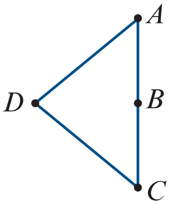
D



E

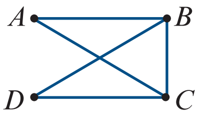


F

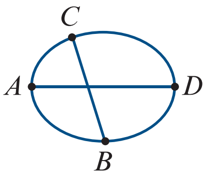


2 Redraw each graph in an equivalent planar form.

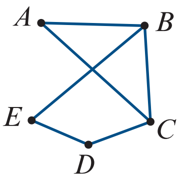
a



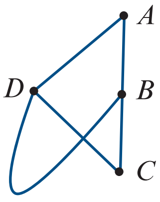
b



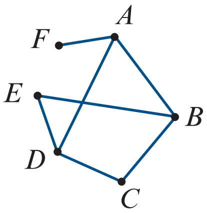
c



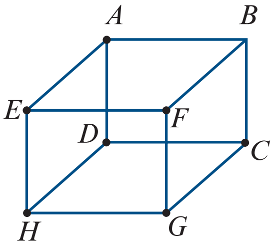
d



e



f

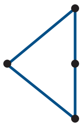


Example 2

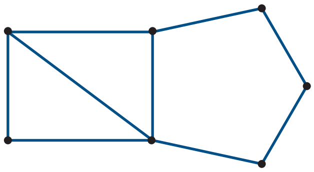
3 For each of the following graphs:

i state the values of v, e and f **ii** verify euler's formula.

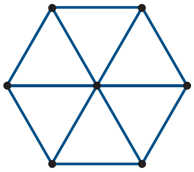
a



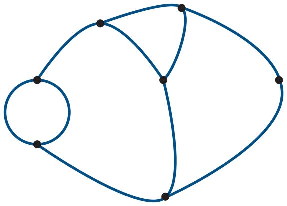
b



c



d



Example 3

4 For a planar connected graph, find:

a f given $v = 4$ and $e = 4$

b v given $e = 3$ and $f = 2$

c e given $v = 3$ and $f = 3$

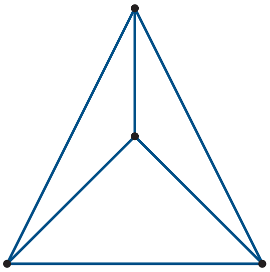
d v given $e = 6$ and $f = 4$

e f given $v = 4$ and $e = 6$

f f given $v = 6$ and $e = 11$

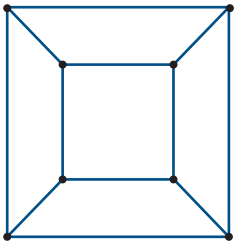
g e given $v = 10$ and $f = 11$

5 The five graphs shown below are known as the platonic (after Plato) solids.



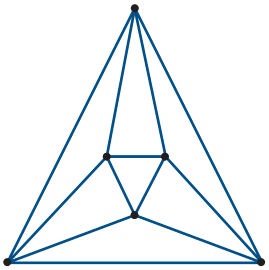
tetrahedron

Graph 1



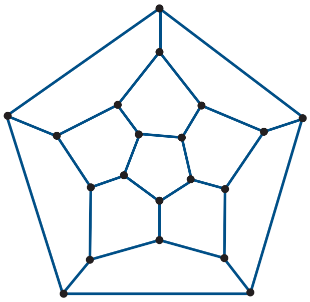
cube

Graph 2



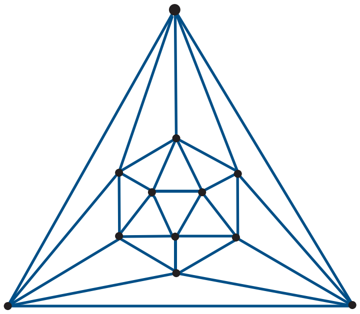
octahedron

Graph 3



dodecahedron

Graph 4



icosahedron

Graph 5

For each graph, write down the values of v, e and f and show that they satisfy euler's formula.

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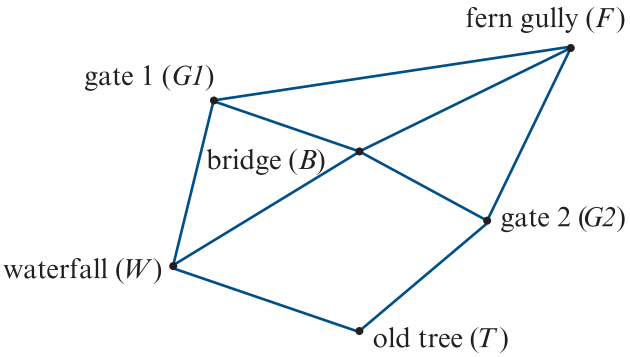
9E Walks, trails, paths, circuits and cycles

Many practical problems that can be modelled by graphs involve moving around a graph, for example, designing a postal delivery route or solving the Königsberg bridge problem. To solve such problems you will need to know about a number of concepts that we use to describe the different ways we can move around a graph. We will use the graph below to explore these ideas.



The graph is used to represent the tracks in Sherbrooke Forest shown above that leads to a suspension bridge (vertex B), a waterfall (vertex W), a very old tree (vertex T), and a fern tree gully (vertex F). People can enter and leave the forest through either gate 1 (vertex $G1$) or gate 2 (vertex $G2$). The edges in the graph represent tracks that connect these places, for example, the edge WB represents the track between the waterfall and the bridge.

▶ Walks, trails and paths

Informally a walk is any route through a graph that moves from one vertex to another along the joining edges. When there is no ambiguity, a walk in a graph can be specified by listing the vertices visited on the walk.

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Walk

A **walk** is a sequence of edges, linking successive vertices, that connects two different vertices in a graph.

A walk in the forest

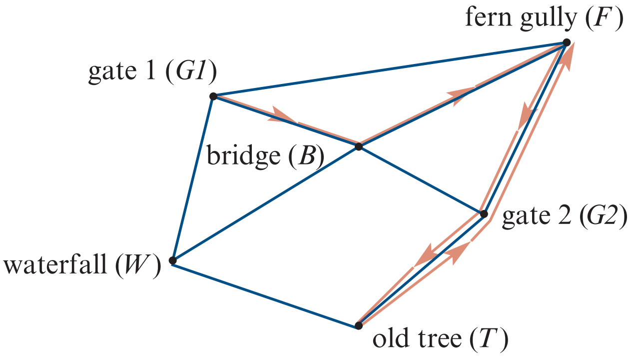
Using the forest track graph (shown in blue), an example of a *walk* is:

G1-B-F-G2-T-G2-F

The red arrows on the graph trace out a walk.

Note1: The double red arrows on the graph indicate that this track is walked along in both directions.

Note2: A walk does not require all of its edges or vertices to be different.



Trail

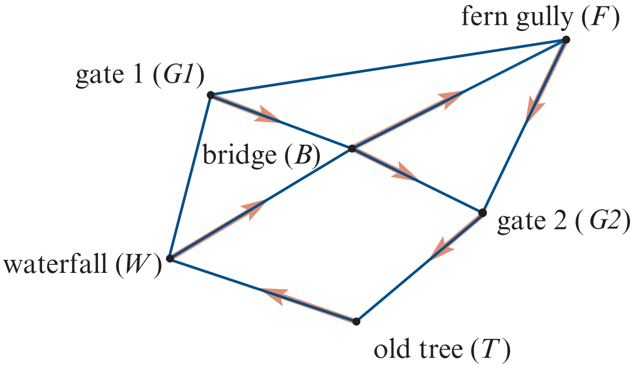
A **trail** is a walk with no repeated edges.

A forest trail

Using the forest track graph, an example of a *trail* is:

G1-B-F-G2-T-W-B-G2

The red arrows on the graph trace out this trail. It has *no repeated edges*. However, there is one repeated vertex *G2*. This is permitted on a trail.



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Path

A **path** is a walk with no repeated vertices.

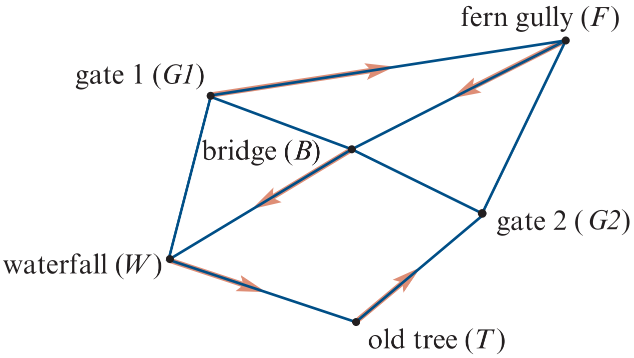
Note: Because a path has no repeated vertices, there can be no repeated edges.

A path in the forest

Using the forest track graph, an example of a *path* is:

G1-F-B-W-T-G2

The red arrows on the graph trace out this path. There are *no repeated edges* or *vertices*.



Circuits and cycles

There is nothing to stop a walk, trail or path beginning and ending at the same vertex. When this happens we say that the walk, trail or path is closed. Because *closed trails* and *closed paths* are so important in practice, we give them special names. We call them *circuits* and *cycles*.

Circuit

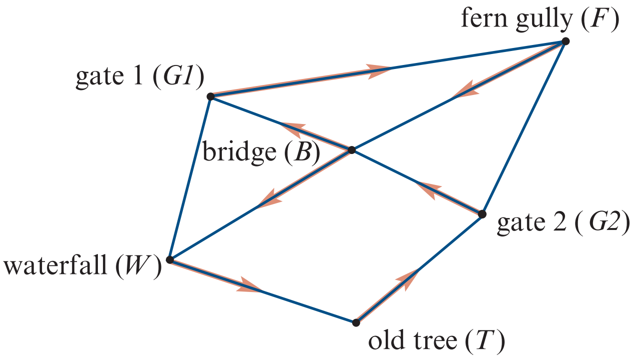
A **circuit** is a walk that has no repeated edges and starts and ends at the same vertex.

A circuit in the forest

Using the forest track graph, an example of a *circuit* is:

G1-F-B-W-T-G2-B-G1

This circuit *begins* and *ends* at the *same* vertex (*G1*). The red arrows on the graph trace out this circuit.



There are *no repeated edges*.
However, the circuit passes through vertex *B* twice.

Cycle

A **cycle** is a walk that has no repeated vertices and starts and ends at the same vertex.

Note: Because a cycle has no repeated vertices, there can be no repeated edges.

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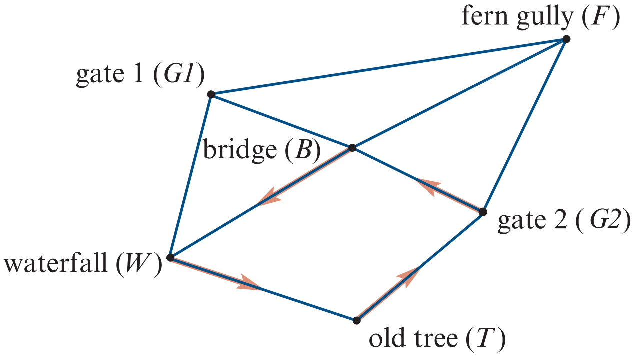
A cycle in the forest

Using the forest track graph, an example of a *cycle* is:

G2-B-W-T-G2

The red arrows on the graph trace out this cycle. Except for the first and last there are no repeated edges or vertices in a cycle.

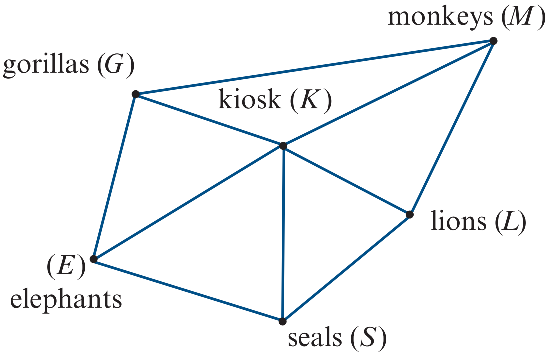


Summary of the properties of walks, trails, circuits and cycles

Type of route	Are repeated edges permitted?	Are repeated vertices permitted?
walk (open)	yes	yes
trail (open)	no	yes
path (open)	no	no
closed walk	yes	yes
circuit (closed trail)	no	yes
cycle (closed path)	no	no (except for the first and last)

Exercise 9E

1 The graph opposite shows the pathways linking five animal enclosures in a zoo to each other and to the kiosk.



a Which of the following represents a trail in the graph?

- **i** *S-L-K-M-K*
- **ii** *G-K-L-S-E-K-M*
- **iii** *E-K-L-K*

b Which of the following represents a path in the graph?

i *K-E-G-M-L* **ii** *E-K-L-M* **iii** *K-S-E-K-G-M*

c Which of the following represents a circuit in the graph?

i *K-E-G-M-K-L-K* **ii** *E-S-K-L-M-K-E* **iii** *K-S-E-K-G-K*

d Which of the following represents a cycle in the graph?

i *K-E-G-K* **ii** *G-K-M-L-K-G* **iii** *L-S-E-K-L*

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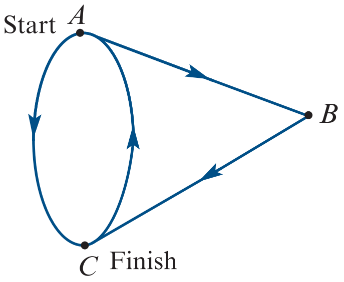
9F Traversable graphs

Many practical problems involve finding a trail in a graph that includes *every* edge. The Königsberg bridge problem is one such problem. Graphs that have this property are called traversable graphs.

A **traversable graph** has a trail that includes *every edge* in the graph.

The graph opposite is an example of a *traversable graph*. It has a trail that includes every edge.

The trail *A-B-C-A-C* is one such example.

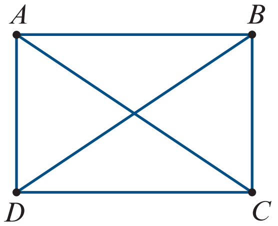


Graph 1: traversable

Not all graphs are traversable.

For example, it is impossible to find a trail in the graph opposite that includes every edge. Try it and see.

This graph is *not traversable*.



Graph 2: non-traversable

Identifying traversable graphs

One way of finding out whether a graph is traversable is to try a number of routes through the graph and see whether they work. When we do this, we are solving the problem by inspection. However, for all but the simplest graphs, this problem-solving method can become very tedious.

Fortunately, there is a more systematic problem-solving method that relates to the degree of the vertices and whether they are *odd* or *even*.

Rules for identifying traversable graphs

For a graph to be traversable, it must first be *connected*.

A connected graph is traversable if:

- *all* vertices are of *even* degree; or
- exactly *two* vertices are of *odd* degree and the rest are of even degree.

From this it follows that:

- a traversable graph will have either a trail, a circuit or a cycle that involves the use of *every edge* in the graph.
- if a graph has *more than two* vertices of odd degree, it is *not* traversable.

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Using these rules, we can see why Graph 1 (above) is traversable: it has *exactly two odd* vertices (A and C) and the remaining vertex (B) is even. Likewise, Graph 2 is *not* traversable: it has four odd vertices (A, B, C and D are *all odd* vertices).

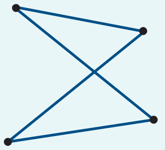
Example 4 Identifying traversable graphs

For each of the following graphs:

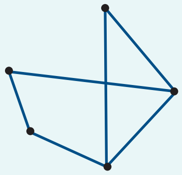
i Determine whether the graph is traversable, and state why.

ii If traversable, check by identifying a trail or circuit that traverses the graph.

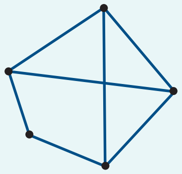
a



b

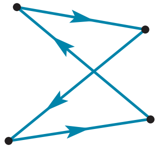


c

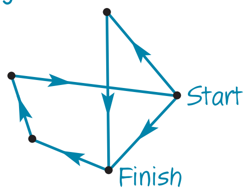


Solution

a Traversable: all even vertices. The graph has a cycle that involves every edge.



b Traversable: two odd vertices. The graph has a trail that involves every edge.



c Not traversable: more than two odd vertices

Exercise 9F

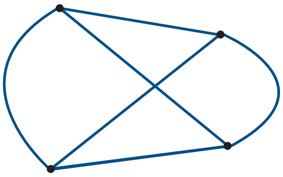
Example 4

For each of the following graphs:

i Determine whether the graph is traversable, and state why.

ii If traversable, check by identifying a trail, path, circuit or cycle that involves every edge in the graph.

1



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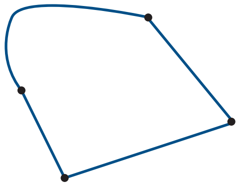

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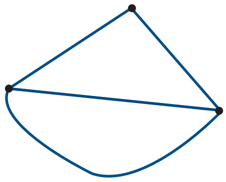

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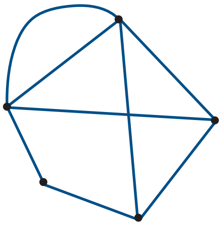

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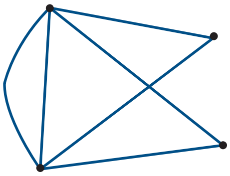

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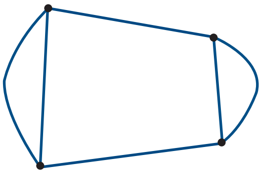

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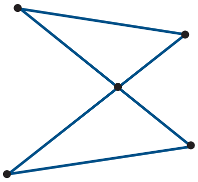

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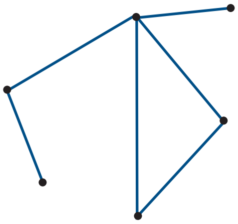

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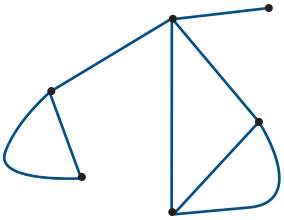

9G Eulerian trails and circuits (optional)

Because of the pioneering work done by Euler, we call a trail that includes every edge of a graph an eulerian trail and a circuit that includes every edge of a graph an eulerian circuit. The reason we are interested in eulerian trails and circuits is because of their practical significance.

Eulerian trails and circuits

- An **eulerian trail** is a trail that includes *every edge* in a graph.
- An **eulerian circuit** is a circuit that includes *every edge* in a graph.

Alternatively, an eulerian circuit is an eulerian trail that starts and finishes at the same vertex.

Only traversable graphs can have an eulerian trail or an eulerian circuit

- To have an eulerian trail or an eulerian circuit, a graph must be *connected*.
- To have an eulerian trail but *not* an eulerian circuit, the graph must have exactly *two odd* degree vertices; an eulerian trail will *start* at one of the odd vertices and *finish* at the other.
- To have an eulerian circuit the graph must have *all even* degree vertices. An eulerian circuit starts and finishes at the same vertex. It can start at any vertex.
- A graph with *more than two odd* degree vertices is *not traversable*. As a consequence, it does not have an eulerian trail or an eulerian circuit.

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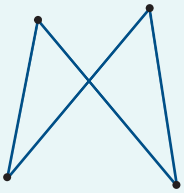
Example 5 Identifying eulerian trails and circuits

For each of the following graphs:

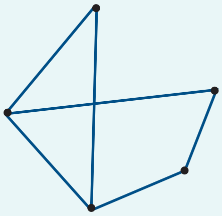
i Determine whether the graph has an eulerian trail, an eulerian circuit or neither, and state why.

ii If the graph has an eulerian trail or an eulerian circuit, show one example.

a



b



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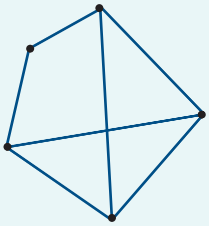
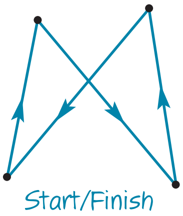
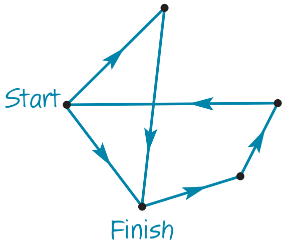


Solution

a Eulerian circuit: all even vertices



b Eulerian trail: two odd vertices, the rest even.



c Neither: more than two odd vertices

Note: In both cases, more than one solution is possible.

▶ Applications of eulerian trails and circuits

In everyday life, *eulerian trails and circuits* relate to situations like the following:

- A postman wants to deliver mail without travelling along any street more than once.
- A visitor to a tourist park wants to minimise the distance they walk to see all of the attractions by not having to retrace their steps at any stage.
- A road inspector wants to inspect the roads linking several country towns without having to travel along each road more than once.

Exercise 9G

Example 5

1 For each of the following graphs:

i Determine whether the network has an eulerian trail, an eulerian circuit or neither, and state why.

ii If the graph has an eulerian trail or circuit, show **one** example.

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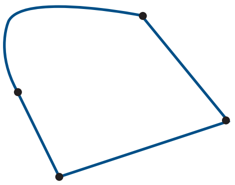
a



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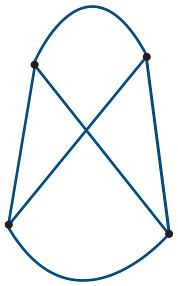

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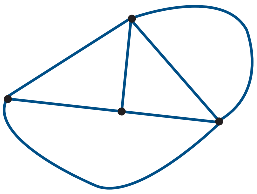

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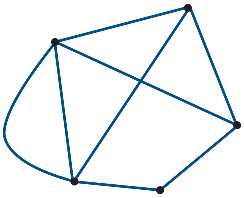

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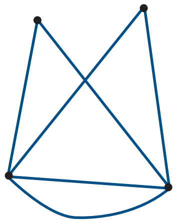

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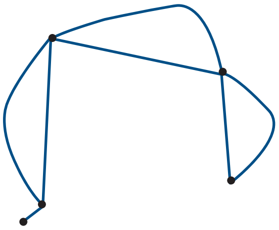

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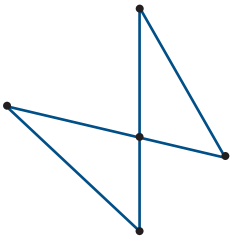

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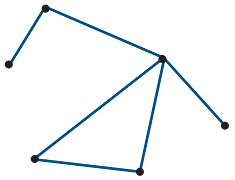
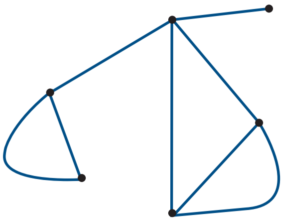
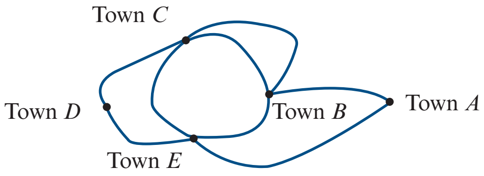


i



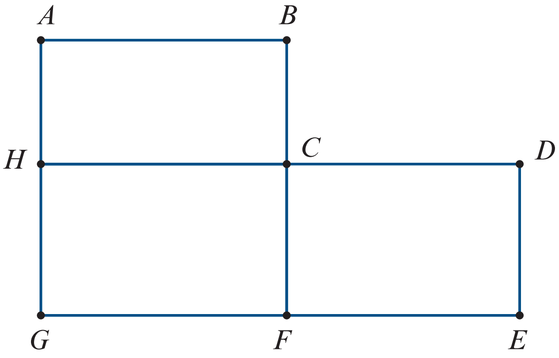
2 A road inspector lives in Town *A* and is required to inspect all roads connecting the neighbouring towns *B*, *C*, *D* and *E*. The network of roads is shown on the right.



a Is it possible for the inspector to set out from Town *A*, carry out his inspection by travelling over every road linking the five towns only once, and return to Town *A*? Explain.

b Show one possible route he can follow.

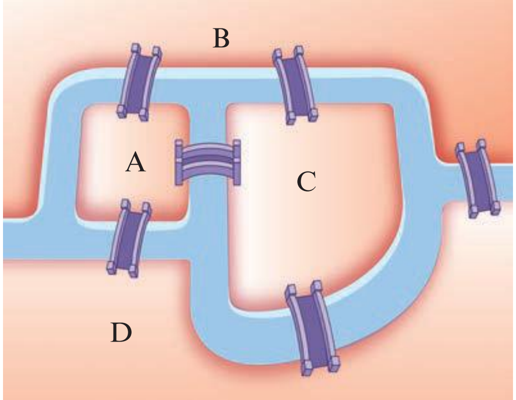
3 A postman has to deliver letters to the houses located on the network of streets shown on the right.



a Is it possible for the postman to start and finish his deliveries at the same point in the network without retracing his steps at some stage? If not, why not?

b It is possible for the postman to start and finish his deliveries at different points in the network without retracing his steps at some stage. Identify one such route.

4 Two islands are connected to the banks of a river by five bridges. See opposite.



a Draw a graph to represent this situation. Label the vertices *A*, *B*, *C* and *D* to represent the river banks and the two islands. Use the edges of the graph to represent the bridges.

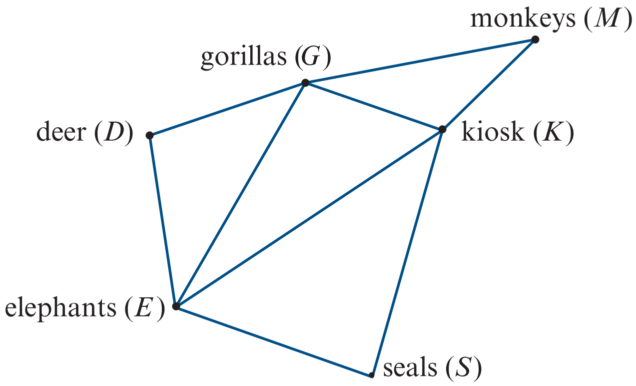
b It is not possible to plan a walking route that passes over each bridge once only. Why not?

c **i** Show where another bridge could be added to make such a walk possible.

ii Draw a graph to represent this situation.

iii Explain why it is now possible to find a walking route that passes over each bridge once only. Mark one such route on your graph.

5 The graph below models the pathways linking five animal enclosures in a zoo to the kiosk and to each other.



a Is it possible for the zoo's street sweeper to follow a route that enables its operator to start and finish at the kiosk without travelling down any one pathway more than once? If so, explain why.

b If so, write down one such route.

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9H Hamiltonian paths and cycles (optional)

Eulerian trails and circuits focus on edges.

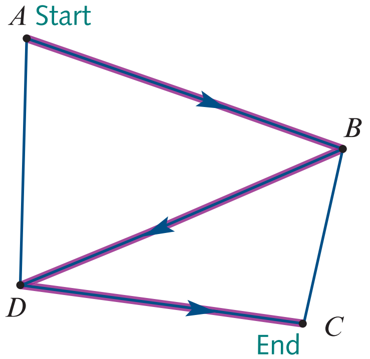
Hamiltonian paths and cycles focus on vertices.

Hamiltonian paths and cycles

- A **hamiltonian path** passes through *every vertex* in a connected graph. It may or may or may not involve all of the edges in the graph.
- A **hamiltonian cycle** is a hamiltonian path that *starts and finishes at the same vertex*.

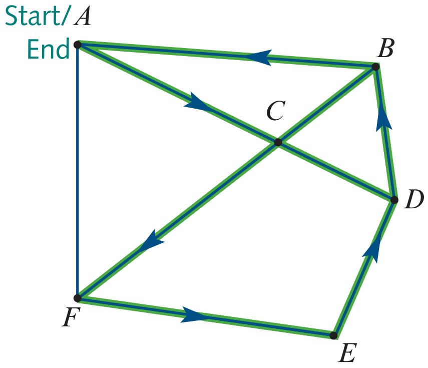
A *hamiltonian path* involves all the vertices but not necessarily all the edges. For example, in the graph opposite, ***A***-***B***-***D***-***C*** is a hamiltonian path. It starts at vertex *A* and ends at vertex *C*. (Follow the arrows.)

Note: A hamiltonian path does not have to involve all edges.



A *hamiltonian cycle* is a hamiltonian path that starts and finishes at the same vertex. For example, in the second graph ***A***-***C***-***F***-***E***-***D***-***B***-***A*** is a hamiltonian cycle. It starts and finishes at vertex *A*. (Follow the arrows.)

Note: A hamiltonian cycle does not have to involve all edges.



Unfortunately, unlike eulerian trails and circuits, there are *no simple rules* for determining whether a network contains a hamiltonian path or cycle. It is just a matter of ‘trial and error’.

▶ Applications of hamiltonian paths and cycles

Hamiltonian paths and cycles have many practical applications. In everyday life, a *hamiltonian path* would apply to situations like the following:

- You plan a trip from Melbourne to Mildura, with visits to Bendigo, Halls Gap, Horsham, Stawell and Ouyen on the way, but do not want to visit any town more than once.

Hamiltonian cycles relate to situations like the following:

- A courier leaves her depot to make a succession of deliveries to a variety of locations before returning to her depot. She does not like to go past each location more than once.
- A tourist plans to visit all of the historic sites in a city without visiting each more than once.
- You are planning a trip from Melbourne to visit Shepparton, Wodonga, Bendigo, Swan Hill, Natimuk, Warrnambool and Geelong before returning to Melbourne. You don’t want to visit any town more than once.

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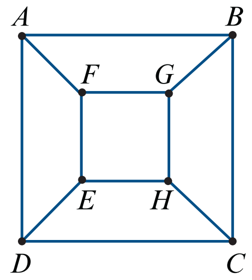
In all these situations, there would be several suitable routes. However other factors, such as time taken or distance travelled may need to be taken into account in order to determine the best route. This is an issue addressed in the next section: weighted graphs and networks.

Exercise 9H

1 List a hamiltonian path for the network shown.

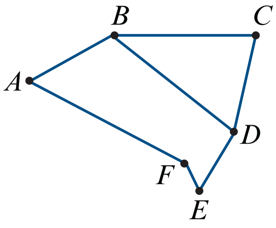
a Starting at A and finishing at D

b Starting at F and finishing at G

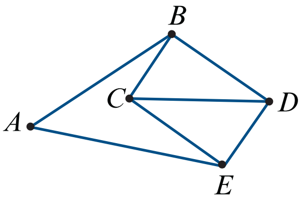


2 Identify a hamiltonian cycle in each of the following graphs (if possible), starting at A each time.

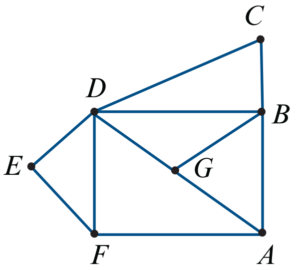
a



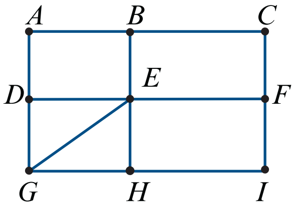
b



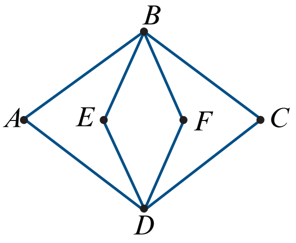
c



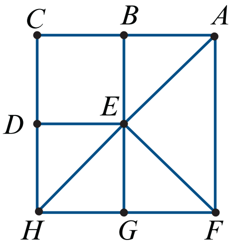
d



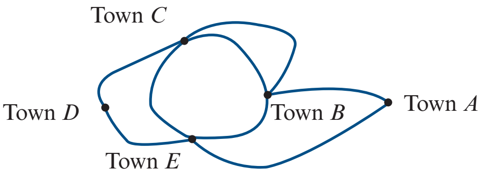
e



f



3 A tourist wants to visit a winery in each of five different towns Apsley (A), Berrigama (B), Cleverland (C), Donsley (D) and Everton (E) in a wine-growing region.



The network of roads connecting the towns is shown opposite.

Can a tourist start a tour that visits each town only once by starting at:

a Cleverland and finishing at Everton? If so, identify one possible route and give its mathematical name.

b Cleverland and finishing at Apsley? If so, identify one possible route and give its mathematical name.

c Everton and finishing at Everton? If so, identify one possible route and give its mathematical name.

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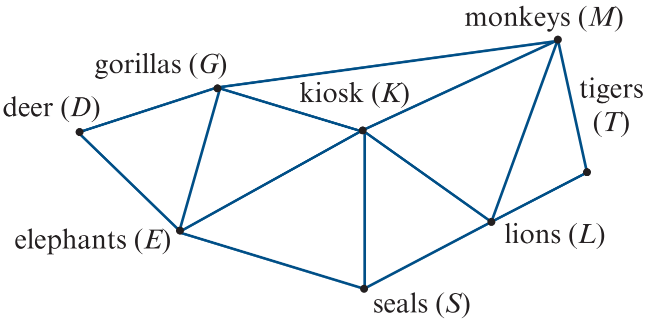
4 The graph opposite models the pathways linking seven animal enclosures in a zoo to the kiosk and to each other.



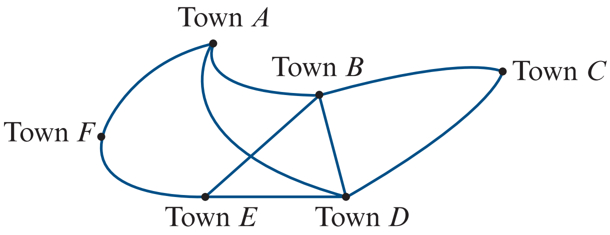
a Is it possible for a visitor to the zoo to start their visit at the kiosk and see all of the animals without visiting any one animal enclosure more than once? If so, identify a possible route and give this route its mathematical name.

b Is it possible for a visitor to the zoo to start their visit at the deer enclosure and finish at the kiosk without visiting the kiosk or any enclosure more than once? If so, identify a possible route and give this route its mathematical name.

9I Weighted graphs, networks and the shortest path problem

Weighted graphs and networks

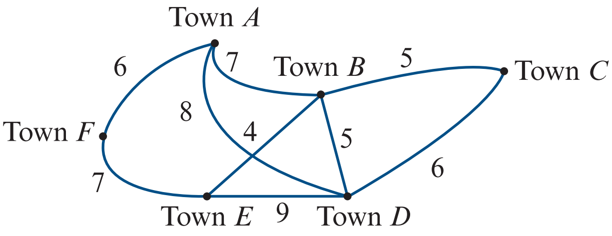
The graph opposite shows how six towns are connected by road.



The towns are represented by the vertices of the graph.

The roads between towns are represented by the edges.

We can give more information about the situation we are representing with the graph by adding numbers to the edges.



When we do this, the resulting graph is called a **weighted graph**.

The weighted graph opposite shows the distances between the towns (in kilometres).

A **network** is a *weighted graph* that represents a real world situation. The weighted graph above could be called a network.

Weighted graphs and networks

- A *weighted graph* is a graph where a number is associated with each edge. These numbers are called weights.
- A *network* is a weighted graph in which the weights are physical quantities, for example, distance, time, cost.

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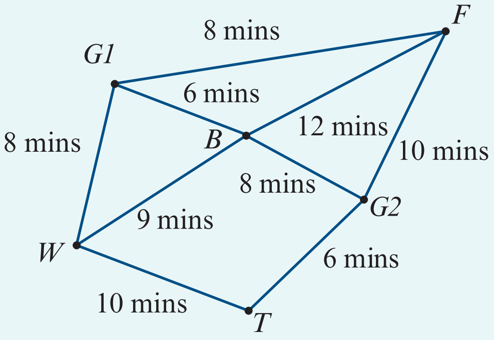
Example 6 Interpreting a network

The network opposite is used to model the tracks in a forest connecting a suspension bridge (B), a waterfall (W), a very old tree (T) and a fern gully (F).

Walkers can enter or leave the forest through either Gate 1 ($G1$) or Gate 2 ($G2$).

The numbers on the edges represent the times (in minutes) taken to walk directly between these places.



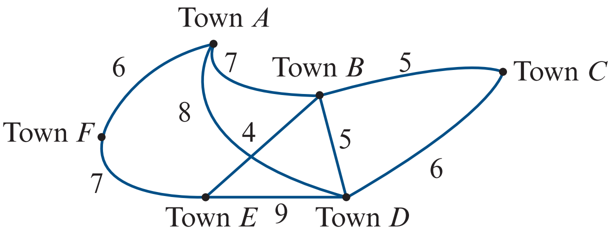
a How long does it take to walk from the bridge directly to the fern gully?

b How long does it take to walk from the old tree to the fern gully via the waterfall and the bridge?

Solution

a Identify the edge that directly links the bridge with the fern gully and read off the time.	The edge is B-F. The time taken is 12 minutes.
b Identify the path that links the old tree to the fern gully visiting the waterfall and the bridge on the way. Add up the times.	The path is T-W-B-F. The time taken is $10 + 9 + 12 = 31$ minutes.

▶ The shortest path problem



Another question we might have when presented with a road network like the one shown is, 'What is the shortest distance between certain towns?'

While this question is easily answered if all of the towns are directly connected by a road, for example, Town A and Town B, the answer is not so obvious if we have to travel through other towns to get there, for example, Town F and Town C.

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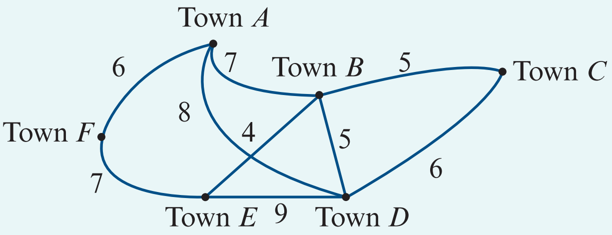
The shortest path problem

Determining the shortest distance or time or the least cost to move around a network is called the **shortest path problem**.

While there are sophisticated techniques for solving the shortest path problem (which you will learn in year 12), the method we use is sometimes called 'by inspection' which involves identifying and comparing the lengths of the likely candidates for the shortest path.

Example 7 Finding the shortest path by inspection

Find the shortest route between Town *C* and Town *F* in the previous network opposite.



Solution

1 Identify all of the likely shortest routes between Town *C* and Town *F* and calculate their lengths.

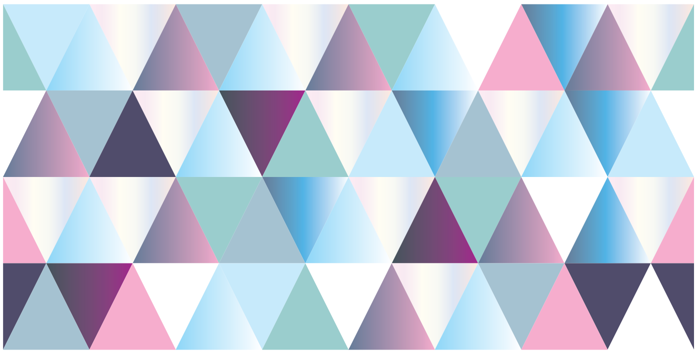
C-D-E-F: The distance is 6 + 9 + 7 = 22 km.
C-B-E-F: The distance is 5 + 4 + 7 = 16 km.
C-B-A-F: The distance is 5 + 7 + 6 = 18 km.

Note: In theory, when using the 'by inspection' method to solve this problem we need to list all possible routes between Town *C* and Town *F* and determine their lengths. However, we can save time by eliminating any route that passes through any town more than once or any road more than once. We can also eliminate any route that 'takes the long way around' rather than using the direct route, for example, when travelling from Town *B* to Town *D* we can ignore the route that goes via Town *A* because it is longer.

2 Compare the different path lengths to identify the shortest path and write your answer.

The shortest path is C-B-E-F.

Note: You should compare the lengths of all likely paths because there can be more than one shortest path in a network.



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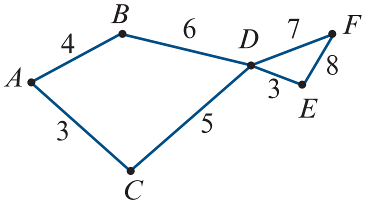
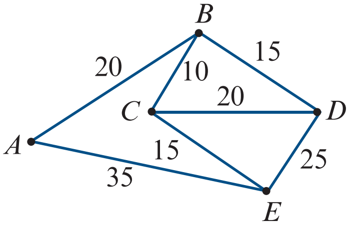
Exercise 9I

Example 6, 7

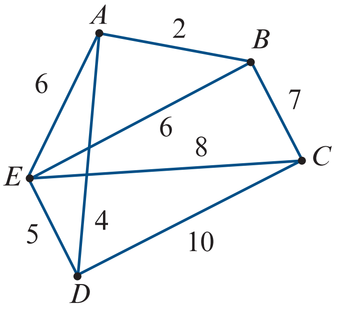
1 Find the shortest path from vertex A to vertex E in this network. The numbers represent time in hours.



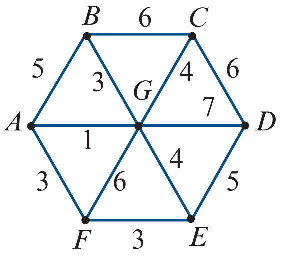
2 Find the shortest path from vertex A to vertex D in this network. The numbers represent lengths in metres.



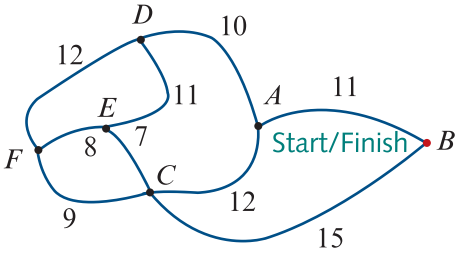
3 Find the shortest path from vertex B to vertex D in this network. The numbers represent cost in dollars.



4 Find the shortest path from vertex B to vertex F in this network. The numbers represent time in minutes.



5 The graph below shows a mountain bike rally course. Competitors must pass through each of the checkpoints, A, B, C, D, E and F. The average times (in minutes) taken to ride between the checkpoints are shown on the edges of the graph.



Competitors must start and finish at checkpoint A but can pass through the other checkpoints in any order they wish.
Which route would have the shortest average completion time?

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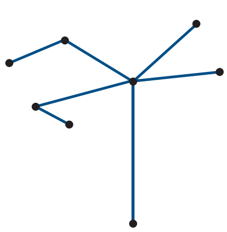
9J Minimum spanning trees

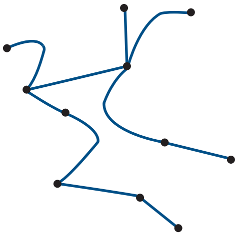
Tree

A **tree** is a connected graph that contains no cycles, multiple edges or loops. A tree may be part of a larger graph.

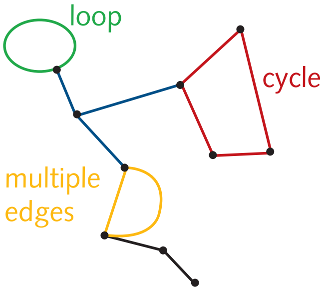
For example, Graphs 1 and 2 below are examples of trees. Graph 3 is *not* a tree.



Graph 1: a tree



Graph 2: a tree



Graph 3: not a tree

Graphs 1 and 2 are trees: they are connected and have no cycles, multiple edges or loops.

Graph 3 is *not* a tree: it has several cycles (loops and multiple edges count as cycles).

For trees, there is a relationship between the number of vertices and the number of edges.

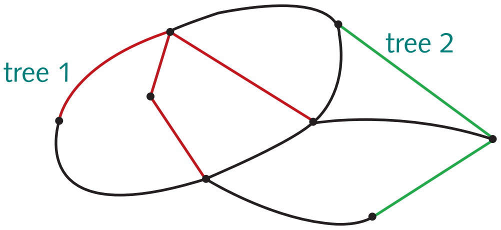
- Graph 1, a tree, has 8 vertices and 7 edges.
- Graph 2, a tree, has 11 vertices and 10 edges.

An inspection of other trees would show that, in general, the number of edges in a tree is one less than the number of vertices.

Rule connecting the number of edges of a graph to the number of vertices

A *tree* with n vertices has $n - 1$ edges.

Another important fact about trees is that every connected graph contains one or more trees. For example, the connected graph opposite contains multiple trees. Two of these are shown superimposed on the graph in colour.



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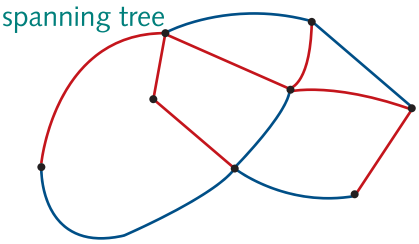
Spanning trees

Spanning tree

A **spanning tree** is a tree that connects *all* of the vertices in a connected graph.

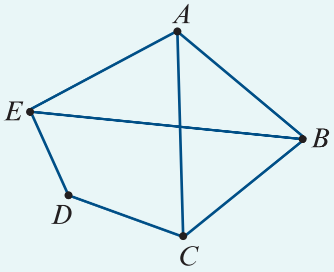
An example of a spanning tree is shown on the graph opposite. There are several other possibilities.

Note that the spanning tree, like the graph, has 8 vertices. From this it can be concluded that the spanning tree will have 8 − 1 = 7 edges, as we can observe.



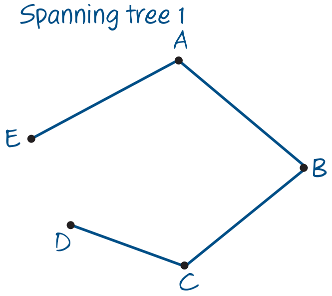
Example 8 Finding a spanning tree in a network

Find two spanning trees for the graph shown opposite.



Solution

1 The graph has five vertices and seven edges.
A spanning tree will have five (n) vertices and four ($n - 1$) edges.



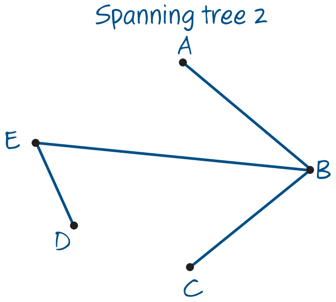
2 To form a spanning tree, remove any *three* edges, provided that:

- all the vertices remain connected
- there are no multiple edges or loops.

Spanning tree 1 is formed by removing edges *EB, ED* and *CA*.

Spanning tree 2 is formed by removing edges *EA, AC* and *CD*.

Note: Several other possibilities exist.



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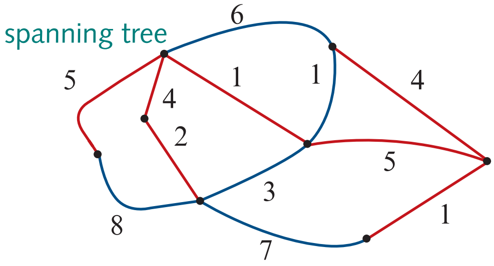
Minimum spanning trees

For weighted graphs or networks, it is possible to determine the 'length' of each spanning tree by adding up the weights of the edges in the tree.

For the spanning tree shown below:

$$\begin{aligned}\text{Length} &= 5 + 4 + 2 + 1 + 5 + 4 + 1\\ &= 22 \text{ units}\end{aligned}$$



Minimum spanning tree

A **minimum spanning tree** is the spanning tree of *minimum* length (may be minimum distance, minimum time, minimum cost, etc.). There may be more than one minimum spanning tree in a weighted graph.

Minimum spanning trees have many real world applications such as planning the layout of a computer network or a water supply system for a new housing estate. In these situations, we might want to minimise the amount of cable or water pipe needed for the job. Alternatively, we might want to minimise the time needed to complete the job or its cost.

Prim's algorithm

Prim's algorithm for finding a minimum spanning tree

Prim's algorithm is a set of rules to determine a minimum spanning tree for a graph.

- Choose a starting vertex (any will do). Inspect the edges starting from this vertex and choose the one with the lowest weight. (If two edges have the same weight, the choice can be arbitrary.) You now have two vertices and one edge.
- Next, inspect the edges starting from the vertices. Choose the edge with the lowest weight. (If two edges have the same weight, the choice can be arbitrary.) You now have three vertices and two edges.
- Repeat the process until all the vertices are connected, and then stop. The result is a minimum spanning tree.

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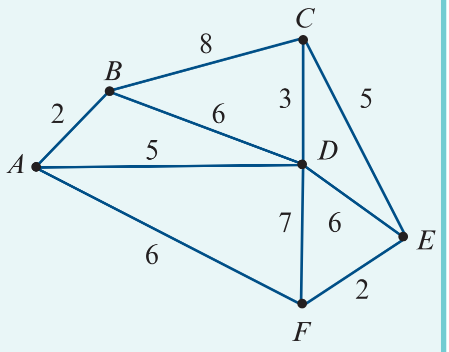
Example 9 Applying Prim's algorithm

Apply Prim's algorithm to obtain a minimum spanning tree for the network shown, and calculate its length.

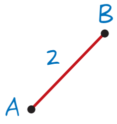


Solution

1 Start at A:

AB is the lowest-weighted edge (2).

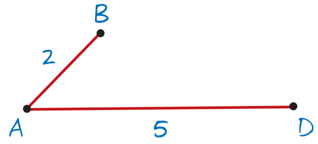
Draw it in.



2 From A or B:

AD is the lowest-weighted edge (5).

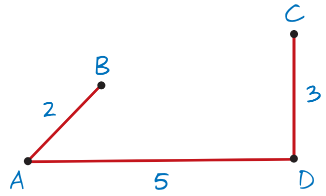
Draw it in.



3 From A, B or D:

DC is the lowest-weighted edge (3).

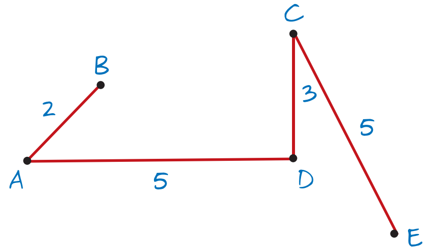
Draw it in.



4 From A, B, C or D:

CE is the lowest-weighted edge (5).

Draw it in.



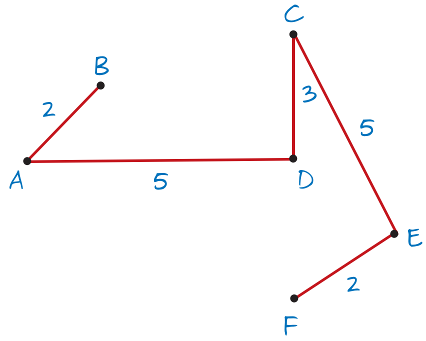
5 From A, B, C, D or E:

EF is the lowest-weighted edge (2).

Draw it in.

All vertices have now been joined.

The minimum spanning tree is determined.



6 Find the length of the minimum spanning tree by adding the weights of the edges.

Minimum spanning tree

Length = 2 + 5 + 3 + 5 + 2 = 17 units

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Exercise 9J

Skillsheet

1 **a** How many edges are there in a tree with 15 vertices?

b How many vertices are there in a tree with 5 edges?

c Draw two different trees with four vertices.

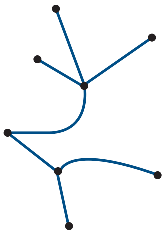
d Draw three different trees with five vertices.

2 A connected graph has eight vertices and ten edges. Its spanning tree has ____ vertices and ____ edges.

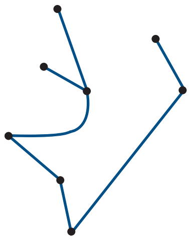
Example 8

3 Which of the following graphs are trees?

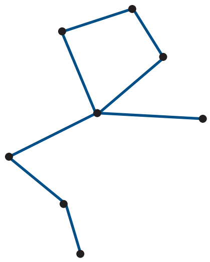
A



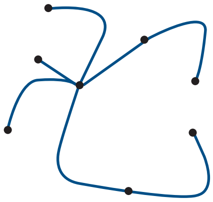
B



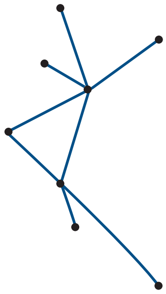
C



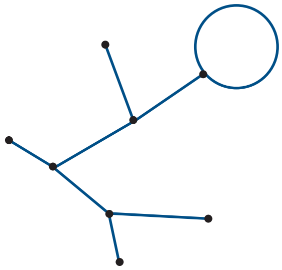
D



E

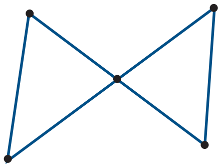


F

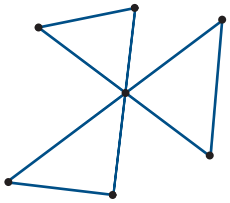


4 For each of the following graphs, draw three different spanning trees.

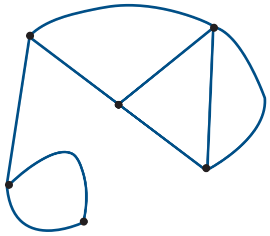
a



b



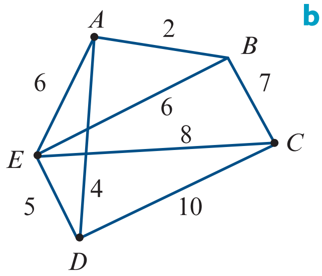
c



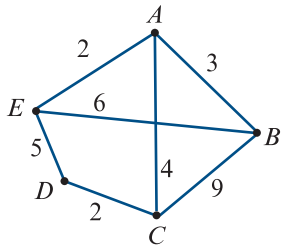
Example 9

5 For each of the following connected graphs, use Prim's algorithm to determine the minimum spanning tree and its length.

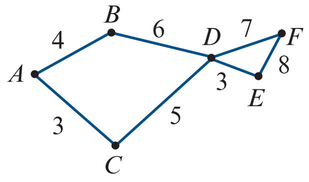
a



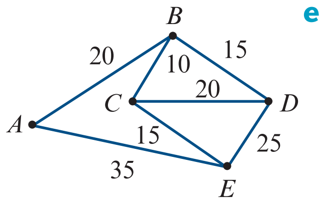
b



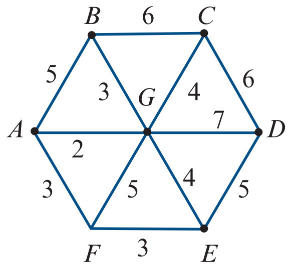
c



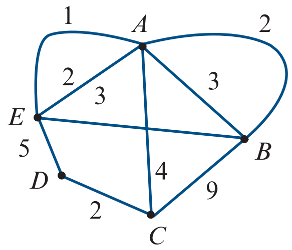
d



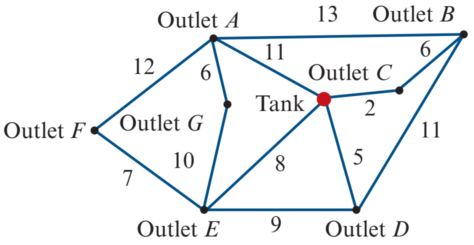
e



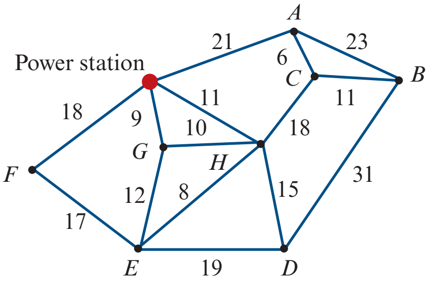
f

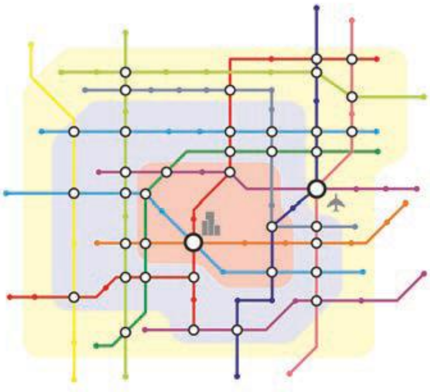


6 Water is to be piped from a water tank to seven outlets on a property. The distances (in metres) of the outlets from the tank and from each other are shown in the network opposite. Starting at the tank, determine the minimum length of pipe needed.



7 Power is to be connected by cable from a power station to eight substations (*A* to *H*). The distances (in kilometres) of the substations from the power station and from each other are shown in the network opposite. Determine the minimum length of cable needed.





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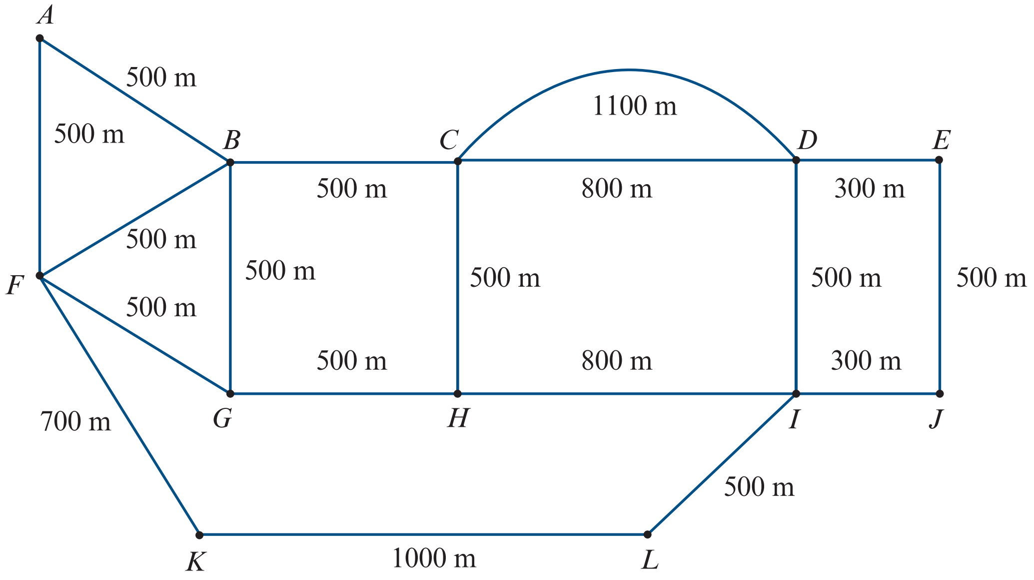
9K Applications, modelling and problem solving

Exercise 9K

1 A network below shows the road layout in a new housing estate. The vertices represent the road intersections and the weighted edges represent roads and their lengths.



a Cars can enter the estate through gates located at either *A* or *J*. What is the shortest distance a car can travel if it enters the estate at gate *A* and leaves at gate *J*?

b The post office is located at intersection *G*. The postman must travel along each road in the estate at least once.

  i Can the postman start and finish his delivery round (which includes every road in the housing estate) at intersection *G*, without travelling along the same road more than once? If not, why not?

  ii Can the postman start his delivery round at intersection *G* and finish at some other intersection in this network of roads, without travelling along the same road more than once? If so, where would they end the delivery round?

  iii If the postman must start and finish his delivery round at the post office at *G*, what is the shortest distance he will cover and still travel along each road at least once?

c Broadband is to be provided to the estate by connecting cables from the ‘exchange’ located at intersection *L* to distribution nodes located at each of the intersections in the estate. What is the shortest length of cable needed to complete this task?

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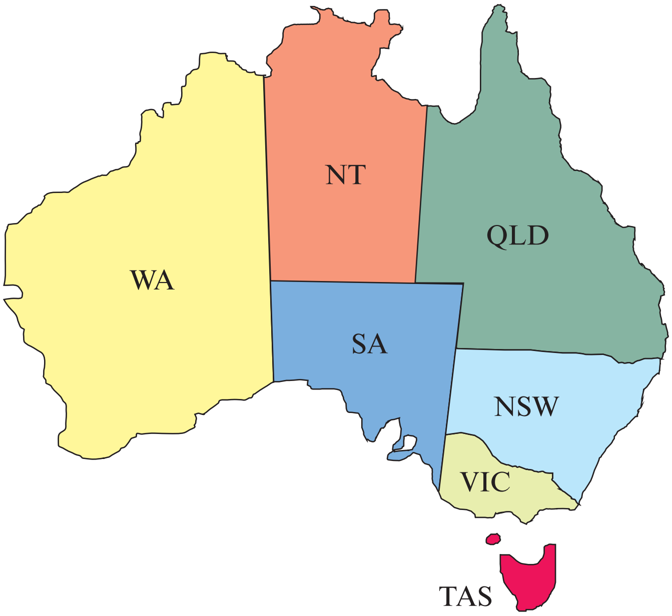
2 Some overseas visitors to Australia wanted to travel through the Northern Territory and each of the states shown on the map. They tried to plan their route so that they crossed each border between neighbouring mainland States and the Northern Territory exactly once. i.e. the WA–NT border, the WA–SA border and so on. The only other condition was that they wanted to visit Tasmania between visiting South Australia and New South Wales, or vice versa.



a Draw a graph to show whether such a journey is possible.

b In which states or territory can the visitors start and finish their journey and in what sequence could they visit the remaining destinations?

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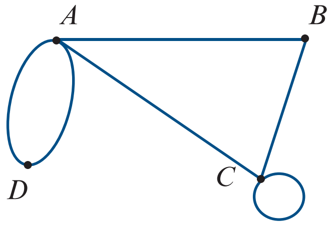
Key ideas and chapter summary

Graph or network

A **graph** is a diagram that consists of a set of points called **vertices** and a set of lines called **edges**. Each edge joins two vertices.



Vertices and edges

In the graph above, A, B, C, and D are **vertices** and the lines AB, AD, AC, and BC are **edges**.

Degree of a vertex

The **degree of vertex** A, written deg(A), is the *number of edges attached to the vertex*. The degree of a loop is two.

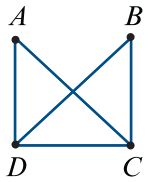
For example, in the graph above: deg(B) = 2 and deg(C) = 4.

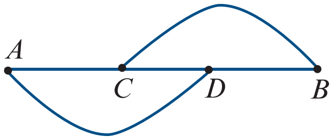
Isomorphic graphs

Two graphs are said to be **isomorphic** (equivalent) if:

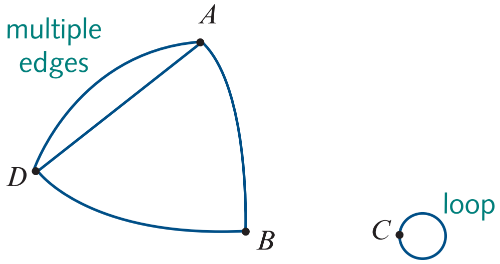
- they both have the same number of edges and vertices
- corresponding vertices have the same degree and the edges connect the same vertices.

For example, the two graphs below are isomorphic or equivalent.





Multiple edges and loops



The graph above is said to have **multiple edges**, as there are two edges joining A and D.

C has one edge, which links C to itself. This edge is called a **loop**.

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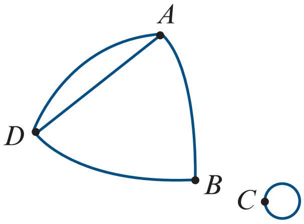
Adjacency matrix

An **adjacency matrix** is a square matrix that uses a zero or an integer to record the number of edges connecting each pair of vertices in the graph.

An example of a graph and its adjacency matrix is shown below.



$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 2 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

Connected graph and bridges

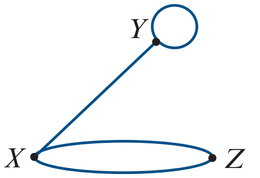
A **graph is connected** if there is a path between each pair of vertices.

A **bridge** is a single edge in a connected graph that if removed leaves the graph disconnected. A graph can have more than one bridge.

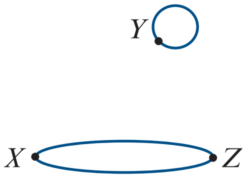
Graph 1 is a connected graph.

Graph 2 is a connected graph.

Edge XY in Graph 1 is a bridge because removing it leaves Graph 1 disconnected.



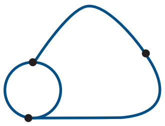
Graph 1



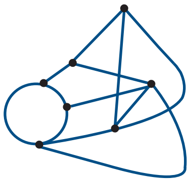
Graph 2

Planar graph

A graph that can be drawn in such a way that no two edges intersect, except at the vertices, is called a **planar graph**.



planar graph



non-planar graph

Euler's formula

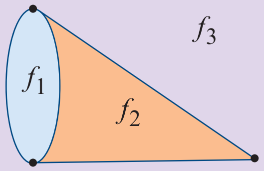
For any **connected planar** graph, **euler's formula** states:

$v - e + f = 2$

v = the number of vertices

e = the number of edges

f = the number of faces



$v = 3,\ e = 4,\ f = 3$

$v - e + f = 3 - 4 + 3 = 2$

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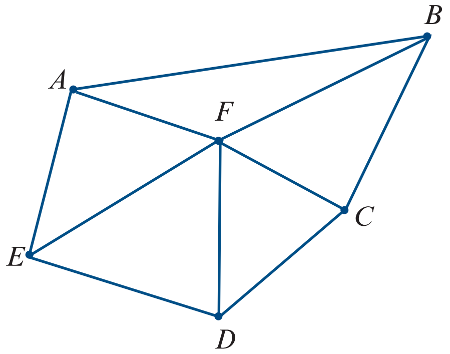
Walk, trail, path, circuit and cycles



A **walk** is a sequence of edges, linking successive vertices, in a graph.
In the graph above *E-A-F-D-C-F-E-A* is a walk.

A **trail** is a walk with no repeated edges.
In the graph above *A-F-D-E-F-C* is a trail.

A **circuit** is a walk that has no repeated edges that starts and ends at the same vertex.
In the graph above *A-F-D-E-F-B-A* is a circuit.

A **path** is a walk with no repeated vertices.
In the graph above *F-A-B-C-D* is a path.

A **cycle** is a walk with no repeated vertices that starts and ends at the same vertex.
In the graph above *B-F-C-B* is a cycle.

Traversable graph

A **traversable graph** has at least one trail that includes every edge.

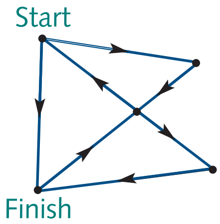
Eulerian trail

A trail that includes every edge just once (but does not start and finish at the same vertex) is called an **eulerian trail**.

Condition for an eulerian trail

To have an eulerian trail (but not an eulerian circuit), a graph must be connected and have exactly two odd vertices, with the remaining vertices being even. The eulerian trail will start at one of the odd vertices and finish at the other.

For example, the graph opposite is connected. It has two odd vertices and three even vertices. It has an eulerian trail that starts at one of the odd vertices and finishes at the other.



Eulerian circuit

An eulerian trail that starts and finishes at the same vertex is called an **eulerian circuit**.

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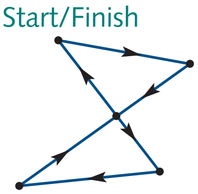
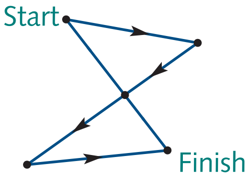
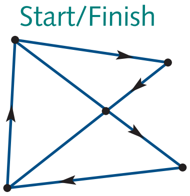
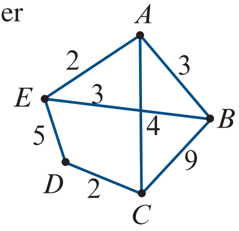
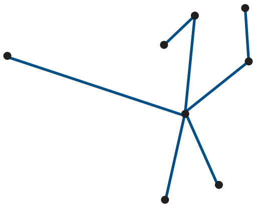
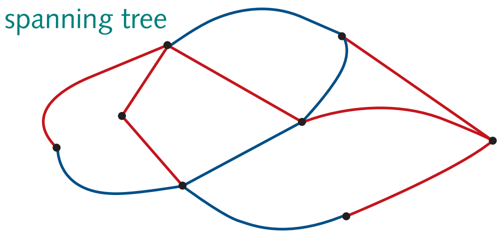
Condition for an eulerian circuit	To have an **eulerian circuit**, a graph must be connected and all vertices must be even. In the network shown, all vertices are even. It has an eulerian circuit. The circuit starts and finishes at the same vertex. 
Hamiltonian path	A **hamiltonian path** is a path through a graph that passes through each vertex exactly once, but does not necessarily start and finish at the same vertex. 
Hamiltonian cycle	A **hamiltonian cycle** is a hamiltonian path that starts and finishes at the same vertex. 
Weighted graphs and networks	A **weighted graph** is one where a number is associated with each edge. These numbers are called weights. When the weights are physical quantities, for example, distance, time, cost, a weighted graph is often called a **network**. 
The shortest path problem	Determining the shortest distance or time or the least cost to move around a network is called the **shortest path problem**.
Tree	A **tree** is a connected graph that contains no cycles, multiple edges or loops. A tree with n vertices has $n - 1$ edges. The tree (right) has 8 vertices and 7 edges. 
Spanning tree	A **spanning tree** is a graph that contains all the vertices of a connected graph, without multiple edges, cycles or loops. 

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Minimum spanning tree	A **minimum spanning tree** is a spanning tree for which the sum of the weights of the edges is as small as possible.
Prim's algorithm	**Prim's algorithm** is a systematic method for determining a minimum spanning tree in a connected graph.

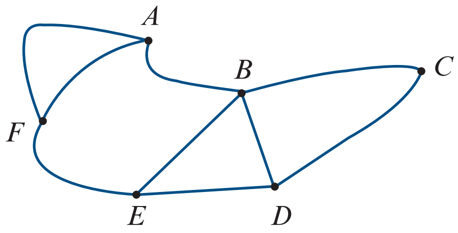
Skills check

Having completed this chapter you should be able to:

- recognise and determine the number of vertices, edges and faces of a graph
- recognise isomorphic graphs, connected graphs, planar graphs, weighted graphs and networks, trees and spanning trees
- construct and interpret an adjacency matrix
- use euler's formula for connected planar graphs
- know and apply the definitions of walks, trails, paths, circuits and cycles
- know and apply the definitions of eulerian trails and circuits
- know the conditions for a graph to have an eulerian trail or an eulerian circuit
- locate an eulerian trail or circuit in a graph
- know and apply the definitions of hamiltonian paths and cycles
- locate a hamiltonian path or cycle in a graph
- determine the shortest path in a weighted graph or a network
- know and apply the definition of a spanning tree
- use Prim's algorithm to determine a minimum spanning tree and its length.

Multiple-choice questions

The following graph relates to Questions 1 to 4



1 The number of vertices in the graph above is:

A 3 **B** 5 **C** 6 **D** 7 **E** 9

2 The number of edges in the graph above is:

A 3 **B** 5 **C** 6 **D** 7 **E** 9

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3 The degree of vertex *B* in the graph above is:

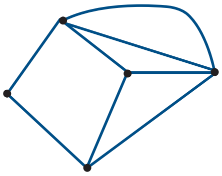
A 1 **B** 2 **C** 3 **D** 4 **E** 5

4 The number of even vertices in the graph above is:

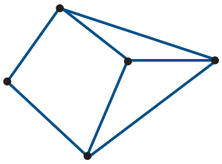
A 1 **B** 2 **C** 3 **D** 4 **E** 5

5 For which graph is the sum of the degrees of the vertices equal to 14?

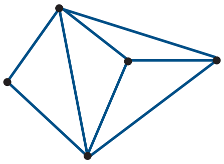
A



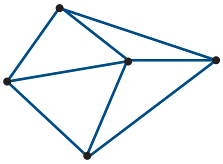
B



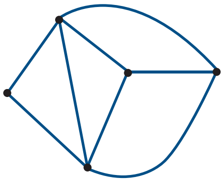
C



D

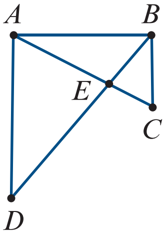


E

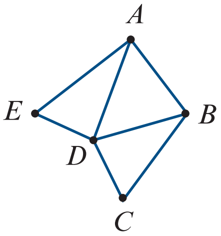


6 The graph that matches the matrix $\begin{array}{c|cccc} & A & B & C & D \\ \hline A & 0 & 1 & 0 & 1 \\ B & 1 & 0 & 1 & 0 \\ C & 0 & 1 & 0 & 1 \\ D & 1 & 0 & 1 & 0 \end{array}$ is:

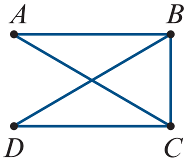
A



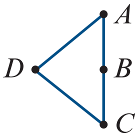
B



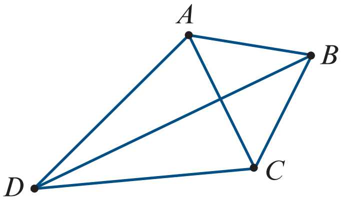
C



D



E



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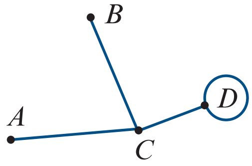
7 The adjacency matrix that matches the graph shown is:



A $\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$

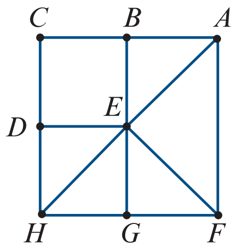
B $\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$

C $\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix} \end{array}$

D $\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix} \end{array}$

E $\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix} \end{array}$

The graph below is to be used when answering Questions 8 to 11



8 The sequence of vertices *C-B-E-A-E-G* represents:

A a walk only

B a trail

C a path

D a circuit

E a cycle

9 The sequence of vertices *D-E-H-G-E-A-B-C-D* represents:

A a walk only

B a trail but not a circuit

C a path but not a cycle

D a circuit

E a cycle

10 The sequence of vertices *C-B-E-A-F-E-G-H* represents:

A a walk only

B a trail but not a circuit

C a path but not a cycle

D a circuit

E a cycle

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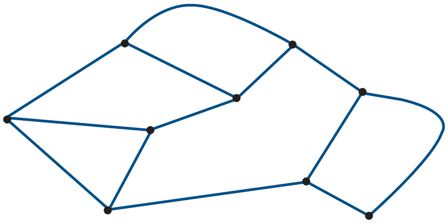
11 The sequence of vertices *D-E-A-F-G-H-D* represents:

A a walk only **B** a trail but not a circuit

C a path but not a cycle **D** a cycle

E none of these

The graph below is to be used when answering Questions 12 and 13



12 The sum of the degrees of the vertices is:

A 22 **B** 23 **C** 24 **D** 25 **E** 26

13 For this graph:

A $v = 9, e = 13, f = 5$ **B** $v = 9, e = 13, f = 6$ **C** $v = 10, e = 13, f = 5$

D $v = 9, e = 14, f = 6$ **E** $v = 11, e = 13, f = 5$

14 Euler's formula for a planar graph is:

A $v - e = f + 2$ **B** $v - e + f = 2$ **C** $v + e + f = 2$

D $e - v + f = 2$ **E** $v - e = f - 2$

15 A connected graph with 10 vertices divides the plane into five faces. The number of edges connecting the vertices in this graph will be:

A 5 **B** 7 **C** 10

D 13 **E** 15

16 For a connected graph to have an eulerian trail but not an eulerian circuit:

A all vertices must be odd **B** all vertices must be even

C there must be exactly two odd vertices and the rest even

D there must be exactly two even vertices and the rest odd

E an odd vertex must be followed by an even vertex

17 For a connected graph to have an eulerian circuit:

A all vertices must be odd **B** all vertices must be even

C there must be exactly two odd vertices and the rest even

D there must be exactly two even vertices and the rest odd

E an odd vertex must be followed by an even vertex

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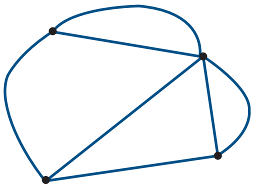
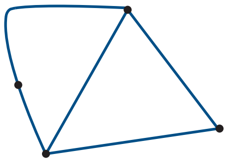
The graphs below are to be used when answering Questions 18 and 19

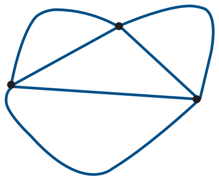
A



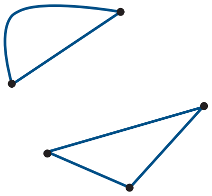
B



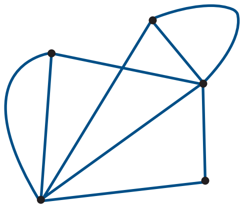
C



D



E

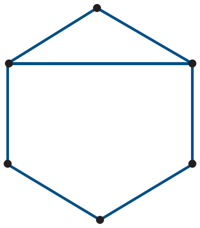


18 Which one of the graphs above has an eulerian trail but not an eulerian circuit?

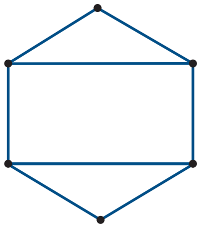
19 Which one of the graphs above has an eulerian circuit?

20 Which one of the following graphs has an eulerian circuit?

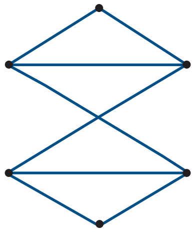
A



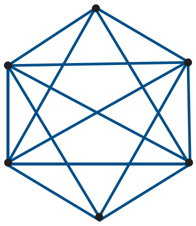
B



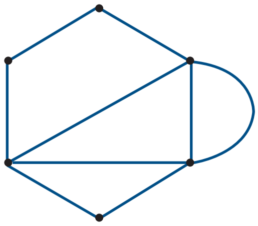
C



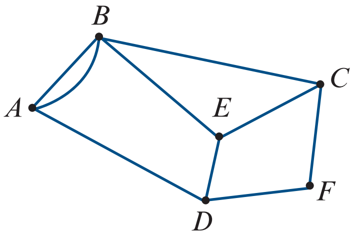
D



E



21 For the graph shown, which additional edge could be added to the network so that the graph formed would contain an eulerian trail?

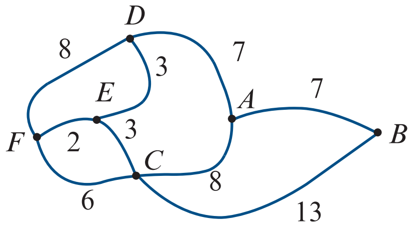


A *AF* **B** *AD*

C *AB* **D** *CF*

E *BF*

22 The length of the shortest path from *F* to *B* in the network shown is:



A 17 **B** 18

C 19 **D** 20

E 21

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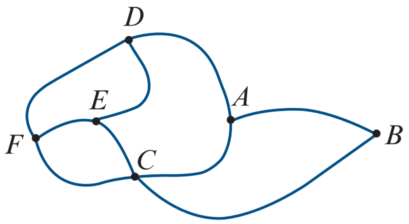
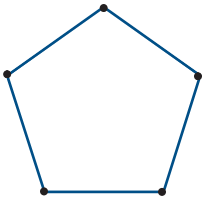
23 Which one of the following paths is a hamiltonian cycle for the graph shown?



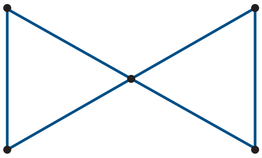
A *F-E-D-F*
B *F-E-D-A-B-C-E-F*
C *F-E-D-A-B-C-F*
D *F-C-A-B-D-E-F*
E *F-D-E-C-A-B-C-F*

24 Of the following, which graph has both an eulerian circuit and hamiltonian cycle?

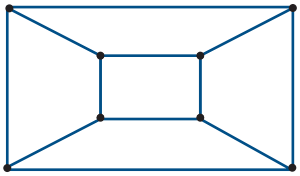
A



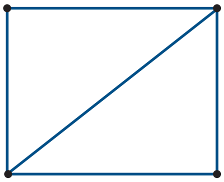
B



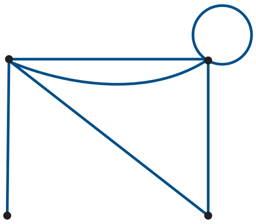
C



D

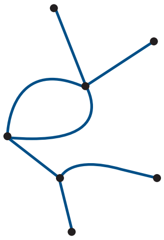


E

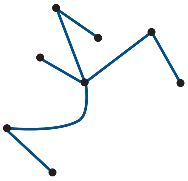


25 Which one of the following graphs is a tree?

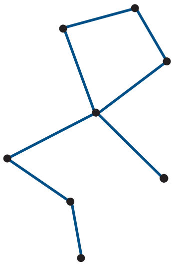
A



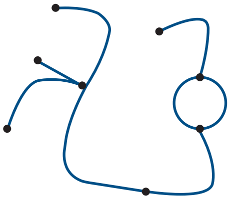
B



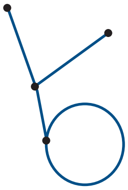
C



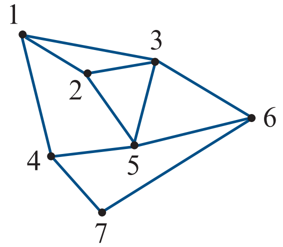
D



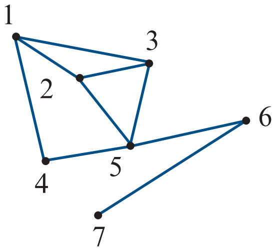
E



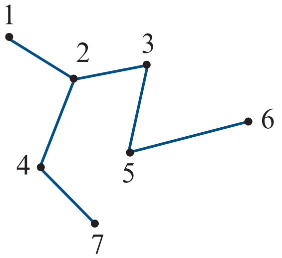
26 Which one of the following graphs is a spanning tree for the graph shown?



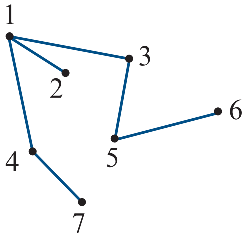
A



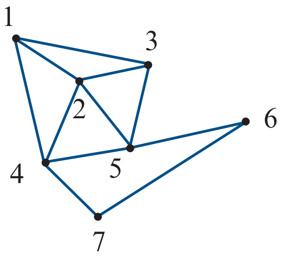
B



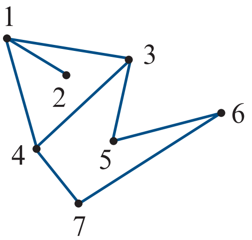
C



D



E



27 A park ranger wants to check all of the walking tracks in a national park, starting at and returning to the park office. She would like to do it without having to travel over the same walking track more than once. If possible, the route she should follow is:

A an eulerian trail but not a circuit
B an eulerian circuit
C a hamiltonian path but not a cycle
D a hamiltonian cycle
E a minimum spanning tree

28 A park ranger wants to check all of the campsites in a national park, starting at and returning to the park office. The campsites are all interconnected with walking tracks. She would like to check the campsites without having to visit each campsite more than once. If possible, the route she should follow is:

A an eulerian trail but not a circuit
B an eulerian circuit
C a hamiltonian path but not a cycle
D a hamiltonian cycle
E a minimum spanning tree

29 The park authorities plan to pipe water to each of the campsites from a spring located in the park. They want to use the least amount of water pipe possible. If possible, the water pipes should follow:

A an eulerian trail but not a circuit
B an eulerian circuit
C a hamiltonian path but not a cycle
D a hamiltonian cycle
E a minimum spanning tree

30 Each week, a garbage collection route starts at the tip and collects the rubbish left at each of the campsites before returning to the tip to dump the rubbish collected. The plan is to visit each campsite only once. If possible, the garbage collection route should follow:

A an eulerian trail but not a circuit
B an eulerian circuit
C a hamiltonian path but not a cycle
D a hamiltonian cycle
E a minimum spanning tree

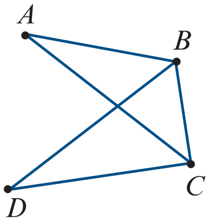
Short-answer questions

1 Draw a connected graph with:

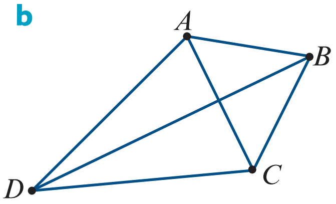
a four vertices, four edges and two faces
b four vertices, five edges and three faces
c five vertices, eight edges and five faces
d four vertices, four edges, two faces and two bridges

2 Redraw each of the following graphs in a planar form.

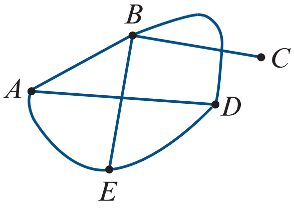
a



b

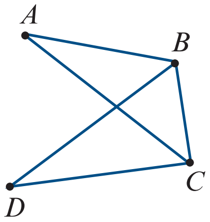


c



3 For the network shown, write down:

a the degree of vertex C
b the numbers of odd and even vertices
c the route followed by an eulerian trail starting at vertex B.



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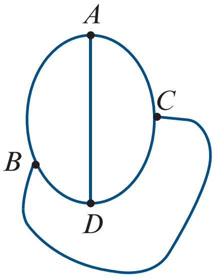
4 Construct an adjacency matrix for the graph opposite.

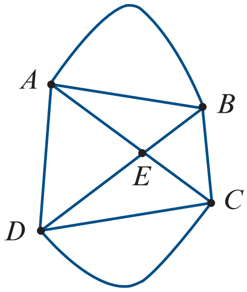


5 For the graph shown, write down:

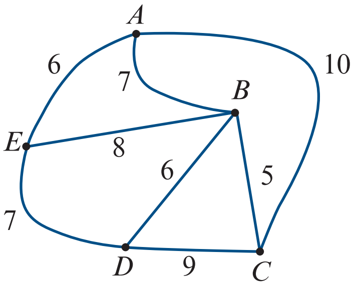
a the degree of vertex C

b the number of odd and even vertices

c the route followed by an eulerian circuit starting at vertex A.



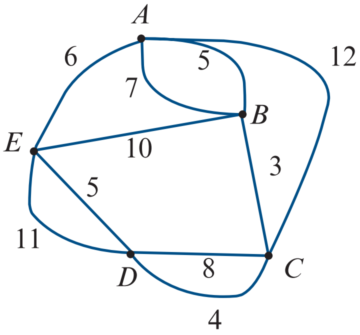
6 For the weighted graph shown, determine the length of the minimum spanning tree.



7 In the network below, the numbers on the edges represent distances in kilometres.
Determine the length of:

a the shortest path between vertex A and vertex D

b the length of the minimum spanning tree.



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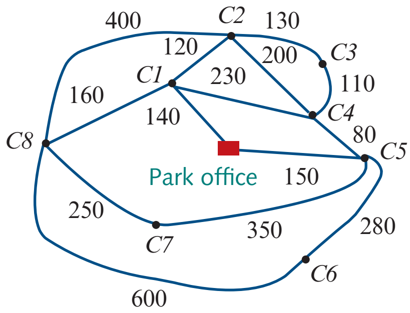
Extended-response questions

1 The diagram opposite shows the network of walking tracks in a small national park. These tracks connect the campsites to each other and to the park office. The lengths of the tracks (in metres) are also shown.



a The network of tracks is planar. Explain what this means.

b Verify euler’s formula for this network.

c A ranger at campsite *C8* plans to visit campsites *C1, C2, C3, C4* and *C5* on her way back to the park office. What is the shortest distance she will have to travel?

d How many even and how many odd vertices are there in the network?

e Each day, the ranger on duty has to inspect each of the tracks to make sure that they are all passable.

  i Is it possible for her to do this starting and finishing at the park office? Explain why.

  ii Identify one route that she could take.

f Following a track inspection after wet weather, the Head Ranger decides that it is necessary to put gravel on some walking tracks to make them weatherproof. What is the minimum length of track that will need to be gravelled to ensure that all campsites and the park office are accessible along a gravelled track?

g A ranger wants to inspect each of the campsites but not pass through any campsite more than once on his inspection tour. He wants to start and finish his inspection tour at the park office.

  i What is the technical name for the route he wants to take?

  ii With the present layout of tracks, he cannot inspect all the tracks without passing through at least one campsite twice. Suggest where an additional track could be added to solve this problem.

  iii With this new track, write down a route he could follow.

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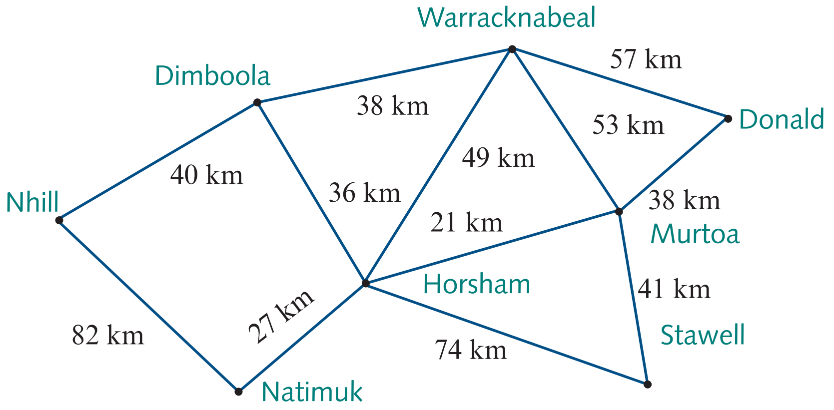
2 The network below shows the major roads connecting eight towns in Victoria and the distances between them in kilometres.



a Find the shortest distance between Nhill and Donald using these roads.

b Verify euler's formula for this network.

c An engineer based in Horsham needs to inspect each road in the network without travelling along any of the roads more than once. He would also like to finish his inspection at Horsham.

- **i** Explain why this cannot be done.
- **ii** The engineer can inspect each road in the network without travelling along any of the roads more than once if he starts at Horsham but finishes at a different town. Which town is that? How far will he have to travel in total?
- **iii** Identify one route, starting at Horsham, that the engineer can take to complete his inspection without travelling along any of the roads more than once.

d A telecommunications company wants to connect all of the towns to a central computer system located in Horsham. What is the minimum length of cable that they will need to complete this task?

e The engineer can complete his inspection in Horsham by only travelling along one of the roads twice. Which road is that?

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10

Shape and measurement

- What is Pythagoras' theorem?
- How do we use Pythagoras' theorem?
- How do we find the perimeter of a shape?
- How do we find the area of a shape?
- What is a composite shape?
- How do we find the volume of a shape?
- How do we find the surface area of a shape?
- What does it mean when we say that two figures are similar?
- What are the tests for similarity for triangles?
- How do we know whether two solids are similar?

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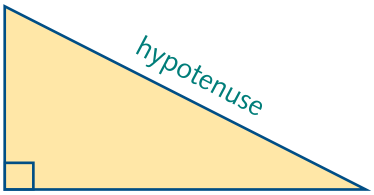
Introduction

The geometry chapter covers perimeter and area of 2D shapes, and surface area and volume of 3D solids. It also covers similarity within 2D shapes and 3D solids.

10A Pythagoras' theorem

Pythagoras' theorem is a relationship connecting the side lengths of a right-angled triangle. In a right-angled triangle, the side opposite the **right angle** is called the **hypotenuse**. The hypotenuse is always the longest side of a right-angled triangle.

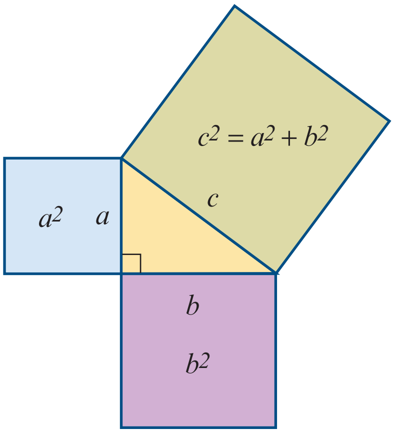


▶ Pythagoras' theorem

Pythagoras' theorem states that, for any right-angled triangle, the sum of the areas of the squares of the two shorter sides (*a* and *b*) equals the area of the square of the hypotenuse (*c*).

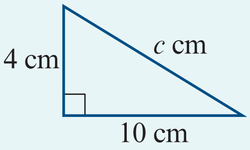
$$c^2 = a^2 + b^2$$

Pythagoras' theorem can be used to find the length of the hypotenuse in a right-angled triangle.



Example 1 Using Pythagoras' theorem to calculate the length of the hypotenuse

Calculate the length of the hypotenuse in the triangle opposite, correct to two decimal places.



Solution

1 Write Pythagoras' theorem.	$c^2 = a^2 + b^2$
2 Substitute known values.	$c^2 = 10^2 + 4^2$
3 Take the square root of both sides, then evaluate.	$c = \sqrt{10^2 + 4^2}$ $= 10.770\ldots$
4 Write your answer correct to two decimal places, with correct units.	The length of the hypotenuse is 10.77 cm, correct to two decimal places.

Hint: To ensure that you get a decimal answer, set your calculator to approximate or decimal mode. (See the Appendix.)

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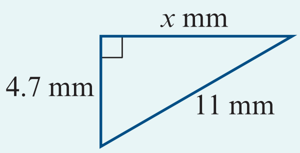
Pythagoras' theorem can also be rearranged to find sides other than the hypotenuse.

Example 2 Using Pythagoras' theorem to calculate the length of an unknown side in a right-angled triangle

Calculate the length of the unknown side, x, in the triangle opposite, correct to one decimal place.



Solution

1 Write Pythagoras' theorem.

$$a^2 + b^2 = c^2$$

2 Substitute known values and the given variable.

$$x^2 + 4.7^2 = 11^2$$

3 Rearrange the formula to make x the subject, then evaluate.

$$x = \sqrt{11^2 - 4.7^2}$$
$$= 9.945\ldots$$

4 Write your answer correct to one decimal place, with correct units.

The length of x is 9.9 mm, correct to one decimal place.

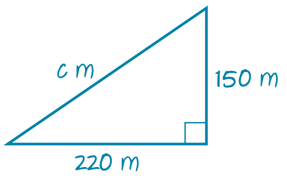
Pythagoras' theorem can be used to solve many practical problems.

Example 3 Using Pythagoras' theorem to solve a practical problem

A helicopter hovers at a height of 150 m above the ground and is a horizontal distance of 220 m from a landing pad. Find the direct distance of the helicopter from the landing pad, correct to two decimal places.

Solution

1 Draw a diagram to show which distance is to be found.



2 Write Pythagoras' theorem.

$$c^2 = a^2 + b^2$$

3 Substitute known values.

$$c^2 = 150^2 + 220^2$$

4 Take the square root of both sides, then evaluate.

$$c = \sqrt{150^2 + 220^2}$$
$$= 266.270\ldots$$

5 Write your answer correct to two decimal places, with correct units.

The helicopter is 266.27 m from the landing pad, correct to two decimal places.

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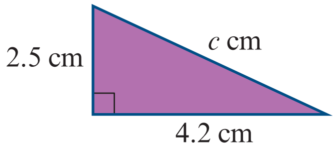
Exercise 10A

Skillsheet

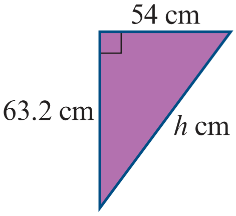
Example 1, 2

1 Find the length of the unknown side in each of these triangles, correct to one decimal place.

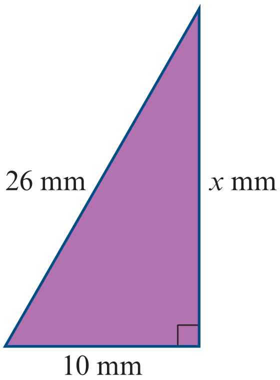
a



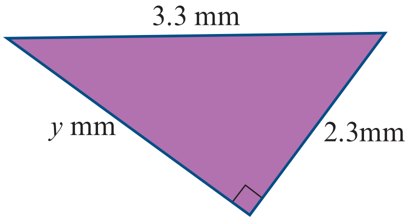
b



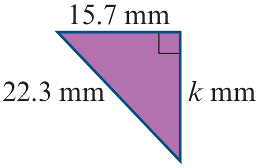
c



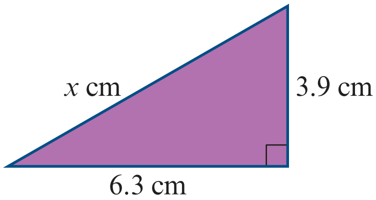
d



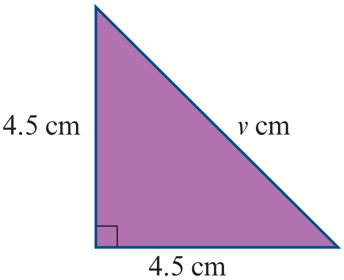
e



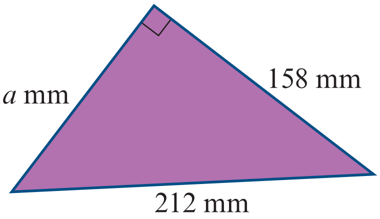
f



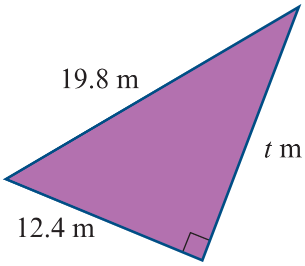
g



h



i



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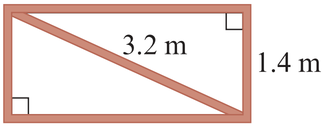
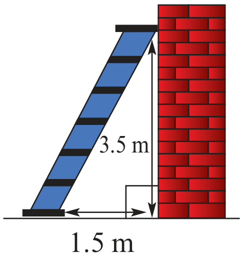
Applications of Pythagoras' theorem

Example 3

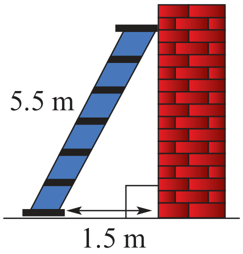
2 A farm gate that is 1.4 m high is supported by a diagonal bar of length 3.2 m. Find the width of the gate, correct to one decimal place.



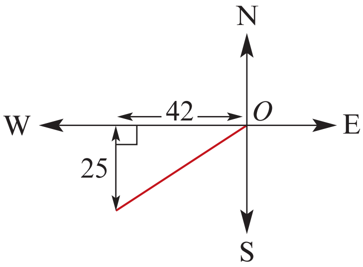
3 A ladder rests against a brick wall as shown in the diagram on the right. The base of the ladder is 1.5 m from the wall, and the top reaches 3.5 m up the wall. Find the length of the ladder, correct to one decimal place.



4 The base of a ladder leaning against a wall is 1.5 m from the base of the wall. If the ladder is 5.5 m long, find how high the top of the ladder is from the ground, correct to one decimal place.



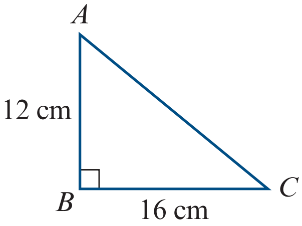
5 A ship sails 42 km due west and then 25 km due south. How far is the ship from its starting point? (Answer correct to two decimal places.)



6 A yacht sails 12 km due east and then 9 km due north. How far is it from its starting point?

7 A hiker walks 10 km due west and then 8 km due north. How far is she from her starting point? (Answer correct to two decimal places.)

8 In a triangle ABC, there is a right angle at B. AB is 12 cm and BC is 16 cm. Find the length of AC.



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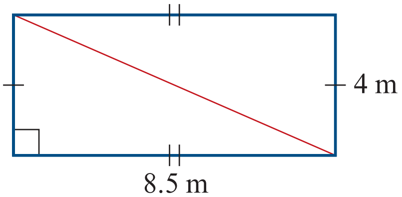
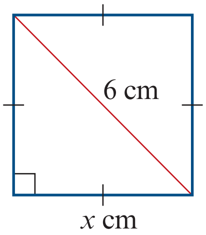
9 Find, correct to one decimal place, the length of the diagonal of a rectangle with dimensions 8.5 m by 4 m.



10 A rectangular block of land measures 28 m by 55 m. John wants to put a fence along the diagonal. How long will the fence be? (Answer correct to three decimal places.)

11 A square has diagonals of length 6 cm. Find the length of its sides, correct to two decimal places.



12 A flying fox on a school camp starts from a tower 25 m high and lands on the ground 100 metres away. What is the distance from the top of the tower to the ground, to the nearest metre?

10B Pythagoras' theorem in three dimensions

When solving three-dimensional problems, it is essential to draw careful diagrams. In general, to find lengths in solid figures, we must first identify the correct right-angled triangle in the plane containing the unknown side. Remember, a plane is a flat surface, such as the cover of a book or a tabletop.

Once it has been identified, the right-angled triangle should be drawn separately from the solid figure, displaying as much information as possible.

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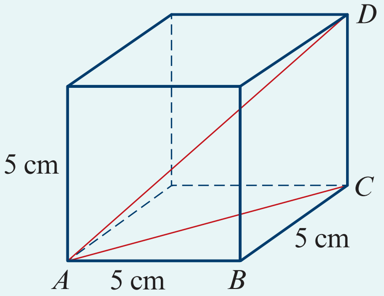
Example 4 Using Pythagoras' theorem in three dimensions

The cube in the diagram on the right has sides of length 5 cm.

Find the length:

a *AC*, correct to two decimal places

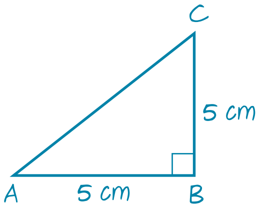
b *AD*, correct to one decimal place.



Solution

a **1** Locate the relevant right-angled triangle in the diagram.

2 Draw the right-angled triangle *ABC* that contains *AC*, and then mark in the known side lengths.



3 Using Pythagoras' theorem, calculate the length *AC*.

$$AC^2 = AB^2 + BC^2$$

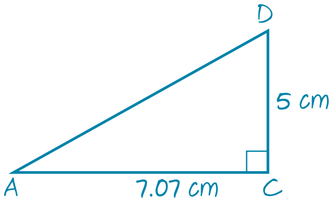
$$\therefore AC = \sqrt{5^2 + 5^2}$$
$$= 7.071\ldots$$

4 Write your answer with correct units and correct to two decimal places.

The length AC is 7.07 cm, correct to two decimal places.

b **1** Locate the relevant right-angled triangle in the diagram.

2 Draw the right-angled triangle *ACD* that contains *AD* and mark in the known side lengths. (From part **a**, *AC* = 7.07 cm, correct to two decimal places.)



3 Using Pythagoras' theorem, calculate the length *AD*.

$$AD^2 = AC^2 + CD^2$$

$$\therefore AD = \sqrt{7.07^2 + 5^2}$$
$$= 8.659\ldots$$

4 Write your answer with correct units and correct to one decimal place.

The length AD is 8.7 cm, correct to one decimal place.

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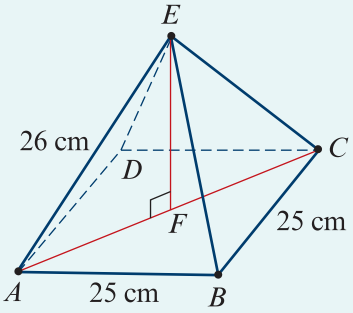
Example 5 Using Pythagoras' theorem in three-dimensional problems

For the square pyramid shown in the diagram, calculate:

a the length *AC*, correct to two decimal places

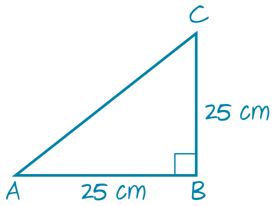
b the height *EF*, correct to one decimal place.



Solution

a **1** Locate the relevant right-angled triangle in the diagram.

2 Draw the right-angled triangle *ABC* that contains *AC*, and mark in known side lengths.



3 Using Pythagoras' theorem, calculate the length *AC*.

$$AC^2 = AB^2 + BC^2$$

$$\therefore AC = \sqrt{25^2 + 25^2}$$

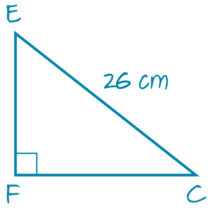
$$= 35.355\ldots$$

4 Write your answer with correct units and correct to two decimal places.

The length AC is 35.36 cm, correct to two decimal places.

b **1** Locate the relevant right-angled triangle in the diagram.

2 Draw the right-angled triangle *EFC* that contains *EF*, and mark in known side lengths.



3 Find *FC*, which is half of *AC*. Use the value of *AC* calculated in part **a**.

$$FC = \frac{AC}{2}$$

$$= \frac{35.36}{2}$$

$$= 17.68 \text{ cm, correct to two decimal places}$$

4 Using Pythagoras' theorem, find *EF*.

$$EF^2 = EC^2 - FC^2$$

$$\therefore EF = \sqrt{26^2 - 17.68^2}$$

$$= 19.065\ldots$$

5 Write your answer with correct units and correct to one decimal places.

The height EF is 19.1 cm, correct to one decimal place.

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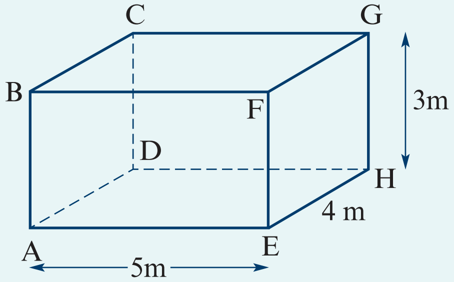
Example 6 Using Pythagoras' theorem in practical three-dimensional problems



A new home entertainment system needs to be set up in a room with dimensions 4 m × 5 m × 3 m as shown in the diagram. Expensive cabling is used to wire the room from corner *A* to corner *G*.

a What length of cabling is required to go from *A* to *E* to *H* to *G*?

b What length of cabling is required to go from *A* to *F* to *G*, correct to two decimal places?

c If cabling costs $9.10 per metre, which is the cheaper option – *A* to *F* to *G* or *A* to *E* to *G*?

Solution

a **1** Add the distances from *A* to *E* (5 m), *E* to *H* (4 m) and *H* to *G* (3 m).

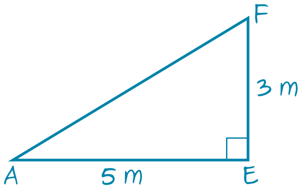
$5 + 4 + 3 = 12$

2 Write your answer with correct units.

The distance is 12 metres.

b **1** First find out the distance from *A* to *F*, by locating the relevant right-angled triangle in the diagram.

2 Draw the right-angled triangle *AFE* that contains *AF*, and then mark in the known side lengths.



3 Using Pythagoras' theorem, calculate the length *AF*.

$AF^2 = AE^2 + EF^2$

$AF^2 = \sqrt{5^2 + 3^2}$

$AF^2 = 5.8309\ldots$

4 Add the lengths *AF* and *FG* and give your final answer with correct units and correct to two decimal places.

The total length (A-F-G)

$= 5.83 + 4$

$= 9.83$ m correct to two decimal places.

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c **1** Work out the cost of cabling from *A* to *F* to *G* by multiplying the length of cabling needed from *A* to *F* to *G* (9.83 m) by the cost of cabling per metre ($9.10)

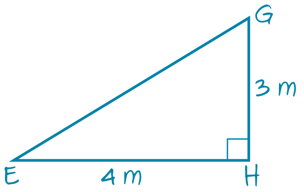
$9.83 \times 9.10 = 89.453$

Cost of cabling from A to F to G is $89.45

2 Work out the distance of cable needed to go from *A* to *E* to *G*. The distance from *A* to *E* is 5 m. Calculate the distance from *E* to *G* by first locating the relevant right-angled triangle in the diagram.

$AE = 5$ m

3 Draw the right-angled triangle *EGH* that contains *EG* and mark in known side lengths.



4 Using Pythagoras' theorem, calculate the length *EG*.

$$EG^2 = EH^2 + HG^2$$
$$EG = \sqrt{4^2 + 3^2}$$
$$EG = 5$$

5 Add the lengths *AE* and *EG* to give total distance from *A* to *E* to *G*.

$$A\text{-}E\text{-}G = 5 + 5$$
$$= 10 \text{ m}$$

6 Work out the cost of cabling from *A* to *E* to *G* by multiplying the length of cabling needed (9 m) by the cost of cabling per metre ($9.10)

$10 \times 9.10 = 91$

Cost of cabling from A to E to G is $91.00.

7 Compare the cost of cabling from *A* to *F* to *G* to the cost of cabling from *A* to *E* to *G* and decide which is the cheaper option.

Cost for A-F-G = $89.45

Cost for A-E-G = $91.00

Thus the cheaper option is to wire cabling from A to F to G.

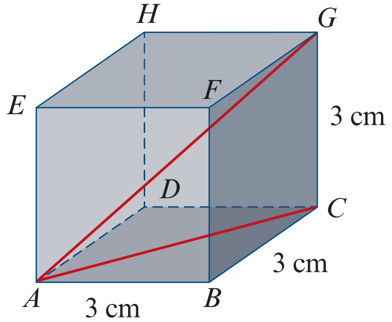
Exercise 10B

Pythagoras' theorem in three dimensions

Example 4 **1** The cube shown in the diagram has sides of 3 cm. Find the length of:

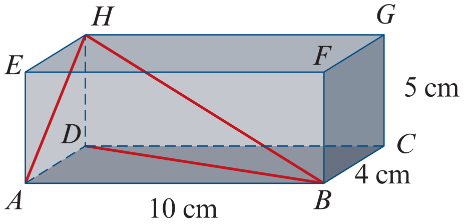
a AC, correct to three decimal places

b AG, correct to two decimal places.



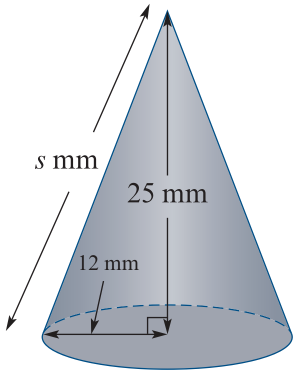
2 For this cuboid, calculate, correct to two decimal places, the length:

a DB **b** BH **c** AH.

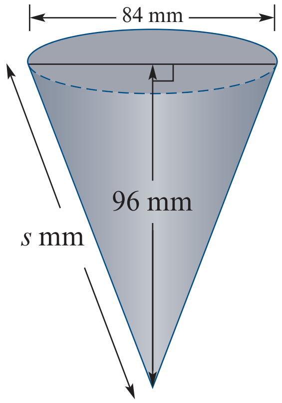


Example 5 **3** Find the sloping height, s, of each of the following cones, correct to two decimal places.

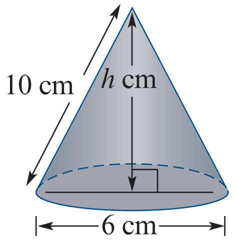
a



b



4 The slant height of this circular cone is 10 cm and the diameter of its base is 6 cm. Calculate the height of the cone, correct to two decimal places.



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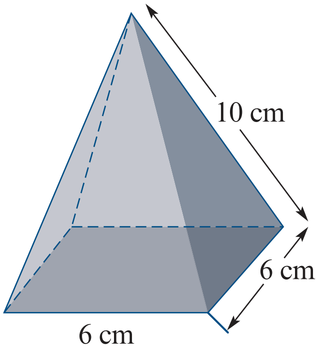
5 For each of the following square-based pyramids find, correct to one decimal place:

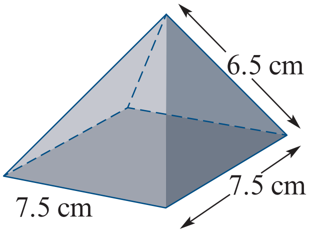
i the length of the diagonal on the base

ii the height of the pyramid.

a



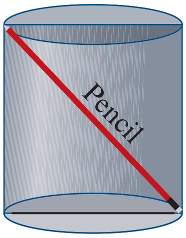
b



Applications of Pythagoras' theorem in three dimensions

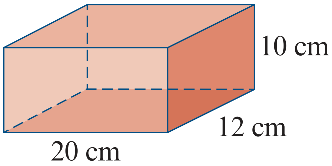
Example 6

6 Find the length of the longest pencil that will fit inside a cylinder with height 15 cm and with circular end surface 8 cm in diameter.



7 Sarah wants to put her pencils in a cylindrical pencil case. What is the length of the longest pencil, correct to two decimal places that would fit inside a cylinder of height 12 cm with a base diameter of 5 cm?

8 Chris wants to use a rectangular pencil box. What is the length of the longest pencil that would fit inside the box shown on the right? (Answer to the nearest centimetre.)



9 A broom is 145 cm long. Would it be able to fit in a cupboard measuring 45 cm by 50 cm and height 140 cm?

10 In order to check the accuracy of the framework and that a room is 'square' a builder often measures the length of the opposing diagonals. What is the distance, correct to two decimal places, from the bottom corner of a room to the top corner diagonally opposite if the room measures 6 m by 4 m by 3.5 m?

11 In the primate enclosure at the zoo, a rope is to be attached from the bottom corner of the enclosure to the opposite top corner for the monkeys to swing and climb on. If the enclosure measures 8 m by 10 m by 12 m, what is the length of the rope? Give your answer correct to two decimal places.

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10C Mensuration: perimeter and area

Mensuration is a part of mathematics that looks at the measurement of length, area and volume. It comes from the Latin word *mensura*, which means 'measure'.

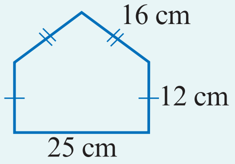
Perimeters of regular shapes

Perimeter

The **perimeter** of a two-dimensional shape is the total distance around its edge.

Example 7 Finding the perimeter of a shape

Find the perimeter of the shape shown.



Solution

To find the perimeter, add up all the side lengths of the shape.

$$\text{Perimeter} = 25 + 12 + 12 + 16 + 16$$
$$= 81 \text{ cm}$$

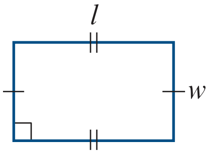
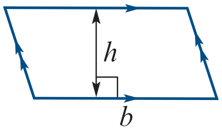
Areas of regular shapes

Area

The **area** of a shape is a measure of the region enclosed by its boundaries.

When calculating area, the answer will be in *square units*, i.e. mm^2, cm^2, m^2, km^2.

The formulas for the areas of some common shapes are given in the table below, along with the formula for finding the perimeter of a rectangle.

Shape	Area	Perimeter
Rectangle 	$A = lw$	$P = 2l + 2w$ or $P = 2(l + w)$
Parallelogram 	$A = bh$	Sum of four sides

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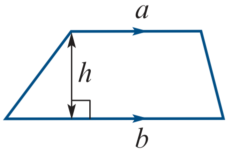
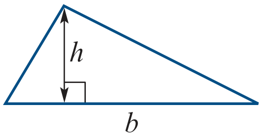
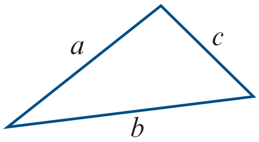
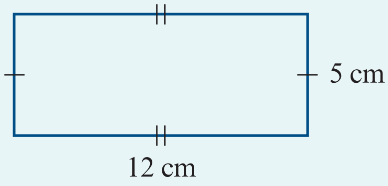
Shape	Area	Perimeter
Trapezium  	$A = \frac{1}{2}(a + b)h$	Sum of four sides
Triangle  	$A = \frac{1}{2}bh$	Sum of three sides
Heron's formula for finding the area of a triangle with three side lengths known.  	$A = \sqrt{s(s-a)(s-b)(s-c)}$ where $s = \frac{a+b+c}{2}$ (s is the half perimeter)	$P = a + b + c$

Example 8 Finding the perimeter of a rectangle

Find the perimeter of the rectangle shown.



Solution

1. Since the shape is a rectangle, use the formula $P = 2l + 2w$.
2. Substitute length and width values into the formula.
3. Evaluate.
4. Give your answer with correct units.

$$P = 2L + 2W$$
$$= 2 \times 12 + 2 \times 5$$
$$= 34 \text{ cm}$$

The perimeter of the rectangle is 34 cm.

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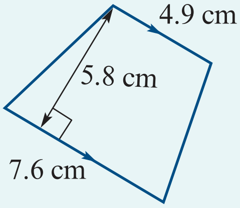
Example 9 Finding the area of a shape

Find the area of the given shape.



Solution

1 Since the shape is a trapezium, use the formula $A = \frac{1}{2}(a + b)h$.

2 Substitute the values for a, b and h.

3 Evaluate.

4 Give your answer with correct units.

$$A = \frac{1}{2}(a + b)h$$

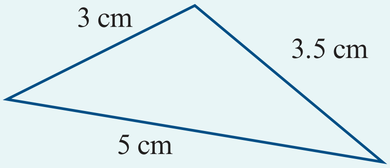
$$= \frac{1}{2}(4.9 + 7.6)5.8$$

$$= 36.25 \text{ cm}^2$$

The area of the shape is 36.25 cm^2.

Example 10 Finding the area of a triangle using Heron's formula

Find the area of the following triangle. Give your answer correct to two decimal places.



Solution

1 Since the height of the triangle is not given, you need to use Heron's formula as the three side lengths are known.

2 Write down Heron's formula.

$$A = \sqrt{s(s - a)(s - b)(s - c)}$$

3 Find the perimeter of the triangle by adding the three side lengths.

$$P = 3 + 3.5 + 5$$

$$= 11.5$$

4 Divide the perimeter by 2 to find s.

$$s = \frac{11.5}{2}$$

$$= 5.75$$

5 Substitute the value for s into Heron's formula to find the area of the triangle.

$$A = \sqrt{5.75(5.75 - 3)(5.75 - 3.5)(5.75 - 5)}$$

$$= 5.16562\ldots$$

6 Give your answer correct to two decimal places and with correct units.

The area of the triangle is 5.17 cm^2, correct to two decimal places.

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The formulas for area and perimeter can be applied to many practical situations.

Example 11 Finding the area and perimeter in a practical problem

A display board for a classroom measures 150 cm by 90 cm.

a If ribbon costs \$0.55 per metre, how much will it cost to add a ribbon border around the display board?

b The display board is to be covered with yellow paper. What is the area to be covered? Give you answer in m^2, correct to two decimal places.

Solution

a	**1**	To find the length of ribbon required, we need to work out the perimeter of the display board. The display board is a rectangle so use the formula $P = 2L + 2w$	$P = 2L + 2W$
	2	Substitute $l = 150$ and $w = 90$ and evaluate.	$= 2(150) + 2(90)$ $= 480$ The length of ribbon required is 480 cm.
	3	Convert from centimetres to metres by dividing the length of ribbon by 100.	$= 480 \div 100$ $= 4.8\text{m}$
	4	To find the cost of the ribbon, multiply the length of the ribbon by \$0.55.	$4.8 \times 0.55 = 2.64$
	5	Evaluate and write down your answer.	Cost of ribbon is \$2.64.
b	**1**	To find the area, use the formula $A = lw$.	$A = LW$
	2	Substitute $l = 150$ and $w = 90$ and evaluate.	$= 150 \times 90$ $= 13\,500 \text{ cm}^2$
	3	Convert your answer to m^2 by dividing by $(100 \times 100 = 10\,000)$.	$A = 13\,500 \div 10\,000$ $= 1.35$
	4	Write your answer with correct units.	Area to be covered with paper is 1.35 m^2.

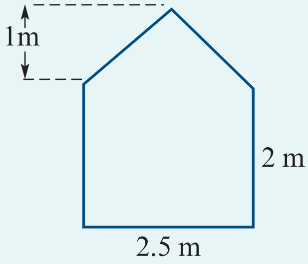
▶ Composite shapes

Composite shape

A *composite shape* is a shape that is made up of two or more basic shapes.

Example 12 Finding the perimeter and area of a composite shape in a practical problem

A gable window at a reception venue is to have LED lights around its perimeter (but not along the bottom of the window). The window is 2.5 m wide and the height of the room is 2 m. The height of the gable is 1 m, as shown in the diagram.

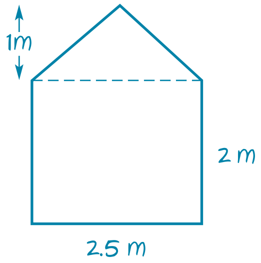


a Calculate the length of LED lights needed, correct to two decimal places.

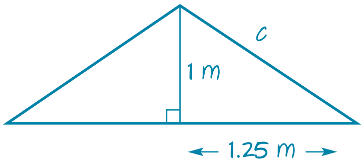
b The glass in the window needs to be replaced. Find the total area of the window, correct to two decimal places.

Solution

a The window is made of two shapes: a rectangle and a triangle.



1 First find the length of the slant edge of the triangle.
Draw a diagram and label the slant edge as c.



Use Pythagoras' theorem to find c.
Note: The length of the base of each triangle is 1.25 m ($\frac{1}{2}$ of 2.5 m).

$c^2 = 1^2 + 1.25^2$

$\therefore c = \sqrt{1^2 + 1.25^2}$

$c = 1.6007\ldots$

$c = 1.60$ m correct to two decimal places

2 Add all the outside edges of the window but do not include the bottom length.

$2 + 2 + 1.60 + 1.60 = 7.2$

3 Write your answer with correct units.

The length of the LED lights is 7.2 m.

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b	**1** To find the total area of the window, first find the area of the rectangle by using the formula $A = bh$.	$A = bh$
	2 Substitute the values for b and h.	$= 2.5 \times 2$
	3 Evaluate and write your answer with correct units.	$= 5\ \text{m}^2$
	4 Find the area of the triangle by using the formula $A = \frac{1}{2}bh$.	$A = \frac{1}{2}bh$
	5 Substitute the values for b and h.	$= \frac{1}{2} \times 2.5 \times 1$
	6 Evaluate and write your answer with correct units.	$= 1.25\ \text{m}^2$
	7 To find the total area of the window, add the area of the rectangle and the area of the triangle.	Total area = area of rectangle + area of triangle $= 5 + 1.25$ $= 6.25\ \text{m}^2$
	8 Give your answer with correct units to two decimal places.	Total area of window is $6.25\ \text{m}^2$ correct to two decimal places.

Exercise 10C

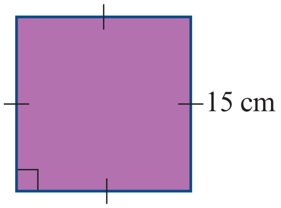
Perimeters and areas

Example 7, 8

1 For each of the following shapes, find, correct to one decimal place:

i the perimeter **ii** the area

a



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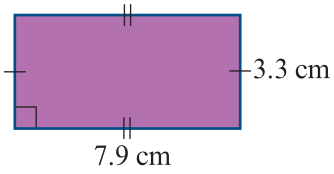
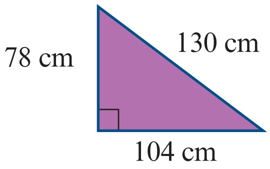
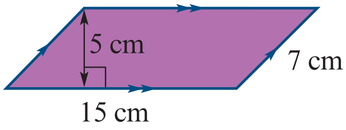


c



d



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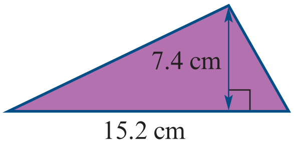
Example 9

2 Find the areas of the given shapes, correct to one decimal place, where appropriate.

a



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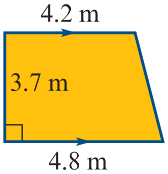

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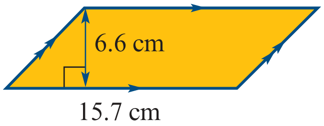
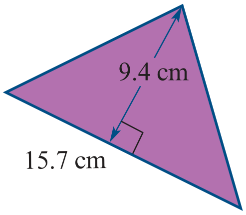
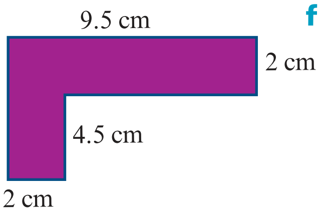


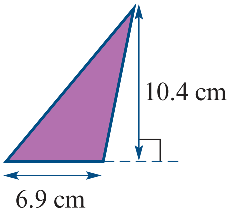
d



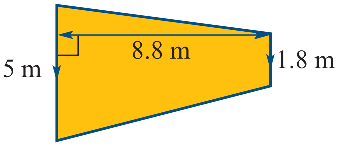
e



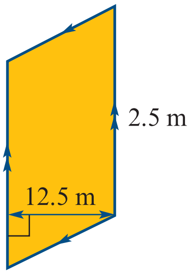
f



g



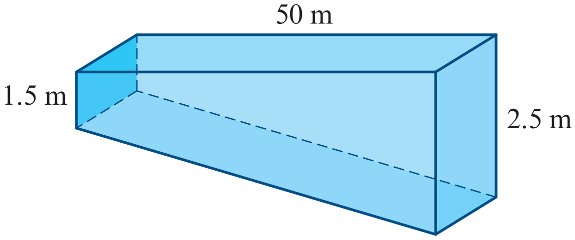
h



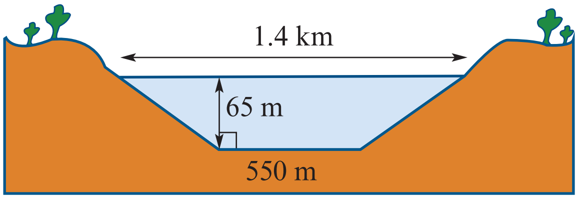
Applications of perimeters and areas

Example 11, 12

3 A 50 m swimming pool increases in depth from 1.5 m at the shallow end to 2.5 m at the deep end, as shown in the diagram (*not* to scale). Calculate the area of a side wall of the pool.



4 A dam wall is built across a valley that is 550 m wide at its base and 1.4 km wide at its top, as shown in the diagram (*not* to scale). The wall is 65 m deep. Calculate the area of the dam wall.



5 Ray wants to tile a rectangular area measuring 1.6 m by 4 m outside his holiday house. The tiles that he wishes to use are 40 cm by 40 cm. How many tiles will he need?

6 One litre of paint covers 9 m^2. How much paint is needed to paint a wall measuring 3 m by 12 m?

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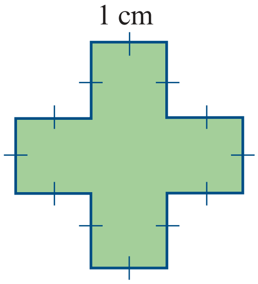
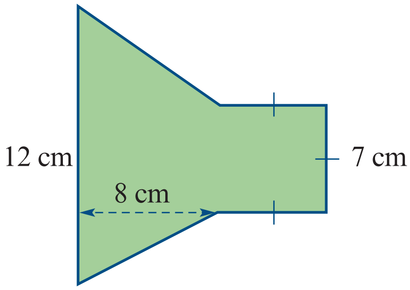
Composite shapes

7 Find the area of the following composite shapes.

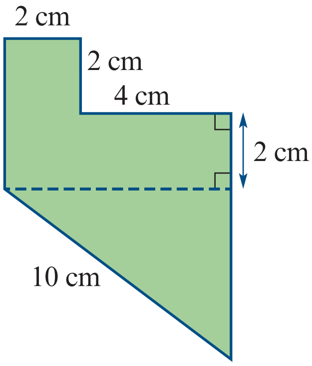
a



b

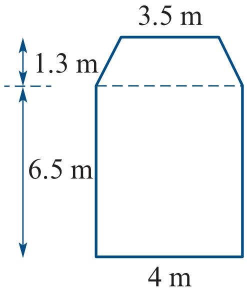


c

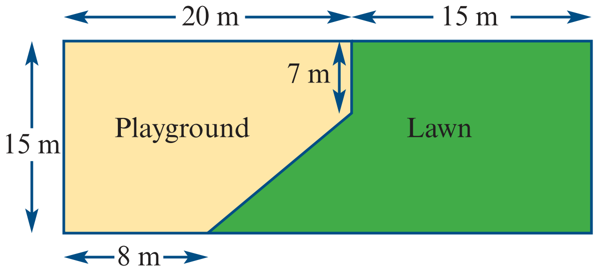


Applications of composite shapes

8 A driveway, as shown in the diagram, is to be paved. What is the area of the driveway, correct to two decimal places?



9 A local council plans to fence a rectangular piece of land to make a children's playground and a lawn as shown. (Not drawn to scale.)



a What is the area of the children's playground?

b What is the area of the lawn?

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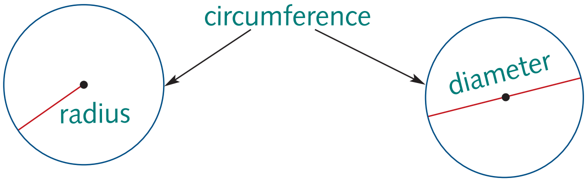
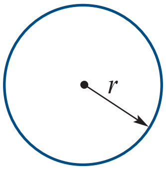
10D Circles

▶ The circumference and area of a circle

The perimeter of a circle is also known as the **circumference** (*C*) of the circle.



The area and the circumference of a circle are given by the following formulas.

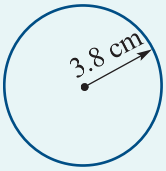
	Area	Circumference
Circle 	$A = \pi r^2$ where r is the **radius**	$C = 2\pi r$ or $C = \pi d$ where d is the **diameter**

Example 13 Finding the circumference and area of a circle

For the circle shown, find:

a the circumference, correct to one decimal place

b the area, correct to one decimal place.



Solution

a **1**	For the circumference, use the formula $C = 2\pi r$.	$C = 2\pi r$
2	Substitute $r = 3.8$ and evaluate.	$= 2\pi \times 3.8$ $= 23.876\ldots$
3	Give your answer correct to one decimal place and with correct units.	The circumference of the circle is 23.9 cm, correct to one decimal place.
b **1**	To find the area of the circle, use the formula $A = \pi r^2$.	$A = \pi r^2$
2	Substitute $r = 3.8$ and evaluate.	$= \pi \times 3.8^2$ $= 45.364\ldots$
3	Give your answer correct to one decimal place and with correct units.	The area of the circle is 45.4 cm^2, correct to one decimal place.

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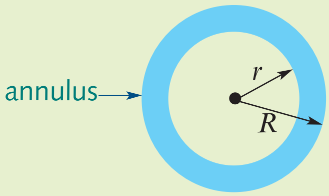
The annulus

Annulus

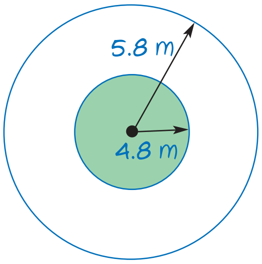
An *annulus* is a flat ring shape bounded by two circles that have the same centre.



Example 14 Finding the area of an annulus in a practical problem

A path 1 metre wide is to be built around a circular lawn of radius 4.8 m. Find the area of the path, correct to two decimal places.

Solution

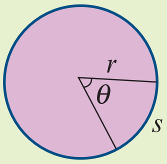
1	Draw a diagram to represent the situation. The two circles form an annulus. The smaller circle has radius of 4.8 m. With the path of width of 1 m, the larger circle has a radius of 5.8 m	
2	Find the area of the larger circle using the formula $A = \pi r^2$.	$A = \pi r^2$
3	Substitute $r = 5.8$ and evaluate.	$A = \pi \times 5.8^2$ $A = 105.68$, correct to two decimal places
4	Find the area of the smaller circle using the formula $A = \pi r^2$.	$A = \pi r^2$
5	Substitute $r = 4.8$ and evaluate.	$A = \pi \times 4.8^2$ $A = 72.38$, correct to two decimal places
6	Subtract the area of the smaller circle from the area of the larger circle to give the required area (the area of the annulus).	Required area = area of large circle − area of small circle $= 105.68 - 72.38$ $= 33.30$
7	Give your answer correct to two decimal places and with correct units.	Area of circular path is 33.30m^2, correct to two decimal places.

Arc length (optional topic)

Arc of a circle

An **arc** is the length of a circle between two points on the circle. The length of the arc, s, is given by:

$$s = r\left(\frac{\theta}{180}\pi\right)$$



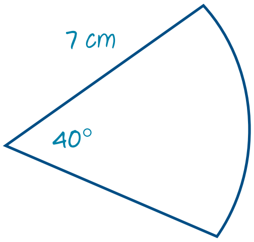
where r is the radius of the circle and $\theta°$ is the angle subtended by the arc at the centre of the circle.

Example 15 Finding the length of an arc

Find, correct to two decimal places, the length of an arc, with an angle of 40° at the centre of the circle and circle radius of 7 cm.

Solution

1 Draw a diagram to represent the situation.



2 Write down the formula for arc length, s.

$$s = r\left(\frac{\theta}{180}\pi\right)$$

3 Substitute $r = 7$ and $\theta = 40$ and evaluate.

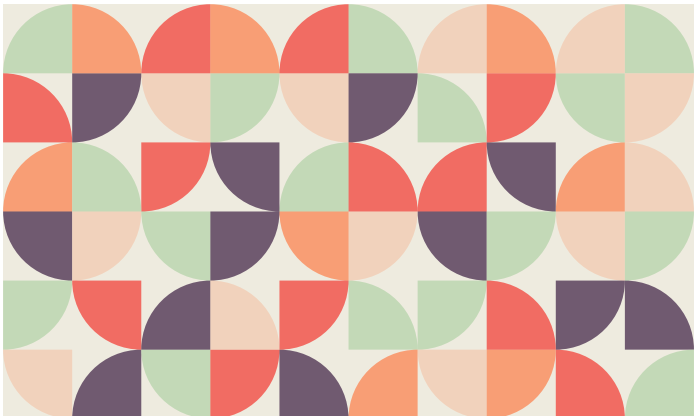
$$s = 7\left(\frac{40}{180}\pi\right)$$

4 Give your answer correct to two decimal places and with correct units.

$s = 4.8869\ldots$

$s = 4.89$ correct to two decimal places

The length of the arc is 4.89 cm.



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Exercise 10D

Finding the circumference and area of a circle

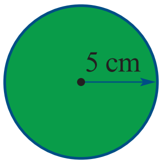
Example 13

1 For each of the following circles, find:

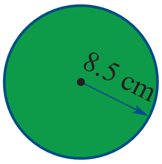
i the circumference, correct to one decimal place

ii the area, correct to one decimal place.

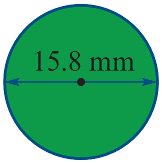
a



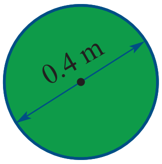
b



c



d



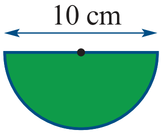
Finding the perimeter and area of shapes involving circles

2 For each of the following shapes, find:

i the perimeter, correct to two decimal places

ii the area, correct to two decimal places.

a



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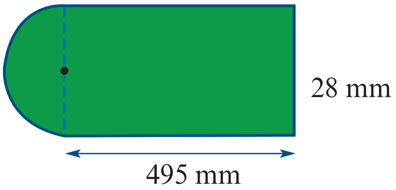

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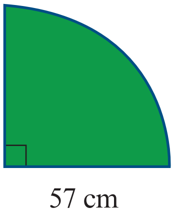
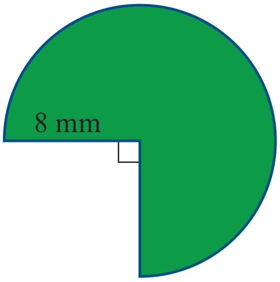
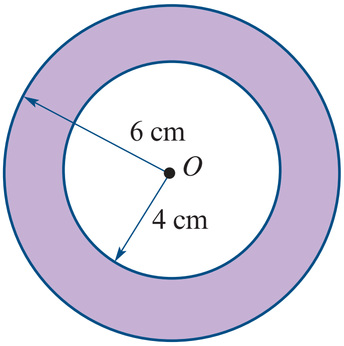


d



Example 14

3 The diagram below shows two circles with centre O. The radius of the inner circle is 4 units and the radius of the outer circle is 6 units. What is the area of the annulus (shaded area) correct to two decimal places?



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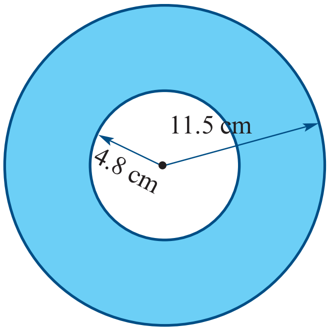
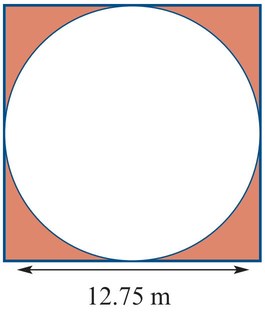
4 Find the shaded areas in the following diagrams, correct to one decimal place.

a



b



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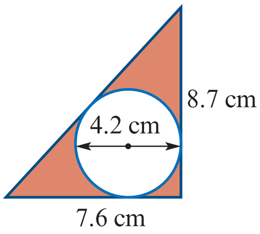
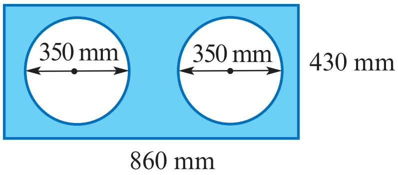


d



Applications of perimeters and areas involving circles

5 A fence needs to be built around an athletics track that has straights 400 m long and semicircular ends of diameter 80 m.

a What length of fencing, correct to two decimal places, is required?

b What area will be enclosed by the fencing, correct to two decimal places?

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6 A couple wish to decorate an arch for their wedding. The width of the arch is 1.4 metres and the semicircle at the top begins at a height of 1.9 metres.

a Material is to be attached around the perimeter of the arch. What is this length, to the nearest metre?

b If material is to cover the whole arch so that you cannot see through the arch, what is the minimum amount of square metres of material required, correct to one decimal place?

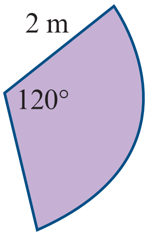
7 Three juggling rings cut from a thin sheet are to be painted. The diameter of the outer circle of the ring is 25 cm and the diameter of the inside circle is 20 cm. If both sides of the three rings are to be painted, what is the total area to be painted? (Ignore the inside and outside edges.) Round your answer to the nearest cm^2.

8 A path 1.2 m wide surrounds a circular garden bed whose diameter is 7 metres. What is the area of the path? Give the answer correct to two decimal places.

Arc length and applications

Example 5 **9** A circle has a radius of 10 cm. An arc of the circle subtends an angle of 50° at the centre. Calculate the arc length correct to two decimal places.

10 Maria wishes to place an edging around the perimeter of her garden bed. The garden bed is in the shape of a sector as shown. What is the perimeter of her garden bed, correct to two decimal places?



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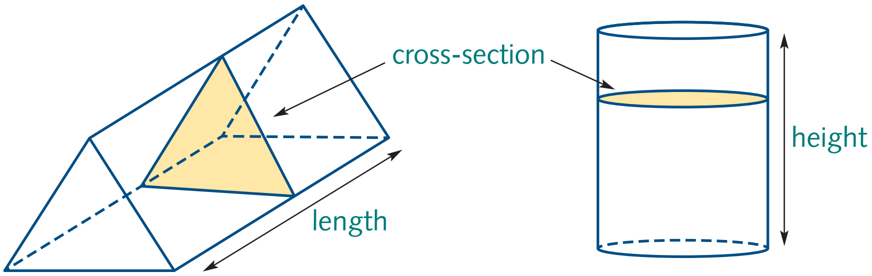
10E Volume

Volume

Volume is the amount of space occupied by a three-dimensional object.

Prisms and cylinders are three-dimensional objects that have a uniform cross-section along their entire length. The volume of a prism or cylinder is found by using its *cross-sectional area*.

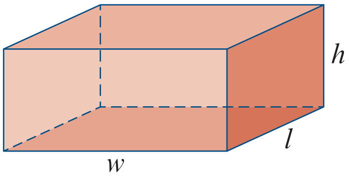
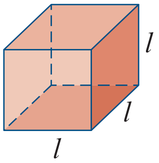


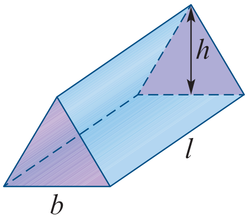
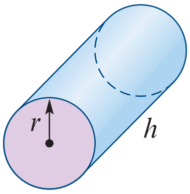
For prisms and cylinders:

$$\text{volume} = \text{area of cross-section} \times \text{height (or length)}$$

When calculating volume, the answer will be in *cubic units*, i.e. mm^3, cm^3, m^3.

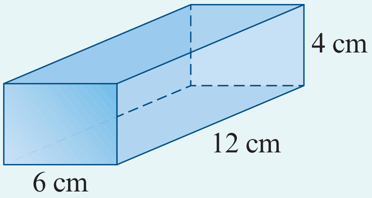
The formulas for the volumes of regular prisms and a cylinder are given in the table below.

Shape	Volume
Rectangular prism (cuboid) 	$V = lwh$
Square prism (cube) 	$V = l^3$

Shape	Volume
Triangular prism 	$V = \frac{1}{2}bhl$
Cylinder 	$V = \pi r^2 h$

Example 16 Finding the volume of a cuboid

Find the volume of the following cuboid.



Solution

1 Use the formula $V = lwh$. $V = lwh$

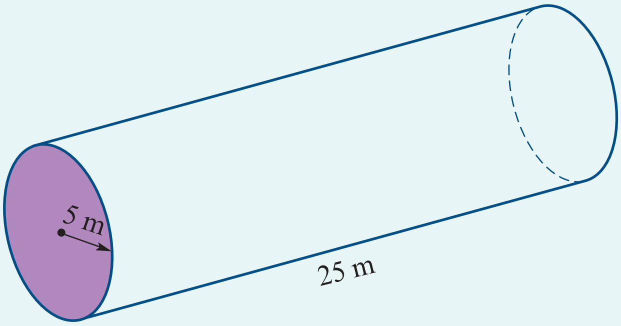
2 Substitute in $l = 12, w = 6$ and $h = 4$. $= 12 \times 6 \times 4$

3 Evaluate. $= 288 \text{ cm}^3$

4 Give your answer with correct units. The volume of the cuboid is 288 cm^3.

Example 17 Finding the volume of a cylinder

Find the volume of this cylinder in cubic metres. Give your answer correct to two decimal places.



Solution

1 Use the formula $V = \pi r^2 h$. $V = \pi r^2 h.$

2 Substitute in $r = 5$ and $h = 25$ and evaluate. $= \pi \times 5^2 \times 25$ $= 1963.495\ldots$

3 Write your answer correct to two decimal places and with correct units. The volume of the cylinder is 1963.50 m^3, to two decimal places.

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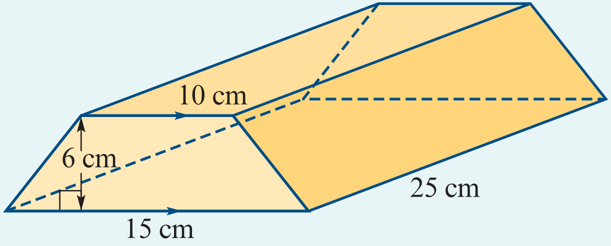
Example 18 Finding the volume of a three-dimensional shape

Find the volume of the three-dimensional shape shown.



Solution

Strategy: To find the volume, find the area of the yellow shaded cross-section and multiply it by the length of the shape.

1 Find the area of the cross-section, which is a trapezium. Use the formula $A = \frac{1}{2}(a + b)h$.

Substitute in $a = 10$, $b = 15$ and $h = 6$ and evaluate.

$$\text{Area of trapezium} = \frac{1}{2}(a + b)h$$
$$= \frac{1}{2}(10 + 15)6$$
$$= 75 \text{ cm}^2$$

2 To find the volume, multiply the area of the cross-section by the length of the shape (25 cm).

$$V = \text{area of cross-section} \times \text{length}$$
$$= 75 \times 25$$
$$= 1875 \text{ cm}^3$$

3 Give your answer with correct units.

The volume of the shape is 1875 cm^3.

▶ Capacity

Capacity

Capacity is the amount of substance that an object can hold.

For example, a bucket might have a capacity of 7 litres.

The difference between volume and capacity is that volume is the space available whilst capacity is the amount of substance that fills the volume.

For example:

- a cube that measures 1 metre on each side has a volume of one cubic metre (m^3) and is able to hold 1000 litres (L) (capacity)
- a bucket of volume 7000 cm^3 can hold 7000 mL (or 7 L) of water.

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The following conversions are useful to remember.

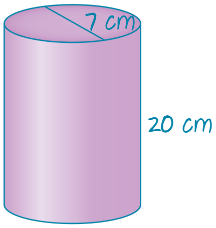
$$1\text{ m}^3 = 1000\text{ litres (L)}$$
$$1\text{ cm}^3 = 1\text{ millilitre (mL)}$$
$$1000\text{ cm}^3 = 1\text{ litre (L)}$$

Example 19 Finding the capacity of a cylinder

A drink container is in the shape of a cylinder. How many litres of water can it hold if the height of the cylinder is 20 cm and the diameter is 7 cm? Give your answer correct to two decimal places.

Solution

1 Draw a diagram to represent the situation.



2 Use the formula for finding the volume of a cylinder $V = \pi r^2 h$.

$V = \pi r^2 h$

3 Since diameter is 7 cm then the radius is 3.5 cm. Substitute $h = 20$ and $r = 3.5$.

$V = \pi \times 3.5^2 \times 20$

4 Evaluate to find the volume of the cylinder.

$V = 769.6902\ldots$

The volume of the cylinder is 769.69 cm^3

5 As there are 1000 cm^3 in a litre, divide the volume by 1000 to convert to litres.

$\dfrac{769.69}{1000} = 0.76969\ldots$

6 Give your answer correct to two decimal places and with correct units.

Cylinder has capacity of 0.77 litres, correct to two decimal places.

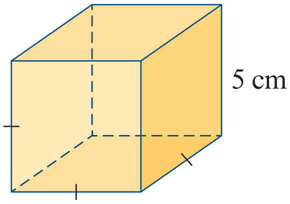
Exercise 10E

Volumes of solids

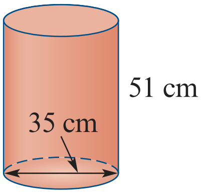
Example 16–18

1 Find the volumes of the following solids. Give your answers correct to one decimal place where appropriate.

a



b



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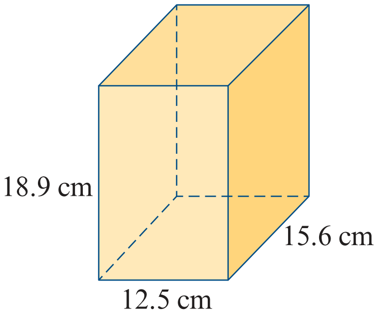
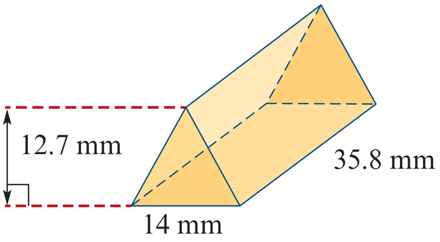
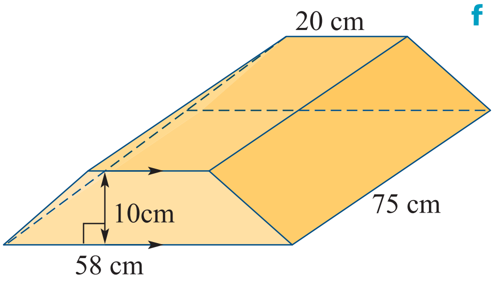
c



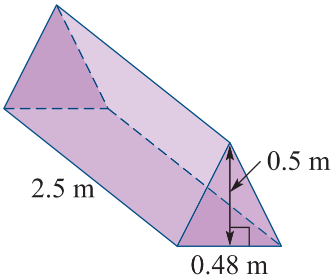
d



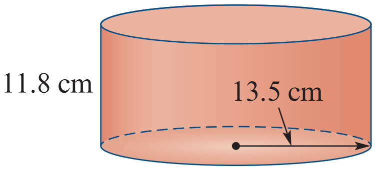
e



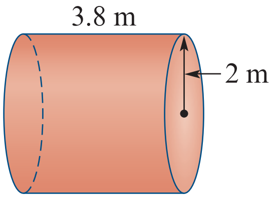
f



g



h



2 A cylindrical plastic container is 15 cm high and its circular end surfaces each have a radius of 3 cm. What is its volume, to the nearest cm^3?

3 What is the volume, to the nearest cm^3, of a rectangular box with dimensions 5.5 cm by 7.5 cm by 12.5 cm?

Applications of volume and capacity

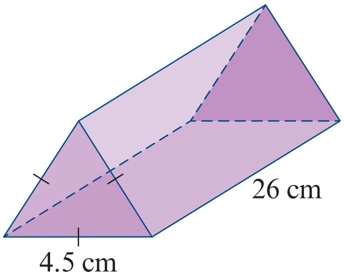
Example 19

4 How many litres of water does a fish tank with dimensions 50 cm by 20 cm by 24 cm hold when full?

5 **a** What is the volume, correct to two decimal places, of a cylindrical paint tin with height 33 cm and diameter 28 cm?

b How many litres of paint would fill this paint tin? Give your answer to the nearest litre.

6 A chocolate bar is made in the shape of an equilateral triangular prism. What is the volume of the box if the length is 26 cm and the side length of the triangle is 4.5 cm? Give your answer to the nearest cm^3.



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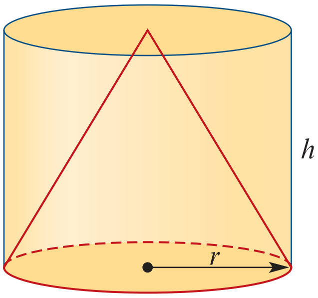
10F Volume of a cone

A cone can fit inside a cylinder, as shown in the diagram. The cone occupies one-third of the volume of the cylinder containing it. Therefore, the formula for finding the volume of a cone is:

$$\text{volume of cone} = \tfrac{1}{3} \times \text{volume of its cylinder}$$

$$\text{volume of cone} = \tfrac{1}{3} \times \text{ area of base} \times \text{height}$$

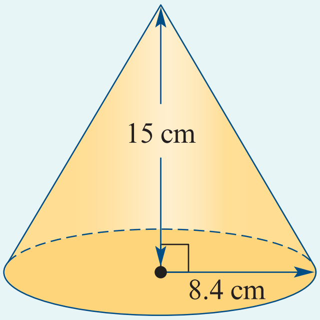
$$V = \tfrac{1}{3}\pi r^2 h$$



The cone in the above diagram is called a right circular cone because a line drawn from the centre of the circular base to the vertex at the top of the cone is perpendicular to the base.

Example 20 Finding the volume of a cone

Find the volume of this right circular cone.
Give your answer to two decimal places.



Solution

1 Use the formula for the volume of a cone, $V = \tfrac{1}{3}\pi r^2 h$.

$$V = \frac{1}{3}\pi r^2 h$$

2 Substitute $r = 8.4$ and $h = 15$ and evaluate.

$$= \frac{1}{3}\pi(8.4)^2 \times 15$$

$$= 1108.353\ldots$$

3 Give your answer correct to two decimal places and with correct units.

The volume of the cone is 1108.35 cm^3, correct to two decimal places.

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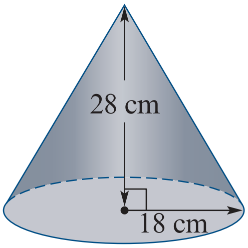
Exercise 10F

Volumes of cones

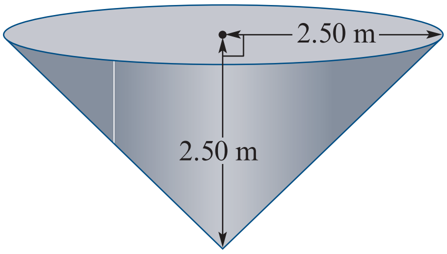
Example 20

1 Find the volume of these cones, correct to two decimal places.

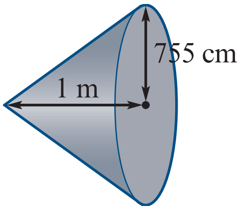
a



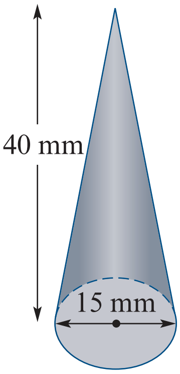
b



c



d



2 Find the volume (to two decimal places) of the cones with the following dimensions.

- **a** Base radius 3.50 cm, height 12 cm
- **b** Base radius 7.90 m, height 10.80 m
- **c** Base diameter 6.60 cm, height 9.03 cm
- **d** Base diameter 13.52 cm, height 30.98 cm

Applications

3 What volume of crushed ice will fill a snow cone level to the top if the snow cone has a top radius of 5 cm and a height of 15 cm? Give your answer to the nearest cm^3.

4 A tepee is a conical shaped tent. What is the volume, correct to two decimal places of a tepee with height 2.6 m and diameter of 3.4 m?

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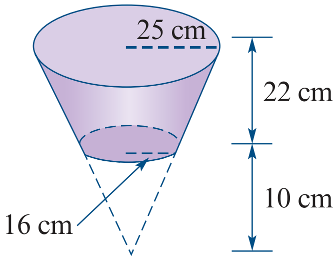
5 How many litres of water can be poured into a conical flask with a diameter 2.8 cm and a height of 10 cm?

6 A solid figure is *truncated* when a portion of the bottom is cut and removed. Find the volume, correct to two decimal places, of the truncated cone shown in the diagram.



7 A plastic cone of height 30 cm and diameter 10 cm is truncated to make a rain gauge. The rain gauge has a height of 25 cm. What is its capacity? Give your answer to the nearest millilitre.

8 A flat-bottomed silo for grain storage is made of a cylinder with a cone on top. The cylinder has a circumference of 53.4 m and a height of 10.8 m. The total height of the silo is 15.3 m. What is the volume of the silo? Give your answer to the nearest m^3.

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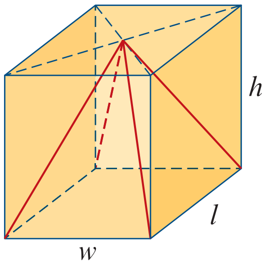
10G Volume of a pyramid

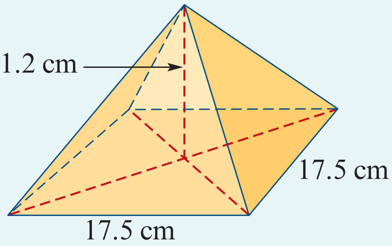
A square pyramid can fit inside a prism, as shown in the diagram. The pyramid occupies one third of the volume of the prism containing it. The formula for finding the volume of a pyramid is therefore:

$$\text{volume of pyramid} = \tfrac{1}{3} \times \text{volume of its prism}$$
$$\text{volume of pyramid} = \tfrac{1}{3} \times \text{area of base} \times \text{height}$$
$$V = \tfrac{1}{3}lwh$$



Example 21 Finding the volume of a square pyramid

Find the volume of a square right pyramid of height 11.2 cm and base 17.5 cm. Give your answer correct to two decimal places.



Solution

1 Use the formula:

$$V = \frac{1}{3} \times \text{area of base} \times \text{height}$$

2 Substitute the values for the area of the base (in this example, the base is a square) and height of the pyramid and evaluate.

$$V = \frac{1}{3} \times \text{area of base} \times \text{height}$$
$$= \frac{1}{3} \times 17.5^2 \times 11.2$$
$$= 1143.333\ldots$$

3 Give your answer correct to two decimal places and with correct units.

The volume of the pyramid is 1143.33 cm^3, correct to two decimal places.

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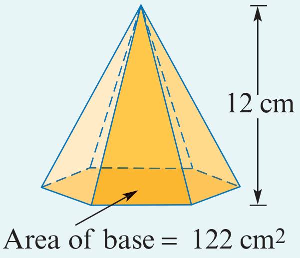
Example 22 Finding the volume of a hexagonal pyramid

Find the volume of this hexagonal pyramid that has a base of area 122 cm^2 and a height of 12 cm.



Solution

1 Use the formula:
$V = \frac{1}{3} \times \text{area of base} \times \text{height}$

$$V = \frac{1}{3} \times \text{area of base} \times \text{height}$$

2 Substitute the values for area of base (122 cm^2) and height (12 cm) and evaluate.

$$= \frac{1}{3} \times 122 \times 12$$
$$= 488 \text{ cm}^3$$

3 Give your answer with correct units.

The volume of the pyramid is 488 cm^3.

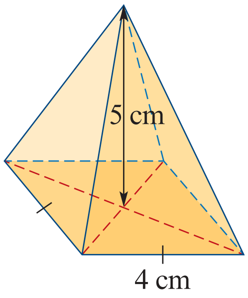
Exercise 10G

Volumes of pyramids

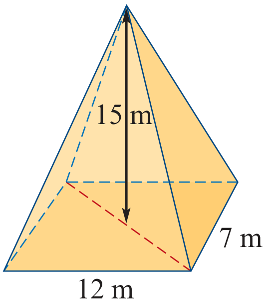
Example 21, 22

1 Find the volumes of the following right pyramids, correct to two decimal places where appropriate.

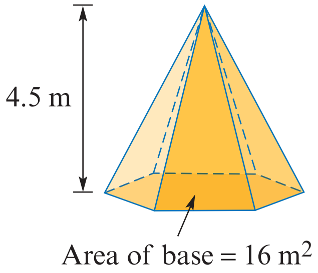
a



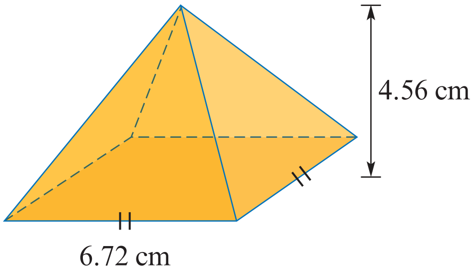
b



c



d



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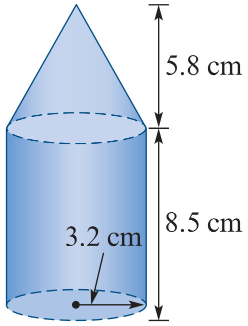
2 A square-based pyramid has a base side length of 8 cm and a height of 10 cm. What is its volume? Answer correct to three decimal places.

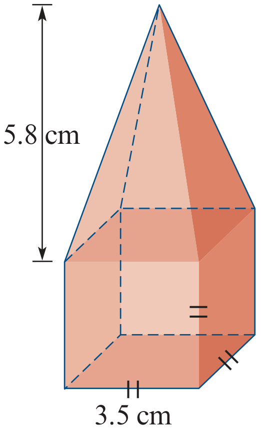
3 The first true pyramid in Egypt is known as the Red Pyramid. It has a square base approximately 220 m long and is about 105 m high. What is its volume?

4 Find the volumes of these composite objects, correct to one decimal place.

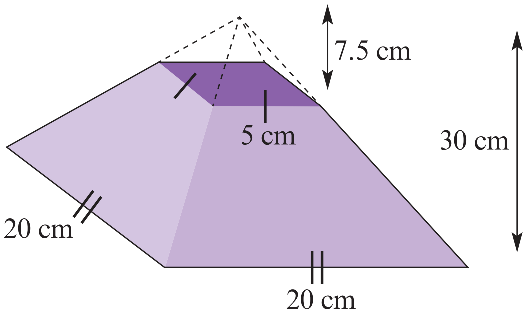
a



b



5 Calculate the volume of the following truncated pyramid, correct to one decimal place.



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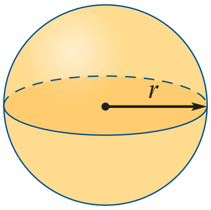
10H Volume of a sphere

The volume of a sphere of radius *r* can be found by using the formula:

$V = \frac{4}{3}\pi r^3$,

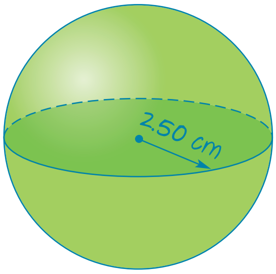


Example 23 Finding the volume of a sphere

Find the volume of this sphere, giving your answer correct to two decimal places.

Solution

1 Use the formula $V = \frac{4}{3}\pi r^3$.



2 Substitute $r = 2.5$ and evaluate.

$$V = \frac{4}{3}\pi r^3$$
$$= \frac{4}{3}\pi \times 2.5^3$$
$$= 65.449\ldots$$

3 Give your answer correct to two decimal places and with correct units.

The volume of the sphere is 65.45 cm^3, correct to two decimal places.

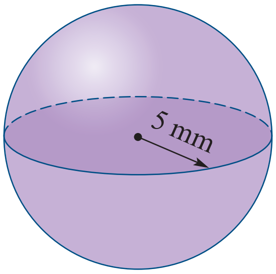
Exercise 10H

Volumes of spheres and hemispheres

Example 23

1 Find the volumes of these spheres, giving your answers correct to two decimal places.

a



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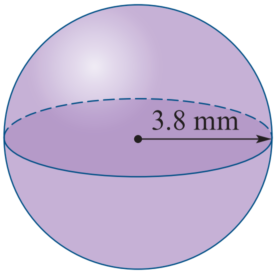

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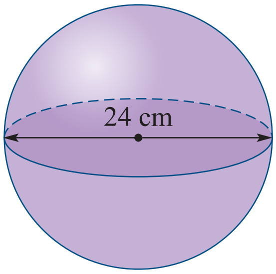

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2 Find the volumes, correct to two decimal places, of the following balls.

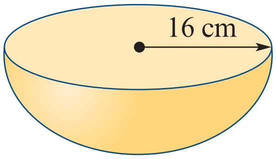
a Tennis ball, radius 3.5 cm

b Basketball, radius 14 cm

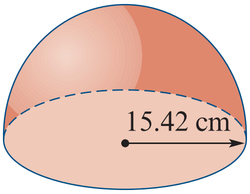
c Golf ball, radius, 2 cm

3 Find the volumes, correct to two decimal places, of the following hemispheres.

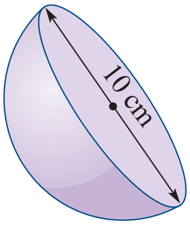
a



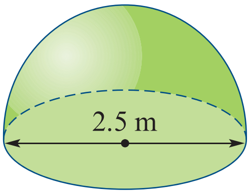
b



c



d



Applications

4 An orange is cut into quarters. If the radius is 35 mm, what is the volume of one quarter to the nearest mm^3?

5 Lois wants to serve punch at Christmas time in her new hemispherical bowl with diameter of 38 cm. How many litres of punch could be served, given that 1 millilitre (mL) is the amount of fluid that fills 1 cm^3? Answer to the nearest litre.

10I Surface area

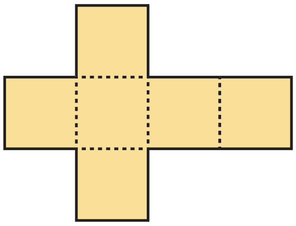
To find the **surface area (*SA*)** of a solid, you need to find the area of each of the surfaces of the solid and then add these all together.

▶ Solids with plane faces (prisms and pyramids)

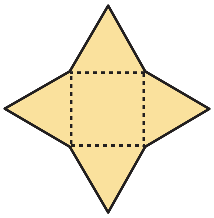
It is often useful to draw the *net* of a solid to ensure that all sides have been added.

A *net* is a flat diagram consisting of the plane faces of a polyhedron, arranged so that the diagram may be folded to form the solid.

For example: The net of a cube and of a square pyramid are shown below.



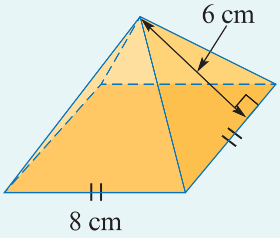
Net of a cube



Net of a square pyramid

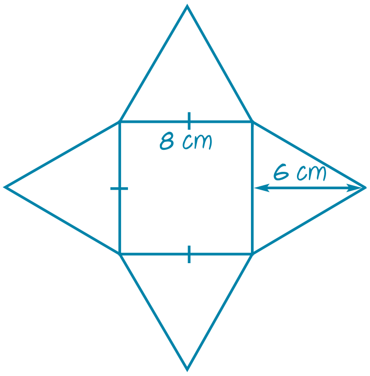
Example 24 Finding the surface area of a pyramid

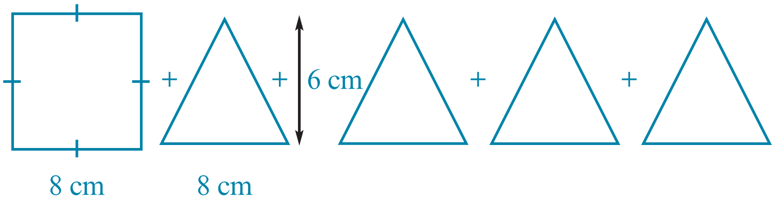
Find the surface area of this square-based pyramid.



Solution

1 Draw a net of the square pyramid. Note that the net is made up of one square and four identical triangles, as shown below.





2 Write down the formula for the total surface area, using the net as a guide, and evaluate.

Total surface area = area of □ + 4 △

$= 8 \times 8 + 4 \times (\frac{1}{2} \times 8 \times 6)$

$= 160$

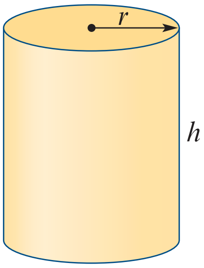
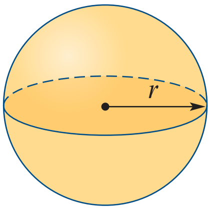
The surface area of the square pyramid is 160 cm^2.

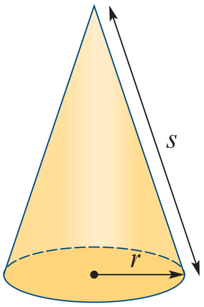
Note: To find the area of the square, multiply the length by the width (8×8). To find the area of the triangles use $A = \frac{1}{2}bh$, where b is 8 and h is 6.

▶ Solids with curved surfaces (cylinder, cone, sphere)

For some special objects, such as the cylinder, cone and sphere, formulas to calculate the surface area can be developed.

The formulas for the surface area of a cylinder, cone and sphere are given below.

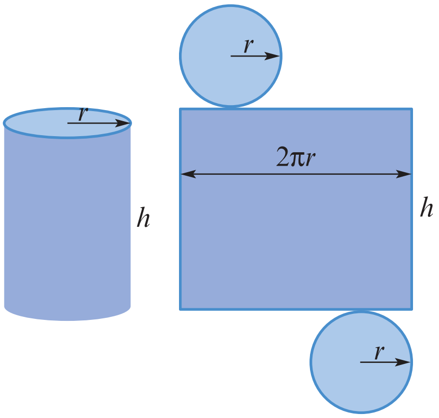
Shape	Surface area
Cylinder 	$$\begin{aligned} SA &= 2\pi r^2 + 2\pi rh \\ &= 2\pi r(r+h) \end{aligned}$$
Sphere 	$SA = 4\pi r^2$

Shape	Surface area
Cone 	$SA = \pi r^2 + \pi rs$

To develop the formula for the surface area of a cylinder, we first draw a net, as shown.

The **total surface area *(TSA)*** of a cylinder can therefore be found using:

$$\begin{aligned} \text{TSA} &= \text{area of ends} + \text{area of curved surface} \\ &= \text{area of 2 circles} + \text{area of rectangle} \\ &= 2\pi r^2 + 2\pi rh \\ &= 2\pi r(r+h) \end{aligned}$$



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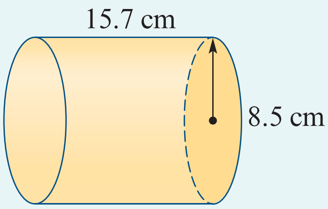
Example 25 Finding the surface area of a cylinder

Find the surface area of this cylinder, correct to one decimal place.



Solution

1 Use the formula for the surface area of a cylinder, $SA = 2\pi r^2 + 2\pi rh$.

$SA = 2\pi r^2 + 2\pi rh$

2 Substitute $r = 8.5$ and $h = 15.7$ and evaluate.

$= 2\pi(8.5)^2 + 2\pi \times 8.5 \times 15.7$

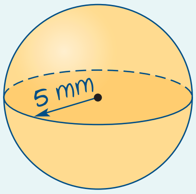
$= 1292.451\ldots$

3 Give your answer correct to one decimal place and with correct units.

The surface area of the cylinder is 12 92.5 cm^2, correct to one decimal place.

Example 26 Finding the surface area of a sphere

Find the surface area of a sphere with radius 5 mm, correct to two decimal places.



Solution

1 Use the formula $SA = 4\pi r^2$.

$SA = 4\pi r^2$

2 Substitute $r = 5$ and evaluate.

$= 4\pi \times 5^2$

$= 314.159\ldots$

3 Give your answer correct to two decimal places and with correct units.

The surface area of the sphere is 314.16 mm^2, correct to two decimal places.

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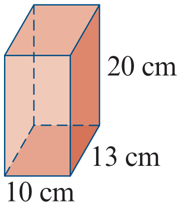
Exercise 10I

Surface areas of prisms and pyramids

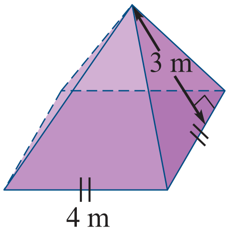
Example 24

1 Find the surface areas of these prisms and pyramids. Where appropriate give your answer correct to one decimal place.

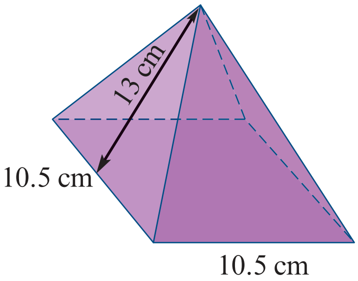
a



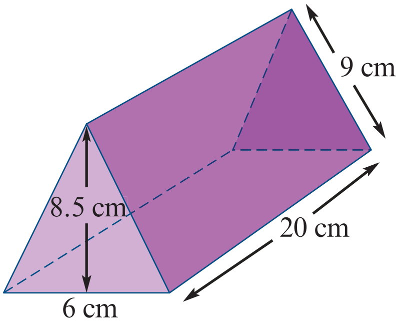
b



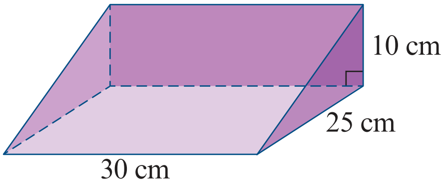
c



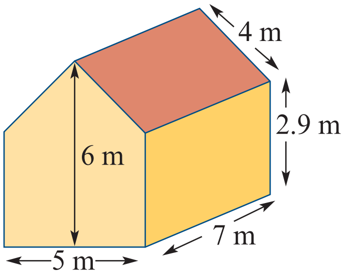
d



e



f

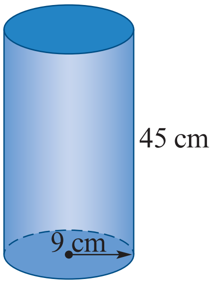


Surface area of curved surfaces

Example 25, 26

2 Find the surface area of each of these solids with curved surfaces, correct to two decimal places.

a



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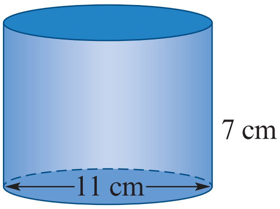

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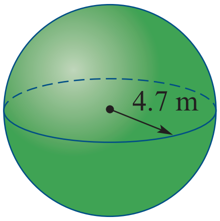

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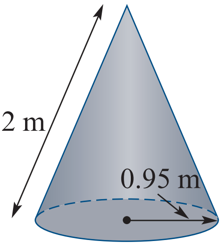
d



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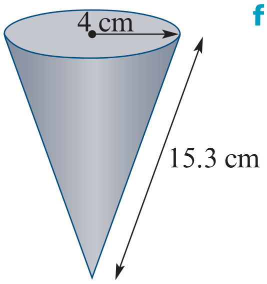

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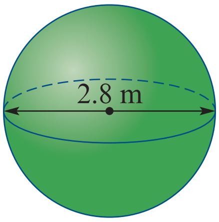
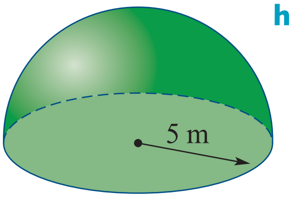
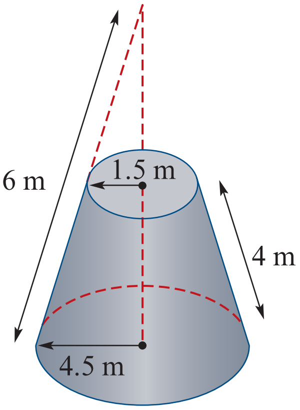


g



h



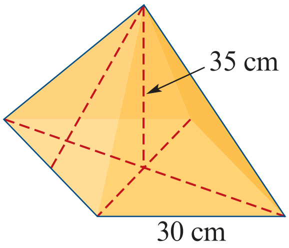
Applications of surface areas

3 A tennis ball has a radius of 3.5 cm. A manufacturer wants to provide sufficient material to cover 100 tennis balls. What area of material is required? Give your answer correct to the nearest cm^2.

4 A set of 10 conical paper hats are covered with material. The height of a hat is 35 cm and the diameter is 19 cm.

- **a** What amount of material, in m^2 will be needed? Give your answer correct to two decimal places.
- **b** Tinsel is to be placed around the base of the hats. How much tinsel, to the nearest metre, is required?

5 For a project, Mark has to cover all sides of a square based pyramid with material (excluding the base). The pyramid has the dimensions as shown in the diagram. How much material will Mark need to cover the sides of the pyramid? Give your answer in metres, correct to two decimal places.



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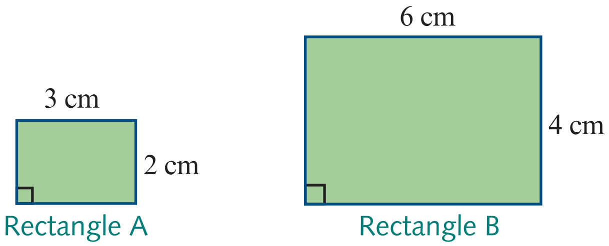
10J Similar figures

Shapes that are similar have the same shape but are different sizes. The three frogs below are **similar figures**.

Polygons (closed plane figures with straight sides), like the rectangles in the diagram below, are similar if:

- corresponding angles are equal
- corresponding sides are proportional (each pair of corresponding side lengths are in the same ratio).



For example, the two rectangles above are similar as their corresponding angles are equal and their side lengths are in the same ratio.

$$\text{Ratio of side length} = 6 : 3 \text{ or } \frac{6}{3} = \frac{2}{1} = 2$$

$$\text{Ratio of side length} = 4 : 2 \text{ or } \frac{4}{2} = \frac{2}{1} = 2$$

When we enlarge or reduce a shape by a **scale factor**, the *original* and the *image* are similar.

In the diagram above, rectangle A has been enlarged by a scale factor, $k = 2$, to give rectangle B.

We can also say that rectangle A has been scaled up to give rectangle B.

We can also compare the ratio of the rectangles' areas.

$$\text{Area of rectangle A} = 6 \text{ cm}^2$$

$$\text{Area of rectangle B} = 24 \text{ cm}^2$$

$$\text{Ratio of areas} = 24 : 6 = \frac{24}{6} = \frac{4}{1} = 4$$

The area of rectangle A has been enlarged by a scale factor, k^2, of 4 to give rectangle B.

We notice that, as the length dimensions are enlarged by a scale factor of 2, the area is enlarged by a scale factor of $2^2 = 4$.

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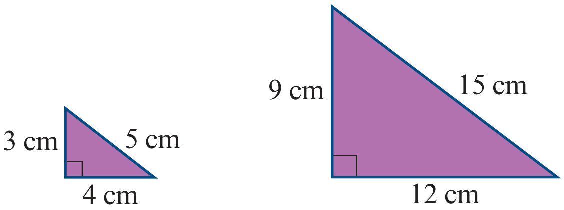
Scaling areas

When all the dimensions are multiplied by a scale factor of k, the area is multiplied by a scale factor of k^2.



For example, the two triangles above are similar as their corresponding side lengths are in the same ratio.

$$\text{Scale factor}, k = \text{Ratio of lengths} = \frac{15}{5} = \frac{9}{3} = \frac{12}{4} = \frac{3}{1} = 3$$

We would expect the area scale factor, k^2, or the ratio of the triangles' areas to be $9(= 3^2)$.

$$\text{Area of small triangle} = 6 \text{ cm}^2$$

$$\text{Area of large triangle} = 54 \text{ cm}^2$$

$$\text{Area scale factor}, k^2 = \text{Ratio of areas} = \frac{54}{6} = \frac{9}{1} = 9.$$

Shapes can be scaled up or scaled down. When a shape is made larger, it is scaled up and when it is made smaller, it is scaled down.

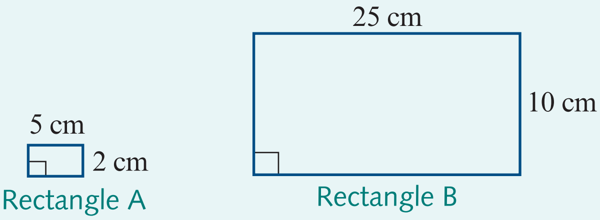
When working out scale factors, the numerator is the length of the second shape and the denominator is the length from the first (or original) shape.

Example 27 Finding the ratio (scale factor) of dimension and area

The rectangles shown are similar.

a Find the ratio of their side lengths.

b Find the ratio of their areas.



Solution

a **1** Since the rectangles are similar, their side lengths are in the same ratio. Compare the corresponding side lengths.

$$\frac{25}{5} = \frac{10}{2} = \frac{5}{1}$$

2 Write your answer.

Note: We can also say that the second rectangle has been scaled up by a scale factor of 5.

The ratio of the side lengths is $\frac{5}{1}$.

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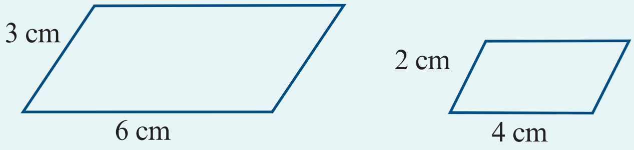
b	**1** Since the dimensions are multiplied by a scale factor of 5, the area will be multiplied by a scale factor of 5^2. Square the ratio of the side lengths.	$5^2 = 25$
	2 Write your answer. Note: We can also say that the area of the second rectangle has been scaled up by a scale factor of 25.	The ratio of the areas is $\frac{25}{1}$.

Example 28 Finding the scale factor

The two shapes shown are similar.

a Determine whether the first shape has been scaled up or down to give the second shape and find the scale factor.

b What is the scale factor for the areas?



Solution

a	**1** Since the shape is made smaller it has been scaled down.	Shape has been scaled down.
	2 The shapes are similar so their side lengths are in the same ratio. Compare the corresponding side lengths.	$\frac{4}{6} = \frac{2}{3}$
	3 Write your answer.	The scale factor is $\frac{2}{3}$
b	**1** The scale factor, k, for the shapes is $\frac{2}{3}$ so the scale factor for the area is k^2. Square the value for k and evaluate.	$k = \frac{2}{3}$ $k^2 = \left(\frac{2}{3}\right)^2$ $= \frac{4}{9}$
	2 Write your answer.	The scale factor for the area is $\frac{4}{9}$.

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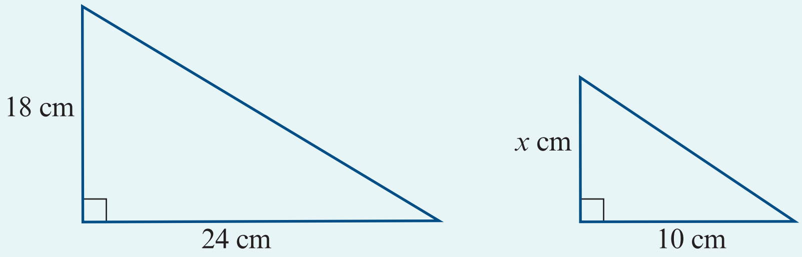
Example 29 Using a scale factor to find unknown values

The following two triangles are similar.

a Find the value of x.

b What is the scale factor?



Solution

a **1** Since the triangles are similar, their side lengths are in the same ratio. Compare the corresponding side lengths.

Note: Make sure that corresponding sides are compared.

$$\frac{x}{18} = \frac{10}{24}$$

2 Solve for x. Multiply by 18.

$$\frac{x}{18} \times 18 = \frac{10}{24} \times 18$$

3 Evaluate and give your answer with correct units.

$x = 7.5$ cm

b **1** Compare corresponding side lengths and simplify fraction.

Remember: The numerator of the fraction is the length from the second shape and the denominator is the length from the first (or original) shape.

$$\frac{7.5}{18} = \frac{10}{24} = \frac{5}{12}$$

2 Write your answer.

Note: In this case triangle has been scaled down.

Triangle has been scaled down by a scale factor of $\frac{5}{12}$.

Example 30 Using scaling in maps

A map has a scale of 1 : 20 000. If the measurement on the map between two towns is 5.4 cm, what is the actual distance between these two towns? Give your answer in kilometres, correct to two decimal places.

Solution

A scale of 1 : 20 000 means that 1 cm on the map represents 20 000 cm (or 200 m) on the ground.

Method 1

1 If the map distance between the two towns is 5.4 cm, then multiply this distance by 20 000 to get the actual distance (in cm) between the two towns.

map distance = 5.4 cm

$$\begin{aligned}\text{actual distance} &= 5.4 \times 20\,000\\ &= 108\,000 \text{ cm}\end{aligned}$$

2 Convert from centimetres to metres by dividing by 100.

$108\,000\text{cm} \div 100 = 1080\text{m}$

3 Convert from metres to kilometres by dividing by 1000.

$1080\text{m} \div 1000 = 1.08\text{km}$

Method 2

1 Let x be the actual distance.
A scale of 1 : 20 000 can also be written as a scale factor of $\frac{1}{20\,000}$.
The ratio of the distance on the map to the actual distance will be the same as the scale factor of $\frac{1}{20\,000}$. Write out the corresponding ratios.

Let x be the actual distance.

$$\frac{5.4}{x} = \frac{1}{20\,000}$$

2 Solve for x. (This can be done by cross-multiplying or by using the solve(function on the CAS calculator.

$$\begin{aligned}5.4 \times 20\,000 &= x\\ x &= 108\,000 \text{ cm}\end{aligned}$$

3 Convert to kilometres by dividing by 100 000.

Note: This is the same as dividing by 100 to convert to metres and then by 1000 to convert to kilometres.

$108\,000 \text{ cm} \div 100\,000 = 1.08 \text{ km}$

4 Write your answer.

The distance between the two towns is 1.08 km.

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Exercise 10J

Skillsheet

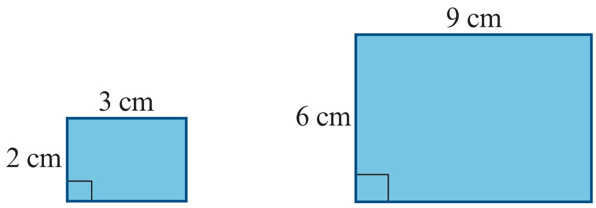
Similarity and ratios

Example 27

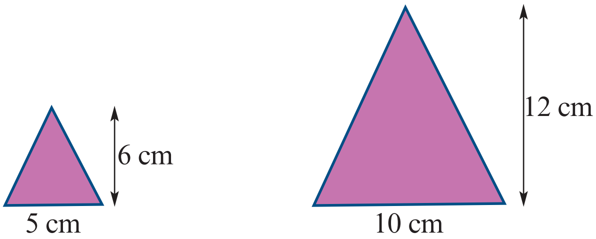
1 The following pairs of figures are similar. For each pair find:

i the ratio of their side lengths **ii** the ratio of their areas.

a

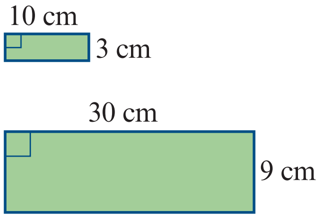


b

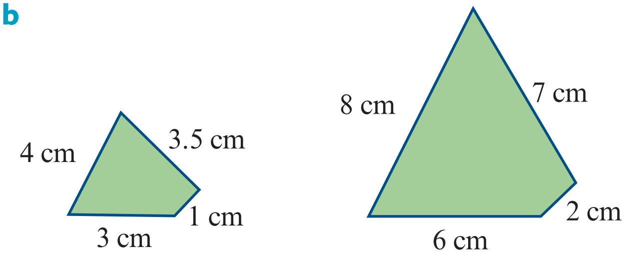


2 Which of the following pairs of figures are similar? For those that are similar, find the ratios of the corresponding sides.

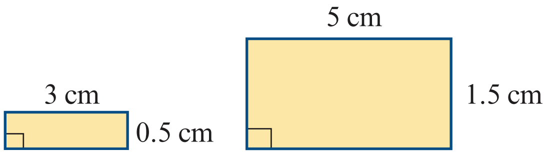
a



b



c



3 Which of the following pairs of figures are similar? State the ratios of the corresponding sides where relevant.

a Two rectangles 8 cm by 3 cm and 16 cm by 4 cm

b Two rectangles 4 cm by 5 cm and 16 cm by 20 cm

c Two rectangles 4 cm by 6 cm and 2 cm by 4 cm

d Two rectangles 30 cm by 24 cm and 10 cm by 8 cm

e Two triangles, one with sides measuring 3 cm, 4 cm and 5 cm and the other 4.5 cm, 6 cm and 7.5 cm.

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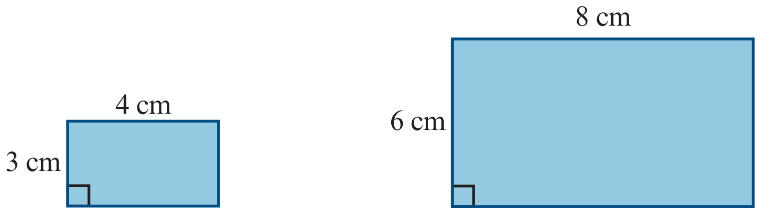
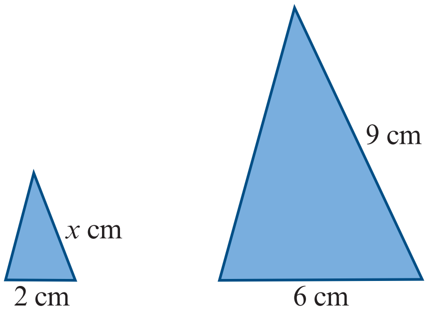
Example 28 **4** The following two rectangles are similar. Find the ratio of their areas.



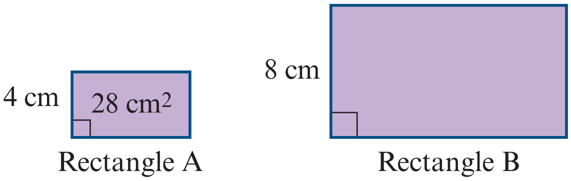
Example 29 **5** The following triangles are similar.

a Find the value of x.

b Find the ratio of their areas.



6 The two rectangles shown below are similar. The area of rectangle A is 28 cm^2. Find the area of rectangle B.



Applications of similarity and ratios

7 A photo is 12 cm by 8 cm. It is to be enlarged and then framed. If the dimensions are tripled, what will be the area of the new photo?

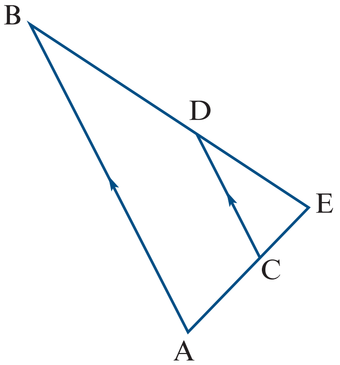
8 What is the scale factor if a photo has been enlarged from 15 cm by 9 cm to 25 cm by 15 cm? Give your answer correct to two decimal places.

Example 30 **9** A scale on a map is 1 : 500 000.

a What is the actual distance between two towns if the distance on the map is 7.2 cm? Give your answer in kilometres.

b If the actual distance between two landmarks is 15 km, what distance would be represented on the map?

10 In triangle ABE, point C lies on side AE and point D lies on side BE. The lines CD and AB are parallel. The length of ED is 5 cm, the length of DB is 7 cm and the length of CD is 6 cm. What is the length of AB?



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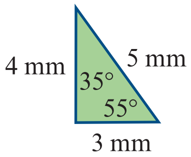
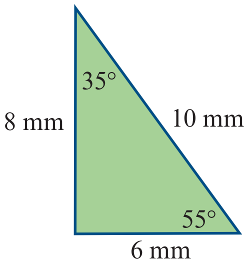
10K Similar triangles

In mathematics, two **triangles** are said to be **similar** if they have the same shape. As in the previous section, this means that corresponding angles are equal and the lengths of the corresponding sides are in the same ratio.

For example, these two triangles are similar.

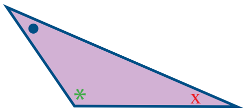


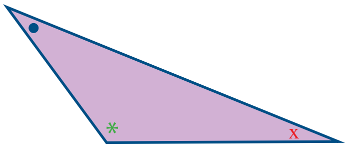


Two triangles can be tested for similarity by considering the following necessary conditions.

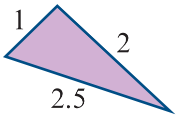
- Corresponding angles are equal (AA).

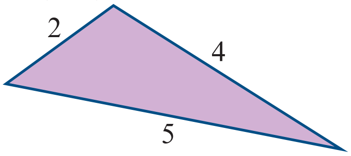
  Remember: If two pairs of corresponding angles are equal, then the third pair of corresponding angles is also equal.





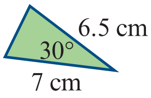
- Corresponding sides are in the same ratio (SSS).

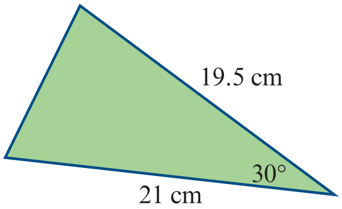




$$\frac{5}{2.5} = \frac{4}{2} = \frac{2}{1} = 2$$

- Two pairs of corresponding sides are in the same ratio and the included corresponding angles are equal (SAS).





$$\frac{19.5}{6.5} = \frac{21}{7} = 3$$

Both triangles have an included corresponding angle of 30°.

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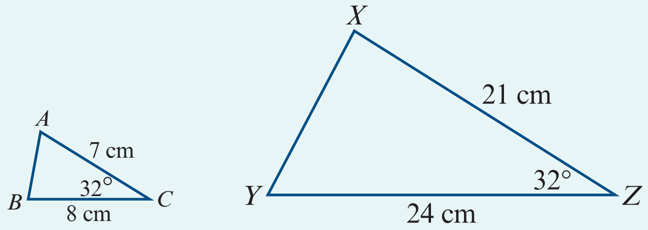
Example 31 Checking if triangles are similar

Explain why triangle *ABC* is similar to triangle *XYZ*.



Solution

1 Compare corresponding side ratios:
AC and *XZ*
BC and *YZ*.

$$\frac{XZ}{AC} = \frac{21}{7} = \frac{3}{1}$$
$$\frac{YZ}{BC} = \frac{24}{8} = \frac{3}{1}$$

2 Triangles *ABC* and *XYZ* have an included corresponding angle.

32° is included and corresponding.

3 Write an explanation as to why the two triangles are similar.

Triangles ABC and XYZ are similar as they have two pairs of corresponding sides in the same ratio and the included corresponding angles are equal (SAS).

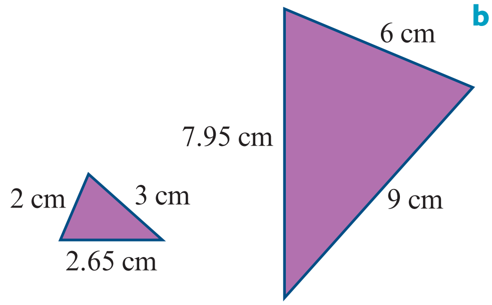
Exercise 10K

Similar triangles

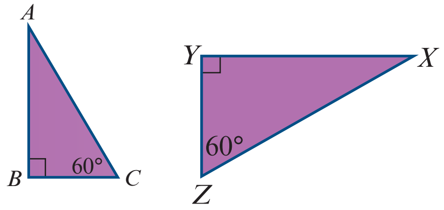
Example 31

1 Three pairs of similar triangles are shown below. Explain why each pair of triangles are similar.

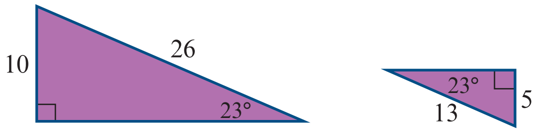
a



b



c



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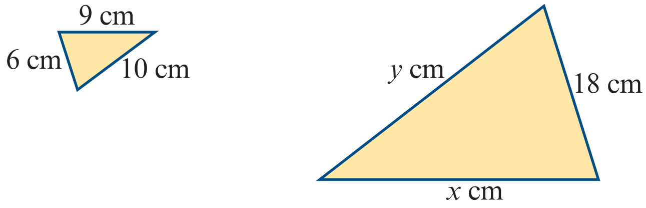
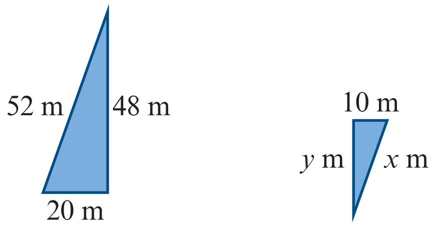
2 Calculate the missing dimensions, marked x and y, in these pairs of similar triangles.

a



b



3 A triangle with sides 5 cm, 4 cm and 8 cm is similar to a larger triangle with a longest side of 56 cm.

a Find the lengths of the larger triangle's other two sides.

b Find the perimeter of the larger triangle.

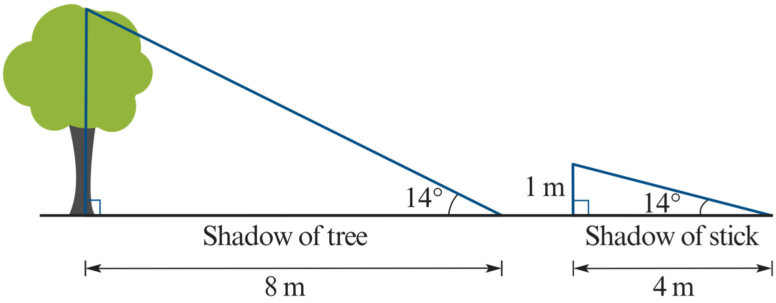
Applications of similar triangles

4 A tree and a 1 m vertical stick cast their shadows at a particular time in the day. The shadow lengths are shown in the diagram below (*not* drawn to scale).

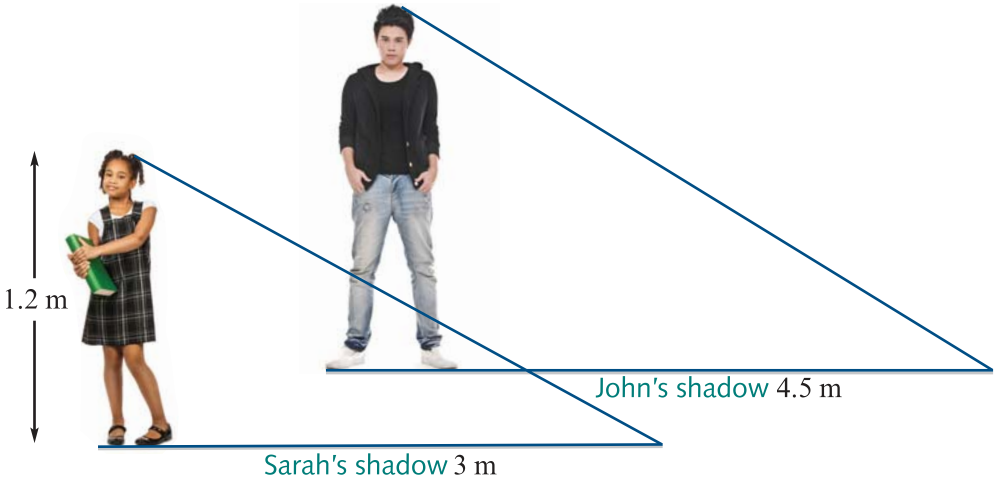
a Give reasons why the two triangles shown are similar.

b Find the scale factor for the side lengths of the triangles.

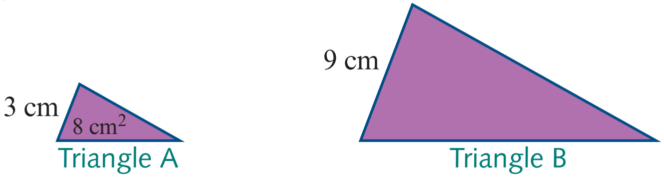
c Find the height of the tree.



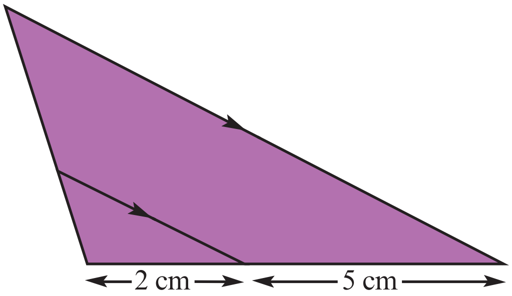
5 John and his younger sister, Sarah, are standing side by side. Sarah is 1.2 m tall and casts a shadow 3 m long. How tall is John if his shadow is 4.5 m long?



6 The area of triangle A is 8 cm^2. Triangle B is similar to triangle A. What is the area of triangle B?



7 Given that the area of the small triangle in the following diagram is 2.4 cm^2, find the area of the larger triangle correct to two decimal places.



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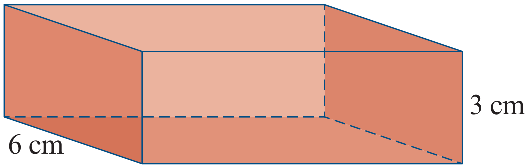
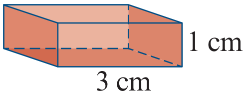
10L Similar solids

Two solids are similar if they have the same shape and the ratios of their corresponding linear dimensions are equal.

▶ Cuboids



Cuboid A



Cuboid B

The two cuboids are similar because:

- they are the same shape (both are cuboids)
- the ratios of the corresponding dimensions are the same.

$$\frac{\text{length of cuboid A}}{\text{length of cuboid B}} = \frac{\text{width of cuboid A}}{\text{width of cuboid B}} = \frac{\text{height of cuboid A}}{\text{height of cuboid B}}$$

$$\frac{6}{2} = \frac{9}{3} = \frac{3}{1} = \frac{3}{1}$$

$$\text{Volume scale factor, } k^3 = \text{Ratio of volumes} = \frac{6 \times 9 \times 3}{2 \times 3 \times 1} = \frac{162}{6} = \frac{27}{1} = \frac{3^3}{1}$$

As the length dimensions are enlarged by a scale factor of 3, the volume is enlarged by a scale factor of $3^3 = 27$.

Scaling volumes

When all the dimensions are multiplied by a scale factor of k, the volume is multiplied by a scale factor of k^3.

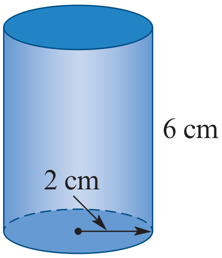
▶ Cylinders

These two cylinders are similar because:

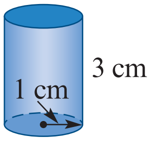
- they are the same shape (both are cylinders)
- the ratios of the corresponding dimensions are the same.

$$\frac{6}{3} = \frac{2}{1}$$

$$\frac{\text{height of cylinder A}}{\text{height of cylinder B}} = \frac{\text{radius of cylinder A}}{\text{radius of cylinder B}}$$



Cylinder A



Cylinder B

$$\text{Volume scale factor, } k^3 = \text{Ratio of volumes} = \frac{\pi \times 2^2 \times 6}{\pi \times 1^2 \times 3} = \frac{24}{3} = \frac{8}{1} = \frac{2^3}{1}$$

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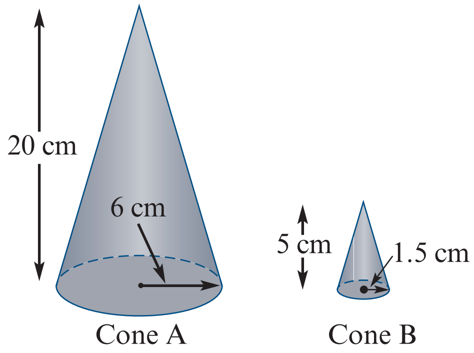
▶ Cones



These two cones are similar because:

- they are the same shape (both are cones)
- the ratios of the corresponding dimensions are the same.

$$\frac{20}{5} = \frac{6}{1.5} = \frac{4}{1}$$

$$\frac{\text{height of cone A}}{\text{height of cone B}} = \frac{\text{radius of cone A}}{\text{radius of cone B}}$$

$$\text{Volume scale factor, } k^3 = \text{Ratio of volumes} = \frac{\frac{1}{3} \times \pi \times 6^2 \times 20}{\frac{1}{3} \times \pi \times 1.5^2 \times 5} = \frac{720}{11.25}$$

$$= \frac{64}{1} = \frac{4^3}{1}$$

Example 32 Comparing volumes of similar solids

Two solids are similar such that the larger one has all of its dimensions three times that of the smaller solid. How many times larger is the larger solid's volume?

Solution

1 Since all of the larger solid's dimensions are 3 times those of the smaller solid, the volume will be 3^3 times larger. Evaluate 3^3.

$3^3 = 27$

2 Write your answer.

The larger solid's volume is 27 times the volume of the smaller solid.

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Exercise 10L

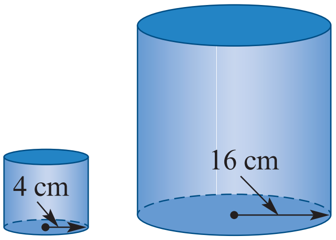
Scaling volumes and surface areas

Example 32

1 Two cylindrical water tanks are similar such that the height of the larger tank is 3 times the height of the smaller tank. How many times larger is the volume of the larger tank compared to the volume of the smaller tank?

2 Two cylinders are similar and have radii of 4 cm and 16 cm, respectively.

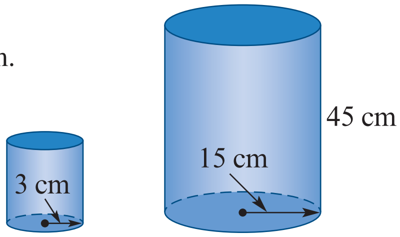
- **a** What is the ratio of their heights?
- **b** What is the ratio of their volumes?



3 Find the ratio of the volumes of two cuboids whose sides are in the ratio $\frac{3}{1}$.

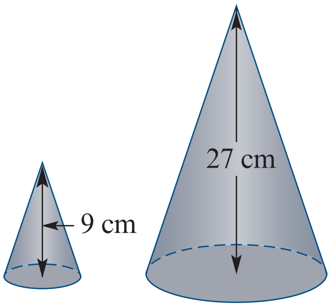
4 The radii of the bases of two similar cylinders are in the ratio $\frac{5}{1}$. The height of the larger cylinder is 45 cm. Calculate:

- **a** the height of the smaller cylinder
- **b** the ratio of the volumes of the two cylinders.



5 Two similar cones are shown at right. The ratio of their heights is $\frac{3}{1}$.

- **a** Determine whether the smaller cone has been scaled up or down to give the larger cone.
- **b** What is the volume scale factor?
- **c** The volume of the smaller cone is 120 cm^3. Find the volume of the larger cone.



6 The radii of the bases of two similar cylinders are in the ratio 3 : 4. The height of the larger cylinder is 8 cm. Calculate:

- **a** the height of the smaller cylinder
- **b** the ratios of the volumes of the two cylinders.

7 A pyramid has a square base of side 4 cm and a volume of 16 cm^3. Calculate:

- **a** the height of the pyramid
- **b** the height and the length of the side of the base of a similar pyramid with a volume of 1024 cm^3.

8 Two spheres have diameters of 12 cm and 6 cm respectively. Calculate:

- **a** the ratios of their surface areas
- **b** the ratio of their volumes.

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10M Problem solving and modelling

Exercise 10M

1 Brandon is building a new rectangular deck that will measure 5 m by 3 m. He has to order concrete for the 20 stump holes that he digs which measure 350 mm by 350 mm by 500 mm.

a What amount of concrete (in m^3) will he need to order? Give your answer correct to two decimal places.

b If the decking boards are 90 mm wide, how many 3 m length boards should he order?

c The decking is to be stained. What is the area?

Brandon builds 4 wooden planter boxes to place on the decking. The boxes have a square base of length 40 cm and a height of 60 cm.

d What is the volume of dirt, in cubic metres, required to fill these boxes if he fills them up to 10 cm from the top?

e Brandon also stains the planter boxes. What is the total surface area in m^2, correct to one decimal place (not counting the base)?

2 Farmer Green owns an irregularly shaped paddock, *APQBRSTA*, as shown in the diagram.

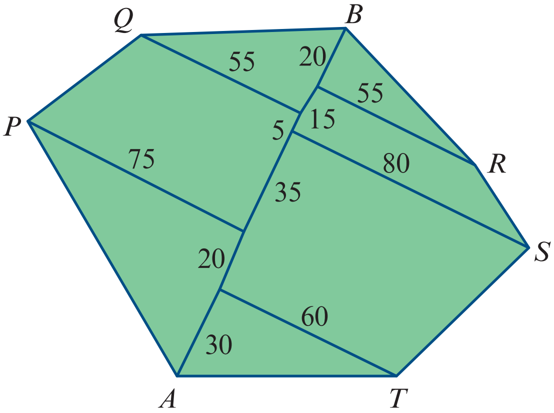
Starting at *A*, he has measured distances in metres along *AB* as well as the distances at right angles from *AB* to each other corner of his paddock.

Use this information to calculate:

a the total area of his paddock in hectares, correct to two decimal places.

(Note: 1 hectare = 10 000 m^2)

b the length of fencing needed to enclose the paddock.



3 A map is drawn to a scale of 1 : 20 000. A park, drawn on the map, has an area of 6 cm^2. Calculate, in m^2 the actual area of the park. If one hectare = 10 000 m^2, give your answer in hectares.

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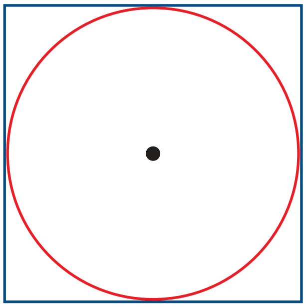
4 An architect uses a scale of 1 cm : 3 m for the plans of a house she is designing. On the plans, a room has an area of 3.3 cm^2.

a Calculate the actual area of the room in m^2.

b Another room in the same house is to have an actual area of 3.5 m^2. What area, in cm^2, would this be on the plans?

5 A 1 m piece of wire is to be cut into two pieces, one of which is bent into a circle (red). The other piece is bent into a square around the circle (blue).



a What is the length of the side of the square (to the nearest centimetre)?

b What are the lengths of the two pieces of wire?

6 A steel pipe has an outside diameter of 100 mm and an inside diameter of 80 mm.

a What is the surface area of its cross section? Give your answer in square centimetres, correct to two decimal places.

b The pipe is 90 cm long.
What is the inside volume, in cubic centimetres, correct to one decimal place?

c What amount of water, in litres, can pass through the pipe at one time?

d The pipe needs to be coated on the outside with a protective material.
What is its surface area, correct to one decimal place?

7 A cubic box has sides of length 10 cm.

a How many times will you have to enlarge the box by a scale factor of two, before it is too big for a room that is 3 m high with length 4 m and width 3 m.

b What is the volume of the enlarged box?

c What is the surface area, in cm^2, of the enlarged box?

d Using the same cube of side length 10 cm, how many times can you reduce it by a scale factor of $\frac{1}{2}$ before it becomes smaller than 1 mm by 1 mm by 1 mm?

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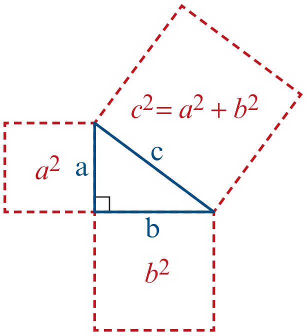
Key ideas and chapter summary

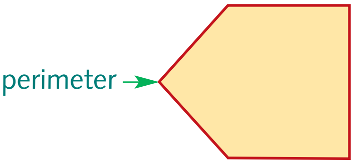
Pythagoras' theorem

Pythagoras' theorem states that: For any right-angled triangle, the sum of the areas of the squares of the two shorter sides (a and b) equals the area of the square of the hypotenuse (c): $c^2 = a^2 + b^2$



Perimeter (P)

Perimeter is the distance around the edge of a two-dimensional shape.



Perimeter of rectangle

$P = 2l + 2w$

Circumference (C)

Circumference is the perimeter of a circle $C = 2\pi r$

Area (A)

Area is the measure of the region enclosed by the boundaries of a two-dimensional shape.

Area formulas

Area of rectangle $= lw$

Area of parallelogram $= bh$

Area of triangle $= \frac{1}{2}bh$

Area of trapezium $= \frac{1}{2}(a + b)h$

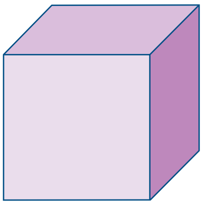
Heron's formula

Area of triangle $= A = \sqrt{s(s-a)(s-b)(s-c)}$ where $s = \dfrac{a+b+c}{2}$ and a, b and c are the sides of the triangle.

Volume (V)

Volume is the amount of space occupied by a three-dimensional object.

- For prisms and cylinders,
  Volume = area of cross-section × height
- For pyramids and cones,
  Volume $= \frac{1}{3} \times$ area of base × height



Volume formulas

Volume of cube $= l^3$

Volume of cuboid $= lwh$

Volume of triangular prism $= \frac{1}{2}bhl$

Volume of cylinder $= \pi r^2 h$

Volume of cone $= \frac{1}{3}\pi r^2 h$

Volume of pyramid $= \frac{1}{3}lwh$

Volume of sphere $= \frac{4}{3}\pi r^3$

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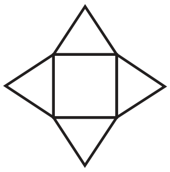
Surface area (*SA*) **Surface area** is the total of the areas of all the surfaces of a solid. When finding surface area, it is often useful to draw the net of the shape.



Surface area formulas

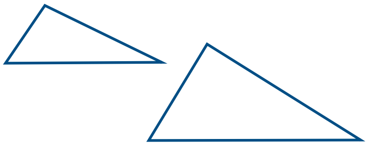
Surface area of cylinder $2\pi r^2 + 2\pi rh$

Surface area of cone $= \pi r^2 + \pi rs$

Surface area of sphere $= 4\pi r^2$

Similar figures or solids **Similar figures or solids** are the same shape but different sizes.

Similar triangles Triangles are shown to be **similar** if:



- corresponding angles are similar (AA)
- corresponding sides are in the same ratio (SSS)
- two pairs of corresponding sides are in the same ratio and the included corresponding angles are equal (SAS).

Ratios of area and volume for similar shapes When all the dimensions of similar shapes are multiplied by a scale factor of k, the areas are multiplied by a scale factor of k^2 and the volumes are multiplied by a scale factor of k^3.

Skills check

Having completed this chapter you should be able to:

- understand and use Pythagoras' theorem to solve two-dimensional and three-dimensional problems
- find the areas and perimeters of two-dimensional shapes
- find the volumes of common three-dimensional shapes
- find the volumes of pyramids, cones and spheres
- find the surface areas of three-dimensional shapes
- use tests for similarity for two-dimensional and three-dimensional figures.

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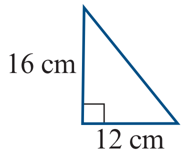
Multiple-choice questions

1 The three side measurements of four different triangles are given below. Which one is a right-angled triangle?

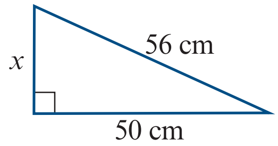
A 1, 2, 3 **B** 15, 20, 25 **C** 10, 10, 15 **D** 9, 11, 15 **E** 4, 5, 12

2 The length of the hypotenuse for the triangle shown is:



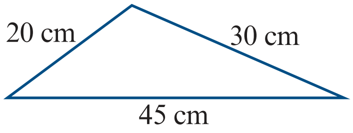
A 10.58 cm **B** 28 cm
C 7.46 cm **D** 20 cm
E 400 cm

3 The value of x in the triangle shown is:



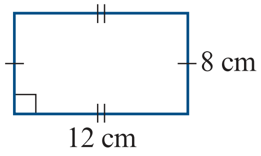
A 636 cm **B** 6 cm
C 75.07 cm **D** 25.22 cm
E 116 cm

4 The perimeter of the triangle shown is:



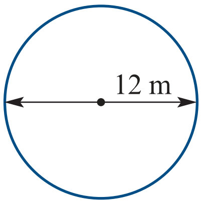
A 450 cm **B** 95 cm
C 95 cm^2 **D** 90 cm
E 50 cm

5 The perimeter of the rectangle shown is:



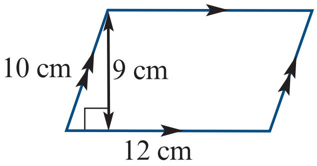
A 40 cm **B** 20 cm
C 96 cm **D** 32 cm
E 28 cm

6 The circumference of a circle with diameter 12 m is closest to:



A 18.85 m **B** 37.70 m
C 453.29 m **D** 113.10 m
E 118.44 m

7 The area of the shape shown is:



A 90 cm^2 **B** 120 cm^2
C 108 cm^2 **D** 44 cm^2
E 180 cm^2

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8 The area of a circle with radius 3 cm is closest to:

A 18.85 cm^2 **B** 28.27 cm^2 **C** 9.42 cm^2 **D** 113.10 cm^2 **E** 31.42 cm^2

9 The volume of a cube with side length 5 cm is:

A 60 cm^3 **B** 30 cm^3 **C** 150 cm^3 **D** 125 cm^3 **E** 625 cm^3

10 The volume of a box with length 11 cm, width 5 cm and height 6 cm is:

A 22 cm^3 **B** 44 cm^3 **C** 330 cm^3 **D** 302 cm^3 **E** 1650 cm^3

11 The volume of a sphere with radius 16 mm is closest to:

A 1072.33 mm^3 **B** 3217 mm^3 **C** 67.02 mm^3
D 268.08 mm^3 **E** 17 157.28 mm^3

12 The volume of a cone with base diameter 12 cm and height 8 cm is closest to:

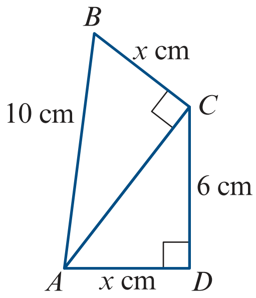
A 1206.37 cm^3 **B** 904.78 cm^3 **C** 3619.11 cm^3
D 301.59 cm^3 **E** 1809.56 cm^3

13 The volume of a cylinder with radius 3 m and height 4 m is closest to:

A 37.70 m^3 **B** 452.39 m^3 **C** 113.10 m^3
D 12 m^3 **E** 12.57 m^3

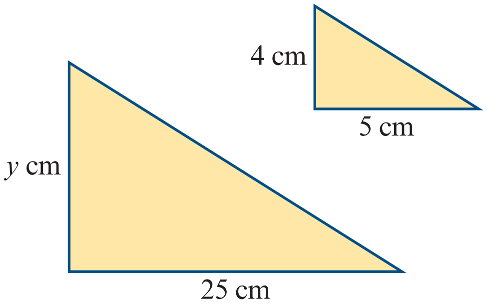
14 In the diagram, angles *ACB* and *ADC* are right angles. If *BC* and *AD* each have a length of *x* cm, then *x* is closest to:

A 8.25 **B** 4 **C** 5
D 5.66 **E** 7.07



15 The two triangles shown are similar.
The value of *y* is:

A 9 cm **B** 24 cm
C 20 cm **D** 21 cm
E 16 cm



16 The diameter of a large sphere is 4 times the diameter of a smaller sphere. It follows that the ratio of the volume of the large sphere to the volume of the smaller sphere is:

A 4 : 1 **B** 8 : 1 **C** 16 : 1 **D** 32 : 1 **E** 64 : 1

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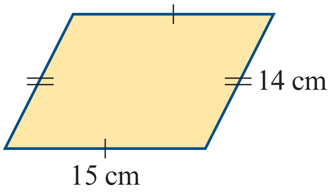
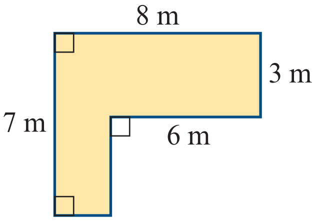
Short-answer questions

1 Find the perimeters of these shapes.

a



b

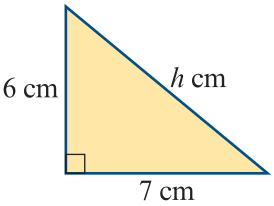


2 Find the perimeter of a square with side length 9 m.

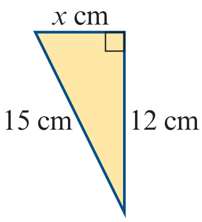
3 Find the perimeter of a rectangle with length 24 cm and width 10 cm.

4 Find the lengths of the unknown sides, correct to two decimal places, in the following triangles.

a

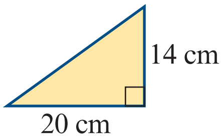


b

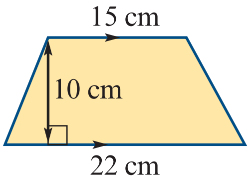


5 Find the areas of the following shapes.

a



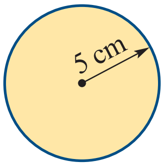
b



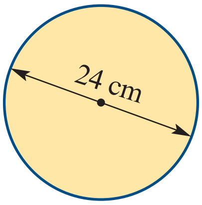
6 Find the surface area of a cube with side length 2.5 m.

7 Find the circumferences of the following circles, correct to two decimal places.

a



b



8 Find the areas of the circles in Question **7**, correct to two decimal places.

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9 A soup can has a diameter of 7 cm and a height of 13.5 cm.

a How much metal, correct to two decimal places, is needed to make the can?

b A paper label is made for the outside cylindrical shape of the can. How much paper, in m^2, is needed for 100 cans? Give your answer correct to two decimal places.

c What is the capacity of one can, in litres, correct to two decimal places?

10 A circular swimming pool has a diameter of 4.5 m and a depth of 2 m. How much water will the pool hold, to the nearest litre?

11 The radius of the Earth is approximately 6400 km. Calculate:

a the surface area in square kilometres

b the volume correct to four significant figures.

12 The diameter of the base of an oilcan in the shape of a cone is 12 cm and its height is 10 cm. Find:

a its volume in square centimetres, correct to two decimal places

b its capacity to the nearest millilitre.

13 A right pyramid with a square base of side length 8 m has a height of 3 m. Find the length of a sloping edge, correct to one decimal place.

14 For the solid shown on the right, find correct to two decimal places:

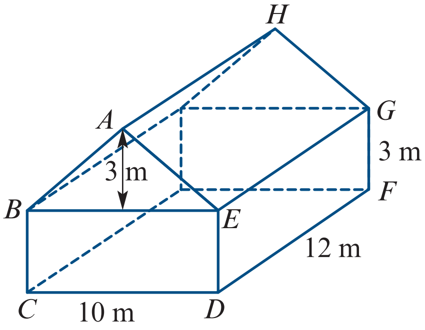
a the area of rectangle *BCDE*

b the area of triangle *ABE*

c the length *AE*

d the area of rectangle *AEGH*

e the total surface area.



15 Find the volume of a rectangular prism with length 3.5 m, width 3.4 m and height 2.8 m.

16 You are given a circle of radius *r*. The radius increases by a scale of factor of 2. By what factor does the area of the circle increase?

17 You are given a circle of diameter *d*. The diameter decreases by a scale factor of $\frac{1}{2}$. By how much does the area of the circle decrease?

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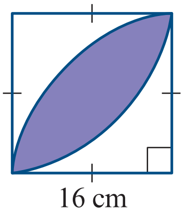
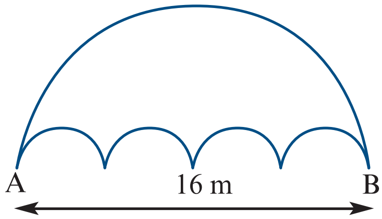
18 For the shaded region, find, correct to two decimal places:

a the perimeter

b the area of the shaded region shown.



19 Which is the shorter path from *A* to *B*? Is it along the four semi-circles or along the larger semi-circle? Give reasons for your answer.

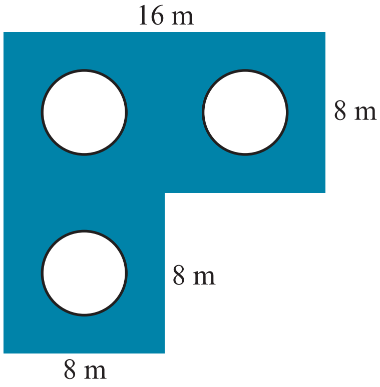


Extended-response questions

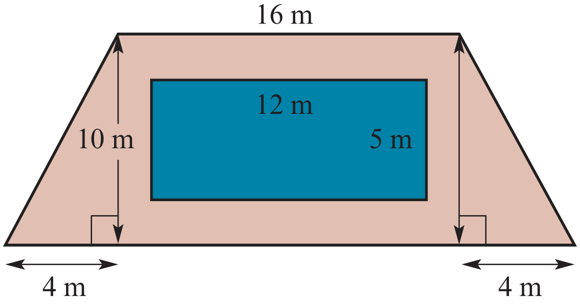
1 A lawn has three circular flowerbeds in it, as shown in the diagram. Each flowerbed has a radius of 2 m. A gardener has to mow the lawn and use a whipper-snipper to trim all the edges. Calculate:

a the area to be mown

b the length of the edges to be trimmed. Give your answer correct to two decimal places.



2 Chris and Gayle decide to build a swimming pool on their new housing block. The pool will measure 12 m by 5 m and it will be surrounded by timber decking in a trapezium shape. A safety fence will surround the decking. The design layout of the pool and surrounding area is shown in the diagram.



a What length of fencing is required? Give your answer correct to two decimal places.

b What area of timber decking is required?

c The pool has a constant depth of 2 m. What is the volume of the pool?

d The interior of the pool is to be painted white. What surface area is to be painted?

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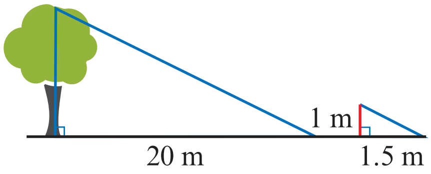
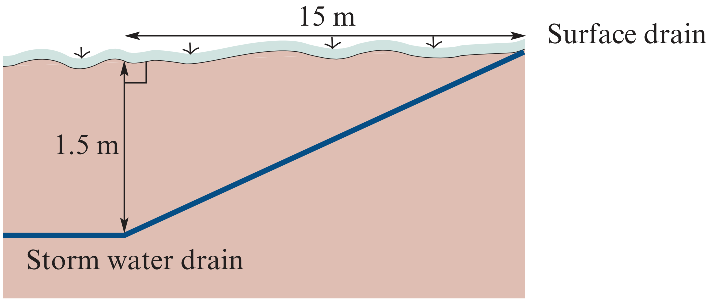
3 A biologist studying gum trees wanted to calculate the height of a particular tree. She placed a 1 m ruler on the ground, which cast a shadow on the ground measuring 1.5 m. The gum tree cast a shadow of 20 m, as shown in the diagram below (*not* to scale). Calculate the height of the tree. Give your answer correct to two decimal places.



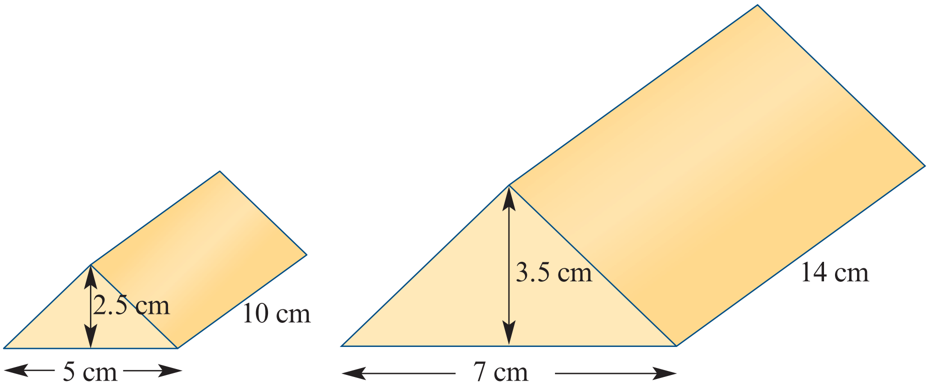
4 A builder is digging a trench for a cylindrical water pipe. From a drain at ground level, the water pipe goes 1.5 m deep, where it joins a storm water drain. The horizontal distance from the surface drain to the storm water drain is 15 m, as indicated in the diagram below (*not* to scale).



a Calculate the length of water pipe required to connect the surface drain to the storm water drain, correct to two decimal places.

b If the radius of the water pipe is 20 cm, what is the volume of the water pipe? Give your answer correct to two decimal places.

5 Two similar triangular prisms are shown below.



a Find the ratio of their surface areas.

b Find the ratio of their volumes.

c What is the volume of the smaller prism to the nearest cm^3?

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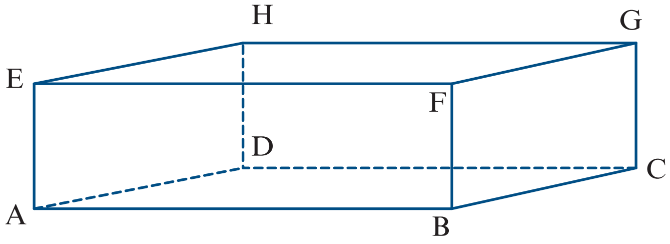
6 The length of a rectangular prism is eight times its height. The width is four times the height. The length of the diagonal between two opposite vertices (AG) is 36 cm. Find the volume of the prism.



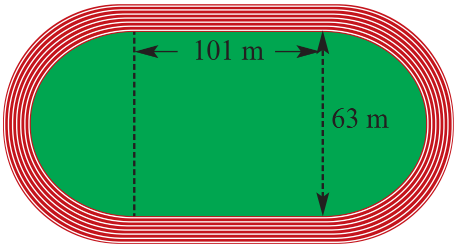
7 The volume of a cone of height 28.4 cm is 420 cm^3. Find the height of a similar cone whose volume is 120 cm^3, correct to two decimal places.

8 An athletics track is made up of a straight stretch of 101 m and two semi-circles on the ends as shown in the diagram. There are 6 lanes each 1 metre wide.

a What is the total distance, to the nearest metre, around the inside lane?

b If 6 athletes run around the track keeping to their own lane, how far, to the nearest metre, would each athlete run?

c Draw a diagram and indicate at which point each runner should start so that they all run the same distance.



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Chapter 11

11

Applications of trigonometry

- How are sin θ, cos θ and tan θ defined using a right-angled triangle?
- How can the trigonometric ratios be used to find the side lengths or angles in right-angled triangles?
- What is meant by an angle of elevation or an angle of depression?
- How are three-figure bearings measured?
- How can the sine and cosine rules be used to solve non-right-angled triangles?
- What are the three rules that can be used to find the area of a triangle?

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Introduction

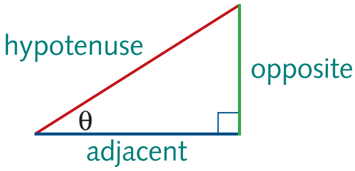
Trigonometry can be used to solve many practical problems. How high is that tree? What is the height of the mountain we can see in the distance? What is the exact location of the fire that has just been seen by fire spotters? How wide is the lake? What is the area of this irregular-shaped paddock?

11A Trigonometry basics

Although you are likely to have studied some trigonometry, it may be helpful to review a few basic ideas.

Naming the sides of a right-angled triangle

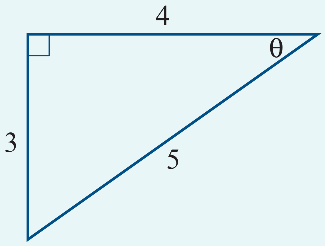
- The *hypotenuse* is the longest side of the right-angled triangle and is always opposite the right angle (90°).
- The *opposite* side is directly opposite the angle θ.
- The *adjacent* side is beside the angle θ, but it is not the hypotenuse. It runs from θ to the right angle.



The opposite and adjacent sides are located in relation to the position of angle θ. If θ was in the other corner, the sides would have to swap their labels. The letter θ is the Greek letter *theta*. It is commonly used to label an angle.

Example 1 **Identifying the sides of a right-angled triangle**

Give the lengths of the hypotenuse, the opposite side and the adjacent side in the triangle shown.



Solution

The hypotenuse is opposite the right angle.	The hypotenuse, $h = 5$
The opposite side is opposite the angle θ.	The opposite side, $o = 3$
The adjacent side is beside θ, but is not the hypotenuse.	The adjacent side, $a = 4$

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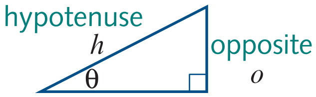
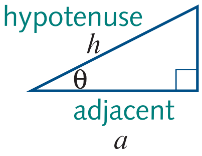
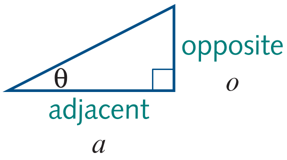
The trigonometric ratios

The **trigonometric ratios sin** θ, **cos** θ and **tan** θ can be defined in terms of the sides of a right-angled triangle.

		
$\sin\theta = \dfrac{\text{opposite}}{\text{hypotenuse}}$ $\sin\theta = \dfrac{o}{h}$	$\cos\theta = \dfrac{\text{adjacent}}{\text{hypotenuse}}$ $\cos\theta = \dfrac{a}{h}$	$\tan\theta = \dfrac{\text{opposite}}{\text{adjacent}}$ $\tan\theta = \dfrac{o}{a}$

“SOH — CAH — TOA”

This mnemonic **SOH-CAH-TOA** is often used by students to help them remember the rule for each trigonometric ratio.

In this mnemonic:

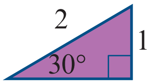
- **SOH** reminds us that sine equals opposite over hypotenuse
- **CAH** reminds us that cosine equals adjacent over hypotenuse
- **TOA** reminds us that tan equals opposite over adjacent

Or you may prefer:

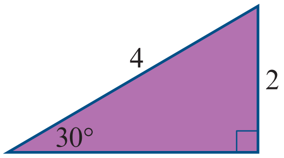
‘**S**ir **O**liver’s **H**orse **C**ame **A**mbling **H**ome **T**o **O**liver’s **A**rms’

The meaning of the trigonometric ratios

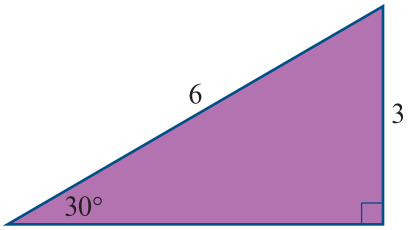
Using a calculator, we find, for example, that sin 30° = 0.5. This means that in *all* right-angled triangles with an angle of 30°, the ratio of the side opposite the 30° to the hypotenuse is always 0.5.



$$\frac{\text{opposite}}{\text{hypotenuse}} = \frac{1}{2} = 0.5$$



$$\frac{\text{opposite}}{\text{hypotenuse}} = \frac{2}{4} = 0.5$$



$$\frac{\text{opposite}}{\text{hypotenuse}} = \frac{3}{6} = 0.5$$

Try drawing any right-angled triangle with an angle of 30° and check that the ratio:

$$\frac{\text{opposite}}{\text{hypotenuse}} = 0.5$$

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Similarly, for *any* right-angled triangle with an angle of 30° the ratios cos 30° and tan 30° always have the same values:

$$\cos 30^\circ = \frac{\text{adjacent}}{\text{hypotenuse}} \text{ is always } \frac{\sqrt{3}}{2} = 0.8660 \text{ (to four decimal places)}$$

$$\tan 30^\circ = \frac{\text{opposite}}{\text{adjacent}} \text{ is always } \frac{1}{\sqrt{3}} = 0.5774 \text{ (to four decimal places).}$$

A calculator gives the value of each trigonometric ratio for any angle entered.

▶ Using you CAS calculator to evaluate trigonometric ratios

Warning!

Make sure that your calculator is set in DEGREE mode before attempting the following example.

See Appendix page 646

Example 2 Finding the values of trigonometric ratios

Use your graphics calculator to find, correct to four decimal places, the value of:

a sin 49° **b** cos 16° **c** tan 27.3°

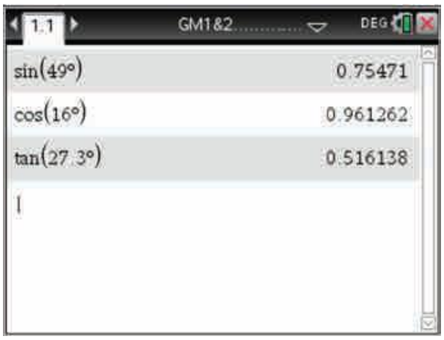
Solution

1 For the **TI-Nspire CAS** ensure that the mode is set in **Degree** and **Approximate (Decimal)**. Refer to Appendix to set mode.

2 In a Calculator page, select **sin** from the ⊞ palette and type 49.

3 Repeat for **b** and **c** as shown on the calculator screen.

Optional: you can add a degree symbol from the $\boxed{\alpha\beta^\circ}$ palette if desired. This will override any mode settings.



```
sin(49°)        0.75471
cos(16°)        0.961262
tan(27.3°)      0.516138
```

4 Write your answer correct to four decimal places.

a sin (49°) = 0.7547
b cos (16°) = 0.9613
c tan (27.3°) = 0.5161

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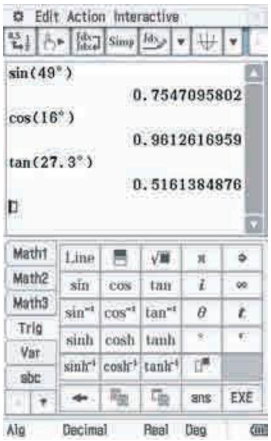
5 For **ClassPad**, in the Main application ensure that the status bar is set to **Decimal** and **Degree** mode.

6 To enter and evaluate the expression:
- Display the **keyboard**
- In the Trig palette select $\boxed{\sin}$
- Type $\boxed{4}\boxed{9}\boxed{^\circ}\boxed{)}$
- Press $\boxed{\text{EXE}}$

7 Repeat for **b** and **c** as shown on the calculator screen.



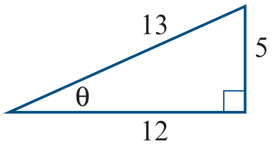
```
sin(49°)
                0.7547095802
cos(16°)
                0.9612616959
tan(27.3°)
                0.5161384876
```

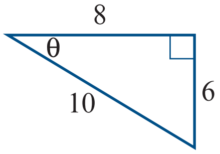
8 Write your answer correct to four decimal places.

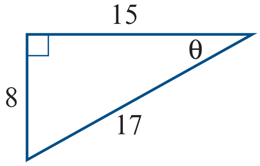
a sin (49°) = 0.7547
b cos (16°) = 0.9613
c tan (27.3°) = 0.5161

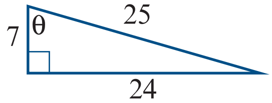
Exercise 11A

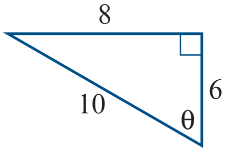
Example 1 **1** State the values of the hypotenuse, the opposite side and the adjacent side in each triangle.

a


b


c


d


e


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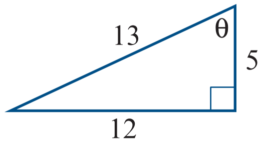

Example 2 **2** Write the ratios for sin θ, cos θ and tan θ for each triangle in Question **1**.

3 Find the values of the following trigonometric ratios, correct to four decimal places.

a sin 27° **b** cos 43° **c** tan 62° **d** cos 79°
e tan 14° **f** sin 81° **g** cos 17° **h** tan 48°
i sin 80° **j** sin 49.8° **k** tan 80.2° **l** cos 85.7°

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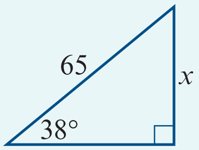
11B Finding an unknown side in a right-angled triangle

The trigonometric ratios can be used to find unknown sides in a right-angled triangle, given an angle and one side. When the unknown side is in the *numerator* (top) of the trigonometric ratio, proceed as follows.

Example 3 Finding an unknown side

Find the length of the unknown side x in the triangle shown, correct to two decimal places.



Solution

1 The sides involved are the opposite and the hypotenuse, so use sin θ.

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

2 Substitute in the known values.

$$\sin 38^\circ = \frac{x}{65}$$

3 Multiply both sides of the equation by 65 to obtain an expression for x. Use a calculator to evaluate.

$$65 \times \sin 38^\circ = x$$
$$x = 65 \times \sin 38^\circ$$
$$= 40.017\ldots$$

4 Write your answer correct to two decimal places.

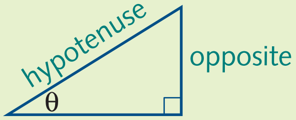
$$x = 40.02$$

Finding an unknown side in a right-angled triangle

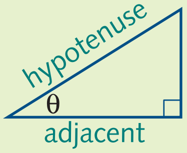
1 Draw the triangle and write in the given angle and side. Label the unknown side as x.

2 Use the trigonometric ratio that includes the given side and the unknown side.

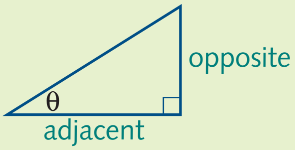
a For the opposite and the hypotenuse, use $\sin\theta = \dfrac{\text{opposite}}{\text{hypotenuse}}$



b For the adjacent and the hypotenuse, use $\cos\theta = \dfrac{\text{adjacent}}{\text{hypotenuse}}$



c For the opposite and the adjacent, use $\tan\theta = \dfrac{\text{opposite}}{\text{adjacent}}$



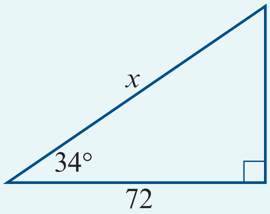
3 Rearrange the equation to make x the subject.

4 Use your calculator to find the value of x to the required number of decimal places.

An extra step is needed when the unknown side is in the *denominator* (at the bottom) of the trigonometric ratio.

Example 4 Finding an unknown side which is in the denominator of the trig ratio

Find the value of x in the triangle shown, correct to two decimal places.



Solution

1 The sides involved are the adjacent and the hypotenuse, so use cos θ.

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

2 Substitute in the known values.

$$\cos 34^\circ = \frac{72}{x}$$

3 Multiply both sides by x.

$$x\cos 34^\circ = 72$$

4 Divide both sides by cos 34° to obtain an expression for x. Use a calculator to evaluate.

$$x = \frac{72}{\cos 34^\circ} = 86.847\ldots$$

5 Write your answer correct to two decimal places.

$$x = 86.85$$

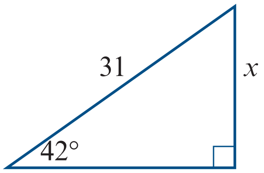
Exercise 11B

Example 3

1 In each right-angled triangle below:

- decide whether the sin θ, cos θ or tan θ ratio should be used
- then find the unknown side x, correct to two decimal places.

a



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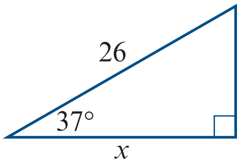

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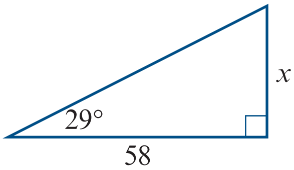

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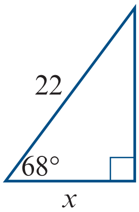

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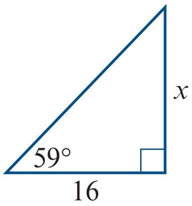

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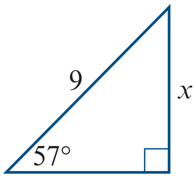

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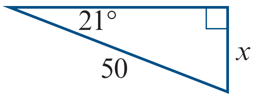

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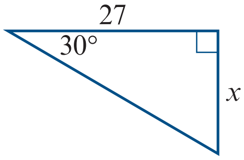

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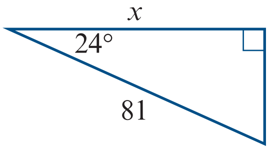

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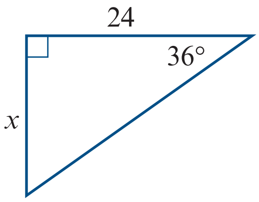

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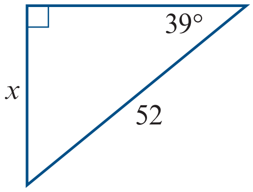

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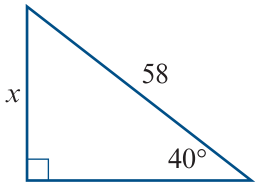
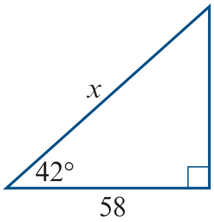


Example 4

2 Find the unknown side x in each right-angled triangle below, correct to two decimal places.

a



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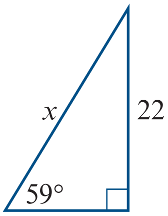

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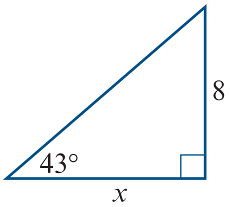

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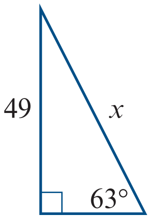

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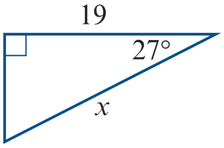

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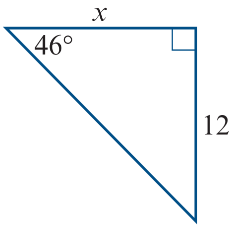

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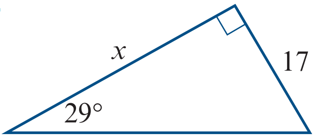

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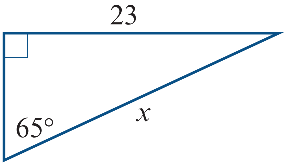

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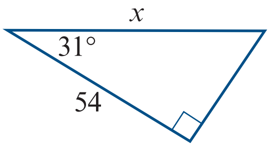

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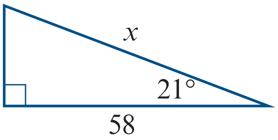

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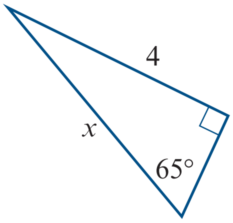

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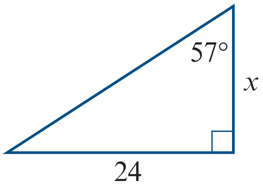
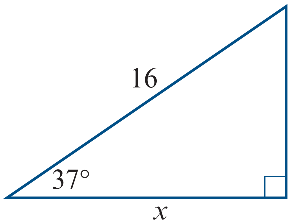


3 Find the length of the unknown side shown in each triangle, correct to one decimal place.

a



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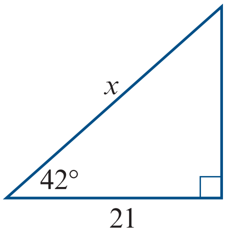

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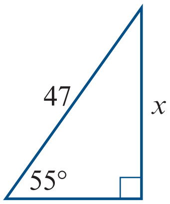

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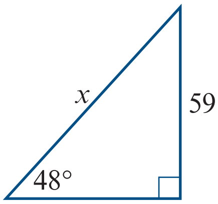

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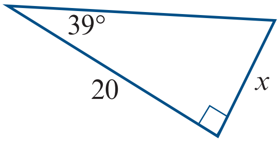

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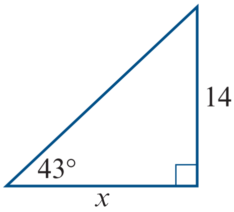

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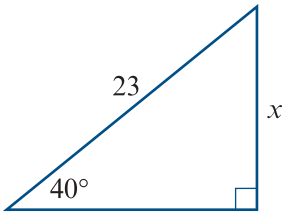

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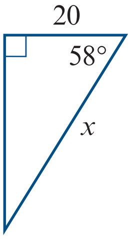

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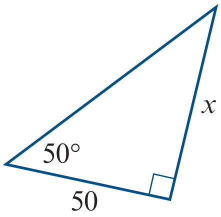

11C Finding an angle in a right-angled triangle

▶ Finding an angle from a trigonometric ratio value

Before we look at how to find an unknown angle in a right-angled triangle, it will be useful to see how to find the angle when we know the value of the trigonometric ratio.

Suppose a friend told you that they found the sine value of a particular angle to be 0.8480 and challenged you to find out the mystery angle that had been used.

This is equivalent to saying:

$\sin \theta = 0.8480$, find the value of angle θ.

To do this, you need to work backwards from 0.8480 by undoing the sine operation to get back to the angle used. It is as if we have to find reverse gear to undo the effect of the sine function.

The reverse gear for sine is called the inverse of sine, written $\sin^{-1}$. The superscript -1 is not a power. It is just saying let us undo, or take one step backwards from using, the sine function.

The request to find θ when $\sin \theta = 0.8480$ can be written as:

$$\sin^{-1}(0.8480) = \theta$$

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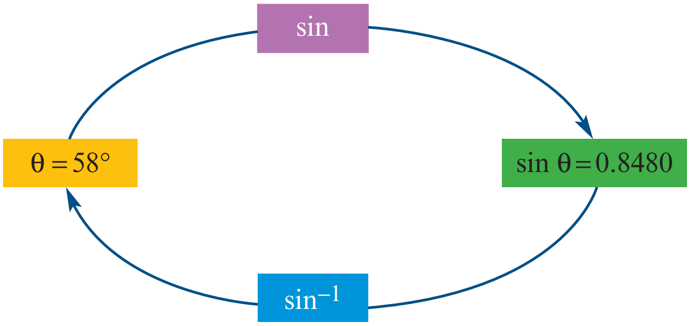
This process is summarised in the following diagram.

- The *top arrow* in the diagram corresponds to: given θ, find sin θ. We use the sine function on our calculator to do this by entering sin 58° into a calculator to obtain the answer 0.8480.
- The *bottom arrow* in the diagram corresponds to: given sin θ = 0.8480, find θ. We use the $\sin^{-1}$ function on our calculator to do this by entering $\sin^{-1}(0.8480)$ to obtain the answer 58°.



Similarly:

- The *inverse of cosine*, written as $\cos^{-1}$, is used to find θ when cos θ = 0.5 (e.g.).
- The *inverse of tangent*, written $\tan^{-1}$, is used to find θ when tan θ = 1.67 (e.g.).

You will learn how to how to use the $\sin^{-1}$, $\cos^{-1}$, $\tan^{-1}$ function of your calculator in the following example.

Example 5 Finding an angle from a trigonometric ratio

Find the angle θ, correct to one decimal place, given:

a $\sin\theta = 0.8480$ **b** $\cos\theta = 0.5$ **c** $\tan\theta = 1.67$

Solution

a We need to find $\sin^{-1}(0.8480)$.

1 For **TI-Nspire CAS**, press trig, select $\sin^{-1}$, then press 0 . 8 4 8 0 enter.

$\sin^{-1}(0.848)$ 57.9948

2 For **ClassPad**, tap $\sin^{-1}$ 0 . 8 4 8 0) EXE.

3 Write your answer, correct to one decimal place

θ = 58.0°

b We need to find $\cos^{-1}$ (0.5).

1 For **TI-Nspire CAS**, press trig, select $\cos^{-1}$, then press 0 . 5 enter.

$\cos^{-1}(0.5)$ 60

2 For **ClassPad**, tap $\cos^{-1}$ 0 . 5) EXE.

3 Write your answer, correct to one decimal place.

$\theta = 60°$

c We need to find $\tan^{-1}$ (1.67).

1 For **TI-Nspire CAS**, press trig, select $\tan^{-1}$, then press 1 . 6 7 enter.

$\tan^{-1}(1.67)$ 59.0867

2 For **ClassPad**, tap $\tan^{-1}$ 1 . 6 7) EXE.

3 Write your answer, correct to one decimal place.

$\theta = 59.1°$

Getting the language right

The language we use when finding an angle from a trig ratio is difficult when you first meet it. The samples below are based of the results of Example 5.

- When you see:

  $\sin(58°) = 0.8480$

  think 'the sine of the angle 58° equals 0.8480'.

- When you see:

  $\sin^{-1}(0.8480) = 58°$

  think 'the angle whose sine is 0.8480 equals 58°'.

- When you see:

  $\cos(60°) = 0.5$

  think 'the cosine of the angle 60° equals 0.5'.

- When you see:

  $\cos^{-1}(0.5) = 60°$

  think 'the angle whose cosine is 0.5 equals 60°'.

- When you see:

  $\tan(59.1°) = 1.67$

  think 'the tan of the angle 59.1° equals 1.67'.

- When you see:

  $\tan^{-1}(1.67) = 59.1°$

  think 'the angle whose tan is 1.67 equals 59.1°'.

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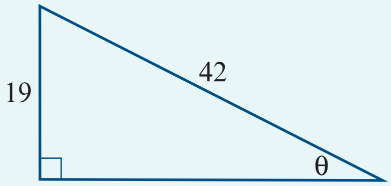
Finding an angle given two sides

Example 6 Finding an angle given two sides in a right-angled triangle

Find the angle θ, in the right-angled triangle shown, correct to one decimal place.



Solution

1 The sides involved are the opposite and the hypotenuse, so use sin θ.

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

2 Substitute in the known values.

$$\sin\theta = \frac{19}{42}$$

3 Write the equation to find an expression for θ. Use a calculator to evaluate.

$$\theta = \sin^{-1}\left(\frac{19}{42}\right) = 26.896\ldots$$

4 Write your answer, correct to one decimal place.

$$\theta = 26.9^\circ$$

The three angles in a triangle add to 180°. As the right angle is 90°, the other two angles must add to make up the remaining 90°. When one angle has been found just subtract it from 90° to find the other angle. In Example 6, the other angle must be $90^\circ - 26.9^\circ = 63.1^\circ$.

Finding an angle in a right-angled triangle

1 Draw the triangle with the given sides shown. Label the unknown angle as θ.

2 Use the trigonometric ratio that includes the two known sides.

- If given the opposite and hypotenuse, use $\sin\theta = \dfrac{\text{opposite}}{\text{hypotenuse}}$
- If given the adjacent and hypotenuse, use $\cos\theta = \dfrac{\text{adjacent}}{\text{hypotenuse}}$
- If given the opposite and adjacent, use $\tan\theta = \dfrac{\text{opposite}}{\text{adjacent}}$

3 Divide the side lengths to find the value of the trigonometric ratio.

4 Use the appropriate inverse function key to find the angle θ.

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Exercise 11C

Example 5 **1** Find the angle θ, correct to one decimal place.

a $\sin\theta = 0.4817$ **b** $\cos\theta = 0.6275$ **c** $\tan\theta = 0.8666$

d $\sin\theta = 0.5000$ **e** $\tan\theta = 1.0000$ **f** $\cos\theta = 0.7071$

g $\sin\theta = 0.8660$ **h** $\tan\theta = 2.500$ **i** $\cos\theta = 0.8383$

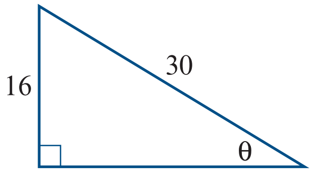
j $\sin\theta = 0.9564$ **k** $\cos\theta = 0.9564$ **l** $\tan\theta = 0.5774$

m $\sin\theta = 0.7071$ **n** $\tan\theta = 0.5000$ **o** $\cos\theta = 0.8660$

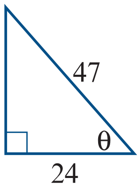
p $\cos\theta = 0.3414$

Example 6 **2** Find the unknown angle θ in each triangle, correct to one decimal place.

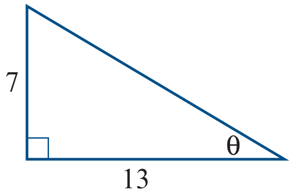
a Use $\sin^{-1}$ for this triangle.



b Use $\cos^{-1}$ for this triangle.



c Use $\tan^{-1}$ for this triangle.



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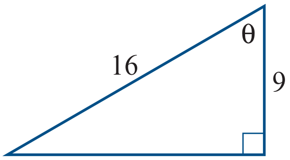

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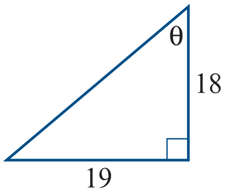

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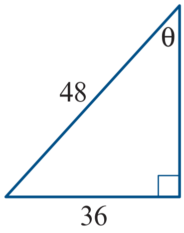

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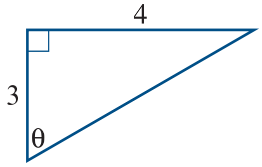

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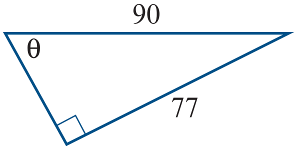

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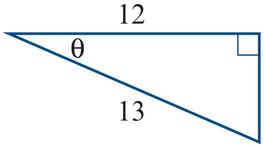

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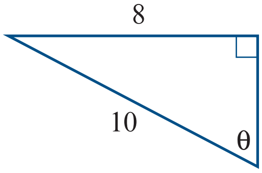

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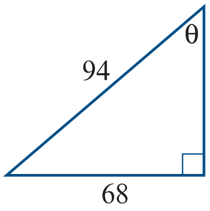

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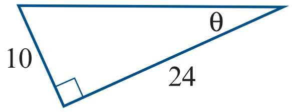

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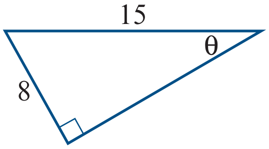

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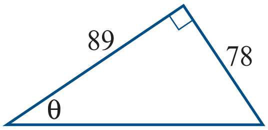

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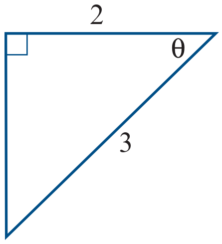

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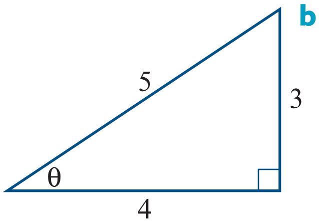
3 Find the value of θ in each triangle, correct to one decimal place.

a



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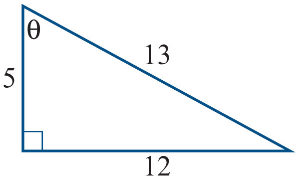

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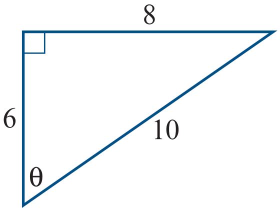

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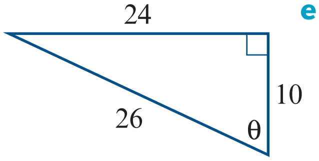

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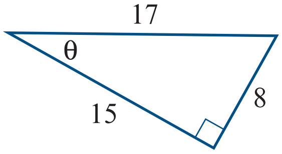

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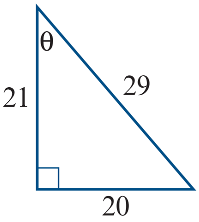
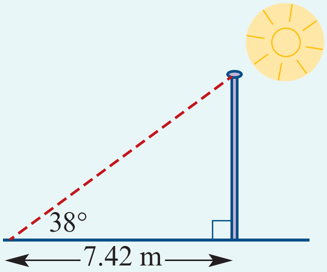


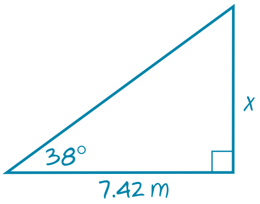
11D Applications of right-angled triangles

Example 7 Application requiring a length

A flagpole casts a shadow 7.42 m long. The sun's rays make an angle of 38° with the level ground. Find the height of the flagpole, correct to two decimal places.



Solution

1 Draw a diagram showing the right-angled triangle. Include all the known details and label the unknown side.	
2 The opposite and adjacent sides are involved so use tan θ.	$\tan\theta = \dfrac{\text{opposite}}{\text{adjacent}}$
3 Substitute in the known values.	$\tan 38° = \dfrac{x}{7.42}$
4 Multiply both sides by 7.42.	$7.42 \times \tan 38° = x$
5 Use your calculator to find the value of *x*.	$x = 5.797\ldots$
6 Write your answer correct to two decimal places.	The height of the flagpole is 5.80 m.

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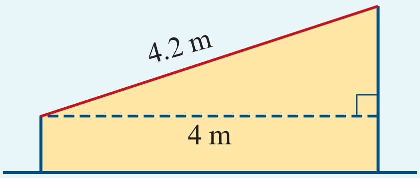
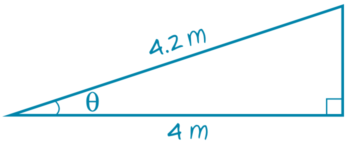
Example 8 Application requiring an angle

A sloping roof uses sheets of corrugated iron 4.2 m long on a shed 4 m wide. There is no overlap of the roof past the sides of the walls. Find the angle the roof makes with the horizontal, correct to one decimal place.



Solution

1 Draw a diagram showing the right-angled triangle. Include all known details and label the required angle.



2 The adjacent and hypotenuse are involved so use cos θ.

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

3 Substitute in the known values.

$$\cos\theta = \frac{4}{4.2}$$

4 Write the equation to find θ.

$$\theta = \cos^{-1}\left(\frac{4}{4.2}\right)$$

$$\theta = 17.752\ldots$$

$$\cos^{-1}\left(\frac{4}{4.2}\right) \qquad 17.7528$$

5 Use your calculator to find the value of θ.

6 Write your answer, correct to one decimal place.

The roof makes an angle of 17.8° with the horizontal.

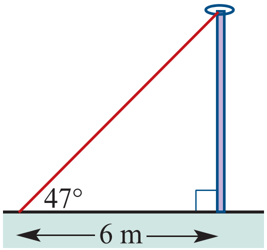
Warning!

Always evaluate a mathematical expression as a whole, rather than breaking it into several smaller calculations. Rounding-off errors accumulate as more approximate answers are fed into the calculations.

Exercise 11D

Example 7

1 A pole is supported by a wire that runs from the top of the pole to a point on the level ground 6 m from the base of the pole. The wire makes an angle of 47° with the ground. Find the height of the pole, correct to two decimal places.



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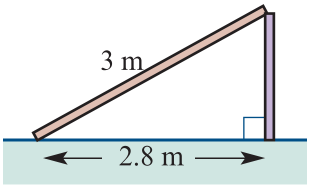
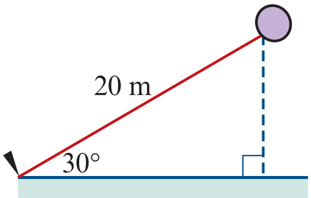
Example 8

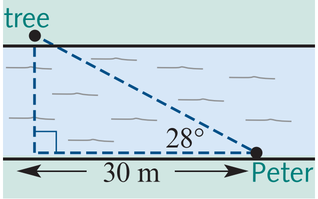
2 A 3 m log rests with one end on the top of a post and the other end on the level ground 2.8 m from the base of the post. Find the angle the log makes with the ground, correct to one decimal place.



3 A balloon is tied to a string 20 m long. The other end of the string is secured by a peg to the surface of a level sports field. The wind blows so that the string forms a straight line making an angle of 30° with the ground. Find the height of the balloon above the ground.



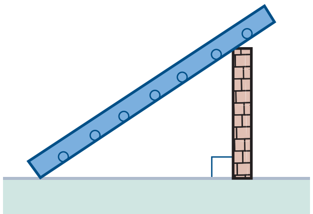
4 Peter noticed that a tree was directly opposite him on the far bank of the river. After he walked 30 m along his side of the river, he found that his line of sight to the tree made an angle of 28° with the riverbank. Find the width of the river, to the nearest metre.



5 A ladder rests on a wall 2 m high. The foot of the ladder is 3 m from the base of the wall on level ground.

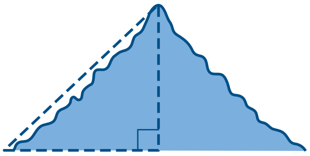
a Copy the diagram and include the given information. Label as θ the angle the ladder makes with the ground.

b Find the angle the ladder makes with the ground, correct to one decimal place.



6 The distance measured up the sloping face of a mountain was 3.8 km. The sloping face was at an angle of 52° with the horizontal.

a Make a copy of the diagram and show the known details. Show the height of the mountain as x.



b Find the height of the mountain, correct to one decimal place.

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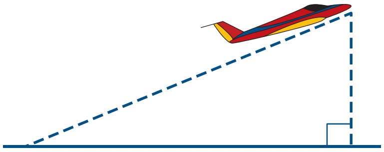
7 An aeroplane maintains a flight path of 17° with the horizontal after it takes off. It travels for 2 km along that flight path.



a Show the given and required information on a copy of the diagram.

b Find, correct to two decimal places:

i the horizontal distance of the aeroplane from its take-off point

ii the height of the aeroplane above ground level.

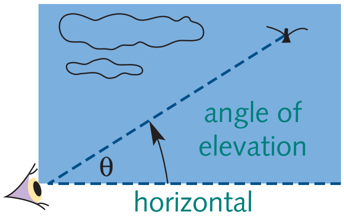
8 A 3 m ladder rests against an internal wall. The foot of the ladder is 1 m from the base of the wall. Find the angle the ladder makes with the floor, correct to one decimal place.

9 The entrance to a horizontal mining tunnel has collapsed, trapping the miners inside. The rescue team decide to drill a vertical escape shaft from a position 200 m further up the hill. If the hill slopes at 23° from the horizontal, how deep does the rescue shaft need to be to meet the horizontal tunnel? Answer correct to one decimal place.

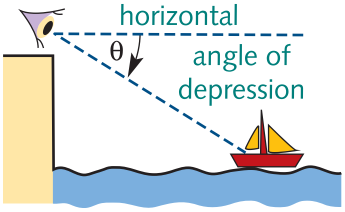
10 A strong rope needs to be fixed with one end attached to the top of a 5 m pole and the other end pegged at an angle of 60° with the level ground. Find the required length of the rope, correct to two decimal places.

11E Angles of elevation and depression

The **angle of elevation** is the angle through which you *raise* your line of sight from the horizontal when you are looking *up* at something.



The **angle of depression** is the angle through which you *lower* your line of sight from the horizontal when you are looking *down* at something.



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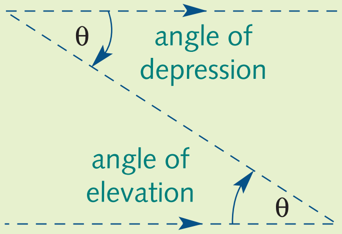
Angles of elevation and depression

angle of elevation = angle of depression

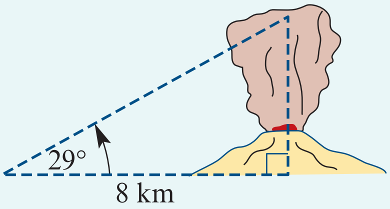
The diagram shows that the angle of elevation and the angle of depression are alternate angles ('Z' angles), so they are equal.



Applications of angles of elevation and depression

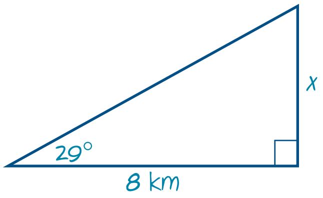
Example 9 Angle of elevation

A park ranger measured the top of a plume of volcanic ash to be at an angle of elevation of 29°. From her map she noted that the volcano was 8 km away. She calculated that the height of the plume to be 4.4 km. Show how she might have done this. Give your answer correct to one decimal place.



Solution

1 Draw a right-angled triangle showing the given information. Label the required height x.



2 The opposite and adjacent sides are involved so use tan θ.

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

3 Substitute in the known values.

$$\tan 29^\circ = \frac{x}{8}$$

4 Multiply both sides by 8.

$$8 \times \tan 29^\circ = x$$

5 Use your calculator to find the value of x.

$$x = 4.434\ldots$$

6 Write your answer correct to one decimal place.

The height of the ash plume was 4.4 km.

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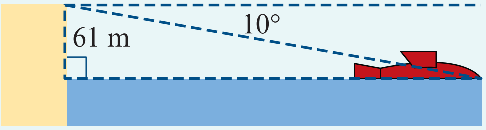
Example 10 Angle of depression

From the top of a cliff 61 m above sea level, Chen saw a capsized yacht. He estimated the angle of depression to be about 10°. How far was the yacht from the base of the cliff, to the nearest metre?

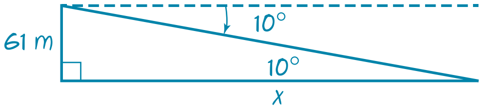


Solution

1 Draw a diagram showing the given information. Label the required distance x.

2 Mark in the angle at the yacht corner of the triangle. This is also 10°, because it and the angle of depression are alternate (or 'Z') angles.

Warning: The angle between the cliff face and the line of sight is *not* 10°.



3 The opposite and adjacent sides are involved so use tan θ.

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

4 Substitute in the known values.

$$\tan 10° = \frac{61}{x}$$

5 Multiply both sides by x.

$$x \times \tan 10° = 61$$

6 Divide both sides by tan 10°.

$$x = \frac{61}{\tan 10°}$$

7 Do the division using your calculator.

$$x = 345.948\ldots$$

8 Write your answer to the nearest metre.

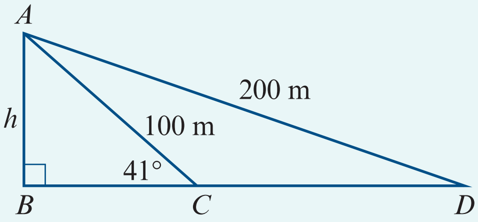
The yacht was 346 m from the base of the cliff.

Example 11 Application with two right-angled triangles

A cable 100 m long makes an angle of elevation of 41° with the top of a tower.

a Find the height, h, of the tower, to the nearest metre.

b Find the angle of elevation, α, to the nearest degree, that a cable 200 m long would make with the top of the tower.



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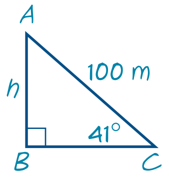
Solution

Strategy: Find h in triangle ABC, then use this value to find α in triangle ABD.

a **1** Draw triangle ABC showing the given and required information.



2 The opposite and hypotenuse are involved, so use sin θ.

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

3 Substitute in the known values.

$$\sin 41^\circ = \frac{h}{100}$$

4 Multiply both sides by 100.

$$h = 100 \times \sin 41^\circ$$
$$h = 65.605\ldots$$

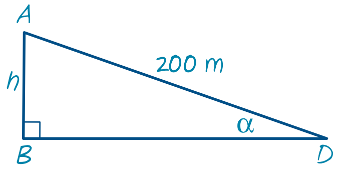
5 Evaluate 100 sin(41°) using your calculator and store the answer as the value of the variable h for later use.

$100 \cdot \sin(41^\circ) \rightarrow h$ 65.6059

6 Write your answer to the nearest metre.

The height of the tower is 66 m.

b **1** Draw triangle ABD showing the given and required information



2 The opposite and hypotenuse are involved, so use sin α.

$$\sin\alpha = \frac{\text{opposite}}{\text{hypotenuse}}$$

3 Substitute in the known values. In part a we stored the height of the tower as h.

$$\sin\alpha = \frac{h}{200}$$

4 Write the equation to find α.

$$\alpha = \sin^{-1}\left(\frac{h}{200}\right)$$
$$\alpha = 19.149\ldots$$

5 Use your calculate to evaluate α.

$100 \cdot \sin(41^\circ) \rightarrow h$ 65.6059

$\sin^{-1}\left(\frac{h}{200}\right)$ 19.1492

6 Write your answer to the nearest degree.

The 200 m cable would have an angle of elevation of 19°.

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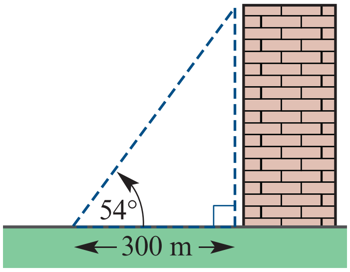
Exercise 11E

Skillsheet

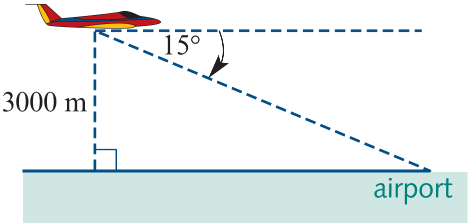
Example 9

1 After walking 300 m away from the base of a tall building, on level ground, Elise measured the angle of elevation to the top of the building to be 54°. Find the height of the building, to the nearest metre.



Example 10

2 The pilot of an aeroplane saw an airport at sea level at an angle of depression of 15°. His altimeter showed that the aeroplane was at a height of 3000 m. Find the horizontal distance of the aeroplane from the airport, to the nearest metre.



3 The angle of elevation measured from ground level to the top of a tall tree was 41°. The distance of the measurer from the base of the tree was 38 m. How tall was the tree? Give your answer to the nearest metre.

4 When Darcy looked from the top of a cliff, 60 m high, he noticed his girlfriend at an angle of depression of 20° on the ground below. How far was she from the cliff? Answer correct to one decimal place.

5 From the top of a mountain, I could see a town at an angle of depression of 1.4° across the level plain. Looking at my map I found that the town was 10 km away. Find the height of the mountain above the plain, to the nearest metre.

6 What would be the angle of elevation to the top of a radio transmitting tower 100 m tall and 400 m from the observer? Answer to the nearest degree.

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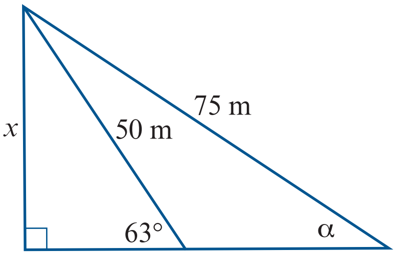
Example 11

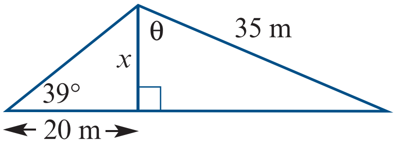
7 **a** Find the length x, correct to one decimal place.

b Find the angle α, to the nearest degree.



8 **a** Find the length x, correct to one decimal place.

b Find the angle θ, to the nearest degree.

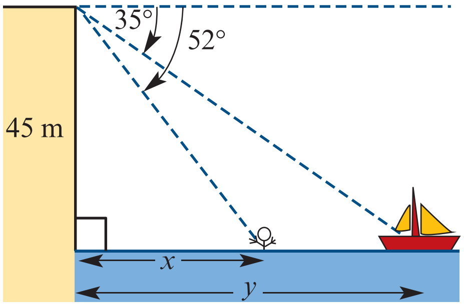


9 From the top of a cliff 45 m high, an observer looking along an angle of depression of 52° could see a man swimming in the sea. The observer could also see a boat at an angle of depression of 35°. Calculate to the nearest metre:

a the distance x of the man from the base of the cliff

b the distance y of the boat from the base of the cliff

c the distance from the man to the boat.



10 A police helicopter hovering in a fixed position at an altitude of 500 m moved its spotlight through an angle of depression of 57° onto a lost child. The pilot sighted the rescue team at an angle of depression of 31°. If the terrain was level, how far, to the nearest metre, was the rescue team from the child?

11F Bearings and navigation

True or three-figure bearings

A **true bearing** is the angle measured clockwise from north around to the required direction. True bearings are also called **three-figure bearings** because they are written using three numbers or figures. For example, 090° is the direction measured 90° clockwise from north, better known as east!

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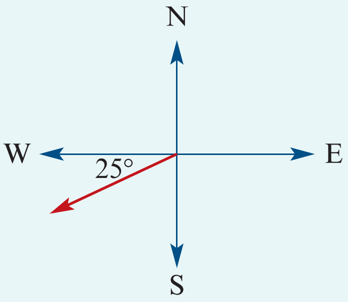
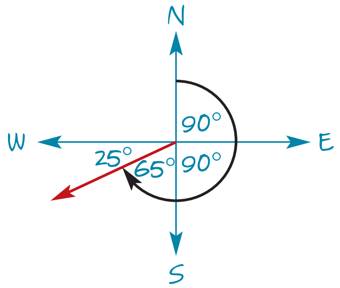
Example 12 Determining three-figure bearings

Give the three-figure bearing for the direction shown.



Solution

1 Calculate the total angles swept out clockwise from north.
There is an angle of 90° between each of the four points of the compass.



2 Write your answer.

The angle from north = 90° + 90° + 65 = 245°
or 270° − 25° = 245°
The three-figure bearing is 245°.

Navigation problems

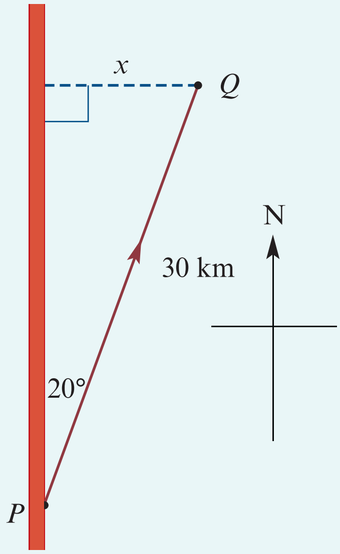
Navigation problems usually involve a consideration of not only the *direction* of travel, given as a bearing, but also the *distance* travelled.

Example 13 Navigating using a three-figure bearing

A group of bushwalkers leave point P, which is on a road that runs north–south, and walk for 30 km on a bearing 020° to reach point Q.

a What is the shortest distance x from Q back to the road correct to one decimal place?

b Looking from point Q, what would be the three-figure bearing of their starting point?



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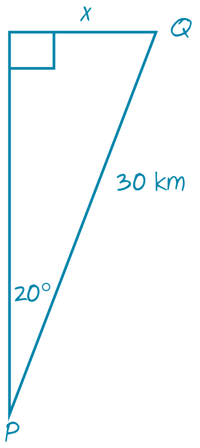
Solution

a **1** Show the given and required information in a right-angled triangle.



2 The opposite and hypotenuse are involved, so use sin θ.

$$\sin \theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

3 Substitute in the known values.

$$\sin 20^\circ = \frac{x}{30}$$

4 Multiply both sides by 30.

$$30 \times \sin 20^\circ = x$$

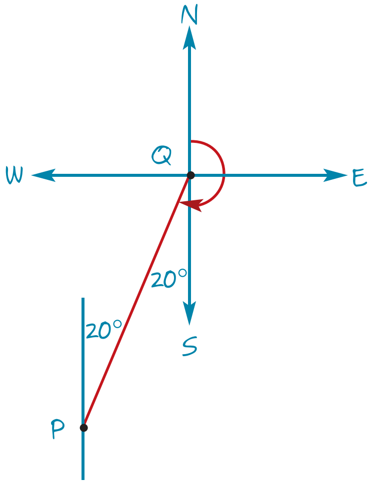
5 Find the value of x using your calculator.

$$x = 10.260\ldots$$

6 Write your answer correct to one decimal place.

The shortest distance to the road is 10.3 km.

b **1** Draw the compass points at Q.



2 Enter the alternate angle 20°.

3 Standing at Q, add all the angles when facing north and then turning clockwise to look at P. This gives the three-figure bearing of P when looking from Q.

The angle from north is

$180^\circ + 20^\circ = 200^\circ$

The three-figure bearing is 200°.

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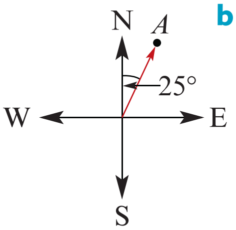
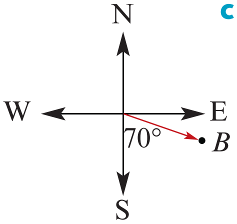
Exercise 11F

Example 12 **1** State the three-figure bearing of each of the points A, B, C and D.

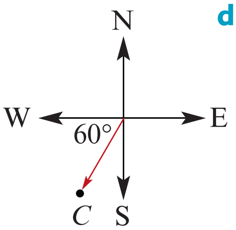
a



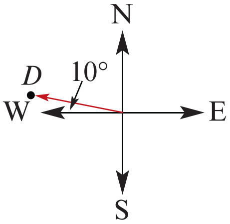
b



c



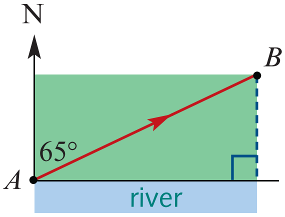
d



Example 13 **2** Eddie camped overnight at point A beside a river that ran east–west. He walked on a bearing of 065° for 18 km to point B.

a What angle did his direction make with the river?

b What is the shortest distance from B to the river, correct to two decimal places?



3 A ship sailed 3 km west, then 2 km south.

a Give its three-figure bearings from an observer who stayed at its starting point, correct to the nearest degree.

b For a person on the ship, what would be the three-figure bearings looking back to the starting point?

4 An aeroplane flew 500 km south, then 600 km east. Give its three-figure bearing from its starting point, to the nearest degree.

5 A ship left port and sailed east for 5 km, then sailed north. After some time an observer at the port could see the ship on a bearing of 050°.

a How far north had the ship travelled? Answer correct to one decimal place.

b Looking from the ship, what would be the three-figure bearing of the port?

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6 A woman walked from point A for 10 km on a bearing of 060° to reach point B. Then she walked for 15 km heading south until she was at point D. Give the following distances correct to one decimal place and directions to the nearest degree.

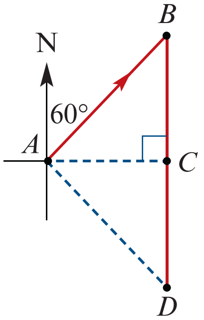
a Find the distances walked from A to B and from B to D.

b How far south did she walk from B to C?

c Find the distance from A to C.

d What is the distance from C to D?

e Find the three-figure bearing and distance she would need to walk to return to her starting point.



7 A ship left port P and sailed 20 km on a bearing of 230°. It then sailed north for 30 km to reach point C. Give the following distances correct to one decimal place and directions to the nearest degree.

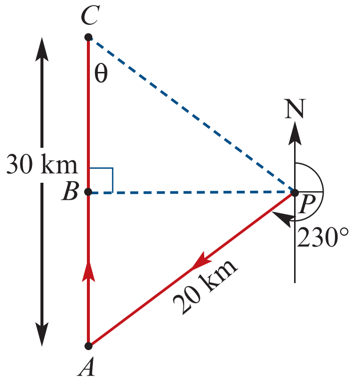
a Find the distance AB.

b Find the distance BP.

c Find the distance BC.

d Find the angle θ at point C.

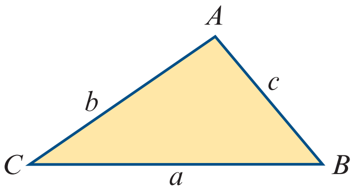
e State the three-figure bearing and distance of the port P from the ship at C.



11G The sine rule

▶ Standard triangle notation

The convention for labelling a non-right-angled triangle is to use the upper case letters A, B, and C for the angles at each corner. The sides are named using lower case letters so that side a is opposite angle A, and so on.



This notation is used for the sine rule and cosine rule (see Section 11H). Both rules can be used to find angles and sides in triangles that do not have a right angle.

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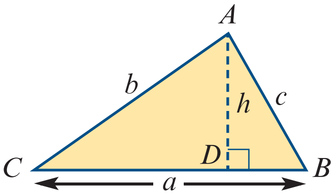
▶ How to derive the sine rule

In triangle *ABC*, show the height *h* of the triangle by drawing a perpendicular line from *D* on the base of the triangle to *A*.



In triangle *ADC*, $\sin C = \dfrac{h}{b}$

So $h = b \times \sin C$

In triangle *ABD*, $\sin B = \dfrac{h}{c}$

So $h = c \times \sin B$

We can make the two rules for *h* equal to each other. $b \times \sin C = c \times \sin B$

Divide both sides by sin *C*. $b = \dfrac{c \times \sin B}{\sin C}$

Divide both sides by sin *B*. $\dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

If the triangle was redrawn with side *c* as the base, then using similar steps we would get: $\dfrac{a}{\sin A} = \dfrac{b}{\sin B}$

We can combine the two rules as shown in the following box.

The sine rule

In any triangle *ABC*:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

The **sine rule** is used to find the sides and angles in a non-right-angled triangle when given:

- two sides and an angle opposite one of the given sides
- two angles and one side.

Note: If neither of the given angles is opposite the given side, find the third angle using $A + B + C = 180°$.

The sine rule can take the form of any of these three possible equations:

$$\frac{a}{\sin A} = \frac{b}{\sin B} \qquad \frac{b}{\sin B} = \frac{c}{\sin C} \qquad \frac{a}{\sin A} = \frac{c}{\sin C}$$

Each equation has two sides and two angles opposite those sides. If we know three of the parts, we can find the fourth.

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So if we know two angles and a side opposite one of the angles, we can find the side opposite the other angle. Similarly, if we know two sides and an angle opposite one of those sides, we can find the angle opposite the other side.

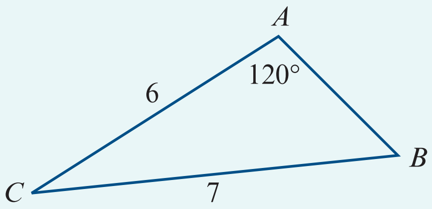
Although we have expressed the sine rule using a triangle ABC, for any triangle, such as PQR, the pattern of fractions consisting of 'side / sine of angle' pairs would appear as:

$$\frac{p}{\sin P} = \frac{q}{\sin Q} \qquad \frac{q}{\sin Q} = \frac{r}{\sin R} \qquad \frac{p}{\sin P} = \frac{r}{\sin R}$$

Using the sine rule

Example 14 Using the sine rule given two sides and an opposite angle

Find angle B in the triangle shown, correct to one decimal place.



Solution

1 We have the pairs $a = 7$ and $A = 120°$
$b = 6$ and $B = ?$
with only B unknown.

So use $\dfrac{a}{\sin A} = \dfrac{b}{\sin B}$.

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

2 Substitute in the known values.

$$\frac{7}{\sin 120°} = \frac{6}{\sin B}$$

3 Cross-multiply.

$$7 \times \sin B = 6 \times \sin 120°$$

4 Divide both sides by 7.

$$\sin B = \frac{6 \times \sin 120°}{7}$$

5 Write the equation to find angle B.

$$B = \sin^{-1}\left(\frac{6 \times \sin 120°}{7}\right)$$

6 Use your calculator to evaluate the expression for B.

$$B = 47.928\ldots°$$

7 Write your answer correct to one decimal place.

Angle B is 47.9°.

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In Example 14, now that we know that $A = 120°$ and $B = 47.9°$, we can use the fact that the angles in a triangle add to 180° to find C.

$$\begin{aligned} A + B + C &= 180° \\ 120° + 47.9° + C &= 180° \\ 167.9° + C &= 180° \\ C &= 180° - 167.9° = 12.1° \end{aligned}$$

As we now know that $A = 120°$, $a = 7$ and $C = 12.1°$, we can find side c using

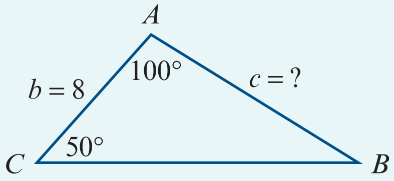
$$\frac{a}{\sin A} = \frac{c}{\sin C}$$

The steps are similar to those in the example.

Finding all the angles and sides of a triangle is called solving the triangle.

Example 15 Using the sine rule given two angles and one side

Find side c in the triangle shown, correct to one decimal place.



Solution

1 Find the angle opposite the given side by using $A + B + C = 180°$

$$\begin{aligned} A + B + C &= 180° \\ 100° + B + 50° &= 180° \\ B + 150° &= 180° \\ B &= 30° \end{aligned}$$

2 We have the pairs $b = 8$ and $B = 30°, c = ?, C = 50°$ with only c unknown. So use $\frac{b}{\sin B} = \frac{c}{\sin C}$.

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$

3 Substitute in the known values.

$$\frac{8}{\sin 30°} = \frac{c}{\sin 50°}$$

4 Multiply both sides by sin 50°.

$$c = \frac{8 \times \sin 50°}{\sin 30°}$$

5 Use your calculator to find c.

$$c = 12.256\ldots$$

6 Write your answer correct to one decimal place.

Side c is 12.3 units long.

▶ Ambiguous case

Sometimes two triangles can be drawn to fit the given information. This can happen when you are given two sides and an angle not between the two given sides. The solution strategy uses the sine rule and the fact that the angles at the base of an isosceles triangle are equal.

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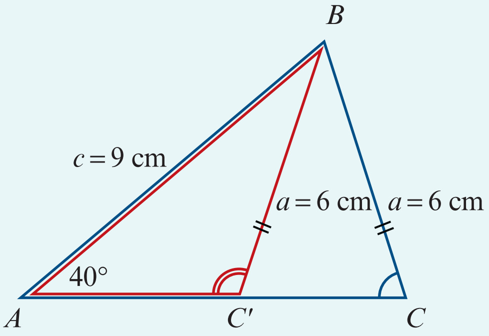
Example 16 Ambiguous case using the sine rule

In triangle *ABC*, $A = 40°$, $c = 9$ cm and $a = 6$ cm.



Side *c* is drawn for 9 cm at 40° to the base. From vertex *B*, side *a* must be 6 cm long when it meets the base of the triangle. When side *a* is measured out with a compass, it can cross the base in two possible places.

There are two possible triangles. *ABC* drawn in blue and *ABC'* drawn in red.

Find the two possible values for angle *C*, shown as $\angle BCA$ and $\angle BC'A$ in the diagram. Give the answers correct to two decimal places.

Solution

1 Using the sine rule, we need two angle-side pairs with only one unknown.
The unknown is angle *C*.

$$\frac{\sin C}{c} = \frac{\sin A}{a}$$

$$\frac{\sin C}{9} = \frac{\sin 40°}{6}$$

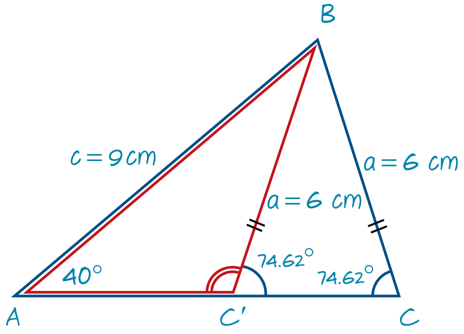
$$\sin C = \frac{9 \times \sin 40°}{6}$$

$$C = \sin^{-1}\left(\frac{9 \times \sin 40°}{6}\right)$$

$$C = 74.62°$$

2 Clearly $\angle BC'A$ is greater than 90°, so it is not the value of angle *C* just calculated.

So $\angle BCA = 74.62°$



3 The two angles at the base of the isosceles triangle *C'BC* are equal.

$\angle BC'C = \angle BCA$

So $\angle BC'C = 74.62°$

4 Two angles on a straight line add to 180°.

$\angle BC'A + 74.62° = 180°$

$\angle BC'A = 180° - 74.62° = 105.38°$

The possible values for angle C are 74.62° and 105.38°

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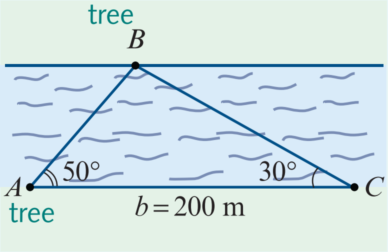
Example 17 Application of the sine rule

Leo wants to tie a rope from a tree at point A to a tree at point B on the other side of the river. He needs to know the length of rope required.

When he stood at A, he saw the tree at B at an angle of 50° with the river-bank. After walking 200 metres east to C, the tree was seen at an angle of 30° with the bank.



Find the length of rope required to reach from A to B, correct to two decimal places.

Solution

1 To use the sine rule, we need two angle-side pairs with only one item unknown.

So one part of the sine rule equation will be:

$$\frac{c}{\sin C}$$

The unknown is the length of the rope, side c. Angle $C = 30°$ is given.

2 We know side $b = 200$ and need to find angle B to use the sine rule equation:

$$\frac{c}{\sin C} = \frac{b}{\sin B}$$

3 Use $A + B + C = 180°$ to find angle B.

$$A+B+C = 180°$$
$$50° + B + 30° = 180°$$
$$B = 100°$$

4 We have the pairs:

$b = 200$ and $B = 100°$

$c =?$ and $C = 30°$

with only c unknown.

So use $\dfrac{c}{\sin \mathrm{C}} = \dfrac{b}{\sin \mathrm{B}}$.

$$\frac{c}{\sin C} = \frac{b}{\sin B}$$

5 Substitute in the known values.

$$\frac{c}{\sin 30°} = \frac{200}{\sin 100°}$$

6 Multiply both sides by sin 30° .

$$c = \frac{200 \times \sin 30°}{\sin 100°}$$

7 Use your calculator to find c.

$$c = 101.542\ldots$$

8 Write your answer correct to two decimal places.

The rope must be 101.54 m long.

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Tips for solving trigonometry problems

- Always make a rough sketch in pencil as you read the details of a problem. You may need to make changes as you read more, but it is very helpful to have a sketch to guide your understanding.
- In any triangle, the longest side is opposite the largest angle. The shortest side is opposite the smallest angle.
- When you have found a solution, re-read the question and check that your answer fits well with the given information and your diagram.
- Round answers for each part to the required decimal places. Keep more decimal places when the results are used in further calculations. Otherwise, rounding off errors accumulate.

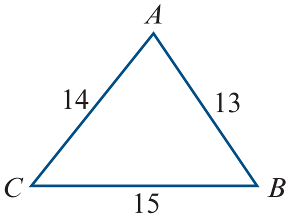
Exercise 11G

In this exercise, calculate lengths correct to two decimal places and angles correct to one decimal place where necessary.

Basic principles

1 In each triangle, state the lengths of sides a, b and c.

a



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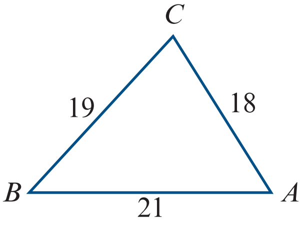
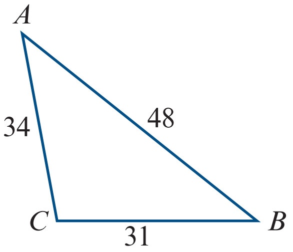
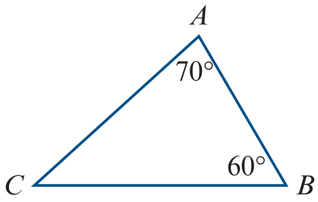


c

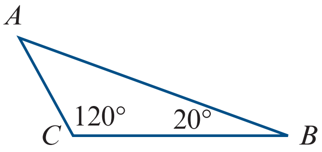


2 Find the value of the unknown angle in each triangle. Use $A + B + C = 180°$.

a



b



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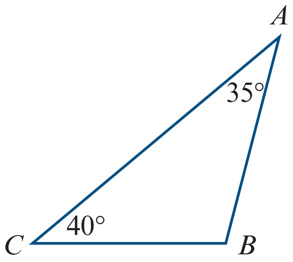

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3 In each of the following a student was using the sine rule to find an unknown part of a triangle, but was unable to complete the final steps of the solution. Find the unknown value by completing each problem.

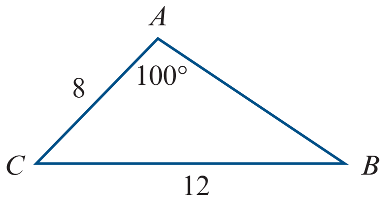
a $\frac{a}{\sin 40^\circ} = \frac{8}{\sin 60^\circ}$ **b** $\frac{b}{\sin 50^\circ} = \frac{15}{\sin 72^\circ}$ **c** $\frac{c}{\sin 110^\circ} = \frac{24}{\sin 30^\circ}$

d $\frac{17}{\sin A} = \frac{16}{\sin 70^\circ}$ **e** $\frac{26}{\sin B} = \frac{37}{\sin 95^\circ}$ **f** $\frac{21}{\sin C} = \frac{47}{\sin 115^\circ}$

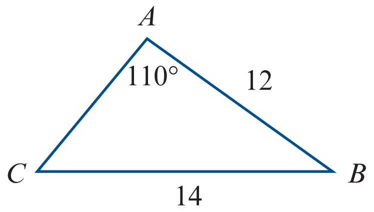
Using the sine rule to find angles

Example 14

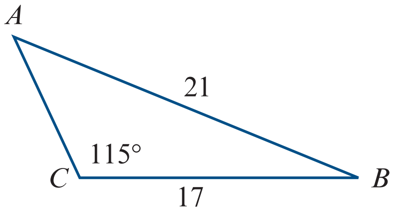
4 **a** Find angle B.



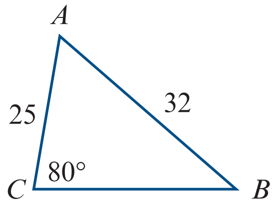
b Find angle C.



c Find angle A.



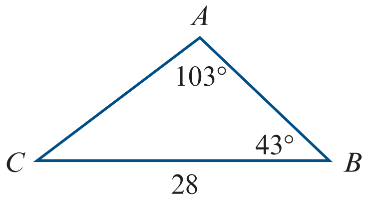
d Find angle B.



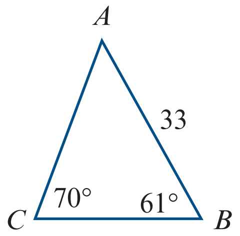
Using the sine rule to find sides

Example 15

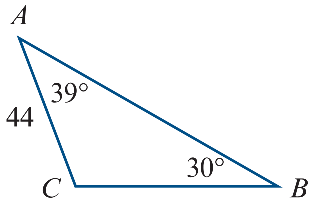
5 **a** Find side b.



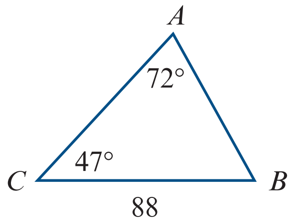
b Find side b.



c Find side a.



d Find side c.



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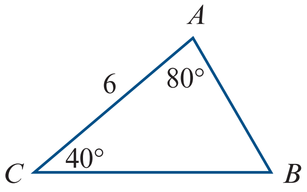
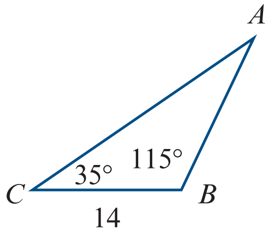
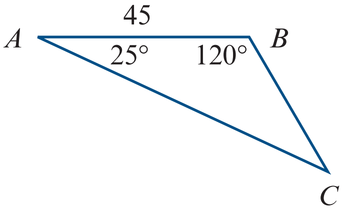
6 **a** Find side c.



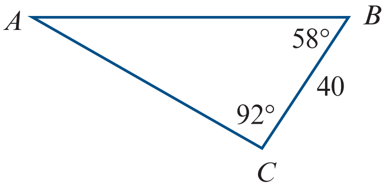
b Find side c.



c Find side b.



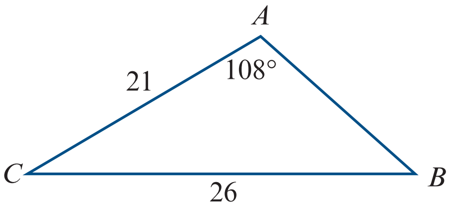
d Find side b.



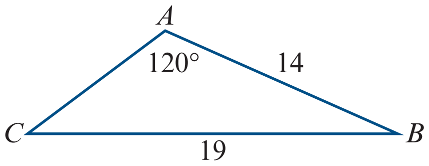
Solving triangles using the sine rule

7 Solve (find all the unknown sides and angles of) the following triangles.

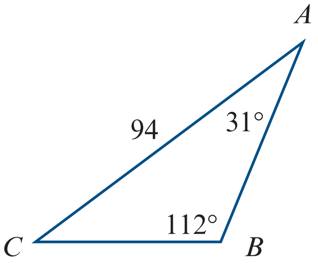
a



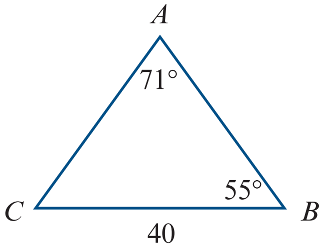
b



c



d



8 In the triangle $ABC, A = 105°, B = 39°$ and $a = 60$. Find side b.

9 In the triangle $ABC, A = 112°, a = 65$ and $c = 48$. Find angle C.

10 In the triangle $ABC,\ B = 50°, C = 45°$ and $a = 70$. Find side c.

11 In the triangle $ABC,\ B = 59°, C = 74°$ and $c = 41$. Find sides a and b and angle A.

12 In the triangle $ABC,\ a = 60, b = 100$ and $B = 130°$. Find angles A and C and side c.

13 In the triangle $ABC,\ A = 130°, B = 30°$ and $c = 69$. Find sides a and b and angle C.

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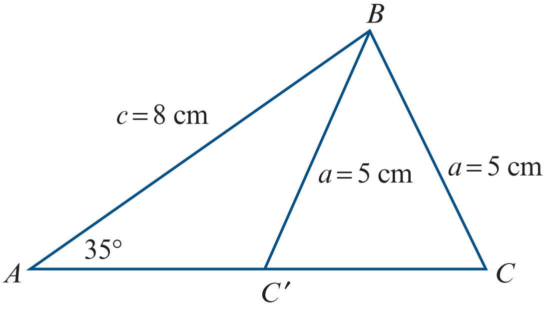
The ambiguous case of the sine rule

Example 16 **14** In triangle ABC, $A = 35°$, $c = 8$ cm and $a = 5$ cm. Two triangles ABC and ABC' can be drawn using the given information. Give the angles to two decimal places.



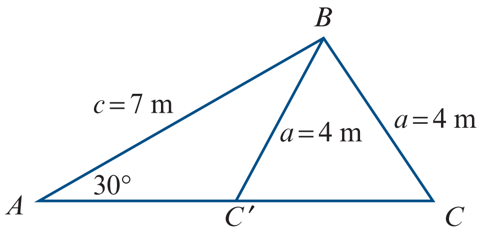
a Use the sine rule to find $\angle BCA$ in triangle ABC.

b Use isosceles triangle $C'BC$ to find $\angle BC'C$.

c Find $\angle AC'B$ by using the rule for two angles on a straight line.

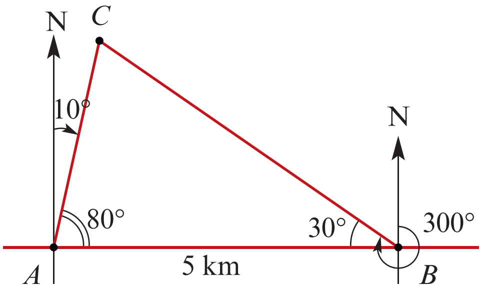
d Give the possible values for C and C', the angle opposite side c.

15 In triangle ABC, $A = 30°$, $c = 7$ cm and $a = 4$ cm. Find the two possible values for angle C, shown as $\angle BCA$ and $\angle BC'A$ in the diagram. Give the angles to two decimal places.



Applications

Example 17 **16** A fire-spotter located in a tower at A saw a fire in the direction 010°. Five kilometres to the east of A another fire-spotter at B saw the fire in the direction 300°. Find the distance of the fire from each tower.



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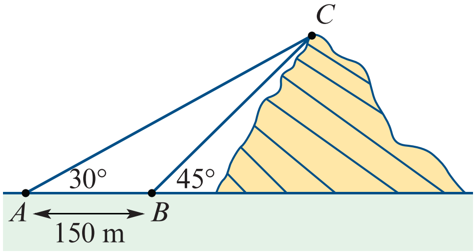
17 A surveyor standing at point *A* measured the angle of elevation to the top of the mountain as 30°. She moved 150 m closer to the mountain and at point *B* measured the angle of elevation to the top of the mountain as 45°.

There is a proposal to have a strong cable from point *A* to the top of the mountain to carry tourists in a cable car. What is the length of the required cable?



18 A naval officer sighted the smoke of a volcanic island on a bearing of 044°. A navigator on another ship 25 km due east of the first ship saw the smoke on a bearing of 342° .

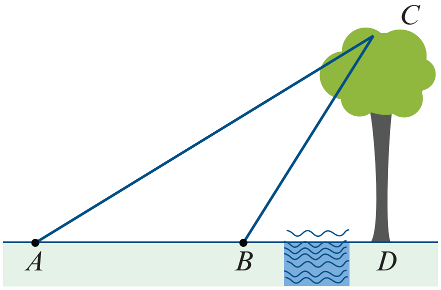
- **a** Find the distance of each ship from the volcano.
- **b** If the ship closest to the volcano can travel at 15 km/h, how long will it take it to reach the volcano?

19 An air-traffic controller at airport *A* received a distress call from an aeroplane low on fuel. The bearing of the aeroplane from *A* was 070°. From airport *B*, 80 km north of airport *A*, the bearing of the aeroplane was 120°.

- **a** Which airport was closest for the aeroplane?
- **b** Find the distance to the closest airport.
- **c** The co-pilot estimates fuel consumption to be 1525 litres per 100 km. The fuel gauge reads 1400 litres. Is there enough fuel to reach the destination?

20 Holly was recording the heights of tall trees in a State forest to have them registered for protection. A river prevented her from measuring the distance from the base of a particular tree.

She recorded the angle of elevation of the top of the tree from point *A* as 25°. Holly walked 80 m towards the tree and recorded the angle of elevation from point *B* as 50°.



- **a** Copy the diagram shown and add the given information.
- **b** Find the angle at *B* in triangle *ABC*.
- **c** Find the angle at *C* in triangle *ABC*.
- **d** Find the length *b* (from *A* to *C*).
- **e** Use the length *b* as the hypotenuse in right-angled triangle *ADC*, and the angle at *A*, to find distance *DC*, the height of the tree.

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11H The cosine rule

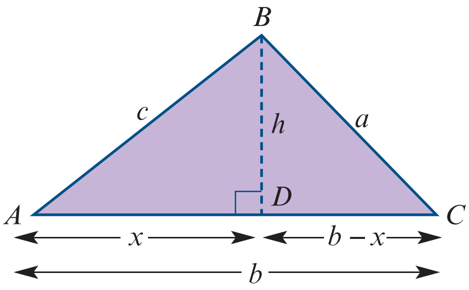
The **cosine rule** can be used to find the length of a side in any non-right-angled triangle when two sides and the angle between them are known. When you know the three sides of a triangle, the cosine rule can be used to find any angle.

▶ How to derive the cosine rule

In the triangle ABC, show the height h of the triangle by drawing a line perpendicular from D on the base of the triangle to B.

Let $AD = x$

As $AC = b$, then $DC = b - x$.



In triangle ABD, $\cos A = \frac{x}{c}$

Multiply both sides by c. $x = c \cos A$ ①

Using Pythagoras’ theorem in triangle ABD. $x^2 + h^2 = c^2$ ②

Using Pythagoras’ theorem in triangle CBD. $(b - x)^2 + h^2 = a^2$

Expand (multiply out) the squared bracket. $b^2 - 2bx + x^2 + h^2 = a^2$

Use ① to replace x with $c \cos A$. $b^2 - 2bc \cos A + x^2 + h^2 = a^2$

Use ② to replace $x^2 + h^2$ with c^2. $b^2 - 2bc \cos A + c^2 = a^2$

Reverse and rearrange the equation. $a^2 = b^2 + c^2 - 2bc \cos A$

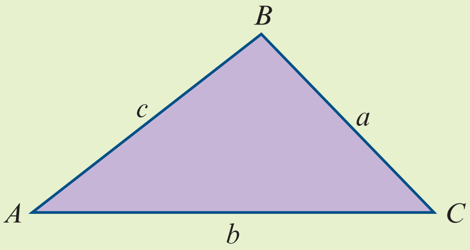
Repeating these steps with side c as the base, we get: $b^2 = a^2 + c^2 - 2ac \cos B$

Repeating these steps with side a as the base, we get: $c^2 = a^2 + b^2 - 2ab \cos C$

The three versions of the cosine rule can be rearranged to give rules for $\cos A$, $\cos B$, and $\cos C$.

The cosine rule

The cosine rule in any triangle ABC:



- when given two sides and the angle between them, the third side can be found using one of the equations:

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$b^2 = a^2 + c^2 - 2ac\cos B$$

$$c^2 = a^2 + b^2 - 2ab\cos C$$

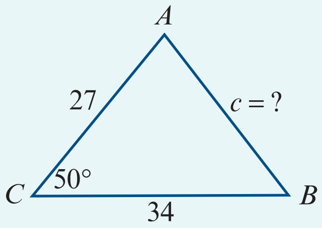
- when given three sides, any angle can be found using one of the following rearrangements of the cosine rule:

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} \qquad \cos B = \frac{a^2 + c^2 - b^2}{2ac} \qquad \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

▶ Using the cosine rule

Example 18 Using the cosine rule given two sides and the angle between them

Find side c, correct to two decimal places, in the triangle shown.



Solution

1 Write down the given values and the required unknown value.

$a = 34, b = 27, c = ?, C = 50°$

2 We are given two sides and the angle between them. To find side c use $c^2 = a^2 + b^2 - 2ab\cos C$

$c^2 = a^2 + b^2 - 2ab\cos C$

3 Substitute the given values into the rule.

$c^2 = 34^2 + 27^2 - 2 \times 34 \times 27 \times \cos 50°$

4 Take the square root of both sides.

$c = \sqrt{(34^2 + 27^2 - 2 \times 34 \times 27 \times \cos 50°)}$

5 Use your calculator to find c.

$c = 26.548\ldots$

6 Write your answer correct to two decimal places.

The length of side c is 26.55 units.

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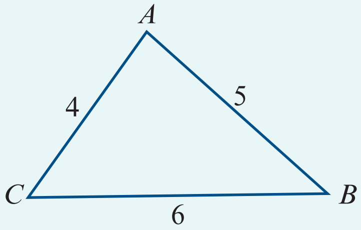
Example 19 Using the cosine rule to find an angle given three sides

Find the largest angle, correct to one decimal place, in the triangle shown.



Solution

1 Write down the given values.

$a = 6, b = 4, c = 5$

2 The largest angle is always opposite the largest side, so find angle A.

$A = ?$

3 We are given three sides. To find angle A use $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

4 Substitute the given values into the rule.

$$\cos A = \frac{4^2 + 5^2 - 6^2}{2 \times 4 \times 5}$$

5 Write the equation to find angle A.

$$A = \cos^{-1}\left(\frac{4^2 + 5^2 - 6^2}{2 \times 4 \times 5}\right)$$

6 Use your calculator to evaluate the expression for A. Make sure that your calculator is in DEGREE mode.

Tip: Wrap all the terms in the numerator (top) within brackets. Also put brackets around all of the terms in the denominator (bottom).

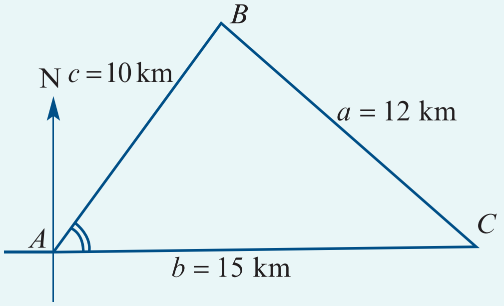
$A = 82.819\ldots^\circ$

7 Write your answer.

The largest angle is 82.8°.

Example 20 Application of the cosine rule: finding an angle and a bearing

A yacht left point A and sailed 15 km east to point C. Another yacht also started at point A and sailed 10 km to point B, as shown in the diagram. The distance between points B and C is 12 km.



a What was the angle between their directions as they left point A? Give the angle correct to two decimal places.

b Find the bearing of point B from the starting point A to the nearest degree.

Solution

a **1** Write the given values.

$a = 12, b = 15, c = 10$

2 Write the form of the cosine rule for the required angle, A.

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

3 Substitute the given values into the rule.

$$\cos A = \frac{15^2 + 10^2 - 12^2}{2 \times 15 \times 10}$$

4 Write the equation to find angle A.

$$A = \cos^{-1}\left(\frac{15^2 + 10^2 - 12^2}{2 \times 15 \times 10}\right)$$

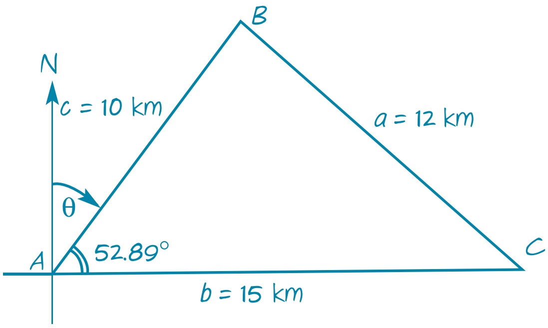
5 Use your calculator to evaluate the expression for A.

$A = 52.891°$

6 Give the answer to two decimal places.

The angle was 52.89°.

b **1** The bearing θ, of point B from the starting point A, is measured clockwise from north.



$$\theta + 52.89° = 90°$$

2 Consider the angles in the right-angle at point A.

$$\theta = 90° - 52.89°$$
$$= 37.11°$$

3 Find the value of θ.

4 Write your answer.

The bearing of point B from point A is 037°.

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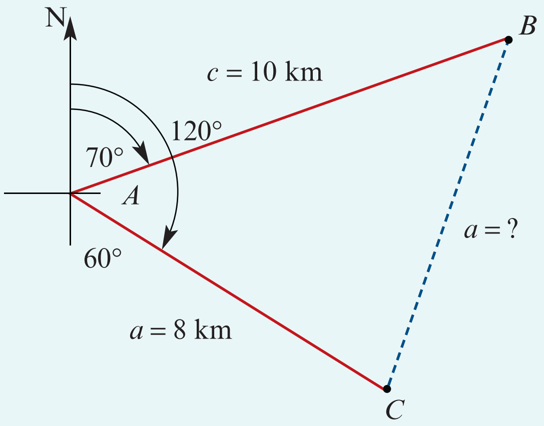
Example 21 Application of the cosine rule involving bearings

A bushwalker left his base camp and walked 10 km in the direction 070° .

His friend also left the base camp but walked 8 km in the direction 120° .

a Find the angle between their paths.

b How far apart were they when they stopped walking? Give your answer correct to two decimal places.



Solution

a **1** Angles lying on a straight line add to 180° .

$$60^\circ + A + 70^\circ = 180^\circ$$
$$A + 130^\circ = 180^\circ$$
$$A = 50^\circ$$

2 Write your answer.

The angle between their paths was 50°.

b **1** Write down the known values and the required unknown value.

$a = ?\ b = 8,\ c = 10,\ A = 50$

2 We have two sides and the angle between them. To find side *a* use $a^2 = b^2 + c^2 - 2bc \cos A$

$$a^2 = b^2 + c^2 - 2bc\cos A$$

3 Substitute in the known values.

$$a^2 = 8^2 + 10^2 - 2 \times 8 \times 10 \times \cos 50^\circ$$

4 Take the square root of both sides.

$$a = \sqrt{(8^2 + 10^2 - 2 \times 8 \times 10 \times \cos 50^\circ)}$$

5 Use your calculator to find the value of *a*.

$$a = 7.820\ldots$$

6 Write your answer correct to two decimal places.

The distance between them was 7.82 km.

Exercise 11H

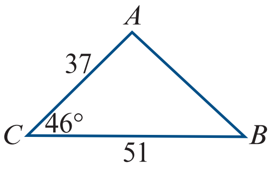
In this exercise, calculate lengths correct to two decimal places and angles correct to one decimal place.

Using the cosine rule to find sides

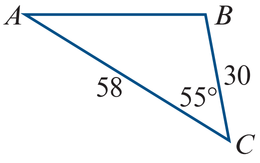
Example 18

1 Find the unknown side in each triangle.

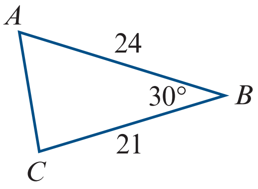
a



b



c



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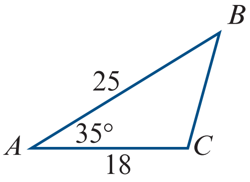

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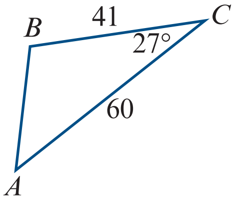

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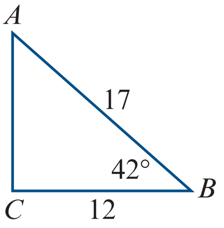


2 In the triangle ABC, $a = 27, b = 22$ and $C = 40°$. Find side c.

3 In the triangle ABC, $a = 18, c = 15$ and $B = 110°$. Find side b.

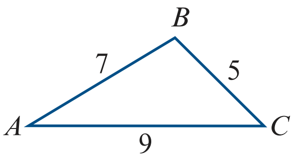
4 In the triangle ABC, $b = 42, c = 38$ and $A = 80°$. Find side a.

Using the cosine rule to find angles

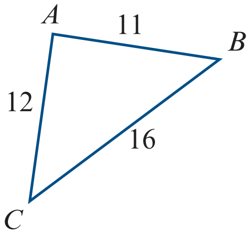
Example 19

5 Find angle A in each triangle.

a



b



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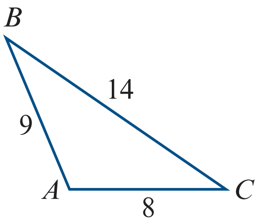

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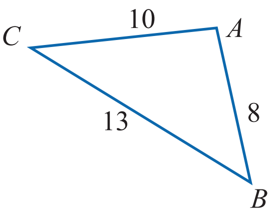

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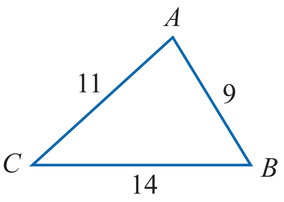

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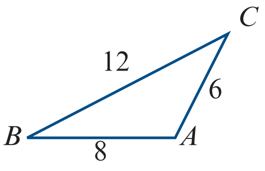

6 In the triangle ABC, $a = 9$, $b = 10$ and $c = 11$. Find angle A.

7 In the triangle ABC, $a = 31$, $b = 47$ and $c = 52$. Find angle B.

8 In the triangle ABC, $a = 66$, $b = 29$ and $c = 48$. Find angle C.

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9 Find the smallest angle in the triangle ABC, with $a = 120, b = 90$ and $c = 105$.

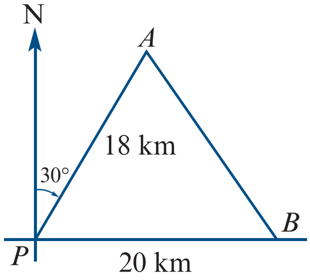
10 In the triangle ABC, $a = 16, b = 21$ and $c = 19$. Find the largest angle.

Applications

Example 20, 21

11 A farm has a triangular shape with fences of 5 km, 7 km and 9 km in length. Find the size of the smallest angle between the fences. The smallest angle is always opposite the smallest side.

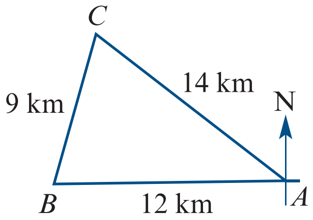
12 A ship left the port P and sailed 18 km on a bearing of 030° to point A. Another ship left port P and sailed 20 km east to point B. Find the distance from A to B, correct to one decimal place.



13 A bushwalker walked 12 km west from point A to point B. Her friend walked 14 km from point A to point C as shown in the diagram. The distance from B to C is 9 km.

a Find the angle at A, between the paths taken by the bushwalkers, correct to one decimal place.

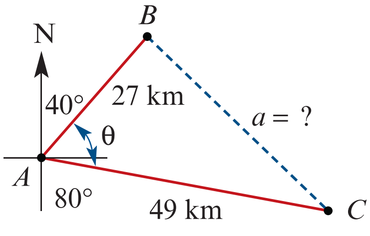
b What is the bearing of point C from A? Give the bearing correct to the nearest degree.



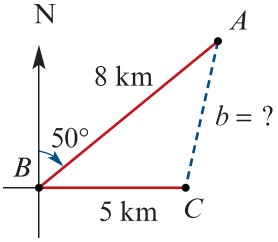
14 A ship left port A and travelled 27 km on a bearing of 40° to reach point B. Another ship left the same port and travelled 49 km on a bearing of 100° to arrive at point C.

a Find the angle θ between the directions of the two ships.

b How far apart were the two ships when they stopped?



15 A battleship B detected a submarine A on a bearing of 050° and at a distance of 8 km. A cargo ship C was 5 km due east of the battleship. How far was the submarine from the cargo ship?



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16 From a lookout tower *A*, a fire-spotter saw a bushfire *B* at a distance of 15 km in the direction 315°. A township *C* was located 12 km on a bearing of 265°. How far was the bushfire from the township?

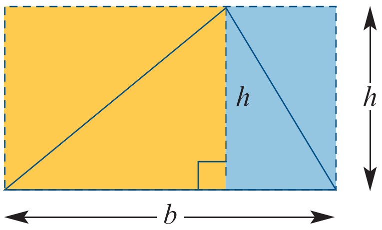
17 Passengers, who are travelling in a car west along a road that runs east–west, see a mountain 9 km away on a bearing of 290°. When they have travelled a further 5 km west along the road, what will be the distance to the mountain?

18 At a point *A* on the ground, the angle of elevation to the top of a radio transmission tower is 60°. From that point a 40 m cable was attached to the top of the tower. At a point *B*, a further 10 m away from the base of the tower, another cable is to be pegged to the ground and attached to the top of the tower. What length is required for the second cable?

11I The area of a triangle

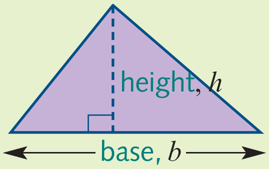
▶ Area of a triangle = $\frac{1}{2}$ × base × height

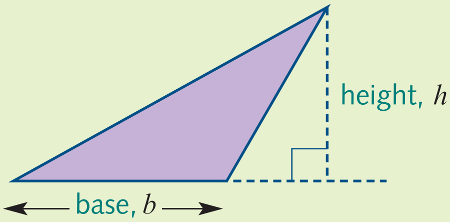
From the diagram, we see that the area of a triangle with a base b and height h is equal to half the area of the rectangle $b \times h$ that it fits within.



Area of a triangle

$$\text{Area of a triangle} = \frac{1}{2} \times \text{base} \times \text{height}$$
$$= \frac{1}{2} \times b \times h$$





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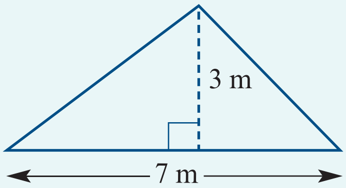
Example 22 Finding the area of a triangle using $\frac{1}{2}$ × base × height

Find the area of the triangle shown, correct to one decimal place.



Solution

1 As we are given values for the base and height of the triangle, use Area = $\frac{1}{2}$ × base × height

Base, $b = 7$
Height, $h = 3$
Area of triangle $= \frac{1}{2} \times b \times h$

2 Substitute the given values.

$= \frac{1}{2} \times 7 \times 3$

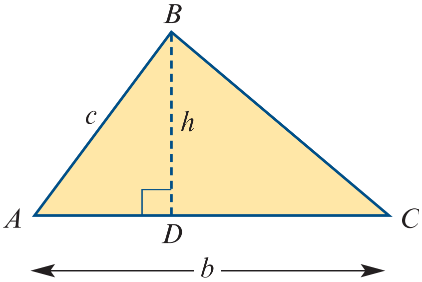
3 Evaluate.

$= 10.5\ \text{m}^2$

4 Write your answer.

The area of the triangle is 10.5 m^2

▶ Area of a triangle = $\frac{1}{2}$ bc sin A



In triangle *ABD*,

$$\sin A = \frac{h}{c}$$

$$h = c \times \sin A$$

So we can replace *h* with *c* × sin *A* in the rule:

$$\text{Area of a triangle} = \frac{1}{2} \times b \times h$$

$$\text{Area of a triangle} = \frac{1}{2} \times b \times c \times \sin A$$

Similarly, using side *c* or *a* for the base, we can make a complete set of three rules:

Area of a triangle

$$\text{Area of a triangle} = \tfrac{1}{2}\, bc \sin A$$

$$\text{Area of a triangle} = \tfrac{1}{2}\, ac \sin B$$

$$\text{Area of a triangle} = \tfrac{1}{2}\, ab \sin C$$

Notice that each version of the rule follows the pattern:

$$\text{Area of a triangle} = \tfrac{1}{2} \times (\text{product of two sides}) \times \mathit{sin}(\text{angle between those two sides})$$

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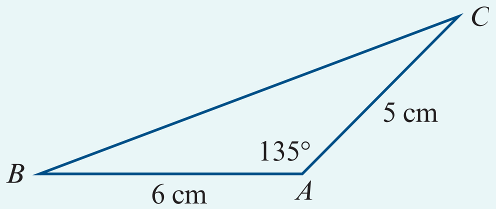
Example 23 Finding the area of a triangle using $\frac{1}{2}$ bc sin A

Find the area of the triangle shown, correct to one decimal place.



Solution

1 We are given two sides b, c and the angle A between them, so use:

Area of a triangle = $\frac{1}{2}\,bc\sin A$

$b = 5, c = 6, A = 135°$

$\text{Area of triangle} = \frac{1}{2}bc\sin A$

2 Substitute values for b, c and A into the rule.

$= \frac{1}{2} \times 5 \times 6 \times \sin 135°$

3 Use your calculator to find the area.

$= 10.606\ldots$

4 Write your answer correct to one decimal place.

The area of the triangle is 10.6 cm^2.

Heron's rule for the area of a triangle

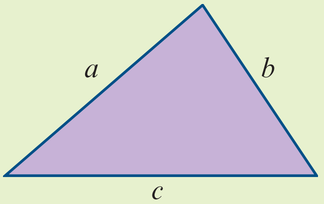
Heron's rule can be used to find the area of any triangle when we know the lengths of the three sides.

Heron's rule for the area of a triangle

Area of a triangle = $\sqrt{s(s-a)(s-b)(s-c)}$

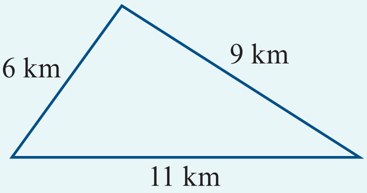
where $s = \frac{1}{2}(a+b+c)$

The variable s is called the *semi-perimeter* because it is equal to half the sum of the sides.



Example 24 Finding the area of a triangle using Heron's formula

The boundary fences of a farm are shown in the diagram. Find the area of the farm, to the nearest square kilometre.



Solution

1 As we are given the three sides of the triangle, use Heron's rule. Start by finding *s*, the semi-perimeter.

Let $a = 6, b = 9, c = 11$

$$s = \frac{1}{2}(a + b + c)$$
$$= \frac{1}{2}(6 + 9 + 11) = 13$$

2 Write Heron's rule.

$$\text{Area of triangle} = \sqrt{s(s - a)(s - b)(s - c)}$$

3 Substitute the values of *s*, *a*, *b* and *c* into Heron's rule.

$$= \sqrt{13(13 - 6)(13 - 9)(13 - 11)}$$
$$= \sqrt{13 \times 7 \times 4 \times 2}$$

4 Use your calculator to find the area.

$$= 26.981\ldots$$

5 Write your answer.

The area of the farm, to the nearest square kilometre, is 27 km^2.

Exercise 11I

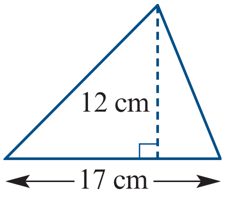
In this exercise, calculate areas correct to one decimal place where necessary.

Finding areas using $\frac{1}{2}$ × base × height

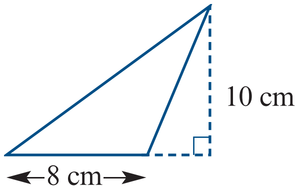
Example 22

1 Find the area of each triangle.

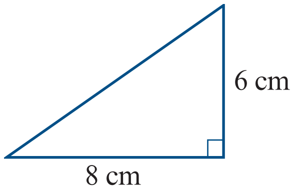
a



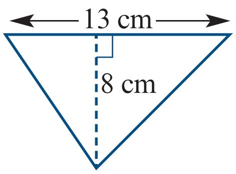
b



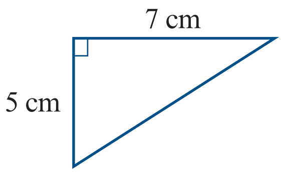
c



d



e



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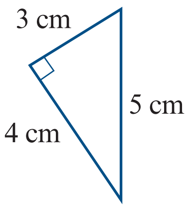

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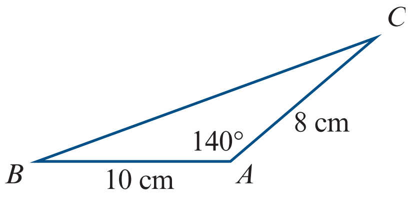
Finding areas using $\frac{1}{2}$ bc sin A

Example 23

2 Find the areas of the triangles shown.

a



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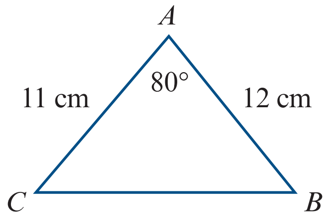

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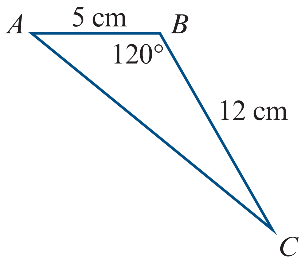

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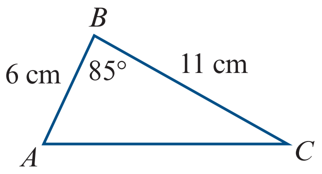

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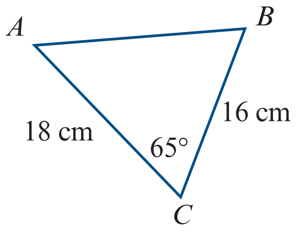

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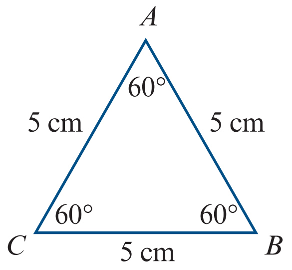
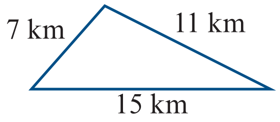


Finding areas using Heron's formula

Example 24

3 Find the area of each triangle.

a



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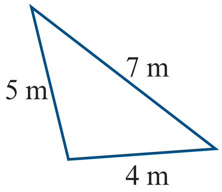

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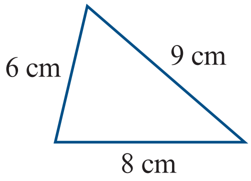

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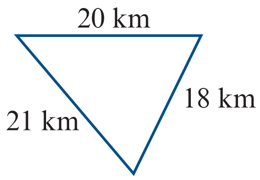

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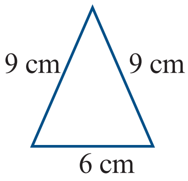

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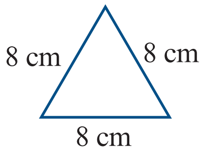

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Mixed problems

4 For each triangle below choose the rule for finding its area from:

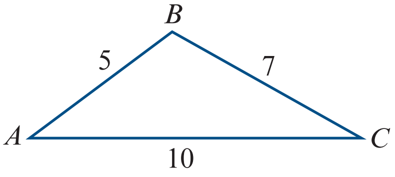
i $\frac{1}{2}\, base \times height$

ii $\frac{1}{2}\, bc\, \sin A$

iii $\frac{1}{2}\, ac\, \sin B$

iv $\sqrt{s(s-a)(s-b)(s-c)}$ where $s = \frac{1}{2}(a+b+c)$

a



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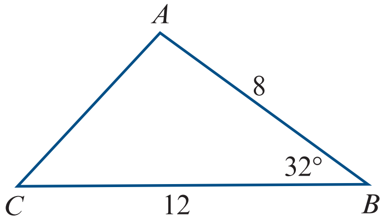

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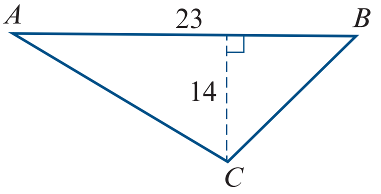

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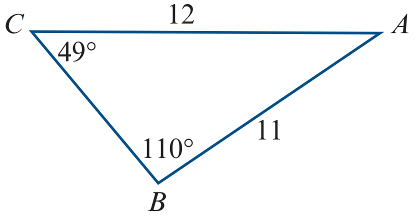
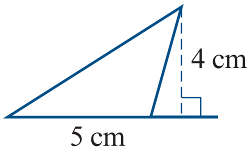


5 Find the area of each triangle shown.

a



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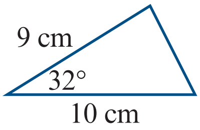

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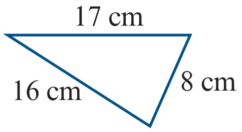

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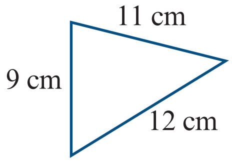

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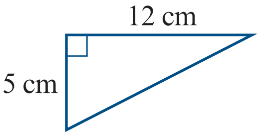

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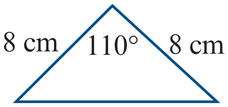

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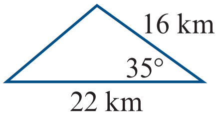

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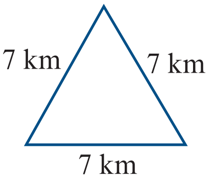
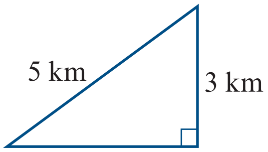


i



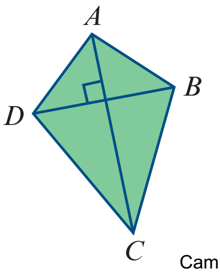
6 Find the area of a triangle with a base of 28 cm and a height of 16 cm.

7 Find the area of triangle *ABC* with side *a* 42 cm, side *b* 57 cm and angle *C* 70°.

8 Find the area of a triangle with sides of 16 km, 19 km and 23 km.

Applications

9 The kite shown is made using two sticks, *AC* and *DB*. The length of *AC* is 100 cm and the length of *DB* is 70 cm. Find the area of the kite.



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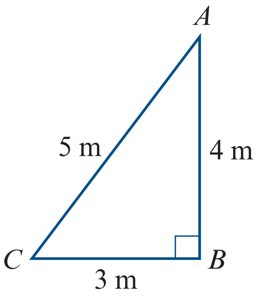
10 Three students *A*, *B* and *C* stretched a rope loop 12 m long into different shapes. Find the area of each shape.

a



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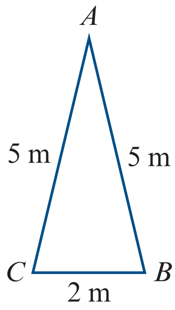
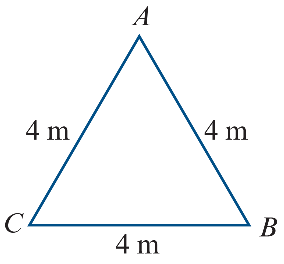


c



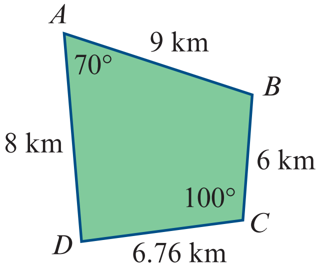
11 A farmer needs to know the area of his property with the boundary fences as shown. The measurements are correct to two decimal places.

Hint: Draw a line from *B* to *D* to divide the property into two triangles.

a Find the area of triangle *ABD*.

b Find the area of triangle *BCD*.

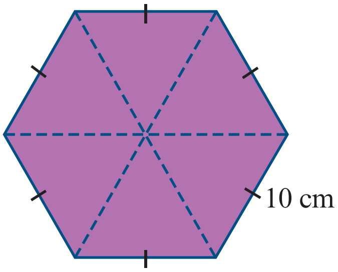
c State the total area of the property.



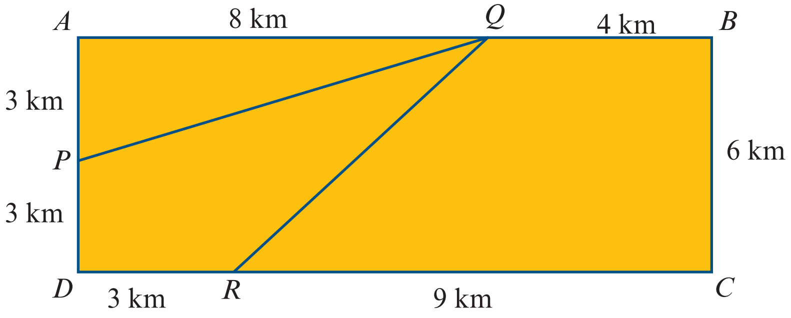
12 A regular hexagon with sides 10 cm long can be divided into six smaller equilateral triangles. (Remember, an equilateral triangle has all sides of equal length.)

a Find the area of each triangle.

b What is the area of the hexagon?



13 A large rectangular area of land, *ABCD* in the diagram, has been subdivided into three regions as shown.



a Find the area of:

i region *PAQ* **ii** region *QBCR* **iii** region *PQRD*.

b Find the size of angle *PQR*, correct to one decimal place.

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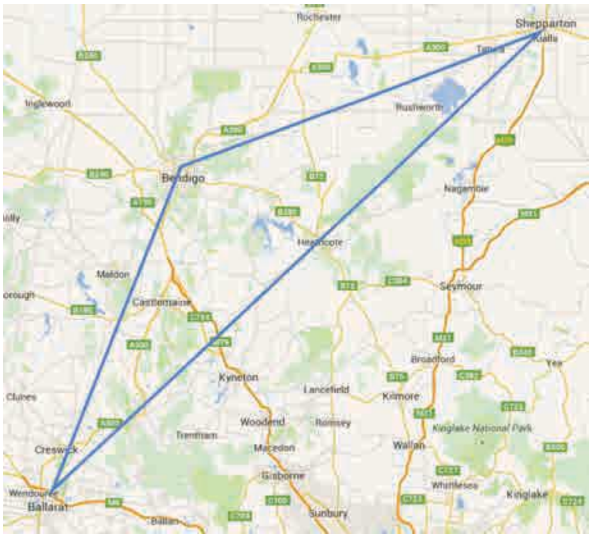
11J Extended application and problem solving task

Exercise 11J

Alice operates a charter flight service for country Victoria. She is planning a round trip from her base at Bendigo to Shepparton, Ballarat and then returning to Bendigo.



a The direct distance from Ballarat to Bendigo is known to be 85 km. By measuring the lengths of the blues lines between the three cities calculate the direct distance from:

- **i** Bendigo to Shepparton
- **ii** Shepparton to Ballarat.

b Her Piper Archer TX has a range of 522 nautical miles. One nautical mile equals 1.85 km. The fuel gauge indicates the tank is one third full. Is there enough fuel to complete the round trip?

c The average cruising speed is 147 miles per hour. One mile equals 1.61 km. Find the total flight time for the round trip.

d Fuel consumption cost is 95 cents per km. What will be her fuel cost for the trip?

e If a vertical line points North use the map to find the bearing that Alice must fly from:

- **i** Bendigo to Shepparton **ii** Shepparton to Ballarat **iii** Ballarat to Bendigo.

f Give the direction and distance flight information that Alice will need to use on the Shepparton to Ballarat stage of the circuit.

g Use your answers to part **e** to find the angle between the flight paths to and from Shepparton.

h Show how a trigonometry distance rule can be applied to the Bendigo–Shepparton and Shepparton–Ballarat distances with the angle at Shepparton to find the Ballarat–Bendigo distance. Confirm that the distance is approximately 85 km.

i Find the area enclosed by the flight paths.

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Key ideas and chapter summary

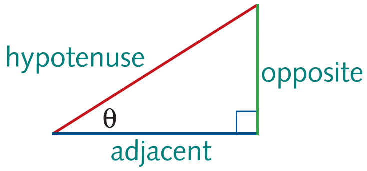
Right-angled triangles

Naming the sides of a right-angled triangle

The **hypotenuse** is the longest side and is always opposite the right angle (90°).

The *opposite* side is directly opposite the angle θ (the angle being considered).

The *adjacent* side is beside angle θ and runs from θ to the right angle.



Trigonometric ratios

The **trigonometric ratios** are sin θ, cos θ and tan θ:

$$\textbf{sin}\ \theta = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \textbf{cos}\ \theta = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \textbf{tan}\ \theta = \frac{\text{opposite}}{\text{adjacent}}$$

SOH-CAH-TOA

This helps you to remember the trigonometric ratio rules.

Degree mode

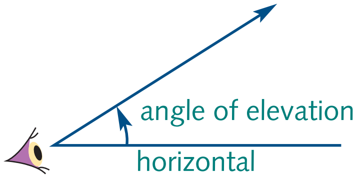
Make sure your calculator is in DEGREE mode when doing calculations with trigonometric ratios.

Applications of right-angled triangles

Always draw well-labelled diagrams showing all known sides and angles. Also label any sides or angles that need to be found.

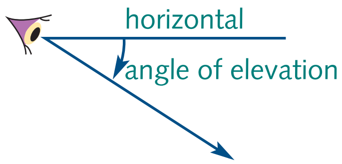
Angle of elevation

The **angle of elevation** is the angle through which you *raise* your line of sight from the horizontal, looking *up* at something.



Angle of depression

The **angle of depression** is the angle through which you *lower* your line of sight from the horizontal, looking *down* at something.

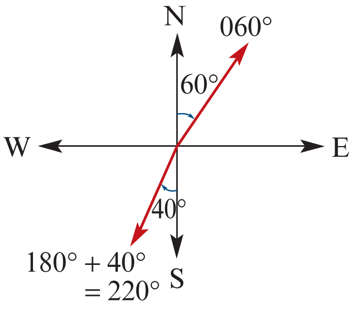


Angle of elevation = angle of depression

The angles of elevation and depression are alternate (‘Z’) angles so they are equal.

Three-figure bearings

Three-figure bearings are measured clockwise from north and always given with three digits, e.g. 060°, 220°.



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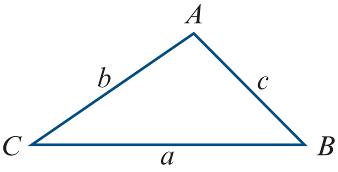
Distance, speed and time Navigation problems usually involve distance, speed and time, as well as direction.

Distance travelled = time taken × speed

Non-right-angled triangles

Labelling a non-right-angled triangle Side a is always opposite angle A, and so on.



Sine rule

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Use the **sine rule** when given:

- two sides and an angle opposite one of those sides
- two angles and one side.

If neither angle is opposite the given side, find the third angle using $A + B + C = 180°$.

Ambiguous case of the sine rule The *ambiguous case of the sine rule* occurs when it is possible to draw two different triangles that *both* fit the given information.

Cosine rule The **cosine rule** has three versions. When given two sides and the angle between them, use the rule that starts with the required side:

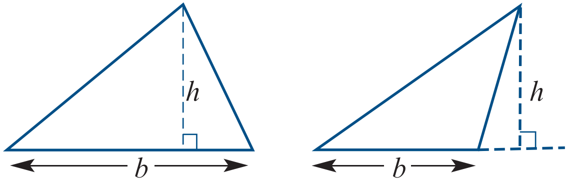
$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$b^2 = a^2 + c^2 - 2ac\cos B$$

$$c^2 = a^2 + b^2 - 2ab\cos C$$

Area of a triangle

- Use the formula area of triangle $= \frac{1}{2} \times b \times h$ if the base and height of the triangle are known:



- Use the formula area of triangle $= \frac{1}{2} \times bc\sin A$ if two sides and the angle between them are known.
- Use Heron's rule if the lengths of the three sides of the triangle are known.

Skills check

Having completed this chapter you should be able to:

- use trigonometric ratios to find an unknown side or angle in a right-angled triangle
- show the angle of elevation or angle of depression on a well-labelled diagram
- show directions on a diagram by using three-figure bearings

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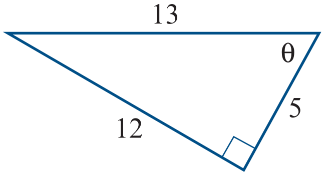
- use the sine rule and cosine rule in non-right-angled triangles to find an unknown side or angle
- use the appropriate rule from the three rules for finding the area of a triangle
- solve practical problems involving right-angled and non-right-angled triangles.

Multiple-choice questions

1 In the triangle shown, $\sin\theta$ equals:

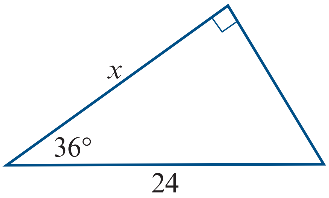


A $\frac{5}{12}$ **B** $\frac{5}{13}$

C $\frac{13}{12}$ **D** $\frac{12}{13}$

E $\frac{12}{5}$

2 The length x is given by:

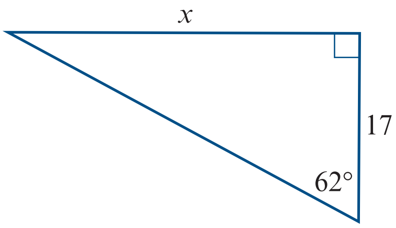


A $24\sin 36°$ **B** $24\tan 36°$

C $24\cos 36°$ **D** $\frac{\sin 36°}{24}$

E $\frac{\cos 36°}{24}$

3 To find length x we should use:

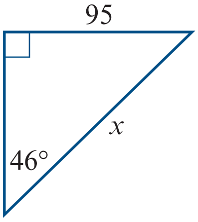


A $17\sin 62°$ **B** $17\tan 62°$

C $17\cos 62°$ **D** $\frac{\tan 62°}{17}$

E $\frac{\sin 62°}{17}$

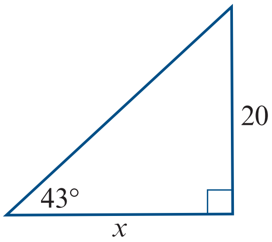
4 The side x is given by:



A $95\tan 46°$ **B** $\frac{95}{\cos 46°}$ **C** $\frac{\sin 46°}{96}$

D $95\sin 46°$ **E** $\frac{95}{\sin 46°}$

5 To find the side x we need to calculate:



A $\frac{\tan 43°}{20}$ **B** $\frac{20}{\tan 43°}$

C $20\tan 43°$ **D** $20\cos 43°$

E $20\sin 43°$

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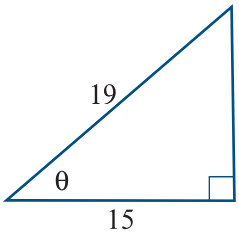
6 To find the angle θ we need to use:

A $\cos^{-1}\left(\frac{15}{19}\right)$ **B** $\cos\left(\frac{15}{19}\right)$

C $\sin^{-1}\left(\frac{15}{19}\right)$ **D** $15\sin(19)$

E $19\cos(15)$

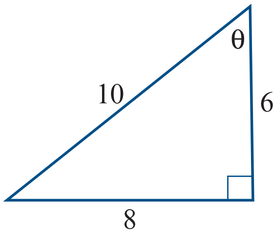


7 The angle θ, correct to one decimal place, is:

A 53.1° **B** 36.9°

C 51.3° **D** 38.7°

E 53.3°

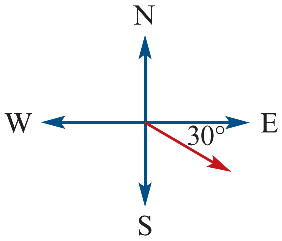


8 The direction shown has the three-figure bearing:

A 030° **B** 060°

C 120° **D** 210°

E 330°

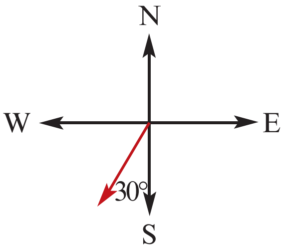


9 The direction shown could be described as the three-figure bearing:

A 030° **B** 060°

C 120° **D** 210°

E −030°

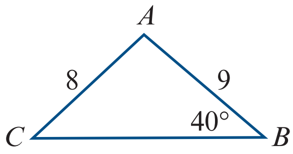


10 In this triangle, angle C equals:

A 34.8° **B** 46.3°

C 53.9° **D** 55.2°

E 86.1°

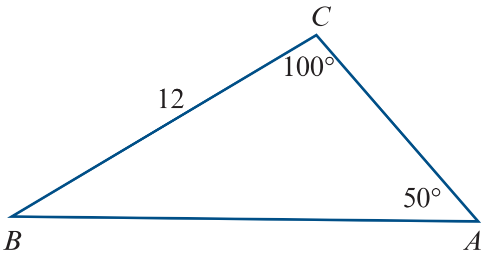


11 To find length c in triangle ABC we should use:

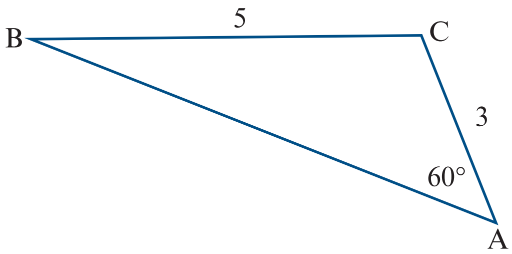
A $\frac{12\sin 100^\circ}{\sin 30^\circ}$ **B** $\frac{12\sin 50^\circ}{\sin 100^\circ}$

C $\frac{\sin 50^\circ}{12\sin 100^\circ}$ **D** $\frac{12\sin 100^\circ}{\sin 50^\circ}$

E $\frac{\sin 100^\circ}{12\sin 50^\circ}$



12 In triangle ABC, $\sin B$ equals:



A $\frac{3}{5}$

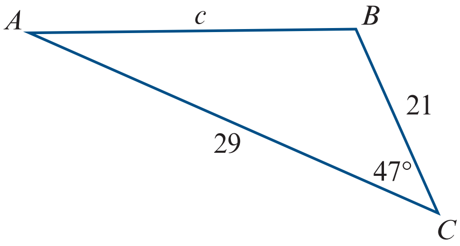
B $\frac{3\sin 60^\circ}{5}$

C $\frac{3}{5\sin 60^\circ}$

D $\frac{5\sin 60^\circ}{3}$

E $\frac{5}{3\sin 60^\circ}$

13 Which expression should be used to find length c in triangle ABC?



A $\frac{1}{2}(21)(29)\cos 47^\circ$

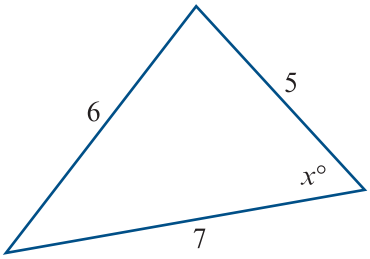
B $\cos^{-1}\left(\frac{21}{29}\right)$

C $\sqrt{21^2 + 29^2}$

D $21^2 + 29^2 - 2(21)(29)\cos 47^\circ$

E $\sqrt{21^2 + 29^2 - 2(21)(29)\cos 47^\circ}$

14 For the given triangle the value of $\cos x$ is given by:



A $\frac{6^2 - 7^2 - 5^2}{2(7)(5)}$

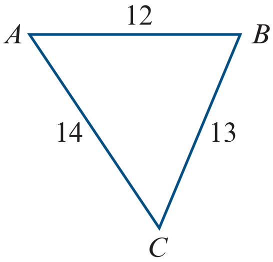
B $\frac{7^2 + 5^2 - 6^2}{2(7)(5)}$

C $\frac{5}{7}$

D $\frac{7^2 - 5^2 - 6^2}{2(5)(6)}$

E $\frac{5^2 - 6^2 - 7^2}{2(5)(6)}$

15 To find angle C we should use the rule:



A $\cos C = \frac{adjacent}{hypotenuse}$

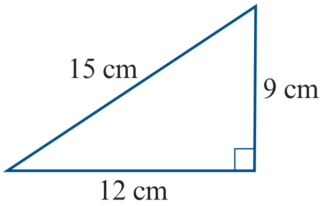
B $\sin C = \frac{opposite}{hypotenuse}$

C $\cos C = \frac{a^2 + c^2 - b^2}{2ac}$

D $\cos C = \frac{a^2 + b^2 - c^2}{2ab}$

E $\frac{b}{\sin B} = \frac{c}{\sin C}$

16 The area of the triangle shown is:



A 108 cm^2

B 54 cm^2

C 36 cm^2

D 90 cm^2

E 67.5 cm^2

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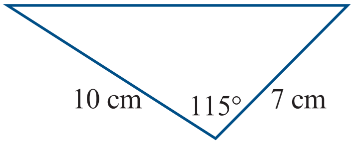
17 The area of the triangle shown, correct to two decimal places, is:

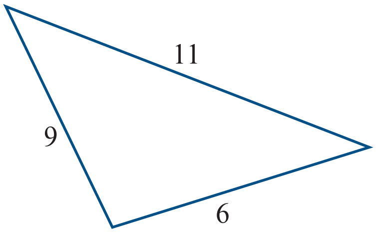


A 35.00 cm^2 **B** 70.00 cm^2

C 14.79 cm^2 **D** 31.72 cm^2

E 33.09 cm^2

18 The area of the triangle shown is given by:



A $26(26-6)(26-9)(26-11)$

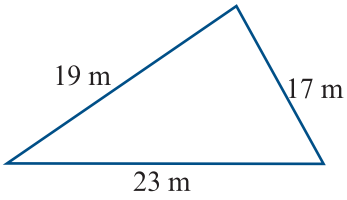
B $\sqrt{26(26-6)(26-9)(26-11)}$

C $\sqrt{13(13-6)(13-9)(13-11)}$

D $\sqrt{6^2+9^2+11^2}$

E $13(13-6)(13-9)(13-11)$

19 The area of the triangle shown, correct to one decimal place, is:

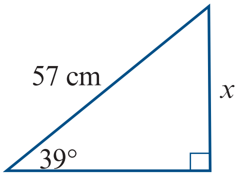


A 29.5 m^2 **B** 218.5 m^2 **C** 195.5 m^2

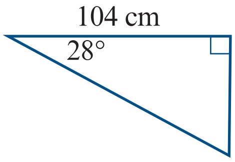
D 161.5 m^2 **E** 158.6 m^2

Short-answer questions

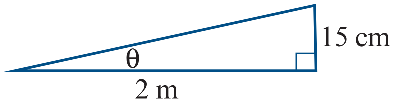
1 Find the length of x, correct to two decimal places.



2 Find the length of the hypotenuse, correct to two decimal places.



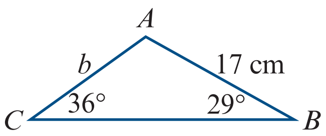
3 A road rises 15 cm for every 2 m travelled horizontally.
Find the angle of slope θ, to the nearest degree.



4 **a** Find the sides of a right-angled triangle for which $\cos\theta = \dfrac{72}{97}$ and $\tan\theta = \dfrac{65}{72}$.

b Hence find $\sin\theta$

5 Find the length of side b, correct to two decimal places.



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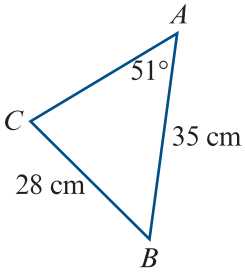
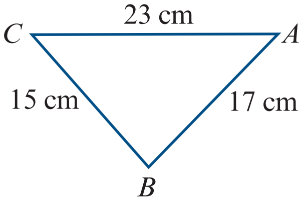
6 Find the angle C, correct to one decimal place.

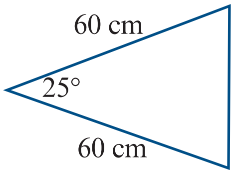


7 Find the smallest angle in the triangle shown, correct to one decimal place.



8 A car travelled 30 km east, then travelled 25 km on a bearing of 070° How far was the car from its starting point? Answer correct to two decimal places.

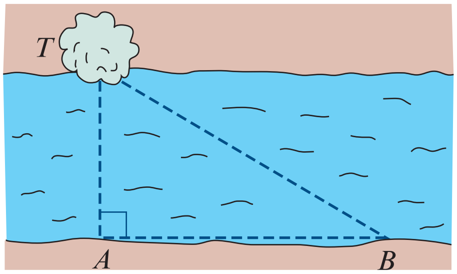
9 A pennant flag is to have the dimensions shown. What area of cloth will be needed for the flag? Answer correct to one decimal place.



10 Find the area of an equilateral triangle with sides of 8 m, correct to one decimal place.

Extended-response questions

1 Tim was standing at point A when he saw a tree T directly opposite him on the far bank of the river. He walked 100 m along the riverbank to point B and noticed that his line of sight to the tree made an angle of 27° with the riverbank. Answer the following correct to two decimal places.



a How wide was the river?

b What is the distance from point B to the tree?

Standing at B, Tim measured the angle of elevation to the top of the tree to be 18°

c Make a clearly labelled diagram showing distance TB, the height of the tree and the angle of elevation, then find the height of the tree.

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2 One group of bushwalkers left a road running north–south to walk along a bearing of 060°. A second group of walkers left the road from a point 3 km further north. They walked on a bearing of 110°. The two groups met at the point C, where their paths intersected.

- **a** Find the angle at which their paths met.
- **b** Find the distance walked by each group, correct to two decimal places.
- **c** If the bushwalkers decided to return to the road by walking back along the path that the second group of walkers had taken, what bearing should they follow?

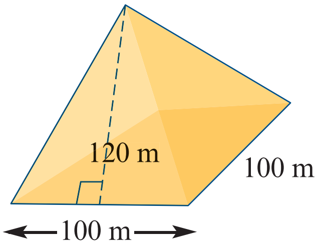
3 A yacht P left port and sailed 45 km on a bearing of 290°.
Another yacht Q left the same port but sailed for 54 km on a bearing of 040°.

- **a** What was the angle between their directions?
- **b** How far apart were they at that stage (correct to two decimal places)?

4 A triangular shade cloth must have sides of 5 m, 6 m and 7 m to cover the required area of a children's playground.

- **a** What angle is required in each of the corners (correct to one decimal place)?
- **b** The manufacturer charges according to the area of the shade cloth. What is the area of this shade cloth (correct to two decimal places)?
- **c** The cost of shade cloth is $29 per square metre. What will be the cost of this shade cloth?

5 The pyramid shown has a square base with sides of 100 m. The line down the middle of each side is 120 m long.



- **a** Find the total surface area of the pyramid. (As the pyramid rests on the ground the area of its base is not part of its surface area.)
- **b** If 1 kg of gold can be rolled flat to cover 0.5 m^2 of surface area, how much gold would be needed to cover the surface of the pyramid?
- **c** At today's prices, 1 kg of gold costs $15 500. How much would it cost to cover the pyramid with gold?

12

Inequalities and linear programming

- What is a linear inequality?
- How do we solve linear inequalities?
- What is linear programming and how is it used?

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Introduction

In Chapter 6, ‘Linear graphs and models’, you learned how linear equations and their graphs are used to model practical situations, such as plant growth, service charges and flow problems. In this chapter you will learn how linear inequalities and their graphs can be used to model a different set of practical situations, such as determining the mix of products in a supermarket to maximise profit, or designing a diet to provide maximum nutrition for minimum cost. This is known as **linear programming**. Linear programming requires you to solve both linear equations and linear inequalities. You have already learned how to solve linear equations in Chapter 3, ‘Linear relations and equations’. You now need to learn how to solve linear inequalities.

12A Review of inequalities

An equality is a statement that one quantity is less than or greater than another.

Making statements like; ‘Walk at least ten thousand steps a day’, ‘Don’t spend more than 20 minutes sitting at a time,’ and ‘Spend no more than four minutes in the shower’ are all examples of inequalities in everyday life.

We write **inequalities** using the symbols shown at right.

Inequality	Symbol
less than	$<$
greater than	$>$
less than or equal to	$\leq$
greater than or equal to	$\geq$

In the table below, a number of everyday statements involving inequalities are translated into symbolic statements using the variables w and h, and the symbols $<$, $\leq$, $>$ and $\geq$.

Everyday statements	Mathematical translation	Symbolic expression w = Rod’s weight in kg h = Meg’s height in cm
Rod weighs *more than* 70 kg. Rod’s weight *exceeds* 70 kg.	Rod’s weight is *greater than* 70 kg.	$w > 70$
Rod weighs *at least* 77 kg.	Rod’s weight is *greater than or equal to* 77 kg.	$w \geq 77$
Meg is *less than* 165 cm tall.	Meg’s height is *less than* 165 cm.	$h < 165$
Meg is *at most* 163 cm tall or Meg is *no more than* 163 cm tall.	Meg’s height is *less than or equal to* 163 cm.	$h \leq 163$

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Exercise 12A

Symbolising statements involving the use of inequalities in everyday language

Use the information in the table above to help you answer the following questions.

1 Let d = distance (in km).
Write down inequalities in terms of the variable d that can be used to symbolise the following statements.

- **a** The distance to the nearest train station is more than 10 km.
- **b** The distance to the nearest shop is less than 5 km.
- **c** The distance to the nearest school is no more than 2 km.
- **d** The distance to the nearest sporting complex is at least 7 km.

2 Let t = time (in hours).
Write down inequalities in terms of the variable t that can be used to symbolise the following statements.

- **a** It will take less than two hours to drive from Corryong to Wangaratta.
- **b** I've been awake for more than 3 hours.
- **c** I spend no more than 5 hours per week watching television.
- **d** I spend less than one hour a night on my school work.
- **e** The maximum amount of time I can afford to go running tonight is one and a half hours.

12B Linear inequalities in one variable

Linear inequalities in one variable and the number line

An expression such as $9 \le 3x \le 21$ is called a *linear inequality* in one variable.

It is an *inequality,* not an equation, because it involves an inequality sign ($\le$) rather than an equals sign (=). Remember that the sign $\le$ means 'less than or equal to'.

Having mastered the language of inequalities, the next step is to learn how to read and plot equalities and inequalities on a number line. The table overpage summarises the conventions we use when doing this.

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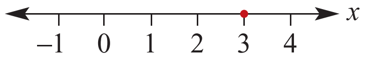
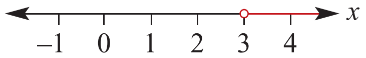
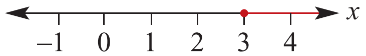
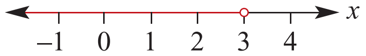
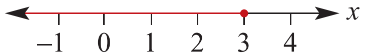
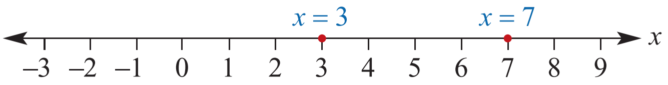
Equality/inequality	read as	plots as
= as in $x = 3$	'x equals 3'	
> as in $x > 3$	'x is greater than 3'	
≥ as in $x \geq 3$	'x is greater than or equal to 3'	
< as in $x < 3$	'x is less than 3'	
≤ as in $x \leq 3$	'x is less than or equal to 3'	

▶ Representing the solution of a linear equation on a number line

The solution to the linear equation $3x = 9$ is $x = 3$, and the solution to the linear equation $3x = 21$ is 7.

We can represent these solutions on a number line by putting a *closed circle* (●) on the number line at $x = 3$ and $x = 7$ as shown.



When solving an *inequality* and graphing its solution on a number line, we need to be careful about whether the end values of the solution are included in the range of possible values.

▶ Representing the solution of a linear inequality on a number line

Case 1: End values included

To solve the linear inequality:

$$9 \leq 3x \leq 21$$

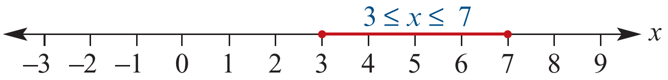
divide through by 3 to make x the subject of the inequality

$$\frac{9}{3} \leq \frac{3x}{3} \leq \frac{21}{3}$$

or $\quad 3 \leq x \leq 7$

There is no single solution to this inequality. Any value of x from 3 to 7 is a solution. For example, $x = 3$, $x = 3.5$, $x = 4.95$ and $x = 7$ are all possible solutions. In fact, it is impossible to list every possible solution, as there are an infinite number of solutions. However, we can represent all the *possible* solutions on a number line.

This is done by marking the points $x = 3$ and $x = 7$ with a closed circle (●) on the number line. These points are then joined by drawing a solid line to indicate that all values between $x = 3$ and $x = 7$ are also solutions, as shown below.



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Case 2: End values not included

To solve the linear inequality:

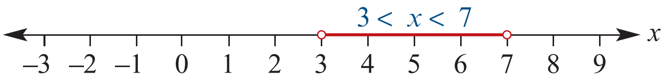
$$9 < 3x < 21$$

divide through by 3 to make x the subject of the inequality

$$3 < x < 7$$

Remember, the sign '<' means 'less than'. This means that $x = 3$ and $x = 7$ are not solutions, but all values between $x = 3$ and $x = 7$ are possible solutions.

To represent this solution on a number line, mark in the points $x = 3$ and $x = 7$ with an open circle (○). These two open circles are then joined by a solid line to indicate that all values between 3 and 7 are solutions, but not $x = 3$ and $x = 7$.



Note that $7 > x > 3$ represents the same values of x as $3 < x < 7$.

Example 1 Representing an inequality on a number line

Represent the inequality:

$$-10 < 5x \le 40$$

for x and display the solution on a number line.

Solution

1 Write the inequality.

$$-10 < 5x \le 40$$

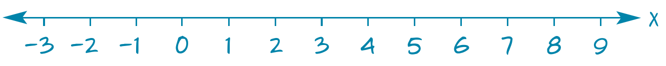
2 Make x the subject of the inequality by dividing through by 5 to make x the subject.

$$\text{or} \quad \frac{-10}{5} < \frac{5x}{5} \le \frac{40}{5}$$

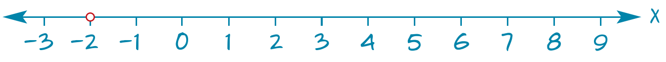
$$\text{or} \quad -2 < x \le 8$$

3 Display the solution on a number line.

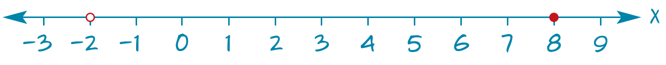
- Draw a number line to include −2 and 8.



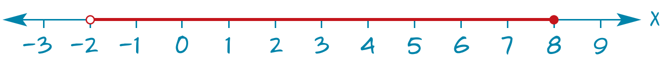
- Mark the point $x = -2$ with an open circle.



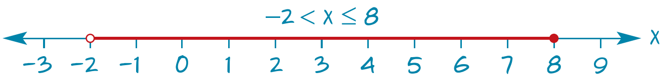
- Mark the point $x = 8$ with a closed circle.



- Join the two points with a solid line.



- Write in the solution inequality on the graph.



Example 2 Solving an inequality and graphing the solution

Solve the inequality:

$$10x > 20$$

for x and display the solution on a number line.

Solution

1 Write the inequality.

$10x > 20$

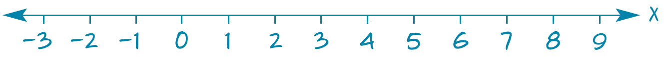
2 Make x the subject of the inequality by dividing through by 10 to make x the subject.

or $\frac{10x}{10} > \frac{20}{10}$

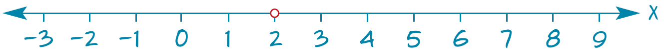
or $x > 2$

3 Display the solution on a number line.

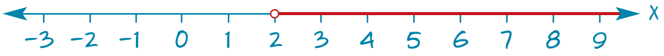
- Draw a number line to include 2.



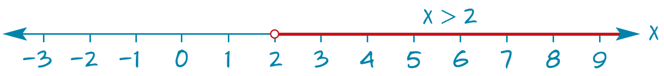
- Mark in the point $x = 2$ with an open circle.



- To indicate all values of x greater than 2, draw a solid line from this point to the right that potentially goes on forever.



- Write in the solution inequality on the graph.



Exercise 12B

Basic properties of inequalities

Skillsheet

1 Which of the symbols < , = or > should be placed in the box in each of the following?

a $7 \square 9$
b $3 \square 2$
c $7 + 1 \square 9 - 1$
d $0.5 \square 1$
e $8 \square 4$
f $-3 \square 1$
g $-2 \square -1$
h $0 \square 0.5$

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Constructing and interpreting number lines

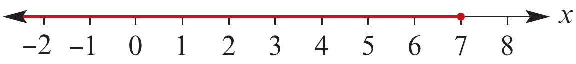
Example 1

2 Represent each of the following inequalities on a number line.

a $1 \le x \le 4$
b $0 < x < 4$
c $x < 4$
d $x \ge 4$
e $-1 \le x < 4$
f $3 < x \le 5$

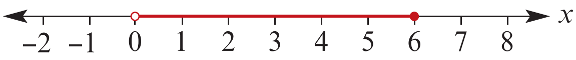
3 Write down an inequality represented by each of the following graphs.

a

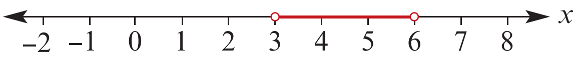


b

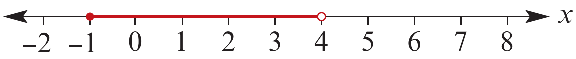
c



d



e



Example 2

4 Solve each of the following inequalities and represent its solution on a number line.

a $3x \ge 15$
b $20x < 100$
c $2x > -4$
d $9x \ge 36$
e $-12 \le 6x < 24$
f $10 < 5x \le 25$

Applications

5 A person becomes a teenager when they turn 13. They stop being a teenager when they turn 20.

Let x represent the variable age (in years).

a Write down an inequality in terms of x that defines a teenager.

b Graph this inequality on a number line.

6 Carry-on luggage in most passenger aircraft can weigh no more than 7 kg.

Let w be the variable weight (in kg).

a Write down an inequality in terms of w that defines the acceptable weight for carry-on luggage.

b Graph this inequality on a number line.

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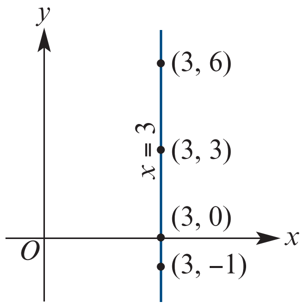
12C Linear inequalities in one variable and the coordinate plane

We can also represent linear inequalities in one variable on the *coordinate plane*.

Plotting an equality ($x = 3$)

The equality $x = 3$ is plotted on a set of axes, as shown.



On a coordinate plane, $x = 3$ is represented by a vertical straight line, located at $x = 3$.

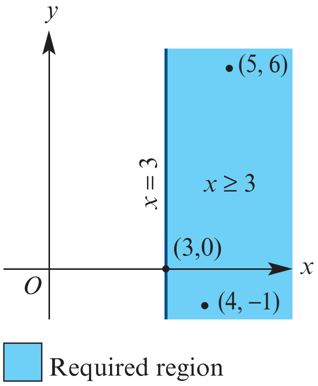
Every point on the line $x = 3$ has the same x-coordinate, 3.

However, the value of y changes along the line, as we can see for the four plotted points.

Plotting an inequality ($x \geq 3$) – boundary included

Just as with graphing the solution of an inequality on a number line, we need to be careful about whether the boundary lines (end values) of the solution are included in the range of possible values.

The inequality $x \geq 3$ is plotted on a set of axes as shown.



On a coordinate plane, $x \geq 3$ is represented by the shaded region that starts at the vertical line $x = 3$ and extends right forever.

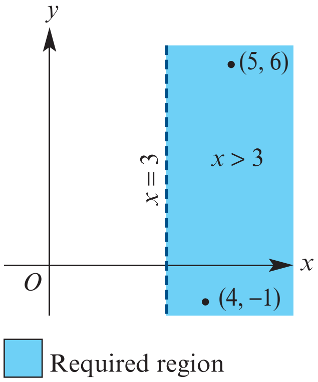
This shaded region includes every point in space with an x-coordinate greater than or equal to three.

The left-hand boundary of the region is the line $x = 3$. There is no right-hand boundary because the region goes on forever.

Some representative points that satisfy the condition $x \geq 3$, and which are found in the shaded region, have also been plotted.

Plotting an inequality ($x > 3$) – boundary not included

The plot of the inequality $x > 3$ is similar to the plot of $x \geq 3$, but the line $x = 3$ is drawn as a *dashed line* to indicate that it is *not* included in the region.



Some representative points that satisfy the condition $x > 3$ have also been plotted.

Note: For $(5, 6),\ 5 > 3$

$(4, -1), 4 > 3$

Therefore both points satisfy the inequality $x > 3$.

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Example 3 Plotting a linear inequality in one variable on the coordinate plane

On the coordinate plane, plot the graphs of:

a $y \leq 4$

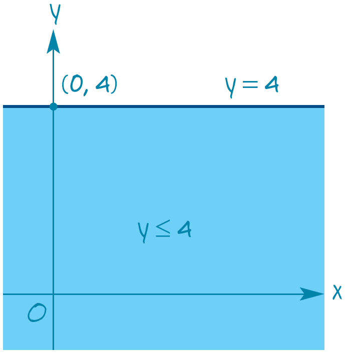
b $-1 < y < 3$

Solution

a $y \leq 4$

1 Draw in a solid line $y = 4$ to define the boundary of the shaded region.

2 Shade the region on and below the line $y = 4$ to represent all the points defined by $y \leq 4$.

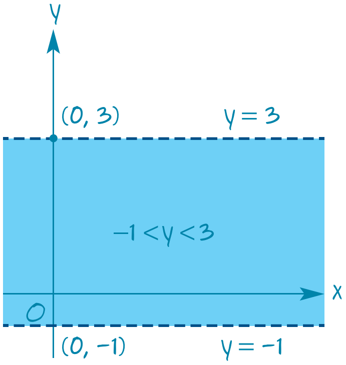


b $-1 < y < 3$

1 Draw in a dashed line $y = 3$ to define the upper boundary of the shaded region.

2 Draw in a dashed line $y = -1$ to define the lower boundary of the shaded region.

3 Shade the region between the lines $y = 3$ and $y = -1$ to represent all the points defined by $-1 < y < 3$.



Exercise 12C

Example 3

1 Graph the following inequalities on the coordinate plane.

a $x \leq 1$ **b** $x > -2$ **c** $y \leq 5$

d $y > 1$ **e** $x < 2$ **f** $-2 \leq y \leq 2$

g $-1 < x < 2$ **h** $3 < x \leq 5$ **i** $-3 \leq y < 0$

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12D Linear inequalities in two variables

▶ Representing the inequality $y - x \geq 2$ graphically

To graph $y - x \geq 2$, we first draw in the line $y - x = 2$.

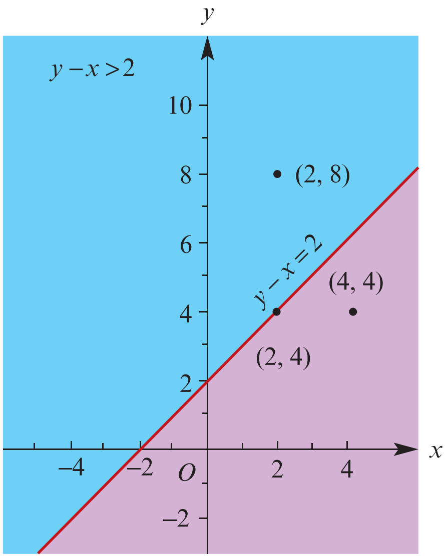
This line has been drawn in red on the graph.

We have now separated the coordinate plane into two regions, shaded blue and purple.

The red line represents all pairs of values of x and y that satisfy $y - x = 2$.

The blue shaded area represents all pairs of values of x and y that satisfy $y - x > 2$.

Together the red line and blue shaded region represent the inequality $y - x \geq 2$.



We can demonstrate this is true by carrying out the following tests.

Test points:

(2, 4): $y - x = 4 - 2 = 2$, so the point (2, 4) lies on the line which forms the boundary of the region. We can also see this from the diagram.

(2, 8): $y - x = 8 - 2 = 6$ as 6 is greater than 2 the inequality $y - x > 2$ is satisfied. We can also see this from the location of the point (2, 8) in the diagram.

(4, 4): $y - x = 4 - 4 = 0$ as 0 is less than 2 the inequality $y - x > 2$ is *not* satisfied. We can also see this from the location of the point (4, 4) in the diagram.

Example 4 Testing if a point lies in a region defined by an inequality

Test to see if the point (1, 4) lies in the region $x + y \geq 6$.

Solution

If the point (1, 4) lies in the region $x + y \geq 6$, the value of the expression $x + y$ when $x = 1$ and $y = 4$ must be greater than or equal to 6.

1 Substitute $x = 1$ and $y = 4$ into the left hand side of the inequality.	Test the point (1, 4): $x + y = 1 + 4 = 5$
2 Test to see if this value is greater than or equal to 6.	$5 \geq 6$ (not true)
3 Draw your conclusion.	Because 5 is less than 6, the point (1, 4) does not lie in the region $x + y \geq 6$.

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▶ Representing the inequality $y - x \le 2$ graphically

To graph $y - x \le 2$, we first draw the line $y - x = 2$.

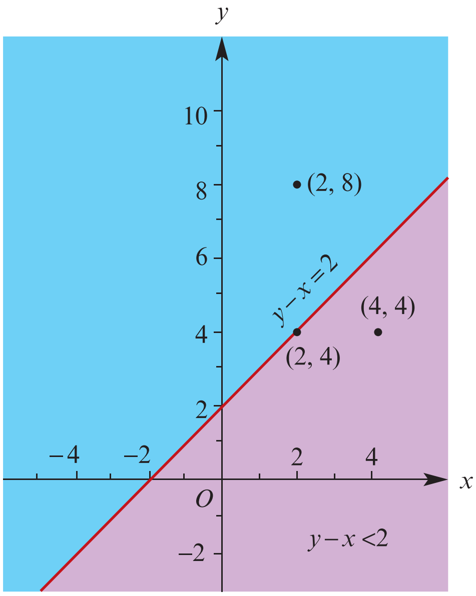
This line has been drawn in red on the graph opposite.

We have now separated the coordinate plane into two regions, shaded blue and purple.

The red line represents all pairs of values of x and y that satisfy $y - x = 2$.

The purple shaded area represents all pairs of values of x and y that satisfy the inequality $y - x < 2$.

Together the red line and purple shaded area represent the inequality $y - x \le 2$.



We can also show this by carrying out the following tests.

Test points:

(2, 4): $y - x = 4 - 2 = 2$, so the point (2, 4) lies on the line that forms the boundary of the region. We can also see this from the diagram.

(2, 8): $y - x = 8 - 2 = 6$, as 6 is greater than 2 the inequality $y - x \le 2$ is *not* satisfied. We can also see this from the location of the point (2, 8) in diagram.

(4, 4): $y - x = 4 - 4 = 0$, as 0 is less than 2 the inequality $y - x \le 2$ is satisfied. We can also see this from the location of the point (4, 4) in diagram. Similar tests would show that all of the points in the purple shaded region satisfy this inequality.

Example 5 Testing if a point lies in a region defined by an inequality

Test to see if the point (2, 3) lies in the region $2x + y \le 12$.

Solution

If the point (2, 3) lies in the region $2x + y \le 12$, the value of the expression $2x + y$ when $x = 2$ and $y = 3$ must be less than or equal to 12.

1 Substitute $x = 2$ and $y = 3$ into the left hand side of the inequality.

Test the point (2, 3):

$2x + y = 2(2) + 3 = 7$

2 Test to see if this value is less than or equal to 12.

$7 \le 12$ (true)

3 Draw your conclusion.

Because 7 is less than 12, the point (2, 3) lies in the region $2x + y \le 12$.

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Linear inequalities

Linear inequalities can be represented by regions in the coordinate plane.

If the inequality sign is:

- $\leq$ or $\geq$, the line defining the region is *included*, and is indicated by using a *solid line* to indicate the boundary
- $<$ or $>$, the line defining the region is not included, and is indicated by using a dashed line to indicate the boundary.

Example 6 Graphing a linear inequality in two variables

Sketch the graph of the region $3x + 2y \leq 18$.

Solution

1 Find the intercepts for the boundary line $3x + 2y = 18$.

- Find the y-intercept. Substitute $x = 0$ into the equation and solve for y.
- Find the x-intercept. Substitute $y = 0$ into the equation and solve for x.

$$3x + 2y = 18$$

When $x = 0$, $2y = 18$

$$y = 9$$

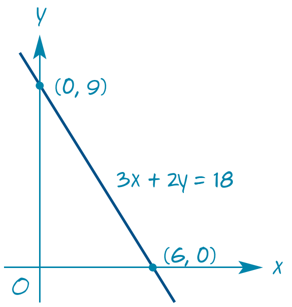
$\therefore$ y-intercept is $(0, 9)$.

When $y = 0$, $3x = 18$

$$x = 6$$

$\therefore$ x-intercept is $(6, 0)$.

2 On a labelled set of axes, draw a straight line through the two intercepts. Use a *solid line* to indicate that the line is included in the region. Label the line.



3 Use a test point to determine whether the required region lies above or below the line.

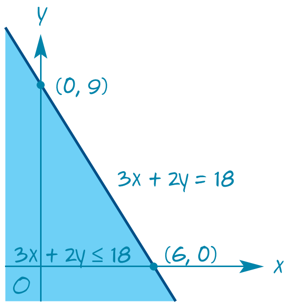
Note: The origin (0, 0) is usually a good point to test.

Test $(0, 0)$:

$3x + 2y = 3(0) + 2(0) = 0$

$0 < 18$, so $(0, 0)$ lies in the region $3x + 2y \leq 18$.

4 As (0, 0) is below the line, the required region lies on and below the line. Shade in the region on and below the line. Label the region.



Example 7 Graphing a linear inequality in two variables

Sketch the graph of the region $4x - 5y > 20$.

Solution

1 Find the intercepts for the boundary line $4x - 5y = 20$.

- Find the y-intercept. Substitute $x = 0$ into the equation and solve for y.
- Find the x-intercept. Substitute $y = 0$ into the equation and solve for x.

$4x - 5y = 20$

When $x = 0, -5y = 20$

$y = -4$

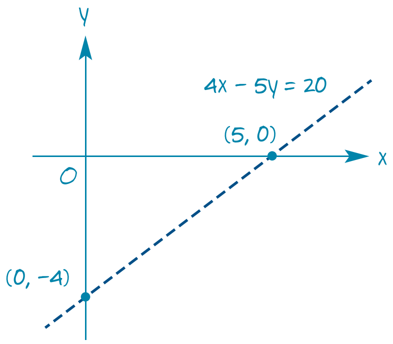
$\therefore$ y-intercept is $(0, -4)$.

When $y = 0, 4x = 20$

$x = 5$

$\therefore$ x-intercept is $(5, 0)$.

2 On a labelled set of axes, draw a straight line through the two intercepts. Use a *dashed line* to indicate that the boundary line is *not* included in the region. Label the line.



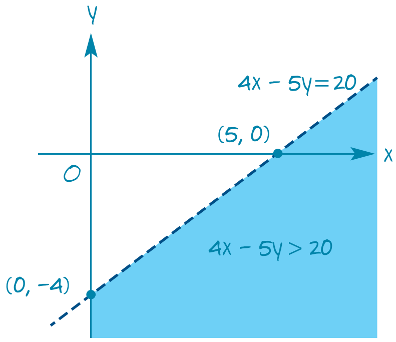
3 Use a test point to determine whether the required region lies above or below the line.

Test $(0, 0)$:

$4x - 5y = 4(0) - 5(0) = 0$

$0 < 20$, so $(0, 0)$ does not lie in the region $4x - 5y > 20$.

4 As $(0, 0)$ is above the line, the required region lies below the line. Shade in the region on and below the line. Label the region.



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To plot a linear inequality

- Graph the inequality as if it contained an equals sign (=).
- Draw a solid line if the inequality is $\le$ or $\ge$.
- Draw a dashed line if the inequality is $<$ or $>$.
- Pick a point *not* on the line to use as a test point. The origin (0, 0) is a good test point, provided the boundary line does not pass through the origin.
- Substitute the test point into the inequality. If the point makes the inequality true, shade the region containing the test point. If not, shade the region *not* containing the test point.

Exercise 12D

Testing points

Example 4, 5

1 Test to see whether the point (0, 0) lies in the following regions.

a $x + y \ge 0$ **b** $x + y < 4$ **c** $2x + y > 2$

d $3x - 2y \ge 3$ **e** $y - 2x > 5$ **f** $x - 3y < 6$

2 Test to see whether the point (1, 2) lies in the following regions.

a $x + y \ge 0$ **b** $x + y < 0$ **c** $2x + y > 2$

d $2x - 2y \ge 3$ **e** $2x + 3y > 5$ **f** $5y - 2x \ge 8$

Graphing inequalities

Example 6, 7

3 Graph the following inequalities.

a $y - x \le 5$ **b** $2x - y \le 4$ **c** $x - y < 3$

d $x + y \ge 10$ **e** $3x + y \le 9$ **f** $5x + 3y \ge 15$

g $3y - 5x < 15$ **h** $2y - 5x > 5$ **i** $y - x > -3$

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12E Feasible regions

In Chapter 3, ‘Linear relations and equations’, you learned how to solve pairs of simultaneous linear equations graphically.

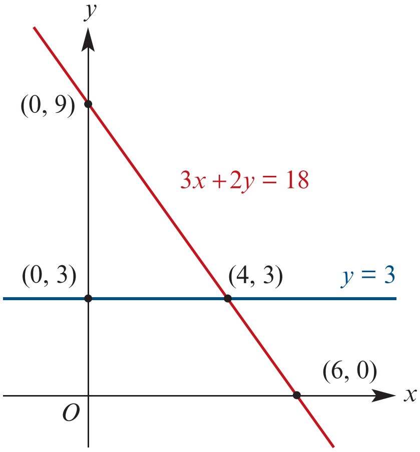
For example, to solve the *pair* of linear equations:

$$3x + 2y = 18$$
$$y = 3$$

graphically, we simply plot their graphs and find the point of intersection. See opposite.

The *solution* is the *point* on the coordinate plane that is *common to both graphs*. This is the point (4, 3), the point where the two lines intersect.

From this, we conclude that $x = 4$ and $y = 3$.



However, when we try to solve the pair of simultaneous linear inequalities:

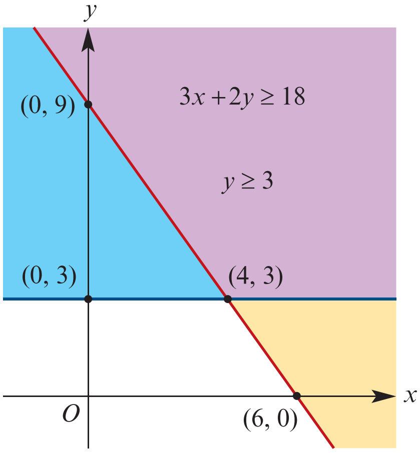
$$3x + 2y \geq 18$$
$$y \geq 3$$

graphically, there is not a single solution, but many solutions.

The *solutions* are all of the points that lie in the *region* in the coordinate plane that is *common to both inequalities*.

In the graph opposite, the purple shaded region (including boundaries) is the **solution region.**

We call this solution region the **feasible region**.



The feasible region

The region common to a set of inequalities is called the *feasible region*.
It is called the feasible region because all the points in this region are possible (or feasible) solutions for this set of inequalities.

The concept of a feasible region is very important when we solve linear programming problems, which you will learn to do in the next section. At present you only need to know how to graph a feasible region. This is explained in the next two worked examples.

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Example 8 Graphing a feasible region

Graph the feasible region for the following four simultaneous inequalities:

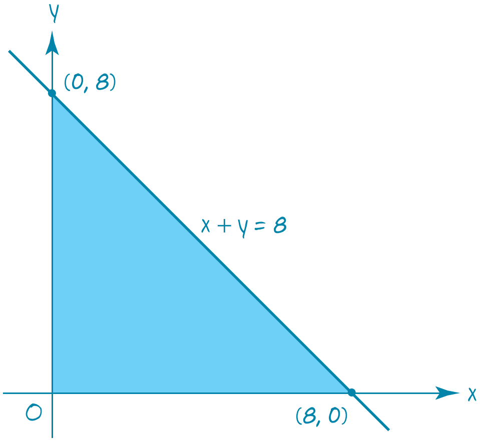
$$x \geq 0, \qquad y \geq 0, \qquad x + y \leq 8, \qquad 3x + 5y \leq 30$$

Solution

Because $x \geq 0$ and $y \geq 0$, the feasible region is restricted to the first quadrant.

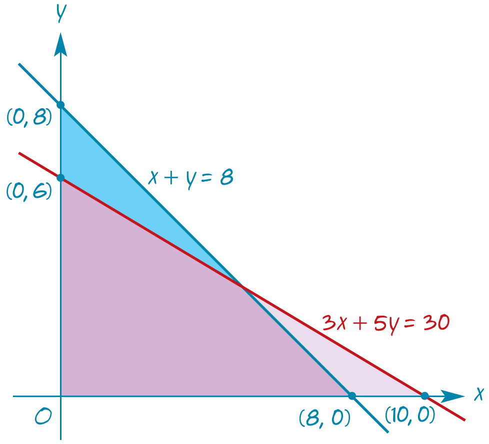
1 Graph the inequality $x + y \leq 8$ in the first quadrant.

- Plot the boundary line $x + y = 8$, marking and labelling the y-intercept (0, 8) and the x-intercept (8, 0).
- Shade in the region bounded by the x and y-axes and the line. Here it has been shaded blue.



2 Graph the inequality $3x + 5y \leq 30$ in the first quadrant.

- Plot the boundary line $3x + 5y = 30$, marking and labelling the y-intercept (0, 6) and the x-intercept (10, 0).
- Shade in the region bounded by the x and y-axes and the line. Here it has been shaded pink, but it becomes purple where it overlaps the blue region.

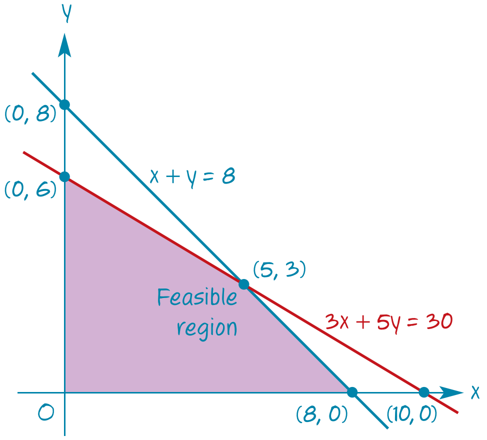


3 The overlap region (purple) is the feasible region.

- Label the overlap region the ‘Feasible region’.
- To complete the feasible region, find the coordinates of the point where the two boundary lines intersect, by solving the simultaneous equations

$$x + y = 8$$
$$3x + 5y = 30$$

The lines intersect at the point (5, 3). Mark this point on the graph.



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Example 9 Graphing a feasible region

Graph the feasible region for the following four simultaneous inequalities:

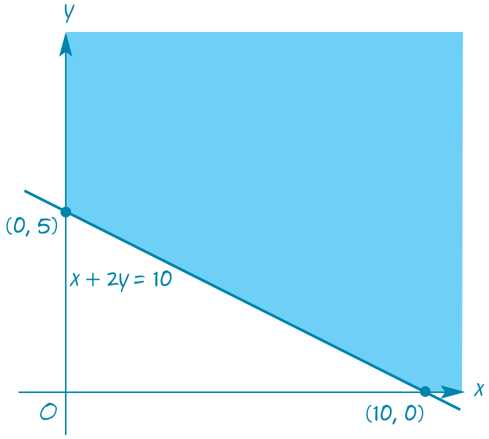
$$x \geq 0, \quad y \geq 0, \quad x + 2y \geq 10, \quad 6x + 4y \geq 36$$

Solution

Because $x \geq 0$ and $y \geq 0$, the feasible region is restricted to the first quadrant.

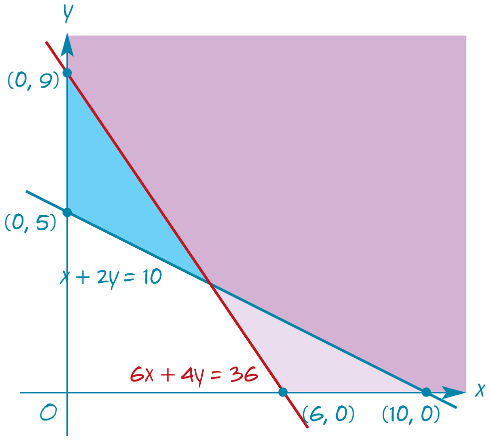
1 Graph the inequality $x + 2y \geq 10$ in the first quadrant.

- Plot the boundary line $x + 2y = 10$, marking and labelling the y-intercept (0, 5) and the x-intercept (10, 0).
- Shade in the region bounded by the x and y-axes and the line. Here it has been shaded blue.



2 Graph the inequality $6x + 4y \leq 36$ in the first quadrant.

- Plot the boundary line $6x + 4y = 36$, marking and labelling the y-intercept (0, 9) and the x-intercept (6, 0).
- Shade in the region bounded by the x and y-axes and the line. Here it has been shaded pink, but it becomes purple where it overlaps the blue region.

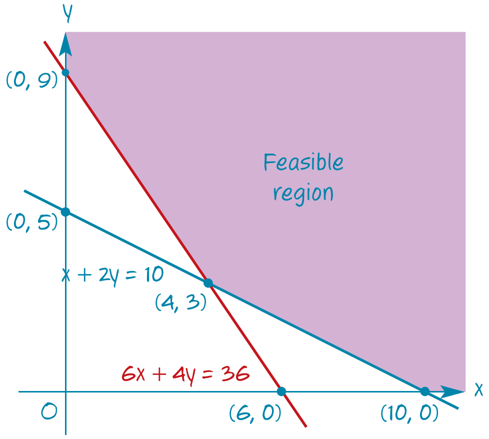


3 The overlap region (purple) is the feasible region.

- Label the overlap region the ‘Feasible region’.
- To complete the feasible region, find the coordinates of the point where the two boundary lines intersect, by solving the simultaneous equations

$$x + 2y = 10$$
$$6x + 4y = 36$$

The lines intersect at the point (4, 3). Mark this point on the graph.



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Exercise 12E

Skillsheet

Graphing feasible regions by hand

Example 8, 9

Graph the feasible region for each of the following sets of linear inequalities.

1 $x \geq 0, \quad y \geq 0, \quad x + y \leq 10$

2 $x \geq 0, \quad y \geq 0, \quad 2x + 3y \leq 12$

3 $x \geq 0, \quad y \geq 0, \quad 3x + 5y \geq 15$

4 $x \geq 0, \quad y \geq 0, \quad x + y \leq 6, \quad 2x + 3y \leq 15$

5 $x \geq 0, \quad y \geq 0, \quad 4x + y \geq 12, \quad 3x + 6y \geq 30$

12F How to use a graphics calculator to graph a feasible region (optional)

Your CAS calculator can be used to graph a feasible region. However, the process is not as straightforward as it might be and should not be seen as a total replacement for graphing a feasible regions from first principles.

How to graph a feasible region using the TI-Nspire CAS

Graph the feasible region for the following four simultaneous inequalities:

$$x \geq 0, \qquad y \geq 0, \qquad x + 2y \geq 10, \qquad 6x + 4y \geq 36$$

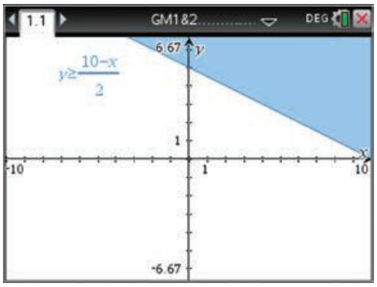
Because $x \geq 0$ and $y \geq 0$, the feasible region is restricted to the first quadrant. We take this into account when setting the viewing window on the calculator.

Steps

1 To graph the inequalities $x + 2y \geq 10$ and $6x + 4y \geq 36$ with a graphics calculator, we first need to rearrange both inequalities so that y is the subject.

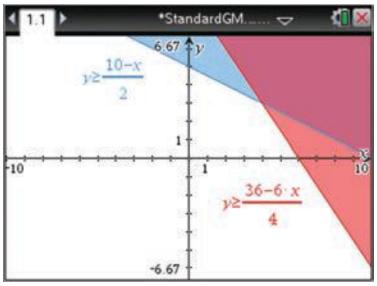
$x + 2y \geq 10$ becomes $y \geq \dfrac{10 - x}{2}$

$6x + 4y \geq 36$ becomes $y \geq \dfrac{36 - 6x}{4}$



2 Open a new document (ctrl + N) and select **Add Graphs**.

a Use the backspace key to delete the = sign and select ≥ from the pop-up list. Type in **(10-x) ÷ 2** Press ▼ to enter the second inequality. The first inequality ($x + 2y \geq 10$) will be plotted.



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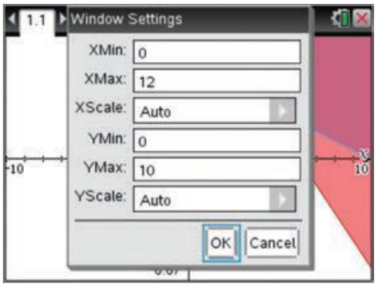
b Repeat the above, but this time type in **(36 − 6x) ÷ 4**.
Press enter. This plots the inequality $6x + 4y \geq 36$.
Hint: If the function entry line is not visible, press tab.

c The inequalities $x \geq 0$ and $y \geq 0$ indicate that the feasible region is restricted to the first quadrant. This is best achieved by resetting the viewing window.

3 Reset the viewing window, using menu > **Window/Zoom**> **Window Settings**. Using tab to move between the entry boxes, enter the following values:

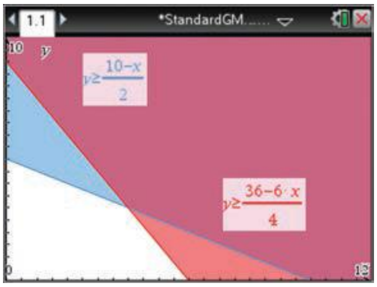
- **XMin = 0**
- **XMax = 12**
- **XScale = Auto**
- **YMin = 0**
- **YMax = 10**
- **YScale = Auto**



Window Settings

XMin: 0
XMax: 12
XScale: Auto
YMin: 0
YMax: 10
YScale: Auto

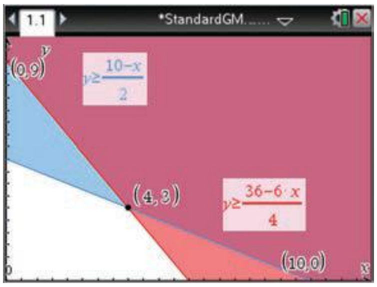
4 Pressing enter confines the plot to the first quadrant. The graphs appear as shown. The feasible region is the more heavily shaded region.
Note: It may be necessary to grab and move the graph labels if they overlap with other labels.



5 To complete the feasible region, we need to know the coordinates of the corner points.

a Press menu **>Geometry>Points & Lines>Intersection Point/s** and then enter.

b Move the cursor to one of the graphs and press ☝, move to the other graph and press ☝.
The point of intersection (4, 3) is displayed.
Press esc to exit the **Intersection Point** tool.
The other two points, (0, 9) and (10, 0), can be determined from the equations of the boundary lines.
Hint: If required to plot equations such as x ≥ 1, enter the equation in a text box in any open area (menu **>Actions>Text)** and drag over either axis.



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How to graph a feasible region using the ClassPad

Graph the feasible region for the following four simultaneous inequalities:

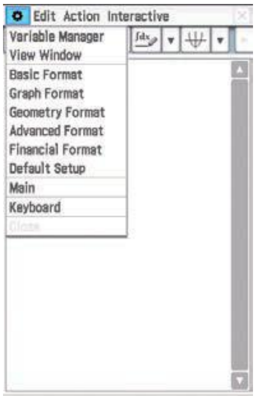
$$x \geq 0, \qquad y \geq 0, \qquad x + 2y \geq 10, \qquad 6x + 4y \geq 36$$

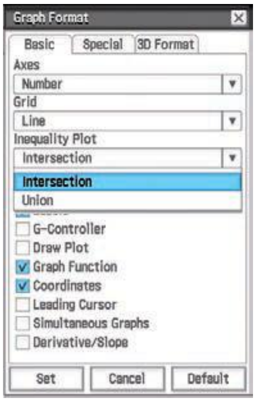
Because $x \geq 0$ and $y \geq 0$, the feasible region is restricted to the first quadrant. We take this into account when setting the viewing window on the calculator.

Steps

1 Set the **Graph Format** to properly sketch the feasible region.

- Tap the Settings menu ⚙.
- Tap **Graph Format**.
- Select **Intersection** in the **Inequality Plot** menu.

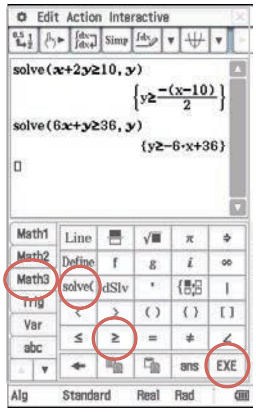




Axes: Number
Grid: Line
Inequality Plot: Intersection
☐ G-Controller
☐ Draw Plot
☑ Graph Function
☑ Coordinates
☐ Leading Cursor
☐ Simultaneous Graphs
☐ Derivative/Slope

2 Rearrange both inequalities so that y is the subject.

- Open the Main menu and bring up the **Keyboard**.
- Tap the **Math3** tab.
- Tap **solve(**.
- Type in $x + 2y \geq 10$ followed by a comma and y.
- Close the bracket and tap **EXE**.
- Using the same process, enter $6x + 4y \geq 36$.



solve(x+2y≥10, y)

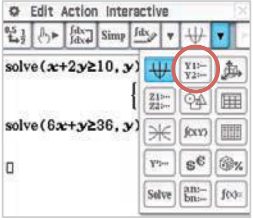
$$\left\{y \geq \frac{-(x-10)}{2}\right\}$$

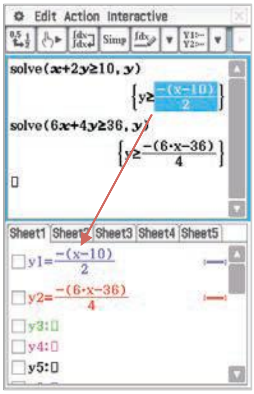
solve(6x+y≥36, y)

$$\{y \geq -6\cdot x+36\}$$

3 Transfer equations from the **Main** to the **Graph Editor** window.

- Tap the right-most down arrow button ▼.
- Tap the **Graph Editor** button $\boxed{\text{Y1:}\text{Y2:}}$.
- Highlight the expression $\dfrac{-(x-10)}{2}$ and drag it into the box on the right of **y1** in the **Graph Editor** window.
- Highlight the expression $\dfrac{-(6x-36)}{4}$ and drag it into the box on the right of **y2** in the **Graph Editor** window.





solve(x+2y≥10, y)

$$\left\{y \geq \frac{-(x-10)}{2}\right\}$$

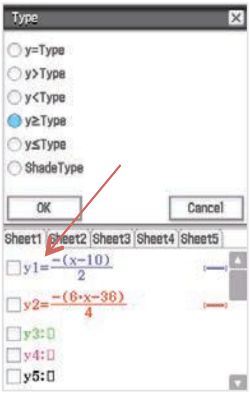
solve(6x+4y≥36, y)

$$\left\{y \geq \frac{-(6\cdot x-36)}{4}\right\}$$

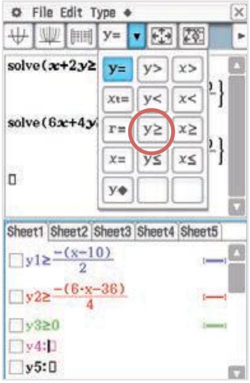
☐ $y1 = \frac{-(x-10)}{2}$
☐ $y2 = \frac{-(6\cdot x-36)}{4}$
☐ y3: □
☐ y4: □
☐ y5: □

4 Select the type of inequality.

- Tap the equals sign directly next to **y1** to bring up the **Type** menu.
- Select the **y≥Type**. Repeat for **y2**.
- Type **0** in the box on the right of **y3** and select the **y≥Type**.
- Click the box on the right of **y4**.
- Go to the tool bar and tap the down arrow ▼ next to the y≥ button and select x≥.
- Type **0** in the box on the right of **x4**.

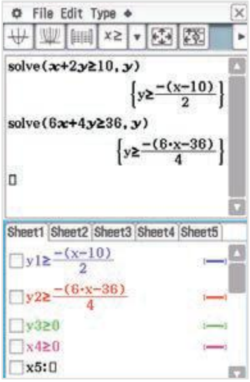


☐ $y1=\frac{-(x-10)}{2}$
☐ $y2=\frac{-(6\cdot x-36)}{4}$
☐ y3:□
☐ y4:□
☐ y5:□



solve(x+2y≥
solve(6x+4y

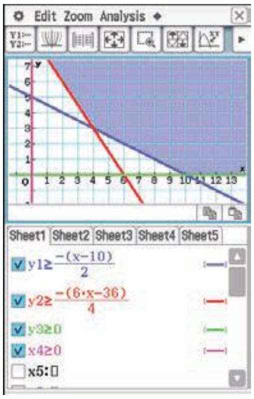
☐ $y1\geq\frac{-(x-10)}{2}$
☐ $y2\geq\frac{-(6\cdot x-36)}{4}$
☐ y3≥0
☐ y4:□
☐ y5:□



solve(x+2y≥10, y)
$\left\{y\geq\frac{-(x-10)}{2}\right\}$
solve(6x+4y≥36, y)
$\left\{y\geq\frac{-(6\cdot x-36)}{4}\right\}$

☐ $y1\geq\frac{-(x-10)}{2}$
☐ $y2\geq\frac{-(6\cdot x-36)}{4}$
☐ y3≥0
☐ x4≥0
☐ x5:□

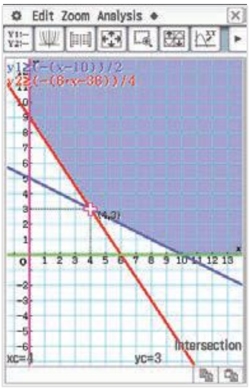
5 Tick the boxes on the left of **y1**, **y2**, **y3** and **x4**.
Then tap the **Graph** button.

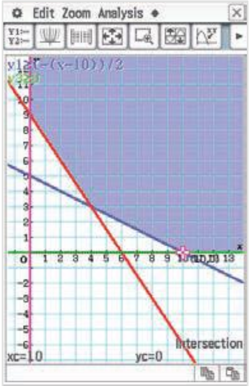


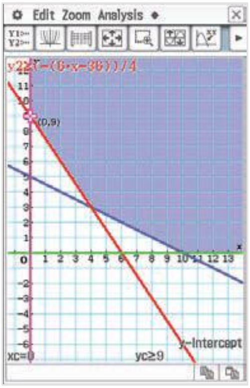
☑ $y1\geq\frac{-(x-10)}{2}$
☑ $y2\geq\frac{-(6\cdot x-36)}{4}$
☑ y3≥0
☑ x4≥0
☐ x5:□

6 To complete the region, the corner points need to be found.

- Tap **Resize**, at the bottom of the screen.
- Go to **Analysis**, select **G-solve** then **Intersection**.
- Select the equations using the up ▲ and ▼ directions from the cursor button.
- Press EXE to confirm the choice.
- Go to **Analysis**>**G-solve**>**y-Intercept**.
- Use the up ▲ and ▼ directions from the cursor button to select the equation.







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Exercise 12F

Using a CAS calculator to graph a feasible region

Use a CAS calculator to graph the feasible region for each of the following sets of linear inequalities.

1 $x \geq 0$, $y \geq 0$, $3x + y \leq 6$, $x + 2y \leq 7$

2 $x \geq 0$, $y \geq 0$, $5x + 2y \geq 20$, $5x + 6y \geq 30$

3 $x \geq 0$, $y \geq 0$, $2x - y \geq 0$, $x + y \leq 30$

12G Linear programming

In many practical situations, there is a need to maximise or minimise some quantity. For example, the profit from running a coffee shop is subject to constraints, such as the number of seats, the number of staff members needed to service the customers, wage costs, power costs. The technique of linear programming, which is the subject area of this section, has been developed to help us solve such problems.

▶ Objective functions and constraints

An **objective function** is a quantity that you are trying to make as large (for example, profits) or as small (for example, the amount of material needed to make a dress) as possible. Of course, there are always factors, such as the resources available or the requirements of the dress pattern, that limit how much profit you can make or how little material you can use to make a dress. These are called *constraints*.

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▶ The linear programming problem

The process of *maximising* or *minimising* a linear quantity, subject to a set of constraints, is at the heart of linear programming.

The linear programming problem

From the mathematical perspective, *linear programming* can be viewed as finding the point, or points, in a *feasible region* that gives the maximum or minimum value of some linear expression, the *objective function*.

At first, this seems like an insurmountable problem, as there are an infinite number of points in the feasible region to choose from. Fortunately, we can make use of the **corner-point principle** to help us solve this problem.

The corner-point principle

In linear programming problems, the maximum or minimum value of a linear objective function will occur at one of the corners of the feasible region.

Note: If two corners give the same maximum or minimum value, then all points along a line joining the two corners will also give the same maximum or minimum value.

The corner-point principle means that we only need to evaluate the objective function at each of the corner points of the feasible region, and find which gives the maximum or minimum value depending as required.

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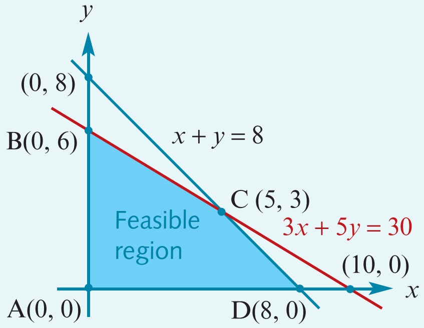
Example 10 Using the corner-point principle to find the maximum value of an objective function

Find the maximum value of the objective function $P = 2x + 3y$, subject to the constraints:

$$x \geq 0, y \geq 0$$
$$x + y \leq 8$$
$$3x + 5y \leq 30$$

as shown by the feasible region opposite.



Solution

1 Set up a table for the objective function.

2 Evaluate the objective function at each of the corners *A*, *B* and *C*.

3 Identify the corner point giving the maximum value and write your answer.

Points	Objective function $P = 2x + 3y$
A (0, 0)	$P = 2 \times 0 + 3 \times 0 = 0$
B (0, 6)	$P = 2 \times 0 + 3 \times 6 = 18$
C (5, 3)	$P = 2 \times 5 + 3 \times 3 = 19$
D (8, 0)	$P = 2 \times 8 + 3 \times 0 = 16$

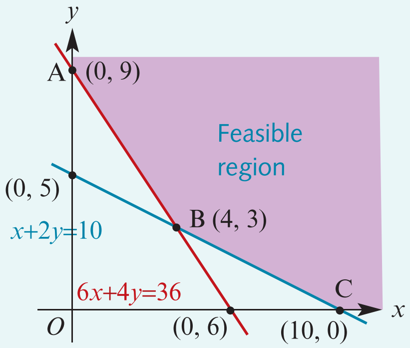
Thus, the maximum value of the objective function, $P = 19$, occurs when $x = 5$ and $y = 3$.

Example 11 Using the corner-point principle to find the minimum value of an objective function

Find the minimum value of the objective function $C = 5x + 2y$, subject to the constraints:

$$x \geq 0, y \geq 0$$
$$x + 2y \geq 10$$
$$6x + 4y \geq 36$$

as shown by the feasible region opposite.



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Solution

1 Set up a table for the objective function.

2 Evaluate the objective function at each of the corners A, B and C.

3 Identify the corner point giving the minimum value and write your answer.

Points	Objective function $C = 5x + 2y$
$A\ (0, 9)$	$C = 5 \times 0 + 2 \times 9 = 18$
$B\ (4, 3)$	$C = 5 \times 4 + 2 \times 3 = 26$
$C\ (10, 0)$	$C = 5 \times 10 + 2 \times 0 = 50$

Thus, the minimum value is $C = 18$, which occurs when $x = 0$ and $y = 9$.

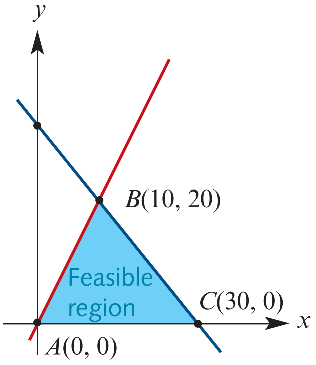
Exercise 12G

For each of the following objective functions and feasible regions, find the maximum or minimum values (as required) and the point at which it occurs.

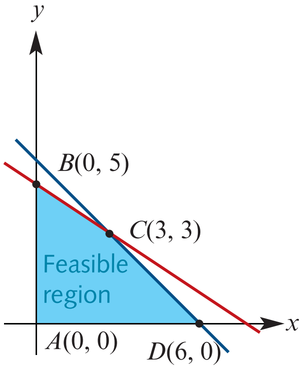
Maximising an objective function

Example 10

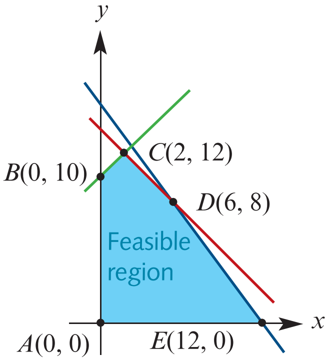
1 Maximise: $P = x + 2y$



2 Maximise: $P = 4x + 2y$



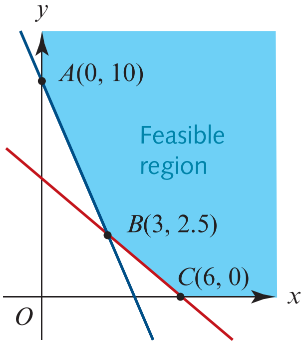
3 Maximise: $P = 3x + 4y$



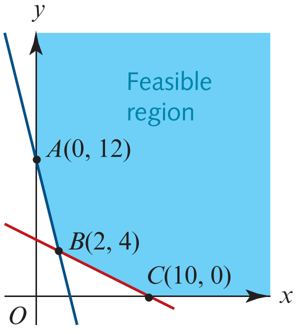
Minimising an objective function

Example 11

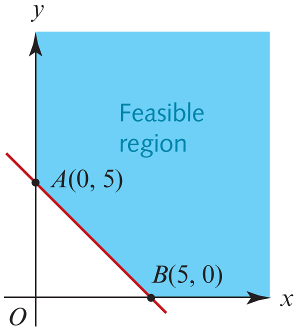
4 Minimise: $C = 3x + 5y$



5 Minimise: $C = x + y$



6 Minimise: $C = 2x + 2y$



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12H Linear programming applications

You now have the necessary sub skills needed to solve a linear programming problem. Two worked examples follow.

Example 12 Setting up and solving a maximising problem

A manufacturer makes two sorts of orange-flavoured chocolates: House Brand and Orange Delights.

The table below shows the amount of chocolate and orange fill (in kg) needed to make each product. Also shown is the total amount of chocolate and orange fill available to make these products each day.

	House Brand	Orange Delight	Total available
Chocolate (kg)	0.3	0.5	300
Orange fill (kg)	0.7	0.5	350

The manufacturer wants to maximise the profit he achieves from making these products.

The table below shows the profit made from selling each of these products.

	House Brand	Orange Delight
Profit per kg($)	7.50	10.00

How many kilograms of House Brand and Orange Delight fill chocolates should be made each day to maximise profit?

Solution

1 Define x and y.

Let x be the amount (in kg) of House Brand made each day.
Let y be the amount (in kg) of Orange Delights made each day.

2 Use the information in the chocolate totals table, to write down the constraints.

Constraints:

- x and y cannot be negative.

$x \geq 0,\ y \geq 0$

- 300 kg of chocolate is available.

$0.3x + 0.5y \leq 300$ (chocolate)

- 350 kg of orange fill is available.

$0.7x + 0.5y \leq 350$ (orange fill)

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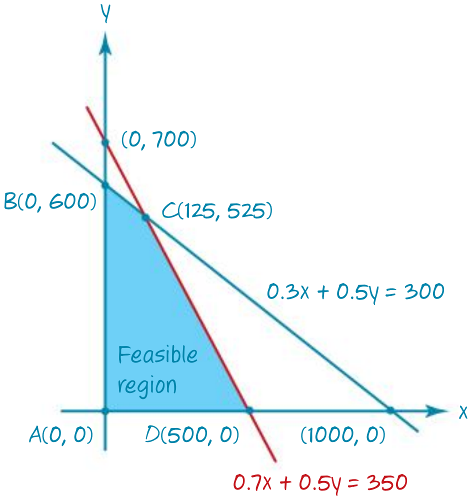
3 Graph the feasible region defined by these constraints. Mark in each of the corner points and label with their coordinates. Use a calculator to determine the point of intersection.



4 Use the information in the profit table to write down the objective function (in dollars). Call it *P*, for profit.

Objective function:

$$P = 7.5x + 10y$$

5 Determine the maximum profit by evaluating the objective function at each corner of the feasible region.

Point	Objective function $P = 7.5x + 10y$
A (0, 0)	$P = 7.5 \times 0 + 10 \times 0 = \$0$
B (0, 600)	$P = 7.5 \times 0 + 10 \times 600$ $= \$6000$
C (125, 525)	$P = 7.5 \times 125 + 10 \times 525$ $= \$6187.50$
D (500, 0)	$P = 7.5 \times 500 + 10 \times 0$ $= \$3750$

6 Write your answer to the question.

The maximum profit is $6187.50, which is obtained by making 125 kg of House Brand and 525 kg of Orange Delights.

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Example 13 Setting up and solving a minimising problem

SpeedGro and Powerfeed are two popular brands of home garden fertiliser. They both contain the nutrients *X*, *Y* and *Z*, needed for healthy plant growth.

The number of units of each nutrient needed to make a kilogram of each type of fertiliser is shown in the table below. Also shown is the amount of each type of nutrient needed in total.

Nutrient type	SpeedGro	Powerfeed	Total needed
X (units/kg)	30	20	160
Y (units/kg)	50	20	200
Z (units/kg)	10	20	80

A gardener wants to ensure that he has the correct amount of each type of nutrient for his vegetable garden at minimum cost.

The cost of buying a kilogram of each type of fertiliser is shown in the table below.

	SpeedGro	Powerfeed
Cost ($/kg)	8	6

How much of each type of fertiliser should he buy to meet his needs at the minimum cost?

Solution

1 Define x and y.

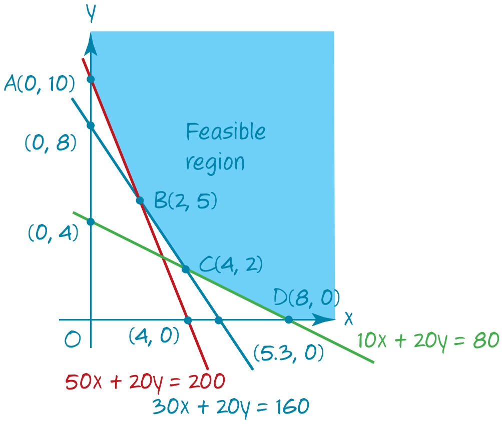
Let x be the amount (in kg) of SpeedGro needed.
Let y be the amount (in kg) of Powerfeed needed.

2 Use the information in Nutrient type table, to write down the constraints.

Constraints:

- x and y cannot be negative. $x \geq 0, y \geq 0$
- At least 160 units of X are needed. $30x + 20y \geq 160$ (nutrient X)
- At least 200 units of Y are needed. $50x + 20y \geq 200$ (nutrient Y)
- At least 80 units of Z are needed. $10x + 20y \geq 80$ (nutrient Z)

3 Graph the feasible region defined by these constraints. Mark in each of the corner points and label with their coordinates. Use a calculator to determine the points of intersection.



4 Use the information in the cost table to write down the objective function (in dollars). Call it *C*, for cost.

Objective function:

$C = 8x + 6y$

5 Determine the minimum cost by evaluating the objective function at each corner of the feasible region.

Point	Objective function $C = 8x + 6y$
A (0, 10)	$C = 8 \times 0 + 6 \times 10 = \$60$
B (2, 5)	$C = 8 \times 2 + 6 \times 5 = \$46$
C (4, 2)	$C = 8 \times 4 + 6 \times 2 = \$44$
D (8, 0)	$C = 8 \times 8 + 6 \times 0 = \$64$

6 Write your answer to the question.

The minimum cost is \$44, which is achieved by buying 4 kg of SpeedGro and 2 kg of Powerfeed.

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Exercise 12H

Solving a linear programming problem step-by-step with guidance

Example 12

1 A factory makes two products: Wigits and Gigits. Two different machines are used. The time spent making each product on each machine (in hours) is shown in the table below.

	Wigit	Gigit	Total machine time available
Machine A	1	1	8
Machine B	2	4	24

The table below shows the profit made from selling each Wigit and Gigit.

	Wigit	Gigit
Profit	\$200	\$360

The factory wants to maximise the profit made from producing Wigits and Gigits. Use this information to fill in the boxes (☐) to complete the following step-by-step solution of this problem.

a Let x be the number of Wigits made each day.
Let y be the number of Gigits made each day.
The constraints for this problem are:

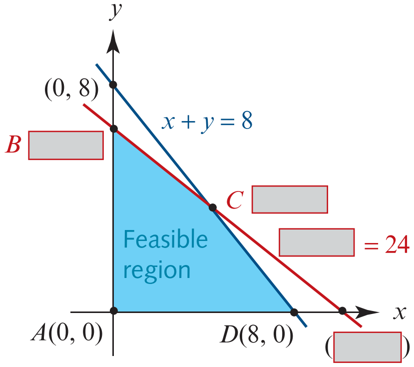
$x \geq 0,\ y \geq \boxed{}$ (minimum number of Wigits and Gigits that can be made)

$x + y \leq 8$ (Machine A time)

$\boxed{}\, x + 4y \leq \boxed{}$ (Machine B time)

Determining the missing information.

b Use these constraints to graph the feasible region. The feasible region is shown on the right. Fill in the information missing in the boxes(☐).



c The objective function is given by $P = 200x + \boxed{}\, y$, where P stands for profit (in dollars). Fill in the box (☐) using the information from the profit made table above.

d How many Wigits and Gigits should be made each day to maximise profit, and what is this profit?

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Example 13

2 An outdoor clothing manufacturer makes two sorts of jackets: Polarbear and Polarfox. The amount of material (in metres) and the time spent making these jackets (in hours) is shown in the table below.

	Polarbear	Polarfox	Total available
Material (in metres)	2	2	520
Time (hours)	2.4	3.2	672

The table below shows the profit made from selling each type of jacket.

	Polarbear	Polarfox
Profit (in dollars)	36	42

The manufacturer wants to maximise the profit made from producing these two types of jackets

Use this information to fill in the boxes ☐ to complete the following step-by-step solution of this problem.

a Let x be the number of Polarbear jackets made.

Let y be the number of Polarfox jackets made.

The constraints for this problem are:

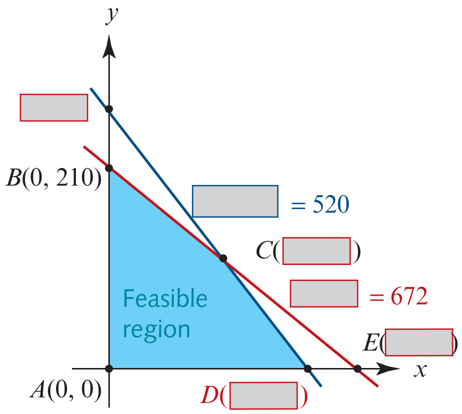
$x \geq$ ☐ , $y \geq$ ☐

☐ $x + 2y \leq$ ☐ (material availability)

☐ $x +$ ☐ $y \leq 672$ (worker time availability)

Determining the missing information.

b Use these constraints to graph the feasible region. The feasible region is shown on the right. Fill in the information missing in the boxes (☐).



c The objective function is given by $P =$ ☐ $x +$ ☐ y, where P stands for profit (in dollars). Fill in the boxes (☐) using the information in the profit made table above.

d What is the maximum profit that can be made, and how many Polarbear jackets and Polarfox jackets should be made each day to achieve this profit?

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3 Following a natural disaster, the army plans to use helicopters to transport medical teams and their equipment into a remote area. They have two types of helicopter: Redhawks and Blackjets.

The carrying capacity of these two helicopters in terms of people and equipment is shown in the table below.

	Redhawks	Blackjets	Total required
People	45	30	450
Equipment (tonnes)	3	4	36

The table below shows the operation cost of these helicopters in dollars per hour.

	Redhawks	Blackjets
Cost (dollars/hour)	3600	3200

The aim is to determine the number of each type of helicopter to complete the operation at least cost. Use this information to fill in the boxes ( ) to complete the following step-by-step solution of this problem.

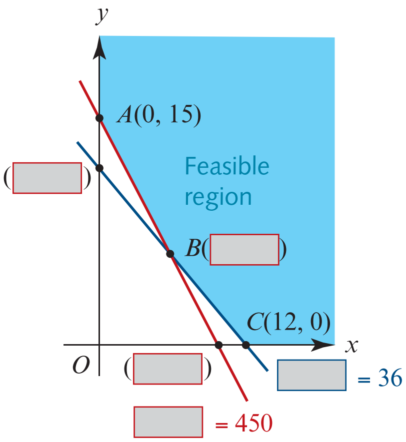
a Let x be the number of Redhawks.
Let y be the number of Blackjets.
The constraints for this problem are:
$x \geq 0, y \geq 0$

______ (people)

______ (equipment)

Fill in the boxes ( ) using the information from carrying capacity table above.

b The feasible region is shown on the right. Some information is missing. Fill in the boxes ( ) using the information from the carrying capacity table above.



c The objective function is given by
$C =$ ______ $x +$ ______ y, where C stands for cost (in dollars). Fill in the boxes ( ) using the information from the cost table above.

d How many Redhawks and Blackjets should be used to minimise the cost per hour, and what is this cost?

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Solving a linear programming problem without guidance

4 A sawmill produces both construction grade and furniture grade timber.

Two operations, sawing and planing, are needed to produce both types of timber. The table below shows the number of hours of sawing and planing needed to produce a cubic metre of each type of timber, and the amount of time available for each activity each day.

	Construction grade	Furniture grade	Time available
Sawing (hours)	2	2	8
Planing (hours)	3	6	18

The sawmill wants to maximise its profit from manufacturing these two types of timber. The table below shows the profit made from selling a cubic metre of each type of timber.

	Construction grade	Furniture grade
Profit ($\$/m^3$)	500	600

a Write down constraints and the profit function for this problem.

b Draw a diagram showing the feasible region for this problem.

c Find how much construction grade and furniture grade timber the sawmill should make each day to maximise its profit. What is this profit?

5 Two breakfast cereal mixes, Healthystart and Wakeup, are available in bulk.

The table below shows the vitamin content (in mg) of each kilogram of Healthystart and Wakeup and the dietary requirement of a purchaser.

	Healthystart	Wakeup	Dietary requirement
Vitamin B1 (mg/kg)	12	20	15
Vitamin B2 (mg/kg)	40	25	30

Let x = amount (in kg) of Healthystart purchased.

Let y = amount (in kg) of Wakeup purchased.

a Use the information to write down the constraints for this problem.

b Healthystart costs \$5 per kilogram and Wakeup costs \$4.50 per kilogram. Write down a cost function for this problem.

c Draw a diagram showing the feasible region for this problem.

d Find the mixture of these two cereals that will meet your needs at minimum cost. What is this cost?

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12I Applications, modelling and problem solving

Exercise 12I

Two types of steel cable, A and B, are made by a company and sent by truck to those businesses which order lengths of it.

Let x be the number of kilometres of cable A in an order.

Let y be the number of kilometres of cable B.

Space limitations on the truck mean that no more than 10 km of cable can be sent per order. Hence the constraints associated with this order system are:

- **i** $x \geq 0$
- **ii** $y \geq 0$
- **iii** $x + y \leq 10$

a A fourth constraint can be expressed as $x \leq 5$. What does this mean?

b Cable A weighs 20 kg/km and cable B weighs 10 kg/km.

- **i** What is the total weight of an order made up of 2 km of cable A and 5 km of cable B?
- **ii** What is the total weight of an order made up of x km of cable A and y km of cable B?
- **iii** Write down the additional constraint arising from an additional total weight limit of 120 kg applied to each order.

c Draw a graph of the complete system of five constraints, clearly indicating the feasible region.

d
- **i** Let \$$P$ be the profit on an order of x km of cable A and y km of cable B. If the profit on cable A is \$150/km, and the profit on cable B is \$100/km, write down a formula for P in terms of x and y.
- **ii** Find the most profitable order consistent with the above constraints, and calculate the corresponding profit.

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Key ideas and chapter summary

Linear inequality	A **linear inequality** involves one or two of the signs $>$, $\geq$, $\leq$ or $\leq$, but *not* an equals sign ($=$).
Displaying linear inequalities in one variable on a number line	A linear inequality in one variable can be represented on a number line by a solid line ending at one or two circles. The line represents all the possible solutions of the inequality. An *open circle* (○) indicates that the end value is not included in the inequality (for $<$ or $>$). A *closed circle* (●) indicates that the end value is included in the inequality (for $\leq$ or $\geq$).
Displaying linear inequalities in one variable on the coordinate plane	Linear inequalities in one variable can be represented on a coordinate plane by a shaded region bounded by one or two lines parallel to the x or y-axes. The region represents all the possible solutions of the inequality. A *dashed line* indicates the line is not included in the inequality. A *solid line* indicates the line is included in the inequality.
Displaying linear inequalities in two variables on the coordinate plane	A linear inequality in two variables can be represented on a coordinate plane by a shaded region bounded by a line. The region represents all the possible solutions of the inequality. The boundary line is dashed if it is not included in the inequality but solid if it is included. A reference point, often the origin (0, 0), can be used to help decide whether the required region lies above or below the line.
Feasible region	When solving simultaneous inequalities, the region in the coordinate plane that is common to all the inequalities is called the **feasible region**.
Linear programming	**Linear programming** involves **maximising** or **minimising** a linear quantity subject to a set of **constraints** represented by a set of linear inequalities.
Objective function	The **objective function** is a linear expression representing the quantity to be maximised (e.g. profit) or minimised (e.g. cost) in a linear programming problem.
Corner-point principle	The **corner-point principle** states that, in linear programming problems, the maximum or minimum value of a linear objective function will occur at one of the corners of the feasible region. If two corners give the same maximum or minimum value, then all of the points along a line joining two corners will have the same maximum or minimum value.

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Skills check

Having completed this topic you should be able to:

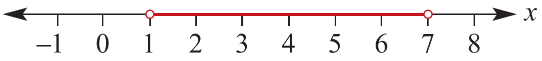
- represent a linear inequality in one variable on a number line
- represent a linear inequality in one or two variables on the coordinate plane
- know the meaning of the terms feasible region, constraint and objective function
- construct a feasible region from a set of linear inequalities
- determine the maximum or minimum value of an objective function for a given feasible region
- set up and solve basic linear programming problems.

Multiple-choice questions

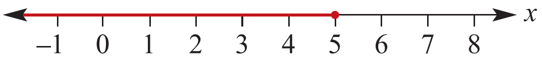
1 The inequality displayed on the number line is:

A $1 \leq x \leq 7$ **B** $1 < x < 7$ **C** $1 \leq x < 7$

D $1 < x \leq 7$ **E** $1 > x > 7$



2 The inequality displayed on the number line is:



A $x < 5$ **B** $x \leq 5$ **C** $x > 5$

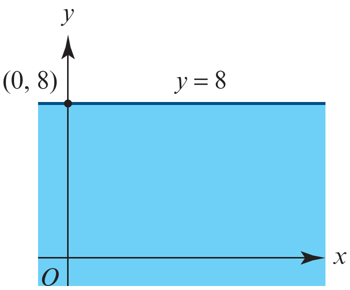
D $x \geq 5$ **E** $0 > x > 5$

3 The inequality displayed on the coordinate plane is:

A $x < 8$ **B** $x \leq 8$

C $y < 8$ **D** $y \leq 8$

E $0 > x > 8$



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4 The point (2, 1) lies in which one of the following regions?

A $x + y \geq 4$ **B** $x + 3y < 5$ **C** $2x - y > 3$

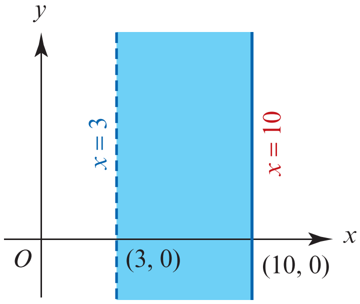
D $3x - 2y \geq 3$ **E** $3 < x + y$

5 The inequality displayed on the coordinate plane is:

A $3 < x < 10$ **B** $3 < x \leq 10$

C $3 \leq x \leq 10$ **D** $3 < y < 10$

E $3 < y \leq 10$

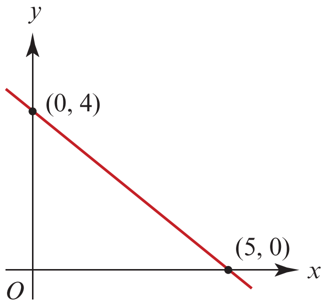


6 The equation of the line displayed is:

A $4x + 5y = 4$ **B** $4x - 5y = 4$

C $5x + 4y = 20$ **D** $4x + 5y = 20$

E $4x - 5y = 20$

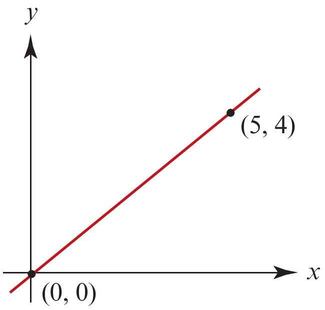


7 The equation of the line displayed is:

A $4x - 5y = 0$ **B** $4x + 5y = 0$

C $5x - 4y = 20$ **D** $5x + 4y = 20$

E $5y = 20x$

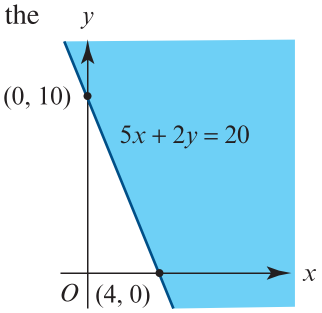


8 The region displayed (including the line) represents the inequality:

A $5x + 2y < 20$ **B** $5x + 2y \leq 20$

C $5x + 2y > 20$ **D** $5x + 2y \geq 20$

E $2x + 5y > 20$



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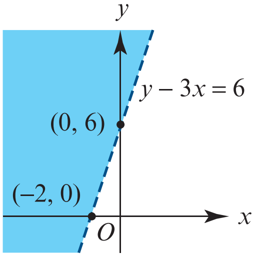
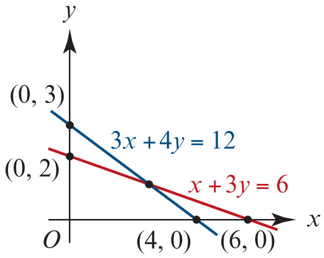
9 The region displayed (not including the line) represents the inequality:

A $y - 3x \le 6$ **B** $y - 3x < 6$
C $y - 3x \ge 6$ **D** $y - 3x > 6$
E $3x - y > 6$



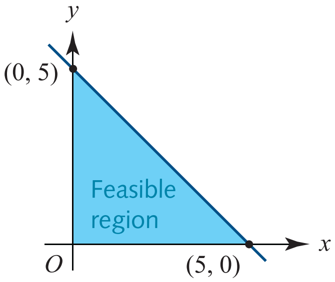
10 The two lines shown intersect at the point:

A (1, 1.3) **B** (2, 1.5)
C (1.2, 1.2) **D** (2.4, 1.2)
E (3, 2.4)



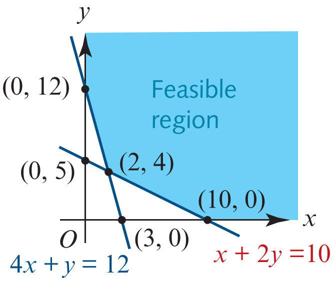
11 The feasible region displayed (including the line) is defined by the inequalities:

A $x \ge 0, \quad y \ge 0, \quad x - y < 5$
B $x \ge 0, \quad y \ge 0, \quad x - y \ge 5$
C $x \ge 0, \quad y \ge 0, \quad x + y < 5$
D $x \ge 0, \quad y \ge 0, \quad x + y \le 5$
E $x \ge 0, \quad y \ge 0, \quad x + y \ge 5$



12 The feasible region displayed (including the lines) is defined by the inequalities:

A $x \ge 0, \quad y \ge 0, \quad x + 2y \ge 10, \quad 4x + y \ge 12$
B $x \ge 0, \quad y \ge 0, \quad x + 2y \le 10, \quad 4x + y \le 12$
C $x \ge 0, \quad y \ge 0, \quad x + 2y > 10, \quad 4x + y > 12$
D $x \ge 0, \quad y \ge 0, \quad x + 2y < 10, \quad 4x + y < 12$
E $x \ge 0, \quad y \ge 0, \quad x + 2y \ge 10, \quad 4x + y \le 12$



13 For the feasible region displayed in Question **12**, the minimum value of the objective function, $C = 2x + y$, is:

A 5 **B** 6 **C** 8
D 12 **E** 20

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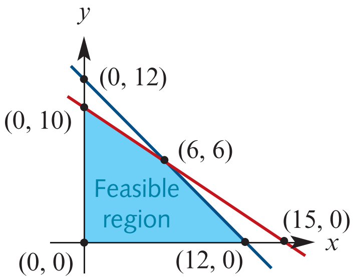
14 For the feasible region displayed, the maximum value of the objective function, $P = 4x + 3y$, is:

A 0 **B** 40

C 42 **D** 48

E 60



The following information relates to Questions 15 to 17

An outdoor clothing manufacturer makes two styles of all-weather coats: long and short.

The amount of material and time taken to manufacture each sort of coat is shown in the table below.

Also shown is the total amount of material and worker time available to make the coats.

	Short coat	Long coat	Total available
Material (in metres)	2	3	450
Time (hours)	2.5	3.5	700

Let x be the number of short coats made.

Let y be the number of long coats made.

15 The constraints that relate to the amount of *material* available are:

A $x \geq 0, \quad y \geq 0, \quad 2x + 3y \leq 450$ **B** $x \geq 0, \quad y \geq 0, \quad 2x + 3y \geq 450$

C $x \geq 0, \quad y \geq 0, \quad 2.5x + 3.5y \leq 700$ **D** $x \geq 0, \quad y \geq 0, \quad 2.5x + 3.5y \geq 700$

E $x \geq 0, \quad y \geq 0, \quad 40x + 48y \geq 700$

16 The constraints that relate to the amount *of time* available are:

A $x \geq 0, \quad y \geq 0, \quad 2x + 3y \leq 450$ **B** $x \geq 0, \quad y \geq 0, \quad 2x + 3y \geq 450$

C $x \geq 0, \quad y \geq 0, \quad 2.5x + 3.5y \leq 700$ **D** $x \geq 0, \quad y \geq 0, \quad 2.5x + 3.5y \geq 700$

E $x \geq 0, \quad y \geq 0, \quad 40x + 48y \geq 700$

17 The manufacturer makes a profit of \$40 for each short coat and \$48 for each long coat. The profit function P is:

A $P = 2.5x + 3.5y$ **B** $P = 2x + 3y$ **C** $P = 3x + 3.5y$

D $P = 40x + 48y$ **E** $P = 450x + 700y$

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Short-answer questions

1 Plot the inequality $-2 \le x < 4$ on a number line.

2 Plot the inequality $1 \le y < 5$ on the coordinate plane.

3 Plot the inequality $5x + 4y < 40$ on the coordinate plane.

4 Plot the region defined by the inequalities:

$$x \ge 0, \qquad y \ge 0, \qquad 3x + 5y \le 60$$

5 Plot the region defined by the inequalities:

$$x \ge 0, \qquad y \ge 0, \qquad 2x + 3y \ge 30, \qquad x + 4y \ge 20$$

Extended-response question

1 A garden products company makes two sorts of fertiliser: Standard Grade and Premium Grade. There are two main ingredients: nitrate and phosphate.
The amounts of nitrate and phosphate to make Standard Grade and Premium Grade fertiliser are shown in the table below. Also shown is the total amount of nitrate and phosphate available.

Ingredients	Standard Grade	Premium Grade	Available
Nitrate (tonnes)	0.8	0.7	56
Phosphate (tonnes)	0.2	0.3	21

The profit from selling a tonne of each sort of fertiliser is given in the table below.

	Standard Grade	Premium Grade
Profit (\$)	600	750

Let x = the amount of Standard Grade fertiliser made.
Let y = the amount of Premium Grade fertiliser made.

a Write the constraints and profit function for this problem.

b Draw a diagram showing the feasible region.

c Find how much of each type of fertiliser the company should make to maximise its profit. What will this profit be?

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Appendix A: TI-Nspire CAS CX with OS4.0

Keystroke actions and short cuts for the TI-Nspire CAS CX

esc : removes menus and dialogue boxes

ctrl + **esc** : undo last move

⇧shift + **esc** : redo last move

tab : move to next entry box (field)

ctrl + **tab** : switch applications in split screen

Navpad (Touchpad)

ctrl : accesses secondary (blue) commands

ctrl + ▲: displays page sorter

ctrl + ◀: displays previous page

ctrl + ▶: displays next page

⌂ on :displays icon page to select applications, mode, My Documents and start a new document

menu :options for each application

ctrl + **menu** :contextual menus (same as right mouse click)

☝ : mouse pointer (cursor). Selects items.

ctrl + ☝ : grab

del : backspace, deletes a character

📖 : catalogue

⊞ : 2D maths template

ctrl + **÷** : adds fraction template

enter : completes commands and displays results

Mode Settings

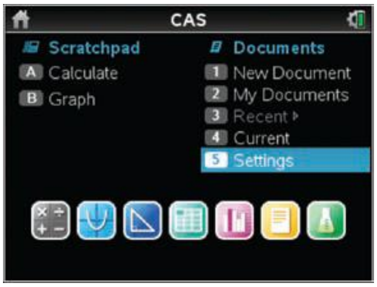
How to set in Degree mode

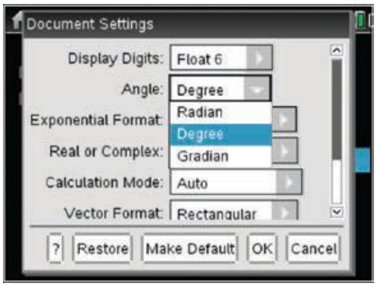
For General Mathematics it is necessary to set the calculator to **Degree** mode right from the start. This is very important for the Trigonometry topic. The calculator will remain in this mode unless you change the setting again or update the operating system.

Press ⌂on and move to **Settings>Document Settings**, arrow down to the down to the **Angle** field, press ▶ and select **Degree** from the list then arrow down to the **Make Default** tab. Select **OK** to accept the change.

Note that there is a separate settings menu for the **Graphs** and **Geometry** pages. These settings are accessed from the relevant pages. For Mathematics it is not necessary for you to change these settings.

Note: when you start your new document you will see **DEG** in the top status line.




Document Settings

Display Digits: Float 6

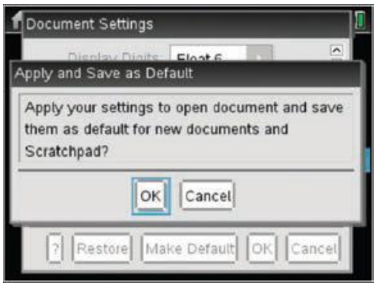
Angle: Degree

Exponential Format:

Real or Complex:

Calculation Mode: Auto

Vector Format: Rectangular


Document Settings

Apply and Save as Default

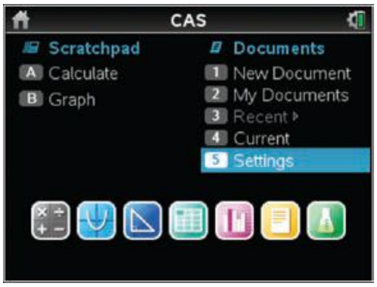
Apply your settings to open document and save them as default for new documents and Scratchpad?

How to set in Approximate (Decimal) mode

For General Mathematics it is useful to set the calculator to **Approximate (Decimal)** mode right from the start. The calculator will remain in this mode unless you change the setting again or update the operating system.

Press ⌂on and move to **Settings>Document Settings**, arrow down to the **Calculation Mode** field, press ▶ and select **Approximate** from the list then arrow down to the **Make Default** tab. Select **OK** to accept the change.

Note that you can make both the **Degree** and **Approximate Mode** selections at the same time if desired.



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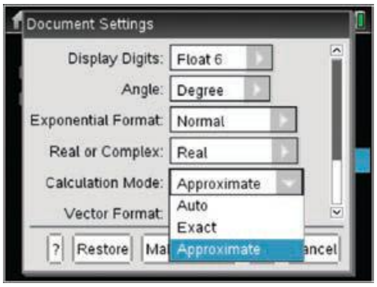
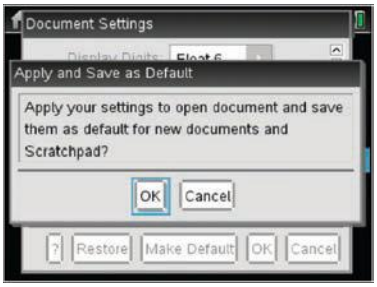


Display Digits: Float 6
Angle: Degree
Exponential Format: Normal
Real or Complex: Real
Calculation Mode: Approximate
Auto
Exact
Approximate
Vector Format:



Apply your settings to open document and save them as default for new documents and Scratchpad?

The home screen is divided into two main areas – **Scratchpad** and **Documents**.

All instructions given in the text, and in the Appendix, are based on the **Documents** platform.

Documents

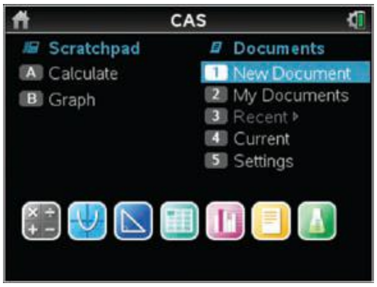
Documents can be used to access all the functionality required for General Mathematics including all calculations, graphing, statistics and geometry.

Starting a new document

1 To start a new document, press ⌂on and select **New Document**.

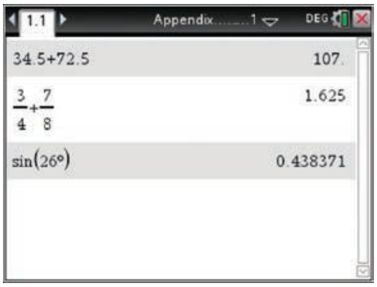
2 If prompted to save an existing document move the cursor to **No** and press enter.

Note: Pressing ctrl+N will also start a new document.



A: Calculator page - this is a fully functional CAS calculation platform that can be used for calculations such as arithmetic, algebra, finance, trigonometry and matrices. When you open a new document select **Add Calculator** from the list.

1 You can enter fractions using the fraction template if you prefer. Press ctrl ÷ to paste the fraction template and enter the values. Use the tab key or arrows to move between boxes. Press enter to display the answer. Note that all answers will be either whole numbers or decimals because the mode was set to approximate (decimal).



```
34.5+72.5                 107.
3/4+7/8                  1.625
sin(26°)              0.438371
```

2 For problems that involve angles (e.g. evaluate sin(26°)) it is good practice to include the degree symbol even if the mode is set to degree (DEG) as recommended.

Note: if the calculator is accidentally left in radian (RAD) mode the degree symbol will override this and compute using degree values.

The degree symbol can be accessed using ?!▸. Alternatively select from the **Symbols** palette ctrl.To enter trigonometry functions such as *sin, cos*, press the trig key or just type them in with an opening parenthesis.

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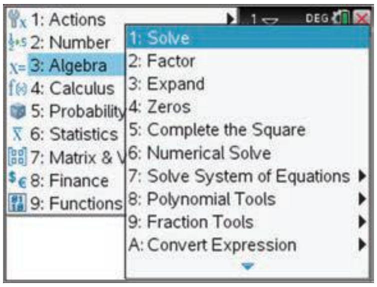
Solving equations

Using the **Solve command** Solve $2y + 3 = 7$ for y.

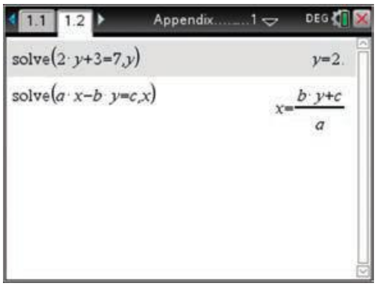
In a **Calculator** page press menu**>Algebra>Solve** and complete the **Solve** command as shown opposite. You must include the variable you are making the subject at the end of the command line.



Hint: You can also type in **solve(** directly from the keypad but make sure you include the opening bracket.

Literal equations such as $ax - by = c$ can be solved in a similar way.

Note that you must use a multiplication sign between two letters.

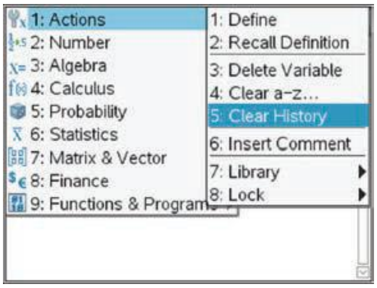


$\text{solve}(2 \cdot y+3=7,y)$ $y=2.$

$\text{solve}(a \cdot x-b \cdot y=c,x)$ $x=\frac{b \cdot y+c}{a}$

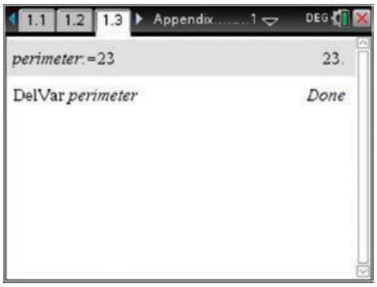
Clearing the history area

Once you have pressed enter the computation becomes part of the **History** area. To clear a line from the history area, press ▲ repeatedly until the expression is highlighted and press enter. To completely clear the History Area, press menu**>Actions>Clear History** and press enter again.



Alternatively press ctrl+menu to access the contextual menu.

It is also useful occasionally to clear any previously stored values. Clearing **History** does not clear stored variables.
Pressing menu**>Actions>Clear a-z. . .** will clear any stored values for single letter variables that have been used.
Use menu**>Actions>Delete Variable** if the variable name is more than one letter. For example, to delete the variable perimeter, then use **DelVar** ***perimeter***.



perimeter:=23 23.

DelVar *perimeter* Done

Note: When you start a new document any previously stored variables are deleted.

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How to construct parallel boxplots from two data lists

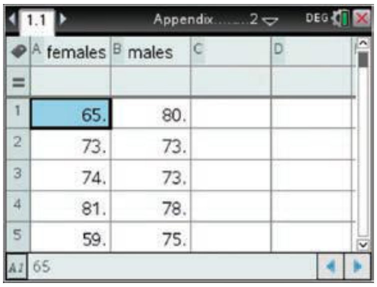
Construct parallel boxplots to display the pulse rates of 23 adult females and 23 adult males.

Pulse rate (beats per minute)	
Females	Males
65 73 74 81 59 64 76 83 95 70 73 79 64 77 80 82 77 87 66 89 68 78 74	80 73 73 78 75 65 69 70 70 78 58 77 64 76 67 69 72 71 68 72 67 77 73

Steps

1 Start a new document: ctrl + N.

2 Select **Add Lists & Spreadsheet**. Enter the data into lists called ***females*** and ***males*** as shown.



1.1 Appendix.......2 DEG

	A females	B males	C	D
=				
1	65.	80.		
2	73.	73.		
3	74.	73.		
4	81.	78.		
5	59.	75.		

A1 65

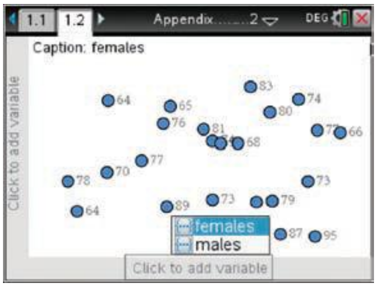
3 Statistical graphing is done through the **Data & Statistics** application.

Press ctrl + I and select **Add Data & Statistics**

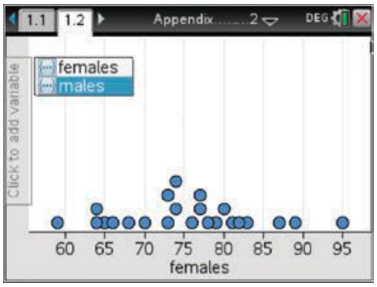
(or press on and arrow to and press enter).

Note: A random display of dots will appear – this is to indicate list data is available for plotting. It is not a statistical plot.

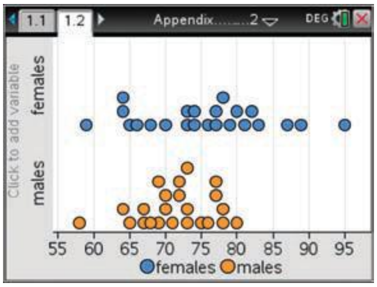
a Press tab, or navigate and click on the "Click to add variable" box to show the list of variables. Select the variable, ***females***. Press enter or to paste the variable to the *x*-axis. A dot plot is displayed by default as shown.



b To add another variable to the *x*-axis press menu**>Plot Properties>Add X Variable**, then enter. Select the variable ***males***. Parallel dot plots are displayed by default.



c To change the plots to box plots press menu**>Plot Type>Box Plot**, then press enter. Your screen should now look like that shown below.



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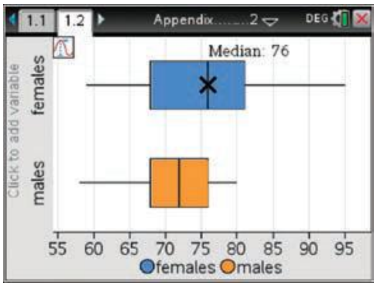
4 Data analysis

Use menu **>Analyze>Graph Trace** and use the cursor arrows to navigate through the key points. Alternatively just move the cursor over the key points. Starting at the far left of the plots, we see that, for females, the

- minimum value is 59: **MinX** = **59**
- first quartile is 68: **Q1**= **68**
- median is 76: **Median** = **76**
- third quartile is 81: **Q3** = **81**
- maximum value is 95: **MaxX** = **95**

and for males, the

- minimum value is 58: **MinX** = **58**
- first quartile is 68: **Q1** = **68**
- median is 72: **Median** = **72**
- third quartile is 76: **Q3** = **76**
- maximum value is 80: **MaxX** = **80**



Use ▼ to trace the other plot. Press esc to exit the **Graph Trace** tool.

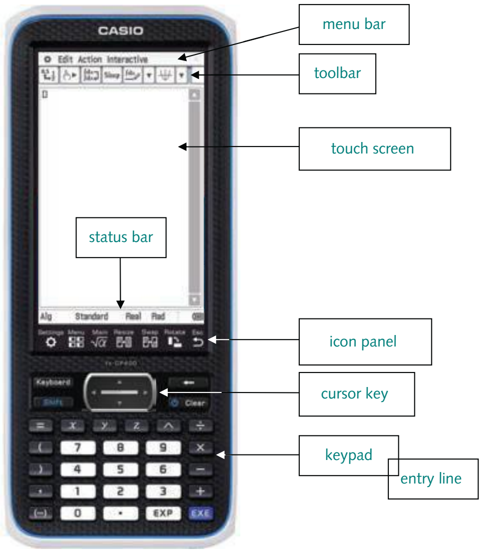
Appendix B: Casio ClassPad II

Operating system

Written for operating system 2.0 or above.

Terminology

Some of the common terms used with the ClassPad are:



The menu bar

The toolbar

The Touch screen contains the work area where the input is displayed on the left and the output is displayed on the right. Use your finger or stylus to tap and perform calculations.

The icon panel contains seven permanent icons that access settings, applications and different view settings. Press **escape** to cancel a calculation that causes the calculator to freeze.

The Cursor key works in a similar way to a computer cursor keys.

The Keypad refers to the hard keyboard.

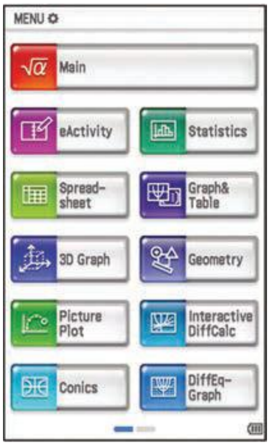
Calculating

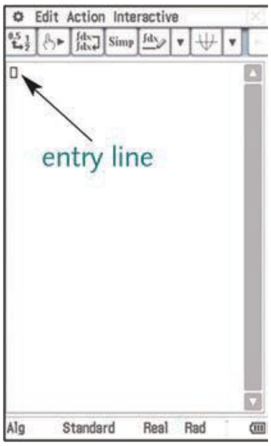
Tap Menu from the **Icon Panel** to display the application menu if it is not already visible.

Tap √α Main to open the **Main** application.

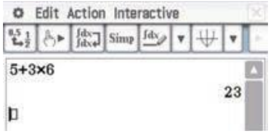
Note: there are two application menus. Alternate between the two by tapping on the screen selector at the bottom of the screen.

1 The main screen consists of an entry line which is recognised by a flashing vertical line (cursor) inside a small square. The history area, showing previous calculations, is above the entry line.



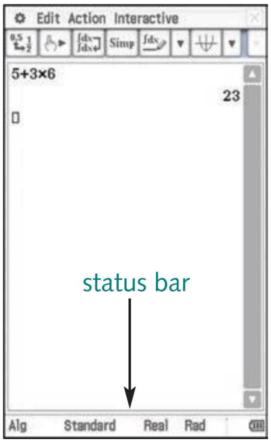


2 To calculate, enter the required expression in the entry line and press **EXE**. For example, if we wish to evaluate 5 + 3 × 6, type the expression in the entry line and press **EXE**.
You can move between the entry line and the history area by tapping or using the cursor keys (i.e. ▼, ◀, ▲, ▶).



```
5+3×6
                23
```

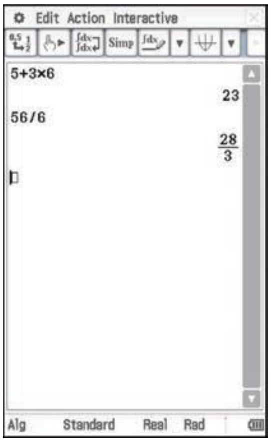
3 The ClassPad gives answers in either exact form or as a decimal approximation. Tapping settings in the **status bar** will toggle between the available options.



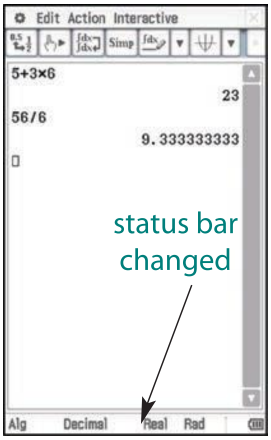
```
5+3×6
                23
Alg  Standard  Real  Rad
```

4 For example, if an exact answer is required for the calculation 56 ÷ 6, the **Standard** setting must be selected.

5 If a decimal approximation is required, change the **Standard** setting to **Decimal** by tapping it and press **EXE**.



```
5+3×6
                23
56/6
                28/3
Alg  Standard  Real  Rad
```



```
5+3×6
                23
56/6
       9.333333333
Alg  Decimal  Real  Rad
```

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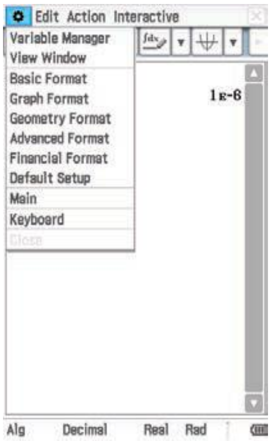
Extremely large and extremely small numbers

When solving problems that involve large or small numbers the calculator's default setting will give answers in scientific form.

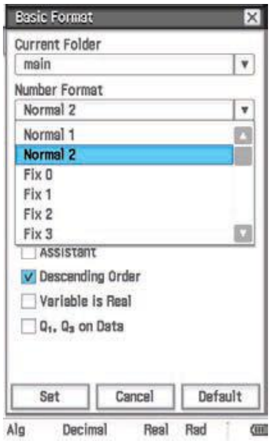
For example, one millionth, or 1/1000000, in scientific form is written as 1×10^{-6} and the calculator will present this as 1E-6.



```
1E-6
```

To change this setting, tap on the settings icon and select **Basic Format.**

Under the Number Format select **Normal 2** and tap SET.



Current Folder: main

Number Format: Normal 2

☑ Descending Order

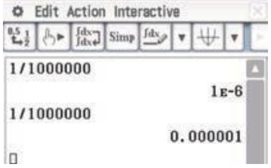
☐ Variable is Real

☐ Q_1, Q_3 on Data

In the Main screen type 1/1000000 and press EXE.

The answer will now be presented in decimal form 0.000001

This setting will remain until the calculator is reset.



```
1/1000000
                1E-6
1/1000000
            0.000001
```

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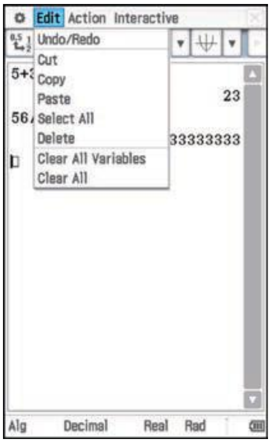
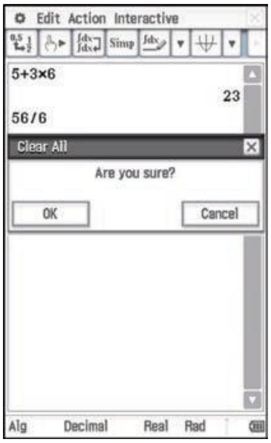
Clearing the history screen

To clear the **Main** application screen, select **Edit** from the menu bar and then tap **Clear All**. Confirm your selection by tapping **OK**. The entire screen is now cleared. To clear the entry line only, press **Clear** on the front calculator.

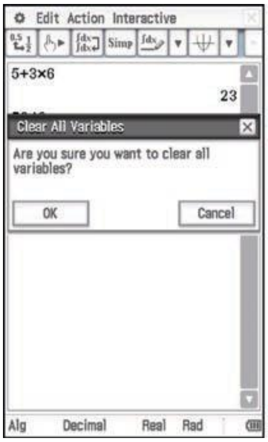


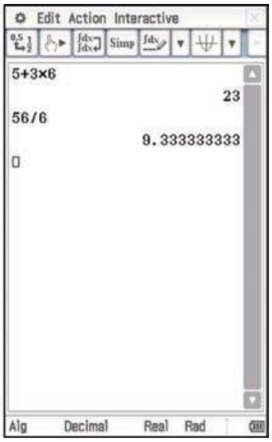


Clearing variables

To clear stored variable values, select **Edit** from the menu bar and then tap **Clear All Variables**. Confirm your selection by tapping OK.

The variables are cleared but the history created on the main screen is kept.





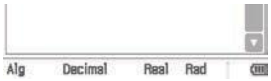
Degree mode

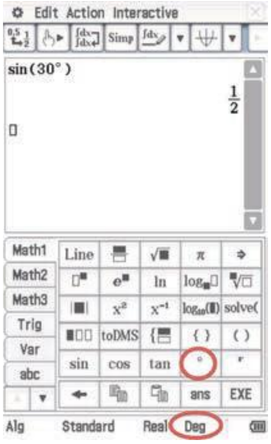
When solving problems in trigonometry, your calculator should be kept in **Degree** mode. In the main screen, the status bar displays the angle mode.

To change the angle mode, tap on the angle unit in the status bar until **Deg** is displayed.

In addition, it is recommended that you always insert the degree symbol after any angle. This overrides any mode changes and reminds you that you should be entering an angle, not a length.

The degree symbol is found in the **Math1** keyboard.





Glossary

A

Adjacency matrix [p. 424] A square matrix that uses a zero or an integer to record the number of edges connecting each pair of vertices in the graph.

Ambiguous case (trigonometry) [p. 566] Occurs when it is possible to draw two different triangles that both fit the given information.

Angle of depression [p. 554] The angle between the horizontal and a direction *below* the horizontal.

Angle of elevation [p. 554] The angle between the horizontal and a direction *above* the horizontal.

Arc [p. 491] The part of a circle between two given points on the circle. The length of the arc of a circle is given by $s = r\left(\frac{\theta}{180}\right)\pi$, where r is the radius of the circle and θ is the angle in degrees subtended by the arc at the centre of the circle.

Area [p. 481] The area of a shape is a measure of the region enclosed by its boundaries, measured in square units.

Area formulas [pp. 481, 489] Formulas used to calculate the areas of regular shapes, including squares, rectangles, triangles, parallelograms, trapeziums, kites, rhombi and circles.

Arithmetic sequence [p. 355] A sequence in which each new term is made by adding a constant amount (d), called the common difference, to the current term. Given the value of the first term in an arithmetic sequence (a) and the common difference, there are rules for finding the nth term.

B

Back-to-back stem plot [p. 98] A type of stem plot used to compare two sets of data, with a single stem and two sets of leaves (one for each group).

Bar chart [p. 53] A statistical graph used to display the frequency distribution of categorical data, using vertical bars.

Bearing [p. 559] *See* Three-figure bearing.

Bivariate data [pp. 276, 307] Data where each observation involves recording information about two variables for each person or thing.

BODMAS [p. 2] An aid for remembering the order of operations: **B**rackets first; **O**rders (powers, square roots) and fractions **O**f numbers; **D**ivision and **M**ultiplication, working left to right; **A**ddition and **S**ubtraction, working left to right.

Boxplot [p. 90] A graphical representation of a five-number summary showing outliers if present. *See* Outliers.

Bridge [p. 422] A single edge in a connected graph that, if removed, leaves the graph disconnected. A graph can have more than one bridge.

C

Categorical data [p. 49] Data generated by a categorical variable. Even if numbers, for example, house numbers, categorical data *cannot* be used to perform meaningful numerical calculations.

Categorical variable [p. 50] Variables that are used to represent characteristics of individuals, for example, place of birth, house number.

Categorical variables come in types, nominal and ordinal.

Causation: [p. 322] Causing or producing an effect. A high correlation between two variables does not necessarily mean that the two variables are causally related.

Circuit [p. 432] A walk with no repeated edges that starts and ends at the same vertex. *See also* Walk.

Circumference [p. 489] The circumference of a circle is the length of its boundary. The circumference, C, of a circle with radius, r, is given by $C = 2\pi r$.

Column matrix [p. 219] A matrix with only one column.

Common difference [p. 356] The fixed amount (d) that is added to make each new term in an arithmetic sequence.

Common ratio [p. 372] The fixed number (r) that multiplies the current term to make the next term in a geometric sequence.

Compound interest [p. 187] Under compound interest, the interest paid on a loan or investment is credited or debited to the account at the end of each period. The interest earned for the next period is based on the principal plus previous interest earned. The amount of interest earned increases each year.

Continuous data [p. 50] Measurements of a variable that can take any value (usually within a range), to any fraction or any number of decimal places.

Corner-point principle [p. 619] States that, in linear programming problems, the maximum or minimum value of a linear objective function will occur at one of the corners of the feasible region. If two corners give the same maximum or minimum value, then all of the points along a line joining two corners will have the same maximum or minimum value.

Correlation coefficient, *r*: [p. 321] A statistical measure of the strength of the linear association between two numerical variables.

Cosine ratio (cos θ) [p. 540] In right-angled triangles, the ratio of the side adjacent to a given angle (θ) to the hypotenuse.

Cosine rule [p. 574] In non-right-angled triangles, a rule used to find:

- the third side, given two sides and an included angle
- an angle, given three sides.

For triangle ABC and side a, the rule is:

$$a^2 = b^2 + c^2 - 2bc\ \cos A$$

$$\text{or } \cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

Similar rules exist for sides b and c.

Credit [p. 194] An advance of money from a financial institution, such as a bank, that does not have to be paid back immediately but which attracts interest after an interest-free period.

Cycle [p. 432] A walk with no repeated vertices that starts and ends at the same vertex. *See also* Walk.

D

Data [p. 49] Information collected about a variable.

Degree of a vertex (deg(*A*)) [p. 414] The number of edges that are attached to the vertex. The degree of a loop is two. The degree of vertex A is written as deg(A).

Diameter [p. 496] The distance from one side of a circle (or sphere) to the opposite side through the centre; it is equal to twice the radius.

Direction of association [p. 315] Linear association between two variables can have a positive or negative direction, determinable by the slope of the line of best fit. For a positive association, the values of the response variable tend to increase with increasing values of the explanatory variable. For a negative association, the values of the response variable tend to decrease.

Discrete data [p. 50] Data that is counted rather than measured and can take only specific numerical values, usually but not always whole numbers.

Discrete variable [p. 50] A numerical variable that represents a quantity that is determined by counting, for example, the number of people waiting in a queue is a discrete variable.

Distribution [p. 52] The pattern in a set of data values. It reflects how frequently different data values occur.

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Dividend [p. 169] The financial return to shareholders on a share of a company. Dividends can be specified as the number of dollars each share receives or as a percentage of the current share price, called the dividend yield.

Dot plot [p. 73] A statistical graph that uses dots to display individual data values on a number line; it is most suitable for small sets of data.

E

Edge [p. 414] A line joining one vertex in a graph or network to another vertex or to itself (a loop).

Effective interest rate [p. 195] For time payment arrangements, the effective rate of interest (r_e) is given by: $r_e = r_f \times \frac{2n}{n+1}$ where n = total number of payments.

Elements of a matrix [p. 217] The numbers or symbols displayed in a matrix.

Eulerian circuit [p. 436] An eulerian trail that starts and finishes at the same vertex. To have an eulerian circuit, the graph must have all even vertices.

Eulerian trail [p. 436] A trail in a connected graph that includes every edge. To have an eulerian trail (but not an eulerian circuit), the graph must have *exactly two* odd vertices. The trail starts at one odd vertex and finishes at the other.

Euler's formula [p. 426] The formula $v - e + f = 2$, which relates the number of vertices (v), edges (e) and faces (f) in a connected planar graph.

Extrapolation [pp. 294, 336] Using a mathematical model to make a prediction *outside* the range of data used to construct a model.

Explanatory variable [p. 307] When investigating associations in bivariate data, the explanatory variable (EV) is the variable used to explain or predict the value of the response variable (RV).

F

Face [p. 426] An area in a graph or network that can only be reached by crossing an edge. One face is always the infinite area surrounding a graph.

Feasible region [p. 611] The region in the coordinate plane that contains all possible solutions to a set of inequalities. Also known as the solution region.

Fibonacci sequence: [pp. 353, 400] The sequence 1, 1, 2, 3, 5, 8, . . . The Fibonacci sequence is generated by the second-order difference equation: $t_n = t_{n-2} + t_{n-1}$ where $t_1 = 1$ and $t_2 = 1$.

Fitting a line by eye [p. 292] A line drawn on a scatterplot with a ruler that aims to capture the linear trend of the data points.

Five-number summary [p. 90] A list of the five key points in a data distribution: the minimum value (Min), the first quartile (Q_1), the median (M), the third quartile (Q_3) and the maximum value (Max).

Flat-rate interest [p. 194] The interest rate calculated from the total interest paid, divided by the term of the loan or investment.

Form of association [p. 317] The description of the association between variables as linear or non-linear.

Formula [p. 115] A mathematical relation connecting two or more variables, for example, $C = 5t + 20, P = 2L + 2W, A = \pi r^2$.

Frequency [p. 52] The number of times a value or a group of values occurs in a data set. Sometimes known as the count.

Frequency table [p. 52] A listing of the values that a variable takes in a data set along with how often (frequently) each value occurs. Frequency can also be recorded as a percentage.

G

Geometric sequence [p. 372] A sequence in which each new term is made by multiplying the current term by a constant called the common ratio (r). Given the value of the first term in a geometric sequence (a) and the common ratio, there are rules for finding the nth term.

Gradient of a straight line [p. 271] *See* Slope of a straight line.

Graph [p. 416] In a specific mathematical sense, as opposed to its common usage, a graph is a diagram that consists of a set of points called vertices that are joined by a set of lines called edges. Each edge joins two vertices.

Grouped data [p. 60] Where there are many different data values, data is grouped in intervals such as 0–9, 10–19, . . .

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GST [p. 168] GST (goods and services tax) is a tax, currently at the rate of 10%, that is added to most purchases of goods and services.

H

Hamiltonian cycle [p. 440] A hamiltonian path that starts and finishes at the same vertex.

Hamiltonian path [p. 440] A path in a connected graph that passes through every vertex in the graph once only. It may or may not start and finish at the same vertex.

Heron's rule (Heron's formula) [pp. 482, 583] A rule for calculating the area of a triangle from its three sides.

$$A = \sqrt{s(s-a)(s-b)(s-c)}$$

where $s = \frac{1}{2}(a+b+c)$ and is called the semi-perimeter.

Hire purchase [p. 194] A financial agreement where the purchaser hires an item and in return makes periodic repayments at an agreed rate of interest. At the end of the agreement, the item becomes the property of the purchaser.

Histogram [p. 63] A statistical graph used to display the frequency distribution of a numerical variable; suitable for medium-to large-sized data sets.

Hypotenuse [p. 470] The longest side in a right-angled triangle.

I

Identity matrix (I) [p. 249] A matrix that behaves like the number one in arithmetic, represented by the symbol I. Any matrix multiplied by an identity matrix remains unchanged.

Inequality [p. 598] A mathematical relation involving the use of $<, \leq, >, \geq$ or $\neq$, for example, $2x \leq 7,\ 3x + 5y > 9,\ -1 \leq x < 2$.

Inflation [p. 205] The tendency of prices to increase with time, resulting in the loss of purchasing power.

Intercept–slope form of the equation of a straight line [p. 276] A linear equation written in the form $y = a + bx$, where a and b are constants. In this equation, a represents the y-intercept and b represents the slope. For example, $y = 5 - 2x$ is the equation of a straight line with y-intercept of 5 and the slope of -2.

Interest [p. 175] An amount of money paid (earned) for borrowing (lending) money over a period of time. It may be calculated on a simple or compound basis.

Interest rate [p. 175] The rate at which interest is charged or paid. It is usually expressed as a percentage of the money owed or lent.

Interpolation [pp. 294, 336] Using a mathematical model to make a prediction *within* the range of data used to construct model.

Interquartile range (IQR) [p. 83] A summary statistic that measures the spread of the middle 50% of values in a data distribution. It is defined as $\text{IQR} = Q_3 - Q_1$.

Inverse matrix (A^{-1}) [p. 249] A matrix that, when multiplied by the original matrix, gives the identity matrix (I). For a matrix A, the inverse matrix is written as A^{-1} and has the property that $A^{-1}A = AA^{-1} = I$.

Isomorphic graphs [p. 419] Equivalent graphs or networks; graphs that have the same number of edges and vertices, identically connected.

L

Least squares method [p. 329] A technique for fitting a line to data that minimises the sum of the squares of the residuals. It works best when there are no outliers.

Line of best fit [p. 292] A line used to approximately model the linear relationship between two variables. It is needed when the data values do not lie exactly on a straight line. Also known as a regression line.

Linear decay [p. 390] When a recurrence rule involves subtracting a fixed amount, the terms in the resulting sequence are said to decay linearly.

Linear equation [p. 123] An equation that has a straight line as its graph. In linear equations, the unknown values are always to the power of 1, for example, $y = 2x - 3, y + 3 = 7, 3x = 8$.

Linear growth [p. 390] When a recurrence rule involves adding a fixed amount, the terms in the resulting sequence are said to grow linearly.

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Linear programming [p. 598] A technique for maximising or minimising the value of a linear quantity subject to a set of constraints.

Linear regression [p. 329] The process of fitting a straight line to bivariate data.

Literal equation [p. 131] An equation with two or more unknowns, for example, $x + 3y = 7, y = 3x - 4, x + 3y + z = 7$.

Location of distribution [p. 72] Two distributions are said to differ in location if the values of the data in one distribution are generally larger than the values of the data in the other distribution.

Logarithm ($\log_{10} x$): [p. 34] A way of writing numbers as a power of ten. For example, if $x = 100 = 10^2$, then $\log_{10} x = 2$.

Loop [p. 416] An edge in a graph or network that joins a vertex to itself.

M

Matrix [p. 216] *A* rectangular array of numbers or symbols set out in rows and columns within square brackets. (Plural – matrices.)

Matrix multiplication [p. 234] The process of multiplying a matrix by a matrix.

Maximum (Max) [p. 90] The largest value in a set of numerical data.

Mean ($\bar{x}$) [p. 78] The balance point of a data distribution. The mean is given by $\bar{x} = \frac{\Sigma x}{n}$, where Σx is the sum of the data values and n is the number of data values.

Median (*M*) [p. 80] The midpoint of an ordered data set that has been divided it into two equal parts, each with 50% of the data values. It is equal to the middle value (for an odd number of data values) or the average of the two middle values (for an even number of data values). It is a measure of the centre of the distribution.

Minimum (Min) [p. 90] The smallest value in a set of numerical data.

Minimum monthly balance [p. 179] The lowest amount that a bank account contains in a given calendar month.

Minimum spanning tree [p. 448] The spanning tree of minimum length in a connected weighted graph or network. A graph may have more than one.

Modal category or interval [pp. 54, 62] The category or data interval that occurs most frequently in a data set.

Mode [p. 54] The most frequently occurring value in a data set. There may be more than one.

Multiple edges [p. 423] Two or more edges that connect the same two vertices in a graph or network.

N

Negative association (bivariate data) [p. 315] A relationship in which large values of the dependent variable are associated with small values of the independent variable, and vice versa.

Negative slope [p. 271] A straight-line graph with a negative slope represents a decreasing y-value as the x-value increases. For the graph of a straight line with a negative slope, y decreases at a constant rate with respect to x.

Negatively skewed distribution [p. 71] A data distribution that has a long tail to the left. In negatively skewed distributions, the majority of data values fall to the right of the mean.

Network [p. 224] A set of points called vertices and connecting lines called edges, enclosing and surrounded by areas called faces.

No association [p. 315] A state of no consistent change in the value of the response variable when the values of the explanatory variable change.

Nominal data [p. 49] Type of categorical data, such as gender, in which categories are given names (‘nominations’) or labels rather than taking a numerical value.

Nominal variable [p. 50] A categorical variable that generates data values that can be used by name only, for example, eye colour: blue, green, brown.

Non-linear equation [p. 137] An equation with a graph that is *not* a straight line. In non-linear equations, the unknown values are not all to the power of 1, for example, $y = x^2 + 5, 3y^2 = 6, b^3 = 27$.

Numerical data [p. 50] Data obtained by measuring or counting some quantity. Numerical data can be discrete (for example, the *number*

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of people waiting in a queue) or continuous (for example, the *amount of time* people spent waiting in a queue).

O

Objective function [p. 618] A linear expression representing the quantity to be maximised or minimised in a linear programming problem. For example, $P = 3x + 5y$ could be an objective function representing profit.

Order of a matrix [p. 217] An indication of the size and shape of a matrix, written as $m \times n$, where m is the number of rows and n is the number of columns.

Ordinal data [p. 49] Type of categorical data, such as clothing size, in which categories are given labels that can be arranged in order, such as numbers or letters.

Ordinal variable [p. 50] A categorical variable that generates data values that can be used to both name and order, for example, house number.

Outliers [p. 75] Data values that appear to stand out from the main body of a data set. Using box plots, possible outliers are defined as data values greater than $Q_3 + 1.5 \times \text{IQR}$ or less than $Q_1 - 1.5 \times \text{IQR}$.

P

Parallel boxplot [p. 98] A statistical graph in which two or more boxplots are drawn side by side. Used to compare distributions in terms of shape, centre and spread.

Path [p. 432] A walk with no repeated vertices or edges. *See also* Walk.

Percentage [p. 16] The number as a proportion of one hundred, indicated by the symbol %. For example, 12% means 12 per one hundred.

Percentage change [pp. 23, 166] The amount of the increase or decrease of a quantity expressed as a percentage of the original value.

Percentage frequency [p. 52] Frequency of a value or group of values, expressed as a percentage of the total frequency.

Perimeter [p. 481] The distance around the edge of a two-dimensional shape.

Personal loan [p. 199] A personal loan is a sum of money borrowed from a financial institution such as a bank for buying an item for personal rather than business or employment-related use. Regular repayments are required to repay the loan.

Piecewise linear graph [p. 297] A graph made up of two or more parts of different straight-line graphs, each representing different intervals on the x-axis. Sometimes called a segmented linear graph.

Planar graph [p. 425] A graph or network that can be drawn in such a way that no two edges intersect, except at the vertices.

Positive association (bivariate data) [p. 315] A relationship in which large values of the dependent variable are associated with large values of the independent variable, and vice versa.

Positive slope [p. 271] A positive slope represents an increasing y-value with increase in x-value. For the graph of a straight line with a positive slope, y increases at a constant rate with respect to x.

Positively skewed distribution [p. 71] A data distribution that has a long tail to the right. In positively skewed distributions, the majority of data values fall to the left of the mean.

Price-to-earnings ratio [p. 170] A measure of the profit of a company, given by the *current share price/profit per share*. A lower value of the price-to-earnings ratio may indicate a better investment.

Prim's algorithm [p. 448] An algorithm (procedure) for determining a minimum spanning tree in a connected graph or network.

Principal (*P*) [p. 175] The initial amount borrowed, lent or invested.

Pronumeral [p. 2] A symbol (usually a letter) that stands for a numerical quantity or variable.

Purchasing power [p. 205] A measure of how much a specific good or service money can buy at different times (due to inflation, for instance), or which different currencies can buy.

Pythagoras' theorem [p. 470] A rule for calculating the third side of a right-angled triangle given the length of the other two sides. In triangle ABC, the rule is: $a^2 = b^2 + c^2$, where a is the length of the hypotenuse.

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Q

Quartiles (Q_1, Q_2, Q_3) [p. 83] Summary statistics that divide an ordered data set into four equal-sized groups, each containing 25% of the scores.

R

Radius [p. 489] The distance from the centre of a circle (or sphere) to any point on its circumference (or surface); equal to half the diameter.

Range (R) [p. 82] The difference between the smallest (minimum) and the largest (maximum) values in a data set: a measure of spread.

Recurrence relation [p. 367] A rule that enables the next term in a sequence to be generated using one or more previous terms. For example, 'Starting with 3, each new term is made by adding 5 to the current term'. Written as: $t_1 = 3$ and $t_{n+1} = t_n + 5$. Gives: 3, 8, 13, 18, . . .

Regression line [p. 329] *See* Line of best fit.

Response variable [p. 307] The variable of primary interest in a statistical investigation. *See also* Explanatory variable.

Right angle [p. 470] An angle equal to 90°.

Rise [p. 272] *See* Slope of a straight line.

Row matrix [p. 219] A matrix with only one row.

Run [p. 272] *See* Slope of a straight line.

S

(s) [pp. 482, 583] *See* Heron's rule.

Scalar multiplication [p. 229] The multiplication of a matrix by a number.

Scale factor (areas) [p. 513] The scale factor, k^2, by which the area of a two-dimensional shape is scaled (increased or decreased) when its linear dimensions are scaled by a factor of k.

Scale factor (volumes) [p. 524] The scale factor, k^3, by which the volume of a solid shape is scaled (increased or decreased) when its linear dimensions are scaled by a factor of k.

Scatterplot [p. 309] A statistical graph used for displaying bivariate data. Data pairs are represented by points on a coordinate plane with the *EV* plotted on the horizontal axis and the *RV* plotted on the vertical axis.

Sequence [p. 352] A list of numbers or symbols written down in succession, for example, 5, 15, 25.

Shares [p. 169] A share is a unit of ownership of a company.

Shortest path [p. 443] The path through a graph or network with minimum length.

Similar figures [p. 513] Figures that have exactly the same shape but differ in size.

Similar triangles [p. 520] Different sized triangles in which the corresponding angles are equal. The ratios of the corresponding pairs of sides are always the same.

Simple interest [p. 175] Interest that is calculated for an agreed period and paid only on the original amount invested or borrowed. Also called flat-rate interest.

Simultaneous linear equations [p. 142] Two or more linear equations in two or more variables, for values that are common solutions to all equations. For example, $3x - y = 7$ and $x + y = 5$ are a pair of simultaneous linear equations in x and y, with the common solution $x = 3$ and $y = 2$.

Sine ratio (sin θ) [p. 540] In right-angled triangles, the ratio of the side opposite a given angle (θ) to the hypotenuse.

Sine rule [p. 563] In non-right-angled triangles, a rule used to find:

- an unknown side, given the angle opposite and another side and its opposite angle
- an unknown angle, given the side opposite and another side and its opposite angle.

For triangle *ABC* the rule is:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

Skewness [p. 71] Lack of symmetry in a data distribution. It may be positive or negative.

Slope of a straight line [p. 271] The ratio of the increase in the dependent variable (y) to the increase in the independent variable (x) in a linear equation. Also known as the gradient. The slope (or gradient) of a straight-line graph is defined to be:

$$\text{slope} = \frac{\text{rise}}{\text{run}}.$$

SOH–CAH –TOA [p. 540] A memory aid for remembering the trigonometric ratio rules.

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Solution [p. 131] A value that can replace a variable and make an equation or inequality true.

Solution region [p. 611] *See* Feasible region.

Spanning tree [p. 447] A subgraph of a connected graph or network that contains all the vertices of the original graph, but without any multiple edges, circuits or loops.

Spread of a distribution [p. 72] A measure of the degree to which data values are clustered around some central point in the distribution. Measures of spread include the standard deviation (s), the interquartile range (IQR) and the range (R).

Square matrix [p. 219] A matrix with the same number of rows as columns.

Standard deviation (*s*) [p. 84] A summary statistic that measures the spread of the data values around the mean. It is given by $s = \sqrt{\dfrac{\Sigma(x - \bar{x})^2}{n - 1}}$.

Stem plot (stem-and-leaf plot) [p. 73] A method for displaying data in which each observation is split into two parts: a ‘stem’ and a ‘leaf’. A stem plot is an alternative display to a histogram; suitable for small-to medium-sized data sets.

Strength of an association (relationship) [p. 318] The degree of association between two variables, classified as weak, moderate or strong. It is determined by observing the degree of scatter in a scatterplot or calculating a correlation coefficient.

Summary statistics [p. 78] Statistics that give numerical values to special features of a data distribution, such as centre and spread. Summary statistics include the mean, median, range, standard deviation and IQR.

Surface area [p. 507] The total of the areas of each of the surfaces of a solid.

Symmetric distribution [p. 71] A data distribution in which the data values are evenly distributed around the mean. In a symmetric distribution, the mean and the median are equal.

T

Tangent ratio (tan θ) [p. 540] In right-angled triangles, the ratio of the side opposite a given angle θ to the side adjacent to the angle.

Term [p. 352] One value in a sequence or series; or one value in an algebraic expression.

Three-figure bearing: [p. 559] An angular direction, measured clockwise from north and written with three digits, for example, 060°, 324°. Also called a true bearing.

Total surface area (TSA) [p. 509] The total surface area (TSA) of a solid is the sum of the surface areas of all of its faces.

Trail [p. 431] A walk with no repeated edges. *See also* Walk.

Traversable graph [p. 434] A graph that has at least one trail that includes every edge.

Tree [p. 446] A connected graph with no circuits, multiple edges or loops.

Trigonometric ratios [p. 540] In right-angled triangles, the ratios

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}, \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}},$$
$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

True bearing [p. 559] *See* Three-figure bearing.

U

Undefined [p. 274] Has no meaning; has no value. The slope, or gradient, of a vertical line is undefined because $\frac{rise}{run}$ gives a zero denominator.

V

Variable [p. 49] A quantity that can have many different values in a given situation. Symbols such as x, y and z are commonly used to represent variables.

Vertex [p. 414] The points in a graph or network. (Plural – vertices).

Volume [p. 495] The volume of a solid is a measure of the amount of space enclosed within it, measured in cubic units.

Volume formulas [pp. 495, 500, 503, 506] Formulas used to calculate the volumes of solids, including cubes, cuboids, prisms, pyramids, cylinders, cones and spheres.

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W

Walk [p. 431] A sequence of edges, linking successive vertices, that connects two different vertices in a graph.

Weighted graph [p. 442] A graph in which a numbers are associated with each edge. These numbers are called weights. When the numbers represent the size of some quantity (such as distance or time), a weighted graph is often called a network.

Y

***y*-intercept** [p. 276] The point at which a graph cuts the y-axis.

Z

Zero matrix (*O*) [p. 227] A matrix that behaves like zero in arithmetic, represented by the symbol O. Any matrix with zeros in every position is a zero matrix.

Zero slope [p. 274] A horizontal line has zero slope. The equation of this line has the form $y = c$ where c is any constant.

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Answers

Chapter 1

Exercise 1A

1 **a** 37 **b** 5 **c** 50 **d** 7

e 7.2 **f** 48 **g** 34.5 **h** 4.5

i 0.6 **j** −0.9

2 **a** 12.53 **b** −27 **c** 31.496 **d** 1

3 **a** 27 **b** $2x - 14$ **c** $50 - 10y$

d $6w$ **e** $k^2 + 8k$ **f** 30 **g** $2x + 14$

h 22 **i** $3x - 5$ **j** $-2 - 2x$

Exercise 1B

1 **a** −1 **b** −4 **c** −16 **d** 3

e −26 **f** −25 **g** −12 **h** 22

i 22 **j** 32 **k** 28 **l** 10

m −6 **n** −13

2 **a** −12 **b** 24 **c** 2.5 **d** −5

e 36 **f** −12 **g** −7 **h** 60

i −1 **j** 60 **k** 6 **l** 19

m −7 **n** −38 **o** 34 **p** 160

Exercise 1C

1 **a** 10 000 **b** 343 **c** 5 **d** 2

e 64 **f** 20 736 **g** 3 **h** 13

i 1000 **j** 4 **k** 2

2 **a** 26 **b** 15 **c** 37

d 5 **e** 79 **f** 4

Exercise 1D

1 **a** 87 **b** 606 **c** 3 **d** 34

2 **a** 6800 **b** 46 800 **c** 80 000 **d** 300

3 **a** 53 467 **b** 3 800 000

c 789 000 **d** 0.009 21

e 0.000 000 103 **f** 2 907 000

g 0.000 000 000 003 8 **h** 21 000 000 000

4 **a** 7.92×10^5 **b** 1.46×10^7

c 5.0×10^{11} **d** 9.8×10^{-6}

e $1.456\,97 \times 10^{-1}$ **f** 6.0×10^{-11}

g $2.679\,886 \times 10^6$ **h** 8.7×10^{-3}

5 **a** 6×10^{24} **b** 4×10^7

c 1×10^{-10} **d** 1.5×10^8

6 **a** 5 **b** 6 **c** 1 **d** 3

e 2 **f** 1 **g** 2 **h** 4

7 **a** 4.9 **b** 0.0787 **c** 1506.9 **d** 6

8 **a** 0.0033 **b** 0.148 68

c 317 **d** 335

9 **a** 1.56 **b** 0.025 **c** 0.03

d 1.8823 **e** 17.668 **f** 0.2875

10 **a** 15.65 **b** 4.69 **c** 39.14

Exercise 1E

1 **a** 570 cm **b** 1587 m **c** 80 mm

d 6.7 m **e** 460 cm **f** 2.89 cm^2

g 52 000 cm^2 **h** 80 000 m^2 **i** 37 cm^2

j 6 000 000 mm^2 **k** 0.5 L

l 700 g **m** 2 300 000 mg

n 0.567 kL **o** 793.4 g **p** 75 500 000 mg

q 500 mL

2 **a** 5.0×10^3 kg **b** 6.0×10^{-3} kg

c $2.71 \times 10^{10}\ \text{m}^2$ **d** $3.3 \times 10^7\ \text{cm}^3$

e $4.87 \times 10^{-4}\ \text{km}^2$ **f** 2.8×10^{-2} L

g 6.0×10^5 cm **h** 1.125×10^{-3} kL

i $5.0 \times 10^{-5}\ \text{km}^3$ **j** $3.4 \times 10^{-4}\ \text{m}^3$

3 **a** 158 mm **b** 589.169 km

c 364.6 cm **d** 13.5 cm^2

4 7.86 m **5** 3000 kg

6 2 250 000 litres **7** 31 trays

Exercise 1F

1 **a** 25% **b** 40% **c** 15% **d** 70%

e 19% **f** 79% **g** 215% **h** 3957%

i 7.3% **j** 100%

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2 a i $\frac{1}{4}$ **ii** 0.25
b i $\frac{1}{2}$ **ii** 0.5
c i $\frac{3}{4}$ **ii** 0.75
d i $\frac{17}{25}$ **ii** 0.68
e i $\frac{23}{400}$ **ii** 0.0575
f i $\frac{34}{125}$ **ii** 0.272
g i $\frac{9}{2000}$ **ii** 0.0045
h i $\frac{3}{10000}$ **ii** 0.0003
i i $\frac{13}{200000}$ **ii** 0.000 065
j i 1 **ii** 1

3 a \$114 **b** \$110 **c** 25.5 m **d** \$1350
e 1.59 cm **f** 2.64 **g** 0.161 **h** \$4570
i \$77 700 **j** \$19 800

4 80% **5** 37.5%
6 95.6% **7** 83.33%
8 20% **9** 37.5%
10 65.08% **11** 150%

Exercise 1G

1 a \$37 **b** \$148

2 a \$4.50; \$85.49 **b** \$18.90; \$170.10
c \$74.85; \$424.15 **d** \$49.80; \$199.20
e \$17.99; \$61.96 **f** \$5.74; \$17.21
g \$164.73; \$434.28 **h** \$19.05; \$44.45
i \$330; \$670

3 a \$425.25 **b** \$699.13 **c** \$227.50
d \$656.25 **e** \$215.25

4 a \$12.95 **b** \$202.95

5 Decreasing \$60 by 8%

6 \$14 840 **7** 21.95%

8 26 880 km

9 a 13% **b** 26% **c** 6%
d 24% **e** 18% **f** 23%

10 7.08%

11 a 19% **b** 33% **c** 45%
d 20% **e** 33% **f** 16%

12 a 25% **b** 40% **c** 7.5%

Exercise 1H

1 35 : 15

2 a 80 : 40 **b** 70 : 9 **c** 80 : 120
d 40 : 4 **e** 40 : 4 : 80

Exercise 1I

1 a 4 : 5 **b** 2 : 9 **c** 2 : 5 : 3 **d** 1 : 3
e 3 : 1 **f** 20 : 3 **g** 9 : 4

2 a 12 : 5 **b** 1 : 20 **c** 3 : 8 **d** 25 : 3
e 3 : 100 : 600 **f** 100 000 : 100 : 1
g 4 : 65 **h** 50 : 10 : 2 : 1

3 a 5 **b** 72 **c** 120 **d** 5000 **e** 24

4 a False **b** False 3 : 4 = 15 : 20
c True **d** False 60 : 12 = 15 : 3 = 5 : 1
e False The girl would be 8. **f** True

5 a 100 : 60 : 175 : 125 : 125
b 20 : 12 : 35 : 25 : 25
c 300 g rolled oats, 180 g coconut, 525 g flour, 375 g brown sugar, 375 g butter, 9 tbsp water, 6 tbsp golden syrup, 3 tsp bicarb soda

Exercise 1J

1 a 32 m and 8 m **b** 5 m and 35 m
c 30 m and 10 m **d** 20 m and 20 m

2 a \$300 and \$200 **b** \$50, \$200 and \$250
c \$50, \$400 and \$50 **d** \$160, \$180 and \$160
e \$250, \$125, \$100 and \$25

3 a 50 bananas **b** 5 mangos
c 75 pieces of fruit

4 a 1.5 litres **b** 6 litres

5 3 kilometres

Exercise 1K

1 a \$15.60 **b** 84 seconds
c \$885 **d** 10 kilometres

2 7 red, 28 yellow

3 a 270 km **b** 225 km **c** 30 km
d 105 km **e** 330 km **f** 67.5 km

4 73 g cone for \$2

5 Brand A

6 51 eggs

7 a 550 kilometres **b** 17 litres

Exercise 1L

1 a $10^3, 3$ **b** $10^6, 6$ **c** $10^{-4}, -4$
d $10^7, 7$ **e** $10^0, 0$ **f** $10^1, 1$
g $10^{-9}, -9$

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2 a 2.477 **b** 3.774 **c** 4.017 **d** −2.328
e −0.222 **f** −1.051 **g** 0.860

3 a 316.23 **b** 0.03 **c** 3.16 **d** 1

Exercise 1M

1 a 4 **b** 2 **c** 9
d 12 **e** 11

2 3 **3** 5

4 a 3 **b** 1

Exercise 1N

1 100 times **2** 2 times **3** 3.16
4 3000 times heavier **5** 1000 times louder
6 10^{-6} times softer

Chapter 1 review

Multiple-choice questions

1 E **2** A **3** B **4** C **5** B
6 C **7** E **8** D **9** C **10** B
11 D **12** A **13** D **14** C **15** E
16 C **17** A **18** C **19** B **20** A

Short-answer questions

1 a 11 **b** 10 **c** 7 **d** 14
e 49 **f** −5 **g** 1 **h** 30
i 2

2 a 125 **b** 3 **c** 6 **d** 2.83
e 4 **f** −4 **g** 8.64 **h** 11.66

3 a 2.945×10^3 **b** 5.7×10^{-2}
c 3.69×10^5 **d** 8.509×10^2

4 a 7500 **b** 0.001 07 **c** 0.456

5 a 8.9 **b** 0.59 **c** 800

6 a 7.15 **b** 598.2 **c** 4.079

7 a 70.7 mm **b** 2.170 km
c 1000 cm^2 **d** 2 500 000 m^2
e 5 cm^2 **f** 0.53 mm^3
g 5 800 000 mg **h** 70 mL

8 a 0.75 **b** 0.4 **c** 0.275

9 a $\frac{1}{10}$ **b** $\frac{1}{5}$ **c** $\frac{11}{50}$

10 a 24 **b** \$10.50 **c** \$13.25

11 a \$51.90 **b** \$986.10

12 \$862.50

13 20%

14 46%

15 Melissa 5.13%, Jody 4.41%

16 a False **b** False **c** False **d** True

17 a \$320 and \$480 **b** \$160 and \$640
c \$160, \$240 and \$400
d \$200, \$200 and \$400

18 18 cups

19 a 27 m **b** 140 m

20 \$20

21 301 km

22 1000 times

Chapter 2

Exercise 2A

1 a Nominal **b** Ordinal
c Ordinal **d** Nominal

2 a Categorical **b** Numerical
c Categorical **d** Numerical
e Categorical **f** Categorical

3 a Nominal **b** Ordinal
c Numerical (discrete)

4 a Discrete **b** Discrete
c Continuous **d** Continuous
e Discrete

Exercise 2B

1 a Nominal
b

	Frequency	
Sex	Number	%
female	5	33.3
male	10	66.7
Total	15	100.0

2 a Ordinal
b

	Frequency	
Shoe size	Number	%
7	3	15
8	7	35
9	4	20
10	3	15
11	2	10
12	1	5
Total	20	100

3 a 69; 8.7%, 26.1% **b** Nominal
c 7 students **d** 10.1%
e Hamburger

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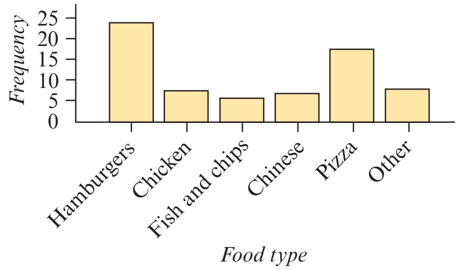
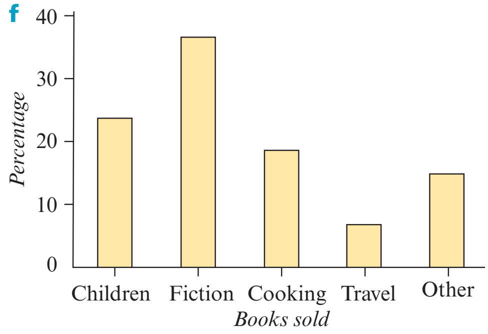
f

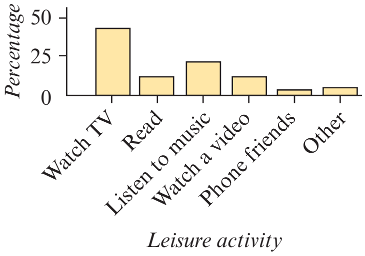


4 **a** 53; 16.4%, 20.7%
b Ordinal
c 21 people
d 50.4%
e Strongly disagree
f

5 **a** 38.4%, 6.5%, 100.0%
b Nominal
c 89 books
d 22.8%
e 232 books
f



6 **a** 200 students
b Nominal
c 4%
d Watch TV
e



Exercise 2C

1 69, hamburgers, 26.1%

2 Strongly disagreed, 20.7%, 16.4%

3 A group of 200 students were asked how they prefer to spend their leisure time. The most popular response was using the internet and digital games (42%), followed by listening to music (23%), reading (13%), watching TV or going to a movie (12%) and phoning friends (4%). The remaining 6% said 'other'. Watching TV for this group of students was clearly the most popular leisure time activity.

4 A group of 579 employees from a large company were asked about the importance to them of the salary that they earned in the job. The majority of employees said that it was important (56.8%), or very important (33.5%). Only a small number of employees said that it was somewhat important (7.8%) with even fewer saying that it was not at all important (1.9%). Salary was clearly important to almost all of the employees in this company.

Exercise 2D

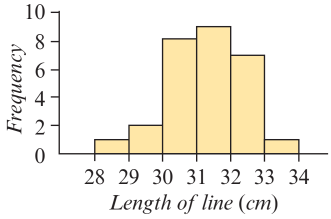
1

Number of magazines	Frequency	
	Number	Percent
0	4	26.7
1	4	26.7
2	3	20.0
3	2	13.3
4	1	6.7
5	1	6.7
Total	15	100.0

2

Amount of money ($)	Frequency	
	Number	Percent
0.00–4.99	13	65
5.00–9.99	3	15
10.0 –14.99	2	10
15.00–19.99	1	5
20.00–24.99	1	5
Total	20	100

3 **a** **i** 2 students **ii** 3 students
iii 8 students
b **i** 32.1% **ii** 39.3% **iii** 89.3%
c



4 **a** 4 students
b 2 children
c 5 students
d 28 students

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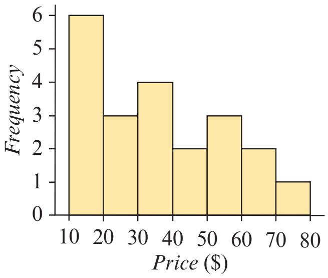
5 **a** 0 students **b** 48 students

c 60–69 marks **d** 33 students

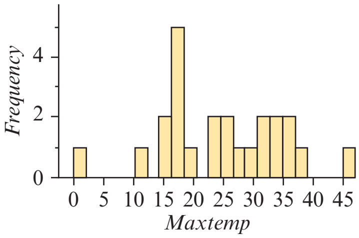
6 **a**



b **i** \$30 to < \$40 **ii** 4 books

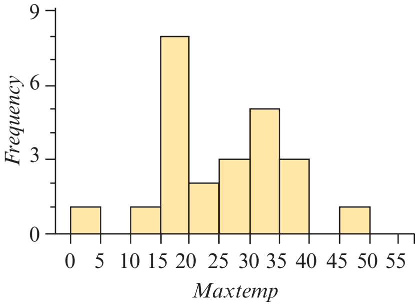
iii \$10 to < \$20

7 **a**



b **i** 11°C **ii** 1 city

c



d **i** 2 cities **ii** 15°C to < 20°C

Exercise 2E

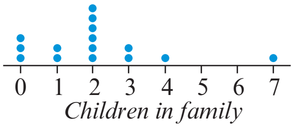
1 **a** Positively skewed **b** Negatively skewed

c Symmetric

2 **a** Location **b** Neither **c** Both

Exercise 2F

1 **a**

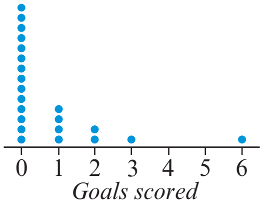


b 2 children

2 **a**

b 7 people

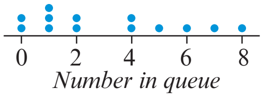
3 **a**



b 0 goals

c Positively skewed with a possible outlier. The player who kicked six goals.

4 **a**



b Around 12:25 p.m.

5 **a** English marks

1	7
2	3 3 6 8
3	2 5 5 8 9
4	3 4 6 6 9
5	0 2 8 9
6	1 4 5 6 9
7	5 8 9 9
8	3 3 4 9
9	2 3 4 7

5|0 represents 50 marks

b 21 students **c** 17 marks

6 **a** 40 people **b** Symmetric

c 21 people

7 **a** Battery time (hours)

0	4
1	7 9
2	0 1 2 4 5 6 6 7 7 8
3	0 0 1 1 3 3 4 7
4	0 1 6

2|5 represents 25 hours

b 9 batteries

8 **a** Homework time (minutes)

0	0
1	0 0 4 5 5 6 9
2	0 0 1 3 7 8 9
3	3 7 9
4	6
5	6
6	3
7	0

4|6 represents 46 minutes.

b 2 students

c Positively skewed

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9 a Price ($)

2	5 8
3	5 6 9
4	5 6 9
5	2
6	8
7	5 5 6 8 9
8	2 4
9	5
10	
11	
12	
13	
14	9
15	

16|4 represents $164 (truncated).

b Approximately symmetric with an outlier ($149)

Exercise 2G

1 a mean = 5; median = 5
b mean = 5; median = 4.5
c mean = 15; median = 15
d mean = 101; median = 99.5
e mean = 2.8; median = 2.1

2 a $M = 9, \text{IQR} = 10.5, R = 21$
b $M = 6.5, \text{IQR} = 8, R = 11$
c $M = 27, \text{IQR} = 7, R = 12$
d $M = 106.5, \text{IQR} = 4.5, R = 8$
e $M = 1.2, \text{IQR} = 1.1, R = 2.7$

3 a $M = 57$ mm; IQR = 59 − 49.5 = 9.5 mm
b $M = 27.5$ hours; IQR = 33 − 23 = 10 hours

4 a $\bar{x} = 12.5$ ha, $M = 7.4$ ha
b The median, as it is typical of more suburbs.

5 a $\bar{x} = \$393\,386$, $M = \$340\,000$
b The median, as it is typical of more apartment prices.

6 $\bar{x} = 365.8$, $s = 8.4$, $M = 366.5$, IQR = 12.5, $R = 31$

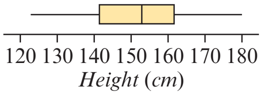
7 $\bar{x} = 214.8$, $s = 35.4$, $M = 207.5$, IQR = 42, $R = 145$

8 $\bar{x} = 3.5$ kg, $s = 0.6$ kg, $M = 3.5$ kg, IQR = 1 kg, $R = 2.4$ kg

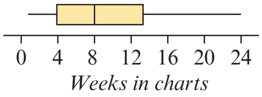
9 a i $\bar{x} = 6.79$, $M = 6.75$
ii IQR = 1.45, $s = 0.93$
b i $\bar{x} = 13.54$, $M = 7.35$
ii IQR = 1.8, $s = 18.79$
c The error does not affect the median or interquartile range very much. It doubles the mean and increases the standard deviation by a factor of 20.

Exercise 2H

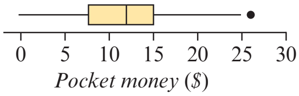
1



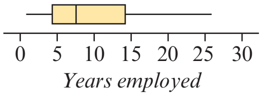
2



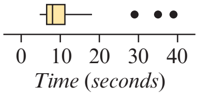
3



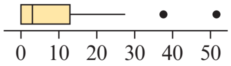
4



5

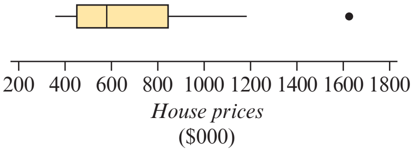


6 a



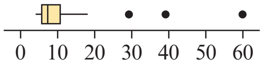
b There are two possible outliers; the people who borrowed 38 and 52 books respectively.

7 a



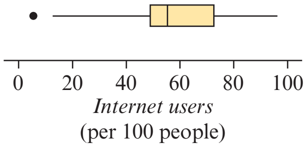
b There was one outlier, the unit which sold for $1 625 000.

8 a



b There are three possible outliers, the three children who took 29, 39, and 60 seconds respectively to complete the puzzle.

9 a



b There is one possible outlier, Afghanistan, that recorded extremely low percentages of internet users (5.45%). At 12.52%, India is just above the outlier cutoff of 12.05%.

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Exercise 2I

Note: The written reports should only be regarded as sample reports. There are many ways of writing the same thing.

1 a Females: M = 34 years, IQR = 28 years
Males: M = 25.5 years, IQR = 13 years

b Report: The median age of the females (M = 34 years) was higher than the median age of males (M = 25.5 years). The spread of ages of the females (IQR = 25 years) was greater than the spread of ages of the males (IQR = 13 years). In conclusion, the median age of the females admitted to the hospital on that day was higher than the males. Their ages were also more variable.

2 a Class A: 6; Class B: 2

b Class A: M = 76.5 marks, IQR = 30.5 marks;
Class B : M = 78 marks, IQR = 12 marks

c Report: The median mark for Class A (M = 76.5) was lower than the median mark for Class B (M = 78). The spread of marks for Class A (IQR = 30.5) was greater than the spread of marks of Class B (IQR = 12). In conclusion, Class B had a higher median mark than Class A and their marks were less variable.

3 a

Japan		*Australia*
3	0	2 3 3 3 4 4
9 7 6 5 5	0	5 5 6 7 7 8 9
4 4 2	1	1 1 4
9 7 5	1	5 7
3 3 2 2	2	1 3
9 8 6	2	
2	3	3
	3	
4	4	

6|2 represents 26 1|5 represents 15

b Japanese A: M = 17 nights,
IQR = 16.5 nights;
Australians : M = 7 nights,
IQR = 10.5 nights

c Report: The median time spent away from home by the Japanese (M = 17 nights) was much higher than the median time spent away from home by the Australians (M = 7 nights). The spread in the time spent away from home by the Japanese (IQR = 16.5 nights) was also greater than the time spent away from home by the Australians (IQR = 10.5). In conclusion, the median time spent away from home by the Japanese was longer than the Australians and the time they spent away from home more variable.

4 a Year 12: M = 5.5 hours, IQR = 4.5 hours;
Year 8: M = 3 hours, IQR = 2.5 hours
(values can vary a little)

b Report: The median homework time for the year 12 students (M = 5.5 hours/week) was higher than the median homework time for year 8 students (M = 3 hours/week). The spread in the homework time for the year 12 students (IQR = 4.5 hours/week) was also greater than the year 8 students (IQR = 2.5 hours/week). In conclusion, the median homework time for the year 12 students was higher than the year 8 students and the homework time was more variable.

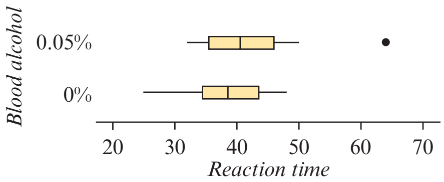
5 a Males: M = 22%, IQR = 15%;
females : M = 19%, IQR = 14% (values can vary a little)

b Report: The median smoking rate for males (M = 22%) was higher than for females (M = 19%). The spread in smoking rates for males (IQR = 15%) was similar to females (IQR = 14%). In conclusion, median smoking rates were higher for males than females but the variability in smoking rates was similar.

6 a Before: M = 26, IQR = 4,
outlier = 45; After : M = 30, IQR = 6,
outliers = 48 & 52 (values can vary a little)

b Report: The median number of sit ups before the fitness class (M = 26) was lower than after the fitness class (M = 30). The spread in number of sit ups before the fitness class (IQR = 6) was less than after the fitness class (IQR = 9). There was one outlier before the fitness class, the person who did 45 sit ups. There were two outliers after the fitness class, the person who did 48 sit ups and the person who did 52 sit ups. In conclusion, the median number of sit ups increased after taking the fitness class and there was more variability in the number of sit ups that could be done.

7 a

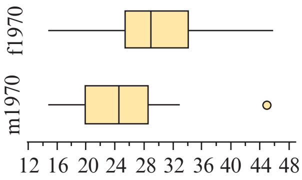


b Report: The median time is slightly higher for the 0.05% blood alcohol group (M = 40.5) than for the 0% blood alcohol group (M = 38.5). The spread in time is also slightly higher for the 0.05% blood alcohol group (IQR = 9.5) than for 0% blood alcohol (IQR = 9.0). There was one outlier, the person with 0.05% blood alcohol who had a very long time of 64 seconds.

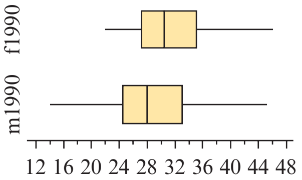
In conclusion, the median time was longer for the 0.05% blood alcohol group than for the 0% blood alcohol group but the variability in times was similar.

Exercise 2J

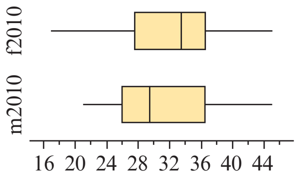
1 a In 1970, the median age for mothers (M = 23.5) was lower than that for fathers (M = 29 years). The spread of ages for the mothers (IQR = 8.5) was the same for fathers (IQR = 9.0). There was one outlier – a mother of age 45 years. This mother was much older than the remainder of the mothers.



b In 1990, the median age for mothers (M = 28) was still lower than the median age for fathers (M = 31). The spread of ages for the mothers (IQR = 9.0) was the same as the 1970 spread and was slightly higher than the spread for fathers (IQR = 9.0).



c In 2010, the median age for mothers (M = 31) was again lower than the median age for fathers (M = 33.5). The spread of ages for the mothers (IQR = 10.5) was higher than the spread for fathers (IQR = 9.5).



d Report

The median age for mothers has increased steadily over the years, from 23.5 in 1970, to 28 in 1990 and 31 in 2010. The spread in ages for mothers was the same in 1970 (IQR = 8.5) and 1990 (IQR = 9), but increased in 2010 (IQR = 10.5). In 1970, a mother of age 45 was considered an outlier, but in 1990 and 2010, the age of 45 was not unusual enough to be an outlier.

The median age for fathers has increased steadily over the years, from 29 in 1970, to 31 in 1990 and to 33.5 in 2010. The spread in ages for fathers has remained reasonably steady during this time (1970: IQR = 9; 1990: IQR = 9; 2010: IQR = 9.5)

The difference in age between mothers and fathers has changed little over time. Fathers have typically been older than mothers by the same amount (1970: 5.5 y; 1990: 3 y; 2010: 4 y)

Chapter 2 review

Multiple-choice questions

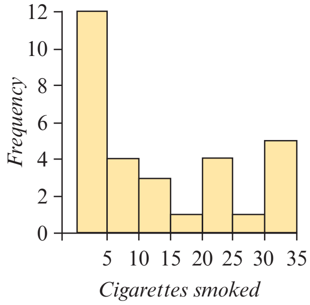
1 D **2** D **3** C **4** B **5** D
6 D **7** D **8** D **9** D **10** E
11 D **12** A **13** C **14** C **15** E
16 B **17** B **18** B **19** C **20** A
21 C

Short-answer questions

1 a Discrete **b** Ordinal

2 a Categorical **b** 7.5%

3



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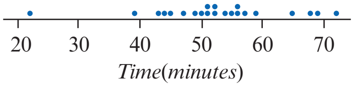
4 **a**



b Time (minutes)

2	2
3	9
4	3 4 5 7 9
5	0 1 1 2 2 4 5 6 6 7 9
6	5 8 9
7	2

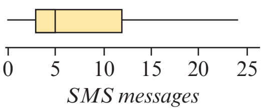
4|7 represents 47 minutes

c $M = 52$ minutes, $Q_1 = 47$ minutes, $Q_3 = 57$ minutes

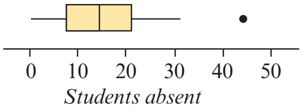
5 $\bar{x} = \$283.57$, $s = \$122.72$, $M = \$267.50$, IQR $= \$90$, $R = \$495$

6 $\bar{x} = 178.89$ minutes, $s = 13.99$ minutes

7



8 **a**



b 14.5 students **c** 27.8%

Extended-response questions

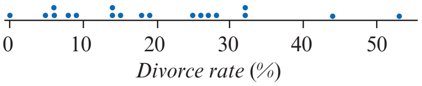
1 **a** Numerical

b Divorce rate (%)

0	0 5 6 6 8 9
1	4 4 5 8 9
2	5 6 7 8
3	2 2
4	4
5	3

3|2 represents 32%

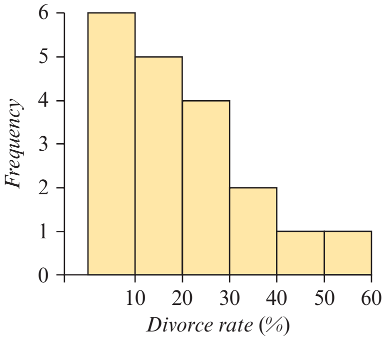
c



d Positively skewed

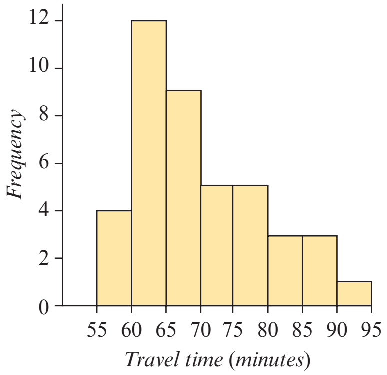
e 21.05%

f $\bar{x} = 20.05\%$, $M = 18\%$



g **i** Positively skewed **ii** 5 countries

2 **a**



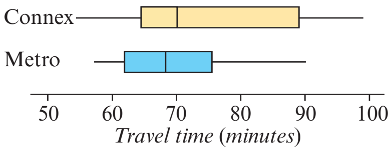
i 9 days **ii** Positively skewed

iii 38.1%

b $\bar{x} = 69.60$ minutes, $s = 9.26$ minutes, Min $= 57$ minutes, $Q_1 = 62$ minutes, $M = 68$ minutes, $Q_3 = 76$ minutes, Max $= 90$ minutes

c **i** 69.60 **ii** 68 **iii** 33, 14 **iv** 76 **v** 9.26

d



e The median travel times for Connex ($M = 70$ minutes) tend to be longer than the median travel times for Metro ($M = 68$ minutes). The spread of times is also longer for Connex (IQR = 24 minutes) compared to Metro (IQR = 14). Both median travel times and variability in travel times was less for Metro than for Connex.

Chapter 3

Exercise 3A

1 **a** \$1400 **b** \$1500 **c** \$1425

2 380 km

3 **a** \$10.50 **b** \$14.40 **c** \$30

4 **a** 157.08 cm **b** 18.85 mm **c** 33.93 cm **d** 45.24 m

5 **a** $P = 14$ **b** $P = 46$ **c** $P = 23$

6 **a** $A = 4$ **b** $A = 14.25$ **c** $A = 16.74$

7 **a** 10°C **b** −17.8°C **c** 100°C **d** 33.3°C

8 a \$2400 **b** \$180
c \$375 **d** \$2014.50

9 a i 15 points **ii** 37 points
iii 68 points
b Greenteam

10 a 13 **b** 23 **c** 101

11 a i 1 h 50 min **ii** 2 h 56 min
iii 3 h 6 min **iv** 2 h 38 min
b 5:15 p.m.

Exercise 3B

1

x	40	41	42	43	44	45
C(\$)	86	88.15	90.3	92.45	94.6	96.75

x	46	47	48	49	50
C(\$)	98.9	101.05	103.2	105.35	107.5

2

r	0	0.1	0.2	0.3	0.4	0.5
C	0	0.628	1.257	1.885	2.513	3.142

r	0.6	0.7	0.8	0.9	1.0
C	3.770	4.398	5.027	5.655	6.283

3

n	50	60	70	80	90	100
C(\$)	49	50.8	52.6	54.4	56.2	58

n	110	115	120	125	130
C(\$)	59.8	60.7	61.6	62.5	63.4

4

M(kg)	60	65	70	75	80	85	90
E(kJ)	650	695	740	785	830	875	920

M(kg)	95	100	105	110	115	120
E(kJ)	965	1010	1055	1100	1145	1190

5

n	3	4	5	6
S	180°	360°	540°	720°

n	7	8	9	10
S	900°	1080°	1260°	1440°

6 a

n	0	1	2	3	4	5
E(\$)	680	740	800	860	920	980

n	6	7	8	9	10
E(\$)	1040	1100	1160	1220	1280

b 6 cars

7

T (years)	1	2	3	4	5
I(\$)	450	900	1350	1800	2250

T (years)	6	7	8	9	10
I(\$)	2700	3150	3600	4050	4500

8

t(years)	5	10	15	20	25
A(\$)	6535	8541	11 162	14 589	19 067

Exercise 3C

1 a $x = 9$ **b** $y = 15$ **c** $t = 5$ **d** $m = 6$
e $g = 6$ **f** $f = 19$ **g** $f = -3$ **h** $v = -5$
i $x = -1$ **j** $g = 1$ **k** $b = 5$ **l** $m = -2$
m $y = 6$ **n** $e = 3$ **o** $h = -5$ **p** $a = -4$
q $t = -10$ **r** $s = -11$ **s** $k = 7$ **t** $n = 4$
u $a = 8$ **v** $b = 21$

2 a $x = 3$ **b** $g = 9$ **c** $n = 4$ **d** $x = -8$
e $j = -4$ **f** $m = 7$ **g** $f = 5.5$ **h** $x = 3.5$
i $y = 5$ **j** $s = -3$ **k** $b = -5$ **l** $d = -4.5$
m $r = 12$ **n** $q = 30$ **o** $x = 48$ **p** $t = -12$
q $h = 40$ **r** $m = 21$ **s** $a = 2$ **t** $f = -2$
u $a = 6$ **v** $y = 40$ **w** $r = 8$ **x** $x = 5$
y $m = 8$ **z** $x = 6.5$

3 a $y = 4$ **b** $x = 11$ **c** $g = 2$ **d** $x = 3$
e $x = 0.5$ **f** $m = 1.2$ **g** $a = 18$ **h** $r = 13$

4 a $x = 5$ **b** $a = 3$ **c** $b = 9$ **d** $y = 3$
e $x = 0$ **f** $c = -4$ **g** $f = -2$ **h** $y = -5$

5 a $a = 2$ **b** $b = 6$ **c** $w = 2$ **d** $c = 2$
e $y = 7$ **f** $f = 2$ **g** $h = 5$ **h** $k = \dfrac{4}{3}$
i $g = 8.5$ **j** $s = 20$ **k** $t = 2.2$ **l** $y = 2$
m $x = -2$ **n** $g = 37$ **o** $p = 2$

Exercise 3D

1 a $P = 27 + x$ **b** $P = 4x$
c $P = 2a + 2b$ **d** $P = 19 + y$

2 a $P = 22 + m$ **b** 8 cm

3 a $P = 4y$ **b** 13 cm

4 a $n + 7 = 15$ **b** 8

5 6 **6** 59

7 a $P = 4x + 12$ **b** 18 cm
c 24 cm, 18 cm

8 78 tickets **9** 25 invitations

10 Anne \$750, Barry \$250

11 30 km

12 a 0.78 h or 47 min
b Ben 9.4 km, Amy 7.8 km

Exercise 3E

1 a $x = 22 - 7y$ **b** $x = 11 + 4y$
c $x = 5 - 6y$ **d** $x = 5y + 12$
e $x = 2y - 5$ **f** $x = 8 - 4y$
g $x = 2y - 3$ **h** $x = \frac{2}{3}y + \frac{5}{3}$
i $x = 5 - 2.5y$ **j** $x = -6 - 3y$
k $x = 6 + 1.5y$ **l** $x = \frac{7}{5}y - 8$
m $x = \dfrac{y - 2}{a}$ **n** $x = \dfrac{9 - y}{m}$

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o $x = \frac{2y+3}{m}$ **p** $x = \frac{d-ny}{m}$

q $x = \frac{y+b}{a}$ **r** $x = \frac{ty-a}{s}$

2 $r = \frac{C}{2\pi}$

3 a $n = \frac{S+360}{180}$ **b** 7 sides

4 a $t = \frac{v-u}{a}$ **b** 3.6 seconds

5 $h = \frac{2A}{b}$

6 $C = \frac{5(F-32)}{9}$

7 a $T = \frac{100I}{PR}$ **b** 5 years

8 a $h = \frac{S}{2\pi r} - r$ **b** 8.0 cm

9 a $c = y - mx$ **b** 13

Exercise 3F

1 a $C = 0.5x + 0.2y$ **b** \$16.50

2 a $C = 40x + 25y$ **b** \$13 875

3 a $C = 1.6x + 1.4y$ **b** \$141.20

4 a $C = 1.75x + 0.7y$ **b** \$52.15

5 a $C = 3.5x + 5y$ **b** \$312

6 a $C = 30x + 60y$ **b** \$3480

7 a $N = x + y$ **b** $V = 0.5x + 0.2y$

c \$37.90

8 5 balls

9 George 16, Maria 21

10 6.67 m

Exercise 3G

1 a 16 **b** 81 **c** 49 **d** 27

e 8 **f** 216 **g** −125 **h** 256

i 10 000

2 a $a = \pm 3.46$ **b** $b = \pm 8.49$

c $c = \pm 23.83$ **d** $d = 4.24$

e $e = 6.69$ **f** $f = -9.12$

3 a $x = \pm 2.83$ **b** $y = \pm 2.24$

c $a = \pm 5.24$ **d** $f = \pm 2.35$

e $h = \pm 3.46$ **f** $c = \pm 2.61$

g $x = 4.33$ **h** $r = 2.62$

i $y = 4.58$ **j** $r = 2.92$

k $m = 2.67$ **l** $b = 1.38$

m $p = \pm 2.12$ **n** $q = 3.17$

o $r = \pm 4.90$ **p** $y = \pm 3$

q $f = 4$ **r** $n = \pm 2.74$

4 1.43 cm **5** 15.7 cm **6 a** 5 cm **b** 10 cm

7 3.27 cm

Exercise 3H

1 a $r = \frac{A}{2\pi h}$ **b** $R = \frac{100I}{PT}$

c $h = \frac{V}{\pi r^2}$ **d** $r = \sqrt{\frac{V}{\pi h}}$

e $a = \frac{2(s-ut)}{t^2}$

2 a $t = \frac{v-5}{2}$ **b** 4.95 sec

3 a $r = \sqrt{\frac{3V}{\pi h}}$ **b** 3.09 cm

Exercise 3I

1 a $(-1, -1)$ **b** $(3, -2)$

c $(1, 1)$ **d** $(1, -1)$

2 a $(2, -4)$ **b** $(-3, 2)$ **c** $(1.5, 2.5)$

d $(2, 1)$ **e** $(0, 6)$ **f** $(7, 2)$

g $(0, 3)$ **h** $(1, 5)$ **i** $(0.4, -2.6)$

j $(7, 25)$ **k** $(-8, -20)$ **l** $(-3, 10)$

m No intersection, lines are parallel

Exercise 3J

1 a $x = 2, y = 1$ **b** $x = 2, y = 5$

c $x = 3, y = 4$ **d** $x = 9, y = 1$

e $x = 3, y = 2$ **f** $x = 1, y = 1$

g $x = 4, y = 3$ **h** $x = 2.4, y = 3.4$

i $p = -1, q = 4$ **j** $a = -2, b = 5$

k $f = -2, g = -1$ **l** $x = 3, y = 2$

2 a $x = 2, y = 4$ **b** $x = -2, y = 3$

c $x = -2, y = 10$ **d** $x = -2, y = 3$

e $x = 4, y = -1$ **f** $x = -7, y = 0$

Exercise 3K

1 a $x = 4, y = -1$ **b** $x = \frac{1}{2}, y = 2$

c $x = -1, y = -2$ **d** $h = \frac{1}{2}, d = -2$

e $p = 3, k = -1$ **f** $t = \frac{32}{17}, s = \frac{28}{17}$

g $m = 2, n = 3$ **h** $x = \frac{4}{3}, y = \frac{7}{2}$

i $a = -1.5, b = 2.25$ **j** $x = \frac{1}{5}, y = -\frac{1}{5}$

k $x = 1.5, y = -0.6$, to 1 d.p.

Exercise 3L

1 a $5t + 6p = 12.75$ and $7t + 3p = 13.80$

b Texta \$1.65, pencil \$0.75

2 a $50p + 5m = 109$ and $75p + 5m = 146$

b Petrol \$1.48/L, motor oil \$7/L

3 a $6a + 10b = 7.10$ and $3a + 8b = 4.60$

b Banana 35c, orange 60c

4 Nails 1.5 kg, screws 1 kg

5 12 emus, 16 wombats

6 6 cm, 12 cm **7** 22, 30 **8** 8, 27

9 Bruce 37, Michelle 33

10 Boy is 9, sister is 3

11 Chocolate thickshake \$5, fruit smoothie \$3

12 Mother 44, son 12

13 77 students

14 10 standard, 40 deluxe

15 252 litres (40%), 448 litres (15%)

16 126 boys, 120 girls

17 7542 litres unleaded, 2458 litres diesel

18 \$10 000 at 5%, \$20 000 at 8%

19 Width 24 m, length 36 m

20 28, 42, 35

Exercise 3M

1 160 m^2, 260 m^2, 460 m^2

2 4 deluxe, 4 standard

3 **a** 31.90 L **b** 47.18 L **c** 176 cm **d** 40.61 L
e 45.65 L
f

weight	60	65	70	75	80
TBW	40.39	42.08	43.76	45.44	47.12

weight	85	90	95	100
TBW	48.80	50.48	52.16	53.84

weight	105	110	115	120
TBW	55.52	57.20	58.89	60.57

4 **a** $V = (55 - 2x)(40 - 2x)x$
b 10 cm or 5.462 cm

Chapter 3 review

Multiple-choice questions

1 C **2** B **3** D **4** B
5 A **6** A **7** D **8** B
9 D **10** B **11** C **12** D
13 C **14** D **15** A **16** C
17 D **18** D **19** D

Short-answer questions

1 **a** $x = 10$ **b** $x = 11$ **c** $x = 8$ **d** $x = 6$
e $x = 1$ **f** $x = 7$ **g** $x = -6$ **h** $x = 11$
i $x = 3$ **j** $x = 3$ **k** $x = 15$ **l** $x = -24$

2 **a** $P = 40$ **b** $P = 130$

3 **a** $A = 30$ **b** $A = 54$

4 94.25 cm

5

x	−20	−15	−10	−5	0
y	−716	−551	−386	−221	−56

x	5	10	15	20	25
y	109	274	439	604	769

a $x = 10$ **b** $x = -5$

6 **a** $a = \pm 7$ **b** $b = \pm 94$ **c** $c = \pm 8$ **d** $d = 3$
e $e = -4$ **f** $f = 3$ **g** $g = \pm 2$ **h** $h = 5$

7 1 **8** 5

9 **a** (1, 3) **b** (4, 1) **c** (5, 1)

10 **a** $x = 2, y = 8$ **b** $x = 3, y = 2.5$
c $p = 5, q = -2$ **d** $p = 5, q = 2$
e $p = 2, q = 1$

Extended-response questions

1 **a** \$57 **b** 7 hours

2 **a**

n	60	70	80	90	100	110
C	55	60	65	70	75	80

n	120	130	140	150	160
C	85	90	95	100	105

b \$105

3 **a** $C = 80 + 45h$ **b** \$215

4 **a** $3a + 5c = 73.5$ **b** \$12 **c** \$7.50
$2a + 3c = 46.5$

5 3 m

6 Indonesian 28; French 42; Japanese 35.

Chapter 4

Exercise 4A

1 **a** 48.8% **b** 24.4% **c** 11.1% **d** 9.2%
e 33.3% **f** 25%

2 **a** \$2600 **b** \$200 **c** \$200 **d** \$1.80
e \$1865 **f** \$20 000

3 **a** \$86.40 **b** \$180 **c** \$0.58 **d** \$73.08

4 **a** \$291.20 **b** \$626.40 **c** \$68 **d** \$6318

5 **a** \$1865.50 **b** \$11.14 **c** \$27.72
d \$10 282 **e** \$847.70 **f** \$2631.20

6 **a** 24% **b** 20%

7 **a** \$60 **b** \$50 **c** \$71.43 **d** \$88

8 **a** \$13.30 **b** \$62.22 **c** \$104.50
d \$4993.90

9 \$212.75 **10** \$92.07

11 **a** \$12.13 **b** \$6.76 **c** \$98.55 **d** \$39.50

12 **a** \$152.90 **b** \$2945.80
c \$10 835 **d** \$1534.50

13 \$2180.91 **14** \$3635.45

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15 **a** $289.97 **b** $29.00

16 **a** 0.5% **b** $12 500

17 **a** A: 8.8, B: 10 **b** A

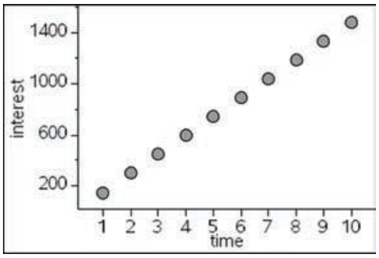
18 **a** $0.50 **b** $1.00
c 500 Alpha oil, 250 Omega mining
d $612.50

19 **a** $250 **b** 10.9%

Exercise 4B

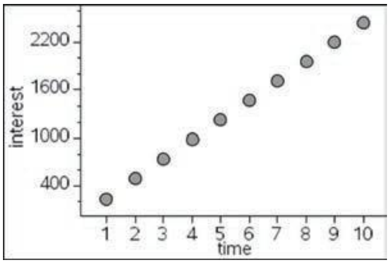
1 **a** $80 **b** $300 **c** $600 **d** $384.38
e $4590 **f** $324.38 **g** $29.95 **h** $14.43
i $6243.75

2 **a**



b $742.5

3 **a**



b $980.50

4 **a** $600 **b** $932.10 **c** $1243.50
d $2411.25 **e** $2832

5 **a** $12 000 **b** $32 000

6 **a** $1950 **b** $11 950

7 $1243.50 **8** $9041.10

9 **a** $118.75 **b** $2750 **c** $2463.19
d $24 000 **e** $1983.63
f $13 617.92

10 **a** $4500 **b** $13.33

11 $3.12 **12** $1.88

13 **a** March $650.72, April $650.72,
May $900.72
b $6.88

Exercise 4C

1 12% **2** 15% **3** 6.5 years
4 354 days **5** $1210 **6** $45 552
7 **a** $180 **b** $780 **c** 3 years
d $1051.60 **e** 7% **f** $1335.15
g $4500 **h** $4650 **i** 5%
j $3698.63 **k** 3 years **l** $220.50
m $1448.28 **n** $1500.78

8 4 years **9** 20 years

10 **a** $18 000 **b** $3000

Exercise 4D

1 **a** $4466.99 **b** $966.99

2 **a** $9523.42 **b** $2523.42

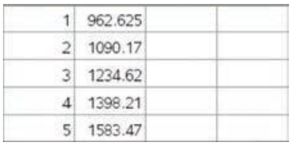
3 Difference: CI – SI = $ 202.61

4 **a** $1552.87 **b** $302.87

5 **a** $1338.23 **b** $338.23

6 Difference: CI – SI = $105.10

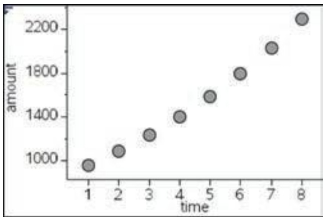
7 **a**



1	962.625
2	1090.17
3	1234.62
4	1398.21
5	1583.47

b $1583.47; $733.47

c The graph curves upwards.

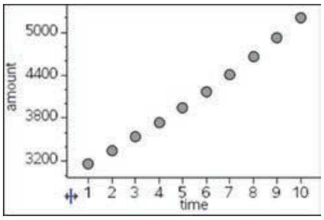


8 **a**

1	3169.5
2	3348.58
3	3537.77
4	3737.66
5	3948.83

b $3737.66 ; $737.66

c The graph curves upwards.



Exercise 4E–1

1 **a** $366 **b** $36

2 **a** $26.40 **b** $324

3 **a** $1210.20 **b** $313.20

4 19.2%

5 23.5%

6 **a** $242 **b** 17.6% **c** 34.1%

7 **a** $41 800 **b** 12.1% **c** 22.9%

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Exercise 4E–2

1 **a** \$17 280 **b** \$2280 **c** 7.6% to 1 d.p.
d 14.6% to 1 d.p.

2 **a** \$27 000 **b** \$2000 **c** 5.3% to 1 d.p.
d 10.1% to 1 d.p.

3 **a** \$49 296 **b** \$9296 **c** 7.7% to 1 d.p.
d 15.3% to 1 d.p.

Exercise 4E–3

1 **a** \$54.57 **b** \$110.29 **c** \$452.00 **d** \$45.99

2 **a** No interest on either card, choose either
b A: no interest, B: \$20.93, choose A
c A: \$36.48, B: \$52.72, choose A
d A: \$229.19, B: \$219.38, choose B

3 **a** \$877.69 **b** \$962.30

4 \$3060.46 **5** 1485.73

Exercise 4F

1 **a** \$3.59 **b** \$3.72

2 **a** \$851.40 **b** \$896.52

3 **a** \$2.62 **b** \$7.10

4 **a** \$680 359.31 **b** \$1 113 531.40

5 **a** \$148 818.78 **b** \$58 917.67

6 **a** \$598.48 **b** \$263.30 **c** \$69.04

Chapter 4 review

Multiple-choice questions

1 B **2** B **3** A **4** D
5 A **6** C **7** E **8** C
9 E **10** A **11** B **12** D
13 B **14** D **15** B **16** B
17 B **18** A **19** E **20** B
21 D **22** C

Short-answer questions

1 **a** Rabbit Easter Eggs **b** 0.7%

2 \$137.50

3 **a** \$1809.09 **b** \$180.91

4 \$791.89

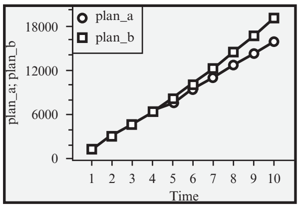
5 **a** \$451.39 **b** \$468.47

6 **a** \$220 **b** 12.9% to 1 d.p.
c 25.0% to 1 d.p.

Extended-response questions

1 **a** \$612.50
b **i** \$437.50 **ii** \$87.50 **iii** \$765.63 **iv** 25%

2 **a & b**



c **i** Plan A **ii** Plan B

3 **a** \$11 000 **b** 9.2%

4 **a** 12.5%
b **i** \$190 **ii** 27.1% **iii** 52.1%
c \$155.20 **d** Credit card

Chapter 5

Exercise 5A

1 **a** 3×4 **b** **i** 16 **ii** 3 **iii** 5
c 22 **d** 18

2 **a** **i** 2×3 **ii** 6, 7
b **i** 1×3 **ii** 2, 6
c **i** 3×2 **ii** −4, 5
d **i** 3×1 **ii** 9, 8
e **i** 2×2 **ii** 15, 12
f **i** 3×4 **ii** 20, 5

3 **a** B **b** D **c** E

4,5 **a** 9 **b** 2 **c** 3 **d** 10
e 8

6 **a** 32 students **b** 3×4
c 22 Year 11 students preferred football.

7 **a** A 4×3, B 2×1, C 1×2, D 2×5
b $a_{32} = 4$, $b_{21} = -5$, $c_{11} = 8$, $d_{24} = 7$

8 **a** **i** 75 ha **ii** 300 ha **iii** 200 ha
b 350 ha
c **i** Farm Y uses 0 ha for cattle.
ii Farm X uses 75 ha for sheep.
iii Farm X uses 150 ha for wheat.
d **i** f_{23} **ii** f_{12} **iii** f_{21}
e 2×3

Exercise 5B

1 **a** **i**

$$\begin{array}{cc} \begin{array}{cc} A & B \end{array} & \\ \left[\begin{array}{cc} 0 & 3 \\ 3 & 0 \end{array}\right] & \begin{array}{c} A \\ B \end{array} \end{array}$$

ii

$$\begin{array}{cc} \begin{array}{ccc} A & B & C \end{array} & \\ \left[\begin{array}{ccc} 0 & 2 & 0 \\ 2 & 0 & 1 \\ 0 & 1 & 0 \end{array}\right] & \begin{array}{c} A \\ B \\ C \end{array} \end{array}$$

iii

$$\begin{array}{cc} \begin{array}{ccc} A & B & C \end{array} & \\ \left[\begin{array}{ccc} 0 & 2 & 0 \\ 2 & 0 & 2 \\ 0 & 2 & 0 \end{array}\right] & \begin{array}{c} A \\ B \\ C \end{array} \end{array}$$

iv

$$\begin{array}{cc} \begin{array}{ccc} A & B & C \end{array} & \\ \left[\begin{array}{ccc} 0 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{array}\right] & \begin{array}{c} A \\ B \\ C \end{array} \end{array}$$

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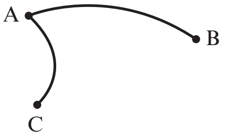
v
$$\begin{array}{cc} \begin{matrix} A & B & C \end{matrix} & \\ \begin{bmatrix} 0 & 2 & 1 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} & \begin{matrix} A \\ B \\ C \end{matrix} \end{array}$$

vi
$$\begin{array}{cc} \begin{matrix} A & B & C \end{matrix} & \\ \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \\ 2 & 2 & 0 \end{bmatrix} & \begin{matrix} A \\ B \\ C \end{matrix} \end{array}$$

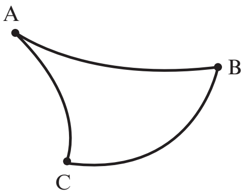
b The number of roads directly connected to B.

2 a Many answers are possible. Examples:

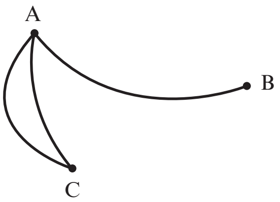
i



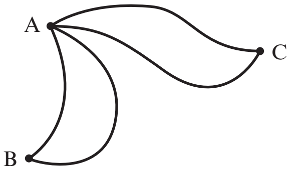
ii



iii



iv



b The number of roads directly connected to town *A*.

3 a
$$\begin{array}{cc} & \begin{matrix} A & B & C & D \end{matrix} \\ \begin{matrix} A \\ B \\ C \\ D \end{matrix} & \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

b Compare the sums of the rows (or columns). The person with the highest total has met the most people.

c Person *B*

d Person *C*

Exercise 5C

1 a $\begin{bmatrix} 9 & 10 \\ 6 & 3 \end{bmatrix}$ **b** $\begin{bmatrix} 7 & 8 \\ 13 & 3 \end{bmatrix}$

c $\begin{bmatrix} 3 & 5 \\ 7 & 0 \end{bmatrix}$ **d** $\begin{bmatrix} 9 \\ 8 \end{bmatrix}$

e $\begin{bmatrix} 3 & 4 \\ 1 & 6 \end{bmatrix}$ **f** $\begin{bmatrix} 4 & -2 \\ 3 & 9 \end{bmatrix}$

g $\begin{bmatrix} 12 & 7 \end{bmatrix}$ **h** $\begin{bmatrix} 0 & 0 \end{bmatrix}$

i $\begin{bmatrix} 0 & 0 \end{bmatrix}$ **j** $\begin{bmatrix} -2 & 2 & 3 & -9 \end{bmatrix}$

2 a $\begin{bmatrix} 8 & 5 \\ 3 & 7 \end{bmatrix}$ **b** $\begin{bmatrix} 8 & 5 \\ 3 & 7 \end{bmatrix}$

c $\begin{bmatrix} -2 & -9 \\ 1 & 1 \end{bmatrix}$ **d** $\begin{bmatrix} 2 & 9 \\ -1 & -1 \end{bmatrix}$

e Not possible **f** $\begin{bmatrix} 3 & 7 \\ 5 & -2 \\ 4 & -1 \end{bmatrix}$

g Not possible **h** $\begin{bmatrix} -9 & 3 \\ 3 & -2 \\ -2 & 15 \end{bmatrix}$

3
$$\begin{array}{cc} & \begin{matrix} \textit{Liberal} & \textit{Labor} & \textit{Democrat} & \textit{Green} \end{matrix} \\ \begin{matrix} \textit{Men} \\ \textit{Women} \end{matrix} & \begin{bmatrix} 43 & 42 & 10 & 5 \\ 37 & 37 & 17 & 9 \end{bmatrix} \end{array}$$

4 a
$$\begin{array}{cc} & \begin{matrix} \textit{Aida} & \textit{Bianca} & \textit{Chloe} & \textit{Donna} \end{matrix} \\ \begin{matrix} \textit{Weight (kg)} \\ \textit{Height (cm)} \end{matrix} & \begin{bmatrix} 6 & 8 & -2 & 7 \\ 5 & 8 & 7 & 6 \end{bmatrix} \end{array}$$

b Bianca

c Bianca

Exercise 5D

1 a $\begin{bmatrix} 14 & -2 \\ 8 & 18 \end{bmatrix}$ **b** $\begin{bmatrix} 0 & -10 \\ 25 & 35 \end{bmatrix}$

c $\begin{bmatrix} -64 & 12 \\ -6 & -14 \end{bmatrix}$ **d** $\begin{bmatrix} 2.25 & 0 \\ -3 & 7.5 \end{bmatrix}$

e $\begin{bmatrix} 18 & 21 \end{bmatrix}$ **f** $\begin{bmatrix} -12 \\ 30 \end{bmatrix}$

g $\begin{bmatrix} 2 & 3 & 0 \\ 0 & 1\frac{1}{2} & \frac{1}{2} \end{bmatrix}$ **h** $\begin{bmatrix} -3 & -6 & 8 \end{bmatrix}$

2 a $\begin{bmatrix} 9 & -12 \\ 6 & 15 \end{bmatrix}$ **b** $\begin{bmatrix} 2 & 28 \\ -6 & -28 \end{bmatrix}$

c $\begin{bmatrix} 1 & -32 \\ 8 & 33 \end{bmatrix}$ **d** $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

e $\begin{bmatrix} 21 & 18 \\ 3 & -12 \end{bmatrix}$

3 a $\begin{bmatrix} 79 & -31 \\ 68 & -36 \end{bmatrix}$ **b** $\begin{bmatrix} -121 & 50 \\ -84 & 103 \end{bmatrix}$

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c $\begin{bmatrix} 13 & -2 \\ 36 & 53 \end{bmatrix}$ **d** $\begin{bmatrix} 69 & -27 \\ 60 & -30 \end{bmatrix}$

4 a $\begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix}$ **b** $\begin{bmatrix} 0 & 5 & 3 & 3 \end{bmatrix}$ **c** $\begin{bmatrix} 6 \\ 14 \\ 8 \end{bmatrix}$

d $\begin{bmatrix} 0 & 15 & 9 & 9 \end{bmatrix}$

5 a

$$\begin{array}{rccc} & \textit{Clothing} & \textit{Furniture} & \textit{Electronics} \\ \textit{Store A} & 6 & 2 & 9 \\ \textit{Store B} & 5 & 1 & 9 \\ \textit{Store C} & 4 & -1 & 5 \end{array}$$

b

$$\begin{array}{rccc} & \textit{Clothing} & \textit{Furniture} & \textit{Electronics} \\ \textit{Store A} & 1.8 & 0.6 & 2.7 \\ \textit{Store B} & 1.5 & 0.3 & 2.7 \\ \textit{Store C} & 1.2 & 0 & 1.5 \end{array}$$

6 a

$$\begin{array}{rc} & \textit{Wins} \\ \textit{Gymnastics rings} & 3 \\ \textit{Parallel bars} & 2 \end{array}$$

b

$$\begin{array}{rc} & \$ \\ \textit{Gymnastics rings} & 150 \\ \textit{Parallel bars} & 100 \end{array}$$

Exercise 5E

1 a Defined, 2 × 1, $\begin{bmatrix} 38 \\ 19 \end{bmatrix}$

b Not defined

c Defined, 3 × 1, $\begin{bmatrix} 17 \\ 32 \\ -10 \end{bmatrix}$

d Not defined

e Defined, 2 × 2, $\begin{bmatrix} 42 & 14 \\ 21 & 7 \end{bmatrix}$

f Not defined **g** Not defined

h Defined, 3 × 2, $\begin{bmatrix} 15 & 5 \\ 24 & 8 \\ -3 & -1 \end{bmatrix}$

2 a 1 × 2 and 2 × 1, [38]

b 1 × 2 and 3 × 1, not defined

c 1 × 3 and 3 × 1, [1]

d 1 × 3 and 2 × 1, not defined

e 1 × 4 and 4 × 1, [2]

f 1 × 4 and 3 × 1, not defined

3 a i $\begin{bmatrix} 6 & 9 \end{bmatrix}$ **ii** $\begin{bmatrix} 10 & 15 \end{bmatrix}$

iii $\begin{bmatrix} 16 & 24 \end{bmatrix}$ **iv** $\begin{bmatrix} 16 & 24 \end{bmatrix}$

b i $\begin{bmatrix} 30 \\ 24 \end{bmatrix}$ **ii** $\begin{bmatrix} 35 \\ 28 \end{bmatrix}$ **iii** $\begin{bmatrix} 35 \\ 28 \end{bmatrix}$

c i $\begin{bmatrix} 4 & 6 \end{bmatrix}$ **ii** $\begin{bmatrix} 15 \\ 12 \end{bmatrix}$ **iii** [22]

iv [132] **v** [132]

4,5 a $\begin{bmatrix} 22 \\ 33 \end{bmatrix}$ **b** $\begin{bmatrix} 64 \\ 53 \end{bmatrix}$ **c** $\begin{bmatrix} 0 & -8 \\ 4 & 2 \end{bmatrix}$

d $\begin{bmatrix} -4 & -3 \\ -14 & -20 \end{bmatrix}$ **e** $\begin{bmatrix} 8 & 3 \\ 5 & 2 \end{bmatrix}$

f $\begin{bmatrix} 16 & 14 \\ 16 & 14 \end{bmatrix}$ **g** $\begin{bmatrix} 31 \\ 35 \\ 21 \end{bmatrix}$ **h** $\begin{bmatrix} 11 \\ 1 \\ 7 \end{bmatrix}$

i $\begin{bmatrix} 83 \end{bmatrix}$ **j** $\begin{bmatrix} 21 \end{bmatrix}$ **k** $\begin{bmatrix} 8 \end{bmatrix}$ **l** $\begin{bmatrix} 4 \end{bmatrix}$

m $\begin{bmatrix} 30 \end{bmatrix}$ **n** $\begin{bmatrix} 36 \end{bmatrix}$ **o** $\begin{bmatrix} 3 & 3 \end{bmatrix}$

6 a $\begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}$ **b** $\begin{bmatrix} 23 & 34 \\ 31 & 46 \end{bmatrix}$

c No

7 a $\begin{bmatrix} 104 & 70 \\ 80 & 54 \end{bmatrix}$ **b** $\begin{bmatrix} 9 & 8 \\ 7 & 6 \end{bmatrix}$

c $\begin{bmatrix} 17 & 17 \\ 13 & 13 \end{bmatrix}$ **d** $\begin{bmatrix} 8 & 6 \\ 4 & 2 \end{bmatrix}$

e $\begin{bmatrix} 14 & 14 \\ 6 & 6 \end{bmatrix}$

8 a $\begin{bmatrix} 376 & 118 & 154 & 420 \\ 643 & 117 & 281 & 523 \end{bmatrix}$ **b** $\begin{bmatrix} 1292 \end{bmatrix}$

c $\begin{bmatrix} -496 & 752 & 976 & -224 \\ -310 & 470 & 610 & -140 \\ -744 & 1128 & 1464 & -336 \\ 558 & -846 & -1098 & 252 \end{bmatrix}$

d $\begin{bmatrix} -131 & -264 & 176 \\ 467 & 62 & 535 \\ 697 & 279 & 406 \end{bmatrix}$

9 a i $\begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix}$ **ii** $\begin{bmatrix} 37 & 54 \\ 81 & 118 \end{bmatrix}$

iii $\begin{bmatrix} 199 & 290 \\ 435 & 634 \end{bmatrix}$

b i $\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$ **ii** $\begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix}$ **iii** $\begin{bmatrix} 8 & 8 \\ 8 & 8 \end{bmatrix}$

c i $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ **ii** $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ **iii** $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

d i $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ **ii** $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ **iii** $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

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e **i** $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ **ii** $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ **iii** $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Exercise 5F

1 5800 kJ

2 $$\begin{array}{ccc} & \textit{Wheels} & \textit{Seats} \\ \textit{Smith} & 14 & 13 \\ \textit{Jones} & 12 & 9 \end{array}$$

3 [110]

4 **a** $$\begin{array}{ccc} \textit{Quiche} & \textit{Soup} & \textit{Coffee} \\ 18 & 12 & 64 \end{array}$$

b $$\begin{array}{cc} & \$ \\ \textit{Quiche} & 5 \\ \textit{Soup} & 8 \\ \textit{Coffee} & 3 \end{array}$$

c $378

5 **a** $$\begin{array}{cccc} \textit{Chips} & \textit{Pastie} & \textit{Pie} & \textit{Sausage roll} \\ 90 & 84 & 112 & 73 \end{array}$$

b $$\begin{array}{cc} & \$ \\ \textit{Chips} & 4 \\ \textit{Pastie} & 5 \\ \textit{Pie} & 5 \\ \textit{Sausage roll} & 3 \end{array}$$

c $1559

6 **a** 1720 **b** $990

7 **a** $$\begin{array}{cc} & \textit{Hrs} \\ I & 10 \\ J & 7 \\ K & 12 \end{array}$$

b $$\begin{array}{cc} & \textit{Av. Hrs} \\ I & 2.5 \\ J & 1.75 \\ K & 3 \end{array}$$

c $$\begin{array}{ccccc} & M & Tu & W & Th \\ \textit{Hrs} & 6 & 11 & 5 & 7 \end{array}$$

d $$\begin{array}{ccccc} & M & Tu & W & Th \\ \textit{Av. Hrs} & 2 & 3.7 & 1.7 & 2.3 \end{array}$$

8 **a** $$\begin{array}{cc} & \textit{Total score} \\ E & 446 \\ F & 415 \\ G & 329 \\ H & 409 \end{array}$$

b $$\begin{array}{cc} & \textit{Av. score} \\ E & 89.2 \\ F & 83 \\ G & 65.8 \\ H & 81.8 \end{array}$$

c $$\begin{array}{cccccc} & T1 & T2 & T3 & T4 & T5 \\ \textit{Test total} & 319 & 307 & 324 & 292 & 357 \end{array}$$

d $$\begin{array}{cccccc} & T1 & T2 & T3 & T4 & T5 \\ \textit{Test av.} & 79.75 & 76.75 & 81 & 73 & 89.25 \end{array}$$

Exercise 5G

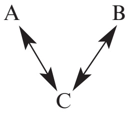
1 D communicates with A, yet A does not communicate with D.

2 **a** $$\begin{array}{cccc} A & B & C & \\ 0 & 1 & 0 & A \\ 1 & 0 & 1 & B \\ 0 & 1 & 0 & C \end{array}$$

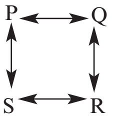
b $$\begin{array}{ccccc} D & E & F & G & \\ 0 & 1 & 1 & 1 & D \\ 1 & 0 & 0 & 0 & E \\ 1 & 0 & 0 & 1 & F \\ 1 & 0 & 1 & 0 & G \end{array}$$

c $$\begin{array}{ccccc} J & K & L & M & \\ 0 & 1 & 1 & 0 & J \\ 1 & 0 & 1 & 1 & K \\ 1 & 1 & 0 & 0 & L \\ 0 & 1 & 0 & 0 & M \end{array}$$

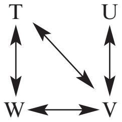
3 **a**



b



c



4 **a** $$Q = \begin{array}{ccccc} C & E & K & R & \\ 0 & 1 & 1 & 1 & C \\ 1 & 0 & 0 & 1 & E \\ 1 & 0 & 0 & 1 & K \\ 1 & 1 & 1 & 0 & R \end{array}$$

b Remy communicates with 3 people.

c **i** $$Q^2 = \begin{array}{ccccc} C & E & K & R & \\ 3 & 1 & 1 & 2 & C \\ 1 & 2 & 2 & 1 & E \\ 1 & 2 & 2 & 1 & K \\ 2 & 1 & 1 & 3 & R \end{array}$$

ii Add column E to get 6 ways.

iii $E \rightarrow R \rightarrow C$
$E \rightarrow R \rightarrow E$
$E \rightarrow C \rightarrow E$
$E \rightarrow R \rightarrow K$
$E \rightarrow C \rightarrow K$
$E \rightarrow C \rightarrow R$

5 **a** $$R = \begin{array}{ccccc} E & F & G & H & \\ 0 & 1 & 1 & 0 & E \\ 1 & 0 & 1 & 1 & F \\ 1 & 1 & 0 & 1 & G \\ 0 & 1 & 1 & 0 & H \end{array}$$

b Three roads directly connected to Fields.

c **i** $$R^2 = \begin{array}{ccccc} E & F & G & H & \\ 2 & 1 & 1 & 2 & E \\ 1 & 3 & 2 & 1 & F \\ 1 & 2 & 3 & 1 & G \\ 2 & 1 & 1 & 2 & H \end{array}$$

ii 7

iii $F \rightarrow G \rightarrow E$
$F \rightarrow E \rightarrow F$
$F \rightarrow G \rightarrow F$
$F \rightarrow H \rightarrow F$
$F \rightarrow E \rightarrow G$
$F \rightarrow H \rightarrow G$
$F \rightarrow G \rightarrow H$

Exercise 5H

1 D

2 a $\begin{bmatrix} 3 & -7 \\ -2 & 5 \end{bmatrix}$ **b** $\begin{bmatrix} 1 & -2 \\ -2 & 4.5 \end{bmatrix}$

c $\begin{bmatrix} -4 & 9 \\ 1 & -2 \end{bmatrix}$ **d** $\begin{bmatrix} -1.5 & 3.5 \\ 1 & -2 \end{bmatrix}$

e $\begin{bmatrix} 0.5 & -0.5 & 0.5 \\ 0.5 & 0.5 & -0.5 \\ -0.5 & 0.5 & 0.5 \end{bmatrix}$

f $\begin{bmatrix} 0.1 & -0.2 & 0.35 \\ 0.4 & 0.2 & -0.6 \\ -0.1 & 0.2 & 0.15 \end{bmatrix}$

g $\begin{bmatrix} 16 & 10 & -15 \\ -8 & -5 & 8 \\ 3 & 2 & -3 \end{bmatrix}$

h $\begin{bmatrix} -0.25 & 0.125 & 0.5 \\ 0 & 0.5 & 0 \\ 0.5 & -0.25 & 0 \end{bmatrix}$

Exercise 5I

1 a $\begin{bmatrix} 23 & 18 & 33 & 77 \\ 17 & 16 & 24 & 50 \end{bmatrix}$ **b** $\begin{bmatrix} 38 & 17 & 51 & 74 \\ 25 & 16 & 35 & 47 \end{bmatrix}$

c $\begin{bmatrix} 8 & 33 & 13 & 59 \\ 7 & 17 & 8 & 32 \end{bmatrix}$ **d** $\begin{bmatrix} 39 & 38 & 43 & 21 \\ 33 & 29 & 29 & 17 \end{bmatrix}$

e $\begin{bmatrix} 55 & 50 & 69 & 28 & 51 & 9 & 44 & 28 \\ 41 & 35 & 42 & 21 & 30 & 8 & 26 & 24 \end{bmatrix}$

f $\begin{bmatrix} 27 & 30 & 37 & 50 & 69 & 10 & 42 & 47 \\ 14 & 25 & 32 & 30 & 42 & 9 & 28 & 33 \end{bmatrix}$

2 a $\begin{bmatrix} A & I & M & \\ H & I & G & H \end{bmatrix}$ **b** $\begin{bmatrix} I & T & S & \\ J & A & N & E \end{bmatrix}$

c $\begin{bmatrix} F & L & Y & \\ H & O & M & E \end{bmatrix}$ **d** $\begin{bmatrix} R & U & N & \\ A & W & A & Y \end{bmatrix}$

e $\begin{bmatrix} D & O & N & T & & G & O & \\ N & O & T & & S & A & F & E \end{bmatrix}$

f $\begin{bmatrix} F & R & E & D & & I & S & \\ N & O & T & & D & E & A & D \end{bmatrix}$

3 a $\begin{bmatrix} 9 & 1 & 9 & 15 & 11 & 6 & 8 & 8 \\ 15 & 1 & 14 & 23 & 20 & 9 & 8 & 15 \end{bmatrix}$

b $\begin{bmatrix} 6 & 1 & 7 & 7 & 4 & 0 & 8 & 3 \\ 1 & 9 & 1 & 6 & 8 & 0 & 4 & 9 \end{bmatrix}$

4 Other student to check

Exercise 5J

1 $x = 2, y = 3$ **2** $x = 1, y = 2$

3 $x = 1, y = -2$ **4** $x = 4, y = 6$

5 $x = 9, y = 2$ **6** $x = -2, y = 3$

Exercise 5K

1 Multiplying by a scalar matrix has the same result as multiplying by a scalar.

2 a $\begin{bmatrix} 52 \\ 64 \\ 44 \end{bmatrix}$ Third quarter costs. **b** $F = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

c 1×4

d Pre-multiply with a 1×3 matrix.

e $\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$

Chapter 5 review

Multiple-choice questions

1 B **2** E **3** C **4** D **5** A
6 D **7** D **8** C **9** A **10** E
11 D **12** B **13** E **14** A **15** D
16 E

Short-answer questions

1 2×4 **2** 1 **3** $\begin{bmatrix} 38 & 34 & 47 & 54 \end{bmatrix}$

4 2×1

5 $\begin{array}{cc} \begin{matrix} P & Q & R \end{matrix} & \\ \begin{bmatrix} 0 & 2 & 0 \\ 2 & 0 & 3 \\ 0 & 3 & 0 \end{bmatrix} & \begin{matrix} P \\ Q \\ R \end{matrix} \end{array}$

6 a $\begin{bmatrix} 9 & 3 \\ 12 & 6 \end{bmatrix}$ **b** $\begin{bmatrix} 3 & 6 \\ 11 & 8 \end{bmatrix}$

c $\begin{bmatrix} -3 & 4 \\ 3 & 4 \end{bmatrix}$ **d** $\begin{bmatrix} 6 & 7 \\ 15 & 10 \end{bmatrix}$

e $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ **f** $\begin{bmatrix} 7 & 21 \\ 14 & 32 \end{bmatrix}$

g $\begin{bmatrix} 20 & 10 \\ 45 & 19 \end{bmatrix}$ **h** $\begin{bmatrix} 1 & -0.5 \\ -2 & 1.5 \end{bmatrix}$

i $\begin{bmatrix} 13 & 5 \\ 20 & 8 \end{bmatrix}$ **j** $\begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix}$

k $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

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Extended-response questions

1 a 40 pigs **b** 320 sheep **c** Farm *A*

2 a 21 pies **b** \$2 **c** $\begin{bmatrix}104\\103\end{bmatrix}$

d Value of sales for each shop

e Shop *A*, \$104

3 a

$$\begin{array}{cc} & \begin{array}{cc} Hours\ walking & Hours\ jogging \end{array} \\ \begin{array}{c} Patsy \\ Geoff \end{array} & \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix} \end{array}$$

b

$$\begin{array}{cc} & \begin{array}{cc} \$ & kJ \end{array} \\ \begin{array}{c} Walking \\ Jogging \end{array} & \begin{bmatrix} 2 & 1500 \\ 3 & 2500 \end{bmatrix} \end{array}$$

c

$$\begin{array}{cc} & \begin{array}{cc} \$ & kJ \end{array} \\ \begin{array}{c} Patsy \\ Geoff \end{array} & \begin{bmatrix} 11 & 8500 \\ 12 & 9500 \end{bmatrix} \end{array}$$

4 a $\begin{bmatrix} 4 & 1 & 8 & 4 & 8 & 5 & 2 & 5 \\ 3 & 1 & 6 & 3 & 5 & 3 & 1 & 4 \end{bmatrix}$

b $\begin{bmatrix} 2 & 3 & 1 & 4 & 0 & 2 & 2 & 1 \\ 1 & 0 & 1 & 1 & 2 & 3 & 0 & 2 \end{bmatrix}$

5 a $6x + 5y = 14$

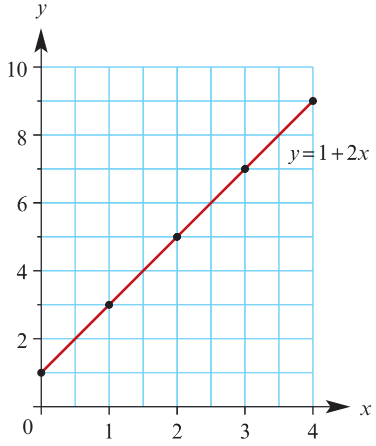
b Apple \$1.50, banana \$1

Chapter 6

Exercise 6A

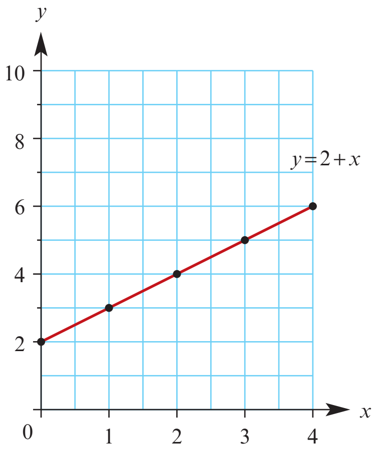
1 a

x	0	1	2	3	4
y	1	3	5	7	9



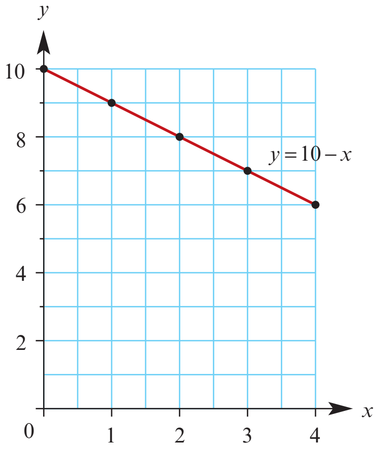
b

x	0	1	2	3	4
y	2	3	4	5	6



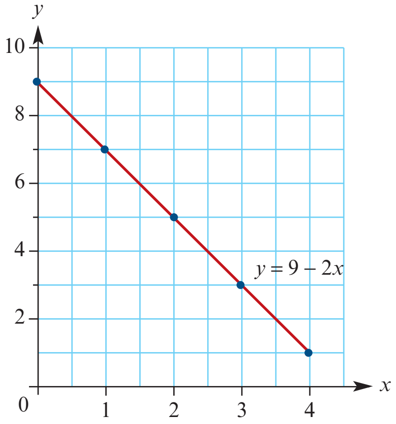
c

x	0	1	2	3	4
y	10	9	8	7	6

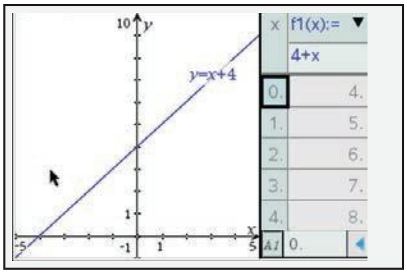


d

x	0	1	2	3	4
y	9	7	5	3	1

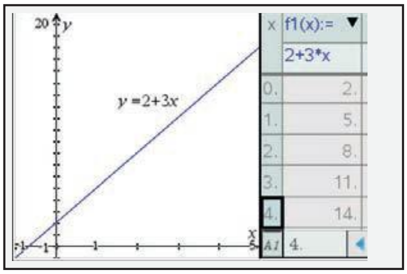


2 a



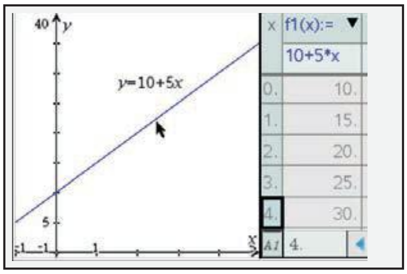
x	f1(x):=
	4+x
0.	4.
1.	5.
2.	6.
3.	7.
4.	8.
A1	0.

b



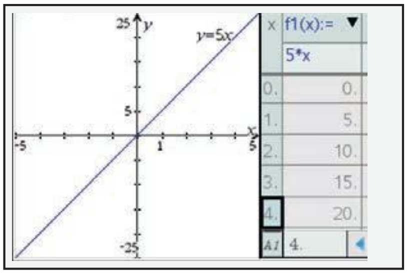
x	f1(x):=
	2+3*x
0.	2.
1.	5.
2.	8.
3.	11.
4.	14.
A1	4.

c



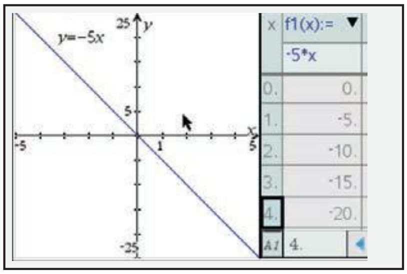
x	f1(x):=
	10+5*x
0.	10.
1.	15.
2.	20.
3.	25.
4.	30.
A1	4.

d



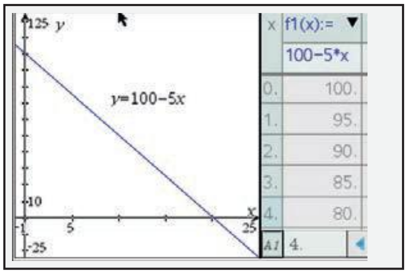
x	f1(x):=
	5*x
0.	0.
1.	5.
2.	10.
3.	15.
4.	20.
A1	4.

e



x	f1(x):=
	-5*x
0.	0.
1.	-5.
2.	-10.
3.	-15.
4.	-20.
A1	4.

f



x	f1(x):=
	100-5*x
0.	100.
1.	95.
2.	90.
3.	85.
4.	80.
A1	4.

3 a (0, 4), (2, 6), (3, 7), (5, 9)
b (0, 8), (1, 6), (2, 4), (3, 2)

Exercise 6B

1 *A* negative, *B* positive, *C* not defined, *D* zero

2 *A* − 2.3, *B* 1.75, *C* 1

3 *A* 2, *B* − 3, *C* 0

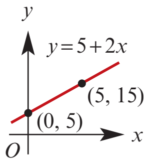
4 a 2 **b** −1 **c** 2 **d** 0.6
e 2 **f** −1

Exercise 6C

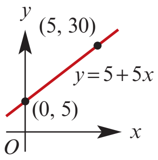
1 a y-intercept = 5, slope = 2
b y-intercept = 6, slope = −3
c y-intercept = 15, slope = −5
d y-intercept = 10, slope = −3
e y-intercept = 0, slope = 3
f y-intercept = −5, slope = −2
g y-intercept = 4, slope = 1
h y-intercept = 3, slope = 0.5
i y-intercept = −5, slope = 2
j y-intercept = 10, slope = 5
k y-intercept = 10, slope = −1
l y-intercept = 0, slope = 2
m y-intercept = 6, slope = −3
n y-intercept = −4, slope = 2
o y-intercept = −3, slope = 0.8
p y-intercept = −2, slope = 3

2 a $y = 2 + 5x$ **b** $y = 5 + 10x$
c $y = -2 + 4x$ **d** $y = 12 - 3x$
e $y = -2 - 5x$ **f** $y = 1.8 - 0.4x$
g $y = 2.9 - 2x$ **h** $y = -1.5 - 0.5x$

3 a



b



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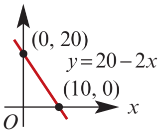
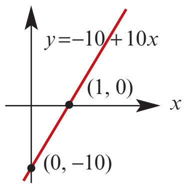
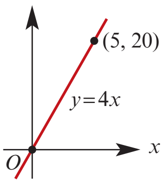
c



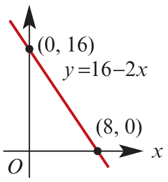
d



e



f



Exercise 6D

1 A: $y = 10 - 2.25x$ B: $y = 2 + 1.75x$
C: $y = x$

2 A: $y = 4 + 2x$ B: $y = 8 - 1.5x$
C: $y = 2 + 0.6x$

Exercise 6E

1 A: $y = 14.5 - 4.5x$ B: $y = -5 + 5x$
C: $y = -5 + 3x$

2 A: $y = 11.5 - 1.5x$ B: $y = -10 + 10x$
C: $y = 2 + 1.2x$

Exercise 6F

1 Same as Exercise 6D, Question 1

2 Same as Exercise 6D, Question 2

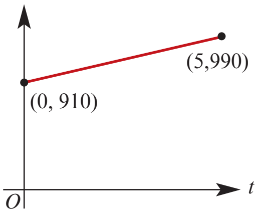
3 Same as Exercise 6E, Question 1

4 Same as Exercise 6E, Question 2

Exercise 6G-1

1 a $h = 910 + 16t$ for $0 \le t \le 5$

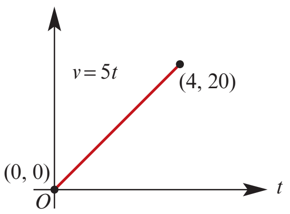
b



c 982 cm

2 a $V = 0 + 5t$ for $0 \le t \le 4$

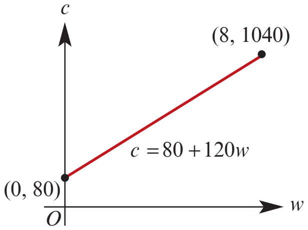
b



c 16 litres

3 a $c = 80 + 120w$

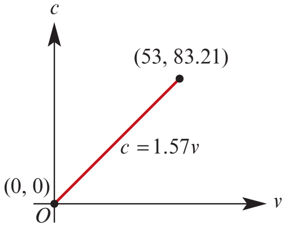
b



c \$680

4 a $c = 1.57v$, for $0 \le v \le 53$

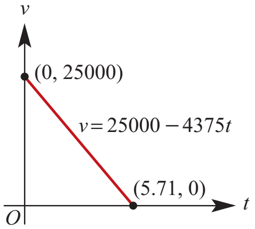
b



c \$83.21

5 a $c = 25\,000 - 4375t$, for $0 \le t \le 5$

b

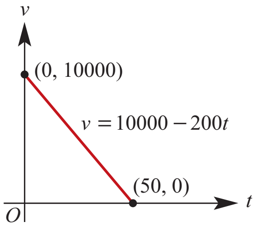


c \$13 625

6 a 50 days

b $V = 10\,000 - 200t$, for $0 \le t \le 50$

c



d 4000 litres

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Exercise 6G-2

1 **a** \$10 **b** \$17.50 **c** $C = 10 + 0.075n$
d \$32.50 **e** \$0.075 (7.5 cents)

2 **a** 500 mL **b** Slightly above 400 mL
c 200 minutes **d** $V = 500 - 2.5t$
e 212.5 mL **f** 2.5 mL/min

3 **a** $F = 32 + 1.8C$ (or as more commonly written: $F = \frac{9}{5}C + 32$)
b **i** 122°F **ii** 302°F **iii** −40°F
c 1.8

Exercise 6G-3

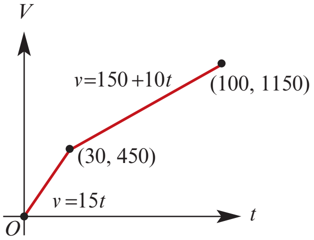
1 **a** $T = 40 + 0.08d$
b **i** 240 minutes **ii** 400 minutes
c **i** Interpolating; prediction made within data
ii Extrapolating; prediction made outside data
d 8 minutes

2 **a** $I = 160 - 1.5F$
b **i** 100 per 100 000 **ii** 70 per 100 000
iii 17.5 per 100 000
c Interpolating; all predictions made within data
d 1.5

3 **a** $H = 50 + 2.7B$
b **i** 158 mg **ii** 104 mg
c **i** Interpolating; prediction made within data
ii Extrapolating; prediction made outside data
d 2.7 mg

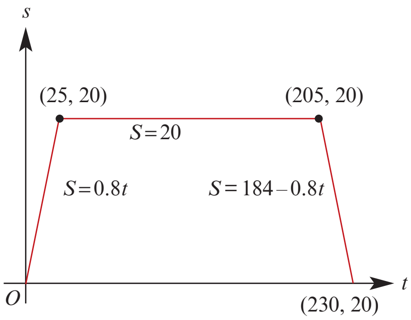
Exercise 6G-4

1 **a** **i** 300 L **ii** 450 L **iii** 750 L
iv 1150 L
b



2 **a** **i** 8 m/s **ii** 20 m/s **iii** 20 m/s
iv 16 m/s

b



Chapter 6 review

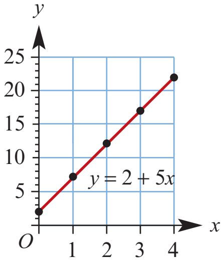
Multiple-choice questions

1 E	**2** D	**3** A	**4** D	**5** C
6 C	**7** E	**8** C	**9** B	**10** E
11 C	**12** B	**13** B	**14** A	**15** C

Short-answer questions

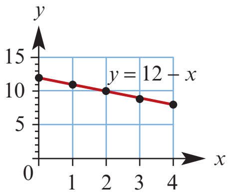
1 **a**

x	0	1	2	3	4
y	2	7	12	17	22



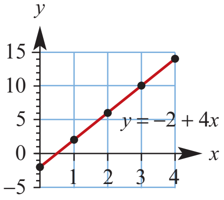
b

x	0	1	2	3	4
y	12	11	10	9	8



c

x	0	1	2	3	4
y	−2	2	6	10	14



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2 A −1.2, B 0.6

3 A 2.25, B −2.67

4 **a** \$755 **b** \$110

Extended-response questions

1 **a** \$200 000

b After 60 months (5 years)

c $V = 300 - 5t$ **d** \$120 000

e \$5000

2 **a** \$80 billion

b $A = 0.16N$ (with A in billions, N in thousands)

c \$96 billion **d** \$240 billion

e \$0.16 billion

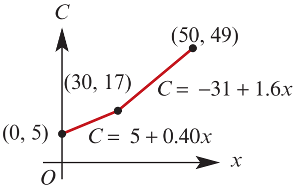
3 **a** $H = 80 + 6.25A$ **b** 98.75 cm

c 6.25

4 **a** **i** \$13 **ii** \$17 **iii** \$49

b **i** \$0.40 (40 cents) **ii** \$1.60

c



Chapter 7

Exercise 7A

1 **a** EV: *age* RV: *diameter*

b EV: *weeks* RV: *weight loss*

c EV: *age* RV: *price*

d EV: *hours* RV: *amount*

e EV: *balls bowled* RV: *runs*

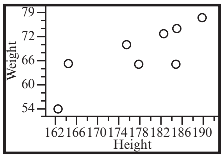
Exercise 7B

1 D

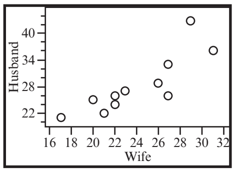
2 **a** EV: *hours of sleep*, RV: *mistakes*

b 12 people **c** 9 hours, 6 mistakes

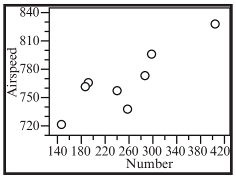
3



4

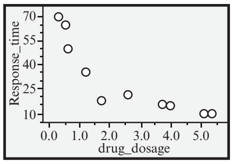


5



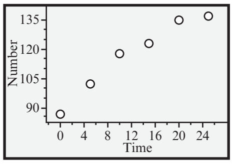
6 **a** *drug dosage*

b



7 **a** *time*

b



Exercise 7C

1 **a** Positively associated

b Positively associated

c Negatively associated

d No association

e Positively associated

f Positively associated

2 Note: Estimates of strength can vary by one level and still be appropriate

a Strong positive linear association

b Strong negative linear association

c Moderate positive linear association

d Strong positive non-linear association

e Moderate negative linear association

f Strong positive linear association with possible outlier

Exercise 7D

1 Numerical data, linear association

2 **a** 0.9 **b** 0.7 **c** −0.6

d −0.1 (estimates could vary by ± 0.2)

3 **a** None **b** Weak negative

c Strong negative **d** Weak positive

e Strong positive **f** Moderate negative

g Moderate positive **h** None

i Weak negative **j** Weak positive

k Perfect positive or strong positive

l Perfect negative or strong negative

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4 A likely non-causal explanation is that older children are better at mathematics, and also taller. This would be an example of a common response.

5 Since the number of bars and the number of school teachers will both increase with the size of the city, then size of the city is likely to explain this correlation. This would be an example of a common response.

6 No; possible confounding variables include the general wealth of a country, level of post-natal care, and female literary levels.

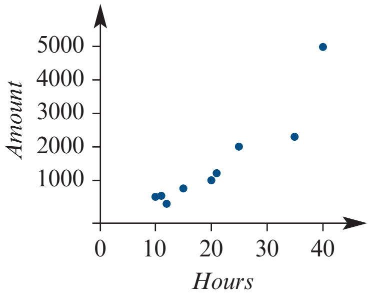
Exercise 7E

1 $r = 0.570$ **2** $r = 0.722$ **3** $r = -0.396$

4 **a** EV: *hours* or the number of hours spent gambling

RV: *amount* or amount spent gambling

b

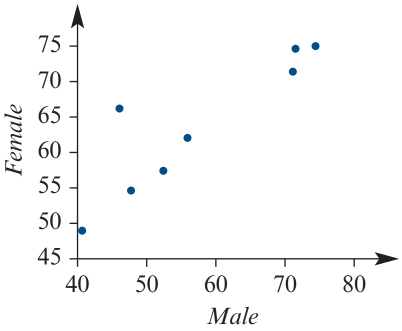


c $r = 0.922$

d Strong positive linear association: Those who gambled for longer tended to spend more on gambling.

5 **a** Either variable could be the EV.

b

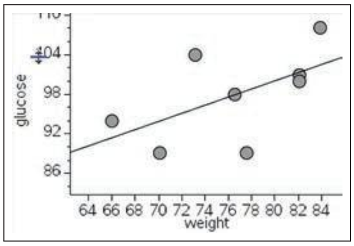


c $r = 0.894$

d Strong positive linear association with an outlier: Those countries with high percentages of males with eye disease also tended to have a high percentage of females with eye disease.

Exercise 7F

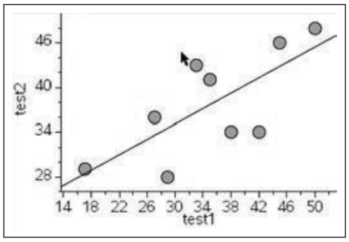
1 **a & b**



c $glucose = 50.8 + 0.616 \times weight$

d $r = 0.570$

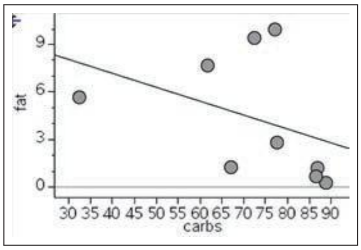
2 **a & b**



c $test\ 2 = 19.6 + 0.515 \times test\ 1$

d $r = 0.722$

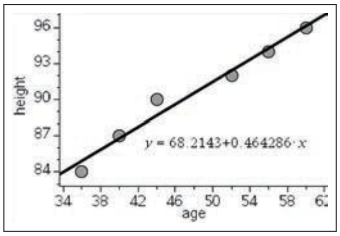
3 **a & b**



c $fat = 10.7 - 0.088 \times carbs$

d $r = -0.396$

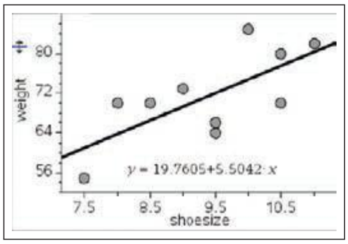
4 **a & b**



c $height = 68.2 + 0.464 \times age$

d $r = 0.985$

5 **a & b**



c $weight = 19.8 + 5.50 \times shoe\ size$

d $r = 0.702$

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Exercise 7G

1 **a** Interpolation **b** Extrapolation

2 **a** 88 cm, interpolation
b 94 cm, interpolation
c 100 cm, extrapolation

3 **a** 59 kg, extrapolation **b** 66 kg, interpolation
c 72 kg, interpolation

4 **a** $175, extrapolation **b** $523, interpolation
c $691, interpolation

5 **a** 171 cm, interpolation
b 197 cm, extrapolation
c 159 cm, interpolation

Exercise 7H

1 **a** **i** 37 650
ii The price of a new car of this type is predicted to be $37 650.
b **i** −4200
ii The slope predicts that the price of second hand car of this type decreases by $4200 each year.
c $16 650

2 **a** **i** 40
ii The intercept predicts that the flavour rating of yoghurt with zero fat content fat is 40.
b **i** 2.0
ii On average, the flavour rating of a yoghurt increases by 2.0 for each 1% increase in fat content.
c 100

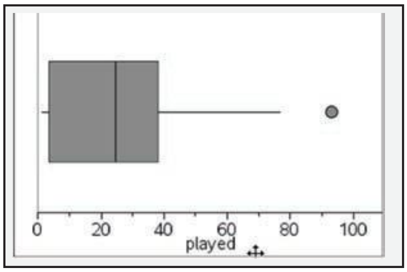
3 **a** **i** 15.7
ii The intercept predicts that students obtaining a zero mark on exam 1 obtained a mark of 15.7 for exam 2 .
b **i** 0.65
ii The slope predicts that the exam 2 marks increase by 0.65 marks for each additional mark obtained on exam 1.
c 29

4 **a** **i** 51
ii The intercept does not have a meaningful interpretation in this example.
b **i** 0.62
ii On average, the blood glucose level of a healthy adult increases by 0.62 mg/100 mL for each kilogram increase in weight.
c 97.5 mg/100 mL

Exercise 7I

1 **a** Tests played key: 1|7 = 17

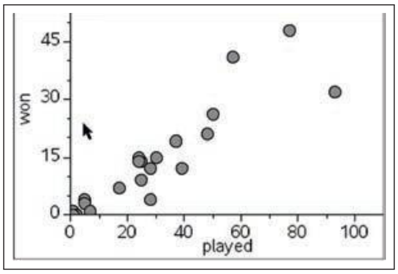
0	1 1 1 1 2 2 5 5 7
1	7
2	4 4 5 5 8 8
3	0 7 9
4	8
5	0 7
6	7
7	7
8	
9	



b The distribution is positively skewed with an outlier. The median number of tests played by these captains is $M = 24.5$. This means that 50% of the captains played 24 or less tests. The number of tests captained is quite variable with 50% captaining between 4 tests ($Q_1 = 3.5$) and 38 tests ($Q_3 = 38$). There is one outlier, Allan Border who captained 93 tests.

c Allan Border

2 **a**



The estimated value of the correlation coefficient r is (*your estimate*)

b There is a strong positive linear association between the number of tests played as

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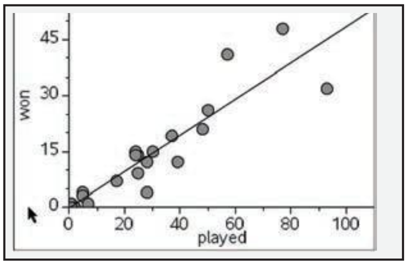
captain and the number of matches won. There are no clear outliers.

c



d The equation of the least squares line is: *won* = −0.25 + 0.49 × *played* (to 2 sig. figs.) slope = 0.49: on average, the number of tests won by a captain increased by 0.49 for each additional match played as captain. The predicted number of matches won by the captain who played 5 matches is 2.2. The actual number of matches won by the captain who played in 5 matches was four matches, so the error of prediction was 1.8 matches (4 − 2.2 = 1.8)

e *r* = 0.9 (to 1 d.p.) – use this answer to check your estimate in **a**.

Chapter 7 review

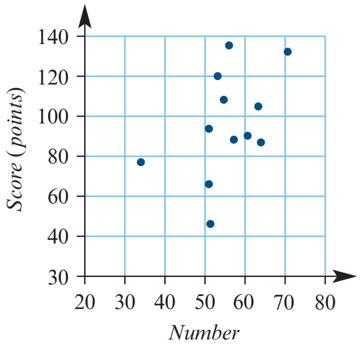
Multiple-choice questions

1 D **2** B **3** D **4** B **5** C
6 E **7** D **8** E **9** D **10** A
11 B **12** E **13** C **14** C **15** B

Short-answer questions

1 a *score*

b



c Weak linear positive association

2 a *r* = 0.9271

b *distance* = 1.6 − 0.12 × *time*

3 a *magnesium content*

b On average, taste scores increases by 7.3 points for each one mg/litre increase in magnesium content.

c 94.8 milligrams/litre

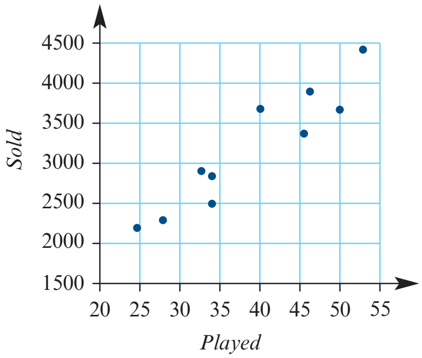
4 a *errors* = 15 − 0.53 × *time*

b *r* = −0.94

Extended-response questions

1 a EV**:** *played*
RV**:** *weekly sales*

b



c *r* = 0.9458

d Strong, positive, linear relationship

e *weekly sales* = 293 + 74.3 × *played*

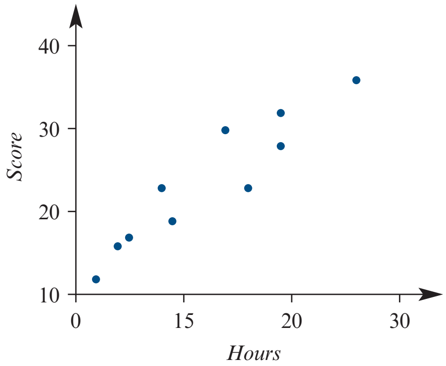
f Slope: on average, the number of down loads increases by 74.3 for each additional time the song is played on the radio in the previous week.
Intercept: predicts 293 downloads of the song if it is not played on radio in the previous week.

g 7723

h Extrapolating

2 a EV: *hours* RV: score

b



c *r* = 0.9375

d Positive strong, linear

e *score* = 12.3 + 0.930 × *hours*

f Slope: on average, test scores of learner drivers increases by 0.93 marks when instruction time increases by one hour.
Intercept: on average, test scores of learner drivers who received no instruction prior to taking the test is 12.3 marks.

g 22

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3 a The statement is questionable because it implies causality. The existence of even a strong relationship between two variables is not, by itself, sufficient information to conclude that one variable causes a change in the other.

b *exam mark* = 25 + 0.70 × *assignment mark* (correct to 2 sig. figures)

c Intercept: on average those who score 0 on the assignment scored 25 marks on the final exam (or equivalent).

Slope: on average, exam marks increase by 0.7 for each additional mark obtained on the assignment.

d 60

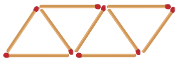
e Reliable: The prediction made in part **d** falls well within the range of the data (interpolating).

Chapter 8

Exercise 8A

1 a 19 **b** 6 **c** 64 **d** March
e 1 **f** 3 **g** 3 **h** 125
i i **j** ♣ **k** 81 **l** ⇒
m 162 **n** ↑ **o** F (week days)
p F (counting words)
q I (letters written using straight lines)

2 a



b 3, 5, 7, . . .

c 9, 11

d Start at 3 and add 2 to make each new term.

3 a

b

Pattern number	1st	2nd	3rd	4th	5th
Number of dots	1	4	7	10	13

c Start with 1 and add 3 to make each new term.

4 a Add 3; 17, 20 **b** Add 9; 55, 64
c Subtract 4; 22, 18 **d** Subtract 8; 34, 26
e Multiply by 2; 48, 96
f Multiply by 3; 324, 972
g Divide by 2; 8, 4
h Multiply by −2; 48, −96
i Add the previous two terms; 8, 13

5 a 6 **b** 16 **c** 11 **d** 21
e 26 **f** 31

6 a i 6 **ii** 18 **iii** 22
b i 2 **ii** 128 **iii** 512
c i 29 **ii** 8 **iii** 1
d i 96 **ii** 12 **iii** 6

Exercise 8B

1 a 3 **c** −4 **d** −3 **f** 0
b, e, Are not arithmetic

2 a 3, 5, 7, . . . **b** 2 **c** 9, 11

3 a 7, 12, 17, . . . **b** Yes, $d = 5$
c 22, 27 matchsticks

4 a $d = 6, t_5 = 29$ **b** $d = -4, t_5 = 1$
c $d = 4, t_5 = 27$ **d** $d = -4, t_5 = -8$
e $d = -5, t_5 = 15$ **f** $d = 0.5, t_5 = 3.5$

5 a 41, 47 **b** 2, −1 **c** 0, −0.5 **d** 59, 67
e −15, −27 **f** 2, 2.3

6 a 28, 33 **b** −10, −16 **c** 24, 33
d 13, 8 **e** 11, 19 **f** 13, 21 **g** 29, 18
h 29, 15 **i** 23, 39, 55

7 a 3, 8, 13, 18, 23
b 16, 9, 2, −5, −12
c 1.6, 3.9, 6.2, 8.5, 10.8
d 8.7, 5.6, 2.5, −0.6, −3.7
e 293, 226, 159, 92, 25

8 a 1, 6, 11, 16, 21, 26; $t_6 = 26$
b 45, 43, 41, 39, 37, 35, 33, 31, 29, 27, 25, 23; $t_{12} = 23$
c 15, 14, 13, 12, 11, 10, 9, 8, 7, 6; $t_{10} = 6$
d 0, 3, 6, 9, 12, 15, 18, 21, 24, 27, 30, 33, 36, 39, 42; $t_{15} = 42$

9 a 5
b i 31 **ii** 51

10 After 9 weeks

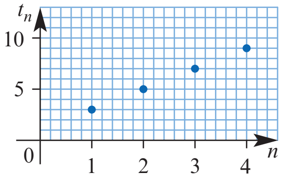
11 a 100, 107, 114, 121 **b** 177

12 a i 9

ii

n	1	2	3	4
t_n	3	5	7	9

iii



iv Points lie on a line with positive slope.

b i 2

ii

n	1	2	3	4
t_n	11	8	5	2

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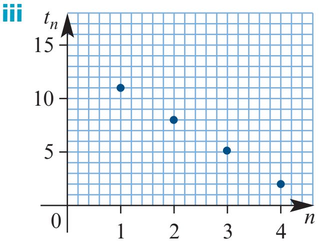
iii



iv Points lie on a line with negative slope.

Exercise 8C

1 a $a = 7, d = 4$ **b** $a = 8, d = -3$
c $a = 14, d = 9$ **d** $a = 62, d = -27$
e $a = -9, d = 5$ **f** $a = -13, d = -6$

2 a 120 **b** 306 **c** −441 **d** 436
e −119 **f** 25.3 **g** −198 **h** −7.4

3 323 **4** 1908

5 −33 **6** $123

7 15.25 m

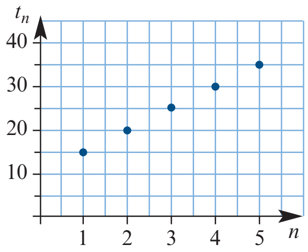
8 68.4, 68.1, 67.8, 67.5 seconds

9 $43

10 a 10 logs **b** 61 logs

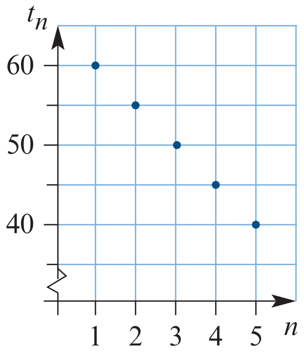
Exercise 8D

1 a 15, 20, 25, 30, 35



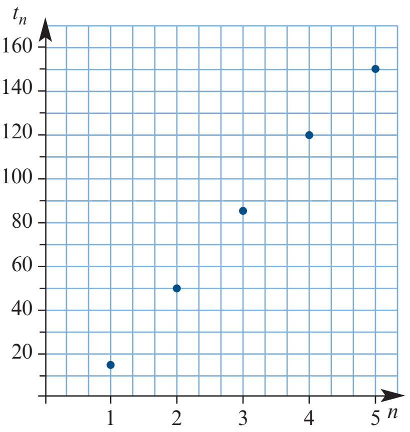
b 235

2 a 60, 55, 50, 45, 40



b 15

3 a 15, 50, 85, 120, 155



b 505

4 a 7, 11, 15 **b** 7, 11, 15, 19, 23
c 43

5 a 16, 22, 28 **b** 16, 22, 28, 34
c Yes **d** 70

Exercise 8E

1 a Yes, 2 **b** Yes, 3 **c** No **d** Yes, 3
e Yes, $\frac{1}{2}$ **f** No **g** No **h** Yes, $\frac{1}{3}$
i Yes, 2

2 a 2 **b** $\frac{1}{4}$ **c** 5 **d** 4
e $\frac{1}{2}$ **f** 6 **g** 10 **h** 7
i 8

3 a 56, 112 **b** 375, 1875 **c** 36, 108
d 5, 10 **e** 8, 512 **f** 9, 81

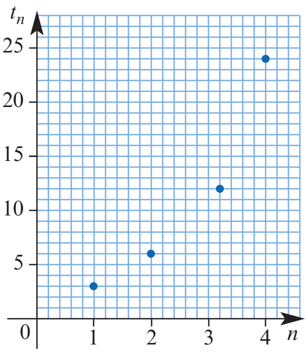
4 a 21 875 **b** 23 328 **c** 3
d 67 228 **e** 5 **f** 649 539

5 a i 24

ii

n	1	2	3	4
t_n	3	6	12	24

iii



iv The graph is a curve with increasing values.

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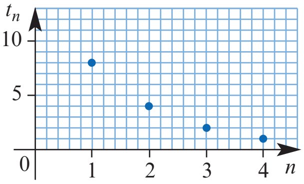
b **i** 1

ii

n	1	2	3	4
t_n	8	4	2	1

iii



iv The graph is a curve with values decreasing and approaching zero.

Exercise 8F

1 **a** $a = 12, r = 2$ **b** $a = 6, r = 3$
c $a = 2, r = 4$ **d** $a = 56, r = \frac{1}{2}$
e $a = 36, r = \frac{1}{3}$ **f** $a = 8, r = 7$
g $a = 1, r = 10$ **h** $a = 100, r = 0.1$
i $a = 17, r = 13$

2 **a** 78 732 **b** 1536 **c** 39 366
d 262 144 **e** 196 830 **f** 1

3 **a** $a = 3, r = 4$ **b** $a = 5, r = 2$
c $a = 8, r = 7$ **d** $a = 200, r = 1.10$
e $a = 600, 0.90$ **f** $a = 3, r = 2$

4 **a** 1 **b** 1152 **c** 524 288
d 531 441 **e** $\frac{1}{3}$ **f** 0.0001

5 3072

6 3172.62

7 5.73

8 **a** $t_n = 9 \times 2^{n-1}$ **b** $t_n = 54 \times (\frac{1}{3})^{n-1}$
c $t_n = 4 \times 5^{n-1}$ **d** $t_n = 6 \times 7^{n-1}$
e $t_n = 5 \times 4^{n-1}$ **f** $t_n = 8 \times 3^{n-1}$

9 1 cm, 3 cm, 9 cm, 27 cm

10 **a** \$50, \$100, \$200 **b** \$350

11 **a** 5, 25, 125, 625 letters
b 780 letters

12 **a** 100, 95, 90.25, 85.74
b 100, 120, 144, 172.8
c 5000, 5150, 5304.50, 5463.64
d 7000, 6720, 6451.20, 6193.15

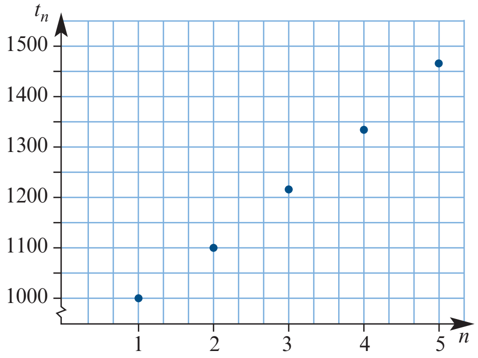
13 \$4267.25

14 \$95 212.46

15 **a** $a = 2000, r = 0.90$ **b** $t_n = 2000 \times 0.90^{n-1}$
c 774 people **d** 14 years

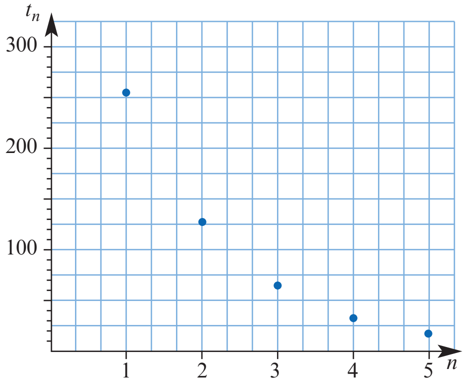
Exercise 8G

1 **a** 1000, 1100, 1210, 1331, 1464.1



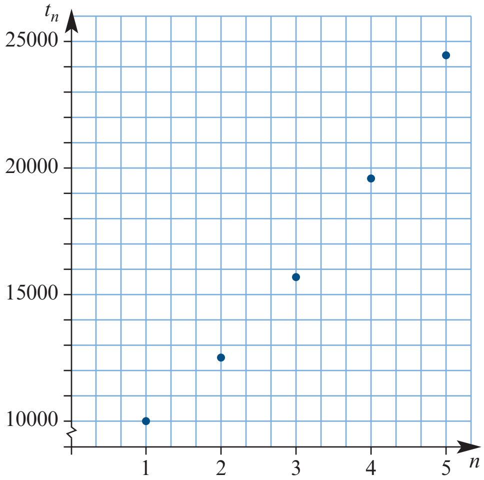
b 3138.43

2 **a** 256, 128, 64, 32, 16



b 0.5

3 **a** 10 000, 12 500, 15 625, 19 531, 24 414



b 2 117 582 to nearest whole number

4 **a** 2, 4, 8, 16 **b** See **a** **c** 1024
d See **c**

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Exercise 8H

1 a 10 450, 10 900, 11 350
  b 10 years

2 a 8400, 8800, 9200 **b** 10 years
  c 8000 **d** 5% per year

3 a $63 000 **b** $45 000 **c** 10 years
  d $V_0 = 95\,000, V_{n+1} = V_n - 11\,400$

4 a $2400 **b** $300 **c** 12.5%
  d 6 years **e** 4 years

5 a i 800 **ii** 44 000 **iii** 10
  b i $500 **ii** $2000
    iii $V_n = 0$ after 8 years

6 a 10 450, 10 920.25, 11 411.66
  b 5 years

7 a 8000 **b** 4 years **c** 7.5%

8 a 65 025 **b** 17 718.7

9 a $1200 **b** 4 years **c** 2 years
  d 44%

10 a $50 260.46
  b $1853

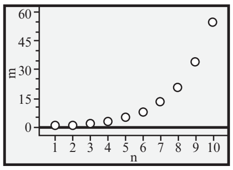
Exercise 8I

1 Second order

2 a 34 **b** 13 **c** 21 **d** 55

3 a 199 **b** 76 **c** 123 **d** 322

4 a 1, 1, 2, 3, 5, 8, 13, 21, 34, 55
  b



5 1, 3, 4, 7, 11, 18, 29, 47, 76, 123

6 a 3, 5, 8, 13, 21 **b** Yes

Chapter 8 review

Multiple-choice questions

1 B **2** C **3** C **4** A **5** B
6 C **7** A **8** C **9** C **10** E
11 D **12** B **13** A

Short-answer questions

1 83

2 45, 75, 105, 135 seconds

3 1536

4 48, 24, 12, 6 metres

5 112

6 659.08

7 a B **b** D **c** A **d** C

Extended-response questions

1 a 37 squares **b** 21st

2 a $a = 300, r = 2$ **b** 38 400 bacteria
  c Day 12

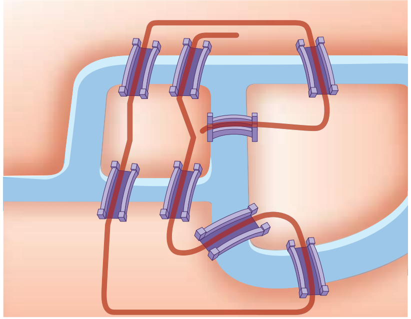
3 a $a = 65\,000, r = 1.09$
  b $514 220.41 **c** 6th year

4 a $\frac{2}{3}$ **b** 144 cm **c** 1170 cm

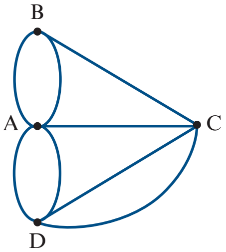
Chapter 9

Exercise 9A

1 a i Two solutions exist. One example is:



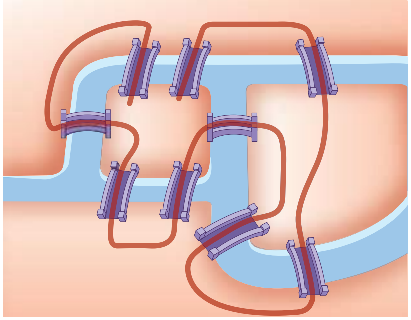
  ii



  iii See graph above – the odd degree vertices are *B* and *A*

  iv No answers – exploration

  b i Many solutions exist. One example is:



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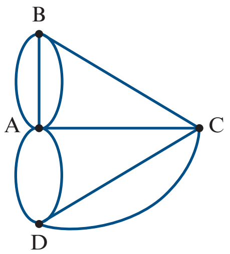
ii



iii See graph above

iv No answers – exploration

Exercise 9B

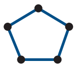
1 a **i** 5 **ii** 6 **iii** 0 **iv** 2
v 3 **vi** 4 **vii** 1

b **i** 4 **ii** 7 **iii** 1 **iv** 6
v 2 **vi** 2 **vii** 2

c **i** 4 **ii** 7 **iii** 0 **iv** 3
v 4 **vi** 2 **vii** 2

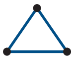
d **i** 8 **ii** 14 **iii** 2 **iv** 5
v 3 **vi** 8 **vii** 0

2 a 10;



many graphs are possible

b 6;



many graphs are possible

c 2;

many graphs are possible

3 Because each edge must start and end at a vertex. It is a bit like shaking hands, there must be two hands at the end of each shake, even if you are shaking hands with yourself (a loop).

4 a Increase by two **b** Increase by one

5 Game of sprouts; an activity with no answers

Exercise 9C-1

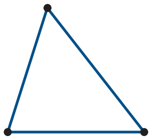
1 a Graphs 1 and 3 **b** Graphs 2 and 3
c Graphs 1 and 2 **d** Graphs 1 and 3
e Graphs 1 and 3

Exercise 9C-2

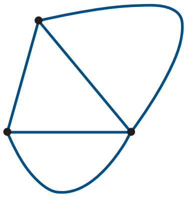
1 A, D, F

2 Many graphs are possible.
Examples include:

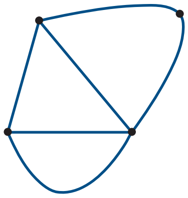
a



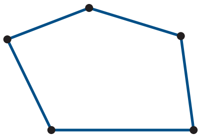
b



c

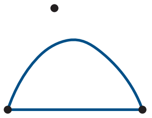


d

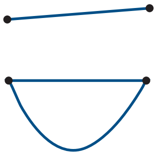


3 Many graphs are possible.
Examples include:

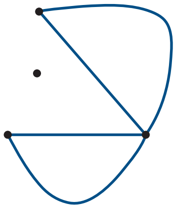
a



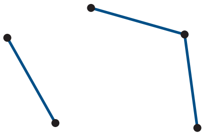
b



c



d



4 3 edges – try it and see

5 a *BD* **b** *CB* and *AB*
c *XW* and *WV*

6

Exercise 9C-3

1 a

$$\begin{array}{cc} & \begin{array}{ccc} A & B & C \end{array} \\ \begin{array}{c} A \\ B \\ C \end{array} & \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \end{array}$$

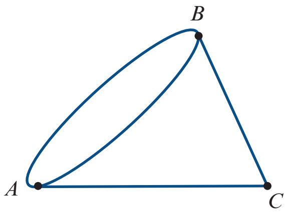
b

$$\begin{array}{cc} & \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \end{array} & \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

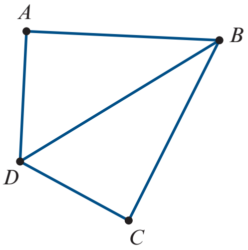
c

$$\begin{array}{cc} & \begin{array}{ccc} X & Y & Z \end{array} \\ \begin{array}{c} X \\ Y \\ Z \end{array} & \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \\ 2 & 1 & 0 \end{bmatrix} \end{array}$$

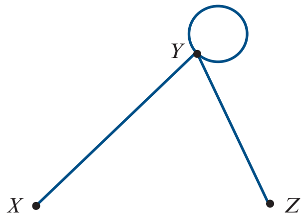
2 a



b



c

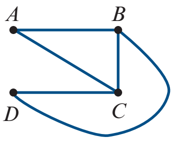


Exercise 9D

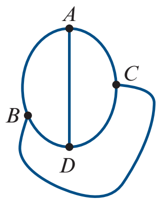
1 A, B, D, F

2 Many solutions are possible.
Examples include:

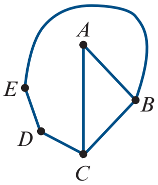
a



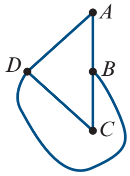
b



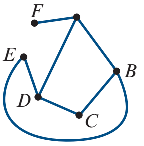
c



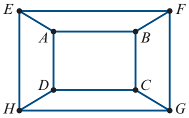
d



e



f



3 **a** **i** $v = 4, e = 4, f = 2$
ii $v - e + f = 4 - 4 + 2 = 2$
b **i** $v = 7, e = 9, f = 4$
ii $v - e + f = 7 - 9 + 4 = 2$
c **i** $v = 7, e = 12, f = 7$
ii $v - e + f = 7 - 12 + 7 = 2$
d **i** $v = 7, e = 10, f = 5$
ii $v - e + f = 7 - 10 + 5 = 2$

4 **a** $f = 2$ **b** $v = 3$ **c** $e = 4$ **d** $v = 4$
e $f = 4$ **f** $f = 7$ **g** $e = 19$

5 Graph 1: $v = 4, e = 6, f = 4;$
$v - e + f = 4 - 6 + 4 = 2$
Graph 2: $v = 8, e = 12, f = 6;$
$v - e + f = 8 - 12 + 6 = 2$
Graph 3: $v = 6, e = 12, f = 8;$
$v - e + f = 6 - 12 + 8 = 2$
Graph 4: $v = 20, e = 30, f = 12;$
$v - e + f = 20 - 30 + 12 = 2$
Graph 5: $v = 12, e = 30, f = 20;$
$v - e + f = 12 - 30 + 20 = 2$

Exercise 9E

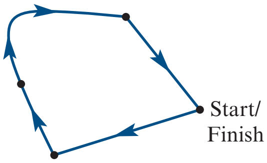
1 **a** ii **b** i & ii **c** ii **d** i & iii

Exercise 9F

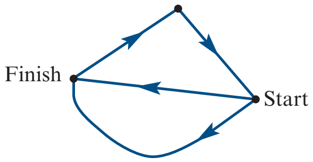
Other trails, circuits or cycles are possible in each case.

1 Not traversable; more than two vertices odd

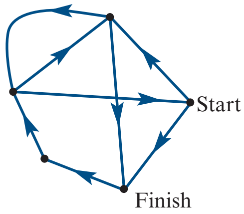
2 Traversable; all vertices even.



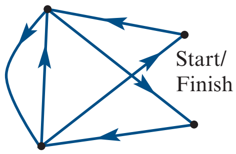
3 Traversable; two vertices odd, the other even



4 Traversable; two vertices odd, the rest even

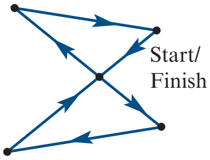


5 Traversable; all vertices even

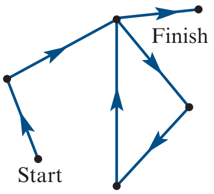


6 Not traversable; more than two vertices odd

7 Traversable; all vertices even



8 Traversable; two vertices odd, the rest even

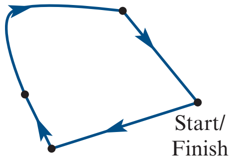


9 Not traversable; more than two vertices odd

Exercise 9G

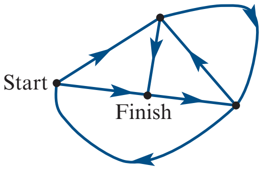
1 Other trails are possible in each case.

a Eulerian circuit: all even vertices

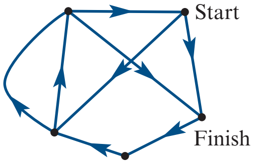


b Neither: more than two odd vertices

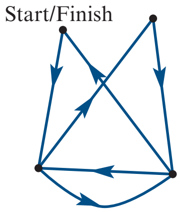
c Eulerian trail: two odd vertices, rest even



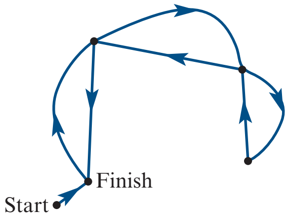
d Eulerian trail: two odd vertices, the rest even



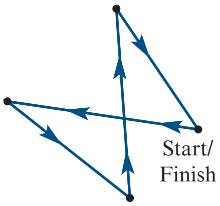
e Eulerian circuit: all even vertices



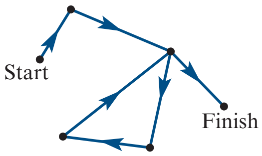
f Eulerian trail: two odd vertices, the rest even



g Eulerian circuit: all even vertices



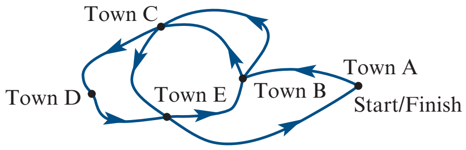
h Eulerian trail: two odd vertices, the rest even



i Neither: more than two odd vertices

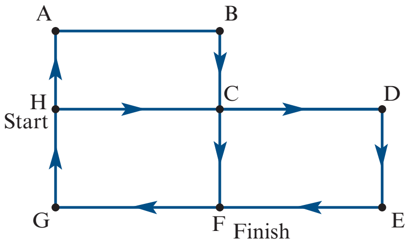
2 a Yes, all vertices even.

b Other routes are possible.

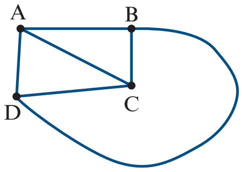


3 a No, not all even vertices

b Several routes are possible. One is shown below.

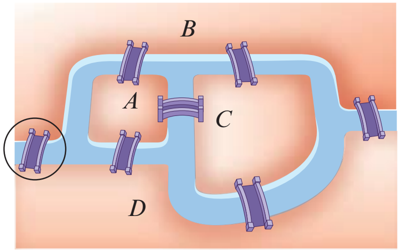


4 a

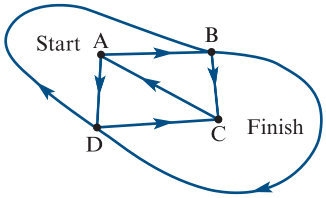


b An eulerian trail does not exist. The graph has more than two odd vertices.

c i One possible solution circled is shown.



ii



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iii The bridges can now be crossed only once in a single walk because an eulerian trail now exists. The graph has two odd vertices and the rest are even. See the graph above for a possible route.

5 a Yes; the graph has an eulerian circuit because it has only even vertices.

b *K-M-G-D-E-G-K-E-S-K*

Exercise 9H

1 Other answers are possible.

a *A-F-G-B-C-H-E-D* **b** *F-A-B-C-D-E-H-G*

2 Other answers are possible.

a *A-B-C-D-E-F-A* **b** *A-B-C-D-E-A*

c *A-F-E-D-C-B-G-A*

d *A-B-C-F-I-H-E-G-D-A*

e No hamiltonian cycle exists.

f *A-E-F-G-H-D-C-B-A*

3 a No **b** Yes: *C-D-E-B-A*, hamiltonian path

c Yes: *E-A-B-C-D-E*, hamiltonian cycle

4 a Yes: *K-M-T-L-S-E-D-G-K*, hamiltonian cycle

b Yes: *D-E-S-L-T-M-G-K*, hamiltonian path

Exercise 9I

1 *A-C-D-E*; 11 hours

2 *A-B-D*; 35 m

3 *B-A-D*; $6

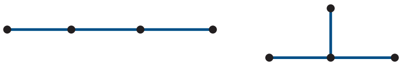
4 *B-G-A-F*; 7 minutes

5 *A-B-C-E-F-D-A;* 63 minutes

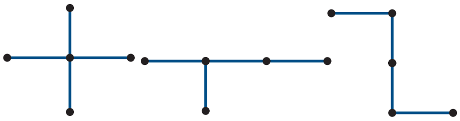
Exercise 9J

1 a 14 **b** 6

c



d

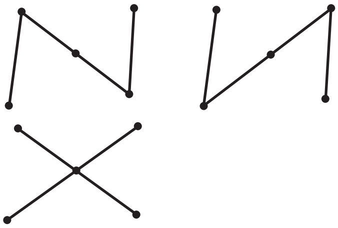


2 8 vertices, 7 edges

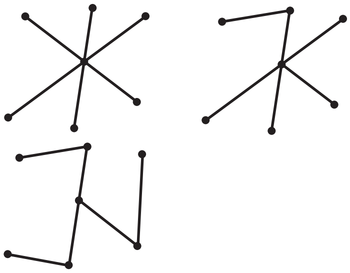
3 A, B, D

4 Other answers are possible.

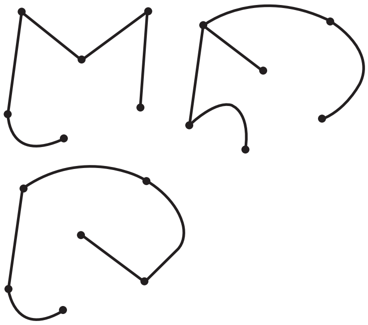
a



b

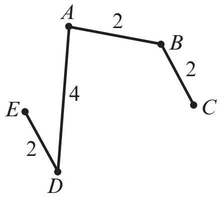


c



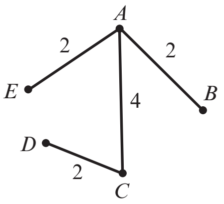
5

a



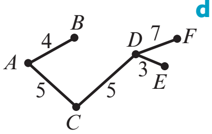
Length = 18

b



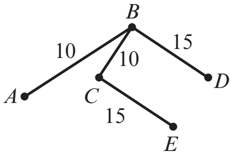
Length = 11

c



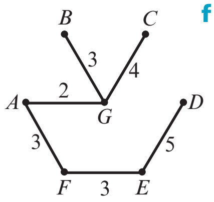
Length = 22

d



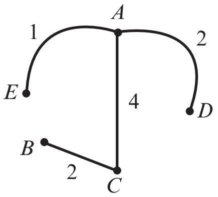
Length = 60

e



Length = 20

f



Length = 9

6 Length = 44 m

7 Length = 94 km

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Exercise 9K

1 a 2600 m

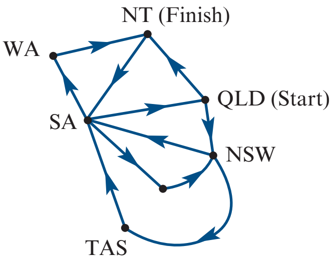
b i No: not all vertices are of even degree

ii Yes: H

iii 11 000 m: *G-F-K-L-I-J-E-D-C-B-A-F-B-G-H-I-D-C-H-G*

c 5600 m

2 a



b Queensland and NT: QLD-NSW-TAS-SA-WA-NT-SA-VIC-NSW-SA-QLD-NT (Other sequences are possible.)

Chapter 9 review

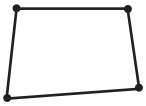
Multiple-choice questions

1 C **2** E **3** D **4** B **5** B
6 D **7** A **8** A **9** D **10** B
11 D **12** E **13** B **14** B **15** D
16 C **17** B **18** B **19** C **20** E
21 B **22** B **23** C **24** A **25** B
26 C **27** B **28** D **29** E **30** D

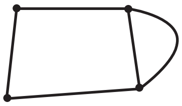
Short-answer questions

1 Many answers are possible. Examples:

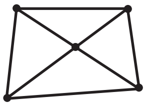
a



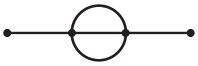
b



c

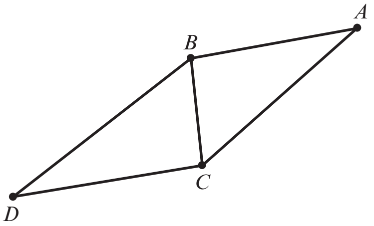


d

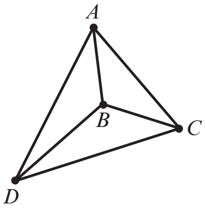


2 Other answers are possible in each case.

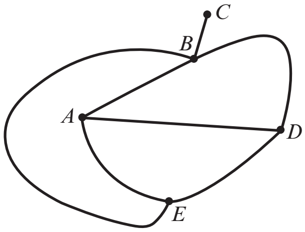
a



b



c



3 a deg(C) = 3

b 2 odd, 2 even

c Other answers are possible, ending at C and tracing each edge once only.
Example: *B-A-C-B-D-C*

4

$$\begin{array}{c|cccc|} & A & B & C & D \\ A & 0 & 1 & 1 & 1 \\ B & 1 & 0 & 1 & 1 \\ C & 1 & 1 & 0 & 1 \\ D & 1 & 1 & 1 & 0 \end{array}$$

5 a deg(C) = 4

b No odd vertices, five even vertices

c Other answers are possible, ending at A and tracing each edge once only.
Example: *A-B-C-D-E-B-A-E-C-D-A*

6 24

7 a 11 km **b** 17 km

Extended-response questions

1 a No edges intersect, except at vertices.

b $v = 9$, $e = 14$, $f = 7$; $9 - 14 + 7 = 2$

c 750 m

d No odd, 9 even

e i Yes, all vertices are even.

ii Many answers are possible. Example: *P-C1-C8-C2-C1-C4-C2-C3-C4-C5-C7-C8-C6-C5-P*

f 1270 m

g i hamiltonian cycle

ii *C7* Park Office

iii *P-C1-C2-C3-C4-C5-C6-C8-C7-P*, or the same route in reverse

2 a 135 km (there are two shortest paths).

b $v = 8$, $e = 12$, $f = 6 : 8 - 12 + 6 = 2$

c i This network does not have an eulerian circuit as it contains two odd vertices

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ii Dimboola, 556 km
iii H-S-M-H-W-Don-M-W-Dim-H-Nat-Nhill-Dim
d 241 km
e The Dimboola/Horsham road

Chapter 10

Exercise 10A

1 **a** 4.9 cm **b** 83.1 cm **c** 24 mm
d 2.4 mm **e** 15.8 mm **f** 7.4 cm
g 6.4 cm **h** 141.4 mm **i** 15.4 m
2 2.9 m **3** 3.8 m **4** 5.3 m
5 48.88 km **6** 15 km **7** 12.81 km
8 20 cm **9** 9.4 m **10** 61.717 m
11 4.24 cm **12** 103 m

Exercise 10B

1 **a** 4.243 cm **b** 5.20 cm
2 **a** 10.77 cm **b** 11.87 cm
c 6.40 cm
3 **a** 27.73 mm **b** 104.79 mm
4 9.54 cm
5 **a** **i** 8.5 cm **ii** 9.1 cm
b **i** 10.6 cm **ii** 3.8 cm
6 17 cm **7** 13 cm **8** 25 cm
9 Yes it will fit **10** 8.02 m **11** 17.55 m

Exercise 10C

1 **a** **i** 60 cm **ii** 225 cm^2
b **i** 22.4 cm **ii** 26.1 cm^2
c **i** 312 cm **ii** 4056 cm^2
d **i** 44 cm **ii** 75 cm^2
2 **a** 56.2 m^2 **b** 16.7 m^2
c 103.6 cm^2 **d** 73.8 cm^2
e 28 cm^2 **f** 35.9 cm^2
g 29.9 m^2 **h** 31.3 m^2
3 100 m^2 **4** 63 375 m^2
5 40 tiles **6** 4 L
7 **a** 5 cm^2 **b** 125 cm^2 **c** 40 cm^2
8 30.88 m^2
9 **a** 252 m^2 **b** 273 m^2

Exercise 10D

1 **a** **i** 31.4 cm **ii** 78.5 cm^2
b **i** 53.4 cm **ii** 227.0 cm^2
c **i** 49.6 mm **ii** 196.1 mm^2
d **i** 1.3 m **ii** 0.1 m^2
2 **a** **i** 25.71 cm **ii** 39.27 cm^2
b **i** 1061.98 mm **ii** 14 167.88 mm^2
c **i** 203.54 cm **ii** 2551.76 cm^2
d **i** 53.70 mm **ii** 150.80 mm^2
3 62.83 cm^2
4 **a** 343.1 cm^2 **b** 34.9 m^2
c 19.2 cm^2 **d** 177 377.4 mm^2
5 **a** 1051.33 m **b** 37 026.55 m^2
6 **a** 6 m **b** 3.4 m^2
7 1060 cm^2 **8** 30.91 m^2
9 8.73 cm **10** 8.19 m

Exercise 10E

1 **a** 125 cm^3 **b** 49 067.8 cm^3
c 3685.5 cm^3 **d** 3182.6 mm^3
e 29 250 cm^3 **f** 0.3 m^3
g 6756.2 cm^3 **h** 47.8 m^3
2 424 cm^3 **3** 516 cm^3 **4** 24 L
5 **a** 20 319.82 cm^3 **b** 20 L
6 228 cm^3

Exercise 10F

1 **a** 9500.18 cm^3 **b** 16.36 m^3
c 59.69 m^3 **d** 2356.19 mm^3
2 **a** 153.94 cm^3 **b** 705.84 m^3
c 102.98 cm^3 **d** 1482.53 cm^3
3 393 cm^3 **4** 7.87 m^3
5 0.02 L **6** 18 263.13 cm^3
7 782 mL **8** 2791 m^3

Exercise 10G

1 **a** 26.67 cm^3 **b** 420 m^3
c 24 m^3 **d** 68.64 cm^3
2 213.333 cm^3 **3** 1 694 000 m^3
4 **a** 335.6 cm^3 **b** 66.6 cm^3
5 3937.5 cm^3

Exercise 10H

1 **a** 523.60 mm^3 **b** 229.85 mm^3
c 7238.23 cm^3
2 **a** 179.59 cm^3 **b** 11 494.04 cm^3
c 33.51 cm^3
3 **a** 8578.64 cm^3 **b** 7679.12 cm^3
c 261.80 cm^3 **d** 4.09 m^3
4 44 899 mm^3 **5** 14 L

Exercise 10I

1 **a** 1180 cm^2 **b** 40 m^2
c 383.3 cm^2 **d** 531 cm^2
e 2107.8 cm^2 **f** 176.1 m^2

2 **a** 3053.63 cm^2 **b** 431.97 cm^2
c 277.59 m^2 **d** 7.37 m^2
e 242.53 cm^2 **f** 24.63 m^2
g 235.62 m^2 **h** 139.02 m^2

3 15 394 cm^2

4 **a** 1.08 m^2 **b** 6 m

5 0.23 m^2

Exercise 10J

1 **a** **i** $\frac{3}{1}$ or $k = 3$ **ii** $\frac{9}{1}$ or $k^2 = 9$
b **i** $\frac{2}{1}$ or $k = 2$ **ii** $\frac{4}{1}$ or $k^2 = 4$

2 **a** Similar, $\frac{3}{1}$ or $k = 3$ **b** Similar, $\frac{2}{1}$ or $k = 2$
c Not similar

3 **a** Not similar **b** Similar, $\frac{4}{1}$ or $k = 4$
c Not similar **d** Similar $\frac{1}{3}$ or $k = \frac{1}{3}$
e Similar $\frac{3}{2}$ or $k = \frac{3}{2}$

4 $\frac{4}{1}$

5 **a** 3 cm **b** $\frac{9}{1}$ **6** 112 cm^2

7 864 cm^2 **8** 1.67

9 **a** 36 km **b** 3 cm **10** 14.4 cm

Exercise 10K

1 **a** SSS **b** AA **c** SAS or SSS or AA

2 **a** $x = 27$ cm, $y = 30$ cm
b $x = 26$ m, $y = 24$ m

3 **a** 28 cm, 35 cm **b** 119 cm

4 **a** AA **b** $\frac{1}{2}$ **c** 2 m

5 1.8 m **6** 72 cm^2 **7** 29.4 cm^2

Exercise 10L

1 27 times **2** **a** $\frac{4}{1}$ **b** $\frac{64}{1}$

3 $\frac{27}{1}$ **4** **a** 9 cm **b** $\frac{125}{1}$

5 **a** Scaled up **b** 27
c 3240 cm^3

6 **a** 6 cm **b** 27: 64

7 **a** 3 cm **b** Height = 12 cm, base = 16 cm

8 **a** 1 : 4 **b** 1 : 8

Exercise 10M

1 **a** 1.23 m^3 **b** 56 boards **c** 15 m^2
d 0.45 m^3 **e** 3.84 m^2

2 **a** 1.21 ha **b** 413.32 m

3 24 hectares

4 **a** 29.7 m^2 **b** 0.39 cm^2

5 **a** 14 cm **b** 56 cm, 44 cm

6 **a** 28.27 cm^2 **b** 4523.9 cm^3
c 4.524 L **d** 2827.4 cm^2

7 **a** 5 times **b** 32 768 000 cm^3
c 614 400 cm^2 **d** 3 times

Chapter 10 review

Multiple-choice questions

1 B	**2** D	**3** D	**4** B	**5** A
6 B	**7** C	**8** B	**9** D	**10** C
11 E	**12** D	**13** C	**14** D	**15** C
16 E				

Short-answer questions

1 **a** 58 cm **b** 30 m

2 36 m **3** 68 cm

4 **a** 9.22 cm **b** 9 cm

5 **a** 140 cm^2 **b** 185 cm^2

6 37.5 cm^2

7 **a** 31.42 cm **b** 75.40 cm

8 **a** 78.54 cm^2 **b** 452.39 cm^2

9 **a** 373.85 cm^2 **b** 2.97 m^2
c 0.52 litres

10 31 809 litres

11 **a** 514 718 540 km^2 **b** 1.098×10^{12} km^3

12 **a** 376.99 cm^3 **b** 377 mL

13 6.4 m

14 **a** 30 m^2 **b** 15 m^2 **c** 5.83 m
d 69.97 m^2 **e** 421.94 m^2

15 33.32 m^3

16 4 **17** $\frac{1}{4}$

18 **a** 50.27 cm **b** 146.12 cm^2

19 both equal 25.13 m

Extended-response questions

1 **a** 154.30 m^2 **b** 101.70 m

2 **a** 61.54 m **b** 140 m^2 **c** 120 m^3 **d** 128 m^2

3 13.33 m

4 **a** 15.07 m **b** 1.89 m^3

5 **a** $\frac{1.96}{1}$ or 1 : 1.96 **b** $\frac{2.744}{1}$ or 1 : 2.744
c 63 cm^3

6 2048 cm^3 **7** 18.71 cm

8 **a** 400 m
b 400 m, 406 m, 412 m, 419 m, 425 m, 431m

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c Each starting point should be 6 m apart except for distance between 3rd and 4th runners which is 7 m.

Chapter 11

Exercise 11A

1 Answers are in order: hypotenuse, opposite, adjacent.
a 13, 5, 12 **b** 10, 6, 8
c 17, 8, 15 **d** 25, 24, 7
e 10, 8, 6 **f** 13, 12, 5

2 Answers are in order: sin θ, cos θ, tan θ.
a $\frac{5}{13}, \frac{12}{13}, \frac{5}{12}$ **b** $\frac{3}{5}, \frac{4}{5}, \frac{3}{4}$
c $\frac{8}{17}, \frac{15}{17}, \frac{8}{15}$ **d** $\frac{24}{25}, \frac{7}{25}, \frac{24}{7}$
e $\frac{4}{5}, \frac{3}{5}, \frac{4}{3}$ **f** $\frac{12}{13}, \frac{5}{13}, \frac{12}{5}$

3 **a** 0.4540 **b** 0.7314 **c** 1.8807 **d** 0.1908
e 0.2493 **f** 0.9877 **g** 0.9563 **h** 1.1106
i 0.9848 **j** 0.7638 **k** 5.7894 **l** 0.0750

Exercise 11B

1 **a** sin θ, 20.74 **b** cos θ, 20.76
c tan θ, 32.15 **d** cos θ, 8.24
e tan θ, 26.63 **f** sin θ, 7.55
g sin θ, 17.92 **h** tan θ, 15.59
i cos θ, 74.00 **j** tan θ, 17.44
k sin θ, 32.72 **l** sin θ, 37.28

2 **a** 78.05 **b** 25.67 **c** 8.58 **d** 54.99
e 21.32 **f** 11.59 **g** 30.67 **h** 25.38
i 63.00 **j** 62.13 **k** 4.41 **l** 15.59

3 **a** 12.8 **b** 28.3 **c** 38.5 **d** 79.4
e 16.2 **f** 15.0 **g** 14.8 **h** 37.7
i 59.6

Exercise 11C

1 **a** 28.8° **b** 51.1° **c** 40.9° **d** 30.0°
e 45.0° **f** 45.0° **g** 60.0° **h** 68.2°
i 33.0° **j** 73.0° **k** 17.0° **l** 30.0°
m 45.0° **n** 26.6° **o** 30.0° **p** 70.0°

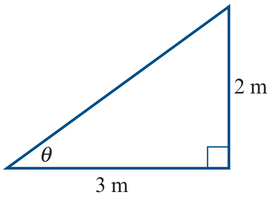
2 **a** 32.2° **b** 59.3° **c** 28.3° **d** 55.8°
e 46.5° **f** 48.6° **g** 53.1° **h** 58.8°
i 22.6° **j** 53.1° **k** 46.3° **l** 22.6°
m 32.2° **n** 41.2° **o** 48.2°

3 **a** 36.9° **b** 67.4° **c** 53.1° **d** 67.4°
e 28.1° **f** 43.6°

Exercise 11D

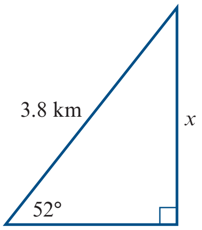
1 6.43 m **2** 21.0° **3** 10 m **4** 16 m

5 **a**



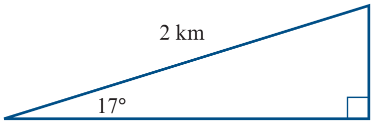
b 33.7°

6 **a**



b 3.0 km

7 **a**



b **i** Horizontal distance 1.91 km
ii Height 0.58 km

8 70.5° **9** 78.1 m **10** 5.77 m

Exercise 11E

1 413 m **2** 11 196 m **3** 33 m
4 164.8 m **5** 244 m **6** 14°
7 **a** 44.6 m **b** 36°
8 **a** 16.2 m **b** 62°
9 **a** 35 m **b** 64 m **c** 29 m
10 507 m

Exercise 11F

1 **a** 025° **b** 110° **c** 210° **d** 280°
2 **a** 25° **b** 7.61 km
3 **a** 236° **b** 056°
4 130°
5 **a** 4.2 km **b** 230°
6 **a** 10 km, 15 km **b** 5 km
c 8.7 km **d** 10 km
e 319°, 13.2 km
7 **a** 12.9 km **b** 15.3 km **c** 17.1 km
d 42° **e** 138°, 23.0 km

Exercise 11G

1 **a** $a = 15, b = 14, c = 13$
b $a = 19, b = 18, c = 21$
c $a = 31, b = 34, c = 48$

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2 **a** $C = 50^\circ$ **b** $A = 40^\circ$ **c** $B = 105^\circ$

3 **a** 5.94 **b** 12.08 **c** 45.11 **d** 86.8°
e 44.4° **f** 23.9°

4 **a** 41.0° **b** 53.7° **c** 47.2° **d** 50.3°

5 **a** 19.60 **b** 30.71 **c** 55.38 **d** 67.67

6 **a** 4.45 **b** 16.06 **c** 67.94 **d** 67.84

7 **a** $c = 10.15$, $B = 50.2^\circ$, $C = 21.8^\circ$
b $b = 7.63$, $B = 20.3^\circ$, $C = 39.7^\circ$
c $a = 52.22$, $c = 61.01$, $C = 37^\circ$
d $b = 34.65$, $c = 34.23$, $C = 54^\circ$

8 39.09 **9** 43.2° **10** 49.69

11 $a = 31.19$, $b = 36.56$, $A = 47^\circ$

12 $A = 27.4^\circ$, $C = 22.6^\circ$, $c = 50.24$

13 $a = 154.54$, $b = 100.87$, $C = 20^\circ$

14 **a** 66.60° **b** 66.60° **c** 113.40°
d 66.60°, 113.40°

15 61.04°, 118.96°

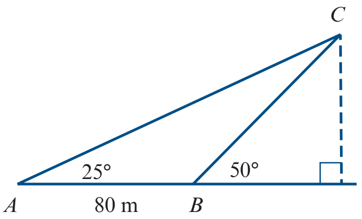
16 2.66 km from A, 5.24 km from B

17 409.81 m

18 **a** 26.93 km from naval ship, 20.37 km from other ship
b 1.36 h (1 h 22 min)

19 **a** Airport A **b** 90.44 km
c Yes

20 **a**



b 130° **c** 25° **d** 145.01 m
e 61.28 m

Exercise 11H

1 **a** 36.72 **b** 47.62 **c** 12.00 **d** 14.55
e 29.95 **f** 11.39

2 17.41 **3** 27.09 **4** 51.51

5 **a** 33.6° **b** 88.0° **c** 110.7° **d** 91.8°
e 88.3° **f** 117.3°

6 50.5° **7** 63.2° **8** 40.9°

9 $B = 46.6^\circ$ **10** $B = 73.2^\circ$ **11** 33.6°

12 19.1 km **13** **a** 39.6° **b** 310°

14 **a** 60° **b** 42.51 km **15** 5.26 km

16 11.73 km **17** 4.63 km **18** 45.83 m

Exercise 11I

1 **a** 102 cm^2 **b** 40 cm^2 **c** 24 cm^2
d 52 cm^2 **e** 17.5 cm^2 **f** 6 cm^2

2 **a** 25.7 cm^2 **b** 65.0 cm^2
c 26.0 cm^2 **d** 32.9 cm^2
e 130.5 cm^2 **f** 10.8 cm^2

3 **a** 36.0 km^2 **b** 9.8 m^2
c 23.5 cm^2 **d** 165.5 km^2
e 25.5 cm^2 **f** 27.7 cm^2

4 **a** iv **b** iii **c** i **d** ii

5 **a** 10 cm^2 **b** 23.8 cm^2 **c** 63.5 cm^2
d 47.3 m^2 **e** 30 m^2 **f** 30.1 m^2
g 100.9 km^2 **h** 21.2 km^2 **i** 6 km^2

6 224 cm^2

7 1124.8 cm^2

8 150.4 km^2

9 3500 cm^2

10 **a** 6 m^2 **b** 4.9 m^2 **c** 6.9 m^2

11 **a** 33.83 km^2 **b** 19.97 km^2
c 53.80 km^2

12 **a** 43.30 cm^2 **b** 259.81 cm^2

13 **a** **i** 12 km^2 **ii** 39 km^2 **iii** 21 km^2
b 29.6°

Exercise 11J

a **i** 106.5 km **ii** 177.9 km
b No **c** 1 h 34 min **d** \$351
e **i** 067° T **ii** 228° T **iii** 024° T
f Fly 177.9 km on a bearing of 228° T
g 19° **h** 84.6 km (using 19°)
i 3084 square km

Chapter 11 review

Multiple-choice questions

1 D **2** C **3** B **4** E
5 B **6** A **7** A **8** C
9 D **10** B **11** D **12** B
13 E **14** B **15** D **16** B
17 D **18** C **19** E

Short-answer questions

1 35.87 cm **2** 117.79 cm **3** 4°
4 **a** 65, 72, 97 **b** $\frac{65}{97}$
5 14.02 cm **6** 76.3° **7** $A = 40.7^\circ$
8 54.17 km **9** 760.7 cm^2 **10** 27.7 m^2

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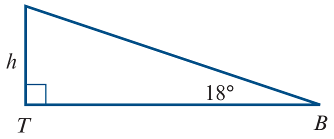
Extended-response questions

1 a 50.95 m **b** 112.23 m

c



Height of tree = 36.47 m

2 a 50°

b First group 3.68 km, second group 3.39 km

c 290°

3 a 110° **b** 81.26 km

4 a 44.4°, 57.1°, 78.5° **b** 14.70 m^2

c $426.30

5 a 24 000 m^2 **b** 48 000 kg

c $744 000 000

Chapter 12

Exercise 12A

1 a $d > 10$ **b** $d < 5$ **c** $d \le 2$ **d** $d \ge 7$

2 a $t < 2$ **b** $t > 3$ **c** $t \le 5$ **d** $t < 1$

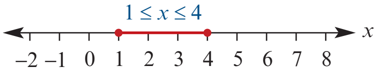
e $t \le 1.5$

Exercise 12B

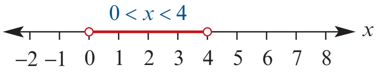
1 a < **b** > **c** = **d** <

e > **f** < **g** < **h** <

2 a

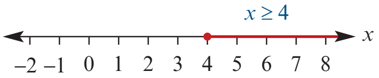


b

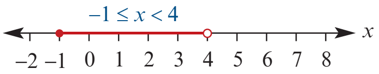


c

d



e



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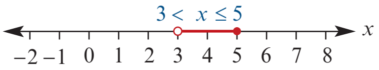
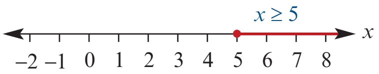


3 a $x \le 7$ **b** $x \ge 1$ **c** $0 < x \le 6$

d $3 < x < 6$ **e** $-1 \le x < 4$

4 a



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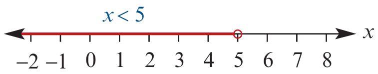

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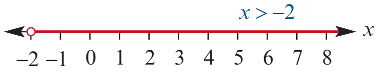

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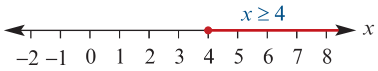

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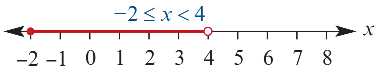

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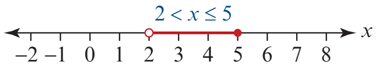
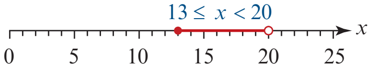


5 a $13 \le x < 20$

b

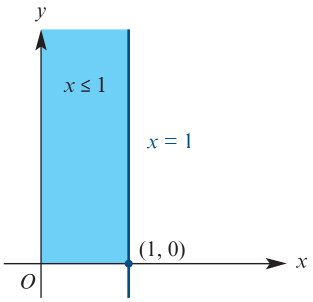


6 a $0 \le x \le 7$ ($x = 0$ allows for a person with no hand luggage.)

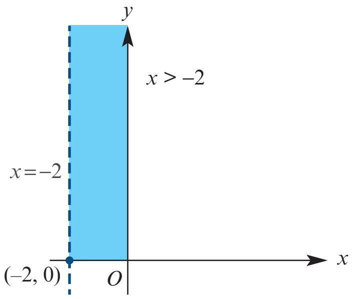
b

Exercise 12C

1 a



b



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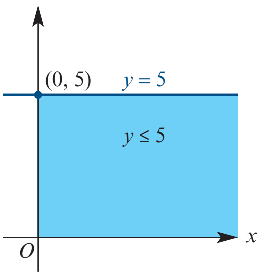
c



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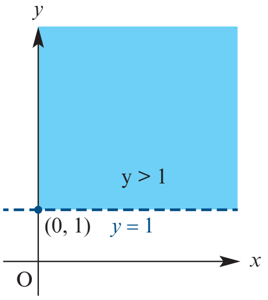

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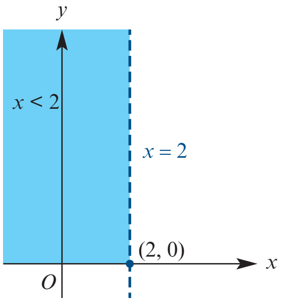

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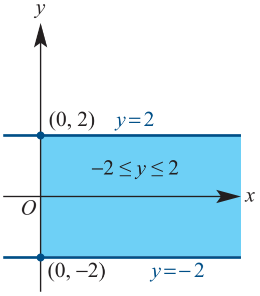

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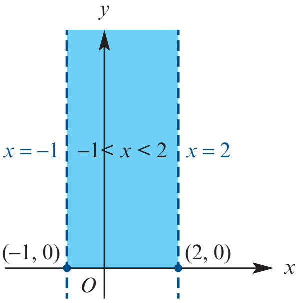

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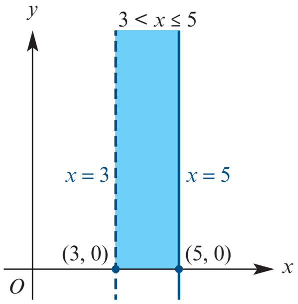
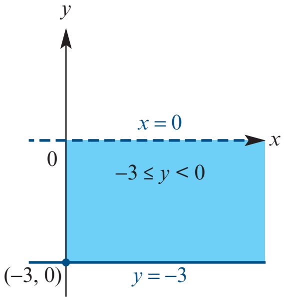


i

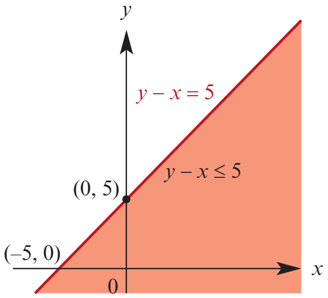


Exercise 12D

1 **a** Yes **b** Yes **c** No **d** No
e No **f** Yes

2 **a** Yes **b** No **c** Yes **d** No
e Yes **f** Yes

3 **a**



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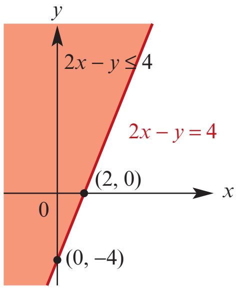

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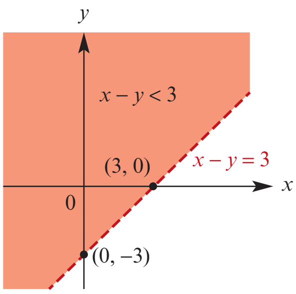
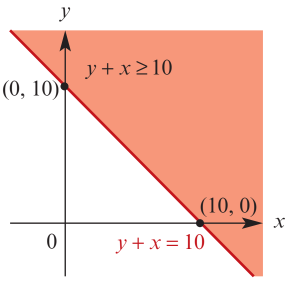
c



d



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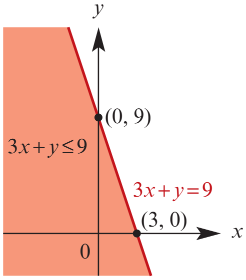

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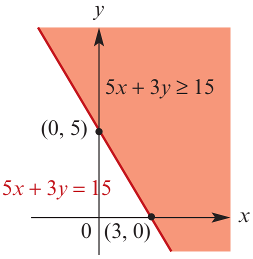

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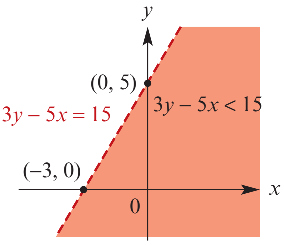

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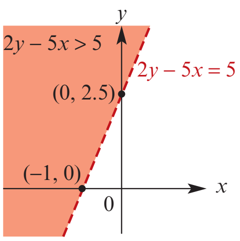

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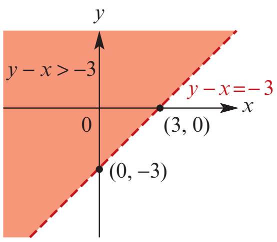
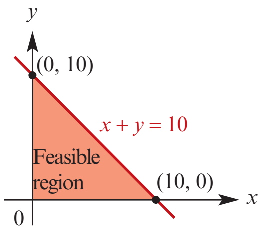


Exercise 12E

1



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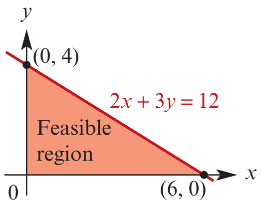

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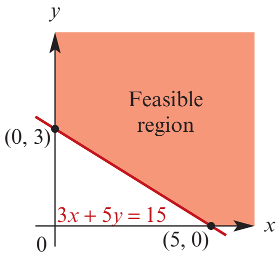

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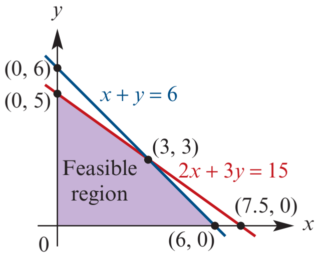

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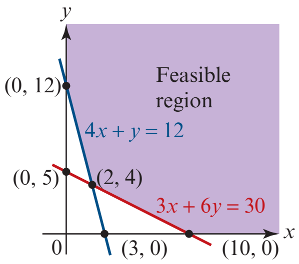

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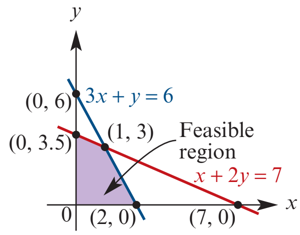
Exercise 12F

1



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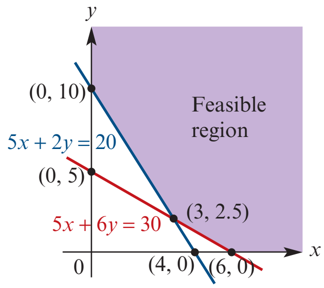

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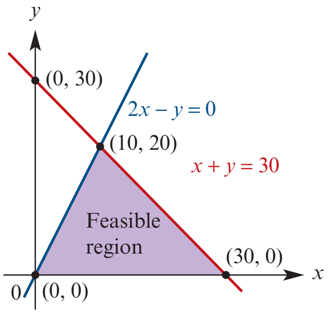


Exercise 12G

1 $P = 50$ at $(10, 20)$ **2** $P = 24$ at $(6, 0)$

3 $P = 54$ at $(2, 12)$ **4** $C = 18$ at $(6, 0)$

5 $C = 6$ at $(2, 4)$

6 $C = 10$ at $(0, 5)$ and $(5, 0)$ and at all points on the line between these two points

Exercise 12H

1 a $y \geq 0$

$2x + 4y \leq 24$ (Machine B time)

b $B(0, 6), 2x + 4y = 24, (12, 0)$; intersection at $C(4, 4)$

c $P = 200x + 360y$

d 4 Wigits and 4 Gigits; Profit = \$2240

2 a $x \geq 0, y \geq 0$

$2x + 2y \leq 520$ (material availability)

$2.4x + 3.2y \leq 672$ (worker time availability)

b $(0, 260)$, $2x + 2y = 520$, $D(260, 0)$; $2.4x + 3.2y = 672$, $E\,(280, 0)$; intersection at $C(200, 60)$

c $P = 36x + 42y$

d \$9720; 200 Polarbear and 60 Polarfox

3 a $45x + 30y \leq 450$ (people)

$3x + 4y \leq 36$ (equipment)

b $(0, 9)$, $3x + 4y = 36$; $(10, 0)$, $45x + 30y = 450$; intersection at $B(8, 3)$

c $C = 3600x + 3200y$

d 8 Redhawks and 3 Blackjets; \$38 400

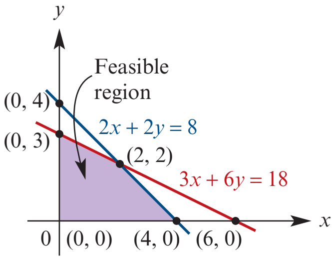
4 a $x \geq 0, y \geq 0$

$2x + 2y \leq 8$ (sawing)

$3x + 6y \leq 18$ (planing)

$P = 500x + 600y$

b



c 2 cubic metres of each; \$2200

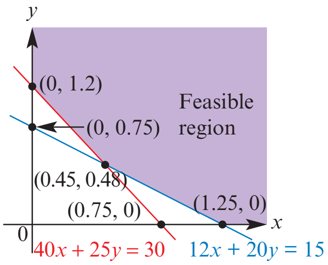
5 a $x \geq 0, \quad y \geq 0$

$12x + 20y \geq 15$ (vitamin B1)

$40x + 25y \geq 30$ (vitamin B2)

b $C = 5x + 4.5y$

c



d 0.45 kg of Healthystart and 0.48 kg of Wakeup; \$4.41

Exercise 12I

a Cable A must not exceed 5 km.

b i 90 kg **ii** $(20x + 10y)$kg

iii $20x + 10y \leq 120$

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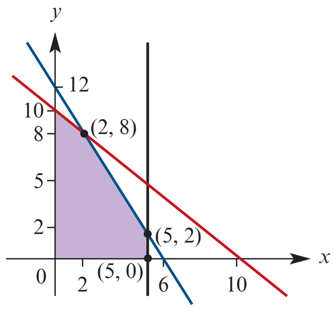
c



d **i** $P = 150x + 100y$

ii $P =$ \$1100 for 2 kg of cable *A* and 8 kg of cable *B*

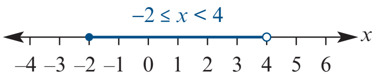
Chapter 12 review

Multiple-choice questions

1 B	**2** B	**3** D	**4** D	**5** B
6 D	**7** A	**8** D	**9** D	**10** D
11 D	**12** A	**13** C	**14** D	**15** A
16 C	**17** D			

Short-answer questions

1



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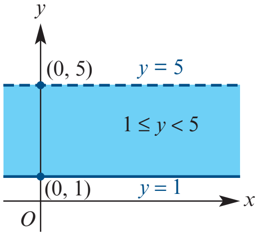

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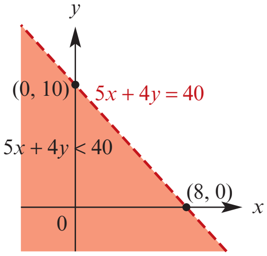

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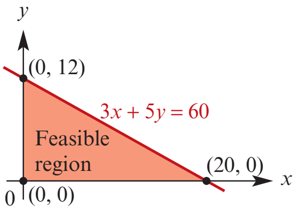

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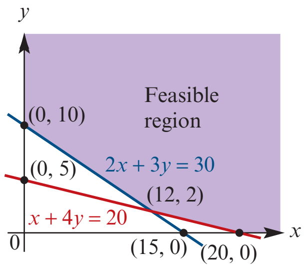


Extended-response questions

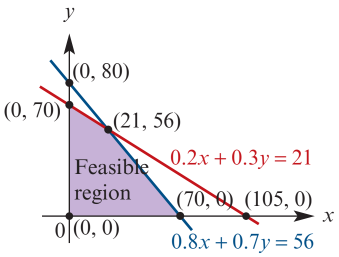
1 **a** $x \ge 0, y \ge 0$

$0.8x + 0.7y \le 56$ (nitrate)

$0.2x + 0.3y \le 21$ (phosphate)

$P = 600x + 750y$

b



c 21 tonnes of Standard Grade and 56 tonnes of Premium Grade; \$54 600

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