KAHNI BURROWS | BEVERLY LANGSFORD WILLING | SUE MICHELL

JACARANDA MATHS QUEST

MATHEMATICAL METHODS 12

FOR QUEENSLAND

UNITS 3&4

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KAHNI BURROWS | BEVERLY LANGSFORD WILLING | SUE MICHELL

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ABOUT THIS RESOURCE

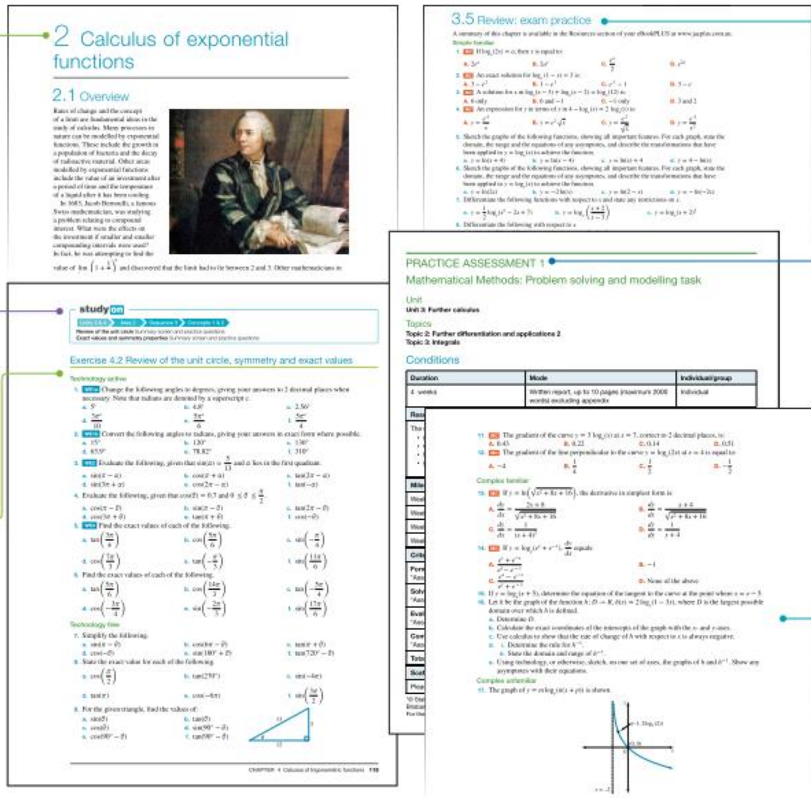
Jacaranda Maths Quest 12 Mathematical Methods Units 3 & 4 for Queensland is expertly tailored to address comprehensively the intent and structure of the new syllabus. The *Jacaranda Maths Quest for Queensland* series provides easy-to-follow text and is supported by a bank of resources for both teachers and students. At Jacaranda we believe that every student should experience success and build confidence, while those who want to be challenged are supported as they progress to more difficult concepts and questions.

Preparing students for exam success

Chapter openers place mathematics in real-world contexts to drive engagement.

FREE access to studyON — our study, revision and exam practice tool — is included with every title. studyON allows you to revise at the concept, chapter, curriculum topic or unit level.

Each subtopic concludes with carefully graded Technology free and Technology active questions.



Every chapter concludes with exam practice questions classified as Simple familiar, Complex familiar and Complex unfamiliar.

Two complete sets of practice assessments modelled on QCAA guidelines — a set for student revision and a quarantined set for teachers — are included. Exemplary responses and worked solutions are provided for teachers.

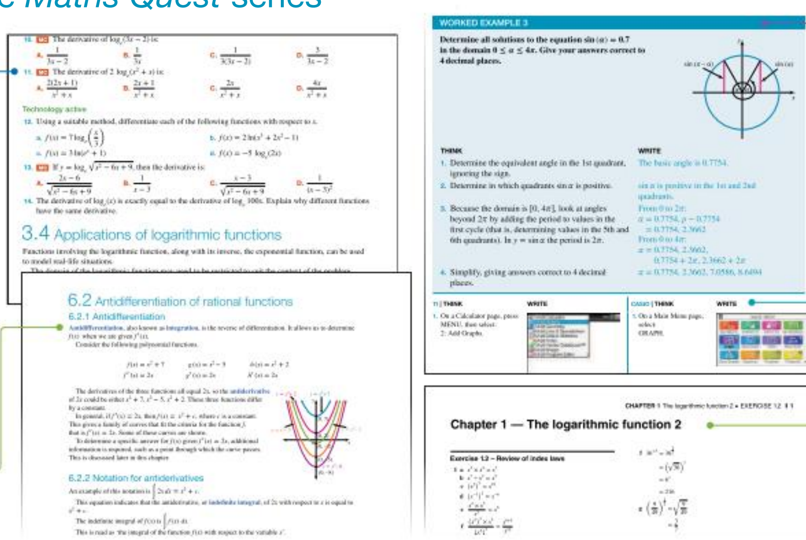
Chapter questions and activities are aligned with Marzano and Kendall's taxonomy of cognitive process — retrieval, comprehension, analysis and knowledge utilisation.

Fully worked examples in the Think/Write format provide guidance and are linked to questions.

Features of the *Maths Quest* series

Questions and chapters are sequenced from lower to higher levels of complexity; ideas and concepts are logically developed and questions are carefully graded, allowing every student to achieve success.

An extensive glossary of mathematical terms is provided in print and as a hover-over feature in the eBookPLUS.



Selected worked examples demonstrate the use of non-CAS calculators.

Free fully worked solutions are provided, enabling students to get help where they need it, whether at home or in the classroom — help at the point of learning is critical. Answers are provided at the end of each chapter in the print and offline PDF.

eBookPLUS features

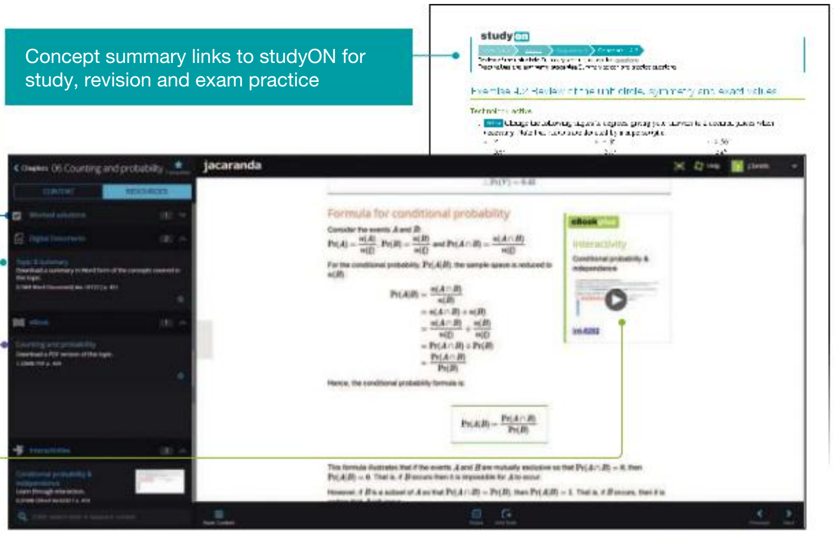
Fully worked solutions for every question

Chapter summaries in downloadable format to assist in study and exam preparation

A downloadable PDF of the entire chapter of the print text

Interactivities and video eLessons placed at the point of learning to enhance understanding and correct common misconceptions

Concept summary links to studyON for study, revision and exam practice

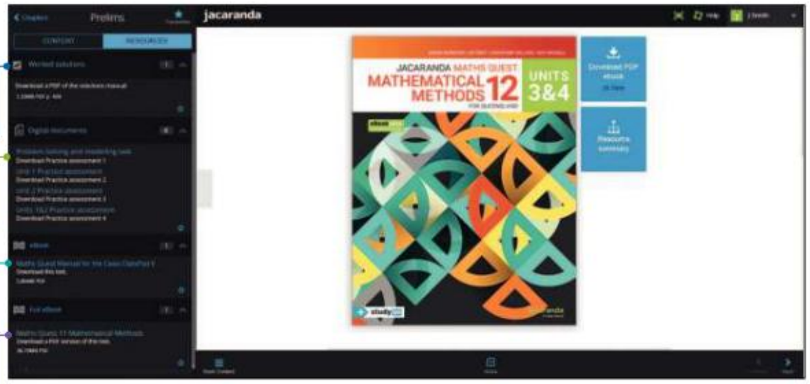


In the Prelims section of your eBooKPLUS

A downloadable PDF of the entire solutions manual, containing worked solutions for every question in the text

A set of four practice assessments: a problem solving and modelling task and three examination-style assessments

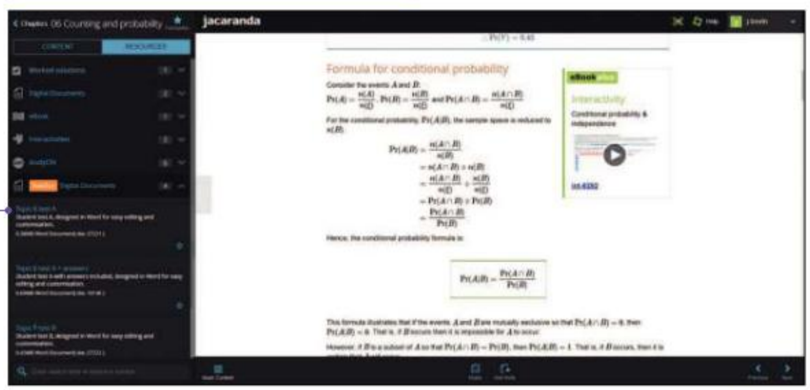
FREE copies of the *Maths Quest Manual for the TI-Nspire CAS calculator* and the *Maths Quest Manual for the Casio Classpad II calculator*



A downloadable PDF of the entire print text

Additional resources for teachers available in the eGuidePLUS

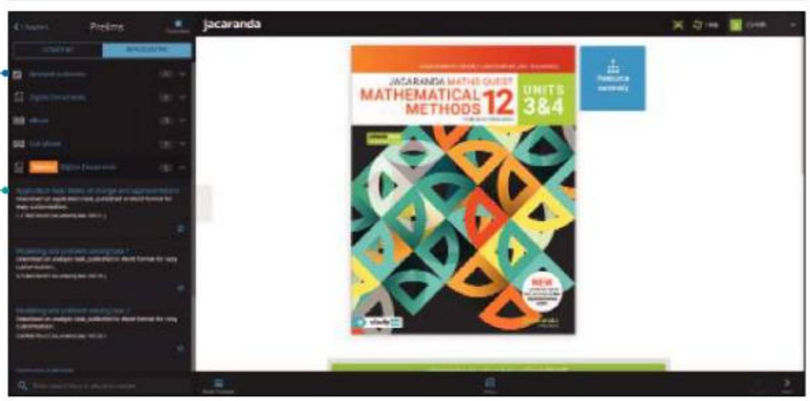
In the Resources tab of every chapter there are two chapter tests in downloadable, customisable Word format with worked solutions.



In the Prelims section of the eGuidePLUS

Work programs are provided to assist with classroom planning.

Practice assessments: in addition to the four provided in the eBookPLUS, teachers have access to a further four quarantined assessments. Modelled on QCAA guidelines, the problem solving and modelling tasks are provided with exemplary responses, and the examination-style assessments include annotated worked solutions. They are downloadable in Word format to allow teachers to customise as they need.



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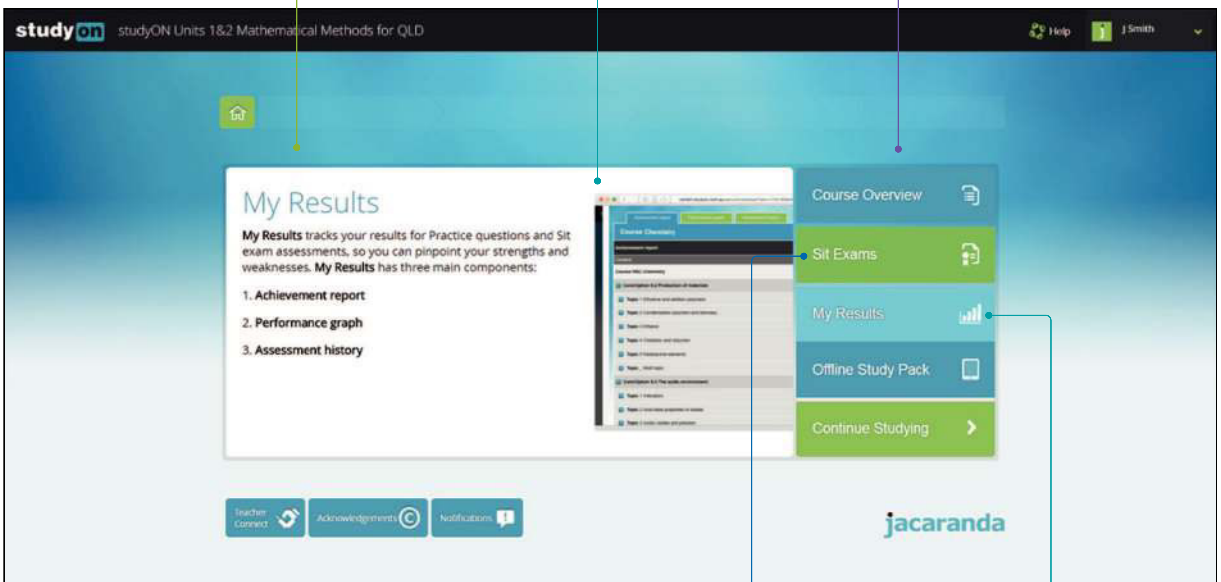
studyON provides a complete study solution. An interactive and highly visual online study, revision and exam practice tool, it is designed to help students and teachers maximise exam results.

Concept summary screens and interactivities summarise key concepts and help prevent misconceptions.

Direct links from the eBookPLUS help scaffold students' understanding and study practices.

The studyON question hierarchy allows students in the *Continue Studying* feature to revise across the entire course, or to drill down to concept level for a more granular set of questions.

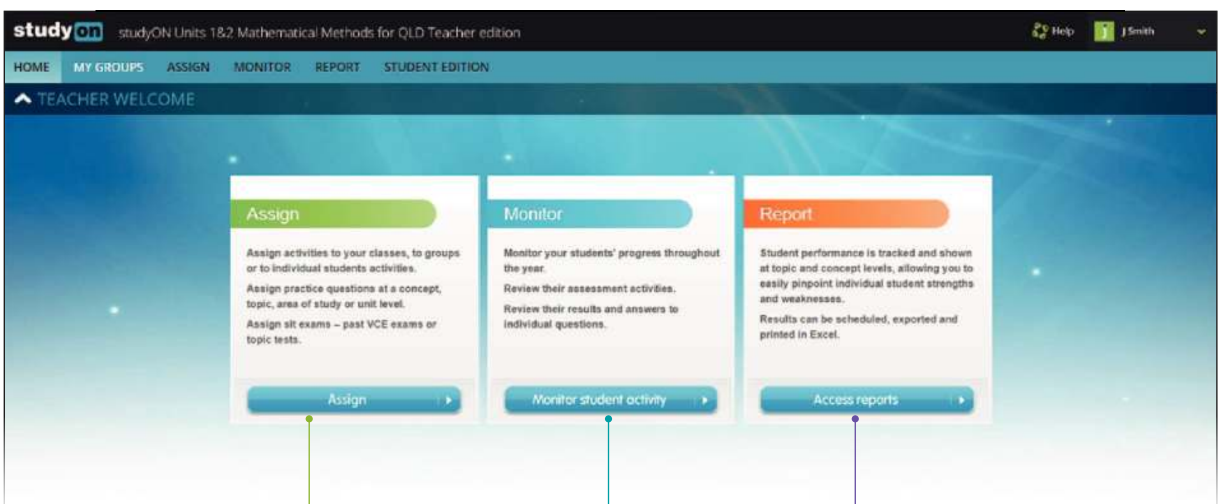


studyON prepares students for actual exams:

- The *Sit Exams* feature allows students to sit timed practice exams.
- Exam-style questions have been authored by our team of highly qualified teachers.
- From 2020, official past QCAA exam questions will be available for Units 3 & 4 with exemplary worked solutions to provide feedback for every question.

studyON's built-in progress tracker enables self-diagnosis of strengths and weaknesses at a topic and concept level, so students know exactly what needs extra revision and can sit their exams with confidence.

studyON Teacher edition is a powerful diagnostic tool



Enables teachers to assign activities for extra revision and practice, and track progress at an individual, group and classroom level

Allows teachers to monitor students' activities and results to pinpoint strengths and weaknesses. Armed with evidence-based insights, teachers can intervene at the right time.

Alignment with the Jacaranda text helps with planning, and instant feedback saves marking time. Built-in reporting functionality lets teachers easily schedule, export and print reports in Excel.

About eBookPLUS and studyON

Access your online Jacaranda resources anywhere, anytime, from any device in three easy steps:

STEP 1 Go to **www.jacplus.com.au** and create a user account.
STEP 2 Enter your registration code.
STEP 3 Instant access!

eBookplus + studyon

eBookPLUS is an electronic version of the textbook, together with a targeted range of supporting multimedia resources.

eBookPLUS features:

- **eBook** — the entire textbook in electronic format
- **Digital documents** designed for easy customisation and editing
- **Interactivities** to reinforce and enhance students' learning
- **eLessons** — engaging video clips and supporting material
- **Weblinks** to relevant support material on the internet

eGuidePLUS features assessment and curriculum material to support teachers.

studyON is an interactive and highly visual online study, revision and exam practice tool designed to help students and teachers maximise exam results.

studyON features:

- **Concept summary screens** provide concise explanations of key concepts, with relevant examples.
- **Access exam questions** that have been written by experienced examiners for practice at a concept, topic or entire course level, and receive immediate feedback. **From 2020, QCAA questions** will be included with exemplary worked solutions.
- **Sit past QCAA exams** (Units 3 & 4) or **topic tests** (Units 1 & 2) in exam-like situations.
- **Video animations and interactivities** demonstrate concepts to provide a deep understanding (Units 3 & 4 only).
- **All results and performance in practice and sit questions** are tracked to a concept level to pinpoint strengths and weaknesses.

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- Visit the JacarandaPLUS Support Centre at **http://jacplus.desk.com** to access a range of step-by-step user guides, ask questions or search for information.
- **Contact** John Wiley & Sons Australia, Ltd.
  Email: support@jacplus.com.au
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Minimum requirements

JacarandaPLUS requires you to use a supported internet browser and version, otherwise you will not be able to access your resources or view all features and upgrades. Please view the complete list of JacPLUS minimum system requirements at **http://jacplus.desk.com**.

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1 The logarithmic function 2

1.1 Overview

Let's say that you were asked to multiply 167 893 by the square root of 283.983.

Chances are that the first thing that you would do is reach for your calculator. But what if you did not have a calculator — could you still do it? And how long would it take you?

Before small hand-held electronic calculators were developed in the early 1970s, calculations that could not be quickly done with pencil, paper and mental arithmetic were performed using a device called a slide rule. Invented in the early seventeenth century, a slide rule is essentially a ruler with a sliding central section.

A normal ruler is marked with numbers that form a linear scale, with the marks for 1, 2 ... 30 cm being equally spaced. In contrast, each of the three ruler sections of a slide rule is marked with numbers that form logarithmic scales. In this type of scale, there are equal distances between the marks for 1, 10, 100, 1000 and so on.

Calculations were done by using a table of logarithms to identify the base 10 logarithms of the numbers you wanted to manipulate, sliding the central scale relative to one of the fixed outer scales until the appropriate numbers lined up with the cursor (a red line fitted in a sliding window), reading a number from a third scale and then using the logarithm table to determine your actual answer.

Sounds like a lot more work than just pressing a few buttons on your calculator, doesn't it?

Yet, with practice, a slide rule (with a log table, a pencil and a piece of paper) can be used to perform calculations in nearly the same time.

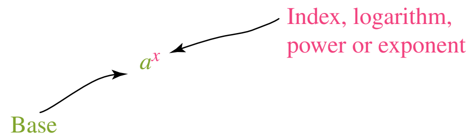
LEARNING SEQUENCE

1.1 Overview
1.2 Review of the index laws
1.3 Logarithmic laws and equations
1.4 Logarithmic scales
1.5 Indicial equations
1.6 Logarithmic graphs
1.7 Applications
1.8 Review: exam practice

Fully worked solutions for this chapter are available in the Resources section of your eBookPLUS at www.jacplus.com.au.

1.2 Review of the index laws

As you may recall from your earlier studies, a number in **index form** has two parts: the base and the index (also called the logarithm, power or **exponent**). Such numbers are written as shown:



The ways in which combinations of numbers written in index form are treated are described by a set of **index laws**.

The index laws

1. When numbers with the same base are multiplied, the indices are added.

$$a^x \times a^y = a^{x+y}$$

2. When numbers with the same base are divided, the indices are subtracted.

$$a^x \div a^y = a^{x-y}$$

$$\text{or } \frac{a^x}{a^y} = a^{x-y}$$

3. When numbers with an index or exponent are raised to another index or exponent, the indices are multiplied.

$$(a^x)^y = a^{xy}$$

4. When numbers have an index of 0, the answer is 1.

$$a^0 = 1$$

5. When a number has a negative index, it becomes a fraction with a positive index.

$$a^{-x} = \frac{1}{a^x} \quad \text{and} \quad \frac{1}{a^{-x}} = a^x$$

6. When a number has a fractional index, the denominator of the fraction becomes the root.

$$a^{\frac{1}{y}} = \sqrt[y]{a} \quad \text{and} \quad a^{\frac{x}{y}} = \sqrt[y]{a^x} \quad \text{or} \quad a^{\frac{x}{y}} = \left(\sqrt[y]{a}\right)^x$$

WORKED EXAMPLE 1

Simplify $\dfrac{\left(2x^2y^3\right)^3 \times 3\left(xy^4\right)^2}{6x^4 \times 2xy^4}$.

THINK

1. Remove the brackets by multiplying the indices.

2. Add the indices of x and add the indices of y. Simplify 2^3 to 8 and multiply the whole numbers.

3. Subtract the indices of x and y. Divide 24 by 12.

WRITE

$$\frac{(2x^2y^3)^3 \times 3(xy^4)^2}{6x^4 \times 2xy^4} = \frac{2^3x^6y^9 \times 3x^2y^8}{12x^5y^4}$$

$$= \frac{24x^8y^{17}}{12x^5y^4}$$

$$= 2x^3y^{13}$$

For negative indices and fractional or decimal indices, the same rules apply.

WORKED EXAMPLE 2

Write the following in simplest form.

a. $64^{\frac{2}{3}}$

b. $32^{-0.4}$

THINK

a. 1. Rewrite using the index law $a^{\frac{x}{y}} = \sqrt[y]{a^x}$.

2. Rewrite using $\sqrt[y]{a^x} = \left(\sqrt[y]{a}\right)^x$.

3. Simplify by taking the cube root of 64.

4. Square 4.

b. 1. Write as a fraction with a positive index.

2. Change 0.4 to $\frac{4}{10}$.

3. Simplify the fractional index.

4. Rewrite using the index law $a^{\frac{x}{y}} = \sqrt[y]{a^x}$.

5. Simplify by taking the 5th root of 32.

6. Square 2.

WRITE

a. $64^{\frac{2}{3}} = \sqrt[3]{64^2}$

$= \left(\sqrt[3]{64}\right)^2$

$= 4^2$

$= 16$

b. $32^{-0.4} = \dfrac{1}{32^{0.4}}$

$= \dfrac{1}{32^{\frac{4}{10}}}$

$= \dfrac{1}{32^{\frac{2}{5}}}$

$= \dfrac{1}{\left(\sqrt[5]{32}\right)^2}$

$= \dfrac{1}{2^2}$

$= \dfrac{1}{4}$

WORKED EXAMPLE 3

Simplify the following, leaving your answers with positive indices.

a. $a^{-2}b^4 \times \left(a^3b^{-4}\right)^{-1}$

b. $\left(\dfrac{a^{\frac{1}{2}}b^{-1}}{3^{-1}b^2}\right)^{-1} \div \left(\dfrac{3a^{-\frac{3}{2}}b^2}{a^{\frac{3}{4}}b^{\frac{1}{2}}}\right)^2$

THINK

WRITE

a. 1. Remove the brackets by multiplying the indices.

a. $a^{-2}b^4 \times (a^3b^{-4})^{-1} = a^{-2}b^4 \times a^3b^4$

2. Add the indices of a and of b.

$= a^{-5}b^8$

3. Place a^5 in the denominator with a positive index.

$= \dfrac{b^8}{a^5}$

b. 1. Remove the brackets by multiplying the indices.

b. $\left(\dfrac{a^{\frac{1}{2}}b^{-1}}{3^{-1}b^2}\right)^{-1} \div \left(\dfrac{3a^{-\frac{3}{2}}b^2}{a^{\frac{3}{4}}b^{\frac{1}{2}}}\right)^2$

$= \dfrac{a^{-\frac{1}{2}}b}{3b^{-2}} \div \dfrac{3^2a^{-3}b^4}{a^{\frac{3}{2}}b}$

2. When dividing by a fraction, invert and multiply.

$= \dfrac{a^{-\frac{1}{2}}b}{3b^{-2}} \times \dfrac{a^{\frac{3}{2}}b}{9a^{-3}b^4}$

3. Add the indices of a and of b in the numerator and add the indices of b in the denominator. Multiply the numbers.

$= \dfrac{ab^2}{27a^{-3}b^2}$

4. Subtract the indices of a and b.

$= \dfrac{a^4}{27}$

WORKED EXAMPLE 4

Simplify $\dfrac{3^n \times 6^{n+1} \times 12^{n-1}}{3^{2n} \times 8^n}$.

THINK

WRITE

1. Write each number as the product of prime factors.

$\dfrac{3^n \times 6^{n+1} \times 12^{n-1}}{3^{2n} \times 8^n}$

$= \dfrac{3^n \times (3 \times 2)^{n+1} \times \left(2^2 \times 3\right)^{n-1}}{3^{2n} \times 2^{3n}}$

2. Remove the brackets.

$= \dfrac{3^n \times 3^{n+1} \times 2^{n+1} \times 2^{2n-2} \times 3^{n-1}}{3^{2n} \times 2^{3n}}$

3. Add the indices of numbers with base 3 in the numerator and add indices of numbers with base 2 in the numerator.

$= \dfrac{3^{3n} \times 2^{3n-1}}{3^{2n} \times 2^{3n}}$

4. Subtract the indices.

$= 3^n \times 2^{-1}$

5. Write the term with a negative index in the denominator with a positive index.

$= 3^n \times \dfrac{1}{2}$

6. Simplify.

$= \dfrac{3^n}{2}$

Index laws can be applied in many situations, such as in exponential modelling. A simple example of an exponential model is presented in the following worked example.

WORKED EXAMPLE 5

An antique chair worth $15 000 is increasing in value by 10% each year.

a. Write an equation for the value of the chair, $\$v$, in terms of the time, t, in years.

b. Hence, find the value of the chair after 10 years. Give your answer correct to the nearest hundred dollars.

THINK	WRITE
a. 1. Find by what percentage the chair appreciates each year.	**a.** The chair appreciates by (100 + 10) % or 110%.
2. Write this as a decimal.	$110\% = \frac{110}{100}$ $= 1.1$
3. Find the value after 1 year.	After 1 year it is worth $\$(15\,000 \times 1.1)$ $= \$16\,500$
4. Find the value after 2 years.	After 2 years it is worth $\$(15\,000 \times 1.1) \times (1.1)$ $= \$\left(15\,000 \times 1.1^2\right)$ $= \$18\,150$
5. Find the value after 3 year.	After 3 years it is worth $\$\left(15\,000 \times 1.1^3\right) = \$19\,965$
6. Hence, find the formula. *Note:* A formula does not include the dollar sign.	$v = 15\,000 \times 1.1^t$
b. 1. Substitute $t = 10$ in the equation.	**b.** $v = 15\,000 \times 1.1^{10}$
2. Evaluate 1.1^{10}.	$= 15\,000 \times 2.593\,742\,5$
3. Calculate v and express your answer correct to the nearest hundred dollars.	$= 38\,906.138 \approx 38\,900$ (to the nearest 100)
4. Write your answer in a sentence.	The value of the chair after 10 years is about $38 900.

studyon

Units 3 & 4 > Area 1 > Sequence 1 > Concept 1 > **Review of the index laws** Summary screen and practice questions

Exercise 1.2 Review of the index laws

Technology free

1. **WE1** Simplify the following.

a. $x^3 \times x^4$
b. $x^7 \div x^2$
c. $\left(x^2\right)^5$
d. $\left(x^{-3}\right)^2$
e. $\dfrac{x^4 \times x^5}{x^3}$
f. $\dfrac{\left(x^2\right)^3 \times x^5}{\left(x^5\right)^2}$
g. $\dfrac{5x^2y^4 \times 4x^5y}{2^2x^3y^2}$
h. $\dfrac{3x^3y^5 \times 10xy^4}{5x^2y^6}$
i. $\dfrac{\left(2xy^2\right)^3 \times 5\left(x^4y\right)^2}{4x^5y^3 \times 3x^2y^3}$
j. $\dfrac{\left(3^2x^3y\right)^2 \times 2\left(xy^3\right)^5}{4x^4y^2 \times 3x^5y}$

2. **WE2** Simplify the following without using a calculator.

a. $27^{\frac{2}{3}}$
b. $16^{\frac{3}{4}}$
c. $25^{-\frac{3}{2}}$
d. $100\,000^{\frac{-3}{5}}$
e. $81^{0.25}$
f. $36^{1.5}$
g. $\left(\dfrac{9}{49}\right)^{\frac{1}{2}}$
h. $\left(\dfrac{27}{64}\right)^{\frac{2}{3}}$
i. $\left(\dfrac{243}{32}\right)^{-\frac{3}{5}}$
j. $\left(\dfrac{256}{81}\right)^{-\frac{3}{4}}$

3. **WE3** Simplify the following, leaving your answers with positive indices.

a. $3x^{-3}y^2 \times \left(x^2y\right)^{-4}$
b. $x^4y^{-1} \times \left(x^{-2}y^3\right)^{-1}$
c. $2x^{\frac{1}{2}}y^{\frac{2}{3}} \times \left(9x^{\frac{3}{2}}y^2\right)^{\frac{1}{2}}$
d. $5x^{-\frac{1}{3}}y^{\frac{3}{4}} \times \left(8^{\frac{1}{3}}x^{\frac{2}{3}}y^{-\frac{1}{2}}\right)^2$
e. $\left(x^{-2}y^{\frac{1}{2}}\right)^{-\frac{3}{2}} \times \left(9x^{-\frac{1}{5}}y^{-\frac{1}{2}}\right)^{\frac{5}{2}}$
f. $16^{\frac{1}{2}}\left(x^{\frac{2}{5}}y^{-\frac{1}{4}}\right)^{-\frac{1}{2}} \times \left(4x^{\frac{2}{5}}y^{\frac{1}{2}}\right)^{\frac{1}{2}}$
g. $\left(\dfrac{a^{\frac{3}{2}}b^{-2}c}{3a^{-\frac{1}{2}}bc^{-2}}\right)^{-2} \div 3\left(\dfrac{a^{\frac{3}{2}}b^3}{a^{-1}c^2}\right)^3$
h. $\left(\dfrac{a^{-\frac{3}{2}}b^{\frac{3}{4}}}{ab^2}\right)^{-2} \div \left(\dfrac{9a^{-3}b}{4a^2b^3}\right)^{\frac{1}{2}}$

4. **WE4** Simplify the following.

a. $2^n \times 4^{n+1} \times 8^{n-1}$
b. $3^n \times 9^{n-1} \times 27^{n+1}$
c. $2^{n-1} \times 3^n \times 6^{n+1}$
d. $2^n \times 3^{n+1} \times 9^n$
e. $\dfrac{3^2 \times 2^{-3}}{9^{\frac{3}{2}}} \times 16$
f. $\dfrac{5^2 \times 3^{-1}}{125 \times 9^{-2}} \div \dfrac{27}{5}$

5. Simplify the following, writing your answers as single fractions with positive indices.

a. $x^{-1} + \dfrac{1}{x^{-1}}$
b. $\left(x^{-1} + x^{-2}\right)^2$
c. $\dfrac{1}{x^{-1}+1} + \dfrac{1}{x^{-1}-1}$
d. $2x\left(x^2 - y^2\right)^{-1} - (x-y)^{-1}$

6. If $a = 2^3, b = 2^{-3}, c = 6^2$ and $d = 3^{-1}$, determine:

a. $\dfrac{a^2b}{c^{\frac{1}{2}}}$
b. $\dfrac{a^{\frac{1}{3}}b^{-1}d}{c^2}$

7. **MC** $3^{-x} + 3^x$ is equal to:

A. 1
B. $\dfrac{1+3^{2x}}{3^x}$
C. 3^{-x^2}
D. 6

Technology active

8. **WE5** A population of organisms is growing so that the number of organisms, N, after t days is given by the formula $N = 500 \times 2^{0.1t}$.
   a. Determine the number of organisms after 10 days.
   b. Determine the size of the population after 15 days. Give your answer to the nearest whole number.

9. **MC** A car worth \$10 000 is depreciating at 20% per annum, so that each year the car is worth 80% of its value the previous year. A model for the value of the car, \$$V$, in terms of the time, t, in years is:
   A. $V = 10\,000 \times 20^t$
   B. $V = 10\,000 \times (0.2)^t$
   C. $V = 0.8 \times 10\,000^t$
   D. $V = 10\,000\,(0.8)^t$

10. A ball is dropped from a window h m above the ground. When it lands on the ground it rebounds to 80% of its height. The equation showing the height of the ball, h metres, after r rebounds is $h = 10 \times (0.8)^r$.
   a. From how far above the ground was the ball dropped?
   b. How far above the ground does the ball reach on the fourth rebound? Give your answer to the nearest centimetre.
   c. What is the total vertical distance that the ball travelled when it hits the ground for the fourth time?

1.3 Logarithmic laws and equations

1.3.1 Writing numbers in logarithmic form

The number 81 can be written as:

$$81 = 3^4$$

That is, given the base 3 and the exponent 4, we can find the number 81 by calculating $3 \times 3 \times 3 \times 3$. Note, however, that we need a calculator to compute $3^{4.5}$:

$$3^4 \text{ gives } 140.296.$$
$$140.296 = 3^{4.5}$$

What do we do if we are given the number and the base, but need to find the power or exponent?

$$100 = 10^x$$

In this case, an easy calculation shows that $100 = 10^2$.
However, how do we find x in an equation such as the following?

$$200 = 10^x$$

This is where logarithms are useful.
If $200 = 10^x$

then $x = \log_{10} 200$

$= 2.301$ (from calculator)

In general, if $N = a^x$

then $x = \log_a N$.

So, a number written in index form as

$$\text{Base number} \rightarrow \overset{\substack{\text{Exponent, index or power} \\ \downarrow}}{10^2} = 100$$

becomes

$$\underset{\substack{\uparrow \\ \text{Base number}}}{\log_{10}} 100 = 2 \leftarrow \text{Exponent, index or power}$$

when written in logarithmic form.

Logarithms can only be used when the base number and the exponent are positive numbers; that is, for $\log_a N$, $a > 0$ and $N > 0$. For any other values, the logarithm is not defined.

1.3.2 Natural logarithms

Any number, provided that it is positive, can be used as the base number. A natural logarithm is one which uses Euler's number (written as e) as its base number.

Named for 18th-century Swiss mathematician Leonhard Euler (pronounced 'oiler'), Euler's number is the base number for many processes and formations in nature that can be described logarithmically, such as the decay of radioactive isotopes, the structure of a spiral galaxy or even the arrangement of leaves on a plant.

Euler's number

Like π, e is an irrational number. It can be found by evaluating the expression $\left(1 + \frac{1}{n}\right)^n$ for increasing values of n:

$$n = 1 \quad \left(1 + \frac{1}{n}\right)^n = \left(1 + \frac{1}{1}\right)^1 = 2$$

$$n = 2 \quad \left(1 + \frac{1}{n}\right)^n = \left(1 + \frac{1}{2}\right)^2 = 2.25$$

$$n = 3 \quad \left(1 + \frac{1}{n}\right)^n = \left(1 + \frac{1}{3}\right)^3 = 2.370\,37$$

$$n = 5 \quad \left(1 + \frac{1}{n}\right)^n = \left(1 + \frac{1}{5}\right)^5 = 2.488\,32$$

$$n = 10\ \left(1 + \frac{1}{n}\right)^n = \left(1 + \frac{1}{10}\right)^{10} = 2.593\,74$$

$$n = 100\ \left(1 + \frac{1}{n}\right)^n = \left(1 + \frac{1}{100}\right)^{100} = 2.704\,81$$

$$n = 1000\ \left(1 + \frac{1}{n}\right)^n = \left(1 + \frac{1}{1000}\right)^{1000} = 2.716\,92$$

$$n = 10\,000\ \left(1 + \frac{1}{n}\right)^n = \left(1 + \frac{1}{10\,000}\right)^{10\,000} = 2.718\,15$$

As n increases, $\left(1 + \frac{1}{n}\right)^n$ becomes closer and closer to 2.718 281 or e.

$$e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n$$

Using a base e, if

$$\mathrm{N} = e^x$$

then

$$x = \log_e(\mathrm{N}) \text{ (this can also be written as } x = \ln(\mathrm{N})\text{).}$$

1.3.3 The logarithmic laws

The indicial laws can be used to derive the logarithmic laws.

1. $a^m \times a^n = a^{m+n} \Leftrightarrow \log_a(m) + \log_a(n) = \log_a(mn)$
   To prove this law:
   Let $x = \log_a(m)$ and $y = \log_a(n)$.
   So $a^x = m$ and $a^y = n$.
   Now $a^m \times a^n = a^{m+n}$.
   Thus, $mn = a^x + a^y = a^{x+y}$.
   By applying the definition of a logarithm to this statement, we get

   $$\log_a(mn) = x + y$$

   or $$\log_a(mn) = \log_a(m) + \log_a(n).$$

2. $a^m \div a^n = a^{m-n} \Leftrightarrow \log_a(m) - \log_a(n) = \log_a\left(\frac{m}{n}\right)$
   To prove this law:
   Let $x = \log_a(m)$ and $y = \log_a(n)$.
   So $a^x = m$ and $a^y = n$.
   Now $\frac{a^x}{a^y} = a^{x-y}$.
   Thus, $\frac{m}{n} = \frac{a^x}{a^y} = a^{x-y}$.

By converting the equation into logarithm form, we get

$$\log_a\left(\frac{m}{n}\right) = x - y$$

or
$$\log_a\left(\frac{m}{n}\right) = \log_a(m) - \log_a(n).$$

Note: Before the first or second law can be applied, each logarithmic term must have a coefficient of 1.

3. $(a^m)^n = a^{mn} \Leftrightarrow \log_a(m^n) = n\log_a(m)$
   To prove this law:
   Let $x = \log_a(m)$.
   So $a^x = m$.
   Now $(a^x)^n = m^n$
   or $a^{nx} = m^n$.
   By converting the equation into logarithm form, we have

$$\log_a(m^n) = nx$$

or
$$\log_a(m^n) = n\log_a(m).$$

Applying these laws, we can also see the following:

4. As $a^0 = 1$, then by the definition of a logarithm, $\log_a(1) = 0$.
5. As $a^1 = a$, then by the definition of a logarithm, $\log_a(a) = 1$.
6. $a^x > 0$; therefore, $\log_a(0)$ is undefined, and $\log_a(x)$ is defined only for $x > 0$ and $a \in R^+\backslash\{1\}$.
   Another important fact related to the definition of a logarithm is
   $a^{\log_a(m)} = m$.
   This can be proved as follows:
   Let $y = a^{\log_a(m)}$.
   Converting index form to logarithm form, we have
   $\log_a(y) = \log_a(m)$.
   Consequently, $a^{\log_a(m)} = m$.

The logarithm laws

1. $\log_a(m) + \log_a(n) = \log_a(mn)$
2. $\log_a(m) - \log_a(n) = \log_a\left(\frac{m}{n}\right)$
3. $\log_a(m^n) = n\log_a(m)$
4. $\log_a(1) = 0$
5. $\log_a(a) = 1$
6. $\log_a(0) = $ undefined
7. $\log_a(x)$ is defined for $x > 0$ and $a \in R^+\backslash\{1\}$.
8. $a^{\log_a(m)} = m$

WORKED EXAMPLE 6

Simplify the following without a calculator.

a. $\mathbf{\log_{10}(5) + \log_{10}(2)}$

b. $\mathbf{\log_4(20) - \log_4(5)}$

c. $\mathbf{\log_2(16)}$

d. $\mathbf{\log_5\left(\sqrt[5]{x}\right)}$

THINK / **WRITE**

a. 1. Rewrite using $\log_a(mn) = \log_a(m) + \log_a(n)$.
   2. Simplify.
   3. Simplify using $\log_a(a) = 1$.

a. $\log_{10}(5) + \log_{10}(2) = \log_{10}(5 \times 2)$
$= \log_{10}(10)$
$= 1$

b. 1. Rewrite using $\log_a\left(\frac{m}{n}\right) = \log_a(m) - \log_a(n)$.
   2. Simplify.
   3. Simplify using $\log_a(a) = 1$.

b. $\log_4(20) - \log_4(5) = \log_4\left(\frac{20}{5}\right)$
$= \log_4(4)$
$= 1$

c. 1. Rewrite 16 as a number with base 2.
   2. Rewrite using $\log_a(m^p) = p\log_a(m)$.
   3. Simplify using $\log_a(a) = 1$.

c. $\log_2(16) = \log_2(2^4)$
$= 4\log_2(2)$
$= 4 \times 1$
$= 4$

d. 1. Rewrite using $\sqrt[x]{a} = a^{\frac{1}{y}}$.
   2. Rewrite using $\log_a(m^p) = p\log_a(m)$.

d. $\log_5\left(\sqrt[5]{x}\right) = \log_5\left(x^{\frac{1}{5}}\right)$
$= \frac{1}{5}\log_5(x)$

WORKED EXAMPLE 7

Simplify the following.

a. $2 + \log_{10}(3)$ **b.** $3\log_3(6) - 3\log_3(18)$ **c.** $\frac{\log_3(9)}{\log_3(27)}$

THINK / **WRITE**

a. 1. Write 2 as $2\log_{10}(10)$ because $\log_{10}(10) = 1$.
   2. Rewrite using $\log_a(m^p) = p\log_a(m)$.
   3. Rewrite using $\log_a(mn) = \log_a(m) + \log_a(n)$.
   4. Write 10^2 as 100.
   5. Multiply the numbers in the brackets.

a. $2 + \log_{10}(3) = 2\log_{10}(10) + \log_{10}(3)$
$= \log_{10}(10^2) + \log_{10}(3)$
$= \log_{10}(10^2 \times 3)$
$= \log_{10}(100 \times 3)$
$= \log_{10}(300)$

b. 1. Rewrite using $\log_a(m^p) = p\log_a(m)$.
   2. Rewrite using $\log_a\left(\frac{m}{n}\right) = \log_a(m) - \log_a(n)$.
   3. Write 6^3 as $6 \times 6 \times 6$ and 18^3 as $18 \times 18 \times 18$.
   4. Simplify.
   5. Write the number with base 3.
   6. Rewrite using $\log_a(m^p) = p\log_a(m)$.
   7. Simplify using $\log_a(a) = 1$.

b. $3\log_3(6) - 3\log_3(18) = \log_3(6^3) - \log_3(18^3)$
$= \log_3\left(\frac{6^3}{18^3}\right)$
$= \log_3\left(\frac{6 \times 6 \times 6}{18 \times 18 \times 18}\right)$
$= \log_3\left(\frac{1}{3^3}\right)$
$= \log_3(3^{-3})$
$= -3\log_3(3)$
$= -3 \times 1$
$= -3$

c. 1. Write the numbers with the same base. It is not possible to cancel the 9 and the 27 because they cannot be separated from the log.

2. Rewrite using $\log_a(m^p) = p\log_a(m)$.

3. Cancel the logs because they are the same.

c. $\dfrac{\log_3(9)}{\log_3(27)} = \dfrac{\log_3(3^2)}{\log_3(3^3)}$

$= \dfrac{2\log_3(3)}{3\log_3(3)}$

$= \dfrac{2}{3}$

1.3.4 Solving logarithmic equations

Solving logarithmic equations involves the use of the logarithm laws as well as converting to index form. As $\log_a x$ is defined only for $x > 0$ and $a \in R^+ \backslash \{1\}$, always check the validity of your solution.

WORKED EXAMPLE 8

Solve the following for x, giving your answer correct to 3 decimal places where appropriate.

a. $\log_e(3) = \log_e(x)$

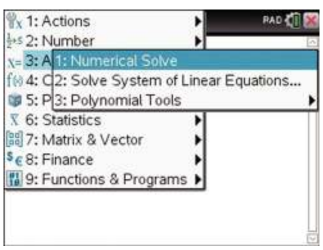
b. $\log_e(x) + \log_e(3) = \log_e(6)$

THINK	WRITE
a. Since the base is the same, equate the numbers.	**a.** $\log_e(3) = \log_e(x)$ $x = 3$
b. 1. Rewrite using $\log_e(mn) = \log_e(m) + \log_e(n)$.	**b.** $\log_e(x) + \log_e(3) = \log_e(6)$ $\log_e(3x) = \log_e(6)$
2. Equate the number parts.	$3x = 6$
3. Solve for x.	$x = 2$

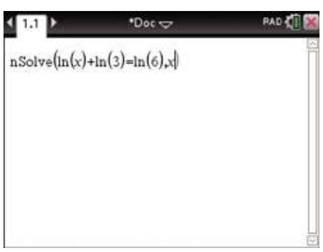
TI | THINK

b. 1. On a Calculator page, press MENU then select:
3: Algebra
1: Numerical Solve.

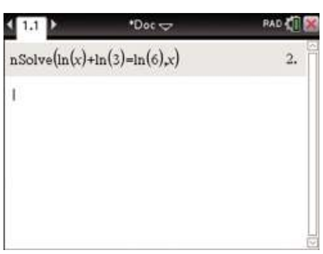
WRITE



2. Complete the entry line as:
$(\ln(x) + \ln(3) = \ln(6), x)$
Then press the ENTER button.



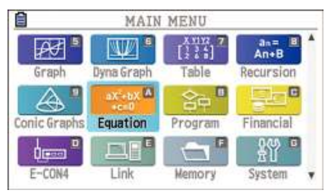
3. The answer appears on the screen.



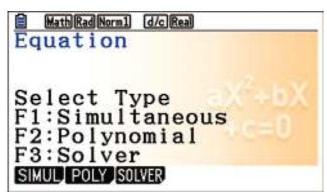
CASIO | THINK

b. 1. On a Main Menu screen, select Equation.

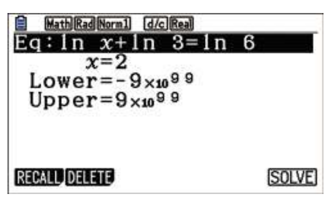
WRITE



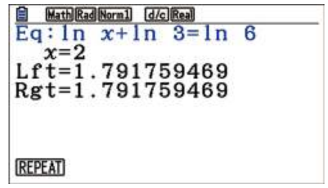
2. On an Equation screen, select Solver by pressing F3.



3. Complete the entry line as:
$(\ln(x) + \ln(3) = \ln(6), x)$
When the equation has been entered, press the SOLVE button.



4. The answer appears on the screen.



WORKED EXAMPLE 9

Solve the following equations for x.

a. $\log_4(64) = x$

b. $\log_2(3x) + 3 = \log_2(x-2)$

c. $\left(\log_2(x)\right)^2 = 3 - 2\log_2(x)$

d. $\log_2(2x) + \log_2(x+2) = \log_2(6)$

THINK	WRITE
a. 1. Convert the equation into index form.	**a.** $\log_4(64) = x$ $4^x = 64$
2. Convert 64 to base 4 and evaluate.	$4^x = 4^3$ $\therefore x = 3$
b. 1. Rewrite 3 in log form, given $\log_2(2) = 1$.	**b.** $\log_2(3x) + 3 = \log_2(x-2)$ $\log_2(3x) + 3\log_2(2) = \log_2(x-2)$
2. Apply the law $\log_a(m^n) = n\log_a(m)$.	$\log_2(3x) + \log_2(2^3) = \log_2(x-2)$
3. Simplify the left-hand side by applying $\log_a(mn) = \log_a(m) + \log_a(n)$.	$\log_2(3x \times 8) = \log_2(x-2)$
4. Equate the logs and simplify.	$24x = x - 2$ $23x = -2$ $x = -\dfrac{2}{23}$
c. 1. Identify the quadratic form of the log equation. Let $a = \log_2(x)$ and rewrite the equation in terms of a.	**c.** $\left(\log_2(x)\right)^2 = 3 - 2\log_2(x)$ Let $a = \log_2(x)$. $a^2 = 3 - 2a$
2. Solve the quadratic.	$a^2 + 2a - 3 = 0$ $(a-1)(a+3) = 0$ $a = 1, -3$
3. Substitute in $a = \log_2(x)$ and solve for x.	$\log_2(x) = 1$ $\quad \log_2(x) = -3$ $x = 2^1$ $\quad x = 2^{-3}$ $\therefore x = 2, \dfrac{1}{8}$
d. 1. Simplify the left-hand side by applying $\log_a(mn) = \log_a(m) + \log_a(n)$.	**d.** $\log_2(2x) + \log_2(x+2) = \log_2(6)$ $\log_2(2x(x+2)) = \log_2(6)$
2. Equate the logs and solve for x.	$2x(x+2) = 6$ $2x^2 + 4x - 6 = 0$ $x^2 + 2x - 3 = 0$ $(x-1)(x+3) = 0$ $x = 1, -3$
3. Check the validity of both solutions.	$x = -3$ is not valid, as $x > 0$.
4. Write the answer.	$\therefore x = 1$

1.3.5 Change of base rule

Although any positive number can be used as the base for a logarithmic or indicial expression, calculators generally only allow you two options — the standard logarithm, $\log_{10}(x)$, or the natural logarithm, $\log_e(x)$. In order to calculate values for bases other than 10 or e, the base must be changed to one of these options.

Suppose $y = \log_a(m)$.

By definition, $a^y = m$.

Take the logarithm to the same base of both sides.

$$\log_b(a^y) = \log_b(m)$$
$$y \log_b(a) = \log_b(m)$$
$$y = \frac{\log_b(m)}{\log_b(a)}$$

Change the base

$$\log_a(m) = \frac{\log_b(m)}{\log_b(a)}$$

WORKED EXAMPLE 10

a. Evaluate the following, correct to 4 decimal places.

i. $\log_7(5)$ **ii.** $\log_{\frac{1}{3}}(11)$

b. If $p = \log_5(x)$**, find the following in terms of** p**.**

i. x **ii.** $\log_x(81)$

THINK	WRITE
a. i. Apply the change of base rule and calculate.	**a. i.** $\log_7(5) = \frac{\log_{10}(5)}{\log_{10}(7)} = 0.8271$
ii. Apply the change of base rule and calculate.	**ii.** $\log_{\frac{1}{3}}(11) = \frac{\log_{10}(11)}{\log_{10}\left(\frac{1}{3}\right)} = -2.1827$
b. i. Rewrite the logarithm in index form to find an expression for x.	**b. i.** $p = \log_5(x)$ $x = 5^p$
ii. 1. Rewrite $\log_x(81)$ using $\log_a(m^n) = n\log_a(m)$.	**ii.** $\log_x(81) = \log_x\left(9^2\right)$ $= 2\log_x(9)$
2. Apply the change-of-base rule so that x is no longer a base. *Note:* Although 9 has been chosen as the base in this working, a different value could be applied, giving a different final answer.	$= 2\frac{\log_9(9)}{\log_9(x)}$ $= 2\frac{1}{\log_9(x)}$
3. Replace x with 5^p and apply the law $\log_a(m^n) = n\log_a(m)$.	$= 2\frac{1}{\log_9(5^p)}$ $= \frac{2}{p\log_9(5)}$

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Units 3 & 4 > Area 1 > Sequence 1 > Concept 2

Logarithmic laws and equations Summary screen and practice questions

Exercise 1.3 Logarithmic laws and equations

Technology free

1. **WE6** Simplify the following without using a calculator.
   a. $\log_6(3) + \log_6(2)$
   b. $\log_{10}(5) + \log_{10}(2)$
   c. $\log_3(6) - \log_3(2)$
   d. $\log_2(10) - \log_2(5)$
   e. $\log_2(32)$
   f. $\log_3(81)$
   g. $\log_5\left(\frac{1}{5}\right)$
   h. $\log_3\left(\frac{1}{27}\right)$
2. Simplify the following.
   a. $\log_2\left(\sqrt{x}\right)$
   b. $\log_3\left(\sqrt[3]{x}\right)$
   c. $3\log_3\left(\sqrt[3]{x}\right)$
   d. $4\log_4\left(\sqrt[4]{x}\right)$
   e. $\log_2\left(\sqrt{\frac{x^4}{y^2}}\right)$
   f. $\log_3\left(\sqrt[5]{\frac{x^5}{y^{10}}}\right)$
3. **WE7** Simplify the following without using a calculator.
   a. $4\log_2(12) - 4\log_2(6)$
   b. $3\log_2(3) - 3\log_2(6)$
   c. $2 + \log_5(10) - \log_5(2)$
   d. $2 + \log_5(2) - \log_5(10)$
   e. $1 + \log_2(5)$
   f. $3 + \log_3(2)$
   g. $\frac{\log_2(64)}{\log_2(8)}$
   h. $\frac{\log_5(125)}{\log_5(25)}$
   i. $\frac{\log_a\left(\sqrt{x}\right)}{\log_a(x)}$
   j. $\frac{\log_a\left(x^2\right)}{\log_a\left(x^3\right)}$
4. Simplify without using a calculator.
   a. $5\log_3(x) + \log_3(x^2) - \log_3(x^7)$
   b. $3\log_2(x) + \log_2(x^3) - \log_2(x^6)$
   c. $3\log_4(x) - 5\log_4(x) + 2\log_4(x)$
   d. $4\log_6(x) - 5\log_6(x) + \log_6(x)$
   e. $\log_{10}(x^2) + 3\log_{10}(x) - 2\log_{10}(x)$
   f. $4\log_{10}(x) - \log_{10}(x) + \log_{10}(x^2)$
   g. $\log_5(x+1) + \log_5(x+1)^2$
   h. $\log_4(x-2)^3 - 2\log_4(x-2)$
5. **WE8** Solve the following for x. Give exact answers when appropriate; otherwise, give answers correct to 3 decimal places.
   a. $\log_e(x) = \log_e(2)$
   b. $\log_e(x) = \log_e(5)$
   c. $\log_e(x) + \log_e(3) = \log_e(9)$
   d. $\log_e(x) + \log_e(2) = \log_e(8)$
   e. $\log_e(x) - \log_e(5) = \log_e(2)$
   f. $\log_e(x) - \log_e(4) = \log_e(3)$
   g. $1 + \log_e(x) = \log_e(6)$
   h. $1 - \log_e(x) = \log_e(7)$
   i. $\log_e(4) - \log_e(x) = \log_e(2)$
   j. $\log_e(5) - \log_e(x) = \log_e(25)$
6. **WE9** Solve the following for x.
   a. $\log_5(125) = x$
   b. $\log_4(x-1) + 2 = \log_4(x+4)$
   c. $3\left(\log_2(x)\right)^2 - 2 = 5\log_2(x)$
   d. $\log_5(4x) + \log_5(x-3) = \log_5(7)$
7. Solve the following for x.
   a. $\log_3(x) = 5$
   b. $\log_3(x-2) - \log_3(5-x) = 2$
8. a. **WE10** Evaluate the following, correct to 4 decimal places.
      i. $\log_7(12)$
      ii. $\log_3\left(\frac{1}{4}\right)$
   b. If $z = \log_3(x)$, find the following in terms of z.
      i. $2x$
      ii. $\log_x(27)$

Technology active

9. Rewrite the following in terms of base 10.
   a. $\log_5(9)$
   b. $\log_{\frac{1}{2}}(12)$
10. Express each of the following in logarithmic form.
   a. $6^3 = 216$
   b. $2^8 = 256$
   c. $3^4 = 81$
   d. $10^{-4} = 0.0001$
   e. $5^{-3} = 0.008$
   f. $7^1 = 7$
11. Find the value of x.
   a. $\log_3(81) = x$
   b. $\log_6\left(\frac{1}{216}\right) = x$
   c. $\log_x(121) = 2$
   d. $\log_2(-x) = 7$
12. Simplify the following.
   a. $\log_2(256 + \log_2(64) - \log_2(128)$
   b. $5\log_7(49) - 5\log_7(343)$
   c. $\log_4\left(\sqrt[6]{\frac{1}{64}}\right)$
   d. $\log_4\left(\frac{16}{256}\right)$
   e. $\dfrac{\log_5(32)}{3\log_5(16)}$
   f. $\dfrac{6\log_2\left(\sqrt[3]{x}\right)}{\log_2\left(x^5\right)}$
13. Simplify the following.
   a. $\log_3(x-4) + \log_3(x-4)^2$
   b. $\log_7(2x+3)^3 - 2\log_7(2x+3)$
   c. $\log_5\left(x^2\right) + \log_5\left(x^3\right) - 5\log_5(x)$
   d. $\log_4(5x+1) + \log_4(5x+1)^3 - \log_4(5x+1)^2$
14. Evaluate the following, correct to 4 decimal places.
   a. $\log_3(7)$
   b. $\log_2\left(\frac{1}{121}\right)$
15. If $n = \log_5(x)$, find the following in terms of n.
   a. $5x$
   b. $\log_5\left(5x^2\right)$
   c. $\log_x(625)$
16. Solve the following for x.
   a. $\log_e(2x-1) = -3$
   b. $\log_e\left(\frac{1}{x}\right) = 3$
   c. $\log_3(4x-1) = 3$
   d. $\log_{10}(x) - \log_{10}(3) = \log_{10}(5)$
   e. $3\log_{10}(x) + 2 = 5\log_{10}(x)$
   f. $\log_{10}\left(x^2\right) - \log_{10}(x+2) = \log_{10}(x+3)$
   g. $2\log_5(x) - \log_5(2x-3) = \log_5(x-2)$
   h. $\log_{10}(2x) - \log_{10}(x-1) = 1$
   i. $\log_3(x) + 2\log_3(4) - \log_3(2) = \log_3(10)$
   j. $\left(\log_{10}(x)\right)\left(\log_{10}\left(x^2\right)\right) - 5\log_{10}(x) + 3 = 0$
   k. $\left(\log_3 x\right)^2 = \log_3(x) + 2$
   l. $\log_6(x-3) + \log_6(x+2) = 1$
17. Express y in terms of x for the following equations.
   a. $\log_{10}(y) = 2\log_{10}(2) - 3\log_{10}(x)$
   b. $\log_4(y) = -2 + 2\log_4(x)$
   c. $\log_9(3xy) = 1.5$
   d. $\log_8\left(\frac{2x}{y}\right) + 2 = \log_8(2)$
18. a. Find the value of x in terms of m for which $3\log_m(x) = 3 + \log_m 27$, where $m > 0$ and $x > 0$.
   b. If $\log_{10}(m) = x$ and $\log_{10}(n) = y$, show that $\log_{10}\left(\dfrac{100n^2}{m^5\sqrt{n}}\right) = 2 + \dfrac{3y}{2} - 5x$.
19. Solve the equation $8\log_x(4) = \log_2(x)$ for x.
20. Solve the following for x, correct to 3 decimal places.
   a. $e^{2x} - 3 = \log_e(2x+1)$
   b. $x^2 - 1 = \log_e(x)$
21. Find x, correct to 4 decimal places, if $\left(3\log_3(x)\right)\left(5\log_3(x)\right) = 11\log_3(x) - 2$.

1.4 Logarithmic scales

Many scientific quantities are measured in terms of scales that are logarithmic rather than linear.

For example, the pH scale used in chemistry to describe the acidity or alkalinity of a substance ranges from 1 for very strong acids through to 7 for neutral substances such as water, up to a pH of 14 for very strong bases or alkalis. This scale is based upon the concentration of hydrogen ions. A substance that has pH 5 is ten times more acidic than one that has a pH of 6, and has ten times the concentration of hydrogen ions. An acid with pH 3 has 10 000 times the acidity of water (pH 7).

Other examples of logarithmic scales include:

- the Richter scale, which describes the amount of energy released by the seismic waves of earthquakes in terms of magnitude
- loudness of sound as a function of the sound's intensity
- the frequencies of musical notes
- the intensity of the brightness of stars.

WORKED EXAMPLE 11

The apparent brightness, B of a star can be found using the formula $B = 6 - 2.5\log_{10}A$, where A is the actual brightness of that star. Find the apparent brightness of a star with actual brightness of 3.16.

THINK	WRITE
1. Write the formula.	$B = 6 - 2.5\ \log_{10}(A)$
2. Substitute 3.16 for A.	When $A = 3.16$,
	$B = 6 - 2.5\ \log_{10}(3.16)$
3. Evaluate $\log_{10}(3.16)$ using a graphics calculator.	$= 6 - 2.5 \times 0.5$
4. Simplify.	$= 4.75$
5. Write your answer in a sentence.	The apparent brightness of the star is 4.75.

WORKED EXAMPLE 12

Loudness, L, in decibels (dB), is related to the intensity, I, of a sound by the equation

$$L = 10\log_{10}\left(\frac{I}{I_0}\right)$$

where I_0 is equal to 10^{-12} watts per square metre $\left(\text{W/m}^2\right)$. (This value is the lowest intensity of sound that can be heard by human ears.)

a. An ordinary conversation has a loudness of 60 dB. Calculate the intensity in W/m^2.

b. If the intensity is doubled, what is the change in the loudness, correct to 2 decimal places?

THINK	WRITE
a. 1. Substitute $L = 60$ and simplify.	**a.** $L = 10\log_{10}\left(\frac{I}{I_0}\right)$
	$60 = 10\log_{10}\left(\frac{I}{10^{-12}}\right)$
	$60 = 10\log_{10}(10^{12}I)$
	$6 = \log_{10}(10^{12}I)$
2. Convert the logarithm to index form and solve for I.	$10^6 = 10^{12}I$
	$I = 10^{-6}\text{ W/m}^2$
b. 1. Determine an equation for L_1.	**b.** $L_1 = 10\log_{10}\left(\frac{I_1}{10^{-12}}\right)$
	$= 10\log_{10}\left(10^{12}I_1\right)$
	$= 10\log_{10}\left(10^{12}\right) + 10\log_{10}(I_1)$
	$= 120\log_{10}(10) + 10\log_{10}(I_1)$
	$= 120 + 10\log_{10}(I_1)$
2. The intensity has doubled; therefore, $I_2 = 2I_1$. Determine an equation for L_2.	$L_2 = 10\log_{10}\left(\frac{2I_1}{10^{-12}}\right)$
	$= 10\log_{10}(2\times 10^{12}I_1)$
	$= 10\log_{10}(2) + 10\log_{10}(10^{12}) + 10\log_{10}(I_1)$
	$= 3.010 + 120\log_{10}(10) + 10\log_{10}(I_1)$
	$= 3.01 + 120 + 10\log_{10}(I_1)$
3. Replace $120 + 10\log_{10}(I_1)$ with L_1.	$= 3.01 + L_1$
Answer the question.	Doubling the intensity increases the loudness by 3.01 dB.

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Units 3 & 4 > Area 1 > Sequence 1 > Concept 3 > **Logarithmic scales** Summary screen and practice questions

Exercise 1.4 Logarithmic scales

Technology active

1. **WE12** The loudness, L, of a jet taking off about 30 metres away is known to be 130 dB. Using the formula $L = 10\log_{10}\left(\frac{I}{I_0}\right)$, where I is the intensity measured in W/m^2 and I_0 is equal to 10^{-12} W/m^2, calculate the intensity in W/m^2 for this situation.

2. The moment magnitude scale measures the magnitude, M, of an earthquake in terms of energy released, E, in joules, according to the formula

$$M = 0.67 \log_{10}\left(\frac{E}{K}\right)$$

where K is the minimum amount of energy used as a basis of comparison.

An earthquake that measures 5.5 on the moment magnitude scale releases 10^{13} joules of energy. Find the value of K, correct to the nearest integer.

3. Two earthquakes, about 10 kilometres apart, occurred in Iran on 11 August 2012. One measured 6.3 on the moment magnitude scale, and the other was 6.4 on the same scale. Use the formula from question 2 to compare the energy released, in joules, by the two earthquakes.
4. An earthquake of magnitude 9.0 occurred in Japan in 2011, releasing about 10^{17} joules of energy. Use the formula from question 2 to find the value of K correct to 2 decimal places.
5. To the human ear, how many decibels louder is a 500 W/m^2 amplifier compared to a 20 W/m^2 model? Use the formula $L = 10 \log_{10}\left(\frac{I}{I_0}\right)$, where L is measured in dB, I is measured in W/m^2 and $I_0 = 10^{-12}$ W/m^2. Give your answer correct to 2 decimal places.
6. Your eardrum can be ruptured if it is exposed to a noise that has an intensity of 10^4 W/m^2. Using the formula $L = 10 \log_{10}\left(\frac{I}{I_0}\right)$, where I is the intensity measured in W/m^2 and I_0 is equal to 10^{-12} W/m^2, calculate the loudness, L, in decibels that would cause your eardrum to be ruptured.

Questions 7–9 relate to the following information.
Chemists define the acidity or alkalinity of a substance according to the formula

$$\text{pH} = -\log_{10}\left[H^+\right]$$

where $\left[H^+\right]$ is the hydrogen ion concentration measured in moles/litre. Solutions with a pH less than 7 are acidic, whereas solutions with a pH greater than 7 are basic. Solutions with a pH of 7, such as pure water, are neutral.

7. Lemon juice has a hydrogen ion concentration of 0.001 moles/litre. Find the pH and determine whether lemon juice is acidic or basic.
8. Find the hydrogen ion concentration for each of the following.
   a. Battery acid has a pH of 0.
   b. Tomato juice has a pH of 4.
   c. Sea water has a pH of 8.
   d. Soap has a pH of 12.
9. Hair conditioner works on hair in the following way. Hair is composed of the protein called keratin, which has a high percentage of amino acids. These acids are negatively charged. Shampoo is also negatively charged. When shampoo removes dirt, it removes natural oils and positive charges from the hair. Positively charged surfactants in hair conditioner are attracted to the negative charges in the hair, so the surfactants can replace the natural oils.

   a. A brand of hair conditioner has a hydrogen ion concentration of 0.000 015 8 moles/litre. Calculate the pH of the hair conditioner.
   b. A brand of shampoo has a hydrogen ion concentration of 0.000 002 75 moles/litre. Calculate the pH of the shampoo.

10. The number of atoms of a radioactive substance present after t years is given by

$$N(t) = N_0 e^{-mt}.$$

a. The half-life is the time taken for the number of atoms to be reduced to 50% of the initial number of atoms. Show that the half-life is given by $\dfrac{\log_e(2)}{m}$.

b. Radioactive carbon-14 has a half-life of 5750 years. The percentage of carbon-14 present in the remains of plants and animals is used to determine how old the remains are. How old is a skeleton that has lost 70% of its carbon-14 atoms? Give your answer correct to the nearest year.

11. A basic observable quantity for a star is its brightness. The apparent magnitudes m_1 and m_2 for two stars are related to the corresponding brightnesses, b_1 and b_2, by the equation

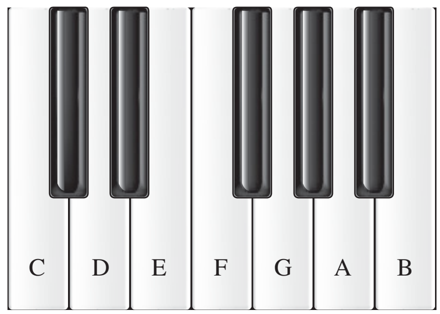
$$m_2 - m_1 = 2.5 \log_{10}\left(\frac{b_1}{b_2}\right).$$

The star Sirius is the brightest star in the night sky. It has an apparent magnitude of −1.5 and a brightness of −30.3. The planet Venus has an apparent magnitude of −4.4. Calculate the brightness of Venus, correct to 2 decimal places.

12. Octaves in music can be measured in cents, n. The frequencies of two notes, f_1 and f_2, are related by the equation

$$n = 1200 \log_{10}\left(\frac{f_2}{f_1}\right).$$

Middle C on the piano has a frequency of 256 hertz; the C an octave higher has a frequency of 512 hertz. Calculate the number of cents between these two Cs.



13. Prolonged exposure to sounds above 85 decibels can cause hearing damage or loss. A gunshot from a 0.22 rifle has an intensity of about $(2.5 \times 10^{13})\, I_0$. Calculate the loudness, in decibels, of the gunshot sound and state if ear protection should be worn when a person goes to a rifle range for practice shooting. Use the formula $L = 10 \log_{10}\left(\frac{I}{I_0}\right)$, where I_0 is equal to 10^{-12} W/m^2, and give your answer correct to 2 decimal places.

14. Early in the 20th century, San Francisco had an earthquake that measured 8.3 on the magnitude scale. In the same year, another earthquake was recorded in South America that was four times stronger than the one in San Francisco. Using the equation $M = 0.67 \log_{10}\left(\frac{E}{K}\right)$, calculate the magnitude of the earthquake in South America, correct to 1 decimal place.

1.5 Indicial equations

An **indicial equation** is an equation in which the power or index contains the unknown.

When we solve an indicial equation such as $3^x = 81$, the technique is to convert both sides of the equation to the same base. For example, $3^x = 3^4$; therefore, $x = 4$.

When we solve an equation such as $x^3 = 27$, we write both sides of the equation with the same index. In this case, $x^3 = 3^3$; therefore, $x = 3$.

If an equation such as $5^{2x} = 2$ is to be solved, then we must use logarithms, as the sides of the equation cannot be converted to the same base or index. To remove x from the power, we take the logarithm of both sides.

$$\log_5\left(5^{2x}\right) = \log_5(2)$$
$$2x = \log_5(2)$$
$$x = \frac{1}{2}\log_5(2)$$

Notes:

1. If $a^x = b$, a solution for x exists only if $b > 0$.
2. $\log_5 2$ can be calculated using a graphics calculator or as $\dfrac{\log_{10} 2}{\log_{10} 5}$ using a standard calculator.

WORKED EXAMPLE 13

Solve the following equations for x, giving your answers in exact form.

a. $4^{3x} \times 16^{3-x} = 256$

b. $7^{x-3} - 3 = 0$

c. $(5^x - 25)(5^x + 1) = 0$

d. $3^{2x} - 9(3^x) + 14 = 0$

THINK	WRITE
a. 1. Convert the numbers to the same base.	**a.** $4^{3x} \times 16^{3-x} = 256$ $4^{3x} \times (4^2)^{3-x} = 4^4$
2. Simplify and add the indices on the left-hand side of the equation.	$4^{3x} \times 4^{6-2x} = 4^4$ $4^{x+6} = 4^4$
3. As the bases are the same, equate the indices and solve the equation.	$x + 6 = 4$ $x = -2$
b. 1. Rearrange the equation.	**b.** $7^{x-3} - 3 = 0$ $7^{x-3} = 3$
2. Take the logarithm of both sides to base 7 and simplify.	$\log_7\left(7^{x-3}\right) = \log_7(3)$ $x - 3 = \log_7(3)$
3. Solve the equation.	$x = \log_7(3) + 3$
c. 1. Apply the Null Factor Law to solve each bracket.	**c.** $(5^x - 25)(5^x + 1) = 0$ $5^x - 25 = 0$ or $5^x + 1 = 0$ $5^x = 25$ $\quad$ $5^x = -1$
2. Convert 25 to base 5. $5^x > 0$, so there is no real solution for $5^x = -1$.	$5^x = 5^2$ $x = 2$
d. 1. Let $a = 3^x$ and substitute into the equation to create a quadratic to solve.	**d.** $3^{2x} - 9(3^x) + 14 = 0$ Let $a = 3^x$. $a^2 - 9a + 14 = 0$

2. Factorise the left-hand side.

$(a-7)(a-2)=0$

3. Apply the Null Factor Law to solve each bracket for a.

$a-7=0 \quad \text{or} \quad a-2=0$

$a=7 \qquad\qquad a=2$

4. Substitute back in for a.

$3^x=7 \text{ or } 3^x=2$

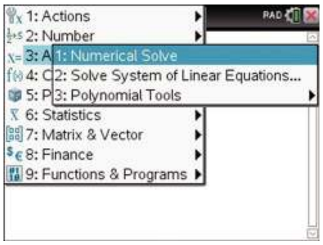
5. Take the logarithm of both sides to base 3 and simplify.

$\log_3(3^x)=\log_3(7) \qquad \log_3(3^x)=\log_3(2)$

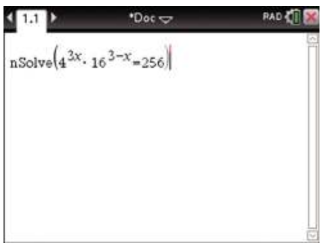
$x=\log_3(7) \qquad x=\log_3(2)$

TI | THINK **WRITE**

a.1. On a Calculator page, press MENU then select:
3: Algebra
1: Numerical Solve.



2. Complete the entry line as:
nSolve
$(4^{3x}\times16^{3-x}=256,x)$
Then press the ENTER button.

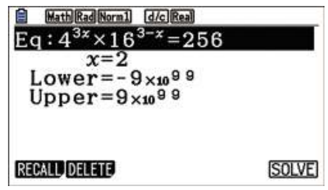


3. The answer appears on the screen.

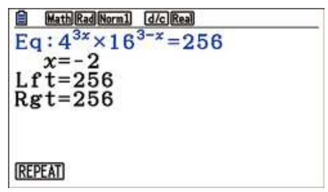
CASIO | THINK **WRITE**

a.1. On an Equation screen, select Solver by pressing F3.

2. Complete the entry line as:
$4^{3x}\times16^{3-x}=256$
When the equation has been entered, press the SOLVE button.



3. The answer appears on the screen.



WORKED EXAMPLE 14

A tennis ball is dropped from a height of 100 cm, bounces and rebounds to 80% of its previous height. The height, h cm, of the ball after n bounces is given by the formula $h=A\times a^n$, where A cm is the height from which the ball is dropped and a is the percentage of the height reached by the ball on the previous bounce.

a. Find the values of A and a, and hence write the formula for h in terms of n.

b. What height will the ball reach after 5 bounces? Give the answer to 1 decimal place.

c. How many bounces will it take before the ball reaches less than 1 cm?

THINK	WRITE
a. 1. The ball is dropped from a height of A cm.	**a.** $A = 100$
2. The percentage of the height reached by the ball on the previous bounce is a.	$a = 80\% = \frac{80}{100} = 0.8$
3. Substitute the values for A and a into the formula $h = A \times a^n$.	$h = A \times a^n$ $h = 100 \times (0.8)^n$
b. 1. Substitute 5 for n.	**b.** $h = 100 \times (0.8)^5$
2. Evaluate using a calculator.	$= 32.768$
3. Write your answer in a sentence.	The ball bounces to 32.8 cm after 5 bounces.
c. 1. Substitute 1 for h.	**c.** $1 = 100 \times (0.8)^n$
2. Divide both sides by 100.	$(0.8)^n = 0.01$
3. Take thet log of both sides to base 10.	$\log_{10}(0.8)^n = \log_{10}(0.01)$
4. Use $\log_a(m^p) = p\log_a(m)$ to simplify.	$n\log_{10}(0.8) = \log_{10}(0.01)$
5. Divide both sides by $\log_{10}(0.8)$.	$n = \dfrac{\log_{10}(0.01)}{\log_{10}(0.8)}$
6. Evaluate.	$= 20.64$
7. Bounces must be in whole numbers.	≈ 21
8. After 20 bounces the ball reaches more than 1 cm, but after 21 bounces the ball reaches less than 1 cm because it bounces to a smaller and smaller height. Write the answer in a sentence.	The ball reaches less than 1 cm after 21 bounces.

studyon

Units 3 & 4 > Area 1 > Sequence 1 > Concept 4 > **Indicial equations** Summary screen and practice questions

Exercise 1.5 Indicial equations

Technology free

1. **WE13** Solve the following equations for x.
   a. $3^{2x+1} \times 27^{2-x} = 81$
   b. $10^{2x-1} - 5 = 0$
   c. $(4^x - 16)(4^x + 3) = 0$
   d. $2\left(10^{2x}\right) - 7(10^x) + 3 = 0$
2. Solve the following equations for x.
   a. $2^{x+3} - \frac{1}{64} = 0$
   b. $2^{2x} - 9 = 0$
   c. $3e^{2x} - 5e^x - 2 = 0$
   d. $e^{2x} - 5e^x = 0$
3. Solve the following equations for x.
   a. $7^{2x-1} = 5$
   b. $(3^x - 9)(3^x - 1) = 0$
   c. $25^x - 5^x - 6 = 0$
   d. $6\left(9^{2x}\right) - 19(9^x) + 10 = 0$
4. Solve the following equations for x.
   a. $16 \times 2^{2x+3} = 8^{-2x}$
   b. $2 \times 3^{x+1} = 4$
   c. $2(5^x) - 12 = -\frac{10}{5^x}$
   d. $4^{x+1} = 3^{1-x}$

5. Solve the following equations for x.

a. $2\left(2^{x-1} - 3\right) + 4 = 0$

b. $2\left(5^{1-2x}\right) - 3 = 7$

6. a. Simplify $x^{-1} - \dfrac{1}{1 - \frac{1}{1+x^{-1}}}$.

b. Solve $2^{3-4x} \times 3^{-4x+3} \times 6^{x^2} = 1$ for x.

7. Solve the following equations for x.

a. $e^{x-2} - 2 = 7$

b. $e^{\frac{x}{4}} + 1 = 3$

c. $e^{2x} = 3e^{x}$

d. $e^{x^2} + 2 = 4$

8. Solve the following equations for x.

a. $e^{2x} = e^{x} + 12$

b. $e^{x} = 12 - 32e^{-x}$

c. $e^{2x} - 4 = 2e^{x}$

d. $e^{x} - 12 = -\dfrac{5}{e^{x}}$

Technology active

9. If $y = m(10)^{nx}$, $y = 20$ when $x = 2$ and $y = 200$ when $x = 4$, find the values of the constants m and n.

10. Solve the following for x, correct to 3 decimal places.

a. $2^{x} < 0.3$

b. $(0.4)^{x} < 2$

11. Solve $\left(\log_3(4m)\right)^2 = 25n^2$ for m.

12. Solve the following for x.

a. $e^{m-kx} = 2n$, where $k \in R\setminus\{0\}$ and $n \in R^{+}$

b. $8^{mx} \times 4^{2n} = 16$, where $m \in R\setminus\{0\}$

c. $2e^{mx} = 5 + 4e^{-mx}$, where $m \in R\setminus\{0\}$

13. **WE14** The diameter of a tree trunk increases according to the formula $D = A \times 10^{0.04t}$, where D cm is the diameter of the trunk t years after it is first measured and A cm is the diameter of the trunk when it is first measured.

a. Write an equation for D in terms of t if the trunk had a diameter of 20 cm when it was first measured.

b. When will the diameter be 25 cm?

c. After how many years will the diameter be greater than 30 cm?

14. If $y = ae^{-kx}$, $y = 3.033$ when $x = 2$ and $y = 1.1157$ when $x = 6$, find the values of the constants a and k. Give your answers correct to 2 decimal places.

15. The compound interest formula $A = Pe^{rt}$ is an indicial equation. If a principal amount of money, P, is invested for 5 years, the interest earned is \$12 840.25, but if this same amount is invested for 7 years, the interest earned is \$14 190.66. Find the integer rate of interest and the principal amount of money invested, to the nearest dollar.

1.6 Logarithmic graphs

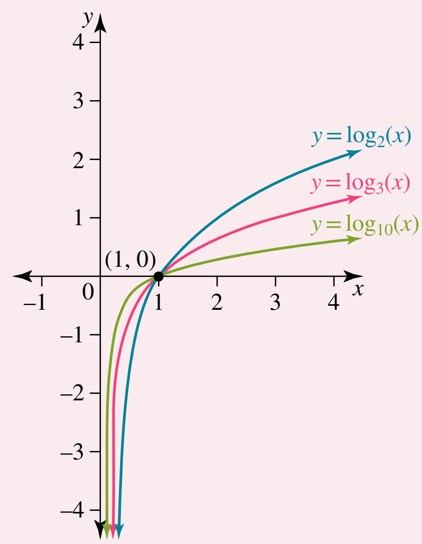
1.6.1 The graph of $y = \log_a(x)$

The graph of the logarithmic function $f: R^+ \rightarrow R, f(x) = \log_a(x), a > 1$ has the following characteristics.

The graph of $y = \log_a(x)$

For $f(x) = \log_a(x), a > 1$:

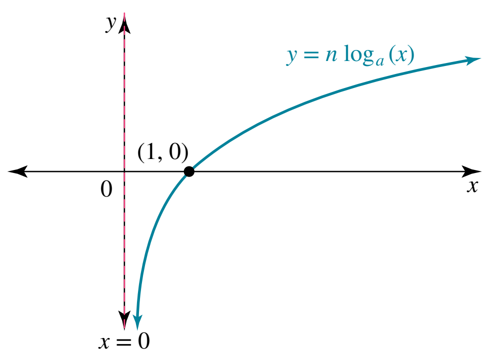
- the domain is $(0, \infty)$
- the range is R
- the graph is an increasing function
- the graph cuts the x-axis at $(1, 0)$
- as $x \rightarrow 0, y \rightarrow -\infty$, so the line $x = 0$ is an asymptote
- as a increases, the graph rises more steeply for $x \in (0, 1)$ and is flatter for $x \in (1, \infty)$.

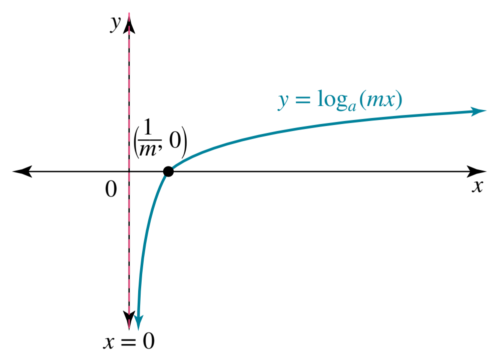


1.6.2 Dilations

Graphs of the form $y = n\log_a(x)$ and $y = \log_a(mx)$

The graph of $y = n\log_a(x)$ is the basic graph of $y = \log_a x$ dilated by factor n parallel to the y-axis or from the x-axis. The graph of $y = \log_a(mx)$ is the basic graph of $y = \log_a(x)$ dilated by factor $\frac{1}{m}$ parallel to the x-axis or from the y-axis. The line $x = 0$ (the y-axis) remains the vertical asymptote and the domain remains $(0, \infty)$.



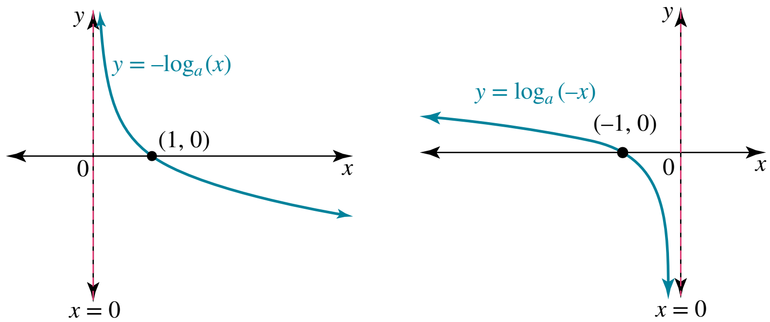


1.6.3 Reflections

Graphs of the form $y = -\log_a(x)$ and $y = \log_a(-x)$

The graph of $y = -\log_a(x)$ is the basic graph of $y = \log_a(x)$ reflected in the x-axis. The line $x = 0$ (the y-axis) remains the vertical asymptote and the domain remains $(0, \infty)$.

The graph of $y = \log_a(-x)$ is the basic graph of $y = \log_a(x)$ reflected in the y-axis. The line $x = 0$ (the y-axis) remains the vertical asymptote, but the domain changes to $(-\infty, 0)$.

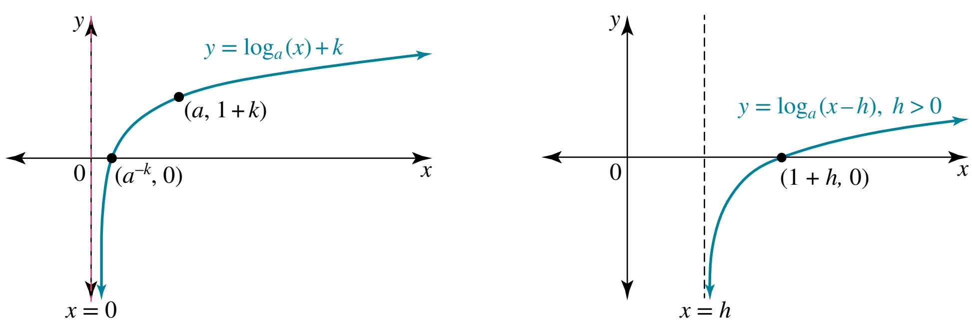


1.6.4 Translations

Graphs of the form $y = \log_a(x) + k$ and $y = \log_a(x - h)$

The graph of $y = \log_a(x) + k$ is the basic graph of $y = \log_a(x)$ translated k units parallel to the y-axis. Thus the line $x = 0$ (the y-axis) remains the vertical asymptote and the domain remains $(0, \infty)$.

The graph of $y = \log_a(x - h)$ is the basic graph of $y = \log_a x$ translated h units parallel to the x-axis. Thus, the line $x = 0$ (the y-axis) is no longer the vertical asymptote. The vertical asymptote is $x = h$ and the domain is (h, ∞).



WORKED EXAMPLE 15

Sketch the graphs of the following, showing all important characteristics. State the domain and range in each case.

a. $y = \log_e(x - 2)$

b. $y = \log_e(x + 1) + 2$

c. $y = \frac{1}{4}\log_e(2x)$

d. $y = -\log_e(-x)$

THINK

a. 1. The basic graph of $y = \log_e(x)$ has been translated 2 units to the right, so $x = 2$ is the vertical asymptote.

WRITE

a. $y = \log_e(x - 2)$
The domain is $(2, \infty)$.
The range is R.

2. Find the x-intercept.

x-intercept, $y = 0$:

$\log_e(x-2) = 0$

$e^0 = x - 2$

$1 = x - 2$

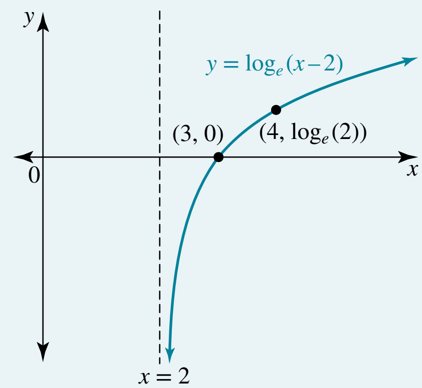
$x = 3$

3. Determine another point through which the graph passes.

When $x = 4$, $y = \log_e(2)$.

The point is $\left(4, \log_e(2)\right)$.

4. Sketch the graph.



b. 1. The basic graph of $y = \log_e(x)$ has been translated 2 units up and 1 unit to the left, so $x = -1$ is the vertical asymptote.

b. $y = \log_e(x+1) + 2$

The domain is $(-1, \infty)$.

The range is R.

2. Find the x-intercept.

The graph cuts the x-axis where $y = 0$.

$\log_e(x+1) + 2 = 0$

$\log_e(x+1) = -2$

$e^{-2} = x + 1$

$x = e^{-2} - 1$

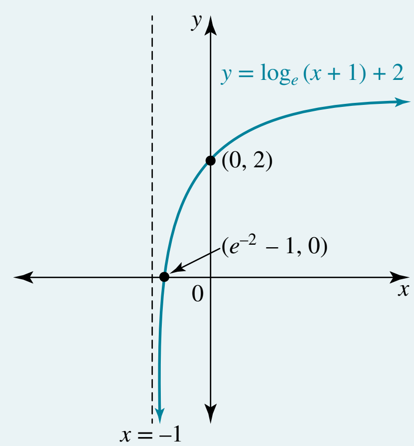
3. Find the y-intercept.

The graph cuts the y-axis where $x = 0$.

$y = \log_e(1) + 2$

$= 2$

4. Sketch the graph.



c. 1. The basic graph of $y = \log_e(x)$ has been dilated by factor $\frac{1}{4}$ from the x-axis and by factor $\frac{1}{2}$ from the y-axis. The vertical asymptote remains $x = 0$.

c. $y = \frac{1}{4}\log_e(2x)$

The domain is $(0, \infty)$.

The range is R.

2. Find the x-intercept.

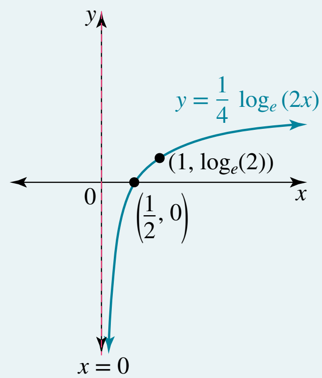
x-intercept, $y = 0$:

$$\frac{1}{4}\log_e(2x) = 0$$
$$\log_e(2x) = 0$$
$$e^0 = 2x$$
$$1 = 2x$$
$$x = \frac{1}{2}$$

3. Determine another point through which the graph passes.

When $x = 1, y = \log_e(2)$.
The point is $\left(1, \log_e(2)\right)$.

4. Sketch the graph.



d. 1. The basic graph of $y = \log_e(x)$ has been reflected in both axes. The vertical asymptote remains $x = 0$.

d. $y = -\log_e(-x)$
The domain is $(-\infty, 0)$.
The range is R.

2. Find the x-intercept.

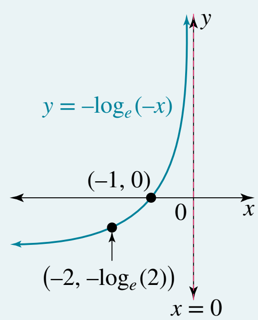
x-intercept, $y = 0$:

$$-\log_e(-x) = 0$$
$$\log_e(-x) = 0$$
$$e^0 = -x$$
$$x = -1$$

3. Determine another point through which the graph passes.

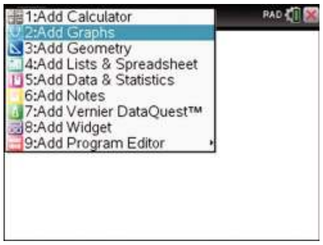
When $x = -2, y = -\log_e(2)$.
The point is $\left(-2, -\log_e(2)\right)$.

4. Sketch the graph.

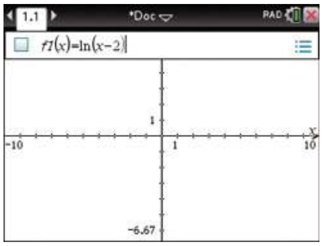


TI | THINK	WRITE

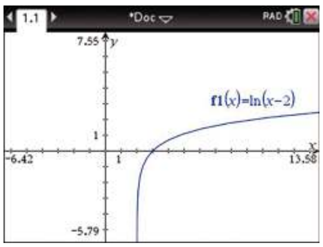
a. 1. On a Calculator page, press MENU then select:
2: Add Graphs.



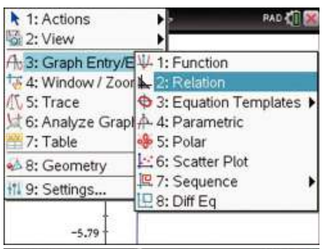
2. Complete the entry line in the $f1(x) =$ tab as:
$\ln(x - 2)$



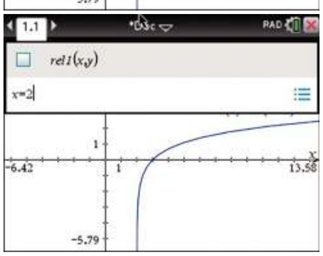
3. Sketch the graph by pressing the ENTER button.



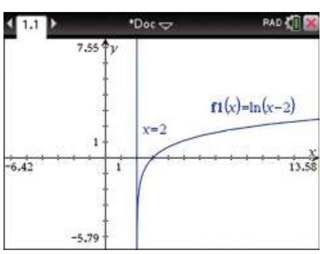
4. To sketch the vertical asymptote, $x = 2$, select:
Menu
3: Graph Entry/Edit
2: Relation.



5. Complete the entry line as:
$x = 2$
then press the ENTER button.

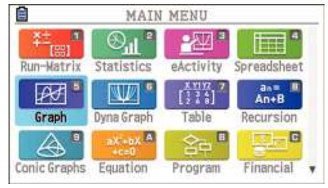


6. The sketch of the graph and asymptote will appear on the screen as one.

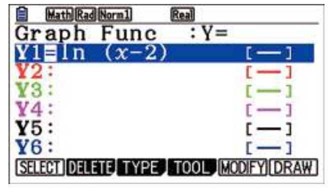


CASIO | THINK	WRITE

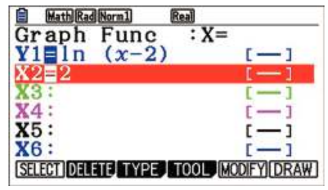
a. 1. On a Main Menu screen, select Graph.



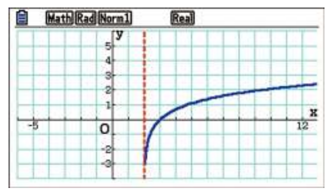
2. Complete the function entry line in the Y1 tab as:
$\ln(x - 2)$



3. The asymptote can be included in the sketch by changing the Y2 line to X2.
To do this, select:
TYPE
X =
Complete the entry line in X = as 2.



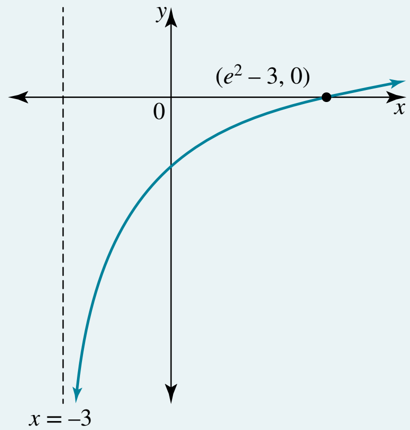
4. Sketch the graph by pressing either the DRAW button or the EXE button.



The situation may arise where you are given the graph of a translated logarithmic function and you are required to find the rule. The information provided to you might include the equation of the asymptote, the intercepts and/or other points on the graph. As a rule, the number of pieces of information is equivalent to the number of unknowns in the equation.

WORKED EXAMPLE 16

The rule for the function shown is of the form $y = \log_e (x - a) + b$. Find the values of the constants a and b.



THINK	WRITE
1. The vertical asymptote corresponds to the value of a.	The vertical asymptote is $x = -3$. Therefore, a must be -3. So $y = \log_e (x + 3) + b$.
2. Substitute in the x-intercept to find b.	The graph cuts the x-axis at $\left(e^2 - 3, 0\right)$. $0 = \log_e \left(e^2 - 3 + 3\right) + b$ $-b = \log_e \left(e^2\right)$ $-b = 2$ $b = -2$ So $y = \log_e (x + 3) - 2$.
3. Write the answer.	$a = -3, b = -2$

on Resources

Interactivity: Logarithmic graphs (int-6418)

studyon

Units 3 & 4 > Area 1 > Sequence 1 > Concept 5 > **Logarithmic graphs** Summary screen and practice questions

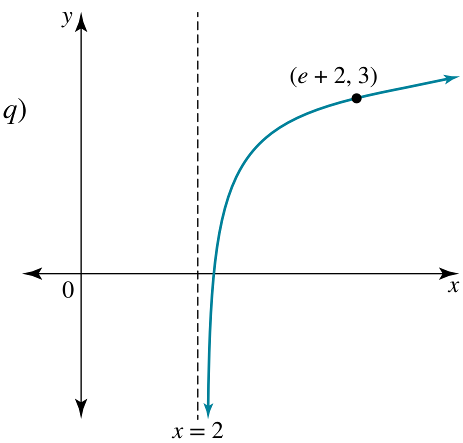
Exercise 1.6 Logarithmic graphs

Technology free

1. **WE15** Sketch the graphs of the following functions, showing all important characteristics. State the domain and range for each graph.
   a. $y = \log_e (x + 4)$
   b. $y = \log_e (x) + 2$
   c. $y = 4 \log_e (x)$
   d. $y = -\log_e (x - 4)$
2. Sketch the graphs of the following functions, showing all important characteristics.
   a. $y = \log_3 (x + 2) - 3$
   b. $y = 3 \log_5 (2 - x)$
   c. $y = 2 \log_{10} (x + 1)$
   d. $y = \log_2 \left(-\dfrac{x}{2}\right)$

Technology active

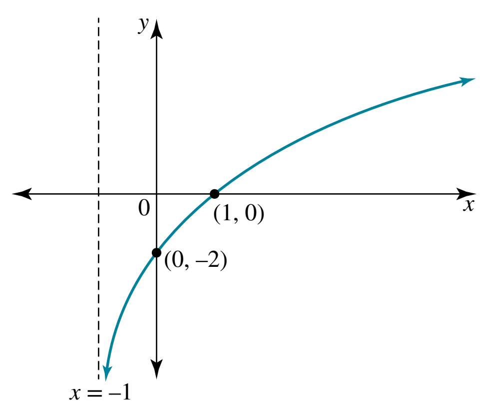
3. WE16 The rule for the function shown is $y = \log_e(x - m) + n$. Find the values of the constants m and n.



4. A logarithmic function with the rule of the form $y = p\log_e(x - q)$ passes through the points $(0, 0)$ and $(1, -0.35)$. Find the values of the constants p and q.
5. Sketch the following graphs, clearly showing any axis intercepts and asymptotes.
   a. $y = \log_e(x) + 3$
   b. $y = \log_e(x) - 5$
   c. $y = \log_e(x) + 0.5$
6. Sketch the following graphs, clearly showing any axis intercepts and asymptotes.
   a. $y = \log_e(x - 4)$ b. $y = \log_e(x + 2)$ c. $y = \log_e(x + 0.5)$
7. Sketch the following graphs, clearly showing any axis intercepts and asymptotes.
   a. $y = \frac{1}{4}\log_e(x)$ b. $y = 3\log_e(x)$ c. $y = 6\log_e(x)$
8. Sketch the following graphs, clearly showing any axis intercepts and asymptotes.
   a. $y = \log_e(3x)$ b. $y = \log_e\left(\dfrac{x}{4}\right)$ c. $y = \log_e(4x)$
9. Sketch the following graphs, clearly showing any axis intercepts and asymptotes.
   a. $y = 1 - 2\log_e(x - 1)$ b. $y = \log_e(2x + 4)$ c. $y = \dfrac{1}{2}\log_e\left(\dfrac{x}{4}\right) + 1$
10. For each of the following functions, state the domain and range. Define the inverse function, f^{-1}, and state the domain and range in each case. (*Hint:* Recall your study of inverse functions from Units 1 and 2.)
    a. $f(x) = 2\log_e(3x + 3)$ b. $f(x) = \log_e(2(x - 1)) + 2$ c. $f(x) = 2\log_e(1 - x) - 2$
11. For each the functions in question 10, sketch the graphs of f and f^{-1} on the same set of axes. Give the coordinates of any points of intersection, correct to 2 decimal places.
12. The equation $y = a\log_e(bx)$ relates x to y. The table shows values for x and y.

x	1	2	3
y	$\log_e(2)$	0	w

    a. Find the integer values of the constants a and b.
    b. Find the value of w correct to 4 decimal places.
13. The graph of a logarithmic function of the form $y = a\log_e(x - h) + k$ is shown. Find the values of a, h and k.



14. The graph of $y = m\log_2(nx)$ passes through the points $(-2, 3)$ and $\left(-\dfrac{1}{2}, \dfrac{1}{2}\right)$. Show that the values of m and n are 1.25 and $-2^{\frac{7}{5}}$ respectively.
15. Solve the following equations for x. Give your answers correct to 3 decimal places,
    a. $x - 2 = \log_e(x)$ b. $1 - 2x = \log_e(x - 1)$
16. Solve the following equations for x. Give your answers correct to 3 decimal places,
    a. $x^2 - 2 < \log_e(x)$ b. $x^3 - 2 \le \log_e(x)$

1.7 Applications

Logarithmic functions can be used to model many real-life situations in which there is continuous growth or decay over time. Examples of this can be as diverse as the rate at which a hot cup of tea cools down, the increasing account balance of a long-term bank investment or the spread of bacteria through a population.

As logarithmic functions are essentially the inverse of exponential functions, they can be used to solve exponential functions of the form $A = A_0 e^{kt}$, where A_0 represents the initial value, t represents the time taken and k represents the rate constant.

In order to determine the value of t in such a situation, the equation is first rearranged:

$$\frac{A}{A_0} = e^{kt}$$

Then we take the natural logarithm of both sides and rearrange for t:

$$\log_e\left(\frac{A}{A_0}\right) = kt$$
$$t = \frac{1}{k}\log_e\left(\frac{A}{A_0}\right)$$

WORKED EXAMPLE 17

If P dollars is invested into an account that earns interest at a rate of r for t years and the interest is compounded continuously, then $A = Pe^{rt}$, where A is the accumulated dollars.

A deposit of \$6000 is invested at the Western Bank, and \$9000 is invested at the Common Bank at the same time. Western offers compound interest continuously at a nominal rate of 6%, whereas the Common Bank offers compound interest continuously at a nominal rate of 5%. In how many years, correct to 1 decimal place, will the two investments be the same?

THINK	WRITE
1. Write the compound interest equation for each of the two investments.	$A = Pe^{rt}$ Western Bank: $A = 6000e^{0.06t}$ Common Bank: $A = 9000e^{0.05t}$
2. Equate the two equations and solve for t. CAS could also be used to determine the answer.	$6000e^{0.06t} = 9000e^{0.05t}$ $\frac{e^{0.06t}}{e^{0.05t}} = \frac{9000}{6000}$ $e^{0.01t} = \frac{3}{2}$ $0.01t = \log_e\left(\frac{3}{2}\right)$ $0.01t = 0.4055$ $t = \frac{0.4055}{0.01}$ $t = 40.5$ years

WORKED EXAMPLE 18

A coroner uses a formula derived from Newton's Law of Cooling to calculate the elapsed time since a person died. The formula is

$$t = -10 \log_e \left(\frac{T - R}{37 - R}\right)$$

where t is the time in hours since the death, T is the body's temperature measured in °C and R is the constant room temperature in °C. An accountant stayed late at work one evening and was found dead in his office the text morning. At 10 am the coroner measured the body temperature to be 29.7 °C. A second reading at noon found the body temperature to be 28 °C. Assuming that the office temperature was constant at 21 °C, determine the accountant's estimated time of death.

THINK	WRITE
1. Determine the time of death for the 10 am information. $R = 21$ °C and $T = 29.7$ °C. Substitute the values into the equation and evaluate.	$t = -10 \log_e \left(\frac{T - R}{37 - R}\right)$ $t = -10 \log_e \left(\frac{29.7 - 21}{37 - 21}\right)$ $= -10 \log_e \left(\frac{8.7}{16}\right)$ $= -10 \log_e (0.54375)$ $= 6.09$ h
2. Determine the time of death for the 12 pm information. $R = 21$ °C and $T = 28$ °C. Substitute the values into the equation and evaluate.	$t = -10 \log_e \left(\frac{T - R}{37 - R}\right)$ $t = -10 \log_e \left(\frac{28 - 21}{37 - 21}\right)$ $= -10 \log_e \left(\frac{7}{16}\right)$ $= -10 \log_e (0.4376)$ $= 8.27$h
3. Determine the estimated time of death for each reading.	$10 - 6.09 = 3.91$ or 3.55 am $12 - 8.27 = 3.73$ or 3.44 am
4. Write the answer.	The estimated time of death is between 3.44 and 3.55 am.

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Units 3 & 4 > Area 1 > Sequence 1 > Concept 6

Applications of logarithms Summary screen and practice questions

Exercise 1.7 Applications

Technology active

1. WE17 A deposit of \$4200 is invested at the Western Bank, and \$5500 is invested at the Common Bank at the same time. Western offers compound interest continuously at a nominal rate of 5%, whereas the Common bank offers compound interest continuously at a nominal rate of 4.5%. In how many years will the two investments be the same? Give your answer to the nearest year.
2. a. An investment triples in 15 years. What is the interest rate that this investment earns if it is compounded continuously? Give your answer correct to 2 decimal places.
   b. An investment of \$2000 earns 4.5% interest compounded continuously. How long will it take for the investment to grow to \$9000? Give your answer to the nearest month.
3. WE18 An elderly person was found deceased by a family member. The two had spoken on the telephone the previous evening around 7 pm. The coroner attended and found the body temperature to be 25 °C at 9 am. If the house temperature had been constant at 20 °C, calculate how long after the telephone call the elderly person died. Use Newton's Law of Cooling, $t = -10\log_e\left(\frac{T-R}{37-R}\right)$, where R is the room temperature in °C and T is the body temperature, also in °C.
4. The number of parts per million, n, of a fungal bloom in a stream t hours after it was detected can be modelled by $n(t) = \log_e\left(t + e^2\right), t \geq 0$.
   a. How many parts per million were detected initially?
   b. How many parts of fungal bloom are in the stream after 12 hours? Give your answer to 2 decimal places.
   c. How long will it take before there are 4 parts per million of the fungal bloom? Give your answer correct to 1 decimal place.
5. If \$1000 is invested for 10 years at 5% interest compounded continuously, how much money will have accumulated after the 10 years?
6. Let $P(t) = 200^{kt} + 1000$ represent the number of bacteria present in a petri dish after t hours. Suppose the number of bacteria trebles every 8 hours. Find the value of the constant k correct to 4 decimal places.

7. An epidemiologist studying the progression of a flu epidemic decides that the function

$$P(t) = \frac{3}{4}\left(1 - e^{-kt}\right), k > 0$$

will be a good model for the proportion, $P(t)$, of the earth's population that will contract the flu after t months. If after 3 months $\frac{1}{1500}$ of the earth's population has the flu, find the value of the constant k, correct to 4 decimal places.
8. Carbon-14 dating works by measuring the amount of carbon-14, a radioactive element, that is present in a fossil. All living things have a constant level of carbon-14 in them.
   Once an organism dies, the carbon-14 in its body starts to decay according to the rule

$$Q = Q_0 e^{-0.000124t}$$

where t is the time in years since death, Q_0 is the amount of carbon-14 in milligrams present at death and Q is the quantity of carbon-14 in milligrams present after t years.

a. If it is known that a particular fossil initially had 100 milligrams of carbon-14, how much carbon-14, in milligrams, will be present after 1000 years? Give your answer correct to 1 decimal place.
b. How long will it take before the amount of carbon-14 in the fossil is halved? Give your answer correct to the nearest year.

9. Glottochronology is a method of dating a language at a particular stage, based on the theory that over a long period of time linguistic changes take place at a fairly constant rate. Suppose a particular language originally has W_0 basic words and that at time t, measured in millennia, the number, $W(t)$, of basic words in use is given by $W(t) = W_0 (0.805)^t$.
a. Calculate the percentage of basic words lost after ten millennia.
b. Calculate the length of time it would take for the number of basic words lost to be one-third of the original number of basic words. Give your answer correct to 2 decimal places.

10. The mass, M grams, of a radioactive element, is modelled by the rule

$$M = a - \log_e (t + b)$$

where t is the time in years. The initial mass is 7.8948 grams, and after 80 years the mass is 7.3070 grams.
a. Find the equation of the mass remaining after t years. Give a correct to 1 decimal place and b as an integer.
b. Find the mass remaining after 90 years.

11. The population, P, of trout at a trout farm is declining due to deaths of a large number of fish from fungal infections.
The population is modelled by the function

$$P = a \log_e (t) + c$$

where t represents the time in weeks since the infection started. The population of trout was 10 000 after 1 week and 6000 after 4 weeks.

a. Find the values of the constants a and c. Give your answers correct to 1 decimal place where appropriate.
b. Find the number of trout, correct to the nearest whole trout, after 8 weeks.
c. If the infection remains untreated, how long will it take for the population of trout to be less than 1000? Give your answer correct to 1 decimal place.

12. In her chemistry class, Hei is preparing a special solution for an experiment that she has to complete. The concentration of the solution can be modelled by the rule

$$C = A \log_e (kt)$$

where C is the concentration in moles per litre (M) and t represents the time of mixing in seconds. The concentration of the solution after 30 seconds of mixing is 4 M, and the concentration of the solution after 2 seconds of mixing was 0.1 M.
a. Find the values of the constants A and k, giving your answers correct to 3 decimal places.
b. Find the concentration of the solution after 15 seconds of mixing.
c. How long does it take, in minutes and seconds, for the concentration of the solution to reach 10 M?

13. Andrew believes that his fitness level can be modelled by the function

$$F(t) = 10 + 2\log_e(t + 2)$$

where $F(t)$ is his fitness level and t is the time in weeks since he started training.

a. What was Andrew's level of fitness before he started training?
b. After 4 weeks of training, what was Andrew's level of fitness?
c. How long will it take for Andrew's level of fitness to reach 15?

14. In 1947 a cave with beautiful prehistoric paintings was discovered in Lascaux, France.

Some charcoal found in the cave contained 20% of the carbon-14 that would be expected in living trees. Determine the age of the paintings to the nearest whole number if

$$Q = Q_0 e^{-0.000124t}$$

where Q_0 is the amount of carbon-14 originally and t is the time in years since the death of the prehistoric material. Give your answer correct to the nearest year.

15. The sales revenue, R dollars, that a manufacturer receives for selling x units of a certain product can be modelled by the function

$$R(x) = 800\log_e\left(2 + \frac{x}{250}\right).$$

Furthermore, each unit costs the manufacturer 2 dollars to produce, and the initial cost of adjusting the machinery for production is \$300, so the total cost in dollars, C, of production is

$$C(x) = 300 + 2x.$$

a. Write the profit, $P(x)$ dollars, obtained by the production and sale of x units.
b. Find the number of units that need to be produced and sold to break even, that is, to reach $P(x) = 0$. Give your answer correct
to the nearest integer.

16. The value of a certain number of shares, \$$V$, can be modelled by the equation

$$V = ke^{mt}$$

where t is the time in months. The original value of the shares was \$10 000, and after one year the value of the shares was \$13 500.

a. Find the values of the constants k and m, giving answers correct to 3 decimal places where appropriate.

b. Find the value of the shares to the nearest dollar after 18 months.

c. After t months, the shares are sold for 1.375 times their value at the time. Find an equation relating the profit made, P, over the time the shares were owned.

d. If the shares were kept for 2 years, calculate the profit made on selling the shares at that time.

1.8 Review: exam practice

A summary of this chapter is available in the Resources section of your eBookPLUS at www.jacplus.com.au.

Simple familiar

1. MC Simplifying $3\log_e(5) + 2\log_e(2) - \log_e(20)$ gives:

A. $\log_e\left(\frac{19}{20}\right)$ **B.** $\log_e(109)$ **C.** $\log_e(480)$ **D.** $2\log_e(5)$

2. MC If $5\log_{10}(x) - \log_{10}\left(x^2\right) = 1 + \log_{10}(y)$, then x is equal to:

A. y **B.** $10y$ **C.** $\sqrt[3]{10y}$ **D.** $\frac{10}{y}$

3. MC The function h has the rule $h(x) = a\log_e(x - m) + k$, where m and k are positive constants and a is a negative constant. The maximal domain of h is:

A. R^+ **B.** $R\backslash\{m\}$ **C.** $R\backslash\{n\}$ **D.** (m, ∞)

4. MC If $7e^{ax} = 3$, then x equals:

A. $\frac{3}{7}\log_e(a)$ **B.** $a\log_e\left(\frac{3}{7}\right)$ **C.** $\frac{\log_e\left(\frac{3}{7}\right)}{a}$ **D.** $\frac{\log_e(3)}{a\log_e(7)}$

5. MC The exact solution of the equation $3^{2x+1} - 4 \times 3^x + 1 = 0$ is:

A. $x = 0, x = -1$ **B.** $x = 0, x = 1$ **C.** $x = -1, x = 1$ **D.** $x = \frac{1}{3}, x = 1$

6. Solve the following equations for x.

a. $2\log_e(x) - \log_e(x - 1) = \log_e(x - 4)$

b. $2\log_e(x + 2) - \log_e(x) = \log_e 3(x - 1)$

c. $2\left(\log_4(x)\right)^2 = 3 - \log_4\left(x^5\right)$

7. Express y in terms of x for the following equations, giving any restrictions for x.

a. $\log_2(y) = 2\log_2(x) - 3$

b. $\log_3(9x) - \log_3\left(x^4 y\right) = 2$

8. Sketch the graphs of each of the following, showing any axis intercepts and the asymptote(s). State the domain and range in each case.

a. $y = \log_e(x - 1) + 3$ **b.** $y = \log_e(x + 3) - 1$ **c.** $y = 2\log_e(-x)$

9. The loudness of plant machinery at a manufacturing business is modelled by the equation $L = 10\log_{10}\left(\frac{I}{I_0}\right)$, where L is the loudness in decibels (dB), I is the intensity in W/m^2 and $I_0 = 10^{-12}$ W/m^2.

a. If the loudness of the plant machinery at this business is known to be 90 dB, calculate the intensity for this situation.

b. Calculate the loudness of the plant machinery if the intensity is 10^{-6} W/m^2.

10. If $\log_2 5 = 2.321$ and $\log_2 9 = 3.17$, find $\log_2\left(\frac{5}{9}\right)$.

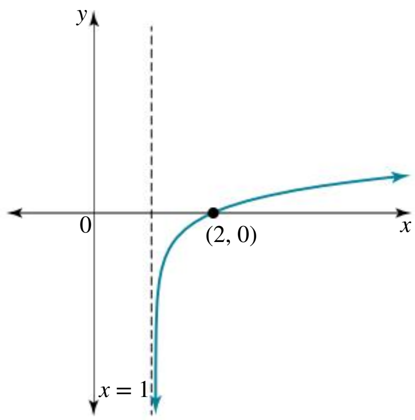
11. Earthquake intensity is often reported on the Richter scale. The magnitude of R is given by $R = \log_{10}\left(\frac{a}{T}\right) + B$, where a is the amplitude of the ground motion in microns at the receiving station, T is the period of the seismic wave in seconds, and B is an empirical factor that allows for the weakening of the seismic wave with the increasing distance from the epicentre of the earthquake.

Find the magnitude of the earthquake if the amplitude of the ground motion is 10 microns, the period is 1 second and the empirical factor is 6.8.

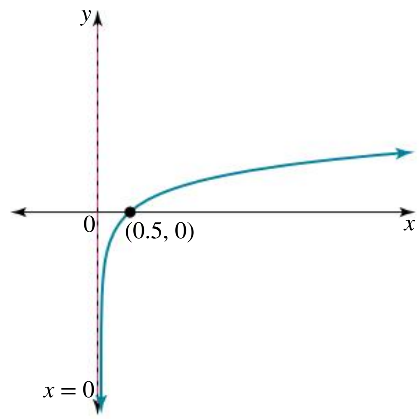
12. Using previous knowledge of the transformation of the graphs of functions, match each equation (a–d) with a suitable graph (P–S).

a. $y = -\log_{10}(x)$
b. $y = 2\log_{10}(x)$
c. $y = \log_{10}(2x)$
d. $y = \log_{10}(x - 1)$

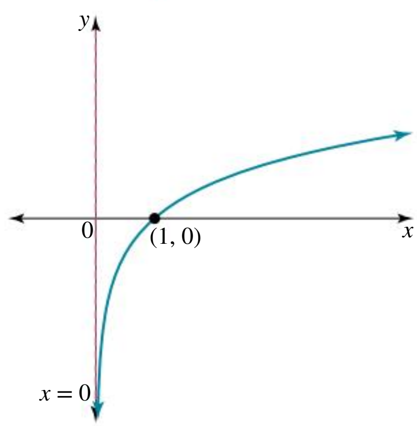
P



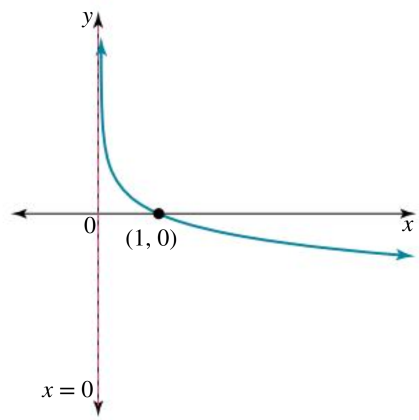
Q



R



S



Complex familiar

13. If $\log_4(p) = x$ and $\log_4(q) = y$, show that $\log_4\left(\dfrac{64q^2}{p^3\sqrt{q}}\right) = 3 - 3x + \dfrac{3y}{2}$.

14. The pH of a substance is a value that defines the acidity or alkalinity of that substance. It depends on the concentration of the hydrogen ion, $[H^+]$ in moles/litre, and is calculated according to the formula

$$\text{pH} = -\log_{10}[H^+].$$

Solutions with a pH less than 7 are acidic, solutions with a pH greater than 7 are basic, and solutions with a pH of 7 are neutral.

a. For each of the following, f
b. ind the pH and state whether the solution is acidic, basic or neutral.
   i. Vinegar has a hydrogen ion concentration of 0.01 moles/litre.
   ii. Ammonia has a hydrogen ion concentration of 10^{-11} moles/litre.
c. Find the hydrogen ion concentration for each of the following.
   i. Apples have a pH of 3.
   ii. Sodium hydroxide has a pH of 14.

15. The table gives values for x and y that relate to the equation $y = a \log_e(bx)$. Find the exact values of a, b and m.

x	1	2	3
y	$-3\log_e(2)$	0	m

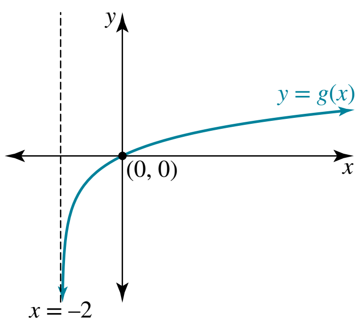
16. An object falls from a high tower. The distance it falls in a certain time is recorded in the table.

t (s)	1	2	3	4	5
d (m)	4.7	18.8	42.3	75.2	117.5

a. If a relationship of the form $d = At^n$ exists, find values for A and n.
b. Use this relationship to predict the value of d after 7 seconds.

Complex unfamiliar

17. The graph shown has the rule $g(x) = \log_e(x - h) + k$, where h and k are constants.



a. State the value of h.
b. Show that $k = -\log_e(2)$.
c. Hence, rewrite the rule in the form $g(x) = \log_e\left(\frac{x-h}{c}\right)$, where c is a constant.

18. Carbon-14 dating measures the amount of radioactive carbon-14 in fossils. This can be modelled by the relationship $Q = Q_0 e^{-0.000\,124t}$, where Q is the amount, in milligrams, of carbon-14 currently present in the fossil of an organism, t is the time in years since the organism's death, and Q_0 is the initial amount, in milligrams, of carbon-14 present.

a. A fossil shell initially has 150 milligrams of carbon-14 present. How much carbon-14 will be present after 2000 years? Give your answer correct to 3 decimal places.
b. Find the number of years it will take for the carbon-14 in the shell to be halved. Give your answer correct to the nearest year.
c. i. Suppose the amount of carbon-14 in the shell is $\frac{Q_0}{n}$. Find an equation relating n to t.
ii. Hence, find how long it will be before the amount of carbon-14 in the fossil shell is $\frac{Q_0}{10}$. Give your answer to the nearest year.

19. The population of quokkas in a small corner of south-western Western Australia is currently described as vulnerable. The once-plentiful population of quokkas was drastically reduced after dingoes, foxes and wild pigs were introduced to Australia.

Conservation efforts and dingo, fox and wild pig control programs have seen quokka populations recovering in some areas. In the Northern Jarrah forest, one of the areas where these conservation practices occur, there were known to be about 150 quokkas in 2008. Conservationists produced a model for the increase in population, P, which was given by

$$P = a \log_e(t) + b$$

where t is the time in years since 2007 and a and b are constants. There were estimated to be about 6000 quokkas present in the forest in 2013.

a. Determine the values of a and b. Give your answers correct to the nearest integer.

b. Calculate the number of quokkas that is expected to be present in 2025. Give your answer correct to the nearest integer.

c. Given that quokkas have a life expectancy of about 10 years, the model for the actual population is revised to

$$P_R = P - 0.25P$$

where P_R is the revised population.

i. Find the equation relating P_R to t, the number of years since 2007.

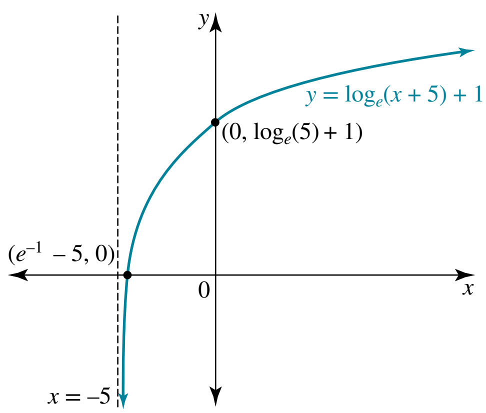
ii. Calculate the revised population prediction for 2025. Give your answer correct to the nearest integer.

20. The graph of the function $f: (-5, \infty) \to R, f(x) = \log_e(x + 5) + 1$ is shown.

a. Find the rule and domain of f^{-1}, the inverse function of f.

b. On the same set of axes, sketch the graph of f^{-1}. Label the axis intercepts with their exact values.

c. Find the coordinates of the point(s) of intersection correct to 3 decimal places.



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Units 3 & 4 Sit exam

Answers

The logarithmic function 2

Exercise 1.2 Review of the index laws

1. a. x^7 b. x^5 c. x^{10}
   d. $\frac{1}{x^6}$ e. x^6 f. x
   g. $5x^4y^3$ h. $6x^2y^3$ i. $\frac{10x^4y^2}{3}$
   j. $\frac{27x^2y^{14}}{2}$
2. a. 9 b. 8 c. $\frac{1}{125}$
   d. $\frac{1}{1000}$ e. 3 f. 216
   g. $\frac{3}{7}$ h. $\frac{9}{16}$ i. $\frac{8}{27}$
   j. $\frac{27}{64}$
3. a. $\frac{3}{x^{11}y^2}$ b. $\frac{x^6}{y^4}$ c. $6x^{\frac{5}{4}}y^{\frac{5}{3}}$
   d. $\frac{20x}{y^{\frac{1}{4}}}$ e. $\frac{243x^{\frac{5}{2}}}{y^2}$ f. $8y^{\frac{3}{8}}$
   g. $\frac{3}{a^9b^3}$ h. $\frac{2a^{\frac{15}{2}}b^{\frac{7}{2}}}{3}$
4. a. 2^{6n-1} b. 3^{6n+1} c. $2^{2n} \times 3^{2n+1}$
   d. $2^n \times 3^{3n+1}$ e. $\frac{2}{3}$ f. 1
5. a. $\frac{1+x^2}{x}$ b. $\frac{(x+1)^2}{x^4}$ c. $\frac{2x}{1-x^2}$
   d. $\frac{1}{x+y}$
6. a. $\frac{4}{3}$ b. $\frac{1}{243}$
7. B
8. a. 1000 b. 1414
9. D
10. a. 10 m b. 4.10 m c. 49.04 m

Exercise 1.3 Logarithmic laws and equations

1. a. 1 b. 1 c. 1
   d. 1 e. 5 f. 4
   g. -1 h. -3
2. a. $\frac{1}{2}\log_2(x)$ b. $\frac{1}{3}\log_3(x)$ c. $\log_3(x)$
   d. $\log_4(x)$ e. $\log_2\left(\frac{x^2}{y}\right)$ f. $\log_3\left(\frac{x}{y^2}\right)$
3. a. 4 b. -3 c. 3
   d. 1 e. $\log_2(10)$ f. $\log_3(54)$
   g. 2 h. $\frac{3}{2}$ i. $\frac{1}{2}$
   j. $\frac{2}{3}$
4. a. 0 b. 0
   c. 0 d. 0
   e. $3\log_{10}(x)$ f. $5\log_{10}(x)$
   g. $\log_5(x+1)^3$ or $3\log_5(x+1)$ h. $\log_4(x-2)$
5. a. 2 b. 5
   c. 3 d. 4
   e. 10 f. 12
   g. 2.207 h. 0.388
   i. 2 j. $\frac{1}{5}$
6. a. 3 b. $\frac{4}{3}$
   c. $2^{-\frac{1}{3}}, 4$ d. $\frac{7}{2}$
7. a. 243 b. $\frac{47}{10}$
8. a. i. 1.2770
   ii. -1.2619
   b. i. $2x = 2 \times 3^z$
   ii. $\frac{3}{z}$
9. a. $\log_5(9) = \frac{\log_{10}(9)}{\log_{10}(5)}$ b. $\log_{\frac{1}{2}}(12) = \frac{\log_{10}(12)}{\log_{10}\left(\frac{1}{2}\right)}$
10. a. $\log_6(216) = 3$ b. $\log_2(256) = 8$
    c. $\log_3(81) = 4$ d. $\log_{10}(0.0001) = -4$
    e. $\log_5(0.008) = -3$ f. $\log_7(7) = 1$
11. a. 4 b. -3
    c. 11 d. -128
12. a. 7 b. -5
    c. $-\frac{1}{2}$ d. -2
    e. $\frac{5}{12}$ f. $\frac{2}{5}$
13. a. $3\log_3(x-4)$ b. $\log_7(2x+3)$
    c. 0 d. $2\log_4(5x+1)$
14. a. 1.7712 b. -6.9189
15. a. 5^{n+1} b. $2n+1$ c. $\frac{4}{n}$
16. a. $\frac{1}{2}\left(e^{-3}+1\right)$ b. e^{-3}
    c. 7 d. 15
    e. 10 f. $-\frac{6}{5}$
    g. 6 h. $\frac{5}{4}$
    i. $\frac{5}{4}$ j. $10^{\frac{3}{2}}$ or 10
    k. 9 or $\frac{1}{3}$ l. 4
17. a. $y = \frac{4}{x^3}$ b. $y = \frac{x^2}{16}$
    c. $y = \frac{9}{x}$ d. $y = 64x$
18. a. $x = 3m$ b. $2 + \frac{3y}{2} - 5x$
19. 16, $\frac{1}{16}$
20. a. $-0.463, 0.675$ b. $0.451, 1$
21. 1.5518, 1.4422

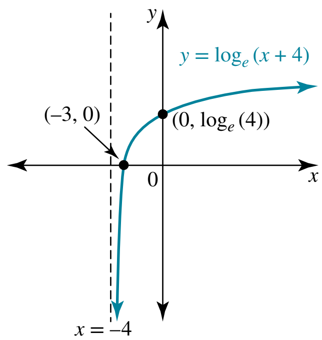
Exercise 1.4 Logarithmic scales

1. $10\ \text{W/m}^2$
2. $K = 61\,808$
3. The 6.4 earthquake is 1.41 times bigger than the 6.3 earthquake.
4. $K = 3691.17$
5. A 500-watt amplifier is 13.98 dB louder than a 20-watt amplifier.
6. 160 dB
7. pH = 3 (acidic)
8. a. $\left[\text{H}^+\right] = 1$ mole/litre
   b. $\left[\text{H}^+\right] = 0.0001$ moles/litre
   c. $\left[\text{H}^+\right] = 10^{-8}$ moles/litre
   d. $\left[\text{H}^+\right] = 10^{-12}$ moles/litre
9. a. 4.8 (acidic)
   b. 5.56 (acidic)
10. a. Sample responses can be found in the worked solutions in the online resources.
    b. 9 988 years old
11. -437.97
12. $n = 361$ cents
13. Ear protection should be worn as $L = 133.98$ dB.
14. 8.7

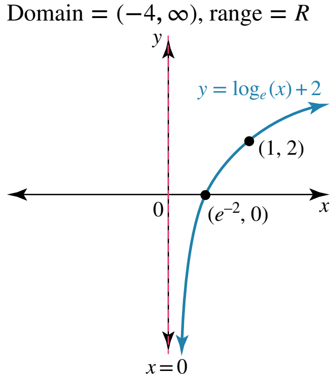
Exercise 1.5 Indicial equations

1. a. 3 b. $\frac{1}{2}\log_{10}(5) + \frac{1}{2}$
   c. 2 d. $\log_{10}\left(\frac{1}{2}\right), \log_{10}(3)$
2. a. -9 b. $\frac{1}{2}\log_2(9)$
   c. $\log_e(2)$ d. $\log_e(5)$
3. a. $\frac{1}{2}\log_7(5) + \frac{1}{2}$ b. 0, 2
   c. $\log_5(3)$ d. $\log_9\left(\frac{2}{3}\right), \log_9\left(\frac{5}{2}\right)$
4. a. $-\frac{7}{8}$ b. $\log_3(2) - 1$
   c. 0, 1 d. $\dfrac{\log_e\left(\frac{3}{4}\right)}{\log_e(12)}$
5. a. 1 b. 0
6. a. $\frac{1}{x} - x - 1$ b. 1, 3
7. a. $2\log_e(3) + 2$ b. $4\log_e(2)$
   c. $\log_e(3)$ d. $\pm\sqrt{\log_e(2)}$
8. a. $2\log_e(2)$ b. $2\log_e(2), 3\log_e(2)$
   c. $\log_e\left(1 + \sqrt{5}\right)$ d. $\log_e\left(6 \pm \sqrt{31}\right)$
9. $n = \frac{1}{2}, m = 2$
10. a. $x < -1.737$ b. $x > -0.756$
11. $m = \dfrac{3^{5n}}{4}, m = \dfrac{1}{4 \times 3^{5n}}$
12. a. $\dfrac{m - \log_e(2n)}{k}, k \in R \setminus \{0\}, n \in R$
    b. $\dfrac{4 - 4n}{3m}, m \in R \setminus \{0\}$
    c. $\dfrac{1}{m}\log_e\left(\dfrac{5 + \sqrt{57}}{4}\right), m \in R \setminus \{0\}$
13. a. $D = 20 \times 10^{0.04t}$
    b. 2 years and 5 months
    c. 5 years
14. $a = 5, k = 0.25$
15. $r = 5\%, P = \$10\,000$

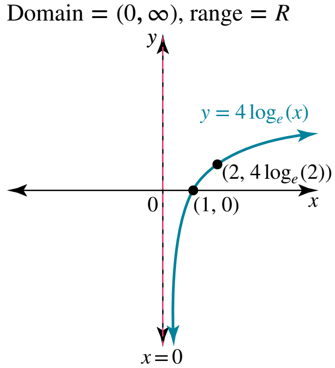
Exercise 1.6 Logarithmic graphs

1.


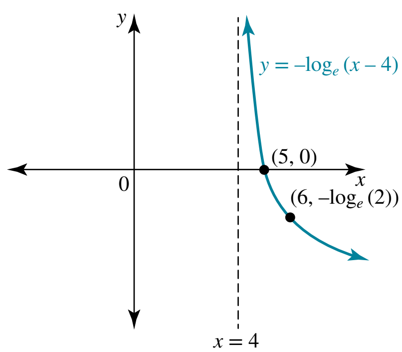
a. Domain = $(-4, \infty)$, range = R

b.


Domain = $(0, \infty)$, range = R

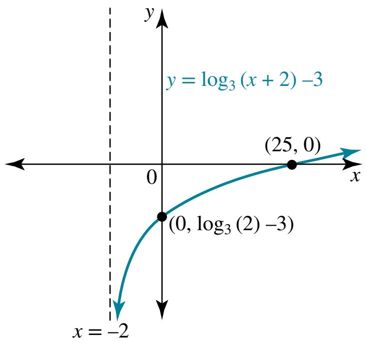
c.


Domain = $(0, \infty)$, range = R

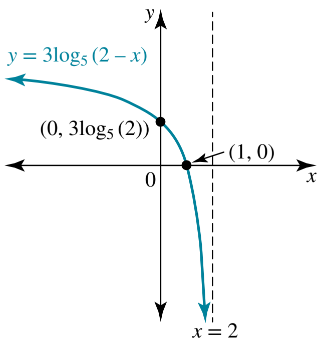
d.


Domain = $(4, \infty)$, range = R

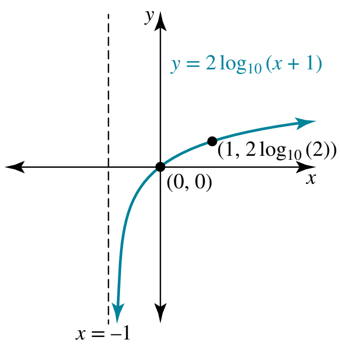
2. a.



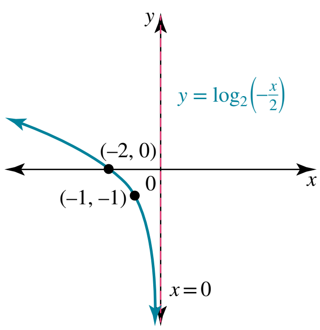
b.



c.



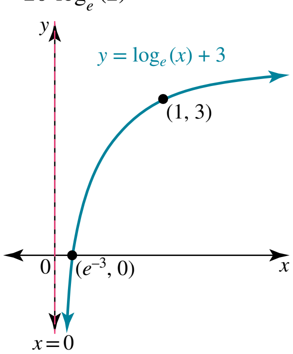
d.



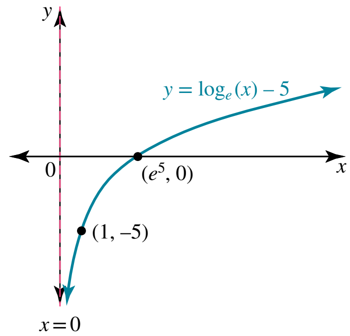
3. $m = 2, n = 2$

4. $p = \dfrac{-7}{20 \log_e (2)}, q = -1$

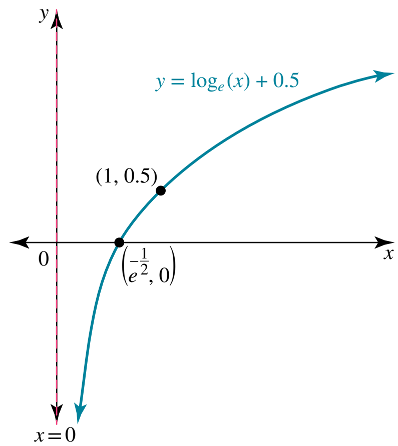
5. a.



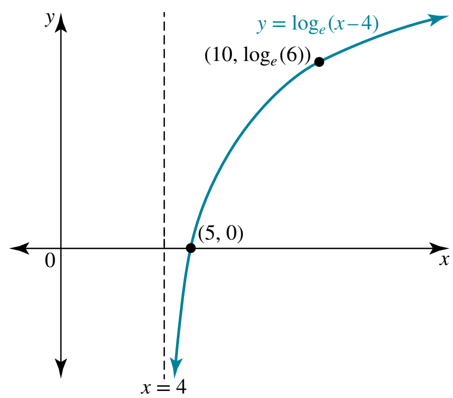
b.



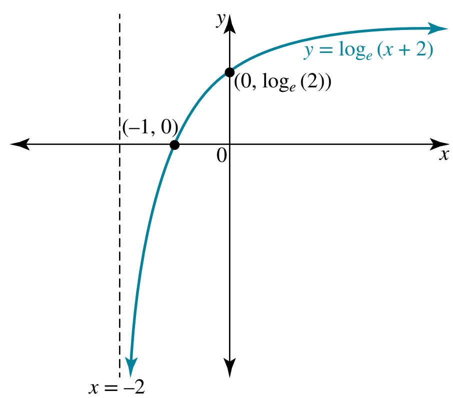
c.



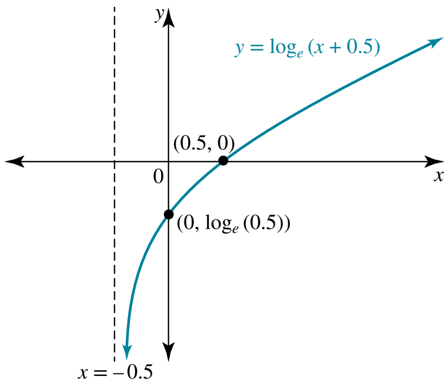
6. a.



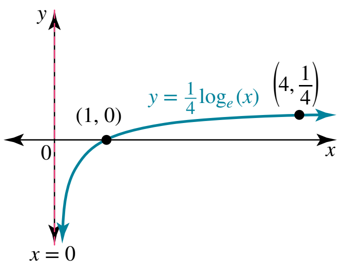
b.



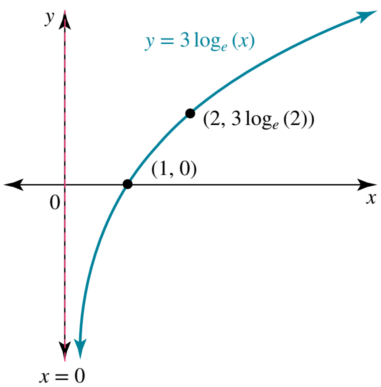
c.



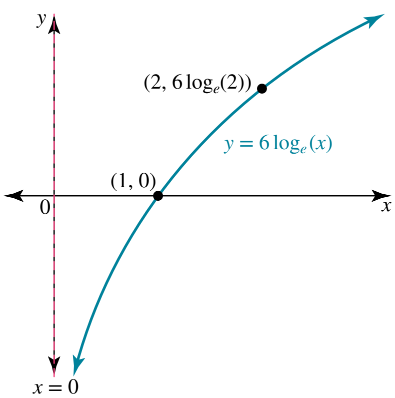
7. a.



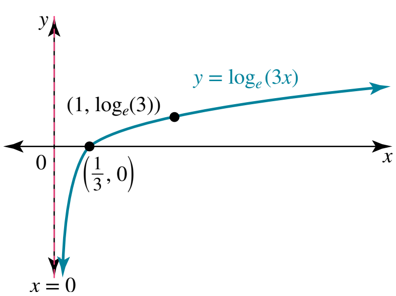
b.



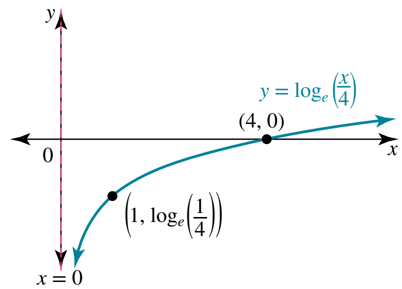
c.



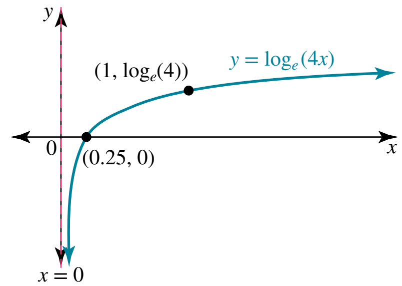
8. a.



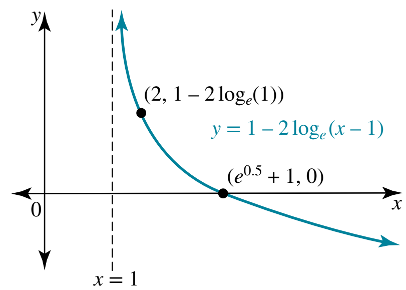
b.



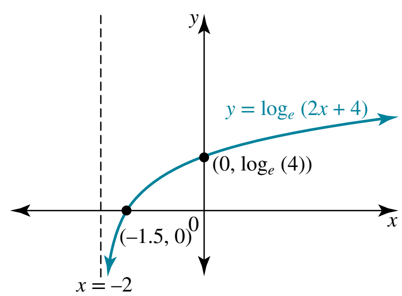
c.



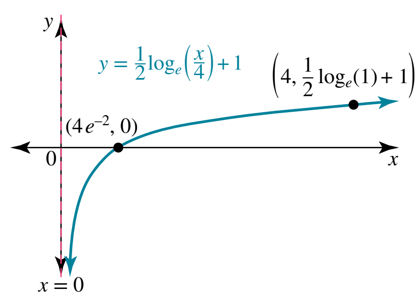
9. a.



b.



c.



10. a. $f(x) = 2\log_e(3(x+1))$, domain $= (-1, \infty)$ and range $= R$

$f^{-1}(x) = \dfrac{1}{3}e^{\frac{x}{2}} - 1$, domain $= R$ and range $= (-1, \infty)$

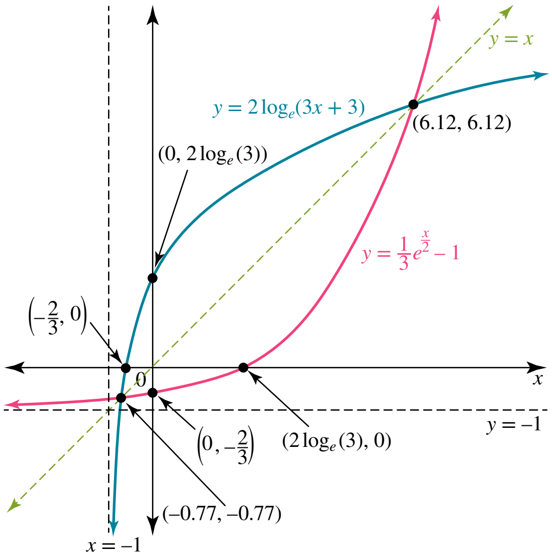
b. $f(x) = \log_e(2(x-1)) + 2$, domain $= (1, \infty)$ and range $= R$

$f^{-1}(x) = \dfrac{1}{2}e^{x-2} + 1$, domain $= R$ and range $= (1, \infty)$

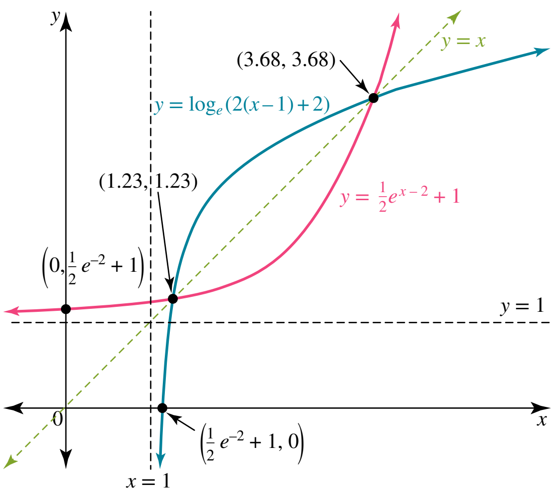
c. $f(x) = 2\log_e(1-x) - 2$, domain $= (-\infty, 1)$ and range $= R$

$f^{-1}(x) = 1 - e^{\frac{1}{2}(x+2)}$, domain $= R$ and range $= (-\infty, 1)$

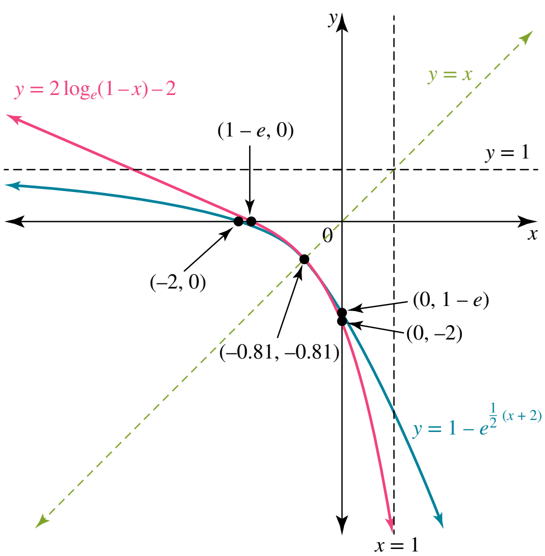
11. a.



b.



c.



12. a. $a = -1, b = \frac{1}{2}$

b. -0.4055

13. a. $a = \dfrac{2}{\log_e(2)}, h = -1, k = -2$

14. $(-2, 3) \Rightarrow 3 = m\log_2(-2n)$ [1]

$\left(-\frac{1}{2}, \frac{1}{2}\right) \Rightarrow \frac{1}{2} = m\log_2\left(-\frac{n}{2}\right)$ [2]

[1] − [2]:

$$\begin{aligned}
3 - \frac{1}{2} &= m\log_2(-2n) - m\log_2\left(-\frac{n}{2}\right)\\
\frac{5}{2} &= m\left(\log_2(-2n) - \log_2\left(\frac{n}{2}\right)\right)\\
&= m\left(\log_2\left(\frac{-2n}{\frac{n}{2}}\right)\right)\\
&= m\log_2(4)\\
&= m\log_2 2^2\\
&= 2m\\
m &= \frac{5}{4}
\end{aligned}$$

Substitute $m = \frac{5}{4}$ into [1]:

$$\begin{aligned}
3 &= \frac{5}{4}\log_2(-2n)\\
\frac{12}{5} &= \log_2(-2n)\\
2^{\frac{12}{5}} &= -2n\\
n &= 2^{\frac{12}{5}} \div -2\\
&= -2^{\frac{7}{5}}
\end{aligned}$$

15. a. 0.159 or 3.146 **b.** 1.2315

16. a. $x \in (0.138, 1.564)$ **b.** $x \in [0.136, 1.315]$

Exercise 1.7 Applications

1. 54 years

2. a. 7.32% **b.** 33 years 5 months

3. 8.46 pm, so the person died $1\frac{3}{4}$ hours after the phone call.

4. a. 2 parts per million

b. 2.96 parts per million

c. 47.2 hours

5. \$1648.72

6. 0.1793

7. 0.0003

8. a. 88.3 mg **b.** 5590 years

9. a. 88.57% lost **b.** 1.87 millennia

10. a. $a = 12.5, b = 100$ **b.** 7.253 g

11. a. $a = -2885.4, c = 10\,000$

b. 4000

c. 22.6 weeks

12. a. $A = 1.440, k = 0.536$

b. 3.00 M

c. 32 minutes 14 seconds

13. a. 11.3863 b. 13.5835 c. 10.18 weeks

14. 12 979 years

15. a. $P(x) = 800 \log_e \left(2 + \frac{x}{250}\right) - 300 - 2x$

b. 750

16. a. $k = 10\,000, m = 0.025$

b. $15 685.58

c. $P = 13\,750e^{0.025t} - 10\,000$

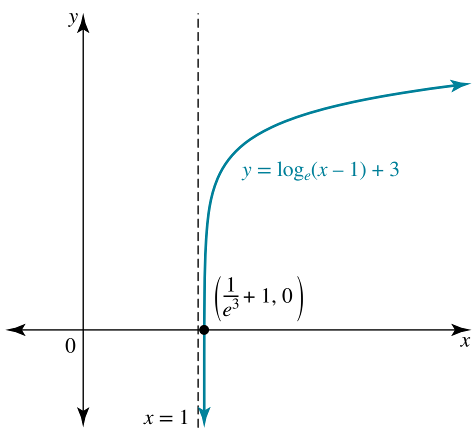
d. $15 054.13

1.8 Review: exam practice

1. D
2. C
3. D
4. C
5. A
6. a. No solution b. 4 c. $\frac{1}{64}$ or 2
7. a. $y = \frac{x^2}{8}$ where $x > 0$

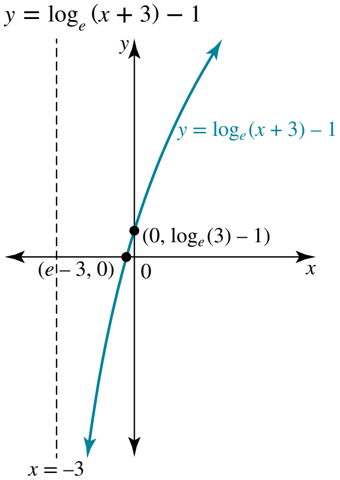
b. $y = \frac{1}{x^3}$ provided that $x > 0$

8. a. $y = \log_e(x - 1) + 3$



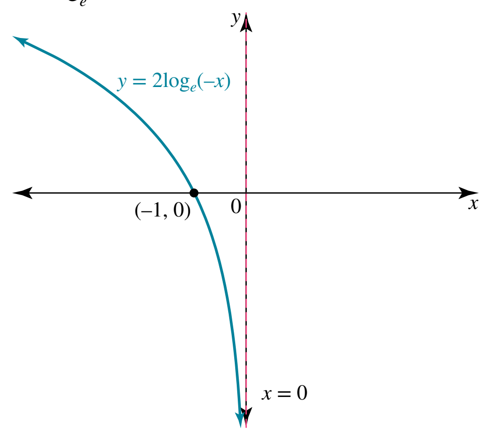
Domain $= (1, \infty)$, range $= R$

b. $y = \log_e(x + 3) - 1$



Domain $= (-3, \infty)$, range $= R$

c. $y = 2\log_e(-x)$



Domain $= (-\infty, 0)$, range $= R$

9. a. 10^{-3} W/m^2 b. 60 dB
10. -0.849
11. 7.8
12. a. S b. R c. Q d. P
13. Sample responses can be found in the worked solutions in the online resources.
14. a. i. pH = 2 (acidic)

ii. pH = 11 (basic)

b. i. 0.001 moles/litre

ii. 10^{-14} moles/litre

15. $a = 3, b = \frac{1}{2}$ and $m = 3\log_e\left(\frac{3}{2}\right)$
16. a. $A = 4.7$ and $n = 2$

b. 230.3 m

17. a. $h = -2$ b. $k = -\log_e(2)$

c. $g(x) = \log_e\left(\frac{x+2}{2}\right)$

18. a. 117.054 mg

b. 5590 years

c. i. $n = e^{0.000\,124t}$

ii. 18 569 years

19. a. $a = 3265, b = 150$

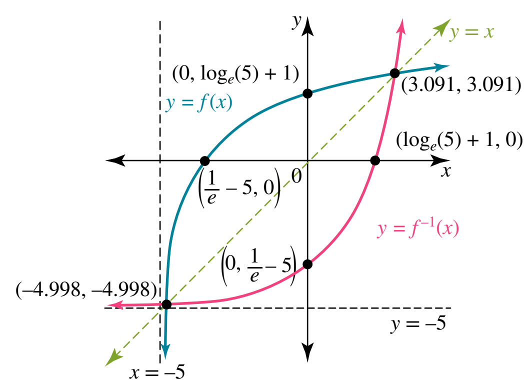
b. 9587 quokkas

c. i. $P_R = 2448.75\log_e(t) + 112.5$

ii. 7 190 quokkas

20. a. $f^{-1}(x) = e^{(x-1)} - 5$, domain $= R$

b. $f^{-1}\left(\log_e(5) + 1, 0\right), \left(0, \frac{1}{e} - 5\right)$

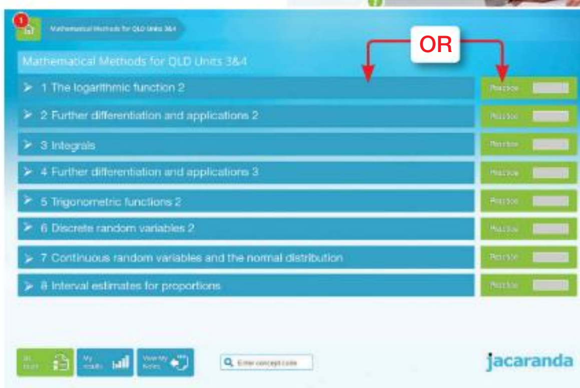


c. $(-4.998, -4.998)$ and $(3.091, 3.091)$

TOPIC 1 The logarithmic function 2

- For revision of this entire topic, go to your **studyON** title in your bookshelf at **www.jacplus.com.au**.
- Select **Continue Studying** to access hundreds of revision questions across your entire course.

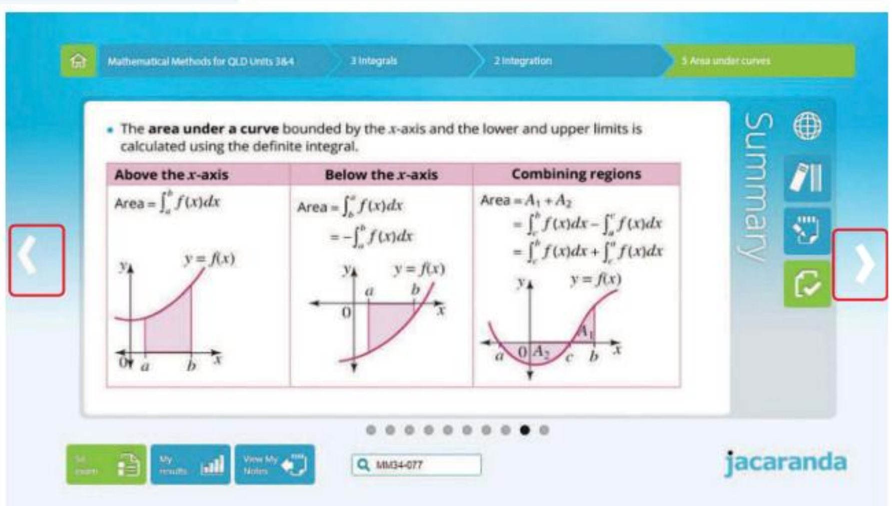
- Select your **course** *Mathematical Methods for Queensland Units 3&4* to see the entire course divided into syllabus topics.
- Select the **Area** you are studying to navigate into the chapter level *OR* select **Practice** to answer all practice questions available for each area.



- Select **Practice** at the sequence level to access all questions in the sequence.

- At **sequence level**, drill down to concept level.

- **Summary screens** provide revision and consolidation of key concepts. Select the **next arrow** to revise all concepts in the sequence and practise questions at the concept level.



2 Calculus of exponential functions

2.1 Overview

Rates of change and the concept of a limit are fundamental ideas in the study of calculus. Many processes in nature can be modelled by exponential functions. These include the growth in a population of bacteria and the decay of radioactive material. Other areas modelled by exponential functions include the value of an investment after a period of time and the temperature of a liquid after it has been cooling.

In 1683, Jacob Bernoulli, a famous Swiss mathematician, was studying a problem relating to compound interest. What were the effects on the investment if smaller and smaller compounding intervals were used? In fact, he was attempting to find the value of $\lim\limits_{n\to\infty}\left(1+\dfrac{1}{n}\right)^n$ and discovered that the limit had to lie between 2 and 3. Other mathematicians in the 17th century were studying similar ideas and came close to finding the limit.

Leonhard Euler, another famous Swiss mathematician, had studied under Bernoulli. While he was endeavouring to solve the problem of limits and investments proposed by Bernoulli, Euler discovered the constant e. He subsequently found many uses of the constant e and, in the mid 18th century published a paper showing all of his findings, including e to 18 decimal places. An approximation for e is 2.718 281 828 459 . . . Like π, Euler's number, e, is irrational.

Euler studied many areas of science, including mechanics, fluid dynamics, astronomy and physics. In mathematics, the influence of Euler is found in geometry, trigonometry, calculus and algebra. He also studied and wrote on the theory of music.

LEARNING SEQUENCE

2.1 Overview
2.2 Review of limits and differentiation
2.3 The exponential function
2.4 Differentiation of exponential functions
2.5 Applications of exponential functions
2.6 Review: exam practice

Fully worked solutions for this chapter are available in the Resources section of your eBookPLUS at www.jacplus.com.au.

2.2 Review of limits and differentiation

2.2.1 Limits

The limit of a function, $y = f(x)$, is the value that the function approaches as x approaches a given value.

Consider the limit of $f(x) = x + 1$ as x approaches 1, using a spreadsheet.

x	$x + 1$
0.9	1.9
0.99	1.99
0.999	1.999
0.9999	1.9999

x	$x + 1$
1.1	2.1
1.01	2.01
1.001	2.001
1.0001	2.0001

As x approaches 1 from the left-hand side, or below, the function approaches 2.
As x approaches 1 from the right-hand side, or above, the function approaches 2.
As both are equal, the limit exists and is written as:

$$\lim_{x\to 1}(x + 1) = 2$$

Since the function $f(x) = x + 1$ is continuous, the limit can be found by direct substitution.

Consider a different function, $g(x) = \dfrac{x^2 - 1}{x - 1}$.

This function is undefined at $x = 1$.

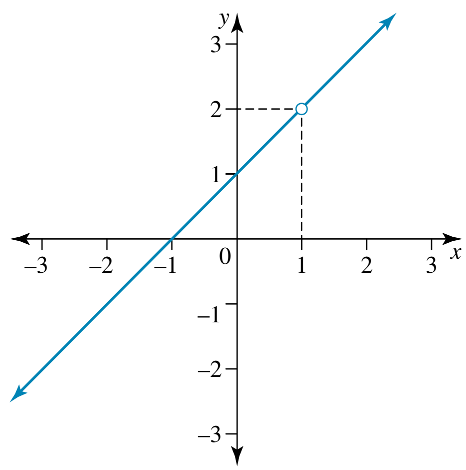
However, $g(x)$ can be simplified to $g(x) = \dfrac{(x - 1)(x + 1)}{(x - 1)}$.

$$g(x) = (x + 1), x \neq 1$$

As shown above, since $\lim_{x\to 1}(x + 1) = 2$, then

$\lim_{x\to 1}\dfrac{x^2 - 1}{x - 1} = 2$ or $\lim_{x\to 1} g(x) = 2$.

The graph of $y = g(x)$ is in fact a linear function with a point discontinuity at $(1, 2)$, as shown.



WORKED EXAMPLE 1

Evaluate the following limits:

a. $\lim_{h\to 4}(3h - 5)$ **b.** $\lim_{x\to 0}\dfrac{x^2 + 5x + 6}{x + 2}$ **c.** $\lim_{x\to -2}\dfrac{x^2 + 5x + 6}{x + 2}$

THINK	WRITE
a. 1. Substitute $h = 4$ as the function is defined.	**a.** $\lim_{h\to 4}(3h - 5)$ $= 3 \times 4 - 5$ $= 7$
2. Answer the question.	$\lim_{h\to 4}(3h - 5) = 7$

b. 1. Substitute $x = 0$ as the function exists for this value.

b. $\lim\limits_{x \to 0} \dfrac{x^2 + 5x + 6}{x + 2}$

$= \dfrac{6}{2}$

$= 3$

2. Answer the question.

$\lim\limits_{x \to 0} \dfrac{x^2 + 5x + 6}{x + 2} = 3$

c. 1. The function is undefined at $x = -2$. Factorise the numerator and simplify.

c. $\lim\limits_{x \to -2} \dfrac{x^2 + 5x + 6}{x + 2}$

$= \lim\limits_{x \to -2} \dfrac{(x+2)(x+3)}{(x+2)}$

$= \lim\limits_{x \to -2} (x + 3)$

2. Substitute $x = -2$.

$= -2 + 3$

$= 1$

3. Answer the question.

$\lim\limits_{x \to -2} \dfrac{x^2 + 5x + 6}{x + 2} = 1$

2.2.2 The derivative as a limit

In Year 11, you were introduced to differentiation, the process of finding the rate of change of a function at any point.

Differentiation from first principles involves finding a limit as h approaches 0.

For the function $y = f(x)$:

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

WORKED EXAMPLE 2

Calculate the derivative of $f(x) = 3x^2 - 4x + 7$ from first principles.

THINK

WRITE

1. State the function

$f(x) = 3x^2 - 4x + 7$

2. The derivative is equal to $\lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$.

$f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

3. Substitute for $f(x)$.

$f'(x) = \lim\limits_{h \to 0} \dfrac{[3(x+h)^2 - 4(x+h) + 7] - [3x^2 - 4x + 7]}{h}$

4. Expand and simplify.

$= \lim\limits_{h \to 0} \dfrac{3x^2 + 6xh + 3h^2 - 4x - 4h + 7 - 3x^2 + 4x - 7}{h}$

$= \lim\limits_{h \to 0} \dfrac{6xh + 3h^2 - 4h}{h}$

5. Factorise and simplify.	$= \lim_{h \to 0} \frac{h(6x + 3h - 4)}{h}$
	$= \lim_{h \to 0}(6x + 3h - 4)$
6. Evaluate the limit as h approaches 0.	$= 6x + 3 \times 0 - 4$ $= 6x - 4$
7. Answer the question.	$f'(x) = 6x - 4$

2.2.3 Estimating a limit

Technology, such as a spreadsheet, can be used to estimate the limit of a given expression.

Consider the limit of $\frac{a^h - 1}{h}$ as $h \to 0$ for various values of $a > 0$.

For $a = 2$:

h	$\frac{(2^h - 1)}{h}$
1	1
0.5	0.828 427 12
0.1	0.717 734 63
0.01	0.695 555 01
0.001	0.693 387 46
0.000 1	0.693 171 2
0.000 01	0.693 149 58

The limit is approaching 0.6931.

For $a = 3$:

h	$\frac{(3^h - 1)}{h}$
1	2
0.5	1.464 101 615
0.1	1.161 231 74
0.01	1.104 669 194
0.001	1.099 215 984
0.000 1	1.098 672 638
0.000 01	1.098 618 323

The limit is approaching 1.0986.

Can we find a value of a where the fraction $\frac{a^h - 1}{h}$ has a limiting value of 1 as $h \to 0$?

The value of a would lie in the interval $2 < a < 3$.

$$\text{If } \frac{a^h - 1}{h} = 1, h \neq 0, \text{ then } a^h - 1 = h \quad \Rightarrow \quad a^h = 1 + h$$

$$a = (1 + h)^{\frac{1}{h}}$$

Consider the value of a as $h \to 0$.

h	$(1 + h)^{\left(\frac{1}{h}\right)}$
1	2
0.5	2.25
0.1	2.593 742 46
0.01	2.704 813 829
0.001	2.716 923 932
0.000 1	2.718 145 927
0.000 01	2.718 268 237
0.000 001	2.718 280 469
0.000 000 1	2.718 281 694
0.000 000 01	2.718 281 786
0.000 000 001	2.718 282 052
1E − 10	2.718 282 053

The fraction is approaching 2.718 282 05 ... as $h \to 0$.

Euler's number, $e \approx 2.718\,28$, is the value of a that gives the limit of the fraction to be 1.

Like π, e is an irrational number.

Scientific and graphics calculators have an e^x function that is treated in the same way as any other function.

An answer given in terms of e is an exact answer.

studyon

Units 3 & 4 > Area 2 > Sequence 1 > Concept 1

Review of limits Summary screen and practice questions

Exercise 2.2 Review of limits and differentiation

Technology free

1. **WE1a** Evaluate the following limits.

   a. $\lim_{h\to 3}(5h + 4)$
   b. $\lim_{h\to -2}(4 - 6h)$
   c. $\lim_{h\to 0}(6h^2 - 3h + 2)$

2. **WE1b** Evaluate the following limits.

   a. $\lim_{x\to 0}\dfrac{2x^2 + 7x + 3}{x - 1}$
   b. $\lim_{x\to 2}\dfrac{x^2 + 4x}{x + 2}$
   c. $\lim_{x\to -3}\dfrac{x^2 + 4x}{x + 1}$

3. **WE1c** Evaluate the following limits.

   a. $\lim_{h\to -3}\dfrac{h^2 - h - 12}{h + 3}$
   b. $\lim_{h\to 0}\dfrac{h^2 + 4h}{h}$
   c. $\lim_{x\to 3}\dfrac{x^2 - x - 6}{3 - x}$

4. Evaluate the following limits.

   a. $\lim_{h\to 0}(4x^2 + 5xh - h^2)$
   b. $\lim_{h\to 0}\dfrac{3x^2h + 4h^2}{h}$

5. **WE2** For the function $f(x) = x^2 - 6x$, calculate the derivative from first principles.
6. Use first principles to differentiate the function $f(x) = 5 + 3x - 2x^2$.
7. Estimate to 5 decimal places, using technology, the limit of $\dfrac{a^h - 1}{h}$ as $h \to 0$, where:

   a. $a = 2.5$
   b. $a = 2.6$
   c. $a = 2.7$
   d. $a = 2.8$
   e. $a = 2.9$
   f. $a = 2.718\,28$

8. Evaluate the following, giving your answers to 4 decimal places.

   a. e^2
   b. e^3
   c. $e^{\frac{1}{2}}$

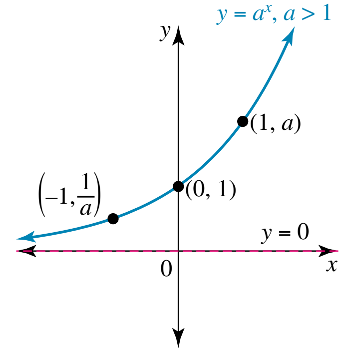
9. Evaluate the following, giving your answers to 3 decimal places.

   a. $2e^{-1}$
   b. $\sqrt[3]{e}$
   c. $\dfrac{4 + e}{e}$

10. a. Use first principles to differentiate the function $y = 8x - x^2$.
    b. Calculate the gradient of the tangent to the curve $y = 8x - x^2$ at the point where $x = 2$.
    c. Hence, determine the equation of the tangent to the curve at $x = 2$.
11. Use differentiation by first principles to determine the gradient of the curve $y = x^3 - 3x^2$ at any point and hence the equation of the tangent at the point where the curve crosses the positive x-axis.
12. Consider the function $f(x) = x^3 - 4x$.
    a. Determine the x-intercepts of this function.
    b. Use first principles to differentiate the function.
    c. Calculate the gradient of the tangents at the points where the function crosses the x-axis.
    d. Deduce that two of the tangents are parallel.
13. a. Simplify the fraction $\dfrac{1}{x + h} - \dfrac{1}{x}$.
    b. Hence, use first principles to determine the derivative of the function $f(x) = \dfrac{1}{x}$, $x \neq 0$.
14. a. Show that $\dfrac{1}{(x + h - 2)} - \dfrac{1}{(x - 2)} = \dfrac{-h}{(x - 2)(x + h - 2)}$.
    b. Evaluate $\lim_{h\to 0}\dfrac{-1}{(x - 2)(x + h - 2)}$.
    c. Hence, use first principles to determine the gradient function for $y = \dfrac{1}{(x - 2)}$, $x \neq 2$.
    d. Determine the x-value(s) on the curve where the tangent is parallel to $9x + y - 7 = 0$.

2.3 The exponential function

2.3.1 Review of exponential functions, $f(x) = a^x$ where $a \in R^+ \setminus \{1\}$



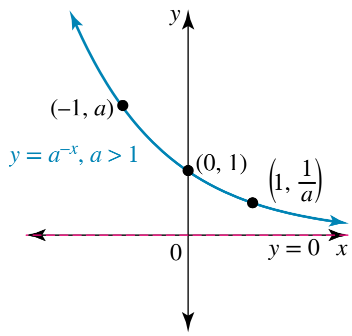
The graph of $f(x) = a^x$, $a > 1$ has the following features:

- The y-intercept is $(0, 1)$.
- The key points are $(1, a)$ and $\left(-1, \frac{1}{a}\right)$.
- The maximal domain is $x \in R$.
- The range is $y \in R^+$ or $y > 0$.
- As $x \to \infty, f(x) \to \infty$.
- As $x \to -\infty, f(x) \to 0$.
- The horizontal asymptote is $y = 0$ (the x-axis).
- It is a one-to-one function.

The graph of $f(x) = a^{-x}$, $a > 1$ is a reflection of $f(x) = a^x$ for $a > 1$ over the y-axis.

The graph of $f(x) = a^{-x}$, $a > 1$ has the following features:

- The y-intercept is $(0, 1)$.
- The key points are $(-1, a)$ and $\left(1, \frac{1}{a}\right)$.
- The maximal domain is $x \in R$.
- The range is $y \in R^+$ or $y > 0$.
- As $x \to \infty, f(x) \to 0$.
- As $x \to -\infty, f(x) \to \infty$.
- The horizontal asymptote is $y = 0$ (the x-axis).
- It is a one-to-one function.



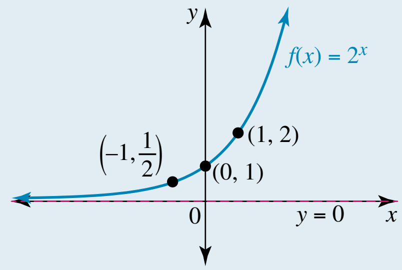
WORKED EXAMPLE 3

Sketch the following exponential functions, showing all important features.

a. $f(x) = 2^x$ **b.** $f(x) = 2^{-x}$ **c.** $f(x) = -2^x$

THINK	WRITE
a. 1. To sketch the graph of $f(x) = 2^x$, first determine the y-intercept, which occurs when $x = 0$.	**a.** $f(x) = 2^x$ $f(0) = 2^0$ $f(0) = 1$ y-intercept $(0, 1)$
2. To help determine the shape of the curve, it is useful to know any two other points on the graph. For example, determine the coordinates of the points at $x = \pm 1$.	$f(1) = 2^1$ and $f(-1) = 2^{-1}$ $f(1) = 2$ $\quad f(-1) = \frac{1}{2}$ $(1, 2)$ and $\left(-1, \frac{1}{2}\right)$
3. State the equation of the horizontal asymptote.	$y = 0$

4. Sketch the graph.



b. 1. To sketch the graph of $f(x) = 2^{-x}$, first determine the y-intercept, which occurs when $x = 0$.

b. $f(x) = 2^{-x}$

$f(0) = 2^{-0}$

$f(0) = 1$

y-intercept $(0, 1)$

2. To help determine the shape of the curve, it is useful to know any two other points on the graph. For example, determine the coordinates of the points at $x = \pm 1$.

$f(1) = 2^{-1}$ and $f(-1) = 2^{-1}$

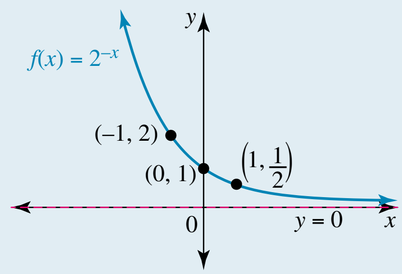
$f(1) = \frac{1}{2}$ $\quad f(-1) = 2$

$\left(1, \frac{1}{2}\right)$ and $(-1, 2)$

3. State the equation of the horizontal asymptote.

$y = 0$

4. Sketch the graph.
Note: This is a reflection of the curve in part **a** over the y-axis.



c. 1. To sketch the graph of $f(x) = -2^x$, first determine the y-intercept, which occurs when $x = 0$.

c. $f(x) = -2^x$

$f(0) = -2^0$

$f(0) = -1$

y-intercept $(0, -1)$

2. To help determine the shape of the curve, it is useful to know any two other points on the graph. For example, determine the coordinates of the points at $x = \pm 1$.

$f(1) = -2^1$ and $f(-1) = -2^{-1}$

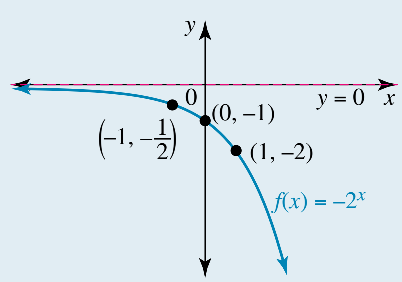
$f(1) = -2$ $\quad f(-1) = -\frac{1}{2}$

$(1, -2)$ and $\left(-1, -\frac{1}{2}\right)$

3. State the equation of the horizontal asymptote.

$y = 0$

4. Sketch the graph.
Note: This is a reflection of the curve in part **a** over the x-axis.



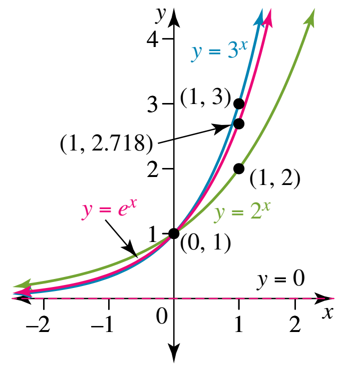
2.3.2 The exponential function, $f(x) = e^x$

It has been established that $e \approx 2.718\,28$, giving $2 < e < 3$.

Therefore, the graph of the exponential function, $y = e^x$, lies between $y = 2^x$ and $y = 3^x$.

The graph of $f(x) = e^x$ has the following features:

- The y-intercept is $(0, 1)$.
- The key points are $(1, e)$ and $\left(-1, \dfrac{1}{e}\right)$.
- The maximal domain is $x \in R$.
- The range is $y > 0$ or $y \in R^+$.
- As $x \to \infty, f(x) \to \infty$.
- As $x \to -\infty, f(x) \to 0$.
- The horizontal asymptote is $y = 0$.
- It is a one-to-one function.



Three exponential functions, $y = 2^x$, $y = e^x$ and $y = 3^x$, are shown.

Graphs of $f(x) = ae^{nx} + b$ where $a, n, b \in R$ can be sketched using your knowledge of transformations.

Sketching $y = e^x$ and its transformations

The graph of $y = ke^x$ is a **dilation** of $y = e^x$ by a factor of k from the x-axis (or perpendicular to the y-axis).

The graph of $y = e^{nx}$ is a **dilation** of $y = e^x$ by a factor of $\dfrac{1}{n}$ from the y-axis (or parallel to the x-axis).

The graph of $y = -e^x$ is a **reflection** of $y = e^x$ in the x-axis.

The graph of $y = e^{-x}$ is a **reflection** of $y = e^x$ in the y-axis.

The graph of $y = e^x + k$ is a **translation** of $y = e^x$ by k units vertically (or parallel to the y-axis).

The graph of $y = e^{(x-h)}$ is a **translation** of $y = e^x$ by h units horizontally (or parallel to the x-axis), giving the vertical asymptote $x = h$.

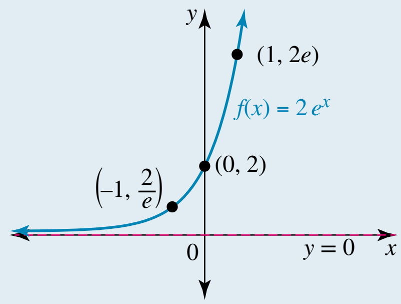
WORKED EXAMPLE 4

a. Sketch the function $f(x) = 2e^x$, showing all important features.

b. State the transformation required to map $f(x) = e^x$ onto $f(x) = 2e^x$.

THINK	WRITE
a. 1. To sketch the graph of $f(x) = 2e^x$, first determine the y-intercept, which occurs when $x = 0$.	**a.** $f(x) = 2e^x$ $f(0) = 2e^0$ $f(0) = 2 \times 1$ $f(0) = 2$ y-intercept $(0, 2)$
2. To help determine the shape of the curve, it is useful to know any two other points on the graph. For example, determine the coordinates of the points at $x = \pm 1$.	$f(1) = 2e^1$ and $f(-1) = 2e^{-1}$ $f(1) = 2e$ $\quad f(-1) = \dfrac{2}{e}$ $(1, 2e) \approx (1, 5.44)$ $\left(-1, \dfrac{2}{e}\right) \approx (-1, 0.74)$
3. State the equation of the horizontal asymptote.	$y = 0$

4. Sketch the function.



b. State the transformation.

b. The transformation required to map $f(x) = e^x$ onto $f(x) = 2e^x$ is a dilation by a factor of 2 from the x-axis.

WORKED EXAMPLE 5

a. Sketch the function $f(x) = 2 + e^{-x}$, showing all important features.

b. State the transformations required to map $f(x) = e^x$ onto $f(x) = 2 + e^{-x}$.

THINK

a. 1. To sketch the graph of $f(x) = 2 + e^{-x}$, first determine the y-intercept, which occurs when $x = 0$.

WRITE

a. $f(x) = 2 + e^{-x}$

$f(0) = 2 + e^{-0}$

$f(0) = 2 + 1$

$f(0) = 3$

y-intercept $(0, 3)$

2. To help determine the shape of the curve, it is useful to know any two other points on the graph. For example, determine the coordinates of the points at $x = \pm 1$.

$f(1) = 2 + e^{-1}$ and $f(-1) = 2 + e^{-1}$

$f(-1) = 2 + e$

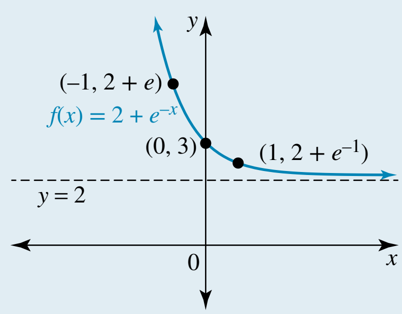
$(1, 2 + e^{-1}) \approx (1, 2.37)$

$(-1, 2 + e) \approx (-1, 4.72)$

3. State the equation of the horizontal asymptote.

$y = 2$

4. Sketch the function.



b. State the transformations.

b. The transformations required to map $f(x) = e^x$ onto $f(x) = 2 + e^{-x}$ are a reflection in the y-axis and a vertical translation upwards by 2 units.

2.3.3 Indicial equations with *e*

Determining the x-intercepts of exponential functions may involve solving equations with e.

- The laws of indices apply in the same way if e is the base.
- Equations involving e are solved using the same methods as any equation involving indices.
- Solving may require the use of logarithms with base e.
- The laws of logarithms apply, using the notation $\log_e(x)$ or $\ln(x)$ as found on your calculator.
- Since $e^x > 0$, not all equations have real solutions. For example, $e^x = -1$ has no real solution.

WORKED EXAMPLE 6

Consider the function $f(x) = 2 - e^{-x}$.

a. Determine the coordinates of any axis intercepts.

b. Sketch the function $f(x) = 2 - e^{-x}$, showing all important features.

c. State the transformations required to map $f(x) = e^x$ onto $f(x) = 2 - e^{-x}$.

THINK

WRITE

a. 1. For the y-intercept, $x = 0$.
- Substitute $x = 0$.
- Evaluate.

a. $f(x) = 2 - e^{-x}$

$f(0) = 2 - e^0$

$f(0) = 2 - 1 = 1$

2. For the x-intercept, $y = 0$.
- Substitute $y = 0$.
- Rearrange the equation.
- Take the log (base e) of both sides.
- Use log laws to simplify.

$2 - e^{-x} = 0$

$2 = e^{-x}$

$\log_e 2 = \log_e e^{-x}$

$-x \log_e e = \log_e 2$

$x = -\log_e 2$

3. State the coordinates of the axis intercepts. (*Hint:* An approximation may be useful.)

Axis intercepts:
$(-\log_e 2, 0)$ and $(0, 1)$
Approximately:
$(-0.693, 0)$ and $(0, 1)$

b. 1. To sketch the graph of $f(x) = 2 - e^{-x}$, use the x- and y-intercepts found in part a.

b. Axis intercepts:
$(-\log_e 2, 0)$ and $(0, 1)$
Approximately:
$(-0.693, 0)$ and $(0, 1)$

2. To help determine the shape of the curve, it is useful to know any two other points on the graph. For example, determine the coordinates of the points at $x = \pm 1$.

$f(1) = 2 - e^{-1}$ and $f(-1) = 2 - e^{-1}$

$f(-1) = 2 - e$

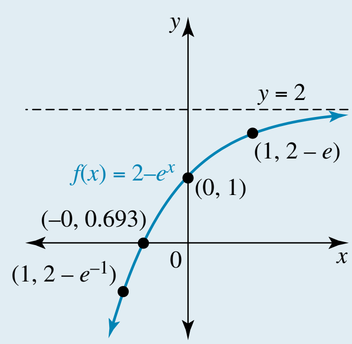
$(1, 2 - e^{-1}) \approx (1, 1.632)$

$(-1, 2 - e) \approx (-1, -0.718)$

3. State the equation of the horizontal asymptote.

$y = 2$

4. Sketch the function.



c. State the transformations.

c. The transformations required to map $f(x) = e^x$ onto $f(x) = 2 - e^{-x}$ are reflections in both the x- and y-axes and a vertical translation upwards by 2 units.

WORKED EXAMPLE 7

Solve $6e^x = 15 + e^x$ for x, giving your answer:

a. as an exact number

b. correct to 3 decimal places.

THINK	WRITE
a. 1. Write the equation.	a. $6e^x = 15 + e^x$
2. Collect like terms.	$5e^x = 15$
3. Find e^x.	$e^x = 3$
4. Take the log of both sides. (*Note*: $\log_e(x) = \ln(x)$)	$\ln(e^x) = \ln(3)$
5. State the solution as an exact number.	$x = \ln(3)$
b. 1. Use your calculator to determine the approximation.	b. $x = 1.098\,61$
2. State the solution correct to 3 decimal places.	$x = 1.099$

WORKED EXAMPLE 8

Solve $e^x - 3e^{-x} = 2$ for x, giving your answer(s) correct to 2 decimal places.

THINK	WRITE
1. Write the equation.	$e^x - 3e^{-x} = 2$
2. Rewrite without negative indices.	$e^x - \frac{3}{e^x} = 2$
3. Multiply each term by e^x.	$(e^x)^2 - 3 = 2e^x$
4. Recognise a quadratic equation in e^x.	$(e^x)^2 - 2e^x - 3 = 0$
5. Let $a = e^x$.	$a^2 - 2a - 3 = 0$
6. Factorise the quadratic expression.	$(a - 3)(a + 1) = 0$
7. Solve for a.	$a = 3$ or $a = -1$
8. Substitute e^x for a.	$e^x = 3$ or $e^x = -1$
9. Solve for x.	$\ln(e^x) = \ln(3),\ e^x \neq -1$ $x = \ln(3)$
10. State the solution correct to 2 decimal places.	$x = 1.10$ (to 2 decimal places)

on Resources

Interactivity: Exponential functions (int-5959)

studyon

Units 3 & 4 > Area 2 > Sequence 1 > Concept 2

The exponential function Summary screen and practice questions

Exercise 2.3 The exponential function

Technology free

1. WE3 Sketch the following exponential functions, showing all important features.
   a. $f(x) = 4^x$ b. $f(x) = 4^{-x}$ c. $f(x) = -4^x$
2. On the same set of axes, sketch the graphs of $y = 10^x$ and $y = 10^{-x}$.
3. WE4 a. Sketch the function $f(x) = 4e^x$, showing all important features.
   b. State the transformations required to map $f(x) = e^x$ onto $f(x) = 4e^x$.
4. a. Sketch the function $f(x) = -5e^x$, showing all important features.
   b. State the transformations required to map $f(x) = e^x$ onto $f(x) = -5e^x$.
5. WE5 a. Sketch the function $f(x) = e^{-x} + 3$, showing all important features.
   b. State the transformations required to map $f(x) = e^x$ onto $f(x) = e^{-x} + 3$.
6. a. Sketch the function $f(x) = e^{2x} + 3$, showing all important features.
   b. State the transformations required to map $f(x) = e^x$ onto $f(x) = e^{2x} + 3$.
7. WE6 Consider the function $f(x) = e^{2x} - 3$.
   a. Determine the coordinates of any axis intercepts for this function.
   b. Sketch the function $f(x) = e^{2x} - 3$, showing all important features.
   c. State the transformations required to map $f(x) = e^x$ onto $f(x) = e^{2x} - 3$.
8. Consider the function $f(x) = 4 - 2e^{-x}$.
   a. Determine the coordinates of any axis intercepts for this function.
   b. Sketch the function $f(x) = 4 - 2e^{-x}$, showing all important features.
   c. State the transformations required to map $f(x) = e^x$ onto $f(x) = 4 - 2e^{-x}$.
9. a. Sketch the function $f(x) = 4e^{\frac{x}{2}}$, showing all important features.
   b. State the transformations required to map $f(x) = e^x$ onto $f(x) = 4e^{\frac{x}{2}}$.

Technology active

10. a. Determine the coordinates of any axis intercepts for the function $y = 3e^{-\frac{x}{2}} - 6$.
    b. Sketch the function, showing all important features.
    c. State the transformation required to map $y = e^x$ onto $y = 3e^{-\frac{x}{2}} - 6$.
11. WE7 Solve $3e^x + 8 = 5e^x$ for x, giving your answer:
    a. as an exact number
    b. correct to 3 decimal places.
12. Solve for x, giving your answer correct to 3 decimal places.
    a. $e^x = 5$ b. $e^x = \frac{1}{2}$ c. $e^x = 2.6$
    d. $e^{-x} = 6$ e. $3 = 2e^x$ f. $3e^{-x} - 10 = 0$
13. Solve for x in each of the following, giving your answer in exact form.
    a. $(e^x - 1)(e^x - 2) = 0$ b. $(e^x - 1)(e^x + 3) = 0$
    c. $(e^{-x} - 1)(e^{2x} - 4) = 0$ d. $(3e^{-x} - 2)(2e^x - 1) = 0$
    e. $(2e^x + 1)(e^x - 4) = 0$ f. $(3e^x - 2)(e^x + 4) = 0$
14. WE8 Solve $e^x - 15e^{-x} = 2$ for x, giving your answer(s) correct to 2 decimal places.
15. Solve for x in each of the following, giving your answers in exact form.
    a. $5e^x - 12e^{-x} - 11 = 0$ b. $3e^x + 6e^{-x} = 11$
    c. $2e^x = 9 + 5e^{-x}$ d. $e^x = 25e^{-x}$

16. Solve for x in each of the following, giving your answers in exact form.

a. $e^x > 1$ b. $e^{-x} < e$ c. $e^{2x} \geq 4$ d. $e^{1-x} \leq 6$

17. a. Sketch the curve $y = 2e^{-x} + 1$.
b. For what values of x is $y < 3$?
c. Discuss why $2e^{-x} + 1 < 0$ has no real solutions.

18. a. Sketch the curve $f(x) = 4 - e^x$, stating all axis intercepts in exact form.
b. For what values of x is $y > 0$?
c. Discuss the range of the function if the domain is $x \geq 0$.

2.4 Differentiation of exponential functions

The derivative of the exponential function can be found using first principles.

If $f(x) = e^x$, then

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0}\frac{e^{x+h}-e^x}{h} \\ &= \lim_{h\to 0}\frac{e^x e^h-e^x}{h} \\ &= \lim_{h\to 0}\frac{e^x(e^h-1)}{h} \\ &= e^x\lim_{h\to 0}\frac{e^h-1}{h} \end{aligned}$$

In subtopic 2.2, it was shown that $\lim_{h\to 0}\frac{a^h-1}{h} = 1$ for $a = e$; that is, $\lim_{h\to 0}\frac{e^h-1}{h} = 1$.

Substituting into $f'(x) = e^x\lim_{h\to 0}\frac{e^h-1}{h}$:

$$f'(x) = e^x$$

The derivative of $y = e^{f(x)}$ can be found using the chain rule, studied in Year 11.

If $y = e^{f(x)}$, let $u = f(x)$.

Then $y = e^u$.

$$\frac{dy}{du} = e^u \text{ and } \frac{du}{dx} = f'(x)$$

The **chain rule** states: $$\frac{dy}{dx} = \frac{dy}{du}\times\frac{du}{dx}$$

Substitute: $$\frac{dy}{dx} = e^u \times f'(x)$$

Replace u as $f(x)$: $$\frac{dy}{dx} = f'(x)e^{f(x)}$$

The differential of an exponential

$$\frac{d(e^x)}{dx} = e^x$$

And with the chain rule:

$$\frac{d(e^{f(x)})}{dx} = f'(x)e^{f(x)}$$

or

$$\frac{d(e^u)}{dx} = e^u \times \frac{du}{dx} \text{ where } u = f(x)$$

WORKED EXAMPLE 9

Using the chain rule, differentiate $y = e^{-5x}$ with respect to x.

THINK	WRITE
1. Write the equation.	$y = e^{-5x}$
2. Substitute $u = -5x$.	$y = e^u$ and $u = -5x$
3. Determine $\frac{dy}{du}$ and $\frac{du}{dx}$.	$\frac{dy}{du} = e^u$ and $\frac{du}{dx} = -5$
4. Use the chain rule to find $\frac{dy}{dx}$.	$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$ $\frac{dy}{dx} = e^u \times (-5)$
5. State the derivative in terms of x.	$\frac{dy}{dx} = -5e^x$
Alternatively, recognise and apply the formula $\frac{d(e^{f(x)})}{dx} = f'(x)e^{f(x)}$ where $f(x) = -5x$.	$y = e^{-5x}$ $\frac{dy}{dx} = -5e^x$

WORKED EXAMPLE 10

Determine the derivative of $y = e^{2x+1}$.

THINK	WRITE
1. Write the equation.	$y = e^{2x+1}$
2. Recognise and apply the formula $\frac{d(e^{f(x)})}{dx} = f'(x)e^{f(x)}$ where $f(x) = 2x + 1$.	$\frac{dy}{dx} = 2e^{2x+1}$

WORKED EXAMPLE 11

Differentiate the the following.

a. $f(x) = e^x(e^x - 3)$

b. $f(x) = \dfrac{e^{3x} - 2e^{-x}}{e^x}$

THINK	WRITE
a. 1. Write the equation.	**a.** $f(x) = e^x(e^x - 3)$
2. Write the equation in expanded form.	$f(x) = e^{2x} - 3e^x$
3. Differentiate each term.	$f'(x) = 2e^{2x} - 3e^x$
b. 1. Write the equation.	**b.** $f(x) = \dfrac{e^{3x} - 2e^{-x}}{e^x}$
2. Separate the terms and simplify.	$f(x) = \dfrac{e^{3x}}{e^x} - \dfrac{2e^{-x}}{e^x}$ $f(x) = e^{2x} - 2e^{-2x}$
3. Differentiate.	$f'(x) = 2e^{2x} + 4x^{-2x}$ $f'(x) = 2e^{2x} + \dfrac{4}{e^{2x}}$

WORKED EXAMPLE 12

Determine the derivative of the function $y = e^{\left(x^3 - x\right)}$.

THINK	WRITE
1. Write the equation.	$y = e^{(x^3 - x)}$
2. Substitute $u = x^3 - x$.	$y = e^u$ and $u = x^3 - x$
3. Determine $\dfrac{dy}{du}$ and $\dfrac{du}{dx}$.	$\dfrac{dy}{du} = e^u$ and $\dfrac{du}{dx} = 3x^2 - 1$
4. Use the chain rule to find $\dfrac{dy}{dx}$.	$\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$ $\dfrac{dy}{dx} = e^u \times \left(3x^2 - 1\right)$
5. State the derivative in terms of x.	$\dfrac{dy}{dx} = (3x^2 - 1)\ e^{(x^3 - x)}$
Alternatively:	
1. Write the equation	$y = e^{(x^3 - x)}$
2. Recognise and apply the formula $\dfrac{d(e^{f(x)})}{dx} = f'(x)e^{f(x)}$ where $f(x) = x^3 - x$.	$\dfrac{dy}{dx} = (3x^2 - 1)\, e^{(x^3 - x)}$

studyon

Units 3 & 1 > Area 2 > Sequence 1 > Concept 3

Differentiation of exponential functions Summary screen and practice questions

Exercise 2.4 Differentiation of exponential functions

Technology free

1. **WE9** Differentiate the following.

a. $y = e^{10x}$ b. $y = e^{\frac{1}{3}x}$ c. $y = e^{\frac{x}{4}}$

d. $y = e^{-x}$ e. $y = 2e^{3x}$ f. $y = 4e^{-5x}$

2. **WE10** Differentiate the following.

a. $y = e^{6x-2}$ b. $y = e^{8-6x}$ c. $y = 2e^{5x+3}$

d. $y = 4e^{7-2x}$ e. $y = -3e^{8x+1}$ f. $y = -2e^{6-5x}$

3. a. $y = 10e^{6-9x}$ b. $y = -5e^{3x+4}$ c. $y = 6e^{-7x}$

d. $y = 2e^{\frac{x}{2}+1}$ e. $y = 3e^{2-\frac{x}{3}}$ f. $y = -4e^{\frac{x}{4}+5}$

4. **MC** The derivative of $y = e^{3x+2}$ is equal to:

A. $3e^{3x+2}$ B. $(3x+2)e^{3x+2}$ C. $3e^{3x}$ D. $3xe^{3x+2}$

5. **WE11** Differentiate the following.

a. $f(x) = 2(e^x + 1)$ b. $f(x) = 3e^{2x}(e^x + 1)$

c. $f(x) = 5(e^{-4x} + 2x)$ d. $f(x) = (e^x + 2)(e^{-x} + 3)$

6. **WE11** Differentiate the following.

a. $f(x) = \dfrac{3e^{3x} + e^{-6x}}{e^x}$ b. $f(x) = \dfrac{4e^{7x} - 2e^{-x}}{e^{-2x}}$

7. **WE12** Determine the derivatives of the following.

a. $y = e^{x^2+3x}$ b. $y = e^{x^2-3x+1}$ c. $y = e^{x^2-2x}$ d. $f(x) = e^{2-5x}$

8. Use the formula $\dfrac{d(e^{f(x)})}{dx} = f'(x)e^{f(x)}$ to differentiate the following functions.

a. $f(x) = e^{6-3x+x^2}$ b. $g(x) = e^{x^3+3x-2}$ c. $h(x) = 3e^{4x^2-7x}$ d. $y = -5e^{1-2x-3x^2}$

9. **MC** The derivative of $6e^{x^3-5x}$ is equal to:

A. $6(3x^2-5)e^{x^3-5x}$ B. $(3x^2-5)e^{x^3-5x}$

C. $6(x^3-5x)e^{x^3-5x}$ D. $6(3x^2-5)e^{3x^2-5}$

10. If $f(x) = 5e^{9-4x}$, determine the exact value of $f'(2)$.

11. If $g(x) = 2e^{x^2-3x+2}$, determine the exact value of $g'(0)$.

12. Calculate the exact value of $h'(-1)$ if $h(x) = -5e^{x^2+3x}$.

13. Determine the equation of the tangent to the curve $y = e^{x^2+3x-4}$ at the point where $x = 1$.

14. Determine the equations of the tangent and the line perpendicular to the curve $y = e^{-3x} - 2$ at the point where $x = 0$.

15. Determine the derivative of the function $f(x) = e^{-2x+3} - 4e$ and hence find:

a. $f'(-2)$ in exact form b. $\{x : f'(x) = -2\}$

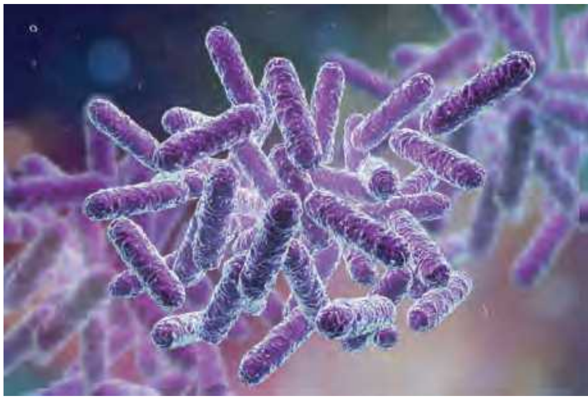
16. Determine the derivative of the function $f(x) = \dfrac{e^{3x} + 2}{e^x}$ and hence find:

a. $f'(1)$ in exact form b. $\{x : f'(x) = 0\}$

2.5 Applications of exponential functions

Functions involving the exponential function $y = e^x$ can be used to model many real-life situations. These include:

- population growth and decay, for example bacteria
- radioactive decay
- growth of investments
- cooling of heated objects.

In modelling that involves exponential growth and decay, the rate of change of the function often implies the change with respect to time, $\frac{dy}{dt}$.

It may be necessary to restrict the domain of the exponential function to suit the context of the problem. For example, time (t) cannot be negative.

WORKED EXAMPLE 13

The number of bacteria on a culture plate, N, can be defined by the rule

$$N(t) = 2000e^{0.3t}, t \geq 0$$

where t is the time, in seconds, the culture has been growing.

a. **How many bacteria were initially present?**

b. **How many bacteria, to the nearest whole number, are present after 10 seconds?**

c. **At what rate is the bacteria population increasing after 10 seconds? Give your answer correct to the nearest whole number.**

THINK	WRITE
a. 1. The initial time is when $t = 0$. Substitute $t = 0$ and evaluate.	a. $N(t) = 2000e^{0.3t}$ $N(0) = 2000e^{0.3(0)}$ $= 2000$
2. Write the answer.	Initially there were 2000 bacteria present.
b. 1. After 10 seconds, $t = 10$. Substitute $t = 10$.	b. $N(t) = 2000e^{0.3t}$ $N(10) = 2000e^{0.3(10)}$ $= 2000e^{3}$ $= 40\,171.074$
2. Write the answer.	After 10 seconds there were 40 171 bacteria present.
c. 1. Differentiate $\left(\frac{d(e^{f(x)})}{dx} = f'(x)\ e^{f(x)}\right)$ to find rate of change with respect to time.	c. $N(t) = 2000e^{0.3t}$ $\frac{dN}{dt} = 2000 \times 0.3e^{0.3t}$ $\frac{dN}{dt} = 600e^{0.3t}$

2. After 10 seconds, $t = 10$. Substitute $t = 10$ and evaluate.

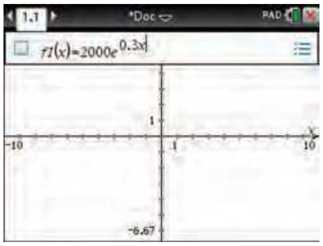
$$\frac{dN}{dt} = 600e^{0.3(10)}$$
$$= 600e^{3}$$
$$= 12\,051.322$$

3. Write the answer with correct units.

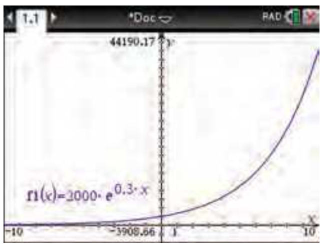
After 10 seconds the bacteria is increasing at a rate of 12 051 bacteria/second.

TI | THINK **WRITE**

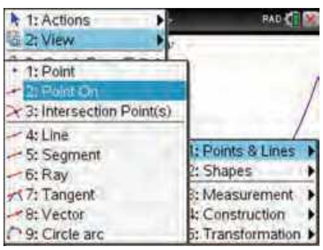
a.1. On a Calculator page, press MENU, then select:
2: Add Graphs.
Complete the entry line in the $f1(x) =$ tab as:
$2000e^{0.3x}$



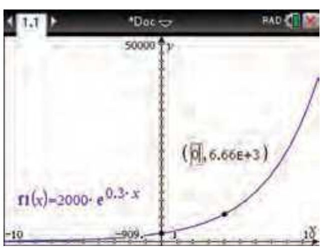
2. Sketch the graph by pressing the ENTER button.



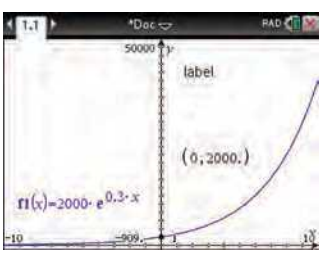
3. To calculate the initial value, select:
Menu
8: Geometry
1: Points & Lines
2: Point On.



4. Move the cursor and select the curve representing $f1\,(x)$.
Press the ESC (escape) button. This allows you to move the text box indicating the coordinates of the point P (x, y).
Complete the entry line in the textbox as 0 for the x-value.

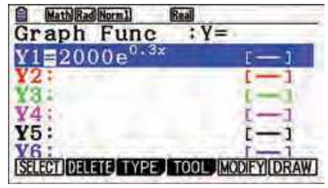


5. The answer appears on the screen.
In this case, when $x = 0,\ y = 2000$.



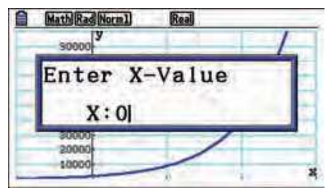
CASIO | THINK **WRITE**

a.1. On a Main Menu screen, select:
Graph.
Complete the entry line in the Y1 tab as:
$2000e^{0.3x}$
Note: The independent variable t has been replaced with x.

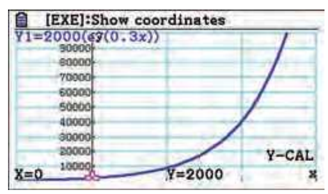


2. Sketch the graph by pressing either the DRAW or EXE button.

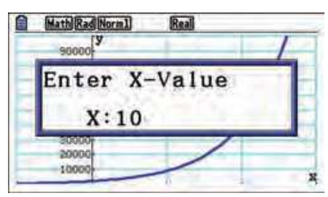
3. To calculate the initial value, select:
G-Solv (SHIFT F5)
Y-CAL.
Complete the entry line in X: as 0.



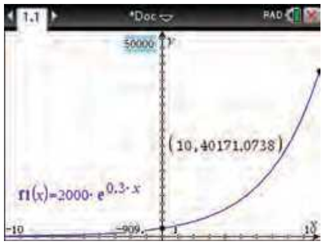
4. The answer appears on the screen.
In this case, when $x = 0,\ y = 2000$.



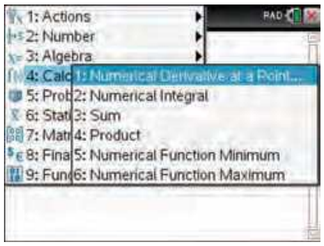
5. To calculate the number of bacteria after 10 seconds, select:
G-Solv (SHIFT F5)
Y-CAL.
Complete the entry line in X: as 10.



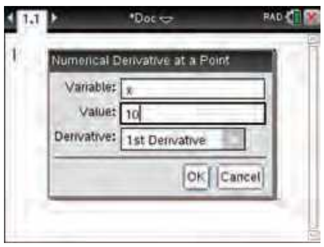
b.1. Complete the entry line in the textbox as 10 for the x-value.
The answer appears on the screen.
In this case, when
$x = 10$,
$y = 40\,171.074$.
Note: You can also calculate an x-value by entering a desired y-value.



c.1. On a Calculator page, press MENU, then select:
4: Calculus
1: Numerical Derivative at a Point.



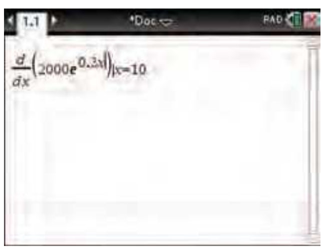
2. Complete the value entry line as:
10
Press the OK button.



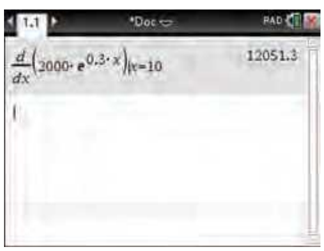
3. Complete the entry line as:

$$\frac{d}{dx}\left(2000e^{0.3x}\right)$$

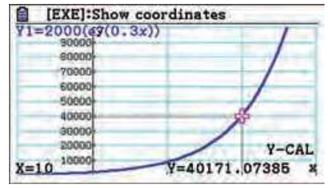
Press the ENTER button to complete the calculation.



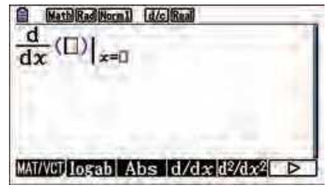
The answer appears on the screen.
In this case, when
$x = 10$,
$\frac{dy}{dx} = 12\,051.322$.



b.1. The answer appears on the screen.
In this case, when
$x = 10$, $y = 40\,171.074$.



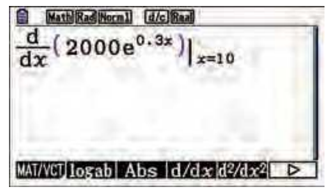
c.1. On a Run Matrix screen, select:
Math $\frac{d}{dx}$.



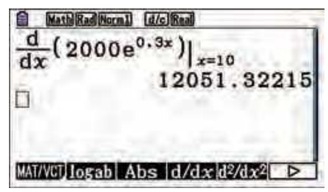
2. Complete the entry line as:

$$\frac{d}{dx}\left(2000e^{0.3x}\right)$$

Enter 10 in the x-value box.
Press the EXE button to complete the calculation.



3. The answer appears on the screen.
In this case, when
$x = 10$,
$\frac{dy}{dx} = 12\,051.322$.



WORKED EXAMPLE 14

The mass, M grams, of a radioactive substance is initially 20 grams. The substance has been stored for 30 years already. The mass in any year is given by

$$M(t) = M_0 e^{-0.00152t}$$

where M_0 is a constant and t is the time in years.

a. **Determine the value of M_0.**

b. **Calculate the amount of the substance remaining after a further 30 years. Give your answer correct to 2 decimal places.**

c. **Determine the rate of decay at this time. Give your answer correct to 2 decimal places.**

THINK	WRITE
a. 1. The mass, M grams, of a radioactive substance is initially 20 grams. Substitute $t = 0, M(0) = 20$.	a. $M(t) = M_0 e^{-0.00152t}$ $M(0) = M_0 e^{-0.00152(0)} = 20$ $M_0 e^0 = 20$ $\Rightarrow M_0 = 20$
2. Write the answer.	$M_0 = 20$
b. 1. Rewrite the equation with $M_0 = 20$.	b. $M(t) = 20e^{-0.00152t}$
2. The mass, M grams, of a radioactive substance is initially 20 grams. After 30 years ($t = 30$), a further 30 years is when $t = 60$. Substitute $t = 60$ and evaluate.	$M(60) = 20e^{-0.00152(60)}$ $= 18.2567$
3. Write the answer.	After a further 30 years, the mass is 18.26 grams.
c. 1. Differentiate to find the rate of change with respect to time.	c. $M(t) = 20e^{-0.001\,52t}$ $M'(t) = 20 \times (-0.00152)\, e^{-0.00152t}$ $M'(t) = -0.0304e^{-0.00152t}$
2. Evaluate the rate of change at $t = 60$ by substitution.	$M'(60) = -0.0304e^{-0.00152(60)}$ $= -0.027\,75$
3. Answer the question with correct units.	The rate of decay after 60 years is 0.03 grams/year. *Note:* The question asked for the rate of decay, so the negative sign is not included in the final answers. The negative indicates a rate that is decreasing.

WORKED EXAMPLE 15

The population of foxes on the outskirts of a city is starting to increase.

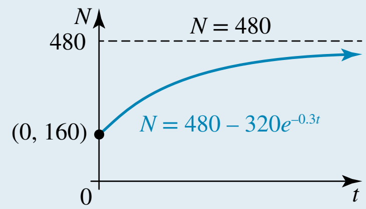
Data collected suggest that a model for the number of foxes is given by $N(t) = 480 - 320e^{-0.3t}$, $t \geq 0$ where N is the number of foxes t years after the observations began.

a. **How many foxes were present initially at the start of the observations?**
b. **By how many had the population of foxes grown at the end of the first year of observations?**
c. **After how many months does the model predict the number of foxes would double its initial population?**
d. **Sketch the graph of N versus t.**
e. **Explain why this model does not predict that the population of foxes will grow to 600.**

THINK	WRITE
a. 1. Calculate the initial number, that is when $t = 0$.	**a.** $N(t) = 480 - 320e^{-0.3t}$ When $t = 0$, $N(0) = 480 - 320e^{0}$ $= 480 - 320$ $= 160$
2. Answer the question.	There were 160 foxes present initially.
b. 1. Calculate the number of foxes, N, after 1 year, $t = 1$.	**b.** When $t = 1$, $N(1) = 480 - 320e^{-0.3}$ ≈ 242.94 After the first year 243 foxes were present.
2. Express the change over the first year in context.	Over the first year the population grew from 160 to 243, an increase of 83 foxes.
c. 1. Calculate the required value of t. *Note:* An algebraic method requiring logarithms has been used here.	**c.** Let $N = 2 \times 160 = 320$. $320 = 480 - 320e^{-0.3t}$ $320e^{-0.3t} = 160$ $e^{-0.3t} = \frac{1}{2}$ $-0.3t = \log_e\left(\frac{1}{2}\right)$ $t = -\frac{1}{0.3}\log_e\left(\frac{1}{2}\right)$ $t \approx 2.31$
2. Answer the question.	$0.31 \times 12 \approx 4$ The population doubles after 2 years and 4 months.

d. Sketch the graph.

d. $N(t) = 480 - 320e^{-0.3t}$
The horizontal asymptote is $N = 480$.
The y-intercept is $(0, 160)$.



e. Give an explanation for the claim that this model will not predict the population of foxes to grow to 600.

e. The presence of an asymptote on the graph shows that as $t \to \infty$, $N \to 480$.
Hence, N can never reach 600.
The population will never exceed 480 according to this model.

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Units 3 & 4 | Area 2 | Sequence 1 | Concept 4

Applications of exponential functions Summary screen and practice questions

Exercise 2.5 Applications of exponential functions

Technology active

1. **WE13** The number of bacteria on a culture plate, N, can be defined by the rule

$$N(t) = 500e^{0.46t}, t \geq 0$$

where t is the time, in hours, that the culture has been growing.

a. How many bacteria were initially present?

b. How many bacteria, to the nearest whole number, are present after 5 hours?

c. At what rate is the bacteria population increasing after 5 hours? Give your answer correct to the nearest whole number.

2. The bilby is an endangered species that can be found in the Kimberley in Western Australia as well as some parts of South Australia, the Northern Territory and Queensland. The gestation time for a bilby is 2–3 weeks and when they are born, they are only about 11 mm in length. The growth of a typical bilby can be modelled by the rule

$$L = L_0e^{0.599t}$$

where L_0 is its length in millimetres at birth and L is the length of the bilby in millimetres t months after its birth.

a. Determine the value of L_0.

b. What is the rate of change of length of the bilby at time t months?

c. At what rate is the bilby growing when it is 3 months old? Give your answer correct to 3 decimal places.

3. WE14 The mass, M grams, of a particular radioactive substance can be modelled by the exponential function

$$M(t) = M_0 e^{-0.005t}$$

where M_0 is a constant and t is the time in days, Initially the substance weighed 50 grams.
   a. Determine the value of M_0.
   b. Calculate the amount of the substance remaining after 10 days. Give your answer correct to 2 decimal places.
   c. Determine the rate of decay at this time. Give your answer correct to 2 decimal places.
4. Changing δ-gluconolactone into gluconic acid can be modelled by the equation $y = y_0 e^{-0.6t}$ where y is the number of grams of δ-gluconolactone present t hours after the process has begun. Suppose 200 grams of δ-gluconolactone is to be changed into gluconic acid.
   a. Determine the value of y_0.
   b. How many grams of δ-gluconolactone will be present after 1 hour? Give your answer correct to the nearest gram.
   c. How long will it take to reduce the amount of δ-gluconolactone to 50 grams? Give your answer correct to the nearest quarter of an hour.
   d. Determine the rate of change in the δ-gluconolactone after 2 hours. Give your answer to 1 decimal place.
5. The decay of radon-222 gas is given by the equation $y = y_0 e^{-0.18t}$, where y is the amount of radon remaining after t days. If initially there was 10 grams of radon-222 gas, determine:
   a. the value of y_0
   b. the amount of gas, to the nearest integer, remaining after 2 days
   c. how many days it will take for the mass to reach half its original mass
   d. the rate of decay after 5 days, correct to 2 decimal places.
6. An amount of $1000 is invested in a building society where the 5% p.a. interest is compounded continuously. The amount in the account after t years can be modelled by the equation $A = A_0 \times e^{rt}$, where A_0 is the initial value of the investment and r is the continuous interest rate expressed as a decimal.
   a. State the value of A_0 and r.
   b. Calculate the amount in the account, correct to the nearest cent, after:
      i. 1 year
      ii. 5 years
      iii. 10 years.
   c. At what rate was the investment increasing, correct to the nearest cent per year, after:
      i. 1 year? ii. 5 years? iii. 10 years?
   d. Estimate how long, to the nearest year, it would take for the investment to double in value.
7. A body that is at a higher temperature than its surroundings cools according to Newton's Law of Cooling, which states that

$$T = T_0 e^{-zt}$$

where T_0 is the original *excess* of temperature, T is the excess of temperature in degrees Celsius after t minutes, and z is a constant.
   a. The original temperature of the body was 95 °C and the temperature of the surroundings was 20 °C. Find the value of T_0.
   b. At what rate is the temperature decreasing after a quarter of an hour if it is known that $z = 0.034$? Give your answer correct to 3 decimal places.

8. The number of people living in Boomerville at any time, t years, after the first settlers arrived can be modelled by the equation $P = P_0e^{kt}$. Initially, on 1 January 1850, Boomerville had a population of 500 people. By 1 January 1860, the population had grown to 675.
   a. Determine the value of P_0.
   b. Calculate the value of k correct to 2 decimal places.
   c. Using this value of k, determine the population on 1 January 1900. Give your answer to the nearest 10 people.
   d. At what rate was the population increasing on 1 January 1900? Give your answer to the nearest whole number.
9. The mass, m kg, of a radioactive isotope remaining in a sample t hours after observations began is given by the rule $m(t) = ae^{-kt}$. Initially there is 2 kg of the isotope. After 3 hours the mass of the isotope has decreased to 1.1 kg.
   a. Determine the values of a and k. Give your answers correct to 1 decimal place where necessary.
   b. Using these values, find the rate of change of the isotope as a function of t.
   c. Calculate the rate of decay of the isotope after 6 hours. Give your answer to 2 decimal places.
   d. Determine the half-life of this isotope, that is, the time it takes for the isotope to reduce to half its original mass. Give your answer correct to 1 decimal place.
10. An unstable gas decomposes in such a way that the amount present, A units, at time t minutes is given by the equation

$$A = A_0e^{-kt}$$

   where k and A_0 are constants. It was known that initially there were 120 units of unstable gas.
   a. Find the value of A_0.
   b. After 2 minutes there were 90 units of the gas left. Find the value of k.
   c. At what rate is the gas decomposing when $t = 5$? Give your answer correct to 3 decimal places.
   d. Will there ever be no gas left? Explain your answer.
11. The population of Australia since 1950 can be modelled by the rule

$$P = P_0e^{0.016t}$$

   where P_0 is the population in millions at the beginning of 1950 and P is the population in millions t years after 1950. It is known that there were 8.2 million people in Australia at the beginning of 1950.
   a. Calculate the population in millions at the beginning of 2015, correct to 1 decimal place.
   b. During which year and month did the population reach 20 million?
   c. Determine the rate of change of population at the turn of the century, namely the year 2000, correct to 2 decimal places.
   d. In which year was the rate of increase of the population predicted to exceed 400 000 people per year?
12. The pressure of the atmosphere, P cm of mercury, decreases with the height, h km above sea level, according to the law

$$P = P_0e^{-kh}$$

   where P_0 is the pressure of the atmosphere at sea level and k is a constant. At 500 m above sea level, the pressure is 66.7 cm of mercury, and at 1500 m above sea level, the pressure is 52.3 cm of mercury.
   a. What are the values of P_0 and k, correct to 2 decimal places?
   b. Find the rate at which the pressure is falling when the height above sea level is 5 km. Give your answer correct to 2 decimal places.

13. The cane toad, originally from South America, is an invasive species in Australia. Cane toads were introduced to Australia from Hawaii in 1935 in an attempt to control cane beetles, though this proved to be ineffective.

In a controlled experiment at a particular waterhole, it was observed that at the beginning of the experiment there were an estimated 30 000 tadpoles (future cane toads) in the water. The number of tadpoles increased by about 60 000 a day over the period of a week. This growth pattern can be defined by the equation

$$T = T_0 e^{kt}$$

where T_0 is the initial number of cane toad tadpoles (in thousands) at the waterhole during the time of the experiment, T is the number of cane toad tadpoles (in thousands) at the waterhole t days into the experiment, and k is a constant.

a. Calculate the value of T_0.

b. How many cane toad tadpoles are in the waterhole after 7 days if it is known that $k = 0.387$? Give your answer to the nearest thousand.

c. Determine the rate at which the cane toad tadpole numbers are increasing after 3 days.

14. **WE 15** The population of possums in an inner city suburb is starting to increase. Observations of the numbers present suggest a model for the number of possums in the suburb given by $P(t) = 83 - 65e^{-0.2t}$, $t \geq 0$, where P is the number of possums observed and t is the time in months since observations began.

a. How many possums were present at the start of the observations?

b. By how many had the population of possums grown at the end of the first month of observations? Give your answer to the nearest whole number.

c. When does the model predict the number of possums would be twice the initial population?

d. Sketch the graph of P versus t.

e. Explain why this model does not predict the population of possums will grow to 100.

15. Manoj pours himself a mug of coffee but gets distracted by a phone call before he can drink the coffee. The temperature of the cooling mug of coffee is given by $T = 20 + 75e^{-0.062t}$, where T is the temperature of the coffee t minutes after it was initially poured into the mug.

a. What was the initial temperature of the coffee when it was first poured?

b. Sketch the graph of temperature, T °C against time, t minutes.

c. To what temperature will the coffee cool if left unattended?

d. How long does it take for the coffee to reach a temperature of 65 °C? Give your answer correct to 2 decimal places.

e. Determine the rate of change in the temperature of the coffee after 10 minutes, correct to 1 decimal place. Explain why the rate of change is negative.

16. Newton's Rule of Cooling states that the rate of change of the temperature of a particle is proportional to the difference between the temperature of the particle and the constant temperature of the surrounding medium. The temperature, T °C, of a particle when placed in a medium with a constant temperature of A °C can be modelled by the equation

$$T = T_0 e^{-kt} + A$$

where t is time and T_0 is a constant.

A metal ball has been heated to a temperature of 200 °C and is placed into a room that is maintained at a constant temperature of 30 °C. After 5 minutes, the temperature of the ball has dropped to 150 °C.

a. State the value of A and hence determine the value of T_0.
b. Calculate the value of k correct to 4 decimal places.
c. Using the values found above, state the equation for this model and sketch its graph.
d. Determine the temperature of the rod, correct to 1 decimal place, after a further 10 minutes.
e. Calculate the rate of change in the temperature, correct to 1 decimal place, at this time.
f. Determine how long it will take for the temperature of the ball to reach 40 °C. Give your answer to 2 decimal places.
g. Verify, with reasons, that the metal ball would never reach 10 °C if left in the room.

2.6 Review: exam practice

A summary of this chapter is available in the Resources section of your eBookPLUS at www.jacplus.com.au.

Simple familiar

1. Evaluate the following limits.

a. $\lim\limits_{x\to 3}(6x-1)$
b. $\lim\limits_{x\to 3}\dfrac{2x^2-6x}{x-3}$
c. $\lim\limits_{x\to 1}\dfrac{2x^2+3x-5}{x^2-1}$
d. $\lim\limits_{x\to 0}\dfrac{3x-5}{2x-1}$

2. Using first principles, calculate $\dfrac{dy}{dx}$ for the following functions.

a. $y=4-x^2$
b. $y=x^2+4x$
c. $y=x(x+1)$

3. Given $f(x)=(x+5)^2$:

a. find $f'(x)$ using first principles
b. calculate $f'(-5)$ and explain its geometric meaning
c. calculate the gradient of the tangent to the curve $y=f(x)$ at its y-intercept
d. calculate the instantaneous rate of change of the function $y=f(x)$ at $(-2, 9)$.

4. Solve the following for x, giving your answers to 3 decimal places.

a. $e^{x+1}=6$
b. $2e^{4-x}-5=0$
c. $e^{-2x}=8$
d. $4-e^{x-2}=0$

5. Solve the following for x, giving your answers in exact form.

a. $e^{2x}-2e^x=0$
b. $(e^x+1)(e^x-3)=0$
c. $e^{2x}+2e^x=8$
d. $2e^{2x}-9e^x+4=0$

6. Consider the function $f(x)=-5^x$.

a. Evaluate $f(2)$.
b. On the same set of axes sketch the graphs of $y=5^x$, $y=-5^x$ and $y=5^{-x}$.
c. Express $y=5^{-x}$ in an equivalent form.

7. Sketch the following graphs and state the domain and range of each graph.

a. $y=2e^x+1$
b. $y=3-3e^{-\frac{x}{2}}$
c. $y=-\dfrac{1}{4}e^{x+1}$

8. Sketch the following graphs and state the domain and range of each graph.

a. $y=-2e^x-3$
b. $y=4e^{-3x}-4$
c. $y=5e^{x-2}$

9. **a.** Sketch the graph of $y=2e^{1-3x}-4$, labelling any intercepts with the coordinate axes with their exact coordinates.
b. Sketch the graph of $y=3\times 2^x-24$ and state its domain and range.

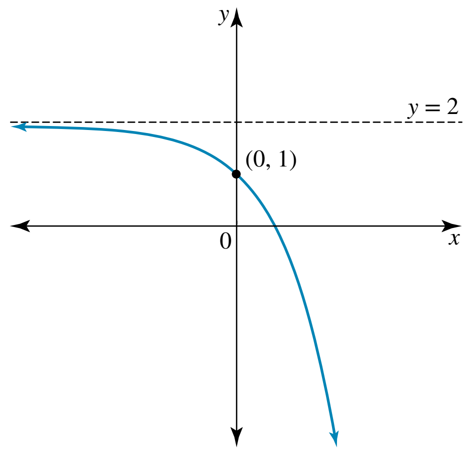
10. MC A possible equation for the graph shown is:

A. $y = 2 - e^{x}$

B. $y = 2 - e^{-x}$

C. $y = 2 + e^{-x}$

D. $y = e^{-x} - 2$



11. Determine the derivative of each of the following functions.

a. $y = e^{-\frac{1}{3}x}$

b. $y = 3x^4 - e^{-2x^2}$

c. $y = \dfrac{4e^x - e^{-x} + 2}{3e^{3x}}$

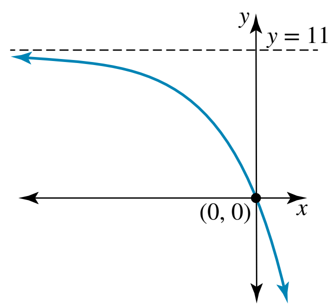
d. $y = (e^{2x} - 3)^2$

12. Consider the function defined by the rule $f(x) = \frac{1}{2}e^{3x} + e^{-x}$.

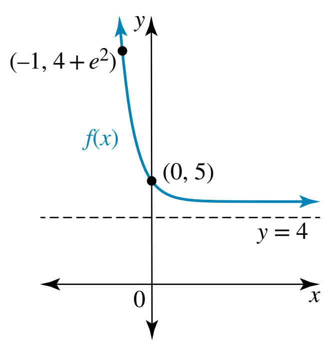
Determine the gradient of the curve when $x = 0$.

Complex familiar

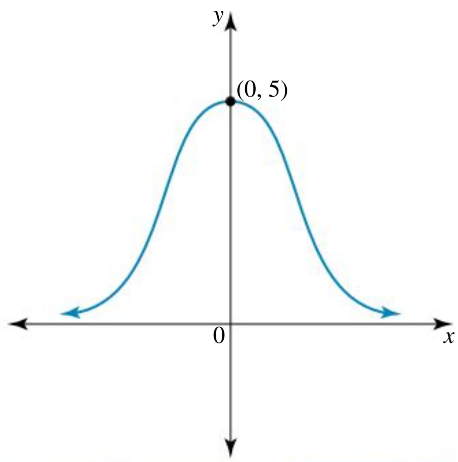
13. a. The graph shown is of the function $f(x) = ae^x + b$. Determine the values of a and b and write the function as a mapping.



b. The graph shown has an equation of the form $y = Ae^{nx} + k$. Determine its equation.



14. The graph of $y = Ae^{-x^2}$, where A is a constant, is shown. Answer the following questions correct to 2 decimal places where appropriate.



a. Determine the value of A.
b. Find $\frac{dy}{dx}$.
c. Determine the gradient of the tangent to the curve at the point where:
   i. $x = -0.5$
   ii. $x = 1$
d. Show that the equation of the tangent at the point where $x = 1$ is given by $10x + ey - 15 = 0$.

15. The mass, m g, of a radioactive isotope remaining in a sample t hours after observations began is given by the rule $m(t) = ae^{-kt}$. Initially there are 4 grams of the isotope. After 6 hours, the mass of the isotope has decreased to 2.8 g.

a. Evaluate the values of a and k. Give your answers correct to 3 decimal places where necessary.
b. Calculate the rate of decay of the isotope as a function of t.
c. Calculate the rate of decay after 6 hours. Give your answer correct to 2 decimal places.

16. The population of a certain town was 250 000 at the beginning of the year 2000 and 400 000 at the beginning of the year 2010. It was found that the relationship between the population, P thousands, and time, t years, could be modelled by the relationship $P(t) = A\,e^{kt}$ where A and k are constants.

a. Determine the values of A and k, giving your answers correct to 3 decimal places where necessary.
b. How many people lived in the coastal town at the beginning of the year 2015? Give your answer to the nearest thousand.
c. The local council has determined that the population should not exceed 750 000. In what year would this likely occur?

Complex unfamiliar

17. Microbiologists have been working with a certain type of bacteria that continues to thrive providing it has a favourable growth medium. For a particular experiment, they started with 500 bacteria and observed that the population doubles every 8 hours. The relationship between the number of bacteria, P, and the time, t hours since the bacteria started multiplying, is given by $P(t) = P_0\,e^{kt}$, where P_0 and k are constants.

a. State the value of P_0 and show that $k = \frac{1}{8}\log_e 2$.
b. The growth phase lasts for 40 hours. How many bacteria are present in the colony at this time?
c. Show that the rate of increase in the colony size after 8 hours is $125\log_e 2$ bacteria/hour.
d. Determine when the rate of increase in the colony size would be double the rate found after 8 hours.

18. The mass, m mg, of a radioactive substance remaining in a sample t hours after observations began is given by the rule $m(t) = ae^{-kt}$. Initially, 30 mg of the substance is present. After 2 hours the scientist observes that 20% had disintegrated.
  a. State the value of a.
  b. Determine the mass of the substance after 2 hours.
  c. Calculate the value of k, correct to 4 decimal places.
  d. Determine the amount of the substance remaining after a further 3 hours. Give your answers correct to 2 decimal places.
  e. Determine when the rate of decay of the substance is 1 mg/hour. Give your answer correct to 1 decimal place.

19. When money is invested in a bank at a constant rate of $r\%$ with continuously compounding interest, the accumulated amount $\$A$ at a time t years after the start of the investment is modelled by the equation

$$A = A_0 e^{rt}$$

where A_0 is the amount to be invested.
  a. Determine the amount to which \$10 000 will grow in 6 years if invested at 4.5%.
  b. If the rate did not change, estimate the time it would take for the investment to triple in value. Give your answer to the nearest month.

20. Newton's Rule of Cooling states that the rate of change of the temperature of a particle is proportional to the difference between the temperature of the particle and the constant temperature of the surrounding medium. The temperature, T °C, of a particle when placed in a medium with a constant temperature of A °C can then be modelled by the equation

$$T = T_0 e^{-kt} + A$$

where t is time and T_0 is a constant.

A saucepan is filled with water and heated until the temperature of the water reaches 100 °C. The saucepan is then placed on a bench in a room that is kept at a constant temperature of 28 °C.

After 3 minutes, the temperature of the water in the saucepan is 76 °C.
  a. State the value of A and determine the value of T_0.
  b. Show that $k = \frac{1}{3}\ln\left(\frac{3}{2}\right)$.
  c. State the equation for this model and sketch a graph to represent it.
  d. Determine the temperature of the water after a further 3 minutes.
  e. Verify, with reasons, that the water would never reach 25 °C if left in the room.

studyon

Units 3 & 4 — Sit exam

Answers

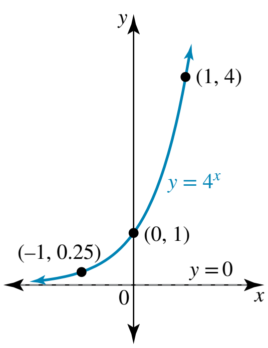
2 Calculus of exponential functions

Exercise 2.2 Review of limits and differentiation

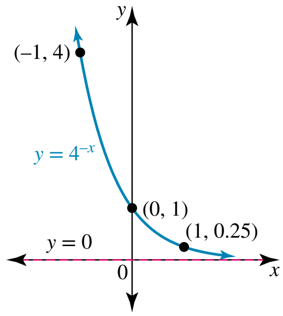
1. a. 19 b. 16 c. 2
2. a. -3 b. 3 c. $\frac{3}{2}$
3. a. -7 b. 4 c. -5
4. a. $4x^2$ b. $3x^2$
5. $f'(x) = 2x - 6$
6. $f'(x) = 3 - 4x$
7. a. 0.916 29 b. 0.955 51 c. 0.993 25
   d. 1.029 62 e. 1.064 71 f. 1.000 00
8. a. 7.3891 b. 20.0855 c. 1.6487
9. a. 0.736 b. 1.396 c. 2.472
10. a. $\frac{dy}{dx} = 8 - 2x$ b. 4 c. $y = 4x + 4$
11. $\frac{dy}{dx} = 3x^2 - 6x$; $y = 9x - 27$
12. a. $(-2, 0)$, $(0, 0)$, $(2, 0)$
    b. $\frac{dy}{dx} = 3x^2 - 4$
    c. At $x = -2$, gradient $= 8$.
       At $x = 0$, gradient $= -4$.
       At $x = 2$, gradient $= 8$.
    d. The gradient of the tangent at $x = -2$ and $x = 2$ is $m = 8$. Therefore, since the gradients are equal, the tangents are parallel.
13. a. $\frac{-h}{(x + h)x}$ b. $\frac{dy}{dx} = \frac{-1}{x^2}$
14. a. Sample responses can be found in the worked solutions in the online resources.
    b. $\frac{-1}{(x - 2)^2}$
    c. $\frac{dy}{dx} = \frac{-1}{(x - 2)^2}$
    d. $x = \frac{5}{3}, \frac{7}{3}$

Exercise 2.3 The exponential function

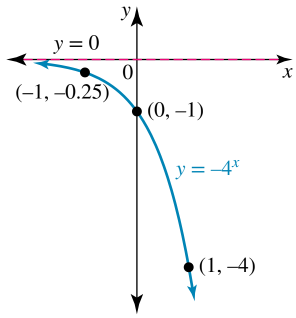
1. a.



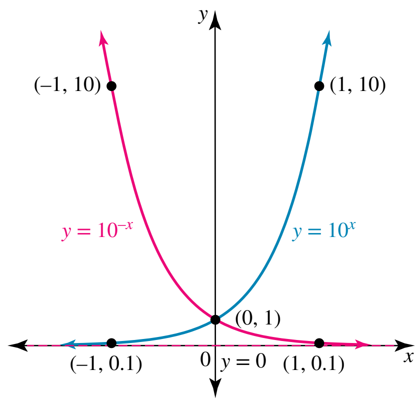
b.



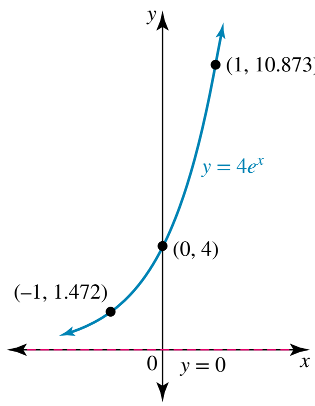
c.



2.

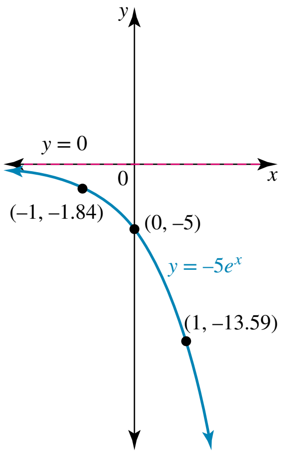


3. a.



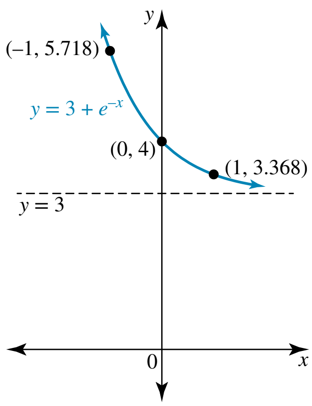
b. The function $f(x) = e^x$ has been dilated by a factor of 4 from the x-axis to give $f(x) = 4e^x$.

4. a.



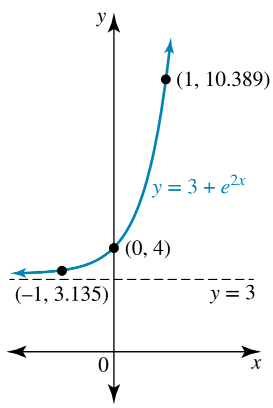
b. The function $f(x) = e^x$ has been reflected in the x-axis and dilated by a factor of 5 from the x-axis to give $f(x) = -5e^x$.

5. a.



b. The function $f(x) = e^x$ has been reflected in the y-axis and translated vertically up by 3 units to give $f(x) = e^{-x} + 3$.

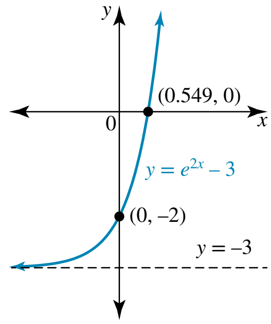
6. a.



b. The function $f(x) = e^x$ has been dilated by a factor of $\frac{1}{2}$ from the y-axis and translated vertically up by 3 units to give $f(x) = e^{2x} + 3$.

7. a. Intercepts: $\left(\frac{1}{2}\ln(3), 0\right)$ and $(0, -2)$

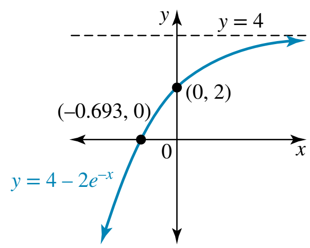
b.



c. The function $f(x) = e^x$ has been dilated by a factor of $\frac{1}{2}$ from the y-axis and translated vertically down by 3 units to give $f(x) = e^{2x} - 3$.

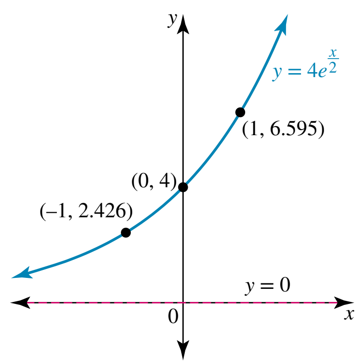
8. a. Intercepts: $(-\ln(2), 0)$ and $(0, 2)$

b.



c. The function $f(x) = e^x$ has been dilated by a factor of 2 from the x-axis, reflected in the x-axis, reflected in the y-axis and translated vertically up by 4 units to give $f(x) = 4 - 2e^{-x}$.

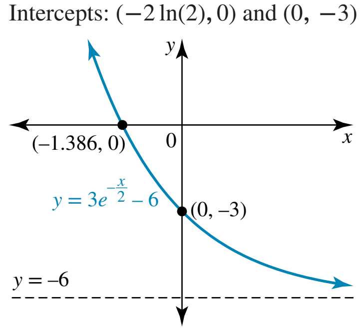
9. a.



b. The function $f(x) = e^x$ has been dilated by a factor of 2 from the y-axis and dilated by a factor of 4 from the x-axis to give $f(x) = 4e^{\frac{x}{2}}$.

10. a. Intercepts: $(-2\ln(2), 0)$ and $(0, -3)$

b.



c. The function $y = e^x$ has been dilated by a factor of 2 from the y-axis and dilated by a factor of 3 from the x-axis, reflected in the y-axis and translated vertically down by 6 units to give $y = 3e^{-\frac{x}{2}} - 6$.

11. a. $x = \ln(4)$ **b.** $x = 1.386$

12. a. $x \approx 1.609$ **b.** $x \approx -0.693$ **c.** $x \approx 0.956$ **d.** $x \approx -1.792$ **e.** $x \approx 0.405$ **f.** $x \approx -1.204$

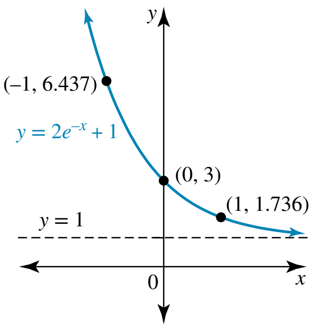
13. a. $x = 0, \ln(2)$ **b.** $x = 0$ **c.** $x = 0, \ln(2)$ **d.** $x = \ln\left(\frac{3}{2}\right), -\ln(2)$ **e.** $x = \ln(4)$ **f.** $x = \ln\left(\frac{2}{3}\right)$

14. $x = 1.61$

15. a. $x = \ln(3)$ **b.** $x = \ln(3), \ln\left(\frac{2}{3}\right)$ **c.** $x = \ln(5)$ **d.** $x = \ln(5)$

16. a. $x > 0$ **b.** $x > -1$ **c.** $x \geq \ln(2)$ **d.** $x \geq 1 - \ln(6)$

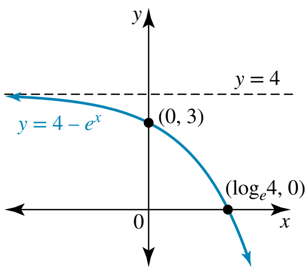
17. a.



b. $x > 0$

c. For $2e^{-x} + 1 < 0$, the curve would be below the x-axis. But $y > 1$ for all x values, so the curve is never below the x-axis. Hence, $2e^{-x} + 1 < 0$ has no real solutions.

18. a.



b. $x < \ln(4)$

c. For $x = 0$, $y = 3$.
Observing the graph:
For $x \geq 0$, $y \leq 3$.
If the domain is restricted to $x \geq 0$, the range is $y \leq 3$ or $y \in (-\infty, 3]$.

Exercise 2.4 Differentiation of exponential functions

1. a. $10e^{10x}$ **b.** $\frac{1}{3}e^{\frac{x}{3}}$ **c.** $\frac{1}{4}e^{\frac{x}{4}}$ **d.** $-e^{-x}$ **e.** $6e^{3x}$ **f.** $-20e^{-5x}$

2. a. $6e^{6x-2}$ **b.** $-6e^{8-6x}$ **c.** $10e^{5x+3}$ **d.** $-8e^{7-2x}$ **e.** $-24e^{8x+1}$ **f.** $10e^{6-5x}$

3. a. $-90e^{6-9x}$ **b.** $-15e^{3x+4}$ **c.** $-42e^{-7x}$ **d.** $e^{\frac{x}{2}+1}$ **e.** $-e^{2-\frac{x}{3}}$ **f.** $-e^{\frac{x}{4}+5}$

4. A

5. a. $2e^{x}$ **b.** $3e^{2x}(3e^{x} + 2)$ **c.** $-10(2e^{-4x} - 1)$ **d.** $3e^{x} - 2e^{-x}$

6. a. $6e^{2x} - 7e^{-7x}$ **b.** $36e^{9x} - 2e^{x}$

7. a. $(2x + 3)e^{x^2+3x}$ **b.** $(2x - 3)e^{x^2-3x+1}$ **c.** $2(x - 1)e^{x^2-2x}$ **d.** $-5e^{2-5x}$

8. a. $(2x - 3)e^{6-3x+x^2}$ **b.** $3(x^2 + 1)e^{x^3+3x-2}$ **c.** $3(8x - 7)e^{4x^2-7x}$ **d.** $10(1 + 3x)e^{1-2x-3x^2}$

9. A

10. $-20e$

11. $-6e^2$

12. $-5e^{-2}$

13. $y = 5x - 4$

14. Tangent: $y = -3x - 1$; perpendicular: $y = \frac{1}{3}x - 1$

15. a. $-2e^7$ **b.** $\frac{3}{2}$

16. a. $2e^2 - \frac{2}{e}$ **b.** 0

Exercise 2.5 Applications of exponential functions

1. a. 500 bacteria **b.** 4987 bacteria **c.** 2294 bacteria/hour

2. a. 11 **b.** $\frac{dL}{dt} = 6.589e^{0.599t}$ **c.** 39.742 mm/month

3. a. 50 **b.** 47.56 grams **c.** 0.24 grams/day

4. a. 200 **b.** 110 grams **c.** $2\frac{1}{4}$ hours **d.** -36.1 grams/hour

5. a. 10 **b.** 7 grams **c.** 4 days **d.** 0.73 grams/day

6. a. $A_0 = 1000$; $r = 0.05$
b. i. \$1051.27
ii. \$1284.03
iii. \$1648.72
c. i. \$52.56/year
ii. \$64.20/year
iii. \$82.44/year
d. 14 years

7. a. 75 **b.** 1.531 °C/ min

8. a. 500 **b.** 0.03 **c.** 2240 **d.** 67 people/year

9. a. $a = 2$; $k = 0.2$ **b.** $\frac{dm}{dt} = -0.4e^{-0.2t}$ **c.** 0.12 kg/h **d.** 3.5 h

10. a. 120
b. $\frac{1}{2}\ln\left(\frac{4}{3}\right)$
c. 8.411 units/min
d. As $t \to \infty, A \to 0$. Technically the graph approaches the line $A = 0$ (asymptotic behaviour). However, the value of A would be so small that in effect, after a long period of time, there is no gas left.

11. a. 23.2 million **b.** August 2005 **c.** 0.29 million/year **d.** 2019

12. a. 75.32 cm; $k = 0.24$
b. 5.45 cm of mercury/km

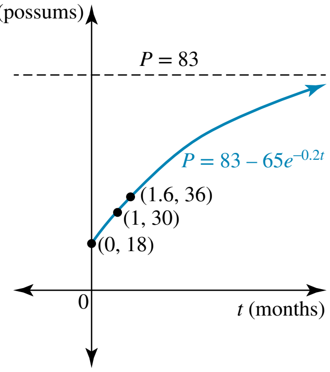
13. a. 30
b. 450 000 tadpoles
c. 37 072.2 tadpoles/day

14. a. 18 possums

b. 12 possums

c. 1.62 months

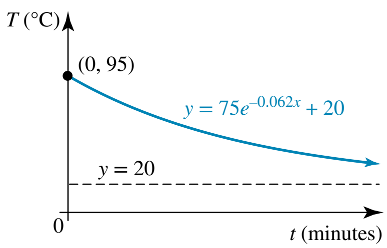
d.



e. The presence of the asymptote at $P = 83$ shows that as $t \to \infty, P \to 83$. The population can never exceed 83, so the population cannot grow to 100 possums.

15. a. 95 °C

b.



c. Approximately 20 °C

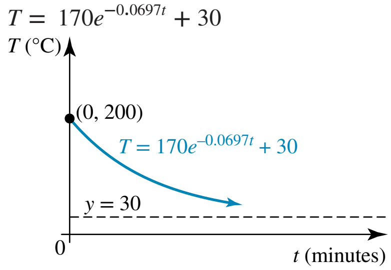
d. 8.24 min

e. After 10 minutes, the coffee is cooling at a rate of 2.5 °C/minute. The temperature is decreasing, so the rate of change will be negative.

16. a. $A = 30; T_0 = 170$

b. $k = 0.0697$

c. $T = 170e^{-0.0697t} + 30$



d. 89.8 °C

e. −4.2 °C/ min

f. 40.65 min

g. From the graph, the temperature of the metal ball is always greater than 30 °C, the temperature of the room, so if left in the room it will never reach a temperature of 10 °C.

2.6 Review: exam practice

1. a. 17 **b.** 6 **c.** $\frac{7}{2}$ **d.** 5

2. a. $-2x$ **b.** $2x + 4$ **c.** $2x + 1$

3. a. $f'(x) = 2x + 10$

b. 0; The function has a stationary point at $x = -5$.

c. 10

d. 6

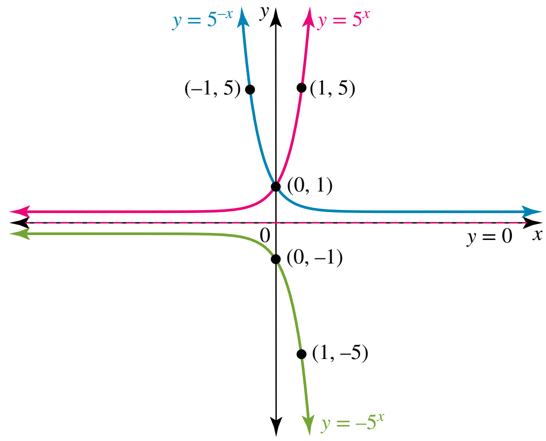
4. a. 0.792 **b.** 3.084 **c.** −1.040 **d.** 3.386

5. a. ln(2) **b.** ln(3)

c. ln(2) **d.** − ln(2), ln(4)

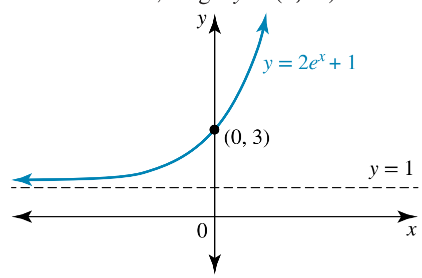
6. a. −25

b.

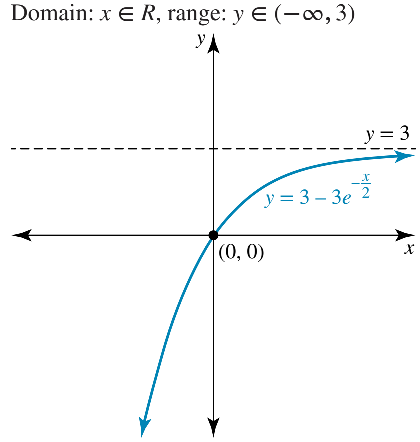


c. $y = \left(\frac{1}{5}\right)^x$ or $y = (0.2)^x$

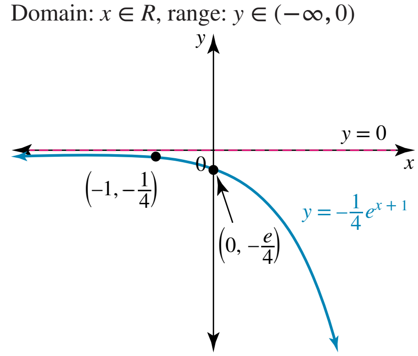
7. a. Domain: $x \in R$, range: $y \in (1, \infty)$



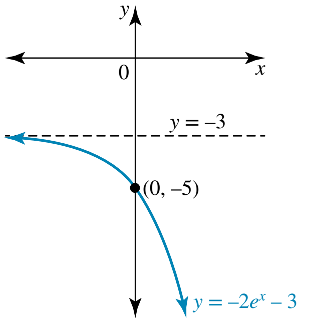
b. Domain: $x \in R$, range: $y \in (-\infty, 3)$



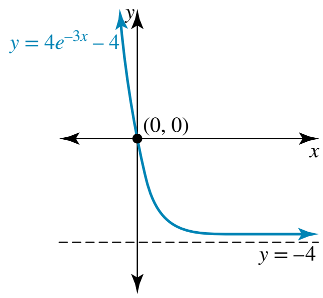
c. Domain: $x \in R$, range: $y \in (-\infty, 0)$



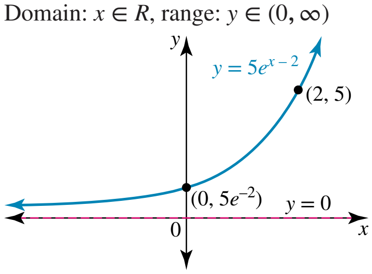
8. a. Domain: $x \in R$, range: $y \in (-\infty, -3)$



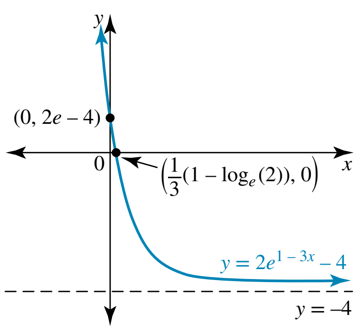
b. Domain: $x \in R$, range: $y \in (-4, \infty)$



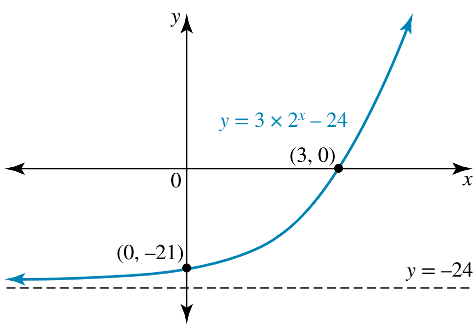
c. Domain: $x \in R$, range: $y \in (0, \infty)$



9. a.



b. Domain R, range $(-24, \infty)$



10. A

11. a. $-\frac{1}{3}e^{-\frac{1}{3}x}$ **b.** $12x^3 + 4xe^{-2x^2}$

c. $-\frac{8}{3}e^{-2x} + \frac{4}{3}e^{-4x} - 2e^{-3x}$ **d.** $4e^{4x} - 12e^{2x}$

12. $\frac{1}{2}$

13. a. $a = -11, b = 11; f: R \to R, f(x) = -11e^x + 11$

b. $y = e^{-2x} + 4$

14. a. 5 **b.** $\frac{dy}{dx} = -10xe^{-x^2}$

c. i. 3.89

ii. −3.68

15. a. $a = 4; k = 0.059$ **b.** $\frac{dm}{dt} = -0.236e^{-0.059t}$

c. 0.17 g/h

16. a. $A = 250; k = 0.047$ **b.** 506 000

c. 2023

17. a. $P_0 = 500$

b. 16 000 bacteria.

c. At $t = 8$, $P'(8) = 125\ln(2)$ bacteria/hour

d. 16 hours

18. a. 30 **b.** 24 mg

c. $k = 0.1116$ **d.** 17.17 mg

e. After 10.8 hours

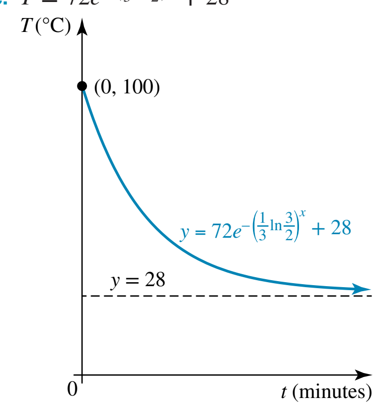
19. a. $13 099.65

b. 24 years and 5 months

20. a. $A = 28; T_0 = 72$

b. Sample responses can be found in the worked solutions in the online resources.

c. $T = 72e^{-\left(\frac{1}{3}\ln\frac{3}{2}\right)t} + 28$



d. 60 °C

e. Since the room is kept at a constant temperature of 28 °C, the water will never cool below this level.

3 Calculus of logarithmic functions

3.1 Overview

The logarithmic function is the inverse of the exponential function studied in the previous chapter, and is fundamental to the study of mathematics. It originated, however, with mathematicians endeavouring to simplify calculations. The first recorded publication on logarithms, in 1614, was by a Scottish mathematician, John Napier (1550–1617). A Swiss mathematician, Joost Bürgi (1552–1632), independently published his theories on logarithms a few years later than Napier. Basically, both mathematicians were trying to find an easier and quicker method for complicated multiplication and division problems. Today, technology allows such problems to be solved easily and accurately. Napier also developed a system for multiplication and division similar to an abacus, called 'Napier's Bones'.

Logarithms have many practical uses in the physical world. They are used to compress large sets of data. The Richter scale is a logarithmic scale that measures the magnitude of an earthquake. In chemistry, logarithms are used to determine the pH levels of various substances. Archaeologists use logarithms to determine the age of artifacts, and actuarial scientists use logarithms to calculate costs and risks. In nuclear medicine, logarithms help technicians determine amounts of radioactive decay. The decibel scale, which measures sound intensity, is another example of a logarithmic scale. The use of logarithms is widespread in many areas of science, engineering and commerce.

Close-up of an old set of Napier's Bones

LEARNING SEQUENCE

3.1 Overview
3.2 The natural logarithm and the function $y = \log_e(x)$
3.3 The derivative of $y = \log_e(x)$
3.4 Applications of logarithmic functions
3.5 Review: exam practice

Fully worked solutions for this chapter are available in the Resources section of your eBookPLUS at www.jacplus.com.au.

3.2 The natural logarithm and the function $y = \log_e(x)$

3.2.1 Defining the natural logarithm

Recall from chapter 1 that the term logarithm is another name for the exponent, index or power, including expressions in e.

Consider the following indicial equations.

$$\text{Base number} \rightarrow 10^{2} = 100 \quad (\text{2: Exponent}) \qquad \text{Base number} \rightarrow e^{p} = q \quad (p\text{: Exponent})$$

Written as logarithms they become:

$$\log_{10} 100 = 2 \leftarrow \text{Exponent} \quad (10\text{: Base number}) \qquad \log_e q = p \leftarrow \text{Exponent} \quad (e\text{: Base number})$$

The logarithm in base e is known as the natural logarithm or the Napierian logarithm.

The natural logarithm is written as $\log_e(x)$ or $\ln(x)$.

The logarithmic laws apply in the normal way when e is the base.

The logarithm laws

1. $\log_a(m) + \log_a(n) = \log_a(mn)$
2. $\log_a(m) - \log_a(n) = \log_a\left(\frac{m}{n}\right)$
3. $\log_a(m^n) = n\log_a(m)$
4. $\log_a(1) = 0$
5. $\log_a(a) = 1$
6. $\log_a(0) = \text{undefined}$
7. $\log_a(x)$ is defined for $x > 0$ and $a \in R^+ \backslash \{1\}$.
8. $a^{\log_a(m)} = m$

WORKED EXAMPLE 1

Solve the following for x, giving your answers correct to 3 decimal places where appropriate.

a. $\log_e(x) = 3$ **b.** $\ln(x) + \ln(2) = \ln(6)$

THINK	WRITE
a. 1. Rewrite the equation as an exponential.	**a.** $\log_e(x) = 3$ $e^3 = x$
2. Evaluate.	$x = 20.085\,536\,9$
3. Answer the question.	$x = 20.086$ (to 3 decimal places)
b. 1. Apply log laws to rewrite the equation.	**b.** $\ln(x) + \ln(2) = \ln(6)$ $\ln(2x) = \ln(6)$ $(2x) = 6$
2. Answer the question.	$x = 3$

WORKED EXAMPLE 2

Solve $2\ln(x) - \ln(3) = 4$ for x, giving your answer correct to 2 decimal places.

THINK	WRITE
1. Apply log laws to combine the left-hand side	$2\ln(x) - \ln(3) = 4$ $\ln\left(x^2\right) - \ln(3) = 4$ $\ln\left(\frac{x^2}{3}\right) = 4$
2. Rewrite as an exponential equation.	$\frac{x^2}{3} = e^4$ $x^2 = 3e^4$
3. Solve for x. Remember: $\log(a)$ is only defined for $a > 0$.	$x = \pm\sqrt{3e^4}$ $x = \sqrt{3e^4}$
4. Evaluate.	$x = 12.798\,220\,58$
5. Answer the question.	$x = 12.80$ (to 2 decimal places)

3.2.2 The inverse relationship of $y = e^x$ and $y = \log_e(x)$

The logarithmic function can also be thought of as the inverse of the exponential function.

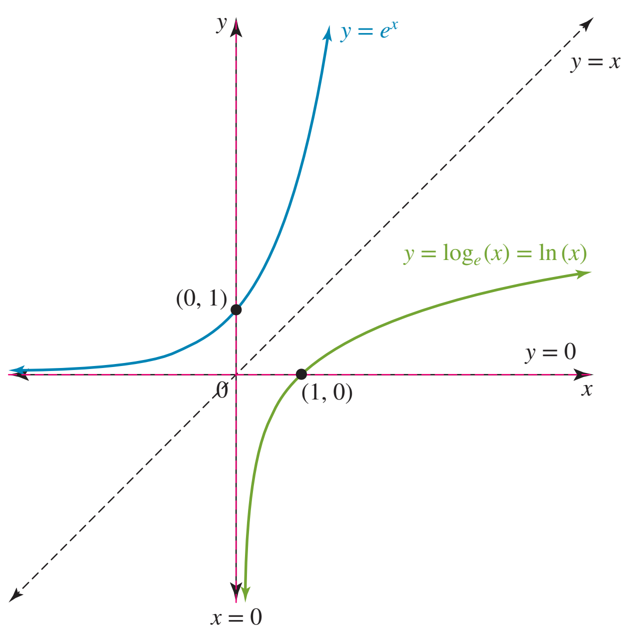
For the inverse of a function to exist, the function needs to be a one-to-one function. The exponential function, $y = e^x$, satisfies this condition.

Inverse relationships are the reflection over the straight line $y = x$. Alternately, $y = x$ is a line of symmetry between a function and its inverse.

To obtain the inverse of the exponential function $y = e^x$:

- interchange the x and y variables, so the function becomes $x = e^y$
- make y the subject: $y = \log_e(x)$ or $y = \ln(x)$.

The graph illustrates the relationship between $y = e^x$ and $y = \log_e(x)$.



Function	Type of function	Domain	Range
$y = e^x$	One-to-one	$x \in R$	$y \in (0, \infty)$ or $y \in R^+$
$y = \ln(x) = \log_e(x)$	One-to-one	$x \in (0, \infty)$ or $x \in R^+$	$y \in R$

3.2.3 Sketching $y = \log_e(x)$ and its transformations

The graph of the function $y = \log_e(x)$ has the following features:

- The graph is an increasing function.
- The graph intersects the x-axis at $(1, 0)$.
- As $x \to 0$, $y \to -\infty$; hence, $x = 0$ is an asymptote.
- The graph of $y = k\log_e(x)$ is a **dilation** of $y = \log_e(x)$ by a factor of k from the x-axis (or parallel to the y-axis.)
- The graph of $y = \log_e(nx)$ is a **dilation** of $y = \log_e(x)$ by a factor of $\frac{1}{n}$ from the y-axis (or parallel to the x-axis.)
- The graph of $y = -\log_e(x)$ is a **reflection** of $y = \log_e(x)$ in the x-axis.
- The graph of $y = \log_e(-x)$ is a **reflection** of $y = \log_e(x)$ in the y-axis.
- The graph of $y = \log_e(x) + k$ is a **translation** of $y = \log_e(x)$ by k units vertically (or parallel to the y-axis).
- The graph of $y = \log_e(x - h)$ is a **translation** of $y = \log_e(x)$ by h units horizontally (or parallel to the x-axis), giving the vertical asymptote, $x = h$.

WORKED EXAMPLE 3

Sketch the graphs of the following, showing all important characteristics. State the domain and range in each case.

a. $y = \log_e(x - 2)$

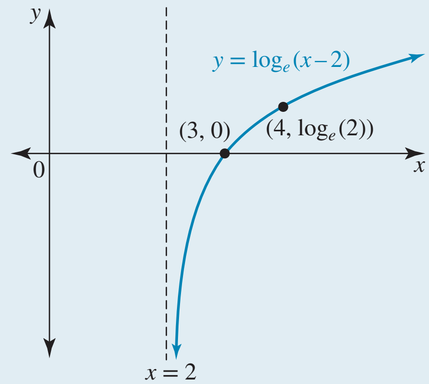
b. $y = \log_e(x + 1) + 2$

c. $y = \frac{1}{4}\log_e(2x)$

d. $y = -\log_e(-x)$

THINK	WRITE
a. 1. The basic graph of $y = \log_e(x)$ has been translated 2 units to the right, so $x = 2$ is the vertical asymptote.	**a.** $y = \log_e(x - 2)$ The domain is $(2, \infty)$. The range is R.
2. Locate the x-intercept, when $y = 0$.	x-intercept, $y = 0$: $\log_e(x - 2) = 0$ $e^0 = x - 2$ $1 = x - 2$ $x = 3$
3. Determine another point through which the graph passes.	When $x = 4$, $y = \log_e(2)$. The point is $(4, \log_e(2))$.

4. Sketch the graph.



b. 1. The basic graph of $y = \log_e(x)$ has been translated up 2 units and 1 unit to the left, so $x = -1$ is the vertical asymptote.

b. $y = \log_e(x + 1) + 2$
The domain is $(-1, \infty)$.
The range is R.

2. Locate the x-intercept, when $y = 0$.

The graph cuts the x-axis where $y = 0$.

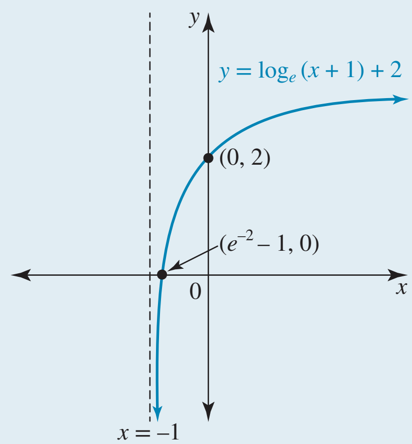
$$\log_e(x + 1) + 2 = 0$$
$$\log_e(x + 1) = -2$$
$$e^{-2} = x + 1$$
$$x = e^{-2} - 1$$

3. Locate the y-intercept, when $x = 0$.

The graph cuts the y-axis where $x = 0$.

$$y = \log_e(1) + 2$$
$$= 2$$

4. Sketch the graph.



c. 1. The basic graph of $y = \log_e(x)$ has been dilated by factor $\frac{1}{4}$ from the x-axis and by factor $\frac{1}{2}$ from the y-axis. The vertical asymptote remains $x = 0$.

c. $y = \frac{1}{4}\log_e(2x)$
The domain is $(0, \infty)$.
The range is R.

2. Locate the x-intercept, when $y = 0$.

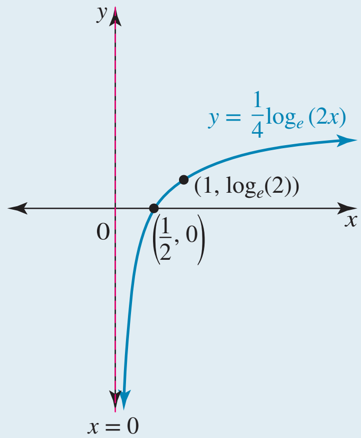
x-intercept, $y = 0$:

$$\frac{1}{4}\log_e(2x) = 0$$
$$\log_e(2x) = 0$$
$$e^0 = 2x$$
$$1 = 2x$$
$$x = \frac{1}{2}$$

3. Determine another point through which the graph passes.

When $x = 1$, $y = \log_e(2)$.
The point is $(1, \log_e(2))$.

4. Sketch the graph.



d. 1. The basic graph of $y = \log_e(x)$ has been reflected in both axes. The vertical asymptote remains $x = 0$.

d. $y = -\log_e(-x)$
The domain is $(-\infty, 0)$.
The range is R.

2. Locate the x-intercept when $y = 0$.

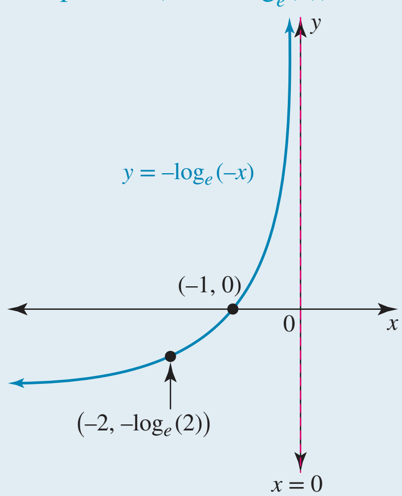
x-intercept, $y = 0$:

$$
\begin{aligned}
-\log_e(-x) &= 0 \\
\log_e(-x) &= 0 \\
e^0 &= -x \\
x &= -1
\end{aligned}
$$

3. Determine another point through which the graph passes.

When $x = -2$, $y = -\log_e(2)$.
The point is $(-2, -\log_e(2))$.

4. Sketch the graph.



on Resources

Interactivity: Logarithmic graphs (int-6418)

studyon

Units 3 & 4 > Area 2 > Sequence 2 > Concept 1

The inverse relationship of $y = e^x$ and $y = \ln(x)$ Summary screen and practice questions

Exercise 3.2 The natural logarithm and the function $y = \log_e(x)$

Technology active

1. **WE1** Solve for x in each of the following. Give exact answers when appropriate; otherwise, give answers correct to 3 decimal places.
   a. $\log_e(x) = 1$
   b. $\log_e(x) = 2$
   c. $\log_e(x) = -2$
   d. $\log_e(x) = -1$
   e. $\log_e(x) = 0.3$
   f. $\log_e(x) = -0.69$
2. Solve the following for x. Give exact answers when appropriate; otherwise, give answers correct to 3 decimal places.
   a. $\log_e(2x) = 2$
   b. $\log_e(3x) = 1$
   c. $\log_e(x^3) = 3$
3. Solve the following for x. Give exact answers when appropriate; otherwise, give answers correct to 3 decimal places.
   a. $\log_e(x - 1) = -1$
   b. $\log_e(2x + 1) = -2$
   c. $\log_e(-x) = 0.36$
   d. $\log_e(-x) = 0.72$
   e. $\log_e(1 - x) = -0.54$
   f. $\log_e(2 + x) = -0.83$
4. **WE2** Solve the following for x. Give exact answers when appropriate; otherwise, give answers correct to 3 decimal places.
   a. $\log_e(x) + \log_e(5) = 8$
   b. $2\ln(x) - \ln(5) = 9$
   c. $1 + \ln(x) = \ln(6)$
   d. $2 - \log_e(x) = \log_e 10$
5. Solve the following for x, giving exact answers.
   a. $\log_e(x) + \log_e(5) - \log_e(10) = \log_e(3)$
   b. $\log_e(x) + \log_e(3) - \log_e(9) = \log_e(4)$
   c. $2\log_e(3) + \log_e(x) - \log_e(2) = \log_e(3)$
   d. $3\log_e(2) + \log_e(x) - \log_e(4) = \log_e(5)$
   e. $\log_e(6) + \log_e(2) - \log_e(x) = \log_e(4)$
   f. $\log_e(4) + \log_e(3) - \log_e(x) = \log_e(2)$
6. Solve the following for x, giving exact answers.
   a. $\log_e(x) + \log_e(x + 1) = \log_e(2)$
   b. $\log_e(x) + \log_e(2x - 1) = \log_e(3)$
   c. $\log_e(x - 1) + \log_e(x + 2) = \log_e(4)$
   d. $\log_e(x + 1) + \log_e(2x - 1) = \log_e(5)$
7. **MC** If $\log_e y = \log_e(x) + \log_e a$, then an equation relating x and y that does not involve logarithms is:
   A. $y = x + a$
   B. $y = ax$
   C. $y = x - a$
   D. $y = \frac{x}{a}$
8. **MC** In the equation $2\log_e(x) - \log_e(3x) = a$, x equals:
   A. $3e^a$
   B. $y = ax$
   C. $3a$
   D. $\log_e(6a)$
9. Write the following equation without logarithms and with y as the subject.

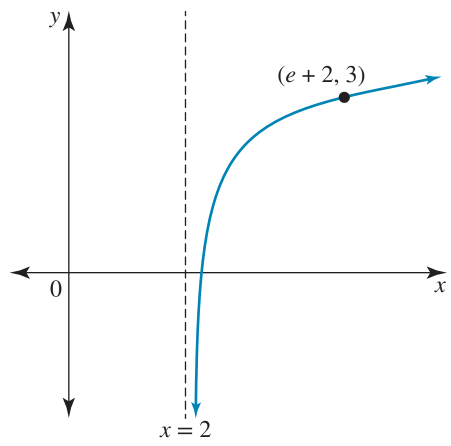
$$2\log_e(x) + 1 = \log_e(y)$$

10. **WE3** Sketch the graphs of the following functions, showing all important characteristics. State the domain and range for each graph.
    a. $y = \log_e(x + 4)$
    b. $y = \log_e(x) + 2$
    c. $y = 4\log_e(x)$
    d. $y = -\log_e(x - 4)$
11. Sketch the following graphs, clearly showing any axis intercepts and asymptotes.
    a. $y = \log_e(x) + 3$
    b. $y = \log_e(x) - 5$
    c. $y = \log_e(x) + 0.5$
12. Sketch the following graphs, clearly showing any axis intercepts and asymptotes.
    a. $y = \log_e(x - 4)$
    b. $y = \log_e(x + 2)$
    c. $y = \log_e(x + 0.5)$
13. Sketch the following graphs, clearly showing any axis intercepts and asymptotes.
    a. $y = \frac{1}{4}\log_e(x)$
    b. $y = 3\log_e(x)$
    c. $y = 6\log_e(x)$
14. Sketch the following graphs, clearly showing any axis intercepts and asymptotes.
    a. $y = \log_e(3x)$
    b. $y = \log_e\left(\frac{x}{4}\right)$
    c. $y = \log_e(4x)$

15. Sketch the following graphs, clearly showing any axis intercepts and asymptotes.

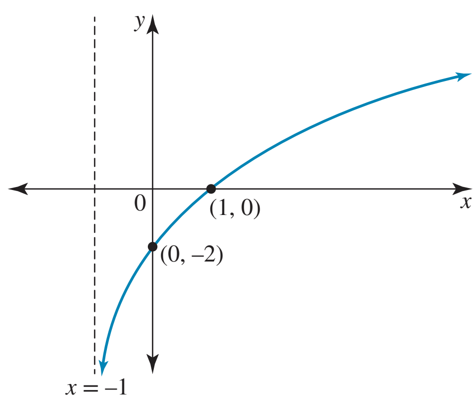
a. $y = 1 - 2\log_e(x - 1)$ b. $y = \log_e(2x + 4)$ c. $y = \frac{1}{2}\log_e\left(\frac{x}{4}\right) + 1$

16. The rule for the function shown is $y = \log_e(x - m) + n$.
Find the values of the constants m and n.



17. A logarithmic function with the rule of the form $y = p\log_e(x - q)$ passes through the points $(0, 0)$ and $(1, -0.35)$. Determine the values of the constants p and q.

18. The graph of a logarithmic function of the form $y = a\log_e(x - h) + k$ is shown. Determine the values of a, h and k.



19. The graph of $y = m\log_2 nx$ passes through the points $(-2, 3)$ and $\left(-\frac{1}{2}, \frac{1}{2}\right)$. Show that the values of m and n are 1.25 and $-2^{\frac{7}{5}}$ respectively.

20. Consider the function $f(x) = 2\log_e(3x + 3)$.

a. State the domain and range of the function, f.

b. Define the inverse function, f^{-1}.

c. State the domain and range of the inverse function, f^{-1}.

d. On the same set of axes, sketch the graphs of f and f^{-1}.

e. Use technology to give the coordinates of any points of intersection, correct to 2 decimal places.

3.3 The derivative of $y = \log_e(x)$

3.3.1 Proof for the derivative of $y = \log_e(x)$

The proof for the derivative of $y = \log_e(x)$ relies on its link to its inverse function $y = e^x$ along with the chain rule.

Let $y = \log_e(x)$.

Then $x = e^y$.

Differentiate both sides with respect to x:

$$\frac{d}{dx}(x) = \frac{d}{dx}(e^y)$$

$$1 = e^y \times \frac{dy}{dx} \text{ (using the formula } \frac{d}{dx}(e^{f(x)}) = f'(x) \times e^{f(x)})$$

$$\frac{dy}{dx} = \frac{1}{e^y}$$

Substitute $x = e^y$: $\frac{dy}{dx} = \frac{1}{x}, x > 0$ (*Note:* The log function $y = \log_e(x)$ has a restricted domain, $x > 0$.)

3.3.2 The derivative of $y = \log_e f(x)$

Let $y = \log_e f(x)$. Then $f(x) = e^y$.

Differentiate both sides with respect to x:

$$\frac{d}{dx}(f(x)) = \frac{d}{dx}(e^y)$$

$$f'(x) = e^y \times \frac{dy}{dx} \text{ (using the formula } \frac{d}{dx}\left(e^{f(x)}\right) = f'(x) \times e^{f(x)})$$

$$\frac{dy}{dx} = \frac{f'(x)}{e^y}$$

Substitute $f(x) = e^y$: $\frac{dy}{dx} = \frac{f'(x)}{f(x)}$

The differential of $\log_e(x)$

$$\frac{d}{dx}(\log_e x) = \frac{1}{x}$$

$$\frac{d}{dx}(\log_e f(x)) = \frac{f'(x)}{f(x)}$$

or

$$\frac{d}{dx}(\log_e u) = \frac{1}{u} \times \frac{du}{dx} \text{ where } u = f(x)$$

Note: The above rules are only applicable for logarithmic functions of base e.

For example, if $y = \log_e 4x$, $\frac{dy}{dx} = \frac{4}{4x} = \frac{1}{x}$.

WORKED EXAMPLE 4

Differentiate the following functions by first simplifying using the log laws.

a. $y = 3\log_e(2x)$

b. $y = 3\log_e\left(\sqrt{x}\right)$

THINK	WRITE
a. 1. Simplify the function by applying the log laws.	**a.** $y = 3\log_e(2x)$ $y = 3(\log_e(2) + \log_e(x))$ $y = 3\log_e(2) + 3\log_e(x)$
2. Differentiate the function.	$\frac{dy}{dx} = 0 + 3 \times \frac{1}{x}$
3. Simplify.	$\frac{dy}{dx} = \frac{3}{x}$
b. 1. Rewrite the function using fractional indices.	**b.** $y = 3\log_e\left(\sqrt{x}\right)$ $y = 3\log_e\left(x^{\frac{1}{2}}\right)$
2. Simplify by applying the log laws.	$y = 3 \times \frac{1}{2} \times \log_e(x)$ $y = \frac{3}{2}\log_e(x)$
3. Differentiate the function.	$\frac{dy}{dx} = \frac{3}{2} \times \frac{1}{x}$
4. Simplify.	$\frac{dy}{dx} = \frac{3}{2x}$

WORKED EXAMPLE 5

Differentiate the following using the formula $\frac{d}{dx}(\log_e f(x)) = \frac{f'(x)}{f'(x)}$.

a. $y = \ln(2x + 1)$

b. $y = 4\ln(-3x), x < 0$

c. $y = 5\ln(x^2)$

d. $y = \log_e(3x^2 - 13x + 15)$

THINK	WRITE
a. 1. Let $y = \ln(2x+1) = \ln(f(x))$. State $f(x)$ and $f'(x)$.	**a.** $f(x) = 2x + 1$ $f'(x) = 2$
2. Substitute into the formula $\frac{d}{dx}(\ln(f(x))) = \frac{f'(x)}{f(x)}$.	$\frac{dy}{dx} = \frac{2}{2x+1}$
b. 1. Let $y = 4\ln(-3x) = 4\ln f(x), x < 0$. State $f(x)$ and $f'(x)$.	**b.** $f(x) = -3x$ $f'(x) = -3$
2. Substitute into the formula $\frac{d}{dx}(\ln(f(x))) = \frac{f'(x)}{f(x)}$.	$\frac{dy}{dx} = 4 \times \frac{-3}{-3x}$
3. Simplify. *Note:* $\frac{d}{dx}(\log_e(kx)) = \frac{1}{x}$ may be useful.	$\frac{dy}{dx} = \frac{4}{x}, x < 0$

c. 1. Let $y = 5\ln(x^2) = 5\ln(f(x))$.
State $f(x)$ and $f'(x)$.

2. Substitute into the formula $\frac{d}{dx}(\ln(f(x))) = \frac{f'(x)}{f(x)}$.

3. Simplify.

Note: This function could have been rewritten as $y = 10\ln(x)$ using log laws.

c. $f(x) = x^2$

$f'(x) = 2x$

$\frac{dy}{dx} = 5 \times \frac{2x}{x^2}$

$\frac{dy}{dx} = \frac{10}{x}$

d. 1. Let $y = \log_e(3x^2 - 13x + 15) = \log_e(f(x))$.
State $f(x)$ and $f'(x)$.

2. Substitute into the formula.

d. $f(x) = 3x^2 - 13x + 15$

$f'(x) = 6x - 13$

$\frac{dy}{dx} = \frac{6x - 13}{3x^2 - 13x + 15}$

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Units 3 & 4 | Area 2 | Sequence 2 | Concept 2

The derivative of $y = \ln(x)$ Summary screen and practice questions

Exercise 3.3 The derivative of $y = \log_e(x)$

Technology free

1. **WE4** Differentiate each of the following with respect to x.
   a. $y = \log_e(10x)$
   b. $y = \log_e(5x)$
   c. $y = \log_e(-x)$
   d. $y = \log_e(-6x)$
   e. $y = 3\log_e(4x)$
   f. $y = -6\log_e 9x$
2. Differentiate each of the following with respect to x.
   a. $y = \ln\left(\frac{x}{2}\right)$
   b. $y = \ln\left(\frac{x}{3}\right)$
   c. $y = 4\ln\left(\frac{x}{5}\right)$
   d. $y = -5\ln\left(-\frac{2x}{3}\right), x < 0$
3. **MC** The derivative of $\log_e(8x)$ is:
   A. $\frac{1}{8}x$
   B. $\frac{8}{x}$
   C. $\frac{1}{x}$
   D. $\log_e 8$
4. **WE5** Differentiate each of the following with respect to x.
   a. $y = \log_e(2x + 5)$
   b. $y = \log_e(6x + 1)$
   c. $y = \log_e(3x - 4)$
   d. $y = \log_e(8x - 1)$
   e. $y = \log_e(3 - 5x)$
   f. $y = \log_e(2 - x)$
5. Differentiate the following with respect to x.
   a. $y = 6\ln(5x + 2)$
   b. $y = 8\ln(4x - 2)$
   c. $y = -4\ln(12x + 5)$
   d. $y = -7\ln(8 - 9x)$
6. Differentiate the following with respect to x.
   a. $y = \log_e(3x^4)$
   b. $y = \log_e(x^2 + 3)$
   c. $y = \log_e(x^2 + 4x)$
   d. $y = \log_e(x^2 - 3x + 2)$
   e. $y = \log_e(x^3 + 2x^2 - 7x)$
   f. $y = \log_e(x^2 - 2x^3 + x^4)$
7. Simplify each of the following using the log laws and differentiate with respect to x.
   a. $y = \ln\left(\sqrt{2x + 1}\right)$
   b. $y = \ln\left(\sqrt{3 - 4x}\right)$
   c. $y = \ln\left(\sqrt{x^2 + 2}\right)$
   d. $y = \ln(x + 3)^{\frac{1}{4}}$
   e. $y = \ln(5x + 2)^{\frac{1}{3}}$
   f. $f(x) = \ln(2 - 3x)^{\frac{1}{5}}$

8. Simplify each of the following using the log laws and differentiate with respect to x.

a. $f(x) = \ln\left(\frac{1}{x+3}\right)$

b. $f(x) = \ln(3x-2)^4$

c. $f(x) = \ln(5x+8)^{-2}$

d. $f(x) = \ln\left(\frac{2}{4+3x}\right)$

9. MC The derivative of $f(x) = \ln(x^2 - 5x + 2)$ is:

A. $\frac{1}{x^2 - 5x + 2}$

B. $2x - 5$

C. $\frac{1}{x(2x-5)}$

D. $\frac{2x-5}{x^2 - 5x + 2}$

10. MC The derivative of $\log_e(3x-2)$ is:

A. $\frac{1}{3x-2}$

B. $\frac{1}{3x}$

C. $\frac{1}{3(3x-2)}$

D. $\frac{3}{3x-2}$

11. MC The derivative of $2\log_e(x^2 + x)$ is:

A. $\frac{2(2x+1)}{x^2+x}$

B. $\frac{2x+1}{x^2+x}$

C. $\frac{2x}{x^2+x}$

D. $\frac{4x}{x^2+x}$

Technology active

12. Using a suitable method, differentiate each of the following functions with respect to x.

a. $f(x) = 7\log_e\left(\frac{x}{3}\right)$

b. $f(x) = 2\ln(x^3 + 2x^2 - 1)$

c. $f(x) = 3\ln(e^x + 1)$

d. $f(x) = -5\log_e(2x)$

13. MC If $y = \log_e\sqrt{x^2 - 6x + 9}$, then the derivative is:

A. $\frac{2x-6}{\sqrt{x^2-6x+9}}$

B. $\frac{1}{x-3}$

C. $\frac{x-3}{\sqrt{x^2-6x+9}}$

D. $\frac{1}{(x-3)^2}$

14. The derivative of $\log_e(x)$ is exactly equal to the derivative of $\log_e 100x$. Explain why different functions have the same derivative.

3.4 Applications of logarithmic functions

Functions involving the logarithmic function, along with its inverse, the exponential function, can be used to model real-life situations.

The domain of the logarithmic function may need to be restricted to suit the context of the problem.

Differentiation of logarithmic functions are used to find:

- the gradient of a logarithmic curve at a given point
- the equation of a tangent to a logarithmic curve at a given point
- maximum and minimum turning points and their applications (discussed in Chapter 9).

WORKED EXAMPLE 6

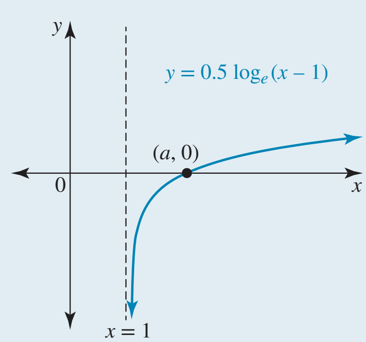
The graph of the function $f(x) = 0.5\log_e(x-1)$ is shown.

a. State the domain and range of f.

b. Find the value of the constant a given that $(a, 0)$ is the x-axis intercept.

c. Find the gradient of the curve at $(a, 0)$.

d. Find the equation of the tangent at $(a, 0)$.

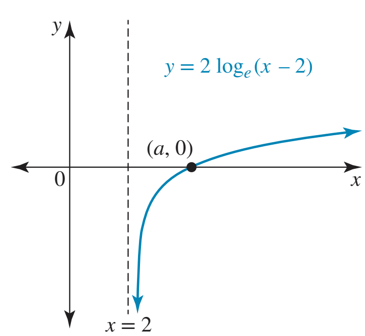


THINK	WRITE
a. State the domain and range of the function $f(x) = 0.5\log_e(x-1)$.	**a.** Domain $= (1, \infty)$ Range $= R$
b. 1. To find the x-intercept, let $f(x) = 0$.	**b.** $0.5\log_e(x-1) = 0$
2. Solve $0.5\log_e(x-1) = 0$ for x.	$\log_e(x-1) = 0$ $e^0 = x - 1$ $1 = x - 1$ $x = 2$
3. Answer the question.	$(a, 0) \equiv (2, 0)$ $\therefore a = 2$
c. 1. Determine the derivative of the function $f(x) = 0.5\log_e(x-1)$.	**c.** $f(x) = 0.5\log_e(x-1)$ $f'(x) = \dfrac{1}{2} \times \dfrac{1}{x-1}$ $= \dfrac{1}{2(x-1)}$
2. Substitute $x = 2$ into the derivative to find the gradient at this point.	$f'(2) = \dfrac{1}{2(2-1)}$ $= \dfrac{1}{2}$ The gradient at $x = 2$ is $\dfrac{1}{2}$.
d. 1. State the general equation for a tangent.	**d.** The equation of the tangent is $y - y_1 = m_T\,(x - x_1)$.
2. State the known information.	The gradient of the tangent at $(x_1, y_1) = (2, 0)$ is $m_T = \dfrac{1}{2}$.
3. Substitute the values into the general equation.	$y - 0 = \dfrac{1}{2}(x - 2)$
4. Simplify.	$y = \dfrac{1}{2}x - 1$

Exercise 3.4 Applications of logarithmic functions

Technology free

1. **WE6** The graph of the function $f: (2, \infty) \to R, f(x) = 2\log_e(x-2)$ is shown.
   a. State the domain and range of f.
   b. Find the value of the constant a, given that $(a, 0)$ is the x-axis intercept.
   c. Find the equations of the tangent at $(a, 0)$.
   d. Find the equation of the line perpendicular to the curve at $(a, 0)$.



Technology active

2. Calculate the gradient of the tangent to the following functions at the specified point.

   a. $y = 2\ln(x), x = 5$ b. $y = \frac{1}{3}\ln(4x + 1), x = 2$ c. $y = \ln(x^2 + 3), x = 3$

3. Find the equation of the tangent to each of the given curves at the specified point.

   a. $y = \log_e(2x - 2)$ at $\left(\frac{3}{2}, 0\right)$ b. $y = 3\log_e(x)$ at $(e, 3)$ c. $y = \frac{1}{2}\log_e x^2$ at $(e, 1)$

4. a. Calculate the gradient of the curve $y = 3\log_e(x - 5)$ at the point where $x = 6$.

   b. Hence, find the equations of the tangent to the curve and the line perpendicular to the curve at the point where $x = 6$.

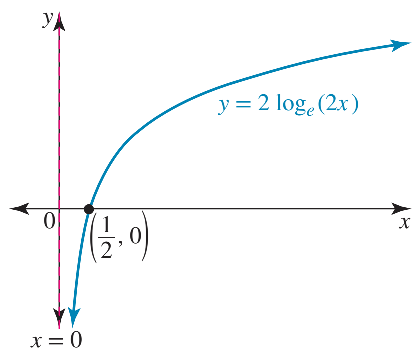
5. Find the equation of the tangent to the curve $y = 4\log_e(3x - 1)$ at the point where the tangent is parallel to the line $6x - y + 2 = 0$.

6. Determine the equations of the tangent and the line perpendicular to the tangent to the curve $y = 7\ln(2x + 3)$ at the point where the tangent is parallel to the line $2x - y + 4 = 0$.

7. The graph of the function defined by the rule $y = 2\log_e 2x$ is shown.

   a. Find the derivative of y with respect to x.

   b. Find the equation of the tangent at $\left(\frac{e}{2}, 2\right)$.



8. The line $y = x$ is a tangent to the curve $y = \log_e(x - 1) + b$, where b is a constant. Find the possible value of b.

9. The equation of a line perpendicular to the curve $y = \log_e(2(x - 1))$ has the equation $y = -2x + k$, where k is a constant. Find the value of k, correct to 1 decimal place.

10. Consider the function defined by the rule $f(x) = \ln(3 - x)$.

    a. State the domain and range of the function f and explain why the domain is restricted.

    b. Calculate, in exact form, the coordinates of any axis intercepts of f.

    c. Determine the equation of the inverse function, f^{-1}, and state the domain and range of the inverse function.

    d. State, in exact form, the coordinates of any axis intercepts of the inverse function, f^{-1}.

    e. On the same set of axes, sketch the graphs of f and f^{-1}, showing all relevant features.

    f. Using technology, determine the coordinates of the point(s) of intersection of f and f^{-1}. Give your answers correct to 3 decimal places.

11. Consider the function defined by the rule $f(x) = \log_e(2x - 1)$.

    a. State the domain and range of the function f and explain why the domain is restricted.

    b. Calculate, in exact form, the coordinates of any axis intercepts of f.

    c. Determine the equation of the inverse function, f^{-1}, and state the domain and range of the inverse function.

    d. State, in exact form, the coordinates of any axis intercepts of the inverse function, f^{-1}.

    e. On the same set of axes, sketch the graphs of f and f^{-1}, showing all relevant features.

    f. Explain why the two graphs do not intersect.

12. Consider the function defined by the rule $f(x) = -2\ln(2 - x) - 1$.

    a. State the domain and range of the function f and explain why the domain is restricted.

    b. Calculate, in exact form, the coordinates of any axis intercepts of f.

    c. Determine the equation of the inverse function, f^{-1}, and state the domain and range of the inverse function.

    d. State, in exact form, the coordinates of any axis intercepts of the inverse function, f^{-1}.

    e. On the same set of axes, sketch the graphs of f and f^{-1}, showing all relevant features.

    f. Using technology, determine the coordinates of the point(s) of intersection of f and f^{-1}. Give your answers correct to 2 decimal places.

13. The function $f: R \rightarrow R, f(x) = 6 \log_e(x^2 - 4x + 8)$ has one stationary point.
    a. State the derived function $f'(x)$.
    b. Use your answer to part **a** to determine the coordinates of this stationary point.
    c. By investigating the derived function, $f'(x)$, determine the nature of this stationary point.
    d. Use technology to graph $y = f(x)$.
14. The number of rats, N, in a derelict house t months after it was last occupied is given by $N = 25 + 95 \log_e(t + 1)$.
    a. Determine the number of rats initially present in the derelict house.
    b. Calculate how long, correct to 1 decimal place, it would take for the number of rats to double.
    c. What is the rate of change in the number of rats after 4 months?

3.5 Review: exam practice

A summary of this chapter is available in the Resources section of your eBookPLUS at www.jacplus.com.au.

Simple familiar

1. **MC** If $\log_e(2x) = a$, then x is equal to:
   A. $2e^a$ **B.** $2a^e$ **C.** $\frac{e^a}{2}$ **D.** e^{2a}
2. **MC** An exact solution for $\log_e(1 - x) = 3$ is:
   A. $3 - e^3$ **B.** $1 - e^3$ **C.** $e^3 - 1$ **D.** $3 - e$
3. **MC** A solution for x in $\log_e(x - 3) + \log_e(x - 2) = \log_e(12)$ is:
   A. 6 only **B.** 6 and −1 **C.** −1 only **D.** 3 and 2
4. **MC** An expression for y in terms of x in $4 - \log_e(x) = 2 \log_e(y)$ is:
   A. $y = \frac{e^4}{x}$ **B.** $y = e^2\sqrt{x}$ **C.** $y = \frac{e^2}{\sqrt{x}}$ **D.** $y = \frac{e^4}{x^2}$
5. Sketch the graphs of the following functions, showing all important features. For each graph, state the domain, the range and the equations of any asymptotes, and describe the transformations that have been applied to $y = \log_e(x)$ to achieve the function.
   a. $y = \ln(x + 4)$ b. $y = \ln(x - 4)$ c. $y = \ln(x) + 4$ d. $y = 4 - \ln(x)$
6. Sketch the graphs of the following functions, showing all important features. For each graph, state the domain, the range and the equations of any asymptotes, and describe the transformations that have been applied to $y = \log_e(x)$ to achieve the function.
   a. $y = \ln(2x)$ b. $y = -2\ln(x)$ c. $y = \ln(2 - x)$ d. $y = -\ln(-2x)$
7. Differentiate the following functions with respect to x and state any restrictions on x.
   a. $y = \frac{1}{2}\log_e(x^2 - 2x + 7)$ b. $y = \log_e\left(\frac{x+2}{x-3}\right)$ c. $y = \log_e(x + 2)^2$
8. Differentiate the following with respect to x.
   a. $y = \log_e\left(\frac{2x+1}{x-5}\right)$ b. $y = \log_e\left(\frac{7}{x-3}\right)$ c. $y = \log_e(9x^2 - 6x + 7)$
9. **MC** If $f(x) = \log_e 3x$, then $f'(1)$ is equal to:
   A. 1 **B.** $\frac{1}{3}$ **C.** $\log_e(3)$ **D.** $3\log_e(3)$
10. **MC** If $y = \log_e\left(\frac{2}{x}\right)$, then $\frac{dy}{dx}$ is equal to:
    A. x **B.** $\frac{1}{x}$ **C.** $\log_e\left(\frac{2}{x}\right)$ **D.** $-\frac{1}{x}$

11. **MC** The gradient of the curve $y = 3\log_e(x)$ at $x = 7$, correct to 2 decimal places, is:

A. 0.43 B. 0.22 C. 0.14 D. 0.51

12. **MC** The gradient of the line perpendicular to the curve $y = \log_e(2x)$ at $x = 4$ is equal to:

A. -4 B. $\frac{1}{4}$ C. $\frac{1}{2}$ D. $-\frac{1}{2}$

Complex familiar

13. **MC** If $y = \ln\left(\sqrt{x^2 + 8x + 16}\right)$, the derivative in simplest form is:

A. $\frac{dy}{dx} = \frac{2x+8}{\sqrt{x^2+8x+16}}$ B. $\frac{dy}{dx} = \frac{x+4}{\sqrt{x^2+8x+16}}$

C. $\frac{dy}{dx} = \frac{1}{(x+4)^2}$ D. $\frac{dy}{dx} = \frac{1}{x+4}$

14. **MC** If $y = \log_e(e^x + e^{-x})$, $\frac{dy}{dx}$ equals:

A. $\frac{e^x + e^{-x}}{e^x - e^{-x}}$ B. -1

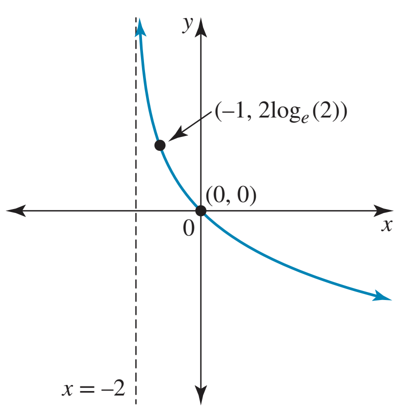
C. $\frac{e^x - e^{-x}}{e^x + e^{-x}}$ D. None of the above

15. If $y = \log_e(x + 5)$, determine the equation of the tangent to the curve at the point where $x = e - 5$.

16. Let h be the graph of the function $h : D \to R$, $h(x) = 2\log_e(1 - 3x)$, where D is the largest possible domain over which h is defined.
   a. Determine D.
   b. Calculate the exact coordinates of the intercepts of the graph with the x- and y-axes.
   c. Use calculus to show that the rate of change of h with respect to x is always negative.
   d. i. Determine the rule for h^{-1}.
      ii. State the domain and range of h^{-1}.
   e. Using technology, or otherwise, sketch, on one set of axes, the graphs of h and h^{-1}. Show any asymptotes with their equations.

Complex unfamiliar

17. The graph of $y = m\log_e(n(x + p))$ is shown.



   a. Calculate the values of the non-zero constants m, n and p.
   b. Describe the transformations that have been applied to $y = \log_e(x)$ to achieve this function.
   c. Determine the equation of the inverse function, $y = f^{-1}(x)$, and state its domain, range and equations of any asymptotes.
   d. Using technology, or otherwise, sketch, on the same set of axes, the graphs of $y = f(x)$ and $y = f^{-1}(x)$. State the coordinates of the point of intersection.

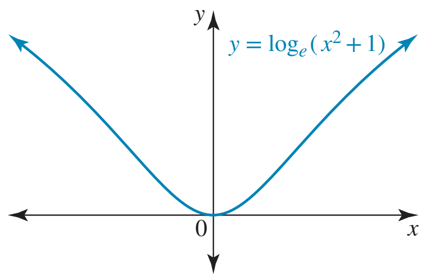
18. The number of people with the flu virus, N, in a particular town t days after a vaccine is introduced is $N = 3000 - 500 \log_e(8t + 1)$.

a. How many people were infected in the town before the vaccine is introduced?
b. Calculate, to the nearest person, the number of people infected after 5 days.
c. Determine the rate of change of the number of people in the town infected with flu.
d. Calculate, to the nearest person, the rate of change after 5 days.

19. Spinal anaesthesia is a form of regional anaesthesia involving an injection into the spine. This type of anaesthesia hastens the recovery for operations such as hip and knee replacements. Five minutes after the end of an operation, a patient shows signs of awakening from the anaesthetic. The alertness of the patient, A units, t minutes after the completion of the operation is given by $A = 15 \log_e(t - 2)$.

a. A patient is awake and allowed to sip water when their alertness is equal to 15. Determine how long this will take. Give your answer correct to 1 decimal place.
b. Find the alertness of the patient after 5 minutes, correct to 1 decimal place.
c. Determine when the rate of increase of alertness is 2 units/minute.

20. The graph of the function $y = \log_e(x^2 + 1)$ is shown.



a. Differentiate the function with respect to x.
b. Points A and B lie on the curve with x values of 2 and -2 respectively. Show that the point of intersection, T, of the tangents at A and B lies on the y-axis.
c. If the tangents at A and B intersect the x-axis at P and Q respectively, show that the length of PQ is less than 0.1 units.

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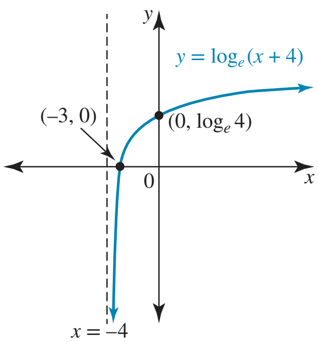
Units 3 & 4 Sit exam

Answers

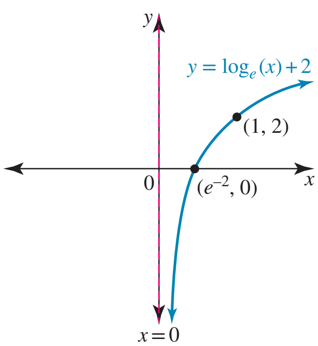
Calculus of logarithmic functions

Exercise 3.2 The natural logarithm and the function $y = \log_e(x)$

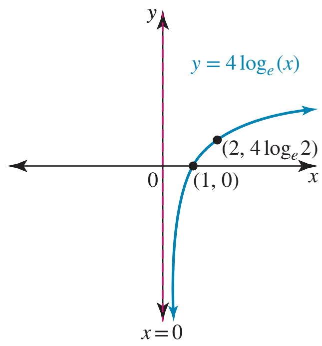
1. a. $x = e$ b. $x = e^2$
   c. $x = 0.135$ d. $x = 0.368$
   e. $x = 1.350$ f. $x = 0.502$
2. a. $x = 3.695$ b. $x = 0.906$ c. $x = e$
3. a. $x = 1.368$ b. $x = -0.432$
   c. $x = -1.433$ d. $x = -2.054$
   e. $x = 0.417$ f. $x = -1.564$
4. a. $x = 596.192$ b. $x = 201.284$
   c. $x = \frac{6}{e}$ d. $x = \frac{e^2}{10}$
5. a. $x = 6$ b. $x = 12$
   c. $x = \frac{2}{3}$ d. $x = \frac{5}{2}$ or $x = 2.5$
   e. $x = 3$ f. $x = 6$
6. a. $x = 1$ b. $x = \frac{3}{2}$ or $x = 1.5$
   c. $x = 2$ d. $x = \frac{3}{2}$ or $x = 1.5$
7. B
8. A
9. $y = ex^2$
10. a. Domain $= (-4, \infty)$, range $= R$



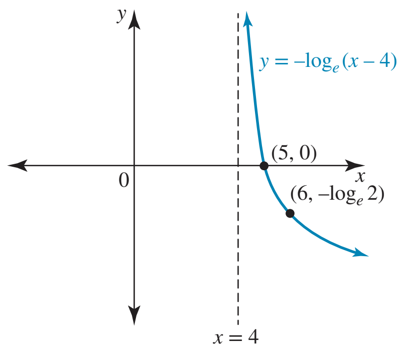
b. Domain $= (0, \infty)$, range $= R$



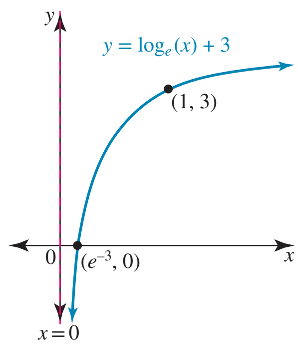
c. Domain $= (0, \infty)$, range $= R$



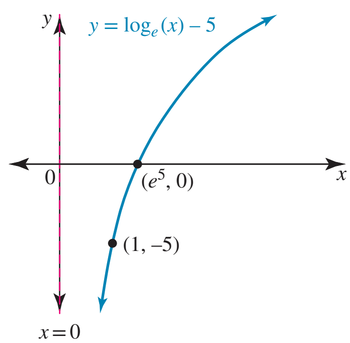
d. Domain $= (4, \infty)$, range $= R$



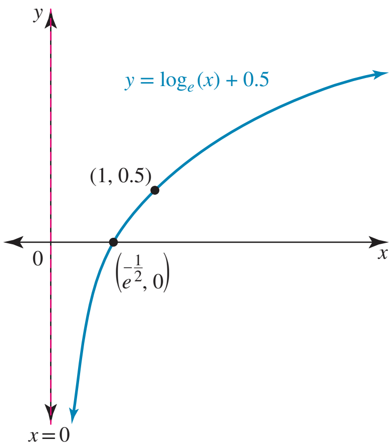
11. a.



b.



c.



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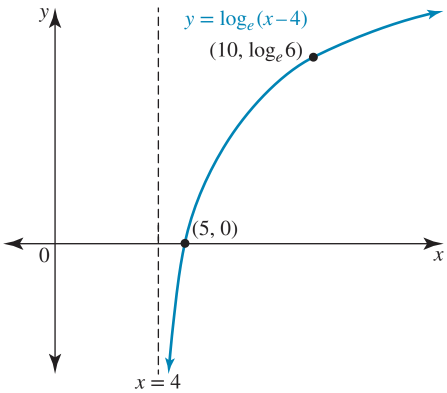
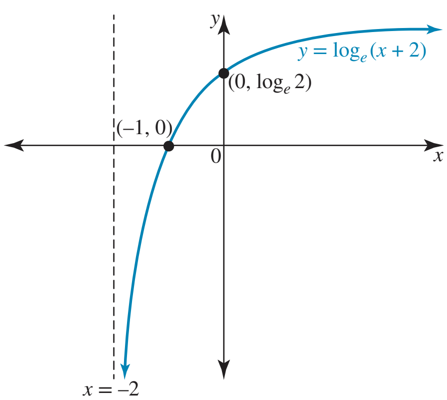
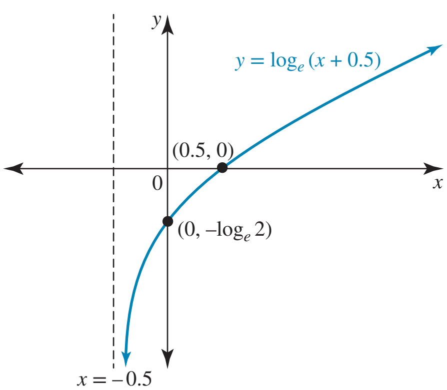


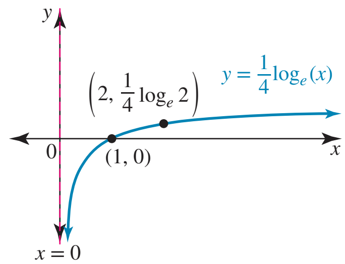
b.



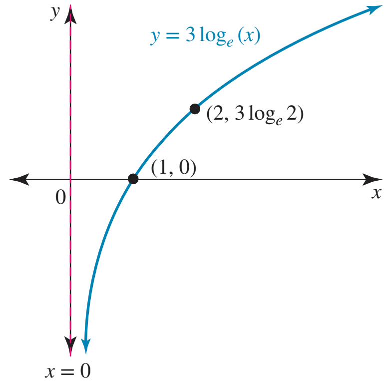
c.



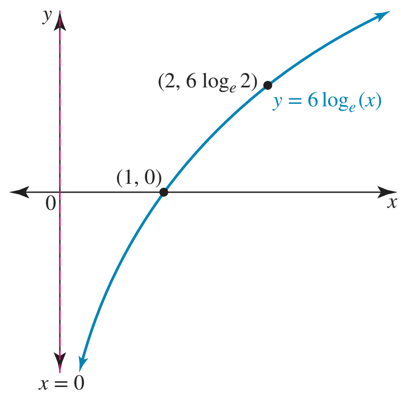
13. a.



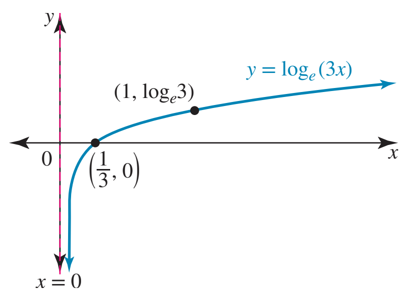
b.



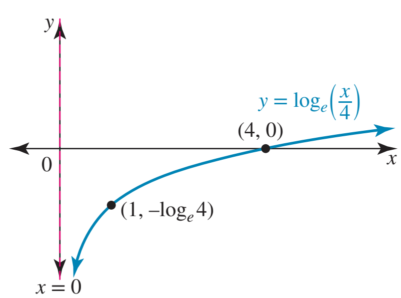
c.



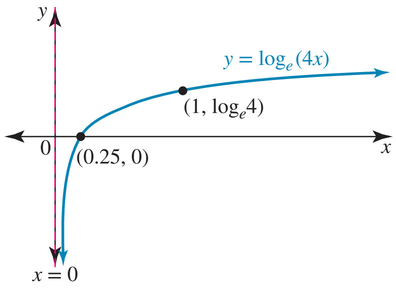
14. a.



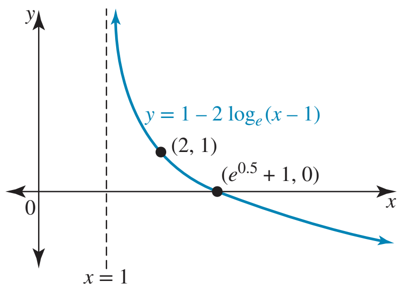
b.



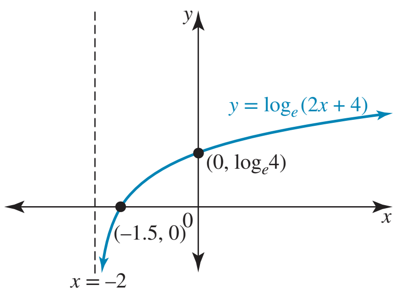
c.



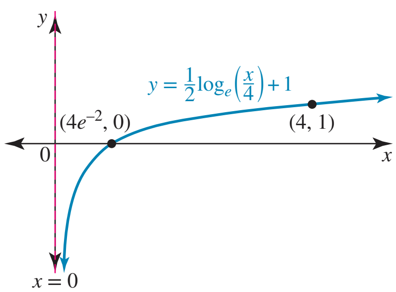
15. a.



b.



c.



16. $m = 2, n = 2$

17. $p = \dfrac{-7}{20\ln(2)}, q = -1$

18. $a = \dfrac{2}{\ln(2)}, h = -1, k = -2$

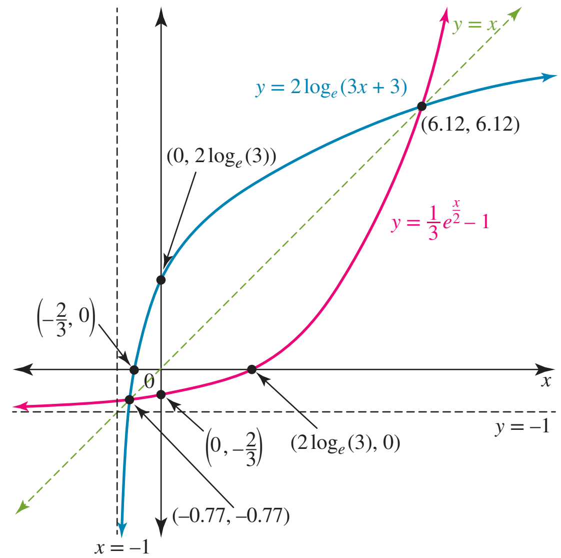
19. Sample responses can be found in the worked solutions in the online resources.

20. a. Domain of $f(x)$: $x > -1$
Range of $f(x)$: $y \in R$

b. $f^{-1}(x) = \dfrac{1}{3}e^{\frac{x}{2}} - 1$

c. Domain of $f^{-1}(x)$: $x \in R$
Range of $f^{-1}(x)$: $y > -1$

d.



e. $(-0.77, -0.77)$ and $(6.12, 6.12)$

Exercise 3.3 The derivative of $y = \log_e(x)$

1. a. $\dfrac{dy}{dx} = \dfrac{1}{x}$ **b.** $\dfrac{dy}{dx} = \dfrac{1}{x}$ **c.** $\dfrac{dy}{dx} = \dfrac{1}{x}$

d. $\dfrac{dy}{dx} = \dfrac{1}{x}$ **e.** $\dfrac{dy}{dx} = \dfrac{3}{x}$ **f.** $\dfrac{dy}{dx} = \dfrac{-6}{x}$

2. a. $\dfrac{dy}{dx} = \dfrac{1}{x}$ **b.** $\dfrac{dy}{dx} = \dfrac{1}{x}$

c. $\dfrac{dy}{dx} = \dfrac{4}{x}$ **d.** $\dfrac{dy}{dx} = \dfrac{-5}{x}$

3. C

4. a. $\dfrac{dy}{dx} = \dfrac{2}{2x + 5}$ **b.** $\dfrac{dy}{dx} = \dfrac{6}{6x + 1}$

c. $\dfrac{dy}{dx} = \dfrac{3}{3x - 4}$ **d.** $\dfrac{dy}{dx} = \dfrac{8}{8x - 1}$

e. $\dfrac{dy}{dx} = \dfrac{5}{5x - 3}$ **f.** $\dfrac{dy}{dx} = \dfrac{-1}{2 - x}$ or $\dfrac{1}{x - 2}$

5. a. $\dfrac{dy}{dx} = \dfrac{30}{5x + 2}$ **b.** $\dfrac{dy}{dx} = \dfrac{16}{2x - 1}$

c. $\dfrac{dy}{dx} = \dfrac{-48}{12x + 5}$ **d.** $\dfrac{dy}{dx} = \dfrac{63}{8 - 9x}$

6. a. $\dfrac{dy}{dx} = \dfrac{4}{x}$ **b.** $\dfrac{dy}{dx} = \dfrac{2x}{(x^2 + 3)}$

c. $\dfrac{dy}{dx} = \dfrac{2(x + 2)}{x(x + 4)}$ **d.** $\dfrac{dy}{dx} = \dfrac{2x - 3}{x^2 - 3x + 2}$

e. $\dfrac{dy}{dx} = \dfrac{3x^2 + 4x - 7}{x(x^2 + 2x - 7)}$ **f.** $\dfrac{dy}{dx} = \dfrac{2(2x - 1)}{x(x - 1)}$

7. a. $\dfrac{dy}{dx} = \dfrac{1}{2x + 1}$ **b.** $\dfrac{dy}{dx} = \dfrac{-2}{3 - 4x}$

c. $\dfrac{dy}{dx} = \dfrac{x}{x^2 + 2}$ **d.** $\dfrac{dy}{dx} = \dfrac{1}{4(x + 3)}$

e. $\dfrac{dy}{dx} = \dfrac{5}{3(5x + 2)}$ **f.** $\dfrac{dy}{dx} = \dfrac{-3}{5(2 - 3x)}$

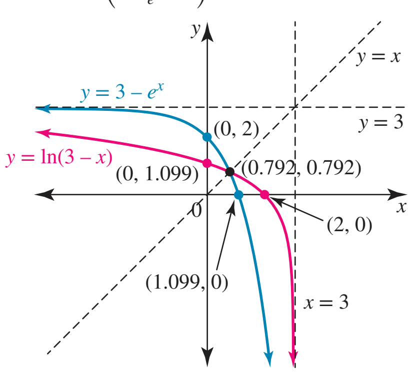
8. a. $f'(x) = \dfrac{-1}{x + 3}$ **b.** $f'(x) = \dfrac{12}{3x - 2}$

c. $f'(x) = \dfrac{-10}{5x + 8}$ **d.** $f'(x) = \dfrac{-3}{4 + 3x}$

9. D
10. D
11. A
12. a. $f'(x) = \frac{7}{x}$ b. $f'(x) = \frac{2x(3x+4)}{x^3 + 2x^2 - 1}$
c. $f'(x) = \frac{3e^x}{e^x + 1}$ d. $f'(x) = \frac{-5}{x}$
13. B
14. The function $y = \ln(100x)$ can be simplified to $y = \ln(100) + \ln(x)$. The differential of a constant is zero, hence the differential of $y = \ln(100x)$ and $y = \ln(x)$ will be the same. This would be true for any logarithmic function of the form $y = \ln(kx)$ where k is a constant.

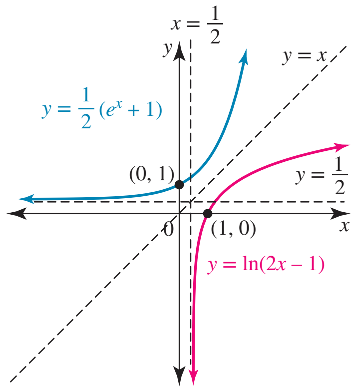
Exercise 3.4 Applications of logarithmic functions

1. a. Domain: $\{x : x \in (2, \infty)\}$; range: $\{y : y \in R\}$
b. $a = 3$
c. $y = 2x - 6$
d. $y = -\frac{1}{2}x + \frac{3}{2}$
2. a. $\frac{dy}{dx} = \frac{2}{5}$ b. $\frac{dy}{dx} = \frac{4}{27}$ c. $\frac{dy}{dx} = \frac{1}{2}$
3. a. $y = 2x - 3$
b. $y = \frac{3}{e}x$ or $3x - ey = 0$
c. $y = \frac{1}{e}x$ or $x - ey = 0$
4. a. $\frac{dy}{dx} = 3$
b. Tangent: $y = 3x - 18$; perpendicular: $x + 3y - 6 = 0$
5. $y = 6x + 4\log_e 2 - 6$
6. Tangent: $y = 2x - 4 + 7\log_e 7$
Perpendicular: $y = \frac{-1}{2}x + 1 + 7\log_e 7$ or
$x + 2y = 2 + 14\log_e 7$
7. a. $\frac{dy}{dx} = \frac{2}{x}$ b. $y = \frac{4x}{e}$
8. $b = 2$
9. $k = 7.4$
10. a. Domain: $x < 3$; range: $y \in R$
b. $(2, 0)$, $(0, \log_e 3)$
c. $f^{-1}(x) = 3 - e^x$; domain of $f^{-1}(x)$: $x \in R$; range of $f^{-1}(x)$: $y < 3$
d. $(0, 2)$ and $(\log_e 3, 0)$
e.

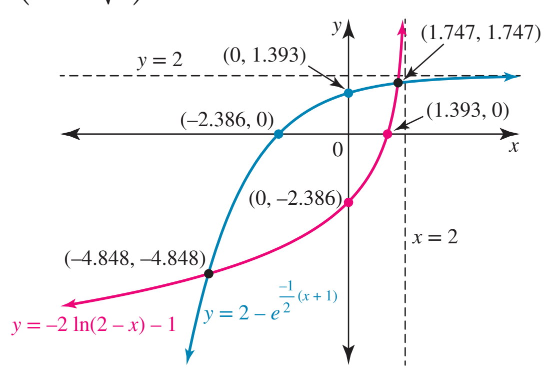


f. $(0.792, 0.792)$

11. a. Domain: $x > \frac{1}{2}$; range: $y \in R$
b. $(1, 0)$
c. $f^{-1}(x) = \frac{1}{2}(e^x + 1)$; domain of $f^{-1}(x)$: $x \in R$; range of $f^{-1}(x)$: $y > \frac{1}{2}$
d. $(0, 1)$
e.

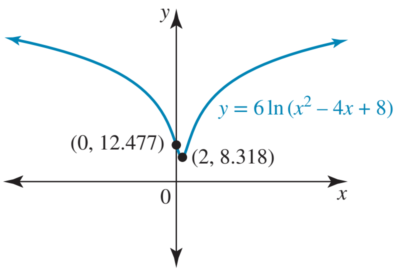


f. Inverse functions are reflections in the line $y = x$. The functions $y = \log_e(x)$ and $y = e^x$ lie on either side of $y = x$, so they do not intersect. The functions $f(x) = \log_e(2x - 1)$ and $f^{-1}(x) = \frac{1}{2}(e^x + 1)$ have been translated further away from the line of symmetry, so there is no point of intersection.
12. a. Domain: $x < 2$; range: $y \in R$
b. $\left(2 - \frac{1}{\sqrt{e}}, 0\right)$ and $(0, -1 - 2\log_e 2)$ (approximately $(1.4, 0)$, $(0, -2.4)$)
c. $f^{-1}(x) = 2 - e^{\frac{-1}{2}(x+1)}$; domain of $f^{-1}(x)$: $x \in R$; range of $f^{-1}(x)$: $y < 2$
d. $\left(0, 2 - \frac{1}{\sqrt{e}}\right)$, $(-1 - 2\log_e 2, 0)$
e.



f. $(-4.85, -4.85)$ and $(1.75, 1.75)$
13. a. $f'(x) = \frac{12(x-2)}{(x^2 - 4x + 8)}$
b. $(2, 6\log_e 4)$
c. Local minimum stationary point

d.



14. a. 25 rats

b. 0.3 months

c. 19 rats/month

3.5 Review: exam practice

1. C

2. B

3. A

4. C

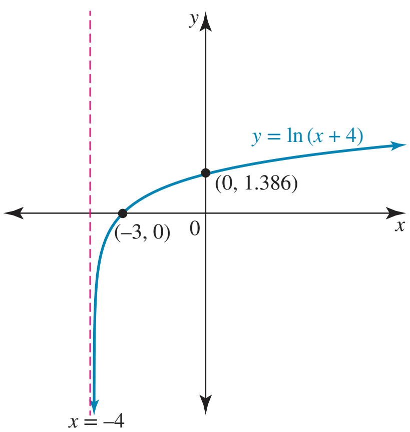
5. a. $y = \ln(x + 4)$

Domain: $x > -4$

Range: $y \in R$

Asymptote: $x = -4$

Transformation: horizontal translation of 4 units to the left (in the negative direction)



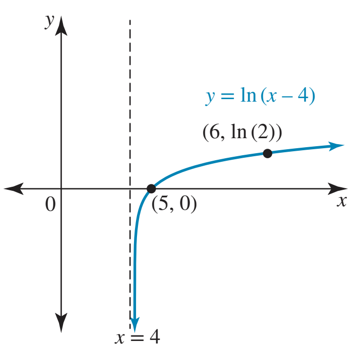
b. $y = \ln(x - 4)$

Domain: $x > 4$

Range: $y \in R$

Asymptote: $x = 4$

Transformation: horizontal translation of 4 units to the right (in the positive direction)



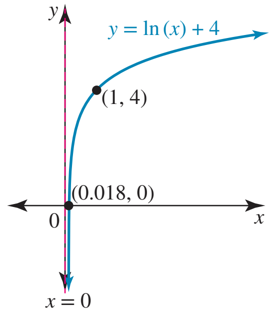
c. $y = \ln(x) + 4$

Domain: $x > 0$

Range: $y \in R$

Asymptote: $x = 0$

Transformation: vertical translation of 4 units upwards (in the positive direction)



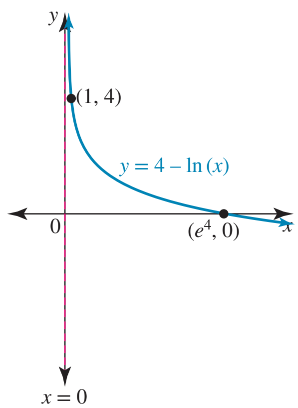
d. $y = 4 - \ln(x)$

Domain: $x > 0$

Range: $y \in R$

Asymptote: $x = 0$

Transformation: vertical translation of 4 units upwards (in the positive direction) and reflection in the x axis



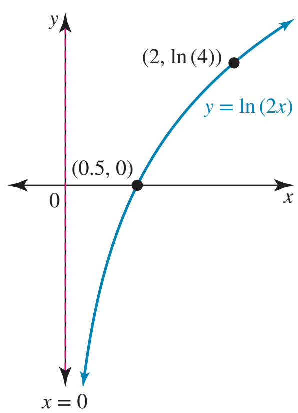
6. a. $y = \ln(2x)$

Domain: $x > 0$

Range: $y \in R$

Asymptote: $x = 0$

Transformation: dilation of half from the y-axis



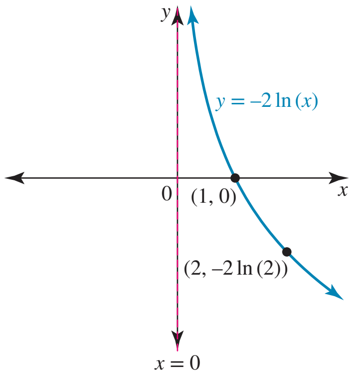
b. $y = -2\ln(x)$

Domain: $x > 0$

Range: $y \in R$

Asymptote: $x = 0$

Transformation: dilation of 2 from the x-axis and a reflection in the x-axis



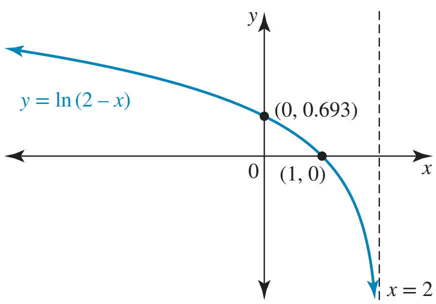
c. $y = \ln(2 - x)$

Domain: $x < 2$

Range: $y \in R$

Asymptote: $x = 2$

Transformation: horizontal translation of 2 units to the right (in the positive direction) and a reflection in the y-axis



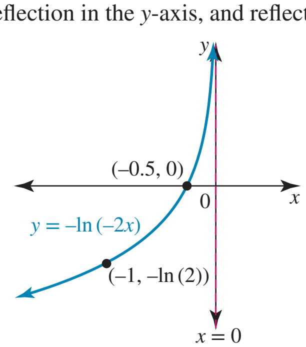
d. $y = -\ln(-2x)$

Domain: $x < 0$

Range: $y \in R$

Asymptote: $x = 0$

Transformation: dilation of half from the y-axis, reflection in the y-axis, and reflection in the x-axis



7. a. $\dfrac{dy}{dx} = \dfrac{x-1}{x^2 - 2x + 7}$; no restrictions on x. Domain: $x \in R$

b. $\dfrac{dy}{dx} = \dfrac{-5}{(x+2)(x-3)}$; $x < -2$ or $x > 3$

c. $\dfrac{dy}{dx} = \dfrac{2}{x+2}$; $x \in R, x \neq -2$

8. a. $\dfrac{dy}{dx} = \dfrac{-11}{(2x+1)(x-5)}$

b. $\dfrac{dy}{dx} = \dfrac{-1}{x-3}$

c. $\dfrac{dy}{dx} = \dfrac{6(3x-1)}{9x^2 - 6x + 7}$

9. A

10. D

11. A

12. A

13. D

14. C

15. $y = \dfrac{1}{e}x + \dfrac{5}{e}$ or $x - ey + 5 = 0$

16. a. $D = \left\{x\colon\, x \in \left(-\infty, \dfrac{1}{3}\right)\right\}$

b. $(0, 0)$

c. $\dfrac{dh}{dx} = 2 \times \dfrac{1}{1-3x} \times -3$

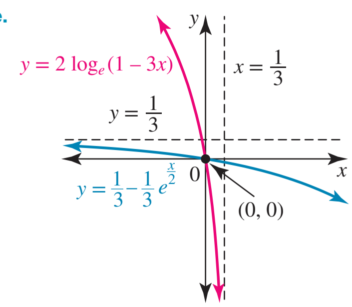
$\dfrac{dh}{dx} = \dfrac{-6}{1-3x}$

For $x < \dfrac{1}{3}$, $1 - 3x > 0$; hence, the rate of change $\dfrac{dh}{dx}$ is always negative.

d. i. $f^{-1}(x) = \dfrac{1}{3} - \dfrac{1}{3}e^{\frac{x}{2}}$

ii. Domain: $x \in R$; range: $y < \dfrac{1}{3}$

e.



17. a. $m = -2, n = \dfrac{1}{2}, p = 2$

b. Dilation of 2 from the y-axis, dilation of 2 from the x-axis, reflection in the x-axis, and horizontal translation in the negative direction of 2 units

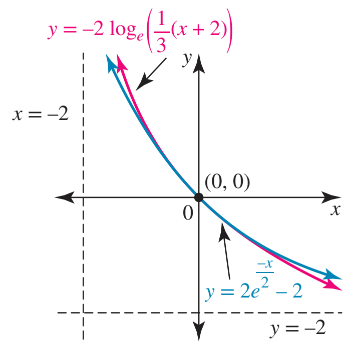
c. $f^{-1}(x) = 2e^{\frac{-x}{2}} - 2$

Domain: $x \in R$

Range: $y > -2$

Asymptote: $y = -2$

d.



18. a. 3000 people
b. 1143 people
c. $\frac{dN}{dt} = \frac{-4000}{(8t + 1)}$
d. Decrease of 98 people/day

19. a. 4.7 minutes
b. 16.5 units
c. 9.5 minutes

20. a. $\frac{dy}{dx} = \frac{2x}{(x^2 + 1)}$
b, c. Sample responses can be found in the worked solutions in the online resources.

4 Calculus of trigonometric functions

4.1 Overview

Early studies of triangles can be traced back to Egyptian and Babylonian mathematics around 4000 BC. They understood the ratio of sides of similar triangles but not angles. Babylonian astronomers kept detailed records of the rising and setting of stars as well as the motion of planets. Greek mathematicians and Indian astronomers investigated the trigonometric functions. The study of trigonometry and geometry is found in the documents of Islamic mathematicians from the Middle Ages. The development of trigonometry as we know it today began with Isaac Newton, reaching its current form when Leonhard Euler published his analysis of trigonometric functions in 1748.

In the past, the principal application of trigonometry for many cultures was in astronomy. Today, trigonometric functions are used to model many physical phenomena that are cyclical or periodic in nature. Examples include the motion of a pendulum and electrical currents, which are both periodic. GPS and cell phones rely on formulas involving sine and cosine; TV and radios transmit images and sounds modelled on sine and cosine functions. In music, sound waves can be modelled by the sine function. Earthquakes create seismic waves; the sine and cosine functions are used to measure the lengths and speed of the waves, allowing seismologists to activate warning devices if necessary. Other examples of trigonometric functions used as models for practical situations include the position and velocity of a particle oscillating in simple harmonic motion, sunlight intensity and the length of daylight hours, average temperature during the day, and the rise and fall of the tides.

In this chapter, you will study the shape of the sine, cosine and tangent functions together with their transformations. You will also investigate the gradient of the functions and apply your knowledge of the trigonometric functions to practical situations.

LEARNING SEQUENCE

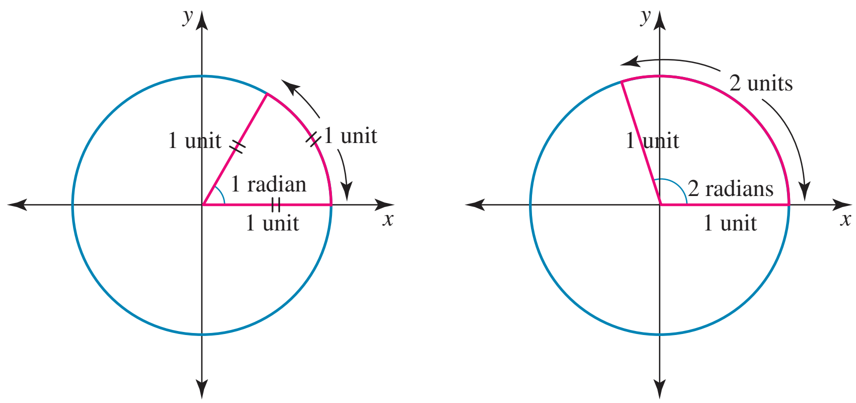
4.1 Overview
4.2 Review of the unit circle, symmetry and exact values
4.3 Review of solving trigonometric equations with and without the use of technology
4.4 Review of graphs of trigonometric functions of the form $y = A\sin(B(x + C)) + D$ and $y = A\cos(B(x + C)) + D$
4.5 Derivatives of the sine and cosine functions
4.6 Applications of trigonometric functions
4.7 Review: exam practice

Fully worked solutions for this chapter are available in the Resources section of your eBookPLUS at www.jacplus.com.au.

4.2 Review of the unit circle, symmetry and exact values

4.2.1 The unit circle and radians

The radius of the unit circle is 1 unit. Remember that angles, including those in the unit circle, can be measured in degrees, minutes and seconds or in radians. Recall that 1 degree is equal to 60 minutes and that 1 minute is equal to 60 seconds. An angle of 1 radian, written 1^c, is equal to the angle formed at the centre of the unit circle by an arc of length 1 unit.

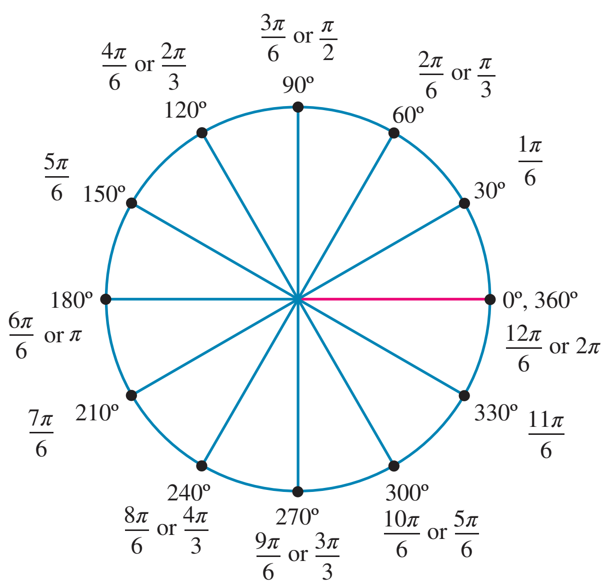


The circumference of the unit circle is 2π, so $360° = 2\pi$ radians.

The relationship between radian and degree measure

$$\pi \text{ radians} = 180°$$

$$1^c = \left(\frac{180}{\pi}\right) \text{degrees or } 1° = \left(\frac{\pi}{180}\right) \text{radians}$$



Angles in the unit circle are measured from the positive x-axis, with positive angles formed when moving anticlockwise around the circle and negative angles formed when moving clockwise around the circle.

The radian symbol is often omitted when the angle is written. An angle without a symbol is therefore assumed to be in radians.

WORKED EXAMPLE 1

a. Convert the following angles to radians, correct to 2 decimal places where necessary.

i. 40° **ii. 125.5°**

b. Convert the following angles to degrees, correct to the nearest minute where necessary.

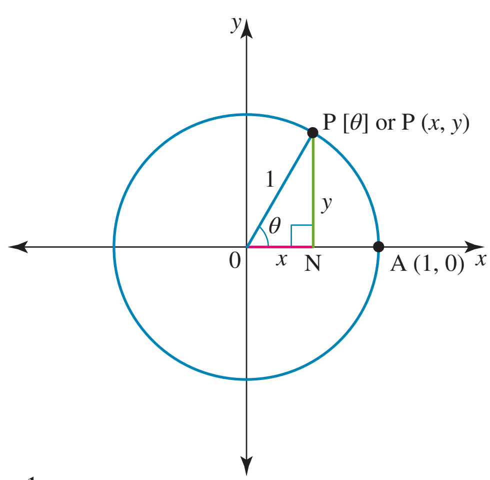
i. $\left(\frac{7\pi}{12}\right)$ **ii. 2.5**

THINK	WRITE
a. i. 1. Substitute $1° = \left(\frac{\pi}{180}\right)$.	**a. i.** $40°$
2. Simplify.	$= 40 \times \frac{\pi}{180}$ $= \frac{2\pi}{9}$
ii. 1. Substitute $1° = \left(\frac{\pi}{180}\right)$.	**ii.** $125.5°$
2. Simplify.	$= 125.5 \times \frac{\pi}{180}$ $= 2.190\,388$ $= 2.19$
b. i. 1. Substitute $\pi = 180°$.	**b. i.** $\frac{7\pi}{12} = \frac{7 \times 180}{12}$
2. Simplify.	$= 105°$
ii. 1. Substitute $1^c = \frac{180}{\pi}$ degrees.	**ii.** 2.5
2. Simplify.	$= 2.5 \times \frac{180}{\pi}$ $= 143.239\,44\,...$
3. Convert to degrees and minutes.	$= 143°14'$

4.2.2 The unit circle and the sine, cosine and tangent ratios

The point P lies on the unit circle. It can be found by rotating the point $(1, 0)$ through an angle of θ, giving the point $P(\theta)$, or by using the right-angled triangle, with Cartesian coordinates $P(x, y)$.

From the right-angled triangle ONP:



$$\cos(\theta) = \frac{x}{1} = x$$

$$\sin(\theta) = \frac{y}{1} = y$$

$$\tan(\theta) = \frac{y}{x} = \frac{\sin(\theta)}{\cos(\theta)}$$

$x^2 + y^2 = 1$, giving the Pythagorean identity $\sin^2(\theta) + \cos^2(\theta) = 1$.

The tangent ratio

$$P(\theta) = (x, y) = (\cos(\theta),\ \sin(\theta))$$

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$$

The Pythagorean identity

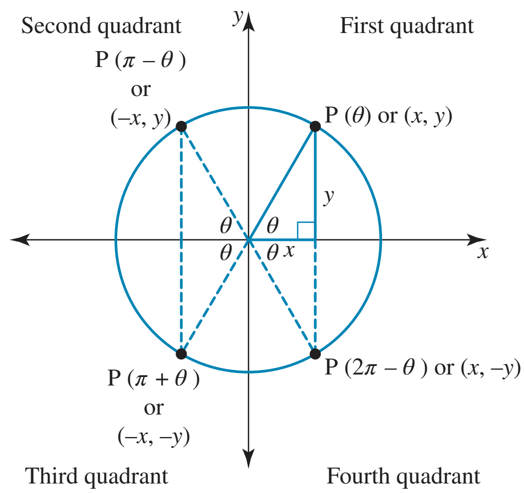
$$\sin^2(\theta) + \cos^2(\theta) = 1$$

4.2.3 Symmetry and the unit circle

Using the symmetry of the unit circle, the following relationships for angles in other quadrants can be determined.

In the second quadrant, the point $P(\pi - \theta)$ is equivalent to the point $(-x, y)$, giving:

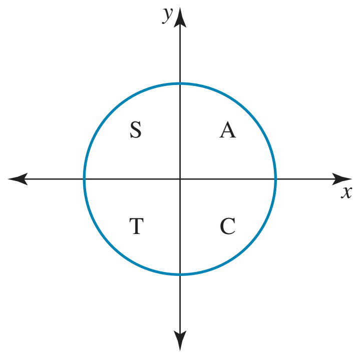
$$\cos(\pi - \theta) = -x = -\cos(\theta)$$
$$\sin(\pi - \theta) = y = \sin(\theta)$$
$$\tan(\pi - \theta) = \frac{y}{-x} = \frac{\sin(\theta)}{-\cos(\theta)} = -\tan(\theta)$$



Similar results can be found for the third and fourth quadrants and are given in the table below.

2nd quadrant $\sin(\pi - \theta) = \sin(\theta)$ $\cos(\pi - \theta) = -\cos(\theta)$ $\tan(\pi - \theta) = -\tan(\theta)$ **S** **Sin positive**	**1st quadrant** $\sin(\theta)$ $\cos(\theta)$ $\tan(\theta)$ **A** **All positive**
T **Tan positive** $\sin(\pi + \theta) = -\sin(\theta)$ $\cos(\pi + \theta) = -\cos(\theta)$ $\tan(\pi + \theta) = \tan(\theta)$ **3rd quadrant**	**C** **Cos positive** $\sin(2\pi - \theta) = -\sin(\theta)$ $\cos(2\pi - \theta) = \cos(\theta)$ $\tan(2\pi - \theta) = -\tan(\theta)$ **4th quadrant**

Or simply:



This can be remembered using mnemonics such as:

Add	**A**ll
Sugar	**S**tations
To	**T**o
Coffee	**C**entral

Angles measured in the clockwise direction are called negative angles.

$$\cos(-\theta) = x = \cos(\theta)$$
$$\sin(-\theta) = -y = -\sin(\theta)$$
$$\tan(-\theta) = \frac{-y}{x} = \frac{-\sin(\theta)}{\cos(\theta)} = -\tan(\theta)$$

Negative angles

$$\cos(-\theta) = \cos(\theta)$$
$$\sin(-\theta) = -\sin(\theta)$$
$$\tan(-\theta) = -\tan(\theta)$$

Note: These relationships are true no matter which quadrant the negative angle is in.

WORKED EXAMPLE 2

If $\sin(\alpha) = \frac{3}{5}$ and α is in the first quadrant, determine the exact values of the following.

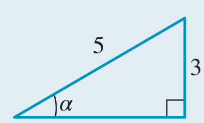
a. $\cos(\alpha)$ **b.** $\tan(\alpha)$ **c.** $\sin(\pi + \alpha)$
d. $\cos(2\pi - \alpha)$ **e.** $\tan(\pi - \alpha)$ **f.** $\cos(-\alpha)$

THINK

a. 1. Draw a right-angled triangle where the opposite is 3 and the hypotenuse is 5, showing $\sin(\alpha) = \frac{3}{5}$ with the adjacent side of 4 (found using Pythagoras' theorem).

WRITE

a.



$a^2 + b^2 = c^2$

$3^2 + b^2 = 5^2$

$b^2 = 16$

$b = 4$

2. State the answer.

Note: An alternative method is to use the Pythagorean identity, as shown.

$\cos(\alpha) = \frac{4}{5}$

$$\begin{aligned}\sin^2(\alpha) + \cos^2(\alpha) &= 1\\ \cos^2(\alpha) &= 1 - \sin^2(\alpha)\\ \cos^2(\alpha) &= 1 - \left(\frac{3}{5}\right)^2\\ \cos^2(\alpha) &= 1 - \frac{9}{25}\\ \cos^2(\alpha) &= \frac{16}{25}\\ \cos(\alpha) &= \pm\frac{4}{5}\end{aligned}$$

But α is in the first quadrant, so $\cos(\alpha) = \frac{4}{5}$.

b. Use the right-angled triangle to find $\tan(\alpha)$.

b. $\tan(\alpha) = \frac{3}{4}$

c. 1. $\pi + \alpha$ is in the third quadrant, where sine is negative.

c. $\sin(\pi + \alpha) = -\sin(\alpha)$

2. Substitute.

$\sin(\pi + \alpha) = -\frac{3}{5}$

d. 1. $(2\pi - \alpha)$ is in the fourth quadrant, where cosine is positive.

d. $\cos(2\pi - \alpha) = \cos(\alpha)$

2. Substitute.

$\cos(2\pi - \alpha) = \frac{4}{5}$

e. 1. $(\pi - \alpha)$ is in the second quadrant, where tangent is negative.

e. $\tan(\pi - \alpha) = -\tan(\alpha)$

2. Substitute.

$\tan(\pi - \alpha) = -\frac{3}{4}$

f. 1. $(-\alpha)$ is in the fourth quadrant, where cosine is positive.

f. $\cos(-\alpha) = \cos(\alpha)$

2. Substitute.

$\cos(-\alpha) = \frac{4}{5}$

4.2.4 Special values and the unit circle

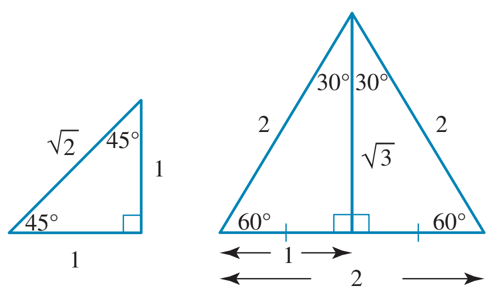
Using the axis intercepts of the unit circle, the values of sine, cosine and tangent for the angles 0, $\frac{\pi}{2}$, π, $\frac{3\pi}{2}$ and 2π can be determined.

In summary:

Angle θ	Point P on the unit circle	$\sin(\theta) = y$ value	$\cos(\theta) = x$ value	$\tan(\theta) = \frac{y}{x}$ value
0	$(1, 0)$	0	1	0
$\frac{\pi}{2}$	$(0, 1)$	1	0	Undefined
π	$(-1, 0)$	0	-1	0
$\frac{3\pi}{2}$	$(0, -1)$	-1	0	Undefined
2π	$1, 0$	0	1	0

4.2.5 Exact values

Exact values for 30°, 45°and 60° or $\frac{\pi}{6}$, $\frac{\pi}{4}$ and $\frac{\pi}{3}$ can be obtained using an equilateral triangle and a right-angled isosceles triangle.



The table below provides a summary of these angles and their ratios.

Angle (θ)	$\sin(\theta)$	$\cos(\theta)$	$\tan(\theta)$
$30°$ or $\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
$45°$ or $\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
$60°$ or $\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$

WORKED EXAMPLE 3

Give exact values for each of the following trigonometric expressions.

a. $\cos\left(\frac{2\pi}{3}\right)$ **b.** $\tan\left(\frac{7\pi}{4}\right)$ **c.** $\cos\left(-\frac{\pi}{6}\right)$ **d.** $\sin\left(\frac{11\pi}{3}\right)$ **e.** $\sin\left(\frac{7\pi}{2}\right)$

THINK	WRITE
a. 1. Rewrite the angle in terms of π and find the corresponding angle in the 1st quadrant.	**a.** $\cos\left(\frac{2\pi}{3}\right) = \cos\left(\pi - \frac{\pi}{3}\right)$
2. The angle is in the 2nd quadrant, so cosine is negative.	$= -\cos\left(\frac{\pi}{3}\right)$
3. Write the answer.	$= -\frac{1}{2}$
b. 1. Rewrite the angle in terms of 2π and find the corresponding angle in the 1st quadrant.	**b.** $\tan\left(\frac{7\pi}{4}\right) = \tan\left(2\pi - \frac{\pi}{4}\right)$
2. The angle is in the 4th quadrant, so tangent is negative.	$= -\tan\left(\frac{\pi}{4}\right)$
3. Write the answer.	$= -1$
c. 1. Rewrite the negative angle as $\cos(-\theta) = \cos(\theta)$.	**c.** $\cos\left(-\frac{\pi}{6}\right) = \cos\left(\frac{\pi}{6}\right)$
2. Write the answer.	$= \frac{\sqrt{3}}{2}$
d. 1. Rewrite the angle in terms of a multiple of 2π.	**d.** $\sin\left(\frac{11\pi}{3}\right) = \sin\left(4\pi - \frac{\pi}{3}\right)$
2. Subtract the extra multiple of 2π so the angle is within 1 revolution of the unit circle.	$= \sin\left(2\pi - \frac{\pi}{3}\right)$
3. The angle is in the 4th quadrant, so sine is negative.	$= -\sin\left(\frac{\pi}{3}\right)$
4. Write the answer.	$= -\frac{\sqrt{3}}{2}$
e. 1. Rewrite the angle in terms of a multiple of 2π.	$\sin\left(\frac{7\pi}{2}\right) = \sin\left(4\pi - \frac{\pi}{2}\right)$
2. Subtract the 2 revolutions of the unit circle.	$= \sin\left(\frac{-\pi}{2}\right)$
3. The angle corresponds to the point $(0, -1)$, and sine is the y-value.	
4. Write the answer.	$= -1$

on Resources

Interactivities: Trigonometric ratios (int-2577)
The unit circle (int-2582)
All sin cos tan (int-2583)
Symmetry points and quadrants (int-2584)

studyon

Units 3 & 4 > Area 2 > Sequence 3 > Concepts 1 & 2

Review of the unit circle Summary screen and practice questions
Exact values and symmetry properties Summary screen and practice questions

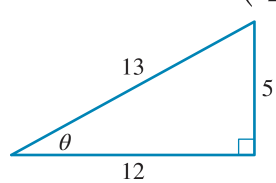
Exercise 4.2 Review of the unit circle, symmetry and exact values

Technology active

1. **WE1a** Change the following angles to degrees, giving your answers to 2 decimal places when necessary. Note that radians are denoted by a superscript c.
   a. 5^c b. 4.8^c c. 2.56^c
   d. $\frac{3\pi^c}{10}$ e. $\frac{5\pi^c}{6}$ f. $\frac{5\pi^c}{4}$
2. **WE1b** Convert the following angles to radians, giving your answers in exact form where possible.
   a. $15°$ b. $120°$ c. $130°$
   d. $63.9°$ e. $78.82°$ f. $310°$
3. **WE2** Evaluate the following, given that $\sin(\alpha) = \frac{5}{13}$ and α lies in the first quadrant.
   a. $\sin(\pi - \alpha)$ b. $\cos(\pi + \alpha)$ c. $\tan(2\pi - \alpha)$
   d. $\sin(3\pi + \alpha)$ e. $\cos(2\pi - \alpha)$ f. $\tan(-\alpha)$
4. Evaluate the following, given that $\cos(\theta) = 0.7$ and $0 \le \theta \le \frac{\pi}{2}$.
   a. $\cos(\pi - \theta)$ b. $\sin(\pi - \theta)$ c. $\tan(2\pi - \theta)$
   d. $\cos(3\pi + \theta)$ e. $\tan(\pi + \theta)$ f. $\cos(-\theta)$
5. **WE3** Find the exact values of each of the following.
   a. $\tan\left(\frac{3\pi}{4}\right)$ b. $\cos\left(\frac{5\pi}{6}\right)$ c. $\sin\left(-\frac{\pi}{4}\right)$
   d. $\cos\left(\frac{7\pi}{3}\right)$ e. $\tan\left(-\frac{\pi}{3}\right)$ f. $\sin\left(\frac{11\pi}{6}\right)$
6. Find the exact values of each of the following.
   a. $\tan\left(\frac{5\pi}{6}\right)$ b. $\cos\left(\frac{14\pi}{3}\right)$ c. $\tan\left(-\frac{5\pi}{4}\right)$
   d. $\cos\left(-\frac{3\pi}{4}\right)$ e. $\sin\left(-\frac{2\pi}{3}\right)$ f. $\sin\left(\frac{17\pi}{6}\right)$

Technology free

7. Simplify the following.
   a. $\sin(\pi - \theta)$ b. $\cos(6\pi - \theta)$ c. $\tan(\pi + \theta)$
   d. $\cos(-\theta)$ e. $\sin(180° + \theta)$ f. $\tan(720° - \theta)$
8. State the exact value for each of the following.
   a. $\cos\left(\frac{\pi}{2}\right)$ b. $\tan(270°)$ c. $\sin(-4\pi)$
   d. $\tan(\pi)$ e. $\cos(-6\pi)$ f. $\sin\left(\frac{3\pi}{2}\right)$
9. For the given triangle, find the values of:
   a. $\sin(\theta)$ b. $\tan(\theta)$
   c. $\cos(\theta)$ d. $\sin(90° - \theta)$
   e. $\cos(90° - \theta)$ f. $\tan(90° - \theta)$



10. Consider $\sin(x) = \frac{5}{6}$.

a. Show that $\sin^2(x) + \cos^2(x) = 1$.

b. Show that $1 + \tan^2(x) = \frac{1}{\cos^2(x)}$.

c. Explain why you didn't need to consider the quadrant in which x was lying for your answers to parts **a** and **b**.

11. If $x = \frac{\pi}{12}$, evaluate $3\sin(2x)$.

12. Calculate the exact values of the following.

a. $\cos\left(\frac{7\pi}{6}\right) + \cos\left(\frac{2\pi}{3}\right)$ **b.** $2\sin\left(\frac{7\pi}{4}\right) + 4\sin\left(\frac{5\pi}{6}\right)$ **c.** $\sqrt{3}\tan\left(\frac{5\pi}{4}\right) - \tan\left(\frac{5\pi}{3}\right)$

d. $\sin^2\left(\frac{8\pi}{3}\right) + \sin\left(\frac{9\pi}{4}\right)$ **e.** $2\cos^2\left(-\frac{5\pi}{4}\right) - 1$ **f.** $\frac{\tan\left(\frac{17\pi}{4}\right)\cos(-7\pi)}{\sin\left(-\frac{11\pi}{6}\right)}$

13. The weight on a spring moves in such a way that its speed, v cm/s, is given by the rule $v = 12 + 3\sin\left(\frac{\pi t}{3}\right)$.

a. Determine the initial speed of the weight.

b. Calculate the exact value of the speed of the weight after 5 seconds.

c. Calculate the exact value of the speed of the weight after 12 seconds.

14. The height, $h(t)$ metres, that the water reaches up the side of the bank of the Brisbane river is determined by the rule

$$h(t) = 0.5\cos\left(\frac{\pi t}{12}\right) + 1.0$$

where t is the number of hours after 6 am.

Find the height of the water up the side of the bank at the following times, giving your answers in exact form.

a. 6 am **b.** 2 pm **c.** 10 pm

4.3 Review of solving trigonometric equations with and without the use of technology

Trigonometric equations involve working with the special angles that have exact values as well as angles that can be analysed using technology.

To solve the basic trigonometric equation, follow these steps:

1. Adjust the domain if required.
2. Look at the sign to identify the quadrants in which the solution(s) lie.
3. Obtain the base angle or first quadrant value.
4. Use the base angle to generate the values for the quadrants required from their symmetrical forms.

To solve trigonometric equations, you may need to use algebraic techniques or the relationships between the functions to reduce the equations to basic forms. These relationships include:

- equations of the form $\sin(x) = a\cos(x)$, which can be converted to $\tan(x) = a$
- equations of the form $\sin^2(x) = a$, which can be converted to $\sin(x) = \pm\sqrt{a}$
- equations of the form $a\sin^2(x) + b\sin(x) + c = 0$, which can be converted to quadratic equations by using a substitution for $\sin(x)$.

Since $-1 \le \sin(x) \le 1$ and $-1 \le \cos(x) \le 1$, some equations may have no solutions.

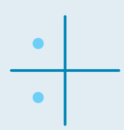
WORKED EXAMPLE 4

Solve the following equations.

a. $\sqrt{2}\cos(x) + 1 = 0, 0 \le x \le 2\pi$

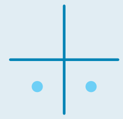
b. $2\sin(x) = -1.5, 0 \le x \le 720°$**, correct to 2 decimal places**

c. $\tan(\theta) - 1 = 0, -\pi \le \theta \le \pi$

THINK	WRITE
a. 1. Express the equation with the trigonometric function as the subject.	**a.** $\sqrt{2}\cos(x) + 1 = 0$ $\sqrt{2}\cos(x) = -1$ $\cos(x) = -\dfrac{1}{\sqrt{2}}$
2. Identify the quadrants in which the solutions lie.	Cosine is negative in quadrants 2 and 3. 
3. Use knowledge of exact values to state the first quadrant base.	The base is $\dfrac{\pi}{4}$, since $\cos\left(\dfrac{\pi}{4}\right) = \dfrac{1}{\sqrt{2}}$.
4. Generate the solutions using the appropriate quadrant forms.	$x = \pi - \dfrac{\pi}{4}, \pi + \dfrac{\pi}{4}$
5. Calculate the solutions from their quadrant forms.	$x = \dfrac{3\pi}{4}, \dfrac{5\pi}{4}$
b. 1. Express the equation with the trigonometric function as the subject.	**b.** $2\sin(x) = -1.5$ $\sin(x) = -0.75$

2. Identify the quadrants in which the solutions lie.

Sine is negative in quadrants 3 and 4.



3. Calculate the base using technology, as an exact value is not possible.

The base is $\sin^{-1}(0.75) = 48.59°$.

4. Generate the solutions using the appropriate quadrant forms. As $x \in [0°, 720°]$, there will be 4 positive solutions from 2 anticlockwise rotations.

$x = 180° + 48.59°, 360° - 48.59°,$
$540° + 48.59°, 720° - 48.59°$

5. Calculate the solutions from their quadrant forms. Alternatively, the solve function on technology can be used to find the solutions (but remember to define the domain).

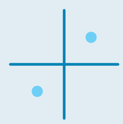
$x = 228.59°, 311.41°, 588.59°, 671.41°$

c. 1. Express the equation with the trigonometric function as the subject.

c. $\tan(\theta) - 1 = 0$
$\tan(\theta) = 1$

2. Identify the quadrants in which the solutions lie.

Tangent is positive in quadrants 1 and 3.



3. Use knowledge of exact values to state the first quadrant base.

The base is $\frac{\pi}{4}$, since $\tan\left(\frac{\pi}{4}\right) = 1$.

4. Generate the solutions using the appropriate quadrant forms. As the domain is $x \in [-\pi, \pi]$, there will be 1 positive solution and 1 negative solution.

$x = \frac{\pi}{4}, -\pi + \frac{\pi}{4}$

5. Calculate the solutions from their quadrant forms.

$x = \frac{\pi}{4}, \frac{-3\pi}{4}$

WORKED EXAMPLE 5

Solve the following equations for x.

a. $2\sin(2x) - 1 = 0, 0 \le x \le 2\pi$

b. $2\cos(2x - \pi) - 1 = 0, -\pi \le x \le \pi$.

THINK

WRITE

a. 1. Change the domain to be that for the given multiple of the variable.

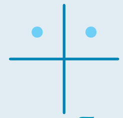
a. $2\sin(2x) - 1 = 0, 0 \le x \le 2\pi$
Multiply each value by 2:
$2\sin(2x) - 1 = 0, 0 \le 2x \le 4\pi$

2. Express the equation with the trigonometric function as the subject.

$2\sin(2x) - 1 = 0$
$2\sin(2x) = 1$
$\sin(2x) = \frac{1}{2}$

3. Solve the equation for $2x$. As $2x \in [0,\ 4\pi]$, each of the 2 revolutions will generate 2 solutions, giving a total of 4 values for $2x$.

Sine is positive in quadrants 1 and 2.
The base is $\frac{\pi}{6}$.



$$2x = \frac{\pi}{6}, \pi - \frac{\pi}{6}, 2\pi + \frac{\pi}{6}, 3\pi - \frac{\pi}{6}$$
$$2x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}$$

4. Calculate the solutions for x.
Note: Dividing by 2 at the very end brings the solutions back within the domain originally specified, namely $0 \le x \le 2\pi$.

$$x = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}$$

b. 1. Change the domain to that for the given multiple of the variable.

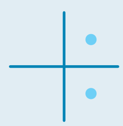
b. $2\cos(2x - \pi) - 1 = 0, -\pi \le x \le \pi$
Multiply each value by 2:
$2\cos(2x - \pi) - 1 = 0, -2\pi \le 2x \le 2\pi$
Subtract π from each value:
$2\cos(2x - \pi) - 1 = 0, -3\pi \le 2x - \pi \le \pi$

2. Express the equation with the trigonometric function as the subject.

$$2\cos(2x - \pi) - 1 = 0$$
$$2\cos(2x - \pi) = 1$$
$$\cos(2x - \pi) = \frac{1}{2}$$

3. Solve the equation for $(2x - \pi)$. The domain of $[-3\pi, \pi]$ involves 2 complete rotations of the unit circle, so there will be 4 solutions, 3 of which will be negative and 1 of which will be positive.

Cosine is positive in quadrants 1 and 4.
The base is $\frac{\pi}{3}$.



$$2x - \pi = \frac{\pi}{3}, -\frac{\pi}{3}, -2\pi + \frac{\pi}{3}, -2\pi - \frac{\pi}{3}$$
$$2x - \pi = \frac{\pi}{3}, -\frac{\pi}{3}, -\frac{5\pi}{3}, -\frac{7\pi}{3}$$

4. Calculate the solutions for x.

$$2x = \frac{\pi}{3} + \pi, -\frac{\pi}{3} + \pi, -\frac{5\pi}{3} + \pi, -\frac{7\pi}{3} + \pi$$
$$= \frac{4\pi}{3}, \frac{2\pi}{3}, -\frac{2\pi}{3}, -\frac{4\pi}{3}$$
$$x = \frac{2\pi}{3}, \frac{\pi}{3}, -\frac{\pi}{3}, -\frac{2\pi}{3}$$

WORKED EXAMPLE 6

Solve the following equations.

a. $\sin(2x) = \cos(2x), 0 \le x \le 2\pi.$
b. $2\sin^2(\theta) + 3\sin(\theta) - 2 = 0, 0 \le x \le 2\pi.$
c. $\cos^2(2\alpha) - 1 = 0, -\pi \le \alpha \le \pi$

THINK | **WRITE**

a. 1. Change the domain to that for the given multiple of the variable.

a. $0 \le x \le 2\pi$
Multiply by 2:
$0 \le 2x \le 4\pi$

2. Reduce the equation to one trigonometric function by dividing by $\cos(2x)$.

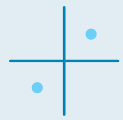
$\sin(2x) = \cos(2x)$

$\dfrac{\sin(2x)}{\cos(2x)} = \dfrac{\cos(2x)}{\cos(2x)}$ providing $\cos(2x) \neq 0$

$\tan(2x) = 1$

3. Solve the equation for $2x$.

Tangent is positive in quadrants 1 and 3.



The base is $\dfrac{\pi}{4}$.

$2x = \dfrac{\pi}{4}, \pi + \dfrac{\pi}{4}, 2\pi + \dfrac{\pi}{4}, 3\pi + \dfrac{\pi}{4}$

$= \dfrac{\pi}{4}, \dfrac{5\pi}{4}, \dfrac{9\pi}{4}, \dfrac{13\pi}{4}$

4. Calculate the solutions for x. Note that the answers are within the prescribed domain of $0 \le x \le 2\pi$.

$x = \dfrac{\pi}{8}, \dfrac{5\pi}{8}, \dfrac{9\pi}{8}, \dfrac{13\pi}{8}$

b. 1. Use substitution to form a quadratic equation.

b. $2\sin^2(\theta) + 3\sin(\theta) - 2 = 0$
Let $A = \sin(\theta)$.
$2A^2 + 3A - 2 = 0$

2. Solve the quadratic equation.

$(2A - 1)(A + 2) = 0$

$A = \dfrac{1}{2}$ or $A = -2$

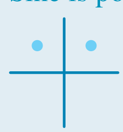
But $A = \sin(\theta)$.

$\sin(\theta) = \dfrac{1}{2}$ or $\sin(\theta) = -2$

3. Solve each trigonometric equation separately.

$\sin(\theta) = \dfrac{1}{2}$

Sine is positive in quadrants 1 and 2.



The base is $\dfrac{\pi}{6}$.

$\theta = \dfrac{\pi}{6}, \pi - \dfrac{\pi}{6}$

$\theta = \dfrac{\pi}{6}, \dfrac{5\pi}{6}$

$\sin(\theta) = -2$
There is no solution as $-1 \le \sin(\theta) \le 1$.

4. Write the answer.

$\theta = \dfrac{\pi}{6}, \dfrac{5\pi}{6}$

c. 1. Change the domain to that for the given multiple of the variable.	**c.** $-\pi \le \alpha \le \pi$ Multiply by 2: $-2\pi \le 2\alpha \le 2\pi$
2. Use substitution to form a quadratic equation and factorise by applying the difference of perfect squares method.	$\cos^2(2\alpha) - 1 = 0$ Let $A = \cos(2\alpha)$. $A^2 - 1 = 0$
3. Solve the quadratic equation.	$(A-1)(A+1) = 0$ $A = 1, -1$ But $A = \cos(2\alpha)$. $\therefore \cos(2\alpha) = 1$ or $\cos(2\alpha) = -1$
4. Solve each trigonometric equation separately.	$\cos(2\alpha) = 1$ $2\alpha = -2\pi, 0, 2\pi$ $\alpha = -\pi, 0, \pi$ $\cos(2\alpha) = -1$ $2\alpha = -\pi, \pi$ $\alpha = -\frac{\pi}{2}, \frac{\pi}{2}$
5. Write the answers in numerical order.	$\therefore \alpha = -\pi, -\frac{\pi}{2}, 0, \frac{\pi}{2}, \pi$

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Units 3 & 4 > Area 2 > Sequence 3 > Concept 2

Solving trigonometric equations Summary screen and practice questions

Exercise 4.3 Review of solving trigonometric equations with and without the use of technology

Technology free

1. **WE4** Solve the following equations.
   a. $2\cos(\theta) + \sqrt{3} = 0$ for $0 \le \theta \le 2\pi$
   b. $\tan(x) + \sqrt{3} = 0$ for $0° \le x \le 720°$
   c. $2\cos(\theta) = 1$ for $-\pi \le \theta \le \pi$
2. a. Solve the equation $2\sin(\theta) + 1 = 0$, $0° \le \theta \le 360°$.
   b. Solve $\sin(x) = 1$, $-2\pi \le x \le 2\pi$.
3. **WE5** Solve the following equations.
   a. $2\cos(3\theta) - \sqrt{2} = 0$ for $0 \le \theta \le 2\pi$
   b. $2\sin(2x + \pi) + \sqrt{3} = 0$ for $-\pi \le x \le \pi$
4. Solve $2\cos\left(3\theta - \frac{\pi}{2}\right) + \sqrt{3} = 0$, $0 \le \theta \le 2\pi$.
5. **WE6** Solve the equation $\cos^2(\theta) - \sin(\theta)\cos(\theta) = 0$ for $0 \le \theta \le 2\pi$.
6. Solve $\{\theta : 2\cos^2(\theta) + 3\cos(\theta) = -1, 0 \le \theta \le 2\pi\}$.
7. Solve the following trigonometric equations for $0 \le \theta \le 2\pi$.
   a. $\sqrt{2}\sin(\theta) = -1$
   b. $2\cos(\theta) = 1$
   c. $\tan(3\theta) - \sqrt{3} = 0$
   d. $\tan\left(\theta - \frac{\pi}{2}\right) + 1 = 0$

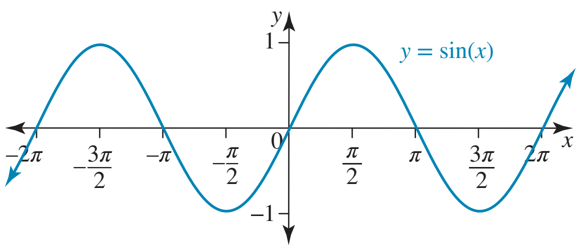
Technology active

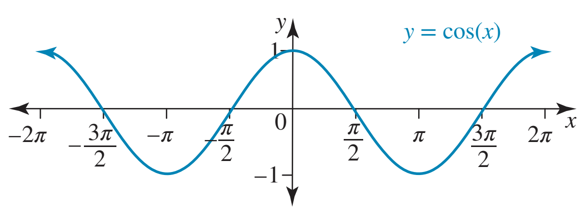
8. Solve the following trigonometric equations for $0° \leq x \leq 360°$.
   a. $2\cos(x) + 1 = 0$
   b. $2\sin(2x) + \sqrt{2} = 0$
9. Solve the following, correct to 2 decimal places.
   a. $3\sin(\theta) - 2 = 0$ given that $0 \leq \theta \leq 2\pi$.
   b. $7\cos(x) - 2 = 0$ given that $0° \leq x \leq 360°$.
10. Solve the following for θ given that $-\pi \leq \theta \leq \pi$.
    a. $2\sin(2\theta) + \sqrt{3} = 0$
    b. $\sqrt{2}\cos(3\theta) = 1$
    c. $\tan(2\theta) + 1 = 0$
11. Solve the following for x given that $-\pi \leq x \leq \pi$.
    a. $2\sin\left(2x + \dfrac{\pi}{4}\right) = \sqrt{2}$
    b. $2\cos(x + \pi) = \sqrt{3}$
    c. $\tan(x - \pi) = -1$
12. Solve the following for θ given that $0 \leq \theta \leq 2\pi$.
    a. $\tan^2(\theta) - 1 = 0$
    b. $4\sin^2(\theta) - (2 + 2\sqrt{3})\sin(\theta) + \sqrt{3} = 0$
13. Solve the following for α where $-\pi \leq \alpha \leq \pi$.
    a. $\sin(\alpha) - \cos^2(\alpha)\sin(\alpha) = 0$
    b. $\sin(2\alpha) = \sqrt{3}\cos(2\alpha)$
    c. $\sin^2(\alpha) = \cos^2(\alpha)$
    d. $4\cos^2(\alpha) - 1 = 0$
14. A particle moves in a straight line so that its distance, x metres, from a point O is given by the equation $x = 3 + 4\sin(2t)$, where t is the time in seconds after the particle begins to move.
    a. Calculate the distance from O when the particle begins to move.
    b. Determine the time when the particle first reaches the point O. Give your answer correct to 2 decimal places.
15. a. Using technology, sketch on the same axes the graphs of $y = \sin x$ and $y = \cos 2x$.
    b. Hence, solve the equation $\sin x = \cos 2x$ for $0 \leq x \leq 2\pi$. Give your answers correct to 2 decimal places.
    c. Discuss why technology was useful in solving this equation.

4.4 Review of graphs of trigonometric functions of the form $y = A\sin(B(x + C)) + D$ and $y = A\cos(B(x + C)) + D$

4.4.1 Graphs of $y = \sin(x)$ and $y = \cos(x)$

The graphs of the sine and cosine functions are shown.





Both graphs have a wave shape that repeats itself every 2π units, oscillating about the line $y = 0$ (the x-axis), rising and falling by up to 1 unit.

The two graphs are said to be 'out of phase' by $\dfrac{\pi}{2}$. That is, a horizontal shift of the cosine graph by $\dfrac{\pi}{2}$ to the right gives the sine graph; likewise, a horizontal shift of the sine graph by $\dfrac{\pi}{2}$ to the left gives the cosine graph.

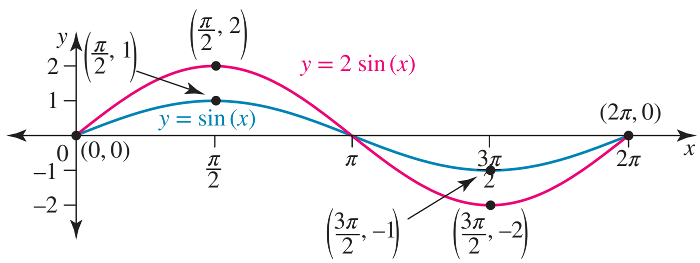
The properties of both sine and cosine functions are summarised below.

Properties of the graphs of the sine and cosine functions
- Period: 2π
- Amplitude: 1
- Line of oscillation (mean position): $y = 0$
- Domain: $x \in R$
- Range: $y \in [-1, 1]$

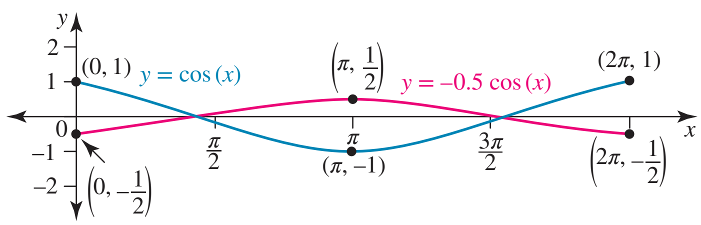
4.4.2 Graphs of $A\sin(Bx) + D$ and $A\cos(Bx) + D$

For graphs in the form $y = A\sin(Bx) + D$ or $y = A\cos(Bx) + D$, **the value of A** affects the amplitude and direction of the sine and cosine functions:
- If $A > 0$, the amplitude is A.
- If $A < 0$, the amplitude is A and the graph is also reflected in the x-axis.

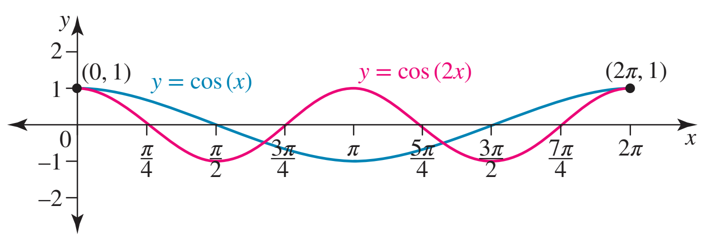
$y = 2\sin(x)$
Amplitude $= 2$
Note: The graph shows a dilation of 2 from the x-axis.



$y = -\frac{1}{2}\cos(x)$
Amplitude $= \frac{1}{2}$
Reflection in x-axis
Note: The graph shows a dilation of $\frac{1}{2}$ from the x-axis.



The value of B affects the period of the sine and cosine functions:
- The period is $\frac{2\pi}{B}$.
- If $B < 0$, the function is reflected over the y-axis.

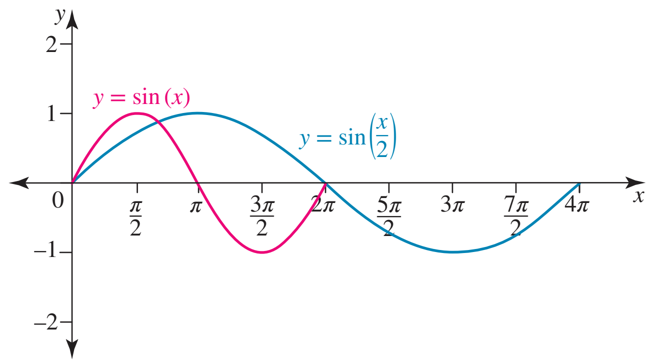


$y = \cos(2x)$
Period $= \frac{2\pi}{2} = \pi$
Note: The graph shows a dilation of $\frac{1}{2}$ from the y-axis.

$y = \sin\left(\frac{x}{2}\right)$

Period $= 2\pi \div \frac{1}{2} = 4\pi$

Note: The graph shows a dilation of 2 from the y-axis.



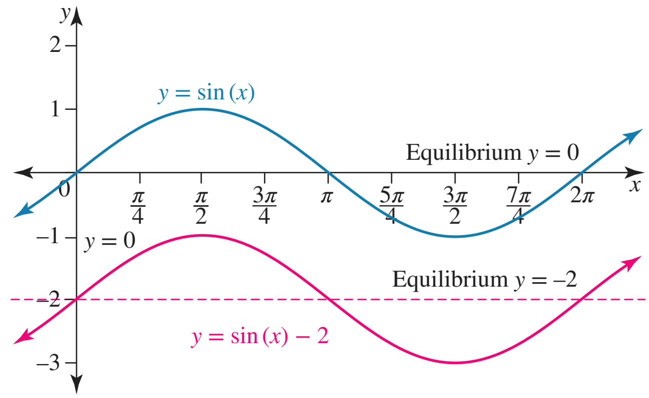
The value of D affects the equilibrium or mean position about which the sine and cosine functions oscillate:

- The graph oscillates about the line $y = D$.
- The range of the function changes.

$y = \sin(x) - 2$

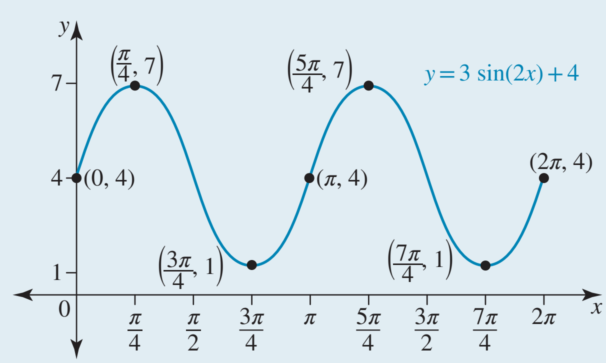
Line of oscillation: $y = -2$

Note: This can be described as a vertical translation down by 2 units or a translation of 2 in the negative direction parallel to the y-axis.



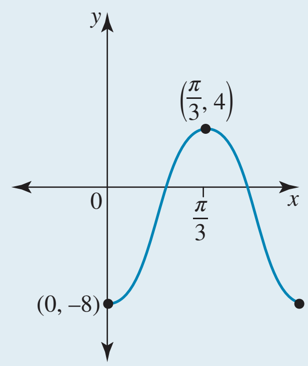
WORKED EXAMPLE 7

Sketch the graph of $y = 3\sin(2x) + 4, 0 \le x \le 2\pi$.

THINK	WRITE
1. State the period and amplitude of the graph.	$y = 3\sin(2x) + 4, 0 \le x \le 2\pi$ The period is $\frac{2\pi}{2} = \pi$. The amplitude is 3.
2. State the mean position and the range.	The mean position is $y = 4$. The range of the graph is $[4 - 3, 4 + 3] = [1, 7]$.
3. Construct appropriate scales on the axes and sketch the graph.	

WORKED EXAMPLE 8

The diagram shows the graph of a cosine function. State its mean position, amplitude and period, and give a possible equation for the function.



THINK	WRITE
1. Deduce the mean position.	The minimum value is -8 and the maximum value is 4, so the mean position is $y = \frac{-8+4}{2} = -2$.
2. State the amplitude.	The amplitude is the distance from the mean position to either its maximum or minimum. The amplitude is $A = \frac{4 - -8}{2} = 6$.
3. State the period.	At $x = \frac{\pi}{3}$, the graph is halfway through its cycle, so its period is $\frac{2\pi}{3}$.
4. Determine a possible equation for the given graph.	Let the equation be $y = A\cos(Bx) + D$. The graph is an inverted cosine shape, so $A = -6$. The period is $\frac{2\pi}{B}$. $\frac{2\pi}{B} = \frac{2\pi}{3}$ $B = 3$ The mean position is $y = -2$, so $D = -2$. The equation is $y = -6\cos(3x) - 2$.

4.4.3 Graphs of $y = A\sin(B(x + C)) + D$ and $y = A\cos(B(x + C)) + D$

Below is a summary of the transformations of $y = \sin(x)$ to $y = A\sin(B(x + C)) + D$.

- A where $A > 0$: **dilation** by a factor of A from the x-axis
- $A < 0$: **reflection** in the x-axis
- B where $B > 0$: **dilation** by a factor of $\frac{1}{B}$ from the y-axis
- $B < 0$: **reflection** in the y-axis
- C: **translation** horizontally, or parallel to the x-axis, of C to the left if $C > 0$ or C to the right if $C < 0$
- D: **translation** vertically, or parallel to the y-axis, of D

Transformations of $y = \cos(x)$ to $y = A\cos(B(x + C)) + D$ follow the same patterns as those for $y = \sin(x)$.

Properties of these trigonometric functions are summarised below.

Properties of graphs in the form $y = A\sin(B(x + C)) + D$ or $y = A\cos(B(x + C)) + D$

- Period: $\dfrac{2\pi}{B}$
- Amplitude: $A,\ A > 0$
- Line of oscillation (mean position): $y = D$
- Domain: $x \in R$
- Range: $y \in [D - A,\ D + A]$

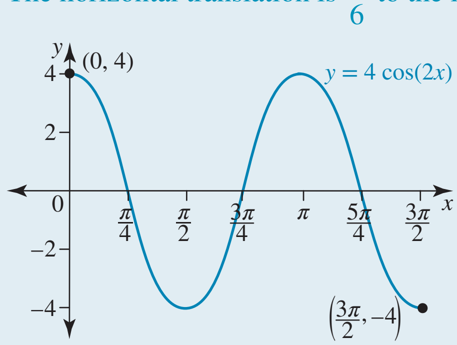
When sketching the trigonometric functions, the following steps may be useful:

1. State the period, amplitude, line of oscillation (or mean position) and range.
2. Sketch the graph without any horizontal translation.
3. Calculate the coordinates of the endpoints for the given domain.
4. Sketch the graph with the horizontal translation.
5. Apply the vertical translation to the graph.
6. Calculate the coordinates of the x-intercepts and the y-intercept if included.

WORKED EXAMPLE 9

a. **Sketch the graph of the function $f: \left[0, \dfrac{3\pi}{2}\right] \to R,\ f(x) = 4\cos\left(2x + \dfrac{\pi}{3}\right)$.**

b. **Hence, sketch the graph of the function $g: \left[0, \dfrac{3\pi}{2}\right] \to R,\ g(x) = 6 - 4\cos\left(2x + \dfrac{\pi}{3}\right)$.**

THINK	WRITE
a. 1. State the period, amplitude, mean position and horizontal translation by rewriting the function in the form $y = A\cos(B(x + C)) + D$. *Note:* It is a common error not to factorise to find B.	a. $f: \left[0, \dfrac{3\pi}{2}\right] \to R, f(x) = 4\cos\left(2x + \dfrac{\pi}{3}\right)$ $f(x) = 4\cos\left(2\left(x + \dfrac{\pi}{6}\right)\right)$ The period is $\dfrac{2\pi}{B}$; in this case, $\dfrac{2\pi}{2} = \pi$. The amplitude is 4. The mean position is $y = 0$. The horizontal translation is $\dfrac{\pi}{6}$ to the left.
2. Sketch the graph without the horizontal translation: $y = 4\cos(2x)$.	

3. Calculate the coordinates of the endpoints of the domain of the given function.

$$f(0) = 4\cos\left(\frac{\pi}{3}\right)$$
$$= 4 \times \frac{1}{2}$$
$$= 2$$
$$f\left(\frac{3\pi}{2}\right) = 4\cos\left(3\pi + \frac{\pi}{3}\right)$$
$$= 4 \times \frac{-1}{2}$$
$$= -2$$

The endpoints of the graph are $(0, 2)$ and $\left(\frac{3\pi}{2}, -2\right)$.

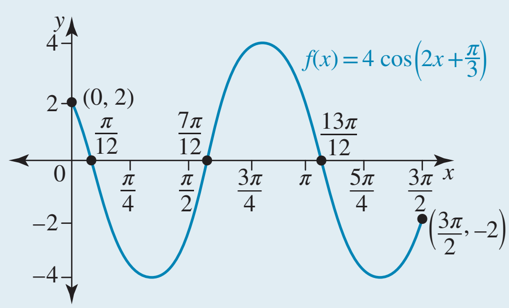
4. Calculate or deduce the positions of the x-intercepts.

Each x-intercept on $y = 4\cos(2x)$ is translated $\frac{\pi}{6}$ units to the left.

Alternatively, let $y = 0$.

$$4\cos\left(2x + \frac{\pi}{3}\right) = 0$$
$$\cos\left(2x + \frac{\pi}{3}\right) = 0, \frac{\pi}{3} \le 2x + \frac{\pi}{3} \le 3\pi + \frac{\pi}{3}$$
$$2x + \frac{\pi}{3} = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$$
$$2x = \frac{\pi}{6}, \frac{7\pi}{6}, \frac{13\pi}{6}$$
$$x = \frac{\pi}{12}, \frac{7\pi}{12}, \frac{13\pi}{12}$$

5. Apply the horizontal translation to key points on the graph already sketched and hence sketch the function over its given domain.



b. 1. State the period, amplitude, line of oscillation and range.
Note: $g(x)$ is related to $f(x)$ from part **a**. The curve $f(x)$ has been reflected in the x-axis and translated vertically upwards by 6.

$$g: \left[0, \frac{3\pi}{2}\right] \to R, g(x) = 6 - 4\cos\left(2x + \frac{\pi}{3}\right)$$
$$g(x) = -4\cos\left(2\left(x + \frac{\pi}{6}\right)\right) + 6$$

Period $= \pi$
Amplitude $= 4$
Line of oscillation: $y = 6$
Range $= [2, 10]$

2. State the translations.

Horizontal translation $\frac{\pi}{6}$ to the left

Vertical translation upwards by 6 units

3. Calculate the endpoints for the given domain.
Note: Since $g(x)$ is a reflection and vertical translation of $f(x)$, the endpoints could be obtained easily using the endpoints of $f(x)$.

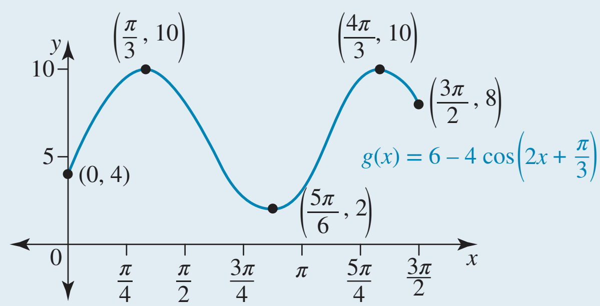
$$g(0) = -4\cos\left(\frac{\pi}{3}\right) + 6$$
$$g(0) = -4 \times \frac{1}{2} + 6$$
$$g(0) = 4$$
$$g\left(\frac{3\pi}{2}\right) = -4\cos\left(2 \times \frac{3\pi}{2} + \frac{\pi}{3}\right) + 6$$
$$g\left(\frac{3\pi}{2}\right) = -4\cos\left(\frac{10\pi}{3}\right) + 6$$
$$g\left(\frac{3\pi}{2}\right) = -4 \times \frac{-1}{2} + 6$$
$$g\left(\frac{3\pi}{2}\right) = 8$$

The endpoints are $(0, 4)$ and $\left(\frac{3\pi}{2}, 8\right)$.

4. Using the transformations of $f(x)$, the function $g(x)$ will not have any x-intercepts.

There are no x-intercepts since $\cos\left(2x + \frac{\pi}{3}\right) \neq \frac{3}{2}$.

5. Sketch the function $y = g(x)$ using the graph of $f(x)$ along with the extra information and transformations.



Resources

Interactivities: Sin and cosine graphs (int-2976)
The unit circle, sine and cosine graphs (int-6551)
Oscillation (int-2977)
Complementary properties of sin and cos (int-2979)

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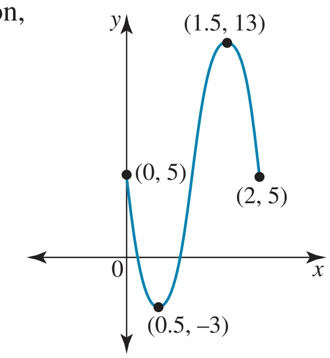
Units 3 & 4 › Area 2 › Sequence 3 › Concept 4

Graphs of sine and cosine functions Summary screen and practice questions

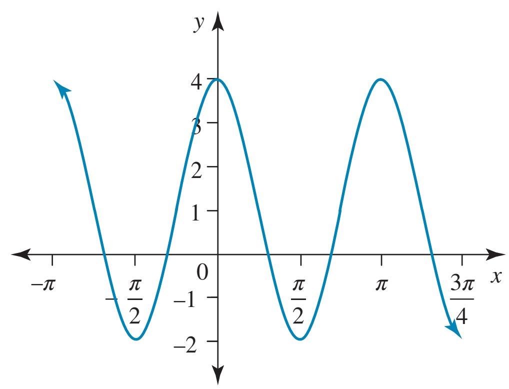
Exercise 4.4 Review of graphs of trigonometric functions of the form $y = A\sin(B(x+C)) + D$ and $y = A\cos(B(x+C)) + D$

Technology free

1. Sketch the following graphs for $0 \le x \le 2\pi$.
   a. $y = \sin(x)$ b. $y = \sin(2x)$ c. $y = 2\sin(x)$ d. $y = \sin(x) + 2$ e. $y = 2 - \sin(x)$
2. Sketch the following graphs for $-2\pi \le x \le 2\pi$.
   a. $y = \cos(x)$ b. $y = \cos\left(\dfrac{x}{2}\right)$ c. $y = 3\cos(x)$ d. $y = 3 - \cos(x)$ e. $y = \cos(x) - 3$
3. **WE7** Sketch the graph of $y = 2\cos(4x) - 3, 0 \le x \le 2\pi$.
4. Sketch the graph of $y = 2 - 4\sin(3x), 0 \le x \le 2\pi$.
5. Sketch the graph of $y = -7\cos(4x)$ for $0 \le x \le \pi$, stating any axis intercepts.
6. For $-\pi \le x \le 2\pi$, sketch the function $y = \dfrac{1}{2}\cos(2x) + 3$.
7. Sketch the graph of $f\colon [0, 2\pi] \to R, f(x) = 1 - 2\sin\left(\dfrac{3x}{2}\right)$, locating any intercepts with the coordinate axes.
8. **WE8** The diagram shows the graph of a sine function. State its mean position, amplitude and period, and give a possible equation for the function.



9. The diagram shows the graph of a cosine function. State its line of oscillation, amplitude and period, and give a possible equation for the function.



10. a. **WE9** Sketch the graph of the function $f\colon \left[0, \dfrac{3\pi}{2}\right] \to R, f(x) = -6\sin\left(3x - \dfrac{3\pi}{4}\right)$.
    b. Hence or otherwise, sketch the function $g\colon \left[0, \dfrac{3\pi}{2}\right] \to R, g(x) = 7 - 6\sin\left(3x - \dfrac{3\pi}{4}\right)$, showing all important features.
11. a. Sketch the function $y = 2\sin\left(x + \dfrac{\pi}{4}\right), 0 \le x \le 2\pi$.
    b. Hence or otherwise, sketch the function $y = 2\sin\left(x + \dfrac{\pi}{4}\right) - 1, 0 \le x \le 2\pi$, showing all important features.

Technology active

12. a. Sketch the function $f: \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right] \to R, f(x) = 4\cos\left(3x - \frac{\pi}{2}\right)$, showing all axis intercepts.

    b. Hence or otherwise, sketch the function $g: \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right] \to R, g(x) = 4 - 4\cos\left(3x - \frac{\pi}{2}\right)$.

    c. Explain how the functions in parts **a** and **b** are related.

13. a. Sketch the graph of $y = 2\cos(3x)$ for one complete cycle.

    b. Sketch the graph that would result from the function $y = 2\cos(3x)$ being translated $\frac{\pi}{3}$ units to the right and 3 units vertically up.

    c. State an equation for the graph formed in part **b**.

14. State the maximum value of the function $f(x) = 2 - 3\cos\left(x + \frac{\pi}{12}\right)$ and give the first positive value of x for when this maximum occurs.

15. a. Sketch $y = \sin(x), 0 \le x \le 4\pi$

    b. Hence, sketch $y = \sin^2(x), 0 \le x \le 4\pi$. Check your graph using technology.

16. a. Solve the equation $2\sin(2x) + \sqrt{3} = 0$ for $x \in [0, 2\pi]$.

    b. Sketch the graph of $y = \sin(2x)$ for $x \in [0, 2\pi]$.

    c. Hence, find $\left\{x: \sin 2x < -\frac{\sqrt{3}}{2}, 0 \le x \le 2\pi\right\}$.

4.5 Derivatives of the sine and cosine functions

4.5.1 Investigating the differential of $y = \sin(x)$ using technology

The derivative of a function gives the slope of the curve at any point.

By considering the slope of the tangent at points such as $x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$ and 2π, the general shape of the derived function can be observed. Only one cycle needs to be considered.

Use the 'Analyze graph' menu on your calculator to find $\frac{dy}{dx}$ at any point on the curve $y = \sin(x)$. Any number of points may be considered.

The table below summarises the gradients of $y = \sin(x)$ from $x = 0$ to $x = 2\pi$, giving values to 1 decimal place where necessary.

***x* radians**	**$\frac{dy}{dx}$, gradient at the point**
0	1
$\frac{\pi}{4}$	0.7
$\frac{\pi}{2}$	0
$\frac{3\pi}{4}$	−0.7

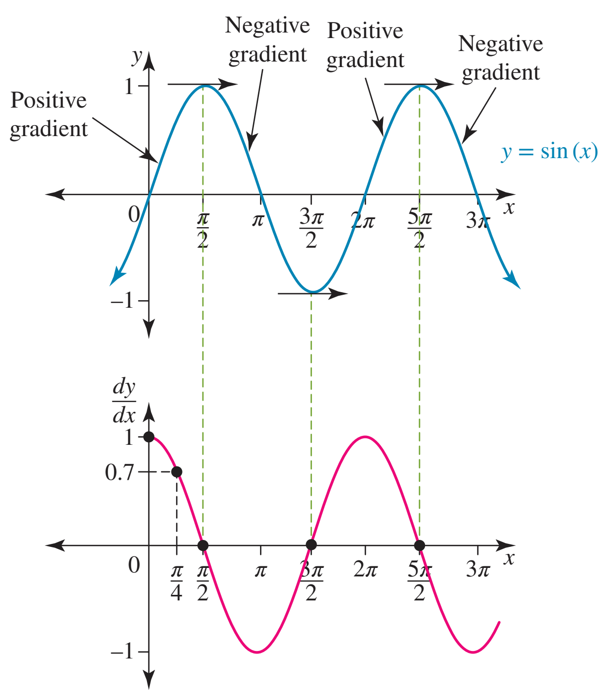
π	-1
$\frac{5\pi}{4}$	-0.7
$\frac{3\pi}{2}$	0
$\frac{7\pi}{4}$	0.7
2π	1

Consider also the sign of the gradient of the sine curve for various intervals.

x intervals	Sign of gradient
$0 < x < \frac{\pi}{2}$	Positive
$\frac{\pi}{2} < x < \frac{3\pi}{2}$	Negative
$\frac{3\pi}{2} < x < 2\pi$	Positive

This information is illustrated in the diagram. It can be seen that the derived function has the shape of cosine graph.

Note: Adding more points gives a better shape.



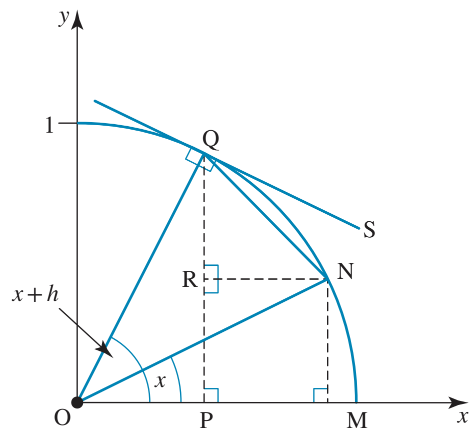
4.5.2 Investigating the derivative of the sine function using first principles

The derivative of $y = \sin(x)$ can be investigated using differentiation from first principles.

Consider $f\colon R \to R, f(x) = \sin(x)$ where x is an angle measurement in radians.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ f(x) &= \sin(x) \\ f(x+h) &= \sin(x+h) \\ \frac{f(x+h) - f(x)}{h} &= \frac{\sin(x+h) - \sin(x)}{h} \\ f'(x) &= \lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h} \end{aligned}$$

To evaluate this limit, we must look at the unit circle.



$$\begin{aligned} \angle NOM = x, \angle QOM &= x + h \\ \angle PQO &= \frac{\pi}{2} - (x+h) \\ \angle RQS &= \frac{\pi}{2} - \left(\frac{\pi}{2} - (x+h)\right) \\ &= x + h \end{aligned}$$

By definition,

$$\begin{aligned} \sin(x) &= MN \\ \sin(x+h) &= PQ \\ \sin(x+h) - \sin(x) &= PQ - MN = QR \\ \frac{\sin(x+h) - \sin(x)}{h} &= \frac{QR}{h} \end{aligned}$$

From the diagram, it can be seen that $\angle RQS = x + h$ and the arc QN has length h.

As $h \to 0$, $\angle RQS$ approaches $\angle RQN$, which approaches x. Furthermore, the arc QN approaches the chord QN.

Consequently, $\frac{QR}{h} \to \frac{QR}{QN}$, but by definition, $\frac{QR}{QN} = \cos(x)$.

Hence,

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h} \\ &= \lim_{h\to 0} \frac{QR}{h} \\ &= \cos(x) \end{aligned}$$

The derivative of cosine can also be investigated geometrically or graphically, using the same methods as shown for the sine function. To differentiate trigonometric functions, the angles need to be in radian measure.

4.5.3 Differentiation of the sine and cosine functions

The derivatives for sine and cosine are summarised below.

Derivatives of sin(*x*) and cos(*x*)

$$\frac{d}{dx}\sin(x) = \cos(x)$$

$$\frac{d}{dx}\cos(x) = -\sin(x)$$

Using the chain rule, introduced in Year 11 and studied further in the next chapter, the following results can also be obtained.

Derivatives of sin(*f*(*x*)) and cos(*f*(*x*))

$$\frac{d}{dx}\sin(f(x)) = \cos(f(x)) \times f'(x)$$

$$\frac{d}{dx}\cos(f(x)) = -\sin(f(x)) \times f'(x)$$

Or simply, where $u = f(x)$:

$$\frac{d}{dx}\sin(u) = \cos(u) \times \frac{du}{dx}$$

$$\frac{d}{dx}\cos(u) = -\sin(u) \times \frac{du}{dx}$$

WORKED EXAMPLE 10

Determine the derivative of each of the following functions.

a. $y = 7\sin(5x)$

b. $y = \cos(x^2 + 2x - 3)$

THINK	WRITE
a. 1. State $f(x)$ and $f'(x)$.	$f(x) = 5x$ $f'(x) = 5$
2. Substitute into the formula $\frac{d}{dx}\sin(f(x)) = \cos(f(x)) \times f'(x)$.	$\frac{dy}{dx} = 7\cos(5x) \times 5$
3. Simplify and answer the question.	$\frac{dy}{dx} = 35\cos(5x)$
b. 1. State u and $\frac{du}{dx}$.	$u = x^2 + 2x - 3$ $\frac{du}{dx} = 2x + 2$
2. Substitute into the formula $\frac{d}{dx}\cos(u) = -\sin(u) \times \frac{du}{dx}$.	$\frac{dy}{dx} = -\sin(x^2 + 2x - 3) \times (2x + 2)$
3. Simplify and answer the question.	$\frac{dy}{dx} = -(2x + 2)\sin\left(x^2 + 2x - 3\right)$ $\frac{dy}{dx} = -2(x + 1)\sin\left(x^2 + 2x - 3\right)$

WORKED EXAMPLE 11

Determine the derivative of the function $y = \sin(6x^\circ)$.

THINK	WRITE
1. The function cannot be differentiated as the angle is not measured in radians. Convert the angle to radian measure.	$\sin(6x^\circ) = \sin\left(6 \times \frac{\pi}{180}x\right)$ $= \sin\left(\frac{\pi}{30}x\right)$
2. Differentiate by applying the formula $\frac{d}{dx}\sin f(x) = \cos f(x) \times f'(x)$.	$y = \sin\left(\frac{\pi}{30}x\right)$ $\frac{dy}{dx} = \frac{\pi}{30}\cos\left(\frac{\pi}{30}x\right)$
3. Simplify and answer the question.	$\frac{dy}{dx} = \frac{\pi}{30}\cos\left(\frac{\pi}{30}x\right)$

WORKED EXAMPLE 12

Determine the equation of the tangent to the curve $y = \sin(3x) + 1$ at the point where $x = \frac{\pi}{3}$.

THINK	WRITE
1. First find the coordinates of the point; that is, determine the y-value when $x = \frac{\pi}{3}$.	When $x = \frac{\pi}{3}$, $y = \sin\left(3 \times \frac{\pi}{3}\right) + 1$ $= \sin(\pi) + 1$ $= 0 + 1$ $= 1$ The point is $\left(\frac{\pi}{3}, 1\right)$.
2. Find the derivative of the function.	$\frac{dy}{dx} = 3\cos(3, x)$
3. Determine the gradient at the point where $x = \frac{\pi}{3}$.	$x = \frac{\pi}{3}, \frac{dy}{dx} = 3\cos\left(3 \times \frac{\pi}{3}\right)$ $= 3\cos(\pi)$ $= 3(-1)$ $= -3$
4. Substitute the appropriate values into the rule $y - y_1 = m(x - x_1)$ to find the equation of the tangent.	$m = -3, (x_1, y_1) = \left(\frac{\pi}{3}, 1\right)$ $y - y_1 = m(x - x_1)$ $y - 1 = -3\left(x - \frac{\pi}{3}\right)$
5. Simplify and answer the question.	$y - 1 = -3x + \pi$ $y = -3x + \pi + 1$ The equation of the tangent is $y = 1 + \pi - 3x$.

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Derivatives of the sine and cosine functions Summary screen and practice questions

Exercise 4.5 Derivatives of the sine and cosine functions

Technology free

1. **WE10** Determine the derivative of each of the following functions.
   a. $y = \sin 8x$
   b. $y = \sin(-6x)$
   c. $y = \sin x$
   d. $y = \sin \frac{x}{3}$
   e. $y = \sin\left(-\frac{x}{2}\right)$
   f. $y = \sin \frac{2x}{3}$
2. Differentiate each of the following.
   a. $y = \cos 3x$
   b. $y = \cos(-2x)$
   c. $y = \cos \frac{x}{3}$
   d. $y = \cos 21x$
   e. $y = \cos(-7x)$
   f. $y = \cos \frac{\pi x}{4}$
3. Differentiate each of the following.
   a. $y = \sin(2x + 3)$
   b. $y = \sin(6 - 7x)$
   c. $y = \sin(5x - 4)$
   d. $y = \sin\left(\frac{3x + 2}{4}\right)$
   e. $y = \sin\left(\frac{8 - 7x}{3}\right)$
   f. $y = 5\pi \sin 2\pi x$
4. Differentiate each of the following.
   a. $y = \cos(8 - x)$
   b. $y = \cos(6 - 5x)$
   c. $y = \cos\left(\frac{2x + 3}{3}\right)$
   d. $y = \cos\left(\frac{4x - 1}{5}\right)$
   e. $y = 4\pi \cos 10\pi x$
   f. $y = -6\cos(-2x)$
5. Determine the derivative of each of the following.
   a. $y = \cos(x^2 - 4x + 3)$
   b. $y = \sin(10 - 5x + x^2)$
   c. $y = \sin(e^x)$
   d. $y = \cos(x^2 + 7x)$
   e. $y = \cos(4x - x^2)$
   f. $y = \sin(x^2 + 3x)$
6. **WE11** Determine the derivative of the function $y = 9\cos\left(10x^0\right)$.
7. For each of the following functions, find $\frac{dy}{dx}$.
   a. $y = 2\cos(3x)$
   b. $y = \cos(x°)$
   c. $y = 3\,\cos\left(\frac{\pi}{2} - x\right)$
   d. $y = -4\,\sin\left(\frac{x}{3}\right)$
   e. $y = \sin(12x°)$
   f. $y = 2\,\sin\left(\frac{\pi}{2} + 3x\right)$

Technology active

8. **WE12** Determine the equation of the tangent to the curve $y = -\cos(x)$ at the point where $x = \frac{\pi}{2}$.
9. Determine the equation of the tangent to the curve with equation $y = 3\,\cos(x)$ at the point where $x = \frac{\pi}{6}$.
10. Determine the point on the curve with equation $y = -2\,\sin\left(\frac{x}{2}\right), x \in [0, 2\pi]$ where the gradient is equal to $\frac{1}{2}$.
11. Consider the function $f: [0, 2\pi] \to R, f(x) = \sin(x) - \cos(x)$. Find:
    a. $f(0)$
    b. $\{x: f(x) = 0\}$
    c. $f'(x)$
    d. $\{x: f'(x) = 0\}$

12. Consider the function $f: [-\pi, \pi] \to R, f(x) = \sqrt{3}\cos(x) + \sin(x)$. Find:
a. $f(0)$
b. $\{x: f(x) = 0\}$
c. $f'(x)$
d. $\{x: f'(x) = 0\}$

13. Determine the x-values over the domain $x \in [-\pi,\ \pi]$ for which the gradients of the functions $f(x) = \sin(2x)$ and $f(x) = \cos(2x)$ are equal.

14. For the function $f(x) = x - \sin(2x), -\frac{\pi}{2} \le x \le \frac{\pi}{2}$, determine the point(s) where the gradient is 0. Give your answer correct to 3 decimal places.

15. For the function $f(x) = 2x + \cos(3x),\ 0 \le x \le \frac{\pi}{2}$, determine the point(s) where the gradient is 0. Give your answer correct to 3 decimal places.

4.6 Applications of trigonometric functions

The trigonometric functions are used to model many practical situations that include periodic behaviour or cyclic phenomena. Examples of these include the rise and fall of the tides, temperature fluctuations, and the heights of moving objects about a fixed point.

In solving practical problems:

- read the question carefully
- state clearly the model being considered
- sketch clear graphs if required
- note any restrictions, particularly on the domain, that apply to the problem
- differentiate the function to determine the rate of change or slope of the curve
- consider the units of measurement
- answer the question.

WORKED EXAMPLE 13

While out in his trawler, John North, a fisherman, notes that the height of the tide in the harbour can be found by using the equation

$$h = 5 + 2\cos\frac{\pi}{6}t$$

where h metres is the height of the tide and t is the number of hours after midnight.

a. What is the height of the high tide and when does it occur in the first 24 hours?
b. What is the difference in height between high and low tides?
c. Sketch the graph of h for $0 \le t \le 24$.
d. John North knows that his trawler needs a depth of 6 metres to enter the harbour. Between what hours is he able to bring his boat back into the harbour?

THINK	WRITE
a. 1. Write the given equation.	a. $h = 5 + 2\cos\frac{\pi}{6}t$
2. For high tide, find the maximum value of h.	For maximum h, $\cos\frac{\pi}{6}t = 1$ So $h = 5 + 2 \times 1$ $= 7$

3. Find when high tide occurs.

$\frac{\pi}{6}t = 0, 2\pi, 4\pi, ...$

$t = 0, 12, 24, ...$

A high tide of height 7 m occurs at midnight, noon the next day, and midnight the next night.

b. 1. Find the minimum value of h.

b. For minimum h,

$\cos\frac{\pi}{6}t = 1$

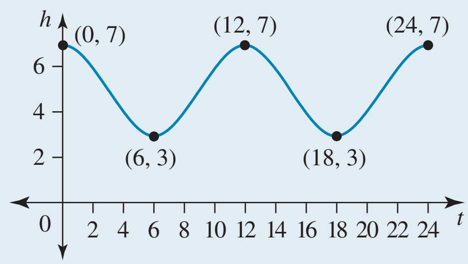
So $h = 5 + 2 \times -1$
$= 3$

2. Find the difference between high and low tides.

The difference between high and low tides is $7 - 3 = 4$ metres.

c. Use the information from parts **a** and **b** to sketch the graph.

c.



d. 1. Find t using the equation when $h = 6$.

d. When $h = 6$,

$5 + 2\cos\frac{\pi}{6}t = 6$

$2\cos\frac{\pi}{6}t = 1$

$\cos\frac{\pi}{6}t = \frac{1}{2}$

$\frac{\pi}{6}t = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}, ...$

$t = 2, 10, 14, 22 ...$

2. Write the answer in words.

From the graph we can see that John North can bring his boat back into harbour before 2 am, between 10 am and 2 pm, and between 10 pm and 2 am the next morning.

WORKED EXAMPLE 14

The temperature on a particular day can be modelled by the function

$$T(t) = -3\cos\left(\frac{\pi t}{9}\right) + 18, 0 \le t \le 18$$

where t is the time in hours after 5:00 am and T is the temperature in degrees Celsius. For the remaining 6 hours of the 24-hour period, the temperature remains constant.

a. **Calculate the temperature at 8:00 am.**

b. **At what time(s) of the day is the temperature 20 °C? Give your answer correct to the nearest minute.**

c. **Find $\frac{dT}{dt}$.**

d. **What is the rate of change of temperature at the time(s) found in part b, correct to 2 decimal places?**

THINK

WRITE

a. At 8:00 am $t = 3$. Substitute this value into the equation.

a.
$$T(3) = -3\cos\left(\frac{3\pi}{9}\right) + 18$$
$$= -3\cos\left(\frac{\pi}{3}\right) + 18$$
$$= -3 \times \frac{1}{2} + 18$$
$$= -1.5 + 18$$
$$= 16.5°\text{C}$$

b. 1. Substitute $T = 20$ into the equation.

b.
$$20 = -3\cos\left(\frac{\pi t}{9}\right) + 18$$

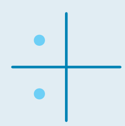
2. Solve the equation for $0 \le t \le 18$.

$$\cos\left(\frac{\pi t}{9}\right) = -\frac{2}{3}$$

3. Interpret your answers and convert the t-values to times of the day.

$\frac{2}{3}$ suggests 0.841 069, and cosine is negative in quadrants 2 and 3.

4. Write the answer.



$$\frac{\pi t}{9} = \pi - 0.841069, \pi + 0.841069$$
$t = 6.5905, 11.4095$ after 5 am

The temperature is 20 °C at 11:35 am and 4:25 pm.

c. Determine $\frac{dT}{dt}$.

c.
$$\frac{dT}{dt} = -3 \times \frac{\pi}{9}\left(-\sin\left(\frac{\pi t}{9}\right)\right)$$
$$= \frac{\pi}{3}\sin\left(\frac{\pi t}{9}\right)$$

d. 1. Substitute $t = 6.6$ (11:36 am) and $t = 11.4$ (4:24 pm) into $\frac{dT}{dt}$.

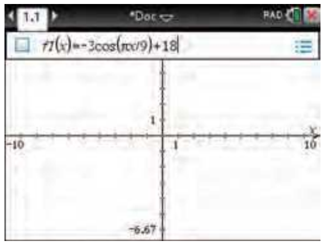
d. When $t = 6.6$ (11:36 am),
$$\frac{dT}{dt} = \frac{\pi}{3}\sin\left(\frac{6.6 \times \pi}{9}\right)$$
$$= 0.78$$
When $t = 11.4$(4.24 am),
$$\frac{dT}{dt} = \frac{\pi}{3}\sin\left(\frac{11.4 \times \pi}{9}\right)$$
$$= -0.78$$

2. Write the answer.

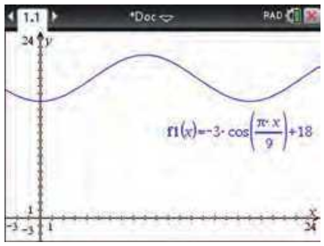
At 11:36 am the temperature is increasing at a rate of 0.78 °C/h.
At 4:24 pm the temperature is decreasing at a rate of 0.78 °C/h.

TI | THINK | **WRITE**

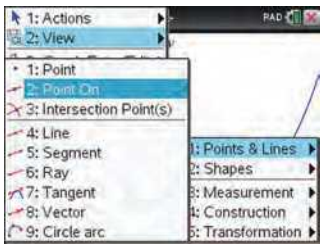
b. 1. On a Calculator page, press MENU, then select:
2: Add Graphs.
Complete the entry line in the $f1\,(x) =$ tab as:
$-3\cos\left(\dfrac{\pi x}{9}\right) + 18$
Note: The independent variable t has been replaced with x.



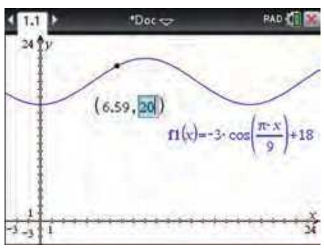
2. Sketch the graph by pressing the ENTER button.



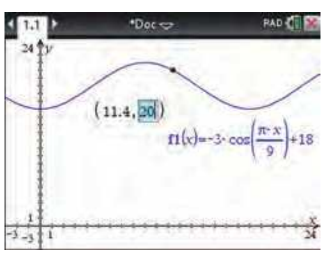
3. To calculate the x-value(s) for when $y = 20$, select:
Menu
8: Geometry
1: Points & Lines
2: Point On.



4. Move the cursor and select the curve representing
$f1\,(x) = -3\cos\left(\dfrac{\pi x}{9}\right) + 18$
Press the ESC (escape) button. This allows you to move the textbox indicating the coordinates of the point P(x, y).
Complete the entry line in the textbox as 20 for the y-value. Press the ENTER button to perform the calculation.

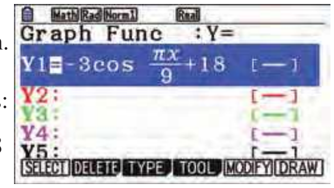


5. Determine the next solution by moving the cursor point on the line to a position that is close to the desired solution.
Complete the entry line in the textbox as 20 for the y-value.
Press the ENTER button to perform the calculation.
The second solution exists at $x = 11.4$.

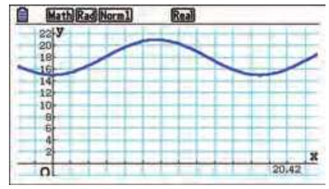


CASIO | THINK | **WRITE**

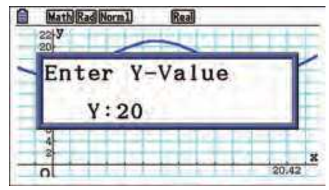
b. 1. On a Main Menu screen, select Graph. Complete the entry line in the Y1 tab as:
$-3\cos\left(\dfrac{\pi x}{9}\right) + 18$
Note: The independent variable t has been replaced with x.



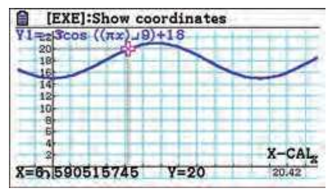
2. Sketch the graph by pressing either the DRAW or EXE button.



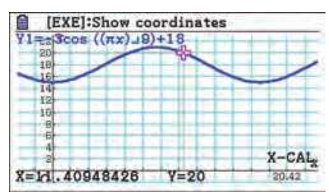
3. To calculate the x-value(s) for when $y = 20$, select:
G-Solv (SHIFT F5)
X-CAL.
Complete the entry line in Y: as 20.
Press the EXE button to perform the calculation.



4. The answer appears on the screen.
The first solution exists at $x = 6.6$.
Note: The family of solutions can be calculated by pressing the directional cursor button either left or right.



5. Determine the next solution by pressing the directional cursor button to the right.
The second solution exists at $x = 11.4$.



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Applications of trigonometric functions Summary screen and practice questions

Exercise 4.6 Applications of trigonometric functions

Technology free

1. **WE13** Fred Greenseas and John North are competing to catch the most fish. Fred Greenseas decides to fish in an inlet several kilometres east of the place where John North fishes. There is a sandbar at the entrance to the inlet, and the depth of water in metres on the sandbar is modelled by the function $d(t) = 6 + 2.5 \sin \frac{\pi t}{6}$, where t is the number of hours after 12 noon.

   a. What is the greatest depth of the water on the sandbar and when does it first occur?
   b. How many hours pass before there is once again the maximum depth of water on the sandbar?
   c. What is the least amount of water on the sandbar?
   d. Sketch the graph of d for $0 \leq t \leq 24$.
   e. Fred Greenseas needs a depth of 7.25 metres to cross the sandbar. Between what hours is he able to enter and leave the inlet?

2. A student wanting to catch fish to sell at a local market on Sunday has discovered that more fish are in the water at the end of the pier when the depth of water is greater than 8.5 metres. The depth of the water (in metres) is given by $d = 7 + 3 \sin \frac{\pi}{6} t$, where t hours is the number of hours after midnight on Friday.

   a. What is the maximum and minimum depth of the water at the end of the pier?
   b. Sketch a graph of d against t from midnight on Friday until midday on Sunday.
   c. When does the water first reach maximum depth?
   d. Between what hours should the student be on the pier in order to catch the most fish?
   e. If the student can fish for only 2 hours at a time, when should she fish in order to sell the freshest fish at the market from 10:00 am on Sunday morning?

3. The mean daily maximum temperature in Tarabon, an experimental town in a glass dome, is modelled by the function $T(m) = 18 + 7 \cos \frac{\pi}{6} m$, where T is in degrees Celsius and m is the number of months after 1 January 2017.
   a. What was the mean daily maximum temperature in March 2017 and August 2017?
   b. What is the highest mean daily maximum temperature in Tarabon? In which months does it occur?
   c. What would the mean daily maximum temperature be in February 2018?
   d. If the pattern continued, how many months would pass before the mean daily maximum temperature would be the same again as it was in February 2018?

4. The height above the ground of the middle of a skipping rope as it is being turned in a child's game is found by using the equation $h = a\sin(nt) + c$, where t is the number of seconds after the rope has begun to turn. During the game, the maximum height the rope reaches is 1.8 metres, and it takes 2 seconds for the rope to complete a full turn.
   a. Find the values of a, n and c, and hence write the equation of h in terms of t.
   b. Sketch the graph of h against t for $0 \le t \le 5$.
   c. After how much time will the rope be 25 cm above the ground? Give your answer correct to the nearest tenth of a second.
5. The population of a colony of frogs rises and falls according to the breeding season. The population can be modelled by the equation $P(t) = 100\sin\frac{\pi t}{2} + 500$, where t is the number of months since the beginning of the year.
   a. Determine the population at the beginning of the year.
   b. Sketch the graph to represent this population of frogs for the year.
   c. Determine the first time at which the population is greatest.

6. **WE14** A mass oscillates up and down at the end of a metal spring. The length of the spring, L cm, after time t seconds is modelled by the function $L(t) = 2\sin(\pi t) + 10$ for $t \ge 0$.
   a. What is the length of the spring when the mass is not oscillating, that is, when it is at the mean position, P?
   b. Find $\frac{dL}{dt}$.
   c. Find the exact value of $\frac{dL}{dt}$ after 1 second.

Technology active

7. The temperature in °C on a particular winter's day in an inland town can be modelled by the function

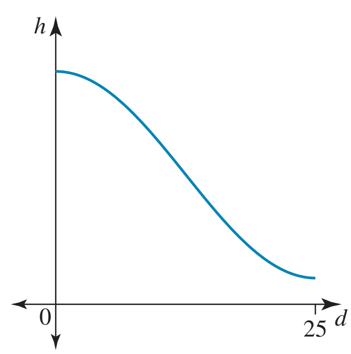
$$T = 2\sin\left(\frac{\pi t}{9}\right) + 12, 0 \le t \le 24$$

   where t is the time in hours after 8:00 am.
   a. Calculate the temperature, correct to the nearest degree, at 12 noon.
   b. Find $\frac{dT}{dt}$.
   c. What is the rate of change of the temperature at midnight? Give your answer correct to 3 decimal places.
8. A section of a rollercoaster track at a local fun park is shown.

   The track can be described by the rule

$$h = 4\cos\left(\frac{\pi d}{25}\right) + 5, 0 \le d \le 25$$

   where h is the height in metres above ground level and d is the horizontal distance in metres from the top of the descent. Note that the d-axis represents the ground.



a. How high is a rollercoaster car from the ground at the beginning of its descent?

b. Find $\dfrac{dh}{dd}$.

c. What is the gradient of the rollercoaster track, correct to 3 decimal places, when:

i. $d = 5$ m? **ii.** $d = 15$ m?

9. Between 6 am and 6 pm on a given day the height, H metres, of the tide in a harbour is given by

$$H(t) = 1.5 + 0.5\sin\left(\frac{\pi t}{6}\right), 0 \le t \le 12.$$

a. What is the period of the function?

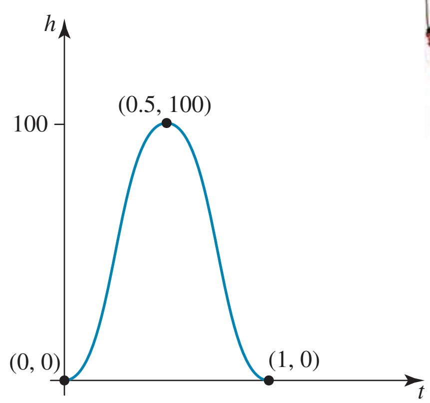
b. What is the value of H at low tide and when does low tide occur?

c. Find $\dfrac{dH}{dt}$.

d. Find the exact value of $\dfrac{dH}{dt}$ at 7:30 am.

e. Find the second time during the given time interval that $\dfrac{dH}{dt}$ equals the value found in part **d**.

10. A young girl is learning to skip. The graph showing this skipping for one cycle is given.



The general equation for this graph is given by $h = a\ \cos(nt) + c$, where h is the height in millimetres of the girl's feet above the ground and t is the time in seconds the girl has been skipping.

a. Find the values of the constants a, n and c, and hence restate the equation for one cycle of the skipping.

b. Find $\dfrac{dh}{dt}$.

c. What is the value of $\dfrac{dh}{dt}$ when $t = 0.25$ seconds?

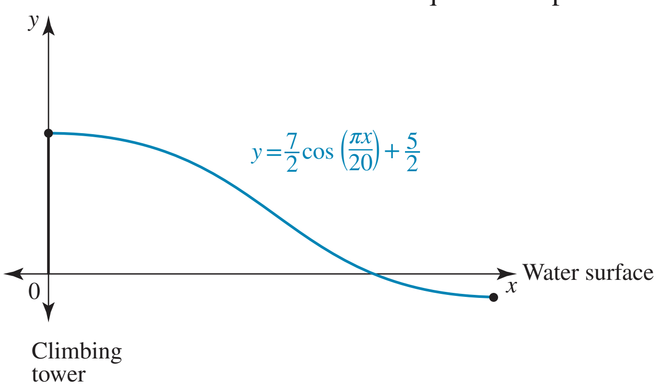
11. The height, h metres, above ground level of a chair on a rotating Ferris wheel is modelled by the function

$$h = 5 - 3.5\ \cos\left(\frac{\pi t}{30}\right)$$

where t is measured in seconds.

a. People can only enter a chair when it is at its lowest position, at the bottom of the rotation. They enter the chair from a platform. How high is the platform above ground level?
b. What is the highest point reached by the chair?
c. How long does 1 rotation of the wheel take?
d. During a rotation, for how long is a chair higher than 7 m off the ground? Give your answer to 3 decimal place.
e. Find $\frac{dh}{dt}$.
f. Find the first two times, correct to 2 decimal places, when a chair is descending at a rate of 0.2 m/s.

12. A section of a water slide at a local aquatic complex is shown.



The water slide can be defined by the rule

$$y = \frac{7}{2}\cos\left(\frac{\pi x}{20}\right) + \frac{5}{2}, 0 \leq x \leq 20$$

where y is the height in metres of the water slide above the water surface and x is the horizontal distance in metres between the start of the slide and the end of the slide. (*Note*: The x-axis represents the water surface.)

a. How high must a person climb in order to reach the top of the water slide?
b. Find $\frac{dy}{dx}$.
c. What is the exact gradient of the water slide:
   i. when $x = 5$?
   ii. when $x = 10$?
d. i. How far from the climbing tower, to the nearest whole metre, does the slide come into contact with the water surface?
   ii. What obtuse angle does the slide make with the water surface at this point? Give your answer correct to 2 decimal places.

13. The depth of water in an inlet has been monitored over a 24-hour period from 4 am on Monday. It was observed that the depth, D metres, was modelled by the function

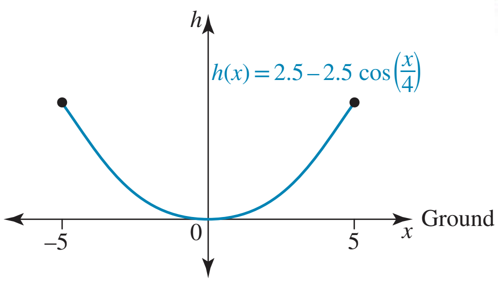
$$D(t) = 2.5 + 0.5\sin\left(\frac{\pi t}{3}\right), 0 \leq t \leq 24$$

where t is time in hours.

a. Determine the depth of the water at 4 am on Monday.
b. Calculate the depth of the water at midday on Monday, correct to 2 decimal places.
c. What was the maximum depth of the water during the 24-hour period and when did this first occur?
d. Sketch the function, $D(t)$.
e. Calculate the rate of change of the depth of water, $D'(t)$, at any time, and sketch the function $y = D'(t)$.
f. Determine when during the 24-hour period the flow of water into the inlet was the greatest.

14. a. Sketch the graph $y = 2.5 - 2.5\cos\left(\frac{x}{4}\right), -4\pi \le x \le 4\pi.$

b. At a skateboard park, a new skateboard ramp has been constructed. A cross-section of the ramp is shown.



The equation that approximately defines this curve is given by

$$h(x) = 2.5 - 2.5\cos\left(\frac{x}{4}\right), \quad -5 \le x \le 5$$

where h is the height in metres above the ground level and x is the horizontal distance in metres from the lowest point of the ramp to each end of the ramp.

i. Determine the maximum depth of the skateboard ramp, giving your answer correct to 1 decimal place.

ii. Find $\frac{dh}{dt}$.

iii. Calculate the gradient of the ramp when it is 3 metres from its lowest point.

iv. Where is the gradient of the ramp equal to 0.58?

15. An industrial process is known to cause the production of two separate toxic gases that are released into the atmosphere. At a factory where this industrial process occurs, the technicians work a 12-hour day from 6:00 am until 6:00 pm.

The emission of the toxic gas X can be modelled by the rule

$$x(t) = 1.5\sin\left(\frac{\pi t}{3}\right) + 1.5, 0 \le t \le 12$$

and the emission of the toxic gas Y can be modelled by the rule

$$y(t) = 2.0 - 2.0\cos\left(\frac{\pi t}{3}\right), 0 \le t \le 12.$$

a. Use technology to sketch, on the same set of axes, the graphs that represent these two toxic gas emissions.

b. From the graph, determine at what time of the day the emissions are the same for the first time. Give your answer to the nearest minute.

c. Calculate, to 2 decimal places, the amount of each gas emitted at the time found in part b.

d. The Environment Protection Authority (EPA) has strict rules about the emissions of toxic gases. The total emission of toxic gases for this particular industrial process is given by

$$T(t) = x(t) + y(t).$$

i. Use technology to sketch the graph of the function $T(t)$.
ii. Determine the maximum and minimum emissions in a 12-hour working day and the times at which these occur.
iii. If the EPA rules state that all toxic emissions from any one company must lie within the range of 0 to 7 units at any one time, indicate whether this company works within the guidelines.

4.7 Review: exam practice

A summary of this chapter is available in the Resources section of your eBookPLUS at www.jacplus.com.au.

Simple familiar

1. Convert the following angles to degrees, giving your answers to the nearest minute where necessary.
   a. $\frac{3\pi}{4}$ b. $\frac{13\pi}{12}$ c. 2.1 d. 1.76
2. Convert the following angles to radians, giving your answers correct to 2 decimal places where necessary.
   a. $35°$ b. $280°$ c. $128.5°$ d. $230°48'$
3. Evaluate the following, given that $\cos(\theta) = \frac{2}{3}$ and $0 \leq \theta \leq \frac{\pi}{2}$.
   a. $\cos(\pi - \theta)$ b. $\sin(\pi - \theta)$ c. $\tan(\pi + \theta)$ d. $\sin(3\pi + \theta)$
   e. $\tan(\pi - \theta)$ f. $\cos(-\theta)$
4. State the exact values of the following.
   a. $\sin(120°)$ b. $\cos(135°)$ c. $\tan(330°)$ d. $\cos(225°)$
   e. $\sin(210°)$ f. $\tan(150°)$
5. State the exact values of the following.
   a. $\sin\left(\frac{3\pi}{4}\right)$ b. $\cos\left(\frac{5\pi}{6}\right)$ c. $\tan\left(\frac{2\pi}{3}\right)$ d. $\cos\left(\frac{4\pi}{3}\right)$
   e. $\sin\left(\frac{5\pi}{4}\right)$ f. $\tan\left(\frac{7\pi}{6}\right)$ g. $\sin\left(\frac{11\pi}{6}\right)$ h. $\cos\left(\frac{5\pi}{3}\right)$
   i. $\tan\left(\frac{7\pi}{4}\right)$ j. $\cos\left(\frac{9\pi}{4}\right)$ k. $\sin\left(\frac{13\pi}{6}\right)$ l. $\tan\left(\frac{7\pi}{6}\right)$
6. State the exact values of the following.
   a. $\tan\left(-\frac{\pi}{4}\right)$ b. $\cos\left(-\frac{3\pi}{4}\right)$ c. $\sin\left(-\frac{2\pi}{3}\right)$ d. $\tan\left(-\frac{5\pi}{6}\right)$
   e. $\sin\left(-\frac{7\pi}{6}\right)$ f. $\cos\left(-\frac{5\pi}{4}\right)$
7. Find all solutions to the following equations in the domain $0 \leq \theta \leq 2\pi$.
   a. $\cos(\theta) = 0$ b. $\sin(\theta) = -\frac{1}{\sqrt{2}}$ c. $\cos(\theta) = \frac{1}{\sqrt{2}}$
   d. $\sin(\theta) = -1$ e. $\cos(\theta) = -\frac{\sqrt{3}}{2}$

8. Determine all the values of θ between 0° and 360° for which:

a. $\sin(\theta) = 1$ b. $\cos(\theta) = \frac{1}{2}$ c. $\sin(\theta) = \frac{\sqrt{3}}{2}$

d. $\cos(\theta) = -1$ e. $\sin(\theta) = \frac{1}{\sqrt{2}}$

9. Determine all the solutions to the following equations in the domain $0 \le x \le 2\pi$.

a. $2\sin(x) = 1$ b. $3\cos(x) = 0$ c. $2\sin(x) = -\sqrt{3}$

d. $\sqrt{2}\cos(x) = 1$ e. $\sqrt{3}\tan(x) + 1 = 0$

10. Solve the following equations for $-\pi \le x \le \pi$, giving your answers in exact form.

a. $4\sin(x) + 2 = 6$ b. $3\cos(x) - 3 = 0$ c. $2\sin(3x) - 5 = -4$

d. $\sqrt{2}\cos(3x) + 2 = 3$ e. $2\cos(2x) + \sqrt{3} = 0$

11. Determine all values between 0 and 2π for the following equations. Give exact answers for parts a–d; give answers correct to 4 decimal places for parts e and f.

a. $\sin(x) = \cos(x)$ b. $\sin(2x) = \cos(2x)$ c. $\sin(2x) = \sqrt{3}\cos(2x)$

d. $\sqrt{3}\sin(3x) = \cos(3x)$ e. $\sin(3x) + 2\cos(3x) = 0$ f. $\sin(x) + 3\cos(x) = 0$

12. Solve the following for α where $0 \le \alpha \le 2\pi$. Give answers in exact form.

a. $\sin^2(2\alpha) + \sin(2\alpha) - 2 = 0$

b. $2\cos^2(3\alpha) + \cos(3\alpha) - 1 = 0$

c. $2\sin^2\left(\alpha - \frac{\pi}{2}\right) = \sin\left(\alpha - \frac{\pi}{2}\right)$

Complex familiar

13. Sketch the graph of the function $y = 2\sin(2x)$ for $-\pi \le x \le \pi$. State the amplitude, period and range.

14. Sketch the graphs of the following for $0 \le x \le 2\pi$. State the period, amplitude and range, and the coordinates of the endpoints. Calculate any axis intercepts, giving your answers in exact form where applicable.

a. $y = 2\sin(2x + \pi)$ b. $y = 3\cos(3x + \pi)$

c. $y = 2\sin\left(x - \frac{\pi}{4}\right) - 1$ d. $y = \cos\left(\frac{1}{2}(x - \pi)\right) + 1$

15. Determine $\frac{dy}{dx}$ for the following functions.

a. $y = \cos(8x - 3)$ b. $y = 4 - 3\sin(2x + 1)$

c. $y = 6\sin(2x) + 3\cos(2x)$ d. $y = \cos\left(x^2 + 2x + 1\right)$

e. $y = 2\sin(4 - 3x)$ f. $y = \sin(-x) - \cos(2x)$

16. Determine the equation of the tangent and the line perpendicular to the tangent to the curve $y = 3\cos(x)$ at the point where $x = \pi$.

Complex unfamiliar

17. The depth of water, D metres, at an inlet can be modelled using the equation $D = 14 + 5\sin\frac{4\pi t}{13}$, where t is the hours since high tide at midnight on January 1.

a. What is the maximum depth of water in the inlet?

b. What is the minimum depth of water in the inlet?

c. What is the period of the function?

d. What is the amplitude of the function?

e. Sketch a graph of D against t for two cycles.

18. During one day in October, the temperature, T°C, is given by $T = 19 - 3\sin\left(\frac{\pi}{12}t\right)$, where t is the time in hours after midnight.

a. What was the temperature at midnight?
b. What was the maximum temperature during the day and at what time did it occur?
c. Over what interval did the temperature vary that day?
d. State the period and sketch the graph of the temperature for $t \in [0, 24]$.
e. If the temperature was below k degrees for 3 hours, find the value of k, correct to 1 decimal place.
f. Determine the fastest rate the temperature is rising, correct to 3 decimal places, and when this rate occurs.

19. The height, h metres, of the tide above mean sea level is given by $h = 4\sin\left(\frac{\pi(t-2)}{6}\right)$, where t is the time in hours since midnight.

a. How far below mean sea level was the tide at 1 am?
b. State the high tide level and show that this first occurs at 5 am.
c. How many hours are there between high tide and the following low tide?
d. Sketch the graph of h versus t for $t \in [0, 12]$.
e. What is the height of the tide predicted to be at 2 pm?
f. How much higher than low tide level is the tide at 11:30 am? Give the answer to 2 decimal places.

20. During a particular day in a Mediterranean city, the temperature inside an office building between 10 am and 7:30 pm fluctuates so that t hours after 10 am the temperature, T °C, is given by

$$T = 19 + 6\sin\left(\frac{\pi t}{6}\right).$$

a. i. State the maximum temperature and the time it occurs.
   ii. State the minimum temperature and the time it occurs.
b. i. What is the temperature in the building at 11:30 am? Answer to 1 decimal place.
   ii. What is the temperature in the building at 7:30 pm? Answer to 1 decimal place.
c. Sketch the graph of the temperature against time from 10:00 am to 7:30 pm.
d. When the temperature reaches 24 °C, an air conditioner in the boardroom is switched on. It is switched off when the temperature in the rest of the building falls below 24 °C. For how long is the air conditioner on in the boardroom?
e. The office workers who work the shift between 11:30 am and 7:30 pm complain that the temperature becomes too cool towards the end of their shift. If management agrees that heating can be used for the coldest two-hour period of their shift, at what time and at what temperature would the heating be switched on? Express the temperature in both exact form and to 1 decimal place.

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Units 3 & 4 Sit exam

Answers

4 Calculus of trigonometric functions

Exercise 4.2 Review of the unit circle, symmetry and exact values

1. a. $286.48°$ b. $275.02°$ c. $146.68°$
   d. $54°$ e. $150°$ f. $225°$
2. a. $\frac{\pi}{12}$ b. $\frac{2\pi}{3}$ c. $\frac{13\pi}{18}$
   d. 1.12 e. 1.38 f. $\frac{31\pi}{18}$
3. a. $\frac{5}{13}$ b. $\frac{-12}{13}$ c. $\frac{-5}{12}$
   d. $\frac{-5}{13}$ e. $\frac{12}{13}$ f. $\frac{-5}{12}$
4. a. $\frac{-7}{10}$ b. $\frac{\sqrt{51}}{10}$ c. $\frac{-\sqrt{51}}{7}$
   d. $\frac{-7}{10}$ e. $\frac{\sqrt{51}}{7}$ f. $\frac{7}{10}$
5. a. -1 b. $-\frac{\sqrt{3}}{2}$ c. $-\frac{1}{\sqrt{2}}$
   d. $\frac{1}{2}$ e. $-\sqrt{3}$ f. $-\frac{1}{2}$
6. a. $-\frac{1}{\sqrt{3}}$ b. $-\frac{1}{2}$ c. -1
   d. $-\frac{1}{\sqrt{2}}$ e. $-\frac{\sqrt{3}}{2}$ f. $\frac{1}{2}$
7. a. $\sin(\theta)$ b. $\cos(\theta)$ c. $\tan(\theta)$
   d. $\cos(\theta)$ e. $-\sin(\theta)$ f. $-\tan(\theta)$
8. a. 0 b. Undefined c. 0
   d. 0 e. 1 f. -1
9. a. $\frac{5}{13}$ b. $\frac{5}{12}$ c. $\frac{12}{13}$
   d. $\frac{12}{13}$ e. $\frac{5}{13}$ f. $\frac{12}{5}$
10. a, b. Sample responses can be found in the worked solutions in the online resources.
    c. Since the ratios are squared, there is no need to consider the quadrant for the angle.
11. $\frac{3}{2}$
12. a. $\frac{-\left(1+\sqrt{3}\right)}{2}$ b. $2-\sqrt{2}$ c. $2\sqrt{3}$
    d. $\frac{3+2\sqrt{2}}{4}$ e. 0 f. -2
13. a. 12 cm/s b. $\frac{24-3\sqrt{3}}{2}$ cm/s c. 12 cm/s
14. a. 1.5 m b. 0.75 m c. 0.75 m

Exercise 4.3 Review of solving trigonometric equations with and without the use of technology

1. a. $\frac{5\pi}{6}, \frac{7\pi}{6}$
   b. $120°, 300°, 480°, 660°$
   c. $\frac{-\pi}{3}, \frac{\pi}{3}$
2. a. $210°, 330°$
   b. $\frac{-3\pi}{2}, \frac{\pi}{2}$
3. a. $\frac{\pi}{12}, \frac{7\pi}{12}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{17\pi}{12}, \frac{23\pi}{12}$
   b. $\frac{-5\pi}{6}, \frac{-2\pi}{3}, \frac{\pi}{6}, \frac{\pi}{3}$
4. $\frac{4\pi}{9}, \frac{5\pi}{9}, \frac{10\pi}{9}, \frac{11\pi}{9}, \frac{16\pi}{9}, \frac{17\pi}{9}$
5. $\frac{\pi}{4}, \frac{\pi}{2}, \frac{5\pi}{4}, \frac{3\pi}{2}$
6. $\frac{2\pi}{3}, \pi, \frac{4\pi}{3}$
7. a. $\frac{5\pi}{4}, \frac{7\pi}{4}$
   b. $\frac{\pi}{3}, \frac{5\pi}{3}$
   c. $\frac{\pi}{9}, \frac{4\pi}{9}, \frac{7\pi}{9}, \frac{10\pi}{9}, \frac{13\pi}{9}, \frac{16\pi}{9}$
   d. $\frac{\pi}{4}, \frac{5\pi}{4}$
8. a. $120°, 240°$
   b. $112.5°, 157.5°, 292.5°, 337.5°$
9. a. $0.73, 2.41$
   b. $73.40°, 286.60°$
10. a. $\frac{-\pi}{3}, \frac{-\pi}{6}, \frac{2\pi}{3}, \frac{5\pi}{6}$
    b. $\frac{-3\pi}{4}, \frac{-7\pi}{12}, \frac{-\pi}{12}, \frac{\pi}{12}, \frac{7\pi}{12}, \frac{3\pi}{4}$
    c. $\frac{-5\pi}{8}, \frac{-\pi}{8}, \frac{3\pi}{8}, \frac{7\pi}{8}$
11. a. $-\pi, \frac{-3\pi}{4}, 0, \frac{\pi}{4}, \pi$
    b. $\frac{-5\pi}{6}, \frac{5\pi}{6}$
    c. $\frac{-\pi}{4}, \frac{3\pi}{4}$
12. a. $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$ b. $\frac{\pi}{6}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{5\pi}{6}$
13. a. $-\pi, 0, \pi$ b. $\frac{-5\pi}{6}, \frac{-\pi}{3}, \frac{\pi}{6}, \frac{2\pi}{3}$
    c. $\frac{-3\pi}{4}, \frac{-\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}$ d. $\frac{-2\pi}{3}, \frac{-\pi}{3}, \frac{\pi}{3}, \frac{2\pi}{3}$
14. a. 3 m b. 1.99 s

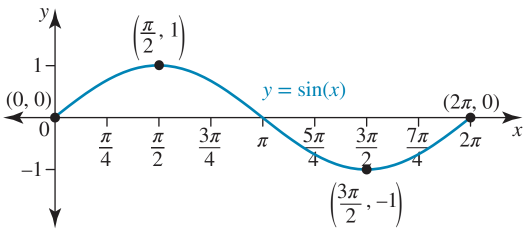
15. a. *See the figure at the bottom of the page.

b. $x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2} \approx 0.52, 2.62, 4.71$

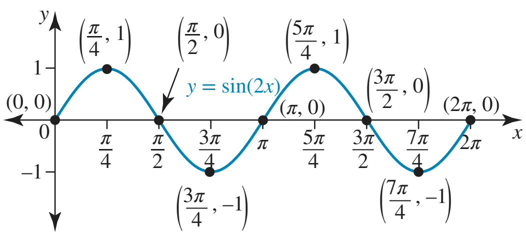
c. The trigonometric ratios involved different angles, so they could not be combined to solve.

Exercise 4.4 Review of graphs of trigonometric functions of the form $y = A\sin(B(x + C)) + D$ and $y = A\cos(B(x + C)) + D$

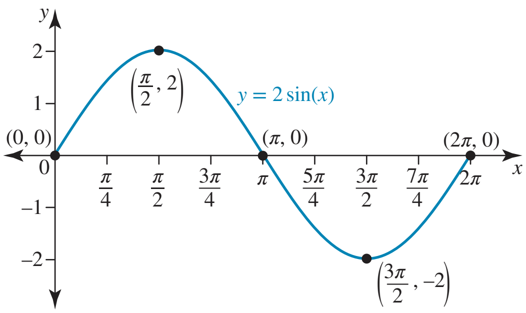
1. a.



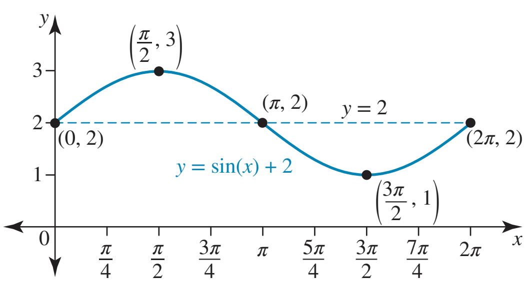
b.



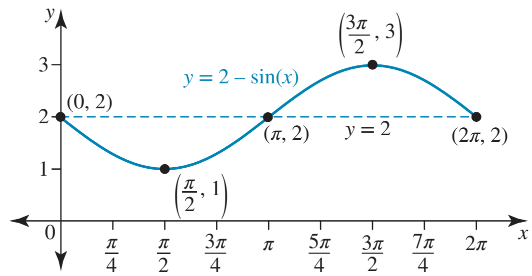
c.



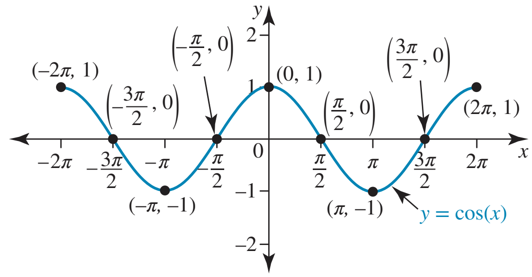
d.



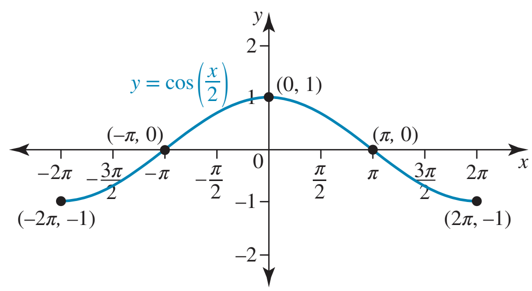
e.



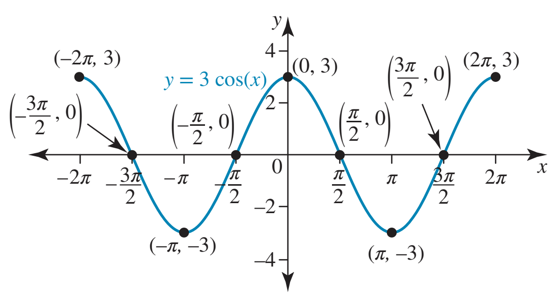
2. a.



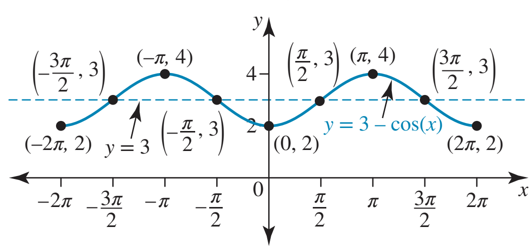
b.



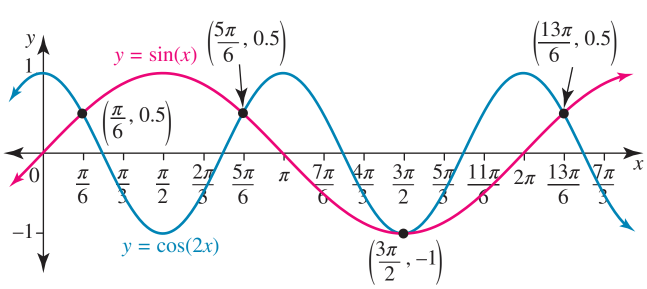
c.



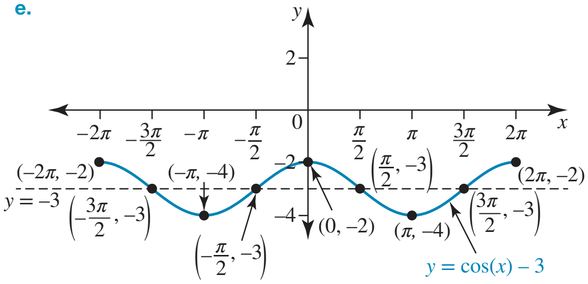
d.



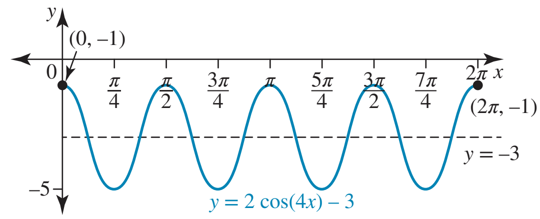
*15. a.



e.

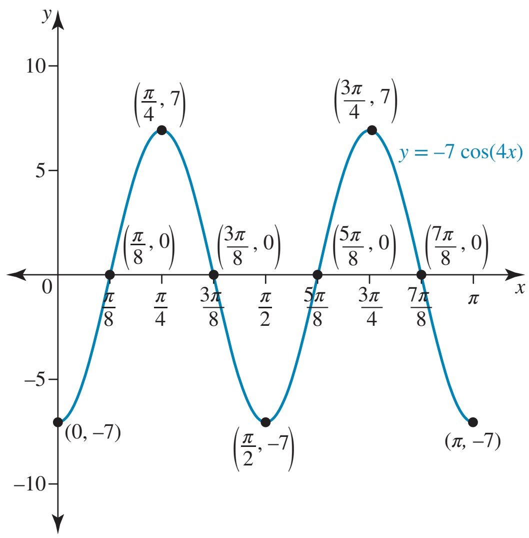


3.

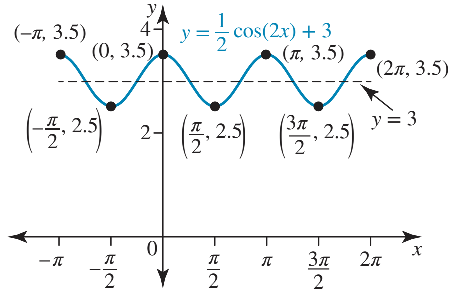


4. *See the figure at the bottom of the page.

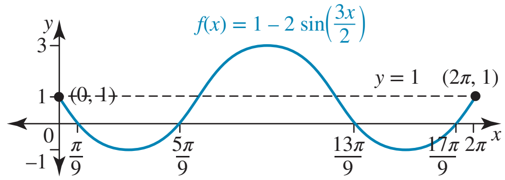
5.



6.



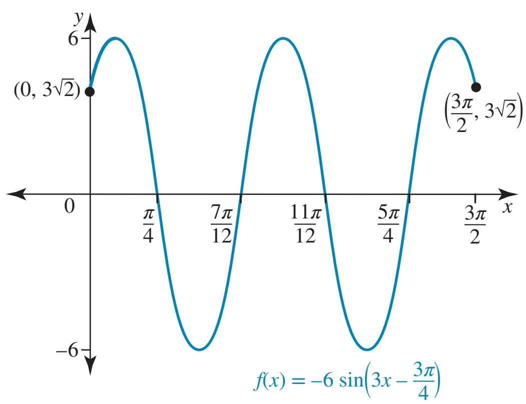
7.



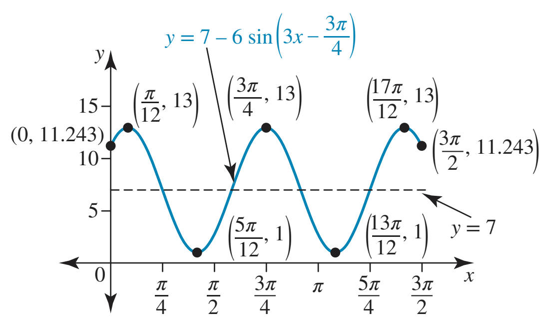
8. Mean position: $y = 5$; amplitude: 8; period: 2; $y = -8\sin(\pi x) + 5$

9. Line of oscillation (mean position): $y = 1$; amplitude: 3; period: π; $y = 3\cos(2x) + 1$

10. a.



b.



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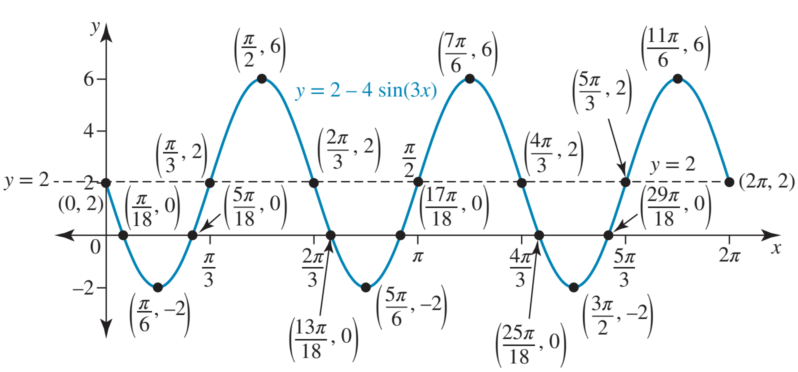
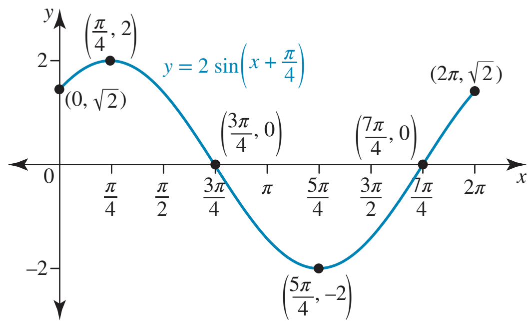
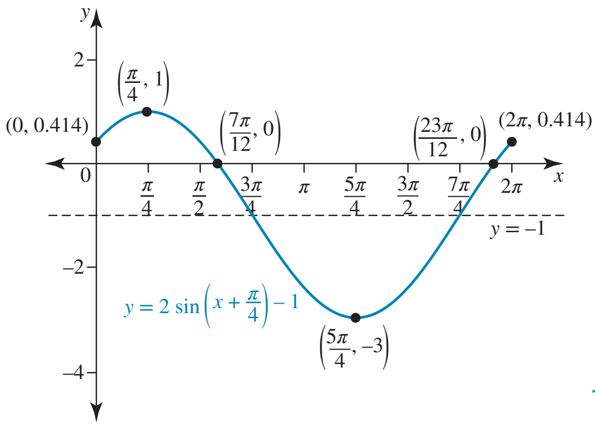


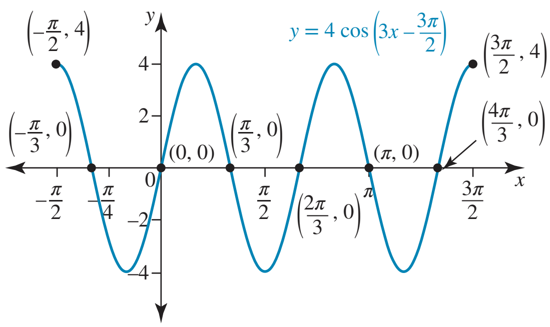
11. a.



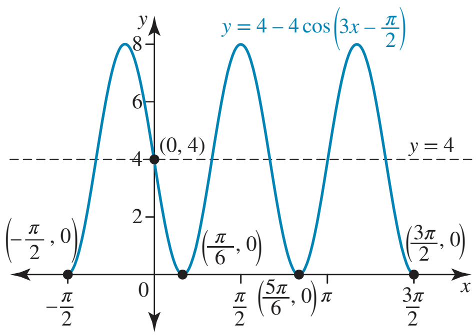
b.



12. a.

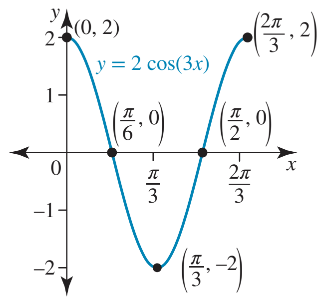


b.

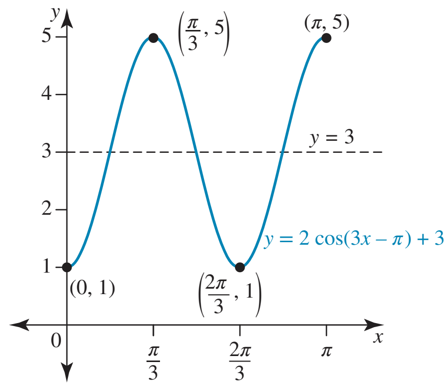


c. The curve $g(x)$ is $f(x)$ reflected in the x-axis (or inverted) and translated vertically up by 4 unit, oscillating around $y = 4$. Neither the period nor the amplitude have changed.

13. a.



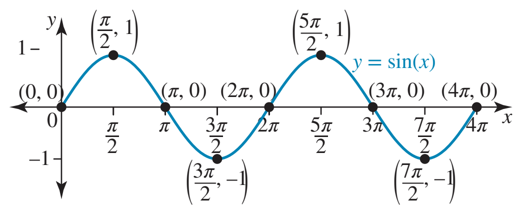
b.



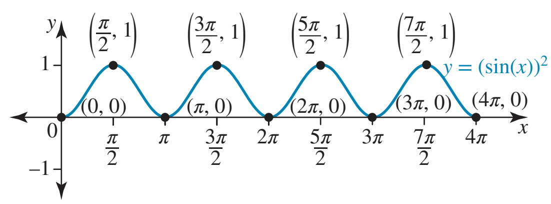
c. $y = 2\cos(3x - \pi) + 3$

14. Maximum: 5; $x = \frac{11\pi}{12}$

15. a.

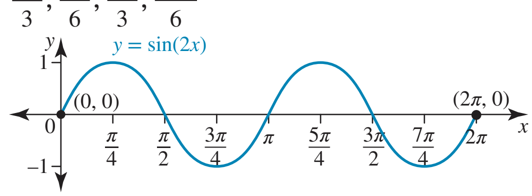


b.



16. a. $\frac{2\pi}{3}, \frac{5\pi}{6}, \frac{5\pi}{3}, \frac{11\pi}{6}$

b.



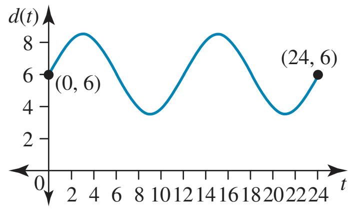
c. $\left\{x : \frac{2\pi}{3} < x < \frac{5\pi}{6}\right\} \cup \left\{x : \frac{5\pi}{3} < x < \frac{11\pi}{6}\right\}$

Exercise 4.5 Derivatives of the sine and cosine functions

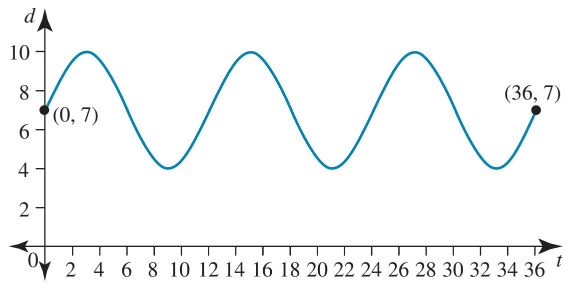
1. a. $8\cos(x)$ b. $-6\cos(-6x)$ c. $\cos(x)$
   d. $\frac{1}{3}\cos\left(\frac{x}{3}\right)$ e. $-\frac{1}{2}\cos\left(-\frac{x}{2}\right)$ f. $\frac{2}{3}\cos\left(\frac{2x}{3}\right)$
2. a. $-3\sin(3x)$ b. $2\sin(-2x)$ c. $-\frac{1}{3}\sin\left(-\frac{x}{3}\right)$
   d. $-21\sin(21x)$ e. $7\sin(-7x)$ f. $-\frac{\pi}{4}\sin\left(\frac{\pi x}{4}\right)$
3. a. $2\cos(2x+3)$ b. $-7\cos(6-7x)$
   c. $5\cos(5x-4)$ d. $\frac{3}{4}\cos\left(\frac{3x+2}{4}\right)$
   e. $-\frac{7}{3}\cos\left(\frac{8-7x}{3}\right)$ f. $10\pi^2\cos(2\pi x)$
4. a. $\sin(8-x)$ b. $5\sin(6-5x)$
   c. $-\frac{2}{3}\sin\left(\frac{2x+3}{3}\right)$ d. $-\frac{4}{5}\sin\left(\frac{4x-1}{5}\right)$
   e. $-40\pi^2\sin(10\pi x)$ f. $-12\sin(-2x)$
5. a. $2(2-x)\sin\left(x^2-4x+3\right)$
   b. $(2x-5)\cos(10-5x+x^2)$
   c. $e^x\cos(e^x)$
   d. $-(2x+7)\sin(x^2+7x)$
   e. $2(x-2)\sin(4x-x^2)$
   f. $(2x+3)\cos\left(x^2+3x\right)$
6. $\frac{-\pi}{2}\sin\left(\frac{\pi}{18}x\right)$
7. a. $\frac{dy}{dx}=-6\sin(3x)$
   b. $\frac{dy}{dx}=-\frac{\pi}{180}\sin\left(\frac{\pi x}{180}\right)$
   c. $\frac{dy}{dx}=3\sin\left(\frac{\pi}{2}-x\right)$
   d. $\frac{dy}{dx}=-\frac{4}{3}\cos\left(\frac{x}{3}\right)$
   e. $\frac{dy}{dx}=\frac{\pi}{15}\cos\left(\frac{\pi x}{15}\right)$
   f. $\frac{dy}{dx}=6\cos\left(\frac{\pi}{2}+3x\right)$
8. $y=x-\frac{\pi}{2}$
9. $y=-\frac{3}{2}x+\frac{\pi}{4}+\frac{3\sqrt{3}}{2}$
10. $\left(\frac{4\pi}{3},-\sqrt{3}\right)$
11. a. $f(0)=-1$ b. $\frac{\pi}{4},\frac{5\pi}{4}$
    c. $f'(x)=\cos(x)+\sin(x)$ d. $\frac{3\pi}{4},\frac{7\pi}{4}$
12. a. $f(0)=\sqrt{3}$
    b. $\frac{-\pi}{3},\frac{2\pi}{3}$
    c. $f'(x)=-\sqrt{3}\sin(x)+\cos(x)$
    d. $\frac{-5\pi}{6},\frac{\pi}{6}$
13. $\frac{-5\pi}{8},\frac{-\pi}{8},\frac{3\pi}{8},\frac{7\pi}{8}$
14. $(-0.524, 0.342)$ and $(0.524, -0.342)$
15. $(0.243, 1.232)$ and $(0.804, 0.863)$

Exercise 4.6 Applications of trigonometric functions

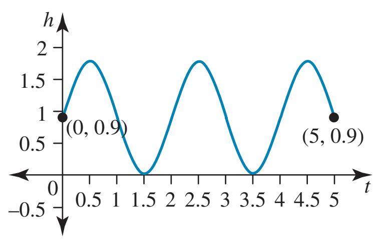
1. a. 8.5 m at 3 pm
   b. 12 hours
   c. 3.5 m
   d.



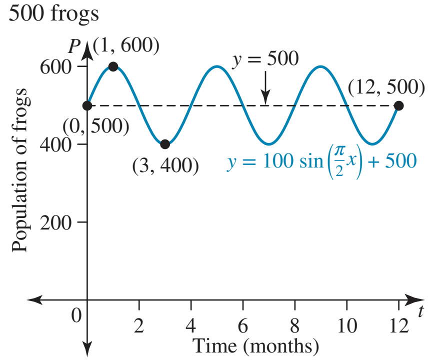
   e. Between 1 pm and 5 pm, and again between 1 am and 5 am the next day
2. a. 10 m; 4 m
   b.



   c. 3 am
   d. Between 1 am and 5 am on Saturday, between 1 pm and 5 pm on Saturday, and between 1 am and 5 am on Sunday
   e. Between 3 am and 5 am on Sunday morning
3. a. 18 °C; 14.5 °C
   b. January and December
   c. 21.5 °C
   d. October
4. a. $a=0.9; n=\pi; c=0.9$
   $h=0.9\sin(\pi t)+0.9$
   b.



   c. 1.3 s
5. a. 500 frogs
   b.



   c. After 1 month

6. a. 10 cm

b. $\frac{dL}{dt} = 2\pi \cos(\pi t)$

c. -2π cm/s

7. a. 14 °C

b. $\frac{dT}{dt} = \frac{2\pi}{9} \cos\left(\frac{\pi}{9} t\right)$

c. 0.535 °C/h

8. a. 9 m

b. $\frac{dh}{dd} = -\frac{4\pi}{25} \sin\left(\frac{\pi}{25} d\right)$

c. i. -0.295 m/m

ii. -0.478 m/m

9. a. 12 h

b. 1 m at 3 pm

c. $\frac{dH}{dt} = \frac{\pi}{12} \cos\left(\frac{\pi t}{6}\right)$

d. $\frac{\pi\sqrt{2}}{24}$

e. 4:30 pm

10. a. $h = -50\cos(2\pi t) + 50$

b. $\frac{dh}{dt} = 100\pi \sin(2\pi t)$

c. 100π mm/s

11. a. 1.5 m

b. 8.5 m

c. 60 s

d. 18.4 s

e. $\frac{dh}{dt} = \frac{7\pi}{60} \sin\left(\frac{\pi t}{30}\right)$

f. 35.51 s; 54.49 s

12. a. 6 m

b. $\frac{dy}{dx} = -\frac{7\pi}{40} \sin\left(\frac{\pi t}{20}\right)$

c. i. $-\frac{7\sqrt{2}\pi}{80}$ m/m

ii. $-\frac{7\pi}{40}$ m/m

d. i. 15 m

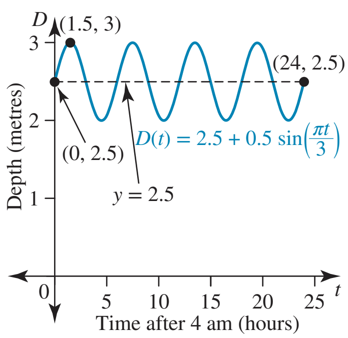
ii. 158.96°

13. a. 2.5 m

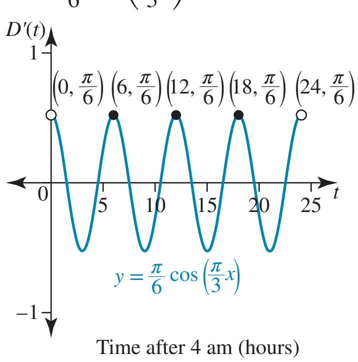
b. 2.93 m

c. 3 m at 5:30 am

d.

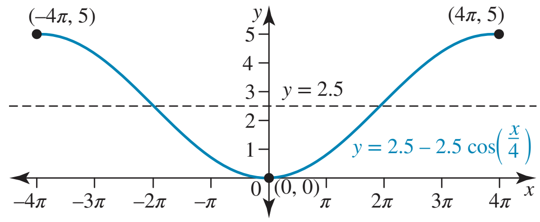


e. $D'(t) = \frac{\pi}{6} \cos\left(\frac{\pi}{3} t\right)$



f. 10 am, 4 pm and 10 pm

14. a.



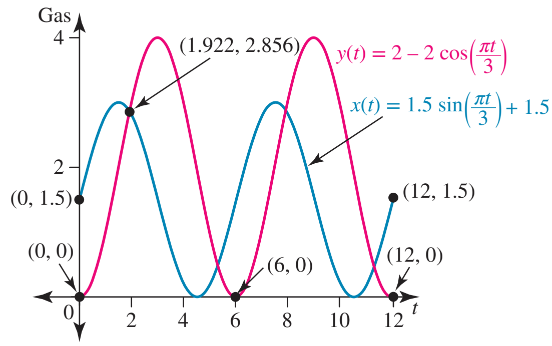
b. i. 1.7 m

ii. $\frac{dh}{dt} = \frac{5}{8} \sin\left(\frac{x}{4}\right)$

iii. 0.426 m/m

iv. 4.756 m horizontally

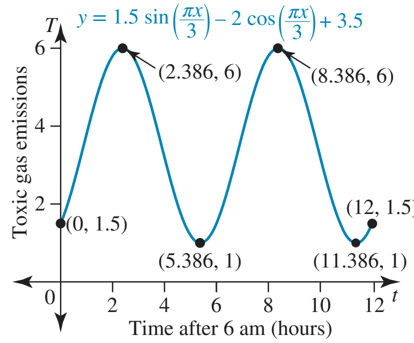
15. a.



b. 7:55 am

c. 2.856 units

d. i.



ii. Maximum gas emissions = 6 units, at 8:23 am and 2:23 pm
Minimum gas emissions = 1 unit, at 11:23 am and 5:23 pm

iii. The maximum gas emission is 6 units and the minimum is 1 unit. They lie within the range of 0 to 7 units, so the company works within the guidelines.

4.7 Review: exam practice

1. a. 135° b. 195° c. 120°19′ d. 100°50′

2. a. $\frac{7\pi}{36}$ b. $\frac{14\pi}{9}$ c. 2.24 d. 4.03

3. a. $-\frac{2}{3}$ b. $\frac{\sqrt{5}}{3}$ c. $\frac{\sqrt{5}}{2}$

d. $-\frac{\sqrt{5}}{3}$ e. $-\frac{\sqrt{5}}{2}$ f. $\frac{2}{3}$

4. a. $\frac{\sqrt{3}}{2}$
b. $-\frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}$
c. $-\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$
d. $-\frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}$
e. $\frac{-1}{2}$
f. $-\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$

5. a. $\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$
b. $\frac{-\sqrt{3}}{2}$
c. $-\sqrt{3}$
d. $\frac{-1}{2}$
e. $-\frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}$
f. $\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$
g. $\frac{-1}{2}$
h. $\frac{1}{2}$
i. -1
j. $\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$
k. $\frac{1}{2}$
l. $\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

6. a. -1
b. $-\frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}$
c. $\frac{-\sqrt{3}}{2}$
d. $\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$
e. $\frac{1}{2}$
f. $-\frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}$

7. a. $\frac{\pi}{2}, \frac{3\pi}{2}$
b. $\frac{5\pi}{4}, \frac{7\pi}{4}$
c. $\frac{\pi}{4}, \frac{7\pi}{4}$
d. $\frac{3\pi}{2}$
e. $\frac{5\pi}{6}, \frac{7\pi}{6}$

8. a. $90°$
b. $60°, 300°$
c. $60°, 120°$
d. $180°$
e. $45°, 135°$

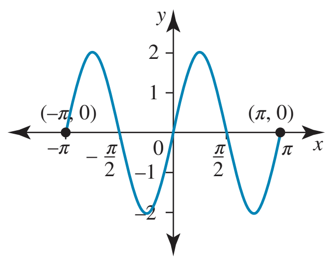
9. a. $\frac{\pi}{6}, \frac{5\pi}{6}$
b. $\frac{\pi}{2}, \frac{3\pi}{2}$
c. $\frac{4\pi}{3}, \frac{5\pi}{3}$
d. $\frac{\pi}{4}, \frac{7\pi}{4}$
e. $\frac{5\pi}{6}, \frac{11\pi}{6}$

10. a. $\frac{\pi}{2}$
b. 0
c. $-\frac{11\pi}{18}, -\frac{7\pi}{18}, \frac{\pi}{18}, \frac{5\pi}{18}, \frac{13\pi}{18}, \frac{17\pi}{18}$
d. $\frac{-3\pi}{4}, \frac{-7\pi}{12}, \frac{-\pi}{12}, \frac{\pi}{12}, \frac{7\pi}{12}, \frac{3\pi}{4}$
e. $\frac{-7\pi}{12}, \frac{-5\pi}{12}, \frac{5\pi}{12}, \frac{7\pi}{12}$

11. a. $\frac{\pi}{4}, \frac{5\pi}{4}$
b. $\frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}, \frac{13\pi}{8}$
c. $\frac{\pi}{6}, \frac{2\pi}{3}, \frac{7\pi}{6}, \frac{5\pi}{3}$
d. $\frac{\pi}{18}, \frac{7\pi}{18}, \frac{13\pi}{18}, \frac{19\pi}{18}, \frac{25\pi}{18}, \frac{31\pi}{18}$
e. 0.6782, 1.7254, 2.7726, 3.8198, 4.8670, 5.9142
f. 1.8926, 5.0342

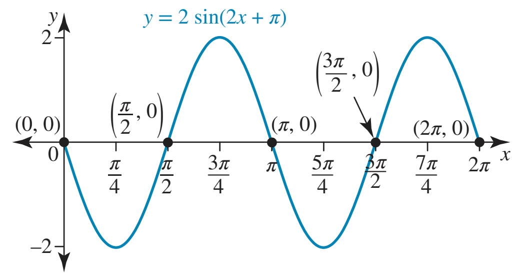
12. a. $\frac{\pi}{4}, \frac{5\pi}{4}$
b. $\frac{\pi}{9}, \frac{\pi}{3}, \frac{5\pi}{9}, \frac{7\pi}{9}, \pi, \frac{11\pi}{9}, \frac{13\pi}{9}, \frac{5\pi}{3}, \frac{17\pi}{9}$
c. $\frac{\pi}{2}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{3\pi}{2}$

13. $y = 2\sin 2x, -\pi \le x \le \pi$; amplitude $= 2$; period $= \pi$; range $-2 \le y \le 2$



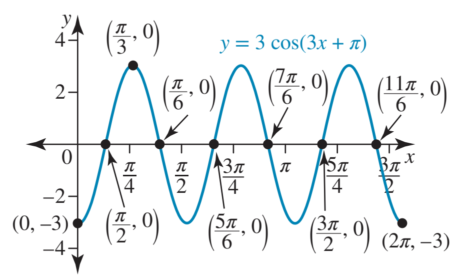
14. a. $y = 2\sin(2x + \pi)$ for $0 \le x \le 2\pi$; period: π; amplitude: 2; range: $[-2, 2]$; endpoints: $(0, 0)$ and $(2\pi, 0)$

For x-intercepts: $x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$



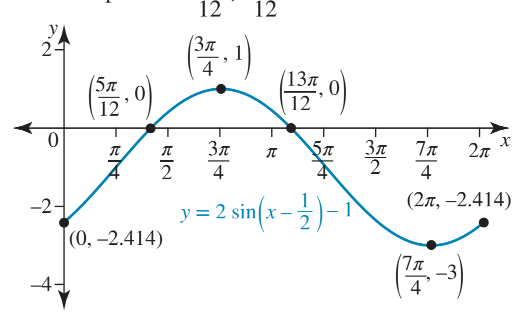
b. $y = 3\cos(3x + \pi)$; period: $\frac{2\pi}{3}$; amplitude: 3; range: $[-3, 3]$; endpoints: $(0, -3)$ and $(2\pi, -3)$

For x-intercepts: $x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6}$

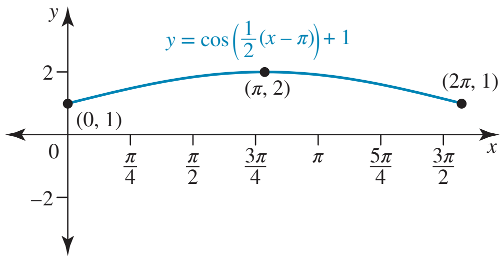


c. $y = 2\sin\left(x - \frac{\pi}{4}\right) - 1$; period: 2π; amplitude: 2; range: $[-3, 1]$; endpoints: $\left(0, -\sqrt{2}\right)$ and $\left(2\pi, -\sqrt{2}\right)$

For x-intercepts: $x = \frac{5\pi}{12}, \frac{13\pi}{12}$



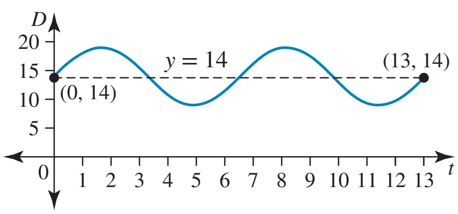
d. $y = \cos\left(\frac{1}{2}(x - \pi)\right) + 1$; period: 4π; amplitude: 1; range: $[0, 2]$; endpoints: $(0, 1)$ and $(2\pi, 1)$
For x-intercepts: there are no solutions for these restricted x-values.



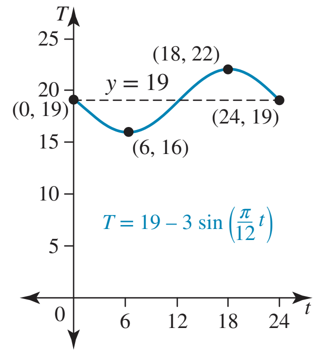
15. a. $\frac{dy}{dx} = -8\sin(8x - 3)$
b. $\frac{dy}{dx} = -6\cos(2x + 1)$
c. $\frac{dy}{dx} = 12\cos(2x) - 6\sin(2x)$
d. $\frac{dy}{dx} = -(2x + 2)\sin\left(x^2 + 2x + 1\right)$
e. $\frac{dy}{dx} = -6\cos(4 - 3x)$
f. $\frac{dy}{dx} = -\cos(-x) + 2\sin(2x)$

16. Tangent: $y = -3$; perpendicular: $x = \pi$

17. a. 19 m
b. 9 m
c. 6.5 h
d. 5
e.

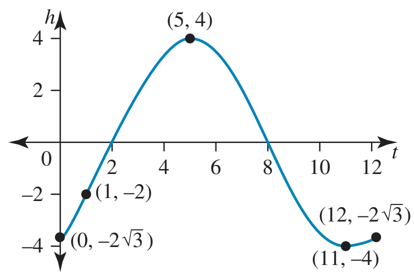


18. a. 19°
b. 22° at 6 pm
c. 16° to 22°
d. 24 h



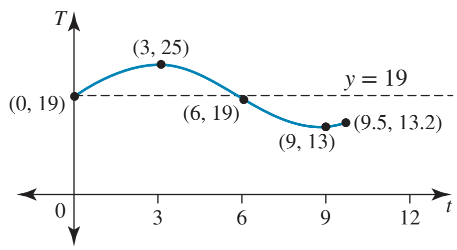
e. $k = 16.2$
f. The temperature is rising the fastest at midday, at a rate of 0.785 degrees/hour.

19. a. 2 m
b. 4 m above mean sea level
c. 6 h
d.



e. At mean sea level
f. 0.14 m higher than mean sea level

20. a. i. 25° at 1 pm
ii. 13° at 7 pm
b. i. 23.2°
ii. 13.2°
c.



d. 2.24 h
e. The heating is switched on at 5:30 pm when the temperature is $(19 - 3\sqrt{2})$ °C or approximately 14.8 °C.

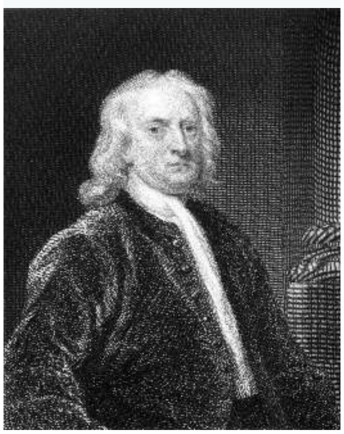
5 Further differentiation and applications

5.1 Overview

Differential calculus is a branch of mathematical analysis concerned with determining how a change in one variable will affect another related variable. Calculus is the study of change, the slopes of curves, and the rate of change between two variables. It is generally thought that Sir Isaac Newton, in England, and Gottfried Leibniz, in Germany, independently discovered calculus in the mid 17th century. Both of them were building on earlier studies of motion and areas: Newton was investigating the laws of motion and gravity as well as geometry, whereas Leibniz was focused on understanding tangents to curves. Although Leibniz was the first to publish his results, controversy remained between the two as to who invented the notation which is still used today.

Although early study of differential calculus involved ratios and geometry, during the 18th century it became more algebraic in nature. Today, calculus is used in many different areas. In economics and commerce, examples of rates of change include marginal costs, the increase or decrease in production costs if another unit is produced, and predictions on the stock market. In science, the rate of growth of bacteria or the rate of decay of a substance can be expressed as a differential equation. In engineering, optimisation — determining the value of one variable that would either maximise or minimise a related variable — is used extensively along with graphing curves.

Sir Isaac Newton
(1643–1727)



Gottfried Wilhelm Leibniz
(1646–1716)

LEARNING SEQUENCE

5.1 Overview
5.2 The chain rule
5.3 The product rule
5.4 The quotient rule
5.5 Applications of differentiation
5.6 Review: exam practice

Fully worked solutions for this chapter are available in the Resources section of your eBookPLUS at www.jacplus.com.au.

5.2 The chain rule

5.2.1 Composite functions

A **composite function**, also known as a function of a function, consists of two or more functions nested within each other.

Consider $g(x) = x^4$ and $h(x) = 2x + 1$. If $f(x) = g(h(x))$, then:

$$\begin{aligned} f(x) &= g(2x + 1) \\ &= (2x + 1)^4 \end{aligned}$$

The function $f(x)$ can be differentiated if it is expanded.

$$\begin{aligned} f(x) &= 16x^4 + 32x^3 + 24x^2 + 8x + 1 \\ f'(x) &= 64x^3 + 96x^2 + 48x + 8 \\ &= 8(8x^3 + 12x^2 + 6x + 1) \\ &= 8(2x + 1)^3 \end{aligned}$$

The chain rule allows us to reach this same outcome without having to expand. The chain rule also allows us to differentiate composite functions that we cannot expand.

In complex functions, the chain rule may need to be applied more than once. For an example of this, see Worked example 3.

5.2.2 The chain rule

The chain rule

If $y = f(g(x))$,

$$\frac{dy}{dx} = f'(g(x)) \times g'(x)$$

Alternatively, if $y = f(u)$ and $u = g(x)$,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

5.2.3 Proof of the chain rule

The proof of the chain rule is as follows.

If $f(x) = m(n(x))$,

then $f(x + h) = m(n(x + h))$.

Therefore, $\dfrac{f(x + h) - f(x)}{h} = \dfrac{m(n(x + h)) - m(n(x))}{h}$.

Multiply the numerator and the denominator by $n(x + h) - n(x)$, as it is expected that at some stage $n'(x)$ will appear somewhere in the rule.

$$\frac{f(x+h)-f(x)}{h} = \frac{n(x+h)-n(x)}{h} \times \frac{m(n(x+h))-m(n(x))}{n(x+h)-n(x)}$$

$$f'(x) = \lim_{h \to 0}\left[\frac{m(n(x+h))-m(n(x))}{n(x+h)-n(x)} \times \frac{n(x+h)-n(x)}{h}\right]$$

$$= \lim_{h \to 0}\left[\frac{m(n(x+h))-m(n(x))}{n(x+h)-n(x)}\right] \times \lim_{h \to 0}\left[\frac{n(x+h)-n(x)}{h}\right]$$

By definition, $n'(x) = \lim_{h \to 0} \frac{n(x+h)-n(x)}{h}$. Also, if we let $n(x) = A$ and $n(x+h) = A+B$, then $n(x+h)-n(x) = A+B-A$, so that

$$\frac{m(n(x+h))-m(n(x))}{n(x+h)-n(x)} = \frac{m(A+B)-n(A)}{B}.$$

Also, as $h \to 0$, $B \to 0$.

Consequently, $\lim_{B \to 0} \frac{m(A+B)-m(A)}{B} = m'(A)$.

Therefore, $\lim_{h \to 0}\left[\frac{m(n(x+h))-m(n(x))}{n(x+h)-n(x)}\right] = m'(n(x))$.

Bringing this all together, we can see that

The derivative of $f(x) = m(n(x))$

If $f(x) = m(n(x))$,

$$f'(x) = m'(n(x)) \times n'(x).$$

Using Leibniz notation, this becomes

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$, where $y = f(u)$ and u is a function of x.

Consider again $y = f(x) = (2x+1)^4$. The chain rule can be used to find the derivative of this function.

Let $u = 2x+1$. $\quad \therefore \frac{du}{dx} = 2$

Also let $y = u^4$. $\quad \therefore \frac{dy}{du} = 4u^3$.

By the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$= 4u^3 \times 2$$
$$= 8u^3$$

Since $u = 2x+1$, $\frac{dy}{dx} = 8(2x+1)^3$.

WORKED EXAMPLE 1

Use the chain rule to determine the derivative of $y = (x^2 + 3x + 5)^7$.

THINK	WRITE
1. Write the function to be derived.	$y = (x^2 + 3x + 5)^7$
2. Let u equal the inner function and rewrite.	$y = u^7$ and $u = x^2 + 3x + 5$
3. Differentiate to determine $\frac{dy}{du}$ and $\frac{du}{dx}$.	$\frac{dy}{du} = 7u^6$ and $\frac{du}{dx} = 2x + 3$
4. Apply the chain rule.	$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$ $\frac{dy}{dx} = 7u^6 \times (2x + 3)$
5. Substitute for u and simplify.	$\frac{dy}{dx} = 7(2x + 3)(x^2 + 3x + 5)^6$

WORKED EXAMPLE 2

Determine $\frac{dy}{dx}$ for the function $y = (4x - 7)^{\frac{2}{3}}$.

THINK	WRITE
1. Write the function to be derived.	$y = (4x - 7)^{\frac{2}{3}}$
2. Let u equal the inner function and rewrite.	$y = u^{\frac{2}{3}}$ and $u = 4x - 7$
3. Differentiate to determine $\frac{dy}{du}$ and $\frac{du}{dx}$.	$\frac{dy}{du} = \frac{2}{3}u^{\frac{-1}{3}}$ and $\frac{du}{dx} = 4$
4. Apply the chain rule.	$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$ $\frac{dy}{dx} = \frac{2}{3}u^{\frac{-1}{3}} \times 4$
5. Substitute for u and simplify.	$\frac{dy}{dx} = \frac{8}{3\sqrt[3]{4x - 7}}$

WORKED EXAMPLE 3

a. **Determine the derivative of $y = \cos^2(e^{2x})$.**
b. **Evaluate the derivative when $x = 0$, giving your answer correct to 4 decimal places.**

THINK	WRITE
a. 1. Write the function to be derived.	$y = \cos^2(e^{2x})$ $y = [\cos(e^{2x})]^2$
2. Let u equal the inner function.	$y = u^2$ and $u = \cos(e^{2x})$

3. Use the chain rule to differentiate this inner function.

$\frac{du}{dx} = -\sin(e^{2x}) \times 2e^{2x}$

$\frac{du}{dx} = -2e^{2x}\sin(e^{2x})$

4. Consider the outer function.

$y = u^2$

5. Differentiate.

$\frac{dy}{du} = 2u$

6. Apply the chain rule.

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$\frac{dy}{dx} = 2u(-2e^{2x}\sin(e^{2x}))$

7. Substitute for u and simplify

$\frac{dy}{dx} = -4e^{2x}\cos(e^{2x})\sin(e^{2x})$

b. 1. Substitute $x = 0$ into $\frac{dy}{dx}$.

$\frac{dy}{dx} = -4e^{0}\cos(e^{0})\sin(e^{0})$

$\frac{dy}{dx} = -4\cos(1)\sin(1)$

$\frac{dy}{dx} = -1.81859485$

2. Answer the question.

$\frac{dy}{dx} = -1.8186$ to 4 decimal places.

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Units 3 & 4 > Area 2 > Sequence 4 > Concept 1 > **The chain rule** Summary screen and practice questions

Exercise 5.2 The chain rule

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1. **WE1** Differentiate each of the following functions.

a. $y = (5x - 4)^3$ **b.** $y = \sqrt{3x + 1}$ **c.** $y = \frac{1}{(2x + 3)^4}$

d. $y = \frac{1}{7 - 4x}$ **e.** $y = (5x + 3)^{-6}$ **f.** $y = (4 - 3x)^{\frac{4}{3}}$

2. **WE2** Determine $\frac{dy}{dx}$ for each of the following functions.

a. $y = (3x + 2)^2$ **b.** $y = (7 - x)^3$ **c.** $y = \frac{1}{2x - 5}$

d. $y = \frac{1}{(4 - 2x)^4}$ **e.** $y = \sqrt{5x + 2}$ **f.** $y = \frac{3}{\sqrt{3x - 2}}$

3. Determine the derivatives of the following functions.

a. $f(x) = (4 - 3x)^5$ **b.** $y = \sqrt{3x^2 - 4}$ **c.** $f(x) = (x^2 - 4x)^{\frac{1}{3}}$

d. $g(x) = (2x^3 + x)^{-2}$ **e.** $g(x) = \left(x - \frac{1}{x}\right)^6$ **f.** $y = (x^2 - 3x)^{-1}$

4. WE3 Use the chain rule to determine the derivatives of the following.

a. $y = \sin^2(x)$

b. $y = e^{\cos(3x)}$

5. If $y = \sin^3(x)$, determine the exact value of $\frac{dy}{dx}$ when $x = \frac{\pi}{3}$.

6. Determine the derivatives of the following functions.

a. $g(x) = 3(x^2 + 1)^{-1}$

b. $g(x) = e^{\cos(x)}$

c. $g(x) = \sqrt{(x + 1)^2 + 2}$

d. $g(x) = \frac{1}{\sin^2(x)}$

e. $f(x) = \sqrt{x^2 - 4x + 5}$

7. Simplify each of the following functions and use the chain rule to determine $g'(x)$.

a. $g(x) = \frac{\sqrt{6x - 5}}{6x - 5}$

b. $g(x) = \frac{(x^2 + 2)^3}{\sqrt{x^2 + 2}}$

8. For each of the following functions, use the chain rule to determine $f'(x)$.

a. $f(x) = 3\cos(x^2 - 1)$

b. $f(x) = 5e^{3x^2 - 1}$

c. $f(x) = \left(x^3 - \frac{2}{x^2}\right)^{-2}$

d. $f(x) = \frac{\sqrt{2 - x}}{2 - x}$

e. $f(x) = \cos^3(2x + 1)$

9. If $f(x) = e^{\sin^2(x)}$, determine $f'\left(\frac{\pi}{4}\right)$.

10. Differentiate the following functions and hence determine the gradients at the given x-values.

a. $f(x) = (2 - x)^{-2}$; determine $f'\left(\frac{1}{2}\right)$.

b. $f(x) = e^{2x^2}$; determine $f'(-1)$.

c. $f(x) = \sqrt[3]{(3x^2 - 2)^4}$; determine $f'(1)$.

d. $f(x) = (\cos(3x) - 1)^5$; determine $f'\left(\frac{\pi}{2}\right)$.

11. If $f(x) = \sin^2(2x)$, determine the points where $f'(x) = 0$ for $x \in [0, \pi]$.

12. MC If $y = e^{3\cos(5x)}$, then $\frac{dy}{dx}$ is:

A. $15\sin(5x)e^{3\cos(5x)}$

B. $-15\sin(5x)e^{3\cos(5x)}$

C. $e^{-15\sin(5x)}$

D. $-15\cos(5x)e^{-3\sin(5x)}$

13. MC If $y = \sin^2(5x)$, then $\frac{dy}{dx}$ is:

A. $2\sin(5x)$

B. $-2\sin(5x)\cos(5x)$

C. $-10\sin(5x)\cos(5x)$

D. $10\sin(5x)\cos(5x)$

14. MC Let $f: R \rightarrow R$ be a differentiable function. For all real values of x, the derivative of $f(e^{4x})$ with respect to x will be:

A. $4e^{4x}f'(x)$

B. $e^{4x}f'(x)$

C. $4e^{4x}f'(e^{4x})$

D. $4f'(e^{4x})$

15. For $y = \sqrt{7 - 2f(x)}$, $\frac{dy}{dx}$ is equal to:

A. $\frac{2f'(x)}{\sqrt{7 - 2f(x)}}$

B. $\frac{-1}{2\sqrt{7 - 2f(x)}}$

C. $\frac{1}{2}\sqrt{7 - 2f'(x)}$

D. $\frac{-f'(x)}{\sqrt{7 - 2f'(x)}}$

16. a. If the function f has a rule $f(x) = \sqrt{x^2 - 1}$ and the function g has the rule $g(x) = x + 3$, calculate the integers m and n such that $f(g(x)) = \sqrt{(x + m)(x + n)}$.

b. If $h(x) = f(g(x))$, determine $h'(x)$.

5.3 The product rule

5.3.1 Differentiation using the product rule

Many functions are the product of two or more functions, such as $f(x) = x\sin(x)$ or $y = e^{2x}(3x + 1)$. See Worked example 4.

To differentiate such functions, the product rule is applied.

The product rule

If $y = f(x) \times g(x)$

$$\frac{dy}{dx} = f(x) \times g'(x) + g(x) \times f'(x)$$

or

If $y = uv$

$$\frac{dy}{dx} = u \times \frac{dv}{dx} + v \times \frac{du}{dx}$$

5.3.2 Proof of the product rule

Let $f(x) = u(x)v(x)$

so $f(x + h) = u(x + h)v(x + h)$.

$$\frac{f(x+h) - f(x)}{h} = \frac{u(x+h)v(x+h) - u(x)v(x)}{h}$$

Add and subtract $u(x)v(x + h)$, as it is expected that at some stage $v'(x)$ will appear somewhere in the rule.

$$\begin{aligned}
\frac{f(x+h) - f(x)}{h} &= \frac{u(x+h)v(x+h) - u(x)v(x+h) + u(x)v(x+h) - u(x)v(x)}{h} \\
&= \frac{[u(x+h) - u(x)]v(x+h) + u(x)[v(x+h) - v(x)]}{h} \\
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{[u(x+h) - u(x)]v(x+h) + u(x)[v(x+h) - v(x)]}{h} \\
&= \lim_{h \to 0} \left[\frac{u(x+h) - u(x)}{h} \times v(x+h)\right] + \lim_{h \to 0} \left[\frac{v(x+h) - v(x)}{h} \times u(x)\right] \\
&= \lim_{h \to 0} \frac{u(x+h) - u(x)}{h} \times \lim_{h \to 0} v(x+h) + \lim_{h \to 0} \frac{v(x+h) - v(x)}{h} \times \lim_{h \to 0} u(x) \\
&= u'(x)v(x) + v'(x)u(x) \\
&= u(x)v'(x) + v(x)u'(x)
\end{aligned}$$

Using Leibnitz notation, if $y = uv$,

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}.$$

WORKED EXAMPLE 4

Differentiate the following functions.

a. $f(x) = x \sin(x)$

b. $y = e^{2x}(3x + 1)$

THINK

a. 1. Define u and v as functions of x.

$f(x) = x\sin(x)$

$u = x$ and $v = \sin(x)$

2. Differentiate with respect to x.

$\frac{du}{dx} = 1$ and $\frac{dv}{dx} = \cos(x)$

3. Apply the product rule and simplify.

$$\frac{dy}{dx} = u \times \frac{dv}{dx} + v \times \frac{du}{dx}$$

$$\frac{dy}{dx} = x \times \cos(x) + \sin(x) \times 1$$

$$\frac{dy}{dx} = x\cos(x) + \sin(x)$$

b. 1. Define u and v as functions of x.

$y = e^{2x}(3x + 1)$

$u = e^{2x}$ and $v = (3x + 1)$

2. Differentiate with respect to x.

$\frac{du}{dx} = 2e^{2x}$ and $\frac{dv}{dx} = 3$

3. Apply the product rule and simplify.

$$\frac{dy}{dx} = u \times \frac{dv}{dx} + v \times \frac{du}{dx}$$

$$\frac{dy}{dx} = e^{2x} \times 3 + (3x + 1) \times 2e^{2x}$$

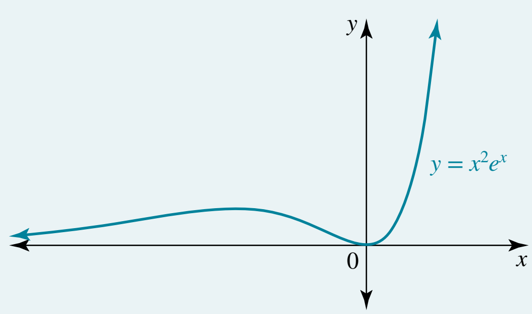
$$\frac{dy}{dx} = 3e^{2x} + 6xe^{2x} + 2e^{2x}$$

$$\frac{dy}{dx} = e^{2x}(6x + 5)$$

WRITE (solutions shown alongside each step above)

WORKED EXAMPLE 5

The graph of $f: R \to R$, $f(x) = x^2e^x$ is shown. Using calculus, determine the coordinates where $f'(x) = 0$.



THINK

1. Define u and v as functions of x.

$f(x) = x^2e^x$

Let $u(x) = x^2$ and $v(x) = e^x$.

2. Differentiate u and v with respect to x.

$u'(x) = 2x$

$v'(x) = e^x$

WRITE (solutions shown alongside each step above)

3. Apply the product rule to determine $f'(x)$.

$$f'(x) = u(x)v'(x) + v(x)u'(x)$$
$$= x^2 \times e^x + e^x \times 2x$$
$$= x^2e^x + 2xe^x$$

4. Solve $f'(x) = 0$.

$$x^2e^x + 2xe^x = 0$$
$$e^x x(x + 2) = 0$$

$e^x > 0$ for all values of x.
Either $x = 0$ or $x + 2 = 0$.

$\therefore x = 0, -2$

5. Substitute the x-values to determine the corresponding y-values.

When $x = -2$,
$$y = (-2)^2e^{-2}$$
$$= 4e^{-2}$$
When $x = 0$,
$$y = (0)^2e^0$$
$$= 0$$

6. Write the answer.

The coordinates where the gradient is zero are $(0, 0)$ and $(-2, 4e^{-2})$.

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Exercise 5.3 The product rule

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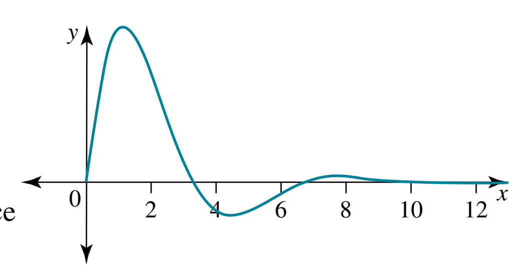
1. **WE4** For each of the following functions, determine the derivative function.
   a. $f(x) = \sin(3x)\cos(3x)$ b. $f(x) = x^2e^{3x}$ c. $f(x) = (x^2 + 3x - 5)e^{5x}$
2. Determine $\frac{dy}{dx}$ for the following functions.
   a. $y = x^2(x + 1)^5$ b. $y = x^3(2x - 1)^4$ c. $y = (4x + 1)^3(3x - 2)^5$
3. Determine the derived functions for the following.
   a. $f(x) = (x + 1)^5\sqrt{x}$ b. $f(x) = x\sqrt{x + 1}$ c. $f(x) = e^{4x}\sqrt{x}$
4. Differentiate the following.
   a. x^2e^{5x} b. $x^{-2}(2x + 1)^3$ c. $x\cos(x)$ d. $2\sqrt{x}(4 - x)$
5. Differentiate the following.
   a. $3x^{-2}e^{x^2}$ b. $e^{2x}\sqrt{4x^2 - 1}$ c. $x^2\sin^3(2x)$ d. $(x - 1)^4(3 - x)^{-2}$
6. If $f(x) = 2x^4\cos(2x)$, determine $f'\left(\frac{\pi}{2}\right)$.
7. **WE5** Given the function $f(x) = (x + 1)\sin(x)$, determine $f'(x)$ and hence determine the gradient of the function when $x = 0$.
8. Given $f(x) = 2x^2(1 - x)^3$, use calculus to determine the coordinates where $f'(x) = 0$.

9. The graph of $f: R^+ \to R, f(x) = e^{-\frac{x}{2}} \sin(x)$ is shown.



a. Calculate the values of x when $f(x) = 0$ for $x \in [0, 3\pi]$.

b. Use calculus to determine the values of x when $f'(x) = 0$ for $x \in [0, 3\pi]$. Give your answers correct to 2 decimal places.

10. Determine the derivative of the following functions and hence determine the gradients at the given points.

a. $f(x) = xe^x$; determine $f'(-1)$.

b. $f(x) = x(x^2 + x)^4$; determine $f'(1)$.

c. $f(x) = \sqrt{x}\sin^2(2x^2)$; determine $f'(\sqrt{\pi})$.

11. **MC** If $f(x) = (x - a)^3 g(x)$, the derivative of $f(x)$ is equal to:

A. $3(x - a)g'(x)$

B. $3(x - a)^2 g'(x)$

C. $3g'(x)$

D. $3(x - a)^2 g(x) + (x - a)^3 g'(x)$

12. **MC** The derivative of $12p(1 - p)^8$ with respect to p is equal to:

A. $96p(1 - p)^8$

B. $-96p(1 - p)^8$

C. $12(1 - p)^7(1 - 9p)$

D. $12(1 - p)^8(1 - 9p)$

13. **MC** Let $y = 2x^3 \sin(x)$. The derivative $\dfrac{dy}{dx}$ when $x = \dfrac{\pi}{2}$ is:

A. $\dfrac{3\pi^2}{2}$

B. $4\pi^2$

C. $\dfrac{3\pi}{2}$

D. $\dfrac{\pi^2}{2}$

14. Evaluate $f'(2a)$ if $f(x) = (x - a)^2 g(x)$, given $g(2a) = 6$ and $g'(2a) = 3$.

15. If $f(x) = g(x)\sin(2x)$ and $f'\left(\dfrac{\pi}{2}\right) = -3\pi$, calculate the constant a if $g(x) = ax^2$.

5.4 The quotient rule

5.4.1 Differentiation using the quotient rule

When one function is divided by a second function, such as $f(x) = \dfrac{x}{x^2 + 1}$ or $y = \dfrac{e^x}{\cos(x)}$, we have the quotient of the two functions. To differentiate such functions, the quotient rule is applied. See Worked example 6.

Before applying the quotient rule, always check if the function can be first simplified. For example, $y = \dfrac{5x^2 - 2x}{x^3}$ can be simplified to $y = 5x^{-1} - 2x^{-2}$ and differentiated by the basic differentiation rule.

The quotient rule

If $y = \dfrac{f(x)}{g(x)}$ where $g(x) \neq 0$,

$$\frac{dy}{dx} = \frac{g(x) \times f'(x) - f(x) \times g'(x)}{[g(x)]^2}.$$

or

If $y = \dfrac{u}{v}$ where $v(x) \neq 0$,

$$\frac{dy}{dx} = \frac{v \times \frac{du}{dx} - u \times \frac{dv}{dx}}{v^2}.$$

5.4.2 Proof of the quotient rule

This rule can be proven as follows by using the product rule.

If $f(x) = \dfrac{u(x)}{v(x)}$, then $f(x) = u(x) \times [v(x)]^{-1}$.

$$\begin{aligned} f'(x) &= u(x) \times -1 \times [v(x)]^{-2} \times v'(x) + [v(x)]^{-1} \times u'(x) \\ &= -\frac{u(x)v'(x)}{[v(x)]^2} + \frac{u'(x)}{[v(x)]} \\ &= \frac{u'(x)v(x)}{[v(x)]^2} - \frac{u(x)v'(x)}{[v(x)]^2} \\ &= \frac{v(x)u'(x) - u(x)v'(x)}{[v(x)]^2} \end{aligned}$$

In Leibnitz notation, the quotient rule states that if $y = \dfrac{u}{v}$, then $\dfrac{dy}{dx} = \dfrac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$.

WORKED EXAMPLE 6

Determine the derivatives of the following functions.

a. $f(x) = \dfrac{x}{x^2 + 1}$

b. $y = \dfrac{e^x}{\cos(x)}$

THINK	WRITE
a. 1. Define u and v as functions of x.	$f(x) = \dfrac{x}{x^2+1}$ $u = x$ and $v = x^2 + 1$
2. Differentiate u and v with respect to x.	$\dfrac{du}{dx} = 1$ and $\dfrac{dv}{dx} = 2x$
3. Apply the quotient rule and simplify.	$\dfrac{dy}{dx} = \dfrac{v \times \frac{du}{dx} - u \times \frac{dv}{dx}}{v^2}$ $\dfrac{dy}{dx} = \dfrac{(x^2+1) \times 1 - x \times 2x}{(x^2+1)^2}$ $\dfrac{dy}{dx} = \dfrac{1 - x^2}{(x^2+1)^2}$
b. 1. Define u and v as functions of x.	$y = \dfrac{e^x}{\cos(x)}$ $u = e^x$ and $v = \cos(x)$
2. Differentiate u and v with respect to x.	$\dfrac{du}{dx} = e^x$ and $\dfrac{dv}{dx} = -\sin(x)$
3. Apply the quotient rule and simplify.	$\dfrac{dy}{dx} = \dfrac{v \times \frac{du}{dx} - u \times \frac{dv}{dx}}{v^2}$ $\dfrac{dy}{dx} = \dfrac{\cos(x) \times e^x - e^x \times (-\sin(x))}{(\cos(x))^2}$
4. Factorise the numerator.	$\dfrac{dy}{dx} = \dfrac{e^x(\cos(x) + \sin(x))}{\cos^2(x)}$

WORKED EXAMPLE 7

Determine the derivative of $y = \dfrac{\sin(2t)}{t^2}$ with respect to t.

THINK	WRITE
1. Define u and v as functions of t.	$y = \dfrac{\sin(2t)}{t^2}$ Let $u = \sin(2t)$ and $v = t^2$.
2. Differentiate u and v with respect to t.	$\dfrac{du}{dt} = 2\cos(2t)$ $\dfrac{dv}{dt} = 2t$
3. Apply the quotient rule to determine $\dfrac{dy}{dt}$ and simplify.	$\dfrac{dy}{dt} = \dfrac{v\frac{du}{dt} - u\frac{dv}{dt}}{v^2}$ $= \dfrac{t^2 2\cos(2t) - \sin(2t) \times 2t}{(t^2)^2}$ $= \dfrac{2t(t\cos(2t) - \sin(2t))}{t^4}$ $= \dfrac{2(t\cos(2t) - \sin(2t))}{t^3}$

WORKED EXAMPLE 8

Determine the derivative of $f(x) = \dfrac{\cos(3x)}{2e^x - x}$ and hence determine the gradient at the point where $x = 0$.

THINK	WRITE
1. Define u and v as functions of x.	$f(x) = \dfrac{\cos(3x)}{2e^x - x}$ Let $u(x) = \cos(3x)$ and $v(x) = 2e^x - x$.
2. Differentiate u and v with respect to x.	$u'(x) = -3\sin(3x)$ $v'(x) = 2e^x - 1$
3. Apply the quotient rule to determine $\dfrac{dy}{dx}$ and simplify.	$f'(x) = \dfrac{v(x)u'(x) - u(x)v'(x)}{v^2}$ $= \dfrac{(2e^x - x) \times -3\sin(3x) - \cos(3x) \times (2e^x - 1)}{(2e^x - x)^2}$ $= \dfrac{-3(2e^x - x)\sin(3x) - (2e^x - 1)\cos(3x)}{(2e^x - x)^2}$
4. Evaluate $f'(0)$.	$f'(0) = \dfrac{-3(2e^0 - 0)\sin(0) - (2e^0 - 1)\cos(0)}{(2e^0 - 0)^2}$ $= \dfrac{0 - 1}{4}$ $= -\dfrac{1}{4}$

5.4.3 The derivative of tan(x)

To determine a rule for the derivative of the tangent function, we can write $\tan(x) = \dfrac{\sin(x)}{\cos(x)}$ and apply the quotient rule.

Let $u = \sin(x)$ and $v = \cos(x)$.

$\dfrac{du}{dx} = \cos(x)$ and $\dfrac{dv}{dx} = -\sin(x)$.

By the quotient rule,

$$\begin{aligned}\frac{d}{dx}(\tan(x)) &= \frac{d}{dx}\left(\frac{\sin(x)}{\cos(x)}\right)\\ &= \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}\\ &= \frac{\cos(x) \times \cos(x) - \sin(x) \times -\sin(x)}{(\cos(x))^2}\\ &= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)}\\ &= \frac{1}{\cos^2(x)} \qquad \text{(by the Pythagorean identity)}\end{aligned}$$

WORKED EXAMPLE 9

Find the derivative of $y = \tan(3x)$.

THINK	WRITE
1 Write the equation.	$y = \tan(3x)$
2 Express u as a function of x and find $\dfrac{du}{dx}$.	Let $u = 3x$ so $\dfrac{du}{dx} = 3$.
3 Express y as a function of u and find $\dfrac{dy}{du}$.	$y = \tan(u)$ so $\dfrac{dy}{du} = \dfrac{1}{\cos^2(u)}$.
4 Find $\dfrac{dy}{dx}$ using the chain rule.	$\dfrac{dy}{dx} = \dfrac{3}{\cos^2(u)} = \dfrac{3}{\cos^2(3x)}$

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Units 3 & 4 > Area 2 > Sequence 4 > Concept 3 > **The quotient rule** Summary screen and practice questions

Exercise 5.4 The quotient rule

Technology free

1. **WE6** If $y = \frac{x+3}{x+7}$ is expressed as $y = \frac{u}{v}$, determine:

   a. u and v
   b. $\frac{du}{dx}$ and $\frac{dv}{dx}$
   c. $\frac{dy}{dx}$

2. If $f(x) = \frac{x^2+2x}{5-x}$ is expressed as $f(x) = \frac{u}{v}$, determine:

   a. u and v
   b. $\frac{du}{dx}$ and $\frac{dv}{dx}$
   c. $f'(x)$

3. Determine the derivative of each of the following.

   a. $y = \frac{2x}{x^2-4}$
   b. $y = \frac{x^2+7x+6}{3x+2}$
   c. $f(x) = \frac{4x-7}{10-3x}$

4. **MC** If $h(x) = \frac{8-3x^2}{x}$, then $h'(x)$ equals:

   A. $\frac{8-9x^2}{x^2}$
   B. $\frac{-3x^2+8}{x^2}$
   C. $\frac{-3x^2-8}{x^2}$
   D. $\frac{-3x^2+8}{x}$

5. **WE7** Use the quotient rule to determine the derivatives of:

   a. $\frac{e^{2x}}{e^x+1}$
   b. $\frac{\cos(3t)}{t^3}$

6. Determine the derivative of $\frac{x+1}{x^2-1}$.

7. **WE8** If $y = \frac{\sin(x)}{e^{2x}}$, determine the gradient of the function at the point where $x = 0$.

8. If $y = \frac{5x}{x^2+4}$, calculate the gradient of the function at the point where $x = 1$.

9. Differentiate the following.

   a. $y = \frac{\sin^2(x^2)}{x}$
   b. $y = \frac{3x-1}{2x^2-3}$
   c. $\frac{e^x}{\cos(2x+1)}$
   d. $\frac{e^{-x}}{x-1}$

10. Differentiate the following.

    a. $\frac{\sin(x)}{\sqrt{x}}$
    b. $f(x) = \frac{(5-x)^2}{\sqrt{5-x}}$
    c. $f(x) = \frac{x-4x^2}{2\sqrt{x}}$
    d. $y = \frac{3\sqrt{x}}{x+2}$

11. **WE9** Differentiate each of the following.

    a. $y = \tan(2x)$
    b. $y = \tan(-4x)$
    c. $y = \tan\left(\frac{x}{5}\right)$
    d. $y = \tan\left(\frac{-3x}{4}\right)$

12. Calculate the gradient at the stated point for each of the following functions.

    a. $y = \frac{2x}{x^2+1}, x = 1$
    b. $y = \frac{\sin(2x+\pi)}{\cos(2x+\pi)}, x = \frac{\pi}{2}$
    c. $y = \frac{x+1}{\sqrt{3x+1}}, x = 5$
    d. $y = \frac{5-x^2}{e^x}, x = 0$

13. Calculate the gradient of the tangent to the curve with equation $y = \frac{2x}{(3x+1)^{\frac{3}{2}}}$ at the point where $x = 1$.

14. Show that $\frac{d}{dx}\left(\frac{1+\cos(x)}{1-\cos(x)}\right) = -\frac{-2\sin(x)}{(\cos(x)-1)^2}$.

15. a. Show that $\frac{d}{dx}\left(\frac{\sin(x)}{\cos(x)}\right) = \frac{1}{\cos^2(x)}$.

b. Hence, determine the gradient of the curve $y = \tan(x)$ at the point where $x = \frac{\pi}{4}$.

16. Given that $f(x) = \frac{\sqrt{2x-1}}{\sqrt{2x+1}}$, calculate m such that $f'(m) = \frac{2}{5\sqrt{15}}$.

5.5 Applications of differentiation

Differentiation can be applied to many different situations. These include:

- finding tangents to curves at specific points
- curve sketching
- optimisation — finding where maximum or minimum values occur within given constraints
- kinematics, the study of motion
- rates of change — investigating how a change in one variable affects another related variable.

This section introduces some of these concepts. Chapter 8 covers these situations in more detail.

The various rules of differentiation may have to be used first before an application problem can be solved.

5.5.1 Tangents and curve sketching

The derivative of a function gives the gradient of the tangent to the curve at any point.

The derivative also shows whether the function has a stationary point, or, as x increases, if the function is increasing or decreasing.

- If $\frac{dy}{dx} = 0$, the function has a stationary point.
- If $\frac{dy}{dx} > 0$, the function is increasing.
- If $\frac{dy}{dx} < 0$, the function is decreasing.

Identifying stationary points provides information that assists curve sketching.

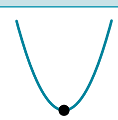
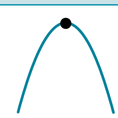
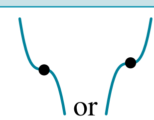
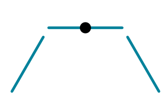
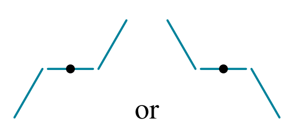
Stationary points are classified as:

- local minimum turning points
- local maximum turning points
- stationary points of inflection (or horizontal points of inflection).

The nature of a stationary point is determined by examining the slope of the tangent to the curve immediately before and after the stationary point.

The word ‘local’ means that the point is a minimum or maximum in a particular locality or neighbourhood. Beyond this section of the graph, there could be other points on the graph that are lower than the local minimum or higher than the local maximum.

The nature of a stationary point is summarised in the following table.

	Minimum turning point	Maximum turning point	Stationary point of inflection
Stationary point			
Slope of tangent			

WORKED EXAMPLE 10

Given that $y = e^{2x}(x+1)^2$, evaluate $\frac{dy}{dx}$ and hence determine the equation of the tangent to the curve at the point $(0, 1)$.

THINK	WRITE
1. Define u and v as functions of x.	$y = e^{2x}(x+1)^2$ Let $u = e^{2x}$ and $v = (x+1)^2$.
2. Differentiate u and v with respect to x.	$\frac{du}{dx} = 2e^{2x}$ $\frac{dv}{dx} = 2(x+1)$
3. Apply the product rule to determine $\frac{dy}{dx}$ and simplify.	$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$ $\frac{dy}{dx} = e^{2x} \times 2(x+1) + (x+1)^2 \times 2e^{2x}$ $= 2e^{2x}(x+1) + 2e^{2x}(x+1)^2$ $= 2e^{2x}(x+1)(1+x+1)$ $= 2e^{2x}(x+1)(x+2)$
4. Evaluate $\frac{dy}{dx}$ when $x = 0$.	When $x = 0$, then $\frac{dy}{dx} = 2e^0(0+1)(0+2)$ $= 4$
5. Determine the equation of the tangent.	If $m = 4$ and $(x_1, y_1) = (0, 1)$, $y - y_1 = m(x + x_1)$ $y - 1 = 4(x - 0)$ $y - 1 = 4x$ $y = 4x + 1$

WORKED EXAMPLE 11

Consider the function $f(x) = e^x(x-2)^3$.

a. **Calculate $f'(x)$ and hence determine the coordinates of the stationary points.**
b. **By investigating the sign of $f'(x)$, state the nature of these stationary points.**
c. **Investigate the values of $f(x)$ as $x \to \pm\infty$. State the equations of any asymptotes.**
d. **Calculate any axis intercepts.**
e. **Sketch the curve of $y = f(x)$, showing all important features.**
f. **State the domain and range of the function.**

THINK	WRITE
a. 1. Define u and v as functions of x.	$f(x) = e^x(x-2)^3$ $u = e^x$ and $v = (x-2)^3$
2. Differentiate with respect to x.	$\frac{du}{dx} = e^x$ and $\frac{dv}{dx} = 3(x-2)^2$

3. Apply the product rule and simplify by factorising.

$\frac{dy}{dx} = u \times \frac{dv}{dx} + v \times \frac{du}{dx}$

$\frac{dy}{dx} = e^x \times 3(x-2)^2 + (x-2)^3 \times e^x$

$\frac{dy}{dx} = e^x(x-2)^2[3 + (x-2)]$

$f'(x) = e^x(x-2)^2(x+1)$

4. Stationary points exist when $\frac{dy}{dx} = 0$.

$e^x(x-2)^2(x+1) = 0$

$e^x = 0$ is undefined, so $x = 2$ or $x = -1$

5. Determine y-values for $x = 2, x = -1$.

The stationary points are $\left(-1, \frac{-27}{e}\right)$ and $(2, 0)$.

b. 1. Construct a table of values for $f'(x)$ for suitable values of x.

$f'(x) = e^x(x-2)^2(x+1)$

x	-2	-1	0	2	3
$f'(x)$	$-16e^{-2}$	0	4	0	$4e^3$
	\	—	/	—	/

2. State the nature of the stationary points by considering the direction of the tangents.

$\left(-1, \frac{-27}{e}\right)$ is a minimum stationary point; $(2, 0)$ is a horizontal point of inflexion (or stationary point of inflexion)

c. 1. Consider the behaviour of $f(x)$ as $x \to \infty$ (or as x becomes very large).

$f(x) = e^x(x-2)^3$

$e^x \to \infty; (x-2)^3 \to \infty$

$\therefore$ as $x \to \infty, f(x) \to \infty$

2. Consider the behaviour of $f(x)$ as $x \to -\infty$ (or as x becomes very small).

$e^x \to 0; (x-2)^3 \to -\infty$

$\therefore$ as $x \to -\infty, f(x) \to 0$ from the negative side.

3. State equations of asymptotes

Note: $y = 0$ is only an asymptote for small values of x.

$y = 0$ is an asymptote.

d. 1. For x-intercepts, $y = 0$.

$e^x(x-2)^3 = 0$

$\therefore x = 2$ or $(2, 0)$

2. For y-intercepts, $x = 0$.

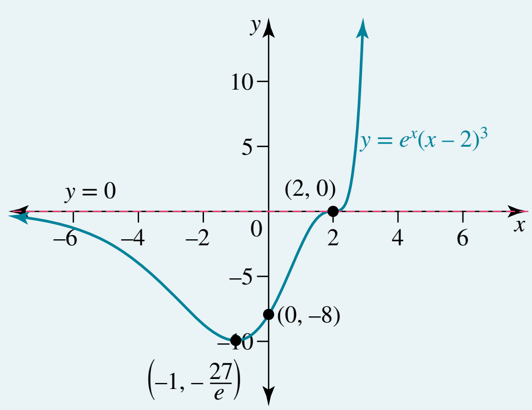
$f(0) = e^0(-2)^3$

3. State the axis intercepts.

$\therefore y = -8$ or $(0, -8)$

The axis intercepts are $(2, 0)$ and $(0, -8)$.

e. 1. Draw axes and plot the axis intercepts and stationary points, noting their nature.



2. Remember the x-axis is an asymptote on the left, as $x \to -\infty$.

f. 1. State the domain.

$f(x) = e^x(x-2)^3$

$f(x)$ is defined for all values of x.

The domain is $x \in R$.

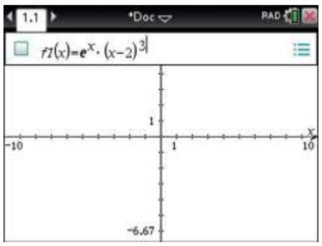
2. State the range.

The minimum y-value is $\dfrac{-27}{e}$

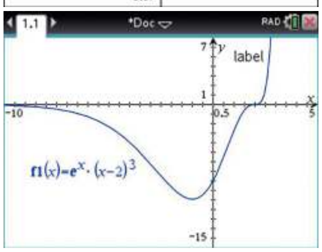
The range is $y \geq \dfrac{-27}{e}$ or $y \in \left[\dfrac{-27}{e}, \infty\right)$.

TI | THINK — **WRITE**

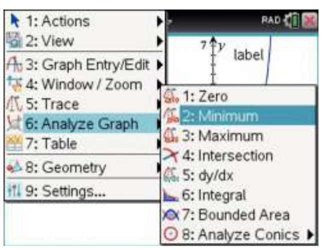
a.1. The graphing function can be used to calculate the location of any stationary point(s).
On a Calculator page, press MENU, then select:
2: Add Graphs.



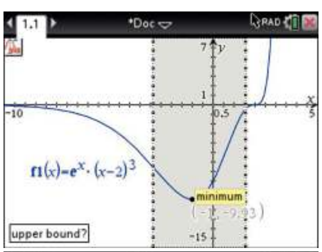
2. Complete the entry line in the $f1(x) =$ tab as:
$e^x\,(x-2)^3$
Press ENTER to sketch the graph.



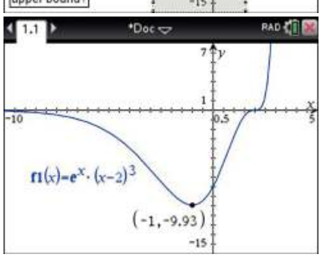
3. On a Calculator page, press MENU, then select:
6: Analyze Graph
2: Minimum.



4. Inspect the graph and set lower and upper bounds to either side of the minimum value.

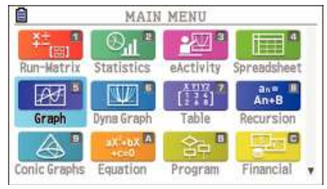


5. The answer appears on the screen.

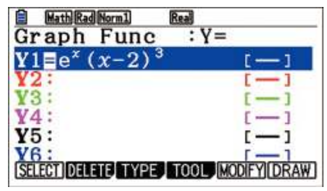


CASIO | THINK — **WRITE**

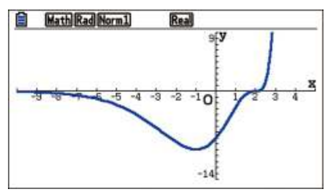
a.1. On a Main Menu screen, select: Graph.



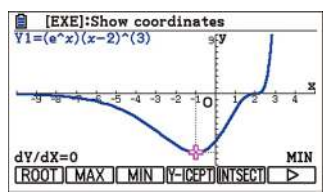
2. Complete the function entry line in the Y1 tab as:
$\ln(x-2)$



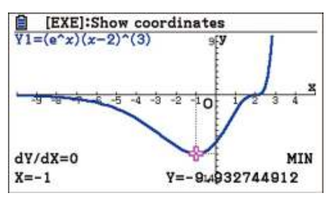
3. Press the DRAW button to sketch the graph.



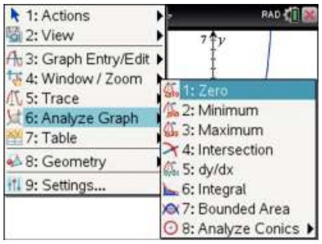
4. Determine the minimum value by selecting:
SHIFT F5 (G-Solv) MIN.



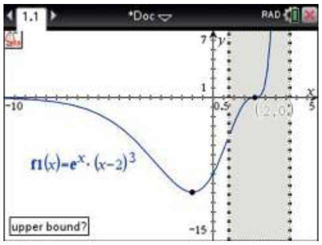
5. The answer appears on the screen.



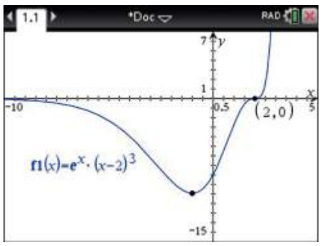
d.1. On a Calculator page, press MENU, then select:
6: Analyze Graph
2: Minimum.



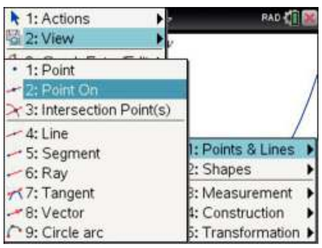
2. Inspect the graph and set lower and upper bounds to either side of the x-intercept.



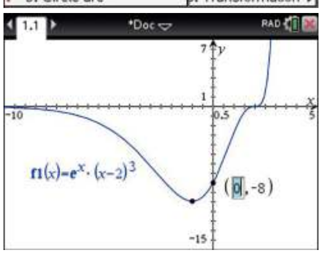
3. The answer appears on the screen.



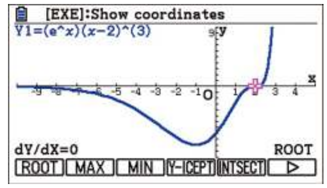
4. On a Calculator page, select:
Menu
8: Geometry
1: Points & Lines
2: Point On.



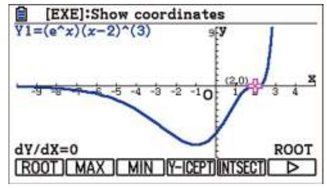
5. Move the cursor and select the curve representing $f1(x)e^{x}(x-2)^{3}$.
Press the ESC (escape) button.
Complete the entry line in the textbox as 0 for the x-value.
Press ENTER to perform the calculation.
The answer appears on the screen.



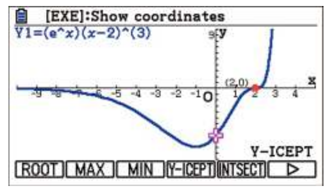
d.1. Determine the x-intercept by selecting:
SHIFT F5 (G-Solv)
ROOT.



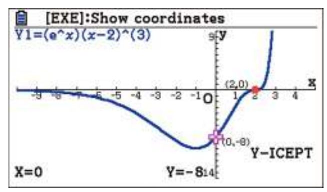
2. The answer appears on the screen.



3. Determine the y-intercept by selecting:
SHIFT F5 (G-Solv)
ROOT.



4. The answer appears on the screen.



5.5.2 Maximum and minimum problems

To solve optimisation problems, apply the following steps.

1. Draw diagrams when necessary.
2. Determine the variables and any connection between them.
3. State or determine the function to be optimised.
4. Differentiate the function and determine the stationary points and their nature.
5. Reject any unrealistic solution.
6. Answer the question.

WORKED EXAMPLE 12

The profit, $P, per item that a store makes by selling n items of a certain type each day is $P = 40\sqrt{n + 25} - 200 - 2n$.

a. Determine the number of items that need to be sold to maximise the profit on each item.

b. Calculate:

i. the maximum profit per item

ii. the total profit per day made by selling this number of items.

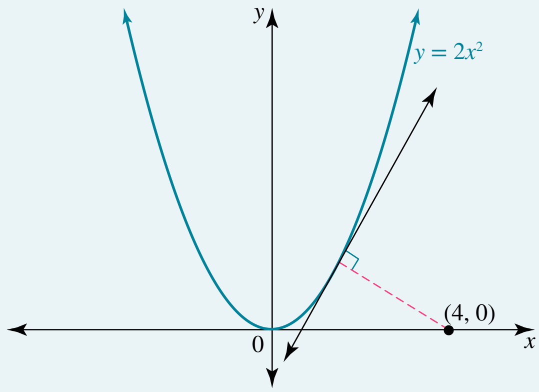
THINK	WRITE
a. 1. Rewrite with powers.	$P = 40\sqrt{n + 25} - 200 - 2n$ $P = 40\,(n + 25)^{\frac{1}{2}} - 200 - 2n$
2. Differentiate with respect to n.	$\frac{dP}{dn} = 40 \times \frac{1}{2} \times (n + 25)^{\frac{-1}{2}} \times 1 - 2$
3. Simplify.	$\frac{dP}{dn} = \frac{20}{\sqrt{n + 25}} - 2$
4. Solve $\frac{dP}{dn} = 0$ for n.	$\frac{20}{\sqrt{n + 25}} - 2 = 0$ $\frac{20}{\sqrt{n + 25}} = 2$ $20 = 2\sqrt{n + 25}$ $\sqrt{n + 25} = 10$ $n + 25 = 100$ $n = 75$
5. Draw a sign diagram to justify your answer.	(see table below)

n	70	75	80
$\frac{dp}{dn}$	≈ 0.052	0	≈ -0.048
slope	/	—	\

THINK	WRITE
6. State the answer.	The maximum profit per item is obtained when 75 items are sold each day.
b. i. 1. Calculate $P(75)$.	$P = 40\sqrt{n + 25} - 200 - 2n$ $P = 40\sqrt{75 + 25} - 200 - 2 \times 75$ $P = 40 \times 10 - 200 - 150$ $P = 50$
2. State the answer.	The maximum profit per item is \$50.
ii. 1. Calculate the total profit for selling 75 items per day.	Total profit = \$50 × 75
2. State the answer.	The maximum total profit per day for selling 75 items is \$3750.

WORKED EXAMPLE 13

Find the minimum distance from the curve $y = 2x^2$ to the point (4, 0), correct to 2 decimal places. You do not need to justify your answer.



THINK	WRITE
1. Let P be the point on the curve such that the distance from P to the point (4, 0) is a minimum.	$P = (x, y)$
2. Write the formula for the distance between the two points.	$d(x) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ $= \sqrt{(x-4)^2 + (y-0)^2}$ $= \sqrt{(x-4)^2 + y^2}$
3. Express the distance between the two points as a function of x only.	$y = 2x^2$ $\therefore d(x) = \sqrt{(x-4)^2 + (2x^2)^2}$ $= (x^2 - 8x + 16 + 4x^4)^{\frac{1}{2}}$
4. Differentiate $d(x)$.	$d'(x) = \frac{1}{2} \times (4x^4 + x^2 - 8x + 16)^{-\frac{1}{2}} \times (16x^3 + 2x - 8)$ $= \frac{16x^3 + 2x - 8}{2\sqrt{4x^4 + x^2 - 8x + 16}}$ $= \frac{8x^3 + x - 4}{\sqrt{4x^4 + x^2 - 8x + 16}}$
5. Solve $d'(x) = 0$ using technology.	$0 = \frac{8x^3 + x - 4}{\sqrt{4x^4 + x^2 - 8x + 16}}$ $0 = 8x^3 + x - 4$ $x = 0.741$
6. Evaluate $d(0.741)$.	$d(0.741) = \sqrt{4(0.741)^4 + (0.741)^2 - 8(0.741) + 16}$ $= 3.439$
7. Write the answer.	The minimum distance is 3.44 units (to 2 decimal places).

WORKED EXAMPLE 14

A new window is to be made to allow more light into a room. The window will have the shape of a rectangle surmounted by a semicircle. The frame of the window will be made from aluminium measuring 336 cm.

a. **Show that the area, A cm^2, of the window is $A = 336x - \frac{1}{2}(4+\pi)x^2$, where x is the radius of the semicircle in cm.**

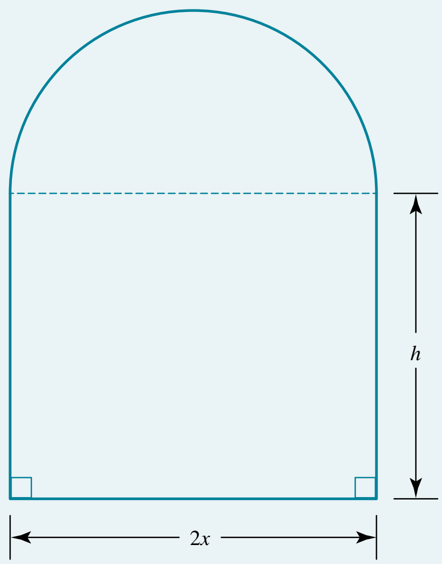
b. **Hence, determine the width of the window for which the area is greatest. Give your answer to the nearest cm.**

c. **Structural limitations mean that the width of the window should not exceed 84 cm. What should the dimensions of the window of maximum area now be? Give your answer to the nearest cm.**

THINK

WRITE

a. 1. Draw a diagram to illustrate the window where radius of the semicircle is x cm and the height of the rectangle is h cm.



2. Use the perimeter to form an expression connecting the two variables, x and h.

$$P = \pi x + 2x + 2h$$
$$336 = \pi x + 2x + 2h$$

3. Express h in terms of x.

$$2h = 336 - \pi x - 2x$$
$$h = \frac{1}{2}(336 - \pi x - 2x)$$

4. Express the area as a function of x by substituting for h.

$$A = 2xh + \frac{1}{2}\pi x^2$$
$$A = 2x \times \frac{1}{2}(336 - \pi x - 2x) + \frac{1}{2}\pi x^2$$
$$A = 336x - \pi x^2 - 2x^2 + \frac{1}{2}\pi x^2$$
$$A = 336x - 2x^2 - \frac{1}{2}\pi x^2$$

5. Express the area in the required form.

$$A = 336x - \frac{1}{2}(4+\pi)x^2$$

b. 1. Differentiate.

$$\frac{dA}{dx} = 336 - (4+\pi)x$$

2. Determine the stationary point and its nature.

$$336 - (4+\pi)x = 0$$
$$x = \frac{336}{4+\pi}$$
$$x \approx 47.05$$

x	40	$\frac{336}{(4+\pi)}$	50
$\frac{dA}{dx}$	positive	0	negative
slope	/	—	\

The maximum area occurs when $x = \dfrac{336}{4+\pi}$ cm or $x \approx 47$ cm (to the nearest cm).

c. 1. State the restrictions for the window.

The width of the window is not to exceed 84 cm.

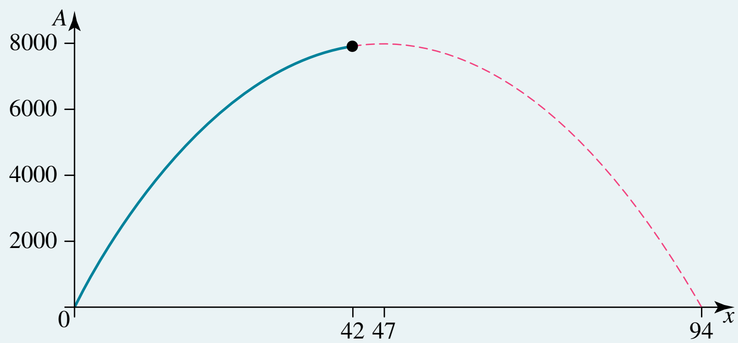
$2x \le 84$

$x \le 42$

2. State the restricted domain for $A(x)$.

Restricted domain of area: $x \in (0, 42]$

3. Sketch the graph of $A(x)$.



4. Determine, for the restricted domain, the value of x for the greatest area.

The maximum area occurs when $x = 42$ cm.

5. Calculate the height, h.

$h = \frac{1}{2}(336 - \pi \times 42 - 2 \times 42)$

$h = 60.0266$

6. Calculate the dimensions.

Width $= 2x = 84$

Total height $= h + x = 60 + 42 = 102$

7. State the answer.

With the restrictions, the area of the window will be greatest if the width is 84 cm and the total height is 102 cm.

5.5.3 Rates of change and kinematics

The instantaneous rate of change, or simply the **rate of change**, of the function $y = f(x)$ is given by the derivative, $\dfrac{dy}{dx}$ or $f'(x)$.

The derivative $\dfrac{dV}{dt}$ could be the rate of change of volume with respect to time.

Kinematics, the study of the motion of a particle moving in a straight line, involves determining rates of change of displacement and velocity with respect to time.

Displacement, x, gives the position of a particle by specifying both its distance and direction from a fixed point, the origin.

Commonly used conventions for motion in a horizontal straight line are as follows:

- If $x > 0$, the particle is to the right of the origin.
- If $x < 0$, the particle is to the left of the origin.
- If $x = 0$, the particle is at the origin.

Common units for displacement are cm, m and km.

Velocity, v, measures the rate of change of displacement with respect to time, so $v = \frac{dx}{dt}$.

For a particle moving in a horizontal straight line, the sign of the velocity indicates direction:

- If $v > 0$, the particle is moving to the right.
- If $v < 0$, the particle is moving to the left.
- If $v = 0$, the particle is stationary, or instantaneously at rest.

Common units for velocity are cm/s, m/s and km/h.

Acceleration, a, measures the rate of change of velocity with respect to time, so $a = \frac{dv}{dt}$. Common units for acceleration include m/s^2.

The term 'initially' means at the start, or when $t = 0$.

WORKED EXAMPLE 15

The number of mosquitoes, N, around a dam on a certain night can be modelled by the equation

$$N = \frac{400}{2t + 1} + 100t + 1000$$

where t equals hours after sunset. Find:

a. the initial number of mosquitoes

b. the rate of change at any time, t

c. the rate of change when $t = 4$ hours.

THINK	WRITE
a. 1. Write the rule.	a. $N = \frac{400}{2t+1} + 100t + 1000$
2. Calculate N when $t = 0$.	$N = \frac{400}{1} + 0 + 1000$ $N = 1400$
3. Answer the question.	Initially, there were 1400 mosquitoes.
b. 1. Differentiate N with respect to t.	b. $N = 400\,(2t+1)^{-1} + 100t + 1000$ $\frac{dN}{dt} = 400 \times -1\,(2t+1)^{-2} \times 2 + 100$
2. Simplify.	$\frac{dN}{dt} = \frac{-800}{(2t+1)^2} + 100$
c. 1. Calculate $\frac{dN}{dt}$ when $t = 4$.	c. $\frac{dN}{dt} = \frac{-800}{(2 \times 4 + 1)^2} + 100$ $= \frac{-800}{(9)^2} + 100$ $= \frac{7300}{81}$ ≈ 90.1

2. Answer the question.

After 4 hours, the rate of change is approximately 90.1 mosquitoes per hour.

WORKED EXAMPLE 16

The displacement, x metres, of a particle after t seconds is given by the equation $x = 4\sin(2t) + 3$.

a. Derive an expression for the velocity, v m/s, of the particle.

b. Determine the time at which the particle is first at rest and its position at this time.

c. Derive an expression for acceleration, a m/s^2, of the particle and its initial acceleration.

THINK	WRITE
a. 1. State the displacement function.	a. $x = 4\sin(2t) + 3$
2. Differentiate with respect to t.	$v = \frac{dx}{dt}$ $v = 4\cos(2t) \times 2$
3. State the expression for velocity.	$v = 8\cos(2t)$
b. 1. At rest, $v = 0$.	b. $8\cos(2t) = 0$
2. Solve for t.	$\cos(2t) = 0$ $2t = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$ $t = \frac{\pi}{4}, \frac{3\pi}{4}, \ldots$
3. The first time the particle is at rest is the lowest value of t.	$t = \frac{\pi}{4}$ seconds
4. Substitute to determine $x\left(\frac{\pi}{4}\right)$.	$x = 4\sin\left(2 \times \frac{\pi}{4}\right) + 3$ $= 4\sin\left(\frac{\pi}{2}\right) + 3$ $= 4 + 3$ $= 7$
5. Answer the question.	The particle is first at rest after $\frac{\pi}{4}$ seconds and it is 7 metres to the right of the origin.
c. 1. State the velocity function.	c. $v = 8\cos(2t)$
2. Differentiate with respect to t.	$a = \frac{dv}{dt}$ $= 8 \times (-\sin(2t)) \times 2$ $= -16\sin(2t)$
3. For initial acceleration, substitute $t = 0$.	$a = -16\sin(2 \times 0)$ $= -16\sin(0)$ $= 0$
4. Answer the question.	The acceleration of the particle is given by the equation $a = -16\sin(2t)$ and the initial acceleration is 0 m/s^2.

WORKED EXAMPLE 17

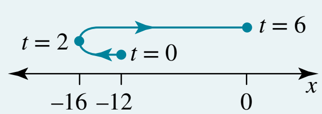
A particle moves in a straight line such that its displacement, x metres, from a fixed origin at time t seconds is modelled by $x = t^2 - 4t - 12, t \geq 0$.

a. **Identify its initial position.**

b. **Determine its velocity function and hence state its initial velocity and describe its initial motion.**

c. **At what time and position is the particle momentarily at rest?**

d. **Show the particle is at the origin when $t = 6$, and calculate the distance it has travelled to reach the origin.**

THINK	WRITE/DRAW
a. Calculate the value of x when $t = 0$.	a. $x = t^2 - 4t - 12, t \geq 0$ When $t = 0$, $x = -12$. Initially the particle is 12 metres to the left of the origin.
b. 1. Calculate the rate of change required.	b. $v = \frac{dx}{dt}$ $v = 2t - 4$
2. Calculate the value of v at the given instant.	When $t = 0$, $v = -4$. The initial velocity is -4 m/s.
3. Describe the initial motion.	Since the initial velocity is negative, the particle starts to move to the left with an initial speed of 4 m/s.
c. 1. Calculate when the particle is momentarily at rest. *Note:* This usually represents a change of direction of motion.	c. The particle is momentarily at rest when its velocity is zero. When $v = 0$, $2t - 4 = 0$ $t = 2$ The particle is at rest after 2 seconds.
2. Calculate where the particle is momentarily at rest.	The position of the particle when $t = 2$ is $x = (2)^2 - 4(2) - 12$ $= -16$ Therefore, the particle is momentarily at rest after 2 seconds at the position 16 metres to the left of the origin.
d. 1. Calculate the position to show the particle is at the origin at the given time.	d. When $t = 6$, $x = 36 - 24 - 12$ $= 0$ The particle is at the origin when $t = 6$.
2. Track the motion on a horizontal displacement line and calculate the required distance.	The motion of the particle for the first 6 seconds is shown.  Distances travelled are 4 metres to the left, then 16 metres to the right.

The total distance travelled is the sum of the distances in each direction.
The particle has travelled a total distance of 20 metres.

on Resources

Interactivities: Stationary points (int-5963)
Rates of change (int-5960)
Kinematics (int-5964)

studyon

Units 3 & 4 > Area 2 > Sequence 4 > Concepts 4, 5 & 6

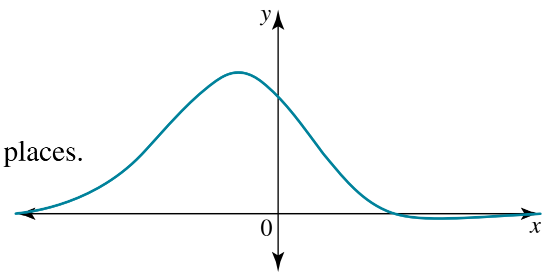
Equations of tangents Summary screen and practice questions
Curve sketching Summary screen and practice questions
Maximum and minimum problems Summary screen and practice questions

Exercise 5.5 Applications of differentiation

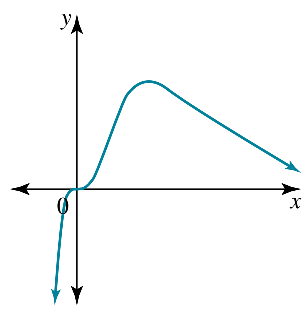
Technology free

1. **WE10** For the function $y = \sqrt{3x^2 + 2x}$, determine:
   a. $\dfrac{dy}{dx}$
   b. the equation of the tangent to the curve at $x = 2$.
2. Use the chain rule to determine the derivative of $y = \dfrac{1}{(2x-1)^2}$ and hence determine the equation of the tangent to the curve at the point where $x = 1$.
3. a. Evaluate $f'(-1)$ if $f(x) = \dfrac{3}{\sqrt{5-4x}}$.
   b. Hence, determine the equation of the tangent to the curve $y = f(x)$ at the point where $x = -1$.
4. For the function with the rule $y = xe^x$, determine the equations of the tangent and the line perpendicular to the curve at the point where $x = 1$.
5. The function h has a rule $h(x) = \sqrt{x^2 - 16}$ and the function g has the rule $g(x) = x - 3$.
   a. Determine the integers m and n such that $h(g(x)) = \sqrt{(x+m)(x+n)}$.
   b. State the maximal domain of $h(g(x))$.
   c. Determine the derivative of $h(g(x))$.
   d. Determine the gradient of the function $h(g(x))$ at the point when $x = -2$.
6. **WE11** Consider the function $f(x) = \dfrac{x}{x^2+1}$.
   a. Calculate $f'(x)$ and hence determine the coordinates of the stationary points.
   b. By investigating the sign of $f'(x)$, state the nature of these stationary points.
   c. Investigate the values of $f(x)$ as $x \to \pm\infty$. State the equations of any asymptotes.
   d. Calculate any axis intercepts.
   e. Sketch the curve of $y = f(x)$, showing all important features.
   f. State the domain and range of the function.

7. Consider the function $f(x) = \ln(x^2 + 1)$.
   a. Calculate $f'(x)$ and hence determine the coordinates of the stationary point.
   b. By investigating the sign of $f'(x)$, state the nature of the stationary point.
   c. Investigate the values of $f(x)$ as $x = \pm 1, \pm 2, \pm 3$. Explain why, in this logarithmic function, x-values can be negative.
   d. Sketch the curve of $y = f(x)$, showing all important features.
   e. State the domain and range of the function.
8. Consider the function $y = (x - 2)^2(x + 3)^2$.
   a. Differentiate the function with respect to x.
   b. Determine the coordinates of any stationary points and their nature.
   c. Sketch the function, clearly showing all important features.
   d. State the domain and range of the function.
9. The graph of $y = e^{-x^2}(1 - x)$ is shown.



   a. Determine the coordinates of the points where the graph cuts the x- and y-axes.
   b. Determine the coordinates of the points where the gradient is 0, giving your answers correct to 3 decimal places.
   c. Determine the equation of the tangent to the curve at the point where the curve intersects the x-axis.
   d. Determine the equation of the line perpendicular to the curve where the curve crosses the y-axis.
   e. Where do the tangent and the perpendicular line from parts **c** and **d** intersect? Give your answer correct to 2 decimal places.
10. The graph of the function $f: R \to R, f(x) = 3x^3e^{-2x}$ is shown. The derivative may be written as $f'(x) = ae^{-2x}(bx^2 + cx^3)$ where a, b and c are constants.



   a. Calculate the exact values of a, b and c.
   b. Calculate the exact coordinates where $f'(x) = 0$.
   c. Determine the equation of the tangent to the curve at $x = 1$.

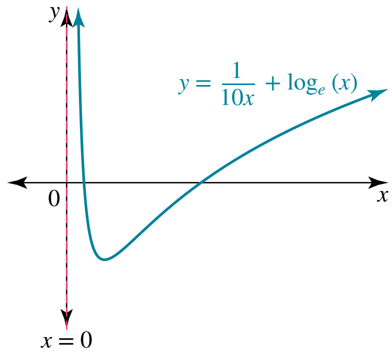
Technology active

11. The length of a snake, L cm, at any time t weeks after it is born is modelled as:

$$L = 12 + 6t + 2\sin\frac{\pi t}{4}, 0 \le t \le 20.$$

   Calculate:
   a. the length at:
      i. birth
      ii. 20 weeks
   b. R, the rate of growth, at any time, t
   c. the maximum and minimum growth rate.

12. The graph of the function $f: R^+ \to R, f(x) = \dfrac{1}{10x} + \log_e(x)$ is shown.



Use calculus to determine the coordinates of the minimum turning point.

13. **WE12** The profit, \$$P$, per item that a store makes by selling n items of a certain type each day is given by $P = 80\sqrt{n+8} - 15 - 5n$.
   a. Determine the number of items that need to be sold to maximise the profit on each item.
   b. Calculate:
      i. the maximum profit per item
      ii. the total profit per day made by selling this number of items.

14. The population of cheetahs, P, in a national park in Africa since 1 January 2010 can be modelled as $N = 100te^{-\frac{t}{12}} + 500$ where t is the number of years.

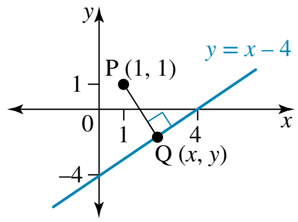
   a. When does this model predict that the maximum population will be reached?
   b. What is the maximum population of cheetahs that will be reached?
   c. How many cheetahs will there be on 1 January in:
      i. 2034?
      ii. 2094?

15. The amount of money in a savings account t years after the account was opened on 1 January 2009 is given by the equation

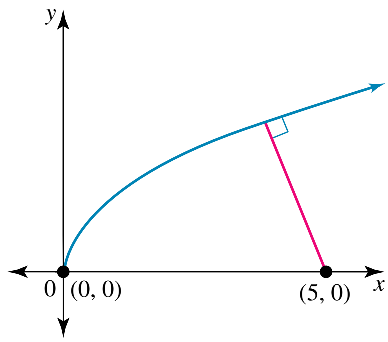
$$A(t) = 1000 - 12te^{\frac{4-t^3}{8}} \quad \text{for } t \in [0, 6].$$

   a. How much money was in the account when the account was first opened?
   b. What was the least amount of money in the account?
   c. When did the account contain its lowest amount? Give the year and month.
   d. How much money was in the account at the end of the six years?

16. **WE13** Determine the minimum distance from the point $(1, 1)$ to the straight line $y = x - 4$.



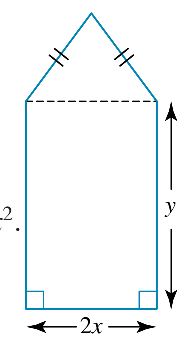
17. Find the minimum distance from the line $y = 2\sqrt{x}$ to the point $(5, 0)$.



18. **WE14** A rectangle with its base on the x-axis is inscribed in the semicircle $y = \sqrt{4 - x^2}$.
   a. Show that the area, A, of the rectangle is $A = 2x\sqrt{4 - x^2}$.
   b. Hence, determine the dimensions of the largest rectangle that can be inscribed in this semicircle.
   c. State the maximum area of the rectangle.

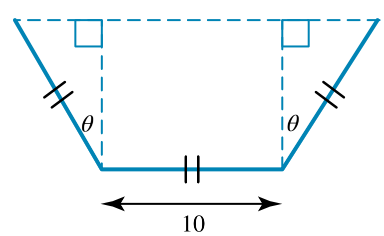
19. The owner of an apartment wants to create a stained glass feature in the shape of a rectangle surmounted by an isosceles triangle of height equal to half its base. This will be adjacent to a door opening on to a balcony.

   The owner has 150 cm of plastic edging to place around the perimeter of the figure, and wants to determine the dimensions of the figure with the greatest area.



   a. Show that the area, A in cm^2, of the stained glass feature is $A = 150x - \left(2\sqrt{2} + 1\right)x^2$.
   b. Hence, determine the width and the height of the figure for which the area is greatest. Give your answers correct to 1 decimal place.
   c. Due to structural limitations, the width of the feature should not exceed 30 cm. What should the dimensions of the stained glass feature of maximum area now be? Give your answer correct to 1 decimal place.

20. A metal gutter is to be formed from a sheet of metal 30 cm wide and 5 m long. The three sides of the gutter are to be equal in length, forming a trapezoidal cross section. The sides are folded so the angle between the vertical and the side is θ, as shown in the diagram.



   a. Show that the area, A, of the cross section is given by $A = 100\cos(\theta)(1 + \sin(\theta))$.
   b. State the restrictions on the value of θ for this metal gutter.
   c. Determine the value of θ that gives a maximum area of the cross section.
   d. Hence, calculate the maximum volume of the gutter that can be formed from this sheet of metal.

21. **WE15** A colony of viruses can be modelled by the rule

$$N(t) = \frac{2t}{(t + 0.5)^2} + 0.5$$

where N hundred thousand is the number of viruses on a nutrient plate t hours after they started multiplying.

a. How many viruses were present initially?
b. Find $N'(t)$.
c. What is the maximum number of viruses, and when will this maximum occur?
d. At what rate were the virus numbers changing after 10 hours?

22. A population of butterflies in an enclosure at a zoo is modelled by

$$N = 220 - \frac{150}{t+1}, t \geq 0$$

where N is the number of butterflies t years after observations of the butterflies commenced.
a. How long did it take for the butterfly population to reach 190 butterflies, and at what rate was the population growing at that time?
b. At what time was the growth rate 12 butterflies per year? Give your answer correct to 2 decimal places.
c. Determine the growth rate after 10 years. Give your answer correct to 2 decimal places.
d. Sketch the graphs of population versus time and rate of growth versus time, and explain what happens to each as $t \to \infty$.

23. **WE16** The displacement, x metres, of a particle after t seconds is given by the equation $x = 2\cos(4t) - 5$.
a. Derive an expression for the velocity, v m/s, of the particle.
b. The particle is initially at rest. Determine the next time the particle is at rest and its position at this time.
c. Derive an expression for acceleration, a m/s^2, of the particle and its initial acceleration.

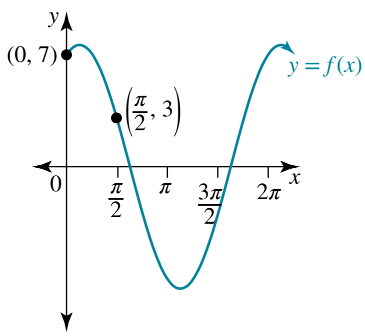
24. The displacement, x metres from the origin, of a particle moving in a straight line after t hours is given by the equation $x(t) = 6 - 4\sin\left(\frac{\pi}{6}t\right)$ for $0 \leq t \leq 24$.
a. State the period and amplitude for the function.
b. Determine the initial position of the particle.
c. Derive an expression for velocity, v m/h.
d. Determine the position of the particle when it is first at rest.
e. Sketch the function, $x(t)$. What observations can you make from the graph?

25. **WE17** A particle moves in a straight line such that its displacement, x metres, from a fixed origin at time t seconds is given by $x = 2t^2 - 8t, t \geq 0$.
a. Identify its initial position.
b. Derive an expression for its velocity and hence state its initial velocity and describe its initial motion.
c. At what time and position is the particle momentarily at rest?
d. Show that the particle is at the origin when $t = 4$ and calculate the distance it has travelled to reach the origin.

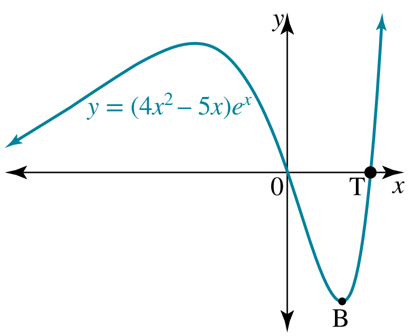
26. The position, in metres, of a particle after t seconds is given by $x(t) = -\frac{1}{3}t^3 + t^2 + 8t + 1, t \geq 0$.
a. Find its initial position and initial velocity.
b. Calculate the distance travelled before it changes its direction of motion.
c. What is its acceleration at the instant it changes direction?

27. The position, x m, relative to a fixed origin of a particle moving in a straight line at time t seconds is $x = \frac{2}{3}t^3 - 4t^2, t \geq 0$.
a. Show the particle starts at the origin from rest.
b. At what time and at what position is the particle next at rest?
c. When does the particle return to the origin?
d. What are the particle's speed and acceleration when it returns to the origin?

28. A ball is thrown vertically upwards into the air so that after t seconds its height h metres above the ground is $h = 50t - 4t^2$.
   a. At what rate is its height changing after 3 seconds?
   b. Calculate its velocity when $t = 5$.
   c. At what time is its velocity -12 m/s and in what direction is the ball then travelling?
   d. When is its velocity zero?
   e. What is the greatest height the ball reaches?
   f. At what time and with what speed does the ball strike the ground?
29. The profile of water waves produced by a wave machine in a scientific laboratory is modelled by the trigonometric function $f(x) = a\sin(x) + b\cos(x)$.



   a. Given that the graph of the wave profile passes through the points $(0, 7)$ and $\left(\frac{\pi}{2}, 3\right)$, calculate the constants a and b.
   b. Determine the maximum and minimum swells for the wave profile, correct to 1 decimal place. Hence, state the range of the function.
   c. Determine $\{x : f(x) = 0, 0 \le x \le 2\pi\}$, giving your answers correct to 4 decimal places.
   d. Evaluate, correct to 3 decimal places, the gradient at the x-values found in part **c**. Comment on your results.
30. A country town has decided to construct a new road. The x-axis is also the position of the railway line that connects Sydney with Brisbane. The road can be approximated by the equation $y = (4x^2 - 5x)e^x$.



   a. The post office for the town is positioned at $(-2, 3.5)$. They want the new road to be adjacent to the post office. Have they made a sensible decision regarding the placement of the road?
   b. Determine the coordinates of the point T where the road crosses the railway line.
   c. Use calculus to determine the coordinates of the point B. Give your answer correct to 3 decimal places.

5.6 Review: exam practice

A summary of this chapter is available in the Resources section of your eBookPLUS at www.jacplus.com.au.

Simple familiar

1. Determine $\frac{dy}{dx}$ for each of the following functions.
   a. $y = 3(2x^2 + 5x)^5$
   b. $y = (4x - 3x^2)^{-2}$
   c. $y = \left(x + \frac{1}{x}\right)^6$
   d. $y = 4(5 - 6x)^{-4}$
2. Use the product rule to differentiate each of the following.
   a. $y = x^2 \sin(x)$
   b. $y = 3x \sin(x)$
   c. $y = x^5 \cos(3x + 1)$
   d. $y = \sin(x)\cos(x)$
   e. $y = 8\sin(5x)\log_e(5x)$
   f. $y = 5\cos(2x)\sin(x)$
3. Differentiate each of the following, expressing your answer in simplest form.
   a. $y = \sin\left(\frac{4x}{3}\right)\cos(x)$
   b. $y = 2x^{-3}\sin(2x + 3)$
   c. $y = 4e^{-5x}\sin(2 - x)$
   d. $y = \frac{1}{\sqrt{x}}\cos(6x)$
   e. $y = \sin x \log_e(x)$
   f. $y = \pi x \cos(2\pi x)$
4. Determine the derivative of each of the following.
   a. $y = \frac{\sin(x)}{x}$
   b. $y = \frac{\sin(4x)}{\cos(2x)}$
   c. $y = \frac{\cos(x)}{x}$
   d. $y = \frac{\cos(x)}{e^x}$
   e. $y = \frac{\sin\left(\sqrt{x}\right)}{x}$
   f. $y = \frac{2\cos(3 - 2x)}{x^2}$
5. **MC** The derivative of $f(x) = x^2\sin(2x)$ is:
   A. $f'(x) = 4x\cos(2x)$
   B. $f'(x) = 2x\sin(2x) + x^2\cos(2x)$
   C. $f'(x) = 2x\sin(2x) + 2x^2\cos(2x)$
   D. $f'(x) = 2x\sin(x) + 2x^2\cos(x)$
6. **MC** The derivative of $f(x) = \frac{\sin(4x)}{4x + 1}$ is:
   A. $f'(x) = \frac{4(4x + 1)\cos(x) - 4\sin(4x)}{(4x + 1)^2}$
   B. $f'(x) = \frac{(4x - 1)\cos(4x) - 4\sin(4x)}{(4x + 1)^2}$
   C. $f'(x) = \frac{4(4x - 1)\cos(4x) - 4\sin(4x)}{(4x + 1)^2}$
   D. $f'(x) = \frac{4\sin(4x) - 4(4x - 1)\cos(4x)}{(4x + 1)^2}$
7. Given that $y = (x^2 + 1)e^{3x}$, determine the equation of the tangent to the curve at $x = 0$.
8. A curve is represented by the equation $y = ax\cos(3x)$ where a is a constant.
   a. If $\frac{dy}{dx} = -5$ when $x = \pi$, what is the value of a?
   b. Determine the equation of the line perpendicular to the curve at $x = \frac{\pi}{3}$.
9. The function $f: R \to R, f(x) = 6\log_e(x^2 - 4x + 8)$ has one stationary point.
   a. Use calculus to determine the coordinates of this stationary point.
   b. Determine the nature of this stationary point.
   c. Use technology to sketch the function, f.
10. Sketch the following functions by determining their stationary points and any axis intercepts. State the range of each function.
    a. $y = x^4 - 4x^3$
    b. $y = \frac{4}{x^2 + 1}$
11. A particle moves in a straight line so that at time t seconds its displacement, x metres, from a fixed origin O is given by $x(t) = t^3 - 6t^2 + 9t, t \geq t \geq 0$.

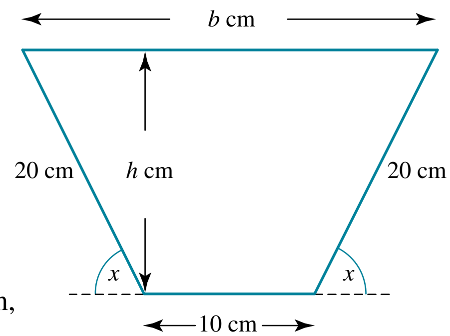
a. How far is the particle from O after 2 seconds?
b. What is the velocity of the particle after 2 seconds?
c. After how many seconds does the particle reach the origin again, and what is its velocity at that time?
d. What is the particle's acceleration when it reaches the origin again?

12. A particle moves in a straight line so that its displacement a point, O, at any time, t, is $x = \sqrt{3t^2 + 4}$. Determine:
a. the velocity as a function of time
b. the acceleration as a function of time
c. the velocity and acceleration when $t = 2$.

Complex familiar

13. Metal box guttering has to be formed on a common wall between two adjacent town houses. The cross section of the box guttering is shown.

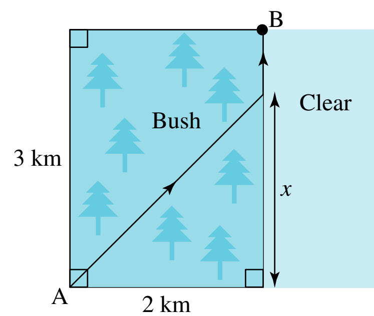
For the most efficient elimination of rain water, this box guttering needs to have a maximum cross-sectional area within the given dimensions.



a. Determine an expression for h, the height of the trapezium, in terms of the angle x in radians, as shown.
b. Determine an expression for b, the base length of the trapezium, in terms of x.
c. Show that the cross-sectional area of the box guttering, A cm^2, is given by $A = 200\sin(x)(2\cos(x) + 1)$.
d. Determine, correct to 3 decimal places, the value of x that gives maximum cross-sectional area, and find this maximum area correct to the nearest cm^2.

14. A bushwalker can walk at a rate of 5 km/h through clear land and 3 km/h through bushland. She has to get from point A to point B following a route indicated on the diagram.

$\left(\textit{Note:}\ \text{Time} = \dfrac{\text{distance}}{\text{speed}}\right)$



a. Determine the distance walked in terms of x:
i. through the bush
ii. through clear land.
b. If the total time taken is T hours, express T as a function of x.
c. Derive $\dfrac{dT}{dx}$.
d. Hence, determine the value of x so that the route is covered in a minimum time.
e. Calculate the minimum time to complete this route. Give your answer in hours and minutes.

15. The line perpendicular to the graph $y = g(f(x))$, where $f(x) = \dfrac{1}{x}$ and $g(x) = x - \dfrac{1}{x^2}$, is given by $y = -x + a$, where a is a real constant. Calculate the possible value(s) of a.

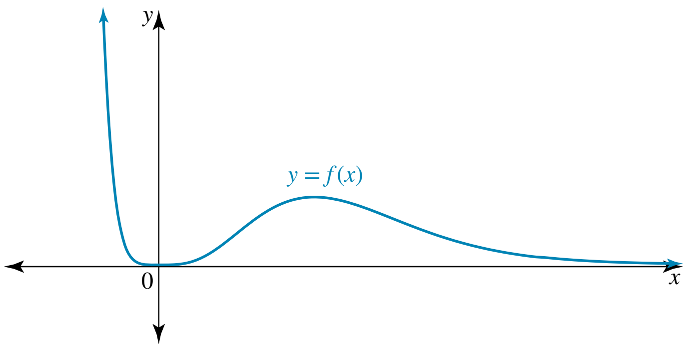
16. Consider the functions $f(x) = 2\sin(x)$ and $h(x) = e^x$.
a. State the rule for:
i. $m(x) = f(h(x))$
ii. $n(x) = h(f(x))$
b. Determine when $m'(x) = n'(x)$ over the interval $x \in [0, 3]$, correct to 3 decimal places.

Complex unfamiliar

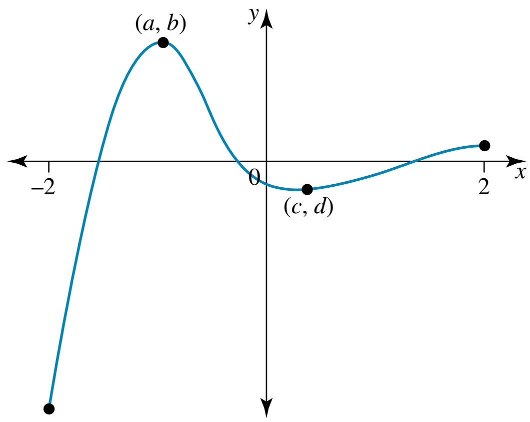
17. If $\dfrac{d}{dx}\left(\dfrac{e^{-3x}}{e^{2x}+1}\right) = \dfrac{e^{-x}(a + be^{-2x})}{(e^{2x}+1)^2}$, calculate the exact values of a and b.

18. a. Consider the function $f: R \to R\, f(x) = x^4 e^{-3x}$. The derivative $f'(x)$ may be written in the form $f'(x) = e^{-3x}(mx^4 + nx^3)$, where m and n are real constants. Calculate the exact values of m and n.

b. The graph of f is shown. Determine the coordinates of the stationary points.



19. Let $f: [-2, 2] \rightarrow R,\ f(x) = \dfrac{\sin(2x - 3)}{e^x}$. The graph of this function is shown.



a. The stationary points occur at (a, b) and (c, d). Calculate the values of a, b, c and d, giving your answers correct to 3 decimal places.

b. Determine the gradient of the tangent to the curve at the point where $x = 1$, correct to 3 decimal places.

20. The population of rabbits on a particular island t weeks after a virus is introduced is modelled by $P = 1200e^{-0.1t}$, where P is the number of rabbits. Determine:

a. the time taken for the population to halve (to the nearest week)

b. the rate of decrease of the population after:

i. 2 weeks **ii.** 10 weeks.

After 15 weeks the virus has become ineffective and the population of rabbits starts to increase again according to the model

$$p = p_0 + 10(t - 15)\log_e(2t - 29)$$

where t is the number of weeks since the virus was first introduced.

Calculate:

c. the value of P_0

d. the population after 30 weeks

e. the rate of change of the population after

i. 20 weeks **ii.** 30 weeks

f. how many weeks the population takes to get back to its original number.

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Units 3 & 4 Sit exam

Answers

5 Further differentiation and applications

Exercise 5.2 The chain rule

1. a. $\frac{dy}{dx} = 15(5x-4)^2$ b. $\frac{dy}{dx} = \frac{3}{2\sqrt{3x+1}}$
c. $\frac{dy}{dx} = \frac{-8}{(2x+3)^5}$ d. $\frac{dy}{dx} = \frac{4}{(7-4x)^2}$
e. $\frac{dy}{dx} = \frac{-30}{(5x+3)^7}$ f. $\frac{dy}{dx} = -4\sqrt[3]{4-3x}$
2. a. $\frac{dy}{dx} = 6(3x+2)$ b. $\frac{dy}{dx} = -3(7-x)^2$
c. $\frac{dy}{dx} = \frac{-2}{(2x-5)^2}$ d. $\frac{dy}{dx} = \frac{8}{(4-2x)^5}$
e. $\frac{dy}{dx} = \frac{5}{2\sqrt{5x+2}}$ f. $\frac{dy}{dx} = \frac{-9}{2(3x-2)^{\frac{3}{2}}}$
3. a. $\frac{dy}{dx} = -15(4-3x)^4$
b. $\frac{dy}{dx} = \frac{3x}{\sqrt{3x^2-4}}$
c. $\frac{dy}{dx} = \frac{2}{3}(x-2)(x^2-4x)^{-\frac{2}{3}}$
d. $\frac{dy}{dx} = -2(6x^2+1)(2x^3+x)^{-3}$
e. $\frac{dy}{dx} = 6\left(1+\frac{1}{x^2}\right)\left(x-\frac{1}{x}\right)^5$
f. $\frac{dy}{dx} = -(2x-3)(x^2-3x)^{-2}$
4. a. $\frac{dy}{dx} = 2\ \cos(x)\sin(x)$ b. $\frac{dy}{dx} = -3\ \sin(3x)e^{\cos(3x)}$
5. $\frac{9}{8}$
6. a. $-\frac{6x}{(x^2+1)^2}$ b. $-\sin(x)e^{\cos(x)}$ c. $\frac{x+1}{\sqrt{x^2+2x+3}}$
d. $\frac{-2\cos(x)}{\sin^3(x)}$ e. $\frac{x-2}{\sqrt{x^2-4x+5}}$
7. a. $g'(x) = \frac{-3}{(6x-5)^{\frac{3}{2}}}$ b. $g'(x) = 5x(x^2+2)^{\frac{3}{2}}$
8. a. $-6x\ \sin(x^2-1)$ b. $30xe^{3x^2-1}$
c. $-\frac{6x^5+8}{x^3\left(x^3-\frac{2}{x^2}\right)^3}$ d. $\frac{1}{2(2-x)^{\frac{3}{2}}}$
e. $-6\ \sin(2x+1)\cos^2(2x+1)$
9. $\sqrt{e}$
10. a. $\frac{16}{27}$ b. $-4e^2$ c. 8 d. 15
11. $(0,0), \left(\frac{\pi}{4},1\right), \left(\frac{\pi}{2},0\right), \left(\frac{3\pi}{4},1\right), (\pi,0)$
12. B
13. D
14. C
15. D
16. a. $m=2, n=4$
b. $h'(x) = \frac{(x+3)}{\sqrt{(x+2)(x+4)}}$

Exercise 5.3 The product rule

1. a. $f'(x) = 3\cos^2(3x) - 3\sin^2(3x)$
b. $f'(x) = 3x^2e^{3x} + 2xe^{3x}$
c. $f'(x) = (5x^2+17x-22)e^{5x}$
2. a. $\frac{dy}{dx} = x(x+1)^4(7x+2)$
b. $\frac{dy}{dx} = x^2(2x-1)^3(14x-3)$
c. $\frac{dy}{dx} = 3(4x+1)^2(3x-2)^4(32x-3)$
3. a. $\frac{dy}{dx} = \frac{(x+1)^4(11x+1)}{2\sqrt{x}}$
b. $\frac{dy}{dx} = \frac{(3x+2)}{2\sqrt{x+1}}$
c. $\frac{dy}{dx} = \frac{e^{4x}(1+8x)}{2\sqrt{x}}$
4. a. $\frac{dy}{dx} = 5x^2e^{5x} + 2xe^{5x}$
b. $\frac{dy}{dx} = \frac{2(x-1)(2x+1)^2}{x^3}$
c. $\frac{dy}{dx} = -x\sin(x) + \cos(x)$
d. $\frac{dy}{dx} = \frac{4-3x}{\sqrt{x}}$
5. a. $\frac{dy}{dx} = \frac{6e^{x^2}(x^2-1)}{x^3}$
b. $\frac{dy}{dx} = \frac{2e^{2x}(4x^2+2x-1)}{\sqrt{4x^2-1}}$
c. $\frac{dy}{dx} = 2x\sin^2(2x)[3x\cos(2x) + \sin(2x)]$
d. $\frac{dy}{dx} = \frac{2(x-5)(x-1)^3}{(x-3)^3}$
6. $-\pi^3$
7. $f'(x) = \sin(x) + (x+1)\cos(x)$; gradient $= 1$
8. $(0,0), (1,0), \left(\frac{2}{5}, \frac{216}{3125}\right)$
9. a. $x = 0, \pi, 2\pi, 3\pi$
b. $x = 1.11, 4.25, 7.39$
10. a. 0 b. 112 c. 0
11. D
12. C
13. A
14. $f'(2a) = 3a(a+4)$
15. $a = \frac{6}{\pi}$

Exercise 5.4 The quotient rule

1. a. $u = x+3;\ v = x+7$ b. $\frac{du}{dx} = 1; \frac{dv}{dx} = 1$
c. $\frac{dy}{dx} = \frac{4}{(x+7)^2}$

2. a. $u = x^2 + 2x$; $v = 5 - x$

b. $\dfrac{du}{dx} = 2x + 2$; $\dfrac{dv}{dx} = -1$

c. $f'(x) = \dfrac{10 + 10x - x^2}{(5 - x)^2}$

3. a. $\dfrac{dy}{dx} = \dfrac{-2(x^2 + 4)}{(x^2 - 4)^2}$

b. $\dfrac{dy}{dx} = \dfrac{3x^2 + 4x - 4}{(3x + 2)^2}$

c. $f'(x) = \dfrac{19}{(10 - 3x)^2}$

4. C

5. a. $\dfrac{dy}{dx} = \dfrac{e^{2x}(2 + e^x)}{(e^x + 1)^2}$

b. $\dfrac{dy}{dt} = \dfrac{-3(t\sin(3t) + \cos(3t))}{t^4}$

6. $\dfrac{-1}{(x-1)^2}$

7. 1

8. $\dfrac{3}{5}$

9. a. $\dfrac{dy}{dx} = \dfrac{4x^2\sin(x^2)\cos(x^2) - \sin^2(x^2)}{x^2}$

b. $\dfrac{dy}{dx} = \dfrac{-6x^2 + 4x - 9}{(2x^2 - 3)^2}$

c. $\dfrac{dy}{dx} = \dfrac{e^x\cos(2x + 1) + 2e^x\sin(2x + 1)}{\cos^2(2x + 1)}$

d. $\dfrac{dy}{dx} = -\dfrac{xe^{-x}}{(x-1)^2}$

10. a. $\dfrac{dy}{dx} = \dfrac{2x\cos(x) - \sin(x)}{2x\sqrt{x}}$

b. $\dfrac{dy}{dx} = \dfrac{-3}{2}\sqrt{5 - x}$

c. $f'(x) = \dfrac{1}{4\sqrt{x}} - 3\sqrt{x}$

d. $\dfrac{dy}{dx} = \dfrac{6 - 3x}{2\sqrt{x}(x + 2)^2}$

11. a. $\dfrac{dy}{dx} = \dfrac{2}{\cos^2(2x)}$

b. $\dfrac{dy}{dx} = \dfrac{-4}{\cos^2(-4x)}$

c. $\dfrac{dy}{dx} = \dfrac{1}{5\cos^2\left(\frac{x}{5}\right)}$

d. $\dfrac{dy}{dx} = \dfrac{-3}{4\cos^2\left(\frac{-3x}{4}\right)}$

12. a. 0 b. 2 c. $\dfrac{7}{64}$ d. -5

13. $\dfrac{-1}{32}$

14. Sample responses can be found in the worked solutions in the online resources.

15. a. Sample responses can be found in the worked solutions in the online resources.

b. 2

16. $m = 2$

Exercise 5.5 Applications of differentiation

1. a. $\dfrac{dy}{dx} = \dfrac{(3x + 1)}{\sqrt{3x^2 + 2x}}$

b. $y = \dfrac{7}{4}x + \dfrac{1}{2}$ or $7x - 4y + 2 = 0$

2. $y = -4x + 5$

3. a. $\dfrac{2}{9}$

b. $y = \dfrac{2}{9}x + \dfrac{11}{9}$ or $2x - 9y + 11 = 0$

4. Tangent: $y = 2e\,x - e$; normal: $y = -\dfrac{1}{2e}x + \left(\dfrac{1 + 2e^2}{2e}\right)$

5. a. $m = -7, n = 1$

b. $\{x : x \le -1\} \cup \{x : x \ge 7\}$

c. $\dfrac{x - 3}{\sqrt{x^2 - 6x - 7}}$

d. $\dfrac{-5}{3}$

6. a. $f'(x) = \dfrac{1 - x^2}{(x^2 + 1)^2}$

Stationary points: $\left(-1, -\dfrac{1}{2}\right)$ and $\left(1, \dfrac{1}{2}\right)$

b. Local minimum stationary point at $\left(-1, -\dfrac{1}{2}\right)$

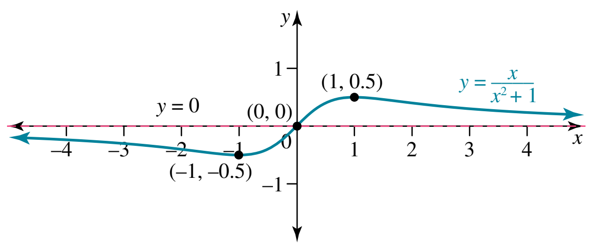
Local maximum stationary point at $\left(1, \dfrac{1}{2}\right)$

c. As $x \to \infty$, $\dfrac{x}{x^2 + 1} \to 0$ (positive side).

As $x \to -\infty$, $\dfrac{x}{x^2 + 1} \to 0$ (negative side). Equation of asymptote: $y = 0$

d. Intercept at $(0, 0)$

e.



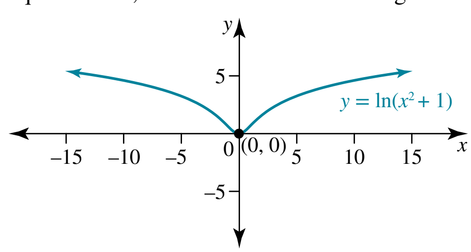
f. Domain: $x \in R$; range $-\dfrac{1}{2} \le y \le \dfrac{1}{2}$

7. a. $f'(x) = \dfrac{2x}{x^2 + 1}$; stationary point: $(0, 0)$

b. Local minimum turning point

c. For all x-values, $x^2 \ge 0$ and $x^2 + 1 \ge 1$. Hence, this logarithmic function is defined and is greater than or equal to zero, even when x-values are negative.

d.

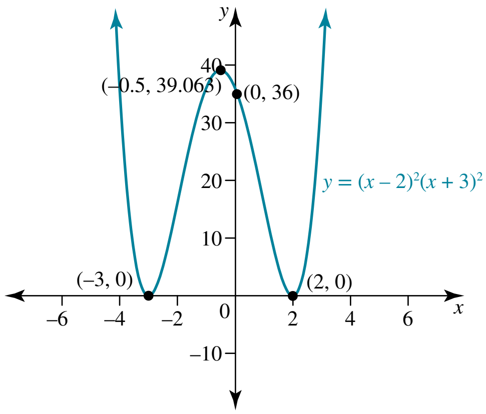


e. Domain: $x \in R$; range: $y \ge 0$

8. a. $\dfrac{dy}{dx} = 2(x - 2)(x + 3)(2x + 1)$

b. Local minimum stationary points at $(-3, 0)$ and $(2, 0)$

Local maximum stationary point at $\left(\dfrac{-1}{2}, \dfrac{625}{16}\right)$

c. 

d. Domain: $x \in R$; range: $y \geq 0$

9. a. $(0, 1)$ and $(1, 0)$

b. $(-0.366, 1.195)$ and $(1.366, -0.057)$

c. $y = -\frac{1}{e}x + \frac{1}{e}$ or $x + ey - 1 = 0$

d. $y = x + 1$

e. $(-0.46, 0.54)$

10. a. $a = 3, b = 3, c = -2$

b. Stationary point of inflection at $(0, 0)$; local maximum turning point at $\left(\frac{3}{2}, \frac{81}{8e^3}\right)$

c. $y = \frac{3}{e^2}x$ or $3x - e^2y = 0$

11. a. i. 12 cm

ii. 132 cm

b. $\frac{dL}{dt} = 6 + \frac{\pi}{2}\cos\left(\frac{\pi t}{4}\right)$

c. Maximum rate of growth $= 6 + \frac{\pi}{2} \approx 7.571$ cm/week; minimum rate of growth $= 6 - \frac{\pi}{2} \approx 4.429$ cm/week

12. $\left(0.1, 1 - \log_e(10)\right)$

13. a. 56 items

b. i. \$345 ii. \$19 320

14. a. 12 years, 1 January 2022

b. 941 cheetahs

c. i. 824 cheetahs ii. 507 cheetahs

15. a. \$1000 b. \$980.34

c. May 2010 d. \$1000

16. $2\sqrt{2}$ units

17. 4 units

18. a. Sample responses can be found in the worked solutions in the online resources.

b. Base: $2\sqrt{2}$ units; height: $\sqrt{2}$ units

c. 4 square units

19. a. Sample responses can be found in the worked solutions in the online resources.

b. Width: 39.2 cm; height: 47.3 cm

c. Width: 30 cm; height: 53.8 cm

20. a. Sample responses can be found in the worked solutions in the online resources.

b. $0 \leq \theta < \frac{\pi}{2}$

c. $\theta = \frac{\pi}{6}$

d. $37\,500\sqrt{3}$ cm^3

21. a. 50 000 viruses

b. $\frac{0.5 - 2t^2}{(t + 0.5)^4}$

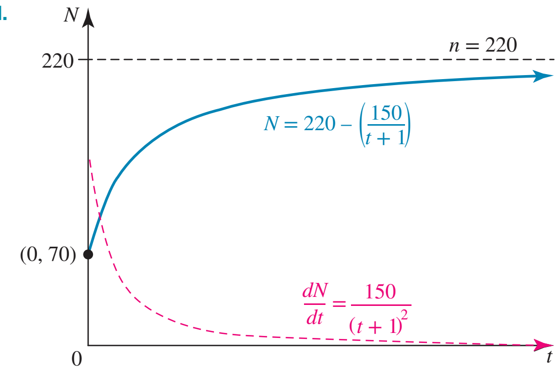
c. 150 000 viruses after half an hour

d. −1641 viruses/hour

22. a. 4 years; 6 butterflies/year

b. 2.54 years

c. 1.24 butterflies/year

d. 

As $t \to \infty$, $N \to 220$ and $\frac{dN}{dt} \to 0$.

23. a. $v = -8\sin(4t)$

b. $\frac{\pi}{4}$ s; −7 m

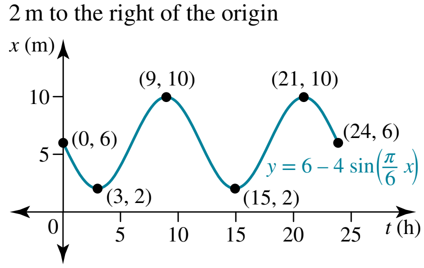
c. $a = -32\cos(4t)$; -32 m/s^2

24. a. 12 h; 4

b. 6 m to the right of the origin

c. $v = -\frac{2\pi}{3}\cos\left(\frac{\pi}{6}t\right)$

d. 2 m to the right of the origin

e. 

The particle is at rest at the turning points of the curve where the displacement is 2 m and 10 m. The particle oscillates between these two positions.

25. a. At the origin

b. −8 m/s; travelling to the left of the origin at 8 m/s

c. 2 s; −8 m (8 m to the left of the origin)

d. 16 m

26. a. 1 m; 8 m/s b. $27\frac{2}{3}$ m c. -6 m/s^2

27. a. Sample responses can be found in the worked solutions in the online resources.

b. 4 s; $21\frac{1}{3}$ m to the left of the origin

c. 6 s

d. 24 m/s; 16 m/s^2

28. a. 26 m/s

b. 10 m/s

c. 7.75 s travelling downwards

d. 6.25 s

e. 156.25 m

f. 12.5 s; 50 m/s

29. a. $a = 3, b = 7$
b. Maximum swell = 7.6 units;
minimum swell = −7.6 units;
range = [−7.6, 7.6]
c. $x = 1.9757$ or 5.1173
d. $f'(1.9757) = -7.616; f'(15.1173) = 7.616$
The gradients are equal in magnitude (size), just differing in direction. One is when the swell is going down, and the other is when the swell is rising. This is due to the symmetry of the curve representing the swell.

30. a. Yes
b. $T = \left(\frac{5}{4}, 0\right)$
c. $B = (0.804, -3.205)$

5.6 Review: exam practice

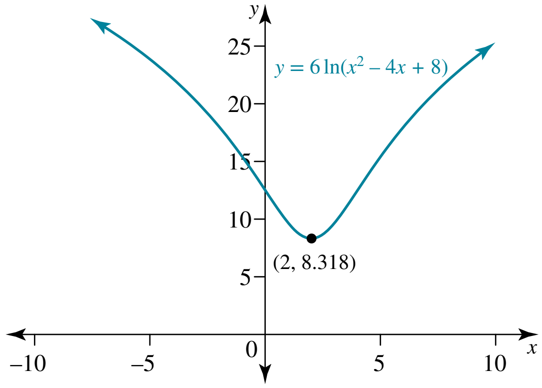
1. a. $15(4x + 5)(2x^2 + 5x)^4$
b. $-4(2 - 3x)(4x - 3x^2)^{-3}$
c. $6\left(x + \frac{1}{x}\right)^5\left(1 - \frac{1}{x^2}\right)$
d. $96(5 - 6x)^{-5}$

2. a. $x^2\cos(x) + 2x\sin(x)$
b. $3x\cos(x) + 3\sin(x)$
c. $5x^4\cos(3x + 1) - 3x^5\sin(3x + 1)$
d. $\cos^2(x) - \sin^2(x)$
e. $\frac{8}{x}\sin(5x) + 40\cos(5x)\log_e(5x)$
f. $5\cos(x)\cos(2x) - 10\sin(x)\sin(2x)$

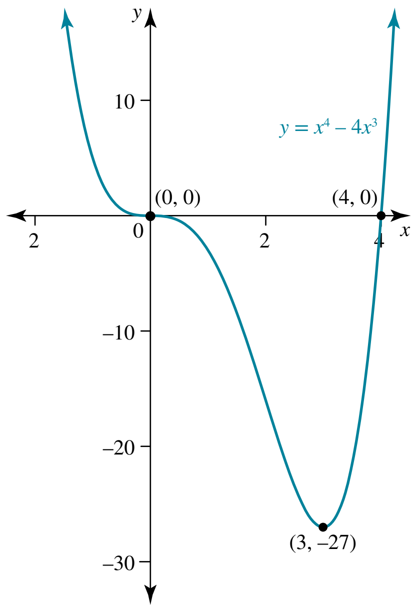
3. a. $\frac{4}{3}\cos(x)\cos\left(\frac{4x}{3}\right) - \sin(x)\sin\left(\frac{4x}{3}\right)$
b. $4x^{-3}\cos(2x + 3) - 6x^{-4}\sin(2x + 3)$
c. $-4e^{-5x}\cos(2 - x) - 20e^{-5x}\sin(2 - x)$
d. $\frac{-6\sin(6x)}{\sqrt{x}} - \frac{\cos(6x)}{2\sqrt{x^3}}$
e. $\frac{1}{x}\sin(x) + \cos(x)\log_e(x)$
f. $\pi\cos(2\pi x) - 2\pi^2 x\sin(2\pi x)$

4. a. $\frac{x\cos(x) - \sin(x)}{x^2}$
b. $\frac{4\cos(2x)\cos(4x) + 2\sin(2x)\sin(4x)}{\cos^2(2x)}$
c. $\frac{-x\sin(x) - \cos(x)}{x^2}$
d. $\frac{-\left(\sin(x) + \cos(x)\right)}{e^x}$
e. $\frac{\sqrt{x}\cos\left(\sqrt{x}\right) - 2\sin\left(\sqrt{x}\right)}{2x^2}$
f. $\frac{4x\sin(3 - 2x) - 4\cos(3 - 2x)}{x^3}$

5. C
6. A
7. $y = 3x + 1$
8. a. 5
b. $y = \frac{1}{5}x - \frac{26\pi}{15}$
9. a. $\left(2, 12\log_e(2)\right)$
b. Local minimum
c.

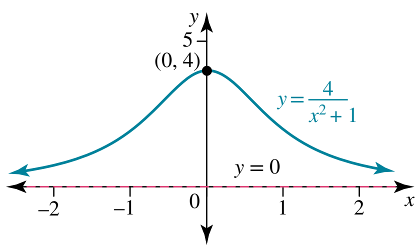


10. a.



Stationary points: $(0, 0)$ and $(3, -27)$
Axis intercepts: $(0, 0)$ and $(4, 0)$
Range: $y \geq -27$

b.



Stationary point: $(0, 4)$
Axis intercept: $(0, 4)$
Range: $0 < y \leq 4$

11. a. 2 m
b. −3 m/s
c. 3 s; at rest ($v = 0$)
d. 6 m/s^2

12. a. $v = \frac{3t}{\sqrt{3t^2 + 4}}$
b. $a = \frac{12}{\left(\sqrt{3t^2 + 4}\right)^3}$
c. $v = 1.5; a = \frac{3}{16}$

13. a. $h = 20\sin(x)$
b. $b = 10 + 40\cos(x)$
c. Sample responses can be found in the worked solutions in the online resources.
d. For a maximum, the angle x is 0.936^c and the maximum area is 352 cm^2.

14. a. i. $\sqrt{4+x^2}$ km

ii. $(3-x)$ km

b. $T = \frac{3}{5} - \frac{x}{5} + \frac{1}{3}\sqrt{4+x^2}$

c. $\frac{dT}{dx} = \frac{x}{3\sqrt{4+x^2}} - \frac{1}{5}$

d. 1.5 km

e. 1 h 8 min

15. $a = -3$

16. a. i. $m(x) = 2\sin(e^x)$

ii. $n(x) = e^{2\sin(x)}$

b. $x = 1.555, 2.105, 2.372, 2.844$

17. $a = -5, b = -3$

18. a. $m = -3, n = 4$

b. $(0, 0), \left(\frac{4}{3}, \frac{256}{81e^4}\right)$

19. a. $a = -1.088, b = 2.655, c = 0.483, d = -0.552$

b. 0.707

20. a. 7 weeks

b. i. 98.25 rabbits/week

ii. 44.15 rabbits/week

c. 267

d. 782 rabbits

e. i. 33 rabbits/week

ii. 44 rabbits/week

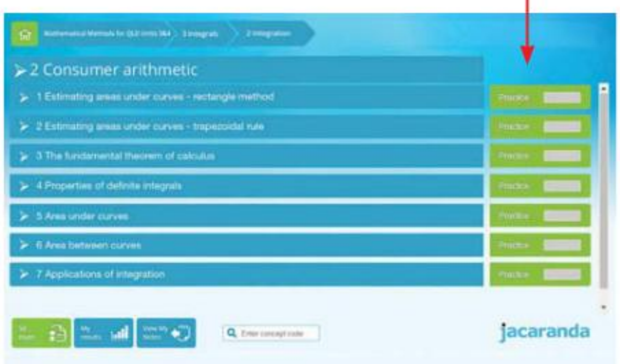
f. 39 weeks

TOPIC 2 Further differentiation and applications 2

- For revision of this entire topic, go to your **studyON** title in your bookshelf at **www.jacplus.com.au**.
- Select **Continue Studying** to access hundreds of revision questions across your entire course.

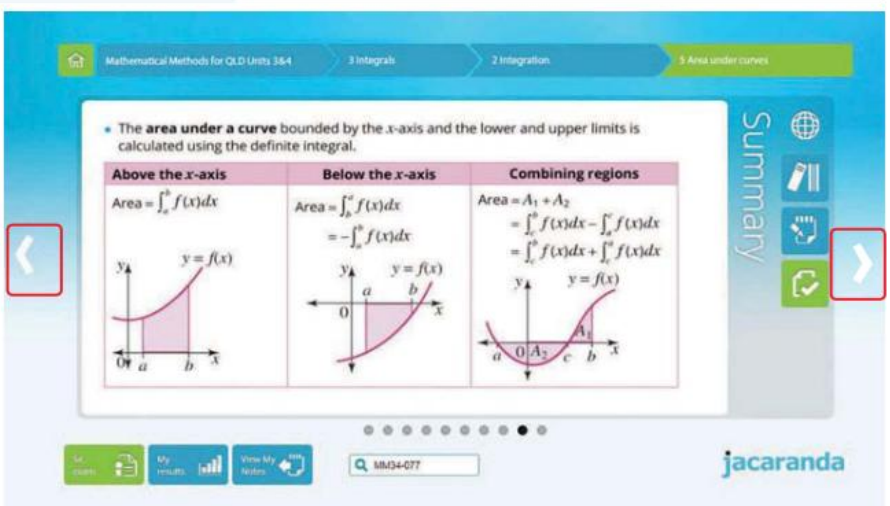
- Select your **course** *Mathematical Methods for Queensland Units 3&4* to see the entire course divided into syllabus topics.
- Select the **Area** you are studying to navigate into the chapter level ***OR*** select **Practice** to answer all practice questions available for each area.

- Select **Practice** at the sequence level to access all questions in the sequence.



- At **sequence level**, drill down to concept level.

- **Summary screens** provide revision and consolidation of key concepts. Select the **next arrow** to revise all concepts in the sequence and practise questions at the concept level.



6 Antidifferentiation

6.1 Overview

In the previous chapter, you studied differential calculus along with some of its applications. Calculus involves the study of change, how a change in one variable will affect another related variable. Antidifferentiation, also known as integration, is the reverse process of differentiation. It, too, involves the study of change. If the rate of change between two variables, the derivative, is known, the relationship connecting the variables may be found using antidifferentiation.

Historically, integration began with mathematicians attempting to find the area under curves. The Italian mathematician Bonaventura Cavalieri (1598–1647) is thought to have been the first to connect areas with integration. In the seventeenth century, Sir Isaac Newton, in England, and Gottfried Leibniz, in Germany, were working independently on areas and integration. They were the first mathematicians to recognise and prove that differentiation and integration were inverse, or reverse, processes. The notation that is commonly used for antidifferentiation was introduced by Leibniz. He based his symbol for the integral on a long 'S' character, standing for 'sum', as he thought of the integral as an infinite sum of the area of rectangles of infinitesimal width.

Calculus has many real-life applications. For example, if you know the velocity of a particle, its displacement can be found using antidifferentiation. Another example is if the rate of change of the temperature of an object is known, the temperature of the object at any time can be found. Economics, meteorology, construction, electronics and epidemiology (the study of the spread of infectious diseases) are just some of the areas where differential and integral calculus are used today.

The techniques of antidifferentiation, along with some useful formulas, are introduced in this chapter. The applications considered are finding a particular function given the gradient function, and the application to motion in a straight line.

LEARNING SEQUENCE

6.1 Overview
6.2 Antidifferentiation of rational functions
6.3 Antidifferentiation of exponential functions
6.4 Antidifferentiation of logarithmic functions
6.5 Antidifferentiation of sine and cosine functions
6.6 Further integration
6.7 Review: exam practice

Fully worked solutions for this chapter are available in the Resources section of your eBookPLUS at www.jacplus.com.au.

6.2 Antidifferentiation of rational functions

6.2.1 Antidifferentiation

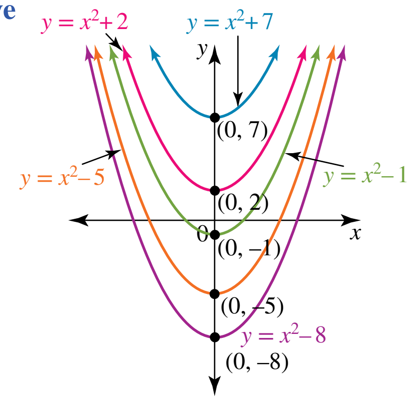
Antidifferentiation, also known as **integration**, is the reverse of differentiation. It allows us to determine $f(x)$ when we are given $f'(x)$.

Consider the following polynomial functions.

$$f(x) = x^2 + 7 \qquad g(x) = x^2 - 5 \qquad h(x) = x^2 + 2$$
$$f'(x) = 2x \qquad g'(x) = 2x \qquad h'(x) = 2x$$

The derivatives of the three functions all equal $2x$, so the **antiderivative** of $2x$ could be either $x^2 + 7$, $x^2 - 5$, $x^2 + 2$. These three functions differ by a constant.

In general, if $f'(x) = 2x$, then $f(x) = x^2 + c$, where c is a constant. This gives a family of curves that fit the criteria for the function f, that is $f'(x) = 2x$. Some of these curves are shown.



To determine a specific answer for $f(x)$ given $f'(x) = 2x$, additional information is required, such as a point through which the curve passes. This is discussed later in this chapter.

6.2.2 Notation for antiderivatives

An example of this notation is $\int 2x\,dx = x^2 + c$.

This equation indicates that the antiderivative, or **indefinite integral**, of $2x$ with respect to x is equal to $x^2 + c$.

The indefinite integral of $f(x)$ is $\int f(x)\,dx$.

This is read as 'the integral of the function $f(x)$ with respect to the variable x'.

The dx indicates that the variable is x.

Another example is $\int g(t)\,dt$, which reads as 'the integral of the function $g(t)$ with respect to the variable t'.

6.2.3 The antiderivative of x^n, $n \neq -1$

Consider the following.

$f(x) = x^3$	$f(x) = x^4$	$f(x) = x^5$	$f(x) = x^6$
$f'(x) = 3x^2$	$f'(x) = 4x^3$	$f'(x) = 5x^4$	$f'(x) = 6x^5$
$\therefore \int 3x^2\,dx = x^3 + c$	$\therefore \int 4x^3\,dx = x^4 + c$	$\therefore \int 5x^4\,dx = x^5 + c$	$\therefore \int 6x^5\,dx = x^6 + c$
So:	So:	So:	So:
$\int x^2 dx = \frac{1}{3}x^3 + c$	$\int x^3 dx = \frac{1}{4}x^4 + c$	$\int x^4 dx = \frac{1}{5}x^5 + c$	$\int x^5 dx = \frac{1}{6}x^6 + c$

This shows the general formula for integration is as follows.

The general formula for integration

$$\int x^n dx = \frac{1}{n+1}x^{n+1} + c, n \neq -1$$

where c is a constant.

This formula is true for all real values of n except for $n = -1$.

Consider the following.

$$f(x) = \sqrt{x}$$
$$= x^{\frac{1}{2}}$$
$$f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$$
$$\therefore \int \frac{1}{2}x^{-\frac{1}{2}}\,dx = \sqrt{x} + c$$

So:

$$\int x^{-\frac{1}{2}}\,dx = 2\sqrt{x} + c$$

$$f(x) = \frac{1}{x^2}$$
$$= x^{-2}$$
$$f'(x) = -2x^{-3}$$
$$\therefore \int -2x^{-3}dx = x^{-2} + c$$

So:

$$\int x^{-3}\,dx = \frac{-1}{2}x^{-2} + c$$

6.2.4 Properties of integration

Integration, being the reverse of differentiation, has the corresponding properties.

Properties of integration

$$\int (f(x) \pm g(x))\,dx = \int f(x)dx \pm \int g(x)dx$$

$$\int kf(x)\,dx = k\int f(x)\,dx$$

$$\int k\,dx = k\int 1\,dx = kx + c$$

where k and c are constants.

WORKED EXAMPLE 1

Determine the following.

a. $\int 6x^2 dx$ **b.** $\int \left(10x^4 - 5x^2 + 7\right) dx$

THINK	WRITE
a. 1. Apply the formula.	**a.** $\int 6x^2dx$ $= 6 \times \frac{x^3}{3} + c$

2. Simplify. $= 2x^3 + c$

b. 1. Integrate each term separately by applying the formula.

b. $\int (10x^4 - 5x^2 + 7)\,dx$

$= 10 \times \frac{1}{5}x^5 - 5 \times \frac{1}{3}x^3 + 7x + c$

2. Simplify. $= 2x^5 - \frac{5}{3}x^3 + 7x + c$

WORKED EXAMPLE 2

Determine y in terms of x if $\frac{dy}{dx} = 2\sqrt{x} + \frac{3}{x^3} - 4$.

THINK / **WRITE**

1. Express as powers of x.

$\frac{dy}{dx} = 2\sqrt{x} + \frac{3}{x^3} - 4$

$\frac{dy}{dx} = 2x^{\frac{1}{2}} + 3x^{-3} - 4$

2. Integrate each term separately by applying the formula.

$y = \int (2x^{\frac{1}{2}} + 3x^{-3} - 4)\,dx$

$= 2 \times \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + 3 \times \frac{x^{-2}}{-2} - 4x + c$

3. Simplify.

$= \frac{4}{3}x\sqrt{x} - \frac{3}{2x^2} - 4x + c$

WORKED EXAMPLE 3

Determine:

a. **an antiderivative of $(2x - 3)(4 - x)$**

b. $\int \left(\frac{x^4 - 2x^3 + 5}{x^3}\right) dx$

THINK / **WRITE**

a. 1. Expand the expression and simplify.

a. $\int (2x - 3)(4 - x)\,dx$

$= \int (8x - 2x^2 - 12 + 3x)\,dx$

$= \int (-2x^2 + 11x - 12)\,dx$

2. Integrate each term separately by applying the formula.

$= -2 \times \frac{x^3}{3} + 11 \times \frac{x^2}{2} - 12x + c$

3. Simplify.

$= -\frac{2}{3}x^3 + \frac{11}{2}x^2 - 12x + c$

b. 1. Expand the expression and simplify.

b. $\int \left(\frac{x^4 - 2x^3 + 5}{x^3}\right) dx$

$= \int (x^4 - 2x^3 + 5) \times x^{-3}\, dx$

$= \int (x - 2 + 5x^{-3})\, dx$

2. Integrate each term separately by applying the formula.

$= \frac{x^2}{2} - 2x + 5 \times \frac{x^{-2}}{-2} + c$

3. Simplify.

$= \frac{x^2}{2} - 2x - \frac{5}{2x^2} + c$

on Resources

Interactivity: Integration of ax^n (int-6419)

studyon

Units 3 & 4 > Area 3 > Sequence 1 > Concepts 1 & 2

Antidifferentiation Summary screen and practice questions
Antidifferentiation of rational functions Summary screen and practice questions

Exercise 6.2 Antidifferentiation of rational functions

Technology free

1. **WE1** Determine the following.

a. $\int x^7\, dx$ **b.** $\int (8x^3 + 4x)dx$

c. $\int (3x^2 + 5x - 8)\, dx$ **d.** $\int (2x^3 + 3x^2 - 6x - 9)\, dx$

2. Determine the following indefinite integrals.

a. $\int (2x + 5)\,\mathrm{d}x$ **b.** $\int \left(3x^2 + 4x - 10\right)\mathrm{d}x$ **c.** $\int \left(10x^4 + 6x^3 + 2\right)\mathrm{d}x$

d. $\int \left(-4x^5 + x^3 - 6x^2 + 2x\right)\mathrm{d}x$ **e.** $\int \left(x^3 + 12 - x^2\right)\mathrm{d}x$

3. **WE2** Determine y in terms of x if:

a. $\frac{dy}{dx} = 4\sqrt{x} - \frac{1}{x^2}$ **b.** $\frac{dy}{dx} = 6\sqrt{x} + \frac{3}{\sqrt{x}} + 8$

4. Integrate the following, expressing your answers with positive powers.

a. $\frac{x^4}{5}$ **b.** $\frac{x^3}{2}$ **c.** $\frac{x^{-4}}{3}$ **d.** $\sqrt{x}$ **e.** $x^{\frac{2}{3}}$ **f.** $4x^{\frac{3}{4}}$

5. Determine the antiderivatives of the following, expressing your answers with positive powers.

a. $x^{\frac{-3}{7}}$ **b.** $\frac{5}{x^3}$ **c.** $\frac{9}{x^2}$ **d.** $\frac{-10}{x^6}$ **e.** $\frac{8}{\sqrt{x}}$ **f.** $\frac{-6}{x\sqrt{x}}$

6. **WE3** Determine:

a. $\int (x + 3)(x - 7)\,dx$

b. $\int 5\left(x^2 + 2x - 1\right)\,dx$

c. $\int \left(x^2 + 4\right)(x - 7)\,dx$

d. $\int x(x - 1)(x + 4)\,dx$

7. **WE3** Determine:

a. $\int \frac{x^2 + x^4}{x}\,dx$

b. $\int \frac{x^2 + 2x - 1}{\sqrt{x}}\,dx$

c. $\int \frac{10 - x + 2x^4}{x^3}\,dx$

8. Given that $f'(x) = x^2 - \frac{1}{x^2}$, determine the rule for f.

9. Determine:

a. $\int x^3 dx$

b. $\int 7x^2 - \frac{2}{5x^3}\,dx$

c. $\int (4x^3 - 7x^2 + 2x - 1)dx$

d. $\int \left(2\sqrt{x}\right)^3 dx$

10. Determine:

a. $f(x)$ if $f'(x) = \frac{3}{2}x - 4x^2 + 2x^3$

b. an antiderivative of $\frac{3}{\sqrt{x}} - 4x^3 + \frac{2}{5x^3}$

c. $\int x(x - 3)(2x + 5)dx$

d. $\int \frac{3x^3 - x}{2\sqrt{x}}dx$

11. Calculate:

a. $\int \left(\frac{2}{\sqrt{x}} + \frac{3}{x^2} - \frac{1}{2x^3}\right)dx$

b. $\int (x + 1)(2x^2 - 3x + 4)dx$

12. Determine an antiderivative for each of the following functions.

a. $(2x + 3)(3x - 2)$

b. $\frac{x^3 + x^2 + 1}{x^2}$

c. $2\sqrt{x} - \frac{4}{\sqrt{x}}$

d. $\left(x^3 - \frac{2}{x^3}\right)^2$

13. The gradient function for a particular curve is given by $\frac{dy}{dx} = x^3 - 3\sqrt{x}$. Determine the general rule for the function, y.

14. Determine the general equation of the curve whose gradient at any point is given by $\frac{x^3 + 3x^2 - 3}{x^2}$.

15. Determine the general equation of the curve whose gradient at any point on the curve is given by $\sqrt{x} + \frac{1}{\sqrt{x}}$.

6.3 Antidifferentiation of exponential functions

As you have learned in Chapter 2:

for $y = e^x$ and $y = e^{ax}$,

$\frac{dy}{dx} = e^x$ and $\frac{dy}{dx} = ae^{ax}$.

Therefore, it follows that:

$$\int e^x\,dx = e^x + c \quad \text{and} \quad \int e^{ax}\,dx = \frac{1}{a}e^{ax} + c$$

where c is a constant.

Antidifferentiation of exponential functions

$$\int e^x\,dx = e^x + c$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + c$$

where a and c are constants.

Generally:

Antidifferentiation of exponential functions including constants

$$\int ke^{ax}\,dx = k\int e^{ax}\,dx$$

$$= k \times \frac{1}{a}e^{ax} + c$$

$$= \frac{k}{a}e^{ax} + c$$

where a, c and k are constants.

WORKED EXAMPLE 4

Determine the following.

a. $\int 8e^x\,dx$ **b.** $\int 8e^{2x}\,dx$

THINK	WRITE
a. Apply the formula.	**a.** $\int 8e^x\,dx$ $= 8\int e^x\,dx$ $= 8e^x + c$
b. 1. Apply the formula.	**b.** $\int 8e^{2x}\,dx$ $= 8\int e^{2x}dx$ $= 8 \times \frac{1}{2}e^{2x} + c$
2. Simplify.	$= 4e^{2x} + c$

WORKED EXAMPLE 5

Determine y if it is known that $\frac{dy}{dx} = (e^x + e^{-x})^2$.

THINK	WRITE
1. Expand the brackets.	$\frac{dy}{dx} = (e^x + e^{-x})^2$ $\frac{dy}{dx} = (e^x + e^{-x})(e^x + e^{-x})$ $\frac{dy}{dx} = (e^{2x} + e^0 + e^0 + e^{-2x})$
2. Simplify.	$\frac{dy}{dx} = (e^{2x} + 2 + e^{-2x})$
3. Integrate each term separately by applying the formula.	$y = \int (e^{2x} + 2 + e^{-2x})dx$ $y = \frac{1}{2}e^{2x} + 2x + \frac{1}{-2}e^{-2x} + c$
4. Simplify.	$y = \frac{1}{2}e^{2x} + 2x - \frac{1}{2}e^{-2x} + c$

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Units 3 & 4 > Area 3 > Sequence 1 > Concept 3

Antidifferentiation of exponential functions Summary screen and practice questions

Exercise 6.3 Antidifferentiation of exponential functions

Technology free

1. **WE4** Determine the following.
   a. $\int e^{2x}dx$ b. $\int e^{4x}dx$ c. $\int e^{-x}dx$ d. $\int e^{-3x}dx$ e. $\int 5e^{5x}dx$ f. $\int 7e^{4x}dx$
2. Determine the antiderivatives of the following.
   a. $e^{\frac{x}{3}}$ b. $0.1e^{\frac{x}{4}}$ c. $3e^{\frac{x}{2}}$ d. $3e^{\frac{-x}{3}}$ e. $e^x + e^{-x}$ f. $\frac{e^x - e^{-x}}{2}$
3. **WE5** Determine y if it is known that $\frac{dy}{dx} = (e^x - e^{-x})^2$.
4. Determine y if it is known that $\frac{dy}{dx} = \left(1 + e^{2x}\right)^2$.
5. If $\frac{dy}{dx} = \left(e^{3x} + 6\right)^2$, determine y as a function of x.
6. Determine:
   a. $\int (x^4 - e^{-4x})dx$ b. $\int \left(\frac{1}{2}e^{2x} - \frac{2}{3}e^{-\frac{x}{2}}\right) dx$

7. Determine:

a. $\int \frac{e^{2x}+3e^{-5x}}{2e^{x}}\,dx$ b. $\int (e^{x}-e^{2x})^{2}dx$

8. Determine the indefinite integral of $\left(e^{\frac{x}{2}}-\frac{1}{e^{x}}\right)^{2}$.

9. The gradient function of a curve is given by $f'(x)=4e^{2x}+8$. Determine the general rule for the function $f(x)$.

10. Determine the general rule for the function $y=f(x)$ if it is known that $\frac{dy}{dx}=e^{2x}\left(e^{2x}-e^{-2x}\right)$.

11. Determine the general equation of the curve whose gradient at any point is given by $6e^{3x}+9x^{2}-2\sqrt{e^{x}}$.

12. Determine $\int (e^{2x}-e^{-3x})^{3}dx$.

13. A curve has a gradient function $f'(x)=4e^{-2x}+k$, where $k\in R$. The function has a stationary point when $x=0$.

    a. Determine the value of k.

    b. Hence, determine the general rule for the function $f(x)$.

14. If it is known that $\int ae^{bx}dx=-3e^{3x}+c$, determine the exact values of the constants a and b.

15. It is known that $\int (me^{nx}+px+q)\,dx=5e^{2x}+2x^{2}-3x+c$. Determine the exact values of the constants m, n, p and q.

6.4 Antidifferentiation of logarithmic functions

As you have learned in Chapter 3:

for $y=\ln(x)$ and $y=\ln(ax+b)$,

$\frac{dy}{dx}=\frac{1}{x}$ and $\frac{dy}{dx}=\frac{a}{(ax+b)}$.

Therefore, it follows that:

$\int \frac{1}{x}\,dx=\ln(x)+c$ and $\int \frac{1}{(ax+b)}\,dx=\frac{1}{a}\ln(ax+b)+c$

where c is a constant.

Antidifferentiation of logarithmic functions

$$\int \frac{1}{x}\,dx=\ln(x)+c \text{ for } x>0$$

$$\int \frac{1}{(ax+b)}\,dx=\frac{1}{a}\ln(ax+b)+c$$

where a, b and c are constants.

Notes:

1. This allows us to determine the antiderivative of x^{n}, $n=-1$ as $\int x^{-1}dx=\int \frac{1}{x}dx=\ln(x)$, $x>0$.
2. Remember that $\log_{e}(x)$ and $\ln(x)$ are equivalent expressions for logarithmic functions of base e.
3. In the formula $\int \frac{1}{(ax+b)}\,dx=\frac{1}{a}\ln(ax+b)+c$, the a in the fraction $\frac{1}{a}$ is the derivative of the linear function $(ax+b)$.

WORKED EXAMPLE 6

Determine:

a. $\displaystyle\int \frac{3}{2x}\,dx$ **b.** $\displaystyle\int \frac{4}{2x+1}\,dx$

THINK	WRITE
a. 1. Remove the factor of $\frac{3}{2}$.	**a.** $\displaystyle\int \frac{3}{2x}\,dx$ $\displaystyle= \frac{3}{2}\int \frac{1}{x}\,dx$
2. Apply the formula.	$= \frac{3}{2}\ln(x) + c$
b. 1. Remove the factor of 4.	**b.** $\displaystyle\int \frac{4}{2x+1}\,dx$ $\displaystyle= 4\int \frac{1}{(2x+1)}\,dx$
2. Apply the formula.	$= 4 \times \frac{1}{2}\ln(2x+1) + c$
3. Simplify.	$= 2\ln(2x+1) + c$

WORKED EXAMPLE 7

Determine $\displaystyle\int \frac{(2x+3)^2}{x}\,dx$.

THINK	WRITE
1. Expand the numerator.	$\displaystyle\int \frac{(2x+3)^2}{x}\,dx$ $\displaystyle= \int \frac{4x^2+12x+9}{x}\,dx$
2. Express with separate fractions and simplify.	$\displaystyle= \int \left(\frac{4x^2}{x} + \frac{12x}{x} + \frac{9}{x}\right)dx$ $\displaystyle= \int \left(4x + 12 + \frac{9}{x}\right)dx$
3. Integrate each term and simplify.	$= 4 \times \frac{x^2}{2} + 12x + 9\ln(x) + c$ $= 2x^2 + 12x + 9\ln(x) + c$

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Antidifferentiation of logarithmic functions Summary screen and practice questions

Exercise 6.4 Antidifferentiation of logarithmic functions

Technology free

1. **WE6** Determine the following.

   a. $\int \frac{3}{x}dx$ b. $\int \frac{8}{x}dx$ c. $\int \frac{6}{5x}dx$ d. $\int \frac{7}{3x}dx$ e. $\int \frac{4}{7x}dx$

2. Antidifferentiate the following.

   a. $\int \frac{1}{x+3}dx$ b. $\int \frac{3}{x+3}dx$ c. $\int \frac{-2}{x+4}dx$ d. $\int \frac{-6}{x+5}dx$ e. $\int \frac{4}{3x+2}dx$

3. Determine the following.

   a. $\int \frac{8}{5x+6}dx$ b. $\int \frac{3}{2x-5}dx$ c. $\int \frac{-5}{3+2x}dx$ d. $\int \frac{-2}{6+7x}dx$

4. Determine the following.

   a. $\int \frac{1}{5-x}dx$ b. $\int \frac{3}{6-11x}dx$ c. $\int \frac{-2}{4-3x}dx$ d. $\int \frac{-8}{5-2x}dx$

5. **WE7** Determine $\int \frac{(2x+5)^2}{x}dx$.

6. Determine $\int \frac{(3x+2)^2}{x^2}dx$.

7. Antidifferentiate the following.

   a. $\frac{3-4x}{x}$ b. $\frac{2x^2-3x+4}{x^2}$ c. $\frac{(4-3x)^2}{2x}$ d. $\frac{9+\sqrt{x}}{x}$

8. The gradient function of a curve is given by $f'(x) = x - \frac{4}{x}$. Determine the general rule for the function $f(x)$.

9. Determine the general equation of the curve whose gradient at any point is given by $2x + 3 - \frac{4}{5-x}$.

10. Determine the general rule for the function $y = f(x)$ if it is known that $\frac{dy}{dx} = x\left(1 - \frac{1}{x}\right)^2$.

11. a. Show that $\frac{x-3}{x+1} = 1 - \frac{4}{x+1}$.

    b. Hence, determine $\int \frac{x-3}{x+1}dx$.

12. a. Show that $\frac{2x-5}{x-3} = 2 + \frac{1}{x-3}$.

    b. Hence, determine $\int \frac{2x-5}{x-3}dx$.

13. a. Show that $\frac{(x+2)^2}{x-2} = x + 6 + \frac{16}{x-2}$.

    b. Hence, determine $\int \frac{(x+2)^2}{x-2}dx$.

14. Determine the values of a and b if $\int \frac{a}{bx+3}dx = 6\ln(2x+3) + c$, where $a, b, c \in R$.

15. A curve has a gradient function $f'(x) = \frac{k}{2x+3}$, where $k \in R$. It is known that the function has a gradient of 2 when $x = 1$.

    a. Determine the value of k.

    b. Hence, determine the general rule for the function $f(x)$.

6.5 Antidifferentiation of sine and cosine functions

As you have learned in Chapter 4:

for $y = \sin(x)$ and $y = \sin(ax + b)$,

$\frac{dy}{dx} = \cos(x)$ $\quad$ $\frac{dy}{dx} = a\cos(ax + b)$.

For $y = \cos(x)$ and $y = \cos(ax + b)$,

$\frac{dy}{dx} = -\sin(x)$ $\quad$ $\frac{dy}{dx} = -a\sin(ax + b)$.

Therefore, it follows that:

$\int \sin(x)\,dx = -\cos(x) + c$ and $\int \cos(x)\,dx = \sin(x) + c$

where c is a constant.

Antidifferentiation of sine and cosine functions

$$\int \sin(x)\,dx = -\cos(x) + c$$

$$\int \cos(x)\,dx = \sin(x) + c$$

$$\int \sin(ax + b)\,dx = -\frac{1}{a}\cos(ax + b) + c$$

$$\int \cos(ax + b)\,dx = \frac{1}{a}\sin(ax + b) + c$$

where a, b and c are constants.

Notes:

1. The a in the fraction $\frac{1}{a}$ is the derivative of the linear function $(ax + b)$.
2. The formulas apply for sine and cosine of linear functions only.

WORKED EXAMPLE 8

Antidifferentiate the following.

a. $\sin(6x)$ $\quad$ **b.** $8\cos(4x)$ $\quad$ **c.** $3\sin\left(-\frac{x}{2}\right)$

THINK	WRITE
a. Integrate by rule.	**a.** $\int \sin(6x)\,dx = -\frac{1}{6}\cos(6x) + c$
b. 1. Integrate by rule.	**b.** $\int 8\cos(4x)\,dx = \frac{8}{4}\sin(4x) + c$
2. Simplify the result.	$= 2\sin(4x) + c$

c. 1. Integrate by rule.

c $\int 3\sin\left(-\frac{x}{2}\right)dx = \frac{-3}{-\frac{1}{2}}\cos\left(-\frac{x}{2}\right) + c$

2. Simplify the result.

$= 6\cos\left(-\frac{x}{2}\right) + c$

WORKED EXAMPLE 9

Determine the indefinite integral of $2e^{4x} - 5\sin(2x) + 4x$.

THINK

1. Integrate each term separately.

2. Simplify.

WRITE

$\int \left(2e^{4x} - 5\sin(2x) + 4x\right) dx$

$= 2 \times \frac{1}{4}e^{4x} - 5 \times \frac{-1}{2}\cos(2x) + 4 \times \frac{x^2}{2} + c$

$= \frac{1}{2}e^{4x} + \frac{5}{2}\cos(2x) + 2x^2 + c$

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Antidifferentiation of sine and cosine functions Summary screen and practice questions

Exercise 6.5 Antidifferentiation of sine and cosine functions

Technology free

1. **WE8** Antidifferentiate the following.
   - a. $\sin(3x)$
   - b. $\sin(4x)$
   - c. $\cos(7x)$
   - d. $\frac{\cos(2x)}{3}$
   - e. $\sin(-2x)$
   - f. $\cos(-3x)$
2. Antidifferentiate the following.
   - a. $\frac{4\sin(6x)}{3}$
   - b. $8\sin(4x)$
   - c. $-6\sin(3x)$
   - d. $-2\cos(-x)$
   - e. $\sin\left(\frac{x}{3}\right)$
   - f. $\cos\left(\frac{x}{2}\right)$
3. Determine the indefinite integrals of the following.
   - a. $3\sin\left(\frac{-x}{4}\right)$
   - b. $-2\sin\left(\frac{x}{5}\right)$
   - c. $4\cos\left(\frac{x}{4}\right)$
   - d. $-6\cos\left(\frac{-x}{2}\right)$
   - e. $4\sin\left(\frac{2x}{3}\right)$
   - f. $6\cos\left(\frac{3x}{4}\right)$
4. **WE9** Determine the indefinite integral of:
   - a. $e^{4x} + \sin(2x) + x^3$
   - b. $3x^2 - 2\cos(2x) + 6e^{3x}$

5. Determine:

a. $\int (\sin(x) + \cos(x))\,dx$

b. $\int (\sin(2x) - \cos(x))\,dx$

c. $\int (\cos(4x) + \sin(2x))\,dx$

d. $\int \left(\sin\left(\frac{x}{2}\right) - \cos(2x)\right)dx$

6. Determine:

a. $\int \left(4\cos(4x) - \frac{1}{3}\sin(2x)\right)dx$

b. $\int (5x + 2\sin(x))\,dx$

c. $\int \left(3\sin\left(\frac{\pi x}{2}\right) + 2\cos\left(\frac{\pi x}{3}\right)\right)dx$

d. $\int \left(3e^{6x} - 4\sin(8x) + 7\right)dx$

7. a. Determine the indefinite integral of $\frac{1}{2}\cos(3x + 4) - 4\sin\left(\frac{x}{2}\right)$.

b. Determine an antiderivative of $\cos\left(\frac{2x}{3}\right) - \frac{1}{4}\sin(5 - 2x)$.

8. a. Determine $\int \left(\sin\left(\frac{x}{2}\right) - 3\cos\left(\frac{x}{2}\right)\right)dx$.

b. If $f'(x) = 7\cos(2x) - \sin(3x)$, determine a general rule for f.

9. Determine the indefinite integral of:

a. $e^{\frac{x}{3}} + \sin\left(\frac{x}{3}\right) + \frac{x}{3}$

b. $\cos(4x) + 3e^{-3x}$

10. Determine an antiderivative of $\frac{1}{4x^2} + \sin\left(\frac{3\pi x}{2}\right)$.

11. The gradient of a tangent to a curve is given by $\frac{dy}{dx} = \cos(2x) - e^{-3x}$. Determine a possible general rule for the curve y.

12. A curve has a gradient function $f'(x) = k\sin(3x)$, where $k \in R$. It is known that the function has a gradient of 2 when $x = \frac{\pi}{2}$.

a. Determine the value of k.

b. Hence, determine the general rule for the function $f(x)$.

13. The gradient function of a curve is $f'(x) = 4\cos(2x) + k$ where $k \in R$. The gradient at the point when $x = \frac{5\pi}{6}$ is -3.

a. Determine the value of k.

b. Hence, determine the general rule for the function $f(x)$.

14. A curve has a gradient function $\frac{dy}{dx} = k\cos\left(2x + \frac{\pi}{3}\right)$, where $k \in R$. If $\frac{dy}{dx} = 5$ when $x = \frac{\pi}{2}$, determine:

a. the value of k

b. the general rule for the function.

15. If it is known that $\int (3\sin(2x) + 8\cos(2x))\,dx = p\sin(2x) + q\cos(2x)$, where $p, q \in R$, determine the values of p and q.

6.6 Further integration

6.6.1 Integration of functions of the form $f(ax + b)$

Consider the function: $f(x) = (ax+b)^{n+1}$

Apply the chain rule: $f'(x) = (n+1)(ax+b)^n \times a$

$$= a(n+1)(ax+b)^n$$

Hence: $\int a(n+1)(ax+b)^n \, dx = (ax+b)^{n+1}$

So: $a(n+1)\int (ax+b)^n \, dx = (ax+b)^{n+1}$

This gives us the general rule:

$$\int (ax+b)^n \, dx = \frac{1}{a(n+1)}(ax+b)^{n+1} + c, n \neq -1$$

Integration of functions of the form $f(ax + b)$

$$\int (ax+b)^n \, dx = \frac{1}{a(n+1)}(ax+b)^{n+1} + c, n \neq -1$$

$$\int e^{(ax+b)} \, dx = \frac{1}{a}e^{(ax+b)} + c$$

$$\int \frac{1}{(ax+b)} \, dx = \frac{1}{a}\ln(ax+b) + c$$

$$\int \sin(ax+b) \, dx = -\frac{1}{a}\cos(ax+b) + c$$

$$\int \cos(ax+b) \, dx = \frac{1}{a}\sin(ax+b) + c$$

where a, b and c are constants.

The rules described above only apply if the expression inside the brackets is linear. If the expression is of any other kind, it may need to be expanded before integrating, or another method may need to be used, such as technology.

WORKED EXAMPLE 10

Antidifferentiate each of the following.

a. $(2x+3)^5$ **b.** $2\sqrt{5x+4}$ **c.** $\left(e^{2x-1}+3\right)^2$

THINK	WRITE
a. 1. Recognise the linear function inside the brackets.	**a.** $\int (2x+3)^5 \, dx$

2. Integrate the expression. $= \frac{1}{2 \times 6}(2x+3)^6$

3. Simplify. $= \frac{1}{12}(2x+3)^6 + c$

b. 1. Take out the factor of 2 and write the square root as a linear expression to a power of $\frac{1}{2}$.

b. $\int 2\sqrt{5x+4}\,dx$

$= 2\int (5x+4)^{\frac{1}{2}}dx$

2. Integrate. $= 2 \times \frac{1}{5} \times \frac{(5x+4)^{\frac{3}{2}}}{\frac{3}{2}}$

3. Simplify. $= \frac{4}{15}(5x+4)^{\frac{3}{2}} + c$

c. 1. Expand the brackets.

c. $\int \left(e^{2x-1}+3\right)^2 dx$

$= \int (e^{4x-2} + 6e^{2x-1} + 9)\,dx$

2. Integrate each term separately. $= \frac{1}{4}e^{(4x-2)} + 6 \times \frac{1}{2}e^{(2x-1)} + 9x$

3. Simplify. $= \frac{1}{4}e^{(4x-2)} + 3e^{(2x-1)} + 9x + c$

6.6.2 Initial conditions

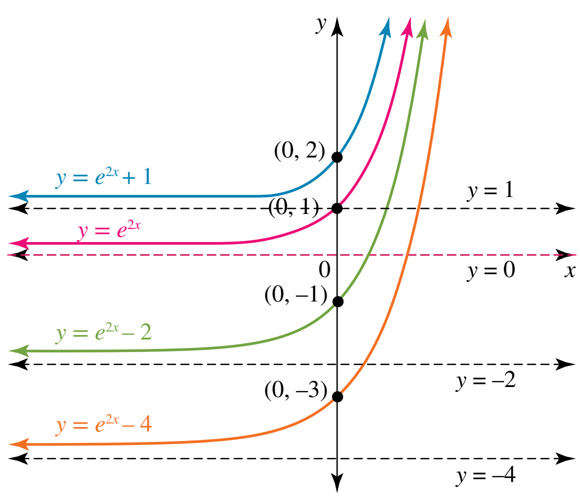
Integration of functions with the constant c gives a family of curves. A specific function can only be found if we are given some additional information to allow us to evaluate the constant, c. This additional information is referred to as an initial condition.

Consider: if $\frac{dy}{dx} = 2e^{2x}$,

then $y = e^{2x} + c$.

This is a set of exponential functions with a horizontal asymptote of $y = c$.

Four functions that belong to this family of curves are shown.



If you were told that the curve passes through the origin, then you would know that when $x = 0$, $y = 0$ or $f(0) = 0$.

Then, for $y = e^{2x} + c$,

substitute $(0, 0)$: $0 = e^{2(0)} + c$

$0 = 1 + c$

$c = -1$

Therefore, $y = e^{2x} - 1$.

Application questions such as those involving rates of change may also be given in terms of the derivative function. Integrating the equation for the rate of change, together with the initial conditions, allows us to determine the original function.

WORKED EXAMPLE 11

Determine the equation of the curve that passes through the point (1, 0) if the gradient is given by $\frac{dy}{dx} = 3x^2 - 2x + 2$.

THINK	WRITE
1. Write the gradient rule and antidifferentiate to determine y.	$\frac{dy}{dx} = 3x^2 - 2x + 2$ $y = \int (3x^2 - 2x + 2)dx$ $= x^3 - x^2 + 2x + c$
2. Substitute the known point into the equation.	When $x = 1, y = 0$: $0 = (1)^3 + (1)^2 + 2(1) + c$ $0 = 1 - 1 + 2 + c$ $c = -2$
3. State the rule for y.	$y = x^3 - x^2 + 2x - 2$

WORKED EXAMPLE 12

A young boy bought an ant farm. It is known that the ant population is changing at a rate defined by $\frac{dN}{dt} = 20e^{0.2t}$, $0 \le t \le 20$, where N is the number of ants in the colony and t is the time in days since the ant farm has been set up.

a. Determine a rule relating N to t if initially there were 50 ants.

b. How many ants make up the colony after 8 days?

THINK	WRITE
a. 1. Write the rate rule and antidifferentiate to determine the function for N.	a. $\frac{dN}{dt} = 20e^{0.2t}$ $N = \int (20e^{0.2t})dt$ $= \frac{20}{0.2}e^{0.2t} + c$ $= 100e^{0.2t} + c$
2. Use the initial condition to determine the value of c.	When $t = 0, N = 50$: $50 = 100e^{0.2(0)} + c$ $50 = 100 + c$ $c = -50$
3. State the equation for N.	$N = 100e^{0.2t} - 50$
b. 1. Substitute $t = 8$ into the population equation.	b. When $t = 8$: $N = 100e^{0.2(8)} - 50$ $= 100e^{1.6} - 50$ $= 445.3$

2. Answer the question.
Note: It is reasonable to round down when counting elements from the natural world.

There are 445 ants after 8 days.

6.6.3 Integration by recognition

Recall that if $\frac{d}{dx}[f(x)] = g(x)$, then $\int g(x)\,dx = f(x) + c$.

This result can be used to determine integrals of functions that are too difficult to antidifferentiate, by first differentiating a related function instead.

WORKED EXAMPLE 13

Given that $y = e^{x^2}$, determine $\frac{dy}{dx}$ and hence determine an antiderivative of xe^{x^2}.

THINK	WRITE
1. Use the chain rule to differentiate the given function.	$y = e^{x^2}$ Let $y = e^u$ and $u = x^2$. $\frac{dy}{du} = e^u$ and $\frac{du}{dx} = 2x$ $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$ $= e^u \times 2x$ $= 2xe^{x^2}$ $\frac{dy}{dx} = 2xe^{x^2}$
2. Rewrite the result as an integral.	$\int 2xe^{x^2}\,dx = e^{x^2}$
3. Adjust the left-hand side so that it matches the expression to be integrated.	$2\int xe^{x^2}\,dx = e^{x^2}$ $\frac{1}{2} \times 2\int xe^{x^2}\,dx = \frac{1}{2} \times e^{x^2}$
4. Write the answer.	$\int xe^{x^2}\,dx = \frac{1}{2}e^{x^2}$

WORKED EXAMPLE 14

Differentiate $y = \ln(x^2 + 4)$ and hence determine $\int \frac{6x}{(x^2+4)}dx$.

THINK	WRITE
1. Differentiate using the chain rule.	$y = \ln(x^2 + 4)$ Let $y = \ln(u)$ and $u = x^2 + 4$. $\frac{dy}{du} = \frac{1}{u}$ and $\frac{du}{dx} = 2x$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$= \frac{1}{u} \times 2x$$
$$= \frac{2x}{\left(x^2+4\right)}$$

2. Simplify.

$$\frac{dy}{dx} = \frac{1}{(x^2+4)} \times 2x$$
$$= \frac{2x}{\left(x^2+4\right)}$$

3. Rewrite the result as an integral.

$$\therefore \int \frac{2x}{(x^2+4)}\,dx = \ln(x^2+4)$$

4. Adjust the left-hand side so that it matches the expression to be integrated.

$$3 \times \int \frac{2x}{(x^2+4)}\,dx = 3 \times \ln(x^2+4)$$

5. Write the answer.

$$\therefore \int \frac{6x}{(x^2+4)}\,dx = 3\ln(x^2+4) + c$$

WORKED EXAMPLE 15

Differentiate $x\cos(x)$ and hence determine an antiderivative of $x\sin(x)$.

THINK	WRITE
1. Write the expression as a function.	Let $y = x\cos(x)$.
2. Differentiate using the product rule.	$\frac{dy}{dx} = x \times (-\sin(x)) + \cos(x) \times 1$
3. Simplify.	$\frac{dy}{dx} = \cos(x) - x\sin(x)$
4. Rewrite the result as an integral.	$\therefore \int (\cos(x) - x\sin(x))\,dx = x\cos(x)$
5. Express as separate integrals.	$\int \cos(x)\,dx - \int x\sin(x)\,dx = x\cos(x)$
6. Simplify by integrating.	$\sin(x) - \int x\sin(x)\,dx = x\cos(x)$
7. Rearrange the equation to make the expression to be integrated the subject.	$\sin(x) - x\cos(x) = \int x\sin(x)dx$
8. Write the answer.	$\therefore \int x\sin(x)dx = \sin(x) - x\cos(x) + c$

6.6.4 Linear motion

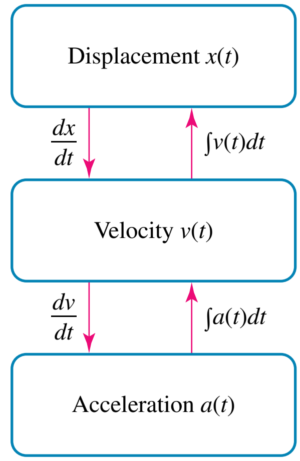
Kinematics, the study of the motion of a particle in a straight line, was introduced in Chapter 5.

Displacement, x, gives the position of a particle, specifying both its distance and direction from the origin, $x = 0$.

Velocity, v, measures the rate of change of displacement with respect to time, t, which means that $v = \frac{dx}{dt}$. It follows that displacement is the antiderivative of velocity.

Acceleration, a, measures the rate of change of velocity with respect to time, t, which means that $a = \frac{dv}{dt}$. It follows that velocity is the antiderivative of acceleration.

These relationships are summarised in the diagram.



WORKED EXAMPLE 16

In each of the following cases, determine the displacement as a function of t if initially the particle is at the origin.

a. $v = t^3 - t$ **b.** $v = (2t - 3)^3$

THINK	WRITE
a. 1. Write the velocity equation and antidifferentiate to determine the displacement function, x.	**a.** $v = \frac{dx}{dt} = t^3 - t$ $x = \int (t^3 - t)dt$ $x = \frac{1}{4}t^4 - \frac{1}{2}t^2 + c$
2. Substitute the initial condition into the formula for x and determine c.	When $t = 0$, $x = 0$: $0 = 0 + c$ $c = 0$
3. State the rule.	$x = \frac{1}{4}t^4 - \frac{1}{2}t^2$
b. 1. Write the velocity equation and antidifferentiate to determine the displacement function, x.	**b.** $v = \frac{dx}{dt}$ $= (2t - 3)^3$ $x = \int (2t - 3)^3 dt$ $= \frac{(2t - 3)^4}{2(4)} + c$ $= \frac{1}{8}(2t - 3)^4 + c$
2. Substitute the initial condition into the formula for x and determine c.	When $t = 0$, $x = 0$: $0 = \frac{1}{8}(-3)^4 + c$ $0 = \frac{81}{8} + c$ $c = -\frac{81}{8}$

3. State the rule. $x = \frac{1}{8}(2t-3)^4 - \frac{81}{8}$

WORKED EXAMPLE 17

The velocity of a particle moving in a straight line along the x-axis is given by

$$v = \frac{dx}{dt} = 9 - 9e^{-3t}$$

where t is the time in seconds and x is the displacement in metres.

a. **Show that the particle is initially at rest.**

b. **Determine the equation relating x to t if it is known that initially the particle was 3 metres to the left of the origin.**

THINK

a. 1. Substitute $t = 0$ and evaluate.

2. Answer the question.

b. 1. Write the velocity equation and antidifferentiate to determine the position equation, x.

2. Substitute the initial condition to determine c. Remember, left of the origin means the position is negative.

3. State the equation.

WRITE

a. $v = 9 - 9e^{-3t}$

$t = 0 \Rightarrow v = 9 - 9e^0$
$= 9 - 9 \times 1$
$= 0\text{ m/s}$

Initially the particle is at rest as its velocity is 0 m/s.

b. $v = \frac{dx}{dt}$
$= 9 - 9e^{-3t}$

$x = \int (9 - 9e^{-3t})dt$
$= 9t + 3e^{-3t} + c$

When $t = 0$, $x = -3$:
$-3 = 9 \times 0 + 3e^0 + c$
$-3 = 3 + c$
$c = -6$

$x = 9t + 3e^{-3t} - 6$

on Resources

Interactivity: Families of curves (int-6421)

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Units 3 & 4 > Area 3 > Sequence 1 > Concepts 6 & 7

Using initial conditions Summary screen and practice questions
Integration by recognition Summary screen and practice questions

Exercise 6.6 Further integration

Technology free

1. **WE10** Antidifferentiate each of the following.
   a. $(x+3)^3$ b. $(x-5)^3$ c. $2(2x+1)^4$
   d. $-2(3x-4)^5$ e. $(6x+5)^4$ f. $3(4x-1)^2$
2. Determine the antiderivative for each of the following.
   a. $(4-x)^3$ b. $(7-x)^4$ c. $4(8-3x)^4$
   d. $-3(8-9x)^{10}$ e. $(2x+3)^{-2}$ f. $(6x+5)^{-3}$
3. Antidifferentiate:
   a. $(3x-5)^5$ b. $\dfrac{1}{(2x-3)^{\frac{5}{2}}}$
4. Determine:
   a. $\displaystyle\int (2x+3)^4 dx$ b. $\displaystyle\int (1-2x)^{-5} dx$
5. Determine:
   a. $\displaystyle\int \left(e^{2x+1}-4\right)^2 dx$ b. $\displaystyle\int \left(2e^{3-x}+3e^{2-x}\right)^2 dx$
6. **WE11** Determine the equation of the function $f(x)$ given that:
   a. $f'(x) = 4x + 1$ and the curve passes through $(0, 2)$
   b. $f'(x) = 5 - 2x$ and the curve passes through $(1, -1)$
   c. $f'(x) = x^{-2} + 3$ and the curve passes through $(1, 4)$
   d. $f'(x) = x + \sqrt{x}$ and $f(4) = 10$
   e. $f'(x) = x^{\frac{1}{3}} - 3x^2 + 50$ and $f(8) = -100$
   f. $f'(x) = \dfrac{1}{\sqrt{x}} - 2x$ and $f(1) = -5$.
7. a. Sketch a family of curves related to the derivative function $f'(x) = 3x^2$.
   b. Determine the rule for the function that belongs to this family of curves and passes through the point $(2, 16)$.
8. a. Sketch a family of curves related to the derivative function $f'(x) = -2\cos(2x)$.
   b. Determine the rule for the function that belongs to this family of curves and passes through the point $\left(\dfrac{\pi}{2}, 4\right)$.
9. Determine the equation of the curve that passes through the point $(0, 3)$ if the gradient is given by $\dfrac{dy}{dx} = 2e^{2x} + e^{-x}$.
10. The gradient function of a particular curve is given by $f'(x) = \cos(2x) - \sin(2x)$ Determine the rule for this function if it is known that the curve passes through the point $(\pi, 2)$.
11. **WE12** It is known that the population of a certain species of bugs is changing at a rate defined by

$$\frac{dP}{dt} = 20e^{0.4t},\ 0 \le t \le 10$$

   where P is the number of bugs at any time t days since the monitoring of the bugs commenced.
   a. Determine the relationship between P and t if initially the population consisted of 35 bugs.
   b. Calculate the number of bugs present after 6 days.

12. A population of sea lions on a distant island is growing according to the model

$$\frac{dP}{dt} = 30e^{0.3t}, \; 0 \le t \le 10$$

where P is the number of sea lions present after t years.

a. If initially there were 50 sea lions on the island, determine the rule for the number of sea lions present, P, after t years.
b. Determine the number of sea lions on the island after 10 years. Give your answer correct to the nearest whole sea lion.

13. The rate of change of the depth of water in a canal is modelled by the rule

$$\frac{dh}{dt} = \frac{\pi}{2}\cos\left(\frac{\pi t}{4}\right)$$

where h is the height of the water in metres and t is the number of hours since 6 am.
a. Determine an expression for h in terms of t if the water is 3 metres deep at 6 am.
b. What are the maximum and minimum depths of the water?
c. For how many hours a day is the water level 4 metres or higher?

14. **WE13** Given that $y = \sqrt{x^2 + 1}$, determine $\frac{dy}{dx}$ and hence determine the antiderivative of $\frac{5x}{\sqrt{x^2 + 1}}$.

15. If $y = (5x^2 + 2x - 1)^4$, determine $\frac{dy}{dx}$ and hence determine an antiderivative of $16(5x + 1)(5x^2 + 2x - 1)^3$.

16. **WE14** Differentiate $\ln\left(3x^2 + 4\right)$ and hence determine an antiderivative of $\frac{x}{(3x^2 + 4)}$

17. If $y = \ln(\cos(x))$, determine:
a. $\frac{dy}{dx}$
b. $\int \tan(x)\,dx$

18. **WE15** Differentiate $x\sin(x)$ and hence determine an antiderivative of $x\cos(x)$.
19. Differentiate $x\ln(x)$ and hence determine an antiderivative of $\ln(x)$.
20. Differentiate $y = 2xe^{3x}$ and hence determine an antiderivative of xe^{3x}.
21. **WE16** In each of the following cases, determine the displacement as a function of t if initially the particle is at the origin.
a. $v = (3t + 1)^{\frac{1}{2}}$
b. $v = \frac{1}{(t + 2)^2}$
c. $v = (2t + 1)^3$

22. A particle moves in a straight line so that its velocity, in metres per second, can be defined by the equation $v = 3t^2 + 6t, \; t \ge 0$. Determine:
a. the displacement of the particle, x metres, as a function of t, if it is known that the particle was initially 2 metres to the left of the origin
b. the position of the particle after 5 seconds.

23. Determine the displacement of a particle that starts from the origin and has a velocity defined by:
a. $v = e^{(3t-1)}$
b. $v = -\sin(2t + 3)$

24. **WE17** A particle is oscillating so that its velocity v cm/s, can be defined by

$$v = \frac{dx}{dt} = \sin(2t) + \cos(2t)$$

where t is the time in seconds and x centimetres is its displacement.

a. Show that initially the particle is moving at 1 cm/s.

b. Determine the equation relating x to t if it is known that initially the particle was at the origin.

25. A particle starting at the origin moves in a straight line with a velocity of $\frac{12}{(t-1)^2}+6$ metres per second after t seconds.

a. Determine the rule relating the position of the particle, x metres, to t.

b. Determine the position of the particle after 3 seconds.

6.7 Review: exam practice

A summary of this chapter is available in the Resources section of your eBookPLUS at www.jacplus.com.au.

Simple familiar

1. Antidifferentiate each of the following.

a. $3x^5$
b. $5x^{-2}$
c. $-2x^4$
d. $2\sqrt{x}$
e. $\frac{x^4}{5}$
f. $(3x-8)^{-6}$
g. $(6-5x)^{-3}$
h. $-10(7-5x)^{-4}$

2. Determine the following.

a. $\int\left(x^4+2x+\frac{1}{x}\right)dx$
b. $\int(3x+1)^5\,dx$
c. $\int\frac{3x^2+2x-1}{x^2}\,dx$
d. $\int\frac{3}{2x+1}\,dx$
e. $\int\frac{-5}{6-10x}\,dx$
f. $\int 3(4x+1)^{-3}\,dx$
g. $\int\frac{(x+4)^2}{2x}\,dx$
h. $\int\left(\sqrt{x}+\frac{2}{3-x}\right)dx$

3. Determine the equation of the curve $f(x)$ given that:

a. $f'(x)=(x+4)^3$ and the curve passes through $(-2, 5)$

b. $f'(x)=8(1-2x)^{-5}$ and $f(1)=3$

c. $f'(x)=(x+5)^{-1}$ and the curve passes through $(-4, 2)$

d. $f'(x)=\frac{8}{7-2x}$ and $f(3)=7$.

4. If a curve has a stationary point at $(1, 5)$ and a gradient of $8x+k$ where k is a constant, determine:

a. the value of k
b. the value of y when $x=-2$.

5. Determine an antiderivative of:

a. $(e^x-3)^2$
b. $(1+e^{-x})^3$

6. Antidifferentiate the following.

a. $-2\sin\left(\frac{5x}{2}\right)$
b. $-3\cos\left(\frac{7x}{4}\right)$
c. $5\sin(\pi x)$
d. $3\cos\left(\frac{\pi x}{2}\right)$
e. $-2\cos\left(\frac{\pi x}{3}\right)$
f. $-\sin\left(\frac{-4x}{\pi}\right)$

7. Integrate each of the following with respect to x.

a. $x^3-\frac{1}{2x+3}+e^{2x}$
b. $x^2+4\cos(2x)-e^{-x}$
c. $\sin\left(\frac{x}{3}\right)+e^{\frac{x}{2}}-(3x-1)^4$
d. $\frac{1}{3x-2}+e^{4x}+\cos\left(\frac{x}{5}\right)$
e. $3\sin\left(\frac{x}{2}\right)-2\cos\left(\frac{x}{3}\right)-e^{\frac{-x}{5}}$
f. $\sqrt{x}+2x-2\sin\left(\frac{\pi x}{3}\right)+5$

8. Determine the equation of the curve $f(x)$ given that:

a. $f'(x) = \cos(x)$ and $f\left(\frac{\pi}{2}\right) = 5$

b. $f'(x) = 4\sin(2x)$ and $f(0) = -1$

c. $f'(x) = 3\cos\left(\frac{x}{4}\right)$ and $f(\pi) = 9\sqrt{2}$

d. $f'(x) = \cos\left(\frac{x}{4}\right) - \sin\left(\frac{x}{2}\right)$ and $f(2\pi) = -2$.

9. A curve has a gradient function $f'(x) = 4\cos(2x) + ke^x$, where k is a constant, and a stationary point at $(0, -1)$. Calculate:

a. the value of k

b. the equation of the curve $f(x)$

c. $f\left(\frac{\pi}{6}\right)$ correct to 2 decimal places.

10. Determine $\frac{d}{dx}\left(\ln\left(x^2+3\right)\right)$ and hence determine $\int \frac{12x}{\left(x^2+3\right)}dx$.

11. Differentiate $\frac{\cos(x)}{\sin(x)}$ and hence determine an antiderivative of $\frac{1}{\sin^2(x)}$.

12. a. Show that $\frac{6x-5}{3-2x} = -3 + \frac{4}{3-2x}$.

b. Hence, determine $\int \frac{6x-5}{3-2x}\,dx$.

Complex familiar

13. If $f'(x) = a\sin(mx) - be^{nx}$ and $f(x) = \cos(2x) - 2e^{-2x} + 3$, calculate the exact constants a, b, m and n.

14. When a bus travels along a straight road in heavy traffic from one stop to another stop, the velocity at time t seconds is given by $v = \frac{t}{400}(50 - t)$, where v is the velocity in metres per second.

a. Calculate the greatest velocity reached by the bus.

b. Determine the rule for the position of the bus, x metres from the first stop, in terms of t.

15. a. Sketch a family of curves related to the derivative function $f'(x) = 3e^{-3x}$.

b. Determine the rule for the function that belongs to this family of curves and passes through the point $(0, 1)$.

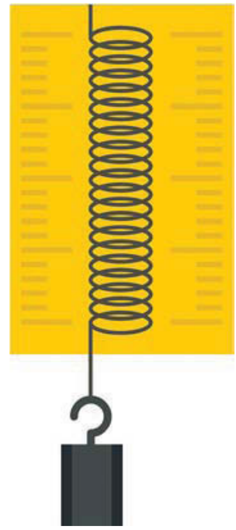
16. Determine $f(x)$ for each of the following.

a. $f'(x) = 5 - 2x$ and $f(1) = 4$

b. $f'(x) = \sin\left(\frac{x}{2}\right)$ and $f(\pi) = 3$

c. $f'(x) = \frac{1}{(1-x)^2}$ and $f(0) = 4$

Complex unfamiliar



17. A particle attached to a spring moves up and down in a straight line so that at time t seconds its velocity, v metres per second, is given by

$$v = 3\pi\,\sin\left(\frac{\pi t}{8}\right), t \geq 0.$$

Initially the particle is stationary and at the origin.

a. Determine the rule relating the position of the particle, x centimetres, to t.
b. What is the maximum displacement of the particle?
c. Where is the particle, relative to the stationary position, after 4 seconds?

18. A newly established suburban area of Brisbane is growing at a rate modelled by the rule

$$\frac{dN}{dt} = 400 + 1000\sqrt{t},\ 0 \leq t \leq 10$$

where N is the number of families living in the suburb t years after the suburb was established in 2015.

a. Determine a rule relating N and t if initially there were 40 families living in this suburb.
b. How many families will be living in the suburb 5 years after its establishment? Give your answer correct to the nearest number of families.

19. A chemical factory has permission from the Environment Protection Authority to release particular toxic gases into the atmosphere for a period of 20 seconds no more than once every 3 hours. This maintains safe levels of the gases in the atmosphere. This rate of emission is given by

$$\frac{dV}{dt} = \left(20t^2 - t^3\right)\ \text{cm}^3/\text{s}$$

where $0 \leq t \leq 20$ and $V\,\text{cm}^3$ is the total volume of toxic gases released over t seconds. Determine the total volume of toxic gases released during a 20-second release period.

20. Over a 24-hour period on a particular March day, starting at 12 am, the rate of change of the temperature for Brisbane was approximately $\frac{dT}{dt} = -\frac{5\pi}{12}\cos\frac{\pi t}{12}$, where T is the temperature in °C and t is the number of hours since midnight. The temperature at midnight was 20 °C.

Determine:

a. the temperature at any time, t

b. whether the temperature reaches 13 °C at any time during the day

c. the maximum temperature and the time at which it occurs

d. the minimum temperature and the time at which it occurs

e. the temperature at:

   i. 2 am

   ii. 3 pm

f. the time when the temperature first reaches 22.5 °C.

studyon

Units 3 & 4 Sit exam

Answers

6 Antidifferentiation

Exercise 6.2 Antidifferentiation of rational functions

1. a. $\frac{x^2}{8} + c$ **b.** $2x^4 + 2x^2 + c$

c. $x^3 + \frac{5}{2}x^2 - 8x + c$ **d.** $\frac{x^4}{2} + x^3 - 3x^2 - 9x + c$

2. a. $x^2 + 5x + c$

b. $x^3 + 2x^2 - 10x + c$

c. $2x^5 + \frac{3}{2}x^4 + 2x + c$

d. $\frac{-2}{3}x^6 + \frac{1}{4}x^4 - 2x^3 + x^2 + c$

e. $\frac{1}{4}x^4 + 12x - \frac{1}{3}x^3 + c$

3. a. $\frac{8}{3}x\sqrt{x} + \frac{1}{x} + c$ **b.** $4x\sqrt{x} + 6\sqrt{x} + 8x + c$

4. a. $\frac{1}{25}x^5 + c$ **b.** $\frac{1}{8}x^4 + c$ **c.** $-\frac{1}{9x^3} + c$

d. $\frac{2}{3}x^{\frac{3}{2}} + c$ **e.** $\frac{3}{5}x^{\frac{5}{3}} + c$ **f.** $\frac{16}{7}x^{\frac{7}{4}} + c$

5. a. $\frac{7}{4}x^{\frac{4}{7}} + c$ **b.** $-\frac{5}{2x^2} + c$ **c.** $-\frac{9}{x} + c$

d. $\frac{2}{x^5} + c$ **e.** $16\sqrt{x} + c$ **f.** $\frac{12}{\sqrt{x}} + c$

6. a. $\frac{1}{3}x^3 - 2x^2 - 21x + c$

b. $\frac{5}{3}x^3 + 5x^2 - 5x + c$

c. $\frac{1}{4}x^4 - \frac{7}{3}x^3 + 2x^2 - 28x + c$

d. $\frac{1}{4}x^4 + x^3 - 2x^2 + c$

7. a. $\frac{1}{4}x^4 + \frac{1}{2}x^2 + c$ **b.** $\frac{2}{5}x^{\frac{5}{2}} + \frac{4}{3}x^{\frac{3}{2}} - 2\sqrt{x} + c$

c. $-5x^{-2} + x^{-1} + x^2 + c$

8. $f(x) = \frac{1}{3}x^3 + \frac{1}{x} + c$

9. a. $\frac{1}{4}x^4 + c$ **b.** $\frac{7}{3}x^3 + \frac{1}{5x^2} + c$

c. $x^4 - \frac{7}{3}x^3 + x^2 - x + c$ **d.** $\frac{16}{5}x^2\sqrt{x} + c$

10. a. $f(x) = \frac{3}{4}x^2 - \frac{4}{3}x^3 + \frac{1}{2}x^4 + c$

b. $6\sqrt{x} - x^4 - \frac{1}{5x^2} + c$

c. $\frac{1}{2}x^4 - \frac{1}{3}x^3 - \frac{15}{2}x^2 + c$

d. $\frac{3}{7}x^3\sqrt{x} - \frac{1}{3}x\sqrt{x} + c$

11. a. $4\sqrt{x} - \frac{3}{x} + \frac{1}{4x^2} + c$

b. $\frac{1}{2}x^4 - \frac{1}{3}x^3 + \frac{1}{2}x^2 + 4x + c$

12. a. $2x^3 + \frac{5}{2}x^2 - 6x + c$ **b.** $\frac{1}{2}x^2 + x - \frac{1}{x} + c$

c. $\frac{4}{3}x\sqrt{x} - 8\sqrt{x} + c$ **d.** $\frac{1}{7}x^7 - 4x - \frac{4}{5x^5} + c$

13. $y = \frac{1}{4}x^4 - 2x\sqrt{x} + c$

14. $y = \frac{1}{2}x^2 + 3x + \frac{3}{x} + c$

15. $y = \frac{2}{3}x\sqrt{x} + 2\sqrt{x} + c$

Exercise 6.3 Antidifferentiation of exponential functions

1. a. $\frac{1}{2}e^{2x} + c$ **b.** $\frac{1}{4}e^{4x} + c$ **c.** $-e^{-x} + c$

d. $-\frac{1}{3}e^{-3x} + c$ **e.** $e^{5x} + c$ **f.** $\frac{7}{4}e^{4x} + c$

2. a. $3e^{\frac{x}{3}} + c$ **b.** $0.4e^{\frac{x}{4}} + c = \frac{2}{5}e^{\frac{x}{4}} + c$

c. $6e^{\frac{x}{2}} + c$ **d.** $-9e^{\frac{-x}{3}} + c$

e. $e^x - e^{-x} + c$ **f.** $\frac{1}{2}e^x + \frac{1}{2}e^{-x} + c$

3. $y = \frac{1}{2}e^{2x} - 2x - \frac{1}{2}e^{-2x} + c$

4. $y = x + e^{2x} + \frac{1}{4}e^{4x} + c$

5. $y = \frac{1}{6}e^{6x} + 4e^{3x} + 36x + c$

6. a. $\frac{1}{5}x^5 + \frac{1}{4}e^{-4x} + c$ **b.** $\frac{1}{4}e^{2x} + \frac{4}{3}e^{-\frac{x}{2}} + c$

7. a. $\frac{1}{2}e^x - \frac{1}{4}e^{-6x} + c$ **b.** $\frac{1}{2}e^{2x} - \frac{2}{3}e^{3x} + \frac{1}{4}e^{4x} + c$

8. $e^x + 4e^{-\frac{x}{2}} - \frac{1}{2}e^{-2x} + c$

9. $f(x) = 2e^{2x} + 8x + c$

10. $y = \frac{1}{4}e^{4x} - x + c$

11. $y = 2e^{3x} + 3x^3 - 4e^{\frac{x}{2}} + c$

12. $\frac{1}{6}e^{6x} - 3e^x - \frac{3}{4}e^{-4x} + \frac{1}{9}e^{-9x} + c$

13. a. -4 **b.** $f(x) = -2e^{-2x} - 4x + c$

14. $a = -9, b = 3$

15. $m = 10, n = 2, p = 4, q = -3$

Exercise 6.4 Antidifferentiation of logarithmic functions

1. a. $3\ln(x) + c$ **b.** $8\ln(x) + c$ **c.** $\frac{6}{5}\ln(x) + c$

d. $\frac{7}{3}\ln(x) + c$ **e.** $\frac{4}{7}\ln(x) + c$

2. a. $\ln(x+3) + c$ **b.** $3\ln(x+3) + c$

c. $-2\ln(x+4) + c$ **d.** $-6\ln(x+5) + c$

e. $\frac{4}{3}\ln(3x+2) + c$

3. a. $\frac{8}{5}\ln(5x+6) + c$ **b.** $\frac{3}{2}\ln(2x-5) + c$

c. $-\frac{5}{2}\ln(3+2x) + c$ **d.** $-\frac{2}{7}\ln(6+7x) + c$

4. a. $-\ln(5-x)+c$

b. $-\frac{3}{11}\ln(6-11x)+c$

c. $\frac{2}{3}\ln(4-3x)+c$

d. $4\ln(5-2x)+c$

5. $2x^2+20x+25\ln(x)+c$

6. $9x+12\ln(x)-\frac{4}{x}+c$

7. a. $3\ln(x)-4x+c$

b. $2x-3\ln(x)-\frac{4}{x}+c$

c. $8\ln(x)-12x+\frac{9}{4}x^2+c$

d. $9\ln(x)+2\sqrt{x}+c$

8. $f(x)=\frac{1}{2}x^2-4\ln(x)+c$

9. $y=x^2+3x+4\ln(5-x)+c$

10. $y=\frac{1}{2}x^2-2x+\ln(x)+c$

11. a. Sample responses can be found in the worked solutions in the online resources.

b. $x-4\ln(x+1)+c$

12. a. Sample responses can be found in the worked solutions in the online resources.

b. $2x+\ln(x-3)+c$

13. a. Sample responses can be found in the worked solutions in the online resources.

b. $\frac{1}{2}x^2+6x+16\ln(x-2)+c$

14. $a=12, b=2$

15. a. 10

b. $f(x)=5\ln(2x+3)+c$

Exercise 6.5 Antidifferentiation of sine and cosine functions

1. a. $-\frac{1}{3}\cos(3x)+c$

b. $-\frac{1}{4}\cos(4x)+c$

c. $\frac{1}{7}\sin(7x)+c$

d. $\frac{1}{6}\sin(2x)+c$

e. $\frac{1}{2}\cos(-2x)+c$

f. $-\frac{1}{3}\sin(-3x)+c$

2. a. $-\frac{2}{9}\cos(6x)+c$

b. $-2\cos(4x)+c$

c. $2\cos(3x)+c$

d. $2\sin(-x)+c$

e. $-3\cos\left(\frac{x}{3}\right)+c$

f. $2\sin\left(\frac{x}{2}\right)+c$

3. a. $12\cos\left(\frac{-x}{4}\right)+c$

b. $10\cos\left(\frac{x}{5}\right)+c$

c. $16\sin\left(\frac{x}{4}\right)+c$

d. $12\sin\left(\frac{-x}{2}\right)+c$

e. $-6\cos\left(\frac{2x}{3}\right)+c$

f. $8\sin\left(\frac{3x}{4}\right)+c$

4. a. $\frac{1}{4}e^{4x}-\frac{1}{2}\cos(2x)+\frac{1}{4}x^4+c$

b. $x^3-\sin(2x)+2e^{3x}+c$

5. a. $\sin(x)-\cos(x)+c$

b. $-\frac{1}{2}\cos(2x)-\sin(x)+c$

c. $\frac{1}{4}\sin(4x)-\frac{1}{2}\cos(2x)+c$

d. $-2\cos\left(\frac{x}{2}\right)-\frac{1}{2}\sin(2x)+c$

6. a. $\sin(4x)+\frac{1}{6}\cos(2x)+c$

b. $\frac{5}{2}x^2-2\cos(2x)+c$

c. $\frac{6}{\pi}\left(\sin\left(\frac{\pi x}{3}\right)-\cos\left(\frac{\pi x}{2}\right)\right)+c$

d. $\frac{1}{2}e^{6x}+\frac{1}{2}\cos(8x)+7x+c$

7. a. $\frac{1}{6}\sin(3x+4)+8\cos\left(\frac{x}{2}\right)+c$

b. $\frac{3}{2}\sin\left(\frac{2x}{3}\right)-\frac{1}{8}\cos(5-2x)+c$

8. a. $f(x)=-2\cos\left(\frac{x}{2}\right)-6\sin\left(\frac{x}{2}\right)+c$

b. $f(x)=\frac{7}{2}\sin(2x)+\frac{1}{3}\cos(3x)+c$

9. a. $3e^{\frac{x}{2}}-3\cos\left(\frac{x}{3}\right)+\frac{1}{6}x^2+c$

b. $\frac{1}{4}\sin(4x)-e^{-3x}+c$

10. $-\frac{1}{4x}-\frac{2}{3\pi}\cos\left(\frac{3\pi x}{2}\right)+c$

11. $y=\frac{1}{2}\sin(2x)+\frac{1}{3}e^{-3x}+c$

12. a. -2

b. $f(x)=\frac{2}{3}\cos(3x)+c$

13. a. -5

b. $f(x)=2\sin(2x)-5x+c$

14. a. -10

b. $y=-5\sin\left(2x+\frac{\pi}{3}\right)+c$

15. $p=4, q=-\frac{3}{2}$

Exercise 6.6 Further integration

1. a. $\frac{1}{3}(x+3)^3+c$

b. $\frac{1}{4}(x-5)^4+c$

c. $\frac{1}{5}(2x+1)^5+c$

d. $-\frac{1}{9}(3x-4)^6+c$

e. $\frac{1}{30}(6x+5)^5+c$

f. $\frac{1}{4}(4x-1)^3+c$

2. a. $-\frac{1}{4}(4-x)^4+c$

b. $-\frac{1}{5}(7-x)^5+c$

c. $-\frac{4}{15}(8-3x)^5+c$

d. $\frac{1}{33}(8-9x)^{11}+c$

e. $-\frac{1}{2}(2x+3)^{-1}+c$

f. $-\frac{1}{12}(6x+5)^{-2}+c$

3. a. $\frac{1}{18}(3x-5)^6+c$

b. $-\frac{1}{3(2x-3)^{\frac{3}{2}}}+c$

4. a. $\frac{1}{10}(2x+3)^5+c$

b. $\frac{1}{8(1-2x)^4}+c$

5. a. $\frac{1}{4}e^{4x+2}-4e^{2x+1}+16x+c$

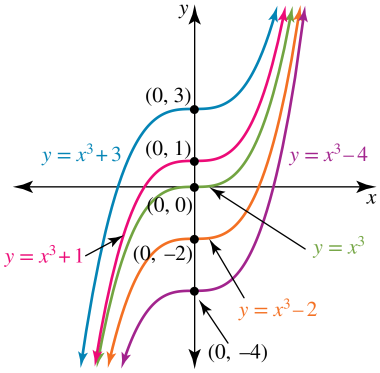
b. $-2e^{6-2x}-6e^{5-2x}-\frac{9}{2}e^{4-2x}+c$

6. a. $f(x) = 2x^2 + x + 2$ b. $f(x) = 5x - x^2 - 5$

c. $f(x) = 3x + 2 - \frac{1}{x}$ d. $f(x) = \frac{1}{2}x^2 + \frac{2}{3}x^{\frac{3}{2}} - \frac{10}{3}$

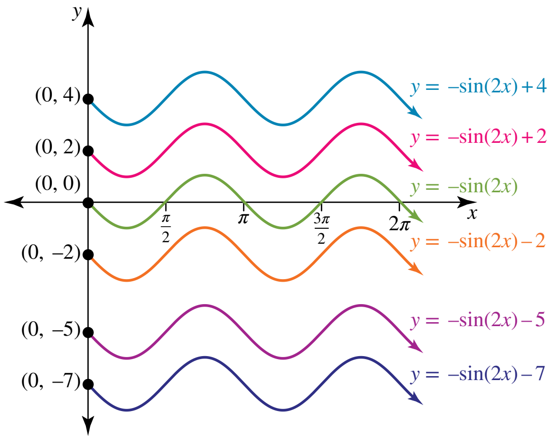
e. $f(x) = \frac{3}{4}x^{\frac{4}{3}} - x^3 + 50x$ f. $f(x) = 2\sqrt{x} - x^2 - 6$

7. a.



b. $f(x) = x^3 + 8$

8. a.



b. $f(x) = 4 - \sin(2x)$

9. $y = e^{2x} - e^{-x} + 3$

10. $f(x) = \frac{1}{2}\sin(2x) + \frac{1}{2}\cos(2x) + \frac{3}{2}$

11. a. $P = 50e^{0.4t} - 15$ b. 536 bugs

12. a. $P = 100e^{0.3t} - 50$ b. 1959 sea lions

13. a. $h = 2\sin\left(\frac{\pi t}{4}\right) + 3$

b. Maximum depth: 5 m; minimum depth: 1 m

c. 8 hours/day

14. $\frac{x}{\sqrt{x^2+1}}$; $5\sqrt{x^2+1} + c$

15. $8(5x+1)\left(5x^2+2x-1\right)^3$; $2(5x^2+2x-1)^4 + c$

16. $\frac{6x}{(3x^2+4)}$; $\frac{1}{6}\ln\left(3x^2+4\right) + c$

17. a. $-\tan(x)$ b. $-\ln(\cos(x)) + c$

18. $\sin(x) + x\cos(x)$; $x\sin(x) + \cos(x) + c$

19. $\ln(x) + 1$; $x\ln(x) - x + c$

20. $2e^{3x} + 6xe^{3x}$; $\frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x} + c$

21. a. $x = \frac{2}{9}\sqrt{(3t+1)^3} - \frac{2}{9}$ b. $x = \frac{1}{2} - \frac{1}{(t+2)}$

c. $x = \frac{1}{8}(2t+1)^4 - \frac{1}{8}$

22. a. $x = t^3 + 3t^2 - 2$ b. 198 m

23. a. $x = \frac{1}{3}e^{(3t-1)} - \frac{1}{3e}$

b. $x = \frac{1}{2}\cos(2t+3) - \frac{1}{2}\cos(3)$

24. a. Sample responses can be found in the worked solutions in the online resources.

b. $x = \frac{1}{2} - \frac{1}{2}\cos(2t) + \frac{1}{2}\sin(2t)$

25. a. $x = 6t - \frac{12}{(t-1)} - 12$

b. At the origin

6.7 Review: exam practice

1. a. $\frac{1}{2}x^6 + c$ b. $-\frac{5}{x} + c$

c. $-\frac{2}{5}x^5 + c$ d. $\frac{4}{3}x^{\frac{3}{2}} + c$

e. $\frac{1}{25}x^5 + c$ f. $-\frac{1}{15(3x-8)^5} + c$

g. $\frac{1}{10(6-5x)^2} + c$ h. $-\frac{2}{3(7-5x)^3} + c$

2. a. $\frac{1}{5}x^5 + x^2 + \ln(x) + c$

b. $\frac{1}{18}(3x+1)^6 + c$

c. $3x + 2\ln(x) + \frac{1}{x} + c$

d. $\frac{3}{2}\ln(2x+1) + c$

e. $\frac{1}{2}\ln(6-10x) + c$

f. $-\frac{3}{8}(4x+1)^{-2} + c$

g. $\frac{1}{4}x^2 + 4x + 8\ln(x) + c$

h. $\frac{2}{3}x\sqrt{x} - 2\ln(3-x) + c$

3. a. $f(x) = \frac{1}{4}(x+4)^4 + 1$

b. $f(x) = (1-2x)^{-4} + 2$

c. $f(x) = \ln(x+5) + 2$

d. $f(x) = -4\ln(7-2x) + 7$

4. a. -8 b. 41

5. a. $\frac{1}{2}e^{2x} - 6e^x + 9x + c$

b. $x - 3e^{-x} - \frac{3}{2}e^{-2x} - \frac{1}{3}e^{-3x} + c$

6. a. $\frac{4}{5}\cos\left(\frac{5x}{2}\right) + c$ b. $-\frac{12}{7}\sin\left(\frac{7x}{4}\right) + c$

c. $-\frac{5}{\pi}\cos(\pi x) + c$ d. $\frac{6}{\pi}\sin\left(\frac{\pi x}{2}\right) + c$

e. $-\frac{6}{\pi}\sin\left(\frac{\pi x}{3}\right) + c$ f. $-\frac{\pi}{4}\cos\left(\frac{-4x}{\pi}\right) + c$

7. a. $\frac{1}{4}x^4 - \frac{1}{2}\ln(2x+3) + \frac{1}{2}e^{2x} + c$

b. $\frac{1}{3}x^3 + 2\sin(2x) + e^{-x} + c$

c. $-3\cos\left(\frac{x}{3}\right) + 2e^{\frac{x}{2}} - \frac{1}{15}(3x-1)^5 + c$

d. $\frac{1}{3}\ln(3x-2) + \frac{1}{4}e^{4x} + 5\sin\left(\frac{x}{5}\right) + c$

e. $-6\cos\left(\frac{x}{2}\right) - 6\sin\left(\frac{x}{3}\right) + 5e^{\frac{-x}{5}} + c$

f. $\frac{2}{3}x\sqrt{x} + x^2 + \frac{6}{\pi}\cos\left(\frac{\pi x}{3}\right) + 5x + c$

8. a. $f(x) = \sin(x) + 4$

b. $f(x) = 1 - 2\cos(2x)$

c. $f(x) = 12\sin\left(\frac{x}{4}\right) + 3\sqrt{2}$

d. $f(x) = 4\sin\left(\frac{x}{4}\right) + 2\cos\left(\frac{x}{2}\right) - 4$

9. a. -4

b. $f(x) = 2\sin(2x) - 4e^x + 3$

c. -2.02

10. $\frac{2x}{(x^2+3)}$; $6\ln\left(x^2+3\right) + c$

11. $\frac{-1}{\sin^2(x)}$; $-\frac{\cos(x)}{\sin(x)} + c$

12. a. Sample responses can be found in the worked solutions in the online resources.

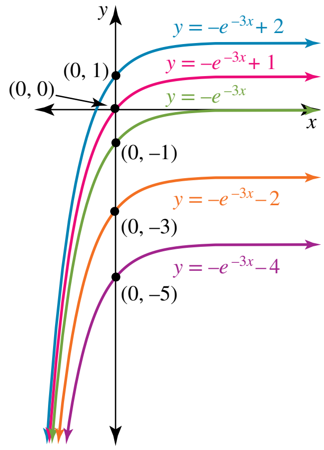
b. $-3x - 2\ln(3-2x) + c$

13. $a = -2, b = -4, m = 2, n = -2$

14. a. 1.5625 m/s

b. $x = \frac{1}{16}t^2 - \frac{1}{1200}t^3$

15. a.



b. $f(x) = 2 - e^{-3x}$

16. a. $f(x) = 5x - x^2$

b. $f(x) = 3 - 2\cos\left(\frac{x}{2}\right)$

c. $f(x) = \frac{1}{(1-x)} + 3$

17. a. $x = 24 - 24\cos\left(\frac{\pi t}{8}\right)$

b. 48 m

c. 24 m above the stationary position

18. a. $N = 400t + \frac{2000}{3}t\sqrt{t} + 40$

b. 9494 families

19. $\frac{40\,000}{3}$ cm^3 or $13\,333\frac{1}{3}$ cm^3

20. a. $T = 20 - 5\sin\left(\frac{\pi t}{12}\right)$

b. No, since $-1 \le \sin(x) \le 1$

c. 25 °C; 6 pm

d. 15 °C; 6 am

e. i. 17.5 °C   ii. 23.5 °C

f. 2 pm

7 Integration

7.1 Overview

In the previous two chapters, you studied various aspects of calculus. Calculus examines how a change in one variable will affect another related variable. Historically, integration began with mathematicians attempting to find the area under curves. Various methods of estimating the areas under curves are investigated in this chapter. This leads to the fundamental theorem of calculus and its applications to the definite integral.

Integration has many real-life applications. In economics and commerce, if marginal costs are known, predictions can be made for total costs incurred. In science, given the rate of change of a substance, such as an oil spill, the total area affected can be ascertained. Motion in a straight line, or kinematics, is yet another application of both differential and integral calculus.

This chapter continues on from the antidifferentiation chapter, studying in particular the definite integral together with its applications in real-life situations.

LEARNING SEQUENCE

7.1 Overview
7.2 Estimating the area under a curve
7.3 The fundamental theorem of calculus and definite integrals
7.4 Areas under curves
7.5 Areas between curves
7.6 Applications of integration
7.7 Review: exam practice

Fully worked solutions for this chapter are available in the Resources section of your eBookPLUS at www.jacplus.com.au.

7.2 Estimating the area under a curve

There are several different ways to estimate or approximate the area between a curve and the x-axis.

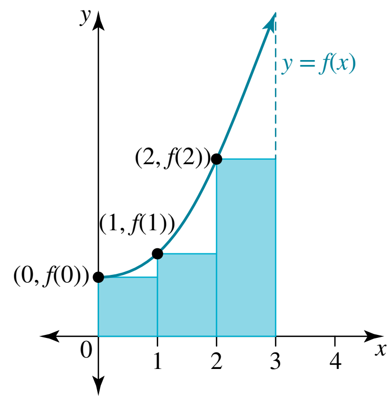
This section considers three methods:

- the left end-point rectangle method
- the right end-point rectangle method
- the trapezoidal method.

7.2.1 The left end-point rectangle method

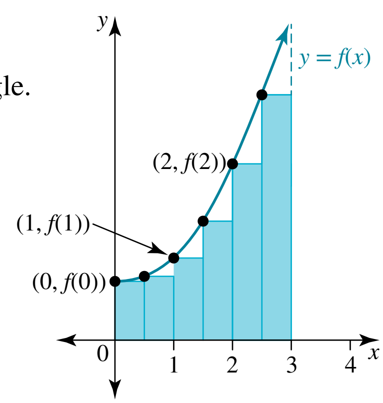
Consider the curve defined by the function $f(x) = x^2 + 2$ and the area between this curve and the x-axis from $x = 0$ to $x = 3$.

- Construct rectangles of width 1 unit.
- The height of the rectangle is given by the function on the left side, so the height of the first rectangle is $f(0)$, the height of the second rectangle is $f(1)$ and so on.
- Approximate area:



$$\begin{aligned} A &= 1 \times f(0) + 1 \times f(1) + 1 \times f(2) \\ &= 2 + 3 + 6 \\ &= 11 \text{ square units} \end{aligned}$$

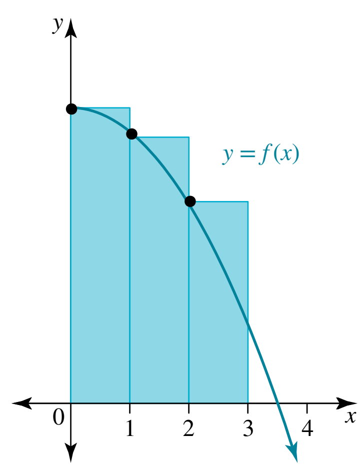
- Construct rectangles of width 0.5 units.
- The height is again given by the function on the left side of the rectangle.
- Approximate area:



$$\begin{aligned} A &= 0.5 \times f(0) + 0.5 \times f(0.5) + 0.5 \times f(1) + 0.5 \times f(1.5) \\ &\quad + 0.5 \times f(2) + 0.5 \times f(2.5) \\ &= 0.5 \times (f(0) + f(0.5) + f(1) + f(1.5) + f(2) + f(2.5)) \\ &= 0.5 \times (2 + 2.25 + 3 + 4.25 + 6 + 8.25) \\ &= 12.875 \text{ square units} \end{aligned}$$

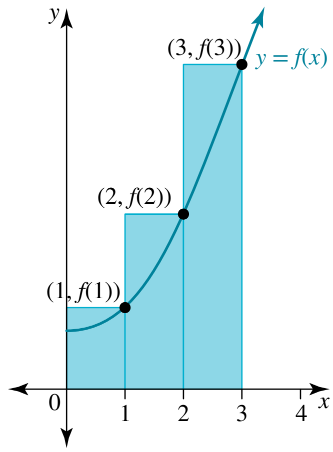
Observe from these examples that:

- the narrower the rectangles, the closer the approximation is to the actual area under the curve
- the approximate area is less than the actual area
- if the function was a decreasing function, for example $f(x) = 10 - x^2$, then the area using the left end-points would be greater than the actual area under the curve.



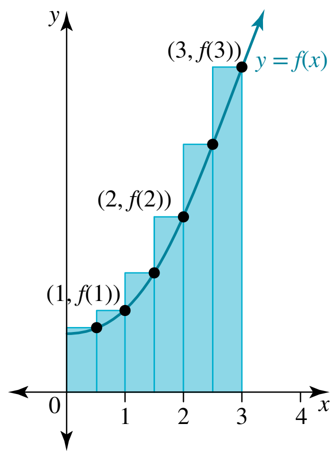
7.2.2 The right end-point rectangle method

Again, consider the curve defined by the function $f(x) = x^2 + 2$ and the area between this curve and the x-axis from $x = 0$ to $x = 3$.



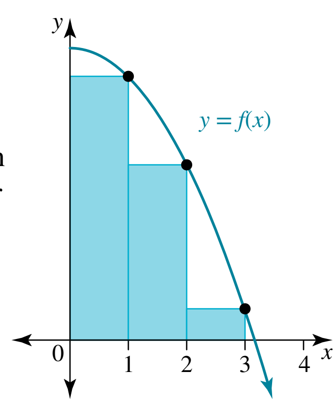
- Construct rectangles of width 1 unit.
- The height of the rectangle is given by the function on the right side, so the height of the first rectangle is $f(1)$, the height of the second rectangle is $f(2)$ and so on.
- Approximate area:

$$\begin{aligned} A &= 1 \times f(1) + 1 \times f(2) + 1 \times f(3) \\ &= 3 + 6 + 11 \\ &= 20 \text{ square units} \end{aligned}$$



- Construct rectangles of width 0.5 units.
- The height is again given by the function on the right side of the rectangle.
- Approximate area:

$$\begin{aligned} A &= 0.5 \times f(0.5) + 0.5 \times f(1) + 0.5 \times f(1.5) + 0.5 \times f(2) \\ &\quad + 0.5 \times f(2.5) + 0.5 \times f(3) \\ &= 0.5 \times (f(0.5) + f(1) + f(1.5) + f(2) + f(2.5) + f(3)) \\ &= 0.5 \times (2.25 + 3 + 4.25 + 6 + 8.25 + 11) \\ &= 17.375 \text{ square units} \end{aligned}$$



Observe from these examples that:

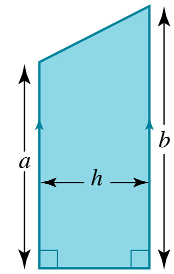
- the narrower the rectangles, the closer the approximation is to the actual area under the curve
- the approximate area is greater than the actual area
- if the function was a decreasing function, for example $f(x) = 10 - x^2$ then the area using the right end-points would be less than the actual area under the curve.

The rectangle methods can also be described as the lower rectangle method (if the tops of the rectangles are below the curve) and the upper rectangle method (if the tops of the rectangles are above the curve).

- If the function is an increasing function, then the left end-point rectangles are lower rectangles and the right end-point rectangles are upper rectangles.
- If the function is a decreasing function, then the left end-point rectangles are upper rectangles and the right end-point rectangles are lower rectangles.

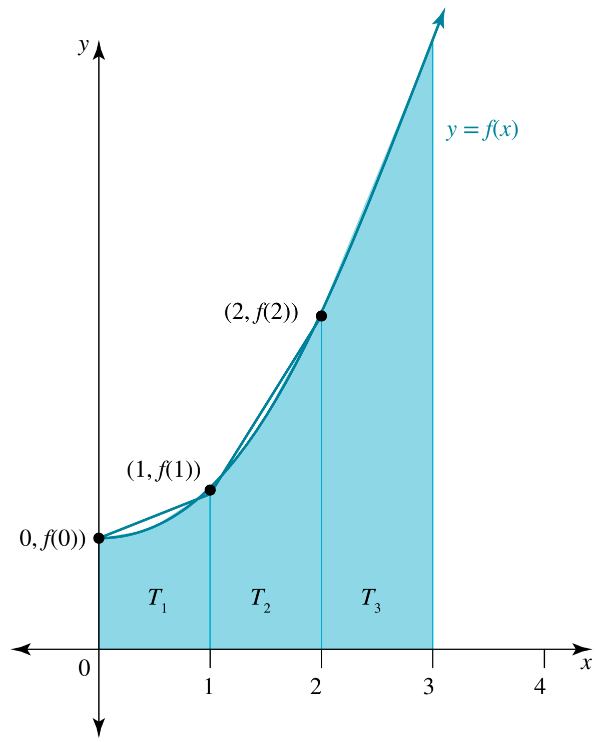
7.2.3 The trapezoidal method

Recall that for a trapezium, area $A = \frac{1}{2}(a + b) \times h$, where a and b are the lengths of the parallel sides and h is the width of the trapezium.



Once again, consider the curve defined by the function $f(x) = x^2 + 2$ and the area between this curve and the x-axis from $x = 0$ to $x = 3$.

- Construct trapeziums of width 1 unit.
- The lengths of the parallel sides are defined by $f(x)$.
- Area of $T_1 = \frac{1}{2}(f(0) + f(1)) \times 1$
  $= \frac{1}{2}(2 + 3)$
  $= \frac{5}{2}$
- Area of $T_2 = \frac{1}{2}(f(1) + f(2)) \times 1$
  $= \frac{1}{2}(3 + 6)$
  $= \frac{9}{2}$
- Area of $T_3 = \frac{1}{2}(f(2) + f(3)) \times 1$
  $= \frac{1}{2}(6 + 11)$
  $= \frac{17}{2}$
- Total area $= \frac{5}{2} + \frac{9}{2} + \frac{17}{2}$
  $= \frac{31}{2}$
  $= 15.5$ square units



Note: When the widths of the rectangles and the trapezia are the same, the average of the areas found using left end-point rectangles and right end-point rectangles gives the same result as the trapezoidal rule.

For example, using a width of 1 unit:

Area using left end-points = 11
Area using right end-points = 20
Average of areas = (11 + 20)/2
= 15.5 square units,
the area of the trapezium.

WORKED EXAMPLE 1

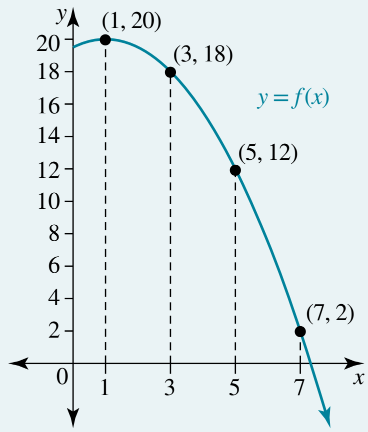
Points on a function, $y = f(x)$, are shown in the diagram. An approximation of the area between the function $y = f(x)$ and the x-axis from $x = 1$ to $x = 7$ is to be found using the given information.

Determine the approximate area using:

a. the left end-point rectangle method

b. the right end-point rectangle method

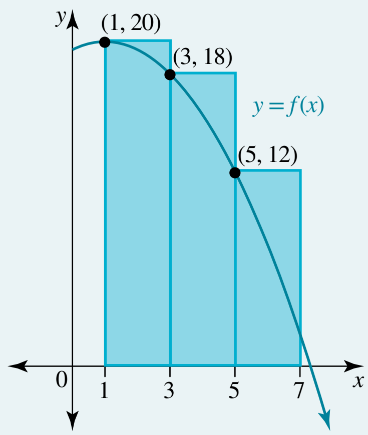
c. the trapezoidal method.



THINK / **WRITE**

a. 1. Draw the left end-point rectangles on the graph. State the width and heights of the rectangles.

a.



The width of each rectangle is 2 units.
The heights of the rectangles are 20, 18 and 12 units.

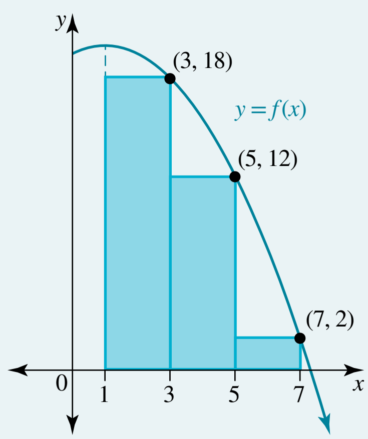
2. Calculate the approximate area by adding the areas of the rectangles.

$A = 2 \times 20 + 2 \times 18 + 2 \times 12$
$= 100$

The area is approximately 100 square units.

b. 1. Draw the right end-point rectangles on the graph. State the widths and heights of the rectangles.

b.



The width of each rectangle is 2 units.
The heights of the rectangles are 18, 12 and 2 units.

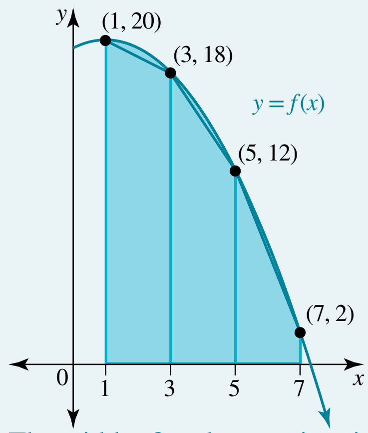
2. Calculate the approximate area by adding the areas of the rectangles.

$A = 2 \times 18 + 2 \times 12 + 2 \times 2$
$= 64$

The area is approximately 64 square units.

c. 1. Draw trapeziums on the graph. State the widths of the trapeziums.

c.



The width of each trapezium is 2 units.

2. Use the formula to calculate the area of each trapezium.

$T_1 = \frac{1}{2}(20 + 18) \times 2 = 38$

$T_2 = \frac{1}{2}(18 + 12) \times 2 = 30$

$T_3 = \frac{1}{2}(12 + 2) \times 2 = 14$

Area $= 38 + 30 + 14$
$= 82$

3. Calculate the approximate area by adding the areas of the trapeziums.

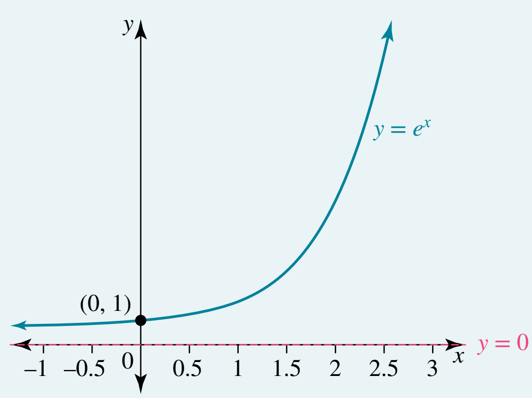
The area is approximately 82 square units.

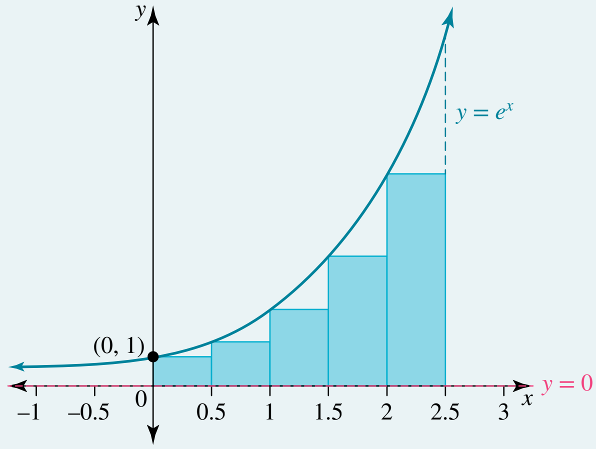
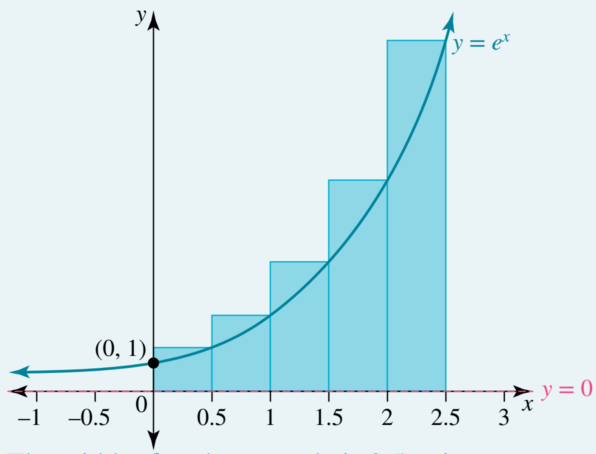
WORKED EXAMPLE 2

The graph of the function defined by the rule $f(x) = e^x$ is shown. Give your answers to the following correct to 2 decimal places.

a. **Use the left end-point rectangle method with rectangles of width 0.5 units to estimate the area between the curve and the x-axis from $x = 0$ to $x = 2.5$.**

b. **Use the right end-point rectangle method with rectangles of width 0.5 units to estimate the area between the curve and the x-axis from $x = 0$ to $x = 2.5$.**



THINK	WRITE
a. 1. Draw the left-end-point rectangles on the graph. State the widths and heights of the rectangles.	**a.**  The width of each rectangle is 0.5 units.The heights of the rectangles are $f(0), f(0.5), f(1), f(1.5)$ and $f(2)$.
2. Determine the approximate area by adding the areas of all the rectangles.	$A = 0.5f(0) + 0.5f(0) + 0.5f(1) + 0.5f(1.5) + 0.5f(2)$ $= 0.5\,[f(0) + f(0.5) + f(1) + f(1.5) + f(2)]$ $= 0.5\,[e^0 + e^{0.5} + e^1 + e^{1.5} + e^2]$ $= 0.5 \times 17.2377$ ≈ 8.62 The area is approximately 8.62 square units.
b. 1. Draw the right end-point rectangles on the graph. State the widths and heights of the rectangles.	**b.**  The width of each rectangle is 0.5 units. The heights of the rectangles are $f(0.5), f(1), f(1.5)$, $f(2)$ and $f(2.5)$.
2. Determine the approximate area by adding the areas of all the rectangles.	$A = 0.5f(0.5) + 0.5f(1) + 0.5f(1.5) + 0.5f(2) + 0.5f(2.5)$ $= 0.5\left[f(0.5) + f(1) + f(1.5) + f(2) + f(2.5)\right]$ $= 0.5\left[e^{0.5} + e^1 + e^{1.5} + e^2 + e^{2.5}\right]$ $= 0.5 \times 28.4202$ ≈ 14.21 The area is approximately 14.21 square units.

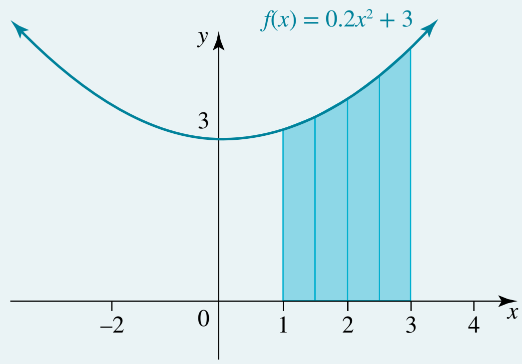
WORKED EXAMPLE 3

Determine an approximation for the area enclosed by the graph of $f(x) = 0.2x^2 + 3$, the x-axis and the lines $x = 1$ and $x = 3$ using the trapezoidal rule with interval widths of 0.5 units.

THINK

a. 1. Sketch the graph of $f(x)$.

WRITE



2. Draw trapeziums of width 0.5 units from $x = 1$ to $x = 3$.

3. Evaluate the height of each vertical side of the trapeziums by substituting the appropriate x-value into $f(x)$.

$f(1) = 0.2(1)^2 + 3 = 3.2$

$f(1.5) = 0.2(1.5)^2 + 3 = 3.45$

$f(2) = 0.2(2)^2 + 3 = 3.8$

$f(2.5) = 0.2(2.5)^2 + 3 = 4.25$

$f(3) = 0.2(3)^2 + 3 = 4.8$

4. Calculate the area of each trapezium.

$T_1 = \frac{1}{2}(3.2 + 3.45) \times 0.5 = 1.6625$

$T_2 = \frac{1}{2}(3.45 + 3.8) \times 0.5 = 1.8125$

$T_3 = \frac{1}{2}(3.8 + 4.25) \times 0.5 = 2.0125$

$T_4 = \frac{1}{2}(4.25 + 4.8) \times 0.5 = 2.2625$

5. Calculate the approximate area by adding the areas of the trapeziums.

Total area of trapeziums
$= 1.6625 + 1.8125 + 2.0125 + 2.2625$
$= 7.75$

Therefore, the area under the curve is approximately 7.75 square units.

on Resources

Interactivity: Estimation of area under a curve (int-6422)

studyon

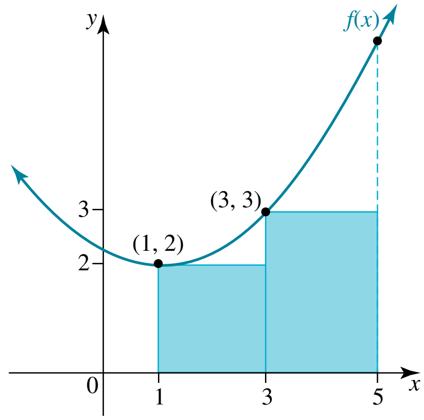
Units 3 & 4 > Area 3 > Sequence 2 > Concepts 1 & 2

Estimating areas under curves — rectangle method Summary screen and practice questions
Estimating areas under curves — trapezoidal method Summary screen and practice questions

Exercise 7.2 Estimating the area under a curve

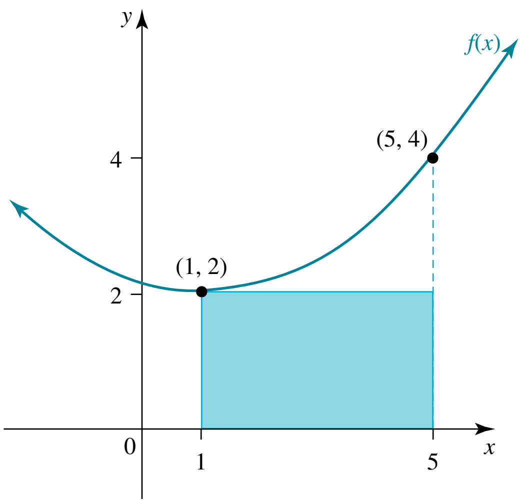
Technology free

1. **WE1** For the curve $f(x)$ shown, determine an approximation for the area between the curve and the x-axis from $x = 1$ to $x = 5$. Use lower rectangles with a width of 2 units.

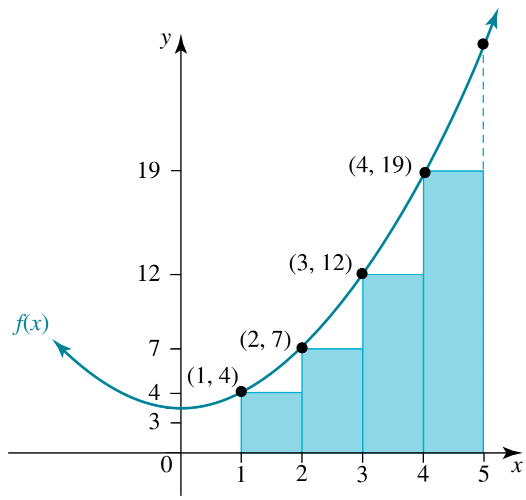


2. For each of the following curves, determine an approximation for the area between the curve and the x-axis from $x = 1$ to $x = 5$ by calculating the areas of the shaded rectangles.

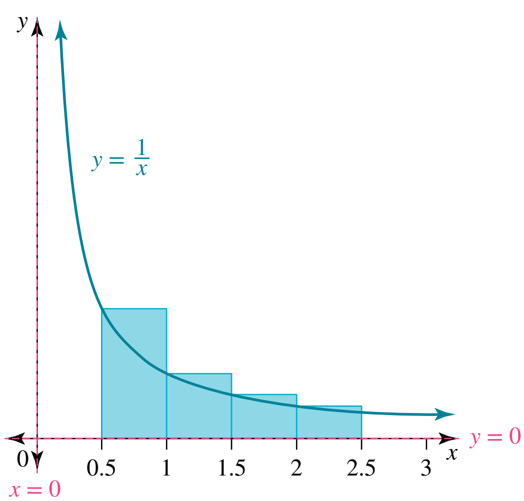
a.

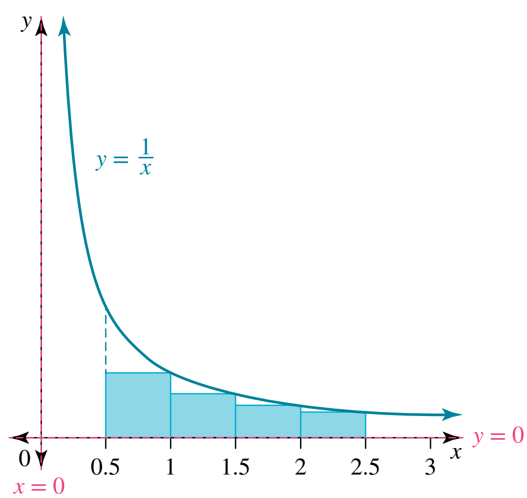


b.



3. **WE2** The left end-point rectangle method and the right end-point rectangle method are shown for the calculation of the approximate area between the curve $f(x) = \dfrac{1}{x}$, $x > 0$, and the x-axis from $x = 0.5$ to $x = 2.5$.



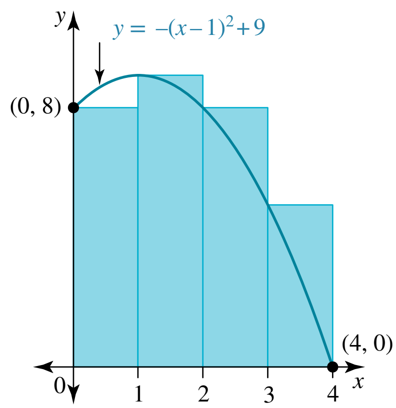


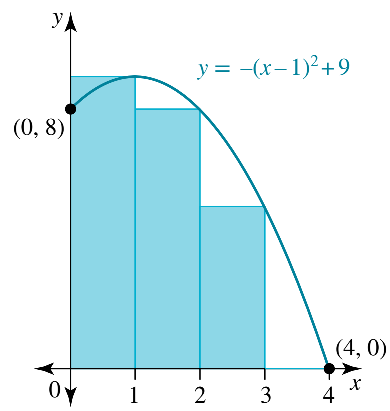
Calculate the approximate area under the curve:

a. using the left end-point rectangle rule

b. using the right end-point rectangle rule.

4. The graph of $f: [0, 4] \to R, f(x) = -(x-1)^2 + 9$ is shown.

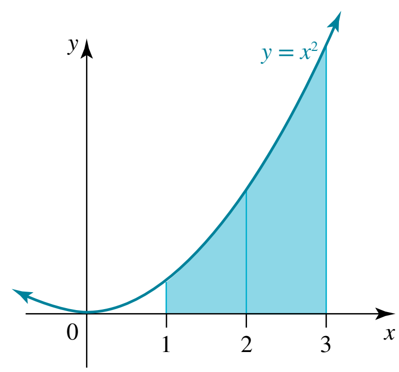




a. Use the left end-point rule with rectangles 1 unit wide to estimate the area between the curve and the x-axis from $x = 0$ to $x = 4$.

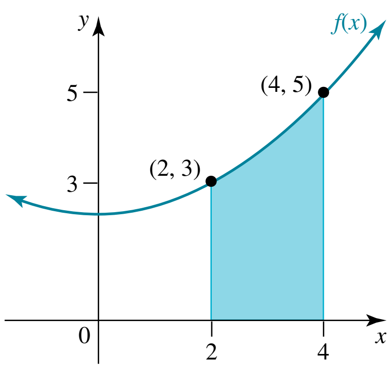
b. Use the right end-point rule with rectangles 1 unit wide to estimate the area between the curve and the x-axis from $x = 0$ to $x = 4$.

5. **WE3** Calculate an approximation for the area enclosed by the graph of $f(x) = x^2$, the x-axis and the lines $x = 1$ and $x = 3$ with interval widths of 1 unit. Use the trapezoidal method.

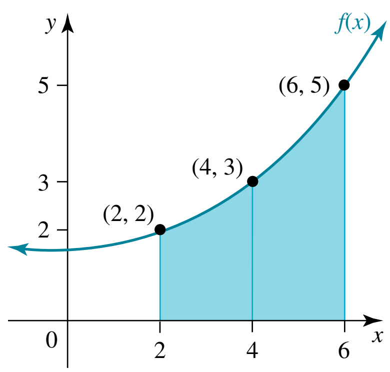


6. For each of the following curves, calculate an approximate area between the curve and the x-axis over the interval indicated by calculating the areas of the shaded trapeziums.

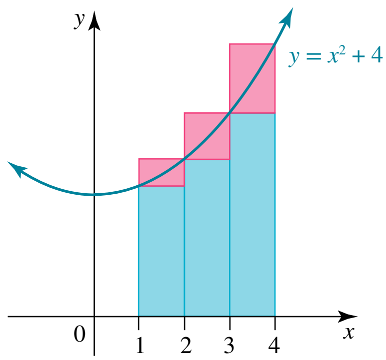
a.



b.

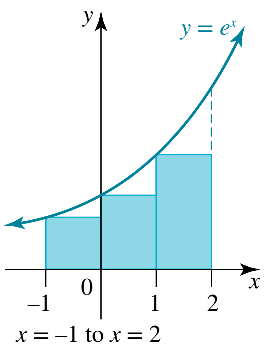


7. Using width intervals of 1 unit, calculate an approximation for the area between the graph of $f(x) = x^2 + 4$ and the x-axis from $x = 1$ to $x = 4$ using:
   a. lower rectangles
   b. upper rectangles
   c. averaging of the lower and upper rectangle areas.

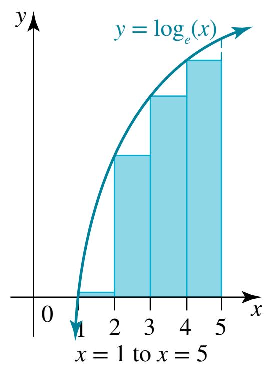


8. For each of the following figures, determine the approximate area between the curve and the x-axis over the interval indicated by calculating the area of the shaded rectangles. Give your answers in exact form.

a.

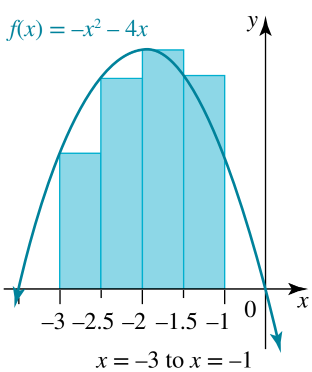


b.

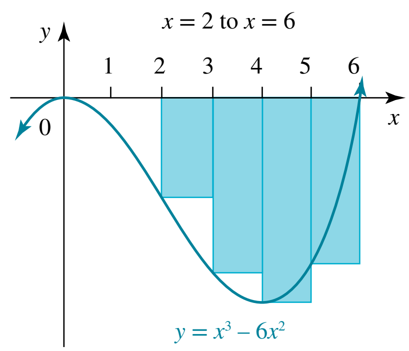


9. For each of the following figures, determine the approximate area between the curve and the x-axis over the interval indicated by calculating the area of the shaded rectangles. Give your answers in exact form.

a.

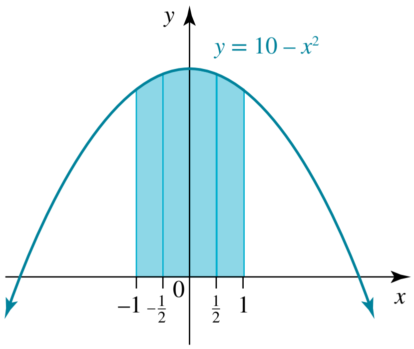


b.

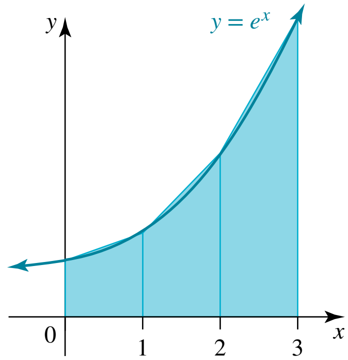


10. For each of the following figures, determine the approximate area between the curve and the x-axis over the interval indicated by calculating the shaded area using the trapezoidal rule. Give your answers in exact form.

a.



b.



Technology active

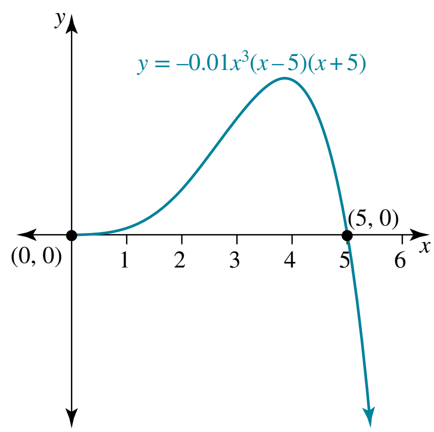
11. Calculate approximations for the area between the graph of $f(x) = (x - 1)^3$ and the x-axis between $x = 1$ and $x = 4$ using the trapezoidal rule and interval widths of:
   a. 1 unit
   b. 0.5 units.

   Give your answers correct to 1 decimal place.

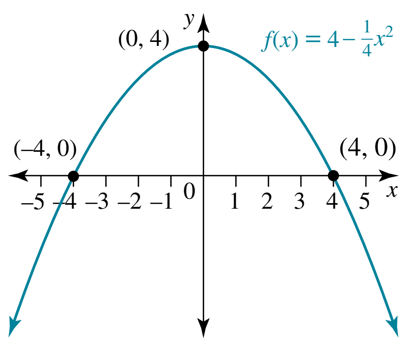
12. Calculate approximations for the area under the graph of $y = \frac{1}{x}$ between $x = 0.5$ and $x = 2.5$ using the trapezoidal rule and interval widths of:
   a. 1 unit
   b. 0.5 units.

   Give your answers correct to 2 decimal places.

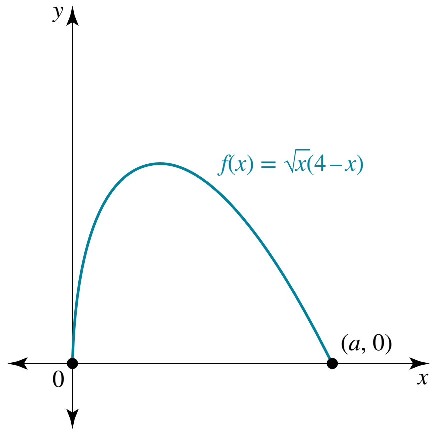
13. Consider the function defined by the rule $f: R \rightarrow R$, $f(x) = -0.01x^3(x - 5)(x + 5), x \geq 0$. The graph of the function is shown. Use the left end-point rule with rectangles 1 unit wide to approximate the area bound by the curve and the x-axis.



14. The graph of the function $f(x) = 4 - \frac{1}{4}x^2$ is shown. Estimate the area bound by the curve and the x-axis using the right end-point method with rectangles of width 1 unit.



15. The graph of $f(x) = \sqrt{x}(4 - x)$ for $x \in [0, a]$ is shown.



a. The graph intersects the x-axis at the point $(a, 0)$ as shown. Calculate the value of the constant a.
b. Use both the left end-point and the right end-point rules to determine the approximate area between the curve and the x-axis from $x = 0$ to $x = a$. Use a rectangle width of 1 and give your answers correct to 2 decimal places.

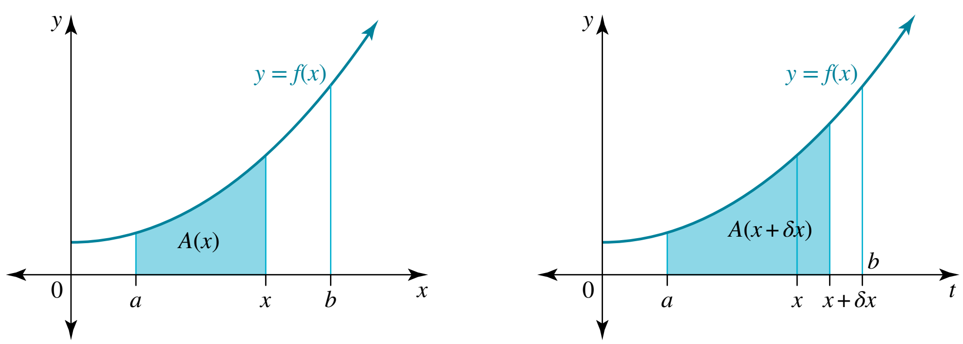
7.3 The fundamental theorem of calculus and definite integrals

7.3.1 The fundamental theorem of calculus

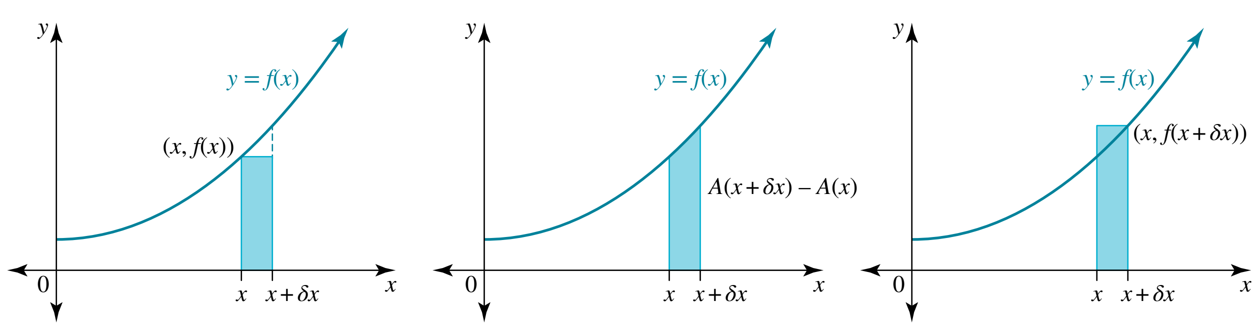
Consider the function $y = f(x)$, where $f(x) \geq 0$, that is continuous for $x \in [a, b]$.

Let $A(x)$ represent the area between the curve and the x-axis from $x = a$ to x.

Similarly, let $A(x + \delta x)$ represent the area between the curve and the x-axis from $x = a$ to $x + \delta x$.



The difference between these areas is $A(x + \delta x) - A(x)$. From the previous section, we know that this area lies between the areas of the left end-point rectangle and the right end-point rectangle.



Left end-point rectangle method   Actual area under the curve   Right end-point rectangle method

Therefore, $f(x) \times \delta x \leq A(x + \delta x) - A(x) \leq f(x, +, \delta, x) \times \delta x$

$$f(x) \leq \frac{A(x + \delta x) - A(x)}{\delta x} \leq f(x + \delta x)$$

As the width of each rectangular strip becomes smaller, the areas become closer together.
This means as $\delta x \to 0, f(x + \delta x) \to f(x)$

or $\lim\limits_{\delta x \to 0} \dfrac{A(x + \delta x) - A(x)}{\delta x} = f(x)$.

By definition of differentiation from first principles, $\lim\limits_{\delta x \to 0} \dfrac{A(x + \delta x) - A(x)}{\delta x} = \dfrac{d}{dx}(A(x))$.

Thus, $\dfrac{d}{dx}(A(x)) = f(x)$.

Integrating both sides with respect to x gives:

$$\int \frac{d}{dx}(A(x))\, dx = \int f(x)\, dx$$

Or simply: $A(x) = \int f(x)\, dx$

Let the antiderivative of $f(x)$ be $F(x)$.

Then: $A(x) = F(x) + c$

Or: $\int f(x)\, dx = F(x) + c$

When $x = a$: $\int f(x)\, dx = F(a) + c$

But: $\int f(x)\, dx = 0$, as the area defined is zero at $x = a$.

$\therefore c = -F(a)$

This gives: $\int f(x)\, dx = F(x) - F(a)$

Let $x = b$: $\int f(x)\, dx = F(b) - F(a)$

That is, the area between the graph of $y = f(x)$, the x-axis and $x = a$ and $x = b$ is given by $F(b) - F(a)$, where $F(x)$ is the antiderivative of $f(x)$.

This is the **fundamental theorem of integral calculus**. It allows areas under graphs to be calculated exactly. The fundamental theorem can be stated as:

$$\begin{aligned} \text{Area} &= \int_a^b f(x)\, dx \\ &= [F(x)]_a^b \\ &= F(b) - F(a) \end{aligned}$$

The fundamental theorem of integral calculus

$$\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a)$$

where $F(x)$ is the antiderivative of $f(x)$.

7.3.2 The definite integral and its properties

- $\int_a^b f(x)\,dx$ is called the **definite integral** as it evaluates to a real number and not a function.
- The indefinite integral, $\int f(x)\,dx$, involves finding only an antiderivative of the function.
- There is no need to add $+c$ as the two c constants in $F(a)$ and $F(b)$ would cancel each other out.
- The variables a and b are called the **terminals** of the definite integral and indicate the range of the values of x over which the integral is taken.
- The function to be integrated, $f(x)$, is called the **integrand**.
- Definite integrals have the following properties, assuming f and g are continuous functions for $x \in [a, b]$ or $a \leq x \leq b$ and k is a real constant.

Properties of definite integrals

$$\int_a^a f(x)\,dx = 0$$

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

$$\int_a^b k f(x)\,dx = k\int_a^b f(x)\,dx$$

$$\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$$

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \text{ provided } a < c < b$$

WORKED EXAMPLE 4

Evaluate the following definite integrals.

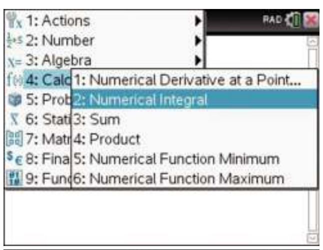
a. $\int_0^3 (3x^2 + 4x - 1)\,dx$

b. $\int_1^2 \frac{4}{(2x+1)^3}\,dx$

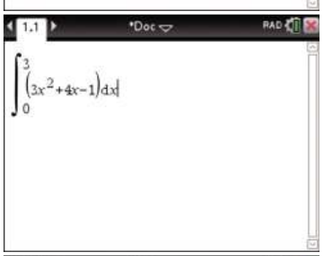
THINK	WRITE
a. 1. Antidifferentiate each term of the integrand and write in the form $[F(x)]_a^b$.	**a.** $\int_0^3 \left(3x^2 + 4x - 1\right) dx$ $= \left[x^3 + 2x^2 - x\right]_0^3$
2. Substitute the values of a and b into $F(b) - F(a)$.	$= \left[3^3 + 2(3)^2 - 3\right] - \left[0^3 + 2(0)^2 - 0\right]$
3. Evaluate the integral.	$= 42 - 0$ $= 42$
b. 1. Express the integrand in simplest index form.	**b.** $\int_1^2 \frac{4}{(2x+1)^3} dx = \int_1^2 4(2x+1)^{-3} dx$
2. Antidifferentiate by rule.	$= \left[\frac{4(2x+1)^{-2}}{2 \times -2}\right]_1^2$ $= \left[-(2x+1)^{-2}\right]_1^2$
3. Express the integral with a positive index number.	$= \left[\frac{-1}{(2x+1)^2}\right]_1^2$
4. Substitute the values of a and b into $F(b) - F(a)$.	$= \left[-\frac{1}{5^2}\right] - \left[\frac{-1}{3^2}\right]$
5. Evaluate the integral.	$= -\frac{1}{25} + \frac{1}{9}$ $= \frac{16}{225}$

TI | THINK — **WRITE**

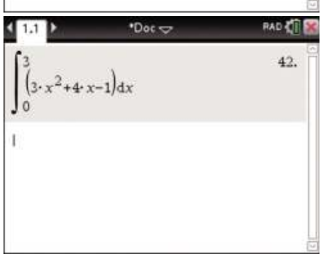
a. 1. On a Calculator page, press MENU, then select:
4: Calculus
2: Numerical Integral.



2. Complete the entry line as:
$\int_0^3 3x^2 + 4x - 1\, dx$
then press the ENTER button.

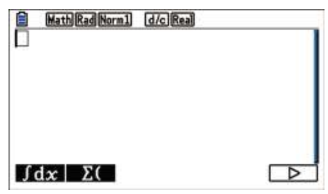


3. The answer appears on the screen.
Note: This calculation can also be performed using the Scratchpad features, as shown in Worked examples 9 and 10.

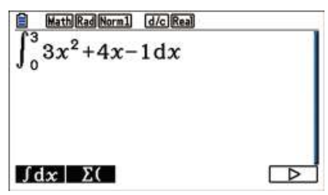


CASIO | THINK — **WRITE**

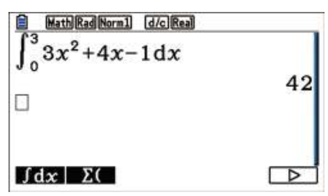
a. 1. On a Run-Matrix page, select:
SHIFT
F5
$\int dx$.



2. Complete the entry line as:
$\int_0^3 3x^2 + 4x - 1\, dx$
then press the EXE button.



3. The answer appears on the screen.
Note: This calculation can also be performed using the GRAPH function, as shown in Worked examples 9 and 10.



WORKED EXAMPLE 5

Evaluate:

a. $\int_0^{\frac{\pi}{2}} \cos(x)\,dx$

b. $\int_0^2 (e^{-x} + 2)\,dx$

THINK	WRITE
a. 1. Antidifferentiate the given function and specify the end points for the calculation using square brackets.	**a.** $\int_0^{\frac{\pi}{2}} \cos(x)dx$ $= [\sin(x)]_0^{\frac{\pi}{2}}$
2. Substitute the upper and lower end points into the antiderivative and calculate the difference between the two values.	$= \sin\left(\frac{\pi}{2}\right) - \sin(0)$ $= 1 - 0$ $= 1$
b. 1. Antidifferentiate the given function and specify the end points for the calculation using square brackets.	**b.** $\int_0^2 (e^{-x} + 2)dx$ $= [-e^{-x} + 2x]_0^2$
2. Substitute the upper and lower end points into the antiderivative and calculate the difference between the two values.	$= (-e^{-2} + 2(2)) - (-e^0 + 2(0))$ $= -\frac{1}{e^2} + 4 + 1$ $= 5 - \frac{1}{e^2}$

Even if the function is unknown, we can use the properties of definite integrals to evaluate the values of related integrals. This is demonstrated in the following worked example.

WORKED EXAMPLE 6

a. Given that $\int_1^3 f(x)dx = 8$, determine:

i. $\int_1^3 2f(x)dx$ **ii.** $\int_1^3 (f(x) + 1)dx$ **iii.** $\int_3^1 f(x)dx$ **iv.** $\int_1^3 (f(x) - x)dx$

b. Calculate k if $\int_1^k (x + 2)dx = 0$.

THINK	WRITE
a. i. Apply the definite integral property $\int_a^b kf(x)dx = k\int_a^b f(x)dx$.	**a. i.** $\int_a^b 2f(x)dx = 2\int_1^3 f(x)dx$ $= 2 \times 8$ $= 16$

ii. 1. Apply the definite integral property $\int_a^b (f(x) \pm g(x))dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$.

2. Integrate the second function and evaluate.

ii. $\int_1^3 (f(x)+1)dx = \int_1^3 f(x)dx + \int_1^3 1dx$

$= 8 + [x]_1^3$
$= 8 + (3-1)$
$= 10$

iii. Apply the definite integral property $\int_a^b f(x)dx = -\int_b^a f(x)dx$.

iii. $\int_3^1 f(x)dx = -\int_1^3 f(x)dx$
$= -8$

iv. 1. Apply the definite integral property $\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$.

2. Integrate the second function and evaluate.

iv. $\int_1^3 (f(x)-x)dx = \int_1^3 f(x)dx - \int_1^3 xdx$

$= 8 - \left[\frac{1}{2}x^2\right]_1^3$
$= 8 - \left(\frac{1}{2}(3)^2 - \frac{1}{2}(1)^2\right)$
$= 8 - \left(\frac{9}{2} - \frac{1}{2}\right)$
$= 8 - 4$
$= 4$

b. 1. Antidifferentiate and substitute the values of 1 and k.

2. Simplify and solve for k.

3. Write the answer.

b. $0 = \int_1^k (x+2)dx$

$0 = \left[\frac{1}{2}x^2 + 2x\right]_1^k$

$0 = \left(\frac{1}{2}k^2 + 2k\right) - \left(\frac{1}{2}(1)^2 + 2(1)\right)$

$0 = \frac{1}{2}k^2 + 2k - \frac{5}{2}$

$0 = k^2 + 4k - 5$
$0 = (k+5)(k-1)$
$k = -5$ or $k = 1$
$k = 1, -5$

on Resources

Interactivity: The fundamental theorem of calculus (int-6423)

studyon

Units 3 & 4 > Area 3 > Sequence 2 > Concepts 3 & 4

The fundamental theorem of calculus Summary screen and practice questions
Properties of definite integrals Summary screen and practice questions

Exercise 7.3 The fundamental theorem of calculus and definite integrals

Technology free

1. **WE4** Evaluate the following definite integrals.

   a. $\int_0^1 x^2\,dx$ b. $\int_0^3 x^3\,dx$ c. $\int_3^4 (x^2 - 2x)\,dx$

   d. $\int_2^6 \frac{1}{x^2}\,dx$ e. $\int_0^2 \left(x^3 + 3x^2 - 2x\right)\,dx$

2. Evaluate the following definite integrals.

   a. $\int_1^3 \frac{2x^3 + 5x^2}{x}\,dx$ b. $\int_1^5 \frac{3}{5x}\,dx$ c. $\int_0^1 \frac{-4}{(3x-4)^5}\,dx$

   d. $\int_3^7 \frac{1}{\sqrt{2x-5}}\,dx$ e. $\int_{-2}^0 \frac{6}{\sqrt{8-3x}}\,dx$

3. **WE5** Evaluate the following definite integrals.

   a. $\int_0^{\frac{\pi}{2}} \sin(x)\,dx$ b. $\int_{\frac{\pi}{2}}^{\pi} 3\sin(4x)\,dx$ c. $\int_0^{\pi} 5\sin\left(\frac{x}{4}\right)\,dx$

   d. $\int_{\pi}^{2\pi} 2\sin\left(\frac{x}{3}\right)\,dx$ e. $\int_{-\pi}^0 \cos(2x)\,dx$ f. $\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} 8\cos(4x)\,dx$

4. **WE5** Determine the exact value of the following definite integrals.

   a. $\int_0^2 e^{4x}\,dx$ b. $\int_{-2}^0 e^{\frac{x}{3}}\,dx$ c. $\int_{-1}^1 -4e^{-2x}\,dx$

   d. $\int_1^2 (3e^{6x} + 5x)\,dx$ e. $\int_1^4 \left(\frac{5}{x} + e^{\frac{x}{2}}\right)\,dx$

5. Evaluate the following.

   a. $\int_0^3 (3x^2 - 2x + 3)\,dx$ b. $\int_1^2 \frac{2x^3 + 3x^2}{x}\,dx$ c. $\int_{-1}^1 \left(e^{2x} - e^{-2x}\right)\,dx$

   d. $\int_{2\pi}^{4\pi} \sin\left(\frac{x}{3}\right)\,dx$ e. $\int_{-3}^{-1} \frac{2}{\sqrt{1-3x}}\,dx$ f. $\int_{\frac{-\pi}{3}}^{\frac{\pi}{2}} \left[\cos(2x) - \sin\left(\frac{x}{2}\right)\right]\,dx$

6. Evaluate:

   a. $\int_{-3}^2 (x+1)^3\,dx$ b. $\int_0^1 (e^x + e^{-x})^2\,dx$

7. **WE6** Given that $\int_2^5 m(x)\,dx = 7$ and $\int_2^5 n(x)\,dx = 3$, calculate:

   a. $\int_2^5 3m(x)\,dx$ b. $\int_2^5 (2m(x) - 1)\,dx$

   c. $\int_5^2 (m(x) + 3)\,dx$ d. $\int_2^5 (2m(x) + n(x) - 3)\,dx$

8. Given that $\int_0^5 f(x)\,dx = 7.5$ and $\int_0^5 g(x)\,dx = 12.5$, determine:

a. $\int_0^5 -2f(x)\ dx$

b. $\int_5^0 g\,(x)\ dx$

c. $\int_0^5 (3f(x) + 2)\ dx$

d. $\int_0^5 (g(x) + f(x))\,dx$

e. $\int_0^5 (8g(x) - 10f(x))\,dx$

f. $\int_0^3 g(x)\,dx + \int_3^5 g(x)\,dx$

9. If $\int_0^k 3x^2\ dx = 8$, determine the value of k.

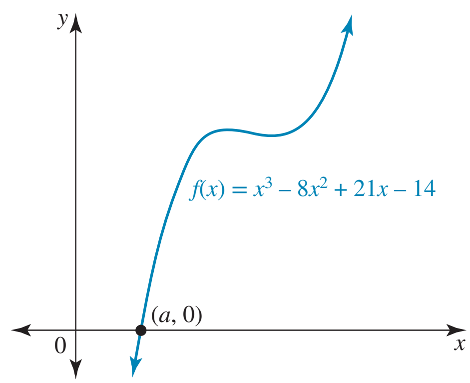
10. If $\int_1^k \frac{2}{x}dx = \log_e(9)$, determine the value of k.

11. Determine the value of a if $\int_0^a e^{\frac{x}{2}}dx = 4$.

12. Determine a if $\int_0^a e^{-2x}\,dx = \frac{1}{2}\left(1 - \frac{1}{e^8}\right)$.

Technology active

13. The graph of the function $f: R \to R, f(x) = x^3 - 8x^2 + 21x - 14$ is shown.



a. The graph cuts the x-axis at the point $(a, 0)$. Calculate the value of the constant a.

b. Evaluate $\int_a^5 (x^3 - 8x^2 + 21x - 14)\,dx$.

14. a. If $y = x\ \sin(x)$, determine $\frac{dy}{dx}$.

b. Hence, determine the value of $\int_{-\pi}^{\frac{\pi}{2}} 2x\cos(x)dx$.

15. a. If $y = e^{x^3 - 3x^2}$, determine $\frac{dy}{dx}$.

b. Hence, determine the value of $\int_0^1 (x^2 - 2x)e^{x^3-3x^2}\ dx$.

7.4 Areas under curves

7.4.1 Definite integrals and areas

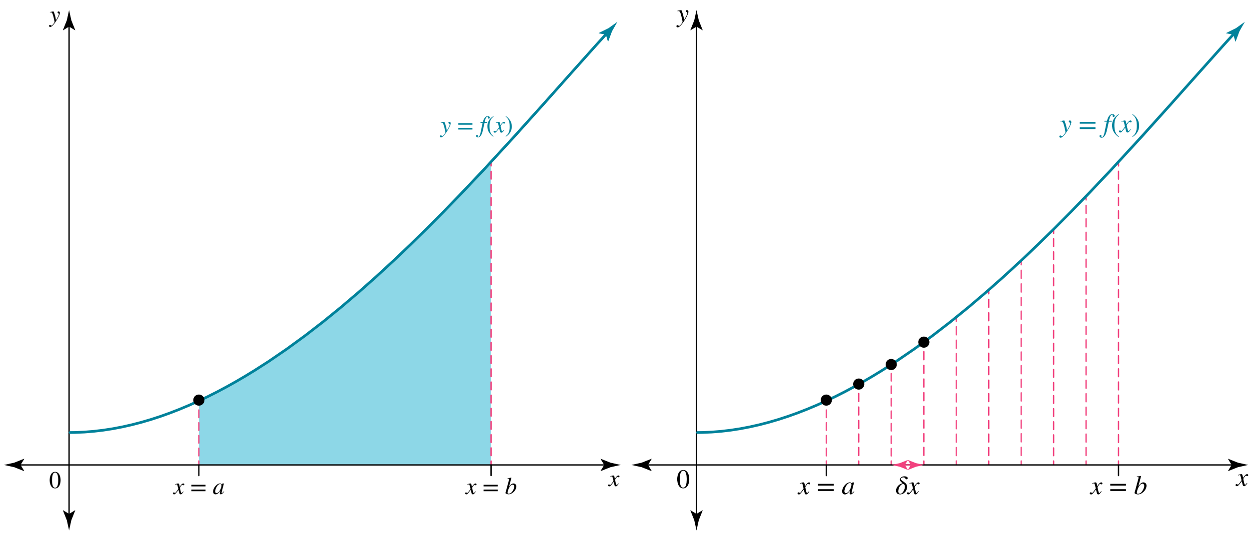
The area between a continuous function, $y = f(x)$, and the x-axis between $x = a$ and $x = b$ is shown in the diagram.

As we have already seen, this area can be approximated by dividing it into a series of thin vertical strips or rectangles of width δx. The approximate value of the area is the sum of the areas of all the rectangles, whether they are left end-point or right end-point rectangles.

As the number of strips increases, $\delta x \to 0.1$

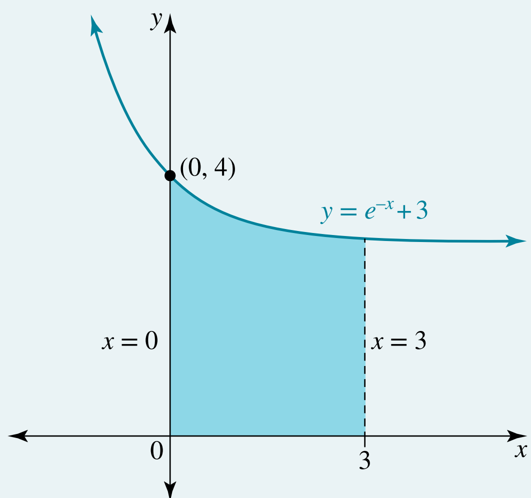
From the fundamental theorem of calculus, the shaded area, A, can be expressed as the limiting sum of the rectangles or the definite integral.

$$A = \lim_{\delta x \to 0} \sum_{x=a}^{x=b} f(x)\,\delta x = \int_a^b f(x)dx$$



WORKED EXAMPLE 7

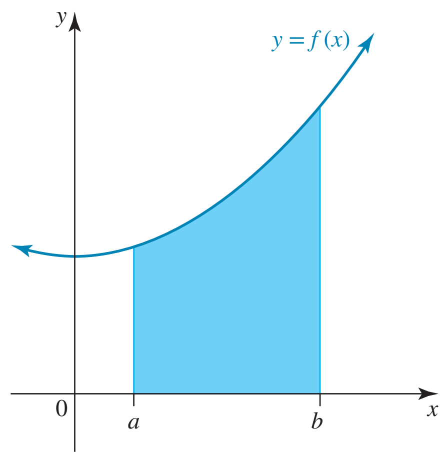
Determine the area bound by the curve defined by the rule $y = e^{-x} + 3$ and the x-axis from $x = 0$ to $x = 3$.

THINK	WRITE
1. Sketch the graph of the given function and shade the required area.	
2. Write the integral needed to determine the area.	$A = \int_0^3 (e^{-x} + 3)\,dx$
3. Antidifferentiate the function and evaluate.	$A = [-e^{-x} + 3x]_0^3$ $= (-e^{-3} + 3(3)) - (-e^0 + 3(0))$ $= -e^{-3} + 9 + 1$ $= -e^{-3} + 10$
4. Write the answer.	The area is $10 - e^{-3}$ square units.

7.4.2 Signed areas

When we calculate the area between the graph of a function, $y = f(x)$, and the x-axis from $x = a$ to $x = b$ using the definite integral $\int_a^b f(x)\ dx$, the area can either be positive or negative.

The definite integral will be positive if $f(x) > 0$, since the curve is above the x-axis and the height of the rectangular strips is positive. The definite integral will be negative if $f(x) < 0$, since the curve is below the x-axis and the height of the rectangular strips is negative. Since area cannot be negative, if $f(x) < 0$, the answer needs to be made positive by either subtraction (that is, $\times(-1)$), reversing the terminals, or taking its absolute value.

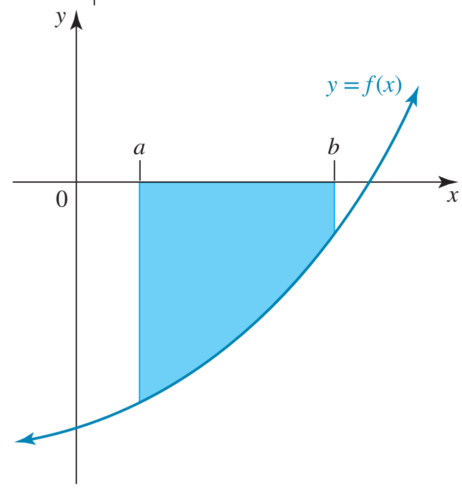


For a region above the x-axis:

Since $f(x) > 0$,

$$\int_a^b f(x)\,dx > 0.$$

Thus, the shaded area $= \int_a^b f(x)dx.$



For a region below the x-axis:

Since $f(x) < 0$,

$$\int_b^a f(x)\,dx < 0.$$

Thus, the shaded area $= -\int_a^b f(x)dx.$

Alternatively, $\int_a^b f(x)\,dx$ (reversing terminals)

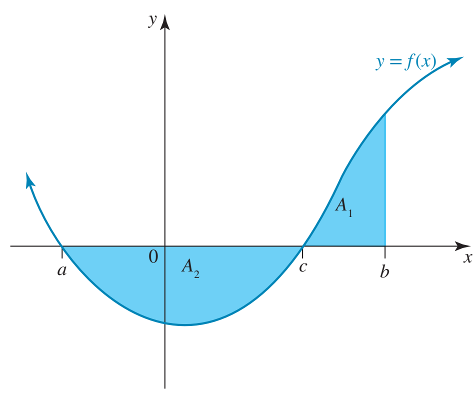
or simply $\left|\int_a^b f(x)\ dx\right|.$

Combining regions

For regions that are combinations of areas above and below the x-axis, calculate separate integrals, one for each area above and one for the area below.

Note that, on the diagram, A_2 would have a negative value.

Shaded area



$$= A_1 + (-A_2)$$

$$= \int_b^c f(x)\,dx + \left(-\int_c^a f(x)\,dx\right)$$

$$\text{or } = \int_b^c f(x)\,dx + \left|\int_c^a f(x)\ dx\right|$$

A sketch of the function, including x-intercepts, is imperative to determine if the function is above or below the x-axis or a combination of both. This will ensure the correct use of the definite integrals required. The procedure is illustrated in the following example.

The definite integral $\int_{-2}^{2} \left(x^3 - 4x\right)\, dx = \left[\frac{x^4}{4} - \frac{4x^2}{2}\right]_{-2}^{2}$

$$= \left[\frac{x^4}{4} - 2x^2\right]_{-2}^{2}$$

$$= \left(\frac{2^4}{4} - 2 \times 2^2\right) - \left(\frac{(-2)^4}{4} - 2 \times (-2)^2\right)$$

$$= (4 - 8) - (4 - 8)$$

$$= (-4) - (-4)$$

$$= 0$$

However, when you draw the function $y = x^3 - 4x$ and shade the region defined by the definite integral $\int_{-2}^{2} \left(x^3 - 4x\right) dx$, you can see the area is not 0 square units.

The function has x-intercepts at $x = -2$, 0 and 2.

The area on the left is above the x-axis, so it is positive, and the area on the right is below the x-axis, so it is negative. The total area equals:

$$\text{Area} = \int_{-2}^{0} \left(x^3 - 4x\right) dx + \left(-\int_{0}^{2} \left(x^3 - 4x\right) dx\right)$$

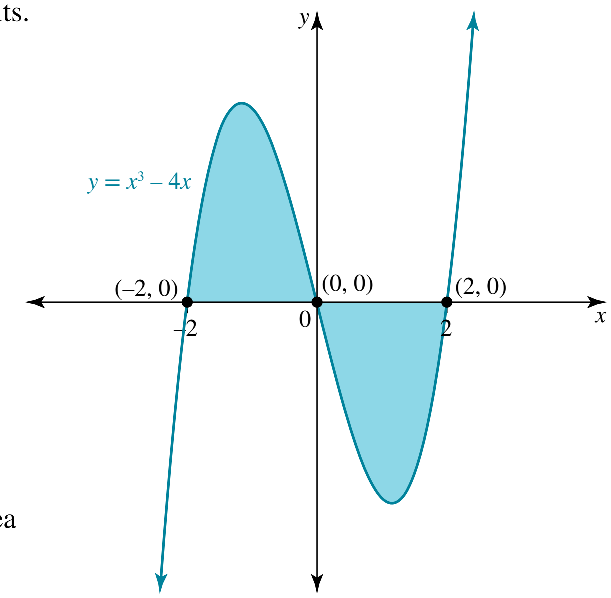
$$= \left[\frac{x^4}{4} - 2x^2\right]_{-2}^{0} - \left[\frac{x^4}{4} - 2x^2\right]_{0}^{2}$$

$$= (0 - (4 - 8)) - ((4 - 8) - 0)$$

$$= (4) - (-4)$$

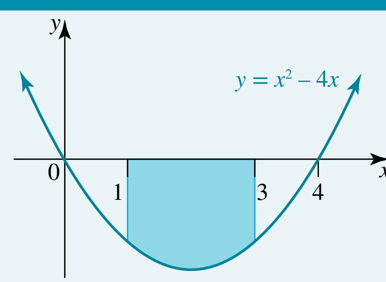
$$= 8 \text{ square units.}$$

Alternatively, using symmetry of this function, the area would be $2\int_{-2}^{0} \left(x^3 - 4x\right) dx$, giving the same result.



WORKED EXAMPLE 8

Calculate the shaded area.



THINK	**WRITE**
1. Express the area in definite integral notation, showing a negative sign in front of the integral as the region is below the x-axis.	$\text{Area} = -\int_{1}^{3} \left(x^2 - 4x\right) dx$

2. Antidifferentiate the integrand.

$$= -\left[\frac{1}{3}x^3 - 2x^2\right]_1^3$$

3. Evaluate.

$$= -\left\{\left[\frac{1}{3}(3)^3 - 2(3)^2\right] - \left[\frac{1}{3}(1)^3 - 2(1)^2\right]\right\}$$

$$= -\left\{[9 - 18] - \left[\frac{1}{3} - 2\right]\right\}$$

$$= -\left[-9 - \left(-1\frac{2}{3}\right)\right]$$

$$= -\left[-9 + 1\frac{2}{3}\right]$$

$$= -\left(-7\frac{1}{3}\right)$$

$$= 7\frac{1}{3}$$

4. State the solution.

The area is $7\frac{1}{3}$ square units.

WORKED EXAMPLE 9

Calculate the area bound by the curve $y = (x^2 - 1)(x^2 - 4)$ and the x-axis from $x = -2$ to $x = 2$.

THINK

1. Make a careful sketch of the given function. Shade the required region.

WRITE

The graph cuts the y-axis where $x = 0$.

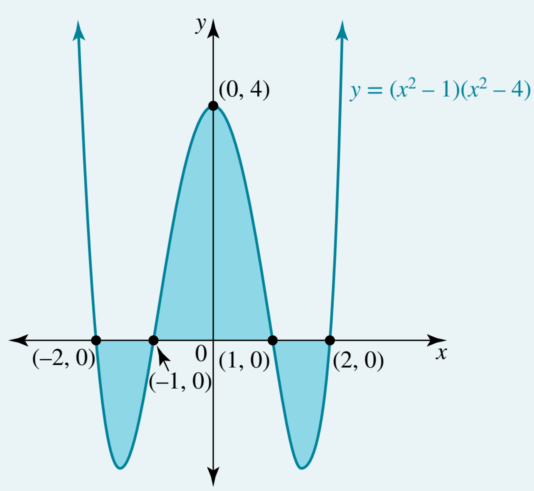
$\therefore$ the y-intercept is $(0, 4)$.

$\therefore$ the graph cuts the x-axis where $y = 0$:

$$(x^2 - 1)(x^2 - 4) = 0$$

$$(x - 1)(x + 1)(x - 2)(x + 2) = 0$$

$$x = \pm 1, x = \pm 2$$



2. Express the area using definite integrals. Account for the negative regions by subtracting these from the positive areas.
Note that the region from $x = -2$ to $x = -1$ is the same as the region from $x = 1$ to $x = 2$ due to the symmetry of the graph.

$$A = \int_{-1}^{1} (x^2 - 1)(x^2 - 4)\, dx - 2\int_{1}^{2} (x^2 - 1)(x^2 - 4)\, dx$$
$$= \int_{-1}^{1} (x^4 - 5x^2 + 5)\, dx - 2\int_{1}^{2} (x^4 - 5x^2 + 5)\, dx$$

3. Antidifferentiate and evaluate.

$$= \left[\frac{1}{5}x^5 - \frac{5}{3}x^3 + 5x\right]_{-1}^{1} - 2\left[\frac{1}{5}x^3 - \frac{5}{3}x^3 + 5x\right]_{1}^{2}$$
$$= \left(\frac{1}{5}(1)^5 - \frac{5}{3}(1)^3 + 5(1)\right) - \left(\frac{1}{5}(-1)^5 - \frac{5}{3}(-1)^3 + 5(-1)\right)$$
$$-2\left[\left(\frac{1}{5}(2)^5 - \frac{5}{3}(2)^3 + 5(2)\right) - \left(\frac{1}{5}(1)^2 - \frac{5}{3}(1)^3 + 5(1)\right)\right]$$
$$= \left(\frac{1}{5} - \frac{5}{3} + 5\right) - \left(-\frac{1}{5} + \frac{5}{3} - 5\right)$$
$$-2\left[\left(\frac{32}{5} - \frac{40}{3} + 10\right) - \left(\frac{1}{5} - \frac{5}{3} + 5\right)\right]$$
$$= \frac{1}{5} - \frac{5}{3} + \frac{1}{5} - \frac{5}{3} + 5 - 2\left(\frac{32}{5} - \frac{40}{3} + 10 - \frac{1}{5} + \frac{5}{3} - 5\right)$$
$$= \frac{2}{5} - \frac{10}{3} + 10 - \frac{64}{5} + \frac{80}{3} - 20 + \frac{2}{5} - \frac{10}{3} + 10$$
$$= -12 + 20$$
$$= 8$$

4. Write the answer.

The area is 8 square units.

on Resources

Interactivity: Area under a curve (int-6424)

studyon

Units 3 & 4 > Area 3 > Sequence 2 > Concept 5 > **Areas under curves** Summary screen and practice questions

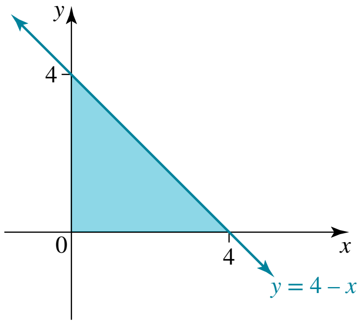
Exercise 7.4 Areas under curves

Technology free

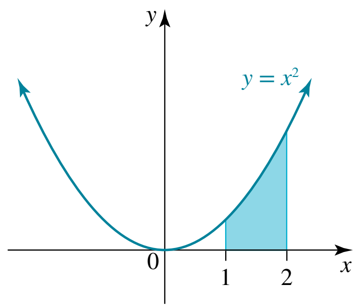
1. **WE7** For each of the following graphs:

- **i.** express the shaded area as a definite integral
- **ii.** hence, calculate the shaded area, giving your answer in exact form.

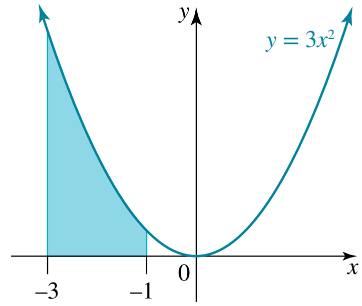
a.



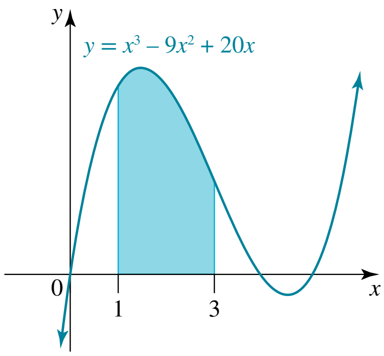
b.



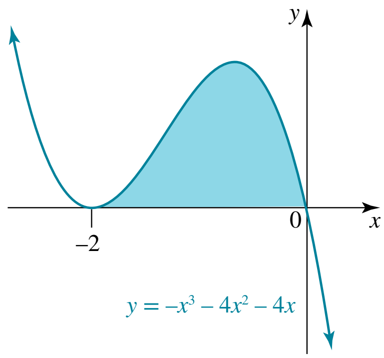
c.



d.



e.

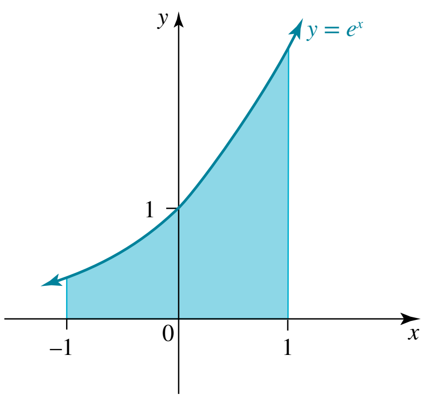


2. For each of the following graphs:

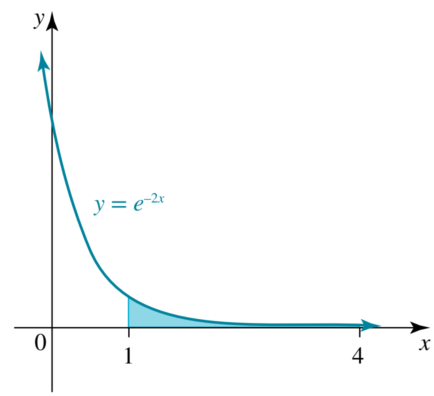
i. express the shaded area as a definite integral

ii. hence, calculate the shaded area, giving your answer in exact form.

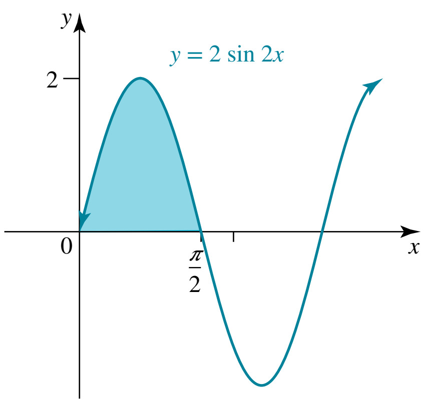
a.



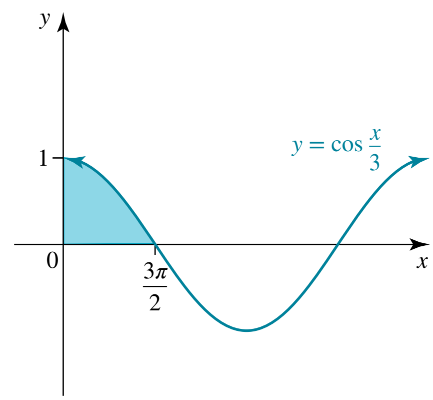
b.



c.

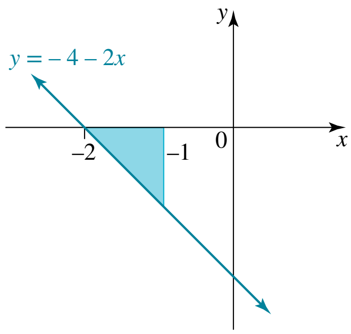


d.

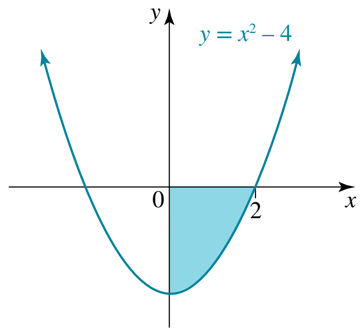


3. **WE8** Express the following shaded areas as definite integrals. Hence, calculate the shaded areas, giving your answers in exact form.

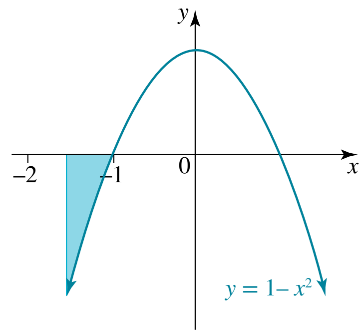
a.



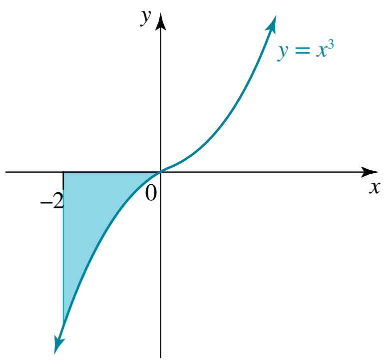
b.



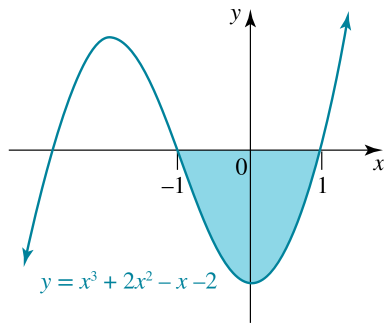
c.



d.

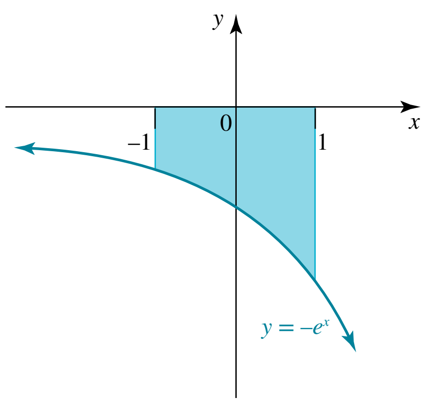


e.

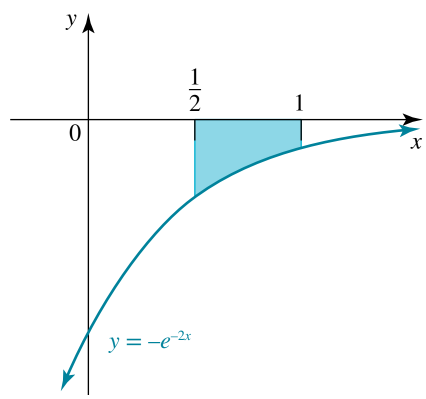


4. Express the following shaded areas as definite integrals. Hence, calculate the shaded areas, giving your answer in exact form.

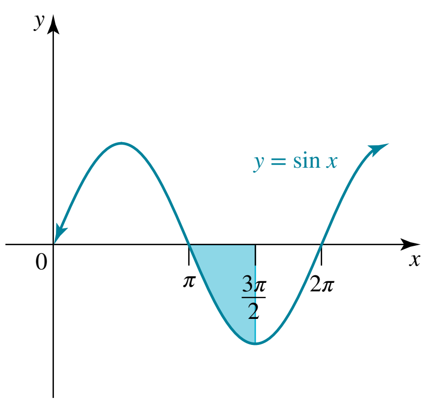
a.



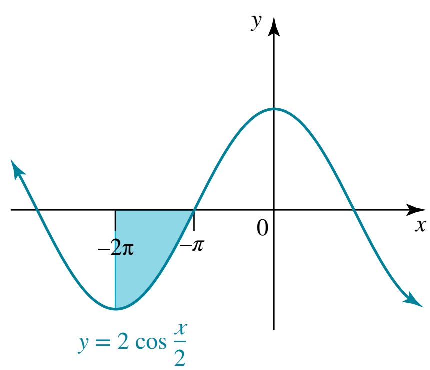
b.



c.



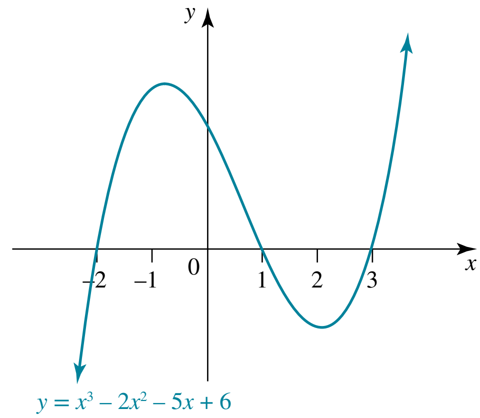
d.



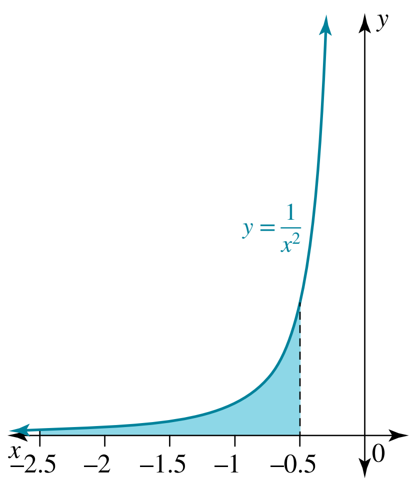
5. **WE9** Consider the function $y = \left(x^2 - 1\right)\left(x^2 - 9\right)$.
   a. Sketch the graph of the function, stating all axis intercepts.
   b. Determine the area enclosed by the function, the lines $x = -3$ and $x = 3$, and the x-axis.

Technology active

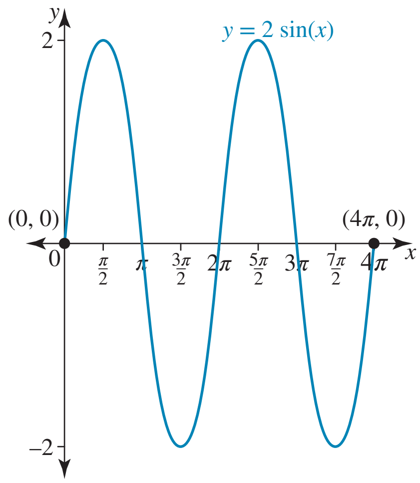
6. For the curve shown, calculate the area between the curve and the x-axis from:
   a. $x = -2$ to $x = 1$
   b. $x = 1$ to $x = 3$
   c. $x = -2$ to $x = 3$.



7. For each of the following functions:
   i. sketch the function, showing clearly all axis intercepts
   ii. calculate the area bounded by the graph of the function and the x-axis.
   a. $g(x) = 8 - x^2$
   b. $g(x) = x^3 - 4x^2$
   c. $f(x) = x(x - 2)(x - 3)$
   d. $f(x) = x^3 - 4x^2 - 4x + 16$
   e. $g(x) = x^3 + 3x^2 - x - 3$
   f. $h(x) = (x - 1)(x + 2)(x + 5)$
8. Determine the exact area between the curve $y = \frac{1}{x}$, the x-axis and the lines $x = \frac{1}{2}$ and $x = 2$. Express your answer in simplest form.
9. Calculate the exact area of the region enclosed by the x-axis, $y = e^{3x}$ and the lines $x = 1$ and $x = 2$
10. Calculate the exact area of the region enclosed by the x-axis, $y = -\cos(x)$ and the lines $x = \frac{\pi}{3}$ and $x = \frac{5\pi}{6}$. (Use a sketch graph to assist your calculation.)
11. Determine the area bound by the curve defined by the rule $y = 2\sqrt{x}$, $x \geq 0$ and the x-axis from $x = 0$ to $x = 25$.
12. Calculate the area bounded by the curve $y = 2\sin(2x) + 3$, the x-axis and the lines $x = 0$ and $x = \pi$.
13. Sketch the graph of $y = 1 - e^{-x}$ and hence calculate the exact area between the curve and the x-axis from $x = -1$ to $x = 1$.
14. The graph of $y = \frac{1}{x^2}$, $x < 0$ is shown. Calculate the area of the shaded region (that is for $-2.5 \leq x \leq -0.5$).

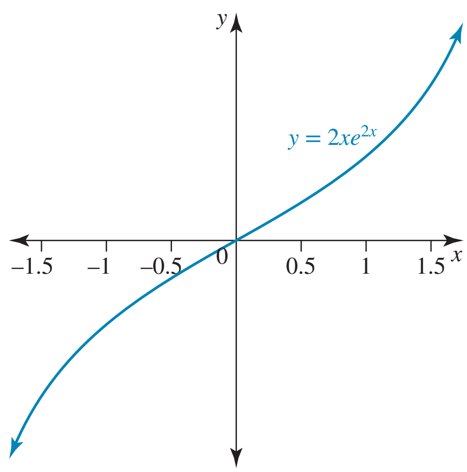


15. The graph of the function $y = 2\sin(x)$ is shown. Using calculus, calculate the area between the curve and the x-axis from $x = 0$ to $x = 4\pi$.

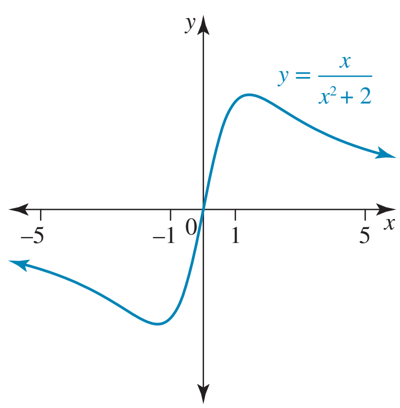


16. a. Differentiate $x\log_e(x)$ for $x > 0$.
b. Hence, determine an antiderivative of $\log_e(x)$.
c. Determine the area bounded by the graph of $\log_e(x)$, the x-axis, $x = 1$ and $x = 4$, giving your answer in exact form.

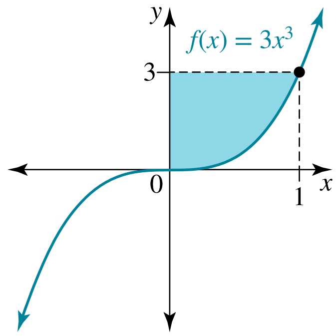
17. The graph of $y = 2xe^{2x}$ is shown.
a. Determine $\dfrac{d}{dx}\left(e^{x^2}\right)$.
b. Hence, calculate the exact area between the curve $y = 2xe^{x^2}$ and the x-axis from $x = -1$ to $x = 1$.



18. a. Differentiate $\log_e(x^2 + 2)$.
b. Hence, determine an antiderivative of $\dfrac{x}{x^2 + 2}$.
c. The graph of $y = \dfrac{x}{x^2 + 2}$ is shown. Calculate the area between the graph, the x-axis, $x = -1$ and $x = 1$. Give your answer in exact form.



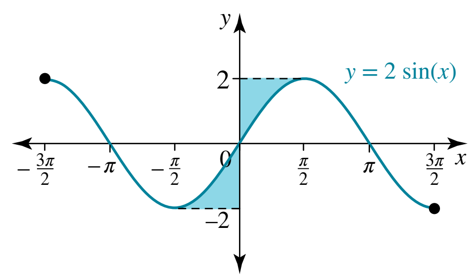
19. The graph of $f: R \to R, f(x) = 3x^3$ is shown.



a. Calculate the area bounded by the curve and the x-axis from $x = 0$ to $x = 1$.
b. Hence, or otherwise, calculate the area of the shaded region.

20. The graph of $y = 2\sin(x), -\frac{3\pi}{2} \le x \le \frac{3\pi}{2}$ is shown.

a. Calculate $\int_a^{\frac{\pi}{2}} 2\sin(x)\,dx$.
b. Hence, or otherwise, calculate the area of the shaded region.



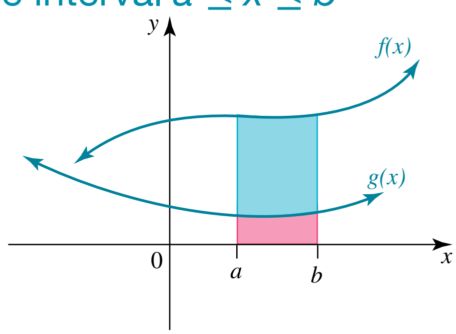
7.5 Areas between curves

When we find areas between two continuous functions, $f(x)$ and $g(x)$, for an interval $a \le x \le b$, our approach depends on whether the curves intersect or not in this interval.

7.5.1 When $f(x)$ and $g(x)$ do not intersect in the interval $a \le x \le b$

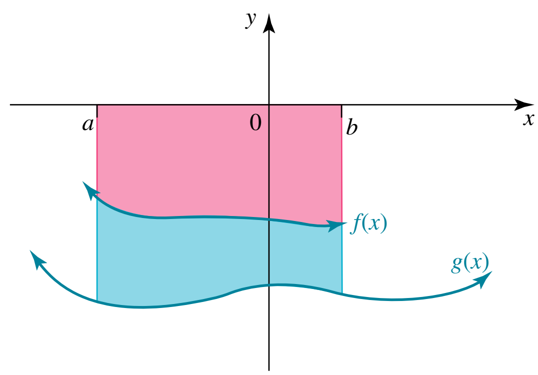
Example 1: If the region is above the x-axis, the lower function is subtracted from the higher function to ensure a positive answer.

$$\text{Blue shaded area} = \int_a^b [f(x) - g(x)]\,dx$$



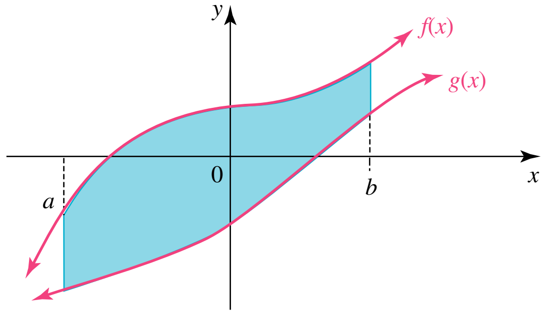
Example 2: If the region is below the x-axis, the lower function is subtracted from the higher function to again ensure a positive answer.

$$\text{Blue shaded area} = \int_a^b [f(x) - g(x)]\,dx$$



Example 3: If the region crosses the x-axis, the lower function is again subtracted from the higher function to ensure a positive answer.

Shaded area $= \int_a^b [f(x) - g(x)]\, dx$



In the case where the region crosses the x-axis, there is no need to consider the x-intercepts because, if the two functions were both translated vertically by the same factor, k units, to ensure they are above the x-axis, the constants would cancel each other. That is,

$$\int_a^b [(f(x) + k) - (g(x) + k)]\, dx = \int_a^b [f(x) - g(x)]\, dx.$$

WORKED EXAMPLE 10

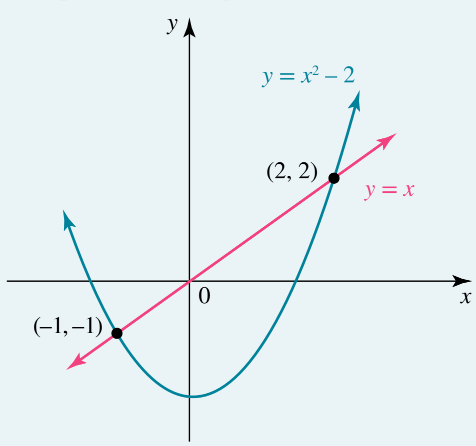
Consider the functions $y = x$ and $y = x^2 - 2$.

a. **Determine the values of x where the functions intersect.**

b. **Sketch the graphs of the functions on the same axes, stating the coordinates of the points of intersection.**

c. **Hence, calculate the area bounded by the two functions.**

THINK	WRITE
a. i. 1. State the two functions.	**a. i.** $y = x$ and $y = x^2 - 2$
2. Determine where the curves intersect.	For points of intersection: $x = x^2 - 2$
3. Solve for x.	$x^2 - x - 2 = 0$ $(x - 2)(x + 1) = 0$ $x = 2$ or $x = -1$
ii. Determine the key points of each function and sketch.	**ii.** For $y = x$, when $x = 0$, $y = 0$; when $x = 2$, $y = 2$; when $x = -1$, $y = -1$. The line passes through $(0, 0)$, $(2, 2)$ and $(-1, -1)$. For $y = x^2 - 2$, when $x = 0$, $y = -2$. Hence, the y-intercept is -2. The parabola also passes through $(2, 2)$ and $(-1, -1)$.



iii. 1. Define $f(x)$ and $g(x)$.	**iii.** Let $f(x) = x$ and $g(x) = x^2 - 2$.
2. Write the area as a definite integral between the values of x at the points of intersection.	Area $= \int_{-1}^{2} [f(x) - g(x)]\,dx$ $= \int_{-1}^{2} [x - (x^2 - 2)]\,dx$ $= \int_{-1}^{2} (x - x^2 + 2)\,dx$
3. Antidifferentiate.	$= \left[\frac{1}{2}x^2 - \frac{1}{3}x^3 + 2x\right]_{-1}^{2}$
4. Evaluate the integral.	$= \left[\frac{1}{2}(2)^2 - \frac{1}{3}(2)^3 + 2(2)\right] - \left[\frac{1}{2}(-1)^2 - \frac{1}{3}(-1)^3 + 2(-1)\right]$ $= \left(2 - \frac{8}{3} + 4\right) - \left(\frac{1}{2} + \frac{1}{3} - 2\right)$ $= \left(3\frac{1}{3}\right) - \left(-1\frac{1}{6}\right)$ $= 3\frac{1}{3} + 1\frac{1}{6}$ $= 4\frac{1}{2}$
5. State the area.	The area is $4\frac{1}{2}$ square units.

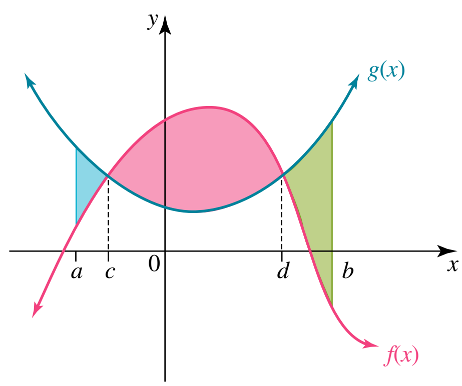
7.5.2 When $f(x)$ and $g(x)$ intersect in the interval $a \le x \le b$

Let the points of intersection of $f(x)$ and $g(x)$ in the interval $a \le x \le b$ be $x = c$ and $x = d$, where $c < d$.

The area is found by considering the separate intervals: $a \le x \le c$, $c \le x \le d$ and $d \le x \le b$

For each section, the lower function is subtracted from the higher function to ensure a positive answer.

$$\text{Shaded area} = \int_a^c [g(x) - f(x)]\,dx + \int_c^d [f(x) - g(x)]\,dx + \int_d^b [g(x) - f(x)]\,dx$$



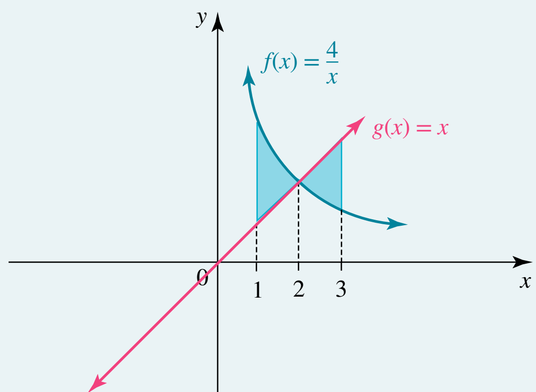
WORKED EXAMPLE 11

Consider the functions $f(x) = \frac{4}{x}$ and $g(x) = x$.

a. **Determine the values of x where the functions intersect.**

b. **Sketch the graphs of the functions on the same axes. Shade the region between the two functions and $x = 1$ and $x = 3$.**

c. **Hence, calculate the area between $f(x)$ and $g(x)$ from $x = 1$ to $x = 3$.**

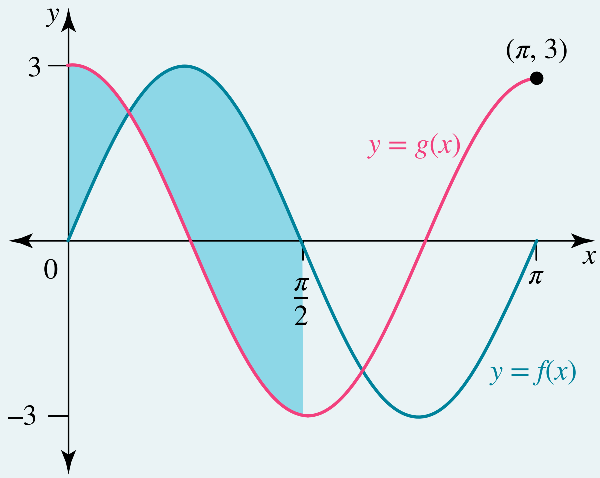
THINK	WRITE
a. 1. State the two functions.	**a.** $f(x) = \frac{4}{x}$, $g(x) = x$
2. Let $f(x) = g(x)$ to calculate the values of x where the graphs intersect.	For points of intersection, $x = \frac{4}{x}$.
3. Solve for x.	$x^2 = 4$ $x^2 - 4 = 0$ $(x + 2)(x - 2) = 0$ $x = -2$ and $x = 2$
b. Sketch $f(x)$ and $g(x)$ on the same axis and shade the region between the two curves from $x = 1$ to $x = 3$.	**b.** 
c. 1. State the area as the sum of two integrals for the two subintervals.	**c.** Area $= \int_1^2 \left(\frac{4}{x} - x\right) dx + \int_2^3 \left(x - \frac{4}{x}\right) dx$
2. Antidifferentiate.	$= \left[4\log_e(x) - \frac{1}{2}x^2\right]_1^2 - \left[\frac{1}{2}x^2 - 4\log_e(x)\right]_2^3$
3. Evaluate the two integrals.	$= \left[4\log_e(2) - \frac{1}{2}(2)^2\right] - \left[4\log_e(1) - \frac{1}{2}(1)^2\right]$ $+ \left\{\left[\frac{1}{2}(3)^2 - 4\log_e(3)\right] - \left[\frac{1}{2}(2)^2 - 4\log_e(2)\right]\right\}$ $= [4\log_e(2) - 2] - \left[4\log_e(1) - \frac{1}{2}\right]$ $+ \left\{\left[\frac{9}{2} - 4\log_e(3)\right] - [2 - 4\log_e(2)]\right\}$ $= 4\log_e(2) - 2 - 0 + \frac{1}{2} + \frac{9}{2}$ $-4\log_e(3) - 2 + 4\log_e(2)$
4. Simplify.	$= 4\log_e\left(\frac{4}{3}\right) + 1$
5. State the area.	The area is $4\log_e\left(\frac{4}{3}\right) + 1$ or approximately 2.151 square units.

WORKED EXAMPLE 12

The graphs of $f(x) = 3\sin(2x)$ and $g(x) = 3\cos(2x)$ are shown for $x \in [0, \pi]$.

a. Determine the coordinates of the point(s) of intersection of f and g for the interval $\left[0, \frac{\pi}{2}\right]$.

b. Using calculus, determine the area enclosed between the curves on the interval $\left[0, \frac{\pi}{2}\right]$.



THINK	WRITE
a. 1. Use simultaneous equations to determine where the graphs intersect, and equate the two equations.	**a.** $3\sin(2x) = 3\cos(2x)$ $\dfrac{3\sin(2x)}{3\cos(2x)} = 1, 0 \le x \le \dfrac{\pi}{2}.$ $\tan(2x) = 1, 0 \le x \le \dfrac{\pi}{2}$
2. Solve for $2x$ for $x \in \left[0, \frac{\pi}{2}\right]$.	$2x = \dfrac{\pi}{4}$ $\therefore x = \dfrac{\pi}{8}$
3. Calculate the corresponding y-value.	$f\left(\dfrac{\pi}{8}\right) = 3\sin\left(\dfrac{\pi}{4}\right)$ $= 3 \times \dfrac{1}{\sqrt{2}}$ $= \dfrac{3\sqrt{2}}{2}$
4. Write the solution.	The coordinates are $\left(\dfrac{\pi}{8}, \dfrac{3\sqrt{2}}{2}\right)$.
b. 1. Determine when $f > g$ and $f < g$.	**b.** When $0 < x < \dfrac{\pi}{8}$, $g > f$. When $\dfrac{\pi}{8} < x < \dfrac{\pi}{2}$, $f > g$.
2. Express each area individually in definite integral notation.	The area is equal to: $A = \int_0^{\frac{\pi}{8}} (3\cos(2x) - 3\sin(2x))dx + \int_{\frac{\pi}{8}}^{\frac{\pi}{2}} (3\sin(2x) - 3\cos(2x))dx$

3. Use calculus to antidifferentiate and evaluate.

$$= \left[\frac{3}{2}\sin(2x) + \frac{3}{2}\cos(2x)\right]_0^{\frac{\pi}{8}} + \left[\frac{3}{2}\cos(2x) - \frac{3}{2}\sin(2x)\right]_{\frac{\pi}{8}}^{\frac{\pi}{2}}$$

$$= \frac{3}{2} - \sin\left(\frac{\pi}{4}\right) + \frac{3}{2}\cos\left(\frac{\pi}{4}\right) - \left(\frac{3}{2}\sin(0) + \frac{3}{2}\cos(0)\right)$$

$$+ \frac{3}{2}\cos(\pi) - \frac{3}{2}\sin(\pi) - \left(-\frac{3}{2}\sin\left(\frac{\pi}{4}\right) + \frac{3}{2}\sin\left(\frac{\pi}{4}\right)\right)$$

$$= \frac{3}{2} \times \frac{\sqrt{2}}{2} + \frac{3}{2} \times \frac{\sqrt{2}}{2} - 0 - \frac{3}{2} + \frac{3}{2} - 0 + \frac{3}{2} \times \frac{\sqrt{2}}{2} + \frac{3}{2} \times \frac{\sqrt{2}}{2}$$

$$= \frac{3\sqrt{2}}{4} + \frac{3\sqrt{2}}{4} + \frac{3\sqrt{2}}{4} + \frac{3\sqrt{2}}{4}$$

$$= 3\sqrt{2}$$

4. Write the answer.

The area is $3\sqrt{2}$ square units.

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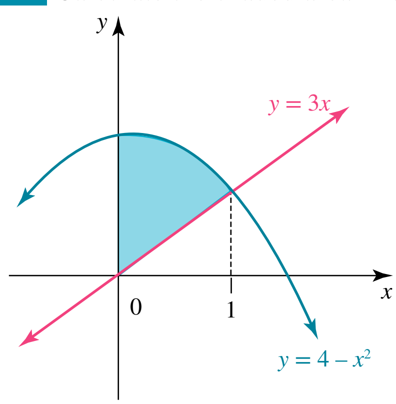
Units 3 & 4 › Area 3 › Sequence 2 › Concept 6 › **Areas between curves** Summary screen and practice questions

Exercise 7.5 Areas between curves

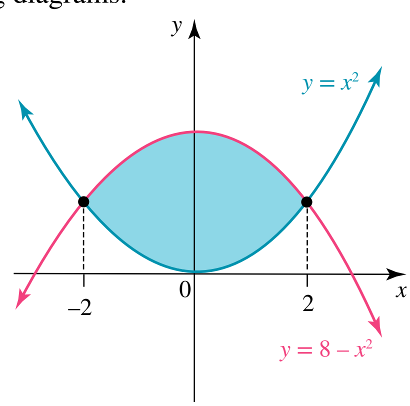
Technology free

1. **WE10** Calculate the shaded area in each of the following diagrams.

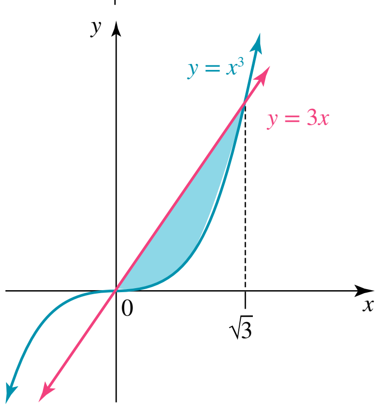
a.



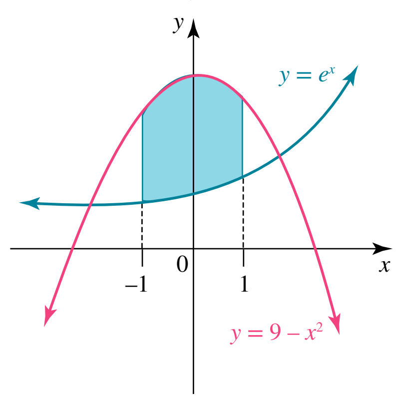
b.



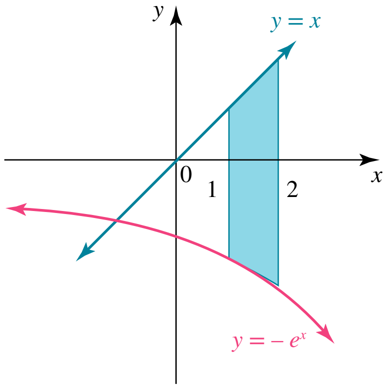
c.



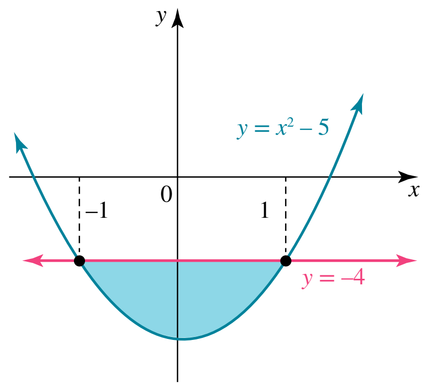
d.



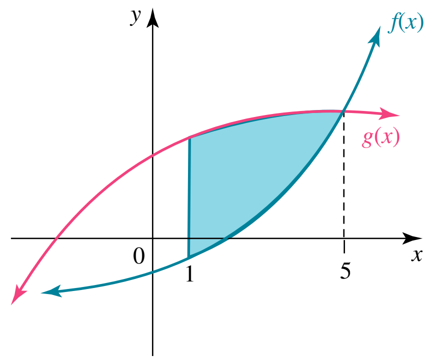
e.



f.



2. MC Which one of the following does not equal the shaded area shown?



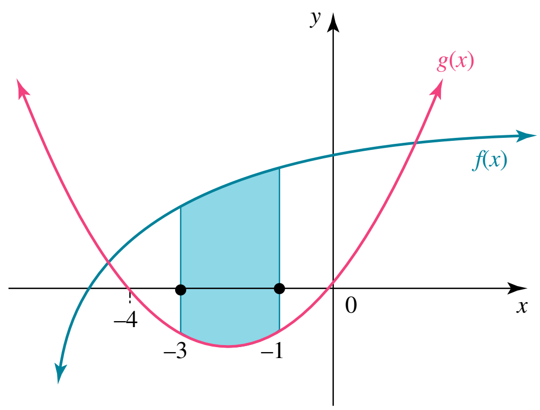
A. $\int_1^5 g(x)\,dx - \int_1^5 f(x)\,dx$

B. $\int_1^5 g(x)\,dx + \int_5^1 f(x)\,dx$

C. $\int_1^5 f(x)\,dx - \int_1^5 g(x)\,dx$

D. $\int_1^5 \left[g(x) - f(x)\right]\,dx$

3. MC The area bounded by the curves $f(x)$, $g(x)$ and the lines $x = -3$ and $x = -1$ is equal to:



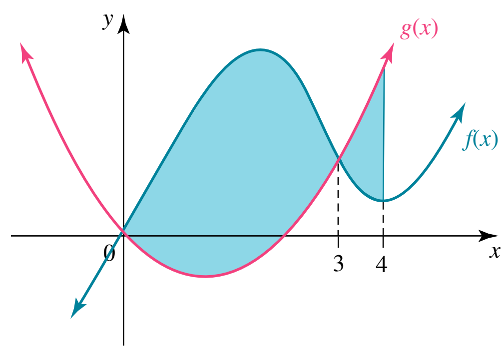
A. $\int_{-1}^{-3} \left[f(x) - g(x)\right]\,dx$

B. $\int_{-3}^{-1} \left[f(x) + g(x)\right]\,dx$

C. $\int_{-3}^{-1} \left[g(x) - f(x)\right]\,dx$

D. $\int_{-3}^{-1} \left[f(x) - g(x)\right]\,dx$

4. MC The shaded area shown is equal to:



A. $\int_0^4 \left[f(x) - g(x)\right]\,dx$

B. $\int_0^3 \left[g(x) - f(x)\right]\,dx + \int_3^4 \left[f(x) - g(x)\right]\,dx$

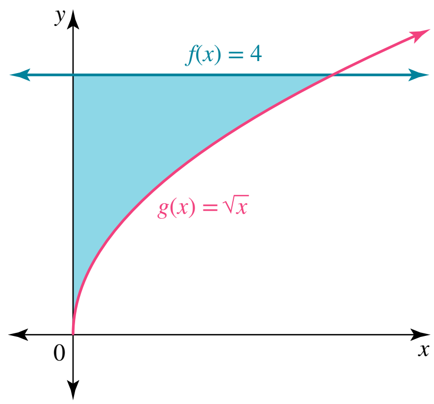
C. $\int_0^4 \left[g(x) - f(x)\right]\,dx$

D. $\int_0^3 \left[f(x) - g(x)\right]\,dx + \int_3^4 \left[g(x) - f(x)\right]$

5. WE11 Consider the functions $f(x) = x^3$ and $g(x) = x$.
 a. Determine the values of x where the functions intersect.
 b. Sketch the graphs of the functions on the same axes.
 c. Hence, calculate the area between $f(x)$ and $g(x)$, giving an exact answer.

6. Consider the functions $f(x) = x^3 + 2x$ and $g(x) = 3x^2$.
 a. Determine the values of x where the functions intersect.
 b. Sketch the graphs of the functions on the same axes.
 c. Hence, calculate the area between $f(x)$ and $g(x)$, giving an exact answer.

7. The graphs of $g(x) = \sqrt{x}$ and the line $f(x) = 4$ are shown. Determine the coordinates of the point of intersection between f and g, and hence calculate the area of the shaded region



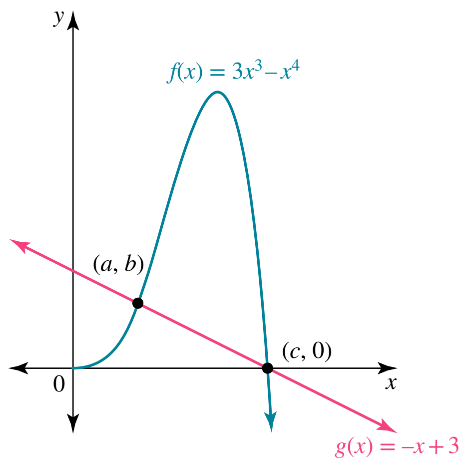
8. Calculate the area enclosed between the curve $f(x) = (x - 3)^2$ and the line $g(x) = 9 - x$.

9. WE12 Consider the functions $f(x) = \sin(x)$ and $g(x) = -\cos(x)$ for $x \in [0, \pi]$.
 a. Determine the coordinates of the points of intersection of f and g for the given domain.
 b. Using calculus, calculate the area enclosed between the curves for the given interval.

10. Calculate the area between the curves $y = \sqrt{3} - \sin 2x$ and $y = \sin 2x$ from $x = 0$ to $x = \frac{\pi}{4}$.

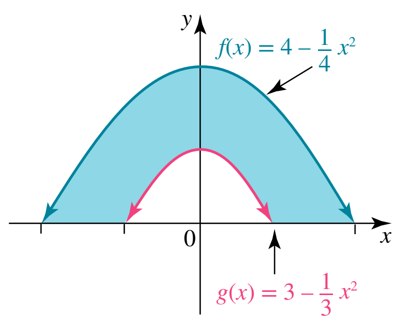
11. Calculate the exact area bounded by the curves $y = e^x$ and $y = 3 - 2e^{-x}$.

12. The graphs of $f(x) = 3x^3 - x^4$ and $g(x) = -x + 3$ are shown.
 a. The graphs intersect at the points (a, b) and $(c, 0)$. Calculate the constants a, b and c.
 b. Calculate the area enclosed between the curves from $x = a$ to $x = c$.



Technology active

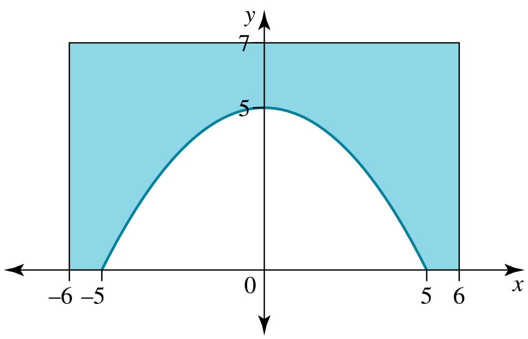
13. The graph shows the cross-section of a bricked archway. (All measurements are in metres.)
 a. Determine the x-intercepts of $f(x)$.
 b. Determine the x-intercepts of $g(x)$.
 c. Determine the cross-sectional area of the brickwork.



14. The edge of a garden bed can be modelled by the rule $y = 0.5\sin\left(\frac{x}{2}\right) + 2$. The bed has edges defined by $y = 0, x = 0$ and $x = 4\pi$. All measurements are in metres.

a. Sketch the graph of $y = 0.5\sin\left(\frac{x}{2}\right) + 2$ along with $y = 0, x = 0$ and $x = 4\pi$ as edges to show the shape of the garden bed.
b. Calculate the area of the garden bed, correct to the nearest square metre.
c. Topsoil is going to be used on the garden bed in preparation for new planting for spring. The topsoil is to be spread so that it is uniformly 50 cm thick. Determine the amount of soil, in cubic metres, that will be needed for the garden bed.

15. A stone footbridge over a creek is shown along with the mathematical profile of the bridge. The arch of the footbridge can be modelled by a quadratic function for $x \in [-5, 5]$, with all measurements in metres.



a. Determine the equation for the arch of the bridge
b. Determine the area between the curve and the x-axis from $x = -5$ to $x = 5$.
c. Determine the area of the side of the bridge represented by the shaded area.
d. The width of the footbridge is 3 metres. Determine the volume of stones used in the construction of the footbridge.

7.6 Applications of integration

7.6.1 Total change as the integral of instantaneous change

If we are given the equation for the instantaneous rate of change of a certain item, such as revenue or area, the amount by which the item has changed over a particular time period would be found by integrating the rate of change equation using the starting and finishing times as the terminals.

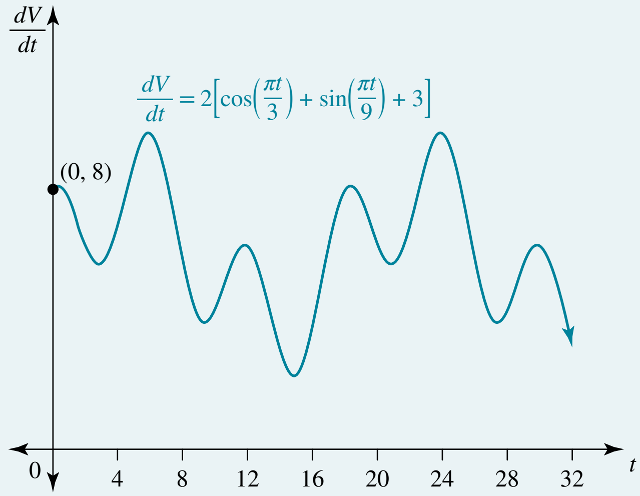
For example, if water is flowing into a holding tank at $\frac{dV}{dt}$ litres per minute, the amount of liquid that has flowed into the tank in the first 30 minutes would be $\int_0^{30}\left(\frac{dV}{dt}\right) dt$ litres.

In economics, the instantaneous rate of change with respect to the number of items is also referred to as the marginal rate of change.

For example, if the revenue function for selling x units is given as $R(x)$ dollars, then the marginal revenue is given by $\frac{dR}{dx}$ dollars per unit. The marginal revenue is the extra revenue received for selling one more unit after a particular number of units have been sold.

WORKED EXAMPLE 13

It is a common practice to include heating in concrete slabs when new residential homes or units are being constructed, because it is more economical than installing heating later. A typical reinforced concrete slab, 10–15 centimetres thick, has tubing installed on top of the reinforcement, then concrete is poured on top. When the system is complete, hot water runs through the tubing. The concrete slab absorbs the heat from the water and releases it into the area above.



The number of litres / minute of water flowing through the tubing over t minutes can be modelled by the rule

$$\frac{dV}{dt} = 2\left[\cos\left(\frac{\pi t}{3}\right) + \sin\left(\frac{\pi t}{9}\right) + 3\right].$$

The graph of this function is shown.

a. **What is the rate of flow of water, correct to 2 decimal places, at:**
   i. **4 minutes?**
   ii. **8 minutes?**

b. **State the period of the given function.**

c. **Determine the volume of water that flows through the tubing during the time period for one whole cycle.**

THINK	WRITE
a. i. Substitute $t = 4$ into the given equation and evaluate.	a. i. $\frac{dV}{dt} = 2\left[\cos\left(\frac{\pi t}{3}\right) + \sin\left(\frac{\pi t}{9}\right) + 3\right]$ When $t = 4$, $\frac{dV}{dt} = 2\left[\cos\left(\frac{4\pi}{3}\right) + \sin\left(\frac{4\pi}{9}\right) + 3\right]$ $= 6.97$ The rate at 4 minutes is 6.97 litres/minute.
ii. Substitute $t = 8$ into the given equation and evaluate.	ii. When $t = 8$, $\frac{dV}{dt} = 2\left[\cos\left(\frac{8\pi}{3}\right) + \sin\left(\frac{8\pi}{9}\right) + 3\right]$ $= 5.68$ The rate at 8 minutes is 5.68 litres/minute.
b. Determine the cycle for the function by analysing the shape of the graph, or the period of the functions	b. Period of $\cos\left(\frac{\pi t}{3}\right) = \frac{2\pi}{\left(\frac{\pi}{3}\right)} = 6$ hours Period of $\sin\left(\frac{\pi t}{9}\right) = \frac{2\pi}{\left(\frac{\pi}{9}\right)} = 18$ hours Period of combined function = 18 hours

c. 1 The area under the curve of the equation of the rate of flow gives the total volume that has flowed through the tubing. To calculate the area, antidifferentiate and evaluate.

c. $A = \int_0^{18} 2\left(\cos\left(\frac{\pi t}{3}\right) + \sin\left(\frac{\pi t}{9}\right) + 3\right) dt$

$= 2\int_0^{18} \left(\cos\left(\frac{\pi t}{3}\right) + \sin\left(\frac{\pi t}{9}\right) + 3\right) dt$

$= 2\left[\frac{3}{p}\sin\left(\frac{pt}{3}\right) - \frac{9}{p}\cos\left(\frac{pt}{9}\right) + 3t\right]_0^{18}$

$= 2\left\{\left(\frac{3}{p}\sin(6p) - \frac{9}{p}\cos(2p) + 54\right) - \left(\frac{3}{p}\sin(0) - \frac{9}{p}\cos(0)\right)\right\}$

$= 2\left\{\left(0 - \frac{9}{p} + 54\right) - \left(0 - \frac{9}{p}\right)\right\}$

$= 2\left(-\frac{9}{p} + 54 + \frac{9}{p}\right)$

$= 2 \times 54$

$= 108$

2. Write the answer.

The volume of water that passes through the tubing during one cycle is 108 litres.

WORKED EXAMPLE 14

A manufacturer of a game knows that the revenue, \$*R*, for selling *x* games is $R = 500\sqrt{x}$. The costs, \$*C*, to produce *x* games is $C = 2000 + x\sqrt{x}$.

a. Calculate the profit made when 25 games were sold.

b. Determine the average profit per game when 25 games were sold.

c. For 100 games, calculate and interpret:

i. the marginal revenue

ii. the marginal cost

iii. the marginal profit.

THINK	WRITE
a. 1. State the revenue equation.	$R = 500\sqrt{x}$
2. Substitute $x = 25$ and evaluate.	$R = 500\sqrt{25} = 2500$
3. State the cost equation.	$C = 2000 + x\sqrt{x}$
4. Substitute $x = 25$ and evaluate.	$C = 2000 + 25\sqrt{25} = 2125$
5. Profit = revenue − cost	Profit = 2500 − 2125 Profit = \$375
b. Average profit per game $= \frac{\text{profit}}{\text{number}}$	Profit per game $= \frac{375}{25}$ = \$15

c. i. 1. State the revenue equation.

$R = 500\sqrt{x}$

2. Differentiate with respect to x.

$R = 500x^{\frac{1}{2}}$

3. Substitute $x = 100$ and evaluate.

$$\frac{dR}{dx} = 500 \times \frac{1}{2}x^{\frac{-1}{2}}$$

$$\frac{dR}{dx} = \frac{250}{\sqrt{x}}$$

$$\frac{dR}{dx} = \frac{250}{\sqrt{100}}$$

$$= 25$$

The marginal revenue at $x = 100$ is \$25, so the approximate revenue from selling the 101st game is \$25.

ii. 1. State the cost equation.

$C = 2000 + x\sqrt{x}$

2. Differentiate with respect to x.

$C = 2000 + x^{\frac{3}{2}}$

3. Substitute $x = 100$ and evaluate.

$$\frac{dC}{dx} = \frac{3}{2}x^{\frac{1}{2}}$$

$$\frac{dC}{dx} = \frac{3\sqrt{x}}{2}$$

$$\frac{dC}{dx} = \frac{3\sqrt{100}}{2}$$

$$\frac{dC}{dx} = 15$$

The marginal cost at $x = 100$ is \$15, so the approximate cost for making the 101st game is \$15.

iii. 1. Write an equation for profit.

2. Differentiate with respect to x.

3. Substitute $x = 100$ and evaluate.

Profit = revenue − costs

$= 500\sqrt{x} - (2000 + x\sqrt{x})$

$= 500\left(x^{\frac{1}{2}} - 2000 - x^{\frac{3}{2}}\right)$

$$\frac{dP}{dx} = 500 \times \frac{1}{2}x^{\frac{-1}{2}} - \frac{3}{2}x^{\frac{1}{2}}$$

$$\frac{dP}{dx} = \frac{250}{\sqrt{x}} - \frac{3\sqrt{x}}{2}$$

$$\frac{dP}{dx} = \frac{250}{\sqrt{100}} - \frac{3\sqrt{100}}{2}$$

$$\frac{dP}{dx} = 25 - 15$$

$$= 10$$

Note: Observe that marginal profit = marginal revenue − marginal cost.

The marginal profit at $x = 100$ is \$10, so the approximate profit for selling the 101st game is \$10.

WORKED EXAMPLE 15

On any day, the cost per item for a machine producing x items is given by $\frac{dC}{dx} = 50 - 4e^{0.02x}$, where $x \in [0, 200]$ and C is the cost in dollars

a. **Use the rate to calculate the marginal cost of producing the 100th item.**
b. **What is the total cost of producing the first 100 items?**
c. **Determine the average cost of production for the first 100 items.**

THINK	WRITE
a. 1. State the marginal rate of change.	$\frac{dC}{dx} = 50 - 4e^{0.02x}$
2. Substitute $x = 100$ and evaluate.	$\frac{dC}{dx} = 50 - 4e^{0.02\times100}$ $= 50 - 4e^2$ $= 20.4438$
3. Answer the question.	The marginal cost of producing the 100th item is approximately \$20.44.
b. 1. Area under the rate of change equation of cost gives the total cost for a given interval.	Total cost $= \int_0^{100} \left(50 - 4e^{0.02x}\right) dx$
2. Antidifferentiate and evaluate.	$= \left[50x - 4 \times \frac{1}{.02} e^{0.02x}\right]_0^{100}$ $= \left[50x - 200e^{0.02x}\right]_0^{100}$ $= \left(50 \times 100 - 200e^2\right) - \left(0 - 200e^0\right)$ $= 5000 - 200e^2 + 200$ $= 3722.19$
3. Answer the question.	The total cost of producing the first 100 items is \$3722.19.
c. 1. Average cost of production = total cost / number produced.	Average cost of production = \$3722.19 / 100 = \$37.22
2. Answer the question.	The average cost of production for the first 100 items is approximately \$37.22 each.

7.6.2 Displacement, velocity and acceleration

The relationships between displacement, velocity and acceleration have been discussed in Chapters 5 and 6. Your knowledge about the definite integral and the area under curves gives you additional skills for the calculation of facts relating to kinematics.

From equations or graphs of velocity as a function of time, the following may be obtained:

- acceleration, the gradient of the velocity function or $\frac{dv}{dt}$
- displacement, the signed area or definite integral
- distance travelled, the area under the curve.

WORKED EXAMPLE 16

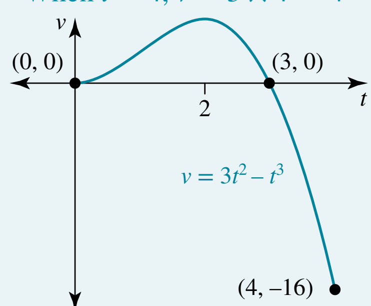
A particle starting from rest accelerates according to the rule $a = 3t(2 - t)$.

a. Determine a relationship between the velocity of the particle, v metres/second, and the time, t seconds.

b. Determine the displacement of the particle after 4 seconds.

c. Sketch the graph of velocity versus time for the first 4 seconds of the motion.

d. Calculate the distance travelled by the particle in the first 4 seconds.

THINK	WRITE
a. 1. Antidifferentiate the acceleration equation to determine the velocity equation.	a. $v = \int a(t)dt$ $= \int (3t(2 - t)dt$ $= \int (6t - 3t^2)dt$ $= 3t^2 - t^3 + c$
2. Apply the initial conditions to determine v in terms of t.	When $t = 0$, $v = 0$, so $c = 0$. $\therefore v = 3t^2 - t^3$
b. 1. Integrate v between $t = 0$ and $t = 4$. As we are finding displacement, there is no need to sketch the graph.	b. $x = \int_0^4 (3t^2 - t^3)dt$ $= \left[t^3 - \frac{1}{4}t^4\right]_0^4$ $= \left(4^3 - \frac{1}{4}(4)^4\right) - \left(0^3 - \frac{1}{4}(0)^4\right)$ $= 0$
2. Write the answer.	After 4 seconds the displacement is 0.
c. Sketch a graph of v versus t.	c. y-intercept: $(0, 0)$ t-intercepts: $0 = 3t^2 - t^3$ $= t^2(3 - t)$ $t = 0, 3$ When $t = 4$, $v = 3 \times 4^2 - 4^3 = -16$. 
d. 1. The area under the curve of a velocity–time graph gives the distance covered. Set up the integrals and subtract the negative region.	d. $D = \int_0^3 (3t^2 - t^3)dt - \int_3^4 (3t^2 - t^3)dt$

2. Antidifferentiate and evaluate.

$$= [t^3 - \frac{1}{4}t^4]_0^3 - [t^3 - \frac{1}{4}t^4]_3^4$$

$$= \left(\left(3^3 - \frac{3^4}{4}\right) - \left(0^3 - \frac{0^4}{4}\right)\right)$$
$$- \left(\left(4^3 - \frac{4^4}{4}\right) - \left(3^3 - \frac{3^4}{4}\right)\right)$$

$$= 27 - \frac{81}{4} - 0 - 64 + 64 + 27 - \frac{81}{4}$$

$$= 54 - \frac{162}{4}$$

$$= 13.5$$

3. Write the answer.

The distance travelled by the particle in 4 seconds is 13.5 metres.

Alternative working for the distance travelled in the first 4 seconds:

1. Determine where the particle is at rest.

At rest: $v = 0$

$3t^2 - t^3 = 0$

$t^2(3 - t) = 0$

$t = 0$ or 3

The particle is at rest initially and after 3 seconds, so it changes direction after 3 seconds.

2. Determine the displacement of the particle initially and at $t = 3$ and 4. *Note:* We were not told where the particle was initially, so we cannot find the constant, c.

$x = t^3 - \frac{1}{4}t^4 + c$

At $t = 0$: $x = c$

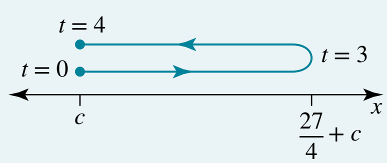
At $t = 3$: $x = 27 - \frac{81}{4} + c$

$x = \frac{27}{4} + c$

At $t = 4$: $x = 64 - 64 + c$

$x = c$

3. Draw a motion diagram to represent the displacement of the particle during the first 4 seconds.



Distance travelled: $= \frac{27}{4} + \frac{27}{4}$

$= \frac{27}{2} = 13.5$ metres

studyon

Units 3 & 4 | Area 3 | Sequence 2 | Concept 7

Applications of integration Summary screen and practice questions

Exercise 7.6 Applications of integration

Technology active

1. **WE13** Heat escapes from a storage tank at a rate of kilojoules per day. This rate can be modelled by

$$\frac{dH}{dt} = 1 + \frac{\pi^2}{9}\sin\left(\frac{\pi t}{45}\right), 0 \leq t \leq 100$$

where $H(t)$ is the total accumulated heat loss in kilojoules, t days after 1 June.

   a. What is the rate of escape of the heat, correct to 2 decimal places, at:
      i. 15 days?
      ii. 60 days?
   b. State the period of the function.
   c. Calculate the total accumulated heat loss after 45 days. Give your answer in exact form.

2. The average rate of increase, in cm/month, in the length of a baby boy from birth until age 36 months is given by the rule

$$\frac{dL}{dt} = \frac{4}{\sqrt{t}}$$

where t is the time in months since birth and L is the length in centimetres. Determine the average total increase in length of a baby boy from 6 months of age until 36 months of age. Give your answer correct to 1 decimal place.

3. A number of apprentice bricklayers are competing in a competition in which they are required to build a fence. The competitors must produce a fence that is straight, neatly constructed and level. The winner will also be judged on how many bricks they have laid during a 30-minute period.

   The winner laid bricks at a rate defined by the rule

$$\frac{dN}{dt} = 0.8t + 2$$

where N is the number of bricks laid after t minutes.

   a. Sketch the graph of the given function for $0 \leq t \leq 30$.
   b. Shade the region defined by $10 \leq t \leq 20$.
   c. How many bricks in total did the winner lay in the 10-minute period defined by $10 \leq t \leq 20$?

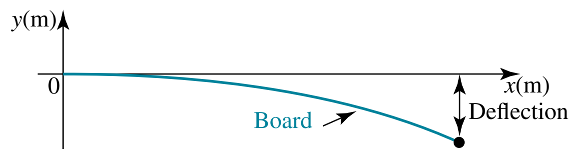
4. The rate of growth of mobile phone subscribers with a particular company in the UK can be modelled by the rule

$$\frac{dN}{dt} = 0.853e^{0.1333t}$$

where N million is the number of subscribers with the company since 1998 and t is the number of years since 1998, the year the company was established. Determine how many millions of mobile phone subscribers have joined the company between 1998 and 2015, correct to 1 decimal place.

5. **WE14** A manufacturer of a game knows that the revenue, $\$R$, for selling x games is $R = 100(\sqrt{x+4} - 2)$ The costs, $\$C$, to produce x games is $C = 50 + x\sqrt{x}$.
   a. Calculate the profit made when 10 games were sold.
   b. Determine the average profit per game when 10 games were sold.
   c. For 20 games, calculate and interpret:
      i. the marginal revenue
      ii. the marginal cost
      iii. the marginal profit.
6. The weekly profit of a factory, P (in hundreds of dollars), is given by $P = 8n - n\sqrt{n}$, where n is the number of employees.
   a. Calculate the weekly profit of a factory with 16 employees.
   b. Determine the average weekly profit per employee when there are 16 employees.
   c. Calculate the marginal weekly profit, in dollars per employee, when the number of employees is:
      i. 10 employees
      ii. 25 employees.
7. **WE15** A manufacturer has found that the cost per item to produce x items is given by $\frac{dC}{dx} = 20 + x + e^{-0.05x}$, where $x \in [0, 50]$ and C is the cost in dollars.
   a. Use the rate to calculate the marginal cost of producing the 10th item.
   b. What is the total cost of producing the first 10 items?
   c. Determine the average cost of production for the first 10 items.
8. On any day the cost per item for a machine producing n items is given by $\frac{dC}{dn} = 40 - 2e^{0.01n}$, where $n \in [0.200]$ and C is the cost in dollars.
   a. Use the rate to determine the cost of producing the 100th item.
   b. Express C as a function of n.
   c. What is the total cost of producing the first 100 items?
   d. Determine the average cost of production for the first 100 items.
9. **WE16** A particle moves in a line so that its velocity, v metres/second, from a fixed point, O, is defined by $v = 1 + 3\sqrt{t+1}$, where t is the time in seconds.
   a. Determine the initial velocity of the particle.
   b. What is the acceleration of the particle when:
      i. $t = 0$?
      ii. $t = 8$?
   c. Sketch the graph of v versus t for the first 10 seconds.
   d. Determine the distance covered by the particle in the first 8 seconds.
10. An object travels in a line so that its velocity, v metres/second, at time t seconds is given by $v = 3\cos\left(\frac{t}{2} - \frac{\pi}{4}\right), t \geq 0.$
    Initially the object is $-3\sqrt{2}$ metres from the origin.
    a. Determine the relationship between the displacement of the object, x metres, and time, t seconds.
    b. What is the displacement of the object when time is equal to 3π seconds?
    c. Sketch the graph of v versus t for $0 \leq t \leq 4\pi$.
    d. Determine the distance travelled by the object after 3π seconds. Give your answer in metres, correct to 2 decimal places.
    e. Determine a relationship between the acceleration of the object, a metres/second^2, and time, t seconds.
    f. What is the acceleration of the object when $t = 3\pi$ seconds?

11. The rate of deflection from a horizontal position of a 3-metre diving board when an 80-kilogram person is x metres from its fixed end is given by $\frac{dy}{dx} = -0.03(x+1)^2 + 0.03$, where y is the deflection in metres.



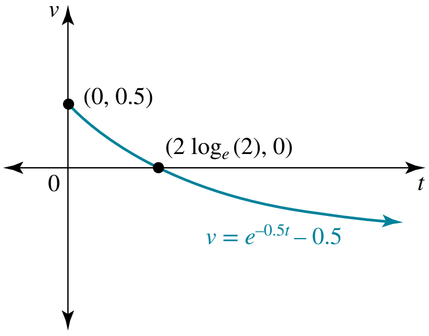
a. What is the deflection when $x = 0$?
b. Determine the equation which measures the deflection.
c. Hence, determine the maximum deflection.

12. The rate of change of position (velocity) of a racing car travelling down a straight stretch of road is given by $\frac{dx}{dt} = t(16 - t)$, where x is measured in metres and t in seconds.

a. Determine the velocity when:
   i. $t = 0$
   ii. $t = 4$.
b. Determine:
   i. when the maximum velocity occurs
   ii. the maximum velocity.
c. Sketch the graph of $\frac{dx}{dt}$ against x for $0 \le t \le 16$.
d. Determine the area under the graph between $t = 0$ and $t = 10$.
e. What does this area represent?

13. The rate of flow of water into a hot water system during a 12-hour period on a certain day is thought to be $\frac{dV}{dt} = 10 + \cos\left(\frac{\pi t}{2}\right)$, where V is in litres and t is the number of hours after 8 am.
a. Sketch the graph of $\frac{dv}{dt}$ against t.
b. Determine the length of time for which the rate is above 10.5 L/h.
c. Determine the volume of water that has followed into the system between:
   i. 8 am and 2 pm
   ii. 3 pm and 8 pm.

14. A particle moves in a straight line. At time t seconds its velocity, v metres per second, is defined by the rule $v = e^{-0.5t} - 0.5$, $t \ge 0$. The graph of the motion is shown.



a. Determine the acceleration of the particle, a m/s^2, in terms of t.
b. Determine the displacement of the particle, x m, if $x = 0$ when $t = 0$.
c. Determine the displacement of the particle after 4 seconds.
d. Calculate the distance covered by the particle in the fourth second. Give your answer correct to 4 decimal places.

15. The maintenance costs for a car increase as the car gets older. It has been suggested that the increase in maintenance costs of dollars per year could be modelled by

$$\frac{dC}{dt} = 15t^2 + 250$$

where t is the age of the car in years and C is the total accumulated cost of maintenance for t years.

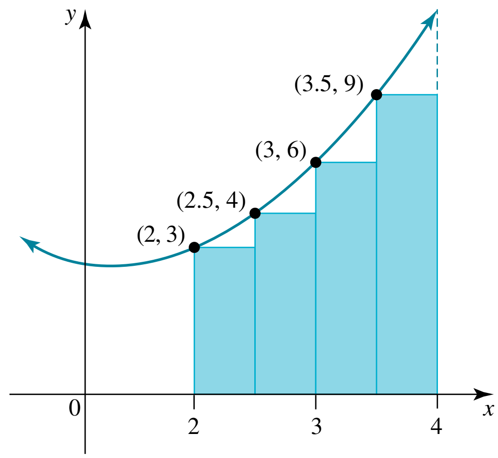
a. Sketch the graph of the given function for $0 \leq t \leq 10$.
b. Determine the total accumulated cost of maintenance for $t = 5$ to $t = 10$ years.

7.7 Review: exam practice

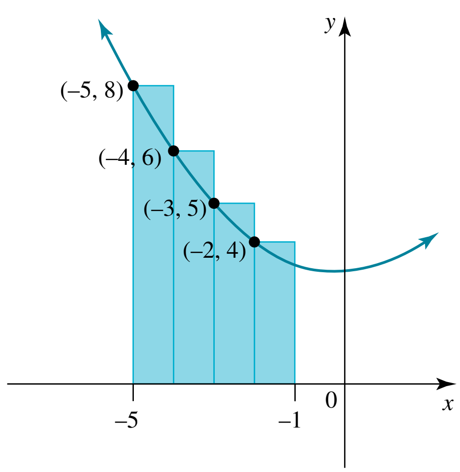
A summary of this chapter is available in the Resources section of your eBookPLUS at www.jacplus.com.au.

Simple familiar

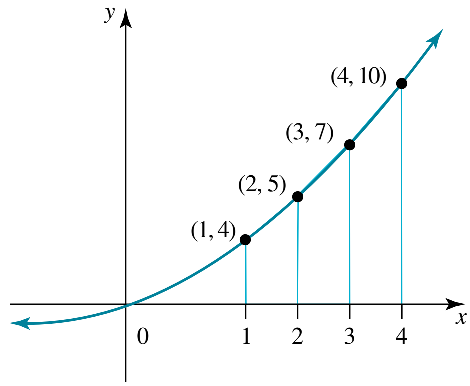
1. Using the figure shown, calculate the approximate area under the curve from $x = 2$ to $x = 4$, using the left-hand, or lower, rectangles.



2. Using the figure shown, calculate the approximate area under the curve from $x = -5$ to $x = -1$, using the upper rectangles.



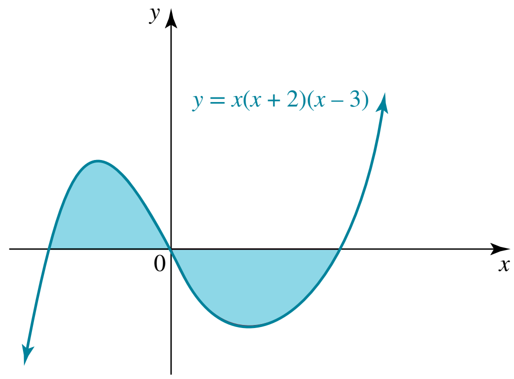
3. A student is using the trapezoidal rule to approximate the area under the curve shown from $x = 0$ to $x = 4$. What would their answer be?



4. Apply the trapezoidal method to determine the area between $y = e^{2x-1}$ and the x-axis from $x = 0$ to $x = 4$, using intervals that are 1 unit wide. (Give your answer correct to 2 decimal places.)
5. Consider the area under the curve $y = \log_e(x)$ from $x = 2$ to $x = 4$. With interval widths of 0.5 units, determine an approximation to the area using:
   a. left end-point rectangles
   b. right end-point rectangles
   c. the average of the rectangles.

   Give your answers in exact form.
6. Determine:
   a. $\displaystyle\int_0^2 \left(3x + 6\sqrt{x} + 1\right) dx$
   b. $\displaystyle\int_0^{\frac{1}{2}} (e^x + 1)(e^x - 1)\, dx$
   c. $\displaystyle\int_{-1}^0 \frac{9}{(2x+3)^4}\, dx$
   d. $\displaystyle\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \cos 2x\, dx$
7. Given that $\displaystyle\int_0^k (4x - 5)\, dx = -2$, determine two possible values for k.
8. Given that $\displaystyle\int_1^5 f(x)\, dx = 4$ and $\displaystyle\int_1^5 g(x)\, dx = 3$, determine:
   a. $\displaystyle\int_1^5 (4f(x) + 1)\, dx$
   b. $\displaystyle\int_1^5 (2f(x) - g(x))\, dx$
   c. $\displaystyle\int_1^5 (3f(x) + 2g(x) - 5)\, dx$
9. a. Sketch the graph of the function $f(x) = \dfrac{1}{x-2}$.
   b. Calculate the exact area between the graph of $f(x)$, the x-axis and the lines $x = 3$ and $x = 6$.

10. Determine the area bounded by the curve $y = x(x + 2)(x - 3)$ and the x-axis.



11. Determine m if $\int_{\frac{1}{2}}^{m} 6\,(2x - 1)^2\,dx = 1$.

12. A particle starts at the origin and travels in a straight line with a velocity, v m/s, modelled by $v = t^2 - t - 2$, where t is the time in seconds.

- **a.** State the equation for the acceleration of the particle.
- **b.** Determine when the particle is at rest.
- **c.** What is the displacement of the particle after 3 seconds?
- **d.** Determine the distance covered by the particle in the first 3 seconds.
- **e.** Hence, determine the average speed of the particle for the first 3 seconds.

Complex familiar

13. **a.** Determine any point(s) of intersection between the two curves $f(x) = x^3 - 3x + 2$ and $g(x) = x + 2$.

- **b.** Sketch $f(x)$ and $g(x)$ on the same set of axes. Label the point(s) of intersection and any x- and y-intercepts.
- **c.** Evaluate the area between the two curves.

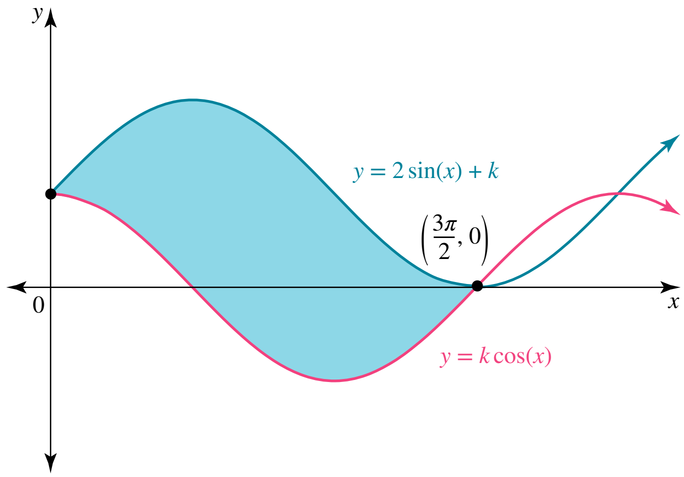
14. A manufacturer of a new gadget knows that the revenue, $\$R$, for selling x gadgets is $R = e^{\frac{x}{20}} - 1$. The costs, $\$C$, to produce x gadgets is $C = 40 + x - \sqrt{x}$.

- **a.** Calculate the profit or loss made when:
  - **i.** 50 gadgets are sold
  - **ii.** 100 gadgets are sold.
- **b.** Determine the average profit per gadget when 100 gadgets are sold.
- **c.** For 120 gadgets, calculate and interpret:
  - **i.** the marginal revenue
  - **ii.** the marginal cost
  - **iii.** the marginal profit.

15. The velocity of a particle is given by $v = 2\sin(2t) + 3$, where x is the displacement in metres and t is the time in seconds. Initially the particle is at the origin.

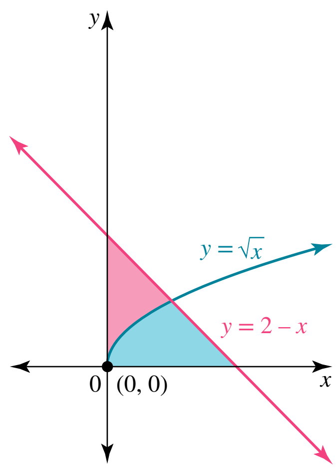
- **a.** Show that the displacement is given by $x = -\cos(2t) + 3t + 1$.
- **b.** Determine the displacement when $t = \frac{\pi}{2}$ seconds.
- **c.** What distance has the particle travelled in the first $\frac{\pi}{2}$ seconds?

16. The graphs of $y = 2\sin(x) + k$ and $y = k\cos(x)$ are shown. The shaded region is equal to $(3\pi + 4)$ square units. Determine the value of the constant k.



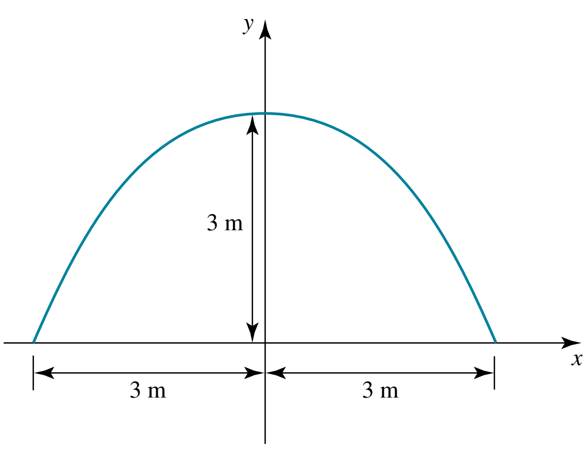
Complex familiar

17. The graphs with equations $y = \sqrt{x}$ and $y = 2 - x$ are shown.



a. Determine the point of intersection of the graphs.
b. Calculate the area of the blue shaded region.
c. Hence, or otherwise, determine the area of the pink shaded region.

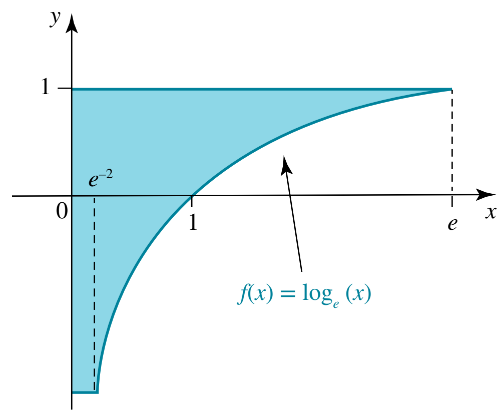
18. An arched window has width 6 metres and height 3 metres, as shown in the diagram. The arch can be approximated to a parabola.



a. Determine the equation of parabola.
b. Calculate the area of the window.
c. Show that the area of the window is $\frac{2}{3}$ the base of the arch times the height.
d. Hence, calculate the area of a similar arched window with width 8 metres and height 4.5 metres.

19. A ground-cover plant can cover the ground at a rate modelled by $\frac{dA}{dt} = 2t + 6t^2 - \frac{1}{4}t^3$, where A is the area in square centimetres and t is the time in weeks.
a. If the plant initially covers $10\,\text{cm}^2$, how long, to the nearest week, will it take to cover $0.6\,\text{m}^2$?
b. What is the maximum area covered by the plant?

20. The cross-section of a platform is shown. (All measurements are in metres.)
a. Determine the derivative of $x \log_e(x)$.
b. Hence, determine an antiderivative of $\log_e(x)$.
c. Calculate the height of the platform.
d. Calculate the cross-sectional area of the platform.
e. Calculate the volume of concrete required to build this platform if it is 20 metres long. Give your answer to the nearest cubic metre.



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Units 3 & 4 Sit exam

Answers

7 Integration

Exercise 7.2 Estimating the area under a curve

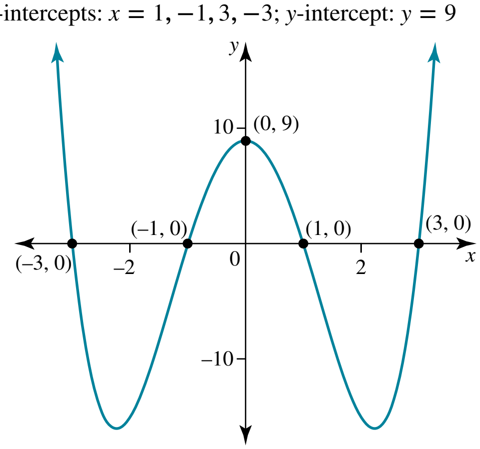
1. 10 sq. units
2. a. 8 sq. units b. 42 sq. units
3. a. $\frac{25}{12}$ sq. units or approx. 2.08 sq. units
   b. $\frac{77}{60}$ sq. units or approx. 1.28 sq. units
4. a. 30 sq. units b. 22 sq. units
5. 9 sq. units
6. a. 8 sq. units b. 13 sq. units
7. a. 26 sq. units b. 41 sq. units
   c. $33\frac{1}{2}$ sq. units
8. a. $(1 + e + e^{-1})$ sq. units b. $\log_e(24)$ sq. units
9. a. 7.25 sq. units b. 100 sq. units
10. a. $19\frac{1}{4}$ sq. units
    b. $\frac{1}{2}\left(1 + 2e + 2e^2 + e^3\right)$ sq. units
11. a. 22.5 sq. units b. 20.8 sq. units
12. a. 1.87 sq. units b. 1.68 sq. units
13. 12 sq. units
14. 21 sq. units
15. a. 4 b. 7.56 sq. units

Exercise 7.3 The fundamental theorem of calculus and definite integrals

1. a. $\frac{1}{3}$ b. $20\frac{1}{4}$ c. $5\frac{1}{3}$ d. $\frac{1}{3}$ e. 8
2. a. $37\frac{1}{3}$ b. $\frac{3}{5}\ln(5) \approx 0.966$
   c. $\frac{85}{256}$ d. 2
   e. 3.65 (to 2 decimal places)
3. a. 1 b. 0
   c. $20 - 10\sqrt{2} \approx 5.86$ d. -6
   e. 0 f. 0
4. a. $\frac{1}{4}\left(e^8 - 1\right)$ b. $3\left(1 - e^{\frac{-2}{3}}\right)$
   c. $2\left(e^{-2} - e^2\right)$ d. $\frac{1}{2}\left(e^{12} + 15 - e^6\right)$
   e. $5\ln(4) + 2e^2 - 2e^{\frac{1}{2}}$
5. a. 27 b. $9\frac{1}{6}$ c. 0
   d. 0 e. $\frac{4}{3}\left(\sqrt{10} - 2\right)$ f. $\sqrt{2} - \frac{3\sqrt{3}}{4}$
6. a. $\frac{65}{4} = 16\frac{1}{4}$ b. $2 + \frac{1}{2}e^2 - \frac{1}{2}e^{-2}$
7. a. 21 b. 11 c. -16 d. 8
8. a. -15 b. -12.5 c. 32.5 d. 20 e. 25 f. 12.5
9. 2
10. 3
11. $2\log_e(3)$
12. 4
13. a. 1 b. $21\frac{1}{3}$
14. a. $\frac{dy}{dx} = x\cos(x) + \sin(x)$ b. $\pi + 2$
15. a. $\frac{dy}{dx} = 3\left(x^2 - 2x\right)e^{\left(x^3 - 3x^2\right)}$
    b. $\frac{1}{3}\left(e^{-2} - 1\right)$

Exercise 7.4 Areas under curves

1. a. i. $\int_0^4 (4 - x)\,dx$ ii. 8 sq. units
   b. i. $\int_1^2 \left(x^2\right)\,dx$ ii. $2\frac{1}{3}$ sq. units
   c. i. $\int_{-3}^{-1} \left(3x^2\right)\,dx$ ii. 26 sq. units
   d. i. $\int_1^3 \left(x^3 - 9x^2 + 20x\right)\,dx$ ii. 22 sq. units
   e. i. $\int_{-2}^{0} \left(-x^3 - 4x^2 - 4x\right)\,dx$
      ii. $1\frac{1}{3}$ sq. units
2. a. i. $\int_{-1}^{1} \left(e^x\right)\,dx$ ii. $\left(e - e^{-1}\right)$ sq. units
   b. i. $\int_1^4 \left(e^{-2x}\right)\,dx$ ii. $\frac{1}{2}\left(e^{-2} - e^{-8}\right)$ sq. units
   c. i. $\int_0^{\frac{\pi}{2}} \left(2\sin(2x)\right)\,dx$ ii. 2 sq. units
   d. i. $\int_0^{\frac{3\pi}{2}} \cos\frac{x}{3}\,dx$ ii. 3 sq. units
3. a. 1 sq. unit b. $5\frac{1}{3}$ sq. units
   c. $1\frac{1}{3}$ sq. units d. 4 sq. units
   e. $2\frac{2}{3}$ sq. units
4. a. $\left(e - e^{-1}\right)$ sq. units b. $\frac{1}{2}\left(e^{-1} - e^{-2}\right)$ sq. units
   c. 1 sq. unit d. 4 sq. units
5. a. x-intercepts: $x = 1, -1, 3, -3$; y-intercept: $y = 9$



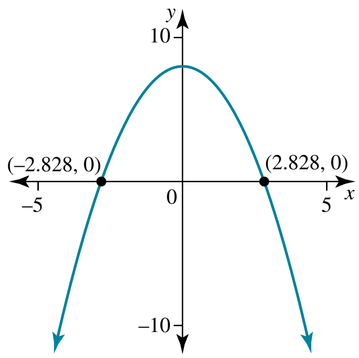
   b. $52\frac{4}{15}$ sq. units

6. a. $15\frac{3}{4}$ sq. units

b. $5\frac{1}{3}$ sq. units

c. $21\frac{1}{12}$ sq. units

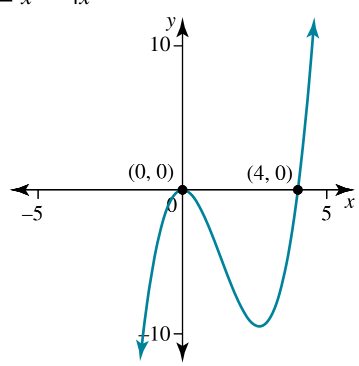
7. a. i. $y = 8 - x^2$



x-intercepts: $x = \pm\sqrt{8}$ or $\pm 2\sqrt{2}$

ii. $\frac{64\sqrt{2}}{3}$ sq. units

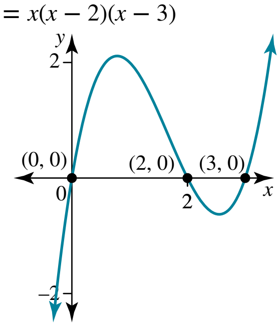
b. i. $y = x^3 - 4x^2$



x-intercepts: $x = 0, 4$

ii. $21\frac{1}{3}$ sq. units

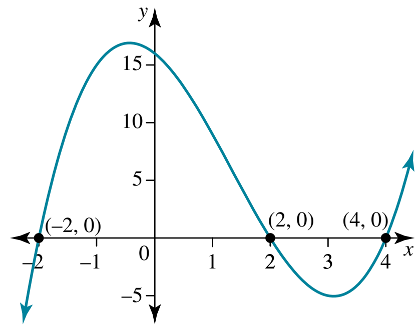
c. i. $y = x(x - 2)(x - 3)$



x-intercepts: $x = 0, 2, 3$

ii. $3\frac{1}{12}$ sq. units

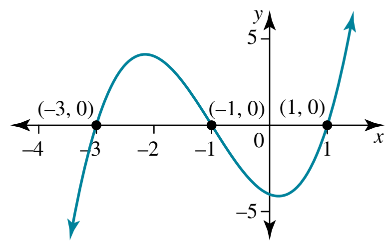
d. i. $y = x^3 - 4x^2 - 4x + 16$



x-intercepts: $x = -2, 2, 4$

ii. $49\frac{1}{3}$ sq. units

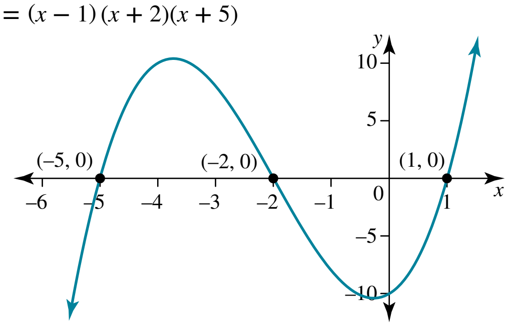
e. i. $y = x^3 + 3x^2 - x - 3$



x-intercepts: $x = -3, -1, 1$

ii. Area = 8 sq. units

f. i. $y = (x - 1)(x + 2)(x + 5)$



x-intercepts: $x = -5, -2, 1$

ii. $40\frac{1}{2}$ sq. units

8. $2\ln(2)$ sq. units

9. $\frac{1}{3}\left(e^6 - e^3\right)$ sq. units

10. $\frac{3 - \sqrt{3}}{2}$ sq. units

11. $166\frac{2}{3}$ sq. units

12. 3π sq. units

13. $\left(e + e^{-1} - 2\right)$ sq. units

14. 1.6 sq. units

15. 16 sq. units

16. a. $\frac{dy}{dx} = 1 + \ln(x)$

b. $\int \ln(x)\,dx = x\ln(x) - x + c$

c. $(4\ln(4) - 3)$ sq. units

17. a. $\frac{dy}{dx} = 2xe^{x^2}$

b. $2(e - 1)$ sq. units

18. a. $\dfrac{dy}{dx} = \dfrac{2x}{(x^2+2)}$

b. $\displaystyle\int \frac{x}{(x^2+2)}\,dx = \frac{1}{2}\ln\left(x^2+2\right) + c$

c. $\left(\ln(3) - \ln(2)\right)$ sq. units, or $\ln\left(\dfrac{3}{2}\right)$ sq. units

19. a. $\dfrac{3}{4}$ sq. units b. $2\dfrac{1}{4}$ sq. units

20. a. 2 b. $2(\pi - 2)$ sq. units

Exercise 7.5 Areas between curves

1. a. $2\dfrac{1}{6}$ sq. units

b. $21\dfrac{1}{3}$ sq. units

c. $2\dfrac{1}{4}$ sq. units

d. $\dfrac{52}{3} + \dfrac{1}{e} - e \approx 14.98$ sq. units

e. $e^2 - e + \dfrac{3}{2} \approx 6.17$ sq. units

f. $1\dfrac{1}{3}$ sq. units

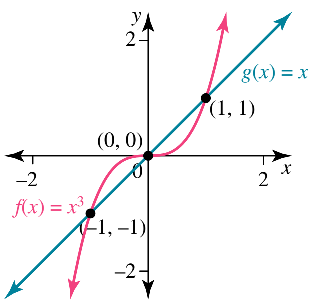
2. C

3. D

4. D

5. a. $x = 0, 1$ or -1

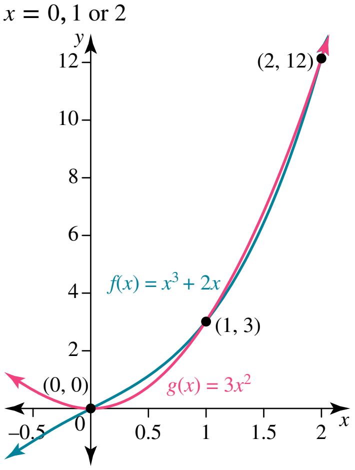
b.



c. $\dfrac{1}{2}$ sq. units

6. a. $x = 0, 1$ or 2

b.



c. $\dfrac{1}{2}$ sq. units

7. $(16, 4)$; $21\dfrac{1}{3}$ sq. units

8. $20\dfrac{5}{6}$ sq. units

9. a. $\left(\dfrac{3\pi}{4}, \dfrac{1}{\sqrt{2}}\right)$ b. $2\sqrt{2}$ sq. units

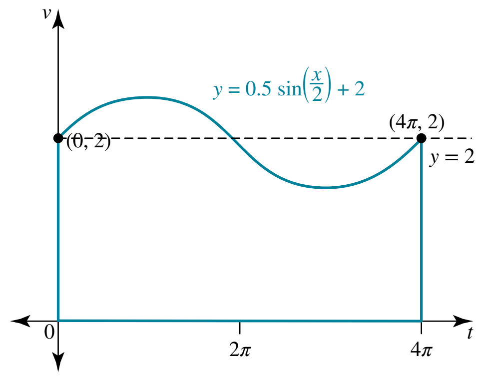
10. $\dfrac{\pi\sqrt{3}}{12}$ sq. units (≈ 0.45 sq. units)

11. $(3\ln(2) - 2)$ sq. units

12. a. $a = 1, b = 2, c = 3$ b. 9.6 sq. units

13. a. $(-4, 0), (4, 0)$ b. $(-3, 0), (3, 0)$ c. $9\dfrac{1}{3}$ m^2

14. a.



b. 25 m^2

c. 12.5 m^3

15. a. $y = 5 - \dfrac{1}{5}x^2$ b. $33\dfrac{1}{3}$ m^2

c. $50\dfrac{2}{3}$ m^2 d. 152 m^3

Exercise 7.6 Applications of integration

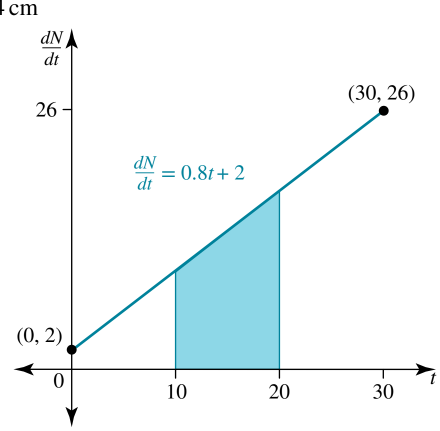
1. a. i. 1.95 kJ/day ii. 0.05 kJ/day

b. 90 days

c. Accumulated heat loss after 45 days is $(45 + 10\pi)$ kJ, or approx. 76.42 kJ.

2. 28.4 cm

3. a, b.



c. 140 bricks

4. 55.3 million

5. a. \$92.54

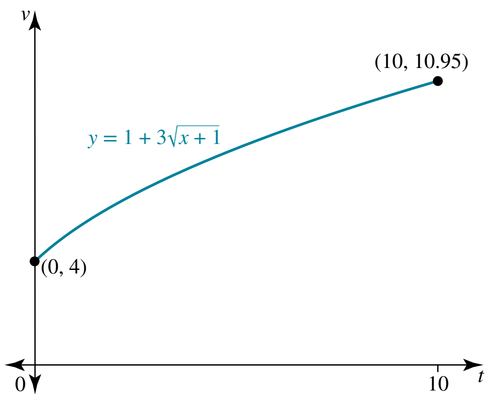
b. \$9.25

c. i. Marginal revenue at $x = 20$ is \$10.21, so the approximate revenue from selling the next game is \$10.21.

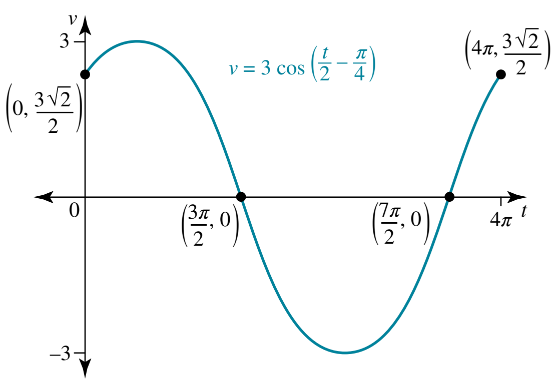
ii. Marginal cost at $x = 20$ is \$6.71, so the approximate cost of manufacturing the next game is \$6.71.

iii. Marginal profit at $x = 20$ is \$3.50, so the approximate profit from selling the next game is \$3.50.

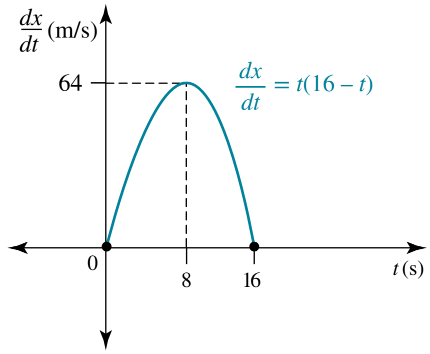
6. a. \$6 400
   b. \$400
   c. i. \$32.57/employee
      ii. \$2/employee
7. a. \$30.61 b. \$257.87 c. \$25.79/item
8. a. \$34.56
   b. $C = 40n - 200\,e^{0.01n} + 200$
   c. \$3 656.34
   d. \$36.56/item
9. a. 4 m/s
   b. i. 1.5 m/s^2
      ii. 0.5 m/s^2
   c.



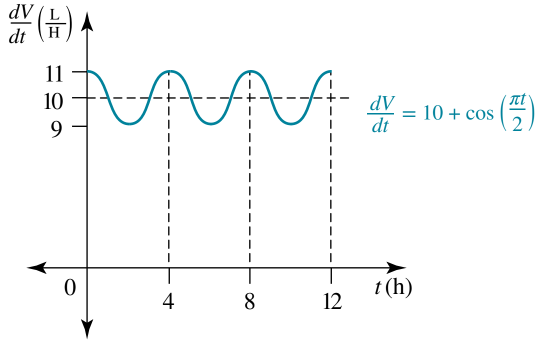
   d. 60 m
10. a. $x = 6\sin\left(\frac{t}{2} - \frac{\pi}{4}\right)$
   b. $-3\sqrt{2}$ m
   c.



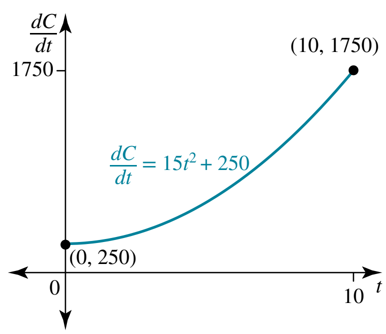
   d. 20.49 m
   e. $a = \frac{dv}{dt} = -\frac{3}{2}\sin\left(\frac{t}{2} - \frac{\pi}{4}\right)$
   f. $\frac{3\sqrt{2}}{4}$ m/s^2
11. a. 0 m
   b. $y = -0.01\,(x+1)^3 + 0.03x + 0.01$
   c. 54 cm downwards
12. a. i. 0 m/s
      ii. 48 m/s
   b. i. 8 s
      ii. 64 m/s
   c.



   d. $466\frac{2}{3}$ m
   e. The area represents the distance travelled in the first 10 s.
13. a.



   b. 4 h
   c. i. 60 L
      ii. 50.6 L
14. a. $a = \frac{dv}{dt} = -0.5\,e^{-0.5t}$
   b. $x = 2 - 2\,e^{-0.5t} - 0.5t$
   c. -0.2707 m
   d. 0.3244 m
15. a.



   b. \$5625

7.7 Review: exam practice

1. 11 sq. units.
2. 23 sq. units.
3. 21 sq. units.
4. 719.72 sq. units
5. a. $\frac{1}{2}\ln\left(\frac{105}{2}\right)$ sq. units

b. $\frac{1}{2}\ln(105)$ sq. units

c. $\frac{1}{4}(2\ln(105) - \ln(2))$ sq. units

6. a. $8 + 8\sqrt{2}$

b. $\frac{1}{2}e - 1$

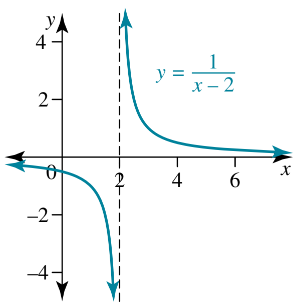
c. $1\frac{4}{9}$

d. $-\frac{\sqrt{3}}{2}$

7. $\frac{1}{2}, 2$

8. a. 20 b. 5 c. -2

9. a.



b. $\log_e(4)$ sq. units

10. $21\frac{1}{12}$ sq. units

11. 1

12. a. $a = \frac{dv}{dt} = 2t - 1$

b. The particle is at rest at $t = 2$ s.

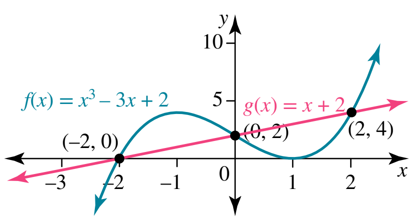
c. After 3 s, the displacement is $-3\frac{1}{3}$ m, or $3\frac{1}{3}$ m to the left of the origin.

d. The distance travelled in first 3 s is $5\frac{1}{6}$ m.

e. The average speed for the first 3 s is $1\frac{13}{18}$ m/s.

13. a. $(-2, 0), (0, 2), (2, 4)$

b.



c. The area between the curves is 8 sq. units.

14. a. i. A loss of \$71.75

ii. A profit of \$17.41

b. \$0.17/gadget

c. i. The marginal revenue at $x = 120$ is \$20.17, so the approximate revenue from selling the next gadget is \$20.17.

ii. The marginal cost at $x = 120$ is \$0.95, so the approximate cost of manufacturing the next gadget is \$0.95.

iii. The marginal profit at $x = 120$ is \$19.22, so the approximate profit from selling the next gadget is \$19.22.

15. a. Sample responses can be found in the worked solutions in the online resources.

b. $\left(2 + \frac{3\pi}{2}\right)$ m

c. $\left(2 + \frac{3\pi}{2}\right)$ m

16. 2

17. a. $(1, 1)$

b. The blue shaded area is $1\frac{1}{6}$ sq. units.

c. The pink shaded area is $\frac{5}{6}$ sq. units.

18. a. $y = -\frac{1}{3}\left(x^2 - 9\right)$

b. 12 m^2

c. Sample responses can be found in the worked solutions in the online resources.

d. 24 m^2

19. a. 19 weeks

b. 0.75 m^2

20. a. $\frac{dy}{dx} = 1 + \log_e(x)$

b. $\int \log_e(x)\,dx = x\log_e(x) - x + c$

c. 3 m

d. $\left(e - e^{-2}\right)$ m^2 or approx. 2.58 m^2

e. 52 m^3

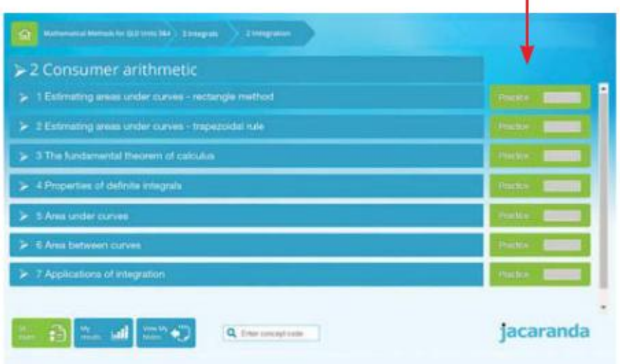
REVISION UNIT 3 Further calculus

TOPIC 3 Integrals

- For revision of this entire topic, go to your **studyON** title in your bookshelf at **www.jacplus.com.au**.
- Select **Continue Studying** to access hundreds of revision questions across your entire course.

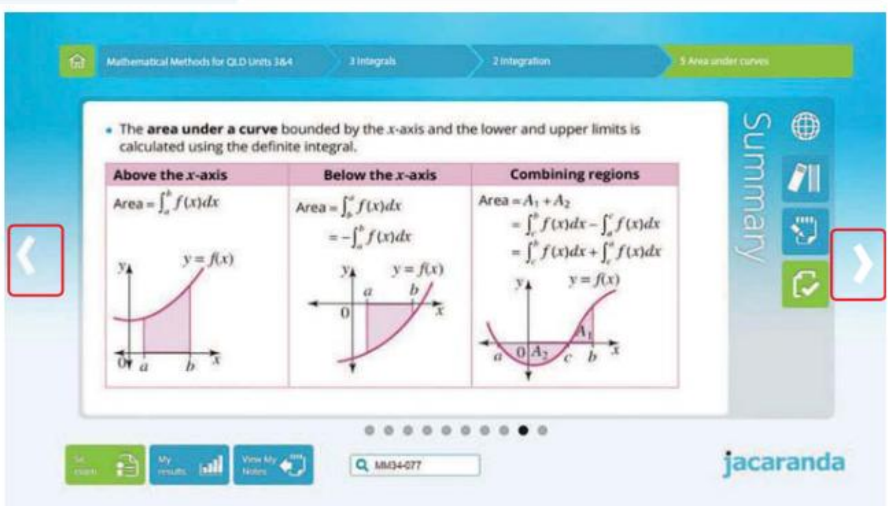
- Select your **course** *Mathematical Methods for Queensland Units 3&4* to see the entire course divided into syllabus topics.
- Select the **Area** you are studying to navigate into the chapter level *OR* select **Practice** to answer all practice questions available for each area.

- Select **Practice** at the sequence level to access all questions in the sequence.



- At **sequence level**, drill down to concept level.

- **Summary screens** provide revision and consolidation of key concepts. Select the **next arrow** to revise all concepts in the sequence and practise questions at the concept level.



PRACTICE ASSESSMENT 1

Mathematical Methods: Problem solving and modelling task

Unit

Unit 3: Further calculus

Topics

Topic 2: Further differentiation and applications 2
Topic 3: Integrals

Conditions

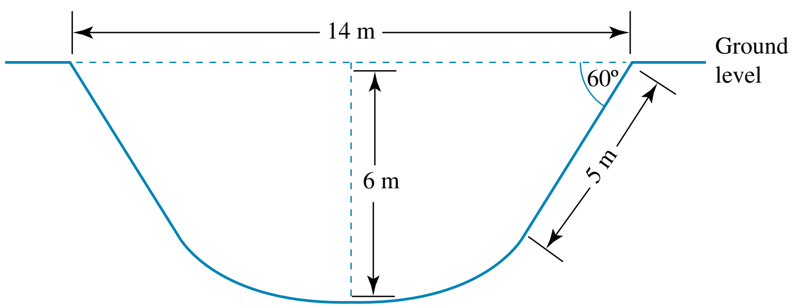
Duration	**Mode**	**Individual/group**
4 weeks	Written report, up to 10 pages (maximum 2000 words) excluding appendix	Individual
Resources permitted		
The use of technology is required, for example: • graphing calculator • computer • internet • graphing packages.		
Milestones		
Week 4		
Week 5		
Week 6		
Week 7 (assessment submission)		
Criterion*	**Marks allocated**	**Result**
Formulate *Assessment objectives 1, 2, 5	4	
Solve *Assessment objectives 1, 6	7	
Evaluate and verify *Assessment objectives 4, 5	5	
Communicate *Assessment objectives 3	4	
Total	20	
Scaffolding		
Please refer to the fourth page of this assessment for an appropriate approach to problem solving and modelling.		

For the most up to date assessment information, please see www.qcaa.qld.edu.au/senior.

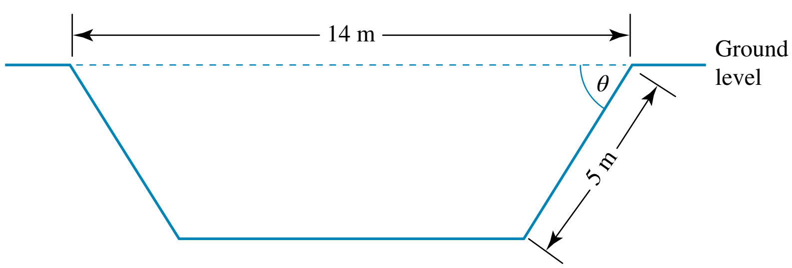
Context

Three engineers are discussing various cross-sections for a proposed new irrigation canal. The canal needs to be 14 metres across at the surface, with a maximum depth of 6 metres. As the canal should be able to carry as much water as possible, the engineers are analysing their proposals to find which gives the greatest area of the cross-section and so the greatest volume. When making calculations, they decided to express all lengths in metres, to 6 decimal places where necessary, and all angles in degrees, correct to 1 decimal place.

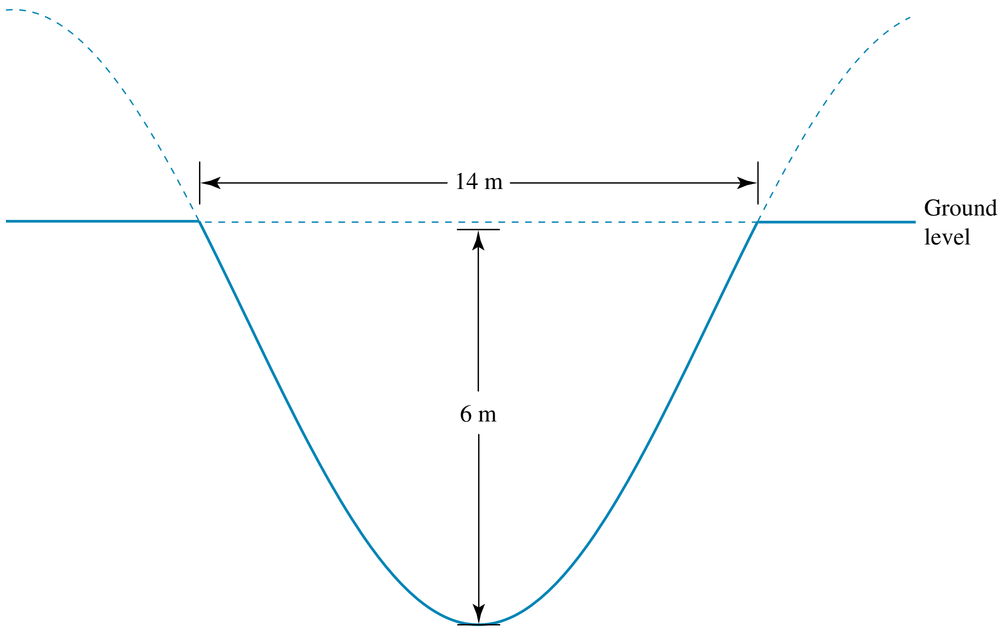
The first engineer proposes a canal with straight sloping sides of length 5 metres with a curved section at the base. He has sketched his proposal, not drawn to scale, and shown it to his colleagues. He maintains that the curved section would be quadratic, cubic, quartic or exponential in shape.



The second engineer proposes a simple trapezoidal canal, as he estimates it would be the most cost effective. He maintains it could easily be constructed with the same area as any other proposal, and with the same length sloping sides of 5 metres.



The third engineer proposes a canal which is the lower half of a sine function. She has stated that the function would be of the form $y = a\sin(\pi nx) + k$ and has presented her sketch to her colleagues.



Task

You are helping the engineers with their calculations to determine the shape of the irrigation canal that gives the greatest volume.

For the first engineer, determine the equations of the curves in his diagram and the areas of the cross-sections.

Determine the angle the second engineer would find for θ if his diagram is to have the same area as the largest area of the first engineer's proposal. Discuss the method used to find this angle.

The third engineer maintains her canal would be greater in area than both of the others. Determine whether she is, in fact, correct.

In making a final decision on the cross-section for the new canal, the three engineers will check all regulations for irrigation canals. One regulation requires, for safety reasons, that the angle made between the side and the surface of the canal needs to be as small as possible. Discuss, with reasons, which of the cross-sections would give the best result for the proposed new irrigation canal when this regulation is considered.

As an engineering student, you are asked to alter the first engineer's proposal to meet the added requirement that the angle between the surface and the side be as small as possible. Prepare your altered design to present to the engineers with justifications for why it is a reasonable model.

Approach to problem-solving and modelling

Formulate

In this task, you will develop equations of various cross-sections from the given information. Using these equations and your knowledge of calculus, you will solve the given problem. Appropriate mathematical procedures and use of technology will be used to develop your response. Your report should include:

- the various types of functions
- algebraic solutions
- solutions derived using technology.

Remember to state all assumptions and variables along with how you made use of technology.

Solve

Select and apply your knowledge of mathematical functions to determine the various equations needed to solve the problem. You must demonstrate how you obtained the functions.

You must use both technological and algebraic solutions efficiently and show detailed calculations demonstrating the procedures used to formulate your solutions. In your solutions, include all diagrams, clearly labelled.

Is it solved?

Evaluate and verify

Justify and explain all the procedures you have used to come to the decision you have made on the most appropriate cross-section. You must interpret the restrictions placed on the irrigation canal to ensure your solution is valid.

Is the solution verified?

Communicate

Once you have completed all necessary calculations, you should consider how you have communicated the aspects of your report. Your response should have:

- used appropriate mathematical language
- included any graphs or diagrams
- shown all calculations or stated how they were obtained
- drawn conclusions, noting particularly how key concepts were addressed.

PRACTICE ASSESSMENT 2

Mathematical Methods: Unit 3 examination

Unit

Unit 3: Further calculus

Topics

Topic 1: The logarithmic function 2
Topic 2: Further differentiation and applications 2
Topic 3: Integrals

Conditions

Technique	**Response Type**	**Duration**	**Reading**
Paper 1: Technology free Paper 2: Technology active	Short response	Paper 1: 60 minutes Paper 2: 60 minutes	5 minutes
Resources		**Instructions**	
• QCAA formula sheet • Notes not permitted • Paper 1: Calculator not permitted • Paper 2: Non-CAS graphics calculator permitted		• Show all working. • Write responses using a black or blue pen. • Unless otherwise instructed, give answers to **two decimal places**.	

Criterion	**Marks allocated Paper 1**	**Marks allocated Paper 2**	**Result**
Foundational knowledge and problem solving *Assessment objectives 1, 2, 3, 4, 5 and 6	47	53	

For the most up to date assessment information, please see www.qcaa.qld.edu.au/senior.

Paper 1 — Technology free — total marks: 47

Part A: Simple familiar — total marks: 27

Question 1 (1 + 3 = 4 marks)

a. If $y = \left(4x^2 - 3x\right)^5$, determine $\frac{dy}{dx}$.

b. If $f(x) = x\sin(2x)$, evaluate $f'(\pi)$.

Question 2 (3 marks)

Solve $2\cos(2\theta) = -\sqrt{3}$ for $0 \leq \theta \leq 2\pi$.

Question 3 (1 + 3 = 4 marks)

a. Determine $\int(3 + \cos(2x - 1))dx$.

b. $\int_3^5 \frac{2}{(x-2)}dx = \ln(k)$. Determine the value of k.

Question 4 (2 + 3 = 5 marks)

Solve the following for x.

a. $\log_e(4x - 3) = 0$

b. $\log_5(3x) - \log_5(x - 2) = 1$

Question 5 (4 marks)

If $f(x) = \dfrac{2x^2 - 3}{x}$, determine the coordinates of the point(s) at which the gradient is 5.

Question 6 (2 + 2 + 3 = 7 marks)

a. On one set of axes, sketch the graphs of $f: [0, \infty) \to R, f(x) = x^2$ and $g: (0, \infty) \to R, g(x) = \dfrac{1}{x}$.

b. Use algebra to find the coordinates of the point of intersection of the two graphs.

c. Determine the exact area between the two curves from their point of intersection to $x = 2$.

Part B: Complex familiar — total marks: 10

Question 7 (3 marks)

Solve the following for the value of x.

$5^{2x} - 4 = 3(5^x)$

Question 8 (2 + 2 + 1 + 2 = 7 marks)

A particle is attached to a spring that moves up and down in a straight line. The velocity of the particle, v cm/s, after t seconds is given by

$$v = 2\pi \sin\left(\frac{\pi t}{4}\right), t \geq 0.$$

a. Determine the rule relating the position of the particle, x cm, to t if the particle starts from rest at the origin.

b. What is the maximum displacement of the particle?

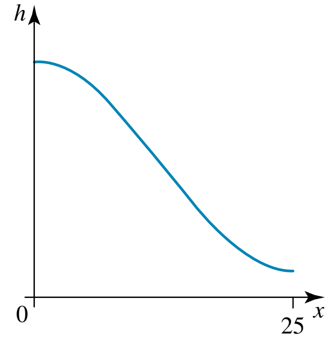
c. Find an expression for the acceleration of the particle at any time t.

d. Determine the particle's displacement and acceleration after 9 seconds? Express your answers in the form $a + b\sqrt{c}$.

Part C: Complex unfamiliar — total marks: 10

Question 9 (2 + 1 + 2 + 2+ 3 = 10 marks)

A section of a rollercoaster track at a local fun park is shown.



The track can be described by the rule

$$h(x) = 4\cos\left(\frac{\pi x}{25}\right) + 5, \quad 0 \le x \le 25$$

where h is the height in metres above ground level and x is the horizontal distance in metres from the top of the descent. Note that the x-axis represents the ground.

a. How high is a rollercoaster car from the ground:

i. at the beginning of its descent?

ii. at the end of the ride?

b. Find $h'(x)$.

c. Sketch the function $y = h'(x)$ for $0 < x < 25$.

d. Determine where the rollercoaster track is steepest, and determine the gradient at this point.

e. The section below the rollercoaster track is to be enclosed on either side for safety reasons. Determine the volume of the enclosure if the track is 2.5 metres wide.

Paper 2 — Technology active — total marks: 53

Part A: Simple familiar — total marks: 33

Question 1 (4 + 2 = 6 marks)

a. Differentiate $\log_e(2x^3 + 4)$ and hence find the value of $\int_{-1}^{1}\left(\frac{x^2}{x^3 + 2}\right)dx$, expressing your answer in the form $k \log_e p$.

b. Given that $\int ae^{bx}dx = -\frac{1}{4}e^{4x} + c$, find the exact values of a and b.

Question 2 (3 + 2 = 5 marks)

Solve the following for x.

a. $\log_6(x - 3) + \log_6(x + 2) = 1$

b. $2^{2x+1} \times 8^{2-x} = 16$

Question 3 (1 + 1 + 1 = 3 marks)

The temperature in °C on a particular winter's day in Toowoomba can be modelled by the function

$$T = 2\sin\left(\frac{\pi t}{9}\right) + 12, 0 \le t \le 24$$

where t is the time in hours after $8{:}00$ am.

a. Calculate the temperature, correct to the nearest degree, at 12 noon.

b. Find $\frac{dT}{dt}$.

c. What is the rate of change of temperature at midnight? Give your answer correct to 3 decimal places.

Question 4 (1 + 1 + 1 + 2 + 1 = 6 marks)

The population of possums in a reserve is starting to increase. The numbers observed currently can be defined by the rule

$$P(t) = 100 - 60e^{-0.1t}$$

where P is the population of possums, t is the time in months since observations began, and $t \ge 0$.
Give all answers to the nearest integer.

a. How many possums were present initially?

b. How many possums were present after:

  i. 1 month?

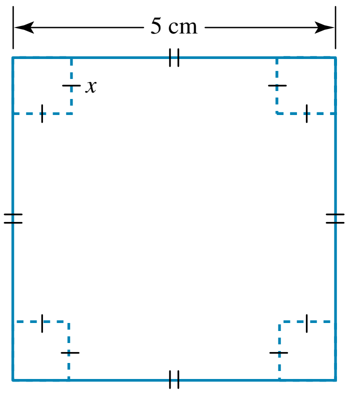
  ii. 1 year?

c. How long does it take for the possum number to double?

d. Sketch the graph of the function representing the possum population for the first year.

Question 5 (1 + 1 + 1 + 2 + 1 = 6 marks)

A square piece of cardboard of side length 5 cm is to be formed into an open-top box by cutting out the corners of length x cm, as shown.



a. Show that the volume, V cm^3, is given by $V = 4x^3 - 20x^2 + 25x$.

b. For which values of x would the function of V be defined?

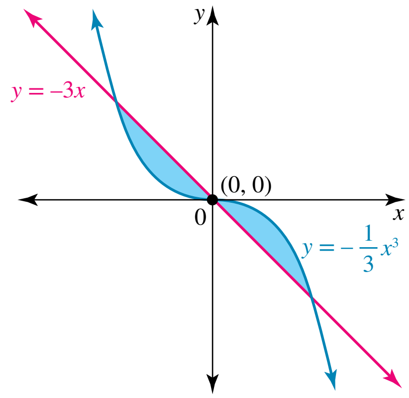
c. State $\frac{dV}{dx}$.

d. Determine the value of x that would give a maximum volume.

e. What is the maximum volume that a box formed from this cardboard could hold? Give your answer correct to 2 decimal places.

Question 6 (3 + 4 = 7 marks)

The graphs of $y = -3x$ and $y = -\frac{1}{3}x^3$ are shown.



a. Determine the coordinates of the points of intersection of the two graphs.

b. Calculate the area of the shaded region shown.

Part B: Complex familiar — total marks: 10

Question 7 (3 + 2 + 1 = 6 marks)

Children are playing on a seesaw. Initially the seesaw is in a horizontal position. The children begin playing, going up and down on the seesaw. The height that a child is above the ground can be modelled by the rule

$$h(t) = a\sin(n\pi t) + k$$

where h is the height in metres a child is above the ground, and t is the time in seconds since the seesaw motion began. The greatest height above the ground that a child can be is 1.5 metres, and the least height is 0.5 metres. It takes 3 seconds for the child to seesaw between the two heights.

a. Find the values of the constants a, n and k.

b. Draw the graph of the function for $0 \leq t \leq 12$.

c. For what length of time during the first 12 seconds is the child 1 or more metres above the ground?

Question 8 (2 + 2 = 4 marks)

The loudness of a sound, S, in decibels (dB), is related to the intensity, I, of the sound by the equation

$$S = 10\log_{10}\left(\frac{I}{10^{-12}}\right)$$

where I is measured in W/m^2.

a. Show that $I = 10^{\frac{S-120}{10}}$.

b. Being in an environment where the intensity of noise is above 0.003 W/m^2 is not recommended for long periods of time, as hearing loss may occur.
If the loudness of a typical rock concert is measured to be 115 dB, is it safe to attend many rock concerts due to the noise level?

Part C: Complex unfamiliar — total marks: 10

Question 9 (1 + 3 + 2 + 1 + 3 = 10 marks)

Consider the graph defined by the function $f(x) = x^2 e^{(1-x)}$.

a. Differentiate to find an expression for $f'(x)$.

b. Hence, determine the coordinates of the stationary point(s).

c. Sketch the function $y = f(x)$, showing clearly all important features, and state the equations of any asymptotes.

d. The point $\mathrm{P}(k, f(k))$ lies on the curve. Determine the gradient of the line joining the point P to the origin, O.

e. Hence, determine the coordinates of any non-stationary point(s) on the curve where the tangent passes through the origin, and determine the equation of this tangent.

Examination marks summary

Question number	Simple familiar	Complex familiar	Complex unfamiliar	Topics
Paper 1 — Technology free				
1	4			2
2	3			2
3	4			1, 3
4	5			1
5	4			2
6	7			3
7		3		1
8		7		2, 3
9			10	2, 3
Paper 2 — Technology active				
1	6			1, 2, 3
2	5			1
3	3			2
4	6			1
5	6			2
6	7			3
7		6		2
8		4		1
9			10	1, 2
Totals	**60**	**20**	**20**	
Percentage	**60%**	**20%**	**20%**	

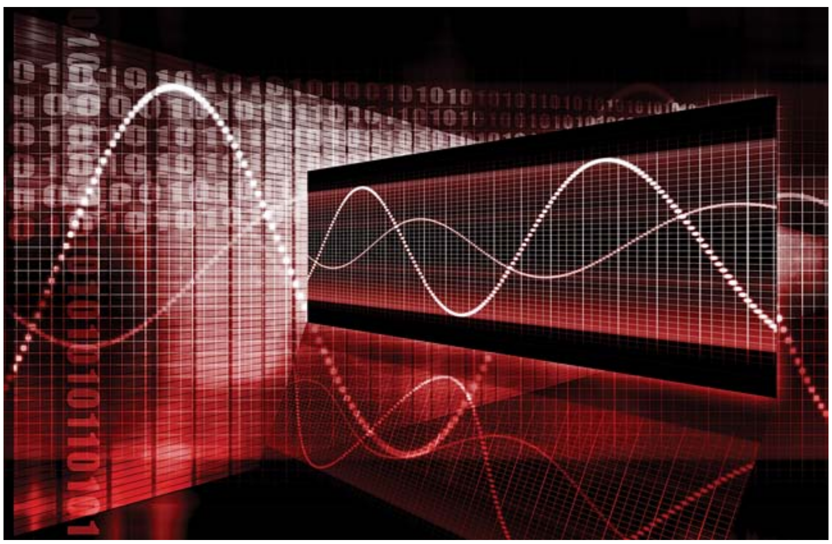
8 The second derivative and applications of differentiation

8.1 Overview

This is the final chapter in the area of differential calculus, the study of how a change in one variable will affect a related variable. You have learned about differentiation and its applications to curve sketching and determining equations of tangents to curves, as well as practical applications. You have also studied antidifferentiation, or integration, along with applications such as determining areas under curves.

In this chapter we will explore the second derivative — the rate of change of the derivative. You will discover that using the second derivative can help in discussing the behaviour of functions. For example, it allows us to answer questions such as: what is the concavity of the function? Is the critical point the largest value of the function?

The second derivative also has many practical applications. In problems of optimisation, the second derivative assists in determining the nature of the critical points. In kinematics, the second derivative of the displacement of a particle with respect to time is the acceleration of the function. Scientists, economists and engineers are all concerned with determining critical values.



LEARNING SEQUENCE

8.1 Overview
8.2 Second derivatives
8.3 Concavity and points of inflection
8.4 Curve sketching
8.5 Applications of the second derivative
8.6 Review: exam practice

Fully worked solutions for this chapter are available in the Resources section of your eBookPLUS at www.jacplus.com.au.

8.2 Second derivatives

From previous chapters, you are familiar with the relationships between displacement, velocity and acceleration as functions of time. If $x = f(t)$ is the displacement of a particle from the origin at time t, then:

- velocity, the rate of change of displacement with respect to time, is $v = \frac{dx}{dt} = f'(t)$
- acceleration, the rate of change of velocity with respect to time, is $a = \frac{dv}{dt} = \frac{d}{dt}\left(\frac{dx}{dt}\right)$.

  This is, in fact, the 'derivative of the derivative' with respect to time.

If $y = f(x)$ is the equation of the curve, then:

- the first derivative, $\frac{dy}{dx} = f'(x)$, is the rate of change of y with respect to x, or in other words, the gradient of the curve
- the rate of change of the derivative with respect to x is the **second derivative**.

Notation for the second derivative

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = f''(x)$$

When we differentiate to determine either the first of second derivatives, we may need to use the rules for differentiation, such as the product or quotient rules.

WORKED EXAMPLE 1

Determine $\frac{d^2y}{dx^2}$ if $y = x^4 + 2x^3 - 4x^2 + 5$.

THINK	WRITE
1. Write the equation.	$y = x^4 + 2x^3 - 4x^2 + 5$
2. Differentiate to determine the first derivative and simplify.	$\frac{dy}{dx} = 4 \times x^3 + 2 \times 3x^2 - 4 \times 2x$ $\frac{dy}{dx} = 4x^3 + 6x^2 - 8x$
3. Differentiate to determine the second derivative and simplify.	$\frac{d^2y}{dx^2} = 4 \times 3x^2 + 6 \times 2x - 8$ $\frac{d^2y}{dx^2} = 12x^2 + 12x - 8$

WORKED EXAMPLE 2

If $f(x) = \frac{4\sqrt{x^5}}{3x^2}$, calculate $f''(9)$.

THINK	WRITE
1. Express the function in simplified form using index laws.	$f(x) = \dfrac{4\sqrt{x^5}}{3x^2}$ $= \dfrac{4x^{\frac{5}{2}}}{3x^2}$ $= \dfrac{4}{3}x^{\frac{5}{2}-2}$ $= \dfrac{4}{3}x^{\frac{1}{2}}$
2. Determine the first derivative, using the basic laws for differentiation, and simplify.	$f'(x) = \dfrac{4}{3} \times \dfrac{1}{2}x^{-\frac{1}{2}}$ $= \dfrac{2}{3}x^{-\frac{1}{2}}$
3. Determine the second derivative, using the basic laws for differentiation, by differentiating the first derivative again; then simplify.	$f''(x) = \dfrac{2}{3} \times \dfrac{-1}{2}x^{-\frac{3}{2}}$ $= \dfrac{-1}{3\sqrt{x^3}}$
4. Substitute in the value for x.	$f''(9) = \dfrac{-1}{3\sqrt{9^3}}$ $= \dfrac{-1}{3 \times 27}$
5. State the final result.	$f''(9) = -\dfrac{1}{81}$

WORKED EXAMPLE 3

Determine the second derivative, $\dfrac{d^2y}{dx^2}$, of $y = x^2 \ln(3x+5)$.

THINK	WRITE
1. Write the equation.	$y = x^2 \ln(3x+5)$
2. Use the product rule and simplify. If $y = uv$, then $y' = uv' + vu'$.	$\dfrac{dy}{dx} = \ln(3x+5) \times 2x + x^2 \times \dfrac{1}{(3x+5)} \times 3$ $= 2x \ln(3x+5) + \dfrac{3x^2}{(3x+5)}$
3. Differentiate with respect to x again, using the product rule for the first term and the quotient rule for the second term of $\dfrac{dy}{dx}$.	$\dfrac{d^2y}{dx^2} = \left\{\ln(3x+5) \times 2 + 2x \times \dfrac{1}{(3x+5)} \times 3\right\}$ $+ \dfrac{(3x+5) \times 6x - 3x^2(3)}{(3x+5)^2}$
4. Simplify.	$\dfrac{d^2y}{dx^2} = 2\ln(3x+5) + \dfrac{6x}{(3x+5)} + \dfrac{(9x^2+30x)}{(3x+5)^2}$

Note: Sometimes you may be able to combine terms with common denominators, but this is not necessary unless you are asked to express your answer in a particular form.

on Resources

Interactivity: Stationary points (int-5963)

study on

Units 3 & 4 > Area 4 > Sequence 1 > Concept 1 > **Second derivatives** Summary screen and practice questions

Exercise 8.2 Second derivatives

Technology free

1. **WE1** Determine $\frac{d^2y}{dx^2}$ for each of the following functions.
   a. $y = x^4 - 5x^3 + x^2 - 9$
   b. $y = x^3 - 4x^2$
   c. $y = 4 - x^2$
   d. $y = x^2(8 - x)$
   e. $y = (2x - 1)^4$
2. Determine the second derivatives of the following.
   a. $x\sqrt{x}$
   b. $\frac{1}{x^2}$
   c. $4e^{2x+3}$
   d. $\cos\left(\frac{2x}{5}\right)$
   e. $3\sin(4x - \pi)$
3. Determine $f''(x)$ if $f(x)$ is given by:
   a. $x\ln(x)$
   b. e^{3x^2}
   c. $\ln(x + 1)$
4. **WE2** If $f(x) = \frac{8\sqrt{x^3}}{3x}$, calculate $f''(4)$.
5. If $f(x) = 8\cos\left(\frac{x}{2}\right)$, calculate $f''\left(\frac{\pi}{3}\right)$.
6. a. If $f(x) = \frac{4x^2}{3\sqrt{x}}$, calculate $f''(4)$.
   b. If $f(x) = \frac{2}{3x - 5}$, calculate $f''(1)$.
7. a. If $f(x) = 4\log_e(2x - 3)$, calculate $f''(3)$.
   b. If $f(x) = e^{x^2}$, calculate $f''(1)$.
8. **WE3** Determine $\frac{d^2y}{dx^2}$ for $y = x^3\log_e\left(2x^2 + 5\right)$.
9. Determine $\frac{d^2y}{dx^2}$ for $y = \frac{x^4}{e^{3x}}$.
10. Determine $\frac{d^2y}{dx^2}$ if:
    a. $y = \log_e\left(x^2 + 4x + 13\right)$
    b. $e^{3x}\cos(4x)$
11. Determine $\frac{d^2y}{dx^2}$ if:
    a. $y = x^3e^{-2x}$
    b. $x^2\cos(3x)$

Technology active

12. Consider the function $f(x) = e^{\sin(x)}$.
    a. Show that the gradient of the function at $x = \pi$ is -1.
    b. Determine $f''(\pi)$.

13. Determine the values of a and b if it is known that, for all x, the second derivative of the function $f(x) = 2\sin(3x) + 4\cos(2x)$ is given by $f''(x) = a\sin(3x) + b\cos(2x)$.

14. Consider the function $y = e^x \sin(x)$.
    a. Show that the function has a stationary point at $x = \frac{3\pi}{4}$.
    b. Evaluate $\frac{d^2y}{dx^2}$ when $x = \frac{3\pi}{4}$. Give your answer correct to 2 decimal places.

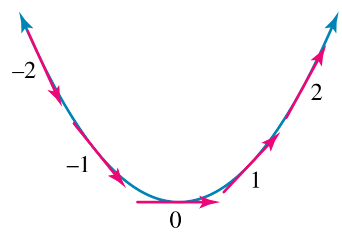
15. The displacement, x metres, of a particle at any time, t seconds, is given by the equation $x = 6\sin\left(\frac{\pi}{4}(2t-1)\right)$.
    a. Where is the particle initially?
    b. When is the particle first at rest?
    c. Calculate the acceleration of the particle at 3.5 seconds.

8.3 Concavity and points of inflection

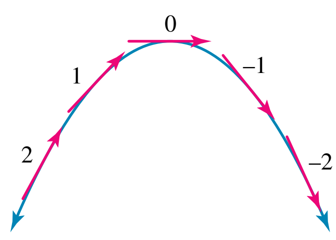
8.3.1 Concavity

The shape of continuous functions is often described in terms of its **concavity**. They are said to be either **concave up** (sometimes referred to as convex) or **concave down**.

A function is concave up when the gradient of the function is increasing, so the rate of change of the gradient is positive.



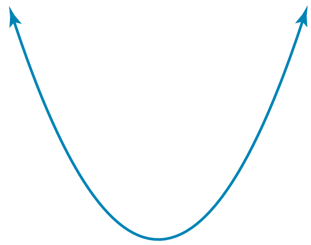
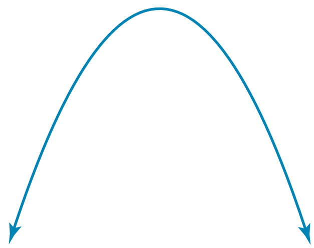
A function is concave down when the gradient of the function is decreasing, so the rate of change of the gradient is negative.



Remember, the rate of change of the gradient with respect to x is the second derivative.

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = f''(x)$$

Concavity may be summarised as follows:

Shape	First derivative	Second derivative
Concave up (or convex) 	$\frac{dy}{dx}$ is increasing	$\frac{d^2y}{dx^2} > 0$
Concave down 	$\frac{dy}{dx}$ is decreasing	$\frac{d^2y}{dx^2} < 0$

8.3.2 Points of inflection

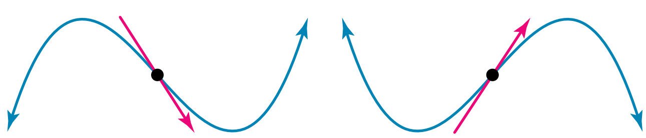
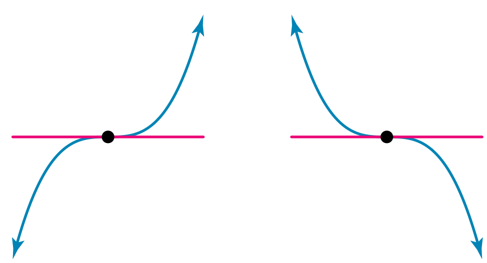
On a continuous curve, the point where the concavity changes is called a **point of inflection**. At this point, the curve changes from being concave up to concave down or vice versa.

The rate of change of the gradient is 0 at a point of inflection, as it has either stopped increasing and started to decrease or vice versa.

For a point of inflection, the following must be true:

$\frac{d^2y}{dx^2} = 0$ **AND** the sign changes either side to show the change in concavity.

A point of inflection occurs when there is a change in concavity, giving two possible shapes.

Points of inflection	Horizontal (or stationary) points of inflection
	

The tangent to the curve at the point of inflection crosses the curve at this point. (In contrast, the tangent to the curve at a maximum or minimum does not cross the curve at that point.)

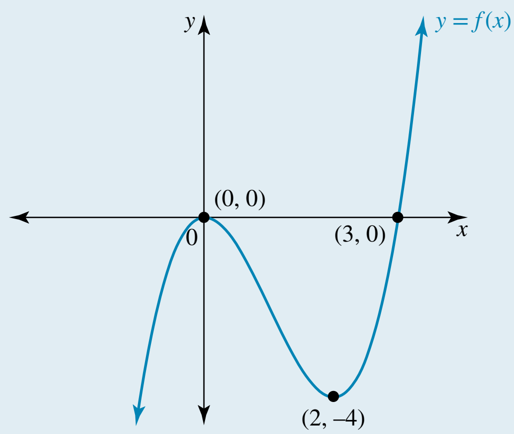
WORKED EXAMPLE 4

a. **Describe the shape of the curve $y = x^3 - 3x^2$ at the point where:**

i. $x = 2$ **ii.** $x = -2$

b. **Determine the coordinates of the point of inflection.**

THINK	WRITE
a. Differentiate twice to determine the second derivative.	$y = x^3 - 3x^2$ $\frac{dy}{dx} = 3x^2 - 6x$ $\frac{d^2y}{dx^2} = 6x - 6$
i. Substitute $x = 2$.	**i.** When $x = 2$: $\frac{d^2y}{dx^2} = 12 - 6$ $= 6$ which is positive The curve is concave up at $x = 2$.
ii. Substitute $x = -2$.	**ii.** When $x = -2$: $\frac{d^2y}{dx^2} = -12 - 6$ $= -18$ which is negative The curve is concave down at $x = -2$.



THINK	WRITE
b **1.** Solve $\frac{d^2y}{dx^2} = 0$.	$\frac{d^2y}{dx^2} = 6x - 6$ $6x - 6 = 0$ $x = 1$
2. Check either side for change in concavity.	see table below

x	1^-	1	1^+
$\frac{d^2y}{dx^2}$	< 0	$= 0$	> 0

The concavity has changed either side of $x = 1$, so the point of inflection is $(1, 2)$.

WORKED EXAMPLE 5

Consider the function $f(x) = 12x^2 - x^3$.

a. Determine where the function is:

i. concave up

ii. concave down.

b. Hence, state the coordinates of the point of inflection.

THINK	WRITE
a. i. 1. For concavity, determine the second derivative.	$f(x) = 12x^2 - x^3$ $f'(x) = 24x - 3x^2$ $f''(x) = 24 - 6x$
2. The function is concave up when $f''(x) > 0$. Solve for x.	$24 - 6x > 0$ $24 > 6x$ $x < 4$ The function is concave up for $x < 4$.
ii. The function is concave down when $f''(x) < 0$. Solve for x.	$24 - 6x < 0$ $24 < 6x$ $x > 4$ The function is concave down for $x > 4$.
b. The point of inflection is where concavity changes. State the point at $x = 4$.	At $x = 4$: $f(4) = 12 \times 16 - 64$ $= 128$ The point of inflection is (4, 128).

WORKED EXAMPLE 6

By investigating the concavity of the curve $y = x^4$, explain why the curve does not have a point of inflection. A sketch of the curve may be useful.

THINK	WRITE
1. For concavity, determine the second derivative.	$y = x^4$ $\frac{dy}{dx} = 4x^3$ $\frac{d^2y}{dx^2} = 12x^2$
2. The point of inflection exists when $\frac{d^2y}{dx^2} = 0$ and changes sign.	$12x^2 = 0$ $x = 0$

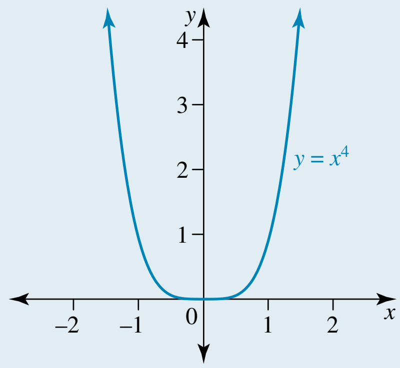
x	0^-	0	0^+
$\frac{d^2y}{dx^2}$	> 0	$= 0$	> 0

3. State a reason for your decision.

The second derivative does not change sign either side of $x = 0$. Therefore, $x = 0$ is not a point of inflection. $\frac{d^2y}{dx^2} \geq 0$ for all x, so the curve is always concave up.

4. Sketch the curve.

A sketch of $y = x^4$ is shown, demonstrating that it is always concave up.



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Units 3 & 4 > Area 4 > Sequence 1 > Concept 2

Concavity and points of inflection Summary screen and practice questions

Exercise 8.3 Concavity and points of inflection

Technology free

1. **WE4** a. Describe the shape of the curve $y = x^3 - 9x^2 + 8$ at the point where:
   i. $x = 4$ ii. $x = -4$
   b. Determine the coordinates of the point of inflection.
2. a. Describe the shape of the curve $y = x^3 + 6x^2$ at the point where:
   i. $x = -3$ ii. $x = 3$
   b. Determine the coordinates of the point of inflection.
3. a. Describe the shape of the curve $y = 4x^2 - x^3$ at the point where:
   i. $x = 0$ ii. $x = 1$
   b. Determine the coordinates of the point of inflection.
4. **WE5** Consider the function $f(x) = x^3 + 9x^2$.
   a. Determine where the function is:
   i. concave up ii. concave down.
   b. Hence, state the coordinates of the point of inflection.
5. a. For the function $y = x^3 + 2x^2 - 3x + 1$, determine where the function is:
   i. concave up ii. concave down.
   b. Hence, state the coordinates of the point of inflection.
6. **WE6** By investigating the concavity of the curve $y = 6 - x^4$, explain why the curve does not have a point of inflection. A sketch of the curve may be useful.
7. By investigating the concavity of the curve $y = 2x^6 - 4$, explain why the curve does not have a point of inflection. A sketch of the curve may be useful.

8. Determine the coordinates of the point(s) of inflection for the following functions.

a. $y = x^3 - 3x^2 - 9x + 5$

b. $y = -x^3 + 9x^2 - 15x - 20$

9. Consider the function $f(x) = x^4 + 4x^3 - 16x + 3$.

a. Determine where the function is:

i. concave up

ii. concave down.

b. State, with reasons, the coordinates of the point(s) of inflection of the function.

10. Determine the point(s) of inflection of the function $f(x) = \frac{1}{2}x^2 - 3x^4$. Hence, state where the function is concave down.

Technology active

11. a. Consider the curve $y = (2x - 3)^3 + 4$.

i. Determine $\frac{d^2y}{dx^2}$.

ii. Solve $\frac{d^2y}{dx^2} = 0$.

iii. Determine the coordinates of any point of inflection.

b. Consider the curve $y = (2x - 3)^4 + 4$.

i. Determine $\frac{d^2y}{dx^2}$.

ii. Solve $\frac{d^2y}{dx^2} = 0$.

iii. Determine the coordinates of any point of inflection.

c. Discuss the similarities and differences between the curves in parts **a** and **b** in relation to any point of inflection. A sketch, using technology, of the curves may be of assistance.

12. Determine the value of k if the function $f(x) = 2x^3 - kx^2 + 3x$ has a point of inflection when $x = 3$.

13. Consider the function $f(x) = x^4 + kx^3$.

a. Determine the value for k if the function f has a point of inflection at $x = 1$.

b. Hence, determine the interval where the function is concave up.

14. Show that the function $f(x) = x\log_e(x), x > 0$ is always concave up.

15. a. Sketch the graph of $y = 2\sin(x) + 3, x \in [0, 2\pi]$.

b. Determine where the function is:

i. concave up

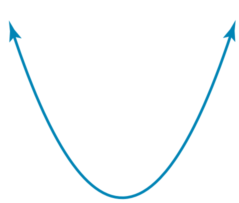
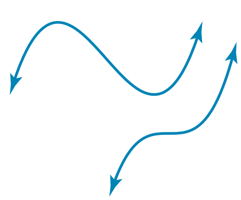
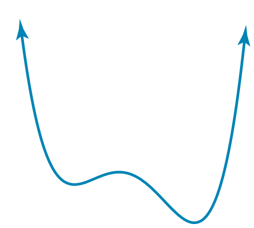
ii. concave down.

c. Hence, state the coordinates of the point(s) of inflection.

8.4 Curve sketching

8.4.1 Reviewing polynomial shapes

You will be familiar with the general shapes of polynomials. These are summarised in the following table. Remember, these graphs may also be inverted.

Quadratics (degree 2)	Cubics (degree 3)	Quartics (degree 4)
		

Knowledge about the first and second derivatives of a function will allow us to make important observations about the function and help us sketch the graph more accurately.

8.4.2 Stationary points, points of inflection and derivatives

A stationary point on a curve is defined as a point where the gradient is 0; that is, where $\frac{dy}{dx} = f'(x) = 0$.

You will be familiar with three types of stationary points:

- relative maximum turning points
- relative minimum turning points
- horizontal (or stationary) points of inflection.

The word 'relative' means that the point is a maximum or a minimum in a particular locality or neighbourhood. Beyond this section of the graph, there could be other points on the graph that are higher than the relative maximum or lower than the relative minimum.

Previously, you have determined the nature of the stationary points by calculating the slope of the tangent, $\frac{dy}{dx}$ or $f'(x)$, on either side of the point. This method is still useful. However, the nature of the turning points can also be determined by considering concavity:

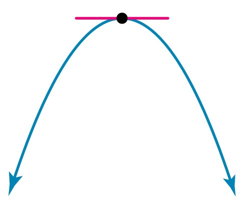
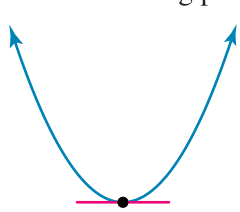
- At a maximum turning point, the curve is concave down.
- At a minimum turning point, the curve is concave up.

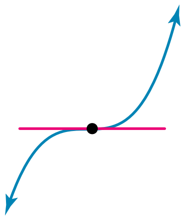
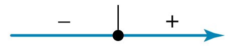
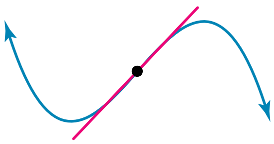
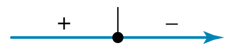
Care needs to be taken to determine the nature of a horizontal (or stationary) point of inflection. At a stationary point of inflection:

- the first derivative equals 0
- the second derivative equals 0, and the sign of the second derivative changes either side of the point, since there is a change in concavity.

Non-stationary points of inflection occur when the first derivative is not 0, and the second derivative equals 0 and changes sign. Determining points of inflection is discussed in the previous section of this chapter.

Turning points and points of inflection are summarised in the following table.

Shape	**First derivative**	**Second derivative**
Maximum turning point 	$\frac{dy}{dx} = 0$	$\frac{d^2y}{dx^2} < 0$
Minimum turning point 	$\frac{dy}{dx} = 0$	$\frac{d^2y}{dx^2} > 0$

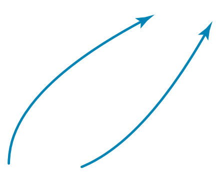
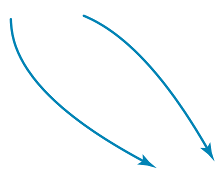
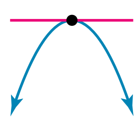
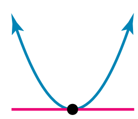
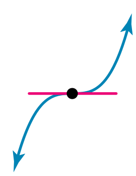
Shape	First derivative	Second derivative
Stationary point of inflection (tangent is horizontal) 	$\frac{dy}{dx} = 0$	$\frac{d^2y}{dx^2} = 0$ and changes sign either side 
Non-stationary point of inflection (non horizontal tangent) 	$\frac{dy}{dx} \neq 0$	$\frac{d^2y}{dx^2} = 0$ and changes sign either side 

8.4.3 Curve sketching

Using the first and second derivatives of a function allows curves to be sketched with greater accuracy.

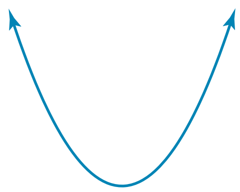
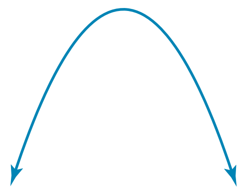
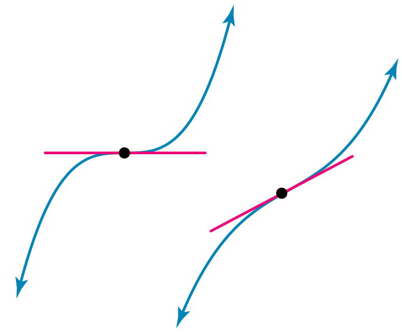
The first derivative of a function gives the gradient at any point:

- If $\frac{dy}{dx} > 0$, the function is increasing.
- If $\frac{dy}{dx} < 0$, the function is decreasing.
- If $\frac{dy}{dx} = 0$, the function has a stationary point.

First derivative		
Increasing function	**Decreasing function**	**Stationary point**
		  

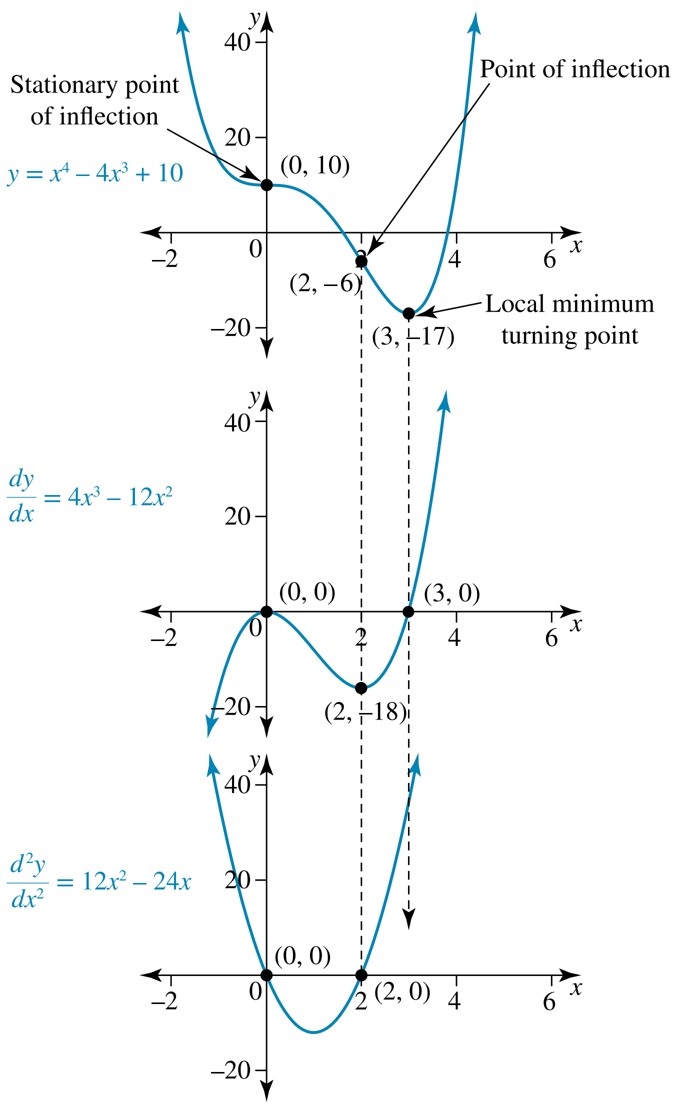
The second derivative of a function gives its concavity:

- If $\frac{d^2y}{dx^2} > 0$, the function is concave up.
- If $\frac{d^2y}{dx^2} < 0$, the function is concave down.
- If $\frac{d^2y}{dx^2} = 0$ and changes sign, there is a point of inflection.

Second derivative		
Concave up	**Concave down**	**Point of inflection**
		

To illustrate the relationships between a function and its derivatives, the following functions are shown.

$$y = x^4 - 4x^3 + 10 \qquad \frac{dy}{dx} = 4x^3 - 12x^2 \qquad \frac{d^2y}{dx^2} = 12x^2 - 24x$$



When sketching the graph of a function, $f(x)$, you may need to consider the following:

- Determine the y-intercept by evaluating $f(0)$.
- Determine the x-intercept(s) by solving $f(x) = 0$, if possible.
- Determine the coordinates of the stationary point(s) and their nature.
- Determine the coordinates of any point(s) of inflection.
- Consider restrictions on the domain.
- Calculate the coordinates of the end points of the domain, where appropriate.
- Identify vertical and horizontal asymptotes, where appropriate.
- Consider the direction of $f(x)$ as $x \to \pm\infty$.

WORKED EXAMPLE 7

Sketch the graph of the function $f: R \to R,\ f(x) = x^3 + 6x^2 + 9x$ by determining the coordinates of all axis intercepts as well as any stationary points and their nature. Include on your sketch the coordinates of any point(s) of inflection.

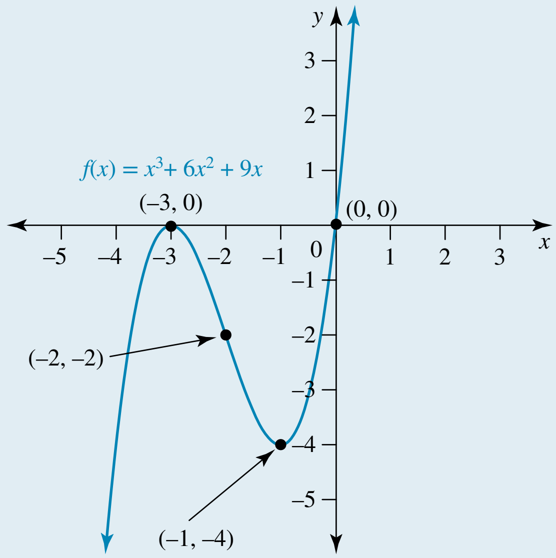
THINK	WRITE
1. State the function and differentiate to determine the first and second derivatives.	$f(x) = x^3 + 6x^2 + 9x$ $f'(x) = 3x^2 + 12x + 9$ $f''(x) = 6x + 12$
2. For x-axis intercepts: • factorise the function • solve for $f(x) = 0$.	$f(x) = x^3 + 6x^2 + 9x$ $= x(x^2 + 6x + 9)$ $x(x + 3)(x + 3) = 0$ $x(x + 3)^2 = 0$ The x-intercepts are $(0, 0)$ and $(-3, 0)$.
3. For stationary points, $f'(x) = 0$.	$f'(x) = 3x^2 + 12x + 9$ $= 3\left(x^2 + 4x + 3\right)$ $3(x + 3)(x + 1) = 0$ $x = -3$ or $x = -1$
4. Determine the nature of the stationary points using the second derivative and determine the corresponding values of $f(x)$.	When $x = -3$: $f''(-3) = -18 + 12 = -6 < 0$, so concave down $f(-3) = 0$ The point $(-3, 0)$ is a maximum turning point. When $x = -1$: $f''(-1) = -6 + 12 = 6 > 0$, so concave up $f(-1) = -1 + 6 - 9 = -4$ The point $(-1, -4)$ is a minimum turning point.
5. For points of inflection, $f''(x) = 0$ and changes sign either side of that value.	$f''(x) = 6x + 12$ $6x + 12 = 0$ $x = -2$ Check for change of sign either side of $x = -2$.

x	-2^-	-2	-2^+
$\dfrac{d^2y}{dx^2}$	< 0	0	> 0

6. Determine $f(x)$ for the point and make a statement.

$f(-2) = -8 + 24 - 18 = -2$
The second derivative has changed sign, so there is a point of inflection at $(-2, -2)$.

7. Sketch the curve, showing all important features.



WORKED EXAMPLE 8

a. Sketch the graph of $y = x^4 - 4x^3$, stating the coordinates of all axis intercepts, any stationary points and their nature, and any point(s) of inflection.

b. State the values of x where the function is:

i. decreasing **ii. concave up.**

THINK

a. 1. State the function and differentiate to determine the first and second derivatives.

2. For x-axis intercepts:
   - factorise the function
   - solve for $y = 0$.

3. For stationary points, solve for $\frac{dy}{dx} = 0$

WRITE

a. $y = x^4 - 4x^3$

$\frac{dy}{dx} = 4x^3 - 12x^2$

$\frac{d^2y}{dx^2} = 12x^2 - 24x$

$y = x^4 - 4x^3$

$x^3(x - 4) = 0$

The x-intercepts are $(0, 0)$ and $(4, 0)$.

$\frac{dy}{dx} = 4x^3 - 12x^2$

$4x^2(x - 3) = 0$

$x = 0, 3$

4. Determine their nature using the second derivative and determine the corresponding y-value.
Remember: If the second derivative is 0, you need to check either side by taking a value of x close to the point. (For example, either side of $x = 0$, take $x = \pm 0.1$)

When $x = 0$:

$$\frac{d^2y}{dx^2} = 12 \times 0 - 24 \times 0 = 0$$

Possible point of inflection; check for change of sign either side of $x = 0$.

x	0^-	0	0^+
$\frac{d^2y}{dx^2}$	> 0	0	< 0

The second derivative has changed sign from concave up to concave down, so there is a horizontal (or stationary) point of inflection at (0, 0).

When $x = 3$:

$$\frac{d^2y}{dx^2} = 12 \times 9 - 24 \times 3 = 36 > 0, \text{ so concave up}$$

$$y = 3^4 - 4 \times 3^3 = -27$$

The point (3, −27) is a minimum turning point.

5. For points of inflection: $\frac{d^2y}{dx^2} = 0$ and changes sign.

$$\frac{d^2y}{dx^2} = 12x^2 - 24x$$

$$12x(x - 2) = 0$$

$$x = 0, 2$$

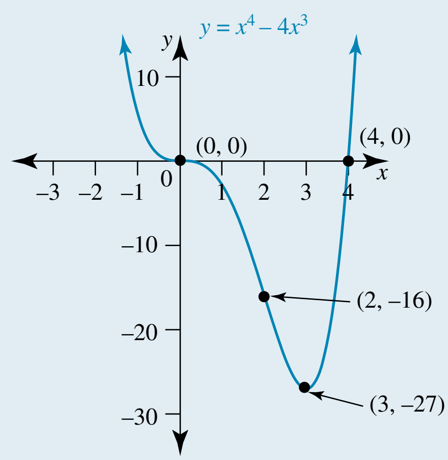
x	2^-	2	2^+
$\frac{d^2y}{dx^2}$	< 0	0	> 0

6. Determine the y-value for the point

$y = 2^4 - 4 \times 2^3 = -16$

The concavity has changed, so there is a point of inflection at (2, −16).

7. Sketch the curve, showing all of the critical points.



b i. For a decreasing function, $\frac{dy}{dx} < 0$.
Read from the graph. Remember the gradient cannot equal 0.

b. i. The function is decreasing when $x < 0$ or $0 < x < 3$.

ii. For concave up, $\frac{d^2y}{dx^2} > 0$.

Read from the graph, noting the points of inflection already found.

ii. There are points of inflection at $(0, 0)$ and $(2, -16)$. The function is concave up when $x < 0$ or $x > 2$.

WORKED EXAMPLE 9

Sketch the graph of $f(x) = 2x\,e^x$, $x \leq 1$ showing the important features, including stationary points and points of inflection. Give your answers correct to 3 decimal places where necessary.

THINK

1. State the function and differentiate using the appropriate rules to determine the first and second derivatives.

WRITE

$f(x) = 2xe^x$

$f'(x) = 2x \times e^x \times 1 + e^x \times 2$

$= 2xe^x + 2e^x$

$f''(x) = (2x \times e^x \times 1 + e^x \times 2) + 2e^x \times 1$

$= 2xe^x + 2e^x + 2e^x$

$= 2xe^x + 4e^x$

2. For x-axis intercepts, solve for $f(x) = 0$.

$f(x) = 2x\,e^x$

$2x\,e^x = 0$

$x = 0$

The x-intercept is $(0, 0)$.

3. For stationary points, solve for $f'(x) = 0$.

$f'(x) = 2xe^x + 2e^x$

$2e^x(x + 1) = 0$

$x = -1$

4. Determine the nature of the stationary points using the second derivative and determine the corresponding value of $f(x)$.

When $x = -1$:

$f''(-1) = -2e^{-1} + 4e^{-1} = 2e^{-1} > 0$, so concave up

$f(-1) = -2\,e^{-1} = -\frac{2}{e}$

The point $\left(-1, -\frac{2}{e}\right) \approx (-1, -0.736)$ is a minimum turning point

5. For points of inflection, $f''(x) = 0$ and changes sign on either side.

$f''(x) = 2xe^x + 4e^x$

$2e^x(x + 2) = 0$

$x = -2$

6. Determine $f(x)$ for the point and make a statement.

x	-2^-	-2	-2^+
$\frac{d^2y}{dx^2}$	< 0	0	> 0

$f(-2) = -4\,e^{-2} = -\frac{4}{e^2}$

The concavity has changed, so the point $\left(-2, -\frac{4}{e^2}\right) \approx (-2, -0.541)$ is a point of inflection.

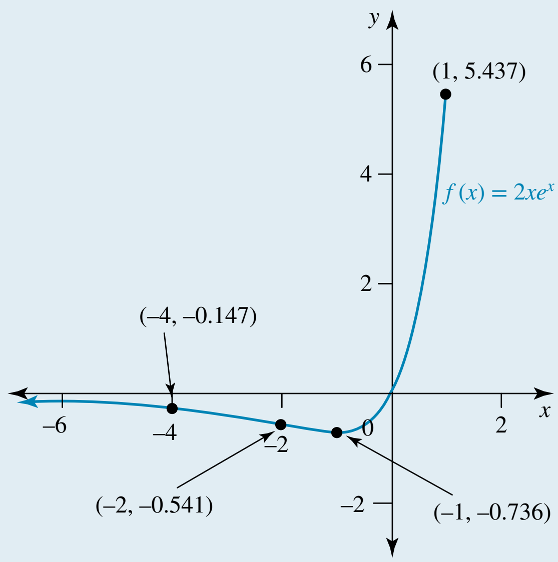
7. Determine the coordinates of the end point of the restricted domain.

$f(1) = 2e$
The end point is $(1, 2e) \approx (1, 5.437)$ correct to 3 decimal places

8. Consider the behaviour of the graph to the left, as x becomes very small.

As $x \to -\infty, e^x \to 0$
$\therefore$ as $x \to -\infty, xe^x \to 0$
The function approaches the x-axis, which will be a horizontal asymptote on the left side of the graph.

9. Sketch the curve, showing all important features.
Include a closed circle at the end point.



on Resources

Interactivities: Graph of a derivative function (int-5961)
Equations of tangents (int-5962)

studyon

Units 3 & 4 | Area 4 | Sequence 1 | Concept 3 | **Curve sketching** Summary screen and practice questions

Exercise 8.4 Curve sketching

Technology free

1. WE7 Sketch the graph of the function $f: R \to R, f(x) = x^3 - 4x^2 + 4x$ by determining the coordinates of all axis intercepts as well as any stationary points and their nature. Include on your sketch the coordinates of any point(s) of inflection.
2. Sketch the graphs of each of the following by determining the coordinates of all axis intercepts and any stationary points, and establishing their nature. Also determine the coordinates of the points of inflection.
   a. $y = x^3 - 27x$
   b. $y = 9x - x^3$
3. Sketch the graphs of each of the following by determining the coordinates of all axis intercepts and any stationary points, and establishing their nature. Also determine the coordinates of the points of inflection. State where the functions are:
   i. increasing
   ii. concave up.
   a. $y = x^3 + 12x^2 + 36x$
   b. $y = -x^3 + 10x^2 - 25x$

4. Sketch the graphs of each of the following by determining the coordinates of all axis intercepts and any stationary points, and establishing their nature. Also determine the coordinates of the points of inflection.
   a. $y = x^3 - 3x^2 - 9x - 5$ b. $y = -x^3 + 9x^2 - 15x - 25$

Technology active

5. WE8 a. Sketch the graph of $y = 8x^3 - x^4$, stating the coordinates of the axis intercepts, any stationary points and their nature, and any point(s) of inflection.
   b. State the values of x where the function is:
      i. decreasing ii. concave up.
6. a. Sketch the graph of the function $f(x) = x^4 - 8x^2 - 9$, stating the coordinates of the axis intercepts, any stationary points and their nature, and any point(s) of inflection.
   b. State the values of x where $f(x)$ is both increasing and concave up.
7. a. Sketch the graph of the function $f(x) = (x - 1)^3 + 8$, showing all important features.
   b. State the values of x where $f(x)$ is both increasing and concave down.
8. Sketch the graphs of each of the following by determining the coordinates of all axis intercepts and any stationary points, and establishing their nature. Also determine the coordinates of the points of inflection. The functions have been given in both factorised and expanded form for ease of calculations.
   a. $y = x^4 + 4x^3 - 16x - 16 = (x - 2)(x + 2)^3$
   b. $y = x^4 - 6x^2 + 8x - 3 = (x - 1)^3(x + 3)$
9. WE9 Sketch the graph of $f(x) = 3x\,e^{-x}$, showing the important features, including stationary points and points of inflection. Give your answers correct to 3 decimal places where necessary.
10. Sketch the graph of $f(x) = 4 - 10xe^x$, $-4 \le x \le 0$, showing the important features, including stationary points and points of inflection. Give your answers correct to 3 decimal places where necessary.
11. The function $f(x) = x^3 + bx^2 + cx + d$ has a stationary point of inflection at $(1, -2)$. Determine the values of b, c and d.
12. A cubic polynomial, $y = x^3 + bx^2 + cx + d$, crosses the y-axis at $y = 5$ and has a point of inflection at $(1, -21)$. Determine the equation of the tangent at the point of inflection.
13. The function $f(x) = x^3 + bx^2 + cx + d$ crosses the x-axis at $x = 3$ and has a point of inflection at $(2, -4)$. Calculate the values of b, c and d.
14. Consider the function $f(x) = \frac{1}{2}\log_e(x^2 + 1)$.
    a. State the domain of $f(x)$.
    b. Determine the coordinates and nature of the stationary point.
    c. By considering the second derivative, show that the function has two points of inflection. State the coordinates of these points.
    d. Sketch the graph of $f(x)$ for a suitable domain.
15. Consider the function $f(x) = \dfrac{10\log_e(x)}{x}$.
    a. State the domain of $f(x)$.
    b. Determine the coordinates and nature of the stationary point.
    c. By considering the second derivative, show that the function has one point of inflection. State the coordinates of this point.
    d. Determine the coordinates of the x-intercept.
    e. By considering various large values of x, discuss the behaviour of the function as x increases.
    f. Sketch the graph of $f(x)$ for a suitable domain.

8.5 Applications of the second derivative

8.5.1 Acceleration

The relationships between displacement, velocity and acceleration have been discussed in Chapters 5, 6 and 7.

Acceleration is the rate of change of velocity with respect to time, $\frac{dv}{dt}$.

Velocity is the rate of change of displacement with respect to time, $\frac{dx}{dt}$.

Acceleration is, in fact, the second derivative of displacement with to time, $\frac{d^2x}{dt^2}$.

Acceleration

$$a = \frac{dv}{dt} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2}$$

WORKED EXAMPLE 10

An object travels in a straight line so that its displacement from the origin, x m/s, at time t seconds is given by $x(t) = te^{-t}, \ \ t \in [0, 5]$.

a. Determine an expression for:

i. velocity as a function of time

ii. acceleration as a function of time.

b. Calculate the initial speed of the object.

c. When is the object at rest? Determine the position of the object at this time.

d. Determine when the acceleration is positive.

THINK	WRITE
a i. 1. Differentiate displacement with respect to time to determine velocity. Use the product rule.	$x(t) = te^{-t}$ $v(t) = \frac{dx}{dt}$ $= e^{-t} \times 1 + t \times (e^{-t} \times -1)$
2. Simplify.	$= e^{-t} - te^{-t}$ $v(t) = e^{-t}(1 - t)$
ii. 1. Differentiate velocity with respect to time to determine acceleration. Use the product rule.	$a(t) = \frac{dv}{dt} = \frac{d^2x}{dt^2}$ $= (1 - t) \times (e^{-t} \times -1) + e^{-t} \times (-1)$
2. Simplify.	$= -e^{-t} + te^{-t} - e^{-t}$ $a(t) = e^{-t}(t - 2)$
b. Substitute $t = 0$ into velocity equation.	$v(0) = e^0(1 - 0)$ $v = 1$ m/s
c. 1. At rest, $v = 0$.	$v(t) = e^{-t}(1 - t)$ $e^{-t}(1 - t) = 0$

2. Solve for t.	$t = 1$ s
3. For position, calculate displacement when $t = 1$.	$x(t) = t e^{-t}$ $x(1) = e^{-1} = \frac{1}{e} \approx 0.368$
4. Answer the question.	The object is at rest after 1 second and its position is $\frac{1}{e}$ metres or approximately 0.368 metres to the right of the origin.
d. 1. State the expression for acceleration.	$a(t) = e^{-t}(t-2)$
2. Solve for $a(t) > 0$.	$e^{-t}(t-2) > 0$ $t > 2$
3. Answer the question and note the restricted domain for time.	Acceleration is positive when $t \in (2, 5]$ seconds.

8.5.2 Optimisation

In many practical situations, it is necessary to determine the maximum or minimum value of the function that describes it. For example, if you were running your own business, you would always want to minimise the production costs while maximising the profits.

Optimisation was discussed in Chapter 5.

Checking the nature of the stationary point(s) using the second derivative may now be easier and quicker instead of using a sign diagram of the first derivative.

When solving optimisation problems, the following steps may be useful:

- Draw a diagram if possible and label it with as few variables as possible.
- Determine a connection between the variables from the information given. These may include:
  - Pythagoras' theorem
  - measurement formulas
  - trigonometry
  - similar triangles.
- Determine an expression for the quantity to be maximised or minimised in terms of the one variable.
- Differentiate the expression to determine the stationary point(s).
- Check the nature of the stationary point(s) by either substituting into the second derivative or using a sign diagram of the first derivative.
- Check whether the answer is the absolute maximum or minimum by evaluating the end points of the domain.
- Sketch the graph the function to check for realistic values.
- Answer the actual question.

WORKED EXAMPLE 11

The sum of two positive numbers is 10. Determine the numbers if the sum of their squares is a minimum.

THINK	WRITE
1. Define the two positive numbers.	Let the two positive numbers be x and y.
2. Write an expression for the sum, S, of the squares of the two numbers.	$S = x^2 + y^2$ [1]

3. State the relationship between the variables. $x + y = 10$ [2]

4. Express y in terms of x using [2]. $y = 10 - x$

5. Express S in terms of one variable, x. $S = x^2 + (10 - x)^2$

6. Simplify the expression.
$$S = x^2 + 100 - 20x + x^2$$
$$= 2x^2 - 20x + 100$$

7. Differentiate to obtain $\frac{dS}{dx}$. $\frac{dS}{dx} = 4x - 20$

8. For the maximum or minimum, solve $\frac{dS}{dx} = 0$.
$$4x - 20 = 0$$
$$x = 5$$

9. Determine the second derivative to verify the minimum.
$$\frac{d^2S}{dx^2} = 4 > 0$$
$S(x)$ is concave up for all x, so S is minimum at $x = 5$.

10 State the value of y. When $x = 5$, $y = 10 - 5 = 5$.

11. Answer the question. The two numbers that add to 10 and have the minimum sum of their squares are 5 and 5.

WORKED EXAMPLE 12

A cuboid container with a base length twice its width is to be made with 48 m^2 of metal.

a. **Show that the height, h m, is $h = \frac{8}{x} - \frac{2x}{3}$, where x m is the width of the base.**

b. **Express the volume, V m^3, in terms of x.**

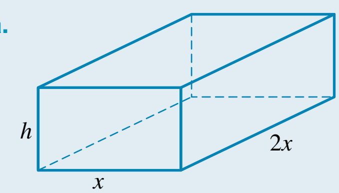
c. **Determine the dimensions of the container with maximum volume.**

d. **Hence, calculate the maximum volume of the container.**

THINK / **WRITE**

a. 1. Draw a diagram of the cuboid.

a.



2. Let x = width of base and hence express length in terms of x.

Let x = width and h = height, so length = $2x$.

3. Calculate the total surface area (TSA) of the cuboid in terms of x and h.

$$\text{TSA} = 2\left[2x(x) + 2x(h) + x(h)\right]$$
$$= 2\left(2x^2 + 3xh\right)$$
$$= 4x^2 + 6xh$$

4. Express h in terms of x.

As TSA $= 48\text{ m}^2$

$$4x^2 + 6xh = 48$$
$$6xh = 48 - 4x^2$$
$$h = \frac{48 - 4x^2}{6x}$$
$$= \frac{48}{6x} - \frac{4x^2}{6x}$$
$$h = \frac{8}{x} - \frac{2x}{3}$$

b. 1. Determine the volume, V, in terms of x and h.

b. $V = x(2x)h$

2. Express the volume in terms of x by substituting for h.

$$V(x) = 2x^2\left(\frac{8}{x} - \frac{2x}{3}\right)$$
$$= 16x - \frac{4x^3}{3}$$

c. 1. Differentiate to obtain $\frac{dV}{dx}$.

c. $$\frac{dV}{dx} = 16 - \frac{4}{3} \times 3x^2$$
$$\frac{dV}{dx} = 16 - 4x^2$$

2. For maximum/minimum values, solve $\frac{dV}{dx} = 0$.

$$16 - 4x^2 = 0$$
$$x^2 = 4$$
$$x = \pm 2$$

Reject $x = -2$, as width cannot be negative.

3. Determine the second derivative to verify the maximum.

$$\frac{d^2V}{dx^2} = -8x$$

When $x = 2$:

$$\frac{d^2V}{dx^2} = -16 < 0$$

$V(x)$ is concave down, so there is a maximum turning point at $x = 2$.

4. Calculate the dimensions by substituting $x = 2$ into expression for h.

$$h = \frac{8}{x} - \frac{2x}{3}$$
$$= \frac{8}{2} - \frac{4}{3}$$
$$= \frac{8}{3}$$

5. State the dimensions of the cuboid.

$x = 2$, $2x = 4$, $h = \frac{8}{3}$

The dimensions of the cuboid with maximum volume are 2 m by 4 m by $\frac{8}{3}$ m.

d. 1. Calculate the volume, $V = lbh$. Alternatively, substitute into the expression for volume.

d. $V = 2 \times 4 \times \frac{8}{3} = \frac{64}{3}$ m^3 or

$$V(x) = 16x - \frac{4}{3}x^3$$
$$V(2) = 16 \times 2 - \frac{4}{3} \times 2^3$$
$$= 32 - \frac{32}{3}$$
$$= \frac{64}{3}$$

2. State the answer.

The maximum volume is $\frac{64}{3}$ m^3.

on Resources

Interactivity: Kinematics (int-5964)

study on

Units 3 & 4 > Area 4 > Sequence 1 > Concept 4

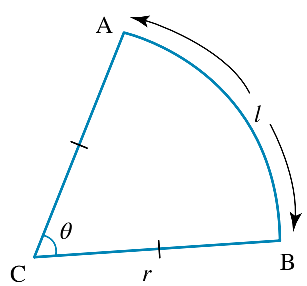
Applications of the second derivative Summary screen and practice questions

Exercise 8.5 Applications of the second derivative

Technology free

1. **WE10** An object travels in a straight line so that its displacement from the origin, x m/s, at time t seconds is given by $x(t) = 8te^{-\frac{t}{2}}, t \in [0, 6]$.
   a. Determine an expression for:
      i. velocity as a function of time
      ii. acceleration as a function of time.
   b. Calculate the initial speed of the object.
   c. When was the object at rest? Determine the position of the object at this time.
   d. Determine when the acceleration is positive.
2. An object travelling in a straight line has its displacement (in metres) after t seconds given by $x(t) = 2\cos(3t - 1) + 3$.
   a. Determine the maximum and minimum displacement.
   b. Determine when the velocity is first equal to 0.
   c. How long after the object is first at rest is it next at rest?
   d. Determine an expression for the acceleration.
3. The velocity of an object which is initially 3 m left of O is given by $v(t) = 3t^2 - 2t - 5$ m/s. Determine:
   a. the displacement from O at any time t
   b. the acceleration at any time t
   c. when the object is at rest
   d. the distance travelled in the first second
   e. the acceleration when the velocity is 0.

4. A particle is travelling in a straight line with its displacement, x metres, at any time, t seconds, given as $x(t) = \dfrac{16}{(t+2)}, t \geq 0$. Determine the acceleration of the particle after 2 seconds.
5. An object initially starts at the origin travelling in a straight line at 15 m/s and speeds up with acceleration, a m/s^2, at any time, t s, given as $a(t) = 12t^2 - 4t + 4, t \geq 0$.
   a. Determine the equation for the velocity of the object with respect to time.
   b. Determine the equation for the position of the object with respect to time.
   c. Calculate the distance travelled in the first 2 seconds.
6. **WE11** The sum of two positive numbers is 32. Determine the numbers if their product is a maximum.
7. The sum of two positive numbers is 8. Determine the numbers if the sum of the cube of one and the square of the other is a minimum.
8. **WE12** The total surface area of a closed cylinder is 200 cm^2. The base radius is r cm and the height is h cm.
   a. Express h in terms of r.
   b. Show that the volume, V cm^3, is $V = 100r - \pi r^3$.
   c. Hence, show that for maximum volume the height must equal the diameter of the base.
   d. Calculate, to the nearest integer, the minimum volume if $2 \leq r \leq 4$.
9. An open rectangular storage bin is to have a volume of 12 m^3. The cost of the materials for its sides is $10 per square metre and the material for the reinforced base costs $25 per square metre. If the dimensions of the base are x and y metres and the bin has a height of 1.5 m, determine, with justification, the cost, to the nearest dollar, of the cheapest bin that can be formed under these conditions.
10. A section of a rose garden is enclosed by edging to form the shape of a sector ABC of radius r metres and arc length l metres. The perimeter of this section of the garden is 8 metres.



   a. If θ is the angle in radian measure subtended by the arc at C, express θ in terms of r.
   b. The formula for the area of a sector is $A_{sector} = \dfrac{1}{2}r^2\theta$. Express the area of a sector in terms of r.
   c. Calculate the value of θ when the area is greatest.
11. A rectangular box with an open top is to be constructed from a rectangular sheet of cardboard measuring 16 cm by 10 cm. The box will be made by cutting equal squares of side length x cm out of the four corners and folding the flaps up.
   a. Express the volume as a function of x.
   b. Determine the dimensions of the box with greatest volume and give this maximum volume.

Technology active

12. A cylinder has a surface area of 220π cm^2. Determine the height and radius of each end of the cylinder so that the volume of the cylinder is maximised, and determine the maximum volume for the cylinder. Give answers correct to 2 decimal places.

13. A colony of blue wrens, also known as superb fairy wrens, survives in a national park because the wooded areas have rich undergrowth and a plentiful supply of insects, the wrens' main food source. Breeding begins in spring and continues until late summer.

The population of the colony any time t months after 1 September can be modelled by the function

$$P(t) = 200te^{-\frac{t}{4}} + 400,\ 0 \leq t \leq 12$$

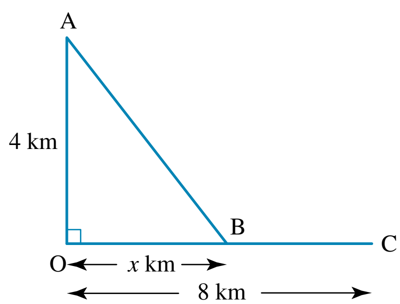
where P is the number of birds in the colony. Determine:

a. the initial population of the birds

b. when the largest number of birds is reached

c. the maximum number of birds, to the nearest bird.

14. A rower is in a boat 4 km from the nearest point, O, on a straight beach. His destination is 8 km along the beach from O. If he is able to row at 5 km/h and walk at 8 km/h, what point on the beach should he row to in order to reach his destination in the least possible time? Give your answer correct to 1 decimal place.



15. A cone is 10 cm high and has a base radius of 8 cm. Determine the radius and height of a cylinder which is inscribed in the cone such that the volume of the cylinder is a maximum. Determine the maximum volume of the cylinder, correct to the nearest cubic centimetre.

8.6 Review: exam practice

A summary of this chapter is available in the Resources section of your eBookPLUS at www.jacplus.com.au.

Simple familiar

1. MC The graph of $y = (x + 2)^3$ has:

A. 1 turning point
B. 2 turning points
C. 1 point of inflection
D. 0 stationary points

2. MC The graph of $x^3 + 2x^2 + x - 2$ has

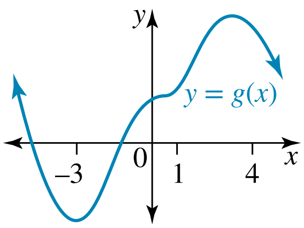
A. 2 points of inflection
B. 1 turning point and 1 point of inflection
C. 3 turning points
D. 2 turning points

3. **MC** The graph of $g(x)$ has the following properties:

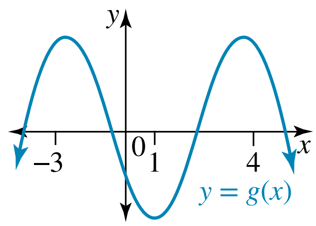
- $g'(x) = 0$ if $x = -3$, 1 and 4
- $g'(x) < 0$ if $x < -3$ and $1 < x < 4$
- $g'(x) > 0$ for all other x.

Which of these diagrams shows the graph of $g(x)$?

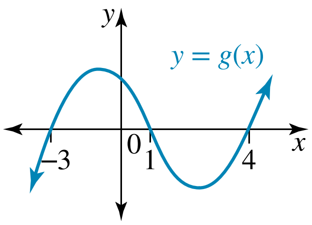
A.



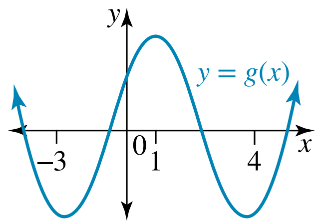
B.



C.

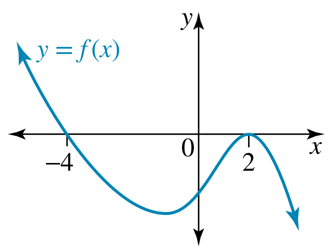


D.



4. **MC** The graph of $f'(x)$ shown indicates that the graph of $f(x)$ has:

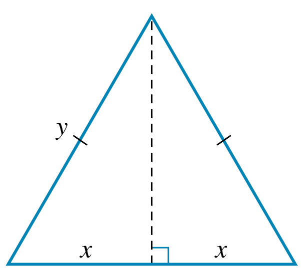
A. a turning point at $x = 2$ and $x = -4$
B. a turning point at $x = 2$ and a point of inflection at $x = -4$
C. a turning point at $x = -4$ and a point of inflection at $x = 2$
D. 2 points of inflection at $x = -4$ and $x = 2$



5. A particle moves in a straight line so that at time t seconds its displacement, x metres, from a fixed origin O is given by $x(t) = t^3 - 6t^2 + 9t, t \geq 0$.

a. How far is the particle from O after 2 seconds?
b. What is the velocity of the particle after 2 seconds?
c. After how many seconds does the particle reach the origin again, and what is its velocity at that time?
d. What is the particle's acceleration when it reaches the origin again?

Questions **6** to **9** relate to the isosceles triangle shown, which has a perimeter of 40 cm.



6. **MC** The value of y in terms of x is:

A. $40 - 2x$
B. $20 - x$
C. $40 - x$
D. $20 - 2x$

7. **MC** The height of the triangle in terms of x is:

A. $\sqrt{400 - 40x}$
B. $20 - \sqrt{40x}$
C. $\sqrt{400 - 40x + 2x^2}$
D. $\sqrt{400 - 40x - x^2}$

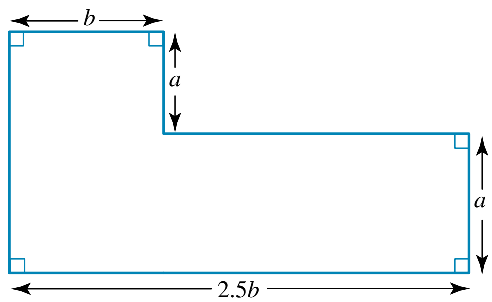
8. **MC** The area in terms of x is:

A. $x\sqrt{400 - 40x + x^2}$
B. $2x\sqrt{400 - 40x + x^2}$
C. $x\sqrt{400 - 40x}$
D. $2x\sqrt{400 - 40x}$

9. **MC** The maximum area of the triangle is obtained if x equals:

A. $6\frac{2}{3}$ cm
B. 10 cm
C. 5 cm
D. $10\frac{2}{5}$ cm

10. A playground is being constructed by the local council. The shape of the playground is shown. All measurements are in metres. The perimeter of the playground is known to be 96 metres.



a. Determine the values of a and b that give a maximum area for the playground.
b. Determine the maximum area.

11. The acceleration of a particle moving in a straight line is given by $\frac{dv}{dt} = 4e^t - 6t + 1$ m/s^2, where v is the velocity at any time. If the particle starts at the origin with a velocity of -1 m/s, determine:
a. the velocity at any time t
b. the displacement, x, from the origin at any time t
c. the displacement from the origin after 1 second.

12. The position of a particle, x metres, from the origin at time t seconds is given by
$x(t) = \frac{1}{4}e^{2t} - 4t^2 - 3t + 10.$
a. Determine the initial position of the particle.
b. Calculate, correct to 2 decimal places, the velocity of the particle after 2 seconds.
c. Derive an expression for the acceleration at any time.
d. Determine when the acceleration of the particle is negative.

Complex familiar

13. The displacement of an object, x metres, from a fixed point at any time, t hours, is given by
$x(t) = 2\cos\left(\frac{\pi t}{12}\right) + 10.$
a. Determine the initial position of the particle.
b. Calculate the velocity of the particle after 3 hours.
c. Show that the particle was initially at rest and determine when the particle is again at rest.
d. Calculate the position and acceleration of the particle at this time.

14. Sketch the graphs of each of the following by determining the coordinates of all axis intercepts and any stationary points, and establishing their nature. Also determine the coordinates of the point(s) of inflection.
a. $y = x^3 - x^2 - 16x + 16$
b. $y = -x^3 - 5x^2 + 8x + 12$
c. $y = x^4 + 6x^3 + 9x^2$

15. A manufacturing company is required to produce cylindrical cans (for tuna) of volume 50 cm^3. The tin used to produce the cans costs 40 cents per 100 cm^2.

   a. Determine the area of tin required, A, in terms of the radius, r.
   b. Calculate the radius of the can (to the nearest tenth) for minimum area.
   c. Hence, calculate the minimum area (to the nearest tenth).
   d. What is the cost of tin to produce 10 000 such cans? Give your answer to the nearest \$20.

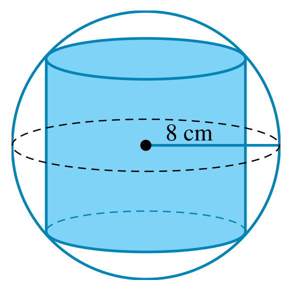
16. Water is being poured into a vase. The volume, V mL, of water in the vase after t seconds is given by $V = \frac{2}{3}t^2(15 - t)$, $0 \le t \le 10$.

   a. What is the volume after 10 seconds?
   b. At what rate is the water flowing into the vase at t seconds?
   c. What is the rate of flow after 3 seconds?
   d. When is the rate of flow the greatest, and what is the rate of flow at this time?

17. **Complex familiar**

   The amount of antibiotic drugs, $A(t)$ milligrams, in the body of a patient after time t hours, where $t \ge 0$, is given by $A(t) = 500te^{-\frac{t}{4}}$.
   a. Determine the expression for the rate of change of the amount of the drug in the patient's body at time t.
   b. Determine the rate of change of the amount of the drug in the patient's body after 2 hours.
   c. Determine the time in hours when the amount of the drug in a patient's body is a maximum.
   d. Determine the maximum amount of the drug in the patient's body. Give your answer in milligrams, correct to 2 decimal places.
   e. Sketch the graph of $A(t)$ versus time.
   f. For how many hours is the amount of the drug in the body in excess of 350 milligrams? Give your answer to the nearest minute.

18. A cylinder of cheese is to be removed from a spherical piece of cheese of radius 8 cm. What is the maximum volume of the cylinder of cheese? Express your answer to the nearest unit.



19. Consider the function $y = \frac{18}{x^2 - 9}$.
   a. State the domain of the function and the coordinates of any axis intercepts.
   b. Show that $\frac{dy}{dx} = \frac{-36x}{\left(x^2 - 9\right)^2}$.
   c. Show that the function does not have any points of inflection.
   d. Determine the coordinates and nature of any stationary points.
   e. By considering extreme values of x, discuss the behaviour of the function as $x \to \pm\infty$.
   f. Sketch the graph to represent the function, showing clearly all important features, including the equation(s) of any asymptotes.

20. a. Sketch the following functions, showing any stationary points or points of inflection.

i. $y = (x - 2)^2 + 1$

ii. $y = (x - 2)^3 + 1$

iii. $y = (x - 2)^4 + 1$

iv. $y = (x - 2)^5 + 1$

b. Discuss the similarities and differences of the functions in part a in terms of their stationary points.

c. Consider the function $y = (x - h)^n + k$, $n \geq 2$ and $n \in Z$. By discussing various values of n, what conclusions can you draw about the shape of functions that can be expressed in this form?

studyon

Units 3 & 4 Sit exam

Answers

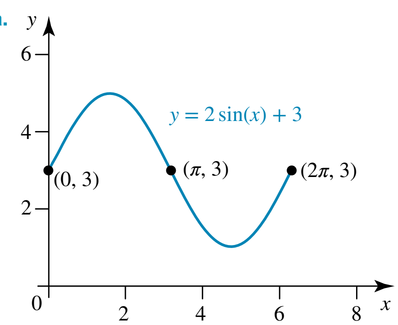
8 The second derivative and applications of differentiation

Exercise 8.2 Second derivatives

1. a. $12x^2 - 30x + 2$
   b. $6x - 8$
   c. -2
   d. $16 - 6x$
   e. $48(2x - 1)^2$
2. a. $\dfrac{3}{4\sqrt{x}}$
   b. $\dfrac{6}{x^4}$
   c. $16e^{2x+3}$
   d. $-\dfrac{4}{25}\cos\left(\dfrac{2x}{5}\right)$
   e. $-48\sin(4x - \pi)$
3. a. $\dfrac{1}{x}$
   b. $6e^{3x^2}\left(6x^2 + 1\right)$
   c. $\dfrac{-1}{(x+1)^2}$
4. $\dfrac{-1}{12}$
5. $-\sqrt{3}$
6. a. $\dfrac{1}{2}$
   b. $\dfrac{-9}{2}$
7. a. $\dfrac{-16}{9}$
   b. $6e$
8. $\dfrac{20x^3\left(2x^2 + 7\right)}{(2x^2 + 5)^2} + 6x\ln(2x^2 + 5)$
9. $3x^2\left(3x^2 - 8x + 4\right)e^{-3x}$
10. a. $\dfrac{-2\left(x^2 + 4x - 5\right)}{(x^2 + 4x + 13)^2}$
    b. $-e^{3x}\left(7\cos(4x) + 24\sin(4x)\right)$
11. a. $2xe^{-2x}\left(2x^2 - 6x + 3\right)$
    b. $\left(2 - 9x^2\right)\cos(3x) - 12x\sin(3x)$
12. a. See worked solutions
    b. 1
13. $a = -18, b = -16$
14. a. Sample responses can be found in the worked solutions in the online resources.
    b. -14.92 (correct to 2 decimal places)
15. a. Initially, the particle has a position of $-3\sqrt{2}$ m, or $3\sqrt{2}$ m to the left of the origin.
    b. The particle is first at rest after 1.5 s.
    c. The acceleration of the particle at 3.5 s is $\dfrac{3\pi^2}{2}$ m/s^2.

Exercise 8.3 Concavity and points of inflection

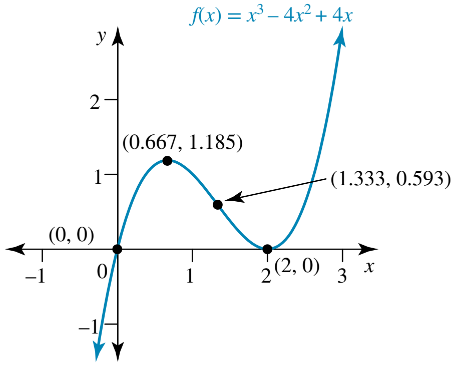
1. a. i. Concave up
      ii. Concave down
   b. $(3, -46)$
2. a. i. Concave down
      ii. Concave up
   b. $(-2, 16)$
3. a. i. Concave up
      ii. Concave up
   b. $\left(\dfrac{4}{3}, 4\dfrac{20}{27}\right)$
4. a. i. $x > -3$
      ii. $x < -3$
   b. $(-3, 54)$
5. a. i. $x > -\dfrac{2}{3}$
      ii. $x < -\dfrac{2}{3}$
   b. $\left(-\dfrac{2}{3}, 3\dfrac{16}{27}\right)$
6. The second derivative does not change sign either side of $x = 0$. In fact, $\dfrac{d^2y}{dx^2} \le 0$ for all x, so the curve is always concave down.
7. The second derivative does not change sign either side of $x = 0$. In fact, $\dfrac{d^2y}{dx^2} \ge 0$ for all x, so the curve is always concave up.
8. a. $(1, -6)$
   b. $(3, -11)$
9. a. i. $x < -2$ or $x > 0$
      ii. $-2 < x < 0$
   b. $(-2, 19), (0, 3)$
10. a. $\left(-\dfrac{1}{6}, \dfrac{5}{432}\right), \left(\dfrac{1}{6}, \dfrac{5}{432}\right)$
    b. $x < -\dfrac{1}{6}$ or $x > \dfrac{1}{6}$
11. a. i. $24(2x - 3)$
       ii. $x = \dfrac{3}{2}$
       iii. $\left(\dfrac{3}{2}, 4\right)$
    b. i. $48(2x - 3)^2$
       ii. $x = \dfrac{3}{2}$
       iii. None
    c. Similarities: Both curves had $\dfrac{d^2y}{dx^2} = 0$ at $x = \dfrac{3}{2}$.
       Differences: Consider the second derivatives:
       For part **a**, the second derivative was a linear function, so it changed sign either side of $x = \dfrac{3}{2}$.
       For part **b**, the second derivative was a perfect square, so it did not change sign.
12. 18
13. a. -2
    b. $x < 0$ or $x > 1$
14. Sample responses can be found in the worked solutions in the online resources.
15. a.



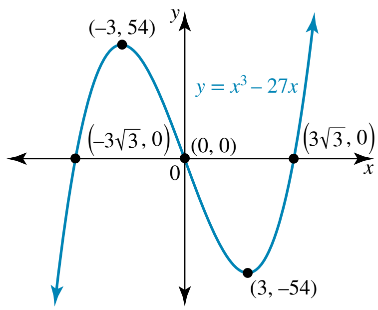
    b. i. $x \in (0, \pi)$
       ii. $x \in (\pi, 2\pi)$
    c. $(\pi, 3)$

Exercise 8.4 Curve sketching

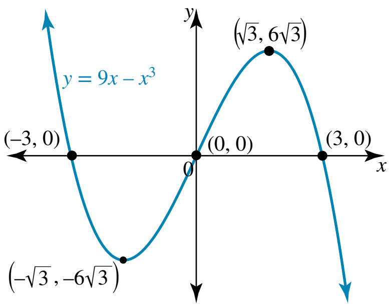
1.



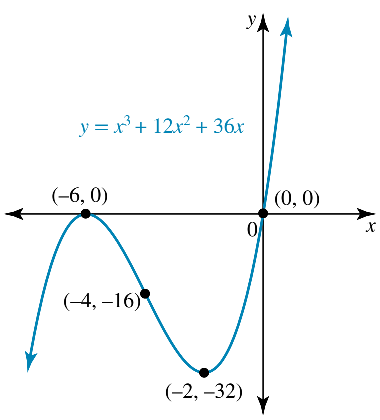
2. a.



b.



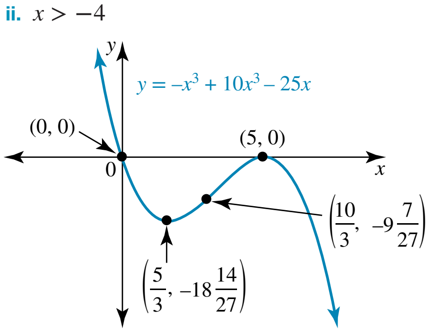
3. a.



i. $x < -6$ or $x > -2$

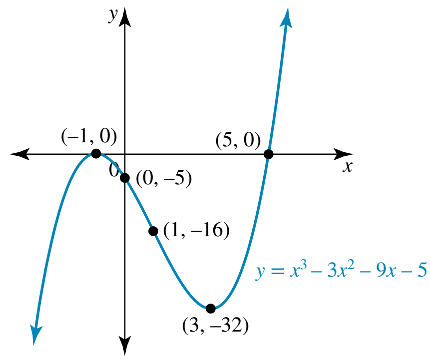
ii. $x > -4$

b.

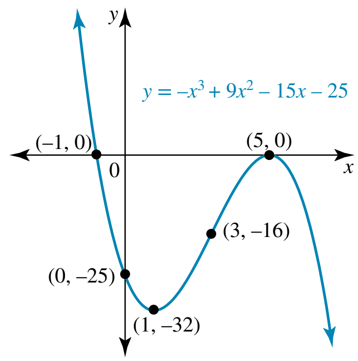


i. $\frac{5}{3} < x < 5$ **ii.** $x < \frac{10}{3}$

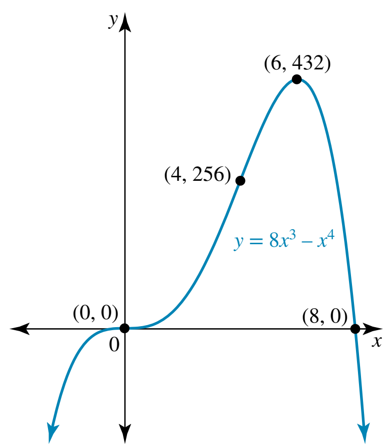
4. a.



b.



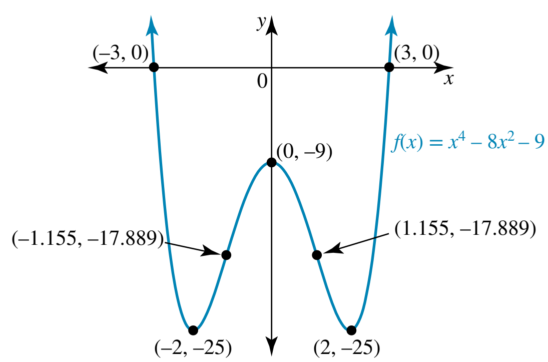
5. a.



b. i. $x > 6$

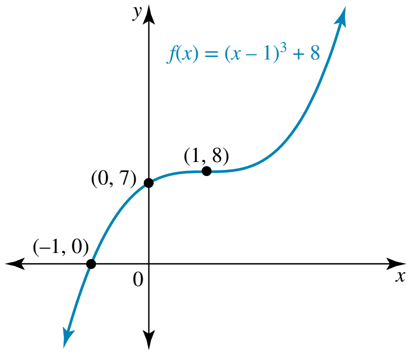
ii. $0 < x < 4$

6. a.



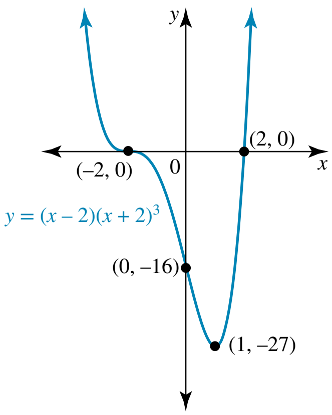
b. $-2 < x < -\frac{2\sqrt{3}}{3}$ or $x > 2$

7. a.

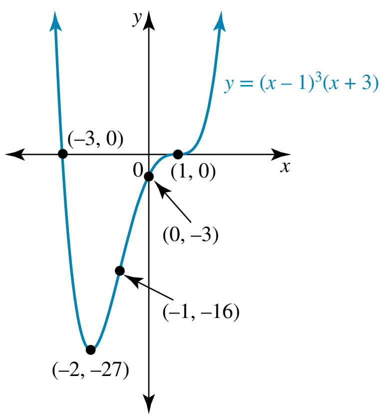


b. $x < 1$

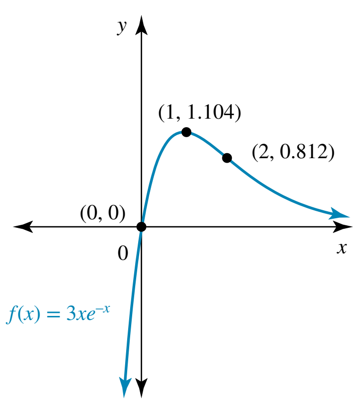
8. a.



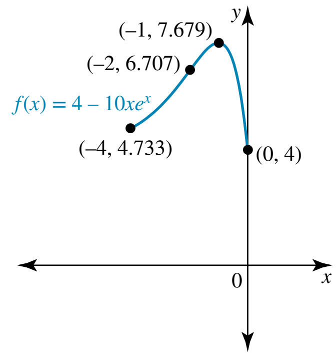
b.



9.



10.



11. $b = -3, c = 3, d = -3$

12. $y = -27x + 6$

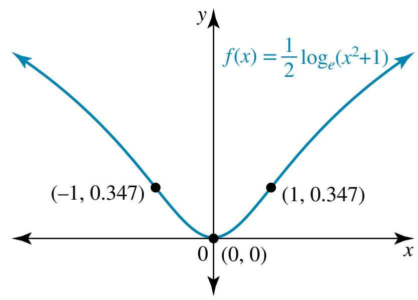
13. $b = -6, c = 15, d = -18$

14. a. $x \in R$

b. $(0, 0)$; relative minimum

c. $\left(-1, \frac{1}{2}\ln(2)\right), \left(1, \frac{1}{2}\ln(2)\right)$ or $(-1, 0.347), (1, 0.347)$

d.



15. a. $x > 0$

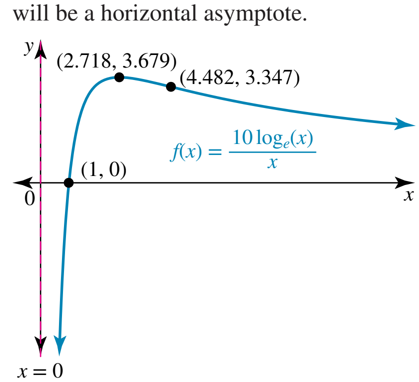
b. $\left(e, \frac{10}{e}\right) \approx (2.718, 3.679)$; relative maximum

c. $\left(e^{\frac{3}{2}}, 15e^{\frac{-3}{2}}\right) \approx (4.482, 3.347)$

d. $(1, 0)$

e. As the values of x increase, the values of $f(x)$ approach 0, the x-axis. Since the curve only crosses the x-axis at $x = 1$, the curve will be approaching the axis. The x-axis will be a horizontal asymptote.

f.

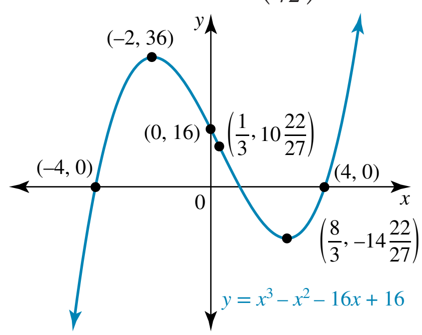


Exercise 8.5 Applications of the second derivative

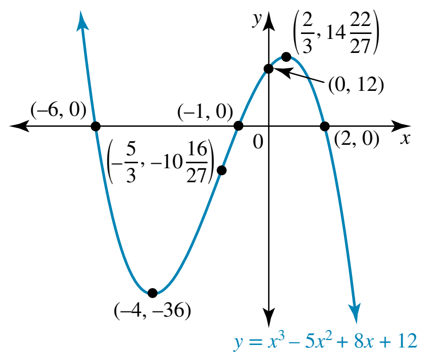
1. a. i. $v(t) = 4e^{-\frac{t}{2}}(2-t)$ ii. $a(t) = 2e^{-\frac{t}{2}}(t-4)$
   b. 8 m/s
   c. 2 s; $\frac{16}{e}$ m ≈ 5.886 m from the origin
   d. $t \in (4, 6]$
2. a. Maximum: 5 m; minimum: 1 m
   b. $\frac{1}{3}$ s
   c. $\frac{\pi}{3}$ s
   d. $a(t) = -18\cos(3t-1)$
3. a. $x(t) = t^3 - t^2 - 5t - 3$ b. $a(t) = 6t - 2$
   c. $1\frac{2}{3}$ s d. 5 m
   e. 8 m/s^2
4. $\frac{1}{2}$ m/s^2
5. a. $v(t) = 4t^3 - 2t^2 + 4t + 15$
   b. $x(t) = t^4 - \frac{2}{3}t^3 + 2t^2 + 15t$
   c. $48\frac{2}{3}$ m
6. 16, 16
7. 2 and 6
8. a. $h = \frac{100 - \pi r^2}{\pi r}$
   b, c. Sample responses can be found in the worked solutions in the online resources.
   d. 175 cm^3
9. \$370
10. a. $\theta = \frac{8-2r}{r}$ b. $A = 4r - r^2$ c. 2^c
11. a. $V = 4x^3 - 52x^2 + 160x$
    b. Length: 12 cm; width: 6 cm; height: 2 cm; volume 144 cm^3
12. Radius: 6.06 cm; height: 12.11 cm; volume: 1395.04 cm^3
13. a. 400 birds b. At the end of December
    c. 694 birds
14. 3.2 km to the right of point O
15. Radius: $5\frac{1}{3}$ cm; height: $3\frac{1}{3}$ cm; volume: 298 cm^3

8.6 Review: exam practice

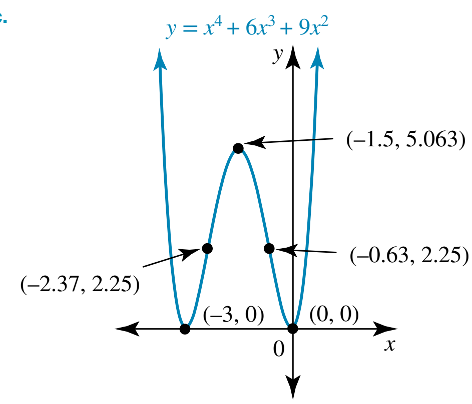
1. C
2. D
3. D
4. C
5. a. 2 m b. −3 m/s
   c. 3 s; 0 m/s d. 6 m/s^2
6. B
7. A
8. C
9. A
10. a. $a = 12$ m, $b = 9.6$ m
    b. 403.2 m^2
11. a. $v(t) = 4e^t - 3t^2 + t - 5$
    b. $x(t) = 4e^t - t^3 + \frac{t^2}{2} - 5t - 4$
    c. $\left(4e - \frac{19}{2}\right)$ m ≈ 1.373 m from the origin
12. a. 10.25 m from the origin
    b. 8.30 m/s
    c. $a(t) = e^{2t} - 8$
    d. $0 \le t < \ln\left(\sqrt{8}\right)$ or $t \in [0, \ln\left(2\sqrt{2}\right))$
13. a. 12 m from the fixed point
    b. $\left(-\frac{\pi\sqrt{2}}{12}\right)$ m/h ≈ −0.370 m/h
    c. After 12 hours
    d. 8 m from the fixed point; $\left(\frac{\pi^2}{72}\right)$ m/h^2 ≈ 0.137 m/h^2
14. a.



b.



c.



15. a. $A = 2\pi r^2 + \frac{100}{r}$ b. 2.0 cm
    c. 75.1 cm^2 d. \$3000
16. a. $333\frac{1}{3}$ mL b. $\frac{dV}{dt} = 20t - 2t^2$
    c. 42 mL/s d. 5 s; 50 mL/s

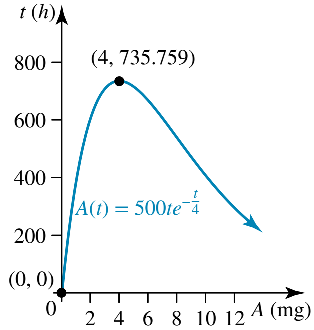
17. a. $A'(t) = 125e^{-\frac{t}{4}}(4 - t)$

b. $\frac{250}{\sqrt{e}}$ mg/h (or 151.63 mg/h to 2 decimal places)

c. After 4 hours

d. 735.76 mg

e.



f. 10 hours 10 minutes

18. 1238 cm^3

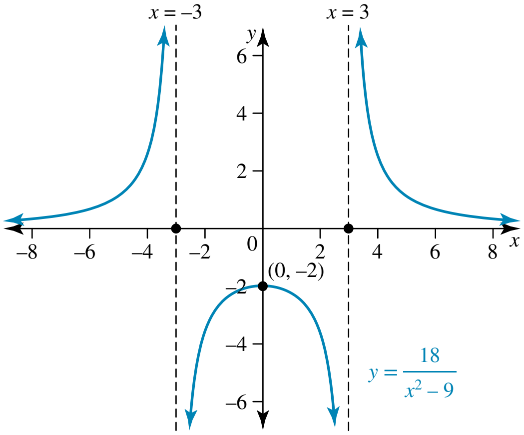
19. a. $x \in R\backslash \pm 3$; axis intercept $(0, -2)$

b, c. Sample responses can be found in the worked solutions in the online resources.

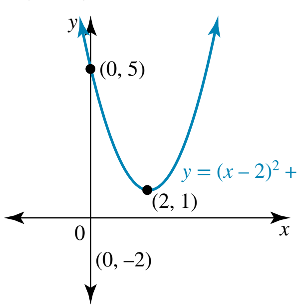
d. Maximum turning point at $(0, -2)$

e. For both large and small values of x, the curve is approaching the x-axis.

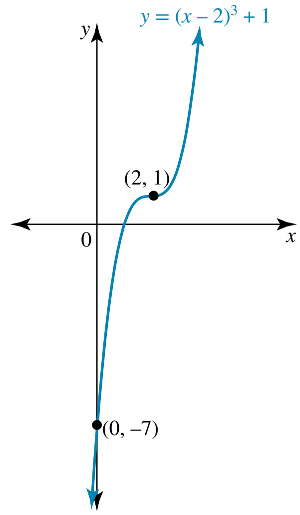
f.



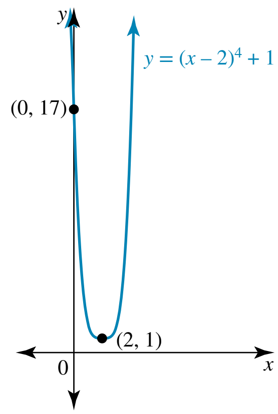
20. a. i. $y = (x - 2)^2 + 1$



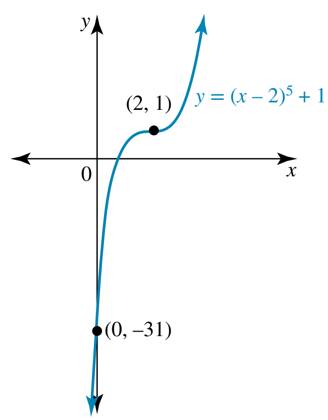
ii. $y = (x - 2)^3 + 1$



iii. $y = (x - 2)^4 + 1$



iv. $y = (x - 2)^5 + 1$



b. Similarities: All have a stationary point at $(2, 1)$, so for each function $\frac{dy}{dx} = 0$ at $x = 2$.

Differences: The functions with even powers are always concave up, so $\frac{d^2y}{dx^2} > 0$.

The functions with odd powers have horizontal points of inflection as their stationary point, so at $x = 2$, both $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} = 0$, and $\frac{d^2y}{dx^2}$ changes sign either side of $x = 2$, changing from concave down to concave up.

c. (h, k) is a minimum turning point when n is even.
(h, k) is a horizontal point of inflection when n is odd.

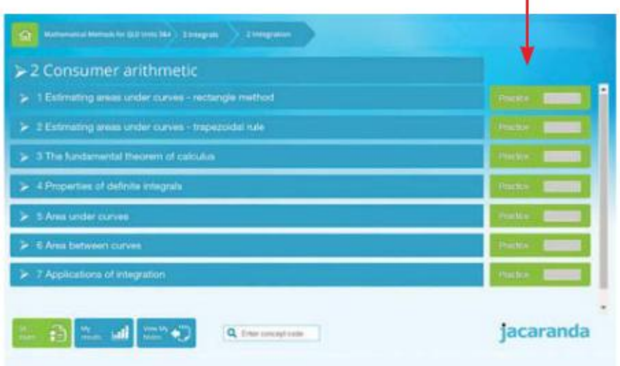
REVISION UNIT 4 Further functions and statistics

TOPIC 1 Further differentiation and applications 3

- For revision of this entire topic, go to your **studyON** title in your bookshelf at **www.jacplus.com.au**.
- Select **Continue Studying** to access hundreds of revision questions across your entire course.

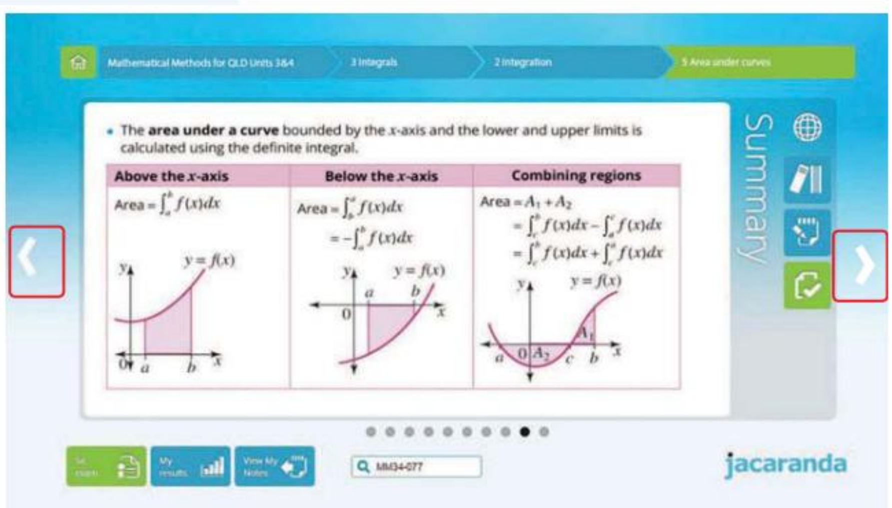
- Select your **course** *Mathematical Methods for Queensland Units 3&4* to see the entire course divided into syllabus topics.
- Select the **Area** you are studying to navigate into the chapter level *OR* select **Practice** to answer all practice questions available for each area.

- Select **Practice** at the sequence level to access all questions in the sequence.



- At **sequence level**, drill down to concept level.

- **Summary screens** provide revision and consolidation of key concepts. Select the **next arrow** to revise all concepts in the sequence and practise questions at the concept level.



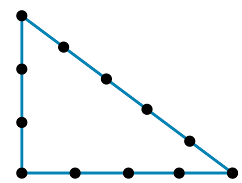
9 Cosine and sine rules

9.1 Overview

Although Pythagoras is given the credit for discovering that the sum of the squares of the sides of a right-angled triangle is equal to the square of its hypotenuse, the Babylonians and the Egyptians were already using this relationship at least two thousand years before he was even born.

The pyramids of Giza were built over four thousand years ago and are remarkable not just for their sheer size, but also for the incredible level of mathematical precision and engineering skill that went into their construction. The right angles of their square bases are exact to within half a degree – no small feat considering that the base of the largest pyramids has sides 230 metres long.

The device they are credited with using to obtain such precise 90-degree angles was a simple piece of rope tied in a loop with twelve evenly spaced knots along its length. The rope was stretched taut to form a triangle with 3 knots to one side, 4 knots to the next side and 5 knots on the last side. Lo and behold, the angle bounded by the 3-knot and 4-knot sides was a perfect 90° — though this is not surprising given that $3^2 + 4^2 = 5^2$, which is consistent with Pythagoras' theorem. Perhaps we should rename it the Egyptian theorem?



LEARNING SEQUENCE

9.1 Overview
9.2 Review of trigonometric ratios and the unit circle
9.3 The sine rule
9.4 The cosine rule
9.5 Area of a triangle
9.6 Applications of the sine and cosine rules
9.7 Review: exam practice

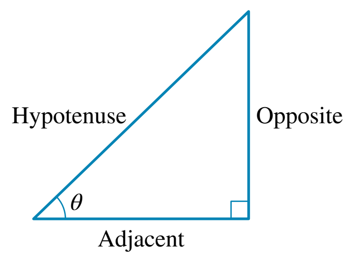
Fully worked solutions for this chapter are available in the Resources section of your eBookPLUS at www.jacplus.com.au.

9.2 Review of trigonometric ratios and the unit circle

9.2.1 Trigonometric ratios

From your previous studies of mathematics, you will be familiar with the geometry of a right-angled triangle and with Pythagoras' theorem. However, before progressing further in trigonometry, we will review the basic concepts.

9.2.2 The right-angled triangle



As you will recall, the longest side of the right-angled triangle is the hypotenuse, and it is positioned opposite the 90° angle as shown. The other two sides of the triangle are labelled as being adjacent or opposite according to their position relative to another angle, θ, as shown in the diagram.

The trigonometric ratios of sine (θ), cosine (θ) and tangent (θ) (written as $\sin(\theta)$, $\cos(\theta)$ and $\tan(\theta)$) are calculated from the relative lengths of the three sides.

The trigonometric ratios sin, cos and tan

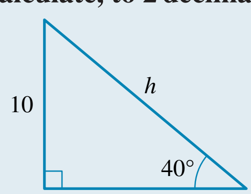
$$\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}}, \cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}}, \text{ and } \tan(\theta) = \frac{\text{opposite}}{\text{adjacent}}$$

These are usually remembered as SOH, CAH, TOA.

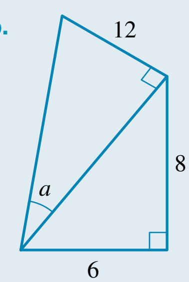
WORKED EXAMPLE 1

Calculate, to 2 decimal places, the value of the pronumeral shown in each diagram.

a.



b.



THINK

a. 1. Choose the appropriate trigonometric ratio.

2. Rearrange to make the required side the subject and evaluate, checking the calculator is in degree mode.

b. 1. Obtain the length of the hypotenuse of the lower triangle.

WRITE

a. Relative to the angle, the sides marked are the opposite and the hypotenuse.

$\sin(40°) = \dfrac{10}{h}$

$h = \dfrac{10}{\sin(40°)}$

$= 15.56$ to 2 decimal places

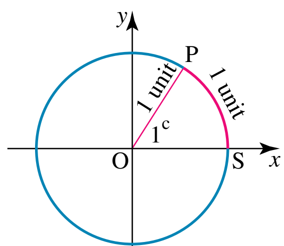
b. From Pythagoras' theorem the sides 6, 8, 10 form a Pythagorean triple, so the hypotenuse is 10.

2. In the upper triangle choose the appropriate trigonometric ratio.	The opposite and adjacent sides to the angle $a°$ are now known. $\tan(a) = \frac{12}{10}$
3. Rearrange to make the required angle the subject and evaluate.	$\tan(a) = 1.2$ $\therefore a = \tan^{-1}(1.2)$ $= 50.19°$ to 2 decimal places

Recall that a Pythagorean triple (mentioned in Worked example 1) is a set of three numbers, a, b and c, that denote the ratio of the lengths of right-angled triangles such that $a^2 + b^2 = c^2$. Examples include 3, 4 and 5, and also 5, 12 and 13.

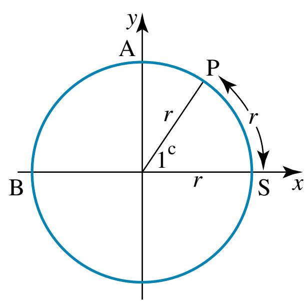
9.2.3 Radians and the unit circle: revision of basic concepts

Angles are measured in degrees or radians. To define a radian we can use a circle which has a radius of one unit. This circle is called the **unit circle**. If we take a piece of string that is the same length as the radius and place it along the circumference of the circle from S to P to form an arc, then the angle formed by joining S and P to O, the centre of the circle, measures one radian.



A unit circle

The radius of the circle can be any length and can still be regarded as a unit. As long as the arc is the same length as the radius, the angle will always measure one radian.



A radian

In general, therefore, a radian is the angle formed at the centre of any circle by radii meeting an arc that is the same length as the radius of the circle. Note the following:

- One radian is written as 1^c (or 1 radian can be written as 1).
- The circumference of a circle is $2\pi r$ units in length.
- If the radius is one unit, as in the case of the unit circle, then the circumference is 2π units, and the angle at the centre of the circle is 2π radians.
- 2π radians $= 360°$
- The length of the semicircle from S through A to B is half the circumference and is π units.
- π radians $= 180°$

- An arc length of r units subtends an angle of 1 radian.
- An arc length of $2\pi r$ units subtends an angle of 2π radians.
- An arc length of a quarter of a circle is $\frac{2\pi r}{4}$ units $\left(\text{that is, } \frac{\pi r}{2} \text{ units}\right)$ and subtends an angle of $\frac{\pi}{2}$ radians.

9.2.4 Determining the number of degrees in one radian

Since $\pi^c = 180°$,

we have $1^c = \frac{180°}{\pi} = 57.296°$ (correct to 3 decimal places).

Radians are converted to degrees using the following equation:

$$1^c = \frac{180°}{\pi}$$

Degrees are converted to radians using the following equation:

$$180° = \pi^c$$

$$1° = \frac{\pi}{180}$$

When writing angles in radians, it is common to omit the small ‘c’ superscript.

WORKED EXAMPLE 2

a. Convert 60° to radian measure.

b. Convert $\frac{2\pi^c}{3}$ to degree measure.

c. Convert $\frac{\pi}{2}$ to degree measure and hence state the value of $\sin\left(\frac{\pi}{2}\right)$.

THINK	WRITE
a. Convert degrees to radians by multiplying by $\frac{\pi}{180}$.	a. $60° = 60 \times \frac{\pi}{180}$ $= \cancel{60} \times \frac{\pi}{\cancel{180}\,3}$ $= \frac{\pi}{3}$
b. Convert radians to degrees by multiplying by $\frac{180}{\pi}$.	b. $\frac{2\pi^c}{3} = \left(\frac{2\pi}{3} \times \frac{180}{\pi}\right)^\circ$ $= \left(\frac{2\cancel{\pi}}{\cancel{3}} \times \frac{\cancel{180}\,60}{\cancel{\pi}}\right)^\circ$ $= 120°$
c. 1. Convert radians to degrees.	c. $\frac{\pi}{2} = \left(\frac{\pi}{2} \times \frac{180}{\pi}\right)^\circ$ $= \left(\frac{\cancel{\pi}}{\cancel{2}} \times \frac{\cancel{180}\,90}{\cancel{\pi}}\right)^\circ$ $= 90°$

2. Calculate the trigonometric value.	$\sin\left(\frac{\pi}{2}\right) = \sin(90°)$ $= 1$

9.2.5 Special angles

Note the following special cases with which you need to be familiar.
Since $180° = \pi$:

$$90° = \frac{\pi}{2} \text{ (dividing both sides by 2)}$$

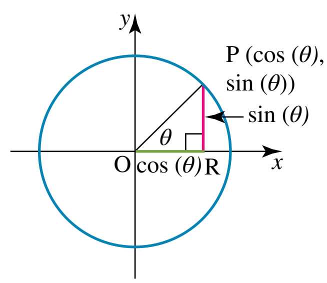
$$60° = \frac{\pi}{3} \text{ (dividing both sides by 3)}$$

$$45° = \frac{\pi}{4} \text{ (dividing both sides by 4)}$$

$$30° = \frac{\pi}{6} \text{ (dividing both sides by 6)}$$

9.2.6 Basic definitions of sine, cosine and tangent

Sine and cosine



In the unit circle the vertical distance PR is defined as sine (θ) (or $\sin(\theta)$) and the horizontal distance OR is defined as cosine (θ) (or $\cos(\theta)$).

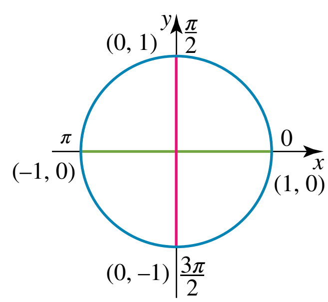
The coordinates of the point P are $(\cos(\theta), \sin(\theta))$ where θ can be in radians or degrees.

The x-coordinate of P is $\cos(\theta)$ and the y-coordinate of P is $\sin(\theta)$.

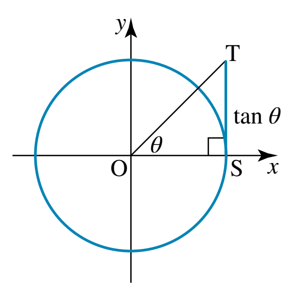
Boundary angles

In Year 11 you learned the values of the trigonometric ratios for the boundary angles.

$\sin(0) = 0$	$\sin(0°) = 0$	$\cos(0) = 1$	$\cos(0°) = 1$
$\sin\left(\frac{\pi}{2}\right) = 1$	$\sin(90°) = 1$	$\cos\left(\frac{\pi}{2}\right) = 0$	$\cos(90°) = 0$
$\sin(\pi) = 0$	$\sin(180°) = 0$	$\cos(\pi) = -1$	$\cos(180°) = -1$
$\sin\left(\frac{3\pi}{2}\right) = -1$	$\sin(270°) = -1$	$\cos\left(\frac{3\pi}{2}\right) = 0$	$\cos(270°) = 0$
$\sin(2\pi) = 0$	$\sin(360°) = 0$	$\cos(2\pi) = 1$	$\cos(360°) = 1$



Tangent



TS is the tangent to the circle that intersects with the x-axis and $\angle TOS = \theta$.

Using the unit circle, the vertical distance TS is defined as $\tan(\theta)$.

Using Pythagoras' theorem in triangle OPR,

$PR^2 + OR^2 = OP^2$.

So, $\sin^2(\theta) + \cos^2(\theta) = 1$.

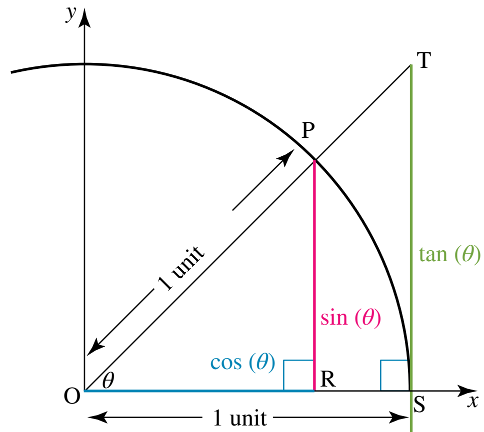
This is called the Pythagorean identity and is used in many trigonometric equations.

In the diagram, ΔOPR is similar to ΔOTS (equiangular).

Therefore: $\dfrac{TS}{OS} = \dfrac{PR}{OR}$

$$\frac{\tan(\theta)}{1} = \frac{\sin(\theta)}{\cos(\theta)}$$

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$$



Identities

Note the boundary angles for the tangent ratio:

$\tan(0) = 0$	$\tan(0°) = 0$
$\tan\left(\dfrac{\pi}{2}\right)$ is undefined.	$\tan(90°)$ is undefined.
$\tan(\pi) = 0$	$\tan(180°) = 0$
$\tan\left(\dfrac{3\pi}{2}\right)$ is undefined.	$\tan(270°)$ is undefined.
$\tan(2\pi) = 0$	$\tan(360°) = 0$

It can be seen that $\tan(90°)$ is undefined because $\tan(90°) = \dfrac{\sin(90°)}{\cos(90°)} = \dfrac{1}{0}$, which is undefined.

9.2.7 Domains and ranges of the trigonometric functions

The domain and range of the unit circle require $-1 \le x \le 1$ and $-1 \le y \le 1$, so $-1 \le \cos(\theta) \le 1$ and $-1 \le \sin(\theta) \le 1$.

Since θ can be any real number, this means that the function f, where f is either sine or cosine, has domain R and range $[-1, 1]$.

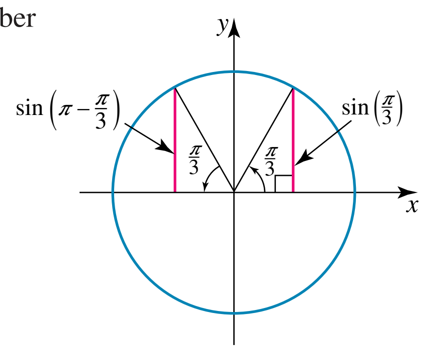
Unlike the sine and cosine functions, the domain of the tangent function is not the set of real numbers, R, since $\tan(\theta)$ is not defined for any value of θ that is an odd multiple of $\dfrac{\pi}{2}$. Excluding these values, intercepts of any size may be cut off on the tangent line, so $\tan(\theta) \in R$.

This means that the function f, where f is tangent, has domain $R\backslash\left\{\pm\dfrac{\pi}{2}, \pm\dfrac{3\pi}{2}, \ldots\right\}$ and range R.

The domain of the tangent function can be written as $R\backslash\left\{(2n+1)\dfrac{\pi}{2}, n \in Z\right\}$, and the tangent function can be written as the mapping $f: R\backslash\left\{(2n+1)\dfrac{\pi}{2}, n \in Z\right\} \to R, f(\theta) = \tan(\theta)$.

Periodicity of trigonometric functions

From the general equation $\sin(x) = a$, we can determine an infinite number of solutions. An example of this general equation is $\sin(x) = \dfrac{\sqrt{3}}{2}$.



One of the solutions is $x = \dfrac{\pi}{3}$, because $\sin\left(\dfrac{\pi}{3}\right) = \dfrac{\sqrt{3}}{2}$.

However, we also know that $\sin\left(\pi - \dfrac{\pi}{3}\right) = \dfrac{\sqrt{3}}{2}$, because sine is positive in the second quadrant.

For this equation there are two solutions between 0 and 2π. They are $\frac{\pi}{3}$ and $\frac{2\pi}{3}$. (There are no solutions in the third and fourth quadrants because here, sine is negative.)

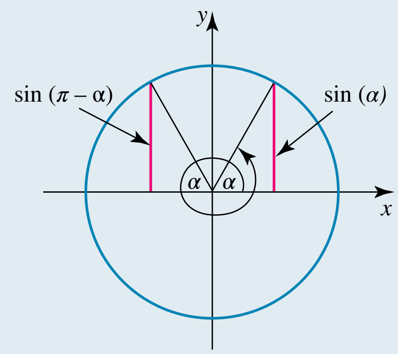
$x = 0$ and $x = 2\pi$ are solutions for $\cos(x) = 1$ over the domain $0 \le x \le 2\pi$.

To determine more solutions, we can go around the unit circle as many times as we wish, determining new solutions each time. Since $\sin\left(2\pi + \frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$ and $\sin\left(3\pi - \frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$, there are four solutions in the domain $0 \le x \le 4\pi$: $\frac{\pi}{3}, \frac{2\pi}{3}, \frac{7\pi}{3}$ and $\frac{8\pi}{3}$.

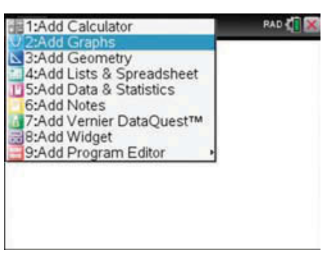
We can also go in a negative direction. In the domain $-2\pi \le x \le 0$ there are two solutions: $\frac{-4\pi}{3}$ and $\frac{-5\pi}{3}$.

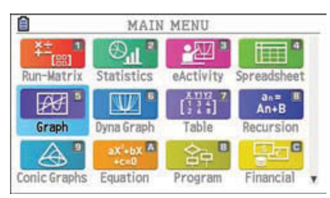
WORKED EXAMPLE 3

Determine all solutions to the equation $\sin(\alpha) = 0.7$ in the domain $0 \le \alpha \le 4\pi$. Give your answers correct to 4 decimal places.

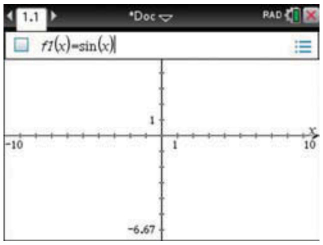


THINK	WRITE
1. Determine the equivalent angle in the 1st quadrant, ignoring the sign.	The basic angle is 0.7754.
2. Determine in which quadrants $\sin\alpha$ is positive.	$\sin\alpha$ is positive in the 1st and 2nd quadrants.
3. Because the domain is $[0, 4\pi]$, look at angles beyond 2π by adding the period to values in the first cycle (that is, determining values in the 5th and 6th quadrants). In $y = \sin\alpha$ the period is 2π.	From 0 to 2π: $\alpha = 0.7754, p - 0.7754$ $= 0.7754, 2.3662$ From 0 to 4π: $\alpha = 0.7754, 2.3662,$ $0.7754 + 2\pi, 2.3662 + 2\pi$
4. Simplify, giving answers correct to 4 decimal places.	$\alpha = 0.7754, 2.3662, 7.0586, 8.6494$

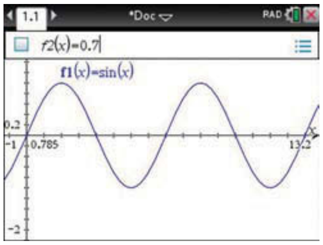
TI | THINK	WRITE
1. On a Calculator page, press MENU, then select: 2: Add Graphs.	

CASIO | THINK	WRITE
1. On a Main Menu page, select: GRAPH.	

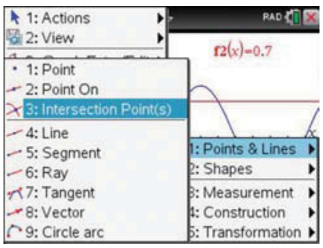
2. Complete the entry line as:
sin (x)
then press the ENTER button.



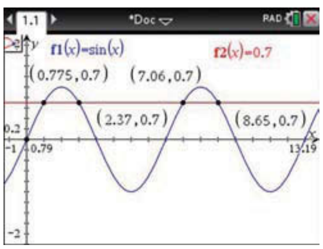
3. Press MENU, then select:
2: Add Graphs.
Complete the entry line as:
0.7
then press the ENTER button.



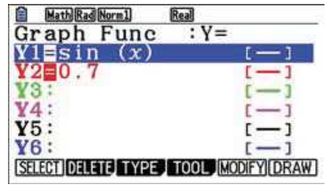
4. Press MENU, then select:
8: Geometry
1: Points & Lines
3: Intersection Points(s).



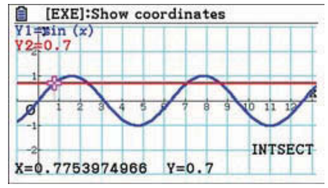
5. Select $f1(x) = \sin(x)$ and $f2(x) = 0.7$ by pressing the ENTER button on both curves.
The coordinate location of each intersection will appear on the screen.



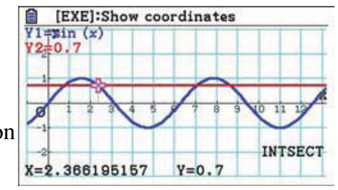
2. Complete the entry line as:
$Y1 = \sin(x)$
$Y2 = 0.7$
then press the EXE or DRAW button.



3. To calculate the intersection points, select:
SHIFT
F5
INTSECT.
The first solution will appear on the screen.



4. Each subsequent solution can be determined by pressing the right arrow button.
The solution will appear on the screen.



Notice that the third solution in Worked example 3 can be found by adding 2π to the first solution, and the fourth solution can be found by adding 2π to the second solution. This is because we have turned through an angle of 2π radians (1 revolution) beyond the original angle.

on Resources

Interactivities: Trigonometric ratios (int-2577)
The unit circle (int-2582)

studyon

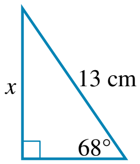
Units 3 & 4 > Area 5 > Sequence 1 > Concept 1

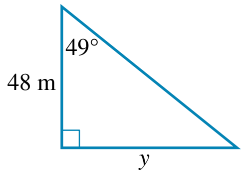
Review of sine, cosine and tangent ratios Summary screen and practice questions

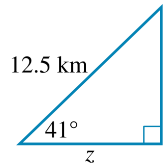
Exercise 9.2 Review of trigonometric ratios and the unit circle

Technology active

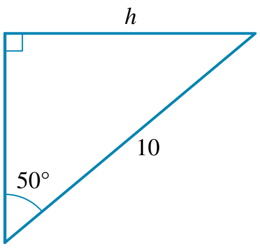
1. The following questions use the tan, sin, or cos ratios in their solutions. Determine the size of the side marked with the pronumeral in each diagram, correct to 3 significant figures.

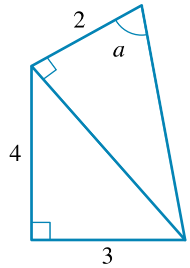
a.


b.


c.


2. **WE1** Calculate, to 2 decimal places, the value of the pronumeral shown in each diagram.

a.


b.


3. **WE2a** Change the following to radians. Give exact answers for **a**, **b**, **c** and **d**. Give other answers correct to 2 decimal places.
   a. $5°$ b. $15°$ c. $120°$ d. $130°$ e. $63.9°$
   f. $78.82°$ g. $235°$ h. $260°$ i. $310°$ j. $350°$

4. **WE2b** Change the following to degrees, giving answers correct to 2 decimal places.
   a. 3^c b. 5^c c. 4.8^c d. 2.56^c
   e. $\frac{7\pi^c}{20}$ f. $\frac{3\pi^c}{10}$ g. $\frac{5\pi^c}{6}$ h. $\frac{5\pi^c}{4}$

5. Evaluate the following using a calculator. Give answers correct to 3 decimal places.
   a. $\sin(0.4)$ b. $\sin(0.8)$ c. $\cos(1.4)$
   d. $\cos(1.7)$ e. $\tan(2.9)$ f. $\tan(2.4)$

6. Evaluate the following using a calculator. Give answers to 3 decimal places.
   a. $\sin(75°)$ b. $\sin(68°)$ c. $\cos(160°)$
   d. $\cos(185°)$ e. $\tan(265°)$ f. $\tan(240°)$

7. Evaluate the following.
   a. $\sin(0)$ b. $\sin(\pi)$ c. $\cos(2\pi)$
   d. $\cos(\pi)$ e. $\tan\left(\frac{3\pi}{2}\right)$ f. $\tan\left(\frac{\pi}{2}\right)$

8. Evaluate the following.
   a. $\sin(90°)$ b. $\sin(360°)$ c. $\cos(180°)$
   d. $\cos(0°)$ e. $\tan(270°)$ f. $\tan(720°)$

9. Evaluate the following without using your calculator.
   a. $\sin^2(20°) + \cos^2(20°)$ b. $\cos^2(50°) + \sin^2(50°)$
   c. $\sin^2(\pi) + \cos^2(\pi)$ d. $\sin^2(2.5) + \cos^2(2.5)$
   e. $\sin^2\left(\frac{\pi}{2}\right) + \cos^2\left(\frac{\pi}{2}\right)$ f. $\sin^2\left(\frac{\theta}{2}\right) + \cos^2\left(\frac{\theta}{2}\right)$
   g. $2\sin^2(\alpha) + 2\cos^2(\alpha)$ h. $5\sin^2(\beta) + 5\cos^2(\beta)$

10. Write the following in order from smallest to largest.
    a. $\sin(35°), \sin(70°), \sin(120°), \sin(150°), \sin(240°)$
    b. $\cos(0.2), \cos(1.5), \cos(3.34), \cos(5.3), \cos(6.3)$

11. If $\sin(\theta) = \frac{8}{17}$ and $\cos(\theta) = \frac{15}{17}$, determine $\tan(\theta)$.

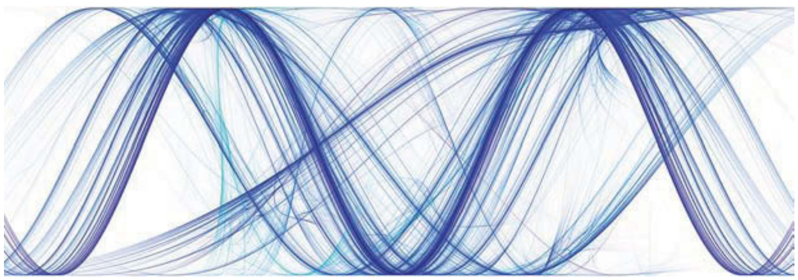
12. If $\sin(A) = 0.6$ and $\cos(A) = 0.8$, calculate $\tan(A)$. Draw a triangle marking in the position of angle A and the possible lengths of the sides.

13. **WE3** For each equation below, calculate all the values of x between 0 and 4π. Give answers correct to 4 decimal places.

a. $\cos(x) = -0.6591$
b. $\sin(x) = 0.9104$
c. $\cos(x) = 0.48$
d. $\sin(x) = -0.371$

14. Calculate all the solutions to the following equations in the domain $0° \leq x \leq 360°$. Give exact answers where possible; otherwise, give answers correct to 2 decimal places.

a. $\cos(2x) = 1$
b. $2\sin(2x) = -1$
c. $2\cos(3x) = -\sqrt{2}$
d. $2\sin(3x) = \sqrt{3}$
e. $\sin(3x) = -0.1254$
f. $3\cos(2x) = 0.5787$
g. $4\sin\left(\frac{x}{2}\right) = 0.913$
h. $\sqrt{2}\cos(x) = -0.2751$



15. Illustrate the following on a unit circle diagram.

a. $\cos(40°)$
b. $\sin(165°)$
c. $\cos(-60°)$
d. $\sin(-90°)$

16. Illustrate the following on a unit circle diagram.

a. $\sin\left(\frac{5\pi}{3}\right)$
b. $\cos\left(\frac{3\pi}{5}\right)$
c. $\cos(5\pi)$
d. $\sin\left(-\frac{2\pi}{3}\right)$

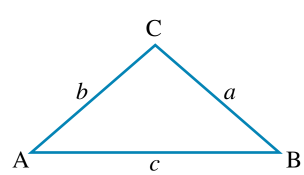
17. a. Calculate the Cartesian coordinates of the trigonometric point $P\left[\frac{\pi}{4}\right]$.

b. Express the Cartesian point P (0, − 1) as two different trigonometric points, one with a positive value for θ and one with a negative value for θ.

9.3 The sine rule

The trigonometry we have studied so far has been applicable to only right-angled triangles. The **sine rule** allows us to calculate the lengths of sides and the size of angles in non-right-angled triangles. Consider the triangle shown.

9.3.1 Derivation of the sine rule

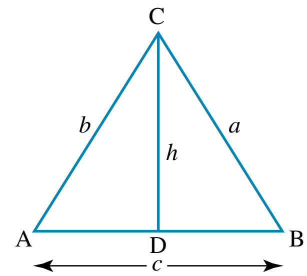


A, B and C represent the three angles in the triangle ABC, and a, b and c represent the three sides. Remember that each side is named with the lower-case letter of the opposite vertex.

Construct a line from C to a point, D, perpendicular to AB. CD is the perpendicular height of the triangle, h.

Now consider ΔACD and ΔBCD separately.

Using the formula for the sine ratio:

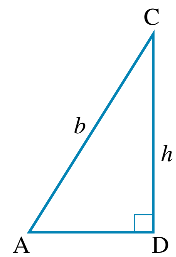


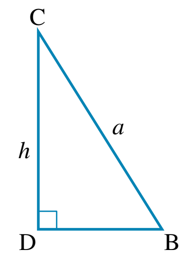
$$\sin(\theta) = \frac{\text{opp}}{\text{hyp}} \qquad \sin(\theta) = \frac{\text{opp}}{\text{hyp}}$$
$$\sin(A) = \frac{h}{b} \qquad \sin(B) = \frac{h}{a}$$
$$h = b\sin(A) \qquad h = a\sin(B)$$

We are now able to equate these two expressions for h.

$$a\sin(B) = b\sin(A)$$

Dividing both sides by $\sin(A)\sin(B)$, we get:





$$\frac{a\sin(B)}{\sin(A)\sin(B)} = \frac{b\sin(A)}{\sin(A)\sin(B)}$$
$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)}$$

Similarly, we are able to show that each of these is also equal to $\frac{c}{\sin(C)}$. Try it!

The sine rule states that in any triangle, ABC, the ratio of each side to the sine of its opposite angle will be equal.

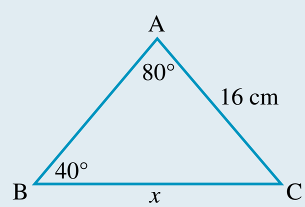
$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}$$

Using the sine rule to calculate side lengths

The sine rule formula allows us to calculate the length of a side in any triangle if we are given the length of one other side and two angles. When using the formula, we need to use only two parts of it.

WORKED EXAMPLE 4

Calculate the length of the side marked x in the triangle shown, correct to 1 decimal place.

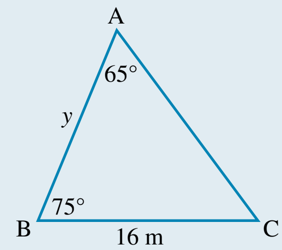


THINK	WRITE
1. Write the formula.	$\frac{a}{\sin(A)} = \frac{b}{\sin(B)}$
2. Substitute $a = x$, $b = 16$, $A = 80°$ and $B = 40°$.	$\frac{x}{\sin(80°)} = \frac{16}{\sin(40°)}$
3. Make x the subject of the equation by multiplying by sin(80°).	$x = \frac{16\sin(80°)}{\sin(40°)}$
4. Calculate and round to 1 decimal place.	$x \approx 24.5$ cm

To use the sine rule we need to know the angle opposite the side we are calculating and the angle opposite the side we are given. In some cases these are not the angles we are given. In such cases we need to use the fact that the angles in a triangle add to 180° to calculate the required angle.

WORKED EXAMPLE 5

Calculate the length of the side labelled y in the figure shown, correct to 4 significant figures.



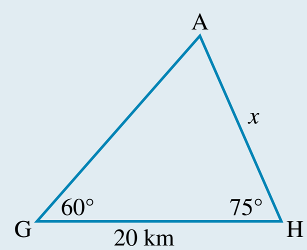
THINK	WRITE
1. Calculate the size of angle C.	$C = 180° - 65° - 75°$ $= 40°$
2. Write the formula.	$\dfrac{a}{\sin(A)} = \dfrac{c}{\sin(C)}$
3. Substitute $a = 16$, $c = y$, $A = 65°$ and $C = 40°$.	$\dfrac{16}{\sin(65°)} = \dfrac{y}{\sin(40°)}$
4. Make y the subject of the equation.	$y = \dfrac{16\sin(40°)}{\sin(65°)}$
5. Calculate and round to 4 significant figures.	$= 11.35\text{m}$

Using the sine rule allows us to solve a number of more complex problems. As with our earlier trigonometry problems, we begin with a diagram and give a written answer to each question.

WORKED EXAMPLE 6

George looks south and observes an aeroplane at an angle of elevation of 60°. Henrietta is 20 km south of where George is, and she faces north to see the aeroplane at an angle of elevation of 75°.

Calculate the distance of the aeroplane from Henrietta's observation point, to the nearest metre.



THINK	WRITE
1. Calculate the size of $\angle GAH$.	$A = 180° - 60° - 75°$ $= 45°$
2. Write the formula.	$\dfrac{g}{\sin(G)} = \dfrac{a}{\sin(A)}$
3. Substitute $g = x$, $a = 20$, $G = 60°$ and $A = 45°$.	$\dfrac{x}{\sin(60°)} = \dfrac{20}{\sin(45°)}$
4. Make x the subject.	$x = \dfrac{20\sin(60°)}{\sin(45°)}$

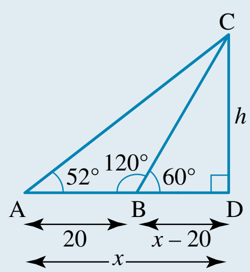
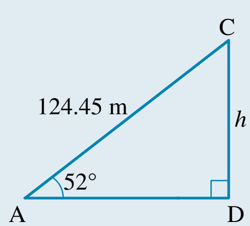
5. Calculate and round to 3 decimal places (i. e. the nearest metre). $x = 24.495$ km

6. Give a written answer. The distance of the aeroplane from Henrietta's observation point is 24 495 m or 24.495 km.

The sine rule can be used to calculate the height of objects that would otherwise be difficult to measure. Problems such as this can be solved by combining the use of the sine rule with the trigonometry of right-angled triangles covered earlier in this chapter.

WORKED EXAMPLE 7

To calculate the height of a building, Kevin measures the angle of elevation to the top as 52°. He then walks 20 m closer to the building and measures the angle of elevation as 60°. How high is the building?

THINK	WRITE
1. Draw a labelled diagram of the situation and fill in the given information.	
2. Check that one of the criteria for the sine rule has been satisfied for triangle ABC.	The sine rule can be used for triangle ABC since two angles and one side length have been given.
3. Determine the value of angle ACB, using the fact that the angle sum of any triangle is 180°.	$\angle ACB = 180° - (52° + 120°)$ $= 8°$
4. Write down the sine rule to determine b.	To determine side length b of triangle ABC: $\frac{b}{\sin(B)} = \frac{c}{\sin(C)}$
5. Substitute the known values into the rule.	$\frac{b}{\sin(120°)} = \frac{20}{\sin(8°)}$
6. Make b the subject of the equation.	$b = \frac{20 \times \sin(120°)}{\sin(8°)}$
7. Calculate and round the answer to 2 decimal places and include the appropriate unit.	≈ 124.45 m
8. Draw a diagram of the situation, that is triangle ADC, labelling the required information. Also label the sides of the triangle.	

9. Choose the sine ratio as we are determining the opposite side and have been given the hypotenuse. $\sin(\theta) = \dfrac{\text{opp}}{\text{hyp}}$

10. Substitute for θ and the hypotenuse. $\sin(52°) = \dfrac{h}{124.45}$

11. Make h the subject of the equation. $124.45 \sin(52°) = h$

$h = 124.45 \sin(52°)$

12. Calculate and round appropriately. ≈ 98.07

13. Give a written answer. The height of the building is 98.07 m.

Using the sine rule to determine angle sizes

Using the sine rule result we are able to calculate angle sizes as well. To do this, we need to be given the length of two sides and the angle opposite one of them. For simplicity, in solving the triangle we invert the sine rule formula when we are using it to determine an angle.

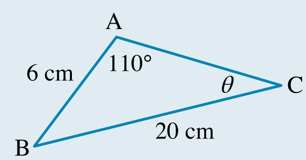
The formula is written:

$$\frac{\sin(A)}{a} = \frac{\sin(B)}{b} = \frac{\sin(C)}{c}$$

As with determining side lengths, we use only two parts of the formula.

WORKED EXAMPLE 8

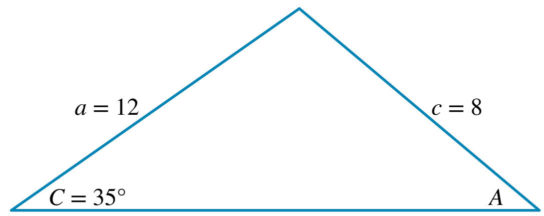
Calculate the size of the angle, θ, in the figure shown, correct to the nearest degree.



THINK	WRITE
1. Write the formula.	$\dfrac{\sin(A)}{a} = \dfrac{\sin(C)}{c}$
2. Substitute $A = 110°$, $C = \theta$, $a = 20$ and $c = 6$.	$\dfrac{\sin(110°)}{20} = \dfrac{\sin(\theta)}{6}$
3. Make $\sin\theta$ the subject of the equation.	$\sin(\theta) = \dfrac{6\sin(110°)}{20}$
4. Calculate a value for $\sin\theta$.	$\sin(\theta) = 0.2819$
5. Calculate $\sin^{-1}(0.2819)$ to determine θ.	$\theta = \sin^{-1}(0.2819)$ $\theta = 16°$

9.3.2 The ambiguous case of the sine rule

When we are given two side lengths of a triangle and an acute angle opposite one of these sides, there are two different triangles we can draw. So far we have only dealt with triangles in which all angles are acute; however, it is also possible to draw triangles with obtuse angles. This is known as the ambiguous case of the sine rule.



For example, take the triangle ABC, where $a = 12$, $c = 8$ and $C = 35°$.

When we solve this for angle A, we get an acute angle as shown:

$$\frac{8}{\sin(35)} = \frac{12}{\sin(A)}$$

$$8\sin(A) = 12\sin(35)$$

$$\sin(A) = \frac{12\sin(35)}{8}$$

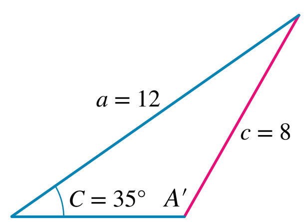
$$A = \sin^{-1}\left(\frac{12\sin(35)}{8}\right)$$

$$= 59.36°$$

However, there is also an obtuse-angled triangle that can be drawn from this given information.

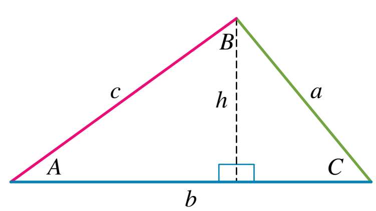
In this case, the size of the obtuse angle is the supplement of the acute angle calculated previously.

$$\begin{aligned} A' &= 180° - A \\ &= 180° - 59.36° \\ &= 120.64° \end{aligned}$$



9.3.3 Determining when to use the ambiguous case

The ambiguous case of the sine rule does not work for every example. This is due to the way the ratios are set in the development of the sine rule, since side length a must be longer than h, where h is the length of the altitude from angle B to the base line b.



For the ambiguous case to be applicable, the following conditions must be met:

- The given angle must be acute.
- The adjacent side must be bigger than the opposite side.
- The opposite side must be bigger than the adjacent side multiplied by the sine of the given angle.

When using the sine rule to calculate a missing angle, it is useful to first identify whether the ambiguous case is applicable to the problem or not.

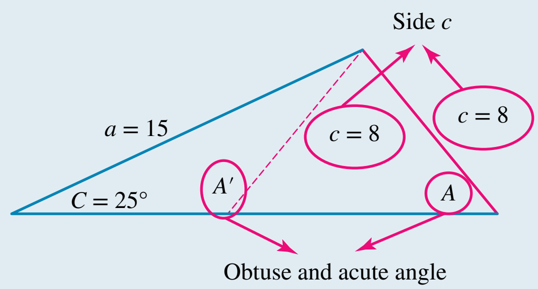
WORKED EXAMPLE 9

Calculate the two possible values of angle A for triangle ABC, given $a = 15, c = 8$ and $C = 25°$.

THINK

1. Draw a non-right-angled triangle, labelling with the given information. Angle A is opposite to side a. Angle C is opposite to side c. Note that two triangles can be drawn, with angle A being either acute or obtuse.

WRITE



2. Substitute the known values into the sine rule.

$$\frac{a}{\sin(A)} = \frac{c}{\sin(C)}$$

3. Rearrange the equation to make $\sin(A)$ the subject and solve. Make sure your calculator is in degree mode. The calculator will only give the acute angle value.

$$\frac{15}{\sin(A)} = \frac{8}{\sin(25°)}$$
$$15\sin(25°) = 8\sin(A)$$
$$\sin(A) = \frac{15\sin(25°)}{8}$$
$$A = \sin^{-1}\left(\frac{15\sin(25°)}{8}\right)$$
$$= 52.41°$$

4. Solve for the obtuse angle A'.

$$A' = 180° - A$$
$$= 180° - 52.41°$$
$$= 127.59°$$

5. Write the answer.

The two possible value for A are 52.41° and 127.59°.

on Resources

Interactivity The sine rule (int-6275)

studyon

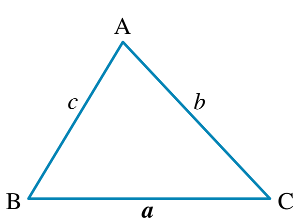
Units 3 & 4 › Area 5 › Sequence 1 › Concept 2 › **The sine rule** Summary screen and practice questions

Exercise 9.3 The sine rule

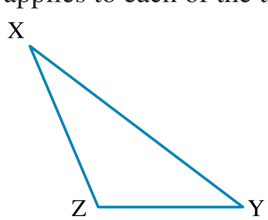
Technology free

1. Write down the sine rule formula as it applies to each of the triangles below.

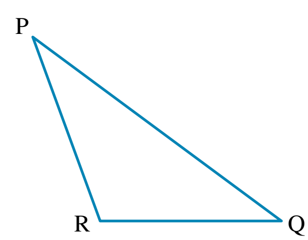
a.



b.

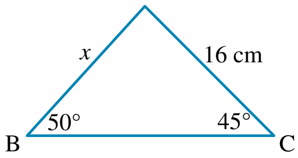


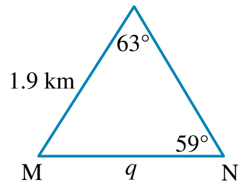
c.

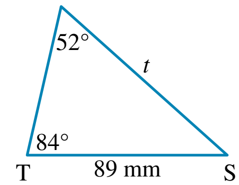


Technology active

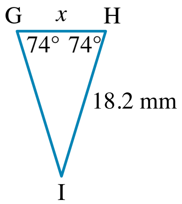
2. **WE4** Use the sine rule to calculate the length of the side marked with the pronumeral in each of the following, correct to 3 significant figures.

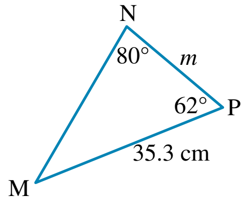
a. 

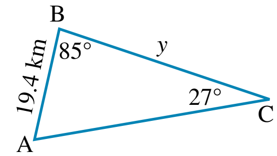
b. 

c. 

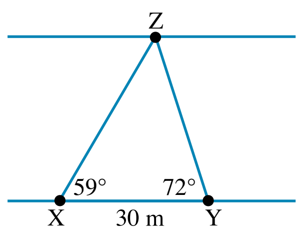
3. **WE5** In each of the following, use the sine rule to calculate the length of the side marked with the pronumeral, correct to 1 decimal place, by first calculating the size of the third angle.

a. 

b. 

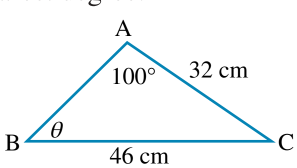
c. 

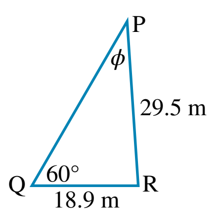
4. ABC is a triangle in which BC = 9 cm, ∠BAC = 54° and ∠ACB = 62°. Calculate the length of side AB, correct to 1 decimal place.
5. XYZ is a triangle in which $y = 19.2$ m, ∠XYZ = 42° and ∠XZY = 28°. Calculate x, correct to 3 significant figures.
6. **WE6** X and Y are two trees 30 m apart on one side of a river, as shown in the diagram. Z is a tree on the opposite side of the river. It is found that ∠XYZ = 72° and ∠YXZ = 59°. Calculate the distance XZ, correct to 1 decimal place.

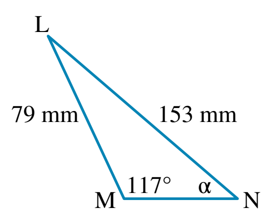


7. From a point, M, the angle of elevation to the top of a building, B, is 34°. From a point, N, 20 m closer to the building, the angle of elevation is 49°.
   a. Draw a diagram of this situation.
   b. Calculate the distance NB, correct to 1 decimal place.
   c. Calculate the height of the building, correct to the nearest metre.
8. **WE7** To calculate the height of a building, Kevin measures the angle of elevation to the top as 48°. He then walks 18 m closer to the building and measures the angle of elevation as 64°. How high is the building?

9. **WE8** Calculate the size of the angle marked with a pronumeral in each of the following, correct to the nearest degree.

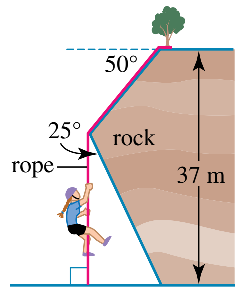
a. 

b. 

c. 

10. In ΔPQR, $q = 12$ cm, $r = 16$ cm and ∠PRQ = 56°. Calculate the size of ∠PQR, correct to the nearest degree.
11. In ΔKLM, LM = 4.2 m, KL = 5.6 m and ∠KML = 27°. Calculate the size of ∠LKM, correct to the nearest degree.
12. A surveyor marks three points, X, Y and Z, in the ground. The surveyor measures XY to be 13.7 m and XZ to be 14.2 m. ∠XYZ is 60°.
    a. Calculate ∠XZY to the nearest degree.
    b. Calculate ∠YXZ to the nearest degree.

13. WE9 In a triangle ABC, $a = 9$ cm, $c = 8$ cm and $A = 42°$. Calculate the two possible values of angle A.
14. Calculate the possible values for angle A in a triangle ABC for which $c = 15$ cm, $b = 12$ cm and $C = 35°$.
15. A cliff is 37 m high. The rock slopes outward at an angle of 50° to the horizontal, then cuts back at an angle of 25° to the vertical, meeting the ground directly below the top of the cliff.
    Carol wishes to abseil from the top of the cliff to the ground as shown in the diagram. Her climbing rope is 45 m long, and she needs 2 m to secure it to a tree at the top of the cliff. Will the rope be long enough to allow her to reach the ground?

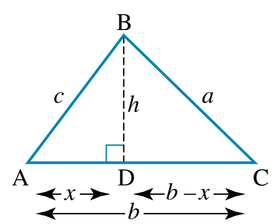


16. Two wires support a flagpole. The first wire is 8 m long and makes a 65° angle with the ground. The second wire is 9 m long. Calculate, to the nearest degree, the angle that the second wire makes with the ground.

9.4 The cosine rule

The sine rule will not allow us to solve all triangles. Depending on the information provided about the triangle, we may need to use the **cosine rule**.

In any non-right-angled triangle, ABC, a perpendicular line can be drawn from angle B to side b. Let D be the point where the perpendicular line meets side b, and let the length of the perpendicular line be h. Let the length AD = x units. The perpendicular line creates two right-angled triangles, ADB and CDB.



Using triangle ADB and Pythagoras' theorem, we obtain:

$$c^2 = h^2 + x^2 \rightarrow [1]$$

Using triangle CDB and Pythagoras' theorem, we obtain:

$$a^2 = h^2 + (b - x)^2 \rightarrow [2]$$

Expanding the brackets in equation [2]:

$$a^2 = h^2 + b^2 - 2bx + x^2$$

Rearranging equation [2] and using $c^2 = h^2 + x^2$ from equation [1]:

$$\begin{aligned} a^2 &= h^2 + x^2 + b^2 - 2bx \\ &= c^2 + b^2 - 2bx \\ &= b^2 + c^2 - 2bx \end{aligned}$$

From triangle ABD, $x = c\cos(A)$. Therefore, $a^2 = b^2 + c^2 - 2bx$ becomes

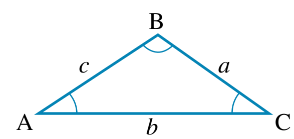
$$a^2 = b^2 + c^2 - 2bc\cos(A)$$

This is called the cosine rule and is a generalisation of Pythagoras' theorem.

In a similar way, if a perpendicular line is drawn from angle A to side a or from angle C to side c, the two right-angled triangles give $c^2 = a^2 + b^2 - 2ab\cos(C)$ and $b^2 = a^2 + c^2 - 2ac\cos(B)$ respectively. From this, the cosine rule can be stated:

In any triangle ABC,

$$\begin{aligned} a^2 &= b^2 + c^2 - 2bc\cos(A) \\ b^2 &= a^2 + c^2 - 2ac\cos(B) \\ c^2 &= a^2 + b^2 - 2ab\cos(C) \end{aligned}$$



Using the cosine rule to calculate side lengths

The cosine rule can be used to solve non-right-angled triangles if we are given either of the following:

- three sides of the triangle
- two sides of the triangle and the included angle (the angle between the given sides).

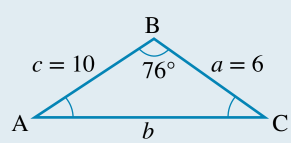
WORKED EXAMPLE 10

Determine the third side of triangle ABC given $a = 6$, $c = 10$ and $B = 76°$.

THINK

1. Draw a labelled diagram of the triangle ABC and fill in the given information.
2. Write the appropriate cosine rule to determine b.
3. Substitute the given values into the rule.
4. Evaluate.
5. Round the answer to 2 decimal places.

WRITE

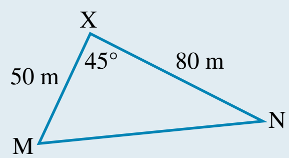


$$\begin{aligned} b^2 &= a^2 + c^2 - 2ac\cos(B) \\ &= 6^2 + 10^2 - 2 \times 6 \times 10 \times \cos(76°) \\ &\approx 106.969\,372\,5 \\ b &= \sqrt{106.969\,372\,5} \\ &\approx 10.34 \end{aligned}$$

The cosine rule also allows us to solve a wider range of practical problems. The important part of solving such problems is marking the correct information on your diagram. If you can identify two side lengths and the included angle, you can use the cosine rule.

WORKED EXAMPLE 11

A surveyor standing at point X sights point M, 50 m away and point N, 80 m away. If the angle between the lines XM and XN is 45°, calculate the distance between the points M and N, correct to 1 decimal place.

THINK	WRITE
1. Draw a diagram and mark all given information on it.	
2. Write the formula with x^2 as the subject.	$x^2 = m^2 + n^2 - 2mn\cos(X)$
3. Substitute $m = 80$, $n = 50$ and $X = 45°$.	$= 80^2 + 50^2 - 2 \times 80 \times 50 \times \cos(45°)$
4. Calculate the value of x^2.	$= 3243.15$
5. Calculate x by taking the square root of x^2.	$x = \sqrt{3243.15}$
6. Give a written answer.	$= 56.9$ m

Using the cosine rule to calculate angle sizes

We can use the cosine rule to calculate the size of the angles within a triangle. Consider the cosine rule formula.

$$a^2 = b^2 + c^2 - 2bc\cos(A)$$

We now make cos (A) the subject of this formula.

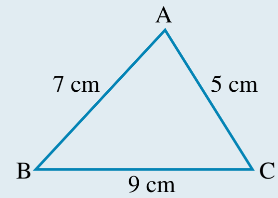
$$\begin{aligned} a^2 &= b^2 + c^2 - 2bc\cos(A) \\ a^2 + 2bc\cos(A) &= b^2 + c^2 \\ 2bc\cos(A) &= b^2 + c^2 - a^2 \\ \cos(A) &= \frac{b^2 + c^2 - a^2}{2bc} \end{aligned}$$

In this form, we can use the cosine rule to calculate the size of an angle if we are given all three side lengths. We can write the cosine rule in three different forms, depending on which angle we wish to calculate.

$$\cos(A) = \frac{b^2 + c^2 - a^2}{2bc}$$
$$\cos(B) = \frac{a^2 + c^2 - b^2}{2ac}$$
$$\cos(C) = \frac{a^2 + b^2 - c^2}{2ab}$$

WORKED EXAMPLE 12

Calculate the size of angle *B* in the triangle shown, correct to the nearest degree.

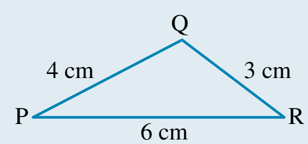


THINK	WRITE
1. Write the formula with cos (B) as the subject.	$\cos(B) = \dfrac{a^2 + c^2 - b^2}{2ac}$
2. Substitute $a = 9$, $b = 5$ and $c = 7$.	$\cos(B) = \dfrac{9^2 + 7^2 - 5^2}{2 \times 9 \times 7}$
3. Calculate the value of cos (B).	$\cos(B) = \dfrac{105}{126}$ $= 0.8333$
4. Make B the subject of the equation.	$B = \cos^{-1}(0.8333)$
5. Calculate B.	$B = 34°$

As we found earlier, the cosine ratio for an obtuse angle will be negative. So, when we get a negative result to the calculation for the cosine ratio, this means that the angle we are calculating is obtuse. Your calculator will give the obtuse angle when you take the inverse.

WORKED EXAMPLE 13

Calculate the size of angle *Q* in the triangle shown, correct to the nearest degree.

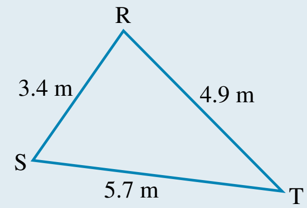


THINK	WRITE
1. Write the formula with cos (Q) as the subject.	$\cos(Q) = \dfrac{p^2 + r^2 - q^2}{2pr}$
2. Substitute $p = 3$, $q = 6$ and $r = 4$.	$\cos(Q) = \dfrac{3^2 \times 4^2 - 6^2}{2 \times 4 \times 3}$
3. Calculate the value of cos (Q).	$\cos(Q) = \dfrac{-11}{24}$ $= -0.4583$
4. Make Q the subject of the equation.	$Q = \cos^{-1}(-0.4583)$
5. Calculate Q.	$Q = 117°$

In some cosine rule questions, you have to work out which angle you need to determine. For example, you could be asked to calculate the size of the largest angle in a triangle. To do this you do not need to calculate all three angles. The largest angle in any triangle will be the one opposite the longest side. Similarly, the smallest angle will lie opposite the shortest side.

WORKED EXAMPLE 14

Calculate the size of the largest angle in the triangle shown, correct to the nearest degree.



THINK / **WRITE**

1. ST is the longest side. Therefore, angle R is the largest angle.
2. Write the formula with cos (R) as the subject.
   $\cos(R) = \dfrac{s^2 + t^2 - r^2}{2st}$
3. Substitute $r = 5.7$, $s = 4.9$ and $t = 3.4$.
   $\cos(R) = \dfrac{4.9^2 + 3.4^2 - 5.7^2}{2 \times 4.9 \times 3.4}$
4. Calculate the value of cos (R).
   $\cos(R) = \dfrac{3.08}{33.32}$
   $= 0.0924$
5. Make R the subject of the equation.
   $R = \cos^{-1}(0.0924)$
6. Calculate R.
   $R = 85°$
7. Give a written answer.
   The largest angle in the triangle is 85°.

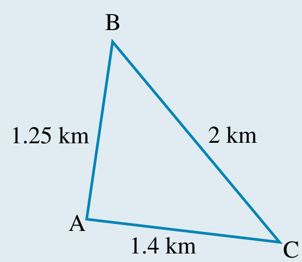
Many problems that require you to determine an angle are solved using the cosine rule. As always, these begin with a diagram and are finished off by giving a written answer.

WORKED EXAMPLE 15

Two paths diverge from point A. The first path goes for 1.25 km to point B. The second path goes for 1.4 km to point C. Points B and C are exactly 2 km apart. Calculate the angle at which the two paths diverge, correct to the nearest degree.

THINK / **WRITE**

1. Draw a diagram.



2. Write the formula with cos (A) as the subject.
   $\cos(A) = \dfrac{b^2 + c^2 - a^2}{2bc}$
3. Substitute $a = 2$, $b = 1.4$ and $c = 1.25$.
   $\cos(A) = \dfrac{1.4^2 + 1.25^2 - 2^2}{2 \times 1.4 \times 1.25}$
4. Calculate the value of cos (A).
   $\cos(A) = \dfrac{-0.4775}{3.5}$
   $= -0.1364$

5. Make A the subject of the equation. $A = \cos^{-1}(-0.1364)$

6. Calculate the value of A. $= 98°$

7. Give a written answer. The roads diverge at an angle of 98°.

on Resources

Interactivities: The cosine rule (int-6276)
Solving non-right-angled triangles (int-6482)

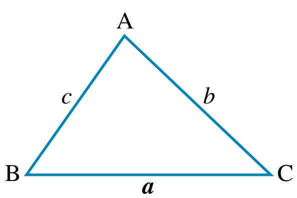
study on

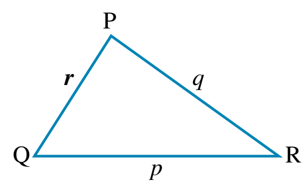
Units 3 & 4 > Area 5 > Sequence 1 > Concept 3 > **The cosine rule** Summary screen and practice questions

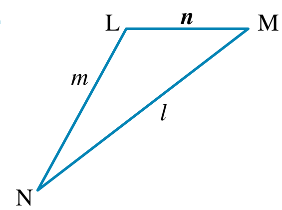
Exercise 9.4 The cosine rule

Technology free

1. Write down the cosine rule formula as it applies to each of the triangles below. In each case, make the boldfaced pronumeral the subject.

a.


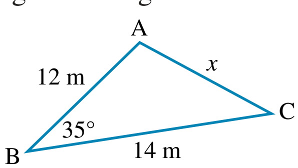
b.


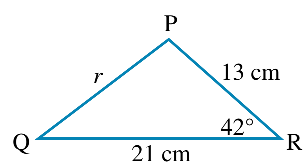
c.


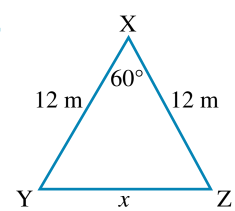
Technology active

2. Calculate the third side of triangle ABC given $a = 3.4$, $b = 7.8$ and $C = 80°$.

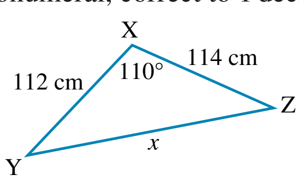
3. **WE10** Calculate the length of the side marked with a pronumeral in each of the following, correct to 3 significant figures.

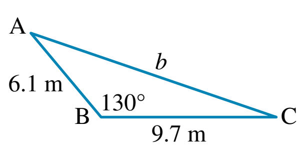
a.


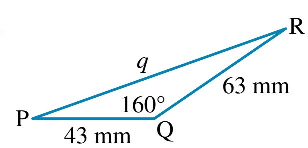
b.


c.


4. In each of the following obtuse-angled triangles, calculate the length of the side marked with the pronumeral, correct to 1 decimal place.

a.


b.


c.


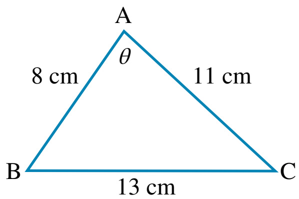
5. In triangle ABC, $b = 64.5$, $c = 38.1$ and $A = 58°34'$. Calculate a.

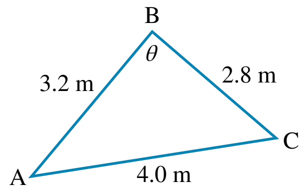
6. In triangle ABC, $a = 17$, $c = 10$ and $B = 115°$. Calculate b, and hence calculate A and C in degrees and minutes.

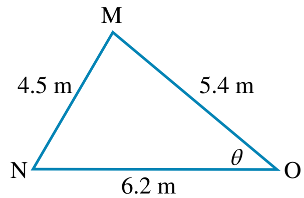
7. **WE11** Len and Morag walk separate paths that diverge from one another at an angle of 48°. After three hours Len has walked 7.9 km and Morag has walked 8.6 km. Calculate the distance between the two walkers at this time, correct to the nearest metre.

8. A cricketer is fielding 20 m from the batsman and at an angle of 35° to the pitch. The batsman hits a ball 55 m and straight behind the bowler. How far must the fieldsman run to field the ball? (Give your answer to the nearest metre.)

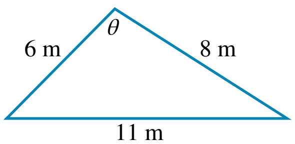
9. **WE12** Calculate the size of the angle marked with the pronumeral in each of the following triangles, correct to the nearest degree.

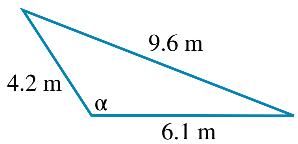
a. 

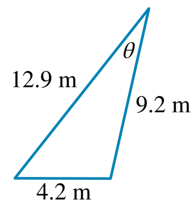
b. 

c. 

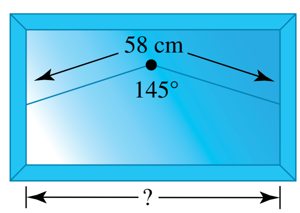
10. **WE13** In each of the obtuse-angled triangles below, calculate the size of the angle marked with the pronumeral, to the nearest degree.

a. 

b. 

c. 

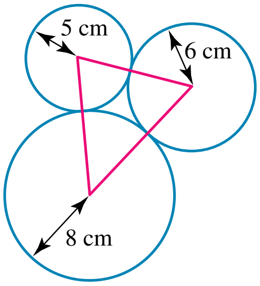
11. **WE14** In triangle ABC, $a = 356$, $b = 207$ and $c = 296$. Calculate the largest angle in degrees and minutes.
12. Calculate, in degrees and minutes, the smallest angle in the triangle with sides 6 cm, 4 cm and 8 cm.
13. In triangle ABC, $a = 23.6$, $b = 17.3$ and $c = 26.4$. Calculate the size of all the angles in degrees and minutes.
14. Calculate the size of all three angles (correct to the nearest degree) in a triangle with side lengths 12 cm, 14 cm and 17 cm.
15. **WE15** Two roads diverge from point P. The first road is 5 km long and leads to point Q. The second road is 8 km long and leads to point R. The distance between Q and R is 4.6 km. Calculate the angle at which the two roads diverge.
16. From the top of a vertical cliff 68 m high, an observer notices a yacht at sea. The angle of depression to the yacht is 47°. The yacht sails directly away from the cliff, and after 10 minutes the angle of depression is 15°. How fast does the yacht sail?
17. The cord supporting a picture frame is 58 cm long. It is hung over a single hook in the centre of the cord and the cord then makes an angle of 145° as shown in the diagram. Calculate the length of the backing of the picture frame, to the nearest centimetre.



18. A hockey goal is 3 m wide. When Sophie is 7 m from one post and 5.2 m from the other, she shoots for goal. Within what angle, to the nearest degree, must the shot be made if it is to score a goal?

19. An advertising balloon is attached to two ropes 120 m and 100 m long. The ropes are anchored to level ground 35 m apart. How high can the balloon fly?

20. Three circles of radii 5 cm, 6 cm and 8 cm are positioned so that they just touch one another. Their centres form the vertices of a triangle. Calculate the largest angle in the triangle in degrees and minutes.



9.5 Area of a triangle

For a triangle that is not right-angled, if two sides and the angle included between these two sides are known, it is also possible to calculate the area of the triangle from that given information.

Area of a right-angled triangle

$$\text{Area} = \frac{1}{2}(\text{base}) \times (\text{height})$$

Consider the triangle ABC shown, where the convention of labelling the sides opposite the angles A, B and C with lower-case letters a, b and c respectively has been adopted in the diagram.

In triangle ABC, construct the perpendicular height, h, from B to a point D on AC. As this is not necessarily an isosceles triangle, D is not the midpoint of AC.

In the right-angled triangle BCD, $\sin(C) = \frac{h}{a} \Rightarrow h = a \sin(C)$.

This means the height of triangle ABC is $a \sin(C)$ and its base is b.

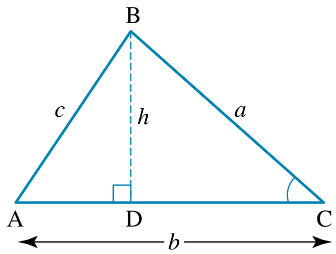
The area of the triangle ABC can now be calculated.

$$\begin{aligned}\text{Area} &= \frac{1}{2}(\text{base}) \times (\text{height}) \\ &= \frac{1}{2}b \times a \sin(C) \\ &= \frac{1}{2}ab \sin(C)\end{aligned}$$

The formula for the area of the triangle ABC, $A_\Delta = \frac{1}{2}ab \sin(C)$, is expressed in terms of two of its sides and the angle included between them.

Alternatively, using the height as $c \sin(A)$ from the right-angled triangle ABD on the left of the diagram, the area formula becomes

$A_\Delta = \frac{1}{2}bc \sin(A)$.



It can also be shown that the area is $A_\Delta = \frac{1}{2}ac \sin(B)$.

Hence, the area of a triangle is $\frac{1}{2}$ × (product of two sides) × (sine of the angle included between the two given sides).

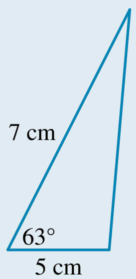
Area of a triangle

$$A_\Delta = \frac{1}{2}ab \sin(C)$$

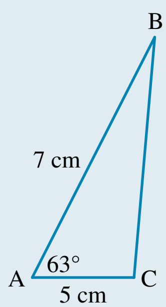
WORKED EXAMPLE 16

Calculate the areas of the following triangles. Give both answers correct to 2 decimal places.

a.

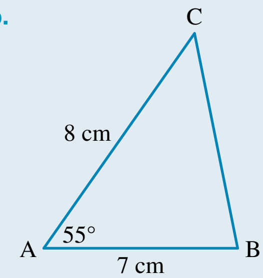


b. A triangle with sides of length 8 cm and 7 cm, and an included angle of 55°

THINK	WRITE
a. 1. Label the vertices of the triangle.	**a.** 
2. Write down the known information.	$b = 5$ cm $c = 7$ cm $A = 63°$
3. Substitute the known values into the formula to calculate the area of the triangle.	Area $= \frac{1}{2}bc \sin(A)$ $= \frac{1}{2} \times 5 \times 7 \times \sin(63°)$ $= 15.592\ldots$ $= 15.59$ (to 2 d.p.)
4. Write the answer, remembering to include the units.	The area of the triangle is 15.59 cm^2, correct to 2 decimal places.

b. 1. Draw a diagram to represent the triangle.

b.



2. Write down the known information.

$b = 8$ cm
$c = 7$ cm
$A = 55°$

3. Substitute the known values into the formula to calculate the area of the triangle.

$$\text{Area} = \frac{1}{2}bc\sin(A)$$
$$= \frac{1}{2} \times 8 \times 7 \times \sin(55°)$$
$$= 22.936\ldots$$
$$= 22.94 \text{ (to 2 d.p.)}$$

4. Write the answer, remembering to include the units.

The area of the triangle is 22.94 cm^2, correct to 2 decimal places.

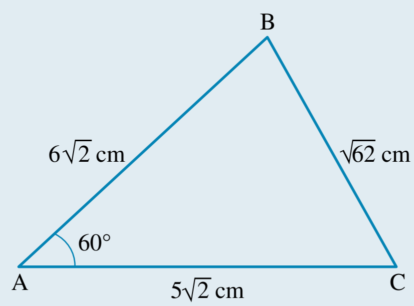
WORKED EXAMPLE 17

Calculate the exact area of the triangle ABC for which $a = \sqrt{62}$, $b = 5\sqrt{2}$, $c = 6\sqrt{2}$ cm and $A = 60°$.

THINK

WRITE

1. Draw a diagram showing the given information.
Note: The naming convention for labelling the angles and the sides opposite them with upper- and lower-case letters is commonly used.



2. State the two sides and the angle included between them.

The given angle A is included between the sides b and c.

3. State the appropriate area formula and substitute the known values.

The area formula is:

$$A_{\Delta} = \frac{1}{2}bc\sin(A),\ b = 5\sqrt{2},\ c = 6\sqrt{2},\ A = 60°$$
$$\therefore A = \frac{1}{2} \times 5\sqrt{2} \times 6\sqrt{2} \times \sin(60°)$$

4. Evaluate, using the exact value for the trigonometric ratio.

$$\therefore A = \frac{1}{2} \times 5\sqrt{2} \times 6\sqrt{2} \times \frac{\sqrt{3}}{2}$$
$$= \frac{1}{2} \times 30 \times 2 \times \frac{\sqrt{3}}{2}$$
$$= 15\sqrt{3}$$

5. State the answer.

The area of the triangle is $15\sqrt{3}\,\text{cm}^2$.

on Resources

Interactivity: Area of triangles (int-6483)

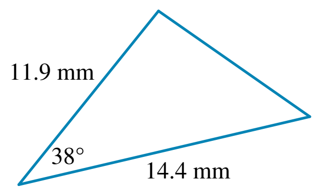
studyon

Units 3 & 4 > Area 5 > Sequence 1 > Concept 4 > **Area of a triangle** Summary screen and practice questions

Exercise 9.5 Area of a triangle

Technology active

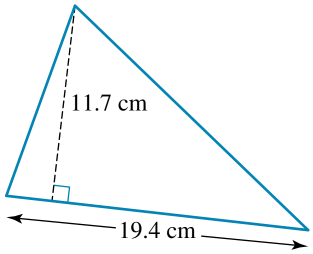
1. **WE16** Calculate the area of the following triangle, correct to 2 decimal places.



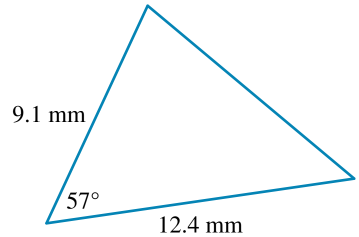
2. Calculate the area of a triangle with sides of length 14.3 mm and 6.5 mm, and an inclusive angle of 32°.
3. **WE17** Calculate the exact area of the triangle ABC for which $a = 10\,\text{cm}$, $b = 6\sqrt{2}\,\text{cm}$, $c = 2\sqrt{13}\,\text{cm}$ and $C = 45°$.
4. Horses graze over a triangular area XYZ where Y is 4 km east of X and Z is 3 km from Y on a bearing of N20°W. Over what area, correct to 2 decimal places, can the horses graze?

5. Calculate the areas of the following triangles.

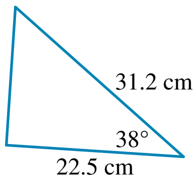
a.



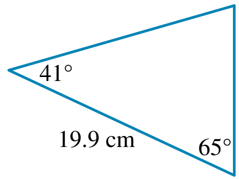
b.



c.



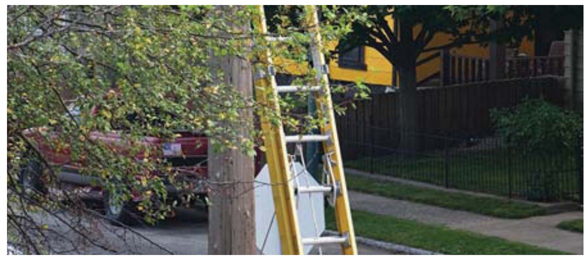
d.



6. a. An isosceles triangle ABC has sides BC and AC of equal length, 5 cm. If the angle enclosed between the equal sides is 20°, calculate:

i. the area of the triangle to 3 decimal places

ii. the length of the third side AB to 3 decimal places.



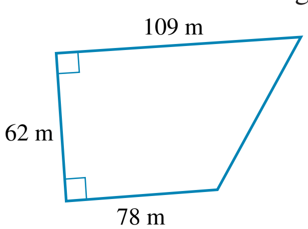
b. An equilateral triangle has a vertical height of 10 cm. Calculate the exact perimeter and area of the triangle.

c. Calculate the exact area of the triangle ABC if, using the naming convention, $a = 4\sqrt{2}$ cm, $b = 6$ cm and $C = 30°$.

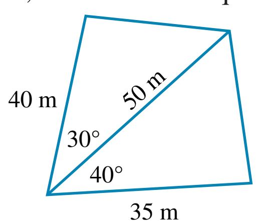
7. A triangle has two sides of length 9.5 cm and 13.5 cm, and one angle of 40.2°. Calculate all three possible areas of the triangle.

8. Calculate the areas of the regions shown, to the nearest square metre. The diagrams are not to scale.

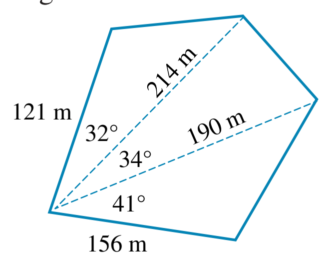
a.



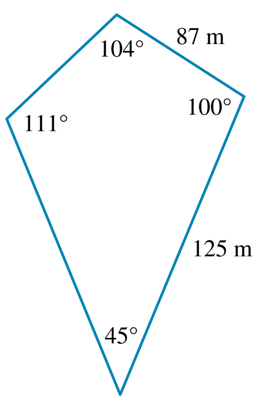
b.



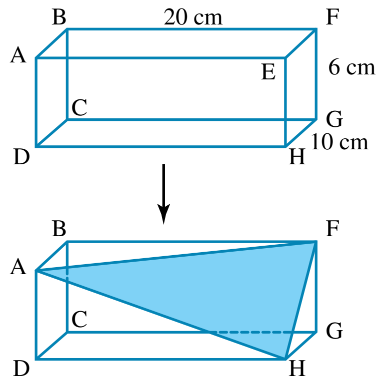
c.



9. A dry field is in the shape of a quadrilateral, as shown in the diagram. How much grass seed is needed to cover the field in 1 mm of grass seed?



10. A section is removed from a block of wood 20 cm in length, 10 cm in width and having a height of 6 cm in such a way that the triangle AFH is formed.



Calculate:

a. the area of the triangle AFH

b. the total surface area of the block of wood after the removal of the section.

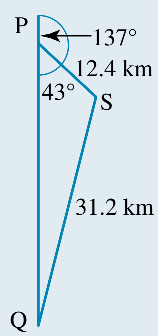
9.6 Applications of the sine and cosine rules

The principles of trigonometry have been used throughout the ages, from the construction of ancient Egyptian pyramids through to modern-day navigation. The sine and cosine rules can be used to calculate distances, heights and bearings in both two-dimensional and three-dimensional contexts.

As with determining side lengths, some questions will be problems that require you to draw a diagram to extract the required information and then give the answer in written form.

WORKED EXAMPLE 18

From point P, a ship (S) is sighted 12.4 km away on a bearing of 137°. Point Q is due south of P and is a distance of 31.2 km from the ship. Calculate the bearing of the ship from Q, correct to the nearest degree.

THINK	WRITE
1. Draw a diagram.	
2. Write the formula.	$\frac{\sin(Q)}{q} = \frac{\sin(P)}{p}$
3. Substitute for p, q and P.	$\frac{\sin(Q)}{12.4} = \frac{\sin(43°)}{31.2}$
4. Make $\sin Q$ the subject.	$\sin(Q) = \frac{12.4\sin(43°)}{31.2}$
5. Calculate a value for $\sin(Q)$.	$\sin(Q) = 0.271$

6. Calculate $\sin^{-1}(0.271)$ to calculate Q. $Q = 16°$

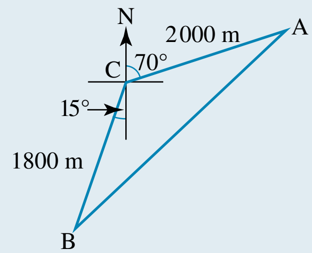
7. Give a written answer. The bearing of the ship from Q is 016° T or N16° E.

WORKED EXAMPLE 19

Two rowers set out from the same point. One rows N70°E for 2000 m and the other rows S15°W for 1800 m. How far apart are the two rowers?

THINK / **WRITE**

1. Draw a labelled diagram of the triangle, call it ABC and fill in the given information.



2. Write down the appropriate cosine rule to calculate side c.
$c^2 = a^2 + b^2 - 2ab\cos(C)$

3. Substitute the given values into the rule.
$= 2000^2 + 1800^2 - 2 \times 2000 \times 1800\cos(125°)$

4. Evaluate.
$\approx 11\,369\,750.342$

$c = \sqrt{11\,369\,750.342}$

5. Round the answer to 2 decimal places.
≈ 3371.91

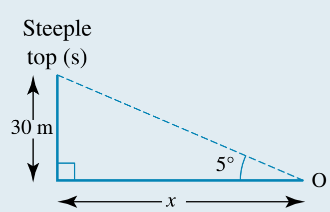
6. Give a written answer. The rowers are 3371.91 m apart.

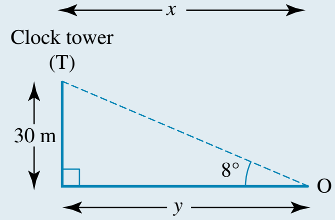
WORKED EXAMPLE 20

A surveyor records that the top of a nearby church steeple has an angle of elevation of 5°, and the top of a clock tower has an 8° angle of elevation. From his position in the middle of a paddock, the church steeple is located at N10°W, while the clock tower is at N15°E. If the tops of the church steeple and the clock tower are both 30 m above the level of the surveyor, what is the distance between them?

THINK / **WRITE**

1. There are two different planes that must be considered in this problem: the vertical plane in which the angles of elevation are taken, and the horizontal plane where the bearings of the steeple and the tower have been observed. First, draw diagrams showing the information in the vertical plane for the two structures, assigning appropriate letters for positions and unknown distances.





2. Use the tan ratio to determine the horizontal distance, x, between the surveyor and the church.

$$\tan(\theta) = \frac{\text{opp}}{\text{adj}}$$
$$\tan(5°) = \frac{30}{x}$$
$$x = \frac{30}{\tan(5°)} = 342.9\text{ m}$$

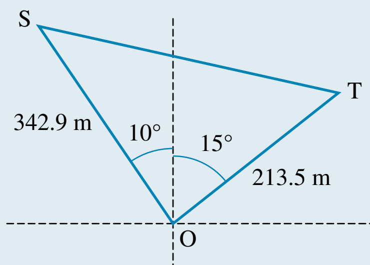
The church is located 342.9 m from the surveyor

3. Similarly, determine the horizontal distance, y, between the surveyor and the clock tower.

$$\tan(\theta) = \frac{\text{opp}}{\text{adj}}$$
$$\tan(8°) = \frac{30}{y}$$
$$y = \frac{30}{\tan(8°)} = 213.5\text{ m}$$

The clock tower is located 213.5 m from the surveyor.

4. Draw a third diagram showing the bearings of the two structures and their horizontal distances from the surveyor.



5. Use the cosine rule to set up an equation to solve for the distance between the church and the clock tower.

$$ST^2 = OS^2 + OT^2 - 2(OS)(OT)\cos(\angle SOT)$$

6. Calculate the distance and round appropriately.

$$x^2 = (342.9)^2 + (213.5)^2 - 2(342.9)(213.5)\cos(25°)$$
$$x^2 = 30\,462.6$$
$$x^2 = \sqrt{30\,462.6}$$
$$x^2 = 174.5$$

7. Write the answer.

The horizontal distance between the church and the clock tower is 174.5 m.

on Resources

Interactivity: Bearings (int-6481)

studyon

Units 3 & 4 > Area 5 > Sequence 1 > Concept 5

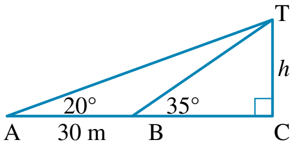
Applications of the sine and cosine rules Summary screen and practice questions

Exercise 9.6 Applications of the sine and cosine rules

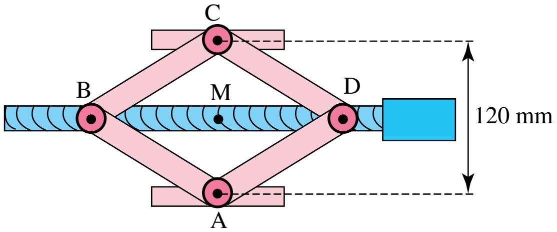
Technology active

1. **WE18** A, B and C are three towns marked on a map. Judy calculates that the distance between A and B is 45 km and the distance between B and C is 32 km. ∠CAB is 45°. Calculate ∠ACB, correct to the nearest degree.
2. A river has parallel banks that run directly east–west. Kylie takes a bearing to a tree on the opposite side. The bearing is 047° T. She then walks 10 m due east, and takes a second bearing to the tree. This is 305° T. Calculate:
   a. her distance from the second measuring point to the tree
   b. the width of the river, to the nearest metre.
3. A cross-country runner runs at 8 km/h on a bearing of 150° T for 45 minutes, then changes direction to a bearing of 053° T and runs for 80 minutes until she is due east of the starting point.
   a. How far was the second part of the run?
   b. What was her speed for this section?
   c. How far does she need to run to get back to the starting point?

4. From a fire tower, A, a fire is spotted on a bearing of N42°E. From a second tower, B, the fire is on a bearing of N12°W. The two fire towers are 23 km apart, and A is N63°W of B. How far is the fire from each tower?
5. An observer sights the top of a building at an angle of elevation of 20°. From a point 30 m closer to the building, the angle of elevation is 35° as shown in the diagram.



   a. Calculate the size of ∠ATB.
   b. Show that the distance BT can be given by the expression
   $$BT = \frac{30\sin(20°)}{\sin(15°)}.$$
   c. Show that the height of the building can be given by the expression $h = \dfrac{30\sin(20°)\sin(35°)}{\sin(15°)}$.
   d. Calculate the height of the building, correct to 1 decimal place.
6. Every car should carry a jack. One type of jack used to raise a car is a scissor-jack. A simple diagram of a scissor-jack is shown. The threaded rod is rotated to increase or decrease the length of the line segment BD.



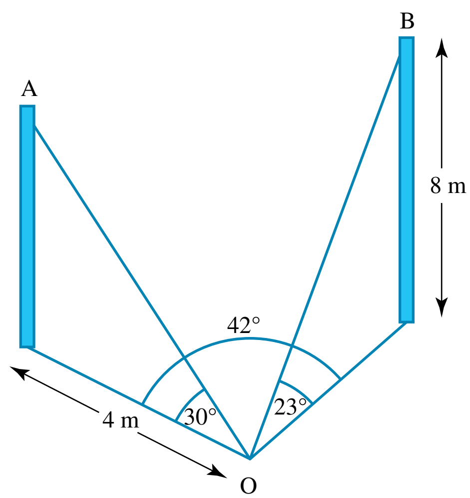
AB = BC = CD = AD = 200 mm

   a. i. In ΔBCD, M is the midpoint of BD. What is the length of CM?
      ii. If ∠BCD = 160°, what is the length of BD, correct to the nearest millimetre?
      iii. What is the size of ∠MBC?
   b. The jack is raised by reducing the length of the line segment BD.
      i. If the height of the jack, AC, is raised to 250 mm, what is the length of BD, correct to the nearest millimetre?
      ii. If ∠MBC is 70°, what is the length of BD and what is the height of the jack?

7. A ship sails on a bearing of S20°W for 14 km, then changes direction and sails for 20 km and drops anchor. Its bearing from the starting point is now N65°W.
   a. How far is it from the starting point?
   b. On what bearing did it sail the 20 km leg?
8. Two rowers set out from the same point. One rows N30°E for 1500 m and the other rows S40°E for 1200 m. How far apart are the two rowers, correct to the nearest metre?
9. **WE19** Ship A is 16.2 km from port on a bearing of 053° T and ship B is 31.6 km from the same port on a bearing of 117° T. Calculate the distance between the two ships, correct to 1 decimal place.
10. Mario cycles 12 km in a direction N68°W, then 7 km in a direction of N34°E.
   a. How far is he from his starting point?
   b. What is the bearing of the starting point from his finishing point?

11. A plane flies in a direction of N70°E for 80 km, then on a bearing of S10°W for 150 km.
   a. How far is the plane from its starting point?
   b. What direction is the plane from its starting point?
12. A plane takes off at 10:00 am from an airfield and flies at 120 km/h on a bearing of N35°W. A second plane takes off at 10:05 am from the same airfield and flies on a bearing of S80°E at a speed of 90 km/h. How far apart are the planes at 10:25 am?
13. **WE20** The pilot of a helicopter hovering 100 metres above the ocean observes a dinghy at a 30° angle of depression on a bearing of N40°E and a yacht at an angle of depression of 5° that is located at S20°E. What is the distance between the yacht and the dinghy? Give your answer to the nearest metre.
14. A mine shaft travels north, slanting down into the earth for 2 km at an angle of 10° to the earth's surface. At the end of the slope the shaft travels horizontally east for 1 km and then veers 45° towards the south for a further 800 m before ending. A new shaft is to be dug that travels in a straight line directly from this point back to the entrance on the surface.
   a. How long will the new mine shaft be?
   b. What angle will the new mine shaft make with the earth's surface at the mine entrance?
15. Seth is beginning to build a large geometric sculpture using steel poles and wire. So far, pole A and pole B have been cemented into position and tensioned wires run from the tops and bases of the poles to an anchor point, O, on the ground. The anchor point is located 4 m from the base of pole A, and the wire connecting the anchor point to the top of pole A makes an angle of 30° with the ground. The wire connecting the anchor point to the top of pole B makes an angle of 23° with the ground. Pole B is 8 m high. The wires connecting the bases of the poles with the anchor point make an angle of 42°.

   A triangular sail will fill the area bounded by the tops of poles A and B and the anchor point. Calculate the area of sail cloth required.



9.7 Review: exam practice

A summary of this chapter is available in the Resources section of your eBookPLUS at www.jacplus.com.au.

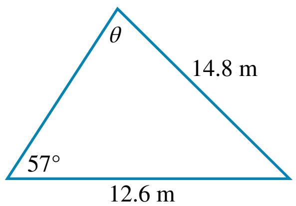
Simple familiar

1. **MC** An angle of 100° has a radian equivalent of:

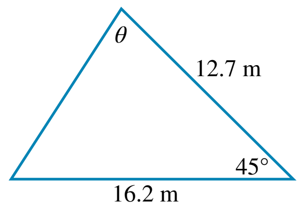
   A. $\dfrac{5\pi}{9}$ **B.** $\dfrac{7\pi}{9}$ **C.** $\dfrac{2\pi}{3}$ **D.** 0.573

2. **MC** In which of the triangles below is the information insufficient to use the sine rule to find θ?

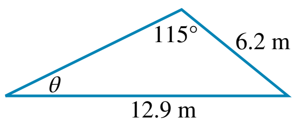
   A.



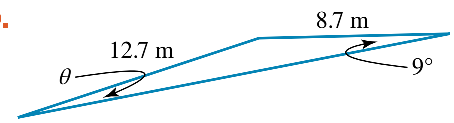
   B.



   C.



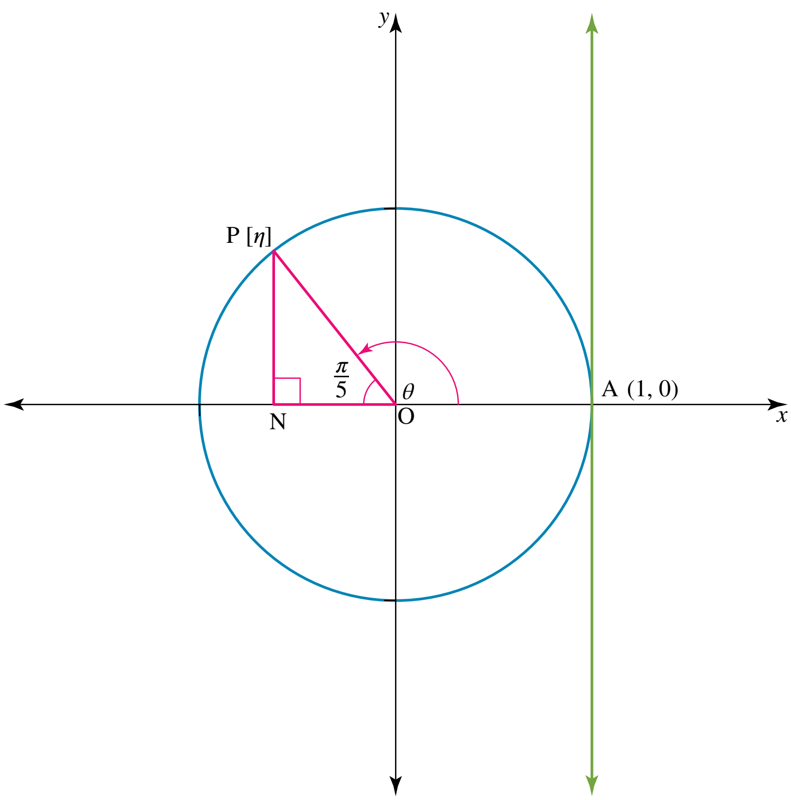
   D.



3. **MC** In a triangle ABC, $a = 5$, $b = 6$ and $c = 105°$. The length of c is:

   A. 3.04 **B.** 5.15 **C.** 8.75 **D.** 7.83

Questions **4** and **5** refer to the following figure.



4. **MC** A possible value of θ for the trigonometric point P $[\theta]$ is:

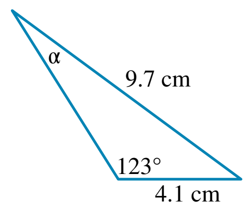
   A. $\dfrac{\pi}{5}$ **B.** $\dfrac{4\pi}{5}$ **C.** $\dfrac{6\pi}{5}$ **D.** $\dfrac{7\pi}{10}$

5. **MC** The value of $\sin(\theta)$ is given by the length of the line segment:

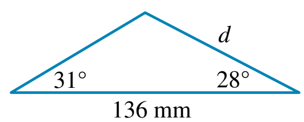
A. OP **B.** PA **C.** ON **D.** NP

6. Express $\dfrac{11\pi^c}{9}$ in degree measure.
7. Determine all solutions to the equation $3\sin(2\theta) = 1.56$ over the domain $0 \le \theta \le \dfrac{\pi}{2}$. Give your answers correct to 3 decimal places.
8. If $\sin(\theta) = -\dfrac{8}{15}$, and $\pi < \theta < \dfrac{3\pi}{2}$, determine:

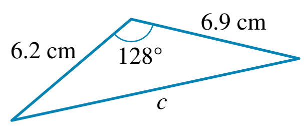
   a. $\cos(\theta)$ **b.** $\tan(\theta)$
9. In ΔXYZ, $x = 9.2$ cm, $\angle XYZ = 56°$ and $\angle YXZ = 38°$. Determine y, correct to 1 decimal place.
10. Use the sine rule to calculate the size of the angle α, correct to the nearest degree.



11. Use the sine rule to calculate d in the figure shown.



12. Use the cosine rule to determine the value of c.



Complex familiar

13. A window ledge 4 metres above the ground can just be reached by a 10-metre ladder. Exactly how far up the ladder does a person of height 1.8 metres need to climb in order for the top of the person's head to be level with the window ledge? (Assume they remain perpendicular to the ground.)
14. Calculate the exact area of an equilateral triangle that has a side length of $\sqrt{12}$ m.
15. During a stunt show two aeroplanes fly side by side until they suddenly diverge at an angle of 160°. After both planes have flown 500 m, what is the distance between the planes, correct to the nearest metre?
16. From point A on level ground, the angle of elevation to a plane is 72°. From point B on the ground, due west of A, the angle of elevation is 47°. If A and B are 3500 m apart, determine the height of the plane off the ground.

Complex unfamiliar

17. A soccer goal is 8 m wide.

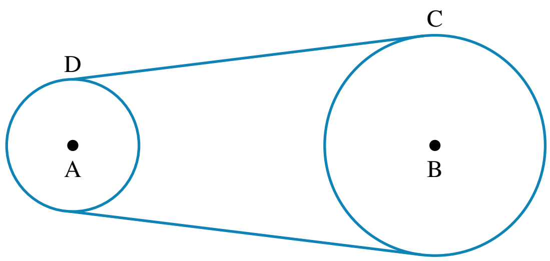
    a. A player is directly in front of the goal such that he is 12 m from each post. Within what angle must he kick the ball to score a goal?

    b. A second player takes an angled shot. This player is 12 m from the nearest post and 17 m from the far post. Within what angle must this player kick to score a goal?

18. Two lighthouses are 20 km apart on a north–south line. The northern lighthouse spots a ship on a bearing of S80°E. The southern lighthouse spots the same ship on a bearing of 040°T. Determine the distance from each lighthouse to the ship.

19. A block of land is known to be in the shape of an isosceles triangle. The unequal side is 4 km in length and the equal angles are β where $\cos(\beta) = \frac{\sqrt{5}}{3}$. Calculate the exact area of this site.

20. Two circular pulleys with radii 3 cm and 8 cm have their centres 13 cm apart. Calculate the length of the belt required to pass tightly around the pulleys, giving the answer to 1 decimal place.



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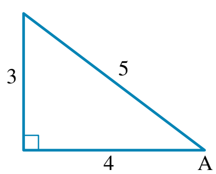
Units 3 & 4 Sit exam

Answers

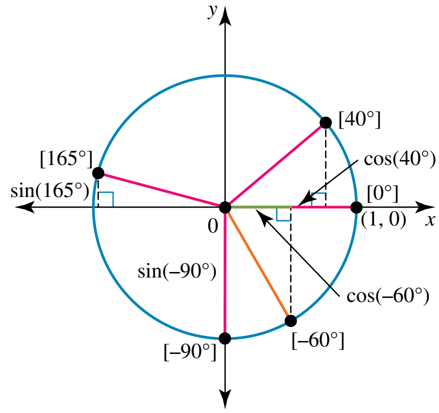
9 Cosine and sine rules

Exercise 9.2 Review of trigonometric ratios and the unit circle

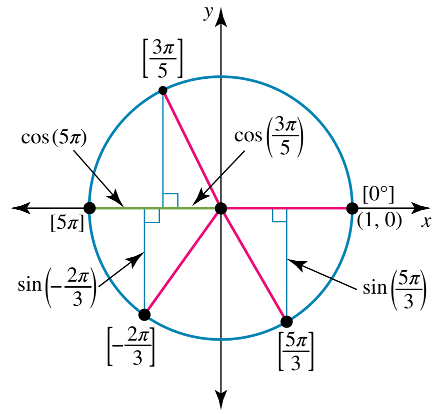
1. a. 12.1 cm b. 55.2 m c. 9.43 km
2. a. 7.66 b. 68.20°
3. a. $\left(\frac{\pi}{36}\right)^c$ b. $\left(\frac{\pi}{12}\right)^c$ c. $\left(\frac{2\pi}{3}\right)^c$ d. $\left(\frac{13\pi}{18}\right)^c$
   e. 1.12^c f. 1.38^c g. 4.10^c h. 4.54^c
   i. 5.41^c j. 6.11^c
4. a. 171.89° b. 286.48° c. 275.02°
   d. 146.68° e. 63° f. 54°
   g. 150° h. 225°
5. a. 0.389 b. 0.717 c. 0.170
   d. −0.129 e. −0.246 f. −0.916
6. a. 0.966 b. 0.927 c. −0.940
   d. −0.996 e. 11.430 f. 1.732
7. a. 0 b. 0 c. 1
   d. −1 e. Undefined f. Undefined
8. a. 1 b. 0 c. −1
   d. 1 e. Undefined f. 0
9. a. 1 b. 1 c. 1 d. 1
   e. 1 f. 1 g. 2 h. 5
10. a. sin(240°), sin(150°), sin(35°), sin(120°), sin(70°)
    b. cos (3.34), cos(1.5), cos(5.3), cos(0.2), cos(6.3)
11. $\frac{8}{15}$
12. $\frac{3}{4}$



13. a. 2.2904, 3.9928, 8.5736, 10.2760
    b. 1.1442, 1.9973, 7.4274, 8.2805
    c. 1.0701, 5.2130, 7.3533, 11.4962
    d. 3.5217, 5.9031, 9.8049, 12.1863
14. a. 0°, 180°, 360°
    b. 105°, 165°, 285°, 345°
    c. 45°, 75°, 165°, 195°, 285°, 315°
    d. 20°, 40°, 140°, 160°, 260°, 280°
    e. 62.40°, 117.60°, 182.40°, 237.6°, 302.40°, 357.60°
    f. 39.44°, 140.56°, 219.44°, 320.56°
    g. 26.39°, 333.61°
    h. 101.22°, 258.78°
15.



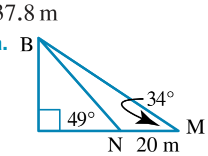
16.



17. a. $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ b. $\left[\frac{3\pi}{2}\right], \left[-\frac{\pi}{2}\right]$ or 270°, –90°

Exercise 9.3 The sine rule

1. a. $\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}$
   b. $\frac{x}{\sin(X)} = \frac{y}{\sin(Y)} = \frac{z}{\sin(Z)}$
   c. $\frac{p}{\sin(P)} = \frac{q}{\sin(Q)} = \frac{r}{\sin(R)}$
2. a. 14.8 cm b. 1.98 km c. 112 mm
3. a. 10.0 mm b. 22.1 cm c. 39.6 km
4. 9.8 cm
5. 27.0 m
6. 37.8 m
7. a.



   b. 43.2 m
   c. 33 m
8. 43.62 m
9. a. 43° b. 34° c. 27°
10. 38°
11. 20°
12. a. 57° b. 63°
13. 48.8° or 131.2°
14. 27.3° or 152.7°
15. 45 m is enough as only 43 m is required.
16. 54°

Exercise 9.4 The cosine rule

1. a. $a^2 = b^2 + c^2 - 2bc\cos(A)$
   b. $r^2 = p^2 + q^2 - 2pq\cos(R)$
   c. $n^2 = l^2 + m^2 - 2lm\cos(N)$
2. 7.95
3. a. 8.05 m b. 14.3 cm c. 12.0 m
4. a. 185.1 cm b. 14.4 m c. 104.4 mm
5. 55.22
6. $b = 23.08$, $A = 41°53'$, $C = 23°7'$
7. 7 km
8. 40 m
9. a. 85° b. 83° c. 45°
10. a. 103° b. 137° c. 10°
11. 88°15′
12. 28°57′
13. $A = 61°15'$, $B = 40°$, $C = 78°45'$
14. 82°, 54°, 44°
15. 32°
16. 1.14 km/h
17. 55 cm
18. 23°
19. 89.12 m
20. 70°49′ (or 70.82°)

Exercise 9.5 Area of a triangle

1. 52.75 mm^2
2. 24.63 mm^2
3. 30 cm^2
4. 5.64 km^2
5. a. 113.49 cm^2 b. 47.32 mm^2
   c. 216.10 cm^2 d. 122.48 cm^2
6. a. i. 4.275 cm^2
      ii. 1.736 cm
   b. Perimeter $= 20\sqrt{3}$ cm; area $= \frac{100\sqrt{3}}{3}$ cm^2
   c. $6\sqrt{2}$ cm^2
7. 41.39 cm^2, 61.41 cm^2, 59.12 cm^2
8. a. 5797 m^2 b. 1062 m^2 c. 27 952 m^2
9. 8.14 m^3
10. a. 120.6 cm^2 b. 690.6 cm^2

Exercise 9.6 Applications of the sine and cosine rules

1. 84°
2. a. 6.97 m b. 4 m
3. a. 8.63 km b. 6.48 km/h c. 9.90 km
4. 22.09 km from A, 27.46 km from B
5. a. 15°
   b, c. Sample responses can be found in the worked solutions in the online resources.
   d. 22.7 m
6. a. i. 60 mm
      ii. 394 mm
      iii. 10°
   b. i. 312 mm
      ii. BD = 137 mm; AC = 376 mm
7. a. 13.11 km b. N20°47′W
8. 2218 m
9. 28.5 km
10. a. 12.57 km b. S 35°1′ E
11. a. 130 km b. S 22°12′ E
12. 74.3 km
13. 1 239 m
14. a. 1 510 m b. 13.3°
15. 28.9 m^2

9.7 Review: exam practice

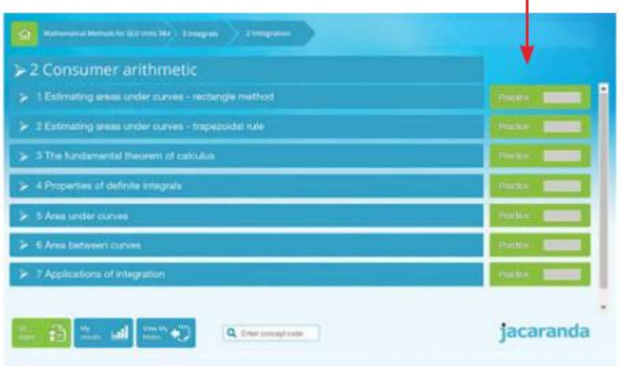
1. A
2. B
3. C
4. B
5. D
6. 220°
7. 0.273, 1.297
8. a. $\cos(\theta) = -\frac{\sqrt{161}}{15}$ b. $\tan(\theta) = \frac{8}{\sqrt{161}}$
9. 12.4 cm
10. 21°
11. 81.7 mm
12. 11.8 cm
13. 5.5 m
14. $3\sqrt{3}$ m^2
15. 949 m apart
16. 2783 m
17. a. 39°
    b. 25°
18. 14.9 km
19. $\frac{8\sqrt{5}}{5}$ km^2
20. 62.5 cm

TOPIC 2 Trigonometric functions 2

- For revision of this entire topic, go to your **studyON** title in your bookshelf at **www.jacplus.com.au**.
- Select **Continue Studying** to access hundreds of revision questions across your entire course.

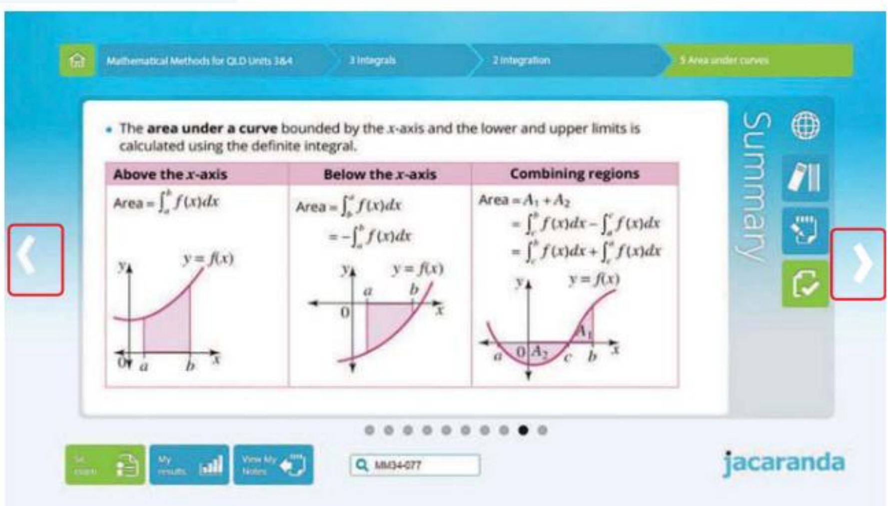
- Select your **course** *Mathematical Methods for Queensland Units 3&4* to see the entire course divided into syllabus topics.
- Select the **Area** you are studying to navigate into the chapter level *OR* select **Practice** to answer all practice questions available for each area.

- Select **Practice** at the sequence level to access all questions in the sequence.



- At **sequence level**, drill down to concept level.

- **Summary screens** provide revision and consolidation of key concepts. Select the **next arrow** to revise all concepts in the sequence and practise questions at the concept level.



10 Discrete random variables

10.1 Overview

Binomial probability involves events made up of trials that have only two outcomes — success or failure. Tossing a coin and rolling a die are two of the most commonly used examples of this type of trial, often with the caveat that they are 'fair' or 'unweighted'. A die is said to be 'loaded' or 'weighted' if it has been engineerped to produce a particular result more often than a fair die would. The most common method of doing this is to drill down into the pips of the opposite face from the one desired, fill the cavities with a heavy material such as lead, and then repaint the pips in the original colour. Loading dice in this way is much easier if they are made of an opaque material. The dice used in casinos are traditionally made from transparent materials so that it is easier to tell if they have been rigged.

Although coins can be made to have one side heavier than the other — usually by etching back the opposite face to remove metal from it — this will only make a difference to the outcome if the coin is spun on its edge rather than tossed. However, this doesn't mean that tossing the coin will remove any bias; scientific studies have found that, when tossed from a flat hand up into the air, nearly 51% of the time the coin will land on the ground with the same face up as when it was originally thrown. This is due to a combination of effects including the rotation rate of the coin as it rises and falls, the average speed at which people toss the coins up into the air, and air resistance.

LEARNING SEQUENCE

10.1 Overview
10.2 Bernoulli distributions
10.3 Bernoulli random variables
10.4 Binomial distributions
10.5 The mean and variance of a binomial distribution
10.6 Applications
10.7 Review: exam practice

Fully worked solutions for this chapter are available in the Resources section of your eBookPLUS at www.jacplus.com.au.

10.2 Bernoulli distributions

In probability theory, the **Bernoulli distribution** is a discrete probability distribution of the simplest kind. It is named after the Swiss mathematician Jacob Bernoulli (1654–1705). The term '**Bernoulli trial**' refers to a single event that has only 2 possible outcomes, a success or a failure, with each outcome having a fixed probability. The following are examples of Bernoulli trials.

- Will a coin land Heads up?
- Will a newborn child be a male or a female?
- Are a random person's eyes blue or not?
- Will a person vote for a particular candidate at the next local council elections or not?
- Will you pass or fail an examination?

WORKED EXAMPLE 1

Determine which of the following can be defined as a Bernoulli trial.

a. Interviewing a random person to see if they have had a flu vaccination this year

b. Rolling a die in an attempt to obtain an even number

c. Choosing a ball from a bag which contains 3 red balls, 5 blue balls and 4 yellow balls

THINK	WRITE
a. Check for the characteristics of a Bernoulli trial.	a. Yes, this is a Bernoulli trial, as there are 2 possible outcomes. A person either has or has not had a flu vaccination this year.
b. Check for the characteristics of a Bernoulli trial.	b. Yes, this is a Bernoulli trial, as there are 2 possible outcomes. The die will show either an odd number or an even number.
c. Check for the characteristics of a Bernoulli trial.	c. No, this is not a Bernoulli trial, as success has not been defined.

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Units 3 & 4 > Area 6 > Sequence 1 > Concept 1 > **Bernoulli distributions** Summary screen and practice questions

Exercise 10.2 Bernoulli distributions

Technology active

1. **WE 1** Determine which of the following can be defined as a Bernoulli trial.
   a. Spinning a spinner with 3 coloured sections
   b. A golfer is at the tee of the first hole of a golf course. As she is an experienced golfer, the chance of her getting a hole in one is 0.15. Will she get a hole in one at this first hole?
   c. A card is drawn from a standard pack of 52 cards. What is the chance of drawing an ace?

2. Determine which of the following can be defined as a Bernoulli trial.
   a. A new drug for arthritis is said to have a success rate of 63%. Jing Jing has just been prescribed the drug to treat her arthritis, and her doctor is interested in whether her symptoms improve or not.
   b. Juanita has just given birth to a baby, and we are interested in the gender of the baby, in particular whether the baby is a girl.
   c. You are asked what your favourite colour is.
   d. A telemarketer rings random telephone numbers in an attempt to sell a magazine subscription and has a success rate of 58%. Will the next person he rings subscribe to the magazine?
3. State clearly why the following are not Bernoulli trials.
   a. A bag contains 12 balls, 5 of which are black, 3 of which are white and 4 of which are red. Paul has just drawn a ball from the bag without returning it. Now it is Alice's turn to draw a ball from the bag. Does she get a red one?
   b. A die is tossed and the outcome is recorded.
   c. A fairy penguin colony at Phillip Island in Victoria is being studied by an ecologist. Will the habitat be able to sustain the colony in the future?

10.3 Bernoulli random variables

Bernoulli distributions are controlled by the probability of success, p. Given that there are only two possible outcomes for a single Bernoulli trial and that the sum of probabilities for that trial is 1, it can be seen that the probability of failure, q, is equal to $1 - p$.

This means that the probability distribution table for a single Bernoulli trial looks like this:

r	0	1
$\mathbf{P}(X = r)$	$1 - p$	p

where r represents the number of successes.

10.3.1 Mean of a Bernoulli distribution

As you will recall from your earlier studies of probability, the mean of a discrete random variable distribution is referred to as the **expected value**, represented by E(X) or μ. The expected value of a discrete random variable is the sum of each value of X in the distribution multiplied by its probability:

$$\mathrm{E}(X) = \sum r\,\mathrm{P}(X = r)$$

In the case of the probability distribution for a Bernoulli trial

$$\begin{aligned}\mathrm{E}(X) &= \sum r\,\mathrm{P}(X = r)\\ &= 0\,(1 - p) + 1 \times p\\ &= p\end{aligned}$$

10.3.2 Variance and standard deviation of a Bernoulli distribution

The **variance** (written as Var(X) or σ^2) and **standard deviation** (written SD(X) or σ) of any distribution are measures of spread used to indicate the range over which an outcome deviates from the expected value.

Substituting the values from the Bernoulli probability distribution table into the equation for variance:

$$\begin{aligned}\mathrm{Var}(X) &= \mathrm{E}\left(X^2\right) - [\mathrm{E}(X)]^2\\ &= \left[0^2(1 - p) + 1^2 p\right] - p^2\\ &= p - p^2\\ &= p(1 - p)\end{aligned}$$

As the standard deviation is calculated by taking the square root of the variance, we can determine an expression for the standard deviation of a Bernoulli distribution:

$$\begin{aligned}\mathrm{SD}(X) &= \sqrt{\mathrm{Var}(X)}\\ &= \sqrt{p(1 - p)}\end{aligned}$$

Note: When calculating Var(X) and SD(X), it is conventional to round to 4 decimal places unless otherwise indicated.

WORKED EXAMPLE 2

A new cream has been developed for the treatment of dermatitis. In laboratory trials the cream was found to be effective in 72% of the cases. Hang's doctor has prescribed the cream for her. Let X be the effectiveness of the cream.

a. Construct a probability distribution table for X.
b. Determine E(X).
c. Determine the variance and the standard deviation of X, correct to 4 decimal places.

THINK	WRITE
a. Construct a probability distribution table and clearly state the value of p.	**a.** p = success with cream = 0.72

r	0	1
$\mathrm{P}(X = r)$	0.28	0.72

THINK	WRITE
b. 1. State the rule for the expected value.	**b.** $\mathrm{E}(X) = \sum_{\text{all } r} r\mathrm{P}(X = r)$

2. Substitute the appropriate values and evaluate.

$E(X) = 0 \times 0.28 + 1 \times 0.72$
$= 0.72$

c. 1. Determine $E(X^2)$.

c. $E(X^2) = 0^2 \times 0.28 + 1^2 \times 0.72$
$= 0.72$

2. Calculate the variance.

$\text{Var}(X) = E(X^2) - [E(X)]^2$

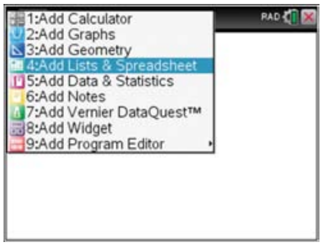
$\text{Var}(X) = 0.72 - (0.72)^2$
$= 0.2016$

3. Calculate the standard deviation.

$\text{SD}(X) = \sqrt{0.2016}$
$= 0.4490$

TI | THINK — **WRITE**

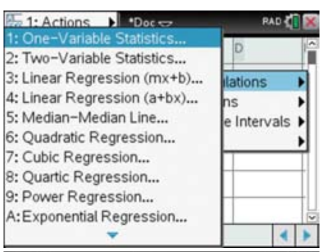
a. 1. On a New Document page, select:
4: Add Lists & Spreadsheets.



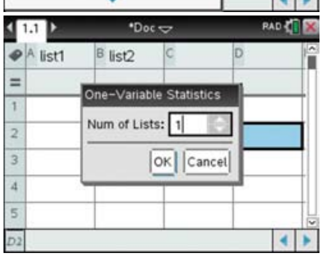
2. Define the list names of each column by completing the entry lines:
list 1
list 2

3. Complete the entries in list 1 as:
0
1
Complete the entries in list 2 as:
0.28
0.72
The probability table has been constructed.

b.1. To complete the statistical calculations, select:
MENU
4: Statistics
1: Stat Calculations
1: One-Variable Statistics.

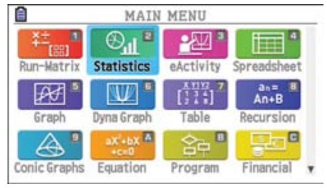


2. Set the number of lists to 1 and press the OK button.

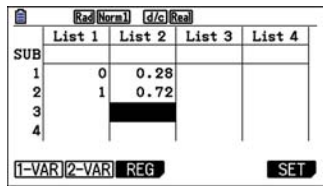


CASIO | THINK — **WRITE**

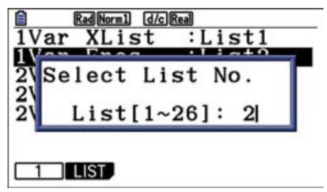
a. 1. On a Main Menu screen, select: Statistics.



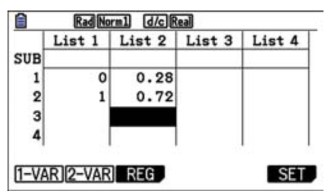
2. Complete the entry line in List 1 as:
0
1
Complete the entry line in List 2 as:
0.28
0.72



3. Select List 2 as a frequency list by pressing the SET button.
Press the LIST button.
Complete the entry line as:
2
Press the EXE button twice.

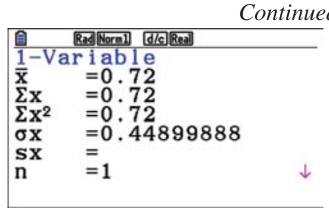


4. The probability table has been constructed.



Continued

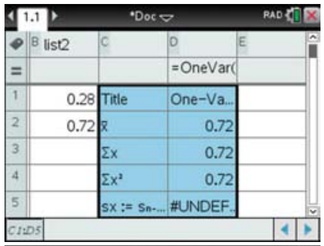
b. To complete the statistical calculations, press the 1-VAR button.
The expected value, $E(X)$, can be read from the screen.



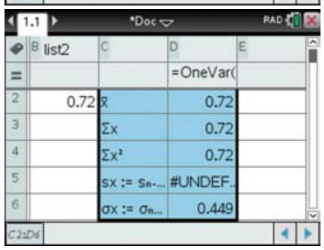
TI | THINK

3. Complete the entry lines as:
X1list: list1
Frequency List: list2
1st Result Column: c[]
Press the OK button.

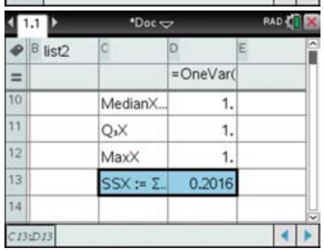
WRITE



4. The expected value E(X) can be read from the screen.



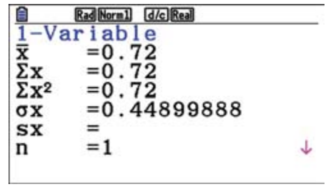
c. The variance can be read by scrolling down the screen.



CASIO | THINK

c. The standard deviation can be read from the screen.
Note: The variance can be calculated using
$\text{SD}(X) = \sqrt{\text{Var}(X)}$.

WRITE



on Resources

Interactivity: Bernoulli distribution (int-6430)

studyon

Units 3 & 4 > Area 6 > Sequence 1 > Concept 2

Bernoulli random variables Summary screen and practice questions

Exercise 10.3 Bernoulli random variables

Technology active

1. **WE2** Caitlin is playing basketball for her local club. The chance that Caitlin scores a goal is 0.42. The ball has just been passed to her and she shoots for a goal. Let X be the random variable that defines Caitlin getting a goal. (Assume X obeys the Bernoulli distribution.)
   a. Set up a probability distribution for this discrete random variable.
   b. Determine E(X).
   c. Determine:
      i. Var(X)
      ii. SD(X)
2. A discrete random variable, Z, has a Bernoulli distribution as follows.

z	0	1
$P(Z = z)$	0.37	0.63

   a. Determine E(Z).
   b. Determine Var(Z).
   c. Determine SD(Z).

3. Eli and Jacinta are about to play a game of chess. As Eli is a much more experienced chess player, the chance that he wins is 0.68. Let Y be the discrete random variable that defines the fact the Eli wins.

   a. Construct a probability distribution table for Y.
   b. Evaluate:
      i. $E(Y)$
      ii. $Var(Y)$
      iii. $SD(Y)$
   c. Calculate $P(\mu - 2\sigma \le Y \le \mu + 2\sigma)$.
4. During the wet season, the probability that it rains on any given day in Cairns in northern Queensland is 0.89. I am going to Cairns tomorrow and it is the wet season. Let X be the chance that it rains on any given day during the wet season.
   a. Construct a probability distribution table for X.
   b. Evaluate:
      i. $E(X)$
      ii. $Var(X)$
      iii. $SD(X)$
   c. Determine $P(\mu - 2\sigma \le X \le \mu + 2\sigma)$.
5. X is a discrete random variable that has a Bernoulli distribution. It is known that the variance for this distribution is 0.21.
   a. Determine the probability of success, p, where $p > 1 - p$.
   b. Determine $E(X)$.
6. Y is a discrete random variable that has a Bernoulli distribution. It is known that the standard deviation for this distribution is 0.4936.
   a. Determine the variance of Y correct to 4 decimal places.
   b. Determine the probability of success, p, if $p > 1 - p$.
   c. Determine $E(Y)$.
7. It has been found that when breast ultrasound is combined with a common mammogram, the rate in which breast cancer is detected in a group of women is 7.2 per 1000. Louise is due for her two-yearly mammography testing, which will involve an ultrasound combined with a mammogram. Let Z be the discrete random variable that breast cancer is detected.
   a. What is the probability that Louise has breast cancer detected at this next test?
   b. Construct a probability distribution table for Z.
   c. Determine $P(\mu - 2\sigma \le Z \le \mu + 2\sigma)$.
8. A manufacturer of sweets reassures their customers that when they buy a box of their 'All Sorts' chocolates there is a 33% chance that the box will contain one or more toffees. Kasper bought a box of 'All Sorts' and selected one. Let Y be the discrete random variable that Kasper chose a toffee.

   a. Construct a probability distribution table for Y.
   b. Determine $E(Y)$.
   c. Determine $P(\mu - 2\sigma \le Y \le \mu + 2\sigma)$.
9. Z is a discrete random variable that has a Bernoulli distribution. It is known that the variance of Z is 0.1075.
   a. Determine the probability of success, correct to 4 decimal places, if P(success) > P(failure).
   b. Construct a probability distribution table for Z.
   c. Evaluate the expected value of Z.
10. Y is a discrete random variable that has a Bernoulli distribution. It is known that the standard deviation of Y is 0.3316.
   a. Calculate the variance correct to 2 decimal places.
   b. Calculate the probability of success correct to 4 decimal places if P(success) > P(failure).

10.4 Binomial distributions

A **binomial distribution** results when a Bernoulli trial is carried out a number of times. A binomial distribution has the following characteristics:

1. The trials must be independent.
2. Only two possible outcomes must exist for each trial — success and failure.
3. The probability for success, p, is the same for each trial.

If a discrete random variable, X, has a binomial distribution, we can say that $X \sim \text{Bi}(n, p)$, where p is the probability of success in a single trial and n is the number of trials.

Consider the case of Cassandra, a Science student who has 5 multiple choice questions left to answer on her exam. These questions are on a topic that she has not studied. Each question has 5 different choices for the correct answer, and she plans to randomly guess the answer for every question. It can be seen that this situation represents a binomial distribution. The five questions can be considered as five independent Bernoulli trials, each having only two possible outcomes: either Cassandra guesses the correct answer to the question (a success, with $p = \frac{1}{5}$), or she guesses incorrectly (a failure, with $1 - p = \frac{4}{5}$).

There are six possible results in this case: Cassandra can get all 5 questions incorrect; 1 correct and 4 incorrect; 2 correct and 3 incorrect; 3 correct and 2 incorrect; 4 correct and 1 incorrect; or all 5 correct. Recall the concept of combinations, as the order of the correct answers out of the 5 questions is not an important consideration in this case.

As you will recall from previous study of the binomial theorem, the probability of getting r successes out of n trials can be calculated using the equation

$$\text{P}(X = r) = \binom{n}{r} p^r (1 - p)^{n-r}$$

where $\binom{n}{r} = {}^nC_r = \dfrac{n!}{(n-r)!\ r!}$.

If X represents the number of questions answered correctly, then we may calculate the probabilities for the distribution as shown in the following table given that $n = 5$, $p = \frac{1}{5}$ and $1 - p = \frac{4}{5}$.

C represents a correctly guessed answer and I represents an incorrect answer.

Number of correct answers, r	Possible outcomes	Probability $\text{P}(X = r) = \binom{n}{r} p^r (1-p)^{n-r}$
0	IIIII	$\text{P}(X = 0) = \binom{5}{0}\left(\frac{1}{5}\right)^0\left(\frac{4}{5}\right)^5 = \frac{1024}{3125} = 0.3277$
1	CIIII, ICIII, IICII, IIICI, IIIIC	$\text{P}(X = 1) = \binom{5}{1}\left(\frac{1}{5}\right)^1\left(\frac{4}{5}\right)^4 = \frac{1280}{3125} = 0.4096$
2	CCIII, ICCII, IICCI, IIICC, CICII, ICICI, IICIC, CIICI, ICIIC, CIIIC	$\text{P}(X = 2) = \binom{5}{2}\left(\frac{1}{5}\right)^2\left(\frac{4}{5}\right)^3 = \frac{640}{3125} = 0.2048$

(Continued)

(Continued)

Number of correct answers, r	Possible outcomes	Probability $P(X = r) = \binom{n}{r} p^r (1-p)^{n-r}$
3	IICCC, ICICC, CIICC, ICCIC, CICIC, CCIIC, ICCCI, CICCI, CCICI, CCCII	$P(X = 3) = \binom{5}{3}\left(\frac{1}{5}\right)^3\left(\frac{4}{5}\right)^2 = \frac{160}{3125} = 0.0512$
4	ICCCC, CICCC, CCICC, CCCIC, CCCCI	$P(X = 4) = \binom{5}{4}\left(\frac{1}{5}\right)^4\left(\frac{4}{5}\right)^1 = \frac{20}{3125} = 0.0064$
5	CCCCC	$P(X = 5) = \binom{5}{5}\left(\frac{1}{5}\right)^5\left(\frac{4}{5}\right)^0 = \frac{1}{3125} = 0.0003$

The results can then be written in the form of a probability distribution table:

r	0	1	2	3	4	5
$P(X = r)$	0.3277	0.4096	0.2048	0.0512	0.0064	0.0003

From this table, we can see that it is most likely that Cassandra will correctly guess the answer to 1 out of the 5 questions and least likely that she will guess all 5 answers correctly.

It should be noted that, if the order is specified for a particular scenario, then the binomial probability distribution rule cannot be used. Although we can calculate the chances of Cassandra getting 3 correct answers, we cannot determine the chances of her specifically getting the first, fourth and fifth questions correct using the binomial distribution.

WORKED EXAMPLE 3

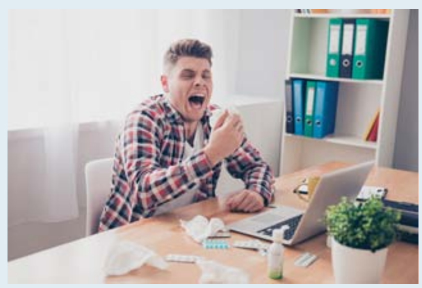
A new drug for hay fever is known to be successful in 40% of cases.

Ten hay-fever sufferers take part in the testing of the drug. Determine the probability that:

a. 4 people are cured

b. no people are cured

c. at least 2 people are cured.



THINK	WRITE
a. 1. Check that all the characteristics have been satisfied for a binomial distribution.	a. This is a binomial distribution with n independent trials and two outcomes, p and $(1 - p)$.
2. Write down the rule for the binomial probability distribution.	$P(X = r) = \binom{n}{r} p^r (1-p)^{n-r}$
3. Define and assign values to variables.	$n = 10$ Let X = number of people cured. Therefore, $r = 4$. $p = 0.4$ $(1 - p) = 0.6$

4. Substitute the values into the rule. $P(X = 4) = \binom{10}{4}(0.4)^4(0.6)^6$

5. Evaluate. $= 210 \times 0.0256 \times 0.046\,656$
$= 0.250\,822\,656$

6. Round the answer off to 4 decimal places. $= 0.2508$

7. Answer the question. The probability that 4 people are cured is 0.2508.

b. 1. Define and assign values to variables.

b. $n = 10$
Let X = number of people cured.
Therefore, $r = 0$.
$p = 0.4$
$(1 - p) = 0.6$

2. Substitute the values into the rule. $P(X = 0) = \binom{10}{0}(0.4)^0(0.6)^{10}$

3. Evaluate. $= 1 \times 1 \times 0.006\,046\,617\,6$
$= 0.006\,046\,6176$

4. Round off the answer to 4 decimal places. $= 0.0060$

5. Answer the question. The probability that no people are cured is 0.0060.

c. 1. The condition that at least 2 people are cured means that the probability will be the sum of probabilities for which $2 \le r \le 10$, that is $P(X \ge 2) = \sum_{r=2}^{10} \binom{10}{r} p^r (1 - p)^{10-r}$.
This expression would require the summation of 9 terms.
It would be easier in this case to determine the probability of the complementary event and subtract it from 1.

c. $P(X \ge 2) = 1 - P(X > 2)$
$P(X \ge 2) = 1 - [P(X = 0) + P(X = 1)]$

2. Define and assign values to variables.

From **b**, $P(X = 0) = 0.0060$.
For $P(X = 1)$:
$n = 10$
Let X = number of people cured.
Therefore, $r = 1$.
$p = 0.4$
$(1 - p) = 0.6$

3. Substitute the values into the expression. $P(X \ge 2) = 1 - \left[(0.0060) + \binom{10}{1}(0.4)^1(0.6)^9\right]$

4. Evaluate. $= 1 - [(0.0060) + (10 \times 0.4 \times 0.010\,077\,696)]$
$= 1 - [0.0060 + 0.040\,310\,784]$
$= 0.953\,689\,216$

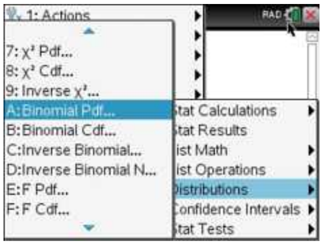
5. Round off the answer to 4 decimal places. $= 0.9537$

6. Answer the question. The probability that at least 2 people are cured is 0.9537.

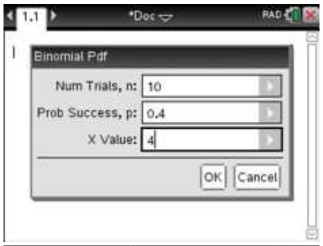
TI | THINK

WRITE

a. 1. On a Calculator page, select:
MENU
6: Statistics
5: Distributions
A: Binomial Pdf ...

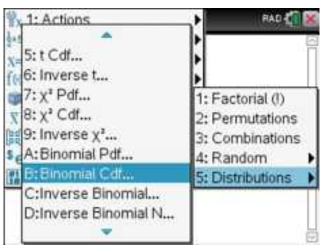


2. Complete the entry lines as:
n: 10
p: 0.4
X Value: 4
then press OK.

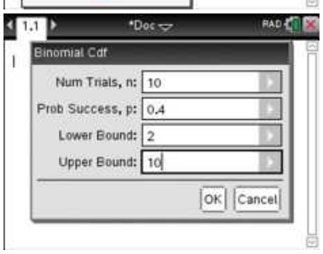


3. The answer appears on the screen.

c. 1. On a Calculator page, select:
MENU
6: Statistics
5: Distributions
B: Binomial Cdf ...



2. Complete the entry lines as:
n: 10
p: 0.4
Lower Bound: 2
Upper Bound: 10
then press the OK button.

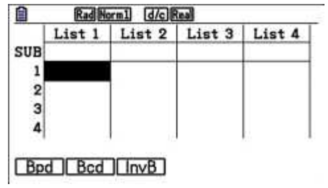


3. The answer appears on the screen.

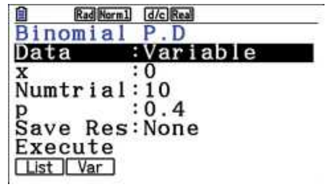
CASIO | THINK

WRITE

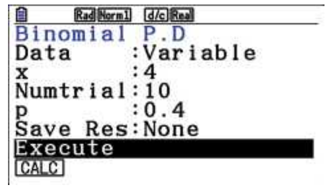
a.1. On a Statistics screen, select:
DIST
BINOMIAL
Bpd.



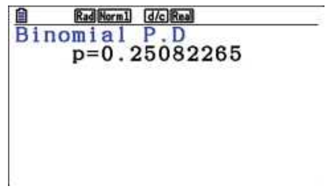
2. Select Variable by pressing the Var button.



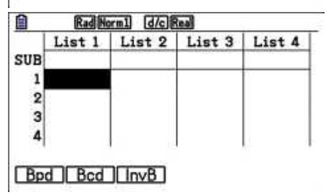
3. Complete the entry lines as:
x: 4
Numtrial: 10
p: 0.4
then press EXE.



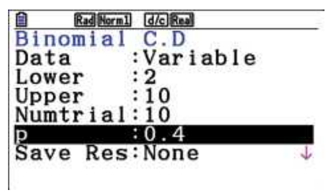
The answer appears on the screen.



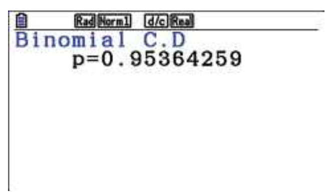
c.1. On a Statistics screen, select:
DIST
BINOMIAL
Bcd.



2. Complete the entry lines as:
Lower: 2
Upper: 10
Numtrial:10
p: 0.4
then press the EXE button.



3. The answer appears on the screen.



WORKED EXAMPLE 4

It is known that 52% of the population participates in sport on a regular basis. Five random individuals are interviewed and asked whether they participate in sport on a regular basis. Let X be the number of people who regularly participate in sport.

a. Construct a probability distribution table for X.

b. Determine the probability that 3 people or fewer play sport.

c. Determine the probability that at least one person plays sport, given that no more than 3 people play sport.

d. Determine the probability that the first person interviewed plays sport but the next 2 do not.

THINK

a. 1. Write the rule for the probabilities of the binomial distribution.

WRITE

a. $X \sim \text{Bi}(5, 0.52)$

$P(X = r) = {}^nC_r(1-p)^{n-r}p^r$

2. Substitute $r = 0$ into the rule and simplify.

$P(X = 0) = {}^5C_0(0.48)^5$
$= 0.025\,48$

3. Substitute $r = 1$ into the rule and simplify.

$P(X = 1) = {}^5C_1(0.48)^4(0.52)$
$= 0.138\,02$

4. Substitute $r = 2$ into the rule and simplify.

$P(X = 2) = {}^5C_2(0.48)^3(0.52)^2$
$= 0.299\,04$

5. Substitute $r = 3$ into the rule and simplify.

$P(X = 3) = {}^5C_3(0.48)^2(0.52)^3$
$= 0.323\,96$

6. Substitute $r = 4$ into the rule and simplify.

$P(X = 4) = {}^5C_4(0.48)(0.52)^4$
$= 0.175\,48$

7. Substitute $r = 5$ into the rule and simplify.

$P(X = 5) = {}^5C_5(0.52)^5$
$= 0.038\,02$

8. Construct a probability distribution table and check that $\sum_{\text{all } r} P(X = r) = 1$.

x	$P(X = r)$
0	0.025 48
1	0.138 02
2	0.299 04
3	0.323 96
4	0.175 48
5	0.038 02

$\sum_{\text{all } r} P(X = r) = 1$

b. 1. Interpret the question and write the probability to be found.

b. $P(X \le 3)$

2. State the probabilities included in $P(X \le 3)$.

$P(X \le 3) = P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3)$
$P(X \le 3) = 1 - (P(X = 4) + P(X = 5))$

3. Substitute the appropriate probabilities and evaluate.

$P(X \leq 3) = 1 - (0.175\,48 + 0.038\,02)$
$P(X \leq 3) = 0.7865$

c. 1. State the rule for conditional probability.

c. $P(X \geq 1 | X \leq 3) = \dfrac{P(X \geq 1 \cap X \leq 3)}{P(X \leq 3)}$

2. Evaluate $P(X \geq 1 \cap X \leq 3)$.

$$\begin{aligned} P(X \geq 1 \cap X \leq 3) &= P(1 \leq X \leq 3) \\ &= P(X = 1) + P(X = 2) + P(X = 3) \\ &= 0.138\,02 + 0.299\,04 + 0.323\,96 \\ &= 0.761\,02 \end{aligned}$$

3. Substitute the appropriate values into the rule.

$$\begin{aligned} P(X \geq 1 | X \leq 3) &= \frac{P(X \geq 1 \cap X \leq 3)}{P(X \leq 3)} \\ &= \frac{0.761\,02}{0.7865} \end{aligned}$$

4. Simplify.

$P(X \geq 1 | X \leq 3) = 0.9676$

d. 1. Order has been specified for this question. Therefore, the binomial probability distribution rule cannot be used. The probabilities must be multiplied together in order.

d. S = plays sport, N = doesn't play sport
$P(SNN) = P(S) \times P(N) \times P(N)$

2. Substitute the appropriate values and evaluate.

$$\begin{aligned} P(SNN) &= 0.52 \times 0.48 \times 0.48 \\ &= 0.1198 \end{aligned}$$

WORKED EXAMPLE 5

The probability of an Olympic archer hitting the centre of the target is 0.7. What is the smallest number of arrows he must shoot to ensure that the probability he hits the centre at least once is more than 0.9?

THINK

1. Write the rule for the probabilities of the binomial distribution.

WRITE

$X \sim \text{Bi}(n, 0.7)$
$P(X \geq 1) > 0.9$

2. The upper limit of successes is unknown, because n is unknown. Therefore, $P(X \geq 1)$ cannot be found by adding up the probabilities. However, the required probability can be found by subtracting from 1 the only probability not included in $P(X \geq 1)$.

$P(X \geq 1) = 1 - P(X = 0)$

3. Substitute in the appropriate values and simplify.

$$\begin{aligned} P(X \geq 1) &= 1 - P(X = 0) \\ 1 - P(X = 0) &> 0.9 \\ 1 -{}^{n}C_{x}(1-p)^{n-r}p^{r} &> 0.9 \\ 1 -{}^{n}C_{0}(0.3)^{n}(0.7)^{0} &> 0.9 \\ 1 - 1 \times (0.3)^{n} \times 1 &> 0.9 \\ 1 - (0.3)^{n} &> 0.9 \end{aligned}$$

4. Rearrange and take the log of both sides to determine the value of n.

$$1 - 0.9 > (0.3)^n$$
$$\log_{10}(0.1) > \log_{10}(0.3)^n$$
$$\log_{10}(0.1) > n\log_{10}(0.3)$$
$$n > \frac{\log_{10}(0.1)}{\log_{10}(0.3)}$$
$$n > 1.91249$$

5. Interpret the result and answer the question.

$n = 2$ (as n must be an integer) The smallest number of arrows the archer needs to shoot in order to guarantee a probability of 0.9 of hitting the centre is 2.

studyon

Units 3 & 4 > Area 6 > Sequence 1 > Concept 3 > **Binomial distributions** Summary screen and practice questions

Exercise 10.4 Binomial distributions

Technology active

1. Which of the following constitutes a binomial probability distribution?
   a. Rolling a die 10 times and recording the number that comes up
   b. Rolling a die 10 times and recording the number of 3s that come up
   c. Spinning a spinner numbered 1 to 10 and recording the number that is obtained
   d. Tossing a coin 15 times and recording the number of Tails obtained
   e. Drawing a card from a fair deck, without replacement, and recording the number of picture cards
   f. Drawing a card from a fair deck, with replacement, and recording the number of black cards
   g. Selecting 3 marbles from a jar containing 3 yellow marbles and 2 black marbles, without replacement
2. Seven per cent of items made by a certain machine are defective. The items are packed and sold in boxes of 50. What is the probability of 5 items being defective in a box?
3. Nadia has a 40% chance of getting a red light on the way to work. What is the probability of Nadia getting a red light on 4 out of 5 days?
4. Peter has 4 chances to knock an empty can off a stand by throwing a ball. On each throw, the probability of success is $\frac{1}{5}$. Determine the probability that he will knock the empty can off the stand:
   a. once
   b. twice
   c. at least once.
5. WE 3 Fifty-five per cent of the local municipality support the local council. If 8 people are selected at random, determine the probability that:
   a. half support the council
   b. all 8 support the council
   c. 5 support the council
   d. 3 oppose the council.
6. The probability of Colin beating Maria at golf is 0.4. If they play once a week throughout the entire year and the outcome of each game is independent of any other, determine the probability that they will have won the same number of matches.
7. It is known that 5 out of every 8 people eat Superflakes for breakfast. Determine the probability that half of a random sample of 20 people surveyed eat Superflakes.

8. On a certain evening, during a ratings period, two television stations put their best shows on against each other. The ratings showed that 39% of people watched Channel 6, while only 30% of people watched Channel 8. The rest watched other channels. A random sample of 10 people were surveyed the next day. Determine the probability that exactly:

a. 6 watched Channel 6

b. 4 watched Channel 8.

9. **WE 4** Jack is an enthusiastic darts player and on average is capable of achieving a bullseye 3 out of 7 times. Jack will compete in a five-round tournament. Let Y be the discrete random variable that defines the number of bullseyes Jack achieves.

a. Construct a probability distribution table for Y, giving your answers correct to 4 decimal places.

b. Determine the probability that Jack will score at most 3 bullseyes.

c. Determine the probability that Jack will score more than 1 bullseye, given that he scored at most 3 bullseyes.

d. Determine the probability that his first shot missed, his second shot was a bullseye and then his next 2 shots missed.

10. At a poultry farm, eggs are collected daily and classified as large or medium. Then they are packed into cartons containing 12 eggs of the same classification. From experience, the director of the poultry farm knows that 42% of all eggs produced at the farm are considered to be large. Ten eggs are randomly chosen from a conveyor belt on which the eggs are to be classified. Let Z be the discrete random variable that gives the number of large eggs.

a. Determine $P(Z = 0)$, $P(Z = 1)$ ... $P(Z = 9)$, $P(Z = 10)$ for this binomial distribution.

b. Construct a probability distribution table for Z.

c. Determine $P(Z \geq 5 | Z \leq 8)$.

11. **WE 5** The probability of winning a prize in a particular competition is 0.2. How many tickets would someone need to buy in order to guarantee them a probability of at least 0.85 of winning at least one prize?

12. Lizzie and Matt enjoy playing card games. The probability that Lizzie will beat Matt is 0.67. How many games do they need to play so that the probability of Matt winning at least one game is more than 0.9?

13. If X has a binomial distribution so that $n = 15$ and $p = 0.62$, calculate:

a. $P(X = 10)$ **b.** $P(X \geq 10)$ **c.** $P(X < 4 | X \leq 8)$

14. A particular medication used by asthma sufferers has been found to be beneficial if used 3 times a day. In a trial of the medication it was found to be successful in 63% of the cases. Eight random asthma sufferers have had the medication prescribed for them.

a. Construct a probability distribution table for the number of sufferers who have benefits from the medication, X.

b. Determine the probability that no more than 7 people will benefit from the medication.

c. Determine the probability that at least 3 people will benefit from the medication, given that no more than 7 will.

d. Determine the probability that the first person won't benefit from the medication, but the next 5 will.

15. Lilly knows that the chance of her scoring a goal during a basketball game is 0.75. What is the least number of shots that Lilly must attempt to ensure that the probability of her scoring at least 1 goal in a match is more than 0.95?

16. The tram that stops outside Maia's house is late 20% of the time. If there are 12 times during the day that the tram stops outside Maia's house, calculate, correct to 4 decimal places:
 a. the probability that the tram is late 3 times
 b. the probability that the tram is late 3 times for at least 6 out of the next 14 days.

10.5 The mean and variance of a binomial distribution

When we are working with a binomial probability distribution, it is very useful to know the mean (or expected value), the variance and the standard deviation. These values are calculated the same way as for other probability distributions that you would have encountered in Year 11.

For a binomial distribution $X \sim \text{Bi}(n, p)$, the mean is calculated simply from the following equation:

$$\mu = \text{E}(X) = np$$

This can be proven as follows.

We have already seen that $\text{P}(X = r) = \binom{n}{r} p^r (1-p)^{n-r}$, where $r = 0, 1, 2, \ldots n$. If we define a value q as being the probability of failure, then $q = 1 - p$ and we can rewrite our expression as

$\text{P}(X = r) = \binom{n}{r} p^r q^{n-r}$ where $\binom{n}{r} = \dfrac{n!}{(n-r)!\,r!}$.

For any probability distribution,

$$\mu = \text{E}(X) = \sum_{r=0}^{n} rP(X = r)$$

$$\mu = \sum_{r=0}^{n} r \binom{n}{r} p^r q^{n-r}$$

$$= \textstyle\sum_{r=0}^{n} r \dfrac{n!}{(n-r)!\,r!} p^r q^{n-r}$$

As the first term of this sum will equal 0,

$$\sum_{r=0}^{n} r \frac{n!}{(n-r)!\; r!} p^r q^{n-r} = \sum_{n}^{r=1} r \frac{n!}{(n-r)!\; r!} p^r q^{n-r}$$

Therefore,

$$\mu = \sum_{r=1}^{n} r \frac{n!}{(n-r)!\,r!} p^r q^{n-r}$$

$$= \sum_{r=1}^{n} \frac{n!}{(n-r)!\,(r-1)!} p^r q^{n-r}$$

Taking the factor np outside the summation gives

$$\mu = np \sum_{r=1}^{n} \frac{(n-1)!}{(n-r)!\,(r-1)!} p^{r-1} q^{n-r}.$$

As $\sum_{r=1}^{n} \frac{(n-1)!}{(n-r)!\,(r-1)!} p^{r-1} q^{n-r}$ is the binomial expansion of $(p+q)^{n-1}$, then

$$\begin{aligned}\mu &= np\ (p+q)^{n-1}\\ &= np\ (1)^{n-1}\\ &= np\end{aligned}$$

Hence, for a binomial distribution, $\mu = \mathrm{E}(X) = np$.
The variance of the binomial distribution $X \sim \mathrm{Bi}\,(n, p)$ is calculated using the equation

$$\sigma^2 = \mathrm{Var}\,(X) = np\,(1-p).$$

Let's look at why.
First, we know that, for any probability distribution

$$\sigma^2 = \mathrm{Var}\,(X) = \mathrm{E}\left(X^2\right) - [\mathrm{E}\,(X)]^2$$

Mathematically, it is clear that
$x^2 = x\,(x-1) + x$, so we can write

$$\begin{aligned}\mathrm{E}\left(X^2\right) &= \mathrm{E}\,[X\,(X-1) + X]\\ &= \mathrm{E}\,[X\,(X-1)] + \mathrm{E}\,(X)\end{aligned}$$

As we know that $\mathrm{E}\,(X) = np$,

$$\begin{aligned}\mathrm{E}\left(X^2\right) &= \mathrm{E}\,[X\,(X-1)] + np\\ &= np + \sum_{r=0}^{n} r\,(r-1)\,P\,(X=r)\\ &= np + \sum_{r=0}^{n} r\,(r-1)\,\frac{n!}{r!\,(n-r)!} p^r q^{n-r}\\ &= np + \sum_{r=0}^{n} \frac{n!}{(r-2)!\,(n-r)!} p^r q^{n-r}\\ &= np + n\,(n-1) \sum_{r=2}^{n} \frac{(n-2)!}{(r-2)!\,(n-r)!} p^r q^{n-r}\\ &= np + n\,(n-1)\,p^2 \sum_{r=2}^{n} \frac{(n-2)!}{(r-2)!\,(n-r)!} p^{r-2} q^{n-r}\\ &= np + n\,(n-1)\,p^2\,(p+q)^{n-2}\\ &= np + n\,(n-1)\,p^2\,(1)^{n-2}\end{aligned}$$

Thus,

$$\mathrm{E}\left(X^2\right) = np + n\,(n-1)\ p^2.$$

We may now substitute this relation into our earlier expression for variance:

$$\begin{aligned}\sigma^2 &= \mathrm{E}\left(X^2\right) - [\mathrm{E}(X)]^2 \\ &= \left[np + n(n-1)p^2\right] - (np)^2 \\ &= np + n^2p^2 - np^2 - n^2p^2 \\ &= np - np^2 \\ &= np(1-p)\end{aligned}$$

Therefore, $\sigma^2 = \mathrm{Var}(X) = np(1-p)$.

Standard deviation

As the standard deviation $\sigma = \mathrm{SD}(X) = \sqrt{\mathrm{Var}(X)}$,

$$\sigma = \mathrm{SD}(X) = \sqrt{np(1-p)}.$$

WORKED EXAMPLE 6

a. A test consists of 20 multiple choice questions, each with 5 alternatives for the answer. A student has not studied for the test, so she chooses the answers at random. Let X be the discrete random variable that describes the number of correct answers.

i. Determine the expected number of correct questions answered.

ii. Determine the standard deviation of the correct number of questions answered, correct to 4 decimal places.

b. A binomial random variable, Z, has a mean of 8.4 and a variance of 3.696.

i. Determine the probability of success, p.

ii. Determine the number of trials, n.

THINK	WRITE
a. i. 1. Write the rule for the expected value.	**a. i.** $\mathrm{E}(X) = np$
2. Substitute the appropriate values and simplify.	$n = 20, p = \frac{1}{5}$ $\mathrm{E}(X) = np$ $= 20 \times \frac{1}{5}$ $= 4$
3. Write the answer.	The expected number of questions correct is 4.
ii. 1. Write the rule for the variance.	**ii.** $\mathrm{Var}(X) = np(1-p)$
2. Substitute the appropriate values and evaluate.	$\mathrm{Var}(X) = 20 \times \frac{1}{5} \times \frac{4}{5}$ $= \frac{16}{5}$ $= 3.2$
3. Write the rule for the standard deviation.	$\mathrm{SD}(X) = \sqrt{\mathrm{Var}(X)}$

4. Substitute the variance and evaluate.

$\sigma = \sqrt{\frac{16}{5}}$
$= 1.7889$

b. i. 1. Write the rules for the variance and expected value.

b. i. $E(Z) = np$
$Var(Z) = np(1-p)$

2. Substitute the known information and label the two equations.

$8.4 = np$ [1]
$Var(Z) = np(1-p)$ [2]
$3.696 = np(1-p)$

3. To cancel out the n, divide equation [2] by equation [1].

$[2] \div [1]: \frac{np(1-p)}{np} = \frac{3.696}{8.4}$

4. Simplify.

$1 - p = 0.44$
$p = 0.56$

5. Write the answer.

The probability of success is 0.56.

ii. 1. Substitute $p = 0.56$ into $E(Z) = np$ and solve for n.

ii. $E(Z) = np$
$8.4 = n \times 0.56$
$n = 15$

2. Write the answer.

There are 15 trials.

on Resources

Interactivity: Effects of *n* and *p* on the binomial distribution (int-6432)

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Units 3 & 4 > Area 6 > Sequence 1 > Concept 4

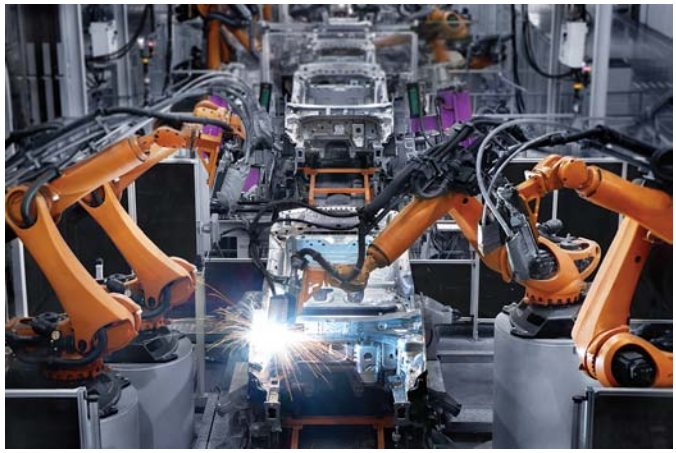
The mean and variance of a binomial distribution Summary screen and practice questions

Exercise 10.5 The mean and variance of a binomial distribution

Technology active

1. **WE6a** A fair die is tossed 25 times. Let X be the discrete random variable that represents the number of ones achieved. Determine, correct to 4 decimal places:
   a. the expected number of ones achieved
   b. the standard deviation of the number of ones achieved.
2. **WE6b** A binomial random variable, Z, has a mean of 32.535 and a variance of 9.021 95.
   a. Determine the probability of success, p.
   b. Determine the number of trials, n.
3. A fair coin is tossed 10 times. Determine:
   a. the expected number of Heads
   b. the variance for the number of Heads
   c. the standard deviation for the number of Heads.
4. A card is selected at random from a standard playing pack of 52 and then replaced. This procedure is completed 20 times. Determine:
   a. the expected number of picture cards
   b. the variance for the number of picture cards
   c. the standard deviation for the number of picture cards.

5. Six out of every 10 cars manufactured are white. Twenty cars are randomly selected. Calculate:
   a. the expected number of white cars
   b. the variance for the number of white cars
   c. the standard deviation for the number of white cars.



6. A fair die is rolled 10 times. Determine:
   a. the expected number of 2s rolled
   b. the probability of obtaining more than the expected number of 2s.
7. Eighty per cent of rabbits that contract a certain disease will die. If a group of 120 rabbits contract the disease, how many would you expect to:
   a. die?
   b. live?
8. A binomial random variable has a mean of 10 and a variance of 5. Determine:
   a. the probability of success, p
   b. the number of trials, n.
9. A binomial random variable has a mean of 12 and a variance of 3. Determine:
   a. the probability of success, p
   b. the number of trials, n.
10. For each of the following binomial random variables, calculate:
   i. the expected value
   ii. the variance.
      a. $X \sim \text{Bi}(45, 0.72)$
      b. $Y \sim \text{Bi}\left(100, \frac{1}{5}\right)$
      c. $Z \sim \text{Bi}\left(72, \frac{2}{9}\right)$
11. Four per cent of pens made at a certain factory do not work. If pens are sold in boxes of 25, determine the probability that a box contains more than the expected number of faulty pens.
12. A statistician estimates the probability that a spectator at a Brisbane Lions versus Collingwood AFL match barracks for Brisbane is $\frac{1}{2}$. At an AFL grand final between these two teams there are 100 000 spectators. Determine:
   a. the expected number of Brisbane supporters
   b. the variance of the number of Brisbane supporters
   c. the standard deviation of the number of Brisbane supporters.
13. Thirty children are given 5 different yoghurts to try. The yoghurts are marked A to E, and each child has to select his or her preferred yoghurt. Each child is equally likely to select any brand. The company running the tests manufactures yoghurt B.
   a. How many children would the company expect to pick yoghurt B?
   b. The tests showed that half of the children selected yoghurt B as their favourite. What does this tell the company manufacturing this product?
14. A binomial experiment is completed 16 times and has an expected value of 10.16.
   a. Determine the probability of success, p.
   b. Determine the variance and the standard deviation.
15. A large distributor of white goods has found that 1 in 7 people who buy goods from them do so by using their lay-by purchasing system. On one busy Saturday morning, 10 customers bought white goods. Let X be the number of people who use the lay-by purchasing system to buy their goods. Determine $E(X)$ and $\text{Var}(X)$.

10.6 Applications

The binomial distribution has important applications in medical research, quality control, simulation and genetics. In this section we will explore some of these areas.

WORKED EXAMPLE 7

It has been found that 9% of the population have diabetes. A sample of 15 people were tested for diabetes. Let X be the random variable that gives the number of people who have diabetes.

a. Determine $P(X \leq 5)$.

b. Determine $E(X)$ and $SD(X)$.

THINK	WRITE
a. 1. Define and assign values to variables.	**a.** $n = 15$ Let X = number of people who have diabetes. Therefore, $0 \leq r \leq 5$. $p = 0.09$ $(1 - p) = 0.91$
2. Substitute the values into the rule.	$P(X \leq 5) = \sum_{r=0}^{5} \binom{15}{r} (0.09)^r (0.91)^{15-r}$
3. Evaluate.	$= \binom{15}{0}(0.91)^{15} + \binom{15}{1}(0.09)^1(0.91)^{14}$ $+ \binom{15}{2}(0.09)^2(0.91)^{13} + \binom{15}{3}(0.09)^3(0.91)^{12}$ $+ \binom{15}{4}(0.09)^4(0.91)^{11} + \binom{15}{5}(0.09)^5(0.91)^{10}$ $= 0.243\,008 + 0.360\,507 + 0.249\,582 + 0.106\,964$ $+ 0.031\,736 + 0.006\,905$ $= 0.998\,702$
4. Round the answer off to 4 decimal places.	$= 0.9987$
5. Answer the question.	The probability that 5 or fewer people of the 15 selected have diabetes is 0.9987.
b. 1. State the rule for the expected value.	**b.** $E(X) = np$
2. Substitute the appropriate values and simplify.	$E(X) = 15 \times 0.09$ $= 1.35$
3. Determine the variance.	$Var(X) = np(1 - 9)$ $= 15 \times 0.09 \times 0.91$ $= 1.2285$
4. Determine the standard deviation.	$SD(X) = \sqrt{Var(X)}$ $= \sqrt{1.2285}$ $= 1.1084$

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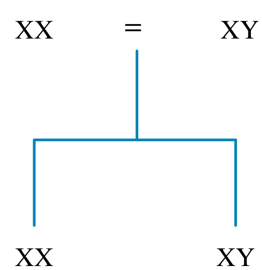
Units 3 & 4 > Area 6 > Sequence 1 > Concept 5 > **Applications** Summary screen and practice questions

Exercise 10.6 Applications

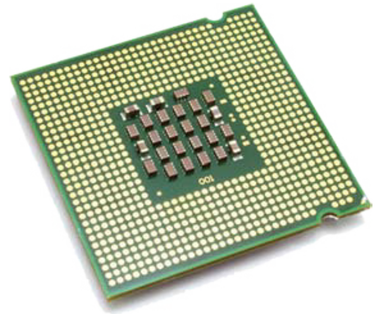
Technology active

1. **WE7** It is thought that about 30% of teenagers receive their spending money from part-time jobs. Ten random teenagers were interviewed about their spending money and how they obtained it. Let Y be the random variable that defines the number of teenagers who obtain their spending money by having a part-time job.
   a. Determine $P(Y \geq 7)$.
   b. Determine $E(Y)$ and $SD(Y)$.

2. In Australia, it is estimated that 30% of the population over the age of 25 have hypertension. A statistician wishes to investigate this, so he arranges for 15 random adults over the age of 25 to be tested to see if they have high blood pressure. Let X be the random variable that defines the number of adults over the age of 25 with hypertension.
   a. Determine $P(X \leq 5)$.
   b. Determine $E(X)$
   c. Determine $SD(X)$.
3. The proportion of defective fuses made by a certain company is 0.02. A sample of 30 fuses is taken for quality control inspection.
   a. Determine the probability that there are no defective fuses in the sample.
   b. Determine the probability that there is only 1 defective fuse in the sample.
   c. How many defective fuses would you expect in the sample?
   d. The hardware chain that sells the fuses will accept the latest batch for sale only if, upon inspection, there is at most 1 defective fuse in the sample of 30. What is the probability that they accept the batch?
   e. Ten quality control inspections are conducted monthly for the hardware chain. Determine the probability that all of these inspections will result in acceptable batches.
4. Suppose that 85% of adults with allergies report systematic relief with a new medication that has just been released. The medication has just been given to 12 patients who suffer from allergies. Let Z be the discrete random variable that defines the number of patients who get systematic relief from allergies with the new medication.
   a. Determine the probability that no more than 8 people get relief from allergies.
   b. Given that no more than 8 people get relief from allergies after taking the medication, determine the probability that at least 5 people do.
   c. Calculate:
      i. $E(Z)$
      ii. $SD(Z)$
5. Consider a woman with the genotype XX and a man with the genotype XY. Their offspring have an equal chance of inheriting one of these genotypes. What is the probability that 6 of their 7 offspring have the genotype XY?



6. Silicon chips are tested at the completion of the fabrication process. Chips either pass or fail the inspection, and if they fail they are destroyed. The probability that a chip fails an inspection is 0.02. What is the probability that in a manufacturing run of 250 chips, only 7 will fail the inspection?



7. A manufacturer of electric kettles has a process of randomly testing the kettles as they leave the assembly line to see if they are defective. For every 50 kettles produced, 3 are selected and tested for any defects. Let X be the binomial random variable that is the number of kettles that are defective, so that $X \sim \text{Bi}(3, p)$.
   a. Construct a probability distribution table for X, giving your probabilities in terms of p.
   b. Assuming $\text{P}(X = 0) = \text{P}(X = 1)$, determine the value of p where $0 < p < 1$.
   c. Determine:
      i. μ
      ii. σ
8. The probability of a person in Australia suffering anaemia is 1.3%. A group of 100 different Australians of differing ages were tested for anaemia.
   a. Determine the probability that at least 5 of the 100 Australians suffer from anaemia. Give your answer correct to 4 decimal places.
   b. Determine the probability that 4 of the 100 Australians suffer from anaemia, given that less than 10 do. Give your answer correct to 4 decimal places.
9. Edie is completing a multiple choice test of 20 questions. Each question has 5 possible answers.
   a. If Edie randomly guesses every question, what is the probability, correct to 4 decimal places, that she correctly answers 10 or more questions?
   b. If Edie knows the answers to the first 4 questions but must randomly guess the answers to the other questions, determine the probability that she correctly answers a total of 10 or more questions. Give your answer correct to 4 decimal places.
10. Six footballers are chosen at random and asked to kick a football. The probability of a footballer being able to kick at least 50 m is 0.7.
   a. Determine the probability, correct to 4 decimal places, that:
      i. only the first three footballers chosen kick the ball at least 50 m
      ii. exactly three of the footballers chosen kick the ball at least 50 m
      iii. at least three of the footballers chosen kick the ball at least 50 m, given that the first footballer chosen kicks it at least 50 m.
   b. What is the minimum number of footballers required to ensure that the probability that at least one of them can kick the ball 50 m is at least 0.95?
11. Lori is a goal shooter for her netball team. The probability of her scoring a goal is 0.85. In one particular game, Lori had 12 shots at goal. Determine the probability, correct to 4 decimal places, that:
   a. she scored at least 9 goals
   b. only her last 9 shots were goals
   c. she scored exactly10 goals, given that her last 9 shots were goals.
12. The chance of winning a prize in the local raffle is 0.08. What is the least number of tickets Siena needs to purchase so that the chance of both her and her sister each winning at least one prize is more than 0.8?

13. A regional community is trying to ensure that their local water supply has fluoride added to it, as a medical officer found that a large number of children aged between 8 and 12 have at least one filling in their teeth. In order to push their cause, the community representatives have asked a local dentist to check the teeth of ten 8– 12-year-old children from the community.

Let X be the binomial random variable that defines the number of 8– 12-year-old children who have at least one filling in their teeth: $X \sim \text{Bi}(10, p)$. Determine the value of p, correct to 4 decimal places, if $\text{P}(X \leq 8) = 0.9$.

10.7 Review: exam practice

A summary of this chapter is available in the Resources section of your eBookPLUS at www.jacplus.com.au.

Simple familiar

1. MC Which of the following does not represent a binomial distribution?
   A. Rolling a die four times and recording the number of 2s
   B. Tossing a coin 10 times and recording the number of Heads
   C. Rolling two dice simultaneously 20 times and recording the outcomes
   D. Drawing a card with replacement and recording the number of aces obtained
2. MC A Bernoulli random variable, X, has a probability of failure of 0.35. The expected value and variance of X are respectively:
   A. 0.35 and 0.2275
   B. 0.35 and 0.65
   C. 0.65 and 0.2275
   D. 0.65 and 0.35
3. MC Suppose that X is a binomial random variable with a mean of 12 and a standard deviation of 3. The probability of success, p, in any trial is:
   A. 0.25
   B. 0.35
   C. 0.50
   D. 0.75
4. MC The probability that the 7:35 am bus arrives on time is 0.45. What is the probability that the bus is on time at least once in the next 5 days?
   A. $1 - (0.55)^5$
   B. $(0.55)^5$
   C. $(0.45)^5$
   D. $1 - (0.45)^5$

5. MC If X is a binomial random variable with $n = 15$ and $p = \frac{1}{5}$, the mean and variance of X are closest to:
   A. $\mu = 3, \sigma^2 = 12$
   B. $\mu = 12, \sigma^2 = 2.4$
   C. $\mu = 2.4, \sigma^2 = 12$
   D. $\mu = 3, \sigma^2 = 2.4$

6. One-quarter of all customers at a particular bookstore buy non-fiction books. If 5 customers purchase a book on a particular day, what is the probability that 3 of them purchased a non-fiction book?

7. One in every 100 new cars is returned with faulty steering. A survey is taken of 300 buyers. Determine the probability that:
   a. none have cars with faulty steering
   b. one has a car with faulty steering.
8. A binomial random variable has a mean of 10 and variance of 8. Determine:
   a. the probability of success, p
   b. the number of trials, n.
9. Six out of every 10 cars manufactured are white. If 20 cars are selected at random, how many would you expect to be white?

10. A random variable X follows a Bernoulli sequence with a probability of success of 0.86. What is the variance of X correct to 2 decimal places?
11. A coin is biased such that the probability of a tail is 0.7. What is the probability that at most 1 Tail will be observed when the coin is tossed 5 times?
12. A binomial experiment is completed 16 times and has an expected value of 10.16.
   a. Determine the probability of success, p.
   b. Determine the variance and the standard deviation.

Complex familiar

13. Five per cent of watches made at a certain factory are defective. Watches are sold to retailers in boxes of 20. Determine:
   a. the expected number of defective watches in each box
   b. the probability that a box contains more than the expected number of defective watches per box
   c. the probability of a 'bad batch', if a 'bad batch' entails more than a quarter of the box being defective.
14. Ten per cent of all Olympic athletes are tested for drugs at the conclusion of their event. One per cent of all athletes use performance-enhancing drugs. Of the 1000 Olympic wrestlers competing from all over the world, Australia sends 10. Determine:
   a. the expected number of Australian wrestlers who are tested for drugs
   b. the probability that half the Australian wrestlers are tested for drugs
   c. the probability that at least 2 Australian wrestlers are tested for drugs
   d. the expected number of drug users among all Olympic wrestlers.

15. One-fifth of Australia's population has a British background. Fifty Australians are randomly selected and questioned about their ancestry. Determine (correct to 4 decimal places) the probability that at least 96% of the selected people have a non-British background.

16. Speedy Saverio's Pizza House claims to cook and deliver 90% of pizzas within 15 minutes of the order being placed. If your pizza is not delivered within this time, it is free. On one busy Saturday night, Saverio has to make 150 deliveries.

   a. How many deliveries are expected to be made within 15 minutes of placing the order?
   b. What is the probability of receiving a free pizza on this night?
   c. If Saverio loses an average of $4 for every late delivery, how much would he expect to lose on late deliveries this night?

Complex unfamiliar

17. An experiment consists of 3 independent trials. Each trial results in a success or failure. The probability of success in a trial is p. Determine in terms of p the probability of exactly 1 success given at least 1 success.

18. Keepers at a zoo are concerned that their herd of 10 giraffes are low in iron. In order to investigate this, they ask the zoo vet to take blood samples from all the giraffes to check the iron levels. Let X be the binomial random variable that defines the number of giraffes that have low iron levels. For this distribution, $X \sim \text{Bi}(10, p)$. Determine the value of p, correct to 4 decimal places, if $\text{P}(X \le 8) = 0.9$.

19. While on holiday at the Gold Coast, Jordan and Bronte play a total of n games of mini-golf. The probability that Jordan wins any game is 0.15. How many games of mini-golf must they play if the probability of Jordan winning exactly 2 games is 0.2759?

20. A barrel contains 100 balls, some of which have a stripe painted on them. Five balls are randomly selected from the barrel with replacement after each ball has been withdrawn. Let p be the proportion of striped balls in the barrel such that $0 < p < 1$. Using technology, determine the value of p for which the probability that exactly 1 of the 5 balls chosen has a stripe will be greatest.

studyon

Units 3 & 4 Sit exam

Answers

10 Discrete random variables

Exercise 10.2 Bernoulli distributions

1. b and c are Bernoulli trials.
2. a, b and d are Bernoulli trials.
3. a. The probability of success is not able to be calculated (as Paul has not replaced the ball he drew).
   b. There are more than two outcomes.
   c. The probability of success is not able to be calculated.

Exercise 10.3 Bernoulli random variables

1. a.

x	0	1
$P(X = x)$	0.58	0.42

b. 0.42
c. i. 0.2436 ii. 0.4936

2. a. 0.63 b. 0.2331 c. 0.4828

3. a.

y	0	1
$P(Y = y)$	0.32	0.68

b. i. 0.68 ii. 0.2176 iii. 0.4665
c. 1

4. a.

x	0	1
$P(X = x)$	0.11	0.89

b. i. 0.89 ii. 0.0979 iii. 0.3129
c. 0.89

5. a. 0.7 b. 0.7
6. a. 0.2436 b. 0.58 c. 0.58
7. a. 0.0072
   b.

z	0	1
$P(Z = z)$	0.9928	0.0072

c. 0.9928

8. a.

y	0	1
$P(Y = y)$	0.67	0.33

b. 0.33
c. 1

9. a. 0.8775
   b.

z	0	1
$P(Z = z)$	0.1225	0.8775

10. a. 0.11 b. 0.8742

Exercise 10.4 Binomial distributions

1. b, d and f are binomial; a, c, e and g are not binomial.
2. 0.1359
3. 0.0768
4. a. 0.4096 b. 0.1536 c. 0.5904
5. a. 0.2627 b. 0.0084
   c. 0.2568 d. 0.2568
6. 0.0381
7. 0.0924
8. a. 0.1023
   b. 0.2001
9. a. *See the table at the bottom of the page.
   b. 0.8891
   c. 0.9315
   d. 0.0800
10. a, b. *See the table at the bottom of the page.
    c. 0.4164
11. 9 tickets
12. 6 games
13. a. 0.1997 b. 0.4665 c. 0.0034
14. a. *See the table at the bottom of the page.
    b. 0.9752
    c. 0.9655
    d. 0.0367
15. 3 shots
16. a. 0.2362 b. 0.0890

Exercise 10.5 The mean and variance of a binomial distribution

1. a. 4.1667
   b. $\text{Var}(X) \approx 3.472$; $\text{SD}(X) = 1.8634$
2. a. 0.7227 b. 45
3. a. 5 b. 2.5 c. 1.58
4. a. 4.62 b. 3.55 c. 1.88
5. a. 12 b. 4.8 c. 2.19
6. a. 1.67 b. 0.2248
7. a. 96 b. 24
8. a. $\frac{1}{2}$ b. 20
9. a. $\frac{3}{4}$ b. 16
10. a. i. 32.4 ii. 9.072
    b. i. 20 ii. 16
    c. i. 16 ii. 12.4
11. 0.2642
12. a. 50 000 b. 25 000 c. 158.11
13. a. 6 children
    b. Yoghurt B is far more popular than expected.
14. a. 0.635
    b. $\text{Var}(X) = 3.7084$; $\text{SD}(X) = 1.9257$
15. $E(X) = 1.4286$; $\text{Var}(X) = 1.2245$

---

*9. a.

y	0	1	2	3	4	5
$P(Y = y)$	0.0609	0.2285	0.3427	0.2570	0.0964	0.0145

*10. a, b.

z	0	1	2	3	4	5	6	7	8	9	10
$P(Z = z)$	0.0043	0.0312	0.1017	0.1963	0.2488	0.2162	0.1304	0.0540	0.0147	0.0024	0.0002

*14. a.

x	0	1	2	3	4	5	6	7	8
$P(X = x)$	0.0004	0.0048	0.0285	0.0971	0.2067	0.2815	0.2397	0.1166	0.0248

Exercise 10.6 Applications

1. a. 0.0106
   b. $E(Y) = 3$; $SD(Y) = 1.4491$
2. a. 0.7216 b. 4.5 c. 1.7748
3. a. 0.5455 b. 0.3340 c. 0.6
   d. 0.8795 e. 0.2769
4. a. 0.0922
   b. 0.9992
   c. i. 10.2
      ii. 1.2369
5. 0.0547
6. 0.1051
7. a.

x	0	1	2	3
$P(X = x)$	$(1-p)^3$	$3(1-p)^2p$	$3(1-p)p^2$	p^3

   b. $\frac{1}{4}$
   c. i. $\frac{3}{4}$
      ii. $\frac{3}{4}$
8. a. 0.0101 b. 0.0319
9. a. 0.0026 b. 0.0817
10. a. i. 0.0093
       ii. 0.1852
       iii. 0.1320
    b. 3 footballers
11. a. 0.9078 b. 0.0008 c. 0.0574
12. 37 tickets
13. 0.6632

10.7 Review: exam practice

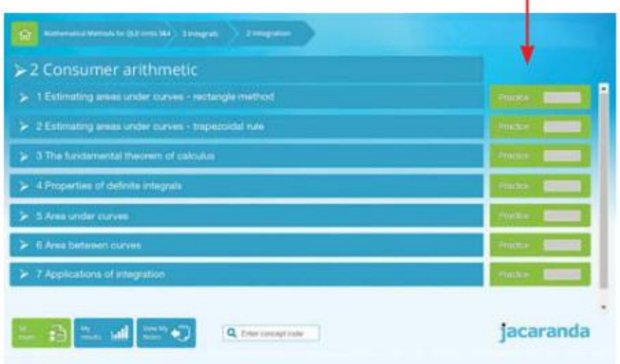
1. C
2. C
3. A
4. A
5. D
6. 0.0879
7. a. 0.0490 b. 0.1486
8. a. 0.2 b. 50
9. 12
10. 0.12
11. 0.030 78
12. a. 0.635 b. 1.9257
13. a. 1 b. 0.2642 c. 0.0003
14. a. 1 b. 0.0015 c. 0.2639 d. 10
15. 0.0013
16. a. 135
    b. 0.1
    c. Expected loss of $60
17. $\frac{3p\left(1-p\right)^2}{1-\left(1-p\right)^3}$
18. 0.6632
19. 10
20. $p = 0.2$

TOPIC 3 Discrete random variables 2

- For revision of this entire topic, go to your **studyON** title in your bookshelf at **www.jacplus.com.au**.
- Select **Continue Studying** to access hundreds of revision questions across your entire course.

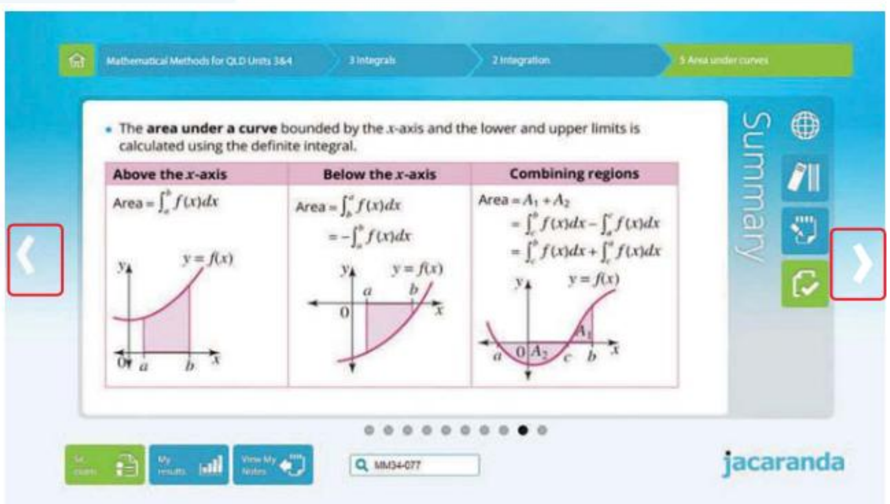
- Select your **course** *Mathematical Methods for Queensland Units 3&4* to see the entire course divided into syllabus topics.
- Select the **Area** you are studying to navigate into the chapter level ***OR*** select **Practice** to answer all practice questions available for each area.

- Select **Practice** at the sequence level to access all questions in the sequence.



- At **sequence level**, drill down to concept level.

- **Summary screens** provide revision and consolidation of key concepts. Select the **next arrow** to revise all concepts in the sequence and practise questions at the concept level.



11 General continuous random variables

11.1 Overview

Continuous random variables such as height have values that depend upon the precision of the devices used to measure them. Modern technology has allowed standardised measurements of length to a reasonable level of precision; however, the first 'standardised' units of length used over five thousand years ago were a bit unreliable. Length was described in terms of cubits, where one cubit was equal to the distance between the elbow and the tips of the finger. Obviously, with so much variation between human beings, an 'official cubit' had to be decided upon, but even then there were problems. Cubit rods found by archaeologists have varied from 47.2 to 52.5 cm in length, with each cubit broken up into 5 to 7 'palms' and a palm broken up into anywhere between 4 and 7 'fingers'. So, depending on the cubit rod used, the same person could be described as being 3 cubits, 2 palms and 1 finger tall; 3 cubits, 5 palms and 5 fingers tall; or somewhere in between.

Today, when people measure their height they can be confident that they will get the same value each time when rounded to the nearest whole centimetre. The size of this centimetre is standardised as one-hundredth of a metre. The metre itself is defined as being the distance travelled by light in a vacuum in $\frac{1}{299\,792\,458}$ of a second.

LEARNING SEQUENCE

11.1 Overview
11.2 Continuous random variables and the probability density function
11.3 Cumulative distribution functions
11.4 Measures of centre and spread
11.5 Review: exam practice

Fully worked solutions for this chapter are available in the Resources section of your eBookPLUS at www.jacplus.com.au.

11.2 Continuous random variables and the probability density function

In Chapter 10, we dealt with discrete random variables, that is, data which is finite or countable. The number of white cars in a car park, the number of students in a class and the number of lollies in a jar are all examples of discrete random variables. A **continuous random variable**, however, can assume any value within a given range.

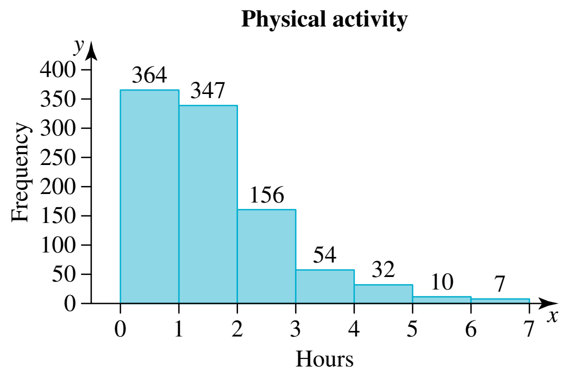
11.2.1 Discrete and continuous variables

For example, imagine that you are measuring the length of a piece of string. Using a metre ruler, you find that its length is somewhere between 9 and 10 centimetres. But is there any way of determining an *exact* value for its length?

This is the fundamental difference between discrete and continuous variables: although you can count the number of white cars in a car park and get an exact number, there is no such thing as an exact value for a continuous variable. You can use more and more precise instruments to measure that string length. For example, you might get 9.5 centimetres using a school ruler, 9.492 centimetres using engineering callipers, or even 9.492 000 000 862 centimetres using a high precision laser. But this is only narrowing the range of values within which a theoretical 'exact' value would lie — determining if it lies between 9 and 10, between 9.4 and 9.5, between 9.49 and 9.50 centimetres and so on. As a result, a continuous random variable is described in terms of the **interval** in which its value lies.

Examples of continuous random variables include time, height and weight — all quantities that must be measured rather than counted.

Consider an Australian health study that was conducted. The study targeted young people aged 5 to 17 years old. They were asked to estimate the average number of hours of physical activity they participated in each week. The results of this study are shown in the following histogram.



The frequencies of individual activity times cannot be determined due to the fact that the times have been grouped into class intervals. This limits the information we are able to extract from the histogram. For example, to determine the number of young people who did less than 2 hours of activity, we simply add the frequencies of the $0 < x < 1$ and $1 < x < 2$ class intervals; that is, $364 + 347 = 711$ people. However, we would not be able to determine the number of people who did less than 1.5 hours of physical activity as this value lies within a class interval rather than being an end point.

Since the value that a continuous random variable can assume is always measured in some way, exact values are unable to be obtained. As there are an infinite number of values that the variable can have within that interval, the probability of a continuous random variable assuming an exact value is zero.

11.2.2 Probability and relative frequency

Because histograms are drawn such that the intervals for each class are of uniform width, the probability of a variable lying within a particular interval is also equal to the fraction of the histogram area that the column for that interval occupies. In other words, the probability that x lies within a particular interval on a histogram is approximately equal to the **relative frequency** of the interval:

$$P(a < x < b) = \frac{f(a < x < b)}{\sum f}$$

So, the probability that a randomly selected person in the study did between 1 and 2 hours of activity can be determined:

$$\begin{aligned} P(1 < x < 2) &= \frac{f(1 < x < 2)}{\sum f} \\ &= \frac{347}{364 + 347 + 156 + 54 + 32 + 10 + 7} \\ &= \frac{347}{970} \\ &= 0.358 \end{aligned}$$

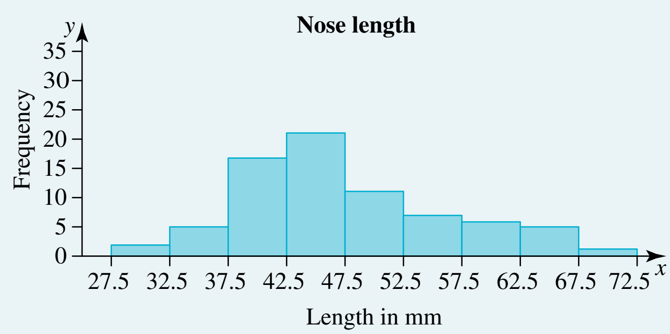
Similarly, we could find the probability that a person did more than 4 hours of activity per week by adding the interval frequencies for all intervals higher than 4:

$$\begin{aligned} P(x > 4) &= \frac{f(4 < x < 5) + f(5 < x < 6) + f(6 < x < 7)}{\sum f} \\ &= \frac{32 + 10 + 7}{970} \\ &= \frac{49}{970} \\ &= 0.051 \end{aligned}$$

WORKED EXAMPLE 1

In a study, the nose lengths, X millimetres, of 75 adults were measured. The results of the study are shown in the table and histogram.

Nose length (mm)	Frequency
$27.5 < X \le 32.5$	2
$32.5 < X \le 37.5$	5
$37.5 < X \le 42.5$	17
$42.5 < X \le 47.5$	21
$47.5 < X \le 52.5$	11
$52.5 < X \le 57.5$	7
$57.5 < X \le 62.5$	6
$62.5 < X \le 67.5$	5
$67.5 < X \le 72.5$	1



Determine the probability of someone in the study having a nose length that is:

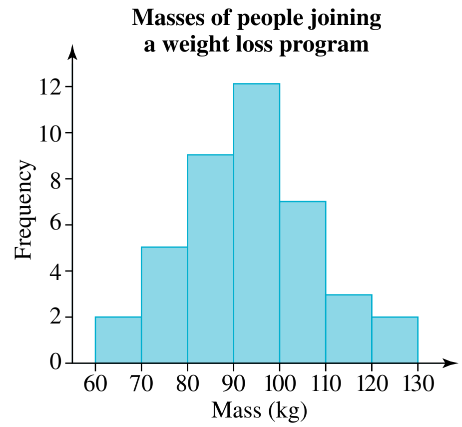
a. between 42.5 mm and 47.5 mm

b. less than or equal to 47.5 mm.

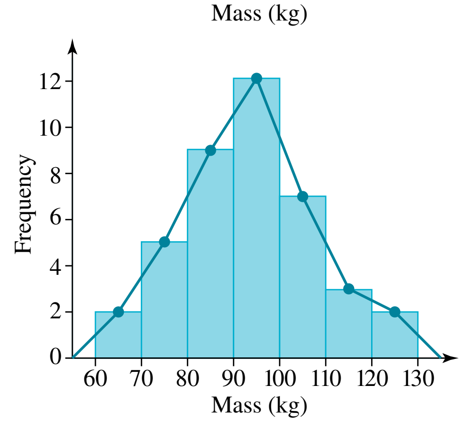
THINK	WRITE
a. 1. Use the table to determine the frequency of the interval and the sum of the frequencies for the study.	**a.** $f(42.5 < x < 47.5) = 21$ $\sum f = 2 + 5 + 17 + 21 + 11 + 7 + 6 + 5 + 1$ $= 75$
2. Substitute values into the equation and evaluate.	$P(42.5 < x < 47.5) = \dfrac{f(42.5 < x < 47.5)}{\sum f}$ $= \dfrac{21}{75}$ $= 0.28$
3. Answer the question.	The probability of a person having a nose length between 42.5 and 47.5 mm is 0.28.
b. 1. Determine the total of the frequencies for all of the intervals such that $x \leq 47.5$.	**b.** $f(x \leq 47.5) = 2 + 5 + 17 + 21 = 45$
2. Substitute values into the equation and evaluate.	$P(x \leq 47.5) = \dfrac{f(x \leq 47.5)}{\sum f}$ $= \dfrac{45}{75}$ $= 0.6$
3. Answer the question.	The probability of a person having a nose length less than or equal to 47.5 mm is 0.6.

11.2.3 Modelling continuous random variables

The histogram shown displays the masses (in kg) of members joining the Toowong branch of a popular weight loss program.

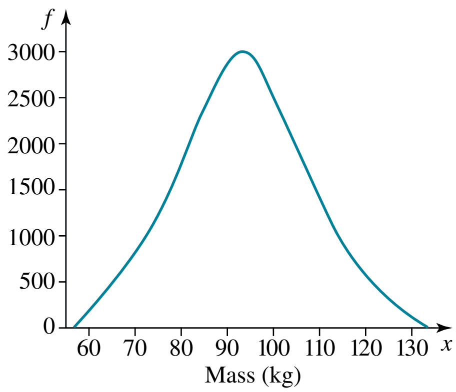


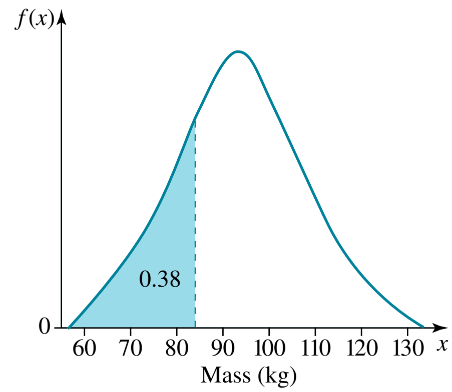
A **frequency polygon** is drawn by joining the midpoints at the top of each bar representing an interval.



This is a relatively small set of data, as the study only involved 40 people. However, if we increased the size of the study so that the members of more branches of the weight loss program were included and the class intervals were made smaller, the frequency polygon would take on the appearance of a smooth curve.

As the shape of a curve may be modelled by a mathematical formula, it should therefore be possible to develop a function, $f(x)$, that will allow us to approximate relative frequency values — and therefore probability values — for intervals regardless of the original interval boundaries that generated the function curve.





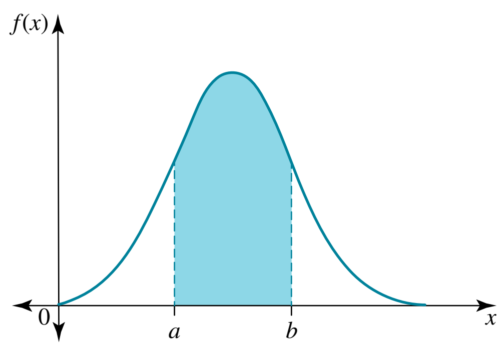
For our weight loss program curve, 38% of the area under the curve is bounded by the x-axis and the line $x = 84$. This represents the relative frequency of members having a mass of 84 kg or less. This also means that the probability of a randomly chosen person in the program having a mass of 84 kg or less is approximately 0.38; that is, $\text{P}(x < 84) = 0.38$.

The absolute area bounded by the curve and the x-axis can have any value greater than 0. However, if the curve is scaled down in such a way that the total area under the curve is equal to 1, it is referred to as the **probability density curve** (or **pdf**) for the continuous random variable.

11.2.4 The probability density curve

As you will recall from Chapter 7, the area under a curve $f(x)$ for any interval may be determined by evaluating the integral of the function over that interval. Thus, the probability that a continuous random variable X lies between the values a and b is given by:

$$\text{P}(a < X < b) = \int_a^b f(x)\,dx$$



Using this relationship, we can clarify why probabilities are assessed for continuous random variables lying within intervals rather than at exact values.

Consider a continuous variable X such that $X = x$ where $x \in R$.

$$\begin{aligned}\text{P}(X = x) &= \int_x^x f(x)\,dx \\ &= [F(x)]_x^x \\ &= F(x) - F(x) \\ \text{P}(X = x) &= 0\end{aligned}$$

Thus, $P(X = x) = 0$ when $x \in [a, b]$.

There are two critical conditions that constrain a probability density function.

First, as probability cannot be a negative number, the probability density function must be greater than or equal to zero over the interval being considered; that is,

$$f(x) \geq 0 \text{ for all } x \in [a, b].$$

Secondly, the sum of probabilities for all possible values of x must be equal to 1:

$$\int_{-\infty}^{\infty} f(x)\, dx = 1$$

(This is analogous to the condition for the discrete probability distribution that $\sum_{\text{all } x} P(X = x) = 1$.)

In the event that an interval is bounded by a and b such that $\int_a^b f(x)\, dx = 1$, then the function must be 0 everywhere else outside this interval.

WORKED EXAMPLE 2

Sketch the graph of each of the following functions and state whether each function is a probability density function.

a. $f(x) = \begin{cases} 2(x-1), & 1 \leq x \leq 2 \\ 0, & \text{elsewhere} \end{cases}$

b. $f(x) = \begin{cases} 0.5, & 2 \leq x \leq 4 \\ 0, & \text{elsewhere} \end{cases}$

c. $f(x) = \begin{cases} 2e^{-x} & 0 \leq x \leq 2 \\ 0, & \text{elsewhere} \end{cases}$

THINK

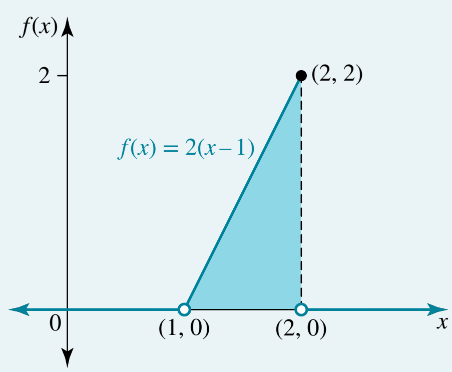
a. 1. Sketch the graph of $f(x) = 2(x-1)$ over the domain $1 \leq x \leq 2$, giving an x-intercept of 1 and an end point of $(2, 2)$. Make sure to include the horizontal lines for $y = 0$ either side of this graph.
Note: This function is known as a triangular probability function because of its shape.

2. Inspect the graph to determine if the function is always positive or zero, that is, $f(x) \geq 0$ for all $x \in [a, b]$.

3. Calculate the area of the shaded region to determine if $\int_1^2 2(x-1)\, dx = 1$.

WRITE

a.



Yes, $f(x) \geq 0$ for all x-values.

Method 1: Using the area of triangles

Area of shaded region $= \frac{1}{2} \times \text{base} \times \text{height}$

$= \frac{1}{2} \times 1 \times 2$

$= 1$

Method 2: Using calculus

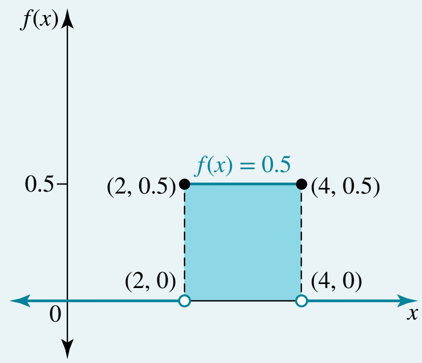
$$\text{Area of shaded region} = \int_2^1 2(x-1)\,dx$$
$$= \int_2^1 (2x-2)dx$$
$$= [x^2 - 2x]_1^2$$
$$= (2^2 - 2(2)) - (1^2 - 2(1))$$
$$= 0 - 1 + 2$$
$$= 1$$

4. Interpret the results.

$f(x) \geq 0$ for all values, and the area under the curve equals 1. Therefore, this is a probability density function.

b. 1. Sketch the graph of $f(x) = 0.5$ for $2 \leq x \leq 4$. This gives a horizontal line with end points of $(2, 0.5)$ and $(4, 0.5)$. Make sure to include the horizontal lines for $y = 0$ on either side of this graph.
Note: This function is known as a uniform or rectangular probability density function because of its rectangular shape.

b.



2. Inspect the graph to determine if the function is always positive or zero, that is, $f(x) \geq 0$ for all $x \in [a, b]$.

Yes, $f(x) \geq 0$ for all x-values.

3. Calculate the area of the shaded region to determine if $\int_2^4 0.5\,dx = 1$.

Again, it is not necessary to use calculus to determine the area.
Method 1:

$$\text{Area of shaded region} = \text{length} \times \text{width}$$
$$= 2 \times 0.5$$
$$= 1$$

Method 2:

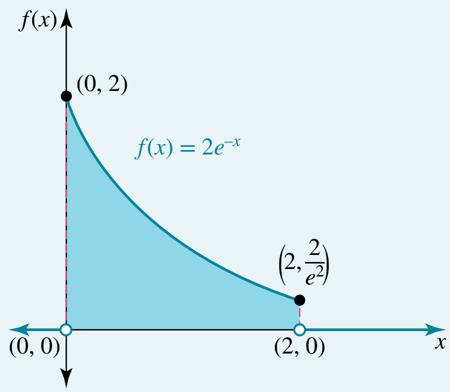
$$\text{Area of shaded region} = \int_4^2 0.5\,dx$$
$$= [0.5x]_2^4$$
$$= 0.5(4) - 0.5(2)$$
$$= 2 - 1$$
$$= 1$$

4. Interpret the results.

$f(x) \geq 0$ for all values, and the area under the curve equals 1. Therefore, this is a probability density function.

c. 1. Sketch the graph of $f(x) = 2e^{-x}$ for $0 \le x \le 2$. The end points will be $(0, 2)$ and $\left(2, e^{-2}\right)$. Make sure to include the horizontal lines for $y = 0$ on either side of this graph.

c.



2. Inspect the graph to determine if the function is always positive of zero, that is, $f(x) \ge 0$ for all $x \in [a, b]$.

Yes, $f(x) \ge 0$ for all x-values.

3. Calculate the area of the shaded region to determine if $\int_0^2 2e^{-x}dx = 1$.

$$\begin{aligned}\int_0^2 2e^{-x}dx &= 2\int_0^2 e^{-x}dx \\ &= 2\left[-e^{-x}\right]_0^2 \\ &= 2\left(-e^{-2} + e^0\right) \\ &= 2\left(-e^{-2} + 1\right) \\ &= 1.7293\end{aligned}$$

4. Interpret the results.

$f(x) \ge 0$ for all values. However, the area under the curve does not equal 1. Therefore, this is not a probability density function.

WORKED EXAMPLE 3

Given that the following functions are probability density functions, determine the value of a in each function.

a. $f(x) = \begin{cases} a(x-1)^2, & 0 \le x \le 4 \\ 0, & \text{elsewhere} \end{cases}$

b. $f(x) = \begin{cases} ae^{-4x}, & x > 0 \\ 0, & \text{elsewhere} \end{cases}$

THINK

WRITE

a. 1. As the function has already been defined as a probability density function, this means that the area under the graph is definitely 1.

a.
$$\int_0^4 f(x)dx = 1$$
$$\int_0^4 a(x-1)^2dx = 1$$

2. Remove a from the integral, as it is a constant.

$$a\int_0^4 (x-1)^2dx = 1$$

3. Antidifferentiate and substitute in the terminals.

$$a\int_0^4 (x-1)^2dx = 1$$

$$a\left[\frac{(x-1)^3}{3}\right]_0^4 = 1$$

$$a\left[\frac{3^3}{3} - \frac{(-1)^3}{3}\right] = 1$$

4. Solve for a.

$$a\left(9 + \frac{1}{3}\right) = 1$$

$$a \times \frac{28}{3} = 1$$

$$a = \frac{3}{28}$$

b. 1. As the function has already been defined as a probability density function, this means that the area under the graph is definitely 1.

b. $$\int_0^\infty f(x)dx = 1$$

$$\int_0^\infty ae^{-4x}dx = 1$$

2. Remove a from the integral, as it is a constant.

$$a\int_0^\infty e^{-4x}dx = 1$$

3. To evaluate an integral containing infinity as one of the terminals, we determine the appropriate limit.

$$a \times \lim_{k\to\infty} \int_0^k e^{-4x}dx = 1$$

4. Antidifferentiate and substitute in the terminals.

$$a \times \lim_{k\to\infty} \int_k^0 e^{-4x}dx = 1$$

$$a \times \lim_{k\to\infty} \left[-\frac{1}{4}e^{-4x}\right]_0^k = 1$$

$$a \times \lim_{k\to\infty} \left(-\frac{e^{-4k}}{4} + \frac{1}{4}\right) = 1$$

$$a \times \lim_{k\to\infty} \left(-\frac{e^{-4k}}{4} + \frac{1}{4}\right) = 1$$

$$a \times \lim_{k\to\infty} \left(-\frac{1}{4e^{4k}} + \frac{1}{4}\right) = 1$$

5. Solve for a. Remember that a number divided by an extremely large number is effectively 0, so $\lim_{k\to\infty}\left(\frac{1}{e^{4k}}\right) = 0$.

$$a\left(0 + \frac{1}{4}\right) = 1$$

$$\frac{a}{4} = 1$$

$$a = 4$$

on Resources

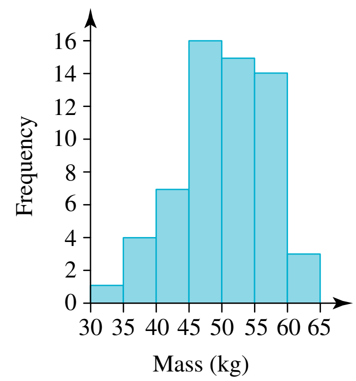
Interactivity: Continuous random variables (int-6433)

Exercise 11.2 Continuous random variables and the probability density function

Technology free

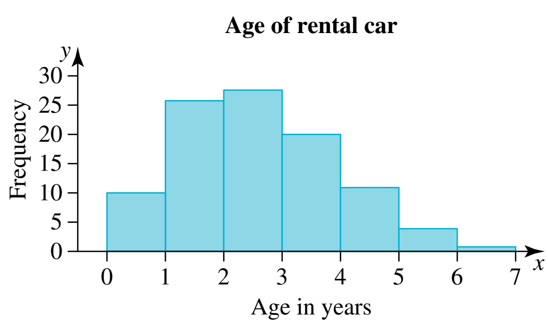
1. Which of the following are continuous random variables?
   a. The population of your town or city
   b. The types of motorbike in a parking lot
   c. The heights of people in an identification line-up
   d. The masses of babies in a group
   e. The languages spoken at home by students in your class
   f. The time spent watching TV
   g. The number of children in the families in your suburb
   h. The air pressure in your car's tyres
   i. The number of puppies in a litter
   j. The types of radio program listened to by teenagers
   k. The times for swimming 50 metres
   l. The quantity of fish caught in a net
   m. The number of CDs you own
   n. The types of shops in a shopping centre
   o. The football competition ladder at the end of each round
   p. The lifetimes of torch batteries
   q. The number of people attending a rock concert
   r. Exam grades
   s. The types of magazine sold at a news agency
   t. Hotel accommodation ratings

2. **WE1** The frequency histogram shows the distribution of masses (in kilograms) of 60 students in Year 7 at Northwood State High School.
   Determine the probability that a random student has a mass:
   a. between 40 and 60 kilograms
   b. less than 45 kilograms
   c. greater than 55 kilograms.



3. A small car-hire firm keeps note of the age and kilometres covered by each of the cars in their fleet. Generally, cars are no longer used once they have either covered 350 000 kilometres or are more than 5 years old. The following information describes the ages of the cars in their current fleet.

Age	Frequency
$0 < x \leq 1$	10
$1 < x \leq 2$	26
$2 < x \leq 3$	28
$3 < x \leq 4$	20
$4 < x \leq 5$	11
$5 < x \leq 6$	4
$6 < x \leq 7$	1



a. Determine:

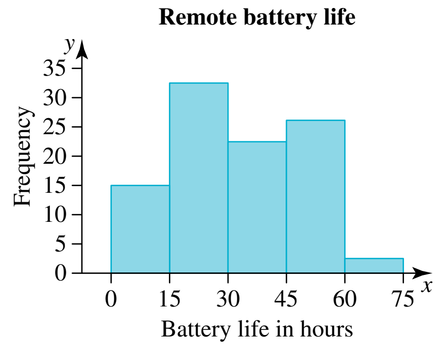
i. $P(X \leq 2)$ **ii.** $P(X > 4)$

b. Determine:

i. $P(1 < X \leq 4)$ **ii.** $P(X > 1|X \leq 4)$

4. The battery life for batteries in television remote controls was investigated in a study. The results are shown in the table and histogram.

Hours of life	Frequency
$0 < x \leq 15$	15
$15 < x \leq 30$	33
$30 < x \leq 45$	23
$45 < x \leq 60$	26
$60 < x \leq 75$	3



a. How many remote control batteries were included in the study?

b. What is the probability that a battery will last more than 45 hours?

c. What is the probability that a battery will last between 15 and 60 hours?

d. A new battery producer is advocating that their batteries have a long life of 60 + hours. If it is known that this is just advertising hype because these batteries are no different from the batteries in the study, what is the probability that these new batteries will have a life of 60 + hours?

5. **WE2** Sketch each of the following functions and determine whether each one is a probability density function.

a. $f(x) = \begin{cases} \frac{1}{4}e^{2x}, & 0 \leq x \leq \log_e(3) \\ 0, & \text{elsewhere} \end{cases}$ **b.** $f(x) = \begin{cases} 0.25, & -2 \leq x \leq 2 \\ 0, & \text{elsewhere} \end{cases}$

6. Sketch each of the following functions and determine whether each one is a probability density function.

a. $f(x) = \begin{cases} \frac{1}{2}\cos(x), & -\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \\ 0, & \text{elsewhere} \end{cases}$ **b.** $f(x) = \begin{cases} \frac{1}{2\sqrt{x}}, & \frac{1}{2} \leq x \leq 4 \\ 0, & \text{elsewhere} \end{cases}$

7. **WE3** Given that the following function is a probability density function, determine the value of n.

$$f(x) = \begin{cases} n(x^3 - 1), & 1 \leq x \leq 3 \\ 0, & \text{elsewhere} \end{cases}$$

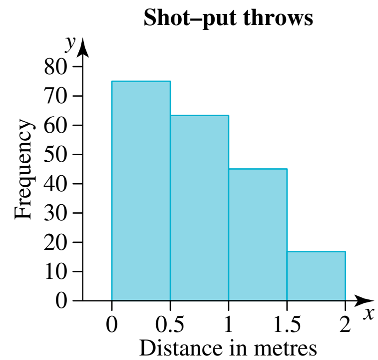
8. Given that the following function is a probability density function, determine the value of a.

$$f(x) = \begin{cases} -ax, & -2 \leq x \leq 0 \\ 2ax, & 0 < x \leq 3 \\ 0, & \text{elsewhere} \end{cases}$$

9. A number of experienced shot-putters were asked to aim for a line 10 metres away.

After each of them put their shot, its distance from the 10-metre line was measured. All of the shots were on or between the 8- and 10-metre lines. The results of the measurements are shown, where X is the distance in metres from the 10-metre line.

Metres	Frequency
$0 < x \le 0.5$	75
$0.5 < x \le 1$	63
$1 < x \le 1.5$	45
$1.5 < x \le 2$	17



a. How many shot-put throws were measured?

b. Calculate:

i. $P(X > 0.5)$ **ii.** $P(1 < X \le 2)$

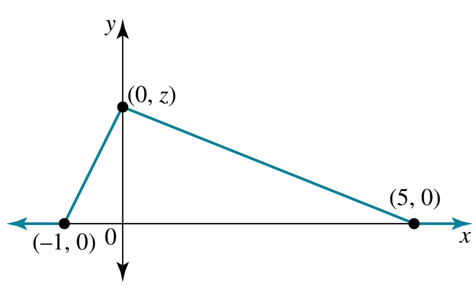
c. A guest shot-putter is visiting the athletics club where the measurements are being conducted. His shot-putting ability is equivalent to the abilities of the club members. Determine the probability that he puts the shot between 50 cm and 1 m of the 10-metre line if it is known that he put the shot within 1 metre of the 10-metre line.

10. The rectangular function, f, is defined by the rule

$$f(x) = \begin{cases} c, & 0.25 < x < 1.65 \\ 0, & \text{elsewhere} \end{cases}.$$

Determine the value of the constant c, given that f is a probability density function

11. The graph of a function, f, is shown. If f is known to be a probability density function, show that the value of z is $\frac{1}{3}$.



Technology active

12. Determine the value of the constant m in each of the following if each function is a probability density function.

a. $f(x) = \begin{cases} m(6-2x), & 0 \le x \le 2 \\ 0, & \text{elsewhere} \end{cases}$

b. $f(x) = \begin{cases} me^{-2x}, & x \ge 0 \\ 0, & \text{elsewhere} \end{cases}$

c. $f(x) = \begin{cases} me^{2x}, & 0 \le x \le \log_e(3) \\ 0, & \text{elsewhere} \end{cases}$

13. Let X be a continuous random variable with the probability density function

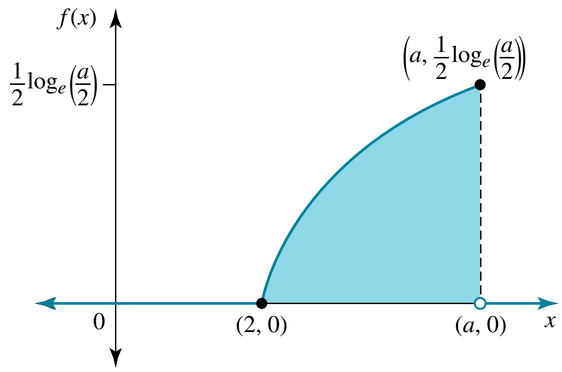
$$f(x) = \begin{cases} x^2 + 2kx + 1, & 0 \le x \le 3 \\ 0, & \text{elsewhere} \end{cases}.$$

Show that the value of k is $-\dfrac{11}{9}$.

14. X is a continuous random variable such that

$$f(x) = \begin{cases} \dfrac{1}{2}\log_e\left(\dfrac{x}{2}\right), & 2 \le x \le a \\ 0, & \text{elsewhere} \end{cases}$$

and $\int_2^a f(x)\,dx = 1$. The graph of this function is shown. Determine the value of the constant a.



11.3 Cumulative distribution functions

11.3.1 Hybrid probability density functions

Given that the probability value of a continuous random variable X lying in the interval between a and b is described by

$$\text{P}(a < x < b) = \int_a^b f(x)\,dx$$

and that the probability of X having an exact value is 0, that is,

$$\text{P}(X = a) = \text{P}(X = b) = 0,$$

it follows that

$$\text{P}(a \le X \le b) = \text{P}(a < X \le b) = \text{P}(a \le X < b) = \text{P}(a < X < b)$$

and

$$\text{P}(a \le X \le b) = \text{P}(a \le X \le c) + \text{P}(c < X \le b), \text{ where } a \le c \le b.$$

These properties are particularly useful when the probability density function is a hybrid function and the required probability encompasses two functions.

WORKED EXAMPLE 4

A continuous random variable, Y, has a probability density function, f, defined by

$$f(y) = \begin{cases} -ay, & -3 \le y \le 0 \\ ay, & 0 < y \le 3 \\ 0, & \text{elsewhere} \end{cases}$$

where a is a positive constant.

a. **Sketch the graph of f.**

b. **Determine the value of the constant, a.**

c. **Determine $P(1 \le Y \le 3)$.**

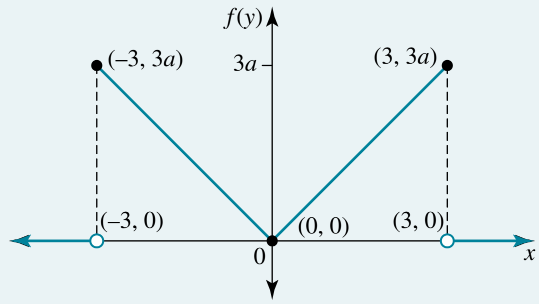
d. **Determine $P(Y < 2 | Y > -1)$**

THINK

a. The hybrid function contains three sections. The first graph, $f(y) = -ay$, is a straight line with end points of $(0, 0)$ and $(-3, 3a)$. The second graph is also a straight line and has end points of $(0, 0)$ and $(3, 3a)$. Don't forget to include the $f(y) = 0$ lines for $x > 3$ and $x < -3$.

WRITE

a. $f(-3) = 3a$ and $f(3) = 3a$



THINK

b. Use the fact that $\int_{-3}^{3} f(y)dy = 1$ to solve for a.

WRITE

b. $\int_{-3}^{3} f(y)dy = 1$

Using the area of a triangle, we find:

$$\frac{1}{2} \times 3 \times 3a + \frac{1}{2} \times 3 \times 3a = 1$$

$$\frac{9a}{2} + \frac{9a}{2} = 1$$

$$9a = 1$$

$$a = \frac{1}{9}$$

THINK

c. Identify the part of the function that the required y-values sit within: the values $1 \le Y \le 3$ are within the region where $f(y) = \frac{1}{9}y$.

WRITE

c.
$$P(1 \le Y \le 3) = \int_{1}^{3} f(y)dy$$

$$= \int_{1}^{3} \left(\frac{1}{9}y\right) dy$$

$$= \left[\frac{1}{18}y^2\right]_1^3$$

$$= \frac{1}{18}(3)^2 - \frac{1}{18}(1)^2$$

$$= \frac{8}{18}$$

$$= \frac{4}{9}$$

Note: The method of finding the area of a trapezium could also be used.

d. 1. State the rule for the conditional probability.

d. $$P(Y < 2|Y > -1) = \frac{P(Y < 2, Y > -1)}{P(Y > -1)}$$
$$= \frac{P(-1 < Y < 2)}{P(Y > -1)}$$

2. Determine $P(-1 < Y < 2)$. As the interval is across two functions, the interval needs to be split.

$$P(-1 < Y < 2) = P(-1 < Y < 0) + P(0 \le Y < 2)$$

3. To calculate the probabilities we need to determine the areas under the curve.

$$= \int_{-1}^{0} -\frac{1}{9} ydy + \int_{0}^{2} \frac{1}{9} ydy$$
$$= -\int_{-1}^{0} \frac{1}{9} ydy + \int_{0}^{2} \frac{1}{9} ydy$$

4. Antidifferentiate and evaluate after substituting the terminals.

$$= -\left[\frac{1}{18}y^2\right]_{1}^{0} + \left[\frac{1}{18}y^2\right]_{0}^{2}$$
$$= -\left(\frac{1}{18}(0)^2 - \frac{1}{18}(-1)^2\right) + \frac{1}{18}(2)^2 - \frac{1}{18}(0)^2$$
$$= \frac{1}{18} + \frac{4}{18}$$
$$= \frac{5}{18}$$

5. Determine $P(Y > -1)$. As the interval is across two functions, the interval needs to be split.

$$P(Y > -1) = P(-1 < Y < 0) + P(0 \le Y \le 3)$$

6. To determine the probabilities we need to determine the areas under the curve. As $P(0 \le Y \le 3)$ covers exactly half the area under the curve, $P(0 \le Y \le 3) = \frac{1}{2}$. (The entire area under the curve is always 1 for a probability density function.)

$$= \int_{-1}^{0} -\frac{1}{9} y\,dy + \int_{0}^{3} \frac{1}{9} y\,dy$$
$$= -\int_{-1}^{0} \frac{1}{9} y\,dy + \frac{1}{2}$$

7. Antidifferentiate and evaluate after substituting the terminals.

$$= -\left[\frac{1}{18}y^2\right]_{-1}^{0} + \frac{1}{2}$$
$$= -\left(\frac{1}{18}(0)^2 - \frac{1}{18}(-1)^2\right) + \frac{1}{2}$$
$$= \frac{1}{18} + \frac{9}{18}$$
$$= \frac{10}{18}$$
$$= \frac{5}{9}$$

8. Now substitute into the formula to determine $P(Y < 2|Y > -1) = \dfrac{P(-1 < Y < 2)}{P(Y > -1)}$.

$$\begin{aligned} P(Y < 2|Y > -1) &= \frac{P(-1 < Y < 2)}{P(Y > -1)} \\ &= \frac{5}{18} \div \frac{5}{9} \\ &= \frac{5}{18} \times \frac{9}{5} \\ &= \frac{1}{2} \end{aligned}$$

11.3.2 The cumulative distribution function

The **cumulative distribution function** $F(x)$ describes the probability that a continuous random variable X has a value less than or equal to x; that is,

$$F(x) = P(x \le c) = \int_{-\infty}^{c} f(x)\, dx.$$

The cumulative distribution function (or **cdf**) of a continuous random variable has three fundamental properties:

- $F(x)$ must be a non-decreasing function; that is, $F(a) \le F(b)$ when $a < b$.
- As x approaches $-\infty$, $F(x)$ approaches or equals 0; that is, $\lim_{x \to -\infty} F(x) = 0$.
- As x approaches ∞, $F(x)$ approaches or equals 1; that is, $\lim_{x \to \infty} F(x) = 1$.

The probability of a variable falling within a particular interval is easily determined by using the cumulative distribution function.

Given that $P(X \le b) = P(X \le a) + P(a < X \le b)$,
$P(a < X \le b) = P(X \le b) - P(X \le a)$.
Thus, $P(a < X \le b) = F(b) - F(a)$.
Consider the probability distribution function $f(x)$ such that

$$f(x) = \begin{cases} 0.5x & 0 \le x \le 2 \\ 0, & \text{elsewhere} \end{cases}.$$

The probabilities for consecutive intervals across the distribution can be tabulated as shown.

$a \le x \le b$	$P(a \le x \le b)$
$0 \le x < 0.25$	0.015 625
$0.25 \le x < 0.5$	0.046 875
$0.5 \le x < 0.75$	0.078 125
$0.75 \le x < 1$	0.109 375
$1 \le x < 1.25$	0.140 625
$1.25 \le x < 1.5$	0.171 875
$1.5 \le x < 1.75$	0.203 125
$1.75 \le x \le 2$	0.234 375

As $\mathrm{P}(a \le X \le b) = \mathrm{P}(a \le X \le c) + \mathrm{P}(c < X \le b)$, $a < c < b$,

$$\begin{aligned}
\mathrm{P}(X \le 0.5) &= \mathrm{P}(0 \le X \le 0.25) + \mathrm{P}(0.25 < X \le 0.5) \\
\mathrm{P}(X \le 0.75) &= \mathrm{P}(X \le 0.5) + \mathrm{P}(0.5 < X \le 0.75) \\
\mathrm{P}(X \le 1) &= \mathrm{P}(X \le 0.75) + \mathrm{P}(0.75 < X \le 1)
\end{aligned}$$

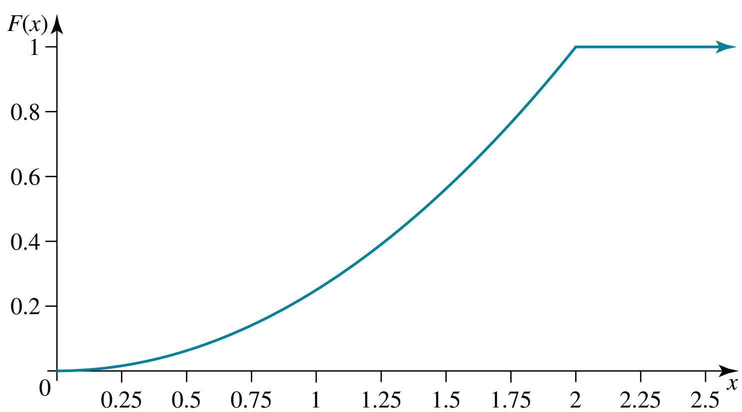
and so on.

We can tabulate the cumulative probability $\mathrm{P}(x \le c)$ across the interval $0 \le c \le 2$:

$x \le c$	$\mathrm{P}(x \le c)$
$x \le 0.25$	0.015 625
$x \le 0.5$	0.062 5
$x \le 0.75$	0.140 625
$x \le 1$	0.25
$x \le 1.25$	0.390 625
$x \le 1.5$	0.562 5
$x \le 1.75$	0.765 625
$x \le 2$	1

Note that as $\int_{-\infty}^{c} f(x)\,dx = \int_{0}^{c} f(x)\,dx$ for this pdf, $\mathrm{P}(x \le c) = \mathrm{P}(0 \le x \le c) = \mathrm{P}(-\infty \le x \le c)$.

When $F(x)$ is graphed against x, the curve formed is continuous across the range of the probability function.



As $F(x) = \int_{0}^{x} \left(\frac{x}{2}\right) dx$ for $0 \le x \le 2$,

$$F(x) = \left[\frac{x^2}{4}\right]_0^x.$$

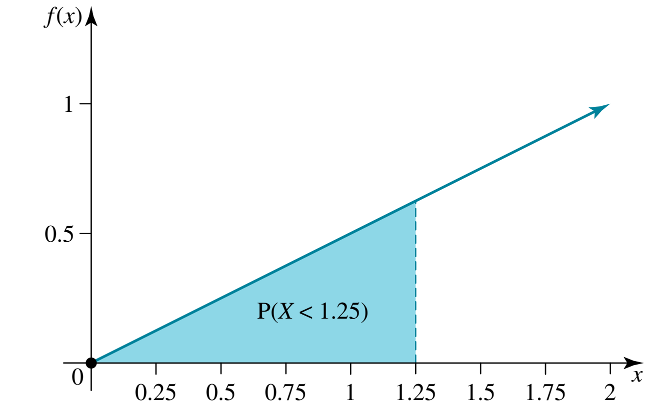
Thus, $F(x) = \dfrac{x^2}{4}$ for $0 \le x \le 2$.

The cumulative distribution function for the continuous random variable X will, in this case, be such that:

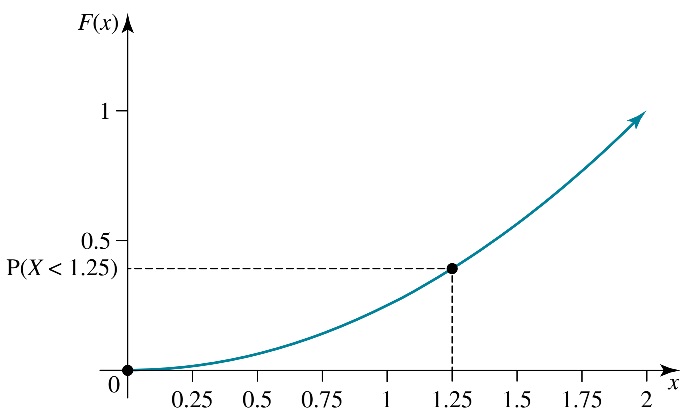
$$F(x) = \begin{cases} 0, & x \le 0 \\ \dfrac{x^2}{4}, & 0 < x \le 2 \\ 1, & x > 2 \end{cases}.$$

It is important to remember that the area under the probability density curve for an interval will be equivalent to a point on the cumulative distribution curve.

Probability density function



Cumulative distribution function



WORKED EXAMPLE 5

A continuous random variable Y has a probability density function such that

$$f(y) = \begin{cases} y, & 0 \le y \le 1 \\ 2 - y, & 1 < y \le 2. \\ 0, & \text{elsewhere} \end{cases}$$

a. **Determine the cumulative distribution function, $F(y)$, of Y.**
b. **Determine:**
   i. **$P(Y \le 0.8)$**
   ii. **$P(0.6 \le Y < 1.2)$**

THINK	WRITE
a. 1. Integrate $f(y)$ to determine $F(y)$ over the domains $0 \leq y \leq 1$ and $1 < y \leq 2$.	**a.** For $0 \leq y \leq 1$, $F(y) = \int_0^y y\,dy = \frac{y^2}{2}$. For $1 < y \leq 2$, $F(y) = P(Y \leq y) = P(Y \leq 1) + P(1 < Y \leq y)$ $F(y) = \int_0^1 y\,dy + \int_1^y (2-y)\,dy$ $= \frac{1}{2} + \left[2y - \frac{y^2}{2}\right]_1^y$ $= \frac{1}{2} + \left[\left(2y - \frac{y^2}{2}\right) - \left(2 \times 1 - \frac{1^2}{2}\right)\right]_1^y$ $= 2y - \frac{y^2}{2} - 1$
2. Write $F(y)$ as a hybrid function for all $y \in R$.	$F(y) = \begin{cases} 0, & y \leq 0 \\ \frac{y^2}{2}, & 0 < y \leq 1 \\ 2y - \frac{y^2}{2} - 1, & 1 < y \leq 2 \\ 1, & y > 2 \end{cases}$
b. i. 1. Identify the appropriate equation and substitute the appropriate values.	**b. i.** $P(Y \leq 0.8) = F(Y \leq 0.8)$ $= \frac{(0.8)^2}{2}$
2. Evaluate.	$= 0.32$
3. Answer the question.	The probability of Y being less than 0.8 is 0.32
ii. 1. Identify the appropriate equation.	**ii.** $P(a \leq Y < b) = F(b) - F(a)$
2. Evaluate.	$P(0.6 \leq Y < 1.2) = F(1.2) - F(0.6)$ $= \left[2(1.2) - \frac{(1.2)^2}{2} - 1\right] - \left[\frac{(0.6)^2}{2}\right]$ $= 0.68 - 0.18$ $= 0.5$
3. Answer the question.	The probability of Y being between 0.6 and 1.2 is 0.5.

on Resources

Interactivity: Cumulative density functions (int-6434)

studyon

Units 3 & 4 > Area 7 > Sequence 1 > Concept 1

Probability density functions Summary screen and practice questions

Exercise 11.3 Cumulative distribution functions

Technology free

1. **WE4** The continuous random variable Z has a probability density function given by

$$f(z) = \begin{cases} -z+1, & 0 \le z < 1 \\ z-1, & 1 \le z \le 2 \\ 0, & \text{elsewhere} \end{cases}.$$

   a. Sketch the graph of f.
   b. Determine $P(Z < 0.75)$.
   c. Determine $P(Z > 0.5)$.

2. The continuous random variable X has a probability density function given by

$$f(x) = \begin{cases} 4x^3, & 0 \le x \le a \\ 0, & \text{elsewhere} \end{cases}$$

   where a is a constant.
   a. Determine the value of the constant a.
   b. Sketch the graph of f.
   c. Determine $P(0.5 \le X \le 1)$.

3. **WE5** The continuous random variable X has a uniform rectangular probability density function defined by

$$f(x) = \begin{cases} \frac{1}{5}, & 1 \le x \le 6 \\ 0, & \text{elsewhere} \end{cases}.$$

   a. Determine the cumulative distribution function, $F(x)$, for X.
   b. Determine:
      i. $P(x \le 4)$
      ii. $P(2.2 < x \le 4.5)$

4. Let X be a continuous random variable with a probability density function defined by

$$f(x) = \begin{cases} \frac{1}{2}\sin(x), & 0 \le x \le \pi \\ 0, & \text{elsewhere} \end{cases}.$$

   a. Determine the cumulative distribution function $F(x)$ for X.
   b. Determine $P\left(X \le \frac{\pi}{2}\right)$.
   c. Determine $P\left(\frac{\pi}{4} < X < \frac{3\pi}{4}\right)$.
   d. Determine $P\left(X > \frac{\pi}{4} | X < \frac{3\pi}{4}\right)$.

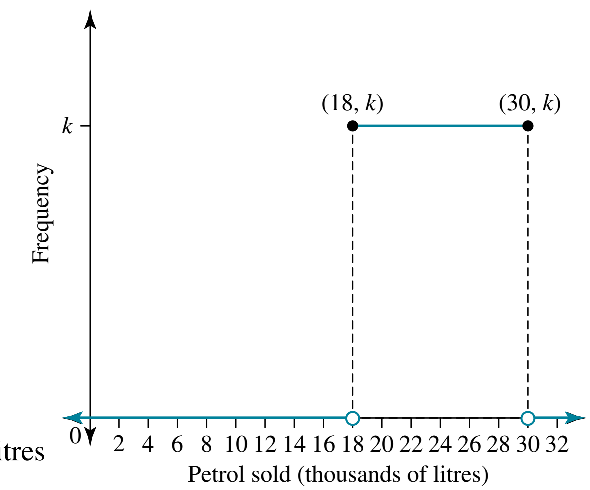
5. A probability density function is defined by the rule

$$f(x) = \begin{cases} k(2 + x), & -2 \le x < 0 \\ k(2 - x), & 0 \le x \le 2 \\ 0, & \text{elsewhere} \end{cases}$$

where X is a continuous random variable and k is a constant.

a. Sketch the graph of f.
b. Determine the value of k.
c. Determine the cumulative distribution function $F(x)$.
d. Determine $\mathrm{P}(-1 \le X \le 1)$.
e. Determine $\mathrm{P}\left(X \ge -1 | X \le 1\right)$.

6. The amount of petrol sold daily by a busy service station is a uniformly distributed probability density function. A minimum of 18 000 litres and a maximum of 30 000 litres are sold on any given day. The graph of the function is shown.



a. Determine the value of the constant k.
b. Determine the formula (including cases) for the probability density function $f(x)$.
c. Determine the formula (including cases) for the cumulative distribution function $F(x)$.
d. Determine the probability that between 20 000 and 25 000 litres of petrol are sold on a given day.
e. Determine the probability that as much as 26 000 litres of petrol were sold on a particular day, given that it was known that at least 22 000 litres were sold.

Technology active

7. The continuous random variable X has a probability density function defined by

$$f(x) = \begin{cases} \frac{3}{8}x^2, & 0 \le x \le 2 \\ 0, & \text{elsewhere} \end{cases}.$$

a. Determine the cumulative distribution function for $f(x)$.
b. Determine $\mathrm{P}(X > 1.2)$.
c. Determine the value of n such that $\mathrm{P}(X \le n) = 0.75$.

8. The continuous random variable Z has a probability density function defined by

$$f(z) = \begin{cases} \frac{1}{2z}, & 1 \le z \le e^2 \\ 0, & \text{elsewhere} \end{cases}.$$

a. Sketch the graph of f and shade the area that represents $\int_1^{e^2} f(z)\,dz$.
b. Determine $\int_1^{e^2} f(z)\,dz$. Explain your result.

The continuous random variable U has a probability function defined by

$$f(u) = \begin{cases} e^{4u}, & u \geq 0 \\ 0, & \text{elsewhere} \end{cases}.$$

c. Sketch the graph of f and shade the area that represents $\int_0^a f(u)\,du$, where a is a constant.

d. Determine the exact value of the constant a if $\int_1^{e^2} f(z)\,dz$ is equal to $\int_0^a f(u)\,du$.

9. The continuous random variable Z has a probability density function defined by

$$f(z) = \begin{cases} \frac{1}{2}\cos(z), & -\frac{\pi}{2} \leq z \leq \frac{\pi}{2} \\ 0, & \text{elsewhere} \end{cases}.$$

Determine $P\left(-\frac{\pi}{6} \leq Z \leq \frac{\pi}{4}\right)$, correct to 3 decimal places.

10. The continuous random variable U has a probability density function defined by

$$f(u) = \begin{cases} 1 - \frac{1}{4}(2u - 3u^2), & 0 \leq u \leq a \\ 0, & \text{elsewhere} \end{cases}$$

where a is a constant. Determine:

a. the value of the constant a
b. $P(U < 0.75)$
c. $P(0.1 < U < 0.5)$
d. $P(U = 0.8)$.

11. The continuous random variable Z has a probability density function defined by

$$f(z) = \begin{cases} e^{-\frac{z}{3}}, & 0 \leq z \leq a \\ 0, & \text{elsewhere} \end{cases}$$

where a is a constant. Determine:

a. the value of the constant a such that $\int_0^a f(z)\,dz = 1$
b. $P(0 < Z < 0.7)$
c. $P(Z < 0.7 | Z > 0.2)$, correct to 4 decimal places
d. the value of α, correct to 2 decimal places, such that $P(Z \leq \alpha) = 0.54$.

12. The continuous random variable X has a probability density function given as

$$f(x) = \begin{cases} 3e^{-3x}, & x \geq 0 \\ 0, & \text{elsewhere} \end{cases}.$$

a. Determine $P(0 \leq X \leq 1)$, correct to 4 decimal places
b. Determine $P(X > 2)$, correct to 4 decimal places.

13. The continuous random variable X has a probability density function defined by

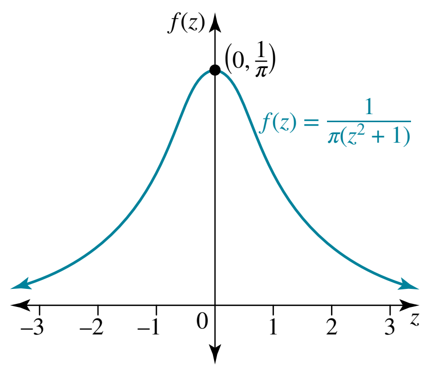
$$f(x) = \begin{cases} \log_e(x^2), & x \geq 1 \\ 0, & \text{elsewhere} \end{cases}.$$

Determine, correct to 4 decimal places:

a. the value of the constant a if $\int_1^a f(x)\,dx = 1$

b. $P(1.25 \leq X \leq 2)$.

14. The graph of the probability function $f(z) = \dfrac{1}{\pi(z^2+1)}$ is shown.



a. Determine, correct to 4 decimal places, $P(-0.25 < Z < 0.25)$.

b. Suppose another probability density function is defined as

$$f(x) = \begin{cases} \dfrac{1}{x^2+1}, & -a \leq x \leq a \\ 0, & \text{elsewhere} \end{cases}.$$

Determine the value of the constant a.

11.4 Measures of centre and spread

The commonly used measures of central tendency and spread in statistics are the mean, median, variance, standard deviation and range. These same measures are appropriate for continuous probability functions.

11.4.1 Measures of central tendency: the mean

Remember that for a discrete random variable,

$$E(X) = \mu = \sum_{x=1}^{x=n} x_n P(X = x_n).$$

This definition can also be applied to a continuous random variable.

We define $E(X) = \mu = \int_{-\infty}^{\infty} x f(x)\,dx$.

The expected value of a continuous random variable

If $f(x) = 0$ everywhere except for $x \in [a, b]$, where the function is defined, then

$$E(X) = \mu = \int_a^b x f(x)\,dx.$$

Consider the continuous random variable, X, which has a probability density function defined by

$$f(x) = \begin{cases} x^2, & 0 \leq x \leq 1 \\ 0, & \text{elsewhere} \end{cases}.$$

For this function,

$$\begin{aligned}
\mathrm{E}(X) = \mu &= \int_0^1 xf(x)\,dx \\
&= \int_0^1 x(x^2)\,dx \\
&= \int_0^1 x^3 dx \\
&= \left[\frac{x^4}{4}\right]_0^1 \\
&= \frac{1^4}{4} - 0 \\
&= \frac{1}{4}
\end{aligned}$$

Similarly, if the continuous random variable X has a probability density function of

$$f(x) = \begin{cases} 7e^{-7x}, & x \geq 0 \\ 0, & \text{elsewhere} \end{cases}$$

then

$$\begin{aligned}
\mathrm{E}(X) = \mu &= \int_0^\infty xf(x)dx \\
&= \lim_{k\to\infty} \int_0^k 7xe^{-7x}dx \\
&= 0.1429
\end{aligned}$$

where technology is required to determine the integral.

The mean of a function of X is similarly found.

The mean of a continuous random variable

The function of X, $g(x)$, has a mean defined by

$$\mathrm{E}(g(x)) = \mu = \int_{-\infty}^{\infty} g(x)f(x)\,dx.$$

So if we again consider

$$f(x) = \begin{cases} x^2, & 0 \leq x \leq 1 \\ 0, & \text{elsewhere} \end{cases}$$

then

$$\begin{aligned}
\mathrm{E}(X^2) &= \int_0^1 x^2 f(x)\,dx \\
&= \int_0^1 x^4\,dx \\
&= \left[\frac{x^5}{5}\right]_0^1 \\
&= \frac{1^5}{5} - 0 \\
&= \frac{1}{5}
\end{aligned}$$

This definition will be important when we investigate the variance of a continuous random variable.

11.4.2 Median and percentiles

The median is also known as the 50th **percentile**, Q_2, the halfway mark or the middle value of the distribution.

The median of a continuous random variable

For a continuous random variable, X, defined by the probability function f, the median, m, can be found by solving $\int_{-\infty}^{m} f(x)\,dx = 0.5$.

Other percentiles that are frequently calculated are the 25th percentile or lower **quartile**, Q_1, and the 75th percentile or upper quartile, Q_3.

The interquartile range

The **interquartile range** is calculated as:

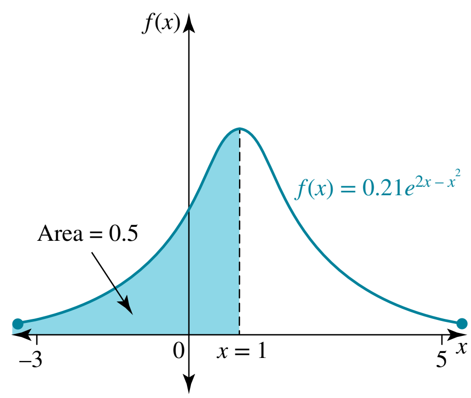
$$\text{IQR} = Q_3 - Q_1$$

Consider a continuous random variable, X, that has a probability density function of

$$f(x) = \begin{cases} 0.21e^{2x-x^2}, & -3 \le x \le 5 \\ 0, & \text{elsewhere} \end{cases}.$$

To determine the median, m, we solve for m as follows:

$$\int_{-3}^{m} 0.21e^{2x-x^2}\,dx = 0.5$$



The area under the curve is equated to 0.5, giving half of the total area and hence the 50th percentile. Solving using technology, the result is that $m = 0.9897 \simeq 1$. This can be seen on a graph as shown.

Consider the continuous random variable X, which has a probability density function of

$$f(x) = \begin{cases} \dfrac{x^3}{4}, & 0 \le x \le 2 \\ 0, & \text{elsewhere} \end{cases}.$$

The median is given by $P(0 \le x \le m) = 0.5$:

$$\begin{aligned} \int_0^m \frac{x^3}{4} dx &= 0.5 \\ \left[\frac{x^4}{16}\right]_0^m &= \frac{1}{2} \\ \frac{m^4}{16} - 0 &= \frac{1}{2} \\ m^4 &= 8 \\ m &= \pm\sqrt[4]{8} \\ m &= 1.6818\,(0 \le m \le 2) \end{aligned}$$

To determine the lower quartile, we make the area under the curve equal to 0.25. Thus the lower quartile is given by $P(0 \le x \le a) = 0.25$:

$$\begin{aligned} \int_0^a \frac{x^3}{4} dx &= 0.25 \\ \left[\frac{x^4}{16}\right]_0^a &= \frac{1}{4} \\ \frac{a^4}{16} - 0 &= \frac{1}{4} \\ a^4 &= 4 \\ a &= \pm\sqrt[4]{4} \\ a = Q_1 &= 1.4142\ (0 \le a \le m) \end{aligned}$$

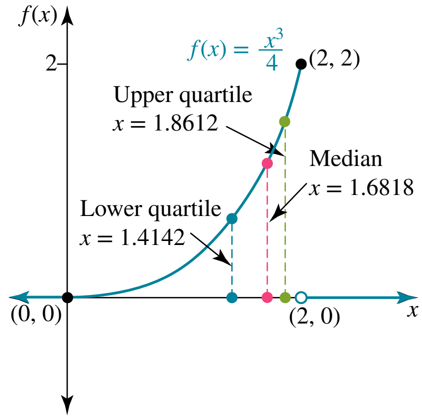
Similarly, to determine the upper quartile, we make the area under the curve equal to 0.75. Thus, the upper quartile is given by $P(0 \le x \le n) = 0.75$:

$$\begin{aligned} \int_0^n \frac{x^3}{4} dx &= 0.75 \\ \left[\frac{x^4}{16}\right]_0^n &= \frac{3}{4} \\ \frac{n^4}{16} - 0 &= \frac{3}{4} \\ n^4 &= 12 \\ n &= \pm\sqrt[4]{12} \\ n = Q_3 &= 1.8612\ (m \le x \le 2) \end{aligned}$$

So, the interquartile range is given by

$$\begin{aligned} Q_3 - Q_1 &= 1.8612 - 1.4142 \\ &= 0.4470 \end{aligned}$$

These values are shown on the following graph.



WORKED EXAMPLE 6

A continuous random variable, Y, has a probability density function, f, defined by

$$f(y) = \begin{cases} ky, & 0 \le y \le 1 \\ 0, & \text{elsewhere} \end{cases}$$

where k is a constant.

a. **Sketch the graph of f.**

b. **Determine the value of the constant k.**

c. **Determine:**

  i. **the mean of Y**   ii. **the median of Y.**

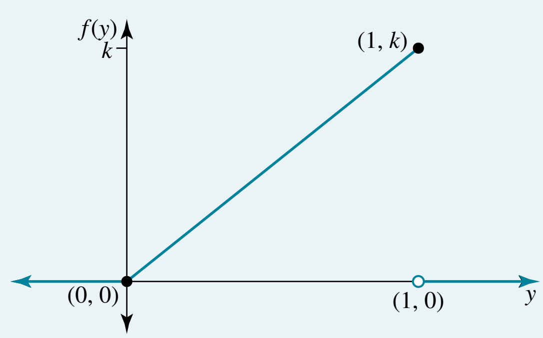
d. **Determine the interquartile range of Y.**

THINK

a. The graph $f(y) = ky$ is a straight line with end points at $(0, 0)$ and $(1, k)$. Remember to include the lines $f(y) = 0$ for $y > 1$ and $y < 0$.

WRITE

a.



b. Solve $\int_0^1 ky\,dy = 1$ to determine the value of k.

b. $\int_0^1 ky\,dy = 1$

$k\int_0^1 y\,dy = 1$

$k\left[\frac{y^2}{2}\right]_0^1 = 1$

$\frac{k(1)^2}{2} - 0 = 1$

$\frac{k}{2} = 1$

$k = 2$

Using the area of a triangle also enables you to find the value of k.

$\frac{1}{2} \times 1 \times k = 1$

$\frac{k}{2} = 1$

$k = 2$

c. i. 1. State the rule for the mean.

c. i. $\mu = \int_0^1 y(2y)\,dy$

$= \int_0^1 2y^2\,dy$

2. Antidifferentiate and simplify.

$= \left[\frac{2}{3}y^3\right]_0^1$

$= \frac{2(1)^3}{3} - 0$

$= \frac{2}{3}$

ii 1. State the rule for the median.

ii $\int_0^m f(y)\,dy = 0.5$

$\int_0^m 2y\,dy = 0.5$

2. Antidifferentiate and solve for m. Note that m must be a value within the domain of the function, so within $0 \le y \le 1$.

$\left[y^2\right]_0^m = 0.5$

$m^2 - 0 = 0.5$

$m = \pm\sqrt{0.5}$

3. Write the answer.

Median = 0.7071 (as $0 < m < 1$)

d. 1. State the rule for the lower quartile, Q_1.

d. $\int_0^a f(y)\,dy = 0.25$

$\int_0^a 2y\,dy = 0.25$

2. Antidifferentiate and solve for Q_1.	$\left[y^2\right]_0^a = 0.25$ $a^2 - 0 = 0.25$ $a = \pm\sqrt{0.25}$ $a = Q_1 = 0.5 \quad (\text{as } 0 < Q_1 < 0.7071)$
3. State the rule for the upper quartile, Q_3.	$\int_0^n f(y)\,dy = 0.75$ $\int_0^n 2y\,dy = 0.75$
4. Antidifferentiate and solve for Q_3.	$\left[y^2\right]_0^n = 0.75$ $n^2 - 0 = 0.75$ $n = \pm\sqrt{0.75}$ $n = Q_3 = 0.8660 \quad (\text{as } 0.7071 < Q_3 < 1)$
5. State the rule for the interquartile range.	$\text{IQR} = Q_3 - Q_1$
6. Substitute the appropriate values and simplify.	$= 0.8660 - 0.5$ $= 0.3660$

11.4.3 Measures of spread: variance, standard deviation and range

The variance and standard deviation are important measures of spread in statistics. From previous calculations for discrete probability functions, we know the following.

Variance and standard deviation for discrete probability functions

$\text{Var}(X) = \text{E}(X^2) - [\text{E}(X)]^2$ and $\text{SD}(X) = \sqrt{\text{Var}(X)}$.

For continuous probability functions,

$$\begin{aligned}
\text{Var}(X) &= \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx \\
&= \int_{-\infty}^{\infty} (x^2 - 2x\mu + \mu^2) f(x)\,dx \\
&= \int_{-\infty}^{\infty} x^2 f(x)\,dx - \int_{-\infty}^{\infty} 2xf(x)\mu\,dx + \int_{-\infty}^{\infty} \mu^2 f(x)\,dx \\
&= \text{E}(X^2) - 2\mu \int_{-\infty}^{\infty} xf(x)\,dx + \mu^2 \int_{-\infty}^{\infty} 1f(x)\,dx \\
&= \text{E}(X^2) - 2\mu \times \text{E}(X) + \mu^2 \\
&= \text{E}(X^2) - 2\mu^2 + \mu^2 \\
&= \text{E}(X^2) - \mu^2 \\
&= \text{E}(X^2) - [\text{E}(X)]^2
\end{aligned}$$

Two important facts were used in this proof: $\int_{-\infty}^{\infty} f(x)\,dx = 1$ and $\int_{-\infty}^{\infty} xf(x)\,dx = \mu = \mathrm{E}(X)$.

Substituting this result into $\mathrm{SD}(X) = \sqrt{\mathrm{Var}(X)}$ gives us

$$\mathrm{SD}(X) = \sqrt{\mathrm{E}(X^2) - [\mathrm{E}(X)]^2}.$$

The range is calculated as the highest value minus the lowest value. So, for the probability density function given by $f(x) = \begin{cases} \frac{1}{5}, & 1 \le x \le 6 \\ 0, & \text{elsewhere} \end{cases}$, the highest possible x-value is 6 and the lowest is 1. Therefore, the range for this function $= 6 - 1 = 5$.

WORKED EXAMPLE 7

For a continuous random variable, X, with a probability density function, f, defined by

$$f(x) = \begin{cases} \frac{1}{2}x + 2, & -4 \le x \le -2 \\ 0, & \text{elsewhere} \end{cases}$$

determine:

a. the mean

b. the median

c. the variance

d. the standard deviation, correct to 4 decimal places.

THINK	WRITE
a. 1. State the rule for the mean and simplify.	a. $\mu = \int_{-4}^{-2} xf(x)\,dx$ $= \int_{-4}^{-2} x\left(\frac{1}{2}x + 2\right)dx$ $= \int_{-4}^{-2} \left(\frac{1}{2}x^2 + 2x\right)dx$
2. Antidifferentiate and evaluate.	$= \left[\frac{1}{6}x^3 + x^2\right]_{-4}^{-2}$ $= \left(\frac{1}{6}(-2)^3 + (-2)^2\right) - \left(\frac{1}{6}(-4)^3 + (-4)^2\right)$ $= -\frac{4}{3} + 4 + \frac{32}{3} - 16$ $= -\frac{8}{3} = -2\frac{2}{3}$
b. 1. State the rule for the median.	b. $\int_{-4}^{m} f(x)\,dx = 0.5$ $\int_{-4}^{m} \left(\frac{1}{2}x + 2\right)dx = 0.5$

2. Antidifferentiate and solve for m. The quadratic formula is needed as the quadratic equation formed cannot be factorised. Alternatively, use technology to solve for m.

$$\left[\frac{1}{4}x^2 + 2x\right]_{-4}^{m} = 0.5$$

$$\left(\frac{1}{4}m^2 + 2m\right) - \left(\frac{(-4)^2}{4} + 2(-4)\right) = 0.5$$

$$\frac{1}{4}m^2 + 2m + 4 = 0.5$$

$$m^2 + 8m + 16 = 2$$

$$m^2 + 8m + 14 = 0$$

So, $$m = \frac{-8 \pm \sqrt{(8)^2 - 4(1)(14)}}{2(1)}$$

$$m = \frac{-8 \pm \sqrt{8}}{2}$$

$$= -4 \pm \sqrt{2}$$

$$\therefore m = -4 + \sqrt{2} \text{ as } m \in [-4, 2].$$

3. Write the answer.

The median is $-4 + \sqrt{2}$.

c. 1. Write the rule for variance.

c. $\text{Var}(X)^2 = \text{E}(X^2) - [\text{E}(X)]^2$

2. Determine $\text{E}\left(X^2\right)$ first.

$$\text{E}\left(X^2\right) = \int_a^b x^2 f(x)\,dx$$

$$= \int_{-4}^{-2} x^2\left(\frac{1}{2}x + 2\right)dx$$

$$= \int_{-4}^{-2} \left(\frac{1}{2}x^3 + 2x^2\right)dx$$

$$= [\frac{1}{18}x^4 + \frac{2}{3}x^3]_{-4}^{-2}$$

$$= \left(\frac{1}{8}(-2)^4 + \frac{2}{3}(-2)^3\right) - \left(\frac{1}{8}(-4)^4 + \frac{2}{3}(-4)^3\right)$$

$$= 2 - \frac{16}{3} - 32 + \frac{128}{3}$$

$$= -30 + \frac{112}{3}$$

$$= \frac{22}{3}$$

3. Substitute $\text{E}(X)$ and $\text{E}\left(X^2\right)$ into the rule for variance.

$$\text{Var}(X) = \text{E}(X^2) - [\text{E}(X)]^2$$

$$= \frac{22}{3} - \left(-\frac{8}{3}\right)^2$$

$$= \frac{22}{3} - \frac{64}{9}$$

$$= \frac{66}{9} - \frac{64}{9}$$

$$= \frac{2}{9}$$

d. 1. Write the rule for standard deviation.

2. Substitute the variance into the rule and evaluate.

d. $SD(X) = \sqrt{Var(X)}$

$= \sqrt{\frac{2}{9}}$

$= 0.4714$

on Resources

Interactivities: Mean (int-6435)
Median and percentiles (int-6436)
Variance, standard deviation and range (int-6437)

studyon

Units 3 & 4 > Area 7 > Sequence 1 > Concepts 4 & 5

Mean and median Summary screen and practice questions
Variance and standard deviation Summary screen and practice questions

Exercise 11.4 Measures of centre and spread

Technology free

1. **WE6** The continuous random variable Z has a probability density function of

$$f(z) = \begin{cases} 4, & 1 \le z \le a \\ 0, & \text{elsewhere} \end{cases}$$

where a is a constant.

a. Determine the value of the constant a.

b. Determine:

i. the mean of Z

ii. the median of Z.

2. The continuous random variable, Y, has a probability density function of

$$f(y) = \begin{cases} 2y, & 0 \le y \le a \\ 0, & \text{elsewhere} \end{cases}$$

where a is a constant.

Determine:

a. the value of the constant a

b. $E(Y)$

c. the median value of Y.

3. **WE7** For the continuous random variable Z, the probability density function is

$$f(z) = \begin{cases} 2z - 4, & 2 \leq z \leq 3 \\ 0, & \text{elsewhere} \end{cases}.$$

Determine the mean, median, variance and standard deviation.

4. The function

$$f(x) = \begin{cases} 3e^{-3x}, & x \geq 0 \\ 0, & \text{elsewhere} \end{cases}$$

defines the probability density function for the continuous random variable X. Determine the mean, median, variance and standard deviation of X.

5. Let X be a continuous random variable with a probability density function of

$$f(x) = \begin{cases} \dfrac{1}{2\sqrt{x}}, & 0 \leq x \leq 1 \\ 0, & \text{elsewhere} \end{cases}.$$

a. Prove that f is a probability density function.
b. Determine $E(X)$.
c. Determine the median value of f.

6. The time in minutes that an individual must wait in line to be served at the local bank branch is defined by

$$f(t) = 2e^{-2t}, t \geq 0$$

where T is a continuous random variable.

a. What is the mean waiting time for a customer in the queue, correct to 1 decimal place?
b. Calculate the standard deviation for the waiting time in the queue, correct to 1 decimal place.
c. Determine the median waiting time in the queue, correct to 2 decimal places.

7. a. Determine the derivative of $\sqrt{4 - x^2}$.
b. Hence, determine the mean value of the probability density function defined by

$$f(x) = \begin{cases} \dfrac{3}{\pi\sqrt{4 - x^2}}, & 0 \leq x \leq \sqrt{3} \\ 0, & \text{elsewhere} \end{cases}.$$

8. Consider the continuous random variable X with a probability density function of

$$f(x) = \begin{cases} h(2 - x), & 0 \leq x \leq 2 \\ h(x - 2), & 2 < x \leq 4 \\ 0, & \text{elsewhere} \end{cases}$$

where h is a positive constant.

a. Determine the value of the constant h.
b. Determine $E(X)$.
c. Determine $\text{Var}(X)$.

9. Consider the continuous random variable X with a probability density function of

$$f(x) = \begin{cases} k, & a \le x \le b \\ 0, & \text{elsewhere} \end{cases}$$

where a, b and k are positive constants.

a. Sketch the graph of the function f.

b. Show that $k = \dfrac{1}{b-a}$.

c. Determine $E(X)$ in terms of a and b.

d. Determine $Var(X)$ in terms of a and b.

Technology active

10. The continuous random variable Y has a probability density function defined by

$$f(y) = \begin{cases} \dfrac{y^2}{3}, & 0 \le y \le \sqrt[3]{9} \\ 0, & \text{elsewhere} \end{cases}.$$

Determine, correct to 4 decimal places:

a. the expected value of Y

b. the median value of Y

c. the lower and upper quartiles of Y

d. the interquartile range of Y.

11. The continuous random variable Z has a probability density function defined by

$$f(y) = \begin{cases} \dfrac{a}{z}, & 1 \le z \le 8 \\ 0, & \text{elsewhere} \end{cases}$$

where a is a constant.

a. Determine the value, correct to 4 decimal places, of the constant a.

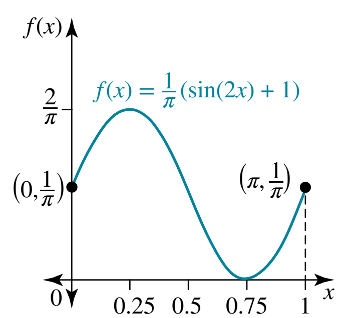
b. Determine $E(Z)$ correct to 4 decimal places.

c. Determine $Var(Z)$ and $SD(Z)$.

d. Determine the interquartile range for Z.

e. Determine the range for Z.

12. X is a continuous random variable. The graph of the probability density function $f(x) = \dfrac{1}{\pi}(\sin(2x) + 1)$ for $0 \le x \le \pi$ is shown.



a. Show that $f(x)$ is a probability density function.

b. Calculate $E(X)$ correct to 4 decimal places.

c. Calculate, correct to 4 decimal places:

i. $Var(X)$ ii. $SD(X)$

d. Determine the median value of f correct to 4 decimal places.

13. The continuous random variable X has a probability density function defined by

$$f(x) = \begin{cases} ax - bx^2, & 0 \le x \le 2 \\ 0, & \text{elsewhere} \end{cases}.$$

Determine the values of the constants a and b if $E(X) = 1$.

14. The continuous random variable Z has a probability density function of

$$f(z) = \begin{cases} \dfrac{3}{z^2}, & 1 \le z \le a \\ 0, & \text{elsewhere} \end{cases}$$

where a is a constant.

a. Show that the value of a is $\dfrac{3}{2}$.
b. Determine the mean value and variance of f correct to 4 decimal places.
c. Determine the median and interquartile range of f.

15. The continuous random variable Y has a probability density function

$$f(y) = \begin{cases} 0.2 \log_e\left(\dfrac{y}{2}\right), & 2 \le y \le 7.9344 \\ 0, & \text{elsewhere} \end{cases}.$$

a. Verify that f is a probability density function.
b. Determine $E(Y)$ correct to 4 decimal places.
c. Determine $\text{Var}(Y)$ and $\text{SD}(Y)$ correct to 4 decimal places.
d. Determine the median value of Y correct to 4 decimal places.
e. State the range.

16. The continuous random variable Z has a probability density function

$$f(z) = \begin{cases} \sqrt{z-1}, & 1 \le z \le a \\ 0, & \text{elsewhere} \end{cases}$$

where a is a constant.

a. Determine the value of the constant a correct to 4 decimal places.
b. Determine, correct to 4 decimal places:
   i. $E(Z)$
   ii. $E(Z^2)$
   iii. $\text{Var}(Z)$
   iv. $\text{SD}(Z)$

11.5 Review: exam practice

A summary of this chapter is available in the Resources section of your eBookPLUS at www.jacplus.com.au.

Simple familiar

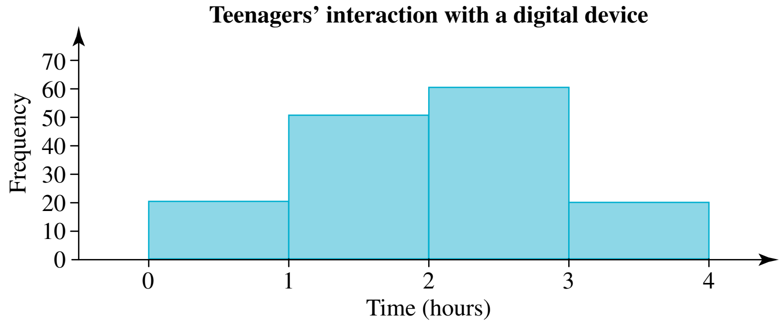
1. Which of the following represent continuous random variables?
   a. The number of goals scored at a football match
   b. The heights of students in a Maths B class
   c. Shoe sizes
   d. The number of girls in a five-child family
   e. The time taken to run a distance of 10 kilometres in minutes
2. X is a continuous random variable with a probability function defined by

$$f(x) = \begin{cases} 2\sin(2x), & a \le x \le b \\ 0, & \text{elsewhere} \end{cases}.$$

Given that $a = 0$ and $0 \le b \le \pi$, what is the value of b?

3. A survey was taken to determine the amount of time, X hours, that teenagers spend interacting with digital devices during a 24-hour period. The table of findings and histogram are shown.

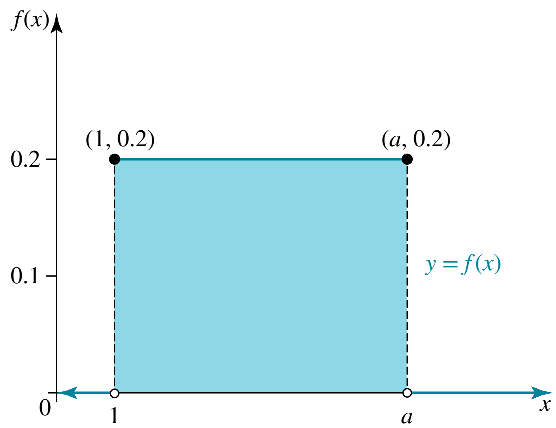
Time in hours	Frequency
$0 \le x \le 1$	20
$1 < x \le 2$	50
$2 < x \le 3$	60
$3 < x \le 4$	20



a. How many teenagers were surveyed?

b. Determine $\mathrm{P}(X \le 3)$.

4. The graph of a rectangular or uniform probability density function, $f(x)$, is shown. What is the value of the constant a?



5. Y is a continuous random variable with a probability density function of

$$f(y) = \begin{cases} 3y^2, & 0 \le y \le 1 \\ 0, & \text{elsewhere} \end{cases}.$$

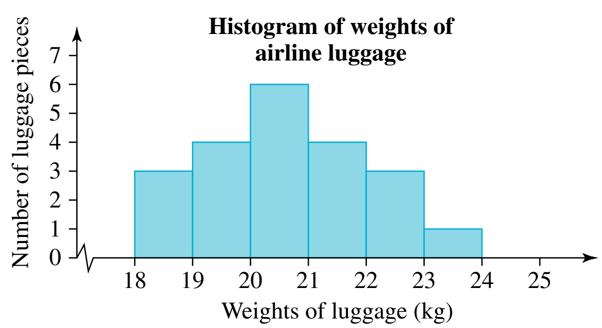
a. Determine the cumulative distribution function for $f(y)$.

b. Determine $\mathrm{P}(0.2 \le Y < 7)$.

6. A histogram was compiled as shown based upon the weights of luggage taken onto a flight from Rockhampton.

a. Explain why it is not possible to determine how many items of luggage had a weight of 19.5 kg.

b. What is the probability that a piece of luggage chosen at random from the flight weighed less than 19 kg?



7. A continuous probability density function is defined by

$$f(x) = \begin{cases} 2x, & 0 \le x \le 1 \\ 0, & \text{elsewhere} \end{cases}.$$

Calculate:

a. the mean of the distribution

b. the variance of the distribution.

8. a. Sketch the graph of

$$f(x) = \begin{cases} \frac{1}{4}(1-x), & -1 \le x < 1 \\ \frac{1}{4}(x-1), & 1 \le x \le 3 \\ 0, & \text{elsewhere} \end{cases}.$$

b. Show that $f(x)$ is a probability density function.

c. Determine E(X).

9. Explain the difference between the probability density function and the cumulative distribution function.

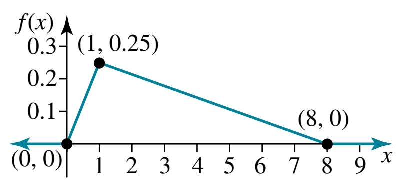
10. X is a continuous random variable with a probability density function defined by

$$f(x) = \begin{cases} \frac{1}{2}x & 0 \le x \le 2 \\ 0 & \text{elsewhere} \end{cases}.$$

a. Determine the cumulative distribution function for $f(x)$.

b. Sketch the cumulative distribution function, indicating on the graph where the value of $P(X < 1.5)$ can be found.

11. Give a formula for the probability density function that is drawn here, given that the function is equal to 0 for $x < 0$ and $x > 8$.



12. Let X be a continuous random variable with a probability density function defined by
$f(x) = ax^2, 0 \le x \le 3$
where a is a constant.

a. Determine the value of a.

b. Determine $P(1 \le X \le 2)$.

Complex familiar

13. X is a continuous random variable with a probability function defined by

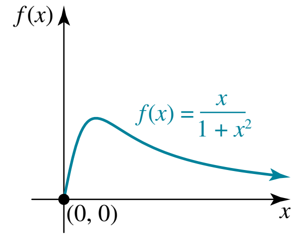
$$f(x) = \begin{cases} \dfrac{2\log_e(x)}{\sqrt{x}}, & 1 \le x \le a \\ 0, & \text{elsewhere} \end{cases}$$

where a is a constant.

a. Determine the value of the constant a. Give your answer correct to 2 decimal places.

b. Determine the mean and the median values of X. Give your answers correct to 3 decimal places.

14. The graph of the probability function $f(x) = \frac{x}{1+x^2}$, $x \geq 0$ for the continuous random variable X is shown.



a. Determine $P(X \leq 2)$ correct to 2 decimal places.
b. If Y is a continuous random variable with a probability density function of $f(y) = \frac{y}{1+y^2}$ for $0 \leq y \leq a$, determine the value of the constant a correct to 1 decimal place.
c. Determine the median value of Y correct to 2 decimal places.

15. Given that the following function is a probability density function, determine the value of a.

$$f(x) = \begin{cases} ax + 0.5, & -2 \leq x < 0 \\ -ax + 1, & 0 \leq x \leq 1 \\ 0, & \text{elsewhere} \end{cases}$$

16. The continuous random variable X has a probability density function defined by

$$f(x) = \begin{cases} k\cos(2x), & 0 \leq x \leq \frac{\pi}{4} \\ 0, & \text{elsewhere} \end{cases}$$

where k is a constant.

a. Determine the value of k.
b. Determine the median of f.

Complex unfamiliar

17. The continuous random variable T has a probability density function defined by

$$f(t) = \begin{cases} 2e^{-2t}, & t \geq 0 \\ 0, & \text{elsewhere} \end{cases}.$$

a. Determine $E(T)$.
b. Determine $SD(T)$.
c. Determine the median value of T.
d. Determine the interquartile range of T, correct to 4 decimal places.

18. X is a continuous random variable such that

$$f(x) = \begin{cases} n\sin(3x)\cos(3x), & 0 < x < \frac{\pi}{12} \\ 0, & \text{elsewhere} \end{cases}.$$

If f is known to be a probability density function, determine the value of the constant, n.

19. Patrick has just spread lawn seed on his nature strip. With constant watering and plenty of sunshine, the time it takes for the lawn seed to germinate, T days after the seeding, can be determined by the probability density function

$$f(t) = \begin{cases} ke^{-0.15t}, & t \geq 0 \\ 0, & \text{elsewhere} \end{cases}$$

where k is a constant.

a. Determine the value of the constant k.

b. What is the expected period of time for the germination of the lawn seed? Give your answer correct to the nearest day.

20. A function f is defined by the rule

$$f(x) = \begin{cases} \log_e(x), & x > 0 \\ 0, & \text{elsewhere} \end{cases}.$$

a. If $\int_1^a f(x)dx = 1$, determine the value of the real constant a.

b. Does this function define a probability density function?

studyon

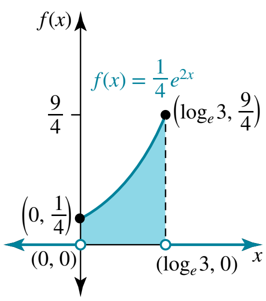
Units 3 & 4 Sit exam

Answers

11 General continuous random variables

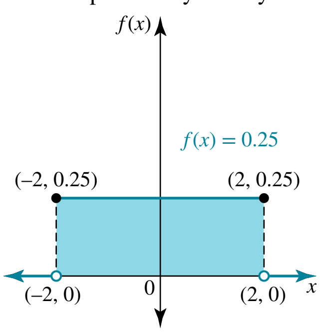
Exercise 11.2 Continuous random variables and the probability density function

1. **c**, **d**, **f**, **h**, **k** and **p** are continuous random variables.
2. a. 0.87 b. 0.17 c. 0.28
3. a. i. $\frac{9}{25}$ ii. $\frac{4}{25}$
   b. i. $\frac{37}{50}$ ii. $\frac{37}{42}$
4. a. 100 batteries b. $\frac{29}{100}$
   c. $\frac{41}{50}$ d. $\frac{3}{100}$
5. a.



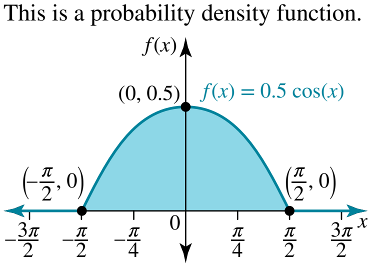
This is a probability density function.

   b.



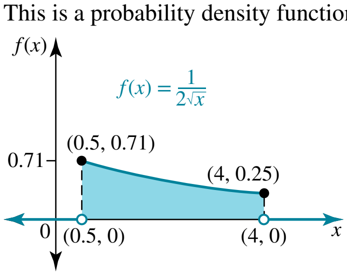
This is a probability density function.

6. a.



This is a probability density function

   b.

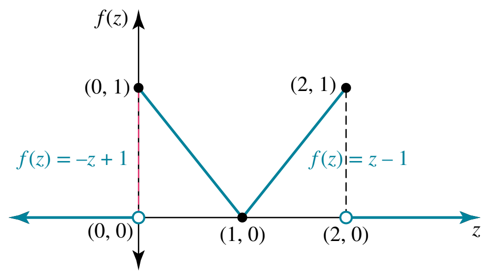


This is not a probability density function.

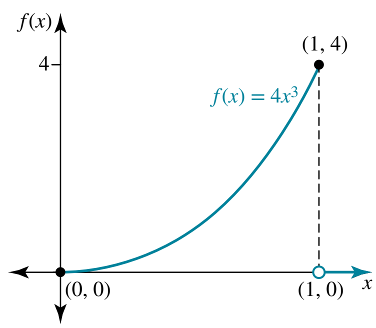
7. $n = \frac{1}{18}$
8. $a = \frac{1}{11}$
9. a. 200 b. i. $\frac{5}{8}$ ii. $\frac{31}{100}$ c. $\frac{21}{46}$
10. $\frac{5}{7}$
11. $\frac{1}{3}$
12. a. $\frac{1}{8}$ b. 2 c. $\frac{1}{4}$
13. Sample responses can be found in the worked solutions in the online resources.
14. $a = 2e \approx 5.4374$

Exercise 11.3 Cumulative distribution functions

1. a.



   b. $\frac{15}{32}$
   c. $\frac{5}{8}$
2. a. 1
   b.

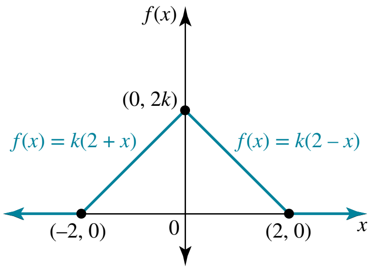


   c. $\frac{15}{16}$
3. a. $F(x) = \begin{cases} 0, & x \le 1 \\ \frac{x}{5} - \frac{1}{5}, & 1 < x \le 6 \\ 1, & x > 6 \end{cases}$
   b. i. 0.6 ii. 0.46
4. a. $F(x) = \begin{cases} 0, & x \le 0 \\ \frac{1}{2}(1 - \cos(x)), & 0 < x \le \pi \\ 1, & x > \pi \end{cases}$
   b. $\frac{1}{2}$

c. $\dfrac{\sqrt{2}}{2}$

d. $2\sqrt{2}-2$

5. a.



b. $\dfrac{1}{4}$

c. $F(x) = \begin{cases} 0, & x \le -2 \\ \dfrac{x}{2} + \dfrac{x^2}{8} + \dfrac{1}{2}, & -2 < x \le 0 \\ \dfrac{1}{2} + \dfrac{x}{2} - \dfrac{x^2}{8}, & 0 < x \le 2 \\ 1 & x > 2 \end{cases}$

d. $\dfrac{3}{4}$

e. $\dfrac{6}{7}$

6. a. $\dfrac{1}{12}$

b. $f(x) = \begin{cases} \dfrac{1}{12}, & 18 \le X < 30 \\ 0, & \text{elsewhere} \end{cases}$

c. $F(x) = \begin{cases} 0, & x < 18 \\ \dfrac{x}{12} - \dfrac{3}{2}, & 18 \le x \le 30 \\ 1, & x > 30 \end{cases}$

d. $\dfrac{5}{12}$

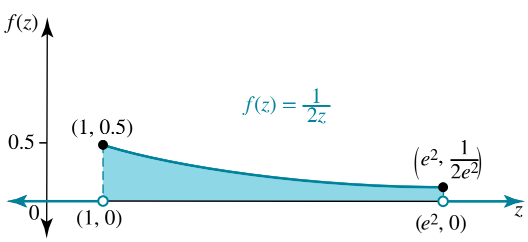
e. $\dfrac{1}{2}$

7. a. $F(x) = \begin{cases} 0, & x < 0 \\ \dfrac{x^3}{8}, & 0 \le x \le 2 \\ 1, & x > 2 \end{cases}$

b. 0.784

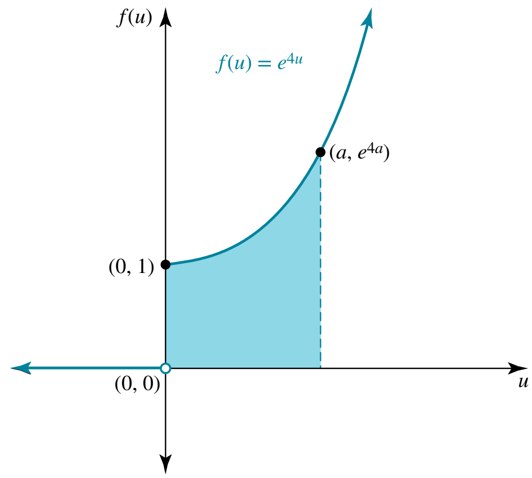
c. 1.817

8. a.



b. Sample responses can be found in the worked solutions in the online resources.

c.



d. $\dfrac{1}{4}\log_e(5)$

9. 0.604

10. a. 1 **b.** 0.715 **c.** 0.371 **d.** 0

11. a. $3\log_e\left(\dfrac{3}{2}\right)$ **b.** 0.6243 **c.** 0.5342 **d.** 0.60

12. a. 0.9502 **b.** 0.0025

13. a. 2.1555 **b.** 0.7147

14. a. 0.1560 **b.** 0.4636

Exercise 11.4 Measures of centre and spread

1. a. $a = \dfrac{5}{4}$

b. i. $\text{E}(Z) = \dfrac{9}{8}$ **ii.** $m = \dfrac{9}{8}$

2. a. $a = 1$ **b.** $\text{E}(Y) = \dfrac{2}{3}$

c. $m = \dfrac{1}{\sqrt{2}}$

3. $\text{E}(Z) = \dfrac{8}{3} = 2\dfrac{2}{3}$; $m = 2 + \dfrac{\sqrt{2}}{2}$; $\text{Var}(Z) = \dfrac{1}{18}$;

$\text{SD}(Z) = \dfrac{1}{3\sqrt{2}}$

4. $\text{E}(X) = \dfrac{1}{3}$; $m = -\dfrac{1}{3}\log_e(0.5)$; $\text{Var}(X) = \dfrac{1}{9}$; $\text{SD}(X) = \dfrac{1}{3}$

5. a. Sample responses can be found in the worked solutions in the online resources.

b. $\text{E}(X) = \dfrac{1}{3}$

c. $m = 0.25$

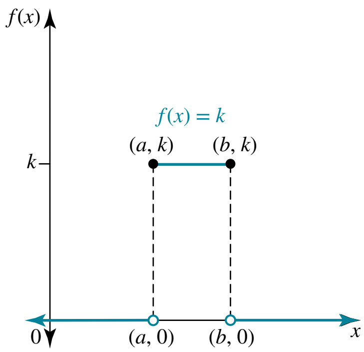
6. a. $\text{E}(T) = 0.5$ min **b.** $\text{SD}(T) = 0.5$ min

c. $m = -\dfrac{1}{2}\log_e(0.5)$

7. a. $\dfrac{dy}{dx} = -\dfrac{x}{\sqrt{4 - x^2}}$ **b.** $\text{E}(X) = \dfrac{3}{\pi}$

8. a. $h = \dfrac{1}{4}$ **b.** $\text{E}(X) = 2$ **c.** $\text{Var}(X) = 2$

9. a.



b. Sample responses can be found in the worked solutions in the online resources.

c. $\mathrm{E}(X) = \dfrac{b+a}{2}$

d. $\mathrm{Var}(X) = \dfrac{(a-b)^2}{12}$

10. a. $\mathrm{E}(Y) = 1.5601$
b. $m = 1.6510$
c. $Q_1 = 1.3104$; $Q_3 = 1.8899$
d. $\mathrm{IQR} = 0.5795$

11. a. $a = 0.4809$
b. $\mathrm{E}(Z) = 3.3663$
c. $\mathrm{Var}(Z) = 3.8195$; $\mathrm{SD}(Z) = 1.9571$
d. $Q_1 = 1.6817$; $Q_3 = 4.7568$; $\mathrm{IQR} = 3.0751$
e. Range $= 7$

12. a. Sample responses can be found in the worked solutions in the online resources.
b. $\mathrm{E}(X) = 1.0708$
c. i. $\mathrm{Var}(X) = 0.5725$ ii. $\mathrm{SD}(X) = 0.7566$
d. $m = 0.9291$

13. $a = \dfrac{3}{2}$; $b = \dfrac{3}{4}$

14. a. Sample responses can be found in the worked solutions in the online resources.
b. $\mathrm{E}(Z) = 1.2164$; $\mathrm{Var}(Z) = 0.0204$
c. $m = \dfrac{6}{5}$; $Q_1 = \dfrac{12}{11}$; $Q_3 = \dfrac{4}{3}$; $\mathrm{IQR} = \dfrac{8}{33}$

15. a. Sample responses can be found in the worked solutions in the online resources.
b. $\mathrm{E}(Y) = 5.7278$
c. $\mathrm{Var}(Y) = 2.1600$; $\mathrm{SD}(Y) = 1.4697$
d. $m = 3.9816$
e. Range $= 5.9344$

16. a. $a = 2.3104$
b. i. $\mathrm{E}(Z) = 1.7863$ ii. $\mathrm{E}\left(Z^2\right) = 3.3085$
iii. $\mathrm{Var}(Z) = 0.1176$ iv. $\mathrm{SD}(Z) = 0.3430$

11.5 Review: exam practice

1. **b** and **e** are continuous random variables.

2. $\dfrac{\pi}{4}$

3. a. 150 b. $\dfrac{13}{15} = 0.8\dot{6}$

4. 6

5. a. $F(y) = \begin{cases} 0, & y < 0 \\ y^3, & 0 \le y \le 1 \\ 1, & y > 1 \end{cases}$

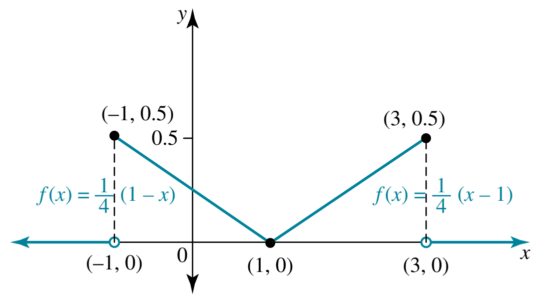
b. 0.992

6. a. The value of 19.5 kg does not lie at the end of an interval. As the weight is a continuous variable, the bags in the interval $19 \le W \le 20$ can have an infinite number of values.

b. $\dfrac{1}{7} \approx 0.1428$

7. a. $\mathrm{E}(X) = \dfrac{2}{3}$ b. $\mathrm{Var}(X) = \dfrac{1}{18} = 0.0\dot{5}$

8. a.



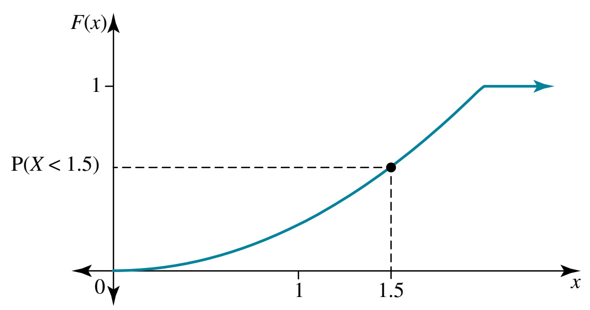
b. Sample responses can be found in the worked solutions in the online resources.

c. 1

9. Sample responses can be found in the worked solutions in the online resources.

10. a. $F(x) = \begin{cases} 0, & x < 0 \\ \dfrac{x^2}{4}, & 0 \le x \le 2 \\ 1, & x > 2 \end{cases}$

b.



11. $f(x) = \begin{cases} \dfrac{1}{4}x, & 0 \le x \le 1 \\ \dfrac{2}{7} - \dfrac{x}{28}, & 1 < x \le 8 \\ 0, & \text{elsewhere} \end{cases}$

12. a. $\dfrac{1}{9}$ b. $\dfrac{7}{27}$

13. a. 2.37
b. $\mathrm{E}(X) = 1.843$; $m = 1.887$

14. a. 0.805 b. 2.5 c. 1.31

15. $\dfrac{2}{5}$

16. a. 2 b. $\dfrac{\pi}{12}$

17. a. 0.5 b. 0.5 c. 0.3465 d. 0.5493

18. 12

19. a. 0.15 b. 7 days

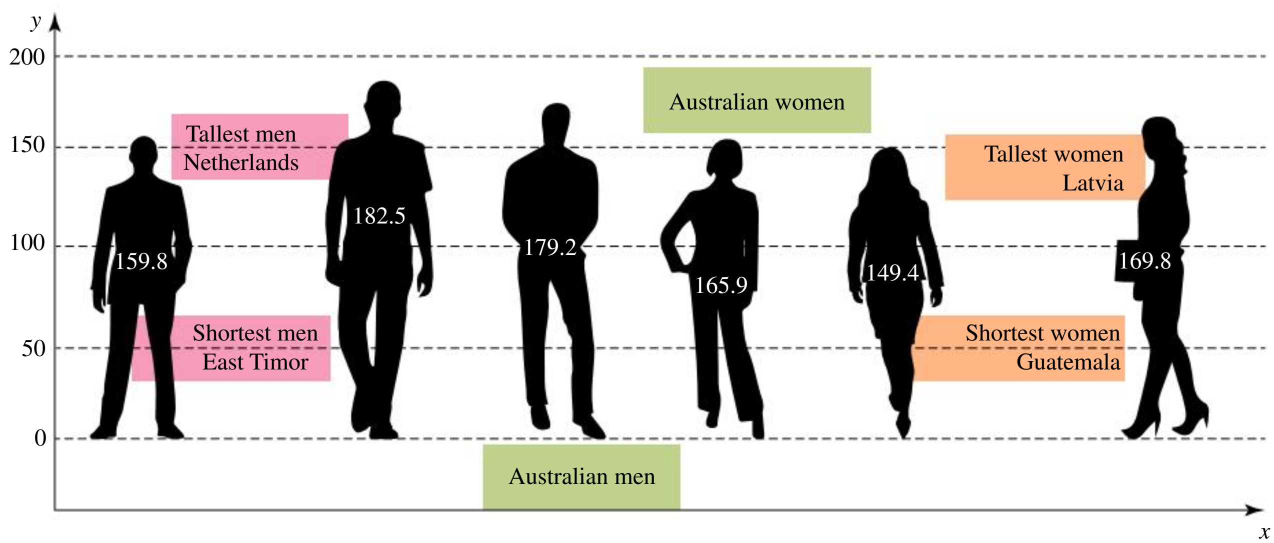
20. a. $a = e$ b. Yes

12 The normal distribution

12.1 Overview

The pattern now referred to as the normal distribution was first referred to by Galileo Galilei in 1632. Writing on the nature of scientific errors in observing astronomical motion, he noted that small errors in astronomical measurements occurred more frequently than larger errors and that the measurements were distributed symmetrically about the true value. In the intervening centuries, it has been found that the bell-shaped curve of the normal distribution turns up everywhere that the measurement of natural phenomena is involved. The heights of human beings in a population, for example, follows a normal distribution. For heights of human beings, it should be noted that the exact geometry of the bell curve varies between population groups, as different genetic and socioeconomic factors result in different values for the average and standard deviation.

Although the normal distribution for human height has no theoretical limits as such, the probability of an adult human being 4 metres tall is so incredibly small that it could be regarded as zero. So far, the tallest height recorded for an adult human is 272 cm and the shortest a mere 55 cm.



LEARNING SEQUENCE

12.1 Overview
12.2 The normal distribution
12.3 Standardised normal variables
12.4 The inverse normal distribution
12.5 Applications of the normal distribution
12.6 Review: exam practice

Fully worked solutions for this chapter are available in the Resources section of your eBookPLUS at www.jacplus.com.au.

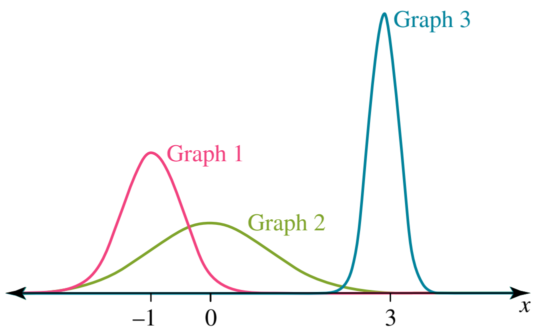
12.2 The normal distribution

12.2.1 Introduction

The probability distribution of many naturally occurring continuous random variables such as heights and weights in large populations has a distinctive bell shape, with the most frequently occurring values clustered closely around the mean. This type of distribution is most commonly referred to as the normal distribution, although you may encounter the terms ‘bell curve’ or ‘Gaussian curve’ being used to describe it.

Apart from continuous random variables such as height and weight, the **normal distribution curve** can be reliably used to model a wide variety of frequency distributions. Examples include examination results, the intelligence quotients of children in a particular age group, the usable lifetimes of lightbulbs and even the ages of stars.

The degree to which a normal curve spreads out depends upon the values of the mean and the standard deviation of the data that it models. The diagram shows three different normal distributions.



Graph 1 has a mean of -1 and a standard deviation of 0.5.
Graph 2 has a mean of 0 and a standard deviation of 1.
Graph 3 has a mean of 3 and a standard deviation of 0.25.

As you can see, the larger the value of the standard deviation is, the more spread-out the bell curve appears. The central peak is always positioned at $x = \mu$.

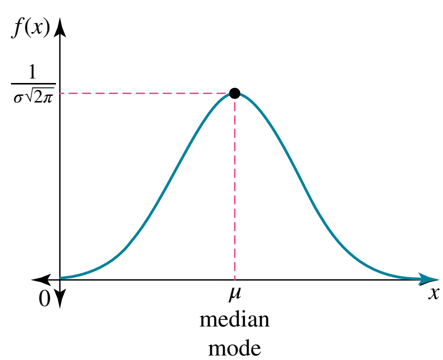
In general, if X is a continuous random variable that follows a normal distribution with a mean of μ and a variance of σ^2, it is written as $X \sim \mathrm{N}(\mu, \sigma^2)$ and its probability density function is given by the equation

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}.$$

12.2.2 Properties of the normal distribution

The normal distribution has five important characteristics.

1. Normal distributions are defined by two parameters — the mean, μ, and the standard deviation, σ.
2. A normal distribution is symmetrical about the mean.
3. The mean, median and mode are equal.
4. The area under the curve is equal to 1. That is, $\int_{-\infty}^{\infty} f(x)\,dx = 1$.
5. The majority of the values cluster around the centre of the curve, with fewer values at the tails of the curve.



As the mean and standard deviation can vary, and the area under the graph must be constant and equal to 1, changing the mean and the standard deviation transforms the normal curve. Changing the standard deviation dilates the curve by a factor of $\frac{1}{\sigma}$ parallel to the y-axis and by a factor of σ parallel to the x-axis. Changing the mean translates the curve horizontally along the x-axis.

12.2.3 Important intervals and their probabilities

Often we are required to find the proportion of a population for a given interval. Using the property that the symmetry of the normal distribution is about the mean, we are able to establish the following facts.

- **Approximately 68%** of the population will fall **within 1 standard deviation of the mean:**

$$\mathrm{P}\left(\mu - \sigma < X < \mu + \sigma\right) \approx 0.68.$$

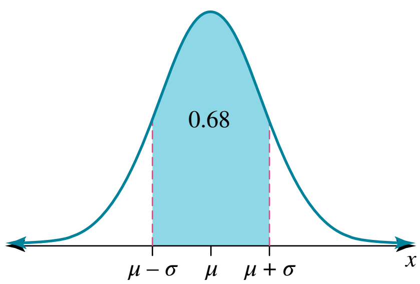
- Approximately **95%** of the population will fall within **2** standard deviations of the mean:

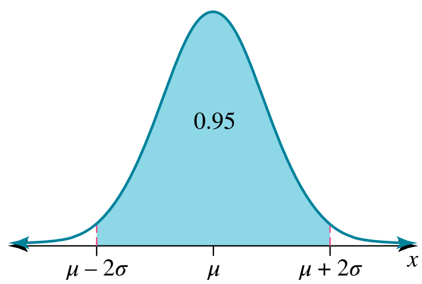
$$\mathrm{P}\left(\mu - 2\sigma < X < \mu + 2\sigma\right) \approx 0.95.$$

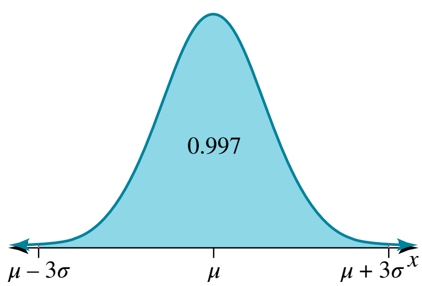
  We say that a randomly chosen member of the population will most probably be or is highly likely to be within **2** standard deviations of the mean.
- Approximately **99.7%** of the population will fall within **3** standard deviations of the mean:

$$\mathrm{P}(\mu - 3\sigma < X < \mu + 3\sigma) \approx 0.997.$$

  We say that a randomly chosen member of the population will almost certainly be within 3 standard deviations of the mean.
  This is shown on the following graphs.







These facts are collectively known as the **empirical rule** (or the 68–95–99.7% rule).

The empirical rule

- Approximately 68% of the population will fall within 1 standard deviation of the mean:

$$\mathrm{P}(\mu - \sigma < X < \mu + \sigma) \approx 0.68.$$

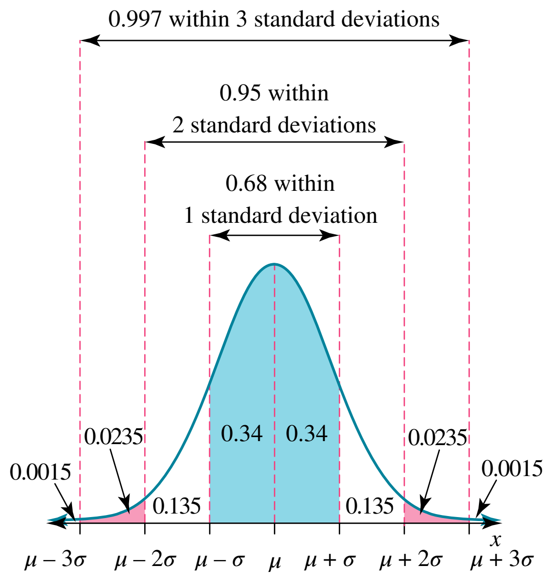
- Approximately 95% of the population will fall within 2 standard deviations of the mean:

$$P(\mu - 2\sigma < X < \mu + 2\sigma) \approx 0.95.$$

- Approximately 99.7% of the population will fall within 3 standard deviations of the mean:

$$P(\mu - 3\sigma < X < \mu + 3\sigma) \approx 0.997.$$

A more comprehensive breakdown of the proportion of the population for each standard deviation is shown on the graph.



WORKED EXAMPLE 1

The probability density function for a normal distribution is given by

$$f(x) = \frac{2}{\sqrt{2\pi}} e^{-\frac{1}{2}(2(x-1))^2}, x \in R.$$

a. **State the mean and standard deviation of the distribution.**
b. **Sketch the graph of the function.**

THINK

a. Use $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$ to determine μ and σ.

WRITE

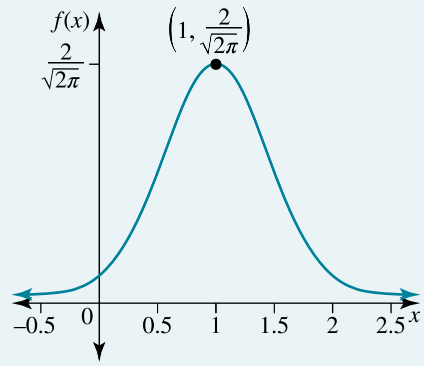
a. $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$

$= \frac{2}{\sqrt{2\pi}} e^{-\frac{1}{2}(2(x-1))^2}$

$\frac{1}{\sigma} = 2$, so $\sigma = \frac{1}{2}$ and $\mu = 1$.

b. Sketch the graph with a mean of 1 and a standard deviation of 0.5. The x-axis needs to be scaled with markings at μ, $\mu \pm \sigma$, $\mu \pm 2\sigma$ and $\mu \pm 3\sigma$. The peak of the graph must also be labelled with its coordinates.

b.



WORKED EXAMPLE 2

The heights of the women in a particular town are normally distributed with a mean of 165 cm and a standard deviation of 9 cm.

a. **What is the approximate probability that a woman chosen at random has a height that is between 156 cm and 174 cm?**

b. **What is the approximate probability that a woman chosen at random is taller than 174 cm?**

c. **What approximate percentage of the women in this particular town are shorter than 147 cm?**

THINK

a. Determine how many standard deviations from the mean the 156–174 cm range is.

WRITE

a. Let X be the height of women in this particular town.

$\mu + \sigma = 165 + 9$
$= 174$

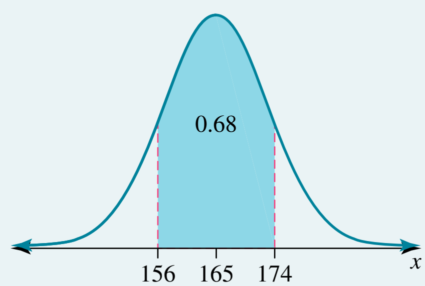
$\mu - \sigma = 165 - 9$
$= 156$

Since the range is 1 standard deviation from the mean,

$P(156 \le X \le 174) \approx 0.68$.

b. Use the fact that $P(156 \le X \le 174) \approx 0.68$ to calculate the required probability. Sketch a graph to help.

b.



Since $P(156 \le X \le 174) \approx 0.68$,

$P(X < 156) \cup P(X > 174) \approx 1 - 0.68$
$= 0.32$

Because of symmetry,

$P(X < 156) = P(X > 174)$
$= \dfrac{0.32}{2}$
$= 0.16$

Thus, $P(X > 174) \approx 0.16$.

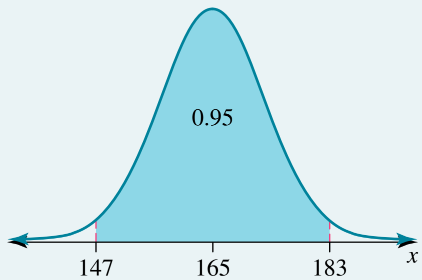
c. 1. Determine how many standard deviations 147 cm is from the mean.

c. $\mu - \sigma = 165 - 9$
$= 156$
$\mu - 2\sigma = 165 - 2 \times 9$
$= 147$

147 cm is 2 standard deviations from the mean. The-corresponding upper value is 183$(165 + 2 \times 9)$.

$P(147 \leq X \leq 183) \approx 0.95$

2. Using symmetry, calculate $P(X < 147)$.



Thus,

$P(X < 147) \cup P(X < 183) \approx 1 - 0.95$
$= 0.05$

and by symmetry,

$P(X < 147) = P(X > 183) \approx \dfrac{0.05}{2}$
$= 0.025$

Thus, approximately 2.5%, of the population of women in this particular town are shorter than 147 cm.

on Resources

Interactivities: The normal distribution (int-6438)
The 68-95-99 rule (int-6439)

studyon

Units 3 & 4 > Area 7 > Sequence 2 > Concept 1 > **The normal distribution** Summary screen and practice questions

Exercise 12.2 The normal distribution

Technology free

1. **WE1** The probability density function of a normal distribution is given by

$$f(x) = \frac{1}{3\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-2}{3}\right)^2}.$$

a. State the mean and the standard deviation of the distribution.
b. Sketch the graph of the probability function.

2. A normal distribution has a probability density function of

$$f(x)=\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(x-3)^2}.$$

a. State μ and σ.
b. Sketch the graph of the probability function.

3. WE2 The results of a Mathematical Methods test are normally distributed with a mean of 72 and a standard deviation of 8.
a. What is the approximate probability that a student who sat the test has a score which is greater than 88?
b. What approximate proportion of the students who sat the test had a score which was less than 48?
c. What approximate percentage of the students who sat the test scored less than 80?

4. The length of pregnancy for a human is normally distributed with a mean of 275 days and a standard deviation of 14 days. A mother gave birth in less than 233 days. What is the approximate probability of this happening for the general population?

5. Consider the normal probability density function

$$f(x)=\frac{1}{4\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x+2}{4}\right)^2}, x\in R.$$

Identify μ.

6. A normal probability density function is defined by

$$f(x)=\frac{10}{3\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{10(x-1)}{3}\right)^2}, x\in R.$$

a. Determine the values of μ and σ.
b. State what effect the mean and standard deviation have on the graph of the normal distribution.
c. Sketch the graph of the function, f.

7. A normal probability density function is given by

$$f(x)=\frac{1}{10\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x+4}{10}\right)^2}, x\in R.$$

a. Determine the values of μ and σ.
b. State what effect the mean and standard deviation have on the graph of the normal distribution.
c. Determine:
i. $\text{Var}(X)$ ii. $\text{E}\left(X^2\right)$
d. Verify that this is a probability density function.

8. $f(x)=\frac{5}{2\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{5(x-2)}{2}\right)^2}$, $x\in R$ defines a normal probability density function.
a. Determine the values of μ and σ.
b. Calculate $\text{E}\left(X^2\right)$.
c. Determine:
i. $\text{E}(5X)$ ii. $\text{E}\left(5X^2\right)$

Technology active

9. Scores on a commonly used IQ test are known to be normally distributed with a mean of 120 and a standard deviation of 20.

a. Determine:

i. $\mu \pm \sigma$ **ii.** $\mu \pm 2\sigma$ **iii.** $\mu \pm 3\sigma$

b. Determine:

i. $P(X < 80)$ **ii.** $P(X > 180)$

10. The results of a Year 12 Biology examination are known to be normally distributed with a mean of 70 and a standard deviation of 6. What approximate percentage of students sitting for this examination can be expected to achieve a score that is greater than 88?

11. A continuous random variable, X, is known to be normally distributed with a mean of 15 and a standard deviation of 5. Determine the range between which approximately:

a. 68% of the values lie

b. 95% of the values lie

c. 99.7% of the values lie.

12. A normal probability density function, X, has a mean of 24 and a standard deviation of 7. Determine the approximate values for:

a. $P(X < 31)$

b. $P(10 < X < 31)$

c. $P(X > 10|X < 31)$.

13. The number of pears harvested from each tree in a large orchard is normally distributed with a mean of 230 and a standard deviation of 25. Determine the approximate probability that the number of pears harvested from a randomly selected tree is:

a. less than 280

b. between 180 and 280

c. is greater than 180, given that less than 280 pears were harvested.

14. The annual rainfall in a particular area of Australia, X mm, is known to be normally distributed with a mean of 305 mm and a standard deviation of 50 mm.

a. Calculate the approximate value of $P(205 < X < 355)$.

b. Determine k such that $P(X < k) \approx 0.025$.

c. Determine h such that $P(X < h) \approx 0.0015$.

15. A normally distributed probability density function is given by

$$f(x) = \frac{5}{\sqrt{2\pi}} e^{-\frac{1}{2}(5(x-1))^2}, x \in R.$$

a. Calculate Var(X), giving your answer correct to 2 decimal places.

b. Calculate $E\left(X^2\right)$, giving your answer correct to 2 decimal places.

c. Determine:

i. $E(2X + 3)$ **ii.** $E((X + 1)(2X - 3))$

16. A continuous random variable, X, is normally distributed with a mean of 72.5 and a standard deviation of 8.4. Determine the approximate values for:

a. $P(64.1 < X < 89.3)$ **b.** $P(X < 55.7)$

c. $P(X > 47.3 \,|\, X < 55.7)$ **d.** m such that $P(X > m) \approx 0.16$.

12.3 Standardised normal variables

12.3.1 The standard normal distribution

Suppose we are comparing the results of two students who took similar Maths tests. Michelle obtained 92 on one test, for which the results were known to be normally distributed with a mean of 80 and a standard deviation of 6. Samara obtained 88 on her test, for which the results were known to be normally distributed with a mean of 78 and a standard deviation of 10. Which student was more successful?

This question is very difficult to answer unless we have some common ground for a comparison. This can be achieved by using a transformed or standardised form of the normal distribution called the **standard normal distribution**. The variable in a standard normal distribution is always denoted by Z, so that it is immediately understood that we are dealing with the standard normal distribution. The standard normal distribution always has a mean of 0 and a standard deviation of 1, so z in the following formula indicates how many standard deviations the corresponding X-value is from the mean. To calculate the value of z, we determine the difference between the x-value and the mean, $x - \mu$. To find how many standard deviations this equals, we divide by the standard deviation, σ. The result is known as the ***z*-value** or z-score.

The *z*-value formula

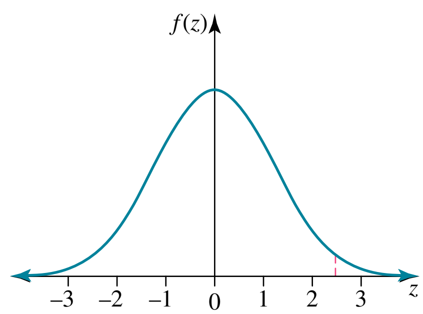
$$z = \frac{x - \mu}{\sigma}$$

Therefore, if $z = \frac{x - \mu}{\sigma}$, $\mu = 0$ and $\sigma = 1$, the probability density function is given by

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}, z \in R.$$

Remember that $\mu \pm 3\sigma$ encompasses approximately 99.7% of the data, so for the standard normal curve, these figures are $0 \pm 3 \times 1 = 0 \pm 3$. Therefore, approximately 99.7% of the data lies between -3 and 3.

For Michelle: $X \sim N\left(80, 6^2\right)$



$$\begin{aligned} z &= \frac{x - \mu}{\sigma} \\ &= \frac{92 - 80}{6} \\ &= \frac{12}{6} \\ &= 2 \end{aligned}$$

For Samara: $X \sim N\left(78, 10^2\right)$

$$\begin{aligned} z &= \frac{x - \mu}{\sigma} \\ &= \frac{88 - 78}{10} \\ &= \frac{10}{10} \\ &= 1 \end{aligned}$$

Michelle's mark lies within 2 standard deviations of the mean, so it lies in the top 2.5%, whereas Samara's mark is 1 standard deviation from the mean, so it is in the top 16%. Hence, Michelle performed better than Samara.

For the standard normal distribution, we say $Z \sim N(0, 1)$.

Let us return to the comparison between Michelle and Samara.

Obviously, not all data values will lie exactly 1, 2 or 3 standard deviations from the mean. In these cases, once the z-value is obtained, the corresponding probability may be found using the cumulative normal distribution (CND) function on your graphics calculator.

WORKED EXAMPLE 3

a. Calculate the values of the following probabilities, correct to 4 decimal places.

i. $P(Z < 2.5)$ **ii. $P(-1.25 \le Z \le 1.25)$**

b. X is a normally distributed random variable such that $X \sim N(25, 3^2)$.

i. Calculate $P(X > 27)$ correct to 4 decimal places.

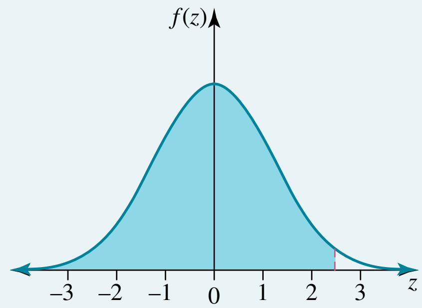
ii. Convert X to a standard normal variable, Z.

THINK

WRITE

a. i. 1. Sketch a graph to help understand the problem.

a. i.



2. Use your graphics calculator to calculate the probability.

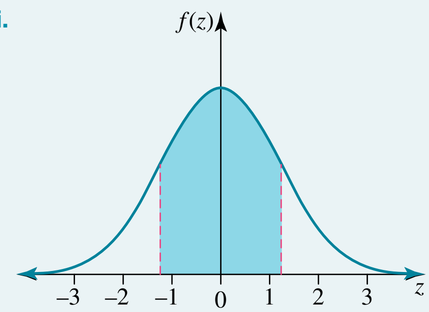
The upper limit is 2.5 and the lower limit is $-\infty$. The mean is 0 and the standard deviation is 1.

$P(z < 2.5) = 0.9938$

ii. 1. Sketch a graph to help understand the problem.

It is important to remember that $P(-1.25 \le Z \le 1.25) = P(-1.25 < Z < 1.25)$ since $P(Z = z) = 0$ for all values of z. This is because we cannot measure a continuous random variable exactly.

ii.



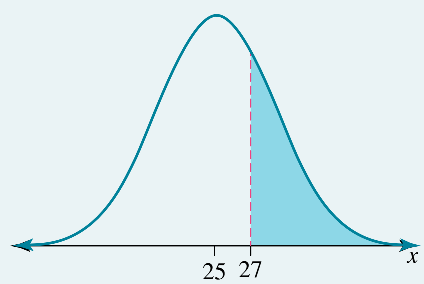
2. Use your graphics calculator to calculate the probability.

The upper limit is 1.25 and the lower limit is -1.25.

$P(-1.25 < Z < 1.25) = 0.7887$

b. i. 1. Sketch a graph to help understand the problem.

b. i.



2. Use your graphics calculator to calculate the probability.
The upper limit is ∞ and the lower limit is 27. The mean is 25 and the standard deviation is 3.

$P(X > 27) = 0.2525$

ii. 1. Write the rule to standardise X.

ii. $z = \dfrac{x - \mu}{\sigma}$

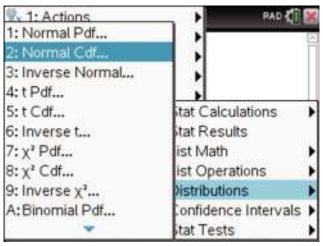
2. Substitute the mean and standard deviation.

$z = \dfrac{27 - 25}{3}$

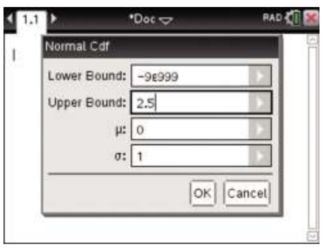
$= \dfrac{2}{3}$

TI | THINK — **WRITE**

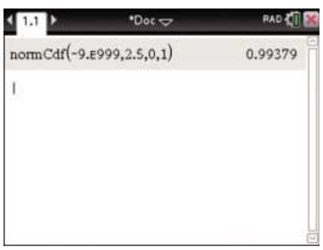
a. 1. On a Calculator page, select:
MENU
6: Statistics
5: Distributions
1: Normal Pdf ...



2. Complete the entry lines as:
Lower bound: −9e999
Upper bound: 2.5
μ: 0
σ: 1
Press the OK button.

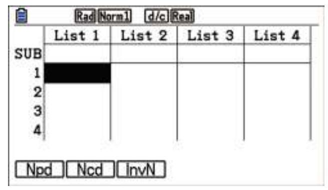


3. The answer appears on the screen.

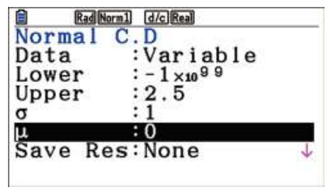


CASIO | THINK — **WRITE**

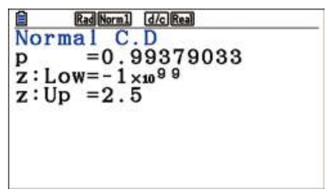
a. 1. On a Statistic screen, select:
DIST
NORM
Ncd



2. Complete the entry lines as:
Variable
Lower: -1×10^{99}
Upper: 2.5
σ: 1
μ: 0
Press the EXE button.

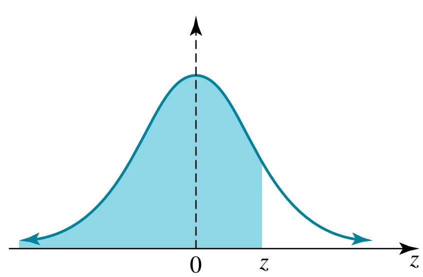


3. The answer appears on the screen.



12.3.2 Calculating probabilities using cumulative normal distribution (CND) tables

If we are working without the aid of a graphics calculator, we can also find the probability corresponding to a particular z-value by using a cumulative normal distribution (CND) table. An example of such a table is shown below. This table represents the probability of observing a result (or area) from $-\infty$ to a particular positive value of the variable z, that is, $P(Z < z)$. If this were to be represented graphically it would correspond to the shaded region shown in the diagram.



Cumulative normal distribution table (CND table)

z	0	1	2	3	4	5	6	7	8	9	Mean differences								
	0			3	4		5	6		8	8		9						
0.0	0.5	0.504	0.508	0.512	0.516	0.5199	0.5239	0.5279	0.5319	0.5359	4	8	12	16	20	24	28	32	36
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753	4	8	12	16	20	24	28	32	35
0.2	0.5793	0.5832	0.5871	0.591	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141	4	8	12	15	19	23	27	31	35
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.648	0.6517	4	8	11	15	19	23	26	30	34
0.4	0.6554	0.6591	0.6628	0.6664	0.67	0.6736	0.6772	0.6808	0.6844	0.6879	4	7	11	14	18	22	25	29	32
0.5	0.6915	0.695	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.719	0.7224	3	7	10	14	17	21	24	27	31
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549	3	6	10	13	16	19	23	26	29
0.7	0.758	0.7611	0.7642	0.7673	0.7703	0.7734	0.7764	0.7793	0.7823	0.7852	3	6	9	12	15	18	21	24	27
0.8	0.7881	0.791	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133	3	6	8	11	14	17	19	22	25
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.834	0.8365	0.8389	3	5	8	10	13	15	18	20	23
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621	2	5	7	9	12	14	16	18	21
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.877	0.879	0.881	0.833	2	4	6	8	10	12	14	16	19
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.898	0.8997	0.9015	2	4	6	7	9	11	13	15	16
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177	2	3	5	6	8	10	11	13	14
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319	1	3	4	6	7	8	10	11	13
1.5	0.9332	0.9345	0.9357	0.937	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441	1	2	4	5	6	7	8	10	11
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545	1	2	3	4	5	6	7	8	9
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633	1	2	3	3	4	5	6	7	8
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706	1	1	2	3	4	4	5	6	6
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.975	0.9756	0.9761	0.9767	1	1	2	2	3	4	4	5	5

2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817	0	1	1	2	2	3	3	4	4
2.1	0.9821	0.9826	0.983	0.9834	0.9838	0.9842	0.9846	0.985	0.9854	0.9857	0	1	1	2	2	2	3	3	4
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.989	0	1	1	1	2	2	2	3	3
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916	0	1	1	1	1	2	2	2	2
2.4	0.9918	0.992	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936	0	0	1	1	1	1	1	2	2
2.5	0.9938	0.994	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952	0	0	0	1	1	1	1	1	1
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.996	0.9961	0.9962	0.9963	0.9964	0	0	0	0	1	1	1	1	1
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.997	0.9971	0.9972	0.9973	0.9974	0	0	0	0	0	1	1	1	1
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.998	0.9981	0	0	0	0	0	0	0	1	1
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986	0	0	0	0	0	0	0	0	0
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.999	0.999	0	0	0	0	0	0	0	0	0
3.1	0.999	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993	0	0	0	0	0	0	0	0	0
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995	0	0	0	0	0	0	0	0	0
3.2	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997	0	0	0	0	0	0	0	0	0
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997		0	0	0	0	0	0	0	0	0
3.5	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0	0	0	0	0	0	0	0	0
3.6	0.9998	0.9998	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0	0	0	0	0	0	0	0	0
3.7	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0	0	0	0	0	0	0	0	0
3.8	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0	0	0	0	0	0	0	0	0
3.9	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0

WORKED EXAMPLE 4

Using the cumulative normal distribution (CND) table, calculate the values of the following.

a. $P(Z < 2)$ **b.** $P(Z < 0.74)$ **c.** $P(Z \le 1.327)$ **d.** $P(Z \le 0.5369)$

THINK

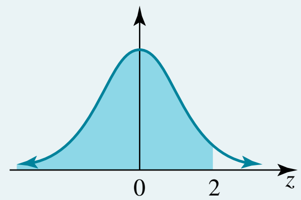
a. 1. Draw a diagram and shade the region required.

2. Using the CND table, go down to the row containing 2.0 and move across to the column headed 0.

3. Write down the required value.

WRITE

a.



$P(Z < 2) = 0.9772$

THINK

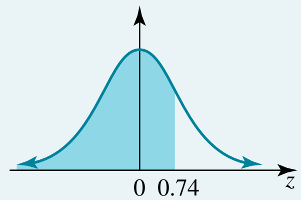
b. 1. Draw a diagram and shade the region required.

2. Using the CND table, go down to the row containing 0.7 and move across to the column headed 4.

3. Write down the required value.

WRITE

b.



$P(Z < 0.74) = 0.7703$

THINK

c. 1. Draw a diagram and shade the region required.

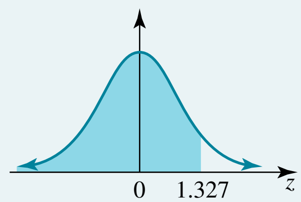
2. Using the CND table, go down to the row containing 1.3 and move across to the column headed 2. Write down the corresponding value.

3. Move across again to the mean difference column headed 7. Add the value from this final column into the last two digits of the previous answer.
Note: The value of 11 really represents 0.0011.

4. Write down the required value.

WRITE

c.



$$\begin{aligned} P(Z \le 1.327) &= 0.9066 + 0.0011 \\ &= 0.9077 \end{aligned}$$

$P(Z \le 1.327) = 0.9077$

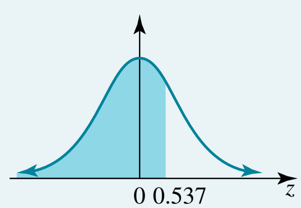
THINK

d. 1. Round off the z-value to 3 decimal places since this is the limit of the CND table.

2. Draw a diagram and shade the region required.

WRITE

d. $P(Z \le 0.5369) = P(Z \le 0.537)$



3. Using the CND table, go down to the row containing 0.5 and move across to the column headed 3. Write down the corresponding value.

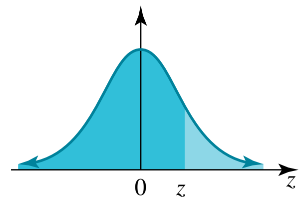
4. Move across again to the mean difference column headed 7. Add the value from this final column onto the last two digits of the previous answer. *Note:* The value of 24 really represents 0.0024.

$$P(Z \le 0.5369) = P(Z \le 0.537)$$
$$P(Z \le 0.537) = 0.7019 + 0.0024$$
$$= 0.7043$$

5. Write down the required value.

$$P(Z \le 0.537) = 0.7043$$

As mentioned previously, the CND table represents the probability of observing a result (or area) from $-\infty$ to a particular positive value of the variable z, that is, $P(Z < z)$. However, it does not directly allow us to observe a result from a particular positive value of the variable z to $+\infty$, that is, $P(Z > z)$. This problem can easily be solved by drawing a diagram of the situation, such as the one shown, and using the fact that the area between the graph and the horizontal axis is 1.



From the graph, it can be seen that $P(Z > z) + P(Z < z) = 1$.

Transposing the equation, we obtain $P(Z > z) = 1 - P(Z < z)$.

WORKED EXAMPLE 5

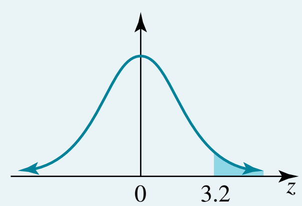
Using the cumulative normal distribution (CND) table, calculate the values of the following.

a. $\mathbf{P(Z < 3.2)}$ **b.** $\mathbf{P(Z \ge 2.3741)}$

THINK / **WRITE**

a. 1. Draw a diagram and shade the region required.

a.



2. Write down the rule for obtaining $P(Z > z)$.

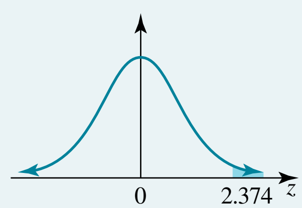
3. Use the CND table to obtain the value required.

4. Evaluate.

$$P(Z > 3.2) = 1 - P(Z < 3.2)$$
$$= 1 - 0.9993$$
$$= 0.0007$$

b. 1. Draw a diagram and shade the region required.

b.



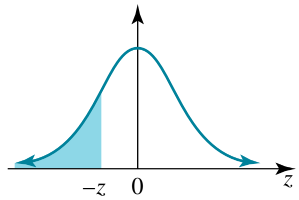
2. Round off the z-value to 3 decimal places since this is the limit of the CND table. Write down the rule for obtaining $P(Z \ge z)$.

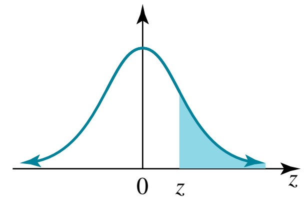
3. Use the CND table to obtain the value required.

4. Evaluate.

$$P(Z \ge 2.3741) = P(Z \ge 2.374)$$
$$= 1 - P(Z < 2.374)$$
$$= 1 - (0.9911 + 0.0001)$$
$$= 1 - 0.9912$$
$$= -0.0088$$

So far we have observed probabilities relating to the positive values of z, that is, values of z greater than the mean. Now we will investigate probabilities associated with negative values of z, that is, values of z less than the mean.

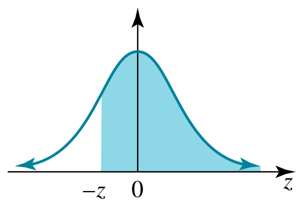


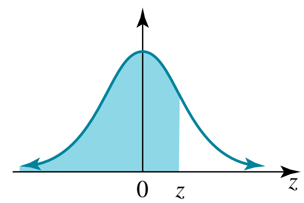


From the figure above, and using the fact that the curve is symmetrical, it can be seen that

$$\begin{aligned}\text{P}(Z < -z) &= \text{P}(Z > z)\\ &= 1 - \text{P}(Z < z) \quad \text{(from previous calculations)}\end{aligned}$$

and the figure below clearly shows that $\text{P}(Z > -z) = \text{P}(Z < z)$.

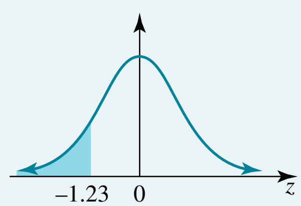
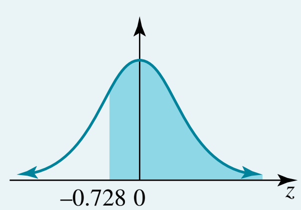




WORKED EXAMPLE 6

Using the cumulative normal distribution (CND) table, calculate the values of the following.

a. $\mathbf{P(Z < -1.23)}$ **b.** $\mathbf{P(Z \geq -0.728)}$

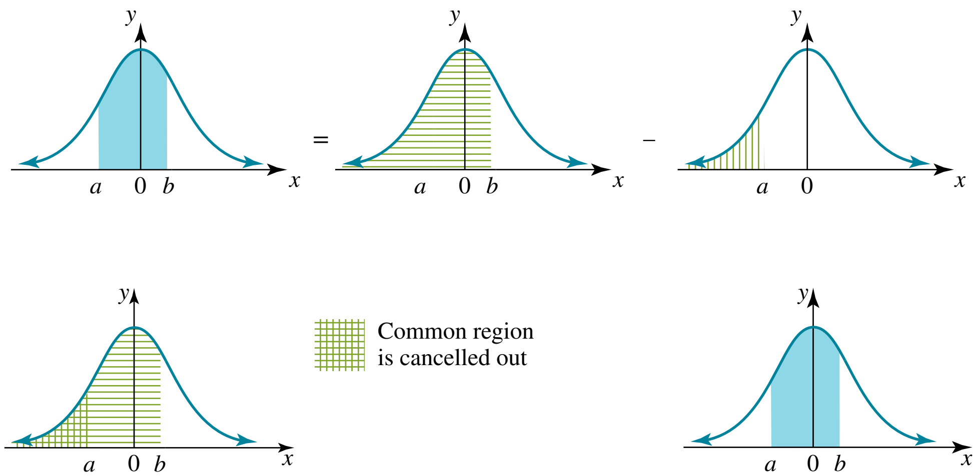
THINK	WRITE
a. 1. Draw a diagram and shade the region required.	**a.** 
2. Write down the rule for obtaining $\text{P}(Z < -z)$.	$\text{P}(Z < -1.23) = \text{P}(Z > 1.23)$ $= 1 - \text{P}(Z < 1.23)$
3. Use the CND table to obtain the value required.	$= 1 - 0.8907$
4. Evaluate.	$= 0.1093$
b. 1. Draw a diagram and shade the region required.	**b.** 
2. Write down the rule.	$\text{P}(Z \geq -0.728) = \text{P}(Z < 0.728)$
3. Use the CND table to obtain the value required.	$= 0.7642 + 0.0024$
4. Evaluate.	$= 0.7666$

Note: The inequality signs > and ≥ can be interchanged, as they give the same probability.

It is also important to be able to determine the probability of z falling between two values, say a and b. As with all of these types of problems, a diagram is essential as it allows us to see the situation clearly and hence solve the problem.

Consider the equation $P(a < Z < b) = P(z < b) - P(z < a)$.

The following figure clearly demonstrates this situation.



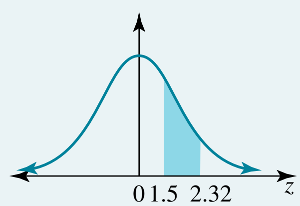
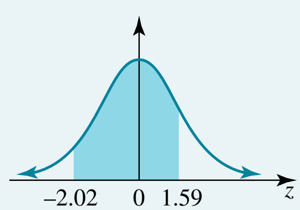
WORKED EXAMPLE 7

Using the cumulative normal distribution (CND) table, calculate the values of the following.

a. $P(1.5 < Z < 2.32)$

b. $P(-2.02 \leq Z \leq 1.59)$

c. $P(-0.235 < Z < -0.108)$

THINK	WRITE
a. 1. Draw a diagram and shade the region required.	**a.** 
2. Write down the rule required.	$P(1.5 < Z < 2.32) = P(Z < 2.32) - P(Z < 1.5)$
3. Use the CND table to obtain the value required and evaluate.	$= 0.9898 - 0.9332$ $= 0.0566$
b. 1. Draw a diagram and shade the region required.	**b.** 

2. Write down the rule for obtaining $P(a \le Z \le b)$.

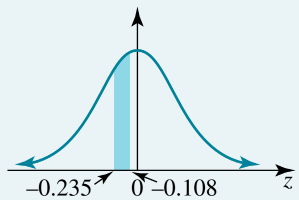
$P(-2.02 \le Z \le 1.59)$
$= P(Z < 1.59) - P(Z < -2.02)$
$= P(Z < 1.59) - P(Z > 2.02)$

3. Using the symmetry of the graph, obtain $P(z < -2.02)$.

$= P(Z < 1.59) - [1 - P(Z < 2.02)]$
$= 0.9441 - [1 - 0.9783]$
$= 0.9441 - 0.0217$
$= 0.9224$

4. Use the CND table to obtain the value required and evaluate.

c. 1. Draw a diagram and shade the region required.



2. Write down the rule.

$P(-0.235 \le Z \le -0.108)$
$= P(Z < -0.108) - P(Z < -0.235)$
$= P(Z > 0.108) - P(Z > 0.235)$

3. Using the symmetry of the graph, obtain $P(z < -0.108)$ and $P(Z < -0.235)$.

$= [1 - P(Z < 0.108)] - [1 - P(Z < 0.235)]$
$= [1 - (0.5398 + 0.0032)] - [1 - (0.5910 + 0.0019)]$
$= [1 - 0.5430] - [1 - 0.5929]$

4. Use the CND table to obtain the values required and evaluate.

$= 0.4570 - 0.4071$
$= 0.0499$

on Resources

Interactivities: Calculation of probabilities (int-6440)
The standard normal distribution (int-6441)

studyon

Units 3 & 4 > Area 7 > Sequence 2 > Concepts 3 & 4

The 65–95–99.7% rule Summary screen and practice questions
Standardised normal variables Summary screen and practice questions

Exercise 12.3 Standardised normal variables

Technology free

1. **WE3** a. Calculate the values of the following probabilities correct to 4 decimal places.
   i. $P(Z < 1.2)$
   ii. $P(-2.1 < Z < 0.8)$

   b. X is a normally distributed random variable such that $X \sim N\left(45, 6^2\right)$.
   i. Calculate $P(X > 37)$ correct to 4 decimal places.
   ii. Convert X to a standard normal variable, Z.

2. If $Z \sim N(0, 1)$, determine:
   a. $P(Z \le 2)$
   b. $P(Z \le -2)$
   c. $P(-2 < Z \le 2)$
   d. $P(Z > 1.95) \cup P(Z < -1.95)$

3. WE4 Use the CND table in section 12.3.2 to determine the values of the following.
   a. $P(Z \leq 1)$
   b. $P(Z < 2.3)$
   c. $P(Z \leq 1.52)$
   d. $P(Z \leq 0.74)$
   e. $P(Z < 1.234)$
   f. $P(Z < 2.681)$
4. WE5 Use the CND table in section 12.3.2 to determine the values of the following.
   a. $P(Z > 2)$
   b. $P(Z \geq 1.5)$
   c. $P(Z \geq 1.22)$
   d. $P(Z > 0.16)$
   e. $P(Z > 1.111)$
   f. $P(Z \geq 2.632)$
5. WE6 Use the CND table in section 12.3.2 to determine the values of the following.
   a. $P(Z \leq -2)$
   b. $P(Z < -1.3)$
   c. $P(Z < -1.75)$
   d. $P(Z > -2.71)$
   e. $P(Z \geq -1.139)$
   f. $P(Z > -0.642)$
6. WE7 Use the CND table in section 12.3.2 to determine the values of the following.
   a. $P(-1.6 \leq Z \leq 1.4)$
   b. $P(-2.21 < Z < 0.34)$
   c. $P(-0.645 \leq Z \leq 0.645)$
   d. $P(-0.72 \leq Z \leq -0.41)$
7. X is a continuous random variable and is known to be normally distributed.
   a. If $P(X < a) = 0.35$ and $P(X < b) = 0.62$, determine:
      i. $P(X > a)$
      ii. $P(a < X < b)$
   b. If $P(X < c) = 0.27$ and $P(X < d) = 0.56$, determine:
      i. $P(c < X < d)$
      ii. $P(X > c | X < d)$
   c. A random variable, X, is normally distributed with a mean of 20 and a standard deviation of 5.
      i. Determine k if $P(X > 32) = P(Z > k)$.
      ii. Determine n if $P(X < 12) = P(Z > n)$.
8. If $X \sim N(20, 25)$, determine:
   a. $P(X > 27)$
   b. $P(X \geq 18)$
   c. $P(X \leq 8)$
   d. $P(7 \leq X \leq 12)$
   e. $P(X < 17 | X \leq 25)$
   f. $P(X < 17 | X < \mu)$
9. Light bulbs have a mean life of 125 hours and a standard deviation of 11 hours. Determine the probability that a randomly selected light bulb lasts:
   a. longer than 140 hours
   b. less than 100 hours
   c. between 100 and 140 hours.
10. The heights jumped by Year 9 high-jump contestants follow a normal distribution with a mean jump height of 152 cm and a variance of 49 cm. Determine the probability that a competitor jumps:
   a. at least 159 cm
   b. less than 150 cm
   c. between 145 cm and 159 cm
   d. between 140 cm and 160 cm
   e. between 145 cm and 150 cm, given that she jumped over 140 cm.

Technology active

11. For a particular type of laptop computer, the length of time, X hours, between charges of the battery is normally distributed such that $X \sim N(50, 15^2)$. Calculate $P(50 < X < 70)$.
12. Convert the variable in each of the following expressions to a standard normal variable, Z, and use it to write an equivalent expression. Use your calculator to evaluate each probability.
   a. $P(X < 61), X \sim N(65, 9)$
   b. $P(X \geq 110), X \sim N(98, 225)$
   c. $P(-2 < X \leq 5), X \sim N(2, 9)$

13. The volume of milk in a 1-litre carton is normally distributed with a mean of 1.000 litres and a standard deviation of 0.006 litres. A randomly selected carton is known to have more than 1.004 litres. Determine the probability that it has less than 1.011 litres.

14. A radar gun is used to measure the speeds of cars on a freeway. The speeds are normally distributed with a mean of 98 km/h and a standard deviation of 6 km/h. What is the probability that a car picked at random is travelling at:
   a. more than 110 km/h
   b. less than 90 km/h
   c. a speed between 90 km/h and 110 km/h?

15. Teresa has taken her pulse each day for a month after going for a brisk walk. Her pulse rate in beats per minute is known to be normally distributed with a mean of 80 beats per minute and a standard deviation of 5 beats per minute. After her most recent walk she took her pulse rate. What is the probability that her pulse rate was:
   a. in excess of 85 beats per minute
   b. equal to or less than 75 beats per minute
   c. between 78 and 82 beats per minute, given that it was higher than 75 beats per minute?

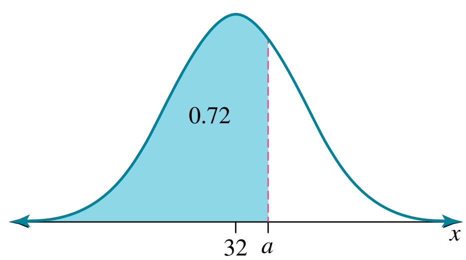
16. The labels on packets of sugar say the bags have a weight of 1 kg. The actual mean weight of the bags is 1.025 kg in order to minimise the number of bags that are underweight. If the weight of the bags is normally distributed with a standard deviation of 10 g, determine the percentage of bags that would be expected to weigh:
   a. more than 1.04 kg
   b. less than 996 g, the legal meaning of underweight.

12.4 The inverse normal distribution

12.4.1 Using technology to determine probability

Technology provides an easy way to determine a Z or X value, given a probability for a normal distribution. Suppose X is normally distributed with a mean of 32 and a standard deviation of 5. We wish to determine $P(X \leq a) = 0.72$.

The key information to enter into your calculator is the known probability, that is, the area under the curve. It is essential to input the correct area so that your calculator knows if you are inputting the 'less than' area or the 'greater than' area.



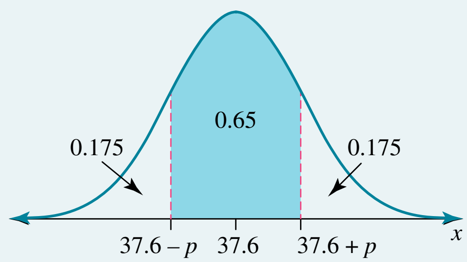
WORKED EXAMPLE 8

If X is a normally distributed random variable, determine:

a. **m given that $P(X \leq m) = 0.85, X \sim N\left(15.2, 1.5^2\right)$**

b. **n given that $P(X > n) = 0.37, X \sim N\left(22, 2.75^2\right)$**

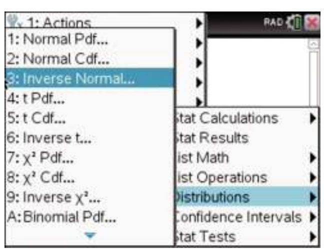
c. **p given that $P(37.6 - p \leq X \leq 37.6 + p) = 0.65, X \sim N\left(37.6, 12^2\right)$.**

THINK	WRITE
a. Use the probability menus on your graphics calculator to determine the required X value.	**a.** $P(X \leq m) = 0.85, \mu = 15.2, \sigma = 1.5\ m = 16.7547$
b. Use the probability menus on the graphics calculator to determine the required X value. *Note:* It may be a requirement to input the 'less than' area, so $P(X < n) = 1 - 0.37 = 0.63$	**b.** $P(X > n) = 0.37, \mu = 22, \sigma = 2.75\ n = 22.9126$
c. 1. Sketch a graph to visualise the problem. Due to symmetry, the probabilities either side of the upper and lower limits can be calculated.	**c.**  $1 - 0.65 = 0.35$ $P(X < 37.6 - p) = P(X > 37.6 + p) = \frac{0.35}{2} = 0.175$
2. Determine p by determining X given that $P(X < 37.6 - p) = 0.175$. *Note:* p could also be found by using the upper limit.	$P(X < 37.6 - p) = 0.175$ $37.6 - p = 26.38$ $= 37.6 - 26.38$ $= 11.22$

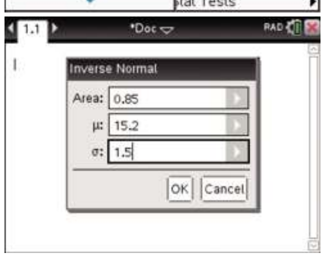
TI | THINK

a. 1. On a Calculator page, select:
MENU
6: Statistics
5: Distributions
3: Inverse Normal ...

WRITE



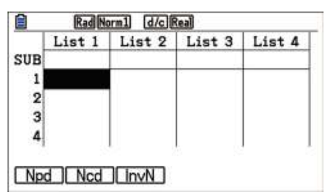
2. Complete the entry lines as:
Area: 0.85
μ:15.2
σ:1.5
Press the OK button.



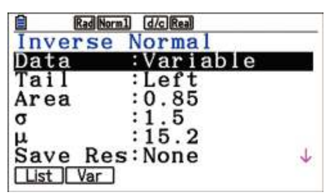
CASIO | THINK

a. 1. On a Statistic screen, select:
DIST
NORM
InvN.

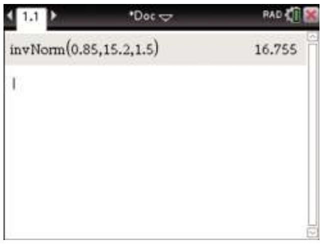
WRITE



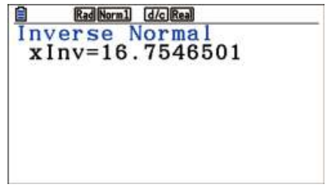
2. Complete the entry lines as:
Data: Variable
Tail: Left
Area: 0.85
σ:1.5
μ:15.2
Press the EXE button.



3. The answer appears on the screen.



3. The answer appears on the screen.



12.4.2 Using a CND table to determine probabilities

WORKED EXAMPLE 9

Determine the value of c in each of the following.

a. $\mathbf{P(Z < c) = 0.57}$ **b.** $\mathbf{P(Z \le c) = 0.25}$ **c.** $\mathbf{P(Z \ge c) = 0.91}$

THINK

a. 1. Draw a diagram of the situation.

2. Using the CND table in section 12.3.2, look up the z-value corresponding to the given probability. Obtain the closest probability value to 0.57 (0.5675 when $c = 0.17$) and then go to the mean differences column headed 6 (which gives 0.0024).

b. 1. Draw a diagram of the situation.

2. This value is not contained in the table, so we must use the symmetry of the standard normal distribution. Redraw the graph displaying the value of c on the opposite side of the mean. Call this c_1 and shade the equivalent section.
Note: $c = -c_1$

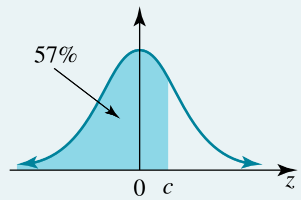
3. Determine the unshaded section of the curve.

4. Use the CND table to determine the z-value of this probability.

5. Using the symmetry of the standard normal distribution, determine the value of z. As the required z-value is to the left of the mean, it will be negative.

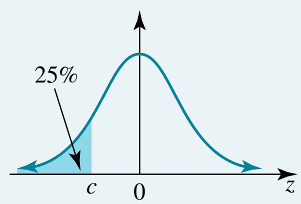
WRITE

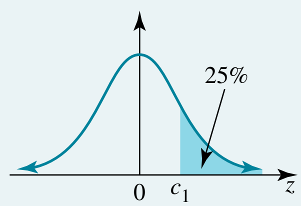
a.



$P(z < c) = 0.57$
$c = 0.17 + 0.006$
$= 0.176$

b.





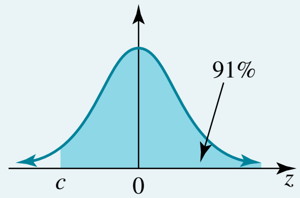
25% is shaded; therefore, 75% is unshaded.

$P(Z < c_1) = 0.75$
$c_1 = 0.67 + 0.004$
$c_1 = 0.674$

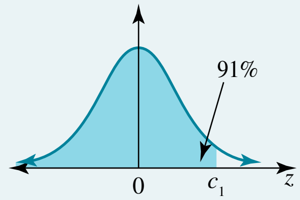
For $P(Z \le c) = 0.25$, $c = -0.674$.

c. 1. Draw a diagram of the situation.

c.



2. Redraw the graph displaying the value of c on the opposite side of the mean. Call this c_1 and shade the equivalent section.
Note: $c = -c_1$.



3. Using the CND table, it is possible to determine the value from $-\infty$ to the positive z-value that corresponds to a probability of 0.91.

$P(Z < c_1) = 0.91$
$c_1 = 1.34$

4. Using the symmetry of the standard normal distribution, determine the value of z. As the required z-value is to the left of the mean, it will be negative.

For $P(Z \geq c) = 0.91$, $c = -1.34$.

12.4.3 Quantiles and percentiles

Quantiles and percentiles are terms that enable us to convey information about a distribution. Quantiles refer to the value below which there is a specified probability that a randomly selected value will fall. For example, to determine the 0.7 quantile of a standard normal distribution, we determine a such that $P(Z < a) = 0.7$.

Percentiles are very similar to quantiles. For the example of $P(Z < a) = 0.7$, we could also be asked to determine the 70th percentile for the standard normal distribution.

WORKED EXAMPLE 10

a. For the normally distributed variable X, the 0.15 quantile is 1.9227 and the mean is 2.7. Determine the standard deviation of the distribution.

b. X is normally distributed so that the 63rd percentile is 15.896 and the standard deviation is 2.7. Determine the mean of X.

THINK

WRITE

a. 1. Write the probability statement.

a. The 0.15 quantile is 1.9227.
$P(X < 1.9227) = 0.15$

2. Determine the corresponding standardised value, Z, by using your calculator.

$P(Z < z) = 0.15$
$z = -1.0364$

3. Write the standardised formula connecting z and x.

$$z = \frac{x - \mu}{\sigma}$$

4. Substitute the appropriate values and solve for σ.

$$-1.0364 = \frac{1.9227 - 2.7}{\sigma}$$
$$-1.0364 = -0.7773$$
$$\sigma = 0.75$$

b. 1. Write the probability statement.

b. The 63rd percentile is 15.896.
$P(X < 15.896) = 0.63$

2. Calculate the corresponding standardised value, Z, by using your calculator.

$P(Z < z) = 0.63$
$z = 0.3319$

3. Write the standardised formula connecting z and x.

$$z = \frac{x - \mu}{\sigma}$$

4. Substitute in the appropriate values and solve for μ.

$$0.3319 = \frac{15.896 - \mu}{2.7}$$
$$0.8960 = 15.896 - \mu$$
$$\mu = 15$$

studyon

Units 3 & 4 > Area 7 > Sequence 2 > Concept 5

The inverse normal distribution Summary screen and practice questions

Exercise 12.4 The inverse normal distribution

Technology free

1. **WE9** Use the CND table in section 12.3.2 to determine the value of c in each of the following.
   a. $P(Z < c) = 0.9$
   b. $P(Z < c) = 0.6$
   c. $P(Z \leq c) = 0.3$
   d. $P(Z < c) = 0.2$
   e. $P(Z \geq c) = 0.8$
   f. $P(Z > c) = 0.54$
2. Use the CND table to determine the value of Z in the following.
   a. $P(Z < z) = 0.39$
   b. $P(Z \geq z) = 0.15$
   c. $P(-z < Z < z) = 0.28$
3. Let $X \sim N(22, 25)$. Calculate k if:
   a. $P(22 - k \leq X \leq 22 + k) = 0.7$
   b. $P(22 - k < X < 22 + k) = 0.24$
   c. $P(X < k | X < 23) = 0.32$
4. If $X \sim N\left(37.5, 8.62^2\right)$, calculate a correct to 2 decimal places such that:
   a. $P(X < a) = 0.72$
   b. $P(X > a) = 0.32$
   c. $P(37.5 - a < X < 37.5 + a) = 0.88$
5. For a standard normal distribution, calculate:
   a. the 0.57 quantile
   b. the 63rd percentile.

Technology active

6. **WE8** Calculate the value of a, correct to 2 decimal places, if X is normally distributed and:
   a. $P(X \leq a) = 0.16, X \sim N\left(41, 6.7^2\right)$
   b. $P(X > a) = 0.21, X \sim N\left(12.5, 2.7^2\right)$
   c. $P(15 - a \leq X \leq 15 + a) = 0.32, X \sim N\left(15, 4^2\right)$
7. Calculate the values of m and n if X is normally distributed and $P(m \leq X \leq n) = 0.92$ when $\mu = 27.3$ and $\sigma = 8.2$. The specified interval is symmetrical about the mean.
8. **WE10** X is distributed normally with a mean of 112, and the 42nd percentile is 108.87. Calculate the standard deviation of the distribution, correct to 1 decimal place.
9. X is a normally distributed random variable such that $X \sim N\left(\mu, 4.45^2\right)$. If the 0.11 quantile is 32.142, calculate the value of μ, correct to 1 decimal place.
10. If X is distributed normally with $\mu = 43.5$ and $\sigma = 9.7$, calculate:
    a. the 0.73 quantile
    b. the 24th percentile.

11. X is distributed normally with a standard deviation of 5.67, and $P(X > 20.952) = 0.09$. Calculate the mean of X, giving your answer correct to 2 decimal places.
12. X is distributed normally with a standard deviation of 3.5, and $P(X < 23.96) = 0.28$. Calculate the mean for X, rounded to the nearest whole number.
13. $X \sim N\left(115, \sigma^2\right)$ and the 76th percentile is 122.42. Calculate the value of σ, giving your answer correct to 1 decimal place.
14. X is distributed normally with $\mu = 41$ and $P(X > 55.9636) = 0.11$. Calculate σ, giving your answer correct to 1 decimal place.
15. X is distributed normally such that $P(X < 33.711) = 0.36$ and $P(X < 34.10) = 0.42$. Calculate the mean and the standard deviation of X, giving your answers correct to 1 decimal place.
16. X is distributed normally such that $P(X > 18.376) = 0.31$ and the 45th percentile is 15.15. Calculate μ and σ for X, giving your answers correct to 1 decimal place.
17. X is normally distributed with a mean of μ and a standard deviation of 3. If 35% of X-values are at least 27, calculate the mean.
18. The time taken for Grade 4 students to complete a small jigsaw puzzle follows a normal distribution with a standard deviation of 30 seconds. If 70% of Grade 4 students complete the puzzle in 4 minutes or less, calculate the mean completion time for Grade 4 students.

12.5 Applications of the normal distribution

Application problems involving the normal distribution cover a wide range of topics. Such questions will not only incorporate theory associated with the normal distribution but may also include other areas of probability you have previously studied.

WORKED EXAMPLE 11

The amount of instant porridge oats in packets packed by a particular machine is normally distributed with a mean of μ grams and a standard deviation of 6 grams. The advertised weight of a packet is 500 grams.

a. **Calculate the proportion of packets that will be underweight (less than 500 grams) when $\mu = 505$ grams.**
b. **Calculate the value of μ required to ensure that only 1% of packets are underweight.**
c. **As a check on the setting of the machine, a random sample of 5 boxes is chosen and the setting is changed if more than one of them is underweight. Calculate the probability that the setting on the machine is changed when $\mu = 505$ grams.**

THINK	WRITE
a. 1. Rewrite the information in the question using appropriate notation.	**a.** X is the amount of instant porridge oats in a packet and $X \sim \text{N}\left(505, 6^2\right)$.
2. Use your graphics calculator to calculate $\text{P}(X < 500)$.	$\text{P}(X < 500) = 0.2023$
b. 1. State the known probability.	**b.** $\text{P}(X < 500) = 0.01$
2. Calculate the corresponding standardised value, Z, by using a graphics calculator or CND table.	$\text{P}(Z < z) = 0.01$ $z = -2.3263$
3. Write the standardised formula connecting z and x.	$z = \dfrac{x - \mu}{\sigma}$
4. Substitute the appropriate values and solve for μ.	$-2.3263 = \dfrac{500 - \mu}{6}$ $-13.9581 = 500 - \mu$ $\mu = 513.96\text{ g}$
c. 1. The wording of the question (sample of 5 boxes) indicates that this is now a binomial distribution. Rewrite the information in the question using appropriate notation.	**c.** Let $Y =$ the number of underweight packets. $Y \sim \text{Bi}(5, 0.2023)$
2. Using a graphics calculator or CND table, calculate the probability.	$\text{P}(Y > 1) = 1 - \text{Pr}(Y \le 1)$ $= 1 - 0.7325$ $= 0.2674$

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Units 3 & 4 > Area 7 > Sequence 2 > Concept 6

Applications of the normal distribution Summary screen and practice questions

Exercise 12.5 Applications of the normal distribution

Technology free

1. **WE11** Packages of butter with a stated weight of 500 g have an actual weight of W g, which is normally distributed with a mean of 508 g.
   a. If the standard deviation of W is 3.0 g, calculate:
      i. the proportion of packages that weigh less than 500 g
      ii. the weight that is exceeded by 99% of the packages.
   b. If the probability that a package weighs less than 500 g is not to exceed 0.01, calculate the maximum allowable standard deviation of W.

2. Chocolate Surprise is a toy that is packed inside an egg-shaped chocolate. A certain manufacturer provides four different types of Chocolate Surprise toy — a car, an aeroplane, a ring and a doll — in the proportions given in the table.

Toy	Proportion
Car	$3k^2 + 2k$
Aeroplane	$6k^2 + 2k$
Ring	$k^2 + 2k$
Doll	$3k$

a. Show that k must be a solution to the equation $10k^2 + 9k - 1 = 0$.
b. Calculate the value of k.
In response to customer demand, the settings on the machine that produce Chocolate Surprise have been changed so that 25% of all Chocolate Surprises produced contain rings. A sample of 8 Chocolate Surprises is randomly selected from a very large number produced by the machine.
c. What is the expected number of Chocolate Surprises in the sample that contain rings? Give your answer correct to the nearest whole number.
d. What is the probability, correct to 4 decimal places, that this sample has exactly 2 Chocolate Surprises that contain rings?

3. A particular brand of car speedometer was tested for accuracy. The error measured is known to be normally distributed with a mean of 0 km/h and a standard deviation of 0.76 km/h. Speedometers are considered unacceptable if the error is more than 1.5 km/h. Calculate the proportion of speedometers that are unacceptable.

4. The heights of adult males in Perth can be taken as normally distributed with a mean of 174 cm and a standard deviation of 8 cm. Suppose the Western Australia Police Force accepts recruits only if they are at least 180 cm tall.
a. What percentage of Perth adult males satisfy the height requirement for the Western Australia Police Force?
b. What minimum height, to the nearest centimetre, would the Western Australia Police Force have to accept if it wanted a quarter of the Perth adult male population to satisfy the height requirement?

Technology active

5. a. Farmer David grows avocados on a farm on Mount Tamborine, Queensland. The average weight of his avocados is known to be normally distributed with a mean weight of 410 g and a standard deviation of 20 g.
   i. Calculate the probability that an avocado chosen at random weighs less than 360 g.
   ii. Calculate the probability that an avocado that weighs less than 360 g weighs more than 340 g.

   b. Farmer Jane grows avocados on a farm next to farmer David's. If Y represents the average weight of Jane's avocados, the weights of which are also normally distributed where $\text{P}(Y < 400) = 0.4207$ and $\text{P}(Y > 415) = 0.3446$, calculate the mean and standard deviation of the weights of Jane's avocados. Give answers correct to the nearest integer.
6. A manufacturer produces metal rods whose lengths are normally distributed with a mean of 145.0 cm and a standard deviation of 1.4 cm.
   a. Calculate the probability, correct to 4 decimal places, that a randomly selected metal rod is longer than 146.5 cm.
   b. A metal rod has a size fault if its length is not within d cm either side of the mean. The probability of a metal rod having a size fault is 0.15. Calculate the value of d, giving your answer correct to 1 decimal place.
   c. A random sample of 12 metal rods is taken from a crate containing a very large number of metal rods. Calculate the probability that there are exactly 2 metal rods with a size fault, giving your answer correct to 4 decimal places.
   d. The sales manager is considering what price, $\$x$, to sell each of the metal rods for, whether they are good or have some kind of fault. The materials cost is \$5 per rod. The metal rods are sorted into three bins. The staff know that 15% of the manufactured rods have a size fault and another 17% have some other fault. The profit, $\$Y$, is a random variable whose probability distribution is shown in the following table.

Bin	Description	Profit ($\$y$)	$\text{P}(Y = y)$
A	Good metal rods that are sold for x dollars each	$x - 5$	a
B	Metal rods with a size fault — these are not sold but recycled.	0	0.15
C	Metal rods with other faults — these are sold at a discount of \$3 each.	$x - 8$	0.17

   i. Calculate the value of a, correct to 2 decimal places.
   ii. Calculate the mean of Y in terms of x.
   iii. Hence or otherwise, calculate, correct to the nearest cent, the selling price of good rods so that the mean profit is 0.
   iv. The metal rods are stored in the bins until a large number is ready to be sold. What proportion of the rods ready to be sold are good rods?
7. A company sells two different products, X and Y, for \$5.00 and \$6.50 respectively. Regular markets exist for both products, with sales being normally distributed and averaging 2500 units (standard deviation 700) and 3000 units (standard deviation 550) respectively each week. It is company policy that if in any one week the sales for a particular product fall below half the average, that product is advertised as a 'special' for the following week.

a. Calculate the probability, correct to 4 decimal places, that product X will be advertised as a 'special' next week.
b. Calculate the probability, correct to 4 decimal places, that product Y will be advertised as a 'special' next week.
c. Calculate the probability, correct to 4 decimal places, that both products will be advertised as a 'special' next week.
d. If 40% of the company's product is product X and 60% is product Y, calculate the probability that:
   i. one product is a 'special'
   ii. if one product is advertised as 'special', then it is product X.

8. The height of plants sold at a garden nursery supplier are normally distributed with a mean of 18 cm and a standard deviation of 5 cm.
   a. Complete the following table by calculating the proportions for each of the three plant sizes, correct to 4 decimal places.

Description of plant	Plant size (cm)	Cost in $	Proportion of plants
Small	Less than 10 cm	2.00	
Medium	10–25 cm	3.50	
Large	Greater than 25 cm	5.00	

   b. Calculate the expected cost, to the nearest dollar, for 150 plants chosen at random from the garden nursery.

9. A fruit grower produces peaches whose weights are normally distributed with a mean of 185 g and a standard deviation of 20 g.

   Peaches whose weights exceed 205 g are sold to the cannery, yielding a profit of 60c per peach. Peaches whose weights are between 165 g and 205 g are sold to wholesale markets at a profit of 45 cents per peach. Peaches whose weights are less than 165 g are sold for jam at a profit of 30c per peach.

   a. Calculate the percentage of peaches sold to the canneries.
   b. Calculate the percentage of peaches sold to the wholesale markets.
   c. Calculate the mean profit per peach.

10. The Lewin Tennis Ball Company makes tennis balls whose diameters are distributed normally with a mean of 70 mm and a standard deviation of 1.5 mm. The tennis balls are packed and sold in cylindrical tins that each hold five tennis balls. A tennis ball fits in the tin if its diameter is less than 71.5 mm.

   a. What is the probability, correct to 4 decimal places, that a randomly chosen tennis ball produced by the Lewin company fits into the tin?

b. The Lewin management would like each ball produced to have a diameter between 68.6 mm and 71.4 mm. What is the probability, correct to 4 decimal places, that a randomly chosen tennis ball produced by the Lewin company is in this range?
c. A tin of five balls is selected at random. What is the probability, correct to 4 decimal places, that at least one ball has a diameter outside the range of 68.6 mm to 71.4 mm?
d. Lewin management wants engineers to change the manufacturing process so that 99.5% of all balls produced have a diameter between 68.6 mm and 71.4 mm. The mean is to stay at 70 mm but the standard deviation is to be changed. What should the new standard deviation be, correct to 4 decimal places?

11. The Apache Orchard grows a very juicy apple called the Fuji apple. Fuji apples are picked and then sorted by diameter in three categories:
- small — diameter less than 60 mm
- jumbo — the largest 15% of the apples
- standard — all other apples.

Diameters of Fuji apples are found to be normally distributed with a mean of 71 mm and a standard deviation of 12 mm.
a. A particular apple is the largest possible whose diameter lies within 2 standard deviations of the mean. What is the diameter? Give your answer correct to the nearest millimetre.
b. Calculate, correct to 4 decimal places, the probability that a Fuji apple, selected at random, has a diameter less than 85 mm.
c. What percentage of apples (to the nearest 1 per cent) is sorted into the small category?
d. Calculate, correct to the nearest millimetre, the minimum diameter of a jumbo Fuji.
e. An apple is selected at random from a bin of jumbo apples. What is the probability, correct to 4 decimal places, that it has a diameter greater than 100 mm?
f. The Apache Orchard receives the following prices for Fuji apples:
- small — 12 cents each
- standard — 15 cents each
- jumbo — 25 cents each.

What is the expected income, correct to the nearest dollar, for a container of 2500 unsorted apples?
g. Some apples are selected before sorting and are packed into bags of six to be sold at the front gate of the orchard. Calculate the probability, correct to 4 decimal places, that one of these bags contains at least two jumbo apples.

12. A brand of disinfectant is sold in two sizes: standard and large. For each size, the contents, in litres, of a randomly chosen bottle is normally distributed with a mean and standard deviation as shown in the following table.

Bottle size	Mean	Standard deviation
Standard	0.765 L	0.007 L
Large	1.015 L	0.009 L

a. Calculate the probability, correct to 4 decimal places, that a randomly chosen standard bottle contains less than 0.75 L.
b. Calculate the probability that a box of 12 randomly chosen large bottles contains at least 4 bottles whose contents are each less than 1 L.

13. Amalie is gathering data on two particular species of yellow butterflies: the lemon emigrant and the yellow emigrant. These two species can be very difficult to tell apart. Both species are equally likely to be caught in a particular area of Australia. One technique for telling them apart is by measuring the lengths of their antennae. For the lemon emigrant, the antennae are distributed normally with a mean of 22 mm and a standard deviation of 1.5 mm.

a. Calculate the probability, correct to 4 decimal places, that a randomly chosen lemon emigrant butterfly will have antennae which are shorter than 18 mm.

b. Amalie knows that 8% of the yellow emigrants have antennae that are shorter than 15.5 mm, and 8% of yellow emigrant butterflies have antennae that are longer than 22.5 mm. Assuming that the antenna lengths are normally distributed, calculate the mean and standard deviation of the antenna length of yellow emigrant butterflies, giving your answers correct to the nearest 0.1 mm.

c. In the region where Amalie is hunting for yellow butterflies, 45% of the yellow butterflies are lemon emigrants and 55% are lemon emigrants. Calculate the probability, correct to 4 decimal places, that a random sample of 12 butterflies from the region will contain 5 yellow emigrant butterflies.

14. The daily error (in seconds) of a particular brand of clock is known to be normally distributed. Only those clocks with an error of less than 3 seconds are acceptable.

a. Calculate the mean and standard deviation of the distribution of error if 2.5% of the clocks are rejected for losing time and 2.5% of the clocks are rejected for gaining time.

b. Determine the probability that fewer than 2 clocks are rejected in a batch of 12 such clocks.

12.6 Review: exam practice

A summary of this chapter is available in the Resources section of your eBookPLUS at www.jacplus.com.au.

Simple familiar

1. The diagram shows two normal distributions with means μ_1 and μ_2 and variances σ_1^2 and σ_2^2.

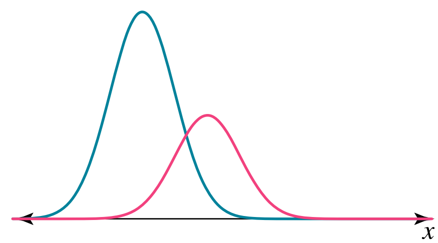
Blue: $X_1 \sim N(\mu_1, \sigma_1^2)$

Pink: $X_2 \sim N(\mu_2, \sigma_2^2)$

Which of the two distributions has:

a. the higher mean

b. the smaller standard deviation?



2. If $X \sim N(20, 25)$, calculate:

a. $P(X > 27)$ b. $P(X \geq 18)$ c. $P(7 \leq X \leq 12)$ d. $P(X < 17 | X \leq 25)$

3. The variable X is normally distributed with mean $\mu = 9$ and standard deviation $\sigma = 3$. Standardise the following X-values.

a. $X = 10$ b. $X = 7.5$ c. $X = 12.4$

4. X is normally distributed with a mean of 10 and a standard deviation of 2. Calculate x_1 if:

a. $P(X \leq x_1) = 0.72$ b. $P(X < x_1) = 0.4$

5. Calculate the value of c given that $P(-c \leq Z \leq c) = 0.38$.

6. Tennis balls are dropped from a height of 2 m. The rebound height of the balls is normally distributed with a mean of 1.4 m and a standard deviation of 0.1 m. What is the probability that a ball rebounds more than 1.25 m?

7. The results by Justine in Chemistry, Mathematical Methods and Physics are shown in the table. The marks, X, the mean, μ, and standard deviation, σ, for each examination are given.

Subject	Mark, X	Mean, μ	Standard deviation, σ	Standardised mark, Z
Chemistry	72	68	5	
Mathematical Methods	75	69	7	
Physics	68	61	8	

Complete the table by calculating Justine's standardised mark for each subject and use this to determine in which subject she did best when compared to her peers.

8. If $Z \sim \text{N}(0, 1)$, identify the 0.35 quantile.
9. The results of a state-wide Science exam are normally distributed. If 68% of entrants scored between 8 and 12 points on the exam, what are the approximate values of the mean and the standard deviation?
10. The heights of Year 9 students are known to be normally distributed with a mean of 160 cm and a standard deviation of 8 cm.
    a. How tall is Theo if he is taller than 95% of Year 9 students?
    b. How tall is Luisa if she is shorter than 80% of Year 9 students?
11. If $X \sim \text{N}\left(20, \sigma^2\right)$ and $\text{P}(X \geq 19) = 0.7$, calculate the standard deviation, σ.
12. Describe the relationship between the mean, mode and median of a standard normal distribution.

Complex familiar

13. Peter has a strawberry farm in Stanthorpe. The average length of a strawberry is normally distributed with a mean of 3.5 cm and a standard deviation of 0.8 cm.

    Strawberries that are longer than 4.5 cm are sold to a restaurant supplier for \$6.50 per kilogram. Strawberries that are between 2.5 cm and 4.5 cm long are sold to a supermarket supplier for \$4.50 per kilogram, and strawberries that are less than 2.5 cm long are sold to a jam manufacturer for \$1.75 per kilogram.
    a. Calculate the percentage of strawberries that are sold to the supermarket supplier.
    b. Calculate the percentage of strawberries that are sold to the jam manufacturer.
    c. Calculate the mean profit for a kilogram of strawberries.
14. Jing Jing scored 85 on the mathematics section of a scholarship examination, the results of which were known to be normally distributed with a mean of 72 and a standard deviation of 9. Rani scored 18 on the mathematics section of a similar examination, the results of which were normally distributed with a mean of 15 and a standard deviation of 4. Assuming that both tests measure the same kind of ability, which student has the higher score?
15. X is a normally distributed variable for which $\text{P}(X < 47) = 0.3694$ and $\text{P}(X > 56) = 0.3385$. Calculate the mean and standard deviation of X.
16. The lengths of fish caught in a certain river follow a normal distribution, with mean 32 cm and standard deviation 4 cm. Fish that are less than 27 cm long are considered to be undersized and must be returned to the river. Calculate the expected number of fish that a fisherman could take home if he catches 20 fish in one afternoon and follows the rules for undersized fish.

Complex unfamiliar

17. The lengths of certain sunflower stems follow a normal distribution with a mean of 75 cm and a standard deviation of 8 cm. Stems are measured and awarded grades depending on their lengths. The top 10% receive an *A* grade, the next 10% a *B* grade and the third 10% a *C* grade. Give a range of sunflower stem lengths to 2 decimal places for which:
    a. an *A* grade is awarded
    b. a *B* grade is awarded
    c. a *C* grade is awarded.

18. Under certain circumstances, a random variable X that follows a binomial distribution with n trials and a probability of success p can be approximated to a normal distribution. The mean of this normal distribution approximation is defined by $\mu = np$, the variance $\sigma^2 = npq$, and standard deviation $\sigma = \sqrt{npq}$.
    Use a normal approximation to determine the probability that an archery contestant will score at least 240 bullseyes out of 500 shots given that she hits the bullseye 49% of the time during practice.
19. $\text{P}(a < X < b) = 0.52$ and the specified interval is symmetrical about the mean. If X is normally distributed with a mean of 42.5 and a standard deviation of 10.3, calculate $\text{P}(X > a | X < b)$.
20. X is distributed normally such that $\text{P}(X < 39.9161) = 0.579$ and $\text{P}(X > 38.2491) = 0.4798$. Calculate the mean and the standard deviation of X, giving your answers correct to 2 decimal places.

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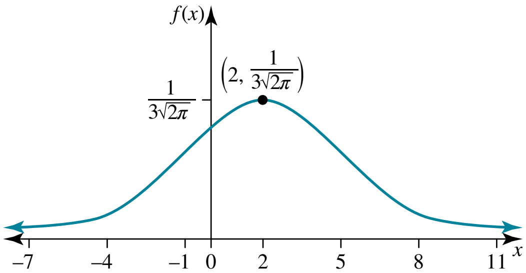
Units 3 & 4 Sit exam

Answers

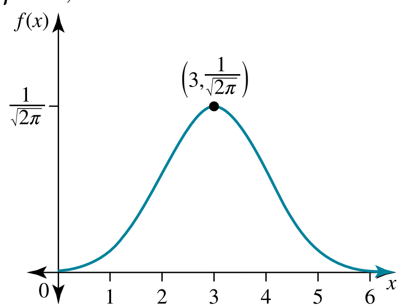
12 The normal distribution

Exercise 12.2 The normal distribution

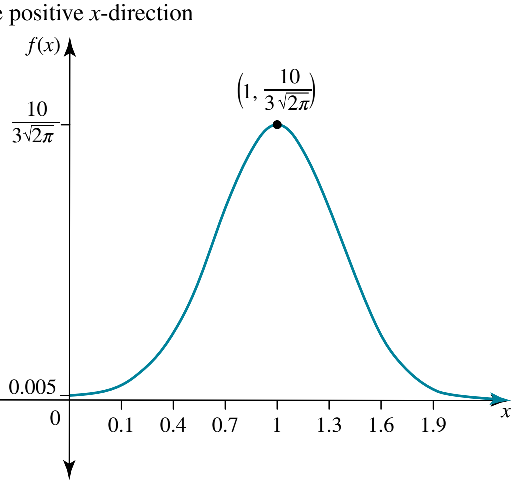
1. a. $\mu = 2; \sigma = 3$
   b.



2. a. $\mu = 3, \sigma = 1$
   b.



3. a. 0.025 b. 0.0015 c. 0.84
4. 0.0015
5. $\mu = -2$
6. a. $\mu = 1; \sigma = 0.3$ or $\frac{3}{10}$
   b. Dilation of factor $\frac{10}{3}$ parallel to the y-axis, dilation of factor $\frac{3}{10}$ parallel to the x-axis, translation of 1 unit in the positive x-direction
   c.



7. a. $\mu = -4; \sigma = 10$
   b. Dilation of factor $\frac{1}{10}$ from the x-axis, dilation of factor 10 from the y-axis, translation of 4 units in the negative x-direction
   c. i. 100
      ii. 116
   d. $\int_{\infty}^{-\infty} \frac{1}{10\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x+4}{10}\right)^2} dx = 0.9999 \approx 1$
      $f(x) \geq 0$ for all values of x, and the area under the curve is 1. Therefore, this function is a probability density function.
8. a. $\mu = 2; \sigma = \frac{2}{5} = 0.4$
   b. $\frac{104}{25} = 4.16$
   c. i. 10 ii. 20.8
9. a. i. 100 and 140
      ii. 80 and 160
      iii. 60 and 180
   b. i. 0.025 ii. 0.0015
10. 0.15%
11. a. 10 and 20 b. 5 and 25 c. 0 and 30
12. a. 0.84 b. 0.815 c. 0.9702
13. a. 0.975 b. 0.95 c. 0.9744
14. a. 0.815 b. 205 c. 155
15. a. 0.04
    b. 1.04
    c. i. 5
       ii. −1.92
16. a. 0.815 b. 0.025 c. 0.94 d. 80.9

Exercise 12.3 Standardised normal variables

1. a. i. 0.8849 ii. 0.7703
   b. i. 0.9088 ii. $-\frac{4}{3}$
2. a. 0.9772 b. 0.0228 c. 0.9545 d. 0.0512
3. a. 0.8413 b. 0.9893 c. 0.9357 d. 0.7703
   e. 0.8914 f. 0.9963
4. a. 0.0228 b. 0.0668 c. 0.1112 d. 0.4364
   e. 0.1333 f. 0.0043
5. a. 0.0228 b. 0.0968 c. 0.0401 d. 0.9966
   e. 0.8727 f. 0.7395
6. a. 0.8644 b. 0.6195 c. 0.4810 d. 0.1051
7. a. i. 0.65 ii. 0.27
   b. i. 0.29 ii. 0.5179
   c. i. 2.4 ii. 1.6
8. a. 0.0808 b. 0.6554 c. 0.0082 d. 0.0501
   e. 0.3260 f. 0.5486
9. a. 0.0863 b. 0.0115 c. 0.9022
10. a. 0.1587 b. 0.3874 c. 0.6826 d. 0.8302
    e. 0.2391
11. 0.4088
12. a. 0.0912 b. 0.2119 c. 0.7501
13. 0.8676
14. a. 0.0228 b. 0.0912 c. 0.8860
15. a. 0.1587 b. 0.1587 c. 0.3695
16. a. 6.68% b. 0.19%

Exercise 12.4 The inverse normal distribution

1. a. 1.282 b. 0.253 c. −0.524 d. −0.842
   e. −0.842 f. −0.100
2. a. −0.2793 b. 1.0364 c. 0.3585
3. a. 5.18 b. 1.525 c. 17.525
4. a. 42.52 b. 41.53 c. 13.40
5. a. 0.1764 b. 0.3319
6. a. 34.34 b. 14.68 c. 1.65

7. $m = 12.9444$; $n = 41.6556$
8. 15.5
9. 37.6
10. a. 49.4443 b. 36.6489
11. 13.35
12. 26
13. 10.5
14. 12.2
15. $\mu = 34.6$; $\sigma = 2.5$
16. $\mu = 15.8$; $\sigma = 5.2$
17. 25.844
18. 3 min 44 s

Exercise 12.5 Applications of the normal distribution

1. a. i. 0.0038 ii. 501.0210 g
   b. 3.4389 g
2. a. $3k^2 + 2k + 6k^2 + 2k + k^2 + 2k + 3k = 1$
   $$10k^2 + 9k - 1 = 0$$
   b. $\frac{1}{10}$
   c. 2
   d. 0.3115
3. 0.0484 or 4.84%
4. a. 22.66% b. 179 cm
5. a. i. 0.0062 ii. 0.9625
   b. $\mu = 405$; $\sigma = 25$
6. a. 0.1420
   b. 2.0
   c. 0.2924
   d. i. 0.68 ii. $0.85x - 4.76$
   iii. $ 5.60 iv. 80%
7. a. 0.0371
   b. 0.0032
   c. 0.0001
   d. i. 0.0167 ii. 0.8856
8. a. *See the table at the bottom of the page.
   b. $531

9. a. 15.87% b. 68.27% c. 45c
10. a. 0.8413 b. 0.6494 c. 0.8845 d. 0.4987
11. a. 95 mm b. 0.8783 c. 18% d. 83 mm
    e. 0.052 f. $399 g. 0.2236
12. a. 0.0161 b. 0.0019
13. a. 0.0038
    b. $\mu = 19.0$ mm; $\sigma = 2.5$ mm
    c. 0.2225
14. a. $\mu = 0$; $\sigma = 1.5306$ b. 0.8816

12.6 Review: exam practice

1. a. Pink b. Pink
2. a. 0.0808 b. 0.6554 c. 0.0501 d. 0.3260
3. a. $\frac{1}{3} = 0.\dot{3}$ b. -0.5 c. $1.1\dot{3}$
4. a. 11.166 b. 9.494
5. 0.496
6. 0.9332
7. Chemistry: 0.8; Mathematical Methods: 0.857; Physics: 0.875. Justine did best in Physics.
8. -0.385
9. $\mu = 10$; $\sigma = 2$
10. a. Theo is 173.16 cm tall.
    b. Luisa is 153.26 cm tall.
11. 1.908
12. Mean = median = mode
13. a. 10.56% b. 10.56% c. $4.42/kg
14. Jing Jing did better.
15. $\mu = 51$; $\sigma = 12$
16. 17
17. a. *A* grade for stems longer than 85.25 cm
    b. *B* grade for stem lengths between 81.73 cm and 85.25 cm
    c. *C* grade for stem lengths between 79.19 cm and 81.73 cm
18. 0.6725
19. 0.6842
20. $\mu = 37.68$; $\sigma = 11.21$

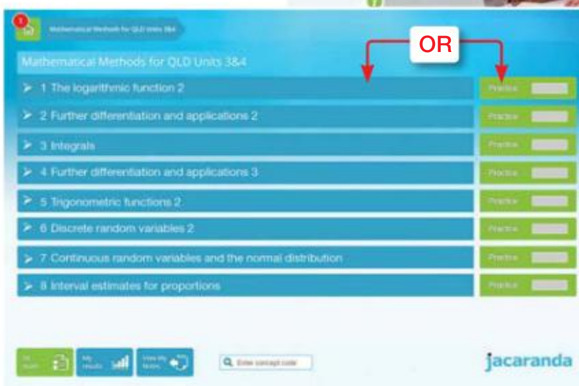
*8. a.

Description of plant	Plant size (cm)	Cost in $	Proportion of plants
Small	Less than 10 cm	2.00	0.0548
Medium	10–25 cm	3.50	0.8644
Large	Greater than 25 cm	5.00	0.0808

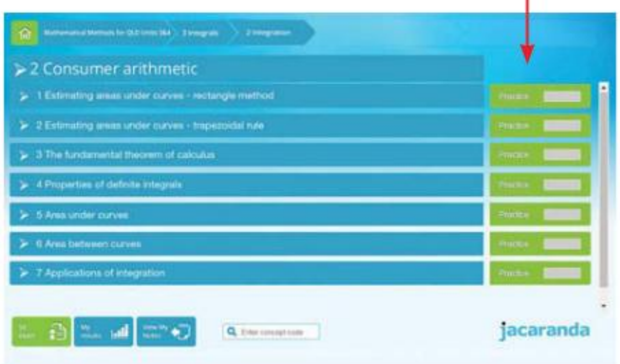
TOPIC 4 Continuous random variables and the normal distribution

- For revision of this entire topic, go to your **studyON** title in your bookshelf at **www.jacplus.com.au**.
- Select **Continue Studying** to access hundreds of revision questions across your entire course.

- Select your **course** *Mathematical Methods for Queensland Units 3&4* to see the entire course divided into syllabus topics.
- Select the **Area** you are studying to navigate into the chapter level ***OR*** select **Practice** to answer all practice questions available for each area.

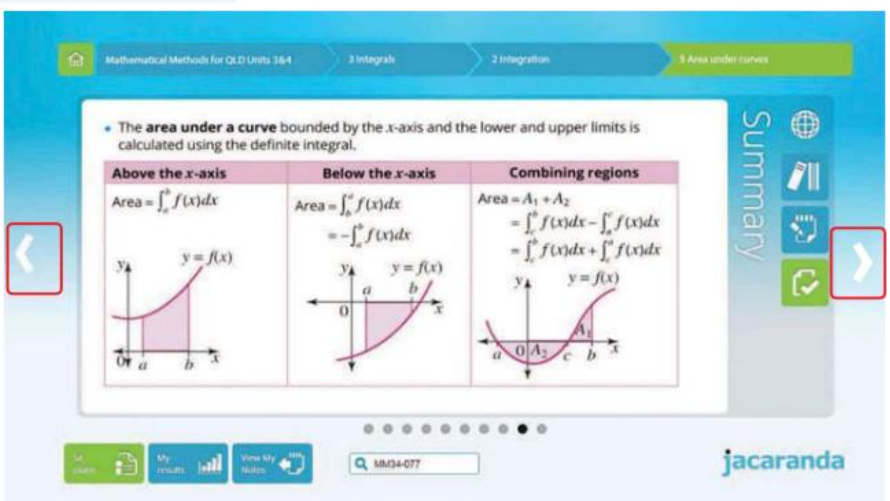


- Select **Practice** at the sequence level to access all questions in the sequence.



- At **sequence level**, drill down to concept level.

- **Summary screens** provide revision and consolidation of key concepts. Select the **next arrow** to revise all concepts in the sequence and practise questions at the concept level.



13 Sampling and confidence intervals

13.1 Overview

Many of the statistics about life in Australia are drawn from the Census of Population and Housing conducted by the Australian Bureau of Statistics every five years. A census is simply a survey of the population of a country. It collects general information such as gender, age and occupation as well as more detailed data such as ethnicity and housing situation. The most recent Australian census revealed many interesting facts about Queensland. Here's a few examples:

- Of the 4.9 million people living in the state, almost half live within the greater Brisbane area.
- The top five languages (other than English) spoken in Queensland are Mandarin, Filipino/Tagalog, Vietnamese, Cantonese and Spanish.
- Aboriginal and Torres Strait Islander People account for 4.0 percent of Queensland's population.
- The top five countries of birth (other than Australia) for people in Queensland are England, New Zealand, India, China and South Africa.

It is quite impractical and sometimes impossible to analyse the entire population for a particular situation, so a sample is usually taken. The sample needs to be representative of the population, and multiple samples should be taken where possible. The study of statistics allows us to gauge the confidence we can have in the validity of our samples for an investigation before we make any predictions or assumptions.

LEARNING SEQUENCE

13.1 Overview
13.2 Sample statistics
13.3 The distribution of $\hat{p}$
13.4 Confidence intervals
13.5 Review: exam practice

Fully worked solutions for this chapter are available in the Resources section of your eBookPLUS at www.jacplus.com.au.

13.2 Sample statistics

Suppose you were interested in the percentage of Year 12 graduates who plan to study Mathematics once they complete school. It is probably not practical to question every student. There must be a way that we can ask a smaller group and then use this information to make generalisations about the whole group.

13.2.1 Samples and populations

A **population** is a group that you want to know something about, and a **sample** is the group within the population that you collect the information from. Normally, a sample is smaller than the population; the exception is a census, where the whole population is the sample.

The number of members in a sample is called the **sample size** (symbol n), and the number of members of a population is called the **population size** (symbol N). Sometimes the population size is unknown.

WORKED EXAMPLE 1

Cameron has uploaded a popular YouTube video. He thinks that the 133 people in his year group at school have seen it, and he wants to know what they think. He decides to question 10 people. Identify the population and sample size.

THINK	WRITE
1. Cameron wants to know what the people in his year at school think. This is the population.	$N = 133$
2. He asks 10 people. This is the sample.	$n = 10$

WORKED EXAMPLE 2

A total of 137 people volunteer to take part in a medical trial. Of these, 57 are identified as suitable candidates and are given the medication. Identify the population and sample size.

THINK	WRITE
1. 57 people are given the medication. This is the sample size.	$n = 57$
2. We are interested in the group of people who might receive the drug in the future. This is the population.	The population is unknown, as we don't know how many people may be given this drug in the future.

on Resources

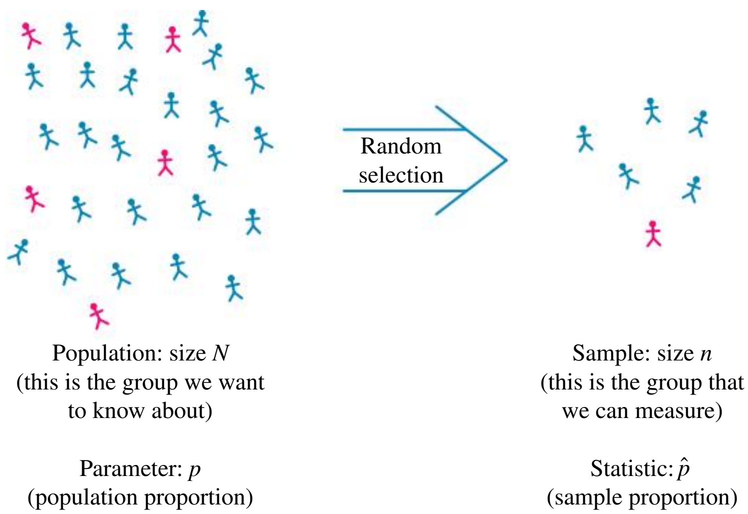
Interactivity: Population parameters and sample statistics (int-6442)

13.2.2 Statistics and parameters

A **parameter** is a characteristic of a population, whereas a **statistic** is a characteristic of a sample. This means that a statistic is always known exactly (because it is measured from the sample that has been selected). A parameter is usually estimated from a sample statistic. (The exception is if the sample is a census, in which case the parameter is known exactly.)

In this unit, we will study binomial data (that means that each data point is either yes/no or success/failure) with special regard to the proportion of successes.

The relationship between populations and samples



WORKED EXAMPLE 3

Identify the following as either sample statistics or population parameters.

a. **Forty-three per cent of voters polled say that they are in favour of banning fast food.**

b. **According to Australian Bureau of Statistics census data, the average family has 1.7 children.**

c. **Between 18% and 23% of Australians skip breakfast regularly.**

d. **Nine out of 10 children prefer cereal for breakfast.**

THINK	WRITE
a. 43% is an exact value that summarises the sample asked.	**a.** Sample statistic
b. The information comes from census data. The census questions the entire population.	**b.** Population parameter

c. 18%–23% is an estimate about the population.

d. Nine out of 10 is an exact value. It is unlikely that all children could have been asked; therefore, it is from a sample.

c. Population parameter

d. Sample statistic

13.2.3 Random samples

A good sample should be representative of the population. If we consider our initial interest in the proportion of Year 12 graduates who intend to study Mathematics once they finished school, we could use a Mathematical Methods class as a sample. This would not be a good sample because it does not represent the population — it is a very specific group.

In a **random sample**, every member of the population has the same probability of being selected. The Mathematical Methods class is not a random sample because students who don't study Mathematical Methods have no chance of being selected; furthermore, students who don't attend that particular school have no chance of being selected.

A **systematic sample** is almost as good as a random sample. In a systematic sample, every kth member of the population is sampled. For example, if $k = 20$, a customs official might choose to sample every 20th person who passes through the arrivals gate. The reason that this is almost as good as random sample is that there is an assumption that the group passing the checkpoint during the time the sample is taken is representative of the population. This assumption may not always be true; for example, people flying for business may be more likely to arrive on an early morning flight. Depending on the information you are collecting, this may influence the quality of the data.

In a **stratified random sample**, care is taken so that subgroups within a population are represented in a similar proportion in the sample. For example, if you were collecting information about students in Years 9–12 in your school, the proportions of students in each year group should be the same in the sample and the population. Within each subgroup, each member has the same chance of being selected.

A **self-selected sample**, that is one where the participants choose to participate in the survey, is almost never representative of the population. For example, television phone polls, where people phone in to answer yes or no to a question, do not accurately reflect the opinion of the population.

WORKED EXAMPLE 4

A survey is to be conducted in a middle school that has the distribution detailed in the table. It is believed that students in different year levels may respond differently, so the sample chosen should reflect the subgroups in the population (that is, it should be a stratified random sample). If a sample of 100 students is required, determine how many from each year group should be selected.

Year level	Number of students
7	174
8	123
9	147

THINK	WRITE
1. Calculate the total population size.	Total population = 174 + 123 + 147 = 444
2. Calculate the number of Year 7s to be surveyed.	$\text{Number of Year 7s} = \frac{174}{444} \times 100$ $= 39.1$ Survey 39 Year 7s.
3. Calculate the number of Year 8s to be surveyed.	$\text{Number of Year 8s} = \frac{123}{444} \times 100$ $= 27.7$ Survey 28 Year 8s.
4. Calculate the number of Year 9s to be surveyed.	$\text{Number of Year 9s} = \frac{147}{444} \times 100$ $= 33.1$ Survey 33 Year 9s.
5. There has been some rounding, so check that the overall sample size is still 100.	Sample size = 39 + 28 + 33 = 100 The sample should consist of 39 Year 7s, 28 Year 8s and 33 Year 9s.

on Resources

Interactivity: Random samples (int-6443)

13.2.4 Using technology to select a sample

If you know the population size, it should also be possible to produce a list of population members. Assign each population member a number (from 1 to *N*). Use the random number generator on your calculator to generate a random number between 1 and *N*. The population member who was allocated that number becomes the first member of the sample. Continue generating random numbers until the required number of members has been picked for the sample. If the same random number is generated more than once, ignore it and continue selecting members until the required number has been chosen.

studyon

Units 3 & 4 > Area 8 > Sequence 1 > Concepts 1 & 2

Samples and proportions Summary screen and practice questions
A random sample Summary screen and practice questions

Exercise 13.2 Sample statistics

Technology free

1. **WE1** On average, Mr Parker teaches 120 students per day. He asks one class of 30 about the amount of homework they have that night. Identify the population and sample size.
2. Bruce is able to sew the hems 100 shirts per day. Each day he checks 5 to make sure that they are suitable. Identify the population and sample size.
3. **WE2** Ms Lane plans to begin her Statistics class each year by telling her students a joke. She tests her joke on this year's class (15 students). She plans to retire in 23 years' time. Identify the population and sample size.

4. Lee-Yin is trying to perfect a recipe for cake pops. She tries 5 different versions before she settles on her favourite. She takes some samples to school and asks 9 friends what they think. Identify the population and sample size.

5. **WE3** Identify the following as either sample statistics or population parameters.
   a. Studies have shown that between 85% and 95% of lung cancers are related to smoking.
   b. About 50% of children aged between 9 and 15 years eat the recommended daily amount of fruit.
6. Identify the following as either sample statistics or population parameters.
   a. According to the 2013 census, the ratio of male births per 100 female births is 106.3.
   b. About 55% of boys and 40% of girls reported drinking at least 2 quantities of 500 mL of soft drink every day.
7. **WE4** A school has 523 boys and 621 girls. You are interested in finding out about their attitudes to sport and believe that boys and girls may respond differently. If a sample of 75 students is required, determine how many boys and how many girls should be selected.
8. In a school, 23% of the students are boarders. For this survey, it is believed that boarders and day students may respond differently. To select a sample of 90 students, how many boarders and day students should be selected?
9. You are trying out a new chocolate pudding recipe. You found 40 volunteers to taste test your new recipe compared to your normal pudding. Half of the volunteers were given a serving the new pudding first, then a serving of the old pudding. The other half were given the old pudding first and then the new pudding. The taste testers did not know the order of the puddings they were trying. The results show that 31 people prefer the new pudding recipe.
   a. What is the population size?
   b. What is the sample size?

10. You want to test a new flu vaccine on people with a history of chronic asthma. You begin with 500 volunteers and end up with 247 suitable people to test the vaccine.
    a. What is the population size?
    b. What is the sample size?
11. In a recent survey, 1 in 5 students indicated that they ate potato crisps or other salty snacks at least four times per week. Is this a sample statistic or a population parameter?
12. Around 25 to 30% of children aged 0–15 years eat confectionary at least four times a week. Is this a sample statistic or a population parameter?
13. According to the Australian Bureau of Statistics, almost a quarter (24%) of internet users did not make an online purchase or order in 2012–13. The three most commonly reported main reasons for not making an online purchase or order were: 'Has no need' (33%); 'Prefers to shop in person/see the product' (24%); and 'Security concerns/concerned about providing credit card details online' (12%). Are these sample statistics or population parameters?

14. According to the 2011 census, there is an average of 2.6 people per household. Is this a sample statistic or a population parameter?

15. A doctor is undertaking a study about sleeping habits. She decides to ask every 10th patient about their sleeping habits.
   a. What type of sample is this?
   b. Is this a valid sampling method?

16. A morning television show conducts a viewer phone-in poll and announces that 95% of listeners believe that Australia should become a republic. Comment on the validity of this type of sample.
17. Tony took a survey by walking around the playground at lunch and asking fellow students questions. Why is this not the best sampling method?

Technology active

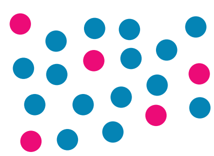
18. A company has 1500 staff members, of whom 60% are male; 95% of the male staff work full time, and 78% of the female staff work full time. If a sample of 80 staff is to be selected, identify the numbers of full-time male staff, part-time male staff, full-time female staff and part-time female staff that should be included in the sample.
19. Use your calculator to produce a list of 10 random numbers between 1 and 100.
20. Use your calculator to select a random sample from students in your Mathematical Methods class.

13.3 The distribution of $\hat{p}$

13.3.1 The sample proportion

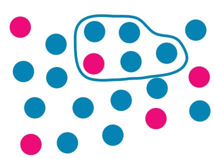
Let us say that we are interested in the following collection of balls. As you can see in Figure 1, there are 20 balls, and $\frac{1}{4}$ of them are red. This means that the population parameter, p, is $\frac{1}{4}$ and the population size, N, is 20.

FIGURE 1



Normally we wouldn't know the population parameter, so we would choose a sample from the population and determine the sample statistic. In this case, we are going to use a sample size of 5, that is, $n = 5$.

FIGURE 2



If our sample is the group shown in Figure 2, then as there is 1 red ball, the **sample proportion** would be $\hat{p} = \frac{1}{5}$.

A different sample could have a different sample proportion. In the case shown in Figure 3, $\hat{p} = \frac{2}{5}$.
In the case shown in Figure 4, $\hat{p} = 0$.

FIGURE 3

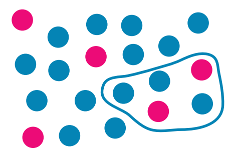
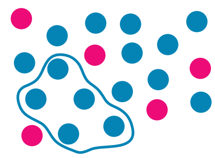


FIGURE 4



It would also be possible to have samples for which $\hat{p} = \frac{3}{5}$, $\hat{p} = \frac{4}{5}$ or $\hat{p} = 1$, although these samples are less likely to occur.

In summary,

Sample proportion

$$\hat{p} = \frac{\text{number of successful outcomes in the sample}}{\text{sample size}}$$

It might seem that using a sample does not give a good estimate about the population. However, the larger the sample size, the more likely that the sample proportions will be close to the population proportion.

on Resources

Interactivity: Distribution of $\hat{p}$ (int-6444)

WORKED EXAMPLE 5

You are trying out a new chocolate tart recipe. You found 40 volunteers to taste test your new recipe compared to your normal one. Half the volunteers were given a serving of the new tart first, then a serving of the original tart. The other half were given the original tart first and then the new one. The taste testers did not know the order of the tarts they were trying. The results show that 31 people prefer the new tart recipe. What is the sample proportion, $\hat{p}$?

THINK	WRITE
1. There are 40 volunteers. This is the sample size.	$n = 40$
2. 31 people prefer the new recipe.	Number of successes = 31
3. Calculate the sample proportion.	$\hat{p} = \frac{31}{40}$

13.3.2 Revision of binomial distributions

In a set of binomial data, each member of the population can have one of two possible values. We define one value as a success and the other value as a failure. (A success isn't necessarily a good thing, it is simply the name for the condition we are counting. For example, a success may be having a particular disease and a failure may be not having the disease).

Binomial distributions

The proportion of successes in a population is called p and is a constant value.

$$p = \frac{\text{number in the population with the favourable attribute}}{\text{population size}}$$

The proportion of failures in a population is called q, where $q = 1 - p$.
The sample size is called n.
The number of successes in the sample is called X.
The proportion of successes in the sample, $\hat{p}$, will vary from one sample to another.

$$\begin{aligned}\hat{p} &= \frac{\text{number in the sample with the favourable attribute}}{\text{sample size}}\\ &= \frac{X}{n}\end{aligned}$$

13.3.3 Sampling distribution of $\hat{p}$

Normally, you would take one sample from a population and make some inferences about the population from that sample. In this section, we are going to explore what would happen if you took lots of samples of the same size. (Assume you return each sample back to the population before selecting again.)

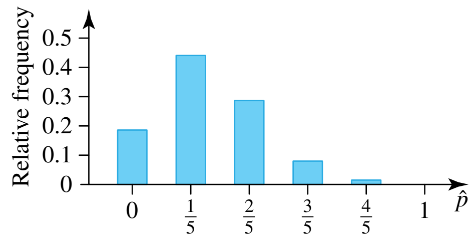
Consider our population of 20 balls (5 red and 15 blue). There are ${}^{20}C_5 = 15\,504$ possible samples that could be chosen. That is, there are 15 504 possible ways of choosing 5 balls from a population of 20 balls. A breakdown of the different samples is shown in the table, where X is the number of red balls in the sample.

X	$\hat{p}$	Number of samples	Relative frequency
0	0	${}^5C_0\,{}^{15}C_5 = 3003$	0.194
1	$\frac{1}{5}$	${}^5C_1\,{}^{15}C_4 = 6825$	0.440
2	$\frac{2}{5}$	${}^5C_2\,{}^{15}C_3 = 4550$	0.293
3	$\frac{3}{5}$	${}^5C_3\,{}^{15}C_2 = 1050$	0.068
4	$\frac{4}{5}$	${}^5C_4\,{}^{15}C_1 = 75$	0.005
5	1	${}^5C_5\,{}^{15}C_0 = 1$	6.450×10^{-5}
	Total number of samples	**15 504**	

Graphing the distribution of $\hat{p}$ against the relative frequency of $\hat{p}$ results in the following.

As the value of $\hat{p}$, the sample proportion, varies depending on the sample, these values can be considered as the values of the random variable, $\hat{P}$.

The graph of the distribution of $\hat{p}$ can also be represented in a probability distribution table. This distribution is called a **sampling distribution**.



$\hat{p}$	0	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{4}{5}$	1
$P(\hat{p} = \hat{p})$	0.194	0.440	0.293	0.068	0.005	6.450×10^{-5}

We can calculate the average value of $\hat{p}$ as shown.

$\hat{p}$	Frequency, f	$f \cdot \hat{p}$
0	3 003	0
$\frac{1}{5}$	6 825	1365
$\frac{2}{5}$	4 550	1820
$\frac{3}{5}$	1 050	630
$\frac{4}{5}$	75	60
1	1	1
Totals	**15 504**	**3876**

$$\text{The average value of } \hat{p} = \frac{3876}{15\,504}$$
$$= 0.25$$

For this distribution, the average value for $\hat{p}$ is equal to the population proportion, p.

Expected value of large samples

It was mentioned earlier that larger samples give better estimates of the population. The proportion of $\hat{p}$ in a large sample conforms to $\hat{P} = \frac{X}{n}$. As the sample is from a large population, X can be assumed to be a binomial variable.

$$\therefore E(\hat{P}) = E\left(\frac{X}{n}\right)$$
$$= \frac{1}{n}E(X) \left(\text{because } \frac{1}{n} \text{ is a constant}\right)$$

$$= \frac{1}{n} \times np$$
$$= p$$

Variance and standard deviation

The variance and standard deviation can be found as follows.

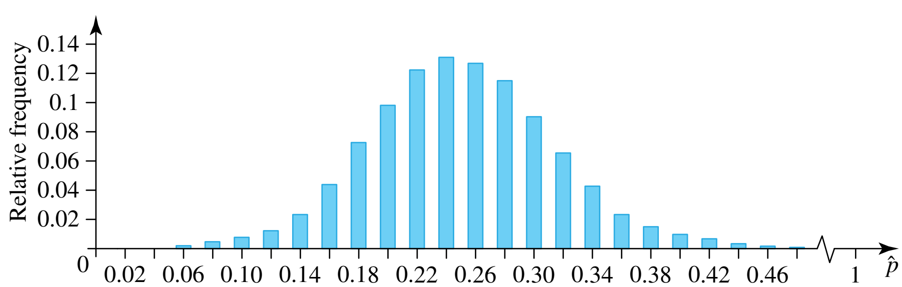
$$\begin{aligned}
\text{Var}(\hat{P}) &= \text{Var}\left(\frac{X}{n}\right) \\
&= \left(\frac{1}{n}\right)^2 \text{Var}(X) \\
&= \frac{1}{n^2} \times npq \\
&= \frac{pq}{n} \\
&= \frac{p(1-p)}{n} \\
\therefore \text{SD}(\hat{P}) &= \sqrt{\frac{p(1-p)}{n}}
\end{aligned}$$

Mean and standard deviation for large samples

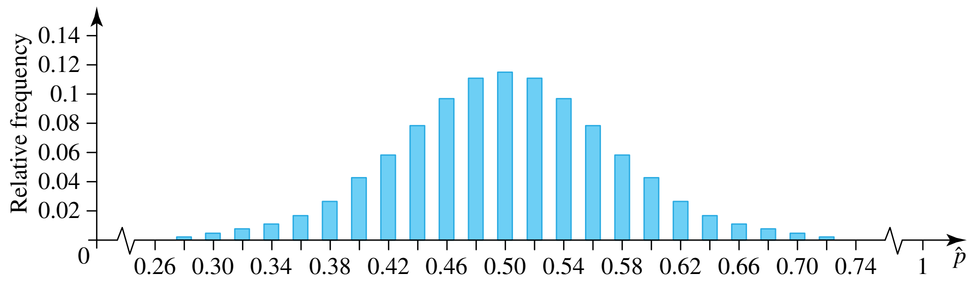
For large samples, the distribution of $\hat{p}$ is approximately normal with a mean or expected value of $\mu_{\hat{p}} = p$ and a standard deviation of $\sigma_{\hat{p}} = \sqrt{\frac{p(1-p)}{n}}$.

There are a number of different ways to decide if a sample is large. One generally accepted method that we will adopt for this section is that if $np \geq 10$, $nq \geq 10$ and $10n \leq N$, then the sample can be called large.

Consider the distribution of $\hat{p}$ when $N = 1000$, $n = 50$ and $p = 0.25$.



Also consider this distribution of $\hat{p}$ when $N = 1000$, $n = 50$ and $p = 0.5$.



As these graphs show, the value of p doesn't matter. The distribution of $\hat{p}$ is symmetrical about p.

WORKED EXAMPLE 6

Consider a population size of 1000 and a sample size of 50. If $p = 0.1$, would this still be a large sample? If not, how big would the sample need to be?

THINK	WRITE
1. Is $10n \le N$?	$n = 50$ and $N = 1000$ $10n = 500$ Therefore, $10n \le N$.
2. Is $np \ge 10$?	$p = 0.1$ $np = 0.1 \times 50$ $= 5$ $5 \not\ge 10$ The sample is not large.
3. Calculate a value for n to make a large sample by solving $np = 10$.	$np = 10$ $0.1n = 10$ $n = 100$
4. Check the other conditions.	$10n = 10 \times 100$ $= 1000$ $= N$ $nq = 100 \times 0.9$ $= 90$ $nq \ge 10$ A sample size of 100 would be needed for a large sample.

WORKED EXAMPLE 7

If $N = 600$, $n = 60$ and $p = 0.3$:
a. calculate the mean of the distribution
b. calculate the standard deviation of the distribution, correct to 2 decimal places.

THINK	WRITE
a. The mean is p.	a. $\mu_{\hat{p}} = p$ $= 0.3$

b. 1. Write the rule for the standard deviation.

2. Substitute the appropriate values and simplify.

b. $\sigma_{\hat{p}} = \sqrt{\frac{p(1-p)}{n}}$

$= \sqrt{\frac{0.3 \times (1-0.3)}{60}}$

$= 0.06$

on Resources

Interactivity: Sampling distribution of $\hat{p}$ (int-6445)

studyon

Units 3 & 4 > Area 8 > Sequence 1 > Concepts 3 & 4

The sample proportion, $\hat{p}$ Summary screen and practice questions
Sampling distribution of $\hat{p}$ Summary screen and practice questions

Exercise 13.3 The distribution of $\hat{p}$

Technology free

1. **WE5** In a 99-g bag of lollies, there were 6 green lollies out of the 15 that were counted. What is the sample proportion, $\hat{p}$?

2. Hang is interested in seedlings that can grow to more than 5 cm tall in the month of her study period. She begins with 20 seedlings and finds that 6 of them are more than 5 cm tall after the month. What is the sample proportion, $\hat{p}$?

3. **WE6** Consider a population size of 1000 and a sample size of 50. If $p = 0.85$, is this a large sample? If not, how big does the sample need to be?
4. If the population size was 10 000 and $p = 0.05$, what would be a large sample size?
5. **WE7** If $N = 500$, $n = 50$ and $p = 0.5$:
   a. calculate the mean of the distribution
   b. calculate the standard deviation of the distribution, correct to 2 decimal places.
6. If $N = 1000$, $n = 100$ and $p = 0.8$:
   a. calculate the mean of the distribution
   b. calculate the standard deviation of the distribution, correct to 2 decimal places.

7. A car manufacturer has developed a new type of bumper that is supposed to absorb impact and result in less damage than previous bumpers. The cars are tested at 25 km/h. If 30 cars are tested and only 3 are damaged, what is the proportion of undamaged cars in the sample?

8. A standard warranty lasts for 1 year. It is possible to buy an extended warranty for an additional 2 years. The insurer decides to use the sales figures from Tuesday to estimate the proportion of extended warranties sold. If 537 units were sold and 147 of them included extended warranties, estimate the proportion of sales that will include extended warranties.

Technology active

9. A Year 12 Mathematical Methods class consists of 12 girls and 9 boys. A group of 4 students is to be selected at random to represent the school at an inter-school Mathematics competition.

   a. What is the value of p, the proportion of girls in the class?
   b. What could be the possible values of the sample proportion, $\hat{p}$, of girls?
   c. Use this information to construct a probability distribution table to represent the sampling distribution of the sample proportion of girls in the small group.
   d. Determine $\mathrm{P}\left(\hat{P} > 0.6\right)$. That is, determine the probability that the proportion of girls in the small group is greater than 0.6.
   e. Determine $\mathrm{P}\left(\hat{P} > 0.5 | \hat{P} > 0.3\right)$.
10. In a particular country town, the proportion of employment in the farming industry is 0.62. Five people aged 15 years and older are selected at random from the town.
   a. What are the possible values of the sample proportion, $\hat{p}$, of workers in the farming industry?
   b. Use this information to construct a probability distribution table to represent the sampling distribution of the sample proportion of workers in the farming industry.
   c. Determine the probability that the proportion of workers in the farming industry in the sample is greater than 0.5.
11. In a population of 1.2 million, it is believed that $p = 0.01$. What would be the smallest sample size that could be considered large?
12. If $N = 1500$, $n = 150$ and $p = 0.15$, calculate the mean and standard deviation for the distribution of $\hat{p}$. Give your answers correct to 3 decimal places where appropriate.
13. If $N = 1200$, $n = 100$ and $p = 0.75$, calculate the mean and standard deviation for the distribution of $\hat{p}$. Give your answers correct to 3 decimal places where appropriate.
14. A distribution for $\hat{p}$ has a mean of 0.12 and a standard deviation of 0.0285. Calculate the population proportion and the sample size.
15. A distribution for $\hat{p}$ has a mean of 0.81 and a standard deviation of 0.0253. Calculate the population proportion and the sample size.
16. If $N = 1500$, $n = 150$ and $p = 0.15$, use technology to graph the distribution for $\hat{p}$.
17. A distribution for $\hat{p}$ has a standard deviation of 0.015. If the sample size was 510 and $\hat{p} > 0.5$, what was the population proportion, correct to 2 decimal places?
18. A distribution for $\hat{p}$ has a standard deviation of 0.0255. If the sample size was 350, what was the population proportion, correct to 2 decimal places?

13.4 Confidence intervals

You have just learned that different samples can have different values for $\hat{p}$. So what can one sample tell us about a population?

Let us say that you are interested in the proportion of the school that buys their lunch. You decide that your class is a reasonable sample and find out that 25% of the class will buy their lunch today. What can you say about the proportion of the whole school that will buy their lunch today? Assuming that your class is in fact a representative sample, you may say that around 25% of the school will buy their lunch. Is it possible to be more specific? By using **confidence intervals**, it is possible to say how confident you are that a population parameter will lie in a particular interval.

13.4.1 The normal approximation to the distribution of $\hat{p}$

We have learned that when we consider the distributions of $\hat{p}$, they are normally distributed with a $\mu_{\hat{p}} = p$ and $\sigma_{\hat{p}} = \sqrt{\dfrac{p(1-p)}{n}}$. As we don't know the exact value for p, the best estimate is $\hat{p}$. This means that the best estimate of the standard deviation is $\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}$.

We know that for normal distributions, $z = \dfrac{x-\mu}{\sigma}$. This means that, to determine the upper and lower values of z, we can use $z = \dfrac{\hat{p} \pm p}{\sigma_{\hat{p}}}$. Rearranging gives us $p = \hat{p} \pm z\sigma_{\hat{p}}$.

Approximate confidence interval

An approximate confidence interval for a population proportion is given by

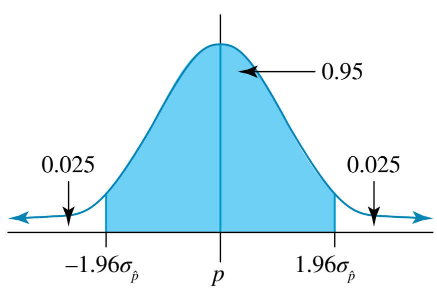
$$(\hat{p} - z\sigma_{\hat{p}}, \hat{p} + z\sigma_{\hat{p}})$$

where $\sigma_{\hat{p}} = \sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}$.

A 95% confidence interval means that 95% of the distribution is in the middle area of the distribution. This means that the tails combined contain 5% of the distribution (2.5% on each end). The z-value for this distribution is 1.96.

The confidence interval for this distribution can be expressed as

$$\left(\hat{p} - 1.96\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}, \hat{p} + 1.96\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right).$$



WORKED EXAMPLE 8

There are 20 people in your class and 25% are planning on buying their lunch. Estimate the proportion of the school population that will purchase their lunch today. Determine a 95% confidence interval for your estimate, given $z = 1.96$.

THINK | **WRITE**

1. There are 20 people in the class. This is the sample size.
   25% are buying their lunch. This is the sample proportion.

$n = 20$
$\hat{p} = 0.25$

2. For a 95% confidence interval, $z = 1.96$.

$z = 1.96$

3. The confidence interval is $\left(\hat{p} - z\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}, \hat{p} + z\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}\right)$.
   Determine $z\sigma_{\hat{p}}$.

$$z\sigma_{\hat{p}} = z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$
$$= 1.96\sqrt{\frac{0.25 \times 0.75}{20}}$$
$$= 0.1898$$

4. Identify the 95% confidence interval by determining the upper and lower values.

$$\hat{p} - z\sigma_{\hat{p}} = 0.25 - 0.1898$$
$$= 0.0602$$
$$\hat{p} + z\sigma_{\hat{p}} = 0.25 + 0.1898$$
$$= 0.4398$$

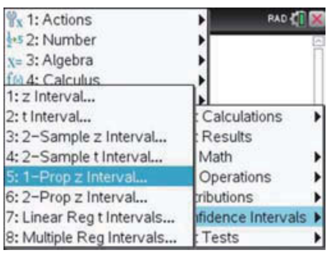
We can be 95% confident that between 6% and 44% of the population will buy their lunch today.

5. Write the answer.

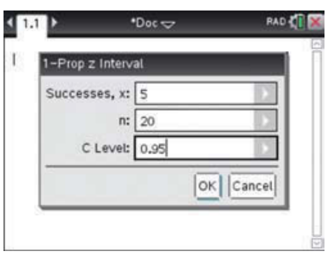
The confidence interval, CI $= (0.06, 0.44)$.

TI | THINK | **WRITE**

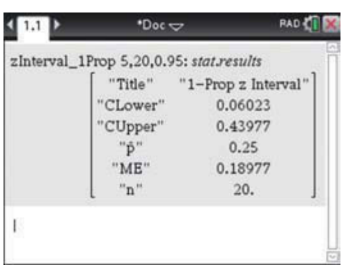
1. On a Calculator page, press MENU then select:
   6: Statistics
   6: Confidence Intervals
   5: 1-Prop z Interval ...



2. Complete the entry lines as:
   x: 5
   n: 20
   C Level: 0.95
   then press the OK button.

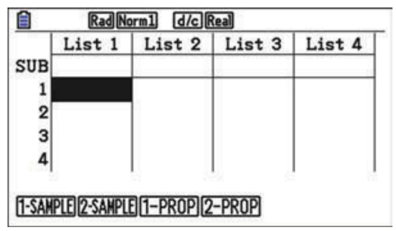


3. The answer appears on the screen.

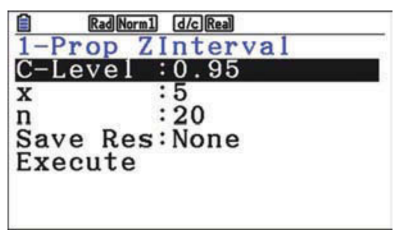


CASIO | THINK | **WRITE**

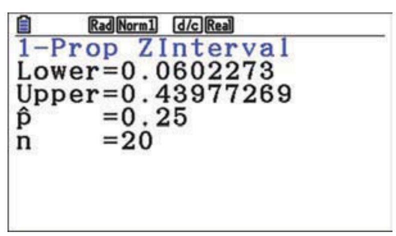
1. On a Statistics screen, press: INTR Z 1-PROP



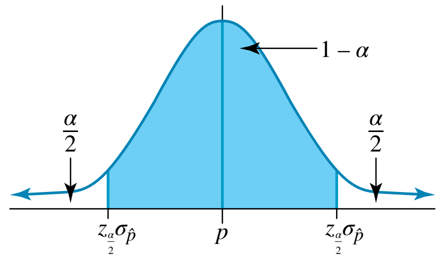
2. Complete the entry lines as:
   C Level: 0.95
   x: 5
   n: 20
   then press the EXE button.



3. The answer appears on the screen.



To determine other confidence intervals, we can talk in general about a $1 - \alpha$ confidence interval. In this case, the tails combined will have an area of α (as each tail has an area of $\frac{\alpha}{2}$). In this case, the z-value that has a tail area of $\frac{\alpha}{2}$ is used.



WORKED EXAMPLE 9

Paul samples 102 people and determines that 18 of them like drinking coconut milk. Estimate the proportion of the population that likes drinking coconut milk. Determine a 99% confidence interval for your estimate, correct to 1 decimal place.

THINK	WRITE
1. There are 102 people in the sample. This is the sample size. 18 like drinking coconut milk.	$n = 102$ $\hat{p} = \frac{18}{102}$ $= 0.18$
2. For a 99% confidence interval, determine the z-value using the inverse standard normal distribution.	For the 99% confidence interval, 1% will be in the tails, so 0.5% in each tail. Therefore, the area under the normal distribution curve to the left of z is 0.995. $z = 2.58$
3. The confidence interval is $(\hat{p} - z\sigma_{\hat{p}}, \hat{p} + z\sigma_{\hat{p}})$. Determine $z\sigma_{\hat{p}}$.	$z\sigma_{\hat{p}} = z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$ $= 2.58\sqrt{\frac{0.18 \times 0.82}{102}}$ $= 0.098$
4. Identify the 99% confidence interval by determining the upper and lower values, correct to 1 decimal place.	$\hat{p} - z\sigma_{\hat{p}} = 0.18 - 0.098$ $= 0.082$ $\hat{p} + z\sigma_{\hat{p}} = 0.18 + 0.098$ $= 0.278$ We can be 99% confident that between 8.2% and 27.8% of the population like drinking coconut milk.

WORKED EXAMPLE 10

Grow Well are 95% sure that 30% to 40% of shoppers prefer their mulch. What sample size was needed for this level of confidence?

THINK	WRITE
1. The confidence interval is symmetric about $\hat{p} : (\hat{p} - z\sigma_{\hat{p}}, \hat{p} + z\sigma_{\hat{p}})$, so the value of $\hat{p}$ must be halfway between the upper and lower values of the confidence interval.	$\hat{p} = \frac{30 + 40}{2}$ $= 35\%$ $= 0.35$

2. State the z-value related to the 95% confidence interval.	$z = 1.96$
3. The lower value of the confidence interval, 30%, is equivalent to $\hat{p} - z\sigma_{\hat{p}}$. Substitute the appropriate values. *Note:* The equation $0.4 = \hat{p} + z\sigma_{\hat{p}}$ could also have been used.	$0.3 = \hat{p} - z\sigma_{\hat{p}}$ $= \hat{p} - z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$ $= 0.35 - 1.96\sqrt{\frac{0.35(1-0.35)}{n}}$
4. Solve for n.	$n = 349.586$
5. Write the answer.	The sample size needed was 350 people.

13.4.2 Margin of error

The distance between the endpoints of the confidence interval and the sample estimate is called the **margin of error**, E.

Worked example 10 considered a 95% confidence interval, $(\hat{p} - z\sigma_{\hat{p}}, \hat{p} + z\sigma_{\hat{p}})$. In this case the margin of error would be $E = z\sigma_{\hat{p}}$.

Margin of error for a 95% confidence interval

For a 95% level of confidence,

$$E = 1.96\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}.$$

Note that the larger the sample size, the smaller the value of E will be. This means that one way to reduce the size of a confidence interval without changing the level of confidence is to increase the sample size.

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Units 3 & 4 > Area 8 > Sequence 1 > Concept 5 > **Confidence intervals** Summary screen and practice questions

Exercise 13.4 Confidence intervals

Technology active

1. **WE8** Of 30 people surveyed, 78% said that they like breakfast in bed. Estimate the proportion of the populations that like breakfast in bed. Determine a 95% confidence interval for the estimate.
2. Of the 53 people at swimming training today, 82% said that their favourite stroke is freestyle. Estimate the proportion of the population whose favourite stroke is freestyle. Determine a 95% confidence interval for the estimate.

3. **WE9** Jenny samples 116 people and finds that 86% of them plan to go swimming over the summer holidays. Estimate the proportion of the population that plan to go swimming over the summer holidays. Determine a 99% confidence interval for your estimate.

4. Yuki samples 95 people and finds that 30% of them eat chocolate daily. Estimate the proportion of the population that eats chocolate daily. Determine a 90% confidence interval for your estimate.
5. **WE10** In a country town, the owners of Edie's Eatery are 95% sure that 35% to 45% of their customers love their homemade apple pie. What sample size was needed for this level of confidence?

6. If Parkers want to be 90% confident that between 75% and 85% of their customers will shop in their store for more than 2 hours, what sample size will be needed?

The following information relates to questions **7**, **8** and **9**.

Teleco is being criticised for its slow response time when handling complaints. The company claims that it will respond within 1 day. Of the 3760 complaints in a given week, a random sample of 250 was selected. Of these, it was found that 20 of them had not been responded to within 1 day.

7. Determine the 95% confidence interval for the proportion of claims that take more than 1 day to resolve.
8. What is the 99% confidence interval for the proportion of claims that take less than 1 day to resolve?
9. Teleco want to be 95% sure that less than 5% of their complaints take more than 1 day to resolve. What sample proportion do they need and how large does the sample need to be to support this claim?
10. A sample of 250 blood donors have their blood types recorded. Of this sample, 92 have Type A blood. What is the 90% confidence interval for the proportion of Australians who have Type A blood?

11. It is believed that 65% of people have brown hair. A random selection of 250 people were asked the colour of their hair. Applying the normal approximation, determine the probability that less than 60% of the people in the sample have brown hair.
12. Nidya is a top goal shooter. The probability of her getting a goal is 0.8. To keep her skills up, each night she has 200 shots on goal. Applying the normal approximation, determine the probability that on Monday the proportion of times she scores a goal is between 0.8 and 0.9, given that it is more than 0.65.
13. Smooth Writing are 95% sure that 25% to 35% of shoppers prefer their pen. What sample size was needed for this level of confidence?
14. An online tutoring company is 99% sure that 20% to 30% of students prefer to use their company. What sample size was needed for this level of confidence?
15. Barton's Dentistry want to be able to claim that 90% to 98% of people floss daily. They would like 99% confidence about their claim. How many people do they need to survey?
16. Tatiana is conducting a survey to estimate the proportion of Year 12 students who will take a gap year after they complete their studies. Previous surveys have shown the proportion to be approximately 15%. Determine the required size of the sample so that the margin of error for the survey is 3% in a confidence interval of approximately 95% for this proportion.
17. Bentons claim that between 85% and 95% of their customers stay for more than 2 hours when they shop. If they surveyed 100 people, how confident are they about that claim?

18. The Brisbane Lions Football Club claim that between 75% and 80% of their members remain members for at least 10 years. If they surveyed 250 people, how confident are they about that claim? Give your answer to the nearest whole number.

13.5 Review: exam practice

A summary of this chapter is available in the Resources section of your eBookPLUS at www.jacplus.com.au.

Simple familiar

1. Identify which of the following are population parameters.
   a. According to the Australian Bureau of Statistics, the unemployment rate is 6.4%.
   b. According to the 2011 census, on average there are 1.7 motor vehicles per dwelling.
   c. According to a poll in the newspaper *The Age*, 54% of Australians will vote Liberal at the next election.
2. Identify the following as either population parameters or sample statistics.
   a. According to data collected at enrolment, 97% of students speak English as their first language.
   b. Between 3% and 15% of children who leave home return again within 12 months.
   c. Three out of every 5 students brings an apple with their lunch.
3. Glen is interested in the proportion of internet users in his local town. He surveys 150 households and determines that 132 of them have internet access. What proportion of households in Glen's town have internet access?
4. A local vet decides to survey every 7th client about their pet care.
   a. What type of sample is this?
   b. Is this a valid method for this situation?

5. If a population proportion is believed to be 0.25 and samples of size 30 are chosen, what is the standard deviation of $\hat{p}$?
6. In a recent voter survey, an approximate 90% confidence interval for the proportion of people who will vote for a particular party is $(0.58, 0.66)$.
   a. What is the value of $\hat{p}$ for this confidence interval?
   b. What is the value of the margin of error?
7. Ronit wants to survey the students at his school. He wants to survey 50 students, but he believes that it is necessary to use a stratified random sample.

Gender	Middle school	Senior school
Male	253	342
Female	287	323

a. What is the population size?
b. What is the sample size?
c. How many from each subgroup should be sampled?

8. The formal committee of Brickwall State High School is trying to choose between strawberry cheesecake and passionfruit cheesecake for dessert at the formal dinner. There will be 423 students at the dinner. Of the 52 senior students sampled, 23 prefer the strawberry cheesecake. What proportion of students chose passionfruit cheesecake?

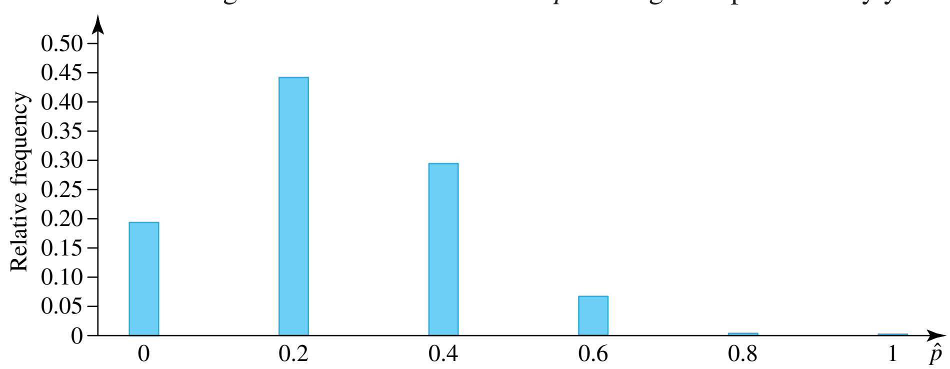
9. If $N = 2\,000$, $n = 120$ and $p = 0.37$, calculate:
a. the mean of the population
b. the standard deviation of the population, correct to 3 decimal places.

10. A textbook publishing company is 95% sure that 65%– 75% of students prefer to use their resources. What sample size was needed for this level of confidence?

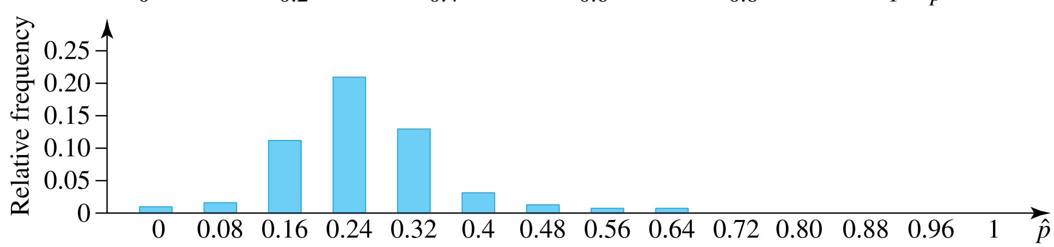
11. On a particular Friday night, 52 000 people went to Suncorp Stadium to watch the football. Every 25th person entering the stadium was asked who they thought would win. Out of the people asked, 1600 people believed that the Queensland Reds would win.
a. What is the population size?
b. What is the sample size?
c. Determine $\hat{p}$.

12. Which of the following could be a distribution for $\hat{p}$ for large samples? Justify your answer.

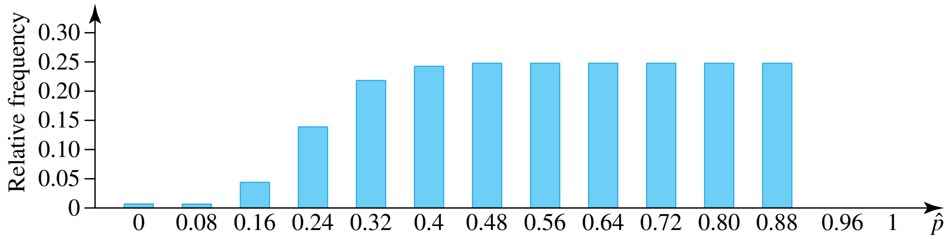
A.



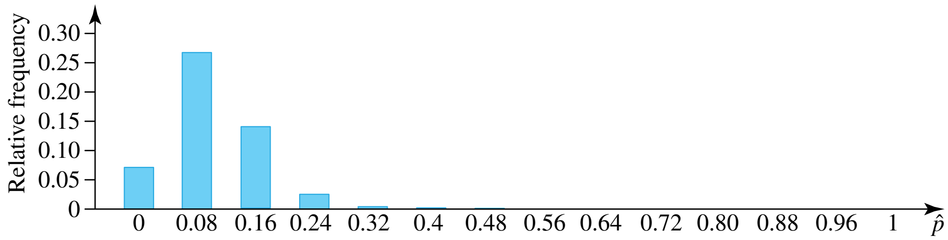
B.



C.



D.



Complex familiar

13. Each year the Year 7 class from Gympie State High visits a theme park. One hundred of the students decided to go on the monster rollercoaster, and 10 of them complained of feeling dizzy afterwards.
    a. What is the value of the sample proportion?
    b. Write an expression for the 95% confidence interval for the likelihood of feeling dizzy.
    c. Determine the margin of error, E, for the 85% confidence interval.
    d. If only 50 people had decided to go on the rollercoaster, what would be the effect on the margin of error?
14. A distribution for $\hat{p}$ has a standard deviation of 0.05. If the sample size is 50, what is the population proportion, correct to 2 decimal places?
15. Maxwell Industries surveyed 100 customers and believe that between 85% and 90% of their customers are satisfied with their level of service. How confident are they about that claim?
16. Krypton Industries are 99% certain that between 67% and 83% of people prefer their product. How many people were sampled for this level of confidence?

Complex unfamiliar

17. Every year, thousands of tourists drive the Great Ocean Road. In a recent survey of 50 people, 87% listed seeing the Twelve Apostles as the highlight of their drive. What proportion of drivers would rate the Twelve Apostles as the highlight of their drive? Give your answer with a 90% confidence level.
18. It is believed that 40% of Australians wear glasses or contact lenses. Four hundred people were randomly selected and asked about their eyesight. Applying the normal distribution, determine the probability that more than 45% of the people in the sample need to wear glasses or contact lenses.

19. The lower limit of a 95% confidence interval is 13%. If 100 people were surveyed, what is the sample proportion, correct to 2 decimal places? Solve this problem graphically.

20. Breanna, Kayley and Teagan spent the day collecting survey results from the same population. They each surveyed 100 people. Breanna found that 23% of people said Yes, Kayley found that 20% of people said Yes, and Teagan found that 19% of people said Yes. They want to obtain an estimate for the population parameter at a 95% confidence interval.

Breanna says they should each work out a confidence interval and then average them out to give the population parameter. Kayley says that they should combine their data into one sample and determine the population parameter using that parameter. Teagan says that it doesn't matter, because they will get the same results either way.

a. Is Teagan correct? (Show all your working.)
b. Who has the most reliable method? Explain your answer.
c. What is the best estimate of the population parameter at a confidence level of 95%?

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Units 3 & 4 Sit exam

Answers

13 Sampling and confidence intervals

Exercise 13.2 Sample statistics

1. $N = 120, n = 30$
2. $N = 100, n = 5$
3. $n = 15$, population size is unknown
4. $n = 9$, population size is unknown
5. a. Population parameter b. Sample statistic
6. a. Population parameter b. Sample statistic
7. 34 boys and 41 girls
8. 21 boarders, 69 day students
9. a. The population size is unknown.
   b. 40
10. a. The population is people who will receive the vaccine in the future. The size is unknown.
    b. 247
11. Sample statistic
12. Population parameter
13. Sample statistics
14. Population parameter
15. a. A systematic sample with $k = 10$
    b. Yes, assuming that the order of patients is random
16. The sample is not random; therefore, the results are not likely to be random.
17. It is probably not random. Tony is likely to ask people who he knows or people who approach him.
18. Full-time male staff: 46
    Part-time male staff: 2
    Full-time female staff: 25
    Part-time female staff: 7
19. Use the random number generator on your calculator to produce numbers from 1 to 100. Keep generating numbers until you have 10 different numbers.
20. First, assign every person in your class a number, e. g. 1 to 25 if there are 25 students in your class. Decide how many students will be in your sample, e.g. 5. Then use the random number generator on your calculator to produce numbers from 1 to 25. Keep generating numbers until you have 5 different numbers. The students that were assigned these numbers are the 5 students in your random sample.

Exercise 13.3 The distribution of $\hat{p}$

1. $\frac{2}{5} = 0.4$
2. $\frac{3}{10} = 0.3$
3. This is not a large sample; $n = 67$ would be a large sample.
4. $n = 200$
5. a. 0.5
   b. 0.07
6. a. 0.8
   b. 0.04
7. $\frac{9}{10} = 0.9$
8. $\frac{147}{537} \approx 0.274$
9. a. $\frac{4}{7} \approx 0.571$
   b. $0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$
   c.

$\hat{p}$	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1
$P(\hat{P} = \hat{p})$	0.021	0.168	0.397	0.331	0.083

   d. 0.414
   e. 0.510
10. a. $0, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 1$
    b.

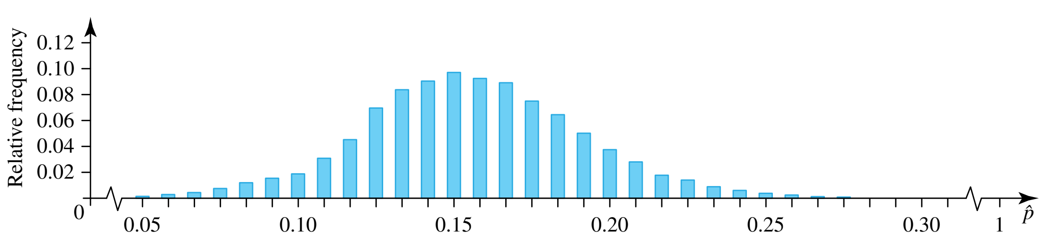
$\hat{p}$	0	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{4}{5}$	1
$P(\hat{P} = \hat{p})$	0.008	0.064	0.211	0.344	0.281	0.092

    c. 0.717
11. 1000
12. $\mu_{\hat{p}} = 0.15, \sigma_{\hat{p}} = 0.029$
13. $\mu_{\hat{p}} = 0.75, \sigma_{\hat{p}} = 0.043$
14. $p = 0.12, n = 130$
15. $p = 0.81, n = 240$
16. *See the figure at the foot of the page.
17. $p = 0.87$
18. $p = 0.35$ or $p = 0.65$

Exercise 13.4 Confidence intervals

1. (0.63, 0.93)
2. (0.72, 0.92)
3. (0.78, 0.94)
4. (0.22, 0.38)
5. 369
6. 173
7. (0.46, 0.114)
8. (0.876, 0.964)

*16. a.



9. $\hat{p} = 2.5\%$; $n = 150$
10. $(0.318, 0.418)$
11. 0.0487
12. 0.4998
13. 323
14. 498
15. 235
16. 544
17. 90%
18. 66%

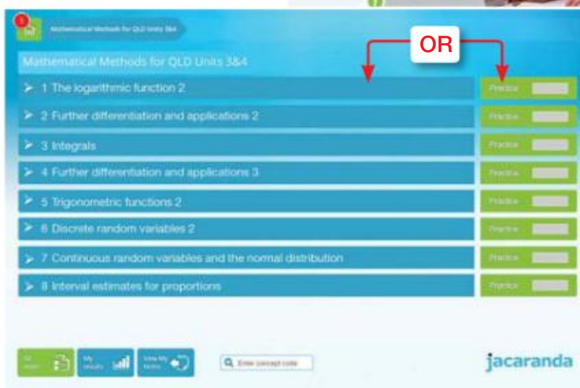
13.5 Review: exam practice

1. **b**
2. a. Population parameter
   b. Population parameter
   c. Sample statistic
3. 0.88
4. a. Systematic sample
   b. Yes, as the clients are likely to be in a random order.
5. 0.079
6. a. 0.62 b. 0.04
7. a. 1205
   b. 50
   c. Middle school: 11 male students, 12 female students; Senior school: 14 male students, 13 female students
8. 0.56
9. a. 0.37 b. 0.044
10. 323
11. a. 52 000 b. 2080 c. 0.77
12. B
13. a. 0.1
    b. 0.1 ± 0.0432
    c. 0.0432
    d. E increases by a factor of $\sqrt{2}$.
14. $p = 0.15$ or 0.85
15. 0.55
16. 195
17. 79%–95%
18. 0.0206
19. 0.21
20. a. Teagan is not correct.
    b. Kayley's method
    c. 16%–26%

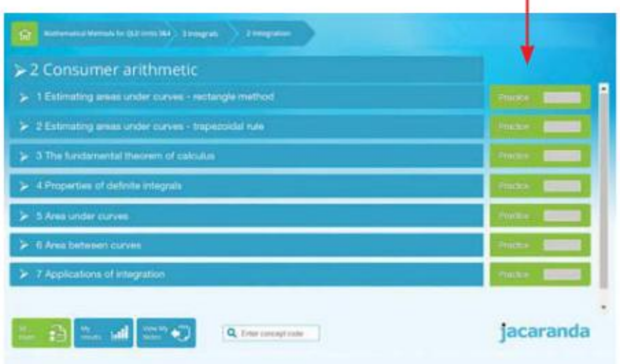
TOPIC 5 Interval estimates for proportions

- For revision of this entire topic, go to your **studyON** title in your bookshelf at **www.jacplus.com.au**.
- Select **Continue Studying** to access hundreds of revision questions across your entire course.

- Select your **course** *Mathematical Methods for Queensland Units 3&4* to see the entire course divided into syllabus topics.
- Select the **Area** you are studying to navigate into the chapter level ***OR*** select **Practice** to answer all practice questions available for each area.

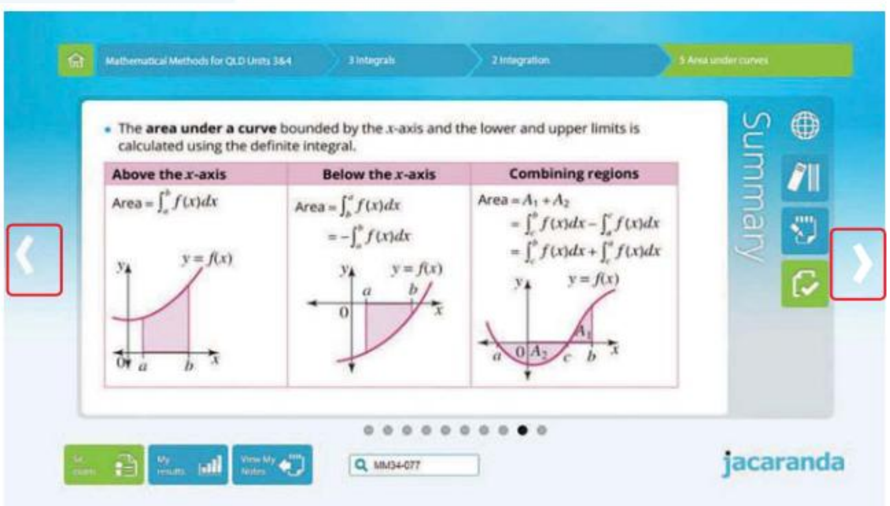


- Select **Practice** at the sequence level to access all questions in the sequence.



- At **sequence level**, drill down to concept level.

- **Summary screens** provide revision and consolidation of key concepts. Select the **next arrow** to revise all concepts in the sequence and practise questions at the concept level



PRACTICE ASSESSMENT 3

Mathematical Methods: Unit 4 examination

Unit

Unit 4: Further functions and statistics

Topics

Topic 1: Further differentiation and applications 3
Topic 2: Trigonometric functions 2
Topic 3: Discrete random variables 2
Topic 4: Continuous random variables and the normal distribution
Topic 5: Interval estimates for proportions

Conditions

Technique	Response type	Duration	Reading
Paper 1: Technology free Paper 2: Technology active	Short response	Paper 1: 60 minutes Paper 2: 60 minutes	5 minutes

Resources	Instructions
• QCAA formula sheet • Notes not permitted • Paper 1: Calculator not permitted • Paper 2: Non-CAS graphics calculator permitted	• Show all working. • Write responses using a black or blue pen. • Unless otherwise instructed, give answers to **two decimal places.**

Criterion	Marks allocated Paper 1	Marks allocated Paper 2	Result
Foundational knowledge and problem solving *Assessment objectives 1, 2, 3, 4, 5 and 6	50	50	

For the most up to date assessment information, please see www.qcaa.qld.edu.au/senior.

Paper 1 — Technology free — total marks: 50

Part A: Simple familiar — total marks: 30

Question 1 (1 + 2 + 2 = 5 marks)

A random variable, X, has the following probability distribution.

x	1	2	3	4	5
$P(X = x)$	0.1	0.2	0.2	a	0.3

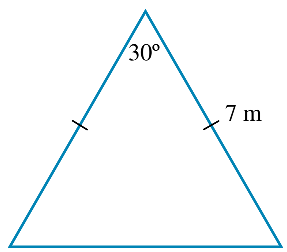
a. Find the value of a.

b. Calculate the expected value of X.

c. Find $P(X = 3 \mid X > 2)$.

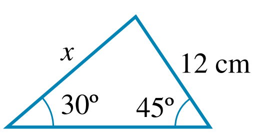
Question 2 (2 marks)

An A-frame house has an angle at the roof apex of $30°$ and two sides each 7 m long as shown. Find the exact area of the front of the house.



Question 3 (3 marks)

In the triangle shown below, find the exact length of side x.



Question 4 (3 marks)

The probability that the 7:35 am bus arrives on time is 0.8. What is the probability that the bus is on time at least once in the next 3 days?

Question 5 (3 marks)

If $f(x) = x\ \sin(x)$, calculate $f''\left(\frac{3\pi}{2}\right)$.

Question 6 (2 + 2 + 2 = 6 marks)

The continuous random variable X has a probability density function defined by

$$f(x) = \begin{cases} kx^2, & 0 \le x \le 2 \\ 0, & \text{elsewhere} \end{cases}$$

where k is a constant.

a. Find the value of k.

b. Determine:

   i. the mean

   ii. the median.

Question 7 ($1 + 3 + 1 + 3 = 8$ marks)

Each year the Year 7 class from Green High School visits a theme park. One hundred of the students decided to go on the monster rollercoaster, and 10 of them complained of feeling dizzy afterwards.

a. What is the value of the sample proportion?

b. Write an expression for the 95% confidence interval for the likelihood of feeling dizzy.

c. Write an expression for the margin of error, M, for the 95% confidence interval.

d. If only 50 people had decided to go on the rollercoaster, what would be the effect on the margin of error?

Part B: Complex familiar — total marks: 10

Question 8 (2 + 1 + 2 = 5 marks)

X has a normal distribution with a mean of 4 and a standard deviation of 2.
Find:

a. $P(4 \leq X \leq 8)$

b. $P(X > 0)$

c. $P(X > 0 \mid X < 4)$

Question 9 (1 + 1 + 1 + 1 + 1 = 5 marks)

An object moves in a straight line with a displacement of x cm from a fixed point, O, at any time, t seconds, where $t \in [0, 6]$. The object is decelerating at a constant rate of 2 cm/s^2. Initially, the object was 12 cm to the right of O, moving away from the point with a speed of 4 cm/s.

a. Determine an expression for:

  i. velocity as a function of time

  ii. displacement as a function of time.

b. When did the object come to rest? Calculate its position at this time.

c. Calculate the distance travelled by the object.

d. Determine the object's average speed during the given time period.

Part C: Complex unfamiliar — total marks: 10

Question 10 (10 marks)

The function $y = \frac{a}{x^2 + b}$, where a and b are non-zero constants, has a point of inflection at $(-1, 2)$. Determine the values of a and b, and hence find the coordinates and the nature of any stationary point(s).

Paper 2 — Technology active — total marks: 50

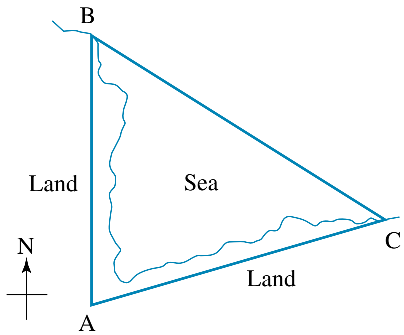
Part A: Simple familiar — total marks: 30

Question 1 (3 marks)

Show that $\frac{d^2}{dx^2}\left[\log_e (ax + b)^n\right] = -\frac{a^2 n}{(ax + b)^2}$.

Question 2 (1 + 2 + 2 = 5 marks)

A triathlon requires the competitors to run 230 m along a shore line from a point marked A directly north to a point B on the shore. From point B, they swim to point C on the edge of shore. They then cycle 230 m from point C directly to the starting point A. The bearing of C from A has been recorded as $074°T$.



a. State the value of angle A.

b. Find the distance the competitors had to swim, correct to the nearest metre.

c. Find the bearing of C from B, correct to the nearest degree.

Question 3 (2 + 1 + 1 = 4 marks)

It is thought that only 40% of puppies adopted come from rescue shelters. Ten random families were surveyed about the places from which they recently adopted their puppies. Let X be the random variable that defines the number of families who adopt their puppy from a rescue shelter. Find, correct to 4 decimal places:

a. the probability that at least 8 families adopt their puppy from a shelter

b. the expected number of families that will adopt from a shelter

c. the standard deviation of X.

Question 4 (3 marks)

Wholistic Enterprises believe that 75% of people prefer their services to those of other companies. If they want their margin of error to be less than 5%, how many people do they need to survey to be 95% confident about their claim?

Question 5 (3 marks)

Krypton Industries are 99% certain that between 67% and 83% of people prefer their product. How many people were sampled for this level of confidence?

Question 6 (2 + 3 + 2 = 7 marks)

A four-wheel-drive vehicle leaves a camp site and travels across a flat sandy plain in a direction of S65°E for a distance of 8.2 km. It then heads due south for 6.7 km to reach a waterhole.

a. **i.** How far is the waterhole from the camp site? Give your answer to 1 decimal place.

ii. What is the bearing of the waterhole from the camp site? Give your answer to the nearest degree.

b. A search plane sets off to find the vehicle. It is on a course that takes it over points A and B, two locations on level ground. At a certain time, the angle of elevation from point A to the plane is 72°. From point B, the angle of elevation is 47°. If A and B are 3500 m apart, find the height of the plane off the ground, to the nearest metre.

Question 7 (2 + 3 = 5 marks)

Consider the function $f(x) = x^4 - 8x^3 + 18x^2 - 8x + 2$.

a. Show that the point $(2, 10)$ is a maximum turning point.

b. Determine the coordinates of the point(s) of inflection, and hence determine the values of x where the function is concave up.

Part B: Complex familiar — total marks: 10

Question 8 (4 marks)

Farmer Jane grows avocados and has found that the average weight of her avocados is normally distributed. Jane has determined that the probability of an avocado weighing less than 400 grams is 0.4207, whereas the probability of an avocado weighing more than 420 grams is 0.2445. Determine the mean and standard deviation of the weights of Jane's avocados. Give your answers correct to the nearest integer.

Question 9 (2 + 1 + 1 + 2 = 6 marks)

The times taken for a class of Year 12 swimmers to swim 50 metres freestyle are normally distributed with a mean of 23.6 seconds and a variance of 0.36 seconds.

a. Determine the approximate percentage of swimmers who can swim the 50 metres freestyle in a time between 23 seconds and 24.2 seconds. Give your answer to 1 decimal place.

b. If a swimmer is chosen at random, what is the probability, correct to 4 decimal places, that they can swim the 50 metres freestyle in less than 23.7 seconds?

c. Find the percentage of swimmers, to the nearest percent, who cannot swim the 50 metres freestyle in less than 23 seconds.

d. What is the probability, correct to 4 decimal places, that one swimmer can swim the 50 metres freestyle in less than 23.6 seconds, given that the swimmer is randomly chosen from the group who can swim the 50 metres freestyle in less than 23.7 seconds?

Part C: Complex unfamiliar — total marks: 10

Question 10 (3 + 2 + 1 + 1 + 2 + 1 = 10 marks)

In a factory manufacturing sweets, candy-covered peanuts are produced from one of two machines: machine A and machine B. The time, X seconds, that machine A takes to produce a candy-covered peanut is distributed normally with a mean of 1.5 seconds and a standard deviation of 0.5 seconds. The time, Y seconds, that machine B takes to produce a candy-covered peanut has the following probability density function:

$$f(y) = \begin{cases} 0, & y < 0 \\ \frac{y}{6}, & 0 \le y \le 1 \\ \frac{1}{2}e^{-0.5(y-1)}, & y > 1 \end{cases}$$

a. Find, correct to 4 decimal places:

  i. $\text{P}(1 \le X \le 3)$

  ii. $\text{P}(1 \le Y \le 3)$

b. Find $\text{E}(Y)$ correct to 4 decimal places.

c. Determine the probability of a candy-covered peanut taking 1 or less seconds to produce if it is made on machine A.

d. Show that the probability of a candy-covered peanut produced on machine B taking 1 or less seconds to produce is $\frac{1}{12}$.

e. The candy-covered peanuts from machine B went into a large bin. A random sample of 20 candy-covered peanuts was chosen from this bin. Determine the probability that at most 2 of these peanuts took 1 seconds or less to produce. Give your answer correct to 4 decimal places.

f. The manufacturer knows that their profits rely on producing the candy-covered peanuts in as little time as possible without compromising the product. Both machines produce excellent results. If only one machine could be used, which machine should the manufacturer use? Give a reason for your answer.

Examination marks summary

Question number	Simple familiar	Complex familiar	Complex unfamiliar	Topics
Paper 1 — Technology free				
1	5			3
2	2			2
3	3			2
4	3			3
5	3			1
6	6			4
7	8			5
8		5		4
9		5		1
10			10	1
Paper 2 — Technology active				
1	3			1
2	5			2
3	4			3
4	3			5
5	3			5
6	7			2
7	5			1
8		4		4
9		6		4
10			10	3, 4
Totals		**20**	**20**	
Percentage	**60%**	**20%**	**20%**	

PRACTICE ASSESSMENT 4

Mathematical Methods: Units 3 & 4 examination

Unit

Unit 3: Further calculus

Topics

Topic 1: The logarithmic function 2
Topic 2: Further differentiation and applications 2
Topic 3: Integrals

Unit

Unit 4: Further functions and statistics

Topics

Topic 1: Further differentiation and applications 3
Topic 2: Trigonometric functions 2
Topic 3: Discrete random variables 2
Topic 4: Continuous random variables and the normal distribution
Topic 5: Interval estimates for proportions

Conditions

Technique	Response Type	Duration	Reading
Paper 1: Technology free Paper 2: Technology active	Short response	Paper 1: 60 minutes Paper 2: 60 minutes	5 minutes
Resources		**Instructions**	
• QCAA formula sheet • Notes not permitted • Paper 1: Calculator not permitted • Paper 2: Non-CAS graphics calculator permitted		• Show all working. • Write responses using a black or blue pen. • Unless otherwise instructed, give answers to **two decimal places**.	

Criterion	Marks allocated Paper 1	Marks allocated Paper 2	Result
Foundational knowledge and problem solving *Assessment objectives 1, 2, 3, 4, 5 and 6	50	50	

For the most up to date assessment information, please see www.qcaa.qld.edu.au/senior.

Paper 1 — Technology free — total marks: 50

Part A: Simple familiar — total marks: 30

Question 1 (1 + 2 + 1 = 4 marks)

a. Determine $\frac{dy}{dx}$ if $y = (4 - 3x)^6$.

b. Consider the function $f(x) = \frac{\ln(x)}{x}$.

i. Find $f'(x)$.

ii. Evaluate $f'(e)$.

Question 2 (2 + 2 = 4 marks)

a. Evaluate $\int_4^9 \frac{1}{\sqrt{x}}\,dx$.

b. If $f'(x) = 1 - \cos(2x)$, determine $f(x)$ if $f(\pi) = 3$.

Question 3 (1 + 2 = 3 marks)

Solve the following for x.

a. $\log_e(2 - 3x) = 0$

b. $\log_{10}(2x) - \log_{10}(x - 1) = 1$

Question 4 (3 + 2 = 5 marks)

Consider the function $f(x) = x^4 - 4x^3$.

a. Determine the coordinates and nature of the stationary point(s).

b. Sketch the function, showing all important features.

Question 5 (3 marks)

Two rowers set out from the same point. One rows on a bearing of N30°E for 2 km and the other rows on a bearing of S30°E for 3 km. How far apart are the two rowers?

Question 6 (1 + 2 + 3 = 6 marks)

A discrete random variable, X, takes the values shown in the table, along with its probabilities.

x	1	2	3	4	5
$P(X = x)$	k	$2k$	$3k$	$4k$	$5k$

Determine:

a. the constant, k

b. $E(X)$, the expected value of X

c. $Var(X)$, the variance of X.

Question 7 (3 + 2 = 5 marks)

Let X be a continuous random variable with a probability function defined by

$$f(x) = 24x^2, 0 \leq x \leq a$$

where a is a constant.
Determine:

a. the value of a

b. $P(0.2 < X < 0.4)$

Part B: Complex familiar — total marks: 10

Question 8 (2 + 2 = 4 marks)

The depth of water, d metres, in a canal changes with the tides according to the rule

$$d = 6 + 2\cos\left(\frac{\pi t}{6}\right)$$

where t is the time in hours after high tide. On a particular day, high tide was at 10:00 am.

a. Calculate the depth of the water at low tide and when it next occurs.

b. Determine the rate of change of the depth of water with respect to time at 2:00 pm.

Question 9 (2 + 1 + 3 = 6 marks)

X has a normal distribution with a mean of 12 and a standard deviation of 3.
Determine:

a. $\text{P}(12 \leq X \leq 21)$

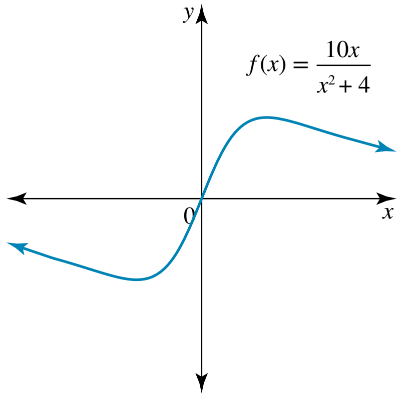
b. $\text{P}(X > 3)$

c. $\text{P}(X > 3 \mid X < 12)$

Part C: Complex unfamiliar — total marks: 10

Question 10 (2 + 1 + 3 + 2 + 2 = 10 marks)

The graph of the function $f(x) = \dfrac{10x}{x^2 + 4}$ is shown.



a. Determine the coordinates of the stationary points.

b. Show that $f''(x) = \dfrac{20x\left(x^2 - 12\right)}{\left(x^2 + 4\right)^3}$.

c. Determine the coordinates of any points of inflection.

d. Differentiate $g(x) = \ln\left(x^2 + 4\right)$ and hence find the antiderivative of $f(x)$.

e. Calculate the area enclosed between the curve, the x-axis and the vertical lines through the points of inflection.

Paper 2 — Technology active — total marks: 50

Part A: Simple familiar — total marks: 30

Question 1 (1 + 3 = 4 marks)

Solve the following equations for x.

a. $3^{(x+3)} - \frac{1}{81} = 0$

b. $2\ln(x+2) - \ln(x) = \ln(3) + \ln(x-1)$

Question 2 (2 + 2 = 4 marks)

A discrete random variable, X, has the following probability distribution.

x	0	1	2	3
$P(x)$	0.12	0.36	0.38	0.14

a. Calculate the expected value of X.

b. Determine $P(X = 3 | X > 1)$.

Question 3 (3 marks)

A cross-country runner starts at checkpoint A and runs for 8 km on a bearing of $050°T$ to reach checkpoint B. The runner then heads directly east for 10 km to checkpoint C. How far is checkpoint C from the starting point, A? Give your answer correct to 2 decimal places.

Question 4 (4 marks)

For the function $y = e^{2x}\sin(3x)$, $\frac{d^2y}{dx^2}$ can be expressed in the form $e^{2x}(a\sin(3x) + b\cos(3x))$ where a and b are constants. Determine the values of a and b.

Question 5 (4 marks)

Smart Industries surveyed 150 people and believe that between 80% and 95% of their customers are satisfied with their level of service. How confident are they about that claim? Give your answer to the nearest percentage.

Question 6 (4 marks)

An object, starting from rest, accelerates according to the equation $a = 4 - t$, where a represents acceleration in m/s^2 and t is time in seconds. Calculate the distance the object has travelled in the first 10 seconds.

Question 7 (1 + 2 + 4 = 7 marks)

For a large population of people, it is fair to assume that dates of birth are evenly distributed throughout the year, so that the probability of a person having their birthday in a particular month is $\frac{1}{12}$.

a. Find the probability that if 8 people are randomly chosen, exactly 3 will have their birthday in August. Give your answer correct to 4 decimal places.

b. Find the probability that of the 8 randomly chosen people, at least 1 will have their birthday in November. Give your answer correct to 4 decimal places.

c. If n people are chosen at random, find the smallest value of n so that the probability that at least 1 will have their birthday in March is greater than 0.9.

Part B: Complex familiar — total marks: 10

Question 8 (1 + 3 = 4 marks)

a. Show that the function $f(x) = 2e^{-2x}, x \geq 0$ is a probability density function.

b. The time, T (in minutes), that an individual must wait in line to be served at the local coffee shop is defined by

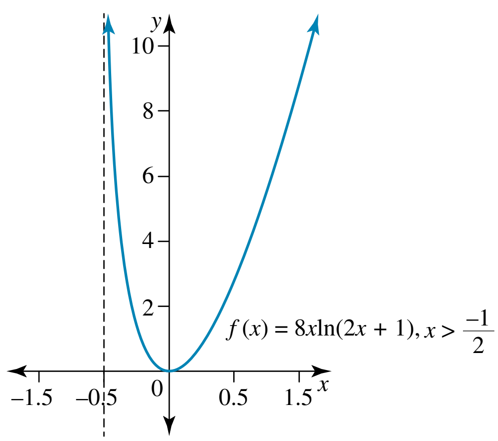
$$f(t) = \begin{cases} 2e^{-2t}, & t \geq 0 \\ 0, & \text{elsewhere} \end{cases}$$

where T is a continuous random variable. Determine the median waiting time in the queue, correct to 2 decimal places.

Question 9 (2 + 4 = 6 marks)

a. Show that if $y = (4x^2 - 1)\ln(2x+1)$, then $\frac{dy}{dx} = 8x\ln(2x+1) + 4x - 2$.

b. The graph of $f(x) = 8x\ln(2x+1), x > -\frac{1}{2}$ is shown. Calculate the area of the region bounded by the curve, the x-axis, $x = 0$ and $x = 1$.



Part C: Complex unfamiliar — total marks: 10

Question 10 (1 + 1 + 1 + 2 + 3 + 2 = 10 marks)

The Apache Orchard grows a very juicy apple called the Fuji apple. Fuji apples are picked and then sorted by diameter into three categories:

- small — diameter less than 60 mm
- jumbo — the largest 15% of the apples
- standard — all other apples.

The diameters of Fuji apples are found to be normally distributed with a mean of 71 mm and a standard deviation of 12 mm.

a. Calculate, correct to 4 decimal places, the probability that a Fuji apple selected at random has a diameter less than 85 mm.

b. Determine the percentage of apples, to the nearest per cent, that are sorted into the small category.

c. What is the minimum diameter, correct to the nearest millimetre, of a jumbo Fuji apple?

d. An apple is selected at random from a bin of jumbo Fuji apples. Determine the probability, correct to 4 decimal places, that it has a diameter greater than 100 mm.

e. The Apache Orchard receives the following prices for Fuji apples:

- small — 12 cents each
- standard — 15 cents each
- jumbo — 25 cents each.

Calculate the expected income, correct to the nearest dollar, for a container of 2500 unsorted apples.

f. Some apples are selected before sorting and are packed into bags of 6 to be sold at the front gate of the orchard. Determine the probability, correct to 4 decimal places, that one of these bags contains at least 2 jumbo apples.

Examination marks summary

Question number	Simple familiar	Complex familiar	Complex unfamiliar	Topics
Paper 1 — Technology free				
1	4			3.2
2	4			3.3
3	3			3.1
4	5			3.2, 4.1
5	3			4.2
6	6			4.3
7	5			4.4
8		4		3.2, 4.1
9		6		4.4
10			10	3.2, 3.3, 4.1
Paper 2 — Technology active				
1	4			3.1
2	4			4.3
3	3			4.2
4	4			3.2, 4.1
5	4			4.5
6	4			3.2, 3.3, 4.1
7	7			4.3
8		4		4.4
9		6		3.3, 4.1
10			10	4.3, 4.4
Totals	**60**	**20**	**20**	
Percentage	**60%**	**20%**	**20%**	

Note: 3.2 = Unit 3, Topic 2

GLOSSARY

antiderivative Where the function $f'(x)$ is a derivative of $f(x)$, $f(x)$ is an antiderivative of $f'(x)$.
antidifferentiation the reverse process to differentiation; allows for $f(x)$ to be found when $f'(x)$ is given
Bernoulli distribution a discrete probability distribution involving one Bernoulli trial
Bernoulli trial a single event that has only two possible outcomes: success or failure. Each outcome has a fixed probability.
binomial distribution a discrete probability distribution involving n Bernoulli trials. Each trial is an independent trial with the same probability, p.
cdf *see* **cumulative distribution function**
chain rule Use the chain rule to differentiate a function that is a composite function of two simpler functions; $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$.
composite function a function that can be expressed as a composition of two simpler functions
concave down describes parabolas that open downwards and have a decreasing gradient
concave up describes parabolas that open upwards and have an increasing gradient, also referred to as convex
concavity a description of the shape of a continuous function; *see* **concave up**, **concave down**
confidence intervals the level of confidence that a population parameter will lie in a particular interval. The 95% confidence interval is given by $\left(\hat{p} - 1.96\sqrt{\frac{\hat{p}(1-\hat{p})}{n}},\ \hat{p} + 1.96\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right)$.
continuous random variable a random variable that assumes quantities that can be measured, such as weights, time, height, in a given range
cosine rule the extended version of Pythagoras' theorem. The cosine rule for a triangle ABC can be expressed as $a^2 = b^2 + c^2 - 2bc\cos(A)$.
cumulative distribution function a function that describes the probability that a continuous random variable X has a value less than or equal to x; $F(x) = \mathrm{P}(x \le c) = \int_{-\infty}^{c} f(x)dx$
definite integral evaluates to a real number not a function; $\int_a^b f(x)dx = F(b) - F(a)$ where $F'(x) = f(x)$, a is the lower terminal and b is the upper terminal
dilation a linear transformation that enlarges or reduces the size of a figure by a scale factor k parallel to either axis or both axes
empirical rule In a normal distribution:

- approximately 68% of the population will fall within 1 standard deviation of the mean
- approximately 95% of the population will fall within 2 standard deviations of the mean
- approximately 99.7% of the population will fall within 3 standard deviations of the mean.

This is also known as the 68–95–99.7% rule.
expected value a measure of the central tendency of the probability distribution of a random variable. For a discrete random variable, X, $\mathrm{E}(X) = \mu = \sum x\mathrm{P}(x)$.
exponent an index or power; for the number $n = a^p$, the base is a and the power, or index, or exponent is p
frequency polygon created by joining the midpoints at the top of each bar representing an interval in a histogram
fundamental theorem of integral calculus allows areas under graphs to be calculated exactly;
$$\int_a^b f(x)dx = F(b) - F(a)$$
indefinite integral $\int f(x)dx = F(x) + c$ where $F'(x) = f(x)$. $F(x)$ is an antiderivative of $f(x)$.

index form the form a^x, where a is the base and x is the index (also called the logarithm, power or exponent)
index laws the mathematical laws that apply to combinations of numbers written in index form
indicial equation an equation in which the unknown variable is an exponent
integrand the function that is to be integrated, $f(x)$
integration (applied to calculate areas under curves) the process of forming the limiting sum of the area of a large number of strips of very small width in order to obtain the total area under the curve
interquartile range the middle 50% of the data; IQR $= Q_3 - Q_1$ (the value of the upper quartile – the value of the lower quartile)
interval a set of numbers described by two numbers where any number that lies between those two numbers is also included in the set
kinematics the study of the motion of a particle in a straight line
logarithm an index or power. If $n = a^x$, then $x = \log_a(n)$ is an equivalent statement.
margin of error the distance between the endpoints of the confidence interval and the sample estimate. For a 95% confidence interval, the margin of error, M, is given by $M = 1.96\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}$.
normal distribution curve a continuous probability distribution characterised by a symmetrical bell-shape curve. It is given by the function $f(x) = \dfrac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$.
parameter a characteristic of a population that stays constant during an analysis of that particular population
pdf *see* **probability density curve**
percentile the value below which there is a specified probability that a randomly selected value will fall. The 70th percentile means that 70% of the data lies below this value.
point of inflection a point on a continuous curve where the concavity changes
population size the number of members included in the total population, symbol N
population the set or group from which information is required
probability density curve or probability density function (pdf); a function that represents the probabilities of a given scenario for the interval $x \in [a, b]$. Two conditions must be met for a function to be classified as a pdf: $\int_a^b f(x)\,dx = 1$ and $f(x) \geq 0$ for all $x \in [a, b]$.
quartiles values that divide an ordered set into four (approximately) equal parts. There are three quartiles: the first (or lower) quartile, Q_1; the second quartile (or median), Q_2; and the third (or upper) quartile, Q_3.
random sample a sample where every member of a population has the same probability of being selected
rate of change the derivative of a function evaluated at an instant
reflection a transformation of a figure defined by the line of reflection where the image point is a mirror image of the pre-image point
relative frequency refers to actual data obtained; relative frequency $= \dfrac{\text{number of times an event has occurred}}{\text{number of trials}}$
sample proportion the proportion of a sample size that has is a successful outcome;
$\hat{p} = \dfrac{\text{number in the sample with the favourable attribute}}{\text{sample size}}$
sample size the number of members included in a sample, symbol n
sample the group or subset within the population that information is collected from
sampling distribution the distribution of the sample proportion, represented in a probability distribution table or a graph
second derivative For a function $f(x)$, the second derivative is the rate of change of the derivative with respect to x, written as $\dfrac{d^2y}{dx^2}$ or $f''(x)$.

self-selected sample a sample where the participants choose to participate; this is rarely representative of the entire population

sine rule the rule that, in a triangle, the ratios of the side lengths to the sines of the opposite angles are equal; expressed as $\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}$

standard deviation a measure of the spread of a data set using the equation $\text{SD}(X) = \sigma = \sqrt{\text{Var}(X)}$. The larger the standard deviation, the more spread out the data. If the standard deviation is small, the data is clumped about the mean.

standard normal distribution the standardised form of the normal distribution where $\mu = 0$ and $\sigma = 1$, commonly written as $Z \sim \text{N}(0, 1)$

statistic a characteristic of a sample. Different samples of the same population can give different results.

stratified random sample a sample where subgroups within a population are represented in a similar proportion. Each member within a subgroup has the same chance of being selected.

systematic sample every k^{th} member of the population is sampled

terminals values that indicate the range over which a definite integral is taken; in $\int_a^b f(x)dx$, the variable a is the lower terminal and b is the upper terminal

translation a transformation of a figure where each point in the plane is moved a given distance in a horizontal or vertical direction

unit circle in trigonometry, the circle $x^2 + y^2 = 1$ with the centre at the origin and a radius of 1 unit

variance a measure of the spread of a data set using the equation $\text{Var}(X) = \sigma^2 = \text{E}(X^2) - [\text{E}(X)]^2$

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