MATHS Quest

YEAR 12

MATHS A

FOR QUEENSLAND

SECOND EDITION

jacaranda

Maths Quest

Lyn Elms ▸ Nick Simpson ▸ Tony MacPherson

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Introduction

Maths Quest Maths A Year 12 for Queensland 2nd edition is one of the exciting Maths Quest resources specifically designed for the Queensland senior Mathematics syllabuses beginning in 2009. It has been written and compiled by practising Queensland Maths A teachers. It breaks new ground in Mathematics textbook publishing.

This resource contains:

- a student textbook with accompanying student website (eBookPLUS)
- a teacher edition with accompanying teacher website (eGuidePLUS)
- a solutions manual containing fully worked solutions to all questions contained in the student textbook.

Student textbook

Full colour is used throughout to produce clearer graphs and headings, to provide bright, stimulating photos and to make navigation through the text easier.

Skills check exercises at the beginning of each chapter enable a quick diagnostic assessment of students' grasp of the basic mathematics needed.

Clear, concise *theory sections* contain *worked examples*, *graphics calculator tips* and *remember boxes*.

Worked examples in a Think/Write format provide clear explanation of key steps and suggest how solutions can be presented.

Exercises contain many carefully graded skills and application problems, including multiple-choice questions. Cross-references to relevant worked examples appear beside the first 'matching' question throughout the exercises.

Sets of *10 Quick Questions* allow students to quickly review the concepts just learnt before proceeding further in the chapter.

Investigations, often suggesting the use of technology, provide further discovery learning opportunities.

Each chapter concludes with a *summary* and *chapter review* exercise containing questions that help consolidate students' learning of new concepts.

A *glossary* of mathematical terms is provided to assist students' understanding of the terminology introduced in each unit of the course. Words in bold type in the theory sections of each chapter are defined in the glossary at the back of the book.

Technology is fully integrated within the resource. To support the use of graphics calculators, instructions for two models of calculator are presented in worked examples and graphics calculator tips throughout the text. The two models of graphics calculator featured are the Casio *fx*-9860G AU and the TI-Nspire CAS. (Note that the screen shots shown in this text for the TI-Nspire CAS calculator were produced using OS 1.7. Screen displays may vary depending on the operating system in use.)

For those students using the TI-89 Titanium model of graphics calculator, an appendix containing matching instructions has been included at the back of the book.

The *Maths Quest for Queensland* series also features the use of spreadsheets with supporting Excel files supplied on the student website.

Student website — eBookPLUS

The accompanying student website contains an electronic version of the textbook as well as the resources listed below.

WorkSHEETs — editable Word 97 documents that may be completed on screen, or printed and completed later.

SkillSHEETs — printable pages that contain additional examples and problems designed to help students revise required concepts.

Test Yourself activities — multiple-choice quizzes for students to test their skills after completing each chapter.

Teacher edition

The teacher edition textbook contains everything in the student textbook and more. To support teachers assisting students in the class, answers appear in red next to most questions in the exercises and investigations. Each chapter is annotated with relevant syllabus information.

Teacher website — eGuidePLUS

The accompanying teacher website contains everything in the student website plus the following resources:

- two tests per chapter (with fully worked solutions)
- fully worked solutions to WorkSHEETs
- a syllabus planning document
- assessment tasks (and answers)
- fully worked solutions to all questions in the student textbook.

Solutions manual

Maths Quest Maths A Year 12 for Queensland Solutions Manual contains the fully worked solutions to every question and investigation in the *Maths Quest Maths A Year 12 for Queensland 2nd edition* student textbook.

Fully worked solutions are available for all titles in the *Maths Quest for Queensland* senior series.

Maths Quest is a rich collection of teaching and learning resources within one package.

About eBookPLUS

Next generation teaching and learning

This book features eBookPLUS: an electronic version of the entire textbook and supporting multimedia resources. It is available for you online at the JacarandaPLUS website (**www.jacplus.com.au**).

Using the JacarandaPLUS website

To access your eBookPLUS resources, simply log on to **www.jacplus.com.au** using your existing JacarandaPLUS login and enter the registration code. If you are new to JacarandaPLUS, follow the three easy steps below.

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- Contact John Wiley & Sons Australia, Ltd.
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Simple and compound interest

1

syllabus reference

Strand
Financial mathematics

Core topic
Managing money 2

In this chapter

Introduction

If, when you were born, your parents invested \$10 000 on your behalf (perhaps to provide funds for a university education) they would want to know the future value of this investment on, say, your seventeenth birthday.

A smart investor would consider the interest rates available at the time, and the terms of the investment. Interest that is calculated more frequently on a fixed investment is of most benefit to the investor, as it would result in higher interest payments. Your \$10 000 would return almost \$23 000 if interest was calculated yearly at 5% per annum (p.a.). If interest was calculated daily at 5% p.a., the return would be almost \$23 400. It is obvious that a higher interest rate results in a better return, all other conditions being equal. The same \$10 000 investment with an interest rate of 10% p.a. would return just over \$50 500 with interest calculated yearly, and almost \$55 000 with interest calculated daily. A smart investor would consider the best combination of these two factors.

The effects of **compounding interest** are quite dramatic. It has been described by many as the eighth wonder of the world. Baron Rothschild recognised its enormous potential. The remarkable rise of the Rothschild Bank stemmed from that family's astute management of money. A search of the World Wide Web reveals this remarkable family that rose from humble beginnings to substantial wealth in the span of just one generation.

It would be virtually impossible to go through life without either investing or borrowing money. A common misconception is that it is best to choose the highest quoted interest rate when investing money and the lowest quoted interest rate when borrowing money. Those quoted rates can be misleading! Throughout this chapter we will address issues such as interest rates, and other factors that affect the amount of interest calculated on an investment or a loan. A graphics calculator can be used to reduce the work involved in many of the lengthy calculations.

SKILLS CHECK

1 Calculate each of the following.

a 5% of \$20 **b** 2.5% of \$350 **c** $3\frac{1}{3}$% of \$45

d $\frac{1}{4}$% of \$100 **e** $12\frac{1}{2}$% of \$32 **f** 0.1% of \$250

2 Express the following as decimals.

a $7\frac{1}{4}$ **b** $7\frac{1}{4}$% **c** $\frac{1}{5}$ **d** $\frac{1}{5}$% **e** $\frac{1}{8}$ **f** 0.1%

3 Express each of the following times in years.

a 3 months **b** 2 months **c** 8 months

d 2 years 1 month **e** 4 years 5 months **f** 30 months

4 Calculate each of the following.

a $50 \cdot 6 \cdot 1\frac{1}{2}$ **b** $300 \cdot 4.2 \cdot \frac{5}{12}$ **c** $6000 \cdot 5\frac{1}{4} \cdot \frac{2}{3}$

d 1.05^3 **e** 1.004^{20} **f** $(1 + \frac{6.5}{100})^{10}$

Investing money

The aim of this investigation is to explore a relationship between the amount of money invested and the interest earned, considering the length of time of the investment and the interest rate applied to the money.

Task 1

Consider making an investment of $10 000 in a fund where the interest rate paid on the money is quoted as 4.5% p.a. This means that, if the money was invested for a whole year, the interest earned would amount to 4.5% of $10 000; that is, $450. If the money was invested for only half a year (6 months), the interest earned would be $\frac{1}{2}$ of 4.5% of $10 000 ($225). What would the interest be if the money was invested for only 1 month (one twelfth of a year), any fraction of a year or any number of years?

Copy and complete the following table showing the interest earned on an investment of $10 000 at 4.5% p.a. over a variable period of time.

Time	Interest calculation	Interest
1 month	$\frac{1}{12}$ of 4.5% of $10 000	$37.50
2 months	$\frac{2}{12}$ of 4.5% of $10 000	
3 months		
4 months		
5 months		
6 months		
7 months		
8 months		
9 months		
10 months		
11 months	$\frac{11}{12}$ of 4.5% of $10 000	
12 months (1 year)	4.5% of $10 000	$450
2 years	Twice 4.5% of $10 000	
3 years	3 times 4.5% of $10 000	

What relationship appears to exist between the amount of interest earned and the length of time of the investment?

(*Continued*)

Task 2

Take again our investment of $10 000 and consider the effect a changing interest rate has on the amount of interest earned. We saw previously that an interest rate of 4.5% p.a. earned $450 each year on the $10 000 invested. If the interest rate fell to 4.4% p.a., the yearly interest earned would be 4.4% of $10 000 ($440). Clearly, the interest rate affects the amount of interest earned.

Copy and complete the table at right showing the interest earned on an investment of $10 000 over a fixed period of a year at a variable interest rate.

Interest rate (p.a.) %	Interest calculation	Interest
4.5	4.5% of $10 000	$450
4.4	4.4% of $10 000	
4.3		
4.2		
4.1		
4		
3.9		
3.5		
3.25		
3		
2.5		
2.125		
2		

What relationship appears to exist between the amount of interest earned and the yearly interest rate?

Task 3

What would be the effect of changing the amount of money invested? Clearly, if we invested $20 000, the money earned should be twice what $10 000 would earn. At an interest rate of 4.5% p.a. the investment should earn 4.5% of $20 000 in a year ($900).

Investment ($)	Interest calculation	Interest
10 000	4.5% of $10 000	$450
20 000	4.5% of $20 000	
1000		
100		
50		
10		

Copy and complete the table showing the interest earned on a variable investment at a fixed interest rate of 4.5% p.a. for a period of a year. How does the value of the investment affect the interest earned?

Conclusion

We have considered separately three variables (time, interest rate and investment) and the effect each has on the amount of interest earned on an investment. Write a relationship in words showing how the interest earned depends on these three variables.

In the next section we will consider this further by developing this relationship as a mathematical equation.

Simple interest

When you lend money for a certain period of time (a **term deposit**) to a bank, building society or other financial institution, you expect to be rewarded by eventually getting your money back, plus an extra amount commonly known as **interest** (I).

Similarly, if you borrow money from any institution by taking out a loan or mortgage, you must pay back the original sum plus interest.

The following examples deal with **simple interest**, that is, interest which is paid only on the original sum of money invested or borrowed.

We saw in the previous investigation that the interest earned from an investment depends on the **rate of interest**, the period or **term** for which it is invested and the amount or **principal** invested. This relationship can be expressed formally by a mathematical equation.

The formula used to calculate simple interest is given by:

$$I = \frac{PRT}{100}$$

where:

I = interest, $
P = principal, $ — that is, the sum of money borrowed or invested
R = rate of interest p.a., % — that is, per annum, (per year)
T = term of interest, years — that is, the period of time for which the sum of money is to be borrowed or invested.

When we invest money, at the end of the time period we collect our investment (principal) and also the interest it has earned.

The sum of the principal, P, and the interest, I, is called the *total amount* and is denoted by the symbol A.

The formula used to calculate the total amount is given by:

$$A = P + I$$

where:

A = total amount at the end of the term, $
P = principal, $
I = simple interest, $.

WORKED Example 1

Calculate the amount of simple interest, I, earned and the total amount, A, at the end of the term, if:

a **$12 000 is invested for 5 years at 9.5% p.a.**
b **$2500 is invested for 3 months at 4.5% p.a.**

THINK	WRITE
a 1 Write down the formula for simple interest.	**a** $I = \frac{PRT}{100}$
2 Write down the known values of the variables.	$P = \$12\,000$ $R = 9.5$ $T = 5$ years

Continued over page

THINK	WRITE
3 Substitute the values into the given formula.	$I = \frac{12\,000 \cdot 9.5 \cdot 5}{100}$
4 Evaluate.	$= \frac{570\,000}{100}$ $= 5700$
5 Answer the question and include the appropriate unit.	The amount of interest earned is $5700.
6 Write down the formula for the total amount.	$A = P + I$
7 Substitute the values for P and I.	$= 12\,000 + 5700$
8 Evaluate.	$= 17\,700$
9 Answer the question and include the appropriate unit.	The total amount at the end of the term is $17 700.
b 1 Write down the formula for simple interest.	**b** $I = \frac{PRT}{100}$
2 Write down the known values of the variables. *Note:* T must be expressed in years, so divide 3 months by 12 months.	$P = \$2500 \quad R = 4.5$ $T = 3$ months $= \frac{3}{12}$ or 0.25 years
3 Substitute the values into the given formula.	$I = \frac{2500 \cdot 4.5 \cdot 0.25}{100}$
4 Evaluate and round off the answer to 2 decimal places.	$= \frac{2812.5}{100}$ $= 28.13$
5 Answer the question and include the appropriate unit.	The amount of interest earned is $28.13.
6 Write down the formula for the total amount.	$A = P + I$
7 Substitute the values for P and I.	$= 2500 + 28.13$
8 Evaluate.	$= 2528.13$
9 Answer the question and include the appropriate unit.	The total amount at the end of the term is $2528.13.

Graphics Calculator tip! Calculating simple interest and amount

A graphics calculator can be used to solve Worked example 1.

For the Casio *fx*-9860G AU

The Casio graphics calculator displays a Financial Calculations facility in its menu. This is accessed by pressing EXE on the TVM screen on the MAIN MENU. Simple interest and compound interest calculations can be performed here using built-in formulas. The values of known variables can be entered and the value of the unknown variable/s can be displayed.

1. Enter the financial calculations section from the menu.
2. Enter the Simple Interest section by pressing the function key (F1) (SMPL).
3. The variables displayed are n (number of interest periods in days — note that the time is expressed in days here rather than years), $I\%$ (annual interest rate) and PV (present value or principal). Enter the following values for each of the variables, pressing (EXE) after each entry.

   $n = 5 \cdot 365$ (number of days of the investment)
   $I\% = 9.5$ (interest rate is 9.5% p.a.)
   $PV = 12\,000$ (principal is \$12 000)
4. Pressing (F1) (SI) displays the simple interest for the period. This key reveals the simple interest for the period amounts to \$5700 (as in our worked example).

5. Press (F1) (REPT) to return to the variables. Then press the (F2) (SFV) function key which displays the simple interest future value/principal plus interest. The key indicates that the amount at the end of the term is \$17 700 (in agreement with our worked example).

Verify the answers to part **b** of Worked example 1 using your calculator.

Note: This calculator uses positive values to represent receipts of money, and negative values to represent payments. For our purposes, this is merely a convention and may be ignored.

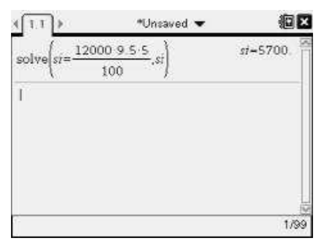
For the TI-Nspire CAS

1. On a calculator page, press:
   - MENU (menu)
   - 3: Algebra (3)
   - 1: Solve (1).

   Complete the entry line as:

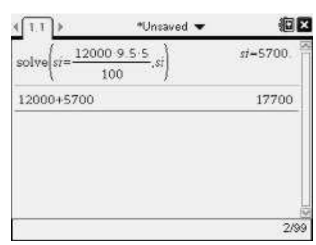
   $$\text{solve}\left(si = \frac{12\,000 \cdot 9.5 \cdot 5}{100}, si\right).$$

   Then press ENTER (enter).



2. $A = P + I$
   $= 12\,000 + 5700$
   $= \$17\,700$

Verify the answers in part **b** of Worked example 1 using your calculator.



WORKED Example 2

After comparing investment options from a variety of institutions, Lynda and Jason decided to invest their \$18 000 in State Government bonds at 7.75% p.a. The investment is for 5 years and the interest is paid semi-annually (every six months). Calculate how much interest:

a **they receive in every payment**

b **will be received in total.**

Continued over page

THINK	WRITE
a 1 Write down the formula for simple interest.	**a** $I = \frac{PRT}{100}$
2 Write down the known values of the variables. *Note:* T must be expressed in years so divide 6 months by 12 months.	$P = \$18\,000$ $R = 7.75$ $T = 6$ months $= \frac{6}{12}$ or 0.5 years
3 Substitute the values into the given formula.	$I = \frac{18\,000 \cdot 7.75 \cdot 0.5}{100}$
4 Evaluate.	$= \frac{69\,750}{100}$ $= 697.5$
5 Answer the question and include the appropriate unit.	Lynda and Jason receive \$697.50 in interest every 6 months.
b Method 1	
1 Write down the formula for simple interest.	**b** $I = \frac{PRT}{100}$
2 Write down the known values of the variables.	$P = \$18\,000$ $R = 7.75$ $T = 5$ years
3 Substitute the values into the given formula.	$I = \frac{18\,000 \cdot 7.75 \cdot 5}{100}$
4 Evaluate.	$= \frac{697\,500}{100}$ $= 6975$
5 Answer the question and include the appropriate unit.	Lynda and Jason will receive a total of \$6975 in interest.
b Alternative method	
1 Multiply the interest received in each 6 month period by the number of 6 month periods in 5 years; that is, multiply \$697.50 by 10.	Interest obtained each 6 months = \$697.50 Number of payments to be received = 10 Total interest received = \$697.50 · 10 = \$6975
2 Answer the question.	Lynda and Jason will receive a total of \$6975 in interest.

Graphics Calculator tip! Simple interest

The following instructions can be used for Worked example 2.

For the Casio *fx*-9860G AU

Use the Financial calculations facility as shown in the steps below.

1. Enter the Simple Interest section by pressing F1 (SMPL).

2. Enter the following values for the variables:
   $n = 365 \mid 2$ (The time is the number of days in half a year.)
   $I\% = 7.75$ (The interest rate is 7.75% p.a.)
   $PV = 18\,000$ (The principal is \$18 000.)
3. Pressing the (F1) (SI) function key displays the simple interest for six months as \$697.50 (the answer to part **a**).
4. To obtain the answer to part **b**, repeat the calculation for part **a**, changing the value for n to $5 \cdot 365$ (the number of days in 5 years). You should confirm the answer of \$6975.

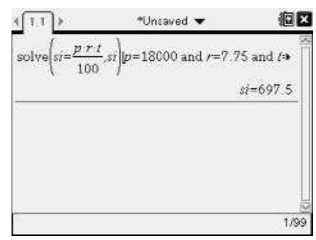
Using the TI-Nspire CAS

1. On a Calculator page, press:
   - MENU (menu)
   - 3: Algebra (3)
   - 1: Solve (1).

   Complete the entry line as:

   $$\text{solve}\left(si = \frac{p \cdot r \cdot t}{100}, si\right) \mid p = 18\,000 \text{ and } r = 7.75 \text{ and } t = 0.5.$$

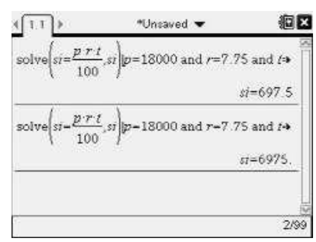
   Then press ENTER (enter).



2. To obtain the answer to part **b**, use the NavPad ▲ to highlight the previous equation and paste it into the entry line. Use the NavPad ◀ to edit the equation as shown.

   $$\text{solve}\left(si = \frac{p \cdot r \cdot t}{100}, si\right) \mid p = 18\,000 \text{ and } r = 7.75$$

   and $t = 5$.
   Then press ENTER (enter).



remember

1. Simple interest is given by $I = \frac{PRT}{100}$.
2. The total amount is given by $A = P + I$.
3. When calculating simple interest, the interest earned is the same for each time period.
4. I = interest (\$)
   P = principal (\$)
   R = rate of interest (% p.a.)
   T = time (years)
   A = amount (\$)
5. A graphics calculator can also be used to shorten simple interest calculations.

EXERCISE 1A Simple interest

The use of a graphics calculator, if available, is encouraged.

WORKED Example 1

1 Find the interest paid on the following deposits for the periods and at the rates given.

a $680 for 4 years at 5% p.a.
b $210 for 3 years at 9% p.a.
c $415 for 5 years at 7% p.a.
d $460 at 12% p.a. for 2 years
e $1020 at $12\frac{1}{2}$% p.a. for 2 years
f $713 at $6\frac{3}{4}$% p.a. for 7 years
g $821 at $7\frac{1}{4}$% p.a. for 3 years
h 11.25% p.a. on $65 for 6 years
i 6.15% p.a. on $21.25 for 9 years
j 9.21% p.a. on $623.46 for 4 years
k $13\frac{3}{4}$% p.a. on $791.35 for 5 years

eBook plus

Digital docs:
SkillSHEET 1.1 Substitution into formulas
SkillSHEET 1.2 Conversion of units of time

2 Find the interest charged or earned on the following loans and investments:

a $690 loaned at 12% p.a. simple interest for 15 months
b $7500 invested for 3 years at 12% per year simple interest
c $25 000 borrowed for 13 weeks at 5.2% p.a. simple interest
d $250 invested at 21% p.a. for $2\frac{1}{2}$ years.

WORKED Example 1

3 Find the amount to which each investment has grown after the investment periods shown in the following examples:

a $300 invested at 10% p.a. simple interest for 24 months
b $750 invested for 3 years at 12% p.a. simple interest
c $20 000 invested for 3 years and 6 months at 11% p.a. simple interest
d $15 invested at $6\frac{3}{4}$% p.a. for 2 years and 8 months
e $10.20 invested at $8\frac{1}{2}$% p.a. for 208 weeks.

4 **multiple choice**

If John had $63 in his bank account and earned 9% p.a. over 3 years, the simple interest earned would be:

A $5.67 B $17.01 C $22.68 D $80.01 E $1701

5 **multiple choice**

If $720 was invested in a fixed deposit account earning $6\frac{1}{2}$% p.a. for 5 years, the interest earned at the end of 5 years would be:

A $23.40 B $216.00 C $234.00 D $954.00 E $23 400.00

6 **multiple choice**

A 4-year bond paid 7.6% p.a. simple interest. If Sonja bought a bond worth $550, the interest she earned would be:

A $16.72 B $167.20 C $717.20 D $1672 E $16 720

7 **multiple choice**

Bodgee Bank advertised a special offer. If a person invests $150 for 2 years, the bank will pay 12% p.a. simple interest on the money. At the expiry date, the investor would have earned:

A $36 B $48 C $186 D $300 E $3600

8 **multiple choice**

Maclay invested \$160 in a bank for 6 years earning 8% simple interest each year. At the end of the 6 years, he will receive in total:

A \$76.80 **B** \$236.80 **C** \$768 **D** \$928 **E** \$2368

9 **multiple choice**

Simple interest was calculated on a term deposit of 4 years at $3\frac{1}{2}$% per year. When Ashleigh calculated the total return on her investment of \$63.50, it was:

A \$7.75 **B** \$8.89 **C** \$71.24 **D** \$72.39 **E** \$88.90

10 **multiple choice**

Joanne asked Sally for a loan of \$125 to buy new shoes. Sally agreed on the condition that Joanne paid it back in two years at 3% p.a. simple interest. The amount Joanne paid Sally at the end of the two years was:

A \$7.50 **B** \$125 **C** \$130.50 **D** \$132.50 **E** \$200

11 **multiple choice**

Betty invests \$550 in an investment account earning 4% p.a. simple interest over 6 years. Ron puts his \$550 in a similar investment earning 5% p.a. simple interest for 5 years. The difference in their earnings at the end of the investment period is:

A \$0 **B** \$5.50 **C** \$7.50 **D** \$55 **E** \$550

12 **multiple choice**

Two banks pay simple interest on short-term deposits. Hales Bank pays 8% p.a. over 5 years and Countrybank pays 10% p.a. for 4 years. The difference between the two banks' final payout figure if \$2000 was invested in each account is:

A \$0 **B** \$150 **C** \$800 **D** \$1200 **E** \$2800

13 Robyn wishes to purchase a new dress worth \$350 to wear to the school formal. If she borrows the total amount from the bank and pays it off over 3 years at 11% p.a. simple interest, what is the total amount Robyn must pay back to the bank?

14 The Sharks Building Society offers loans at $8\frac{1}{2}$% p.a. simple interest for a period of 18 months. Andrew borrows \$200 from Sharks to buy Monique an engagement ring. Calculate the amount of interest Andrew is to pay over the 18 months.

15 Silvio invested the \$1500 he won in Lotto with an insurance company bond that pays $12\frac{1}{4}$% p.a. simple interest provided he keeps the bond for 5 years. What is Silvio's total return from the bond at the end of the 5 years?

16 The insurance company that Silvio used in the previous question allows people to withdraw part or all the money early. If this happens the insurance company will only pay $6\frac{3}{4}$% p.a. simple interest on the amount which is withdrawn over the period it was invested in the bond. The part which is left in the bond receives the original agreed interest. Silvio needed \$700 for repairs to his car 2 years after he had invested the money but left the rest in for the full 5 years. How much interest did he earn from the bond in total?

17 Jill and John decide to borrow money to improve their yacht, but cannot agree which loan is the better value. They would like to borrow \$2550. Jill goes to the Big-4 Bank and finds that they will lend her the money at $11\frac{1}{3}$% p.a. simple interest for 3 years. John finds that the Friendly Building Society will lend the \$2550 to them at 1% per month simple interest for the 3 years.
- **a** Which institution offers the better rate over the 3 years?
- **b** Explain why.

WORKED Example 2

18 Sue and Harry invested \$14 500 in State Government bonds at 8.65% p.a. The investment is for 10 years and the interest is paid semi-annually (that is, every six months). Calculate how much interest:
- **a** they receive every payment
- **b** will be received in total.

19 Anna invested \$85 000 in Ski International debentures. She earns 7.25% p.a. which is paid quarterly for one year.
- **a** Calculate, how much interest:
  - **i** she receives quarterly
  - **ii** will be received in total, over a year.
- **b** Would Anna receive the same amount of interest over a 3-year period if it were paid annually rather than quarterly?

20 Mrs Williams invested \$60 000 in government bonds at 7.49% p.a. with interest paid semi-annually (that is, every 6 months).
- **a** How much interest is she paid each 6 months?
- **b** How much interest is she paid over $3\frac{1}{2}$ years?
- **c** How long would the money need to be invested to earn a total of \$33 705 in interest?

Finding *P*, *R* and *T*

In many cases we may wish to find the principal, interest rate or period of a loan. In these situations it is necessary to rearrange or transpose the simple interest formula after (or before) substitution, as the following example illustrates.

WORKED Example 3

A bank offers 9% p.a. simple interest on an investment. At the end of 4 years the interest earned was \$215. How much was invested?

THINK	WRITE
1 Write the simple interest formula.	$I = \frac{PRT}{100}$
2 List the values of *I*, *R* and *T*.	$I = \$215 \quad R = 9 \quad T = 4$ years

THINK

3. Substitute into the formula.
4. Make P the subject by multiplying both sides by 100 and dividing both sides by $(9 \cdot 4)$.
5. Use a calculator to evaluate.
6. Write your answer.

WRITE

$$I = \frac{P \cdot R \cdot T}{100}$$

$$215 = \frac{P \cdot 9 \cdot 4}{100}$$

$$P = \frac{215 \cdot 100}{9 \cdot 4}$$

$$P = 597.22$$

The amount invested was \$597.22.

Transposed simple interest formula

It may be easier to use the transposed formula when finding P, R or T.

Simple interest formula transposes:

to find the principal $P = \frac{100 \cdot I}{R \cdot T}$

to find the interest rate $R = \frac{100 \cdot I}{P \cdot T}$

to find the period of the loan or investment $T = \frac{100 \cdot I}{P \cdot R}$

WORKED Example 4

When \$720 is invested for 36 months it earns \$205.20 simple interest. Find the yearly interest rate.

THINK

For the Casio *fx*-9860G AU

1. Write the simple interest formula.
2. List the values of P, I and T. T must be expressed in years.
3. Substitute into the formula.
4. Evaluate on a calculator. Remember to bracket $(720 \cdot 3)$.
5. Write your answer.

WRITE/DISPLAY

$$R = \frac{100 \cdot I}{P \cdot T}$$

$P = \$720$, $I = \$205.20$, $T = 36$ months (3 years)

$$R = \frac{100 \cdot 205.20}{720 \cdot 3}$$

The interest rate offered was 9.5% per annum.

Continued over page

THINK

For the TI-Nspire CAS

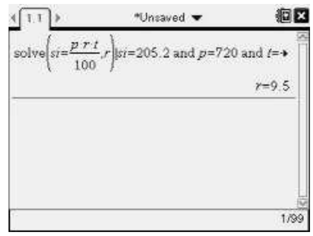
1. On a Calculator page, press:
   - MENU (menu)
   - 3: Algebra (3)
   - 1: Solve (1).

   Complete the entry line as:

   solve $\left(si = \dfrac{p \cdot r \cdot t}{100}, r\right) \mid si = 205.2$ and $p = 720$ and $t = \dfrac{36}{12}$.

   Then press ENTER (enter).
2. Write your answer.

WRITE/DISPLAY



Solve $I = \dfrac{P \cdot R \cdot T}{100}$, for R,

given $I = 205.20$, $P = 720$ and $T = \dfrac{36}{12} = 3$.

The interest rate offered was 9.5% per annum.

WORKED Example 5

An amount of $255 was invested at 8.5% p.a. How long will it take, to the nearest year, to earn $86.70 in interest?

THINK

For the Casio *fx*-9860G AU

1. Write the simple interest formula.
2. Substitute the values of P, I and R.
3. Substitute into the formula.
4. Evaluate on a calculator. Remember to bracket $(255 \cdot 8.5)$.
5. Write your answer.

WRITE/DISPLAY

$T = \dfrac{100 \cdot I}{P \cdot R}$

$P = \$255$, $I = \$86.70$, $R = 8.5$

$T = \dfrac{100 \cdot 86.70}{255 \cdot 8.5}$

The period of the investment was 4 years.

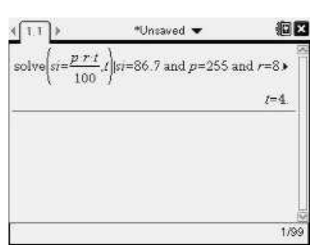
For the TI-Nspire CAS

1. On a Calculator page, press:
   - MENU (menu)
   - 3: Algebra (3)
   - 1: Solve (1).

   Complete the entry line as:

   solve$\left(si = \dfrac{p \cdot r \cdot t}{100}, t\right) \mid si = 86.7$ and $p = 255$ and $r = 8.5$.

   Then press ENTER (enter).



THINK

2 Write your answer.

WRITE/DISPLAY

Solve $I = \frac{P \cdot R \cdot T}{100}$, for T,

given $I = 86.70$, $P = 255$ and $R = 8.5$.

The period of the investment was 4 years.

remember

When finding P, R or T:

1. substitute the given values into the formula and then rearrange to isolate the pronumeral, or
2. transpose the simple interest formula

   (a) to find the principal $P = \frac{100 \cdot I}{R \cdot T}$

   (b) to find the interest rate $R = \frac{100 \cdot I}{P \cdot T}$

   (c) to find the period of the loan or investment $T = \frac{100 \cdot I}{P \cdot R}$

   and substitute the given values into the transposed formula.

EXERCISE 1B Finding P, R and T

WORKED Example 3

eBook plus

Digital doc: SkillSHEET 1.3 Substitution into the simple interest formula

1 For each of the following, find the principal invested.

a Simple interest of 5% p.a., earning \$307 interest over 2 years

b Simple interest of 7% p.a., earning \$1232 interest over 4 years

c Simple interest of 8% p.a., earning \$651 interest over 18 months

d Simple interest of $5\frac{1}{2}$% p.a., earning \$78 interest over 6 years

e Simple interest of 6.25% p.a., earning \$625 interest over 4 years

WORKED Example 4

2 For each of the following, find the interest rate offered. Express rates in % per annum.

a Loan of \$10 000, with a \$2000 interest charge, for 2 years

b Investment of \$5000, earning \$1250 interest for 4 years

c Loan of \$150, with a \$20 interest charge, for 2 months

d Investment of \$1400 earning \$178.50 interest for 6 years

e Investment of \$6250 earning \$525 interest for $2\frac{1}{2}$ years

WORKED Example 5

3 For each of the following, find the period of time (to the nearest month) for which the principal was invested or borrowed.

a Investment of \$1000 at simple interest of 5% p.a. earning \$50

b Loan of \$6000 at simple interest of 7% p.a. with an interest charge of \$630

c Loan of \$100 at simple interest of 24% p.a. with an interest charge of \$6

d Investment of \$23 000 at simple interest of $6\frac{1}{2}$% p.a. earning \$10 465

e Loan of \$1 500 000 at simple interest of 1.5% p.a. with an interest charge of \$1875

4 Lennie Cavan earned \$576 in interest when she invested in a fund paying 9.5% p.a. simple interest for 4 years. How much did Lennie invest originally?

5 Lennie's sister Lisa also earned \$576 interest at 9% p.a. simple interest, but she only had to invest it for 3 years. What was Lisa's initial investment?

6 Jack Kahn put some money away for 5 years in a bank account which paid $3\frac{3}{4}$% p.a. interest. He found from his bank statement that he had earned \$66. How much did Jack invest?

7 James needed to earn \$225. He invested \$2500 in an account earning simple interest at a rate of 4.5% p.a. How many months will it take James to achieve his aim?

8 Carol has \$3000 to invest. Her aim is to earn \$450 in interest at a rate of 5% p.a. Over what term would she invest?

9 **multiple choice**

Peter borrowed \$5000 and intended to pay it back in 3 years. The terms of the loan indicated Peter was to pay $9\frac{3}{4}$% p.a. interest. The interest Peter paid on the loan was:

A \$121.88 **B** \$446.25 **C** \$1462.50 **D** \$3537.50 **E** \$146 250

10 **multiple choice**

Joanne's accountant found that for the past 2 years she had earned a total of \$420 interest in an account paying 6% p.a. simple interest. When she calculated how much she invested the amount was:

A \$50.40
B \$350
C \$3500
D \$5040
E \$7000

11 **multiple choice**

A loan of \$1000 is taken over 5 years. The total amount repaid for this loan is \$1800. The simple interest rate per year on this loan is closest to:

A 5% **B** 8% **C** 9% **D** 16% **E** 36%

12 **multiple choice**

Jarrod decides to buy a motorbike at no deposit and no repayments for 3 years. He takes out a loan of \$12 800 and is charged at 7.5% p.a. simple interest over the 3 years. The lump sum Jarrod has to pay in 3 years time is:

A \$960 **B** \$2880 **C** \$9920 **D** \$13 760 **E** \$15 680

13 **multiple choice**

Chris and Jane each take out loans of \$4500 and are offered $6\frac{1}{4}$% p.a. simple interest over a 3-year period. Chris's interest is paid monthly whereas Jane's is paid yearly. The difference in the total amount of interest each person pays after the 3 years is:

A none **B** \$843.75 **C** \$877.50 **D** \$9652.50 **E** \$10 530

14 Alisha has $8900 that she is able to invest. She has a goal of earning at least $1100 in 2 years or less. Do any of the following investments satisfy Alisha's goal?

a 10% p.a. for 15 months

b $4\frac{1}{4}$% p.a. earning $1200

c After 100 weeks a final payout of $10 500

d After 2 years at 7.2% p.a.

investigation investigation

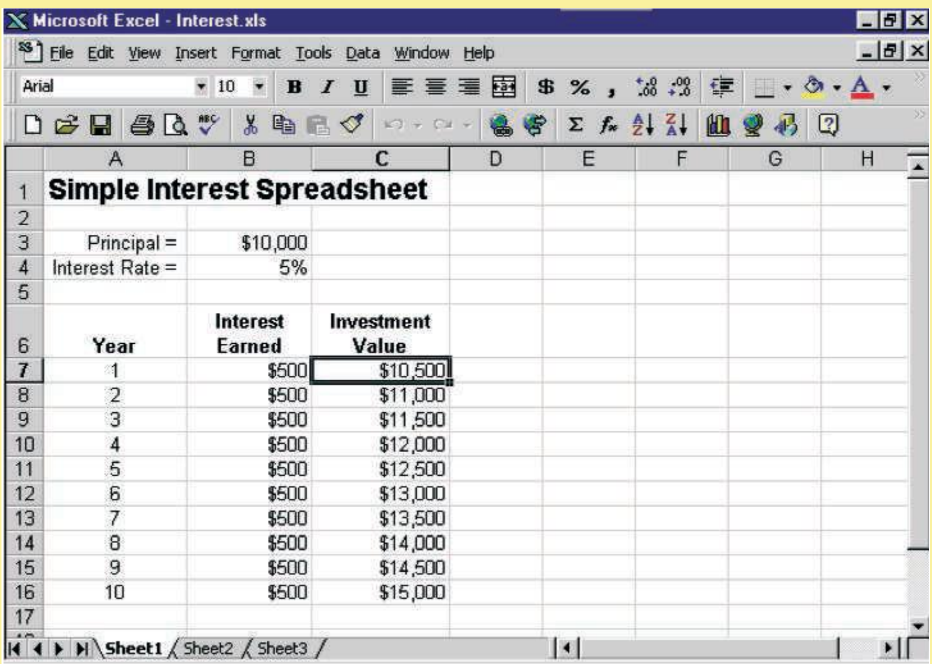
Simple interest spreadsheets

eBook*plus*

Digital doc:
Spreadsheet
205 Simple interest

Throughout this chapter we will use some spreadsheets that allow us to track the growing value of an investment over time.

1 Use the **Simple interest** weblink in your eBookPLUS and download the spreadsheet.



	A	B	C
1	Simple Interest Spreadsheet		
2			
3	Principal =	$10,000	
4	Interest Rate =	5%	
5			
6	Year	Interest Earned	Investment Value
7	1	$500	$10,500
8	2	$500	$11,000
9	3	$500	$11,500
10	4	$500	$12,000
11	5	$500	$12,500
12	6	$500	$13,000
13	7	$500	$13,500
14	8	$500	$14,000
15	9	$500	$14,500
16	10	$500	$15,000
17			

Sheet1 / Sheet2 / Sheet3

2 Sheet 1, titled 'Simple Interest Spreadsheet', models an investment of $10 000 at 5% p.a.

3 Use the graphing function on your spreadsheet to draw a line graph for the amount of interest earned each year and the value of the investment after each year.

4 Change the amount of the principal and the interest rate and note the change in the figures displayed and the chart.

5 Use your cursor to move around the cells under the headings 'Interest Earned' and 'Investment Value'. Note the formulas used in the calculations.

6 Save the spreadsheet as Simple Interest.

7 Print out the spreadsheet.

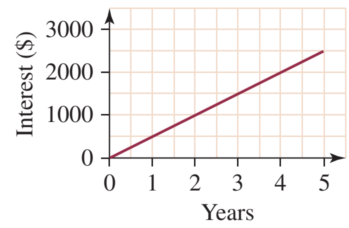
8 Close the spreadsheet. Attempt to reproduce it. Remember to use the 'copy' facility to copy formulas down columns.

Graphing simple interest functions

Suppose that we invest $10 000 at 5% p.a. simple interest. The table below shows the amount of interest that we will receive over various lengths of time.

No. of years	1	2	3	4	5
Interest	$500	$1000	$1500	$2000	$2500

The amount of interest earned can be graphed by the **linear function** at right.



Note that the **gradient** of this graph is 500, which is the amount of one year's interest, or 5% of the principal. This means that for every 1 unit increase on the x-axis (1 year in time) the y-axis (Interest) increases by 500 units ($500).

WORKED Example 6

Leilay invests savings of $6000 at 4% p.a.

a Complete the table below to calculate the interest that will have been earned over 5 years.

No. of years	1	2	3	4	5
Interest					

b Graph the interest earned against the number of years the money is invested.

THINK

a Use the simple interest formula to calculate the interest earned on $6000 at 4% p.a. for 1, 2, 3, 4 and 5 years.

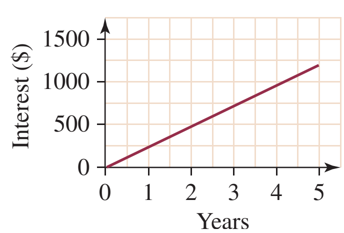
b Draw the graph with Years on the horizontal axis and Interest on the vertical axis.

WRITE

a

No. of years	1	2	3	4	5
Interest	$240	$480	$720	$960	$1200

b



We are able to compare the interest that is earned by an investment at varying interest rates by graphing the interest earned at varying rates on the one set of axes.

WORKED Example 7

Kylie has $12 000 to invest. Three different banks offer interest rates of 4%, 5% and 6%.

a Complete the table below to show the interest that she could earn over 5 years.

No. of years	1	2	3	4	5
Interest (4%)					
Interest (5%)					
Interest (6%)					

b Show this information in graph form.

THINK

a 1 Use the simple interest formula to calculate the interest earned on $12 000 at 4% p.a. for 1, 2, 3, 4 and 5 years.

2 Use the simple interest formula to calculate the interest earned on $12 000 at 5% p.a. for 1, 2, 3, 4 and 5 years.

3 Use the simple interest formula to calculate the interest earned on $12 000 at 6% p.a. for 1, 2, 3, 4 and 5 years.

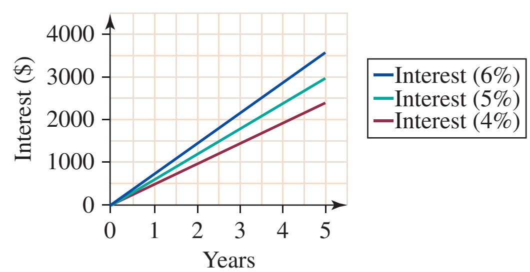
b Draw a line graph for each investment.

WRITE

a

No. of years	1	2	3	4	5
Interest (4%)	$480	$960	$1440	$1920	$2400
Interest (5%)	$600	$1200	$1800	$2400	$3000
Interest (6%)	$720	$1440	$2160	$2880	$3600

b



A graphics calculator greatly shortens the work in calculations involving formulas. Graphs can also be produced displaying the results of tables of values produced from the formulas. The previous worked example can be solved quickly using these capabilities. A variety of approaches can be used. One solution is shown in the next worked example.

WORKED Example 8

Use a graphics calculator to answer the question in Worked example 7.

THINK

For the Casio *fx*-9860G AU

a 1 To create a table, press:
- MENU
- 7: TABLE.

Enter the formulas as:

Y1 = 12 000 · 4 · X | 100

Y2 = 12 000 · 5 · X | 100

Y3 = 12 000 · 6 · X | 100.

Press EXE after each line.

WRITE/DISPLAY

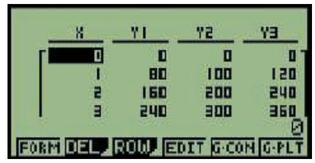
a

Continued over page

THINK | **WRITE/DISPLAY**

2 To set the range, press F5 (SET). Enter the values as shown.

3 To see the table of values, press:
- EXE
- F6 (TABL).

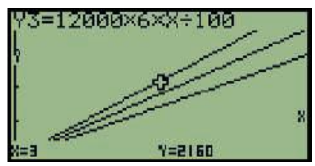


b 1 To adjust the window settings, press:
- SHIFT
- F3 (V-WIN).

Enter the values as shown and then press EXE.

b

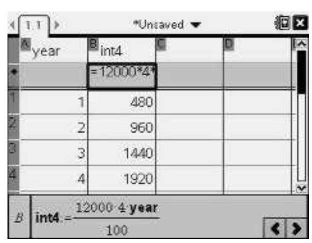
2 To graph the table, press F5 (G-CON).
To trace points on the graphs, press:
- SHIFT
- F1 (TRCE).



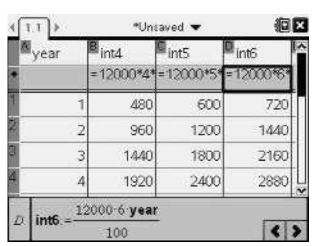
For the TI-Nspire CAS

a 1 Open a Lists & Spreadsheet screen.
Using the NavPad ▲ to highlight the cell, label Column A as 'year'. Enter the values 1 to 5 in this column.
Label Column B as 'int4'. In the header cell below, complete the entry line as:
= 12 000 · 4 · year | 100.
Then press ENTER.

a



2 Label Column C as 'int5'. In the header cell below, complete the entry line as:
= 12 000 · 5 · year | 100.
Then press ENTER.
Label Column D as 'int6'. In the header cell below, complete the entry line as:
= 12 000 · 6 · year | 100.
Then press ENTER.



THINK

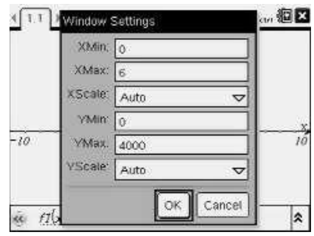
b ① To show this information in graph form, open a Graphs page.
For appropriate window settings for the graph, press:
- MENU (menu)
- 4: Window/Zoom (4)
- 1: Window Settings (1).

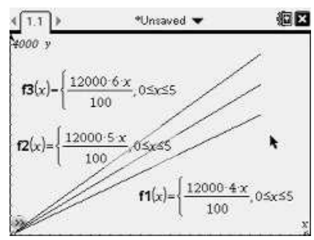
Adjust the window settings as shown, pressing Tab (tab) to move between fields.
Press ENTER (enter).

② Complete the entry lines as:
$f1(x) = 12\,000 \cdot 4 \cdot x \mid 100 \mid 0 \le x \le 5$
$f2(x) = 12\,000 \cdot 5 \cdot x \mid 100 \mid 0 \le x \le 5$
$f3(x) = 12\,000 \cdot 6 \cdot x \mid 100 \mid 0 \le x \le 5.$
Press ENTER (enter) after each line.

WRITE/DISPLAY

b





EXERCISE 1C Graphing simple interest functions

WORKED Example 6, 8

1 An amount of $8000 is invested at 5% p.a.

a Copy and complete the table below to calculate the interest over 5 years.

No. of years	1	2	3	4	5
Interest					

b Draw a graph of the interest earned against the length of the investment.

2 Savings of $20 000 are to be invested at 8% p.a.

a Copy and complete the table below to calculate the interest for various lengths of time.

No. of years	1	2	3	4	5
Interest					

b Draw a graph of the interest earned against the length of the investment.

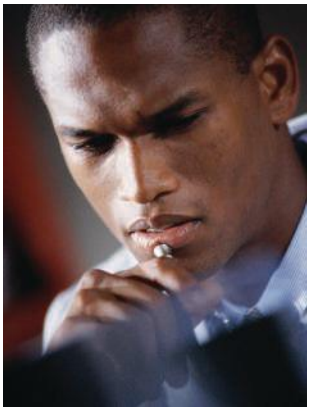
c What is the gradient of the linear graph drawn?

d Use your graph to find the amount of interest that would have been earned after 10 years.

3 Draw a graph to represent the interest earned by each of the following investments over 5 years.
 a $15 000 at 7% p.a.
 b $2000 at 10% p.a.
 c $8600 at 7.5% p.a.
 d $50 000 at 8.2% p.a.

4 A graph is drawn to show the interest earned on $6000 at 4.8% p.a. for various lengths of time. Without drawing the graph, state the gradient.

5 Darren invests $3200 at 2.5% p.a. for 5 years.
 a Graph the amount of interest that Darren would have earned at the end of each year for the 5 years.
 b Graph the total value of Darren's investment at the end of each year.



WORKED Example 7, 8

6 Julian has $25 000 to invest at 5% p.a., 6% p.a. or 8% p.a.
 a Complete the table below to show the interest that he would earn over 5 years.

No. of years	1	2	3	4	5
Interest (5%)					
Interest (6%)					
Interest (8%)					

 b Show this information in graph form.

7 Mark has $5500 to invest at 3% p.a., 3.5% p.a. or 3.75% p.a.
 a Complete the table below to show the interest that he would earn over various lengths of time.

No. of years	1	2	3	4	5
Interest (3%)					
Interest (3.5%)					
Interest (3.75%)					

 b Show this information in graph form.

8 Draw a graph to show the interest earned on an investment of $12 500 at 4.5% p.a., 5% p.a. and 5.2% p.a. Use the graph to find:
 a the amount of interest earned by each investment after 8 years
 b how much more the investment at 5.2% p.a. is worth after 10 years than the 4.5% p.a. investment.

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9 Three banks offer $4000 debentures at rates of 5.2% p.a., 5.8% p.a. and 6.2% p.a. Draw a graph of the value of the debentures at maturity against the number of years of the debenture.

10 QUICK QUESTIONS 1

Find the simple interest on each of the following investments.

1 \$4000 at 5% p.a. for 4 years

2 \$9000 at 7% p.a. for 2 years

3 \$15 000 at 6% p.a. for 3 years

4 \$950 at 0.1% p.a. for 2 years

5 \$40 000 at 3.5% p.a. for 5 years

6 \$1200 at 4.6% p.a. for $2\frac{1}{2}$ years

7 \$5745 at $3\frac{3}{4}$% p.a. for 1 year

8 \$32 500 at 4.1% p.a. for 18 months

9 \$532 at 0.2% p.a. for 6 months

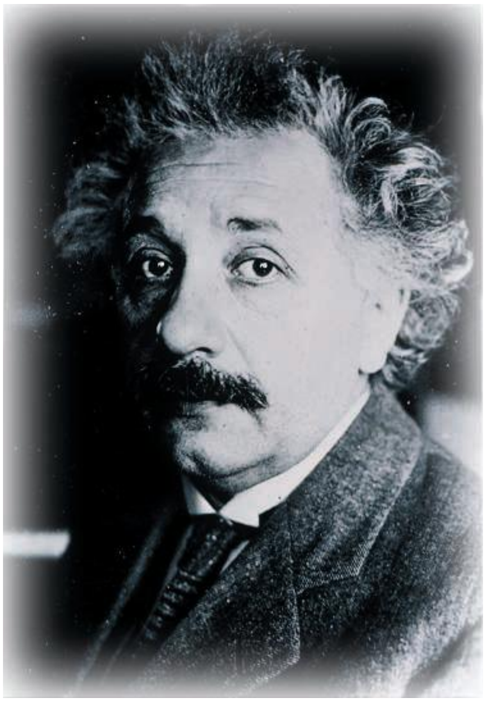
10 \$3330 at 6.95% p.a. for 9 months

Calculation of compound interest

eBook*plus*

Interactivity:
Compound interest
int-0193

The effect of compounding interest (which oil billionaire J. P. Getty called the 'eighth wonder in the world' and theoretical physicist Albert Einstein [at right] described as 'the driving force of the Universe') is a secret of financial wealth creation.



As mentioned previously, when we are dealing with simple interest, the interest is the same for each time period.

In calculating **compound interest**, *the principal on which interest is calculated is increased by adding (or reinvesting) the interest at the end of each interest period during the term*. Therefore, interest is calculated on the principal as well as the interest over each time period. This is the process by which our savings accounts earn interest. Let us consider an amount of \$1000, to be invested for a period of 5 years at an interest rate of 10% p.a. We will compare the interest earned using simple interest and compound interest.

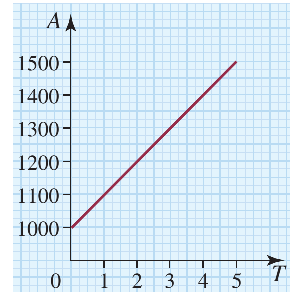
Simple interest	Compound interest
Initial principal, $P = \$1000$	Initial principal, $P = \$1000$
Rate of interest, $R = 10\%$	Rate of interest, $R = 10\%$
Interest for Year 1 10% of \$1000 $I_1 = \$100$	Interest for Year 1 10% of \$1000 $I_1 = \$100$
Principal at the beginning of Year 2 $P_2 = \$1000$	Principal at the beginning of Year 2 $P_2 = \$1000 + \$100$ $= \$1100$
Interest for Year 2 10% of \$1000 $I_2 = \$100$	Interest for Year 2 10% of \$1100 $I_2 = \$110$
Principal at the beginning of Year 3 $P_3 = \$1000$	Principal at the beginning of Year 3 $P_3 = \$1100 + \$110$ $= \$1210$
Interest for Year 3 10% of \$1000 $I_3 = \$100$	Interest for Year 3 10% of \$1210 $I_3 = \$121$
Principal at the beginning of Year 4 $P_4 = \$1000$	Principal at the beginning of Year 4 $P_4 = \$1210 + \$121$ $= \$1331$
Interest for Year 4 10% of \$1000 $I_4 = \$100$	Interest for Year 4 10% of \$1331 $I_4 = \$133.10$
Principal at the beginning of Year 5 $P_5 = \$1000$	Principal at the beginning of Year 5 $P_5 = \$1331 + \$133.10$ $= \$1464.10$
Interest for Year 5 10% of \$1000 $I_5 = \$100$	Interest for Year 5 10% of \$1464.10 $I_5 = \$146.41$
The simple interest earned over a 5-year period is \$500.	The compound interest earned over a 5-year period is \$610.51.

This table has illustrated how, as the principal for compound interest increases periodically, that is, P = \$1000, \$1100, \$1210, \$1331, \$1464.10 . . ., so does the interest, that is, I = \$100, \$110, \$121, \$133.10, \$146.41 . . ., while both the principal for simple interest and the interest earned remain constant, that is, P = \$1000 and I = \$100. The difference of \$110.51 between the compound interest and simple interest earned in a 5-year period represents the interest earned on added interest.

If we were to place the set of data obtained in two separate tables and represent each set graphically — as the total amount of the investment, A, versus the year of the investment, T — we would find that the simple interest investment grows at a constant rate while the compound interest investment grows exponentially.

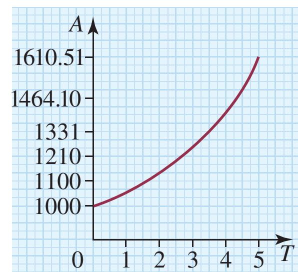
1. The simple interest investment is represented by a straight line as shown at right.

T	A
0	1000
1	1100
2	1200
3	1300
4	1400
5	1500



2. The compound interest investment is represented by a curve as shown at right.

T	A
0	1000
1	1100
2	1210
3	1331
4	1464.10
5	1610.51



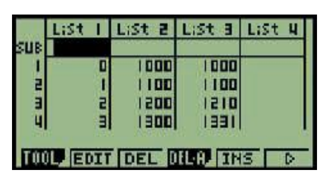
Graphics Calculator tip! Comparing investments

The above data may be entered into a graphics calculator and used to plot each of the graphs.

For the Casio *fx*-9860G AU

1. To enter the data, press:
   - MENU
   - 2: STAT.

   Enter the time in List 1, the simple interest investment amounts in List 2, and the compound interest investment amounts in List 3. Press EXE after each entry.



2. To set the appropriate window settings, press:
   - SHIFT
   - F3 (V-WIN).

   Enter the values as shown and then press EXE.

   Note: The y-scale of 100 is missing from the bottom of the screen.

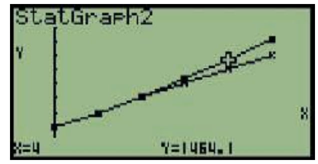
3. To set the type of graph, press:
   - F1 (GRPH)
   - F6 (SET).

   Set the StatGraph1 fields as shown.

4. Set StatGraph2 with XList: List1 and YList: List3. Press EXE when completed.

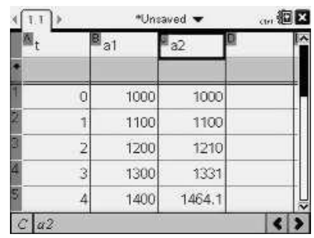
5. To select the graph, press F4 (SEL). Turn on Statgraph1 and StatGraph2. Ensure that Statgraph3 is turned off.

6. To draw the two graphs, press F6 (DRAW). To compare values of the investments, press:
- SHIFT
- F1 (TRCE).

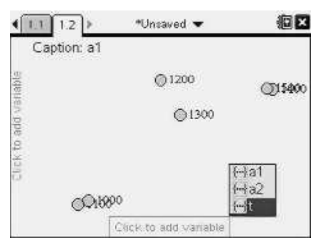


For the TI-Nspire CAS

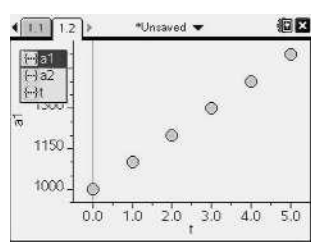
1. Open a Lists & Spreadsheet page. Label column A as 't', column B as 'a1' and column C as 'a2'. Enter the data in the appropriate columns.



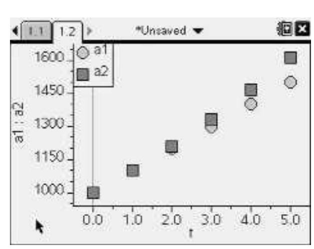
2. To draw scatterplots of this data, open a Data & Statistics page. Press Tab (tab) and select 't' as the x-variable, then press ENTER (enter). Press Tab (tab) again to move into the next field and select 'a1' as the y-variable, then press ENTER (enter).



3. To plot the second scatterplot, press:
- MENU (menu)
- 2: Plot Properties (2)
- 7: Add Y Variable (7).



4. Select 'a2' as the y-variable, and press ENTER (enter). The two scatterplots will appear on the same set of axes.



Formula for compound interest

We will now derive a mathematical relationship for compound interest. As seen from the previous example, the initial amount:

$$A_0 = P$$
$$= \$10\,000$$

After 1 year $\quad A_1 = 10\,000 \cdot 1.1 \quad$ (increasing \$10 000 by 10%)

After 2 years $\quad A_2 = A_1 \cdot (1.1)$

$$= 10\,000 \cdot 1.1 \cdot 1.1 \quad \text{(substituting the value of } A_1\text{)}$$
$$= 10\,000 \cdot 1.1^2$$
$$= 10\,000(1 + \tfrac{10}{100})^2$$

After 3 years $\quad A_3 = A_2 \cdot 1.1$

$$= 10\,000 \cdot 1.1^2 \cdot 1.1$$
$$= 10\,000 \cdot 1.1^3$$
$$= 10\,000(1 + \tfrac{10}{100})^3$$

The pattern then continues such that the value of the investment after T years equals:

$$\$10\,000 \cdot 1.1^T = 10\,000(1 + \tfrac{10}{100})^T$$

We can generalise this example to any investment with yearly compound interest calculations.

$$A = P\left(1 + \frac{R}{100}\right)^T$$

This formula, however, assumes that the interest is compounded yearly (which frequently is not the case). If the interest rate is R% p.a. and interest is compounded quarterly, then the interest rate per period is $R\% \mid 4$. To account for the number of compounding periods per year (n), we must divide the interest rate by the number of periods per year and also adjust the number of terms over which the interest is calculated. If the investment is over T years and the calculation is performed 'n' times per year then the number of terms is $n \cdot T$.

This produces the following general formula for compound interest.

$$A = P\left(1 + \frac{R}{100 \cdot n}\right)^{n \cdot T}$$

where A = amount (or **future value**) of the investment, \$
P = principal (or **present value**), \$
R = interest rate p.a., %
n = number of compounding periods (or rests) per year
T = time of investment, years.

Note: The compound interest formula derives a value for the final amount or future value. To find the compound interest accrued, it is necessary to subtract the original principal from the final amount.

So $\quad CI = A - P$

where CI = compound interest, \$.

In the financial world, the terms future value (FV) or **compounded value** (CV) and present value (PV) are used instead of amount and principal. Hence, the compound interest formula becomes:

$$FV \text{ or } CV = PV\left(1 + \frac{R}{100 \cdot n}\right)^{n \cdot T}$$

WORKED Example 9

Calculate the future value of an investment of $12 000 at 7% p.a. for 5 years, where interest is compounded annually.

THINK	WRITE
1 Write down the formula for the future value.	$FV = PV\left(1 + \frac{R}{100 \cdot n}\right)^{n \cdot T}$
2 Write down the value of *PV*, *R*, *T* and *n*.	$PV = \$12\,000$, $R = 7$, $T = 5$, $n = 1$
3 Substitute into the formula.	$FV = 12\,000\left(1 + \frac{7}{100 \cdot 1}\right)^{1 \cdot 5}$
	$= 12\,000\left(1 + \frac{7}{100}\right)^{5}$
4 Calculate.	$= \$16\,830.62$

Consider an investment of $6000 at 8% p.a. for 2 years with interest compounded quarterly. Interest is paid four times per year and therefore eight times in 2 years. The interest rate must be calculated per quarter. This is done by dividing the annual rate by four. Therefore, in the following example the rate is 2% per quarter.

WORKED Example 10

Calculate the future value of an investment of $6000 at 8% p.a. for 2 years with interest compounded quarterly.

THINK	WRITE
1 Write down the formula for the future value.	$FV = PV\left(1 + \frac{R}{100 \cdot n}\right)^{n \cdot T}$
2 Write down the value of *PV*, *R*, *T* and *n*.	$PV = \$6000$, $R = 8$, $T = 2$, $n = 4$
3 Substitute into the formula.	$FV = 6000\left(1 + \frac{8}{100 \cdot 4}\right)^{4 \cdot 2}$
	$= 6000\left(1 + \frac{8}{400}\right)^{8}$
4 Calculate.	$= \$7029.96$

Graphics Calculator tip!

Financial calculations

The financial calculations sections of later model Casio and TI-Nspire CAS calculators offer built-in formulas to facilitate compound interest calculations. You are advised to consult your calculator manual to become familiar with this section. Worked examples 9 and 10 can be used to demonstrate this facility.

Step 1 Enter the financial calculations section of your calculator (select TVM from the Casio menu; in a calculator screen on the TI-Nspire press MENU, Finance, Finance Solver).

Step 2 The function key (F2) (Compound Interest) accesses the compound interest formula on the Casio, while the Finance Solver provides this facility on the TI-Nspire CAS. The variables displayed are explained below.

n represents the number of compounding periods.

I% is the annual interest rate.

PV is the present value.

PMT is the payment for each instalment (used for regular periodic payments).

FV is the future value.

P/Y or *PpY* represents the number of instalment periods per year.

C/Y or *CpY* represents the number of compounding periods per year.

WORKED Example 11

Use a graphics calculator to calculate the future value after 5 years at 7% p.a. on an investment with the present value of $12 000, where no regular payments are made, there is one instalment period a year and interest is calculated once a year.

THINK — **WRITE/DISPLAY**

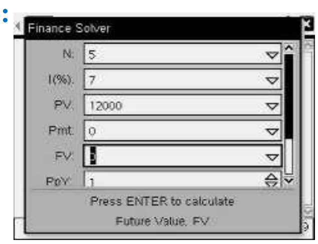
1 Enter the following data (in the Compound Interest screen for the Casio; in the Finance Solver for the TI-Nspire):

$n = 5$	(Interest is calculated for 5 years.)
$I\% = 7$	(Annual interest rate of 7% p.a.)
$PV = 12\,000$	(Present value is $12 000.)
$PMT = 0$	(No regular payments are made.)
$FV = 0$	(This value will be calculated.)
P/Y or $PpY = 1$	(One instalment period per year.)
C/Y or $CpY = 1$	(Interest is calculated once a year.)

Casio:

(Scroll down for C/Y.)

TI-Nspire:

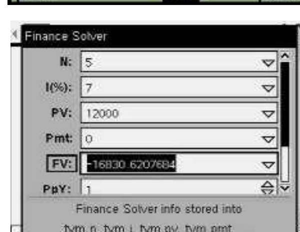


2 The function key (F5) (FV) on the Casio displays the future value as $16 830.62.

Casio:

3 To access the *FV* on the TI-Nspire, tab the cursor to the 'FV' line then press (enter).

TI-Nspire:



4 Write the answer. — Future value is $16 830.60.

WORKED Example 12

Use the financial calculations section of a Casio or TI-Nspire graphics calculator to calculate the future value after 2 years at 8% p.a. on an investment with the present value of $6000, where no regular payments are made, there is one instalment period a year and interest is calculated 4 times a year.

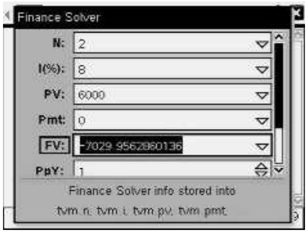
THINK

1. Enter the following data:
   $n = 2$ (Interest is calculated for 2 years.)
   $I\% = 8$ (Annual interest rate of 8% p.a.)
   $PV = 6000$ (Present value is $6000.)
   $PMT = 0$ (No regular payments are made.)
   $FV = 0$ (This value will be calculated.)
   P/Y or $PpY = 1$ (One instalment period per year.)
   C/Y or $CpY = 4$ (Interest is calculated 4 times a year.)
2. The FV line displays the future value as $7029.96.
3. Write the answer.

WRITE/DISPLAY

Casio:

TI-Nspire:



Future value would be $7030.

WORKED Example 13

Ninkasi has $15 000 to invest for 3 years.
She considers the following options:
- **a** **a term deposit at 5.25% p.a. compounded annually**
- **b** **a debenture in a company, paying an interest rate of 5.08% p.a. compounded quarterly**
- **c** **a building society, paying a return of 5.4% p.a. compounded monthly**
- **d** **a business venture with guaranteed return of 7.3% p.a. compounded daily.**

All the investments are equally secure.
Advise Ninkasi which option to take.

THINK

a 1. Write down the formula for compound interest.
2. Write down the known values of the variables.
3. Substitute the values into the given formula.
4. Evaluate.
5. Answer the question and include the appropriate unit.

WRITE

a $A = P\left(1 + \dfrac{R}{100 \cdot n}\right)^{n \cdot T}$

$P = \$15\,000$
$R = 5.25$
$T = 3$
$n = 1$

$A = 15\,000\left(1 + \dfrac{5.25}{100 \cdot 1}\right)^{1 \cdot 3}$
$= 15\,000(1.0525)^3$
$= 17\,488.7018$

The compounded amount is $17 488.70.

THINK	WRITE
b 1 Repeat steps 1 and 2 performed in part **a**. *Note:* The interest rate is given as per annum; however, we need to know the rate per quarter, hence we use n as 4. The interest is compounded quarterly; that is, 4 times in one year. Therefore in 3 years, interest will be compounded 12 times.	**b** $A = P\left(1 + \frac{R}{100 \cdot n}\right)^{n \cdot T}$ $P = \$15\,000$ $R = 5.08$ $T = 3$ $n = 4$
2 Substitute the values into the given formula.	$A = 15\,000\left(1 + \frac{5.08}{100 \cdot 4}\right)^{4 \cdot 3}$
3 Evaluate.	$= 15\,000(1.0127)^{12}$ $= 17\,452.633\,91$
4 Answer the question and include the appropriate unit.	The compounded amount is \$17 452.63.
c 1 Repeat steps 1 and 2 performed in part **a**. *Note:* The interest rate is given as per annum; however, we need to know the rate per month, hence we use n as 12. The interest is compounded monthly; that is, 12 times in one year. Therefore in 3 years, interest will be compounded 36 times.	**c** $A = P\left(1 + \frac{R}{100 \cdot n}\right)^{n \cdot T}$ $P = \$15\,000$ $R = 5.4$ $T = 3$ $n = 12$
2 Substitute the values into the given formula.	$A = 15\,000\left(1 + \frac{5.4}{100 \cdot 12}\right)^{12 \cdot 3}$
3 Evaluate.	$= 15\,000(1.0045)^{36}$ $= 17\,631.494\,99$
4 Answer the question and include the appropriate unit.	The compounded amount is \$17 631.49.
d 1 Repeat steps 1 and 2 performed in part **a**. *Note:* The interest rate is given as per annum; however, we need to know the rate per day, hence we use n as 365. The interest is compounded daily; that is, 365 times in one year. Therefore in 3 years, interest will be compounded 1095 times.	**d** $A = P\left(1 + \frac{R}{100 \cdot n}\right)^{n \cdot T}$ $P = \$15\,000$ $R = 7.3$ $T = 3$ $n = 365$
2 Substitute the values into the given formula.	$A = 15\,000\left(1 + \frac{7.3}{100 \cdot 365}\right)^{365 \cdot 3}$
3 Evaluate.	$= 15\,000(1.0002)^{1095}$ $= 18\,672.060\,28$
4 Answer the question and include the appropriate unit.	The compounded amount is \$18 672.06.
5 Compare answers from options **a** to **d** and answer the question.	The best option for Tamara is **d**, as she receives the greatest amount of interest in the same period of time.

remember

1. The future value of an investment under compound interest can be calculated by calculating the simple interest for each year separately.
2. The formula used to calculate compound interest is:

$$A = P\left(1 + \frac{R}{100 \cdot n}\right)^{n \cdot T}$$

where A = final amount, \$
P = principal, \$
R = interest rate, % p.a.
n = number of compounding periods per year
T = time, years.

3. The compound interest is calculated by subtracting the principal from the final amount.
4. In financial terms, the compound interest formula may be written:

$$FV \text{ or } CV = PV\left(1 + \frac{R}{100 \cdot n}\right)^{n \cdot T}$$

where FV = future value, \$
CV = compounded value, \$
PV = present value, \$

5. A graphics calculator is an aid in these calculations.

EXERCISE 1D Calculation of compound interest

WORKED Example 9, 11

1 Ray has \$5000 to invest. He invests it for 3 years at 10% p.a., with interest paid annually. Calculate the future value of the investment.

2 Suzanne is to invest \$15 000 for 2 years at 7% p.a., with interest paid annually. Calculate the future value of the investment.

3 Kiri has \$2000 to invest. She invests the money at 8% p.a. for 5 years, with interest compounded annually. Calculate the future value of Kiri's investment.

4 Use the formula to calculate the future value of each of the following investments with interest compound annually.
- **a** \$4000 at 5% p.a. for 3 years
- **b** \$8000 at 3% p.a. for 5 years
- **c** \$18 000 at 8% p.a. for 4 years
- **d** \$11 500 at 5.5% p.a. for 3 years
- **e** \$8750 at 6.25% p.a. for 6 years

WORKED Example 10, 12

5 Carla is to invest \$45 000 at 9.2% p.a. for 5 years with interest compounded six-monthly. Calculate the future value of the investment.

6 A passbook savings account pays interest of 0.2% p.a. Luke has \$500 in such an account. Calculate the future value of the account after 2 years, if interest is compounded quarterly.

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201 Compound interest

7 Noel is to invest $12 000 at 8% p.a. for 2 years with interest compounded quarterly. Calculate the future value of the investment.

8 Vicky invests $30 000 in a one-year fixed deposit at an interest rate of 6% p.a., with interest compounding monthly. Calculate the future value of the investment upon maturity.

9 Calculate the compounded value of each of the following investments.
   a $960 for 1 year at 4.50% p.a. with interest compounded six-monthly
   b $7500 for $3\frac{1}{2}$ years at 5.6% p.a. with interest compounded quarterly
   c $152 000 for $2\frac{1}{2}$ years at 7.2% p.a. with interest compounded six-monthly
   d $14 000 for 4 years at 9% p.a. with interest compounded monthly
   e $120 000 for 20 years at 11.95% p.a. with interest compounded quarterly

10 **multiple choice**

A sum of $5000 is invested for 2 years at the rate of 4.75% p.a., compounded quarterly. The interest paid on this investment, to the nearest dollar, is:
A $475 B $495 C $1900 D $5475 E $5495

11 **multiple choice**

After selling their house Mr and Mrs Dengate have $61 800. They plan to invest it at 6% p.a., with interest compounded annually. The value of their investment will first exceed $100 000 after:
A 7 years B 8 years C 9 years D 10 years E 11 years

WORKED Example 13

12 **multiple choice**

Warren wishes to invest $10 000 for a period of 5 years. The following investment alternatives are suggested to him. The best investment would be:
A simple interest at 9% p.a.
B compound interest at 8% p.a. with interest compounded annually
C compound interest at 7.8% p.a. with interest compounded six-monthly
D compound interest at 7.2% p.a. with interest compounded quarterly
E compound interest at 7.7% p.a. with interest compounded daily.

13 **multiple choice**

Which of the following investments, to be invested for 6 years and compounded semi-annually at 8% p.a., will have a future value closest to $15 000?
A $900 B $8500 C $9000 D $9500 E $10 000

14 Brittany has $13 500 to invest. An investment over a 2-year term will pay interest of 8% p.a. Calculate the compounded value of Brittany's investment if the compounding period is:
   a one year b six months c three months d monthly.

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066 Simple & compound interest

15 Kerry invests $100 000 at 8% p.a. for a one-year term. For such large investments interest is compounded daily.
   a Calculate the daily percentage interest rate, correct to 4 decimal places.
   b Calculate the compounded value of Kerry's investment on maturity.
   c Calculate the amount of interest paid on this investment.
   d Calculate the extra amount of interest earned, compared with the interest calculated at a simple interest rate.

16 Simon invests $4000 for 3 years at 6% p.a. simple interest. Monica also invests $4000 for 3 years but her interest rate is 5.6% p.a., with interest compounded quarterly.

a Calculate the value of Simon's investment on maturity.

b Show that the compounded value of Monica's investment is greater than Simon's investment.

c Explain why Monica's investment is worth more than Simon's, despite receiving a lower rate of interest.

Compound interest spreadsheet

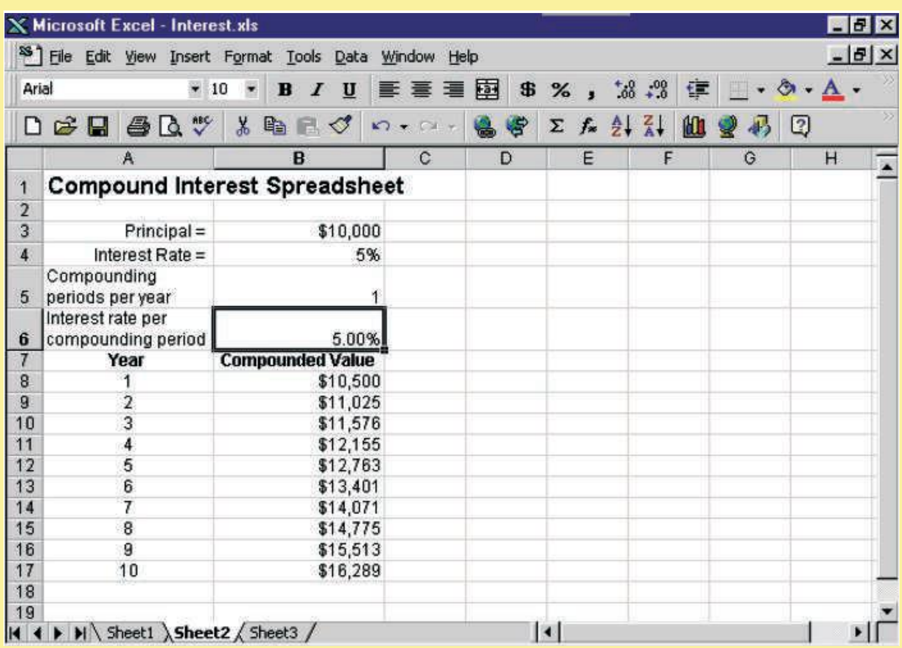
investigation

Earlier we wrote a spreadsheet to show the growth of an investment over a number of years under simple interest. We will now write a similar spreadsheet to show the growth under compound interest.

eBook*plus*

Digital doc:
Spreadsheet
201 Compound interest

1 Use the **Compound interest** weblink in your eBookPLUS and download the spreadsheet.



	A	B
1	Compound Interest Spreadsheet	
2		
3	Principal =	$10,000
4	Interest Rate =	5%
5	Compounding periods per year	1
6	Interest rate per compounding period	5.00%
7	Year	Compounded Value
8	1	$10,500
9	2	$11,025
10	3	$11,576
11	4	$12,155
12	5	$12,763
13	6	$13,401
14	7	$14,071
15	8	$14,775
16	9	$15,513
17	10	$16,289
18		
19		

2 Select Sheet 2, 'Compound Interest'. This spreadsheet models a $10 000 investment at 5% p.a. interest with interest compounded annually (one compounding period per year).

3 Use the graphing function to draw a graph showing the growth of this investment over 10 years. Compare this graph with the graph drawn for the corresponding simple interest investment.

4 Change the number of compounding periods per year to see the change in the value of the investment. Your graph should change as you change the information.
5 Change other information, such as the principal and interest rate, to see the change in your graph.
6 Move your cursor throughout the spreadsheet and observe the formulas in the cells.
7 Save this spreadsheet as Compound Interest.

To investigate calculation of compound interest from a table of compounded values, use the **Compound interest** weblink in your eBookPLUS'.

eBook*plus*

Digital doc:
Spreadsheet
201 Compound interest

Graphing compound interest functions

Earlier we drew graphs of the simple interest earned by various simple interest investments and found that these graphs were linear. This occurred because the amount of interest earned in each interest period was the same.

With compound interest the interest earned in each interest period increases, and so when we graph the future value of the investment, an exponential graph results. The shape of the graph is a smooth curve which gets progressively steeper. We can use the compound interest formula to complete tables that will then allow us to graph a compound interest function.

WORKED Example 14

Pierre invests $5000 at 5% p.a., with interest compounded annually.

a Complete the table below to show the future value at the end of each year.

No. of years	0	1	2	3	4	5
Future value						

b Draw a graph of the future value of the investment against time.

Continued over page

THINK

a Complete the table using the compound interest formula.

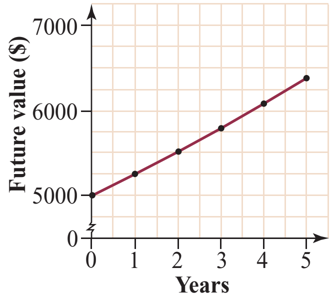
b Draw the graph by drawing a smooth curve between the marked points.

WRITE

a

No. of years	0	1	2	3	4	5
Future value	$5000	$5250	$5512	$5788	$6077	$6381

b



To graph the interest earned, the principal must be subtracted from the future value of the investment. As with simple interest, such graphs can be used to compare investments.

WORKED Example 15

Amy is to invest $2000 at 5% p.a., 6% p.a. or 7% p.a., compounded annually.

a Copy and complete the table at right to find the future value of each investment at the end of each year.

No. of years	1	2	3	4	5
Future value (5%)					
Future value (6%)					
Future value (7%)					

b Draw a graph that will allow the investments to be compared.

THINK

a Use the compound interest formula to complete the table.

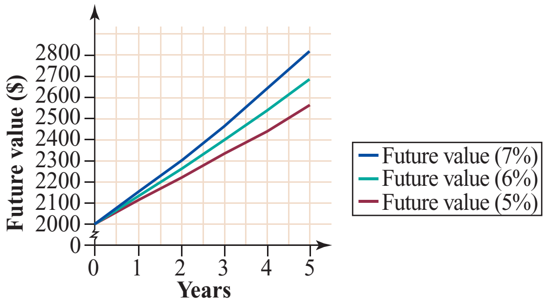
b Draw each graph by joining the points with a smooth curve.

WRITE

a

No. of years	1	2	3	4	5
Future value (5%)	$2100	$2205	$2315	$2431	$2553
Future value (6%)	$2120	$2247	$2382	$2525	$2676
Future value (7%)	$2140	$2290	$2450	$2622	$2805

b



Graphics calculators can speed up these calculations as shown in the following worked example.

WORKED Example 16

Solve Worked example 15 using a graphics calculator.

THINK | **WRITE/DISPLAY**

Casio *fx*-9860G AU

a ① To create a table, press:
- MENU
- 7: TABLE.

Enter the formulas as:
Y1 = 2000(1 + 5 / 100)^X
Y2 = 2000(1 + 6 / 100)^X
Y3 = 2000(1 + 7 / 100)^X.
Where X represents the time.

a

② To define the range of X, press F5 (SET).
To return to the equations screen, press EXE.

③ To display the table, press F6 (TABL).

b ① To return to the equations screen, press EXIT.
To set appropriate window settings, press:
- SHIFT
- F3 (V-WIN).

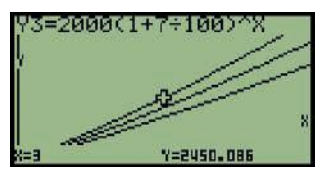
Enter the fields as shown, and then press EXE.

Note: The scale of the *y*-axis is set as 500.

b

② To display the three graphs, press:
- F6 (TABL)
- F5 (G-CON).

To trace points on the graphs, press:
- SHIFT
- F1 (TRCE).

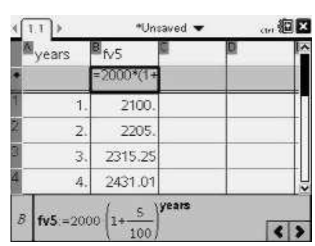


For the TI-Nspire CAS

a ① Open a Lists & Spreadsheet page.
Label Column A as 'years'.
Enter the values 1 to 5 in this column.

② Label Column B as 'fv5'. In the header cell below, complete the entry line as:
= 2000 (1 + 5 / 100)^years.
Then press ENTER.

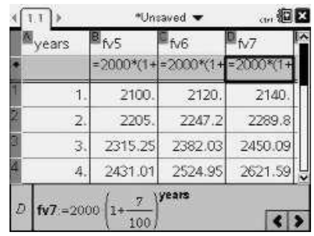
a



Continued over page

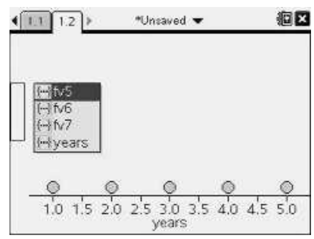
THINK | **WRITE/DISPLAY**

3 Label Column C as 'fv6'. In the header cell below, complete the entry line as:
= 2000 (1 + 6 | 100)^years.
Then press ENTER (enter).
Label Column D as 'fv7'. In the header cell below, complete the entry line as:
= 2000 (1 + 7 | 100)^years.
Then press ENTER (enter).



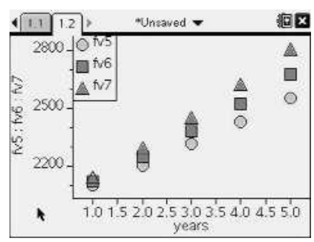
b 1 To draw a scatterplot, open a Data & Statistics page.
Press Tab (tab) to select 'years' as the *x*-variable.
Press Tab (tab) again and select 'fv5' as the *y*-variable, then press ENTER (enter).

b



2 To plot the second scatterplot, press:
- MENU (menu)
- 2: Plot Properties (2)
- 7: Add Y Variable (7).

Select 'fv6' and press ENTER (enter). Repeat the above steps and select 'fv7' to plot the last scatterplot.



investigation: Doubling your money

In financial circles a formula is commonly used to determine the length of time it takes to double a sum of money invested at a compound interest rate of *R*%.

$$\text{Number of years to double money} = \frac{70}{R}$$

Let's investigate the accuracy of this statement.

Task 1

Consider a $10 000 sum of money invested at an interest rate of 10% p.a. compounding monthly. If the above equation is accurate, the number of years required to double this money should be 70 divided by 10; i.e. 7 years. Apply the compound interest formula for a period of 7 years. You should find that the $10 000 amounts to a sum of $20 079.20 (roughly double the original amount).

Copy and complete the following table.

Investment	Interest rate	Compounding period	Years	Amount
\$10 000	10% p.a.	Yearly	7	
\$10 000	10% p.a.	6-monthly	7	
\$10 000	10% p.a.	Quarterly	7	
\$10 000	10% p.a.	Monthly	7	\$20 079.20
\$10 000	10% p.a.	Weekly	7	

What do you conclude about the accuracy of the statement from these calculations?

Task 2

Draw up a table similar to the previous one and vary the investment quantities, the interest rate and the compounding period. Calculate the number of years required for the investment to double from the formula above. Add these figures to the table. Complete the table by calculating the amount in each case.
What conclusion can you draw from the results of your calculations?

Task 3

1 We can use a graphics calculator to determine the time required for a sum of money to double in value.
Let's consider our original investment of \$10 000 at 10% p.a. compounding monthly.

For the Casio *fx*-9860G AU

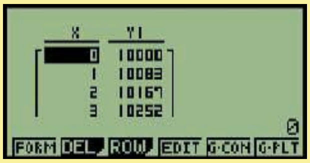
To create a table of values, press:

- MENU
- 7: TABLE.

Enter the formula.
Y1 = 10 000(1 + 10 ÷ 1200)^X.

To set the table with the appropriate domain, press F5 (SET) and enter the values as shown.

To view the table, press F6 (TABL).



To find how long it takes for the investment to reach \$20 000, scroll down the list of values.

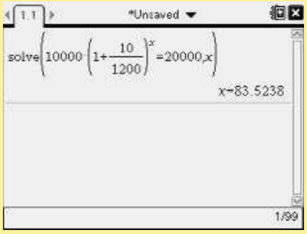
This shows that it takes 84 months (7 years) for the \$10 000 to amount to \$20 000.

(*Continued*)

For the TI-Nspire CAS

The formula is:

$$A = P\left(1 + \frac{R}{100 \cdot n}\right)^{n \cdot T}.$$



Where:

$A = 20\ 000$

$P = 10\ 000$

$R = 10$

$n = 12$

To find T, complete the entry line as:

$$\text{solve}\left(10\ 000 \cdot \left(1 + \frac{10}{1200}\right)^{x} = 20\ 000,\ x\right).$$

The time required to double the value of the investment is 84 months (7 years).

2 Repeat the calculations using a variety of investment values, interest rates and compounding periods.

3 Are your conclusions consistent with those of Tasks 1 and 2? How would you rate the accuracy of the statement?

Determining the interest rate in compound interest investments

The aim of this activity is to investigate some methods that could be used to determine the interest rate required for a given investment to achieve a particular final value in a given period of time. This activity is a challenging one, with only minimal guidance given.

Consider an investment of $1000 with interest compounding yearly. Let us assume that we can invest this money for 5 years and at the end of that time the money would amount to $1200. What interest rate would this require?

Using trial and error

1 Write down your compound interest formula.

2 Using the present value, the number of compounding periods per year and the time, try various values for the interest rate. Continue your trials until you have found an interest rate that produces a value equal to, or slightly greater than $1200 for the future value. Record your trials in a table.

3 What interest rate do you consider is necessary for this investment?

Solving the compound interest formula

1 Write down your compound interest formula.

2 Substitute values for the present value, future value, time and number of compounding periods per year. The only unknown in your equation is the interest rate.

3 Try to solve your equation to find a value for R.

Hint: The method involves taking the fifth root of $\dfrac{\text{future value}}{\text{present value}}$.

4 Compare your answer with that obtained using trial and error.

For the TI-Nspire CAS

1 Write an equation and use the solve command to find the value for the interest rate.

2 Compare your answer with those obtained previously.

Using the financial calculation section on a graphics calculator

1 Enter the financial section of the calculator.

2 In the compound interest section, enter values for all known variables. Your only unknown should be *I*% (which you set to 0).

3 Press the appropriate function key to reveal the unknown interest rate. (For the Casio, press (F2) (I%); for the TI-Nspire CAS calculator, place the cursor on 'I(%):0', and press (enter).) Is your answer consistent with those obtained above?

It is obvious from this activity, the manual calculation of an interest rate from the compound interest formula is quite tedious. A graphics calculator greatly reduces the workload.

Comparing simple and compound interest functions using graphics calculators

Task 1

Your grandfather left you \$20 000 in his will. You have no need to use the money at this stage, so you are looking at investing it for approximately 12 years. Your research has narrowed down your options to 4.25% p.a. simple interest or 3.6% p.a. interest compounding yearly. At this stage, you do not anticipate having to withdraw your money in the short term; however, it may be necessary to do so.

Let us investigate to determine which would be the better option if you were forced to withdraw your money at any period of time within 12 years.

When using graphics calculators, there are often different ways of approaching a problem, as we often find when solving problems using pen and paper. The Casio *fx*-9860G AU calculator and the TI-Nspire CAS calculator allow us to show two different methods which could be used to solve this problem.

For the Casio *fx*-9860G AU

1 To construct a table of values, press:
- (MENU)
- 7: TABLE.

Complete the entry lines as:

Y1 = 20 000 + 20 000 · 4.25 · X ∣ 100 (SI)

Y2 = 20 000(1 + 3.6 ∣ 100)^X. (CI)

2 To set the range of X, press (F5) (SET).

(*Continued*)

3 To view the contents of the table, press (F6) (TABL).

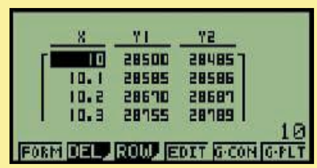
4 Use the down arrow to scroll down the table, comparing the values for Y1 and Y2.
At 11 years, the Y2 value becomes greater than the Y1 value.

5 To refine the time when this change occurs press:

- (EXIT)
- (F5) (SET).

Adjust the values in the Table Setting window as shown, pressing (EXE) after each entry.

6 To display the table, press (F6) (TABL).
It can be seen that the change occurs at 10.1.



7 To graph the functions, the window settings will need to be changed. Press:

- (SHIFT)
- (F3) (V-WIN).

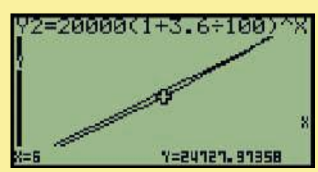
Enter the window settings as shown, and then press (EXE).
Press (F5) to reset the table settings to:
Start: 0 End: 12 Step: 1.

8 From the functions screen, press:

- (F6) (TABL)
- (F5) (G-CON).

To trace the graphs press:

- (SHIFT)
- (F1) (TRCE).



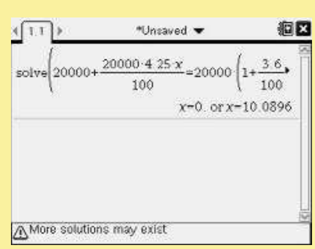
Confirm that this is consistent with the result from the table.

For the TI-Nspire CAS

1 To find an algebraic solution, it is necessary to find the intersection of the two simultaneous equations:

$FV = 20\,000 + \dfrac{20\,000 \cdot 4.25 \cdot x}{100}$ for simple interest

$FV = 20\,000\left(1 + \dfrac{3.6}{100}\right)^x$ for compound interest.



On a Calculator page, complete the entry line as:

$$\text{solve}\left(20\,000 + \frac{20\,000 \cdot 4.25 \cdot x}{100} = 20\,000\left(1 + \frac{3.6}{100}\right)^x, x\right).$$

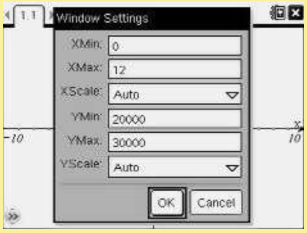
Simple and compound interest funds will have the same value after approximately 10.1 years. At this stage it is unknown which of the two funds provide the better option prior to this time. Looking at their graphs can determine this.

2 To draw the graphs, open a Graphs page. Complete the entry lines as:

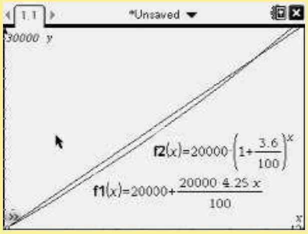
$$f1(x) = 20\,000 + \frac{20\,000 \cdot 4.25 \cdot x}{100}$$

$$f2(x) = 20\,000\left(1 + \frac{3.6}{100}\right)^x.$$

Press ENTER (enter) after each entry.
Adjust the window settings as shown.

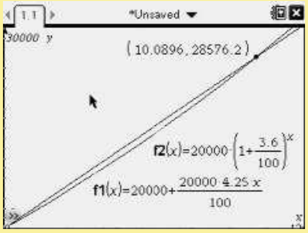


3 The graph shows that the simple interest investment (the straight line) is initially better than the compound interest investment (curved line). The two intersect at a point, after which the compound interest investment is better.



4 To find the point of intersection, press:
- MENU (menu)
- 7: Points & Lines (7)
- 3: Intersection Point(s) (3).

Move the cursor and press CLICK once on each graph. The point of intersection will appear.



After approximately 10.1 years, the two investments both amount to \$28 576.20.

5 Write a paragraph summarising the results of this investigation. Describe which option would be the better one considering that you may be forced to withdraw your money at any time within the 12 years. Justify any statements.

Task 2

If the investment sum had been \$10 000 instead of \$20 000 (half the original amount), would this affect the trends shown in Task 1? Investigate and write your conclusions.

Task 3

If the interest rates were half the quoted ones, (i.e. 2.125% p.a. simple interest and 1.8% p.a. compounding yearly), predict what effect this would have on the outcome of the investigation. Repeat Task 1 and write your conclusions.

investigation: Comparing simple and compound interest functions using a spreadsheet

Let us look at the previous investigation using a spreadsheet.

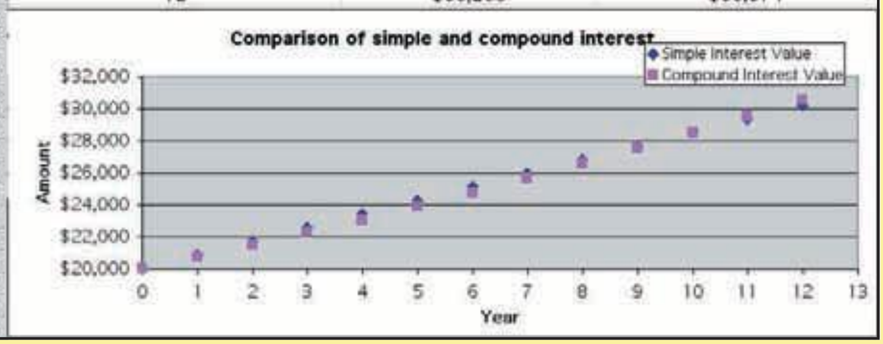
Task 1

Your grandfather left you $20 000 in his will. You have no need to use the money at this stage, so you are looking at investing it for approximately 12 years. Your research has narrowed down your options to 4.25% p.a. simple interest or 3.6% p.a. interest compounding yearly. At this stage, you do not anticipate having to withdraw your money in the short term; however, it may be necessary to do so.

Let us investigate to determine which would be the better option if you were forced to withdraw your money at any period of time within 12 years.

The spreadsheet and graphs we are aiming to produce appear as follows.

	A	B	C
1	Comparison of simple and compound interest		
2			
3	Principal=	$20,000	
4	Simple interest rate (%pa) =	4.25	
5	Compound interest rate (%pa) =	3.6	
6	Compounding periods per year =	1	
7			
8	Year	Simple Interest Value	Compound Interest Value
9	0	$20,000	$20,000
10	1	$20,850	$20,720
11	2	$21,700	$21,466
12	3	$22,550	$22,239
13	4	$23,400	$23,039
14	5	$24,250	$23,869
15	6	$25,100	$24,728
16	7	$25,950	$25,618
17	8	$26,800	$26,540
18	9	$27,650	$27,496
19	10	$28,500	$28,486
20	11	$29,350	$29,511
21	12	$30,200	$30,574



1 Enter the spreadsheet heading shown in cell A1.
2 Enter the side headings in cells A3 to A6.
3 In cell B3 enter the value of 20 000 then format it to currency with 0 decimal places.
4 Enter the numeric values shown in cells B4, B5 and B6.
5 In row 8, enter the column headings shown.
6 In cell A9, enter the value 0.

7 In cell A10, enter the formula **=A9+1**. The value of 1 should appear. Copy this formula down from cell A11 to A21.

8 The formula for the simple interest value is:
principal + interest
principal + principal*rate*time/100.
Enter the formula **=$B$3+$B$3*$B$4*A9/100** in cell position B9, then copy this formula down from B9 to B21. Format these cells to currency with 0 decimal places. Check that the values which appear agree with those on the spreadsheet displayed.

9 The formula for the compound interest value for cell position C9 is:
=$B$3*(1+$B$5/(100*$B$6))^A9
Enter this value, then copy it down from C9 to C21. Format these cells to currency with 0 decimal places. Check the values with those on the spreadsheet displayed.

10 Use the graphing facility of your spreadsheet to produce a graph similar to the one displayed on the previous page.

11 From the table and the shape of the graphs it is obvious that a critical point occurs somewhere between the 10-year and 11-year marks. Modify your spreadsheet by inserting rows between these two years. Enter part-year values; such as 10.2, 10.5 and so on, in column A. Copy the formulas in columns B and C to complete the entries. Continue to investigate until you can find a fairly exact value for the time when these two graphs cross.

12 Write a paragraph summarising the results of your spreadsheet and graphs. Describe which option would be the better one, considering that you may be forced to withdraw your money at any time within the 12 years. Support any conclusions by referring to your spreadsheet and graphs.

Task 2

If the investment sum had been $10 000 instead of $20 000 (half the original amount), would this affect the trends shown in Task 1? Investigate and write your conclusions.

Task 3

If the interest rates were half the quoted ones (2.125% p.a. simple interest and 1.8% p.a. compounding yearly) predict what effect this would have on the outcome of the investigation. Repeat Task 1 and write your conclusions.

remember

1. Graphing the future value of a compound interest investment results in an exponential graph.
2. To graph the amount of compound interest paid, we need to subtract the principal from the future value.
3. By drawing the graphs of several investments on one set of axes, we can compare the investments.

EXERCISE 1E Graphing compound interest functions

WORKED Example 14, 16

1 An amount of $8000 is invested at 5% p.a. with interest compounded annually.

a Copy and complete the table below to calculate the future value of the investment at the end of each year.

No. of years	1	2	3	4	5
Future value					

b Draw a graph of the future value against the length of the investment.

2 An amount of $12 000 is to be invested at 8% p.a. with interest compounded annually.

a Copy and complete the table below to calculate the future value at the end of each year.

No. of years	1	2	3	4	5
Future value					

b Draw a graph of the interest earned against the length of the investment.

c Use your graph to find the future value of the investment after 10 years.

3 Draw a graph to represent the future value of the following investments against time.

a $15 000 at 7% p.a. with interest compounded annually

b $2000 at 10% p.a. with interest compounded annually

4 A graph is drawn to show the future value of an investment of $2000 at 6% p.a. with interest compounding six-monthly.

a Complete the table below.

Years	0.5	1	1.5	2	2.5	3	3.5	4	4.5	5
FV										

b Use the table to draw the graph.

5 An amount of $1200 is invested at 4% p.a. with interest compounding quarterly.

a Graph the future value of the investment at the end of each year for 10 years.

b Graph the compound interest earned by the investment at the end of each year.

WORKED Example 15, 16

6 James has $8000 to invest at either 4% p.a., 6% p.a. or 8% p.a. compounding annually.

a Complete the table below to show the interest that he would earn over 5 years.

No. of years	1	2	3	4	5
Interest (4%)					
Interest (6%)					
Interest (8%)					

b Show this information in graph form.

7 Petra has $4000 to invest at 6% p.a.

a Complete the table below to show the future value of the investment at the end of each year, if interest is compounded **i** annually, **ii** six-monthly.

No. of years	1	2	3	4	5
Annually					
Six-monthly					

b Show this information in graphical form.

10 QUICK QUESTIONS 2

1 Calculate the simple interest earned on an investment of $9240 made at 7.4% p.a. for 3 years.

2 Corey invests $14 200 for 5 years in debentures that pay 4.3% p.a. simple interest. Calculate the total value of Corey's investment at maturity.

3 Emma invests $27 500 in investment bonds which pay 6.25% p.a. simple interest for 2 years. Calculate the interest earned in this investment.

4 The interest that Emma receives is paid in quarterly instalments. Calculate the size of each quarterly interest payment.

5 Vladimir invests $2000 at 5% p.a. for 3 years with interest compounded annually. Calculate the compounded value of Vladimir's investment.

6 Calculate the amount of interest earned by Vladimir.

7 Calculate the compounded value of an investment of $6000 at 6.4% p.a. for 2 years with interest compounded annually.

8 Calculate the compounded value of an investment of $6000 at 6.4% p.a. for 2 years with interest compounded six-monthly.

9 Calculate the compounded value of an investment of $6000 at 6.4% p.a. for 2 years with interest compounded quarterly.

10 Calculate the compounded value of an investment of $13 200 at 7.2% p.a. for 18 months, with interest compounded monthly.

Nominal and effective interest rates

An **interest rate** might be **nominated** at 3.5% p.a. compounding monthly. In reality, we would not receive 3.5% per year on the money invested because the interest is compounding monthly. The actual percentage return on the investment is more than 3.5% per year because the interest calculated each consecutive period is based on a higher principal. The **effective interest rate** or **actual interest rate** represents the actual percentage return per year on an investment. It could also be considered as the simple interest rate that would produce the same return as the nominated compound interest rate.

WORKED Example 17

An interest rate is quoted at 3.5% p.a. compounding monthly. What is the effective interest rate?

THINK	WRITE
1 No principal is quoted, so select a principal of $100.	Let $P = \$100$
2 Select a time period of 1 year.	$T = 1$ year
3 Write values for R and n from the quoted interest rate.	$R = 3.5$ $n = 12$
4 Write down the formula for compound interest.	$A = P\left(1 + \frac{R}{100 \times n}\right)^{n \times T}$
5 Substitute the values into the formula.	$= 100\left(1 + \frac{3.5}{100 \times 12}\right)^{12 \times 1}$
6 Evaluate, rounding to the nearest cent.	$= 100\left(1 + \frac{3.5}{1200}\right)^{12}$ $= \$103.56$
7 Calculate the actual interest.	$CI = A - P$ $= \$103.56 - \$100$ $= \$3.56$
8 Represent this interest as a percentage of the principal.	% interest $= \frac{\$3.56}{\$100} \times 100$ $= 3.56\%$
9 This represents the effective interest rate.	Effective interest rate is 3.56% p.a.

Note: The interest rate in the previous worked example is 3.5% p.a. compounding monthly and this is equivalent to a simple interest rate of 3.56% p.a. Because interest rates may be expressed in a variety of ways, the best way to compare interest rates is to determine the effective interest rate in each case.

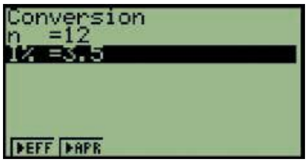
Graphics Calculator tip! Calculating effective interest rates

The effective interest rate can be calculated quickly using a graphics calculator.

For the Casio *fx*-9860G AU

1. To calculate the effective interest rate, press:
   - MENU
   - C: TVM
   - F5 (CNVT).

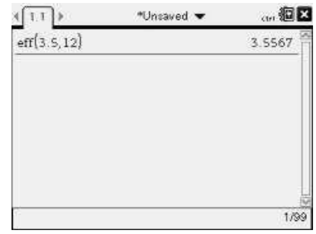
   Enter n and $I\%$ as shown, pressing EXE after each entry.



2. To show the effective interest rate, press F1 (EFF).

This gives an effective interest rate of 3.56% p.a. (to 2 decimal places).

For the TI-Nspire CAS



1. To calculate the effective interest rate, open a Calculator page and press:
   - MENU (menu)
   - 8: Finance (8)
   - 5: Interest Conversion (5)
   - 2: Effective Interest Rate (2).

   Complete the entry line as:

   eff(3.5, 12).

This gives an effective interest rate of 3.56% p.a. (to 2 decimal places).

WORKED Example 18

Jack has the choice of investing his money in three different funds:

a 3.9% p.a. compounding quarterly
b 3.95% p.a. simple interest
c 3.85% p.a. compounding daily.

Which investment would provide him with the greatest return?

THINK	WRITE
a Find the effective interest rate for each account, taking a principal of \$100 and a time of 1 year.	**a** Take a principal of \$100 and a time of 1 year.
1 Write down the compound interest formula.	$A = P\left(1 + \frac{R}{100 \cdot n}\right)^{n \cdot T}$
2 Write down values for P, R, T and n.	$P = \$100$ $R = 3.9$ $T = 1$ $n = 4$
3 Substitute the values into the formula.	$A = 100\left(1 + \frac{3.9}{100 \cdot 4}\right)^{4 \cdot 1}$
4 Evaluate, rounding to the nearest cent.	$= 100\left(1 + \frac{3.9}{400}\right)^{4}$ $= \$103.96$
5 Calculate the actual interest.	$CI = A - P$ $= \$103.96 - \$100$ $= \$3.96$
6 Calculate the effective interest rate.	Effective interest rate $= \frac{\$3.96}{\$100} \cdot 100\%$ $= 3.96\%$ p.a.
b The simple interest rate is the effective interest rate, so no calculations are required in this case.	**b** Effective interest rate = 3.95% p.a.

Continued over page

THINK	WRITE
c 1 Repeat steps as for part **a** to calculate the effective interest rate.	**c** $A = P\left(1 + \frac{R}{100 \cdot n}\right)^{n \cdot T}$ $P = \$100$ $R = 3.85$ $T = 1$ $n = 365$ $A = 100\left(1 + \frac{3.85}{100 \cdot 365}\right)^{365 \cdot 1}$ $= 100\left(1 + \frac{3.85}{36\,500}\right)^{365}$ $= \$103.92$ $CI = A - P$ $= \$103.92 - \$100$ $= \$3.92$ Effective interest rate $= \frac{\$3.92}{\$100} \cdot 100\%$ $= 3.92\%$ p.a.
2 Compare the interest rates and provide an answer.	3.9% p.a. compounding quarterly ≡ 3.96% p.a. effective rate 3.95% p.a. simple interest ≡ 3.95% p.a. effective rate 3.85% p.a. compounding daily ≡ 3.92% p.a. effective rate
3 Answer the question.	So the option which would provide Jack with the greatest return is 3.9% p.a. compounding quarterly.

remember

1. The effective interest rate represents the actual percentage return per year on an investment.
2. To compare interest rates, it is best to determine the effective interest rate in each case.

EXERCISE 1F Nominal and effective interest rates

WORKED Example 17

eBook*plus*

Digital doc: WorkSHEET 1.2

WORKED Example 18

1 Tamara's bank offered her an interest rate of 4% p.a. compounding quarterly. What is the equivalent effective interest rate?

2 If the bank offered Tamara the same interest rate but compounded daily, what effective interest rate would this represent?

3 What simple interest rate would be equivalent to a rate of 3.75% p.a. compounding 6-monthly?

4 Which would provide the better return as an investment?
   a 4.2% p.a. compounding quarterly   **b** 4.175% p.a. compounding monthly

5 Patrick was offered an investment rate of 4.97% p.a. compounding daily or 5% p.a. compounding monthly. Which should he choose?

summary

Simple interest

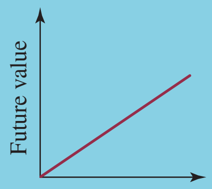
- Simple interest is given by $I = \frac{PRT}{100}$ where I = interest, \$
  P = principal, \$
  R = rate of interest p.a., %
  T = term of interest, years
- The total amount is given by $A = P + I$.
- When calculating simple interest, the interest earned is the same for each time period.

Finding *P*, *R* and *T*

- To find the principal $P = \frac{100 \cdot I}{R \cdot T}$
- To find the interest rate $R = \frac{100 \cdot I}{P \cdot T}$
- To find the period of the loan or investment $T = \frac{100 \cdot I}{P \cdot R}$

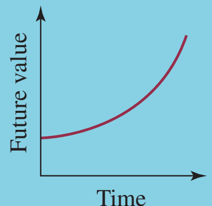
Graphing simple interest functions

- When a simple interest function is graphed, it gives rise to a linear graph.



Compound interest

- The compound interest formula is given by $A = P\left(1 + \frac{R}{100 \cdot n}\right)^{n \cdot T}$
  where A = final value, \$
  P = principal, \$
  R = interest rate, % p.a.
  n = number of compounding periods per year
  T = time of investment, years.
- In the financial world, the formula may be quoted as FV or $CV = PV\left(1 + \frac{R}{100 \cdot n}\right)^{n \cdot T}$
  where FV = future value, \$
  CV = compounded value, \$
  PV = present value, \$.
- The amount of compound interest paid is found by subtracting the principal from the future value of the investment.
- When a compound interest function is graphed, it gives rise to an exponential curve.



Effective interest rates

- The effective interest rate represents the simple interest rate that would produce the same return as the nominal compound interest rate.
- Effective interest rates are used to compare interest rates which are expressed in different terms.

CHAPTER review

1A 1 Calculate the simple interest earned on an investment of \$5000 at 4% p.a. for 5 years.

1A 2 Calculate the simple interest earned on each of the following investments.
- **a** \$3600 at 9% p.a. for 4 years
- **b** \$23 500 at 6% p.a. for 2 years
- **c** \$840 at 2.5% p.a. for 2 years
- **d** \$1350 at 0.2% p.a. for 18 months
- **e** \$45 820 at 4.75% p.a. for $3\frac{1}{2}$ years

1A 3 Dion invests \$32 500 in a debenture paying 5.6% simple interest for 4 years.
- **a** Calculate the interest earned by Dion.
- **b** Calculate the total value of Dion's investment after 4 years.
- **c** If the debenture paid Dion in quarterly instalments, calculate the value of each interest payment.

1A 4 **multiple choice**

Two banks pay simple interest on short-term deposits. Bank A pays 6% p.a. over 4 years and Bank B pays 6.5% p.a. for $3\frac{1}{2}$ years. The difference between the two banks' final payout figure if \$5000 was invested in each account is:

A \$0 **B** \$62.50 **C** \$1137.50 **D** \$1200 **E** \$5062.50

5 **multiple choice**

Clayton invested \$360 in a bank for 3 years at 8% simple interest each year. At the end of the 3 years, the total amount he will receive is:

A \$28.80 **B** \$86.40 **C** \$388.80 **D** \$446.40 **E** \$453.50

6 **multiple choice**

1A Philip borrowed \$7000 and intended to pay it back in 4 years. The terms of the loan indicated Philip was to pay 9% p.a. interest. The interest Philip paid on the loan was:

A \$630 **B** \$2520 **C** \$9520 **D** \$9881 **E** \$25 200

1B 7 Bradley invests \$15 000 for a period of 4 years. Calculate the interest rate, given that Bradley earned a total of \$3900 interest.

1B 8 Kerry invests \$23 500 at 4.6% p.a. If he earned \$1351.25 in interest, calculate the length of time for which the money was invested.

9 **multiple choice**

1B A loan of \$5000 is taken over 5 years. The simple interest is calculated monthly. The interest bill on this loan is \$1125. The simple interest rate per year on this loan is:

A 3% **B** 3.75% **C** 4% **D** $4\frac{1}{2}$% **E** 5%

1B 10 **multiple choice**

The principal invested in an investment bond that will accumulate \$2015 after 6 months invested at $6\frac{1}{2}$% p.a. is:

A \$6000 **B** \$6200 **C** \$50 000 **D** \$60 000 **E** \$62 000

11 **multiple choice** 1B

A loan of $10 000 is taken over 10 years. The total interest bill on this loan is $2000. The simple interest rate per year on this loan is:

A 1.5% **B** 2% **C** 3% **D** $4\frac{1}{2}$% **E** 5%

12 An amount of $7500 is to be invested at 6% p.a. simple interest. 1C

a Copy and complete the table below to calculate the interest over 5 years.

No. of years	1	2	3	4	5
Interest					

b Draw a graph of the interest earned against the length of the investment.

c What is the gradient of the linear graph drawn?

d Use your graph to find the amount of interest that would have been earned after 10 years.

13 Vicky invests $2400 at 5% p.a. for 3 years with interest compounded annually. Calculate the compounded value of the investment at the end of the term. 1D

14 Barry has an investment with a present value of $4500. The investment is made at 6% p.a. with interest compounded six-monthly. Calculate the future value of the investment in 4 years. 1D

15 Calculate the compounded value of each of the following investments. 1D

a $3000 at 7% p.a. for 4 years with interest compounded annually

b $9400 at 10% p.a. for 3 years with interest compounded six-monthly

c $11 400 at 8% p.a. for 3 years with interest compounded quarterly

d $21 450 at 7.2% p.a. for 18 months with interest compounded six-monthly

e $5000 at 2.6% p.a. for $2\frac{1}{2}$ years with interest compounded quarterly

16 Dermott invested $11 500 at 3.2% p.a. for 2 years with interest compounded quarterly. Calculate the total amount of interest paid on this investment. 1D

17 Kim and Glenn each invest $7500 for a period of 5 years. 1D

a Kim invests her money at 9.9% p.a. with interest compounded annually. Calculate the compounded value of Kim's investment.

b Glenn invests his money at 9.6% p.a. with interest compounded quarterly. Calculate the compounded value of Glenn's investment.

c Explain why Glenn's investment has a greater compounded value than Kim's.

18 $20 000 is to be invested at 4% p.a. with interest compounded annually. 1E

a Copy and complete the table below to calculate the future value at the end of each year.

No. of years	1	2	3	4	5
Future value					

b Draw a graph of the interest earned against the length of the investment.

c Use your graph to find the future value of the investment after 10 years.

19 Noel deposited his money in an investment account returning 3.87% p.a. compounding monthly. What is the equivalent effective interest rate? 1F

20 Christof can choose from two investment accounts. One returns 4.1% p.a. compounding monthly while the other returns 4% p.a. compounding daily. Which investment account should he choose? 1F

21 A building society advertises investment accounts at the following rates: 1F

a 3.875% p.a. compounding daily

b 3.895% p.a. compounding monthly

c 3.9% p.a. compounding quarterly.

Which account should Christof choose? Justify your answer by providing evidence for your choice.

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eBook*plus* ACTIVITIES

1A Simple interest

Digital docs

- SkillSHEET 1.1: Practise substitution into formula *(page 10)*
- SkillSHEET 1.2: Practise conversion of units of time *(page 10)*

1B Finding *P*, *R* and *T*

Digital docs

- SkillSHEET 1.3: Practise substitution into the simple interest formula *(page 15)*
- Spreadsheet 205: Investigate simple interest *(page 17)*

1C Graphing simple interest functions

Digital doc

- WorkSHEET 1.1: Calculate interest, principal amounts, duration and rates for simple interest scenarios *(page 22)*

1D Calculation of compound interest

Digital docs

- Spreadsheet 201: Investigate compound interest *(page 33, 35)*
- Spreadsheet 066: Investigate simple and compound interest *(page 33)*
- Spreadsheet 201: Investigate compound interest *(page 34)*

Interactivity

- Compound interest int-0193: Consolidate your understanding of compound interest by playing 'Who wants to be a millionaire?' *(page 23)*

1F Nominal and effective interest rates

Digital doc

- WorkSHEET 1.2: Calculate and compare variables in different investment scenarios *(page 50)*

Chapter review

Digital doc

- Test Yourself: Take the end-of-chapter test to test your progress *(page 53)*.

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Appreciation and depreciation

2

syllabus reference

Strand

Financial mathematics

Core topic

Managing money 2

In this chapter

Introduction

With the passing years, it seems that the cost of goods and services continues to rise. This **appreciation** in price is known as **inflation**. In Australia, these price movements are measured by the **Consumer Price Index (CPI)**. When goods lose value, this is called **depreciation**. While some assets such as houses and land increase in value in the long term, others, such as new cars and computers, continue to decrease. Throughout this chapter we shall look at methods of calculating appreciation and depreciation; how these may be represented graphically; and how they can help us plan for the future.

SKILLS CHECK

1 Calculate the following, giving your answer in decimal form.

a $1 + \frac{10}{100}$ **b** $1 + \frac{1}{100}$ **c** $1 + \frac{3.5}{200}$

d $1 - \frac{3.75}{100}$ **e** $1 - \frac{4.5}{100}$ **f** $1 - \frac{3}{1200}$

2 Calculate each of the following.

a 10% of \$40 **b** 12.5% of \$40 **c** 3.95% of \$100

d 4.125% of \$10 **e** 5.5% of \$20 **f** 3.625% of \$120.50

3 Calculate the following, rounding to the nearest cent.

a Increase \$40 by 10%. **b** Increase \$2.50 by 20%.

c Increase \$7.45 by 2.5%. **d** Decrease \$20 by 10%.

e Decrease \$145 by 6.25%. **f** Decrease \$4000 by 1.5%.

4 a Express \$20 as a percentage of \$50.

b Express \$350 as a percentage of \$400.

c Express \$6.45 as a percentage of \$10.

d Express 45c as a percentage of \$10.

e Express \$1.80 as a percentage of \$3.50.

f Express 12.5c as a percentage of \$8.

5 Calculate the following, expressing your answer in decimal form correct to 2 decimal places.

a $\left(1 + \frac{3}{100}\right)^2$ **b** $\left(1 - \frac{3}{100}\right)^2$ **c** $\left(1 + \frac{4.2}{600}\right)^5$

d $\left(1 - \frac{4.2}{600}\right)^5$ **e** $\left(1 + \frac{3.875}{1200}\right)^{10}$ **f** $\left(1 - \frac{3.875}{1200}\right)^{10}$

Consumer Price Index

The Consumer Price Index (CPI) measures price movements in Australia. Let us investigate this further to gain an understanding of how this index is calculated.

A 'basket' of goods and services, representing a high proportion of household expenditure is selected. The prices of these goods are recorded each quarter. The basket on which the CPI is based is divided into eleven groups, which are further divided into subgroups. The groups are: food, alcohol and tobacco, clothing and footwear, housing, household contents and services, health, transportation, communication, recreation, education, financial and insurance services. Weights are attached to each of these groups to reflect the importance of each in relation to the total household expenditure. The following table shows the weights of the eleven groups.

CPI group	Weight (% of total)
Food	17.7
Alcohol and tobacco	7.4
Clothing and footwear	5.2
Housing	19.8
Household contents and services	8.1
Health	4.7
Transportation	15.2
Communication	2.9
Recreation	12.3
Education	2.7
Financial and insurance services	4.0
Total all groups	100

The weights indicate that a 'typical' Australian household spends 17.7% of its income on food purchases, 5.2% on clothing and footwear and so on. The CPI is regarded as an indication of the cost of living as it records changes in the level of retail prices from one period to another.

Let us consider a simplified example showing how this CPI is calculated, and how we are able to compare prices between one period and another.

Take three items with prices as follows:

- A pair of jeans costing $75
- A hamburger costing $3.90
- A CD costing $25.

Let us say that during the next period of time, the jeans sell for $76, the hamburger for $4.20 and the CD for $29. This can be summarised in a table.

		Period 1		Period 2	
Item	**Weight (W)**	**Price (P)**	**W · P**	**Price (P)**	**W · P**
Jeans	5.2	$75	390	$76	395.2
Hamburger	17.7	$ 3.90	69.03	$ 4.20	74.34
CD	12.3	$25	307.5	$29	356.7
Total			766.53		826.24

The appropriate weight of each item is multiplied by the price of the item to determine what proportion the household would spend on each item in each period. The total weighted expenditure is then calculated.

(*Continued*)

The first period is regarded as the base and is allocated an index number of 100. The second period is compared with the first period and is expressed as a percentage of it. So, period 2 is 826.24 | 766.53 · 100%; that is, 107.8% compared with period 1 as 100%. This means that the average price of these items has risen by 7.8% (the inflation factor) from period 1 to period 2 and the CPI for period 2 is 107.8.

Task 1

1 Take 8 items with which you are quite familiar, each item from a different group (similar to the ones chosen in the previous example — jeans, hamburger, CD). Slot them into their correct categories and note the weighting for each.

2 Draw up a table similar to the previous one, entering the item names and corresponding weightings.

3 Let the period 2 price be the typical prices of the items at this period of time. Enter these values into the table.

4 Let the period 1 price be the prices of these items one year ago (use your knowledge of these items to determine an educated realistic estimate). Enter these values into the table in the period 1 price column.

5 Complete the table as demonstrated.

6 Determine the CPI for period 2 compared with period 1 as the base period.

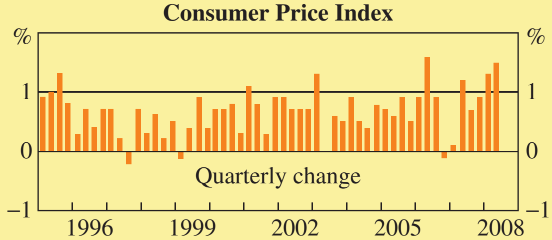
7 What is your inflation factor?

Task 2

1 Conduct a search of the World Wide Web using the words ‘inflation CPI Australia’.

2 Research the history and use of the CPI in Australia. What items are in the subgroups? You should discover that separate CPI indices are calculated for each Australian State. These are then combined to produce an average CPI for the whole of Australia, like that shown in the graph below.

Consumer Price Index



3 Write a report of your findings. Remember to reference the source of your material.

Inflation and appreciation

One of the measures of how an economy is performing is the rate of *inflation*. Inflation is the rise in prices within an economy and is generally measured as a percentage. In Australia this percentage is called the Consumer Price Index (CPI). By looking at the inflation rate we can estimate what the cost of various goods and services will be at some time in the future.

To estimate the future price of an item one year ahead, we increase the price of an item by the rate of inflation. The financial functions of graphics calculators can assist these calculations.

WORKED Example 1

The cost of a new car is $35 000. If the inflation rate is 5% p.a., estimate the price of the car after one year.

THINK	WRITE
Increase $35 000 by 5%.	Future price = 105% of $35 000 = $105 \div 100 \times \$35\,000$ = $36 750

When calculating the future cost of an item several years ahead, the method of calculation is the same as for compound interest. This is because we are adding a percentage of the cost to the cost each year.

Remember the compound interest formula is $A = P\left(1 + \frac{R}{100 \times n}\right)^{n \times T}$ and so in these examples P is the original price, R is the inflation rate expressed as a % p.a., n is the number of rests per year and T is the time in years. Generally the inflation is expressed as a yearly rate, so the value of n is 1.

WORKED Example 2

The cost of a television set is $800. If the average inflation rate is 4% p.a., estimate the cost of the television after 5 years.

THINK	WRITE
1 Write the values of P, R, T and n.	$P = \$800$, $R = 4$, $T = 5$, $n = 1$
2 Write down the compound interest formula.	$A = P\left(1 + \frac{R}{100 \times n}\right)^{n \times T}$ $= 800\left(1 + \frac{4}{100 \times 1}\right)^{1 \times 5}$
3 Substitute the values of P, R, T and n.	$= \$800 \times (1.04)^5$
4 Calculate.	= $973.32

A similar calculation can be made to anticipate the future value of collectable items, such as stamp collections and memorabilia from special occasions. This type of item increases in value over time if it becomes rare, and rises at a much greater rate than inflation. The amount by which an item grows in value over time is known as **appreciation**.

WORKED Example 3

Jenny purchases a rare stamp for $250. It is anticipated that the value of the stamp will rise by 20% per year. Calculate the value of the stamp after 10 years, correct to the nearest $10.

THINK	WRITE
1 Write the values of P, R, T and n.	$P = \$250$ $R = 20$ $T = 10$ $n = 1$
2 Write down the compound interest formula.	$A = P\left(1 + \frac{R}{100 \cdot n}\right)^{n \cdot T}$ $= 250\left(1 + \frac{20}{100 \cdot 1}\right)^{1 \cdot 10}$
3 Substitute the values of P, R, T and n.	$= \$250 \cdot (1.2)^{10}$
4 Calculate and round off to the nearest $10.	$= \$1550$

Modelling appreciation with the aid of a graphics calculator

You have purchased a rare coin that the coin dealer told you should appreciate by 15% each year. You paid $850 for the coin and hope that its value will treble within the next 10 years. The coin dealer is not sure whether this is the case, so you offer to produce a graph for him displaying the value of the coin over the next 10 years.

Using a graphics calculator greatly simplifies the calculations and will produce a graph that can be used to determine the value of the coin at any period of time. The screens displayed and instructions supplied are those of the Casio *fx*-9860G AU calculator and TI-Nspire CAS calculator.

For the Casio *fx*-9860G AU

1 To draw a graph, press:
- MENU
- 5: GRAPH.

Enter the formula:

Y1 = 850(1 + 15 ÷ 100)^X.

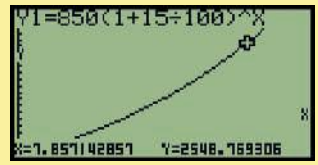
2 To set an appropriate scale, press:
- SHIFT
- F3 (V-WIN).

Enter the values as shown, and then press EXE.

3 To draw the graph, press F6 (DRAW).

To trace the values on the graph, press:
- SHIFT
- F1 (TRCE).

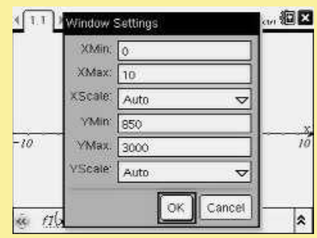
Use the cursor to move along the curve and find a value as close to $850 · 3 as possible.



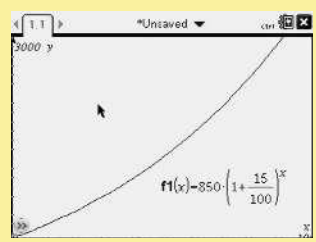
For the TI-Nspire CAS

1 To draw a graph, an appropriate scale must be set. To do this, open a Graphs page and press:
- MENU (menu)
- 4: Window/Zoom (4)
- 1: Window Settings (1).

Complete the fields as shown then select OK.

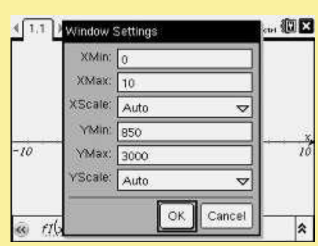


2 Complete the entry line as:
$f1(x) = 850 \cdot (1 + 15 \div 100)$^$x$.
Then press ENTER (enter).



3 To trace the values on the graph, press:
- MENU (menu)
- 5: Trace (5)
- 1: Graph Trace (1).

Use the cursor to move along the curve and find a value as close to $850 · 3 as possible.



4 Investigate further to determine the time when the coin will be 4 times its original value.

5 Prepare a report to the coin dealer explaining how the value of the coin changes over time. Provide a graph to support your figures.

6 The coin dealer asks you if he could use your graph for values other than $850, which was the initial value of the coin. Explain how the graph could be used to determine the time when any sum of money doubles, trebles and so on, as long as the appreciation rate remains at 15% p.a.

remember

1. Inflation is the measure of the rate at which prices increase.
2. The inflation rate is given as a percentage and is called the Consumer Price Index.
3. To estimate the cost of an item after one year, we increase the price by the percentage inflation rate.
4. To estimate the cost of an item after several years, we use the compound interest formula, using the inflation rate as the value of R.
5. Rare items such as collectibles and memorabilia increase in value as time goes on at a rate that is usually greater than inflation.

EXERCISE 2A Inflation and appreciation

WORKED Example 1

1 The cost of a yacht is \$20 000. If the inflation rate is 4% p.a., estimate the cost of the yacht after one year.

eBook plus

Digital doc: SkillSHEET 2.1 Finding a percentage of a quantity

2 For each of the following, estimate the cost of the item after one year, with the given inflation rate.
   a A CD player costing \$600 with an inflation rate of 3% p.a.
   b A toaster costing \$45 with inflation at 7% p.a.
   c A loaf of bread costing \$3.50 with inflation at 6% p.a.
   d An airline ticket costing \$560 with inflation at 3.5% p.a.
   e A washing machine costing \$925 with inflation at 0.8% p.a.

3 An electric guitar is priced at \$850 at the beginning of 2009.
   a If the inflation rate is 3.3% p.a., estimate the cost of the guitar at the beginning of 2010.
   b The government predicts inflation to fall to 2.7% p.a. in 2010. Estimate the cost of the guitar at the beginning of 2011.

4 When the Wilson family go shopping, the weekly basket of groceries costs \$112.50. The inflation rate is predicted to be 4.8% p.a. for the next year. How much should the Wilson's budget per week be for groceries for the next year?

WORKED Example 2

5 The cost of a skateboard is \$550. If the average inflation rate is predicted to be 3% p.a., estimate the cost of a new skateboard in 4 years' time.

6 The cost of a litre of milk is \$1.70. If the inflation rate is an average 4% p.a., estimate the cost of a litre of milk after 10 years.

7 A daily newspaper costs \$2.00. With an average inflation rate of 3.4% p.a., estimate the cost of a newspaper after 5 years (to the nearest 5c).

8 If a basket of groceries costs \$98.50 in 2009, what would the estimated cost of the groceries be in 2016 if the average inflation rate for that period is 3.2% p.a.?

9 **multiple choice**

A bottle of soft drink costs \$2.50. If the inflation rate is predicted to average 2% p.a. for the next five years, the cost of the soft drink in five years will be:

A \$2.60 **B** \$2.65 **C** \$2.70 **D** \$2.75 **E** \$2.76

WORKED Example 3

10 Veronica bought a shirt signed by the Australian cricket team after it won the 2007 World Cup for \$200. If the value of the shirt increases by 20% per annum for the next 5 years, calculate the value of the shirt (to the nearest \$10).

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Digital doc: Extension Inflation and appreciation

11 Ken purchased a rare bottle of wine for \$350. If the value of the wine is predicted to increase at 10% per annum, estimate the value of the wine in 20 years (to the nearest \$10).

12 The 1968 Australian 2c piece is very rare. If a coin collector purchased one in 1999 for \$400 and the value of the coin increases by 15% p.a., calculate its value in 2012 (to the nearest \$10).

Modelling depreciation

An **asset** is an item that has value to its owner. Many assets such as cars and computers lose value over time. This is called *depreciation*.

Consider the case of a new motor vehicle. The value of the car depreciates the moment that you drive the car away from the showroom. This is because the motor vehicle is no longer new and if it were sold it would have to be sold as a used car. The car then continues to lose value steadily each year.

investigation

Depreciation of motor vehicles

Choose a make of car and find out the price for a new vehicle of this make and model. Then go through the classified advertisements in the newspaper and complete the table below or look in the RACQ publication *The Road Ahead*.

Age of car (years)	Price
New (0)	
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	

Draw a graph that shows the price of this car as it ages.

There are two types of depreciation, the **straight line method** and the **declining balance** or **diminishing value method**. Depreciation is a significant factor in the operation of a business. The Australian Taxation Office (ATO) allows depreciation on items necessary for the operation of the business as a legitimate tax deduction.

Straight line depreciation

The straight line method is where the asset depreciates by a constant amount each year. When this type of depreciation is graphed, a straight line occurs and the asset will reduce to a value of 0.

In such a case, a linear function can be derived that will allow us to calculate the value of the item at any time. The function can be found using the gradient–intercept method. The initial value of the item (V_o) will be the vertical intercept and the gradient will be the negative of the amount that the item depreciates, D, each year. The equation of this linear function will be:

$$V = V_o - Dt$$

where V is the present value of the item and t is the age of the asset, in years.

Note that gradients for depreciation will always be negative.

WORKED Example 4

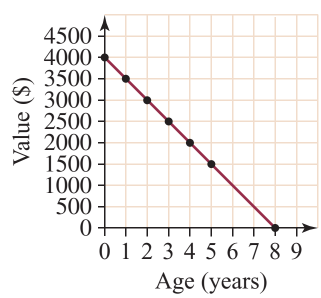
The table below shows the declining value of a computer. Graph the value against time and write an equation for this function.

Age (years)	Value ($)
New (0)	4000
1	3500
2	3000
3	2500
4	2000
5	1500

THINK

1. Draw a set of axes with age on the horizontal axis and value on the vertical.
2. Plot each point given by the table.
3. Join all points to graph the function.
4. Write the initial value as V_o and use the gradient to state D.
5. Write the equation using $V = V_o - Dt$.

WRITE



$V_o = 4000$, $D = 500$

$V = V_o - Dt$
$V = 4000 - 500t$

In the previous worked example, how long does it take for the computer to depreciate to a value of $0? The computer is said to be 'written off' when it reaches this value.

Declining balance or diminishing value depreciation

The other method of depreciation used is the declining balance or diminishing value method of depreciation. Here, the value of the item depreciates each year by a percentage of its current value. Under such depreciation, the value of the item never actually becomes zero.

This type of depreciation is an example of exponential decay. A graph depicting the value over time is **non-linear**, showing a downward-falling curve which never actually reaches a zero value.

WORKED Example 5

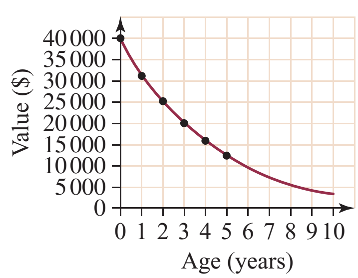
The table at right shows the value of a car that is purchased new for $40 000. Plot the points on a set of axes and graph the depreciation of the car. Use the graph to estimate the value of the car after 10 years.

Age of car (years)	Value ($)
New (0)	40 000
1	32 000
2	25 600
3	20 480
4	16 384
5	13 107

THINK

1. Draw a set of axes with age on the horizontal axis and value on the vertical.
2. Plot the points from the table.
3. Join the points with a smooth curve.
4. Estimate the value after 10 years from the graph you have drawn.

WRITE



From the graph, the approximate value of the car after 10 years is $4000.

A graphics calculator can be used to plot the values in the tables in Worked examples 4 and 5. This can check the shapes of the curves. The two variables (age and value) can be entered as lists. Because no equations are entered, we can not extrapolate values from the graph beyond the limits of the values in the lists. Later in this chapter we will explore a graphics calculator technique which will enable us to extrapolate values from graphs.

WORKED Example 6

With the aid of a graphics calculator, produce a graph showing the relationship between the age and value of the computer in Worked example 4.

THINK | **WRITE/DISPLAY**

For the Casio *fx*-9860G AU

1. To draw a scatterplot, press:
   - MENU
   - 2 STAT.

   Enter the data for Age in List 1 and Value in List 2.

2. To set an appropriate scale, press;
   - SHIFT
   - F3 (V-WIN).

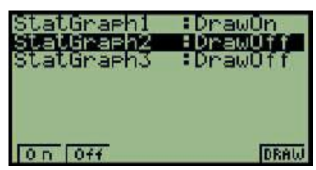
   Enter the values as shown and then press EXE.

3. To set the graph type, press:
   - F1 (GRPH)
   - F6 (SET).

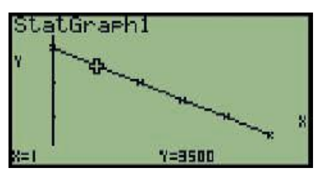
   Enter the settings as shown and then press EXE.

4. To ensure that StatGraph 1 is selected, press F4 (SEL).

   Enter the settings as shown.

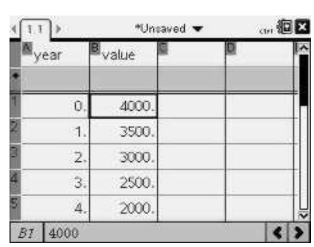


5. To draw the graph, press:
   - F6 (DRAW)
   - SHIFT
   - F1 (TRCE).

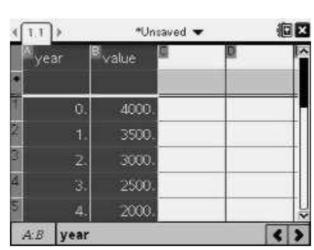


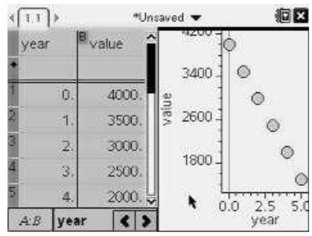
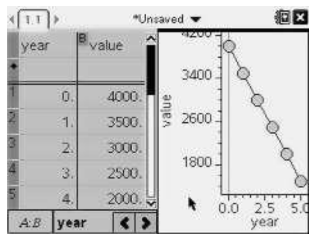
For the TI-Nspire CAS

1. To draw a scatterplot of the data, open a Lists & Spreadsheet page.
   Label column A as 'year' and column B as 'value'. Enter the appropriate data in these columns.



2. To select both columns, place the cursor over the label *year*, then press:
   - NavPad ▲
   - CAPS
   - NavPad ▶.



THINK	WRITE/DISPLAY
3 To draw the scatterplot, press: • MENU (menu) • 3: Data (3) • 6: Quick Graph (6).	
4 To connect the points, press: • MENU (menu) • 2: Plot Properties (2) • 1: Connect Data Points (1).	

remember

1. Depreciation is the loss in the value of an item over time.
2. Depreciation can be of two types:
   (a) straight line depreciation — the item loses a constant amount of value each year
   (b) declining balance or diminishing value depreciation — the value of an item depreciates by a percentage of its value each year.
3. Straight line depreciation can be graphed using a linear function in which the new value of the item is the vertical intercept and the gradient is the negative of the annual loss in value.
4. A declining balance or diminishing value depreciation is an example of exponential decay and is graphed with a smooth curve.
5. A graphics calculator can be used to graph depreciation. Values can be determined at any point along the straight line or curve.

EXERCISE 2B Modelling depreciation

The following questions may be solved with the aid of a graphics calculator.

WORKED Example 4, 6

1 The table below shows the depreciating value of a tractor.

Age (years)	Value ($)
New (0)	100 000
1	90 000
2	80 000
3	70 000
4	60 000
5	50 000

a Draw a graph of the value of the tractor against the age of the tractor.
b Write a function for the value of the tractor.

2 The table at right shows the depreciating value of a tow truck.

Draw a graph of value against age and, hence, write the value as a linear function of age.

Age (years)	Value ($)
New (0)	50 000
1	42 000
2	34 000
3	26 000
4	18 000
5	10 000

3 The function $V = 50\ 000 - 6000A$ shows the value, V, of a car when it is A years old.

a Draw a graph of this function.

b Use the graph to calculate the value of the car after 5 years.

c After how many years would the car be 'written off' (that is, the value of the car becomes $0)?

4 A computer is bought new for $6400 and depreciates at the rate of $2000 per year.

a Write a function for the value, V, of the computer against its age, A.

b Draw the graph of this function.

c After how many years does the computer become written off?

WORKED Example 5, 6

5 The table at right shows the declining value of a new motorscooter.

a Plot the points shown by the table and draw a graph of the value of the motorscooter against age.

b Use your graph to estimate the value of the motorscooter after 8 years.

Age (years)	Value ($)
New (0)	20 000
1	15 000
2	11 250
3	8500
4	6250
5	4750

6 The table at right shows the declining value of a semi-trailer.

a Plot the points as given in the table and then draw a curve of best fit to graph the depreciation of the semi-trailer.

b Use your graph to estimate the value of the semi-trailer after 10 years.

c After what number of years will the value of the semi-trailer fall below $50 000?

Age (years)	Value ($)
New (0)	600 000
1	420 000
2	295 000
3	205 000
4	145 000
5	100 000

7 **a** A gymnasium values its equipment at $200 000. Each year the value of the equipment depreciates by 20% of the value of the previous year. Calculate the value of the equipment after:

i 1 year **ii** 2 years **iii** 3 years **iv** 4 years.

b Plot these points on a set of axes and draw a graph of the value of the equipment against its age.

8 **multiple choice**

Which of the tables below shows a straight line depreciation?

A

Age (years)	Value ($)
New (0)	4000
1	3600
2	3240
3	2916
4	2624
5	2362

B

Age (years)	Value ($)
New (0)	4000
1	3600
2	3200
3	2800
4	2400
5	2000

C

Age (years)	Value ($)
New (0)	4000
1	3600
2	3300
3	3100
4	3000
5	2950

D

Age (years)	Value ($)
New (0)	4000
1	3000
2	2500
3	1500
4	1000
5	500

E

Age (years)	Value ($)
New (0)	4000
1	3500
2	3000
3	2500
4	2000
5	1000

9 A car is bought new for $30 000.

a The straight line method of depreciation sees the car lose $4000 in value each year. Complete the table below.

Age (years)	Value ($)
New (0)	30 000
1	
2	
3	
4	
5	

b Draw a graph of this depreciation.

c The declining balance or diminishing value method of depreciation sees the value of the car fall by 20% of the previous year's value. Complete the table below.

Age (years)	Value ($)
New (0)	30 000
1	
2	
3	
4	
5	

d On the same set of axes draw a graph of this depreciation.

e After how many years is the car worth more under declining balance or diminishing value than under straight line depreciation?

Straight line depreciation

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Interactivity: Reducing balance depreciation int-0194

We have already seen that the straight line method of depreciation is where the value of an item depreciates by a constant amount each year.

The depreciated value of an item is called the **salvage value**, S. The salvage value of an asset can be calculated using the formula:

$$S = V_o - Dn$$

where V_o is the purchase price of the asset, D is the amount of depreciation per period and n is the total number of periods.

WORKED Example 7

A laundry buys dry-cleaning equipment for $30 000. The equipment depreciates at a rate of $2500 per year. Calculate the salvage value of the equipment after 6 years.

THINK	WRITE
1 Write the formula.	$S = V_o - Dn$
2 Substitute the values of V_o, D and n.	$= \$30\,000 - \$2500 \cdot 6$
3 Calculate the value of S.	$= \$15\,000$

By solving an equation we are able to calculate when the value of an asset falls below a particular value.

WORKED Example 8

eBook*plus*

Tutorial:
Worked example 8
int-0901

A plumber purchases equipment for a total of \$60 000. The value of the equipment is depreciated by \$7500 per year. When the value of the equipment falls below \$10 000 it should be replaced. Calculate the number of years after which the equipment should be replaced.

THINK	WRITE
1 Write the formula.	$S = V_o - Dn$
2 Substitute for S, V_o and D.	$10\,000 = 60\,000 - 7500n$
3 Solve the equation to find the value of n.	$7500n = 50\,000$ $n = 6\frac{2}{3}$
4 Give a written answer, taking the value of n up to the next whole number.	The equipment should be replaced after 7 years.

A graphics calculator can provide the answer to Worked example 8 without the need to solve an equation.

WORKED Example 9

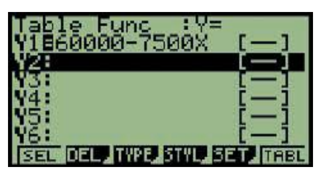
Use a graphics calculator for the problem in Worked example 8.

THINK | **WRITE/DISPLAY**

For the Casio *fx*-9860G AU

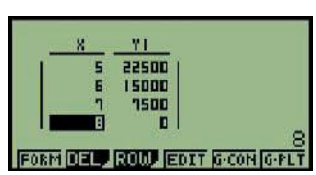
1. To create a table of values, press:
   - MENU
   - 7: TABLE.

   Complete the entry line as:
   Y1 = 60 000 − 7500X.
   Note: The variable x is the number of years.



2. To set the range of the table, press F5 (SET).
   Enter the values as shown.
   Press EXE.

3. To display the table of values, press F6 (TABL). Scroll down to where the value becomes below \$10 000.



4. Write the solution. — The equipment should be replaced after 7 years.

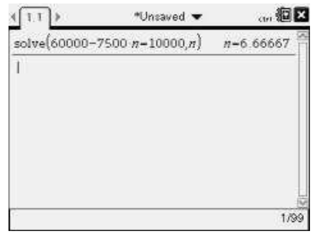
Continued over page

THINK | **WRITE/DISPLAY**

For the TI-Nspire CAS

1. To solve an equation, open a Calculator page and press:
   - MENU (menu)
   - 3: Algebra (3)
   - 1: Solve (1).

   Complete the entry line as:
   solve(60 000 – 7500n = 10 000, n).
   Then press ENTER (enter).



2. Write the solution.

Solve $S = V_0 - Dn$, for n,
given $S = 10\,000$, $V_0 = 60\,000$ *and* $D = 7500$
The equipment should be replaced after 7 years.

remember

1. Straight line depreciation occurs when the value of an asset depreciates by a constant amount each year.
2. The formula to calculate the salvage value, S, of an asset is $S = V_0 - Dn$ where V_0 is the purchase price of the asset, D is the amount of depreciation per period and n is the total number of periods.
3. To calculate a value of V_0, D or n, we substitute all known values and solve the equation that is formed.

EXERCISE 2C Straight line depreciation

WORKED Example 7

1 A car that is purchased for $45 000 depreciates by $5000 each year. Calculate the salvage value of the car after 5 years.

2 Calculate the salvage value:
- a after 5 years of a computer that is purchased for $5000 and depreciates by $800 per year
- b after 7 years of a motorscooter that is purchased for $25 000 and depreciates by $2100 per year
- c after 6 years of a semi-trailer that is purchased for $750 000 and depreciates by $80 000 per year
- d after 2 years of a mobile phone that is purchased for $225 and depreciates by $40 per year
- e after 4 years of a farmer's plough that is purchased for $80 000 and depreciates by $12 000 per year.

3 A bus company buys 15 buses for $475 000 each.
- a Calculate the total cost of the fleet of buses.
- b If each bus depreciates by $25 000 each year, calculate the salvage value of the fleet of buses after 9 years.

4 The price of a new car is \$25 000. The value of the car depreciates by \$300 each month. Calculate salvage value of the car after 4 years.

WORKED Example 8, 9

5 An aeroplane is bought by an airline for \$60 million. If the aeroplane depreciates by \$4 million each year, calculate when the value of the aeroplane falls below \$30 million.

6 Calculate the length of time for each of the following items to depreciate to the value given.
  a A computer purchased for \$5600 to depreciate to less than \$1000 at \$900 per year
  b An electric guitar purchased for \$1200 to depreciate to less than \$500 at \$150 per year
  c An entertainment unit purchased for \$6000 to become worthless at \$750 per year
  d Office equipment purchased for \$12 000 to depreciate to less than \$2500 at \$1500 per year

7 A motor vehicle depreciates from \$40 000 to \$15 000 in 10 years. Assuming that it is depreciating in a straight line, calculate the annual amount of depreciation.

8 Calculate the annual amount of depreciation in an asset that depreciates:
  a from \$20 000 to \$4000 in 4 years
  b from \$175 000 to \$50 000 in 10 years
  c from \$430 000 to \$299 500 in 9 years.

9 A computer purchased for \$3600 is written off in 4 years. Calculate the annual amount of depreciation.

10 A car that is 5 years old has an insured value of \$12 500. If the car is depreciating at a rate of \$2500 per year, calculate its purchase price.

11 Calculate the purchase price of each of the following assets given that:
  a after 5 years the value is \$50 000 and is depreciating at \$12 000 per year
  b after 15 years the value is \$4000 and is depreciating at \$1500 per year
  c after 25 years the value is \$200 and is depreciating at \$50 per year.

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Digital doc: WorkSHEET 2.1

12 An asset that depreciates at \$6500 per year is written off after 12 years. Calculate the purchase price of that asset.

Declining balance or diminishing value method of depreciation

The declining balance or diminishing value method of depreciation occurs when the value of an asset depreciates by a given percentage each period.

Consider the case of a car purchased new for \$30 000, which depreciates at the rate of 20% p.a. Each year the salvage value of the car is 80% of its value at the end of the previous year.

After 1 year: $S = 80\%$ of \$30 000
$= \$24\,000$

After 2 years: $S = 80\%$ of \$24 000
$= \$19\,200$

After 3 years: $S = 80\%$ of \$19 200
$= \$15\,360$

WORKED Example 10

A small truck that was purchased for \$45 000 depreciates at a rate of 25% p.a. By calculating the value at the end of each year, find the salvage value of the truck after 4 years.

THINK	WRITE
1 The salvage value at the end of each year will be 75% of its value at the end of the previous year.	
2 Find the value after 1 year by calculating 75% of \$45 000.	After 1 year: S = 75% of \$45 000 = \$33 750
3 Find the value after 2 years by calculating 75% of \$33 750.	After 2 years: S = 75% of \$33 750 = \$25 312.50
4 Find the value after 3 years by calculating 75% of \$25 312.50.	After 3 years: S = 75% of \$25 312.50 = \$18 984.38
5 Find the value after 4 years by calculating 75% of \$18 984.38.	After 4 years: S = 75% of \$18 984.38 = \$14 238.28

The salvage value under a declining balance can be calculated using the formula:

$$S = V_o\left(1 - \frac{R}{100}\right)^T$$

where S is the salvage value, V_o is the purchase price, R is the annual percentage depreciation and T is the number of years. Note the similarity to the compound interest formula with a minus sign in place of a plus sign (because the value is decreasing). The value of n is 1 since depreciation is usually calculated yearly.

WORKED Example 11

The purchase price of a yacht is \$15 000. The value of the yacht depreciates by 10% p.a. Calculate (correct to the nearest \$1) the salvage value of the yacht after 8 years.

THINK	WRITE
1 Write the formula.	$S = V_o\left(1 - \frac{R}{100}\right)^T$
2 Substitute values for V_o, R and T.	$= 15\,000\left(1 - \frac{10}{100}\right)^8$
3 Calculate the salvage value.	$= \$15\,000 \cdot 0.9^8$ $= \$6457.00$

To calculate the amount by which the asset has depreciated, we subtract the salvage value from the purchase price.

WORKED Example 12

The purchase price of a computer for a music studio is \$40 000. The computer depreciates by 12% p.a. Calculate the amount by which the computer depreciates in 10 years.

THINK

1. Write the formula.
2. Substitute the value of V_o, R and T.
3. Calculate the value of S.
4. Calculate the amount of depreciation by subtracting the salvage value from the purchase price.

WRITE

$S = V_o\left(1 - \frac{R}{100}\right)^T$

$= 40\,000\left(1 - \frac{12}{100}\right)^{10}$

$= \$40\,000 \cdot 0.88^{10}$

$= \$11\,140.04$

Depreciation = \$40 000 − \$11 140.04

= \$28 859.96

remember

1. The declining balance or diminishing value method of depreciation occurs when the value of an asset depreciates by a fixed percentage each year.
2. The salvage value of an asset can be calculated by subtracting the percentage depreciation each year.
3. The salvage value can be calculated using the formula $S = V_o\left(1 - \frac{R}{100}\right)^T$.
4. To calculate the amount of depreciation, the salvage value should be subtracted from the purchase price.

EXERCISE 2D Declining balance or diminishing value method of depreciation

WORKED Example 10

1 The purchase price of a forklift is \$50 000. The value of the forklift depreciates by 20% p.a. By calculating the value of the forklift at the end of each year, find the salvage value of the forklift after 4 years.

2 A trailer is purchased for \$5000. The value of the trailer depreciates by 15% each year. By calculating the value of the trailer at the end of each year, calculate:

a the salvage value of the trailer after 5 years (to the nearest \$10)

b the amount by which the trailer depreciates:

i in the first year **ii** in the fifth year.

3 A company purchases a mainframe computer for \$3 000 000. The value of the computer depreciates by 15% p.a. By calculating the value at the end of each year, find the number of years that it takes for the salvage value of the mainframe to fall below \$1 000 000.

WORKED Example 11

4 Use the declining balance depreciation formula to calculate the salvage value after 7 years of a power generator purchased for \$800 000 that depreciates at a rate of 10% p.a. (Give your answer correct to the nearest \$1000.)

5 Calculate the salvage value of an asset (correct to the nearest \$100) with a purchase price of:

a \$10 000 that depreciates at 10% p.a. for 5 years
b \$250 000 that depreciates at 15% p.a. for 8 years
c \$5000 that depreciates at 25% p.a. for 5 years
d \$2.2 million that depreciates at 30% p.a. for 10 years
e \$50 000 that depreciates at 40% p.a. for 5 years.

6 A plumber has tools and equipment valued at \$18 000. If the value of the equipment depreciates by 30% each year, calculate the value of the equipment after 3 years.

WORKED Example 12

7 A yacht is valued at \$950 000. The value of the yacht depreciates by 22% p.a. Calculate the amount that the yacht will depreciate in value over the first 5 years (correct to the nearest \$1000).

8 A new car is purchased for \$35 000. The owner plans to keep the car for 5 years then trade the car in on another new car. The estimate is that the value of the car will depreciate by 16% p.a. Calculate:

a the amount the owner can expect as a trade-in for the car in 5 years (correct to the nearest \$100)
b the amount by which the car will depreciate in 5 years.

9 **multiple choice**

A shop owner purchases fittings for her store that cost a total of \$120 000. Three years later, the shop owner is asked to value the fittings for insurance. If the shop owner allows for depreciation of 15% on the fittings, which of the following calculations will give the correct estimate of their value?

A $120\,000 \cdot 0.85^3$ **B** $120\,000 \cdot 0.15^3$ **C** $120\,000 \cdot 1.15^3$ **D** $120\,000 \cdot 0.55$
E $120\,000 \cdot 0.45$

10 **multiple choice**

A computer purchased for \$3000 will depreciate by 25% p.a. The salvage value of the computer after 4 years will be closest to:

A \$0 **B** \$10 **C** \$950 **D** \$1000 **E** \$2000

11 An electrician purchases tools of trade for a total of \$8000. Each year the electrician is entitled to a tax deduction for the depreciation of this equipment. If the rate of depreciation allowed is 33% p.a., calculate:

a the value of the equipment at the end of one year (correct to the nearest \$1)
b the tax deduction allowed in the first year
c the value of the equipment at the end of two years (correct to the nearest \$1)
d the tax deduction allowed in the second year.

12 An accountant purchased a computer for \$6000. The value of the computer depreciates by 33% p.a. When the value of the computer falls below \$1000, it is written off and a new one is purchased. How many years will it take for the computer to be written off?

Rates of depreciation

In the previous investigation you chose a make and model of car and researched the salvage value of this car after each year.

1 Calculate the percentage depreciation for each year.

2 Calculate if this percentage rate is approximately the same each year.

3 Using the average annual depreciation, calculate a table of salvage values for the first 10 years of the car's life.

4 Draw a graph showing the depreciating value of the car.

5 Some people claim that it is best to trade your car in on a new model every three years. With reference to your figures and graph, express your views on this statement.

10 QUICK QUESTIONS 1

1 The price of a new DVD player is $1250. The DVD player will depreciate under straight line depreciation at a rate of $200 per year. Calculate the value of the DVD player after 3 years.

2 An asset that was valued at $39 000 when new depreciates to $22 550 in 7 years. Calculate the annual amount of depreciation under straight line depreciation.

3 A computer that is purchased new for $9000 depreciates at a rate of $1350 per year. Calculate the length of time before the computer is written off.

4 A car dealer values a used car at $7000. If the car is 8 years old and the rate of depreciation is $1750 per year, calculate the value of the car when new.

5 Write the formula for depreciation under the declining balance method.

6 A truck is valued new at $50 000 and depreciates at a rate of 32% p.a. Calculate the value of the truck after 5 years (correct to the nearest $50).

7 An asset that has a purchase price of $400 000 depreciates at a rate of 45% p.a. Calculate the asset's value after 6 years (correct to the nearest $1000).

8 For the asset in question **7**, calculate the amount by which it has depreciated in 6 years.

9 Office equipment valued at $250 000 depreciates at a rate of $15% p.a. Calculate the amount by which it depreciates in the first year.

10 Calculate the length of time it will take for the salvage value of the office equipment in question **9** to fall below $20 000.

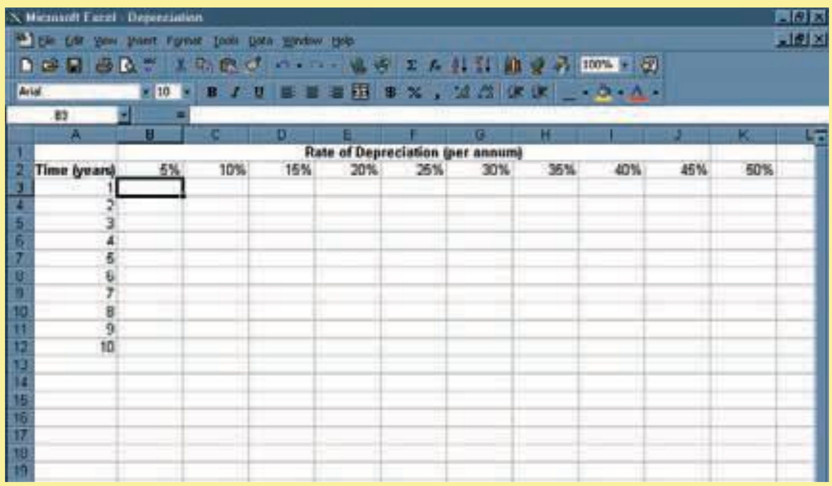
Depreciation tables

The following computer application will prepare a table that will show the depreciated value of an asset with a purchase price of $1 over various periods of time and various rates of depreciation.

investigation

Depreciation table

1 Open a new spreadsheet and type in the headings shown.
2 In cell B3 enter the formula **=(1-B$2)^$A3**.
3 Highlight the range of cells B3 to K12. Select the **Edit** function, and then use (a) the **Fill** and (b) the **Right** and **Fill** and **Down** functions to copy the formula throughout the table.
4 The table that you now have should have the values shown in the table.



Time (years)	Rate of depreciation (per annum)									
	5%	10%	15%	20%	25%	30%	35%	40%	45%	50%
1	0.9500	0.9000	0.8500	0.8000	0.7500	0.7000	0.6500	0.6000	0.5500	0.5000
2	0.9025	0.8100	0.7225	0.6400	0.5625	0.4900	0.4225	0.3600	0.3025	0.2500
3	0.8574	0.7290	0.6141	0.5120	0.4219	0.3430	0.2746	0.2160	0.1664	0.1250
4	0.8145	0.6561	0.5220	0.4096	0.3164	0.2401	0.1785	0.1296	0.0915	0.0625
5	0.7738	0.5905	0.4437	0.3277	0.2373	0.1681	0.1160	0.0778	0.0503	0.0313
6	0.7351	0.5314	0.3771	0.2621	0.1780	0.1176	0.0754	0.0467	0.0277	0.0156
7	0.6983	0.4783	0.3206	0.2097	0.1335	0.0824	0.0490	0.0280	0.0152	0.0078
8	0.6634	0.4305	0.2725	0.1678	0.1001	0.0576	0.0319	0.0168	0.0084	0.0039
9	0.6302	0.3874	0.2316	0.1342	0.0751	0.0404	0.0207	0.0101	0.0046	0.0020
10	0.5987	0.3487	0.1969	0.1074	0.0563	0.0282	0.0135	0.0060	0.0025	0.0010

5 Use the spreadsheet's graphing facility to draw a depreciation graph for each of the depreciation rates shown in the table.
6 Print out a copy of the graph for one of these rates.

The table produced shows the depreciated value of $1 and can be used to make calculations about depreciation.

WORKED Example 13

An item is purchased for $500 and depreciates at a rate of 15% p.a. Use the depreciation table on page 78 to calculate the value of the item after 4 years.

THINK

1. Look up the table to find the depreciated value of $1 at 15% p.a. for 4 years.
2. Multiply the depreciated value of $1 by $500.

WRITE

Depreciated value = 0.5220 · $500
= $261

The computer application on page 78 will produce a general table for a declining balance or diminishing value depreciation. We should be able to use the formula to create a table and graph showing the salvage value of an asset under both straight line and diminishing value depreciation.

WORKED Example 14

A car is purchased new for $20 000. The depreciation can be calculated under straight line depreciation at $2500 per year and under diminishing value at 20% p.a.

a Complete the table at right. (Give all values to the nearest $1.)

b Draw a graph of both the straight line and diminishing value depreciation and use the graph to show the point at which the straight line value of the car falls below the diminishing value.

Age of car (years)	Straight line value ($)	Diminishing value ($)
New (0)	20 000	20 000
1		
2		
3		
4		
5		
6		
7		
8		

THINK

a
1. Copy the table.
2. Complete the straight line column by subtracting $2500 from the previous year's value.
3. Complete the diminishing value by multiplying the previous year's value by 0.8.

WRITE

a

Age of car (years)	Straight line value ($)	Diminishing value ($)
New (0)	20 000	20 000
1	17 500	16 000
2	15 000	12 800
3	12 500	10 240
4	10 000	8192
5	7500	6554
6	5000	5243
7	2500	4194
8	0	3355

Continued over page

THINK

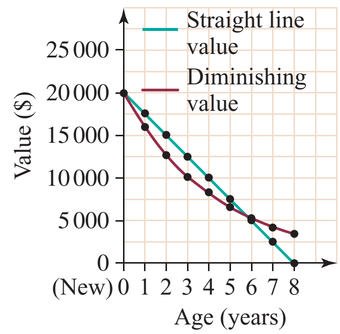
b 1 Plot the points generated by the table.

2 Join the points for the straight line depreciation with a straight line.

3 Join the points for the diminishing value depreciation with a smooth curve.

WRITE

b



4 The graph shows the straight line going below the curve after 6 years.

The straight line depreciation becomes less than the diminishing value depreciation after 6 years.

WORKED Example 15

Use a graphics calculator to solve Worked example 14.

THINK

WRITE/DISPLAY

For the Casio *fx*-9860G AU

a 1 To create a table of values, press:

- MENU
- 7: TABLE
- EXE.

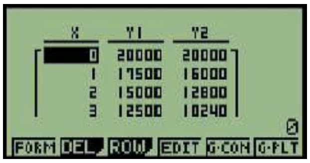
Enter the two formulas as:

Y1 = 20 000 – 2500X
Y2 = 20 000(1 – 20 ÷ 100)^X.

a

2 To set the range of x, press F5 (SET).

3 To display the table, press F6 (TABL).



THINK	**WRITE/DISPLAY**

b 1 To set appropriate window settings, press:

- MENU
- 5: GRAPH.

The functions appear on the screen. Press:

- SHIFT
- F3 (V-WIN).

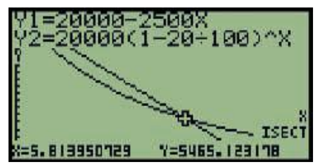
Enter the values as shown.

b 

2 To display the graph, press F6 (DRAW).
To find the points of intersection, press:

- SHIFT
- F5 (G-SLV)
- F5 (ISCT).

The first point of intersection is displayed, (0, 20 000).
To show the second point of intersection, press the right arrow.

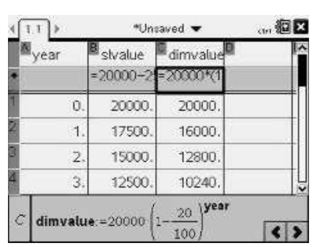


3 Write the solution.

The straight line depreciation becomes less than the diminishing value depreciation after 6 years.

For the TI-Nspire CAS

a To create a table of values using a spreadsheet, open a Lists & Spreadsheet page.
Label column A as 'year', column B as 'slvalue' and column C as 'dimvalue'. Enter the values from 0 to 8 in column A.
In the column B header cell enter the formula as:
= 20 000 – 2500 · year.
Then press ENTER.
In the column C header cell enter the formula as:
= 20 000 · (1 – 20 ÷ 100)^year.
Then press ENTER.

a 

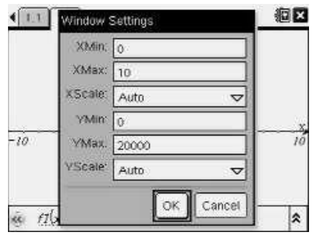
Continued over page

THINK | **WRITE/DISPLAY**

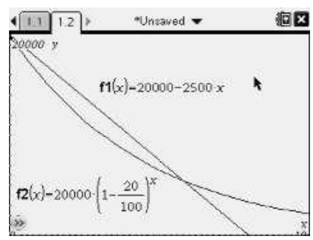
b 1. To draw graphs of the Straight Line Depreciation and the Diminishing Value Depreciation, open a Graphs page.
To adjust window settings, press:
- MENU/Zoom (menu)
- 4: Window (4)
- 1: Window Settings (1).

Enter the values as shown and then select OK.

b

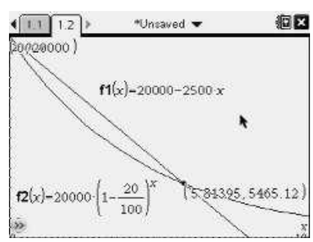


2. Complete the entry lines as:
$f1(x) = 20000 - 2500 \cdot x$
$f2(x) = 20000(1 - 20 \div 100)$^$x$.
Press ENTER (enter) after each entry.



3. To find the intersection point of these two graphs, press:
- MENU (menu)
- 7: Points & Lines (7)
- 3: Intersection Point(s) (3).

Move the cursor and press CLICK once on each graph. The coordinates of the intersection will be displayed.



4. Write the solution.

The straight line depreciation becomes less than the diminishing value depreciation after 6 years.

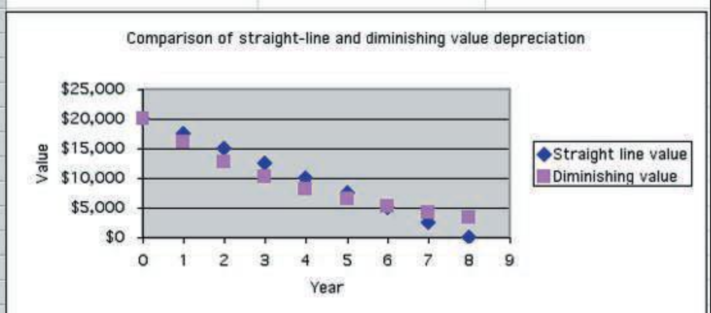
Depreciation is an allowable tax deduction for people in many occupations. A tax deduction for depreciation is allowed when equipment used in earning an income depreciates in value and will eventually need replacing. Depending on the equipment and the occupation, either straight line or diminishing value depreciation may be used.

Under diminishing value depreciation, when the salvage value falls below a certain point the equipment may be written off. This means that the entire remaining balance can be claimed as a tax deduction and as such is considered worthless. From this point on, no further tax deductions can be claimed for this equipment. A visit to the Australian Taxation Office's website provides more detailed information on this subject.

Comparing straight line depreciation and diminishing value depreciation using a spreadsheet

Let us look at a solution to Worked example 14 using a spreadsheet. The value of the new car is $20 000. Depreciation using the straight line method is $2500 per year. Depreciation using the diminishing value method is 20% p.a. Let us draw up a spreadsheet similar to the one following.

	A	B	C
1	**Comparison of straight-line and diminishing value depreciation**		
2			
3	New value=	$20,000	
4	Str line deprec per year =	$2,500	
5	Dimin value rate (%pa) =	20	
6			
7	**Year**	**Straight line value ($)**	**Diminishing value ($)**
8	0	$20,000	$20,000
9	1	$17,500	$16,000
10	2	$15,000	$12,800
11	3	$12,500	$10,240
12	4	$10,000	$8,192
13	5	$7,500	$6,554
14	6	$5,000	$5,243
15	7	$2,500	$4,194
16	8	$0	$3,355
17			
18			



1 Start by entering the main heading in cell A1 and the side headings in cells A3, A4 and A5.
2 Enter the new value, 20 000, in cell B3 and format it as currency with no decimal places.
3 Enter the value 2500 in cell B4 and format as currency with no decimal places.
4 Enter the numeric value of 20 in cell B5.
5 Head up the three columns in row 7.
6 In cell position A8, start the years by entering 0.
7 In cell A9, enter the formula **=A8+1**. Copy this formula down to row 16.
8 In cell B8, enter the formula **=$B$3-$B$4*A8** (the formula for straight line depreciation). Copy this formula down to cell B16.
9 The diminishing value formula to be entered into cell C8 is **=$B$3*(1-$B$5/100)^A8**. Copy this formula down to cell C16.

(Continued)

10 Check that all your figures agree with those in the spreadsheet above. Notice that they are the same as those in the table of Worked example 14.
11 Use the graphing facility of the spreadsheet to produce graphs similar to the ones shown.
12 From the table and the graphs it is evident that a critical point occurs around the 6-year mark. This is consistent with the graph shown in the worked example.
13 Try changing values in cells B3, B4 and B5. Notice how the spreadsheet adjusts.
14 Save your spreadsheet.

WORKED Example 16

A builder has tools of trade that are purchased new for $14 000. He is allowed a tax deduction of 33% p.a. for depreciation of this equipment. When the salvage value of the equipment falls below $3000, the electrician is allowed to write the equipment off on the next year's return. Complete the depreciation table below. (Use whole dollars only.)

Years	Salvage value ($)	Tax deduction ($)
1		
2		
3		
4		
5		

THINK

1. Calculate the salvage value by multiplying the previous year's value by 0.67.
2. Calculate the tax deduction by multiplying the previous year's value by 0.33.
3. When the salvage value is less than $3000, claim the entire amount as a tax deduction.

WRITE

Year	Salvage value ($)	Tax deduction ($)
1	9380	4620
2	6285	3095
3	4211	2074
4	2821	1390
5	0	2821

remember

1. Graphs can be drawn to compare the salvage value of an asset under different rates of depreciation, or to compare diminishing value and straight line depreciation.
2. The amount by which an asset depreciates can, in many cases, be claimed as a tax deduction.

EXERCISE 2E Depreciation tables

WORKED Example 13

1 Use the table of depreciated values of $1 to calculate:

a the value of a computer purchased for $5000 after 5 years, given that it depreciates at 20% p.a.

b the value of a car after 8 years with an initial value of $35 000, given that it depreciates at 15% p.a.

c the value of a boat with an initial value of $100 000 after 10 years, given that it depreciates at 10% p.a.

WORKED Example 14, 15

2 A taxi owner purchases a new taxi for $40 000. The taxi depreciates under straight line depreciation at $5000 per year and under diminishing value depreciation at 20% p.a.

a Copy and complete the table below. Give all values to the nearest $100.

eBook plus

Digital docs:
SkillSHEET 2.2 Converting percentages to decimals
SkillSHEET 2.3 Increase or decrease a percentage

Age of car (years)	Straight line value ($)	Diminishing value ($)
New (0)	40 000	40 000
1		
2		
3		
4		
5		
6		
7		
8		

b Draw a graph of the salvage value of the taxi under both methods of depreciation.

c State when the value under straight line depreciation becomes less than under diminishing value depreciation.

3 A company has office equipment that is valued at $100 000. The value of the equipment can be depreciated at $10 000 each year or by 15% p.a.

a Draw a table that will show the salvage value of the office equipment for the first ten years. (Give all values correct to the nearest $50.)

b Draw a graph of the depreciating value of the equipment under both methods of depreciation.

4 A computer purchased new for $4400 can be depreciated at either 20% p.a. or 35% p.a. Draw a table and a graph that compare the salvage value of the computer at each rate of depreciation.

WORKED Example 16

5 A teacher purchases a laptop computer for $6500. A tax deduction for depreciation of the computer is allowed at the rate of 33% p.a. When the value of the computer falls below $1000, the computer can be written off. Copy and complete the table at right. (Give all values correct to the nearest $1.)

Year	Salvage value ($)	Tax deduction ($)
1		
2		
3		
4		
5		
6		

6 A plumber purchases a work van for $45 000. The van can be depreciated at a rate of 25% p.a. for tax purposes and the van can be written off at the end of 8 years. Copy and complete the depreciation schedule at right. (Give all answers correct to the nearest $1.)

Year	Salvage value ($)	Tax deduction ($)
1		
2		
3		
4		
5		
6		
7		
8		

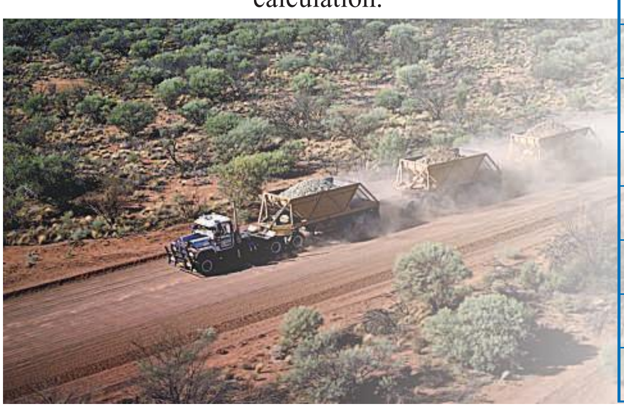
7 A truck is purchased for $250 000. The truck can be depreciated at the rate of $25 000 each year or over 10 years at 20% p.a.

a Copy and complete the table at right. (Give all values correct to the nearest $1.)

b Draw a graph of the depreciating value of the truck under both methods of depreciation.

c Complete a depreciation schedule for each method of calculation.

Age of truck (years)	Straight line value ($)	Diminishing value ($)
New (0)	250 000	250 000
1		
2		
3		
4		
5		
6		
7		
8		
9		
10		



8 Leilay is a cinematographer and on 1 March purchases a camera for work purposes. The cost of the camera is $40 000 and for tax purposes the camera depreciates at the rate of 25% p.a.
- **a** Calculate the amount that the camera will depreciate in the first year.
- **b** The financial year ends on 30 June. For what fraction of the financial year did Leilay own the camera?
- **c** Leilay is allowed a tax deduction for depreciation of her camera. Calculate the amount of tax deduction that Leilay is allowed for the financial year ending on 30 June.

9 Calculate the amount of depreciation on each of the following assets:
- **a** a tractor with an initial value of $80 000 that depreciates at 15% p.a. for 3 months
- **b** a bicycle with an initial value of $600 that depreciates at 25% p.a. for 6 months
- **c** office furniture with an initial value of $8000 that depreciates at 30% p.a. for 8 months
- **d** a photocopier with an initial value of $2500 that depreciates at 40% p.a. for 9 months.

Future and present value of an annuity

An **annuity** is a form of investment involving regular periodic contributions to an account. On such an investment, interest compounds at the end of each period and the next contribution to the account is then made.

Superannuation is a common example of an annuity. Here, people invest in a fund on a regular basis, the interest on the investment compounds, while the principal is added to for each period. The annuity is usually set aside for a person's entire working life and is used to fund retirement. It may also be used to fund a long-term goal, such as a trip in 10 years' time.

To understand the growth of an annuity, we first need to revise compound interest.

The compound interest formula is:

$$A = P\left(1 + \frac{R}{100 \cdot n}\right)^{n \cdot T}$$

where A is the amount that an investment will become, P is the principal, R is the interest rate per annum, n is the number of interest periods per year and T is the time in years.

WORKED Example 17

Calculate the value of a \$5000 investment made at 8% p.a. for 4 years, with interest compounded annually.

THINK / **WRITE**

1. Write the formula. $A = P\left(1 + \frac{R}{100 \cdot n}\right)^{n \cdot T}$
2. Substitute values for P, R, n and T. $= 5000\left(1 + \frac{8}{100 \cdot 1}\right)^{1 \cdot 4}$
   $= \$5000 \cdot (1.08)^4$
3. Calculate the value of A. $= \$6802.44$

An annuity takes the form of a sum of compound interest investments. Consider the case of a person who invests \$1000 at 10% p.a. at *the end* of each year for five years.

To calculate this, we would need to calculate the value of the first \$1000 which is invested for four years, the second \$1000 which is invested for three years, the third \$1000 which is invested for two years, the fourth \$1000 which is invested for one year and the last \$1000 which is added to the investment.

WORKED Example 18

Calculate the value of an annuity in which \$1000 is invested each year at 10% p.a. for 5 years.

THINK / **WRITE**

1. The value of n is 1 in all cases. $n = 1$
2. Use the compound interest formula to calculate the amount to which the first \$1000 will grow. $A = P\left(1 + \frac{R}{100}\right)^T$
   $= \$1000 \cdot 1.1^4$
   $= \$1464.10$
3. Use the compound interest formula to calculate the amount to which the second \$1000 will grow. $A = P\left(1 + \frac{R}{100}\right)^T$
   $= \$1000 \cdot 1.1^3$
   $= \$1331.00$
4. Use the compound interest formula to calculate the amount to which the third \$1000 will grow. $A = P\left(1 + \frac{R}{100}\right)^T$
   $= \$1000 \cdot 1.1^2$
   $= \$1210.00$
5. Use the compound interest formula to calculate the amount to which the fourth \$1000 will grow. $A = P\left(1 + \frac{R}{100}\right)^T$
   $= \$1000 \cdot 1.1$
   $= \$1100.00$
6. Find the total of the separate \$1000 investments, remembering to add the final \$1000. Total value $= \$1464.10 + \$1331.00 + \$1210.00 + \$1100.00 + \$1000$
   $= \$6105.10$

Annuity calculations

The amount to which an annuity grows is called the **future value of an annuity** and can be calculated using the formula:

$$A = M\left\{\frac{(1+r)^n - 1}{r}\right\}$$

M is the amount of each periodical investment made at the end of the period; r is the interest rate per period expressed as a decimal and n is the number of deposits.

The **present value of an annuity**, P, can be calculated using the formula:

$$P = M\left\{\frac{(1+r)^n - 1}{r(1+r)^n}\right\}$$

This formula allows us to calculate the single sum needed to be invested to give the same financial result as an annuity where we are given the size of each contribution.

The financial calculator section of the later model graphics calculators makes short work of annuity calculations. Let us use the Casio *fx*-9860G AU calculator and TI-Nspire CAS calculator to demonstrate this facility.

WORKED Example 19

Christina invests $500 into a fund every 6 months at 9% p.a. interest, compounding six-monthly for 10 years. Calculate the future value of the annuity after 10 years.

THINK

1. Enter the financial section of the calculator.
2. Enter the following data as shown below right.

$n = 10 \cdot 2$	(Interest is calculated twice a year for 10 years.)
$I\% = 9$	(Interest rate is 9% p.a.)
$PV = 0$	(No deposit is made initially – only regular 6-monthly payments.)
$PMT = 500$	(Regular $500 payments are made.)
$FV = 0$	(This value will be calculated.)
PpY or $P/Y = 2$	(Regular payments are made twice a year.)
CpY or $C/Y = 2$	(Interest is calculated twice a year.)

WRITE/DISPLAY

For the Casio *fx*-9860G AU

To enter the data as shown, press:

- MENU
- C: TVM
- F2 (CMPD).

Press EXE to move between fields.

To calculate the future value of the annuity, press F5 (FV).

The future value of annuity after 10 years is $15 685.71.

Continued over page

THINK

3. Solve for *FV*. Take the cursor to the *FV* line and press (F5) (FV) for the Casio *fx*-9860G AU calculator and press ENTER (enter) on the TI-Nspire CAS calculator.

WRITE/DISPLAY

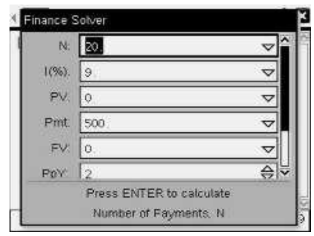
For the TI-Nspire CAS

To enter the data as shown, open a Calculator page.

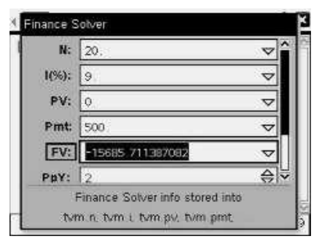
Press:

- MENU (menu)
- 8: Finance (8)
- 1: Finance Solver (1).

Press Tab (tab) to move between fields.



Press the Tab (tab) key to return to the 'FV' line, then press ENTER (enter).



THINK

4. Write the answer.

WRITE/DISPLAY

The future value of the annuity after 10 years is $15 685.71.

WORKED Example 20

Vicky has the goal of saving $10 000 in the next five years. The best interest rate that she can obtain is 8% p.a., with interest compounded annually. Calculate the amount of each annual contribution that Vicky must make.

THINK

1. Enter the following data using the methods described for the previous worked example.

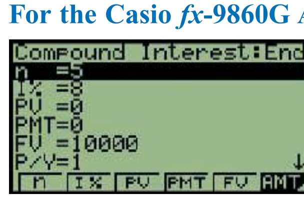
$n = 5 \cdot 1$ (Interest is calculated once a year for 5 years.)

$I\% = 8$ (Interest rate is 8% p.a.)

$PV = 0$ (No deposit is made initially.)

WRITE/DISPLAY

For the Casio *fx*-9860G AU



THINK

$PMT = 0$ (This is the unknown value which will be calculated.)

$FV = 10\,000$ (The future value is \$10 000.)

PpY or $P/Y = 1$ (Vicky makes one payment per year.)

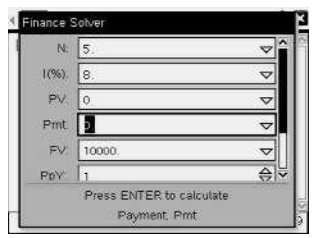
CpY or $C/Y = 1$ (Interest is calculated yearly.)

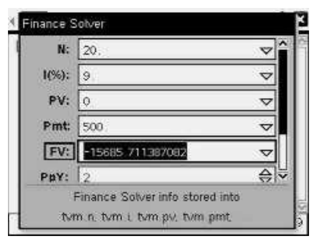
2 Solve for *PMT*. Take the cursor to the *PMT* line and press (F4) (PMT) for the Casio *fx*-9860G AU calculator. For the TI-Nspire CAS calculator, press ENTER (enter).

3 Write the answer.

WRITE/DISPLAY

For the TI-Nspire CAS





A payment of \$1704.56 is required as the annual contribution.

investigation

Annuity calculator

eBook*plus*

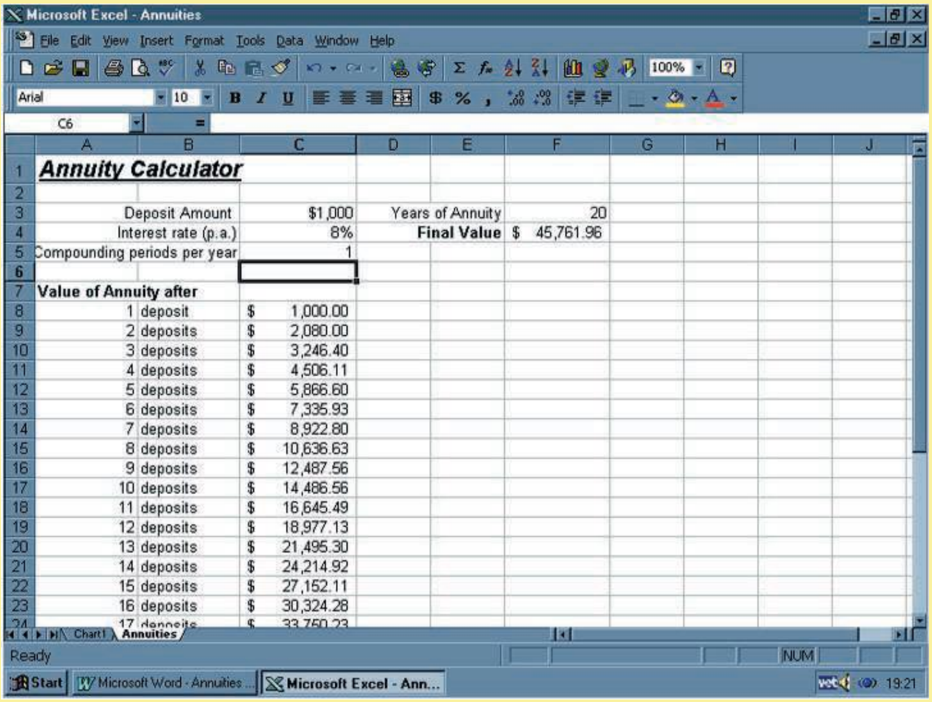
Digital doc:
Spreadsheet
200 Annuity calculator

Use the **Annuity calculator** in your eBookPLUS and download the spreadsheet. The spreadsheet will show you the growth of an annuity where \$1000 is invested at the end of each year for 20 years, at the rate of 8% p.a. interest, compounding annually.

1 The spreadsheet shows that after 20 years the value of this investment is \$45 761.96. Over the page is the growth of the annuity after each deposit is made. This will allow you to see the growth for up to 30 deposits. You can use **Edit**, then **Fill** and **Down** functions on the spreadsheet to see further.

2 Browse the spreadsheet, taking note of the formulas in the various cells.

(*Continued*)



	A	B	C	D	E	F
1	**Annuity Calculator**					
2						
3		Deposit Amount	$1,000	Years of Annuity		20
4		Interest rate (p.a.)	8%		**Final Value**	$ 45,761.96
5	Compounding periods per year		1			
6						
7	**Value of Annuity after**					
8	1	deposit	$ 1,000.00			
9	2	deposits	$ 2,080.00			
10	3	deposits	$ 3,246.40			
11	4	deposits	$ 4,506.11			
12	5	deposits	$ 5,866.60			
13	6	deposits	$ 7,335.93			
14	7	deposits	$ 8,922.80			
15	8	deposits	$ 10,636.63			
16	9	deposits	$ 12,487.56			
17	10	deposits	$ 14,486.56			
18	11	deposits	$ 16,645.49			
19	12	deposits	$ 18,977.13			
20	13	deposits	$ 21,495.30			
21	14	deposits	$ 24,214.92			
22	15	deposits	$ 27,152.11			
23	16	deposits	$ 30,324.28			
24	17	deposits	$ 33,750.23			

3 Click on the Tab ‘Chart 1’. This is a line graph that shows the growth of the annuity for up to 30 deposits.

4 Change the size of the deposit to $500 and the compounding periods to 2. This will show how much benefit can be achieved by reducing the compounding period.

Annuity values using tables

To compare an annuity with a single sum investment, we need to use the present value of the annuity. The *present value of an annuity* is the single sum of money which, invested on the same terms as the annuity, will produce the same financial result.

Problems associated with annuities can be simplified by creating a table that will show either the future value or present value of an annuity of $1 invested per interest period.

Future value of $1

Consider $1 is invested into an annuity each interest period. The table we are going to construct on a spreadsheet shows the future value of that $1.

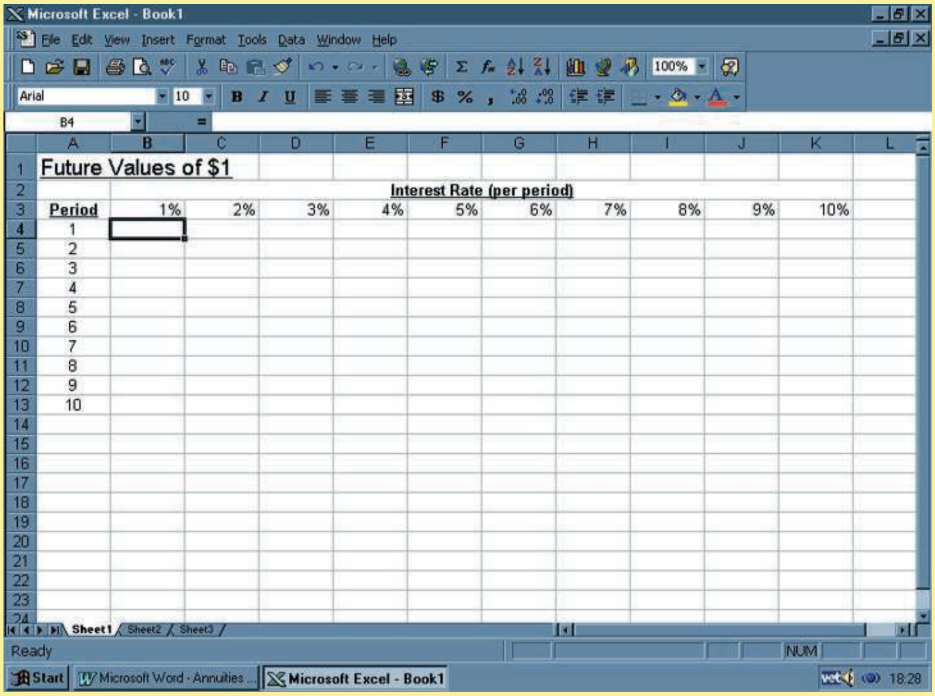
1 Open a new spreadsheet.

2 Type in the headings shown on the following page.

3 In cell B4 enter the formula **=((1+B$3)^$A4-1)/B$3**. (This is the future value formula.) Format the cell, correct to 4 decimal places.

4 Highlight the range of cells B3 to K13. Choose **Edit** and then the **Fill** and **Down** functions followed by the **Fill** and **Right** functions to copy the formula to all other cells in this range.

5 Save the spreadsheet as 'Future value of $1'.



This completes the table. The table shows the future value of an annuity of $1 invested for up to 10 interest periods at up to 10% per interest period. You can extend the spreadsheet further for other interest rates and longer investment periods.

Future values of $1										
	Interest rate (per period)									
Period	1%	2%	3%	4%	5%	6%	7%	8%	9%	10%
1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
2	2.0100	2.0200	2.0300	2.0400	2.0500	2.0600	2.0700	2.0800	2.0900	2.1000
3	3.0301	3.0604	3.0909	3.1216	3.1525	3.1836	3.2149	3.2464	3.2781	3.3100
4	4.0604	4.1216	4.1836	4.2465	4.3101	4.3746	4.4399	4.5061	4.5731	4.6410
5	5.1010	5.2040	5.3091	5.4163	5.5256	5.6371	5.7507	5.8666	5.9847	6.1051
6	6.1520	6.3081	6.4684	6.6330	6.8019	6.9753	7.1533	7.3359	7.5233	7.7156
7	7.2135	7.4343	7.6625	7.8983	8.1420	8.3938	8.6540	8.9228	9.2004	9.4872
8	8.2857	8.5380	8.8923	9.2142	9.5491	9.8975	10.2598	10.6366	11.0285	11.4359
9	9.3685	9.7546	10.1591	10.5828	11.0266	11.4913	11.9780	12.4876	13.0210	13.5795
10	10.4622	10.9497	11.4639	12.0061	12.5779	13.1808	13.8164	14.4866	15.1929	15.9374

The above table is the set of future values of $1 invested into an annuity. This is the table you should have obtained.

A table such as this can be used to find the value of an annuity by multiplying the amount of the annuity by the future value of $1.

WORKED Example 21

Use the previous table to find the future value of an annuity where $1500 is deposited at the end of each year into an account that pays 7% p.a. interest, compounded annually for 9 years.

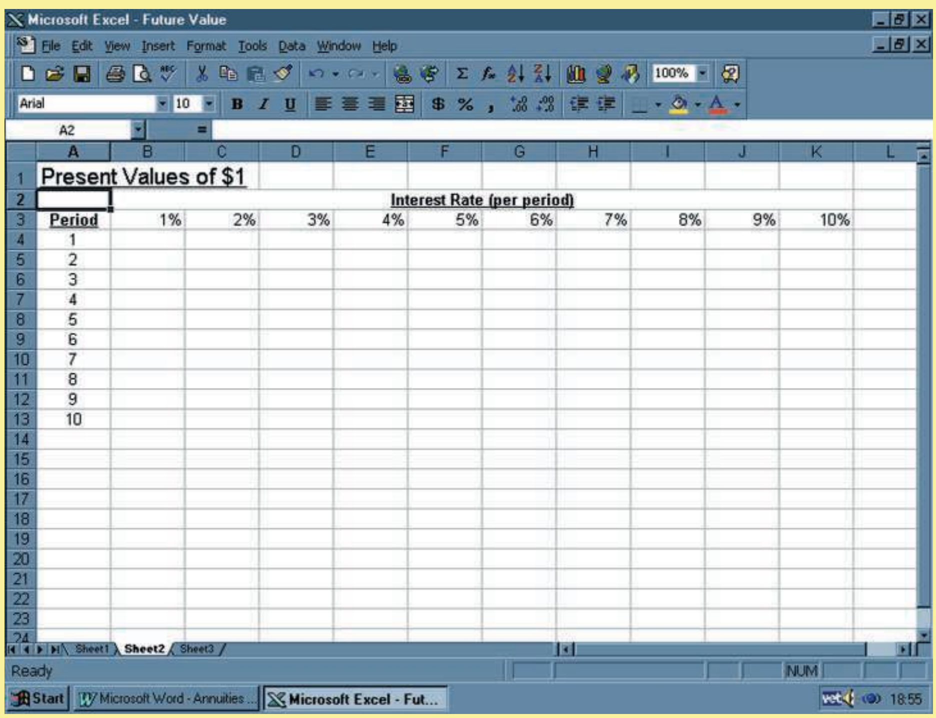
THINK	WRITE
1 Look up the future value of $1 at 7% p.a. for 9 years.	Future value = $1500 · 11.9780
2 Multiply this value by 1500.	= $17 967

Just as we have a table for the future value of an annuity, we can create a table for the present value of an annuity.

investigation — Present value table

The table we are about to make on a spreadsheet shows the present value of an annuity of $1 invested per interest period.

1 Open a new spreadsheet.



2 Enter the headings shown in the spreadsheet.

3 In cell B4 type the formula **=((1+B$3)^$A4-1)/(B$3*(1+B$3)^$A4)**.

4 Drag from cell B4 to K13 and then use the **Edit** and then the **Fill** and **Down** and the **Fill** and **Right** functions to copy this formula to the remaining cells in your table.

5 Save your spreadsheet as 'Present value of $1'.

The table created shows the present value of an annuity of $1 per interest period for up to 10% per interest period and for up to 10 interest periods.

The result that you should have obtained is shown below.

Present values of $1										
	Interest rate (per period)									
Period	1%	2%	3%	4%	5%	6%	7%	8%	9%	10%
1	0.9901	0.9804	0.9709	0.9615	0.9524	0.9434	0.9346	0.9259	0.9174	0.9091
2	1.9704	1.9416	1.9135	1.8861	1.8594	1.8334	1.8080	1.7833	1.7591	1.7355
3	2.9410	2.8839	2.8286	2.7751	2.7232	2.6730	2.6243	2.5771	2.5313	2.4869
4	3.9020	3.8077	3.7171	3.6299	3.5460	3.4651	3.3872	3.3121	3.2397	3.1699
5	4.8534	4.7135	4.5797	4.4518	4.3295	4.2124	4.1002	3.9927	3.8897	3.7908
6	5.7955	5.6014	5.4172	5.2421	5.0757	4.9173	4.7665	4.6229	4.4859	4.3553
7	6.7282	6.4720	6.2303	6.0021	5.7864	5.5824	5.3893	5.2064	5.0330	4.8684
8	7.6517	7.3255	7.0197	6.7327	6.4632	6.2098	5.9713	5.7466	5.5348	5.3349
9	8.5660	8.1622	7.7861	7.4353	7.1078	6.8017	6.5152	6.2469	5.9952	5.7590
10	9.4713	8.9826	8.5302	8.1109	7.7217	7.3601	7.0236	6.7101	6.4177	6.1446

This table can be used in the same way as the future value table.

WORKED Example 22

Liam invests $750 per year into an annuity at 6% per annum for 8 years, with interest compounded annually. Use the above table to calculate the present value of Liam's annuity.

THINK

1. Use the table to find the present value of a $1 annuity at 6% for 8 interest periods.
2. Multiply this value by 750.

WRITE

Present value = $750 · 6.2098
= $4657.35

remember

1. The compound interest formula is $A = P\left(1 + \frac{R}{100 \cdot n}\right)^{n \cdot T}$, where A is the amount that the investment will grow to, R is the interest rate per year, n is the number of periods per year and T is the time of the investment in years.
2. An annuity is a form of investment where periodical equal contributions are made to an account, with interest compounding at the end of each period.
3. The value of an annuity is calculated by adding the value of each amount contributed as a separate compound interest investment.
4. A table of future values shows the future value of an annuity where $1 is invested per interest period.
5. A table of present values shows the present value of an annuity where $1 is invested per interest period.
6. A table of present or future values can be used to compare investments and determine which will give the greater financial return.

EXERCISE 2F Future and present value of an annuity

WORKED Example 17

1 Calculate the value after 5 years of an investment of $4000 at 12% p.a., with interest compounded annually.

2 Calculate the value to which each of the following compound interest investments will grow.
a $5000 at 6% p.a. for 5 years, with interest calculated annually
b $12 000 at 12% p.a. for 3 years, with interest calculated annually
c $4500 at 8% p.a. for 4 years, with interest compounded six-monthly
d $3000 at 9.6% p.a. for 3 years, with interest compounded six-monthly
e $15 000 at 8.4% p.a. for 2 years, with interest compounded quarterly
f $2950 at 6% p.a. for 3 years, with interest compounded monthly

WORKED Example 18

3 At the end of each year for four years Rodney invests $1000 into an investment fund that pays 7.5% p.a. interest, compounded annually. By calculating each investment of $1000 separately, use the compound interest formula to calculate the future value of Rodney's investment after four years.

WORKED Example 19

4 At the end of every six months Jason invests $800 into a retirement fund which pays interest at 6% p.a., with interest compounded six-monthly. Jason does this for 25 years. Calculate the future value of Jason's annuity after 25 years.

5 Calculate the future value of each of the following annuities on maturity.
a $400 invested at the end of every six months for 12 years at 12% p.a. with interest compounded six-monthly
b $1000 invested at the end of every quarter for 5 years at 8% p.a. with interest compounded every quarter
c $2500 invested at the end of each quarter at 7.2% p.a. for 4 years with interest compounded quarterly
d $1000 invested at the end of every month for 5 years at 6% p.a. with interest compounded monthly

6 **multiple choice**

Tracey invests $500 into a fund at the end of each year for 20 years. The fund pays 12% p.a. interest, compounded annually. The total amount of interest that Tracey earns on this fund investment is:

A $1200 B $4323.15 C $4823.23 D $26 026.22 E $36 026.22

WORKED Example 20

7 Thomas has the goal of saving $400 000 for his retirement in 25 years. If the best interest rate that Thomas can obtain is 10% p.a., with interest compounded annually, calculate the amount of each annual contribution that Thomas will need to make.

8 Calculate the amount of each annual contribution needed to obtain each of the following amounts.
a $25 000 in 5 years at 5% p.a., with interest compounded annually
b $100 000 in 10 years at 7.5% p.a., with interest compounded annually
c $500 000 in 40 years at 8% p.a., with interest compounded annually

WORKED Example 21

9 Use the table of future values on page 93 to determine the future value of an annuity of $800 invested per year for 5 years at 9% p.a., with interest compounded annually.

10 Use the table of future values on page 93 to determine the future value of each of the following annuities.
- a $400 invested per year for 3 years at 10% p.a., with interest compounded annually
- b $2250 invested per year for 8 years at 8% p.a., with interest compounded annually
- c $625 invested per year for 10 years at 4% p.a., with interest compounded annually
- d $7500 invested per year for 7 years at 6% p.a., with interest compounded annually

11 Samantha invests $500 every 6 months for 5 years into an annuity at 8% p.a., with interest compounded every 6 months.
- a What is the interest rate per interest period?
- b How many interest periods are there in Samantha's annuity?
- c Use the table on page 93 to calculate the future value of Samantha's annuity.

12 Use the table on page 93 to calculate the future value of each of the following annuities.
- a $400 invested every 6 months for 4 years at 14% p.a., with interest compounded six-monthly
- b $600 invested every 3 months for 2 years at 12% p.a., with interest compounded quarterly
- c $100 invested every month for 5 years at 10% p.a., with interest compounded six-monthly

13 Use the table of future values to determine whether an annuity at 5% p.a. for 6 years or an annuity at 6% p.a. for 5 years will produce the greatest financial outcome. Explain your answer.

14 **multiple choice**

Use the table of future values to determine which of the following annuities will have the greatest financial outcome.
- A 5% p.a. for 9 years, with interest compounded annually
- B 6% p.a. for 8 years, with interest compounded annually
- C 8% p.a. for 6 years, with interest compounded annually
- D 7% p.a. for 7 years, with interest compounded annually
- E 10% p.a. for 5 years, with interest compounded six-monthly

WORKED Example 22

15 Use the table of present values on page 95 to determine the present value of an annuity of $1250 per year for 8 years invested at 9% p.a.

16 Use the table of present values to determine the present value of each of the following annuities.
- a $450 per year for 5 years at 7% p.a., with interest compounded annually
- b $2000 per year for 10 years at 10% p.a., with interest compounded annually
- c $850 per year for 6 years at 4% p.a., with interest compounded annually
- d $3000 per year for 8 years at 9% p.a., with interest compounded annually

eBook*plus*

Digital doc: WorkSHEET 2.2

10 QUICK QUESTIONS 2

1 Calculate the amount of interest earned on \$10 000 invested for 10 years at 10% p.a., with interest compounding annually.

2 Calculate the future value of an annuity of \$1000 invested every year for 10 years at 10% p.a., with interest compounding annually.

3 Calculate the future value of an annuity where \$200 is invested each month for 5 years at 5% p.a., with interest compounding quarterly.

4 Calculate the amount of each annual contribution to an annuity that will have a future value of \$15 000 if the investment is for 8 years at 7.5% p.a., with interest compounding annually.

5 Calculate the amount of each annual contribution to an annuity that will have a future value of \$500 000 in 25 years when invested at 10% p.a., with interest compounding annually.

6 Calculate the present value of an annuity that will have a future value of \$50 000 in 10 years at 10% p.a., with interest compounding annually.

7 Calculate the present value of an annuity that will have a future value of \$1 000 000 in 40 years at 10% p.a., with interest compounding annually.

8 Calculate the present value of an annuity where annual contributions of \$1000 are made at 10% p.a., with interest compounding annually for 20 years.

9 Use the table on page 93 to find the future value of \$1 invested at 16% p.a. for 4 years, with interest compounding twice annually.

10 Use the answer to question 9 to calculate the future value of an annuity of \$1250 every six months for 4 years, with interest of 16% p.a., compounding twice annually.

investigation

A growing investment

Bindi is investing \$20 000 in a fixed term deposit earning 6% p.a. interest. When Bindi has \$30 000 she intends to put a deposit on a house.

1 Write the compound interest function that will model the growth of Bindi's investment.

2 Use your graphics calculator to graph this function

3 Find the length of time (correct to the nearest year) that it will take for Bindi's investment to grow to \$30 000.

4 Suppose that Bindi had been able to invest at 8% p.a. How much quicker would Bindi's investment have grown to the \$30 000 she needs?

5 Calvin has \$15 000 to invest. Find the lowest interest rate at which Calvin must invest his money, if his investment is to grow to \$30 000 in 8 years.

summary

Inflation

- The price of goods and services rises from year to year. To predict the future price of an item we can use the compound interest formula taking the rate of inflation as R.
- The same method is used to predict the future value of collectibles and of memorabilia, which tend to rise at a rate greater than inflation.

Modelling depreciation

- Depreciation can be calculated in two ways. The depreciation can be straight line depreciation or declining balance depreciation.
- Straight line depreciation occurs when the value of an asset decreases by a constant amount each year. The graph of salvage value is a straight line, the vertical intercept is the purchase price and the gradient is the negative of the annual depreciation.
- Declining balance depreciation occurs when the salvage value of the item is a percentage of the previous year's value. The graph of a declining balance depreciation will be an exponential decay graph.

Straight line depreciation

- The salvage value of an asset under straight line depreciation can be calculated using the formula $S = V_o - Dn$, where S is the salvage value, V_o is the purchase price of the asset, D is the amount of depreciation per period and n is the number of periods of depreciation.
- Values of V_o, D or n can be calculated by substitution and solving the equation.

Declining balance method of depreciation

- Under declining balance depreciation the salvage value of an asset can be calculated using the formula $S = V_o\left(1 - \frac{R}{100}\right)^T$, where R is the percentage depreciation per year.
- To calculate the amount by which an asset depreciates in a year we subtract the salvage value at the end of the year from the salvage value at the beginning of the year.

Depreciation tables

- Depreciation can be compared using either a table or a graph.
- Tax deductions are allowed for depreciation of assets that are used as part of earning an income.
- A depreciation schedule is used to calculate tax deductions over a period of years.

Future value of an annuity

- An annuity is where regular equal contributions are made to an investment. The interest on each contribution compounds as additions are made to the annuity.
- The future value of an annuity is the value that the annuity will have at the end of a fixed period of time.

Use of tables

- A table can be used to find the present or future value of an annuity.
- The table shows the present or future value of \$1 under an annuity.
- The present or future value of \$1 must be multiplied by the contribution per period to calculate its present or future value.

CHAPTER review

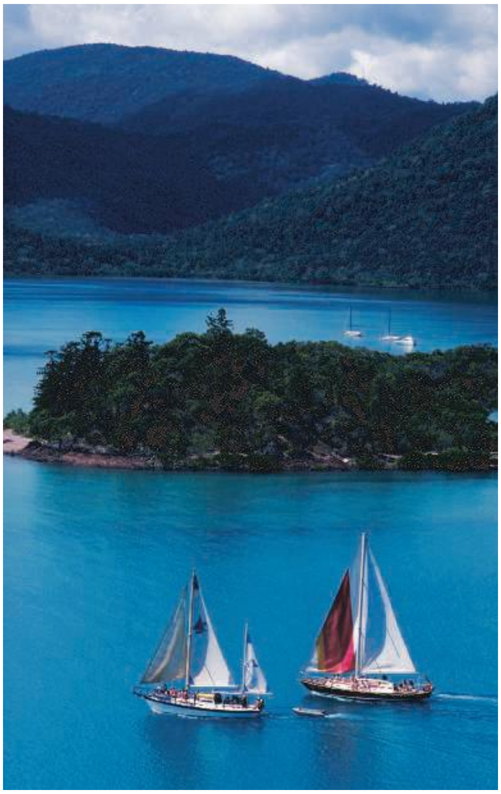
2A **1** An MP3 player is currently priced at $80. If the current inflation rate is 4.3%, estimate the price of the player after one year.

2A **2** It is predicted that the average inflation rate for the next five years will be 3.7%. If a skateboard currently costs $125, estimate the cost of the skateboard after five years.

2A **3** In 1980, Cherie bought a limited edition photograph autographed by Sir Donald Bradman for $120. If the photograph appreciates in value by 15% per annum, calculate the value of the photograph in 2010 (to the nearest $100).

2B **4** The table below shows the depreciating value of a yacht.

Age (years)	Value ($)
New (0)	200 000
1	180 000
2	160 000
3	140 000
4	120 000
5	100 000



a Draw a graph of the value of the yacht against its age.
b Write a function for the value of the yacht.

2B **5** The table below shows the depreciating value of a racing bike.

Age (years)	Value ($)
New (0)	3500
1	3250
2	3000
3	2750
4	2500
5	2250

a Draw a graph of the value of the bike against age.
b Write a function for the straight line depreciation.
c Use your graph to estimate the value of the bike after 9 years.

6 The function $V = 15\ 000 - 900A$ shows the value, V, of a motorcycle when it is A years old. **2B**

a Draw a graph of this function.
b Use the graph to calculate the value of the motorcycle after 5 years.
c After how many years would the motorcycle be 'written off' (the value of the motorcycle become $0)?

7 The table below shows the declining value of a 4-wheel drive. **2B**

Age (years)	Value ($)
New (0)	60 000
1	48 000
2	38 400
3	30 720
4	24 576
5	19 660

a Plot the points as given in the table and then draw a curve of best fit to graph the depreciation of the 4-wheel drive.
b Use your graph to estimate the value of the 4-wheel drive after 10 years.
c After what number of years will the value of the 4-wheel drive fall below $10 000?

8 A laundry buys dry-cleaning equipment for $8000. Each year the equipment depreciates by 25% of the previous year's value. Calculate the value of the equipment at the end of the first five years and use the results to draw a graph of the depreciation. **2B**

9 The purchase price of a car is $32 500. The car depreciates by $3250 each year. Use the formula $S = V_o - Dn$ to calculate the salvage value of the car after 8 years. **2C**

10 Calculate the salvage value of an asset: **2C**

a after 6 years, that was purchased for $4000 and depreciates by $450 each year
b after 10 years, that was purchased for $75 000 and depreciates by $6000 each year
c after 9 years, that was purchased for $640 000 and depreciates by $45 000 each year.

11 A movie projector is purchased by a cinema for $30 000. The projector depreciates by $2500 each year. Calculate the length of time it takes for the projector to be written off. **2C**

12 A camera that was purchased new for $1500 has a salvage value of $500 four years later. Calculate the annual amount of depreciation on the camera. **2C**

2C

13 Arthur buys a car for $25 000. The depreciation on the car is $2250 each year. He decides that he will trade the car in on a new car in the final year before the salvage value falls below $10 000. When will Arthur trade the car in?

2D

14 The purchase price of a mobile home is $40 000. The value of the mobile home depreciates by 15% p.a. By calculating the value of the mobile home at the end of each year, find the salvage value of the mobile home after 4 years. (Give your answer correct to the nearest $1.)

2D

15 Use the declining balance depreciation formula to calculate the salvage value after 7 years of a crop duster that was purchased for $850 000 and depreciates at 8% p.a. (Give your answer correct to the nearest $1000.)

2D

16 Calculate the salvage value of an asset (correct to the nearest $10) with a purchase price of:

a $40 000 that depreciates at 10% p.a. for 5 years
b $1500 that depreciates at 4% p.a. for 10 years
c $180 000 that depreciates at 12.5% p.a. for 15 years
d $4.5 million that depreciates at 40% p.a. for 10 years
e $250 000 that depreciates at $33\frac{1}{3}$% p.a. for 4 years.

2D

17 A company buys a new bus for $600 000. The company keeps buses for 10 years then trades them in on a new bus. The estimate is that the value of the bus will depreciate by 12% p.a. Calculate:

a the amount the owner can expect as a trade-in for the bus in 10 years
b the amount by which the bus will depreciate in 10 years.

2E

18 A company has office equipment that is valued at $100 000. The value of the equipment can be depreciated at $10 000 each year or by 15% p.a.

a Draw a table that will show the salvage value of the office equipment for the first ten years under both methods of depreciation. Give values correct to the nearest $50.
b Draw a graph of the depreciating value of the equipment under both methods of depreciation.

19 A personal computer is purchased for \$4500. A tax deduction for depreciation of the computer is allowed at the rate of 33% p.a. When the value of the computer falls below \$1000, the computer can be written off. Copy and complete the table below. 2E

Year	Salvage value (\$)	Tax deduction (\$)
1		
2		
3		
4		
5		

20 Use the table of future values of \$1 on page 93 to calculate the future value of an annuity of \$4000 deposited per year at 7% p.a. for 8 years, with interest compounded annually. 2F

21 Use the table of future values of \$1 to calculate the future value of the following annuities. 2F

a \$750 invested per year for 5 years at 8% p.a., with interest compounded annually
b \$3500 invested every six months for 4 years at 12% p.a., with interest compounded six-monthly
c \$200 invested every 3 months for 2 years at 16% p.a., with interest compounded quarterly
d \$1250 invested every month for 3 years at 10% p.a., with interest compounded six-monthly

22 Use the table of present values of \$1 on page 95 to calculate the present value of an annuity of \$500 invested per year for 6 years at 9% p.a., with interest compounded annually. 2F

23 Use the table of present values to calculate the present value of each of the following annuities. 2F

a \$400 invested per year for 5 years at 10% p.a., with interest compounded annually
b \$2000 invested every six months for 5 years at 14% p.a., with interest compounded six-monthly
c \$500 invested every three months for $2\frac{1}{2}$ years at 16% p.a., with interest compounded quarterly
d \$300 invested every month for 4 years at 12% p.a., with interest compounded half-yearly

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eBook*plus* ACTIVITIES

2A Inflation and appreciation

Digital docs

- SkillSHEET 2.1: Practise finding a percentage of a quantity *(page 62)*
- Extension: Learn more about inflation and appreciation *(page 63)*

2C Straight line depreciation

Digital doc

- WorkSHEET 2.1: Calculate appreciation and depreciation of items in a variety of scenarios *(page 73)*

Tutorial

- WE8 Int-0901: Watch a worked example on calculating the depreciation of a company car *(page 71)*

Interactivity

- Reducing balance depreciation int-0194: Consolidate your understanding of reducing balance depreciation *(page 70)*

2E Depreciation tables

Digital docs

- SkillSHEET 2.2: Practise converting percentages to decimals *(page 85)*
- SkillSHEET 2.3: Practise increasing or decreasing a percentage *(page 85)*

2F Future and present value of an annuity

Digital docs

- Spreadsheet 200: Investigate using an annuity calculator *(page 91)*
- WorkSHEET 2.2: Calculate annuities, present values, rates and scrap value *(page 97)*

Chapter review

Digital doc

- Test Yourself: Take the end-of-chapter test to test your progress *(page 103)*.

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Consumer credit and investments

3

syllabus reference

Strand
Financial mathematics

Core topic
Managing money 2

In this chapter

SKILLS CHECK

1 Express the following times as fractions of a year.
- a 1 month
- b 18 months
- c 1 day
- d 30 months
- e 1 week
- f 1 fortnight

2 Express the following percentages as decimals.
- a 10%
- b 1%
- c 2.5%
- d $3\frac{3}{4}$%
- e 0.5%
- f 1.25%

3 Calculate the number of months in each of the following.
- a $1\frac{1}{2}$ years
- b $2\frac{3}{4}$ years
- c $\frac{2}{3}$ year
- d $\frac{1}{6}$ year
- e 5 years 9 months
- f $1\frac{1}{4}$ years

4 a How many days in the months of:
- i June?
- ii November?
- iii March?

b How many days are there from:
- i 1 April to 15 May?
- ii 10 July to 12 August?
- iii 5 September to 31 October?

c What is the date 55 days after each of the following?
- i 1 January
- ii 30 June
- iii 17 February

5 Calculate each of the following.
- a The commission on $10 000 at the rate of 5% on the first $8000 and 2% on the remainder
- b Tax on $735 at the rate of $2.50 per $100 or part of $100
- c 75c:$2 as a percentage

Flat rate interest

When students leave school and enter the workforce, the question of how best to invest savings frequently arises. Accumulating funds in a savings account generally attracts a low interest rate. They would be better off placing their funds in other investments, particularly if the money has grown to a substantial sum. Would a short-term deposit in a bank or building society be more suitable? Perhaps purchasing shares would be appropriate. Typically, a sizeable proportion of the family income is consumed by repayments on a personal loan, a housing loan or a car loan. In this era of 'plastic money' most workers readily obtain access to a credit card. Used wisely, credit cards can be an aid in financial budgeting and planning. On the other hand, we have heard many tales of woe, relating stories of misery and accumulating financial debt. This chapter aims to make you aware of the advantages of investing wisely and the pitfalls to avoid in successfully negotiating a path through the minefield of the money market.

In the previous two chapters we calculated the simple interest earned on investments. Flat rate interest is the borrowing equivalent of simple interest. Flat rate interest applies to many personal loans, small loans and hire purchase agreements.

When money is borrowed from a lending institution, such as a bank, at a flat rate of interest, the total amount of interest is calculated as a percentage of the initial amount borrowed and then this is multiplied by the **term of the loan**. The term of the loan is the length of time over which the loan is agreed to be repaid.

The formula for calculating the amount of flat interest to be paid on a loan is the same formula as for simple interest (I):

$$I = \frac{PRT}{100}$$

where P is the principal (or amount borrowed), R is the interest rate per year and T is the number of years over which the loan is to be repaid.

We discussed the use of the graphics calculator for simple interest calculations in chapter 1. It would facilitate calculations here also.

WORKED Example 1

Calculate the interest to be paid on a loan of \$20 000 at 7.5% p.a. flat interest if the loan is to be repaid over 5 years.

THINK	WRITE
1 Write the formula.	$I = \frac{PRT}{100}$
2 Substitute the values of P, R and T.	$= \frac{\$20\,000 \cdot 7.5 \cdot 5}{100}$
3 Calculate.	$= \$7500$

Once the interest has been calculated, we can determine the total amount that must be repaid in a loan. This is calculated by adding the principal and the interest.

WORKED Example 2

Alvin borrows \$8000 to buy a car at a flat rate of 9% p.a. interest. Alvin is to repay the loan, plus interest, over 4 years. Calculate the total amount that Alvin is to repay on this loan.

THINK	WRITE
1 Write the interest formula.	$I = \frac{PRT}{100}$
2 Substitute the values of P, R and T.	$= \frac{\$8000 \cdot 9 \cdot 4}{100}$
3 Calculate the interest.	$= \$2880$
4 Calculate the total repayments by adding the interest and principal.	Total repayments = \$8000 + \$2880 = \$10 880

Most loans are repaid on a monthly basis. Once the total amount to be repaid has been calculated, this can be divided into equal monthly, fortnightly or weekly instalments.

WORKED Example 3

Narelle buys a computer on hire purchase. The cash price of the computer is \$3000, but Narelle must pay a 10% deposit with the balance paid at 8% p.a. flat interest in equal monthly instalments over 3 years.

a Calculate the deposit.
b Calculate the balance owing.
c Calculate the interest on the loan.
d Calculate the total amount to be repaid.
e Calculate the amount of each monthly instalment.

THINK	WRITE
a Find 10% of \$3000.	**a** Deposit = 10% of \$3000 = \$300
b Subtract the deposit from the cash price to find the amount borrowed.	**b** Balance = \$3000 − \$300 = \$2700
c 1 Write the interest formula.	**c** $I = \frac{PRT}{100}$
2 Substitute for P, R and T.	$= \frac{\$2700 \cdot 8 \cdot 3}{100}$
3 Calculate the interest.	= \$648
d Add the interest to the amount borrowed.	**d** Total repayments = \$2700 + \$648 = \$3348
e Divide the total repayments by 36 (the number of monthly instalments in 3 years).	**e** Monthly repayments = \$3348 ÷ 36 = \$93.00

If given the amount to be repaid each month, we can calculate the interest rate. The interest on the loan is the difference between the total repaid and the amount borrowed. This is then calculated as a yearly amount and written as a percentage of the amount borrowed.

WORKED Example 4

Theresa borrows \$12 000 to buy a car. This is to be repaid over 5 years at \$320 per month. Calculate the flat rate of interest that Theresa has been charged.

THINK	WRITE
1 Calculate the total amount that is repaid.	Total repayments = \$320 · 60 = \$19 200
2 Subtract the principal from the total repayments to find the interest.	Interest = \$19 200 − \$12 000 = \$7200
3 Calculate the interest rate using the formula (remember to use brackets on the denominator).	$R = \frac{100I}{PT}$ $= \frac{100 \cdot 7200}{12\,000 \cdot 5}$ = 12
4 Write the answer.	The flat rate of interest is 12% p.a.

remember

1. Flat rate interest is the borrowing equivalent of simple interest. It is calculated based on the initial amount borrowed.
2. The simple interest formula is used to calculate the amount of flat rate interest to be paid on a loan.
3. The total amount to be repaid on a loan is the principal plus interest. To calculate the amount of each instalment, we divide the total amount by the number of repayments.
4. When given the amount of each instalment, we can calculate the flat rate of interest.

EXERCISE 3A Flat rate interest

eBook plus

Digital docs:
SkillSHEET 3.1 Converting percentages to fractions
SkillSHEET 3.2 Converting percentages to decimals
SkillSHEET 3.3 Finding a percentage of a quantity
SkillSHEET 3.4 Writing one quantity as a percentage of another

WORKED Example 1

1 Calculate the amount of flat rate interest paid on each of the following loans.
   a $5000 at 7% p.a. for 2 years
   b $8000 at 5% p.a. for 3 years
   c $15 000 at 10% p.a. for 5 years
   d $9500 at 7.5% p.a. for 4 years
   e $2500 at 10.4% p.a. for 18 months

2 Roula buys a used car that has a cash price of $7500. She has saved a deposit of $2000 and borrows the balance at 9.6% p.a. flat rate to be repaid over 3 years. Calculate the amount of interest that Roula must pay.

WORKED Example 2

3 Ben borrows $4000 for a holiday. The loan is to be repaid over 2 years at 12.5% p.a. flat interest. Calculate the total repayments that Ben must make.

4 Calculate the total amount to be paid on each of the following flat rate interest loans.
   a $3500 at 8% p.a. over 2 years
   b $13 500 at 11.6% p.a. over 5 years
   c $1500 at 13.5% p.a. over 18 months
   d $300 at 33% p.a. over 1 month
   e $100 000 at 7% p.a. over 25 years

WORKED Example 3a–d

5 Mr and Mrs French purchase a new lounge suite, which has a cash price of $5500. They purchase the lounge on the following terms: 30% deposit with the balance to be repaid at 9% p.a. flat interest over 2 years. Calculate:
   a the deposit
   b the balance owing
   c the interest to be paid
   d the total amount that they pay for the lounge.

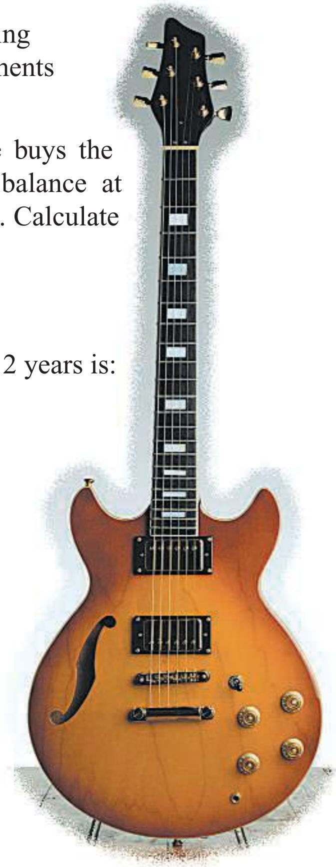
WORKED Example 3c–e

6 Yasmin borrows $5000 from a credit union at a flat interest rate of 8% p.a. to be repaid over 4 years in equal monthly instalments. Calculate:
   a the interest Yasmin pays on the loan
   b the total amount that Yasmin must repay
   c the amount of each monthly repayment.

7 Ian borrows $2000 from a pawnbroker at 40% p.a. interest. The loan is to be paid over 1 year in equal weekly payments.
   a Calculate the interest on the loan.
   b Calculate the total that Ian must repay.
   c Calculate Ian's weekly payment.

8 The Richards family purchase an entertainment system for their home. The total cost of the system is $8000. They buy the system on the following terms: 25% deposit with the balance repaid over 3 years at 12% p.a. flat interest in equal monthly instalments. Calculate:

a the deposit
b the balance owing
c the interest on the loan
d the total repayments
e the amount of each monthly repayment.



9 Sam buys an electric guitar with a cash price of $1200. He buys the guitar on the following terms: one-third deposit, with the balance at 15% p.a. flat interest over 2 years in equal monthly instalments. Calculate the amount of each monthly repayment.

10 **multiple choice**

The amount of flat interest on a loan of $10 000 at 10% p.a. for 2 years is:

A $1000
B $2000
C $4000
D $11 000
E $12 000

11 **multiple choice**

A refrigerator with a cash price of $1800 is bought on the following terms: 20% deposit with the balance paid in 12 equal monthly instalments at 12% p.a. flat interest. The total cost of the refrigerator when purchased on terms is:

A $172.80
B $216.00
C $1612.80
D $1972.80
E $2016.00

WORKED Example 4

12 Andy borrows $4000, which is to be repaid over 4 years at $110 per month. Calculate the flat rate of interest that Andy has been charged.

13 Sandra buys a used car with a cash price of $12 000 on the following terms: 20% deposit with the balance paid at $89.23 per week for 3 years. Calculate:

a the deposit
b the balance owing
c the total cost of the car
d the flat rate of interest charged.

14 Calculate the flat rate of interest charged on a lounge suite with a cash price of $5000 if it is purchased on the following terms: 15% deposit with the balance paid at $230.21 per month for 2 years.

investigation

Flat rate interest loan calculator

Use the **Flat interest** weblink in your eBookPLUS and download the spreadsheet. This spreadsheet will demonstrate how to calculate a deposit, the total repayments on a loan and the size of each repayment.

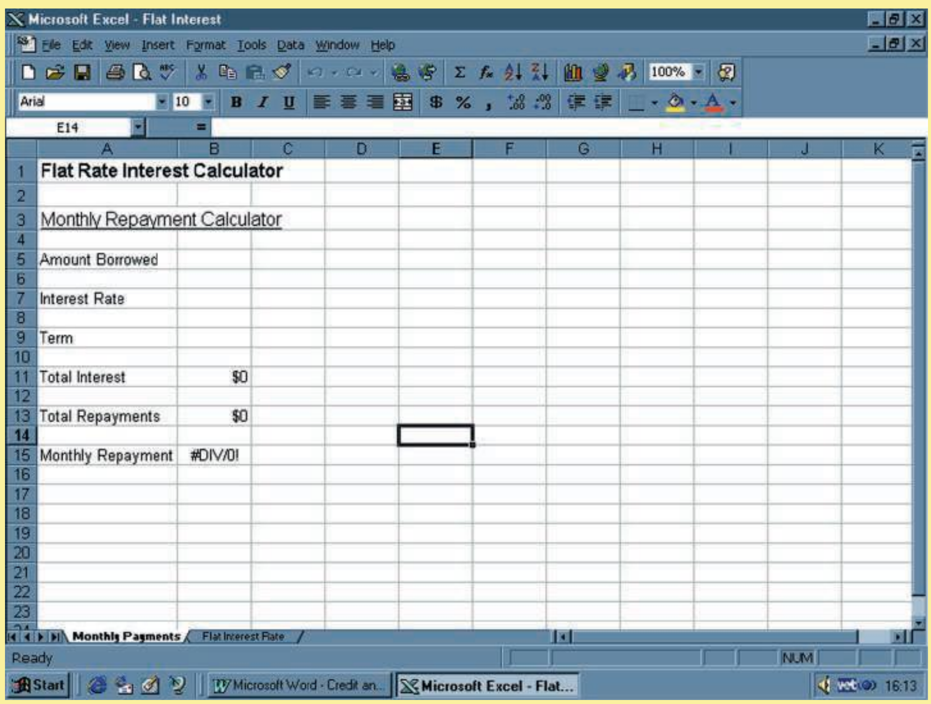
eBook*plus*

Digital doc:
Spreadsheet
202 Flat interest

Monthly payment calculator

Consider a $5000 loan to be repaid at 9% p.a. flat rate interest over 3 years.

1. On the sheet titled 'Monthly payments', in cell B5 enter the amount which has been borrowed ($5000), or the balance owing on a purchase after the deposit has been paid.
2. In cell B7 enter the interest rate as a percentage (9%).
3. In cell B9 enter the number of years over which the loan is to be repaid (3).



4. The total interest paid on the loan will be displayed in cell B11. The formula for this will be displayed in this cell.
5. Cell B13 shows the total amount to be repaid and cell B15 shows the amount of each repayment.

Flat interest rate calculator

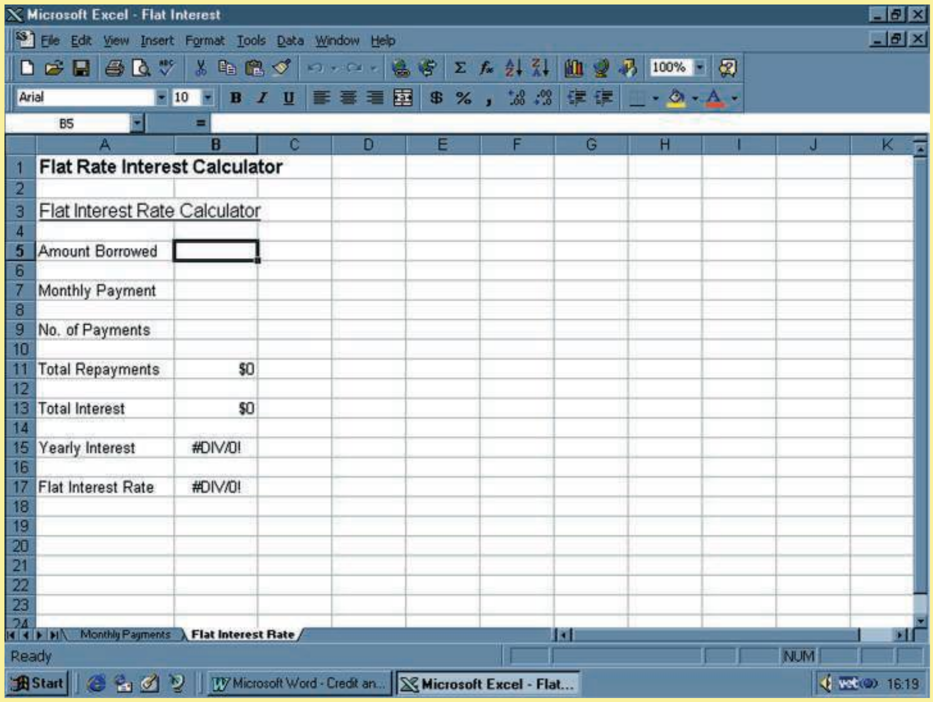
The worksheet 'Flat interest rate' will calculate the flat rate of interest charged given the amount of each repayment. Consider a $15 000 loan that is repaid over 5 years at $350 per month.

(Continued)

1 In cell B5 enter the amount borrowed ($15 000).

2 In cell B7 enter the amount of each monthly payment ($350).

3 In cell B9 enter the total number of monthly payments (60).



4 Displayed will be the total amount to be repaid, the total interest paid on the loan, the amount of interest paid per year and the flat rate of interest.

Check your answers to the previous exercise on this spreadsheet.

Home loans

The biggest loan that most people will ever take out will be for a home. These loans are usually for large amounts of money, often over $300 000 and are taken over long periods of time. Most commonly they are taken over 10, 15, 20 or 25 years but they can be taken over even longer periods of up to 35 years.

Home loans are not charged at a flat rate of interest. The interest on these loans is reducible, which means that the interest is calculated on the amount of money owing on the loan at the time, rather than on the amount initially borrowed. This is known as a **reducing balance loan**.

The interest on a home loan is usually calculated at the beginning of each month, and payments are calculated on a monthly basis. So each month, interest is added to the loan and a payment is subtracted from the balance owing. The balance increases by the amount of interest, then decreases by the amount of each payment.

Consider the case of a person who borrows $200 000 to buy a home at 9% p.a. reducible interest. The monthly repayment on this loan is $1800. The interest rate of 9% p.a. converts to 0.75% per month.

First month's interest = 0.75% of $200 000
= $1500
Balance owing = $200 000 + $1500 − $1800
= $199 700

In the second month the interest is calculated on the balance owing at the end of the first month.

Second month's interest = 0.75% of $199 700
= $1497.75
Balance owing = $199 700 + $1497.75 − $1800
= $199 397.75

The progress of this loan can be followed in the computer application following.

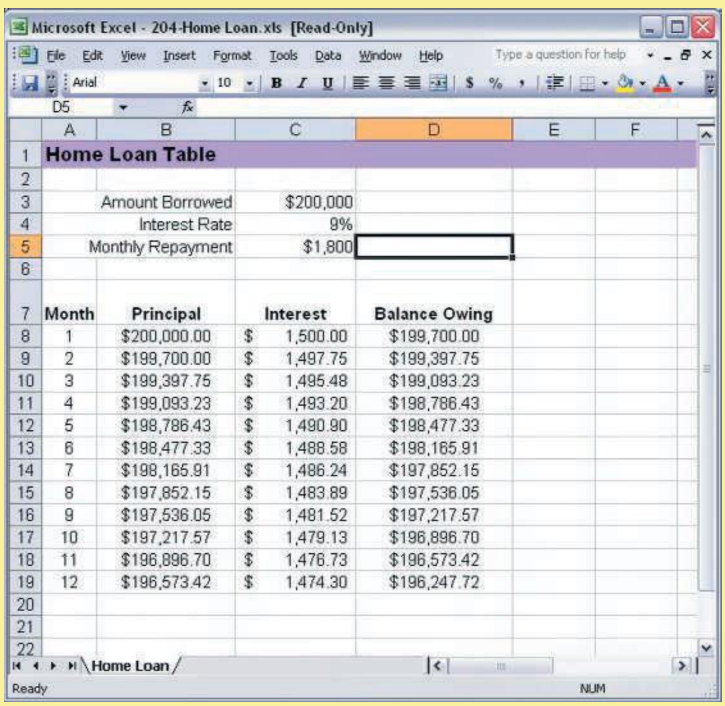
investigation

Home loan calculator

eBook plus

Digital doc:
Spreadsheet
204 Home loan

Download the spreadsheet 'Home loan' from your eBookPLUS and locating the weblinks for this chapter. This spreadsheet will allow you to follow the progress of a home loan as it is paid off.



Microsoft Excel - 204-Home Loan.xls [Read-Only]

	A	B	C	D
1	**Home Loan Table**			
2				
3		Amount Borrowed	$200,000	
4		Interest Rate	9%	
5		Monthly Repayment	$1,800	
6				
7	**Month**	**Principal**	**Interest**	**Balance Owing**
8	1	$200,000.00	$ 1,500.00	$199,700.00
9	2	$199,700.00	$ 1,497.75	$199,397.75
10	3	$199,397.75	$ 1,495.48	$199,093.23
11	4	$199,093.23	$ 1,493.20	$198,786.43
12	5	$198,786.43	$ 1,490.90	$198,477.33
13	6	$198,477.33	$ 1,488.58	$198,165.91
14	7	$198,165.91	$ 1,486.24	$197,852.15
15	8	$197,852.15	$ 1,483.89	$197,536.05
16	9	$197,536.05	$ 1,481.52	$197,217.57
17	10	$197,217.57	$ 1,479.13	$196,896.70
18	11	$196,896.70	$ 1,476.73	$196,573.42
19	12	$196,573.42	$ 1,474.30	$196,247.72

Use the **Edit** and then the **Fill** and **Down** functions on columns A, B, C and D. Look down column D to find when the balance owing becomes 0 or when it becomes negative. At this time the loan will have been fully repaid.

1 What is the term of the loan?
2 Change the amount borrowed. What effect does this have?
3 Experiment by changing the interest rate. What effect does it have?
4 Experiment by changing the monthly repayment.
5 Write a report outlining the results of your investigation.

WORKED Example 5

Mr and Mrs Chan take out a $100 000 home loan at 8% p.a. reducible interest over 25 years. Interest is calculated and added on the first of each month. They make a payment of $775 each month. Calculate:

a the interest added after 1 month

b the balance owing after 1 month.

THINK	WRITE
a 1 Apply the simple interest formula for a time period of 1 month.	**a** $I = \frac{PRT}{100}$
2 Solve the equation.	Interest $= \frac{100\,000 \cdot 8 \cdot \frac{1}{12}}{100}$ $= \$666.67$
b Add the interest to the principal and subtract the repayment.	**b** Balance owing $= \$100\,000 + \$666.67 - \$775$ $= \$99\,891.67$

When interest is calculated every year for such a long period of time, as with many home loans, the amount of money required to pay off such a loan can be a great deal more than the initial loan. Notice how, in the previous worked example, the $100 000 loan has been reduced by only $108.33, even though a payment of $775 has been made.

WORKED Example 6

A loan of $120 000 is paid off at 9% p.a. reducible interest over a period of 25 years. The monthly repayment is $1007.04. Calculate the total repayments on this loan.

THINK	WRITE
1 Calculate the number of repayments by multiplying the number of years by 12.	No. of repayments $= 25 \cdot 12$ $= 300$
2 Multiply the monthly repayment by the number of repayments.	Total repayments $= \$1007.04 \cdot 300$ $= \$302\,112.00$

Notice how the loan was for $120 000 and the total repaid was more than 2.5 times this amount!

remember

1. The interest on home loans is calculated at a reducible rate. This means that the interest is calculated on the balance owing rather than the initial amount borrowed.
2. Interest is calculated each month; this is then added to the principal and a payment is made. The interest next month is then calculated on the new amount owing.
3. To calculate the total amount to be repaid on a home loan, we multiply the monthly payment by the number of repayments made.

EXERCISE 3B Home loans

WORKED Example 5

1 Mr and Mrs Devcich borrow $80 000 to top up their home loan. The interest rate is 12% p.a. and their monthly payment is $850 per month.
   a Calculate the interest for the first month of the loan.
   b Calculate the balance owing at the end of the first month.

2 The repayment on a loan of $50 000 at 7.5% p.a. over a 15-year term is $463.51 per month.
   a Calculate the interest for the first month of the loan and the balance owing at the end of the first month.
   b Calculate the amount by which the balance has reduced in the first month.
   c Calculate the interest for the second month of the loan and the balance at the end of the second month.
   d By how much has the balance of the loan reduced during the second month?

3 The repayment on a loan of $150 000 over a 20-year term at 9.6% p.a. is $1408.01 per month. Copy and complete the table below.

Month	Principal ($)	Interest ($)	Balance owing ($)
1	150 000.00	1200.00	149 791.99
2	149 791.99		
3			
4			
5			
6			
7			
8			
9			
10			

4 Mr and Mrs Roebuck borrow $85 000 as a home loan. The interest rate is 9% p.a. and over a 25-year term the monthly repayment is $764.77.

a Copy and complete the table below.

Month	Principal ($)	Interest ($)	Balance owing ($)
1	85 000.00	637.50	84 872.73
2	84 872.73		
3			
4			
5			
6			
7			
8			
9			
10			
11			
12			

b Mr and Mrs Roebuck decide to increase their monthly payment to $800. Copy and complete the table below.

Month	Principal ($)	Interest ($)	Balance owing ($)
1	85 000.00	637.50	84 837.50
2	84 837.50		
3			
4			
5			
6			
7			
8			
9			
10			
11			
12			

c How much less do Mr and Mrs Roebuck owe at the end of one year by increasing their monthly repayment?

WORKED Example 6

5 The repayments on a loan of $105 000 at 8% p.a. reducible interest over 25 years are $810.41 per month. Calculate the total repayments made over the life of the loan.

6 The Taylors borrow $140 000 over 20 years at 9% p.a.
- a The monthly repayment on this loan is $1259.62. Calculate the total made in repayments.
- b The Taylors attempt to pay the loan off quickly by increasing their monthly payment to $1500. The loan is then paid off in 161 months. Calculate the total repayments made under this plan.
- c How much will the Taylors save by increasing each monthly payment?

7 **multiple choice**

The first month's interest on a $60 000 home loan at 12% p.a. reducible interest is:

A $600 B $7200 C $59 400 D $60 600 E $67 200

8 **multiple choice**

A $95 000 loan at 8% p.a. reducible interest over a 15-year term has a monthly payment of $907.87. The total amount of interest paid on this loan is:

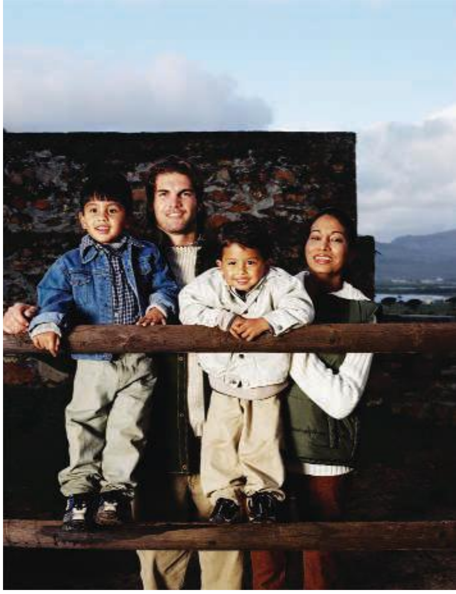
A $7600 B $68 416.60 C $102 600 D $114 000 E $163 416.60

9 Mr and Mrs Chakraborty need to borrow $100 000 to purchase a home. The interest rate charged by the bank is 7% p.a. Calculate the total interest paid if the loan is taken over each of the following terms:
- a $706.78 per month over a 25-year term
- b $775.30 per month over a 20-year term
- c $898.83 per month over a 15-year term
- d $1161.08 per month over a 10-year term.

10 The Smith and Jones families each take out a $50 000 loan at 9.5% p.a. reducible interest. The Smith family repay the loan at $500 per month and the Jones family repay the loan at $750 per month.
- a How much does each family make in repayments in the first year?
- b Complete the following table for each family.

i

Jones family			
Month	Principal ($)	Interest ($)	Balance owing ($)
1	50 000.00	395.83	49 645.83
2	49 645.83		
3			
4			
5			
6			
7			
8			
9			
10			
11			
12			



ii

Jones family			
Month	**Principal ($)**	**Interest ($)**	**Balance owing ($)**
1	50 000.00	395.83	49 645.83
2	49 645.83		
3			
4			
5			
6			
7			
8			
9			
10			
11			
12			

c After one year how much less does the Jones family owe than the Smith family?

10 QUICK QUESTIONS 1

1 Calculate the amount of flat rate interest payable on a loan of \$1500 at 14% p.a. to be repaid over 2 years.

2 Calculate the amount of flat rate interest payable on a loan of \$2365 at 19.2% p.a. to be repaid over $2\frac{1}{2}$ years.

3 Calculate the total repayments on a loan of \$5000 at a flat rate of 13.5% p.a. to be repaid over 3 years.

4 Susan buys a lounge suite on terms. The cash price of the lounge is \$6500 and she pays a 15% deposit. Calculate the amount of the deposit.

5 Calculate the balance that Susan owes on the lounge suite.

6 Calculate the interest that Susan will pay at 17% p.a. flat rate interest for 3 years.

7 Calculate the total amount that Susan will have to repay.

8 Calculate the monthly repayment that Susan will need to make.

9 Harry and Sally borrow \$164 000 to purchase a home. The interest rate is 12% p.a. Calculate the amount of interest payable for the first month.

10 A \$175 000 loan that is repaid over 25 years has a monthly repayment of \$1468.59. Calculate the total amount of interest that is paid on this loan.

The cost of a loan

Because of the different ways that interest can be calculated, the actual interest rate quoted may not be an accurate guide to the cost of the loan. By using a flat rate of interest, a lender can quote an interest rate less than the equivalent reducible interest rate.

To compare flat and reducible rates of interest, we need to calculate the effective rate of interest for a **flat rate loan**. The **effective rate** of interest is the equivalent rate of reducible interest for a flat rate loan.

The formula for effective rate of interest is:

$$E = \frac{(1 + r)^n - 1}{n}$$

where E is the effective rate of interest expressed as a decimal, r is the stated rate of flat interest expressed as a decimal and n is the term of the loan in years.

WORKED Example 7

Andrea takes out an $8000 loan for a car over 5 years at 6% p.a. flat rate interest. Calculate the effective rate of interest charged on the loan.

THINK	WRITE
1 Write the formula.	$E = \frac{(1 + r)^n - 1}{n}$
2 Substitute $r = 0.06$ and $n = 5$.	$= \frac{(1.06)^5 - 1}{5}$
3 Calculate.	$= 0.068$
4 Write the interest rate as a percentage.	The effective rate of interest is 6.8% p.a.

A loan with a reducible rate of interest can be compared to a flat rate of interest if we are able to calculate the total repayments made over the term of the loan.

WORKED Example 8

An $85 000 loan at 10% p.a. reducible interest is to be repaid over 15 years at $913.41 per month.
a Calculate the total repayments on the loan.
b Calculate the total amount of interest paid.
c Calculate the equivalent flat rate of interest on this loan.

THINK	WRITE
a Multiply the monthly repayments by the number of months taken to repay the loan.	a Total repayments = $913.41 · 180 = $164 413.80
b Subtract the initial amount borrowed from the total repayments.	b Interest = $164 413.80 – $85 000 = $79 413.80
c 1 Calculate the interest rate using the formula.	c $R = \frac{100I}{PT}$
2 Substitute into the formula and evaluate.	$R = \frac{100 \cdot 79\,413.80}{85\,000 \cdot 15}$ $= 6.2$
3 Write and answer.	Flat interest rate is 6.2% p.a.

The most accurate way to compare loans is to calculate the total repayments made in the loan.

WORKED Example 9

Allison borrows $6000 and has narrowed her choice of loans down to two options.
Loan A: At 8% p.a. flat rate interest over 4 years to be repaid at $165.00 per month.
Loan B: At 12% p.a. reducible interest over 3 years to be paid at $199.29 per month.
Which of the two loans would cost Allison less?

THINK	WRITE
1 Calculate the total repayments on Loan A.	Loan A repayments = $165.00 · 48 = $7920
2 Calculate the total repayments on Loan B.	Loan B repayments = $199.29 · 36 = $7174.44
3 Write a conclusion.	Loan B would cost $745.56 less than Loan A.

In the previous worked example Allison should take Loan B even though it has a much higher advertised interest rate. This of course would depend upon Allison's ability to meet the higher monthly payments.

Generally the quicker that you can pay off a loan the cheaper the loan will be. The savings are particularly evident when we are examining home loans. Some home loans that offer a lower interest rate allow you to make only the minimum monthly repayment. This will maximise the amount of interest that the customer will pay.

People who can afford to pay more than the minimum amount may be better off over time by paying a slightly higher rate of interest and paying the loan off over a shorter period of time.

There are other factors which must also be considered when we are negotiating the terms of a loan. Interest rates can be fixed or variable. In the latter case, when the rate changes it generally increases rather than decreases. This is usually not accompanied by a pay rise in the workforce. The household budget must then accommodate these higher loan repayments. This can lead to foreclosure of the loan and repossession of the property if these payments cannot be met.

WORKED Example 10

Mr and Mrs Beasley need to borrow $100 000 and have the choice of two home loans.

Loan X: 6% p.a. over 25 years with a fixed monthly repayment of $644.30. No extra repayments are allowed on this loan.

Loan Y: 7% p.a. over 25 years with a minimum monthly payment of $706.78.

Mr and Mrs Beasley believe they can afford to pay $800 per month on Loan Y. If they do, the loan will be repaid in 18 years and 9 months. Which loan would you recommend?

THINK	WRITE
1 Calculate the total repayments on Loan X.	Loan X repayments = $644.30 · 300 = $193 290
2 Calculate the total repayments on Loan Y.	Loan Y repayments = $800 · 225 = $180 000
3 Make a recommendation.	Mr and Mrs Beasley should choose Loan Y as they will save $13 290 provided they can continue to pay $800 per month.

With loans such as the one in Worked example 10, the savings depend upon the ability to make the extra repayments. If this is doubtful, Loan X would have been the safer option.

The other factor to consider when calculating the cost of a loan is fees. Many loans have a monthly management fee attached to them. This will need to be calculated into the total cost and may mean that a loan with a slightly higher interest rate but no fee may be a cheaper option.

remember

1. The actual cost of a loan is calculated by the total cost in repaying the loan. The interest rate is a guide but not the only factor in calculating cost.
2. A loan that is quoted at a flat rate of interest can be compared to a reducible rate of interest only by calculating the effective rate of interest on the flat rate loan. The effective rate of interest is the equivalent reducible rate of interest and is found using the formula:
$$E = \frac{(1+r)^n - 1}{n}$$
3. By calculating the total repayments on a loan, we can calculate the equivalent flat rate of interest paid on the loan.
4. A loan that is repaid over a shorter period of time will usually cost less than one where the repayments are made over the full term of the loan.
5. The flexibility of a loan, which includes factors such as whether extra repayments can be made, is important when considering the cost of a loan.
6. When we are calculating the cost of a loan, any ongoing fees need to be determined.

EXERCISE 3C The cost of a loan

WORKED Example 7

1 A $15 000 loan is to be repaid at 8% p.a. flat rate interest over a 10-year term. Use the formula $E = \frac{(1+r)^n - 1}{n}$ to calculate the effective rate of interest.

2 Calculate the effective rate of interest on each of the following flat rate loans.
- **a** 10% p.a. over 4 years
- **b** 8% p.a. over 2 years
- **c** 12% p.a. over 5 years
- **d** 7.5% p.a. over 10 years
- **e** 9.6% p.a. over 6 years

3 A bank offers loans at 8% p.a. flat rate of interest. Calculate the effective rate of interest for a loan taken over:
- **a** 2 years
- **b** 3 years
- **c** 4 years
- **d** 5 years
- **e** 10 years
- **f** 20 years.

WORKED Example 8

4 An $85 000 home loan at 9% p.a reducible interest is to be repaid over 25 years at $713.32 per month.
- **a** Calculate the total repayments on the loan.
- **b** Calculate the total amount of interest paid.
- **c** Calculate the equivalent flat rate of interest on the loan.

5 Calculate the equivalent flat rate of interest paid on a $115 000 loan at 12% p.a. reducible interest to be repaid over 30 years at $1182.90 per month.

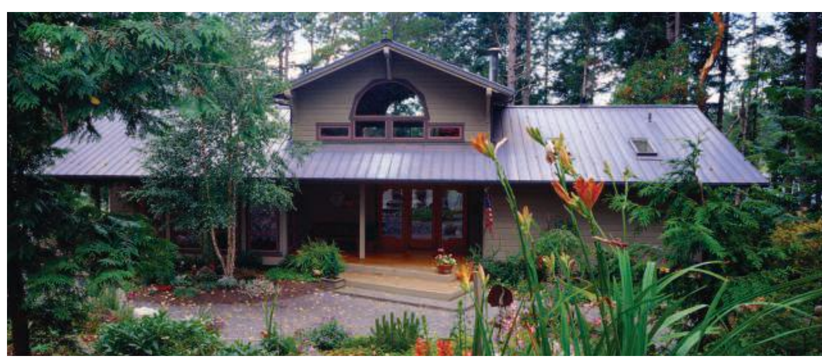
WORKED Example 9

6 Kim borrows $12 000 for a holiday to South-East Asia. She is faced with a choice of two loans.
Loan I: At 10% p.a. flat rate of interest over 2 years to be repaid at $600 per month.
Loan II: At 12.5% p.a. reducible interest over 3 years to be repaid at $401.44 per month.
Which loan will cost Kim the least money?

7 Calculate the total cost of repaying a loan of $100 000 at 8% p.a. reducible interest:
- **a** over 25 years with a monthly repayment of $771.82
- **b** over 20 years with a monthly repayment of $836.44
- **c** over 10 years with a monthly repayment of $1213.28.

WORKED Example 10

8 Masako and Ryu borrow $125 000 for their home. They have the choice of two loans.
Loan 1: A low interest loan at 7% p.a. interest over 25 years with fixed repayments of $833.47 per month.
Loan 2: A loan at 7.5% p.a interest over 25 years with minimum repayments of $923.74 per month.
Masako and Ryu believe they can afford to pay $1000 per month. If they do, Loan 2 will be repaid in 20 years and 4 months.
Which loan should they choose if they could afford to pay the extra each month?



9 **multiple choice**

A loan can be taken out at 8% p.a. flat interest or 9% p.a. reducible interest. Using the formula $E = \frac{(1+r)^n - 1}{n}$, the number of years of the loan (n) after which the effective rate of interest on the flat rate loan becomes greater than the reducible rate loan is:

A 2 years **B** 3 years **C** 4 years **D** 5 years **E** 6 years

10 Glenn and Inge are applying for a \$150 000 loan to be repaid over 25 years.

a Bank A charges 7.8% p.a. interest, no fees, with the loan to be repaid at \$1137.92 per month. Calculate the total cost of this loan.

b Bank B charges 7.6% p.a. interest, a \$600 loan application fee, a \$5 per month management fee and repayments of \$1118.26 per month. Calculate the total cost of this loan.

11 **multiple choice**

A \$50 000 loan is to be taken out. Which of the following loans will have the lowest total cost?

A 5% p.a. flat rate interest to be repaid over 10 years
B 8% p.a. reducible interest to be repaid over 10 years at \$606.64 per month
C 6% p.a. reducible interest to be repaid over 12 years at \$487.93 per month
D 6.5% p.a. reducible interest to be repaid over 10 years at \$567.74 per month, with a \$600 loan application fee and \$8 per month account management fee
E 7% p.a. reducible interest to be repaid over 15 years at \$450 per month

12 A home loan of \$250 000 is taken out over a 20-year term. The interest rate is 9.5% p.a. and the monthly repayments are \$2330.33.

a The mortgage application fee on this loan was \$600 and there is a \$10 per month account management fee. Calculate the total cost of repaying this loan.

b Calculate the equivalent flat rate of interest on the loan. (Consider the extra payments as part of the interest.)

c If the loan is repaid at \$3000 per month, it will take $11\frac{1}{2}$ years to repay the loan. Calculate the equivalent flat rate of interest if this repayment plan is followed.

investigation

Researching home loans

Suppose that you wish to borrow \$100 000 to buy a unit. Go to a bank or other lender and gather the following information:

1 the annual interest rate
2 the loan application fee and any other costs such as stamp duty, legal costs etc. associated with establishing the loan
3 the monthly account-keeping or management fee, if any
4 the monthly repayment if the loan is repaid over:
a 15 years **b** 20 years **c** 25 years.
5 the total cost of repaying the loan in each of the above examples.

There are many ways that people can reduce the overall cost of repaying a mortgage. Research and explain why people are able to save money by adopting the following repayment strategies:

1 repaying the loan fortnightly or weekly instead of monthly
2 using an account where the whole of a person's net pay is deposited on the mortgage and then a redraw is used to meet living expenses.

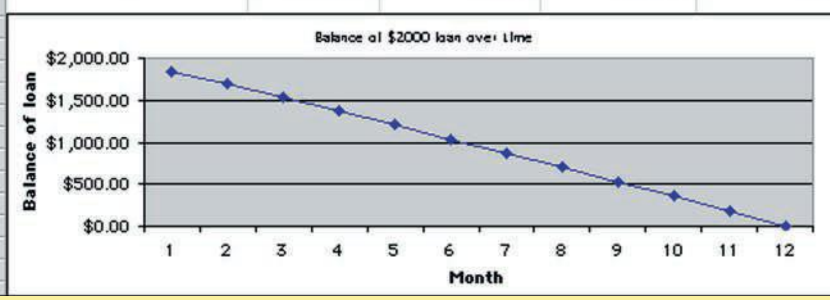
investigation

Constructing a loan repayment schedule using a spreadsheet

You have just obtained a $2000 loan from a bank to purchase a second-hand car. The interest rate on the loan is 15% p.a. on a monthly reducing balance. You agree to repay the loan (principal plus interest) in equal monthly instalments of $180.52 over a period of 1 year.

Let us construct the spreadsheet shown below.

	A	B	C	D	E
1	**Loan repayments spreadsheet**				
2					
3	Principal =	$2,000			
4	Interest rate (%pa) =	15			
5	Repayment	$ 180.52			
6	Repayments per year =	12			
7					
8	**Month**	**Principal**	**Interest**	**Repayment**	**Balance**
9	1	$2,000.00	$25.00	$180.52	$1,844.48
10	2	$1,844.48	$23.06	$180.52	$1,687.02
11	3	$1,687.02	$21.09	$180.52	$1,527.58
12	4	$1,527.58	$19.09	$180.52	$1,366.16
13	5	$1,366.16	$17.08	$180.52	$1,202.72
14	6	$1,202.72	$15.03	$180.52	$1,037.23
15	7	$1,037.23	$12.97	$180.52	$869.67
16	8	$869.67	$10.87	$180.52	$700.03
17	9	$700.03	$8.75	$180.52	$528.26
18	10	$528.26	$6.60	$180.52	$354.34
19	11	$354.34	$4.43	$180.52	$178.25
20	12	$178.25	$2.23	$180.52	-$0.04
21					



1 Enter the spreadsheet heading in cell A1 and the side headings in cells A3 to A6.

2 Enter 2000 in cell B3 then format it to currency. Similarly, enter 180.52 in cell B5 then format as currency.

3 Enter the data shown in cells B4 and B6.

4 Enter the column headings in row 8.

5 In cell A9, we start with the first month. Enter the value 1 here.

6 In order to make this spreadsheet versatile, where possible, the entries from now on should contain references to other cells rather than entries in the form of numbers.

7 In cell A10, enter the formula **=A9+1** (adding 1 to the value in the cell above). Copy this formula down to row 20.

8 In cell B9, enter the formula **=B3**. This reproduces the original value of the principal in this cell.

(Continued)

9 Enter the formula **=E9** in cell B10. This takes the balance at the end of the previous period as the principal for the next period. Copy this formula down from cell B10 to B20. The correct values will not appear in these cells yet, as column E has not been completed. Once the whole spreadsheet has been completed, all the cells containing a formula will display values.

10 The formula in cell C9 is that for simple interest.
Enter **=B9*$B$4/($B$6*100)**. Notice the reference to other cells rather than the entry of numeric values. Copy this formula down from C9 to C20.

11 The repayment is the same amount each month. Enter the formula **=$B$5** in cell D9. Copy this formula down from D9 to D20.

12 The balance at the end of each period is calculated as the principal at the start of the period plus the interest charges minus the payment made for that period. In cell E9, enter the formula **=B9+C9–D9**. Copy this formula down from cell E9 to E20. You should find that the spreadsheet displays all the values now.

13 Use the graphing facility of the spreadsheet to create a graph similar to the one shown.

14 Notice that the principal reduces each month, as does the interest. Change the principal in cell B3. Write down the effect this has on the spreadsheet.

15 Change the interest rate in cell B4. Describe the effect this has on the spreadsheet.

16 Increase and decrease the repayment in cell B5. What effect does this have?

17 What effect does increasing the number of payments per year have on the spreadsheet (increase the value in cell B6)? Decrease the number of yearly payments and note the effect.

18 A method used to decrease the interest payable on housing loans is to increase the repayment amount or to increase the number of yearly payments. Would doubling the monthly repayment halve the life of the loan? Would doubling the number of repayments per year halve the life of the loan? Explain.

19 Imagine you wanted to make an extra large payment one month. What effect would this have on the loan and the shape of the curve? Enter a value of 500 as the repayment in the sixth month. Describe the effect this has on the spreadsheet and the shape of the graph.

This spreadsheet can be used to check the answers to many of the problems on loan calculations. Save it so you have a copy readily available.

Loan repayments

With a reducing balance loan, an amount of interest is added to the principal each month and then a repayment is made which is then subtracted from the outstanding balance. Consider the case of a $2000 loan at 15% p.a. to be repaid over 1 year in equal monthly instalments of $180.52. We constructed the spreadsheet for this loan in the previous investigation and it is summarised below.

Month	Opening balance	Interest	Closing balance
1	$2000.00	$25.00	$1844.48
2	$1844.48	$23.06	$1687.02
3	$1687.02	$21.09	$1527.59
4	$1527.59	$19.09	$1366.17
5	$1366.17	$17.08	$1202.73
6	$1202.73	$15.03	$1037.25
7	$1037.25	$12.97	$ 869.70
8	$ 869.70	$10.87	$ 700.05
9	$ 700.05	$ 8.75	$ 528.29
10	$ 528.29	$ 6.60	$ 354.37
11	$ 354.37	$ 4.43	$ 178.29
12	$ 178.29	$ 2.23	$ 0.00

The actual calculation of the amount to be repaid each month to pay off the loan plus interest in the given period of time is beyond this course. We will look at these calculations in a graphics calculator investigation later. The most practical way to find the amount of each monthly repayment is to use a table of repayments.

Monthly repayment per $1000 borrowed

	Interest rate										
Year	**5%**	**6%**	**7%**	**8%**	**9%**	**10%**	**11%**	**12%**	**13%**	**14%**	**15%**
1	$85.61	$86.07	$86.53	$86.99	$87.45	$87.92	$88.38	$88.85	$89.32	$89.79	$90.26
2	$43.87	$44.32	$44.77	$45.23	$45.68	$46.14	$46.61	$47.07	$47.54	$48.01	$48.49
3	$29.97	$30.42	$30.88	$31.34	$31.80	$32.27	$32.74	$33.21	$33.69	$34.18	$34.67
4	$23.03	$23.49	$23.95	$24.41	$24.89	$25.36	$25.85	$26.33	$26.83	$27.33	$27.83
5	$18.87	$19.33	$19.80	$20.28	$20.76	$21.25	$21.74	$22.24	$22.75	$23.27	$23.79
6	$16.10	$16.57	$17.05	$17.53	$18.03	$18.53	$19.03	$19.55	$20.07	$20.61	$21.15
7	$14.13	$14.61	$15.09	$15.59	$16.09	$16.60	$17.12	$17.65	$18.19	$18.74	$19.30

(*continued*)

Monthly repayment per $1000 borrowed (*continued*)

	Interest rate										
Year	**5%**	**6%**	**7%**	**8%**	**9%**	**10%**	**11%**	**12%**	**13%**	**14%**	**15%**
8	$12.66	$13.14	$13.63	$14.14	$14.65	$15.17	$15.71	$16.25	$16.81	$17.37	$17.95
9	$11.52	$12.01	$12.51	$13.02	$13.54	$14.08	$14.63	$15.18	$15.75	$16.33	$16.92
10	$10.61	$11.10	$11.61	$12.13	$12.67	$13.22	$13.78	$14.35	$14.93	$15.53	$16.13
11	$ 9.86	$10.37	$10.88	$11.42	$11.96	$12.52	$13.09	$13.68	$14.28	$14.89	$15.51
12	$ 9.25	$ 9.76	$10.28	$10.82	$11.38	$11.95	$12.54	$13.13	$13.75	$14.37	$15.01
13	$ 8.73	$ 9.25	$ 9.78	$10.33	$10.90	$11.48	$12.08	$12.69	$13.31	$13.95	$14.60
14	$ 8.29	$ 8.81	$ 9.35	$ 9.91	$10.49	$11.08	$11.69	$12.31	$12.95	$13.60	$14.27
15	$ 7.91	$ 8.44	$ 8.99	$ 9.56	$10.14	$10.75	$11.37	$12.00	$12.65	$13.32	$14.00
16	$ 7.58	$ 8.11	$ 8.67	$ 9.25	$ 9.85	$10.46	$11.09	$11.74	$12.40	$13.08	$13.77
17	$ 7.29	$ 7.83	$ 8.40	$ 8.98	$ 9.59	$10.21	$10.85	$11.51	$12.19	$12.87	$13.58
18	$ 7.03	$ 7.58	$ 8.16	$ 8.75	$ 9.36	$10.00	$10.65	$11.32	$12.00	$12.70	$13.42
19	$ 6.80	$ 7.36	$ 7.94	$ 8.55	$ 9.17	$ 9.81	$10.47	$11.15	$11.85	$12.56	$13.28
20	$ 6.60	$ 7.16	$ 7.75	$ 8.36	$ 9.00	$ 9.65	$10.32	$11.01	$11.72	$12.44	$13.17
21	$ 6.42	$ 6.99	$ 7.58	$ 8.20	$ 8.85	$ 9.51	$10.19	$10.89	$11.60	$12.33	$13.07
22	$ 6.25	$ 6.83	$ 7.43	$ 8.06	$ 8.71	$ 9.38	$10.07	$10.78	$11.50	$12.24	$12.99
23	$ 6.10	$ 6.69	$ 7.30	$ 7.93	$ 8.59	$ 9.27	$ 9.97	$10.69	$11.42	$12.16	$12.92
24	$ 5.97	$ 6.56	$ 7.18	$ 7.82	$ 8.49	$ 9.17	$ 9.88	$10.60	$11.34	$12.10	$12.86
25	$ 5.85	$ 6.44	$ 7.07	$ 7.72	$ 8.39	$ 9.09	$ 9.80	$10.53	$11.28	$12.04	$12.81

The table shows the monthly repayment on a $1000 loan at various interest rates over various terms. To calculate the repayment on a loan, we simply multiply the repayment on $1000 by the number of thousands of dollars of the loan.

WORKED Example 11

Calculate the monthly repayment on a loan of $85 000 at 11% p.a. over a 25 year term.

THINK	WRITE
1 Look up the table to find the monthly repayment on $1000 at 11% p.a. for 25 years.	
2 Multiply this amount by 85.	Monthly repayment = $9.80 · 85 = $833

This table can also be used to make calculations such as the effect that interest rate rises will have on a home loan.

WORKED Example 12

The Radley family borrow $160 000 for a home at 8% p.a. over a 20 year term. They repay the loan at $1400 per month. If the interest rate rises to 9% p.a., will they need to increase their repayment and, if so, by how much?

THINK	WRITE
1 Look up the table to find the monthly repayment on $1000 at 9% p.a. for 20 years.	
2 Multiply this amount by 160.	Monthly repayment = $9.00 · 160 = $1440.00
3 If this amount is greater than $1400, state the amount that the repayment needs to rise.	The Radley family will need to increase their monthly repayments by $40.

remember

1. The amount of each monthly repayment is best determined by using a table of repayments.
2. The amount of each repayment is calculated by multiplying the monthly repayment on a $1000 loan by the number of thousands of the loan.

EXERCISE 3D Loan repayments

WORKED Example 11

1 Use the table of repayments on pages 127–8 to calculate the monthly repayment on a $75 000 loan at 7% p.a. over a 15-year term.

2 Use the table of repayments to calculate the monthly repayment on each of the following loans.
- a $2000 at 8% p.a. over a 2-year term
- b $15 000 at 13% p.a. over a 5-year term
- c $64 000 at 15% p.a. over a 25-year term
- d $100 000 at 12% p.a. over a 20-year term
- e $174 000 at 9% p.a. over a 22-year term

3 Jenny buys a computer for $4000 on the following terms: 10% deposit with the balance paid in equal monthly instalments over 3 years at an interest rate of 14% p.a.
- a Calculate Jenny's deposit.
- b Calculate the balance owing on the computer.
- c Use the table of repayments to calculate the amount of each monthly repayment.

WORKED Example 12

4 Mr and Mrs Dubois borrow $125 000 over 20 years at 10% p.a. to purchase a house. They repay the loan at a rate of $1500 per month. If the interest rate rises to 12% p.a., will Mr and Mrs Dubois need to increase the size of their repayments and, if so, by how much?

5 Mr and Mrs Munro take out a $180 000 home loan at 9% p.a. over a 25-year term.
   a Calculate the amount of each monthly repayment.
   b After 5 years the balance on the loan has been reduced to $167 890. The interest rate then rises to 10% p.a. Calculate the new monthly repayment required to complete the loan within the existing term.

6 A bank will lend customers money only if they believe the customer can afford the repayments. To determine this, the bank has a rule that the maximum monthly repayment a customer can afford is 25% of his or her gross monthly pay. Darren applies to the bank for a loan of $62 000 at 12% p.a. over 15 years. Darren has a gross annual salary of $36 000. Will Darren's loan be approved? Explain your answer.

7 Tracey and Barry have a combined gross income of $84 000.
   a Calculate Tracey and Barry's gross monthly income.
   b Using the rule applied in the previous question, what is the maximum monthly repayment on a loan that they can afford?
   c If interest rates are 11% p.a., calculate the maximum amount (in thousands) which they could borrow over a 25-year term.

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8 Mr and Mrs Yousef borrow $95 000 over 25 years at 8% p.a. interest.
   a Calculate the amount of each monthly repayment on the loan.
   b Mr and Mrs Yousef hope to pay the loan off in a much shorter period of time. By how much will they need to increase the monthly repayment to pay the loan off in 15 years?

investigation

Loan repayments

Using Excel

Use the **Loan repayments** weblink in your eBookPLUS and download the spreadsheet.

This spreadsheet shows the graph of a home loan of $100 000 at 6% p.a. that is repaid over 20 years.

Use the graph to determine how long it takes for the outstanding balance to reduce to:

1 $80 000
2 $50 000
3 $20 000.
4 Next, change the amount borrowed in the spreadsheet to $200 000. Does it take the same length of time for the outstanding balance to be halved?
5 Change the interest rate to 12% p.a. and the amount borrowed back to $100 000. Does it still take the same length of time for the balance to be halved?
6 Experiment with different loans and look for a pattern in the way in which the balance of the loan reduces.

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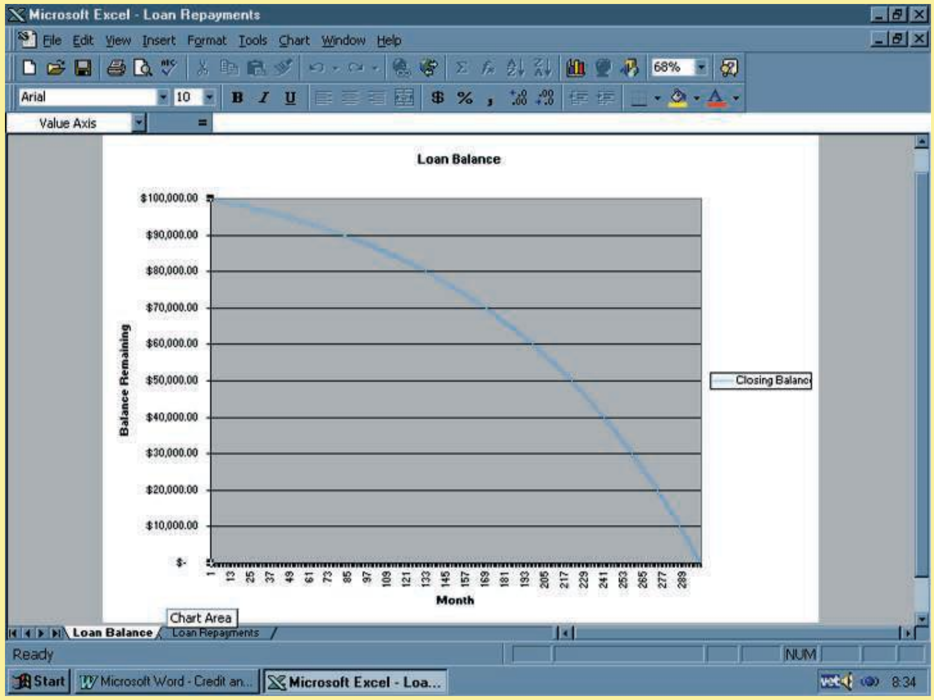
Spreadsheet
206 Loan repayments



Using the Casio *fx*-9860G AU graphics calculator

The features used in this investigation are available in the financial section of the Casio graphics calculator. The instructions given refer to the Casio *fx*-9860G AU version. You are advised to consult with your manual before commencing, so you are familiar with the variables and terms. The compound interest section provides a function enabling a periodic payment on a loan or investment account to be determined.

The amortisation section provides the facility to display:

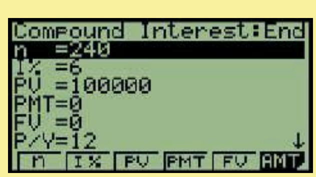
- the interest and principal portion of each instalment
- the balance of the principal after any instalment
- the total interest paid on the loan to any particular point
- a graph showing the progress of the loan.

Imagine you obtain a $100 000 housing loan. The interest rate on the loan is 6% p.a. on a monthly reducing balance. You agree to repay the loan plus interest in equal monthly instalments over a period of 20 years.

Task 1

Enter the compound interest function of the financial section. From the main (MENU), select TVM and press (F2) (CMPD). From the problem posed above, you have the following information:

number of years = 20
interest rate = 6
present value = 100 000
period of payment = monthly
reducing interest period = monthly.



(*Continued*)

This enables you to enter the following data:

$n = 20 \cdot 12 = 240$	(There are monthly payments over 20 years.)
$I\% = 6$	
$PV = 100\,000$	
$PMT = 0$	(There should be no value here at this stage.)
$FV = 0$	(This value is unknown at this stage.)
$P/Y = 12$	(Payments are monthly; that is, 12 per year.)
$C/Y = 12$	(Interest is calculated monthly.)

Press (F4) (PMT) to display the monthly payment required. Your screen should display PMT = –716.4310585.

So, a monthly payment of $716.43 is required to repay the loan over 20 years.

Press (F1) (REPT) to return to the compound interest screen, and then find the future value of the loan by pressing (F5) (FV). Your screen should display FV = 0 (meaning that the loan has been paid off).

Task 2

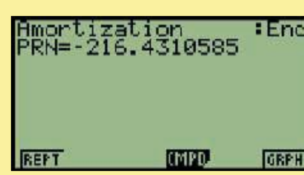
Enter the amortisation section through the (F4) (AMT).

PM1 represents the first instalment period of instalments from 1 through to ‘*n*’.
PM2 represents the second instalment period of instalments from 1 through to ‘*n*’.
Let us say we are interested in the whole term of the loan; that is, periods 1 to 240.
Start with the following data: PM1 = 1 (instalment 1) PM2 = 240 (instalment 240)
The other entries should be complete from the previous calculations.

1 Looking at the interest and principal portion of each instalment

The INT function shows the interest portion of instalment PM1.
The PRN function shows the principal portion of instalment PM1.

Pressing (F2) (INT) accesses the INT function, which indicates that the interest component of instalment 1 is $500. Pressing (F1) (REPT) (to repeat) then (F3) (PRN) accesses the PRN function, which shows that the principal component of instalment 1 is $216.43.



This means that only about 30% ($216.43 | $716.43 · 100) of the first instalment goes towards paying off the principal; the remainder is consumed in interest charges.

Change PM1 to a value of 2. What change is there in the interest and principal components of instalment 2?

Investigate the change in these components as you change the PM1 value through to 240 (the last instalment).

2 Looking at the balance of the principal after any instalment

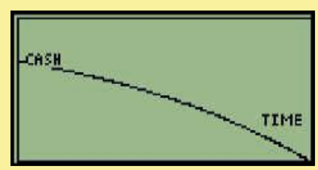
The BAL function displays the balance of principal after PM2.
Enter a value of 1 for PM2. You should find that the balance on the loan after the first payment is $99 783.57; that is, only $216.43 has been paid off the loan.

Change the PM2 value and see what happens over the duration of the loan.

3 Looking at the total interest paid on the loan to any particular point

The **Σ**INT function displays the total interest paid from instalment PM1 to (and including) instalment PM2. Enter a value of 1 for PM1 and 2 for PM2. This would display the total interest paid on instalments 1 and 2. You should find that this value is $998.92. Investigate to determine the total interest payable over other time periods. You should compare the interest paid in the first two instalments with that paid in the last two instalments. What do you notice?

4 Looking at a graph showing the progress of the loan



The screen displaying the above results shows a graphing facility at F6 (GRPH). Press this button to display a graph showing the progress of the loan. Pressing SHIFT F1 (TRCE) will allow you to trace the curve. The interest (INT) and principal (PRN) are displayed for each instalment (n) by pressing ▶.

Using the TI-Nspire CAS graphics calculator

The features used in this investigation are available in the finance section of the TI-Nspire CAS graphics calculator. You are advised to consult with your manual before commencing, so you are familiar with the variables and format of the functions.

The finance section enables us to examine aspects of periodic payments on loans or investment accounts.

The amortisation section provides the facility to display:

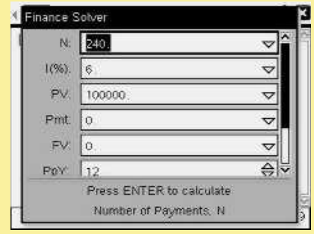
- the interest **Σ**Int(and principal **Σ**Prn(portion of each instalment
- the balance bal(of the principal after any instalment
- the total interest **Σ**Int(paid on the loan to any particular point
- a graph showing the progress of the loan.

Imagine you obtain a $100 000 housing loan. The interest rate on the loan is 6% p.a. on a monthly reducing balance. You agree to repay the loan plus interest in equal monthly instalments over a period of 20 years.

(*Continued*)

For the TI-Nspire

Task 1

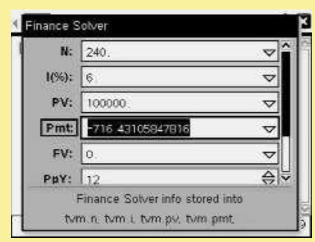


To calculate the monthly repayment required to pay off the loan of $100 000 in 20 years, open a Calculator page.
Press:
- MENU menu
- 8: Finance 8
- 1: Finance Solver 1.

From the information given, enter the data as shown.
$n = 12 \cdot 20 = 240$ (Monthly payments for 20 years)
$I\% = 6$
PV (present value) = 100 000
$PMT = 0$ (this is the value required)
FV (future value) = 0 (loan paid off)
$PpY = 12$ (Payments are monthly — 12 per year)
$CpY = 12$ (Interest is calculated monthly).

Note: CpY: 12 is off screen.

Press Tab tab to move between fields.



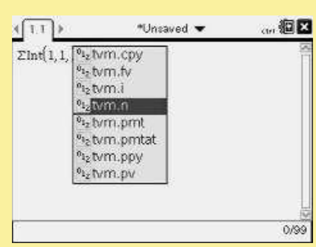
Press Tab tab to move to the 'Pmt' field and then press ENTER enter.
The screen displays a monthly payment of $716.43.
Press Tab tab to move to the 'FV' field and then press ENTER enter.

Task 2

The values for n, I, PV, PMT, FV, PpY and CpY have been stored as variables.
To see any of these values, press:
- Esc esc
- Var sto→ var.

Select any of the options and press ENTER enter. The value will be displayed.
The TI-Nspire CAS calculator has the capacity to calculate the interest paid, principal paid and balance at various points throughout the loan repayment schedule.

1 Looking at the interest and principal portion of each instalment.

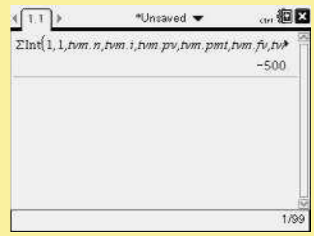


To calculate the interest paid on the first payment, press:
- MENU menu
- 8: Finance 8
- 3: Amortization 3
- 3: Interest Paid 3.

The format of the expression is:
ΣInt $(nPMT1, nPMT2, n, I, PV, PMT, FV, CpY, PpY)$.

To show the interest on the first payment, *nPMT*1 is 1 and *nPMT*2 is 1. To retrieve the values of *n*, *I*, *PV*, *PMT*, *FV*, Cp*Y* and Pp*Y*, press the var key, select them in the correct order, separating each by a comma.

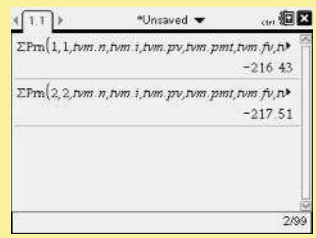
Complete the entry line as:
ΣInt(1, 1, *tvm.n*, *tvm.i*, *tvm.pv*, *tvm.pmt*, *tvm.fv*, *tvm.cpy*, *tvm.ppy*).
Then press ENTER.
The interest paid on the first payment is $500.



To see the principal component of Instalment 1, press:

- MENU
- 8: Finance
- 3: Amortization
- 4: Principal Paid.

Complete the entry line as:
ΣPrn(1, 1, *tvm.n, tvm.i, tvm.pv, tvm.pmt, tvm.fv, tvm.cpy, tvm.ppy*).
Then press ENTER.
This shows only $216.43 of the $716.43 (30%) goes towards paying off the principal; the remainder is interest.



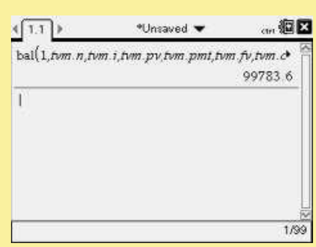
To display the principal component of payment 2, repeat the previous procedure, and enter in the formula:
ΣInt(2, 2, *tvm.n, tvm.i, tvm.pv, tvm.pmt, tvm.fv, tvm.cpy, tvm.ppy*), then press ENTER.
Describe what you notice.
Investigate the change in the interest and principal components as the loan progresses from payment 1 to payment 240 (the last instalment).

2 Looking at the balance of the principal after any instalment

To calculate the balance of the loan after the first payment, press:

- MENU
- 8: Finance
- 3: Amortization
- 2: Balance.

Complete the entry line as:
Σbal(1, *tvm.n*, *tvm.i*, *tvm.pv*, *tvm.pmt*, *tvm.fv*, *tvm.cpy*, *tvm.ppy*).
Then press ENTER.



(*Continued*)

The balance on the loan after the first payment is \$99 783.60. Change the payment number and investigate what happens over the duration of the loan.

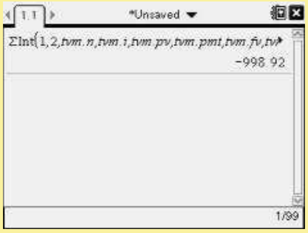
3 Looking at the total interest paid on the loan to any particular point.

To find the total interest paid on instalments 1 and 2, Complete the entry line as:

ΣInt(1, 2, *tvm.n*, *tvm.i*, *tvm.pv*, *tvm.pmt*, *tvm.fv*, *tvm.cpy*, *tvm.ppy*).

The total interest paid on instalments 1 and 2 is \$998.92.

Investigate the total interest payable over other time periods.



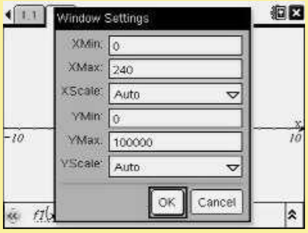
Compare the interest paid in the first two instalments with that paid in the last two instalments. What do you notice?

4 Looking at a graph showing the progress of the loan.

To change the window settings to suitable values, open a Graphs page.
Press:

- MENU (menu)
- 4: Window/Zoom (4)
- 1: Window Settings (1).

Enter the values as shown and select OK.



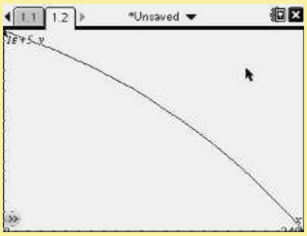
To graph the progress of the loan, press:

- MENU (menu)
- 3: Graph Type (3)
- 2: Parametric (2).

Complete the entry lines as:

$x1(t) = t$
$y1(t)$ = bal(t, 240, 6, 100 000, –716.43, 0, 12, 12)
$0 \le t \le 240$ *tstep* = 10.

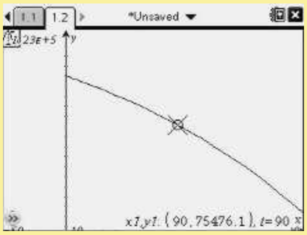
Then press ENTER (enter).



To trace the balance over the life of the loan, press:

- MENU (menu)
- 5: Trace (5)
- 1: Graph Trace (1).

Move the cursor along the curve to trace the balance of the loan as time progresses.



Task 3

You should now be familiar with the functions in the amortisation section of your calculator. Conduct further investigations of your own so you can provide a report on the following situation.

Cathy and Noel Jackson are considering purchasing a home. They would need a loan of $175 000. The best deal they can arrange is at 6.5% p.a. on a monthly reducing balance over a period of 25 years. They have compiled a list of questions they would like answered.

1. What would be the monthly repayment?
2. What total interest would be paid over the period of the loan?
3. If the house is sold after 10 years, how much would be owing on the loan?
4. At what point in time would half the loan have been repaid?
5. If an extra $100 per month is paid, by how much could the term of the loan be reduced?
6. If the interest rate rose to 7% p.a., what difference would this make to the monthly instalment?
7. At what stage of the loan is the interest component of the instalment equal to the principal component?

Compile a report to answer these questions for the Jacksons.

Bonds, debentures and term deposits

Debentures

If a company needs money, one option is for it to offer a **debenture** (a legal document detailing an investment agreement) for sale to the public. An investor will pay an amount of money (principal) to the company, and in return the company agrees to pay the investor interest at regular intervals (monthly, quarterly or yearly). At the end of the agreed term the principal is returned to the investor. The advantage of the debenture is two-fold: first, the company has the use of the money during the agreed period to make more money for the company; and second, the investors know what their return will be for each period and are guaranteed the return of the principal.

Term deposits

Term deposits allow an investor to lend money to a bank or building society for a particular length of time. The money cannot be withdrawn during the agreed period but earns a better interest rate than in a normal savings account. At the end of the term the interest plus the principal is paid back to the investor. The advantage of the term deposit is that the money is secure and the interest rate is better than that on a savings account. The disadvantage, of course, is that if the money is needed during the period it cannot be withdrawn (except under special circumstances agreed to by the bank).

TODAY'S BEST BUYS

Top 5 Term Deposit Rates - 30 days			
Company	Product	Rate %	Apply
LAIKI BANK	Term Deposits	4	Apply
ING DIRECT It's your money	Term Deposits	3.75	Apply
RaboPlus Help yourself	Term Deposit	3.3	Apply
MACQUARIE BANK	Term Deposit	3	Apply
st.george	Term Deposits	1.5	Apply

Need a short term deposit? These 30-days term deposits are ranked by interest rate on balance of $25000

Top 5 Term Deposit Rates - 90 days			
Company	Product	Rate %	Apply
ING DIRECT It's your money	Term Deposits	5	Apply
MACQUARIE BANK	Term Deposit	4.9	Apply
UBank backed by nab	Term Deposit	4.61	Apply
Defcredit	Term Deposit	4.55	Apply
Westpac	Term Deposits	4.2	Apply

These 90-days term deposits are ranked by interest rate on balance of $25000

Source: www.ratecity.com.au

Investment bonds

Investment bonds are another form of investment which is offered to the investor by a bank or the government, and interest is paid on the investment monthly, quarterly, six monthly or annually. The one advantage is that the bond can be sold to someone else during the period before the maturation date. This allows the investor some flexibility if the money is needed during the period of investment.

All the above investment types offer advantages to the investor and to the institution. The institution has the use of the money over a fixed period and the investor receives higher than normal interest. All of these investments carry some risk and individuals must decide on which type to use based on personal circumstances.

Bonds, debentures and term deposits are simple interest accounts.

WORKED Example 13

Jaclyn buys $50 000 worth of debentures in a company. She earns 9.5% p.a. simple interest, paid to her quarterly (that is, every 3 months). If the agreed period of the debenture was 18 months:

a **calculate the amount of interest Jaclyn will earn for each quarter**

b **calculate the total amount of interest collected at the end of the term.**

THINK | **WRITE**

a 1. Write the simple interest formula.

a $I = \frac{PRT}{100}$

2. List the values of P, R and T.

$P = 50\,000$
$R = 9.5$
$T = \frac{1}{4}$

3. Substitute into the formula and evaluate.

$I = \frac{50\,000 \cdot 9.5 \cdot \frac{1}{4}}{100}$
$= 1187.50$

4. Write your answer.

Jaclyn will earn \$1187.50 for each quarter.

b 1. There are 6 quarters in 18 months. Alternatively, use the simple interest formula with the new data.

b Total interest = \$1187.50 · 6
= \$7125

or

$I = \frac{50\,000 \cdot 9.5 \cdot 1.5}{100}$
$= 7125$

2. Write your answer.

The total interest earned is \$7125.

WORKED Example 14

Townbank offers a term deposit account paying investors 12.5% p.a. simple interest on investments over \$100 000 for 2 years or more. Peta decides to invest \$150 000 in this account for 2 years. How much interest will Peta earn at the end of the investment?

THINK | **WRITE**

1. Write the simple interest formula.

$I = \frac{PRT}{100}$

2. List the values of P, R and T.

$P = 150\,000$
$R = 12.5$
$T = 2$

3. Substitute into the formula and evaluate.

$I = \frac{150\,000 \cdot 12.5 \cdot 2}{100}$
$= \$37\,500$

4. Write your answer.

Peta's \$150 000 invested for 2 years will earn \$37 500.

WORKED Example 15

An investment bond is offered to the public at 9% p.a. Louise buys a bond worth $2000 that will mature in 2 years. How much in total will Louise receive at the end of the 2 years?

THINK	WRITE
1 Write the simple interest formula.	$I = \frac{PRT}{100}$
2 List the values of P, R and T.	$P = 2000$ $R = 9$ $T = 2$
3 Substitute into the formula.	$I = \frac{2000 \cdot 9 \cdot 2}{100}$
4 Use a calculator to evaluate.	$I = \$360$
5 Add interest to principal.	$A = P + I$ $A = 2000 + 360$ $= 2360$
6 Write your answer.	The $2000 investment bond will mature at the end of 2 years to a total of $2360 at simple interest of 9% p.a.

remember

1. Simple interest accounts include bonds, debentures and term deposits.
2. Read the question carefully: does it ask for the interest or the final total amount?

EXERCISE 3E Bonds, debentures and term deposits

eBook plus
Digital doc: Spreadsheet 205 Simple interest

WORKED Example 13

1 Spice Clothing company offers debentures paying 8% p.a. interest paid quarterly for a period of 2 years. When $20 000 worth of Spice debentures are purchased, calculate the total return on the investment.

WORKED Example 14

2 Harry decided to invest $2000 in a term deposit for 18 months. The bank offered 10.5% p.a. interest paid each half-year. Calculate the interest Harry would earn on the investment.

WORKED Example 15

3 An investment bond is advertised as paying $10\frac{1}{2}$% p.a. interest on a 3-year investment. Elise purchased a bond for $3000, but needed to sell it after 18 months. How much will Elise receive at the end of her 18-month investment?

4 Rabbit debentures, worth $10 000, were purchased for a period of 15 months. The debenture paid 12% p.a., payable each 3 months. What was the investment worth at the end of the 15 months?

5 JNK Bank offers term deposits on amounts above \$5000 at 12% p.a. simple interest payable each quarter for periods longer than 2 years. Mr Smith invests \$6000 in this term deposit for $2\frac{1}{2}$ years. What is Mr Smith's final return on his money?

6 Mark purchases a \$2500 investment bond earning $12\frac{1}{4}$% p.a. interest paid yearly. The bond matures after 2 years. What interest will Mark earn?

7 **multiple choice**

Debentures in TRADEX are issued at 9% p.a. simple interest. The interest gained on an investment of \$7000 over 3 years would be:

A \$630 **B** \$1890 **C** \$7630 **D** \$8890 **E** \$18 900

8 **multiple choice**

The rate of interest on a term deposit for 3 months is 4.25% per year. If \$10 000 is invested in the term deposit, the amount of interest earned over the 3 months is:

A \$106.25 **B** \$141.67 **C** \$425 **D** \$1062.50 **E** \$1275

9 **multiple choice**

State government bonds pay interest of $7\frac{1}{4}$% p.a. simple interest. Philippa invested \$2500 in the bonds which mature in 5 years. Philippa's income each quarter would be:

A \$45.31 **B** \$135.94 **C** \$181.25 **D** \$725 **E** \$2718.77

10 **multiple choice**

ElCorp offers company debentures earning $8\frac{1}{2}$% p.a. interest for an investment of \$5000 for 2 years. The interest on the investment is:

A \$170 **B** \$212.50 **C** \$825 **D** \$850 **E** \$5850

11 **multiple choice**

A term deposit is advertised stating that if \$2500 is invested for 2 years the interest earned is \$285. The rate of interest per annum is:

A 0.114% **B** 5.7% **C** 10% **D** 11.4% **E** 17.5%

12 **multiple choice**

A principal amount is invested in a bond that will accumulate to a total of \$64 365 after 4 months at $6\frac{1}{2}$% p.a. The principal is:

A \$6300 **B** \$50 000 **C** \$60 000 **D** \$63 000 **E** \$63 336

13 The following term deposit rates were advertised in a magazine.

Toni Ford had \$5500 to invest. Calculate her return if she invested the money in a term deposit with this bank for:

a 35 days
b 120 days
c 1 year.

Hint: Express days as a fraction of a year.

Term	Rate
30–59 days	4.2% p.a.
60–149 days	4.7% p.a.
150–269 days	5.0% p.a.
270–365 days	5.4% p.a.

14 Dennis and Delia have \$7500 to invest. They know that they will need the money in 18 months but are not sure how to invest it. While reading a magazine, they see the following three advertisements:

i investment bonds offered at $12\frac{1}{2}$% p.a. interest paid each 6 months

ii debentures in a company paying 12% p.a. with interest paid each quarter

iii a term deposit paying $11\frac{3}{4}$% p.a. interest paid each 3 months.

a Calculate their total return on each investment.

b What did you notice about the time in which the interest was calculated?

Bank savings accounts

Most banks offer their customers savings accounts with interest that is usually paid on:

1. the minimum monthly balance, or
2. the daily balance.

The interest is added at a specified time — say once or twice a year — as nominated by the bank, for example, on the first day of June and December of each year. The more frequently the interest is added, the better for the customers.

Savings accounts — minimum monthly balances

To calculate interest on a **minimum monthly balance** saving account, the bank looks at the balances of the account for each month and calculates the interest on the smallest balance that appears in each month.

WORKED Example 16

At the beginning of March, Ryan had \$621 in his savings bank account. On 10 March he deposited \$60. If the bank pays 8% p.a. interest paid monthly and calculated on the minimum monthly balance, calculate the interest Ryan earns in March.

THINK	WRITE
1 The smallest balance for March is \$621, as the only other transaction in that month increased the balance.	Minimum monthly balance for March is \$621.
2 Write the simple interest formula.	$I = \frac{PRT}{100}$
3 List the values of P, R and T.	$P = 621$ $R = 8$ $T = \frac{1}{12}$
4 Substitute into the formula and evaluate.	$I = \frac{621 \cdot 8 \cdot \frac{1}{12}}{100}$ $= 4.14$
5 Write your answer.	The interest earned for the month of March was \$4.14.

The minimum monthly balance method is used in the next worked example.

WORKED Example 17

Date	Deposit	Withdrawal	Balance
3/7	$100		$337.50
7/7	$500		$837.50
21/7		$678	$159.50
28/7	$ 50		$209.50

The above bank statement shows the transactions for July. Find the interest that will be earned in July if the bank pays 7% p.a. simple interest on the minimum monthly balance.

THINK / **WRITE**

1. To find the smallest balance for July, look at all the running balances. Also check balances at the start and end of the month. Notice that the balance on 1 and 2 July, if shown, would have been $237.50.
   Minimum monthly balance for July is $159.50.
2. Write the simple interest formula.
   $I = \frac{PRT}{100}$
3. List the values of P, R and T.
   $P = 159.50$
   $R = 7$
   $T = \frac{1}{12}$
4. Substitute into the formula and evaluate.
   $I = \frac{159.50 \cdot 7 \cdot \frac{1}{12}}{100}$
   $= 0.93$
5. Write your answer.
   The interest earned for July was $0.93.

Savings accounts — daily balances

To calculate the interest on a **daily balance** saving account, the bank looks at the balances of the account. The number of days each balance is maintained is used to calculate the interest. When doing these calculations for yourself, you need to set out your workings carefully, for example using tables.

Let's investigate Worked example 17 again, using the daily balance method.

WORKED Example 18

Daily balance method

Use the daily balance method and the bank statement shown in Worked example 17 to find the interest that will be earned in July, if the bank pays 7% p.a. simple interest on the daily balance.

THINK

1. Set up a table showing each new balance and the number of days the balance applies. Look at all running balances including those for 1 and 31 July.
2. Calculate the interest for each balance. As the interest rate is in % per annum, express the number of days as a fraction of a year; for example, 2 days = $\frac{2}{365}$ of a year.

WRITE

Balance $	Number of days the balance applies	Simple interest calculations $	Interest earned $
$237.50	2	$\dfrac{237.50 \cdot 7 \cdot \frac{2}{365}}{100}$	$0.0911
$337.50	4	$\dfrac{337.50 \cdot 7 \cdot \frac{4}{365}}{100}$	$0.2589
$837.50	14	$\dfrac{837.50 \cdot 7 \cdot \frac{14}{365}}{100}$	$2.2486
$159.50	7	$\dfrac{159.50 \cdot 7 \cdot \frac{7}{365}}{100}$	$0.2141
$209.50	4	$\dfrac{209.50 \cdot 7 \cdot \frac{4}{365}}{100}$	$0.1607

3. Sum the interest. The calculations were to hundredths of a cent for accuracy.
   Interest for month = $2.9734
4. Round off to the nearest cent.
   $2.9734 ≈ $2.97
5. Write your answer.
   The interest earned for July was $2.97.

The daily balance method offers more interest than the minimum monthly balance method, as it credits the customer for all monies in the account, including the $500 deposited for 14 days.

Graphics Calculator tip! Calculating the number of days between dates

The Casio *fx*-9860G AU and TI-Nspire CAS graphics calculators both have functions which will calculate the number of days between dates. This can be helpful in interest calculations involving daily balances.

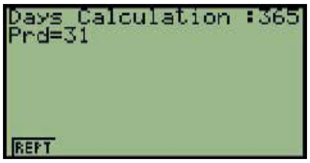
For the Casio *fx*-9860G AU

To find the number of days between
15 December 2008 and 15 Jan 2009.
Press:

- MENU
- C: TVM
- F6 (▶)
- F2 (DAYS).

The dates are entered in the form MMDDYYYY (prompting occurs for each one). Enter values for d1 and d2. This screen shows the days between 15 December 2008 and 15 January 2009

Press F1 (PRD) to show the number of days between these two dates.



For the TI-Nspire

To find the number of days between:

(a) 21 April and 12 October

(b) 15 December 2008 and 15 Jan 2009.

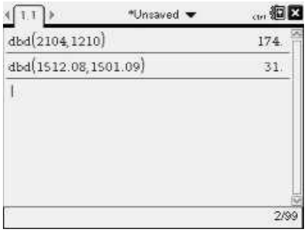
Open a Calculator page.

Complete the entry line as:

dbd(2104,1210)

dbd(1512.08,1501.09).

Press ENTER after each line.
Dates are expressed in the form DDMM (if no year is mentioned) or DDMM.YY (if a year is specified).



remember

1. Two methods used by banks for calculating interest on savings accounts are:
   (a) minimum monthly balances
   (b) daily balances.
2. Daily balances offer the best interest rate for investors.
3. Look at the balances on the first and last day of the month when establishing the minimum monthly balance or daily balances.
4. Express days as a fraction of a year; for example, 1 day = $\frac{1}{365}$ of a year.

EXERCISE 3F Bank savings accounts

WORKED Example 16

1 A bank savings account showed that the opening balance for the month was $2150. That month Paul paid the following bills out of the account:
Electricity $21.60 Telephone $10.30 Rent $52.00
Paul also deposited his wage of $620 for the month into the account.
a What was Paul's minimum monthly balance?
b If the bank pays 5.5% p.a. paid monthly on the minimum monthly balance, how much interest did Paul earn in the month?

eBook*plus*
Digital docs:
SkillSHEET 3.5 Minimum monthly balance
Spreadsheet 204 Home loan calculations

WORKED Example 17

2

Date	Deposit	Withdrawal	Balance
1/5			$27.50
3/5	$12		$39.50
7/5		$16	$23.50
19/5		$ 8	$15.50
27/5	$10		$25.50

Roberta's bank statement shows the above transactions for May. Find the interest Roberta will earn in May if the bank pays 6% p.a. simple interest:
a on the minimum monthly balance
b on the daily balance.

WORKED Example 18

3 For the month of July, Rhonda received $3.20 in interest on her savings account. Rhonda's minimum balance in July was $426.20. What was the per annum simple interest rate offered by the bank?

4 Kristen receives the following statement from her bank. Due to a computer error the interest and balances were not calculated.

Kristen rang the bank and was told that she received interest at a rate of $6\frac{3}{4}$% p.a. paid monthly on her minimum monthly balance. Copy out Kristen's statement and fill in the balances and interest payments.

Date	Transaction	Debit	Credit	Balance
1 May	Balance B/F			2132.20
3 May	Cheq 4217	460.27		
7 May	Deposit		230.16	
17 May	Cheq 4218	891.20		
26 May	Wages		1740.60	
31 May	Interest		______	
2 June	Deposit		415.10	
8 June	Cheq 4220	2217.00		
19 June	Cheq 4219	428.50		
21 June	Cheq 4222	16.80		
23 June	Wages		1740.60	
30 June	Interest		______	
1 July	Deposit		22.80	
4 July	Cheq 4221	36.72		
18 July	Cheq 4223	280.96		
26 July	Wages		1740.60	
31 July	Interest		______	

5 Using the bank statement from question **4**, another bank offers to show Kristen that daily balance interest credited each quarter is more rewarding. The interest is still 6.75% p.a. but is only credited at the end of the quarter, that is, on 31 July. Calculate:

a the interest for the quarter ending July

b the increase in interest earned using the daily balance method.

Hint: This could be done using a spreadsheet.

6 Clark Kent has the following income and expenses for August and September.

Income:	\$1410.20 salary each fortnight beginning 4 August
	\$461.27 income tax refund on 5 September
	\$68.20 cheque from health fund on 10 August
Expenses:	\$620.80 rent on 20 August and 20 September
	\$180.64 telephone account on 2 September
	\$150.26 electricity account on 15 August
	\$180.00 Visa account on 30 August
	\$327.60 health fund on 5 August and 5 September

Draw up a statement (as for question **4**) for Clark, remembering that he receives $7\frac{1}{2}$% interest paid on the last day of each month on the minimum monthly balance in the account.

7 If the savings interest rate is $2\frac{1}{2}$% p.a., calculate the interest credited at the end of each quarter for the following accounts using:

i the minimum monthly balance

ii the daily balance.

Also calculate:

iii the increase in interest earned using the daily rather than the minimum monthly balance method.

a The third quarter statement for July, August and September

Date	Deposit	Withdrawal	Balance
3/7		\$100	\$ 750.00
7/8	\$ 500		\$ 1 250.00
21/8	\$ 670		\$ 1 920.00
28/8		\$420	\$ 1 500.00
20/9	\$10 000		\$11 500.00

b The first quarter statement for January, February and March in 2008

Date	Deposit	Withdrawal	Balance
31/12/2007	\$100		\$400.00
1/2/2008	\$600		?
1/3/2008		\$100	?
28/3/2008	\$ 50		?

c The fourth quarter statement for October, November and December.

Date	Deposit	Withdrawal	Balance
3/10	\$2 100		\$2 450.00
17/12	\$3 500		\$5 950.00
21/12		\$1 900	\$4 050.00
22/12		\$ 400	\$3 650.00
28/12		\$ 650	\$3 000.00

10 QUICK QUESTIONS 2

1 Calculate the amount of flat rate interest payable on a loan of $4500 at 21% p.a. over a 3-year term.

2 A loan of $2000 is repaid over 1 year at a rate of $100 per week. Calculate the rate of interest charged on the loan.

3 A loan of $120 000 at 11% p.a. reducible over 20 years is repaid at $1238.63 per month. The bank also charges an $8 per month account management fee. Calculate the total cost of repaying the loan.

4 A loan of $5000 is advertised at a rate of 9% p.a. flat rate interest for a term of 4 years. Use the formula $E = \frac{(1+r)^n - 1}{n}$ to calculate the effective rate of interest on this loan (correct to 1 decimal place).

5 A loan of $10 000 at 11% p.a. reducible interest is repaid over 4 years at a rate of $258.46 per month. Calculate the equivalent flat rate of interest charged on the loan (correct to 1 decimal place).

6 Calculate the monthly repayment on a $1000 loan over a 2 year term at 8% p.a.

7 What monthly repayment would be necessary on a $20 000 loan over a 10-year term at 10% p.a.?

8 What would be the quarterly earnings on a debenture worth $20 000 earning 7.5% p.a. simple interest?

9 A term deposit of $5000 is invested for 2 years at 8.25% p.a. simple interest. How much would be collected when the term deposit is paid out?

10 How much interest would Tom earn on his bank account for the month of January if his minimum monthly balance was $825 and his bank paid 3.25% p.a. interest?

Investing in real estate

Real estate transactions

When buying or selling property, most people consult a real estate agency. An agent acts on behalf of the seller and receives a fee, called a **commission**. The commission rate can vary slightly, but a common rate used is:

5% of the first $18 000 plus
2.5% of the remainder of the sale price.

A 10% **Goods and Services Tax (GST)** charge is then applied to the agent's commission. This money is a tax applied by the government.

The seller is responsible for the agent's commission and the GST charge. This money is subtracted from the sale price of the property.

WORKED Example 19

How much will Bill receive from the sale of his unit if a real estate agent sells it for $124 220?

THINK	WRITE
1 Calculate the commission on the sale price.	Commission = 5% of $18 000 + 2.5% of ($124 220 – $18 000) = 5% of $18 000 + 2.5% of $106 220 = $900 + $2655.50 = $3555.50
2 Calculate the GST (10% of agent's commission).	GST = 10% of $3555.50 = $355.55
3 Subtract commission and GST from the sale price.	Amount received from sale = sale price – commission – GST = $124 220 – $3555.50 – $355.55 = $120 308.95
4 Write the answer.	Bill will receive $120 308.95.

When a property is transferred from one name to another, **transfer duty** is payable on the transaction. This is a government tax and is paid by the purchaser. Different scales exist for commercial properties and first-home buyers. A transfer for a principal place of residence (where the purchaser lives — not a rented property) for other than a first-home buyer attracts the following transfer duty charges.

Purchase price/value	Transfer duty
Up to $350 000	$1.00 for each $100 or part of $100
$350 001 to $540 000	$3500 + $3.50 for every $100 or part of $100 over $350 000
$540 001 to $980 000	$10 150 + $4.50 for every $100 or part of $100 over $540 000
More than $980 000	$29 950 + $5.25 for every $100 or part of $100 over $980 000

WORKED Example 20

When Hes buys Bill's unit for $124 220, what will be the total cost of his purchase?

THINK	WRITE
1 The buyer is responsible for transfer duty charges. Locate the correct category.	Transfer duty = $1 per $100 or part of $100
2 Find the number of lots of $100. Any part lots must be rounded up to the next whole number.	No. of $100 lots = $124 220 | $100 = 1242.2 = 1243 (rounded up)
3 Calculate the transfer duty.	Transfer duty = $1 · 1243 = $1243
4 Add the transfer duty to the cost of the house.	Total cost of purchase = house cost + transfer duty = $124 220 + $1243 = $125 463

WORKED Example 21

What transfer duty would be paid on a house purchased for $375 410?

THINK	WRITE
1 Locate the correct category for transfer duty.	Transfer duty = $3500 + $3.50 per $100 or part of $100 over $350 000
2 Calculate the amount over $350 000.	Amount over $350 000 = $375 410 − $350 000 = $25 410
3 Calculate the number of lots of $100 in this amount (round up to the next whole number if necessary).	No. of $100 lots = $25 410 ÷ $100 = 254.1 = 255 (rounded up)
4 Calculate the transfer duty.	Transfer duty = $3500 + $3.50 × 255 = $3500 + $892.50 = $4392.50

remember

1. In real estate transactions, the real estate agent acts on behalf of the seller.
2. The seller is responsible for the agent's commission.
3. A GST is also paid by the seller on the agent's commission. This money is collected by the government.
4. The seller receives the sale price minus the commission and GST charges.
   Proceeds from sale = sale price − commission − GST
5. The purchaser is responsible for transfer duty charges in the transfer of the property name.
6. The cost to the purchaser is the sale price plus transfer duty charges.
   Cost to purchaser = sale price + transfer duty.

EXERCISE 3G Investing in real estate

For commission and transfer duty rates, refer to the scales on pages 148 and 149.

WORKED Example 19

1 Calculate the commission payable on the sale of units valued at:
  a $79 950 b $128 250 c $462 000.

2 What GST would be due on the unit in question 1?

3 What proceeds would be obtained from the sale of the units in question 1?

WORKED Example 20

4 What would the purchaser pay for the unit in question 1a?

5 Calculate the transfer duty payable on the purchase of units valued at:
  a $187 250 b $269 240 c $542 120.

WORKED Example 21

6 What would the purchaser pay for the units in question 5?

7 Ian sold his unit for $275 000 and upgraded to a residence with a sale tag of $475 000.
  a What did he receive from the sale of his unit?
  b How much did his new residence cost him?
  c How much did Ian pay in charges for this upgrade?

8 Bob sold his house to Chris for $420 000. Bob was liable for the commission and GST costs while Chris had to pay transfer duty for the transaction. How much more did Chris pay for the house than Bob received from the sale?

9 The Gardner family paid $3637.50 in commission to the real estate agent who sold their unit. What was the sale price of their unit?

10 Jim and Nancy paid $4725 in transfer duty when they purchased their new home. What was the total cost of their home?

11 The transfer duty payable on Gordon and Jenny's new home was $13 750. What was the advertised sale price of the home?

investigation

Rent or buy?

Although the Australian dream is to own your own home, the option of renting may be more financially viable, particularly if the purchase of a home is only for a short period (for example, 3 or 4 years).

Consider the following case scenario.

The Ling family has just moved to Queensland from interstate. They plan to stay in Queensland for only 3 years, after which time Mr Ling will receive a transfer to his company's Sydney office. Should they purchase a house, knowing that they will have to sell in 3 years' time; or should they simply rent for the 3 years?

Mr Ling has jotted down costs associated with purchasing and renting. These are itemised below.

Purchasing

Consider purchasing a home for \$400 000.

Savings to be used for the purchase are \$50 000.

Purchase expenses of \$10 000 are to be allowed. This money will come from savings.

Remaining \$40 000 provides the 10% deposit for the housing loan.

Repayments on the \$360 000 home loan are \$2500 per month.

House maintenance costs (maintenance, rates and insurance) are estimated to be \$3000 per year.

Say the house sells for \$475 000 in 3 years' time.

Real estate sale costs estimated at \$12 000.

Estimate the balance owing on the loan before the sale to be \$340 000.

Renting

Estimated rent on a house valued at \$400 000 is:

\$400 per week for the first year

\$420 per week for the second year

\$450 per week for the third year

Invest the \$50 000 savings in a term deposit for 3 years at 7.5% p.a. compounding monthly.

An extra \$10 000 can be saved each year since rent costs are lower than house repayments. This money can be invested at the end of each year at 6.5% p.a. compounding monthly.

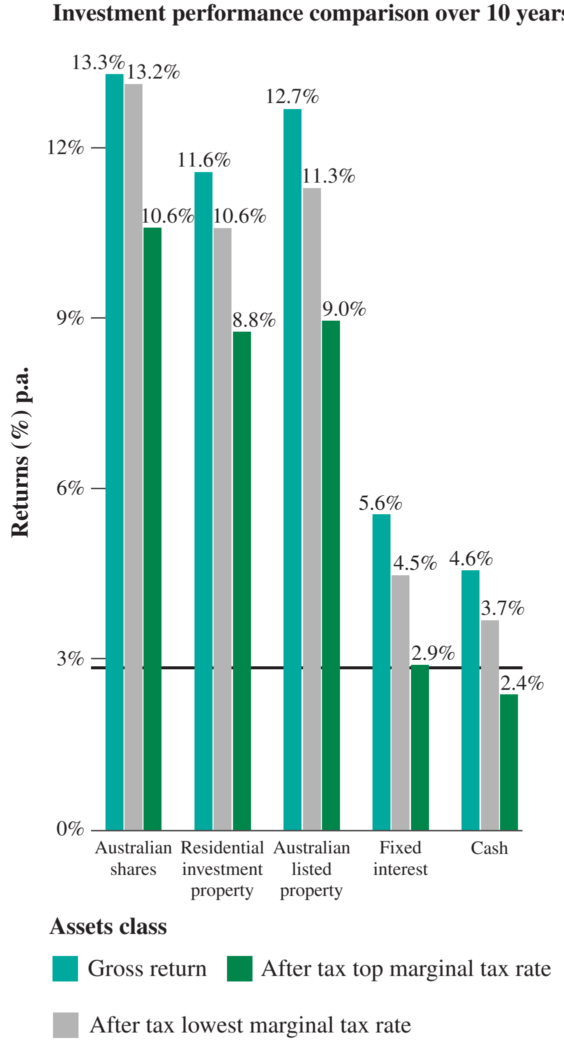
1 Draw up a spreadsheet detailing the costs associated with buying and renting over the 3-year period.

2 Write a report to Mr and Mrs Ling recommending whether they should buy or rent. Support your decision by referring to figures in your spreadsheet.

Investing in the stock market

Investment options

When it comes to investing funds, there are three broad areas to be considered. These are shares, interest-bearing deposits and property. Although there have been short-term fluctuations in these areas, over the long term, domestic shares have been shown to perform well. Consider the following graph.



Source: ASX

Depositing money in banks and similar financial institutions is the most common type of investment, as it is safe and the return can be calculated in advance.

There is an element of risk when investing funds by purchasing **shares**. The shares have the potential to return more money to the investor than depositing money in a bank; however, there is also the chance that the shares may fall in value.

The operations of the share market are quite complex. The following section looks at a simplified version of its operations, providing adequate basic knowledge for the first-time investor.

The share market

Over 40% of Australians have either a direct or indirect involvement in the Australian **share market**. Many of these are recent first-time investors who became involved through participating in **floats** of Australian companies such as Woolworths, the Commonwealth Bank and Telstra. In order to be a wise investor, it is essential to understand the basic operations of the share market and the common terminology used.

The origin of stock exchanges is traced back to thirteenth century Italy. By way of comparison, the first stockbroking operations started in Australia in 1829. The *Sydney Morning Herald* newspaper began publishing daily quotations of share prices in 1837. In 1987, the six Australian state exchanges united to form the Australian Stock Exchange (ASX). Trading on the ASX by the end of the year 2000 had typically increased to record more than 50 000 trades daily. The ASX is accessed through the World Wide Web. Conduct a search to investigate the resources available through the site.

How does a company become listed on the stock exchange? Some companies reach a point where, in order to expand, the company needs a substantial injection of capital. Instead of borrowing this money, companies can raise the funds from the general public. As long as the company complies with prerequisite standards set by the ASX, it can apply to become a publicly listed company. It can then 'float' or 'list' the company on the share market and initially sell shares to the public at a nominated price per share. This initial price is the **face** or **par value** of the share, which remains fixed. After this stage, the shares can be traded readily on the stock exchange. If the company is considered to have potential, and demand for its shares is high, the share price will rise. If, on the other hand, the demand for the shares is low, the price would fall. The price of the share traded on the stockmarket is known as the **market price**. This price typically varies in the market at any time.

When a company has been operating for a while, if a profit is made, the company can decide to distribute some of this profit to shareholders as a **dividend**. The company can keep some of the money for expansion of the business. A dividend is usually quoted as cents per share, for example, 15 cents per share.

In buying and selling shares on the stockmarket, the broker doing the trading for you charges a fee, called **brokerage**. This fee is added to the cost of the shares when you buy and subtracted from the sale price of the shares when you sell. Shares these days are generally bought and sold via the Internet (brokerage still applies). These charges (brokerages) can vary, but a common rate is:

- $29.95 for transactions up to $10 000
- 0.31% of the transaction value for those above $10 000

Two important terms are the **dividend yield** and the **price–earnings ratio**. Expressed mathematically, these can be defined as:

$$\text{Dividend yield} = \frac{\text{dividend per share}}{\text{market price of share}} \cdot 100\%$$

This can be considered as the percent return on the money invested. It can be compared with the effective interest rate on a bank deposit.

$$\text{Price–earnings ratio} = \frac{\text{market price}}{\text{earnings (that is, yearly profit)}}$$

This represents the number of years it would take for the profit earned to pay for the shares.

WORKED Example 22

Margaret bought 2000 XYZ shares at $7.40 each. What would it cost her to purchase them?

THINK	WRITE
1 The cost of the shares is made up of the market price plus the brokerage payable.	Total cost = market price + brokerage
2 Calculate the market price.	Market price = $7.40 · 2000 = $14 800
3 Calculate the brokerage payable, choosing the correct category.	Brokerage = 0.31% of $14 800 = $45.88
4 Find the total cost.	Total cost = $14 800 + $45.88 = $14 845.88

WORKED Example 23

If Margaret (Worked example 22) sells her 2000 shares for $7.44 each, we would expect her to make a small profit. Is this the case? Explain.

THINK	WRITE
1 The money received from a sale is the market price minus the brokerage charges.	Money received from sale = market price − brokerage
2 Calculate the market price.	Market price = $7.44 · 2000 = $14 880
3 Calculate the brokerage payable, choosing the correct category.	Brokerage = $0.31% of $14 880 = $46.13
4 Find the amount received.	Money received from sale = $14 880 − $46.13 = $14 833.87
5 Compare the cost with the amount received from the sale.	Cost of shares = $14 845.88 Therefore, a small loss is made. Loss = $14 845.88 − $14 833.87 = $12.01
6 Write an explanation.	Even though the sale price per share is higher than the buying price per share, the brokerage charged on the two transactions erodes this profit margin.

WORKED Example 24

A company has an after-tax profit of $34.2 million. There are 90 million shares in the company. What dividend will the company declare if all the profits are distributed to the shareholders?

THINK	WRITE
1 Dividend is calculated by dividing the profit by the number of shares.	Dividend = $34 200 000 | 90 000 000 = $0.38
2 Give a written answer.	The dividend is 38c per share.

WORKED Example 25

Paul bought 1000 Cottonworths shares at $5.75 per share. The company paid a yearly dividend of 19.5 cents per share (assume that this is all the profits). Calculate:

a **the total dividend received**
b **the dividend yield**
c **the price–earnings ratio (P–E ratio).**

THINK	WRITE
a Total dividend = dividend per share · number of shares	**a** Total dividend = 19.5c · 1000 = $195
b 1 Quote the rule for dividend yield.	**b** Dividend yield $= \frac{\text{dividend per share}}{\text{market price per share}} \cdot 100$
2 Substitute values (take care with units).	$= \frac{19.5\text{c}}{\$5.75} \cdot 100\%$ $= \frac{19.5\text{c}}{575\text{c}} \cdot 100\%$
3 Calculate the result.	= 3.39%
c 1 Quote the rule for price–earnings ratio.	**c** P–E ratio $= \frac{\text{market price per share}}{\text{yearly profit per share}}$
2 Substitute values (take care with units).	$= \frac{\$5.75}{19.5\text{c}}$ $= \frac{575\text{c}}{19.5\text{c}}$
3 Calculate the result.	= 29.5

remember

1. When shares are purchased, brokerage is added to the cost of the shares.
2. When shares are sold, brokerage is subtracted from the sale price of the shares.
3. Brokerage is calculated depending on the value of the order.
4. An investment in shares earns money through dividend payments and by increasing in value.
5. A dividend is a payment made to shareholders. It is calculated by dividing the profit to be distributed to shareholders by the total number of shares in the company.
6. To calculate the true worth of an investment we calculate the dividend yield. The dividend yield is found by writing the dividend as a percentage of the share price, that is,
$$\text{Dividend yield} = \frac{\text{dividend per share}}{\text{market price of share}} \cdot 100\%.$$
7. The price–earnings ratio represents the number of years it takes for the dividends to pay for the shares.
$$\text{Price–earnings ratio} = \frac{\text{market price per share}}{\text{yearly dividend per share}}$$

EXERCISE 3H Investing in the stock market

Where brokerage calculations are appropriate, use the scale of charges on page 154. Assume that all after-tax profits are paid as dividends.

WORKED Example 22

1 What would be the cost of purchasing 5000 Quintex shares at \$3.75 each?

2 Brisbane Bank shares are selling at \$24.50 each. What would a parcel of 100 cost?

WORKED Example 23

3 How much money would be received from the sale of 200 Reservation Bank shares at \$26 each?

4 Harry Wallman shares are selling at \$3.20 each. After selling a parcel of 5000, how much money would be received?

Unless stated otherwise, for the calculations in this exercise, assume that companies distribute all their profits as dividends.

WORKED Example 24

5 A company has issued 20 million shares and makes an after-tax profit of \$5 million. Calculate the dividend to be declared by the company.

6 A company that has 2 million shares makes a profit of \$3 million. Calculate the dividend that will be declared.

7 A company makes an after-tax profit of \$150 000. If there are 2.5 million shares in the company, calculate the dividend that the company will declare.

8 A company with an after-tax profit of $1.2 million consists of 4.1 million shares. Calculate the dividend the company will declare, in cents, correct to 2 decimal places.

9 A company makes a before-tax (gross) profit of $3.4 million.
a If the company is taxed at the rate of 36%, calculate the amount of tax it must pay.
b What will be the after-tax profit of the company?
c If there are 5 million shares in the company, calculate the dividend that the company will declare.

10 A company makes a gross profit of $14.5 million and there are 8 million shares in the company.
a Calculate the after-tax profit if company tax is paid at the rate of 36%.
b If $3.2 million is to be reinvested in the company, calculate the amount of money that is to be distributed to the shareholders.
c Calculate the dividend that this company will declare.

11 A company declares a dividend of 78c. If there are 4.2 million shares in the company, calculate the after-tax profit of the company.

WORKED Example 25b

12 A company with a share price of $10.50 declares a dividend of 48c per share. Calculate the dividend yield for this company.

13 Copy and complete the table below.

Dividend	Share price	Dividend yield
$0.56	$8.40	
$0.78	$7.40	
$1.20	$23.40	
$1.09	$15.76	
$0.04	$0.76	

14 Hsiang purchased shares in a company for $3.78 per share. The company paid Hsiang a dividend of 11c per share. Calculate the dividend yield, correct to 2 decimal places.

15 **multiple choice**

Which of the following companies paid the highest dividend yield?
A Company A has a share value of $4.56 and pays a dividend of 35c/share.
B Company B has a share value of $6.30 and pays a dividend of 62c/share.
C Company C has a share value of $12.40 and pays a dividend of $1.10/share.
D Company D has a share value of 85c and pays a dividend of 7.65c/share.
E Company E has a share value of $2.50 and pays a dividend of 20c/share.

16 George bought $5600 worth of shares in a company. The dividend yield for that company was 6.5%. Calculate the amount that George receives in dividends.

17 Andrea bought shares in a company for $11.50 each. The company paid a dividend of 76c/share.

a Calculate the dividend yield for this company.

b One year later the share value is $12.12. The company then has a dividend yield of 8.75%. Calculate the dividend per share.

18 A company's prospectus predicts that the dividend yield for the coming year will be 6.7%. Its share price is $21.50.

a Calculate the dividend paid if the dividend yield in the prospectus is paid.

b If there are 5.2 million shares in the company, calculate the after-tax profit of the company.

19 Janice buys shares in a company at $5.76. The company pays a dividend in July of 22.7c and a dividend in February of 26.4c. Calculate the dividend yield for the whole financial year (July to the following June).

20 The dividend paid by a company for the 2008–09 financial year was 5.6c/share, with a share price of $9.50.

a Calculate the dividend yield for 2008–09.

b In the 2009–10 financial year the share price rose by 12%. Calculate the share price for this year.

c In 2009–10 the dividend paid to shareholders increased by 15%. Calculate the dividend paid, in cents, correct to 1 decimal place.

d Calculate the dividend yield for 2009–10.

WORKED Example 25

21 A company's shares are available at $12.80 each. The company paid a yearly dividend of 15.5 cents per share. On a purchase of 500 shares, calculate:

a the total dividend received

b the dividend yield

c the price–earnings ratio.

22 A company's shares are selling at $5 each. They pay a yearly dividend of 6 cents per share.

a What total dividend would a bundle of 1000 return?

b Determine the dividend yield.

c What is the price–earnings ratio of the shares?

23 A company's shares are selling at $6 each and returning an annual dividend of 4.5 cents per share.

a How many shares would need to be purchased to receive an annual dividend of $225?

b What is the dividend yield of the shares?

c What is the price–earnings ratio?

24 A company's shares have a dividend yield of 4.3%. What is their price–earnings ratio?

Graphing share performance

Because shares offer no guaranteed returns, we can only use the past performance of a share to try to predict its future performance. One simple way to do this would be by graphing the value of the share at regular intervals and then drawing a line of best fit to try to monitor the trend.

By continuing the **line of best fit** you can make a prediction for future share prices. This is called **extrapolating** information from the graph. **Interpolate** is the opposite of extrapolate and occurs when drawing a graph using data found between the end points.

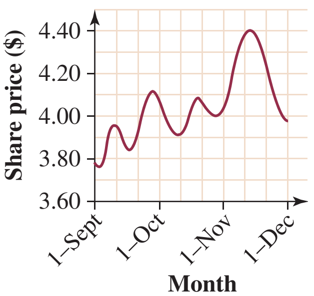
Many share traders use sophisticated analysis of graphs (charts) to determine both short- and long-term movements of share prices, but this 'technical analysis' is beyond the scope of this book.

WORKED Example 26

The graph at right shows the share price of a company over a 3-month period.

a On the graph draw a line of best fit.

b Use your line of best fit to estimate the share price after another three months.

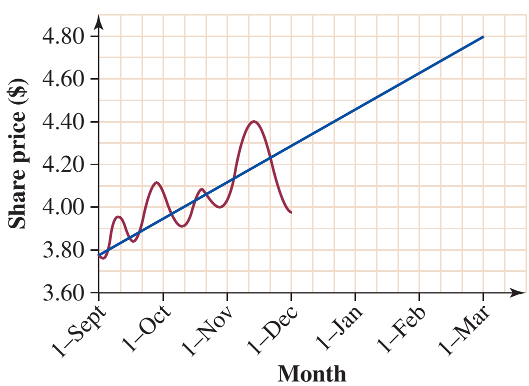


THINK

a Draw a line on the graph, which best fits between the points marked.

WRITE

a



THINK

b Extend the line of best fit for three months and read the predicted share price.

WRITE

b The predicted share price is $4.80.

You should be able to produce your own graph to answer this type of question from a set of data that you have been given or have researched.

WORKED Example 27

Below is the share price of a company taken on the first day of the month for one year.

Month	Share price	Month	Share price
January	\$10.34	July	\$10.98
February	\$10.54	August	\$11.56
March	\$10.65	September	\$11.34
April	\$10.89	October	\$11.23
May	\$10.72	November	\$11.48
June	\$11.10	December	\$11.72

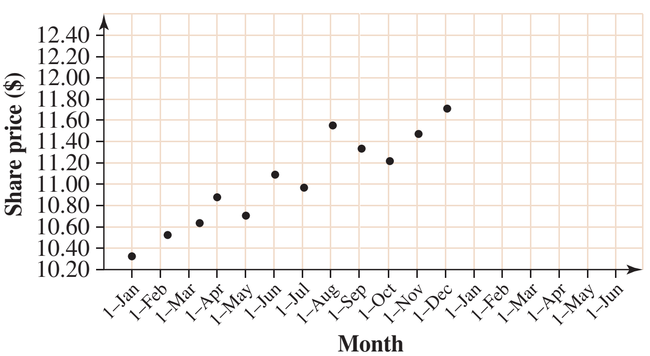
a **On a set of axes plot the share price for each month and draw a line of best fit.**

b **Predict the share price in June of the following year.**

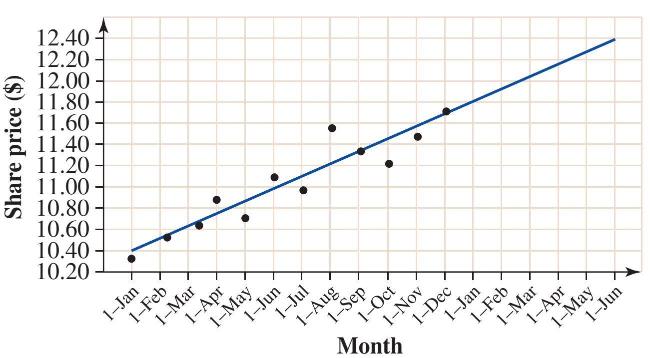
THINK | **WRITE**

a 1 Draw up a set of axes and plot the data.

a



2 Draw a straight line on the graph that best fits in with the marked points.



b 1 Extend the line of best fit for six months.

b

2 Predict the share price by reading from the line of best fit.

The predicted share price is \$12.35.

remember

1. To try to predict possible future movement in share prices, we use the past performance of the share.
2. Graphing the past share price allows us to examine trends by drawing a line of best fit on the graph.
3. We can then use the line of best fit to predict the future price of a share.

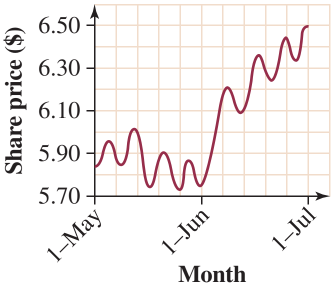
EXERCISE 3I Graphing share performance

WORKED Example 26

1 The graph at right shows the movement in a share price over a 3-month period.

a Copy the graph into your book and on it draw a line of best fit.

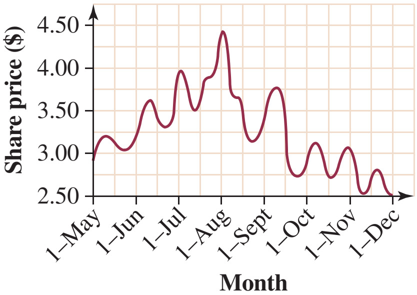
b Use your graph to predict the value of the share after 6 months.



2 The graph at right shows the movement in a share price over a 6-month period.

a Copy the graph into your book and on it draw a line of best fit.

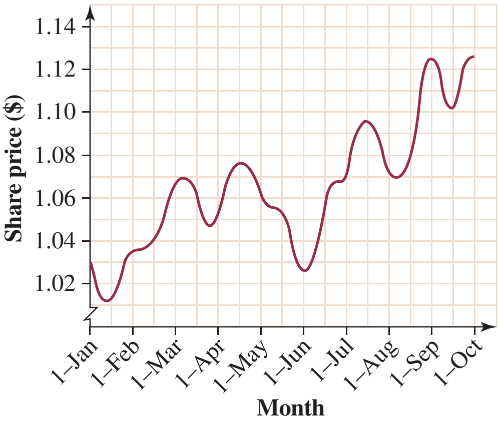
b Predict the value of the share after a further 6 months.



3 The graph at right shows the movement in a share price over a 9-month period.

a Copy the graph into your book and on it draw a line of best fit.

b Use your graph to predict the value of the share after a further 12 months.

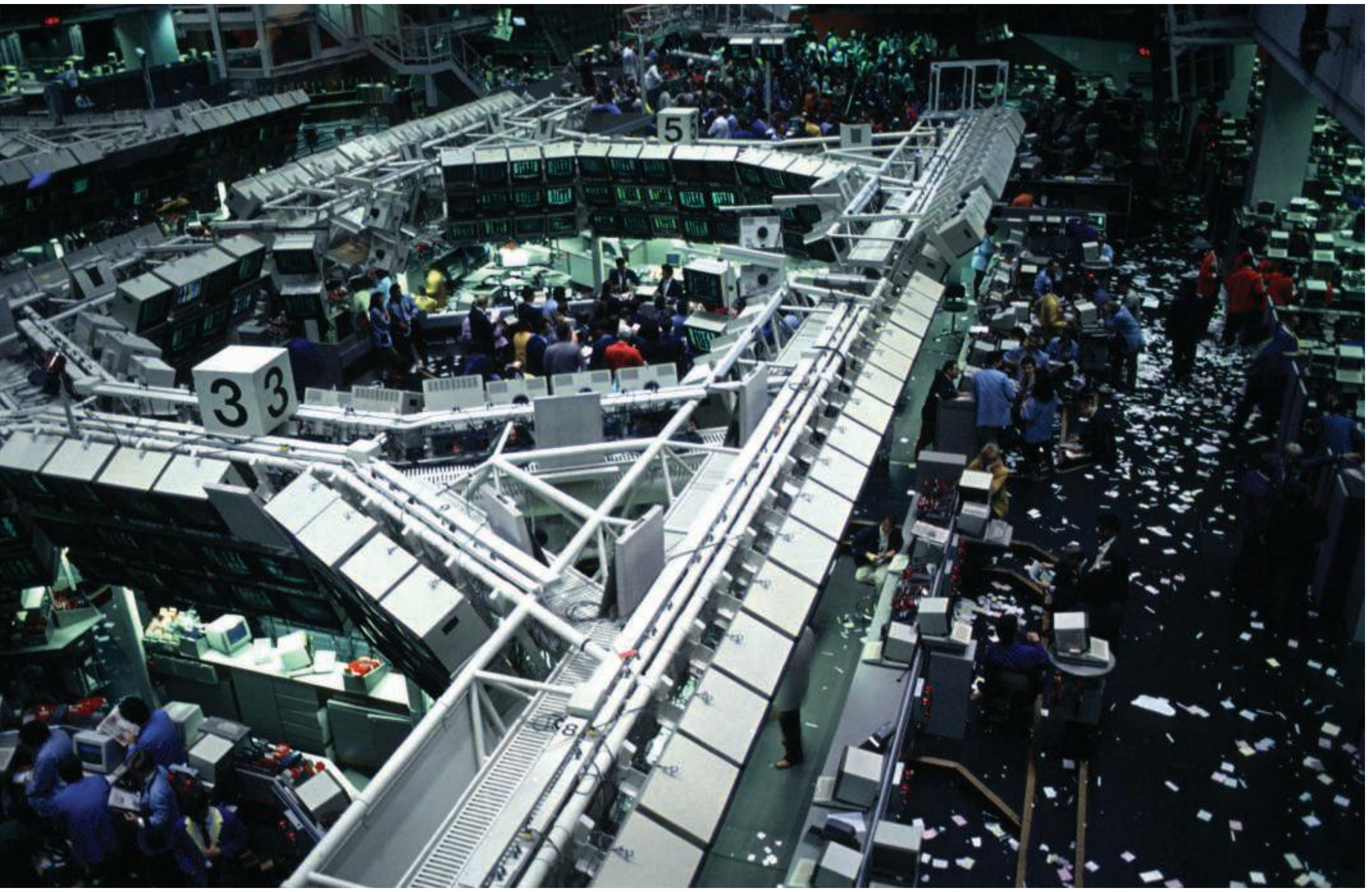


WORKED Example 27

4 The table below shows the share price of a large multinational company over a 12-month period.

Month	Share price	Month	Share price
January	$12.86	July	$13.45
February	$13.43	August	$13.86
March	$11.98	September	$14.40
April	$12.10	October	$13.65
May	$12.11	November	$13.20
June	$12.98	December	$12.86

a Plot the share prices on a set of axes and on your graph draw a line of best fit.
b Use your graph to predict the value of the share after a further 6 months.



5 The table below shows the share price of BigCorp Productions Ltd over a period of one year.

Month	Share price	Month	Share price
January	$12.40	July	$13.17
February	$12.82	August	$13.62
March	$12.67	September	$13.41
April	$13.05	October	$13.30
May	$13.06	November	$13.46
June	$12.89	December	$13.20

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Digital doc: WorkSHEET 3.2

a Graph the share price for each month and show a line of best fit.
b Use your line of best fit to predict the share price after a further year.

History of mathematics

THE WALL STREET CRASH

The sudden and virtually unanticipated Wall Street Crash of 1929 caused crippling widespread share price losses and consequent hardships.

In October 1929, the barely forty-year-old New York Stock Exchange (NYSE) experienced an unprecedented 'collapse' in stocks that were traded in this market and the shares in these stocks were virtually worthless. This event, known as 'The Wall Street Crash', is deemed to be the beginning of the world-wide economic devastation known as 'The Great Depression'. Business activity throughout the world was paralysed, virtually overnight. The prior 'bull market', in which there were more buyers than sellers, ended with a precipitous fall in the market prices of all stocks. The share scripts that were held by business organisations and individual investors were instantaneously substantially devalued or worthless. Stock markets throughout the world had many sellers and virtually no buyers. Banks were closed as depositors tried to withdraw their money. Many of the banks were never to open again. A complete loss of confidence in stock markets prevailed until the beginning of the Second World War in 1939.

Questions

1 What caused the Wall Street Crash?
2 What happened to share prices at the time of the crash?
3 What were the consequences of the crash?

Research

What was the cause and outcome of the Financial Crisis in 2008?

Stockmarket

Share market prices are published in the state and national newspapers daily. They are also available on the ASX website. To conduct a search of the site and see the facilities available, use the **ASX** weblink in your eBookPLUS. You might prefer to conduct this investigation by gathering information from the web.

The Australian Information

Industrials

The Australian portrays industrial information like this:

Industrial shares

Code	Stock	4pm close	Move	Vol 100's	Buy	Sell	Year High	Year Low	Div. yield	P–E ratio
7599	SDI	.23			.23	.27	.325	.15		9
1229	Sedgman	.56	+.01	4043	.555	.56	3.11	.37	13.39f	20
2891	SeekComm	2.90	+.17	6083	2.88	2.90	6.40	2.02	4.97f	11
8198	SelectHvt	3.15		143	3.15	3.22	7	2.30	11.11f	6
8944	SelectVac	.002			.002	.003	.019	.002		
9894	Senetas	.028		1000	.026	.028	.063	.02		56
7032	Servcorp	2.25	+.05	61	2.20	2.25	4.623	2.046	7.78f	5
7728	SevenNwrk	6.37	+.08	4448	6.29	6.37	10.38	5.01	5.34f	25
4098	SevenNwrk respf	80	+.89	26	79.35	.80	97.30	72.50	8.91f	
7858	Shearer	2.35			2.41	2.80	2.40	2.30	6.38f	9

The first column 'Code' contains a four-digit identification code unique to the share. Column 2 identifies the name of the share. The third column, headed '4 pm close', shows the price of the share at 4 pm. The 'Move' column shows the change, in dollars, that this closing price represents, compared with the previous closing price (a negative sign indicating a downward movement in price and a positive sign indicating an upward movement). The column 'Vol 100's' shows how many shares were traded on the day (in lots of 100). The 'Buy' and 'Sell' columns show the prices that buyers are prepared to pay to purchase and the prices that sellers are asking. The year 'High' and 'Low' columns show the highest and lowest prices of the year. The last two columns show the dividend yield and the price–earnings ratio. Blanks occurring in any column indicate that there were no transactions for that particular entry.

In this investigation you are going to track the progress of two sets of shares.

1. Select two shares you feel might be a sound investment from those displayed in a newspaper or on a Web site. Record the price per share of each.
2. Consider buying a bundle of 1000 of each of these shares. Calculate the total cost of each of these shares, including brokerage. Brokerage is calculated separately on each share transaction.
3. Record the 4 pm closing price of both the shares over a period of time (about 1 month).
4. Record the volume traded each day.
5. Record the dividend yield and price-earnings ratio for each stock.

(*Continued*)

6 At the end of your observation period, assume that you have to sell your shares. Compile a report on the progress of your shares. Your report must include for each share:
   a the name of the share
   b calculations showing the cost of purchasing 1000 shares (include brokerage)
   c a graph plotting the share price over the period of time, including a description of the trend shown by your graph
   d calculations showing the money received from the sale of the shares (include brokerage)
   e the result of your trading (in terms of profit or loss)
   f the dividend yield of the share
   g the P–E ratio of the share
   h a prediction of a future value of the share. On your graph, draw the line of best fit to find the trend in the movement of the share. Extrapolate this line to predict the price of the share 6 months from now.
   i any other observations or comments you consider relevant.

History of mathematics

THE DOW JONES INDUSTRIAL AVERAGE

When you see headlines like, *Dow hammered*, the Dow is the Dow Jones Industrial Average (DJIA or the Dow). A share market index created in 1889 by two young Wall Street Journal journalists, Charles Dow and Eddie Jones, it tracked 10 stocks, of which General Electric is the only company remaining in the index today. The index was expanded to 20 stocks in 1916 and 30 in 1928. Before 1928, the index was calculated by adding the prices of the 20 stocks and dividing by the number of stocks. From April 1999, a more sophisticated divisor, 0.2252, has been used. So, if you add up the 30 stock prices and divide by 0.2252, you have the Dow.

The index tracks the performance of 30 large companies: not the largest or best companies in America but representing the diversity of the economy. Stocks are the paid-up capital or fully-paid shares of a particular company. They change over time and include AT&T, Boeing, Coca-Cola, Exxon Oil, General Motors, IBM, McDonalds and retailers.

The DJIA lost 11%, or nearly 1165 points, in the April to June (second) quarter of 2002. Stocks for Coca-Cola and McDonalds did not fall during this period; however, most other stocks fell significantly. The greatest losses (in percentages) were for IBM, which was down 30.8%, AT&T, which lost 31.8%, and Intel 39.9%.

Indexes can give you an idea of how particular types of shares are performing. Other important market indexes are the Nikkei 225 Index, the Nasdaq Composite, Crude Oil and Standard & Poor's 500. Many companies and countries develop their own indexes to keep an eye on the performance of their shares.

Questions

1. Who created the Dow Jones Industrial Average?
2. How many companies are now used to measure this index?
3. How is the index now calculated?
4. Which types of stock contributed most to the second quarter decline of 2002?

Research

1. Use either television news or a newspaper to find out what the current DJIA is.
2. Search the Web to find out what the Nasdaq Composite tracks.

summary

Flat rate interest

- A flat rate loan is one where interest is calculated based on the amount initially borrowed.
- Flat rate loans have the interest calculated using the simple interest formula:

$$I = \frac{PRT}{100}$$

- The total repayments on a flat rate loan are calculated by adding the interest to the amount borrowed.
- The monthly or weekly repayments on a flat rate loan are calculated by dividing the total repayments by the number of weeks or months in the term of the loan.

Home loans

- The interest of home loans is calculated at a reducible rate. This means that the interest is calculated on the outstanding balance at the time and not on the initial amount borrowed.
- The interest on home loans is usually calculated and added monthly, while repayments are calculated on a monthly basis.
- To calculate the total cost of a home loan, we multiply the amount of each monthly payment by the number of payments.

The cost of a loan

- To compare a flat rate loan with a reducing balance loan, the equivalent reducing balance interest rate can be calculated using the formula:

$$E = \frac{(1+r)^n - 1}{n}.$$

- When we are comparing two or more loans, the most accurate comparison is done by calculating the total cost of repaying the loan.
- A loan that is repaid over a shorter period of time will generally cost less even if the interest rate may be slightly higher.
- The flexibility of loan repayments is an important consideration when we are calculating the cost of a loan.
- When we are calculating the cost of a loan, fees such as application fee and account management fees must be considered along with the interest payable.

Loan repayments

- The amount of each monthly repayment is best calculated using a table of monthly repayments.
- The monthly repayment on a $1000 loan at the given rate over the given term is then multiplied by the number of thousands of the loan to find the size of each repayment.

Bonds, debentures and term deposits

- Term investments with governments are called bonds.
- Term investments with companies are called debentures.
- Term investments with banks are called term deposits.
- All three are investments for a fixed period of time offering a simple interest rate.

Bank savings accounts — minimum monthly and daily balances

- Two methods used by banks for calculating interest on savings accounts are:
  1. minimum monthly balances
  2. daily balances.
- Daily balances offer the best interest rate for investors.
- Look at the balances on the first and last day of the month when establishing the minimum monthly balance or daily balances.
- Express days as a fraction of a year; for example, 1 day = $\frac{1}{365}$ of a year.

Investing in real estate

- A real estate agent acts on behalf of the seller.
- The seller is responsible for the agent's commission and GST charges on the commission.
- The buyer is responsible for transfer duty charges in the transfer of documents to the buyer's name.

Investing in the stock market

- When shares are being purchased, brokerage is added to the cost of the shares.
- When shares are being sold, brokerage is subtracted from the sale price of the shares.
- Dividend yield = $\frac{\text{dividend per share}}{\text{market price per share}} \cdot 100\%$
- Price–earnings ratio = $\frac{\text{market price per share}}{\text{yearly profit per share}}$
- When you buy shares you purchase a share in the company. There is no guaranteed return with shares, although there is a greater potential for profit than with investments such as banking and property, but with that comes a higher risk.
- Profit can be made from buying shares in two ways:
  1. The value of the share could rise over time.
  2. The company may pay a dividend to its shareholders. The dividend when written as a percentage of the share price is called the dividend yield.
- To try to predict the future movement in share prices, we can graph the past movement in the share price and draw a line of best fit on the graph. This line of best fit can be extrapolated to estimate the future price.

CHAPTER review

1 Calculate the amount of flat rate interest that will be paid on each of the following loans. 3A

a $8000 at 7% p.a. for 2 years
b $12 500 at 11.5% p.a. for 5 years
c $2400 at 17.8% p.a. for 3 years
d $800 at 9.9% p.a. over 6 months
e $23 400 at 8.75% p.a. over 6 years

2 Calculate the total repayments made on a loan of $4000 at 23% p.a. flat rate interest to be repaid over 3 years. 3A

3 Noel borrows $5600 at 7.6% p.a. to be repaid in monthly instalments over 3 years. Calculate the amount of each monthly instalment. 3A

4 Shane borrows $9500 to purchase a new car. He repays the loan over 4 years at a rate of $246.60 per month. Calculate the flat rate of interest charged on the loan. 3A

5 Mr and Mrs Warne borrow $125 000 to purchase a home unit. The interest rate is 12% p.a. and the monthly repayments are $1376.36. Calculate: 3B

a the first month's interest on the loan
b the balance of the loan after the first month.

6 Mr and Mrs Buckley borrow $130 000 to purchase a home unit. The interest rate is 8% p.a. and over a 20-year term the monthly repayment is $1087.37. 3B

a Copy and complete the table below.

Month	Principal ($)	Interest ($)	Balance owing ($)
1	130 000.00	866.67	129 779.30
2	129 779.30		
3			
4			
5			
6			
7			
8			
9			
10			
11			
12			

b Mr and Mrs Buckley decide to increase their monthly payment to $1500. Complete the table below.

Month	Principal ($)	Interest ($)	Balance owing ($)
1	130 000.00	866.67	129 366.67
2	129 366.67		
3			
4			
5			
6			
7			
8			
9			
10			
11			
12			

c How much less do Mr and Mrs Buckley owe at the end of one year by increasing their monthly repayment?

3B

7 Mr and Mrs Stone borrow $225 000 for their home. The interest rate is 9.6% p.a. and the term of the loan is 25 years. The monthly repayment is $1989.48.

a Calculate the total repayments made on this loan.

b If Mr and Mrs Stone increase their monthly payments to $2000, the loan will be repaid in 24 years and 1 month. Calculate the amount they will save in repayments with this increase.

3C

8 Use the formula $E = \frac{(1+r)^n - 1}{n}$ to calculate the effective interest rate on each of the following flat rate loans (answer correct to 2 decimal places).

a $4000 at 7% p.a. over 2 years

b $12 000 at 11% p.a. over 5 years

c $1320 at 23% p.a. over 2 years

d $45 000 at 9.2% p.a. over 10 years

3C

9 Yu-Ping borrows $13 500 for a holiday to Africa at 12.5% p.a. reducible interest over a 5-year term. The monthly repayments on the loan are $303.72.

a Calculate the total repayments on the loan.

b Calculate the amount of interest that Yu-Ping pays on the loan.

c Calculate the equivalent flat rate of interest on the loan.

10 Kristen and Adrian borrow $150 000 for their home. They have the choice of two loans. Loan 1: At 8% p.a. interest over 25 years with fixed monthly repayments of $1157.72. Loan 2: At 8.25% p.a interest over 25 years with minimum monthly repayments of $1182.68 and an $8 per month account management fee.
Kristen and Adrian believe they can afford to pay $1500 per month. If they do, Loan 2 will be repaid in 14 years and 2 months. Which loan should Kristen and Adrian choose if they can afford to pay the extra each month? **3C**

11 Stephanie has a credit card with an outstanding balance of $423. Calculate the minimum payment that must be made if she must pay 5% of the balance, or $10, whichever is greater. **3C**

12 Use the table of repayments on pages 127–8 to calculate the monthly repayment on each of the following loans. **3D**
- **a** $25 000 at 9% p.a. over a 10-year term
- **b** $45 000 at 14% p.a. over a 15-year term
- **c** $164 750 at 15% p.a. over a 25-year term
- **d** $425 000 at 12% p.a. over a 15-year term

13 Mr and Mrs Rowe take out a $233 000 home loan at 12% p.a. over a 25-year term. **3D**
- **a** Use the table of repayments to calculate the amount of each monthly repayment.
- **b** After 3 years the balance on the loan has been reduced to $227 657. The interest rate then rises to 13% p.a. Calculate the new monthly repayment required to complete the loan within the existing term.

14 **multiple choice** **3E**

A 6-year bond pays $8\frac{1}{2}$% p.a. simple interest. If Rhonda buys a bond worth $500, the interest she would earn would be:

A $233.75 **B** $250
C $255 **D** $755
E $2550

15 **multiple choice** **3E**

Simple interest was calculated on a term deposit of 5 years at $3\frac{3}{4}$% p.a. When Leigh calculated her total return on her investment principal of $350, her return was:

A $61.25 **B** $65.63
C $131.25 **D** $400
E $415.63

16 **multiple choice** **3E**

State government bonds pay interest of $7\frac{3}{4}$% p.a. simple interest. Jess invested $3500 in the bonds which mature in 5 years. Jess's income each quarter would be:

A $67.81 **B** $113.00
C $271.25 **D** $1356.25
E $3567.81

3E

17 Steve invested the $1800 he won at the races in an insurance company bond that pays $12\frac{1}{2}$% p.a. provided he keeps the bond for 4 years. What is Steve's total return from the bond at the end of the 4 years?

3E

18 Jocelyn buys $3500 worth of debentures in a company. She earns 8.5% p.a. simple interest paid to her quarterly. If the agreed period of the debenture was 28 months, calculate the amount of interest Jocelyn will earn.

3E

19 The bank offers a term deposit account paying investors 10.5% p.a. on investments over $10 000 for 2 years. Paul decides to invest $12 000 in this account. How much interest will he earn at the end of the investment?

3E

20 An investment bond is offered to the public at 10% per year. Louis buys a bond worth $4000 that will mature in $2\frac{1}{2}$ years. How much in total will Louis receive at the end of the $2\frac{1}{2}$ years?

3F

21 **multiple choice**

In the bank statement shown below the minimum balance for the month is:

Date	Transaction	Deposit	Withdrawal	Balance
5/4	Transfer from CBR	$100		$456.50
7/4	Salary	$1500		$1956.50
9/4	Cheque — 23456		$1380	$576.50
23/4	ATM — Rowville		$125	$451.50

A $356.50 **B** $451.50 **C** $456.50 **D** $576.50 **E** $1956.50

3F

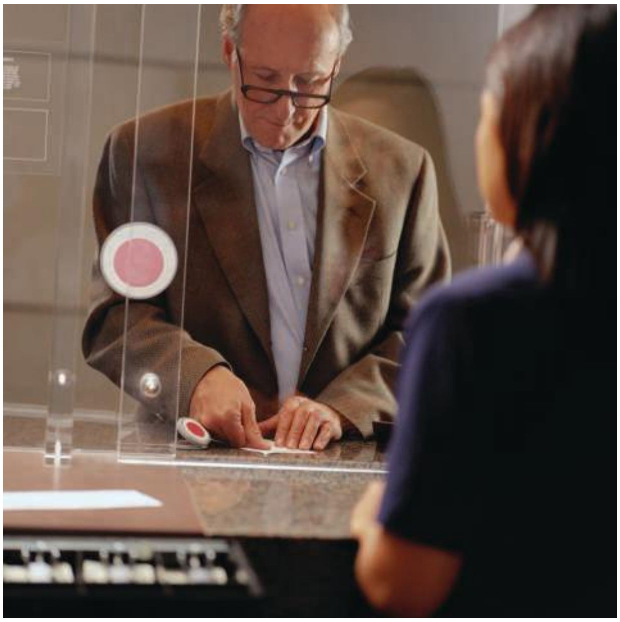
22

Date	Deposit	Withdrawal	Balance
1/5			$302.20
3/5		$28.80	$273.40
7/5	$12		
19/5	$6		
27/5		$10	

Deborah's bank statement shows the above transactions for May. Calculate the interest Deborah will earn in May if the bank pays $4\frac{3}{4}$% p.a. simple interest monthly:

a on the minimum monthly balance

b on the daily balance.



3F

23 At the beginning of July, Ross had $580 in his savings bank account. On 15 July he withdrew $80. If the bank pays 8% p.a. interest paid monthly, calculate the interest Ross earns in July:

a if calculated on the minimum monthly balance

b if calculated on the daily balance.

24 What commission would be payable on a house with a sale price of $405 000? 3G

25 What GST would be payable on a house which sold for $342 650? 3G

26 What proceeds would a seller receive from the sale of his $243 620 home unit through a real estate agent? 3G

27 The Brown family's new home has a price tag of $467 550. When transfer duty is included, what would the Browns pay for their new home? 3G

28 What would you pay for 1000 MMOB shares at $15.50 per share? 3H

29 Comco shares are selling at $9.50 each. How much would you receive from a sale of 5000 of these shares? 3H

30 A company that has 10.9 million shares makes a profit of $21 million. If this entire amount is distributed among the shareholders, calculate the dividend that will be declared. 3H

31 A company which has an after-tax profit of $2.3 billion distributes this among its 156 million shares. Calculate the dividend that this company will declare. 3H

32 A company has a share price of $8.62. It declares a dividend of 45c per share. Calculate the dividend yield on this share. 3H

33 A company with a share price of 45c declares a dividend of 0.7c per share. Calculate the dividend yield on this investment. 3H

34 The dividend yield from a share valued at $19.48 is 4.2%. Calculate the dividend paid by the company, correct to the nearest cent. 3H

35 ANX Banking Corporation shares are selling at $10.50 per share. The company paid a dividend of 26 cents per share. On a purchase of 1000 shares, calculate: 3H

a the total dividend received
b the dividend yield
c the price–earnings ratio.

36 The table below shows the fluctuations in a share's price over a period of 1 year. 3I

Month	Share price	Month	Share price
January	$15.76	July	$16.60
February	$16.04	August	$16.77
March	$16.27	September	$16.51
April	$16.12	October	$16.71
May	$16.49	November	$16.69
June	$16.39	December	$16.98

a On a set of axes plot the share price for each month.
b Draw a line of best fit on your graph and use your line to predict the share price after one year.

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Digital doc: Test Yourself Chapter 3

eBookplus ACTIVITIES

3A Flate rate interest

Digital docs

- SkillSHEET 3.1: Practise converting percentages to fractions *(page 109)*
- SkillSHEET 3.2: Practise converting percentages to decimals *(page 109)*
- SkillSHEET 3.3: Practise finding a percentage of a quantity *(page 109)*
- SkillSHEET 3.4: Practise writing one quantity as a percentage of another *(page 109)*

3B Home loans

Digital docs

- Spreadsheet 202: Investigate flat interest *(page 111)*
- Spreadsheet 204: Investigate home loan costs *(page 113)*

3D Loan repayments

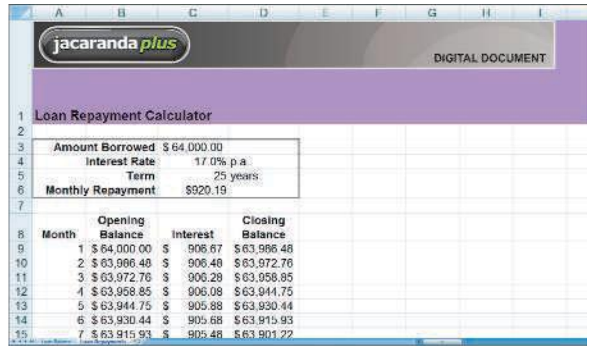
Digital docs

- Spreadsheet 206: Investigate loan repayments *(page 130)*
- WorkSHEET 3.1: Calculate loan repayments, interest and total amounts of investments *(page 130)*

3E Bonds, debentures and term deposits

Digital doc

- Spreadsheet 205: Investigate simple interest *(page 140)*



3F Bank savings accounts

Digital docs

- SkillSHEET 3.5: Practise calculating the minimum monthly balance *(page 146)*
- Spreadsheet 204: Investigate home loan calculations *(page 146)*

3I Graphing share performance

Digital doc

- WorkSHEET 3.2: Calculate consumer credit and investment amounts *(page 164)*

Chapter review

Digital doc

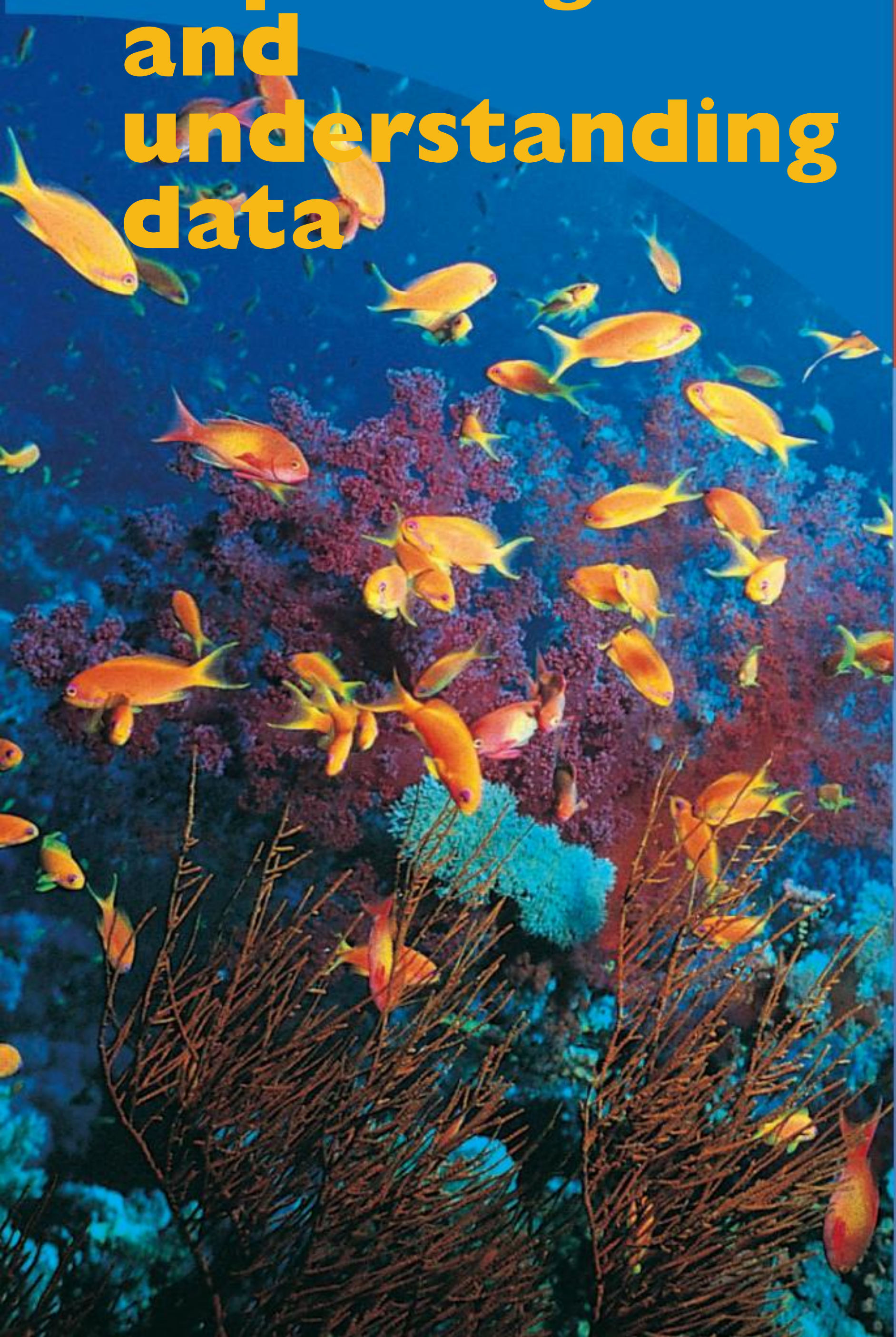
- Test Yourself: Take the end-of-chapter test to test your progress *(page 173)*.

To access eBookPLUS activities, log on to

www.jacplus.com.au

Exploring and understanding data

4



syllabus reference

Strand

Statistics and probability

Core topic

Exploring and understanding data

In this chapter

Populations and samples

Early population counts were musters, where community members were gathered and counted. In 1828, the first Australian **census** was conducted in New South Wales. Each State conducted its own separate census until 1886, five years after the first simultaneous census of the British Empire. In 1901, a common census was conducted throughout Australia; however, the results were not collated to form a total for Australia.

The Census and Statistics Act of December 1905 provided that: '*The Census shall be taken in the year 1911, and in every tenth year thereafter*.'

During the Depression and World War II, no census was taken. The first post-war census took place in Australia in 1947.

The types of questions have changed over time to reflect the changes in our society. The time required to process the responses to the questions has been reduced with the introduction of Optical Mark Reading machines (1991 census) and Intelligent Character Recognition machines which can read handwritten words in the 2001 census. Since 1961, a census has been held every five years, and the fifteenth national Census of Housing and Population was held in 2006.

The 2001 census coincided with Australia's Centenary of Federation. Participants were given the opportunity to place their census forms in a time capsule (to be held by the National Archives) for 99 years. Descendants would then have a glimpse into the lives of their forebears.

SKILLS CHECK

Many of the skills required for this chapter were developed in Year 11 (chapters 9 and 10 of *Maths Quest Maths A Year 11 2nd edition*). Revise the methods by completing the following exercises.

1 Write each of the following as a decimal (correct to 3 decimal places).

a $\frac{3}{8}$ **b** $\frac{1}{12}$ **c** $\frac{65}{80}$ **d** $\frac{124}{210}$

2 Convert each of the following to percentages.

a $\frac{3}{4}$ **b** 0.125 **c** $\frac{85}{200}$ **d** 0.04

3 Use your calculator to generate a set of 10 random integers in the range:

a 1 to 20 inclusive **b** 50 to 100 inclusive.

4 Round the following numbers to integers.

a 3.6 **b** 4.02 **c** 2.91 **d** 6.5 **e** 0.9

5 Find the unknown in each of the following.

a $\frac{1}{4} = \frac{2}{a}$ **b** $\frac{3}{7} = \frac{b}{21}$ **c** $\frac{2}{9} = \frac{5}{c}$ **d** $\frac{5}{d} = \frac{2}{7}$ **e** $\frac{e}{9} = \frac{7}{6}$

6 What types of features on a graph can cause it to be misleading?

7 For the following sets of scores x:

6, 9, 8, 7, 6, 5, 8, 11, 6, 7

Calculate:

a Σx **b** $\bar{x}$ **c** median
d mode **e** lower quartile **f** upper quartile
g range **h** interquartile range.

investigation: Australia's population and housing census

It is important that we understand the reason for recording statistical data accurately. In our society, it is difficult to imagine a world without statistics. Try to imagine a State of Origin football match where no one kept the score! The excitement of the game would probably hold our attention for a while, but if no score was recorded, winning or losing would not be an issue, and we would soon lose interest.

A census is an example of information collected from the whole **population**. It is not always possible or feasible to conduct a **questionnaire** on the whole population, so when this opportunity arises, it is vital to ensure that the questions are carefully worded and that relevant information is sought.

The Australian Bureau of Statistics (ABS) is the government department responsible for administering the Australian census, then collating and analysing the responses. Their website (www.abs.gov.au) details information about their role and it displays statistical data from many areas. Access this site to conduct your research. Prepare a report providing responses to the following:

1 What is a national housing and population census?
2 Who takes part?
3 Is it compulsory to take part?
4 What types of questions are asked in the census? How have they changed over the years?
5 Why should we have a census?
6 Who has access to the information we provide?
7 How is the census conducted?
8 Conclude your report with an expression of your opinion (agreement/disagreement) of the answers gathered from your research. Provide constructive suggestions to improve any aspect of the gathering, collating and analysing of the census data.

Populations

A census represents information or data collected from every member of the population. The term *population* does not necessarily represent a group of people; it is also used to represent a group of objects with the same defined characteristics. So, the population under study may be the wildlife in a national forest, the number of wattle trees in a park, the soil in a farmer's field or the number of cars in a country town.

In some cases it may be possible to determine the exact extent of the population (the number of wattle trees in the park or the number of cars in a country town); however, it is often not possible to obtain an exact figure for the population (the extent of the wildlife in a forest) because circumstances are constantly changing.

Sometimes it is not physically possible to consider the whole population (all the soil in a farmer's field), as it would not be practical. It is often very costly and time consuming to consider the whole population in a study. For these reasons, we need to obtain information about the population by selecting a **sample** that can then be studied.

A *census* is conducted when we obtain information from the whole *population*; however, a **survey** is conducted on a *sample* of the population.

The particular circumstances determine the status of the body being studied (whether it represents the population or a sample of the population). Consider, for example, your Mathematics A class. If we were to try to determine the number of left-handed people in your school who studied Mathematics A, and there was only one such class in your school, then your class would be regarded as the whole population. If, on the other hand, there were several Mathematics A classes in your school, then your class would be considered a sample of the population.

Samples

It is most important when selecting a sample from a population that the sample represents the population as closely as possible. For this to occur, the characteristics of the sample should occur in the same proportions as they do in the population. There is little point in selecting a sample where this is not the case, for analysis of the sample would lead to misleading conclusions. We often see this occurring when polls are conducted prior to an election. Quite frequently they predict a particular outcome while the election results in a different outcome.

WORKED Example 1

In each of the following, state if the information was obtained by census or survey.

a **A school uses the roll to count the number of students absent each day.**
b **The television ratings, in which 2000 families complete a questionnaire on what they watch over a one-week period.**
c **A light globe manufacturer tests every hundredth light globe off the production line.**
d **A teacher records the examination marks of her class.**

THINK	WRITE
a Every student is counted at roll call each morning.	**a** Census
b Not every family is asked to complete a ratings questionnaire.	**b** Survey
c Not every light globe is tested.	**c** Survey
d The marks of every student are recorded.	**d** Census

Samples and sampling

When we select a sample from a population, if it has been chosen carefully, it should, upon analysis of the data, yield the same (or very similar) results to those of the population. A decision must be made regarding the size of the sample. In practice, the size chosen is the smallest one that would be considered appropriate in those circumstances, and the size that would yield a proportion of the elements close to that occurring in the population.

Sample size

The aim of this investigation is to observe how the composition of a sample is affected by the sample size.

1 Take a large packet of mixed coloured jellybeans (200 or more) as the population. (Coloured disks could be substituted.)
2 Place the jellybeans in a container and mix well. Without looking, draw out a sample of 10 in such a way that each jellybean has an equal chance of being selected. This can then be considered a random sample. Count the number of red jellybeans in the sample of 10.
3 Return the sample of 10 jellybeans to the container, mixing them well with the others. Select a random sample of 20 jellybeans, using the same method as before; record the number of red ones.
4 Continue in this manner, returning each sample to the container, mixing them well, then selecting a sample containing 10 more than the previous selection. Record the number of red jellybeans in each of the samples.
5 Generate a table of the format below:

Sample size	Number of red jellybeans	Proportion of red jellybeans (as a decimal)
10		
20		
30		
200		
Whole population		

6 Enter the data in the first and third columns (sample size and proportion) into a spreadsheet or graphics calculator. Graph the sample size against the proportion of red jellybeans. (Alternatively, this could be graphed on graph paper.)
7 Knowing that the proportion of red jellybeans in the whole population (the final row in the table above) represents the true answer, comment on the effect of the sample size on the composition of the sample.
8 For your particular experiment, what would be the minimum sample size which closely resembles the composition of the population?
9 If a sample is used to predict the composition or characteristics of a population, describe what you feel are the desirable qualities of the sample in order to be a reliable predictor of the composition or characteristics of the population.
10 Repeat the experiment. Comment on the similarities/differences in your results.

Random sampling

A simple *random sample* is one for which each element of the population has an equal chance of being chosen. A way in which this can be achieved is by numbering each element of the population then randomly selecting items for the sample by using random digit tables, the random function on a calculator *or* numbers drawn from a container.

Random sampling

The aim of this investigation is to compare different random sampling techniques as methods of selecting a sample that is representative of the population. Consider selecting a random sample of ten (10) students from your Mathematics A class. (Your class is the population in this investigation. You may adjust the sample size if you wish.)

1. Select a characteristic that is present in some of your class members such as brown eyes, fair hair, height above 175 cm and so on.
2. Calculate and record the proportion of the population in your class with this characteristic.
3. Have the students number off 1, 2, 3, until all students have a number. This number for our purposes may be regarded as the population number.

Task 1 Using random digit tables to select a sample

Tables of randomly generated digits are published. Below is a sample of a two-digit random number table. These tables are generally much larger than the extract shown. For our purposes, this size will be sufficient.

Two-digit random number table

16	79	43	59	41	16	39	29	11	12
13	54	24	09	46	24	93	53	28	82
25	56	61	15	97	82	65	77	94	82
85	41	99	74	09	05	98	89	72	10
71	51	35	29	52	52	89	02	92	96
02	81	92	89	17	08	04	63	43	03
84	67	19	23	43	11	05	17	08	07
36	36	72	21	86	99	28	41	24	22
23	04	78	05	33	01	66	06	04	57
80	22	99	14	89	15	65	19	06	25

The following rules apply to the use of random digit tables.

Step 1 Begin at any position in the table (this position being chosen randomly).

Step 2 Move in any direction (vertically, horizontally) along a column or row.

Step 3 Continue moving in this direction, recording the numbers as you go.

Step 4 If the same number is repeated, do not record the number a second time.

Step 5 Continue recording until the required total has been reached.

1 Use the two-digit random number table to select ten numbers within the range of numbers in your class.

2 Determine the students in your class to whom these numbers refer.

3 Calculate and record the proportion of these students with the characteristic you chose. How closely does it match the population proportion which you have previously calculated?

Task 2 Using the random function on a calculator

1 Many scientific and graphing calculators can be set to generate random integers in the range of your population number. (Your teacher will show you, if you are unsure.)

2 Use your calculator to generate ten different random integers.

3 Relate these numbers to specific students in your class.

4 Calculate and record the proportion of students in your sample with your chosen characteristic.

5 Compare this value with the population proportion.

Task 3 Using lot sampling

This type of sampling is used in drawing lotto winning numbers.

1 Write numbers (up to and including your population number) on small, equally sized pieces of paper and place them in a container. Mix well.

2 Draw the numbers one at a time (without replacing them) until ten numbers have been drawn.

3 Relate these numbers to the relevant students, as before.

4 Calculate and record the proportion of students with your chosen characteristic in this sample.

5 Compare the value with the population proportion.

Conclusions

1 Draw up a table to display the results of all your experiments.

2 Compare the results obtained using the various techniques.

3 Did you find one method better than any other?

4 How did the results using these three methods compare with the population result?

investigation

Generating random integers using a spreadsheet

This activity creates a spreadsheet to generate random integers (whole numbers) within a given range. Consider the spreadsheet below.

	A	B	C	D	E	F	G	H	I	J	K
1	Generating random integers										
2											
3	This spreadsheet will generate 100 random integers										
4	in a given range										
5											
6	Enter the range in cells B7 and B8										
7	What is the smallest integer?	3									
8	What is the largest integer?	8									
9											
10	Here are your 100 random integers										
11		6	7	8	3	8	3	3	8	8	7
12		6	7	7	3	5	3	5	5	7	7
13		4	6	4	3	8	5	8	4	3	3
14		7	3	6	7	4	6	5	3	4	6
15		4	6	6	4	4	8	5	4	7	7
16		4	4	3	6	7	3	4	5	8	5
17		8	4	6	8	3	7	3	4	8	7
18		3	5	7	7	4	3	4	3	8	6
19		7	4	5	5	4	8	6	8	8	5
20		3	3	5	3	4	7	6	8	5	6
21											
22	Press the function key F9 to recalculate										

1. Enter the headings in cells A1, A3, A4, A6, A7, A8 and A10.
2. Leave cells B7 and B8 blank. You will enter values in these cells once you run the spreadsheet.
3. In cell B11, enter the formula **=INT(RAND()*($B$8-$B$7+1))+$B$7**. This formula will generate a random integer in the range of the value entered in cell B7 to the value entered in cell B8 inclusive. (You will not find a correct value appears until you enter values in cells B7 and B8.)
4. Copy this formula to the region B11 to K20. This will generate 100 random integers in this region.
5. The function **F9** will recalculate different sets of random integers. Add this instruction to cell A22.
6. Enter values in B7 and B8. Notice the set of integers produced. Press the **F9** key to generate a different set. Continue to generate new sets, making sure that the numbers generated are within the range of those entered in cells B7 and B8. You will find that if you generate large integers you may have to widen columns B to K.
7. Save your spreadsheet and obtain a printout.
8. You may wish to use this spreadsheet for generating a two-digit random number table for your own use.

WORKED Example 2

Use the two-digit random number table on page 180 to select ten students from a numbered class of 30 according to the following rules.

Rule 1 Start in the bottom left-hand corner.

Rule 2 Snake up and down the columns.

THINK	WRITE
1 The selected numbers must be in the range 1 to 30 inclusive.	
2 Moving up the first column on the left. The numbers in the range are: 23, 2, 25, 13, 16. Continue by snaking down the second column and so on, until 10 numbers have been selected (ignore the second occurrence of a number).	
3 Give 10 student numbers.	Students selected have numbers 23, 2, 25, 13, 16, 4, 22, 19, 24 and 9.

investigation

ABS interviewer survey

The ABS conducts a census every five years. To monitor changes that might occur between these times, surveys are conducted on samples of the population. The ABS selects a representative sample of the population and interviewers are allocated particular households. It is important that no substitutes occur in the sampling. The interviewer must persevere until the selected household supplies the information requested. It is a legal requirement that selected households cooperate.

The following questionnaire is reproduced from the ABS website (www.abs.gov.au). It illustrates the format and types of questions asked by an interviewer collecting data regarding employment from a sample.

MINIMUM SET OF QUESTIONS WHEN INTERVIEWER USED — Q1 to Q17

Q.1. I WOULD LIKE TO ASK ABOUT LAST WEEK, THAT IS, THE WEEK STARTING MONDAY THE AND ENDING (LAST SUNDAY THE /YESTERDAY).

Q.2. LAST WEEK DID DO ANY WORK AT ALL IN A JOB, BUSINESS OR FARM?

Yes	☐ **Go to Q.5**
No	☐
Permanently unable to work	☐ **No More Questions**
Permanently not intending to work (if aged 65+ only)	☐ **No More Questions**

(Continued)

Q.3. LAST WEEK DID DO ANY WORK WITHOUT PAY IN A FAMILY BUSINESS?

Yes	☐ **Go to Q.5**
No	☐
Permanently not intending to work (if aged 65+ only)	☐ **No More Questions**

Q.4. DID HAVE A JOB, BUSINESS OR FARM THAT WAS AWAY FROM BECAUSE OF HOLIDAYS, SICKNESS OR ANY OTHER REASON?

Yes	☐
No	☐ **Go to Q.13**
Permanently not intending to work (if aged 65+ only)	☐ **No More Questions**

Q.5. DID HAVE MORE THAN ONE JOB OR BUSINESS LAST WEEK?

Yes	☐
No	☐ **Go to Q.7**

Q.6. THE NEXT FEW **QUESTIONS** ARE ABOUT THE JOB OR BUSINESS IN WHICH USUALLY WORKS THE MOST HOURS.

Q.7. DOES WORK FOR AN EMPLOYER, OR IN OWN BUSINESS?

Employer	☐
Own business	☐ **Go to Q.10**
Other/Uncertain	☐ **Go to Q.9**

Q.8. IS PAID A WAGE OR SALARY, <u>OR</u> SOME OTHER FORM OF PAYMENT?

Wage/Salary	☐ **Go to Q.12**
Other/Uncertain	☐

Q.9. WHAT ARE (WORKING/PAYMENT) ARRANGEMENTS?

Unpaid voluntary work	☐ **Go to Q.13**
Contractor/Subcontractor	☐
Own business/Partnership	☐
Commission only	☐
Commission with retainer	☐ **Go to Q.12**
In a family business without pay	☐ **Go to Q.12**
Payment in kind	☐ **Go to Q.12**
Paid by the piece/item produced	☐ **Go to Q.12**
Wage/salary earner	☐ **Go to Q.12**
Other	☐ **Go to Q.12**

Q.10. DOES HAVE EMPLOYEES (IN THAT BUSINESS)?

Yes	☐
No	☐

Q.11. IS THAT BUSINESS INCORPORATED?

Yes	☐
No	☐

Q.12. HOW MANY HOURS DOES USUALLY WORK EACH WEEK IN (THAT JOB/ THAT BUSINESS/ALL JOBS)?

1 hour or more	☐ **No More Questions**
Less than 1 hour/no hours	☐

Insert occupation questions if required
Insert industry questions if required

Q.13. AT ANY TIME DURING THE LAST 4 WEEKS HAS BEEN LOOKING FOR FULL-TIME OR PART-TIME WORK?

Yes, full-time work	☐	
Yes, part-time work	☐	
No	☐	**No More Questions**

Q.14. AT ANY TIME IN THE LAST 4 WEEKS HAS

Written, phoned or applied in person to an employer for work?	☐	
Answered an advertisement for a job?	☐	
Looked in newspapers?		
Yes	☐	
No	☐	
Checked factory notice boards, or used the touchscreens at Centrelink offices?	☐	

AT ANY TIME IN THE LAST 4 WEEKS HAS

Been registered with Centrelink as a jobseeker?	☐	
Checked or registered with an employment agency?	☐	
Done anything else to find a job?	☐	
Advertised or tendered for work	☐	
Contacted friends/relatives	☐	
Other	☐	**No More Questions**
Only looked in newspapers	☐	**No More Questions**
None of these	☐	**No More Questions**

Q.15. IF HAD FOUND A JOB COULD HAVE STARTED WORK LAST WEEK?

Yes	☐
No	☐
Don't know	☐

Remaining questions are only required if Duration of Unemployment is needed for output or to derive the long-term unemployed.

Q.16. WHEN DID BEGIN LOOKING FOR WORK?

Enter Date	......./......./.......
Less than 2 years ago	DD MM YY
	......./.......
2 years or more ago	MM YY
	....
5 years or more ago	YY
Did not look for work	☐

Q.17. WHEN DID LAST WORK FOR TWO WEEKS OR MORE?

Enter Date	......./......./.......	
Less than 2 years ago	DD MM YY	
	......./.......	
2 years or more ago	MM YY	
	....	
5 years or more ago	YY	
Has never worked (for two weeks or more)	☐	**No More Questions**

(*Continued*)

Reading the questionnaire carefully you will note that, although the questions are labelled 1 to 17, there are only fifteen (15) questions requiring answers (two are introductory statements to be read by the interviewer). Because of directions to forward questions, no individual would be asked all fifteen questions.

1 How many questions would be asked of those who have a job?

2 How many questions would unemployed individuals answer?

3 How many questions apply to those not in the labour force?

Choose a topic of interest to you and conduct a survey

1 Design an interview questionnaire of a similar format to the ABS survey, using directions to forward questions.

2 Decide on a technique to select a representative sample of the students in your class.

3 Administer your questionnaire to this sample.

4 Collate your results.

5 Draw conclusions from your results.

6 Prepare a report which details the:
   a aim of your survey
   b design of the survey
   c sample selection technique
   d results of the survey collated in table format
   e conclusions.

remember

1. Before beginning a statistical investigation it is important to identify the target population.
2. The information can be obtained either by:
   (a) Census — the entire target population is questioned, or
   (b) Survey — a population sample is questioned such that those selected are representative of the entire target population.
3. A random sample is one where chance is the only factor in deciding who is surveyed. This can be done using a random number generator, or lot sampling.

EXERCISE 4A Populations and samples

WORKED Example 1

1 Copy and complete the following:
When we obtain data from the whole population, we conduct a ________________; however, a survey obtains data from a ________________ of the population.

2 A school conducts an election for a new school captain. Every teacher and student in the school votes. Is this an example of a census or a survey? Explain your answer.

3 A questionnaire is conducted by a council to see what sporting facilities the community needs. If 500 people who live in the community are surveyed, is this an example of a census or a survey?

4 For each of the following, state whether a census or a survey has been used.
  a Two hundred people in a shopping centre are asked to nominate the supermarket where they do most of their grocery shopping.
  b To find the most popular new car on the road, 500 new car buyers are asked what make and model car they purchased.
  c To find the most popular new car on the road, the make and model of every new car registered is recorded.
  d To find the average mark in the mathematics half-yearly examination, every student's mark is recorded.
  e To test the quality of tyres on a production line, every 100th tyre is road tested.

5 For each of the following, recommend whether you would use a census or a survey to find:
  a the most popular television program on Monday night at 7.30 pm
  b the number of cars sold during a period of one year
  c the number of cars that pass through the tollgates on the Brisbane Gateway Bridge each day
  d the percentage of defective computers produced by a company.

6 An opinion poll is conducted to try to predict the outcome of an election. Two thousand people are telephoned and asked about their voting intention. Is this an example of a census or a survey?

WORKED Example 2

7 Use the two-digit random number table on page 180. Start at the bottom left-hand corner then snake up and down the columns selecting 10 numbers in the range 50 to 99.

8 Use your calculator to generate 10 random integers in the range 50 to 99.

eBook*plus*

Digital doc: Extension Sampling methods

9 Use your calculator to generate a set of random two-digit integers in the range 01 to 99. Write these numbers in table format. Use your table (and some random selection technique) to select 10 random integers in the range 50 to 99.

10 Compare your answers to questions 7, 8 and 9. Does it appear that three different sets of random numbers resulted?

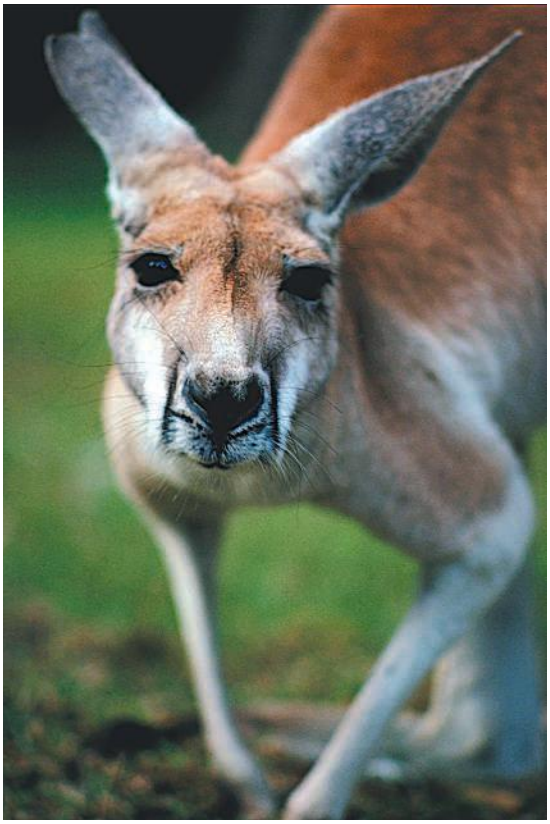
Bias

No doubt you have heard the comment, 'There are lies, damned lies and statistics'. This implies that we should be wary of statistical figures quoted. Indeed, we should always make informed decisions of our own and not simply accept the mass of statistics that bombards us through the media.

Bias can be introduced into statistics by:

1. questionnaire design
2. sampling bias
3. the interpretation of results.

Bias in questionnaire design



Consider a survey designed to collect data on opinions relating to culling kangaroo numbers in Australia.

The questions may be designed to be *emotive* in nature. Respondents in these situations feel obliged to show compassion. Posing a question in the form, 'The kangaroo is identified as a native Australian animal, not found anywhere else in the world. Would you be in favour of culling kangaroos in Australia?', would almost certainly encourage a negative response.

Using a *leading* question (one which leads the respondent to answer in a particular way) can cause bias to creep into responses. Rephrasing the question in the form, 'As you know, kangaroos cause massive damage on many farming properties. You'd agree that their numbers need culling, wouldn't you?', would encourage a positive response.

Using *terminology* that is unfamiliar to a large proportion of those being surveyed would certainly produce unreliable responses. 'Do you think we need to cull herbivorous marsupial mammals in Australia?', would cause most respondents to answer according to their understanding of the terms used. If the survey was conducted by an interviewer, the term could be explained. In the case of a self-administered survey, there would be no indication of whether the question was understood or not.

Sampling bias

As discussed previously, an ideal sample should reflect the characteristics of the population. Statistical calculations performed on the sample would then be a reliable indication of the population's features.

Selecting a sample using a *non-random method*, generally tends to introduce an element of bias.

Particular responses can be selected from all those received. In collecting information on a local issue, an interviewer on a street corner may record responses from many passers-by. From all the data collected, a sample could be chosen to support the issue, or alternatively another sample could be chosen to refute the same issue.

A sample may be selected under *abnormal conditions*. Consider a survey to determine which lemonade was more popular – Kirks or Schweppes. Collecting data one week when one of the brands was on special at half price would certainly produce misleading results.

Data are often collected by radio and television stations via *telephone polls*. A 'Yes' response is recorded on a given phone-in number, while the 'No' respondents are asked to ring a different phone-in number. This type of sampling does not produce a representative sample of the population. Only those who are highly motivated tend to ring and there is no monitoring of the number of times a person might call, recording multiple votes.

When data are collected from mailing surveys, bias results if the *non-response rate* is high (even if the selected sample was a random one). The responses received often represent only those with strong views on the subject, while those with more moderate views tend to lack representation in their correct proportion.

Statistical interpretation bias

Once the data have been collected, collated and subjected to statistical calculations, bias may still occur in the interpretation of the results.

Misleading graphs can be drawn leading to a biased interpretation of the data. Graphical representations of a set of data can give a visual impression of 'little change' or 'major change' depending on the scales used on the axes (we learned about misleading graphs in Year 11).

The use of terms such as 'majority', 'almost all' and 'most' are open to interpretation. When we consider that 50.1% 'for' and 49.9% 'against' represents a 'majority for' an issue, the true figures have been hidden behind words with very broad meanings. Although we would probably not learn the real facts, we should be wary of statistical issues quoted in such terms.

investigation

Bias in statistics

The aim of this investigation is to study statistical data that you suspect to be biased.

Conduct a search of newspapers, magazines or any printed material to collect instances of quoted statistics that you believe to be biased. There are occasions when television advertisements quote statistical figures as a result of questionable sampling techniques. For each example, discuss:

1 the purpose of the survey
2 how the data might have been collected
3 the question(s) that may have been asked (try to pose the question(s) in a variety of ways to influence different outcomes)
4 ways in which bias might be introduced
5 variations in interpretation of the data.

investigation

Biased sampling

Discuss the problem that would be caused by each of the following biased samples.

1 A survey is to be conducted to decide the most popular sport in a local community. A sample of 100 people was questioned at a local football match.
2 A music store situated in a shopping centre wants to know the type of music that it should stock. A sample of 100 people was surveyed. The sample was taken from people who passed by the store between 10 and 11 am on a Tuesday.
3 A newspaper conducting a Gallup poll on an election took a sample of 1000 people from the Gold Coast.

WORKED Example 3

Discuss why the following selected samples could provide bias in the statistics collected.

a In order to determine the extent of unemployment in a community, a committee phoned two households (randomly selected) from each page of the local telephone book during the day.

b A newspaper ran a feature article on the use of animals to test cosmetics. A form beneath the article invited responses to the article.

THINK	WRITE
a 1 Consider phone book selection.	a Phoning two randomly selected households per page of the telephone directory is possibly a representative sample.
2 Consider those with no phone contact.	However, those without a home phone and those with unlisted numbers could not form part of the sample.
3 Consider the hours of contact.	An unanswered call during the day would not necessarily imply that the resident was at work.
b 1 Consider the newspaper circulation.	b Selecting a sample from a circulated newspaper excludes those who do not have access to the paper.
2 Consider the urge to respond.	In emotive issues such as these, only those with strong views will bother to respond, so the sample will represent extreme points of view.

remember

Bias can be introduced at each of the following stages:
1. questionnaire design
2. sampling bias
3. interpretation of results.

EXERCISE 4B Bias

1 Rewrite the following questions, removing any elements or words that might contribute to bias in responses.

a The poor homeless people, through no fault of their own, experience great hardship during the freezing winter months. Would you contribute to a fund to build a shelter to house our homeless?

b Most people think that, since we've developed as a nation in our own right and broken many ties with Great Britain, we should adopt our own national flag. You'd agree with this, wouldn't you?

c You'd know that our Australian 50 cent coin is in the shape of a dodecagon, wouldn't you?

d Many in the workforce toil long hours for low wages. By comparison, politicians seem to get life pretty easy when you take into account that they only work for part of the year and they receive all those perks and allowances. You'd agree, wouldn't you?

2 Rewrite parts **a** to **d** in question **1** so that the expected response is reversed.

WORKED Example 3

3 What forms of sampling bias can you identify in the following samples?

a Choosing a sample from students on a bus travelling to a sporting venue to answer a questionnaire regarding sporting facilities at their school

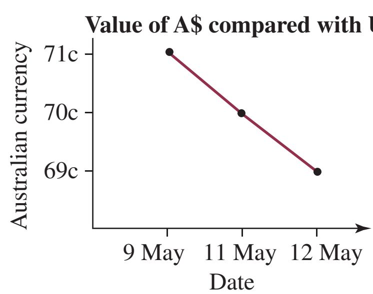
b Sampling using 'phone-in' responses to an issue viewed on a television program

c Promoting the results of a mail-response survey when fewer than half the selected sample replied

d Comparing the popularity of particular chocolate brands when one brand has a 'two for the price of one' special offer

e Choosing a Year 8 class and a Year 12 class to gather data relating to the use of the athletics oval after school

4 Why does this graph produce a biased visual impression?



5 Comment on the following statement:

'University tests have demonstrated that *Double-White* toothpaste is consistently used by the majority of teenagers and is more effective than most other toothpastes.'

6 Surveys are conducted on samples to determine the characteristics of the population. Discuss whether the samples selected would provide a reliable indication of the population's characteristics.

	Sample	Population
a	Year 11 students	Student drivers
b	Year 12 students	Students with part-time jobs
c	Residents attending a neighbourhood watch meeting	Residents of a suburb
d	Students in the school choir	Music students in the school
e	Cars in a shopping centre car park	Models of Holden cars on the road
f	Males at a football match	Popular TV programs
g	Users of the local library	Popular teenage magazines

investigation

Bias

It is important that a sample is chosen randomly to avoid bias.
Consider the following situation.

The government wants to improve sporting facilities in Brisbane. They decide to survey 1000 people about what facilities they would like to see improved. To do this, they choose the first 1000 people through the gate at a football match at Suncorp Stadium.

In this situation it is likely that the results will be biased towards improving facilities for football. It is also unlikely that the survey will be representative of the whole population in terms of equality between men and women, age of the participants and ethnic backgrounds.

Questions can also create bias. Consider asking the question, 'Is football your favourite sport?' The question invites the response that football is the favourite sport rather than allowing a free choice from a variety of sports by the respondent.

Consider each of the following surveys and discuss:

a any advantages, disadvantages and possible causes of bias
b a way in which a truly representative sample could be obtained.

1. Surveying food product choices by interviewing customers of a large supermarket chain as they emerge from the store between 9.00 am and 2.00 pm on a Wednesday
2. Researching the popularity of a government decision by stopping people at random in a central city mall
3. Using a telephone survey of 500 people selected at random from the phone book to find if all Australian States should have Daylight Saving Time in summer
4. A bookseller using a public library database to survey for the most popular novels over the last three months
5. An interview survey about violence in sport taken at a rugby league football venue as spectators leave

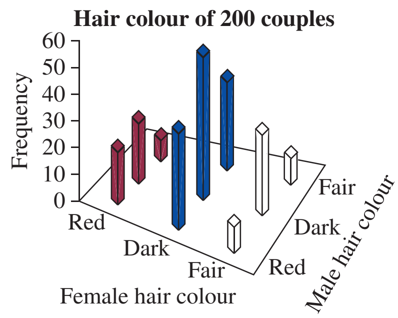
Contingency tables

When sample data are collected, it is often useful to break the data into categories. A two-way frequency table or **contingency table** displays data that have been classified into different types.

Consider, for example, data collected on the hair colour of 200 couples. It may be represented in a table such as the one below.

These data could be represented as a 3-dimensional bar chart, as shown below.

		Female			
		Red	Dark	Fair	Total
Male	Fair	11	25	9	45
	Dark	19	51	28	98
	Red	17	27	13	57
	Total	47	103	50	200

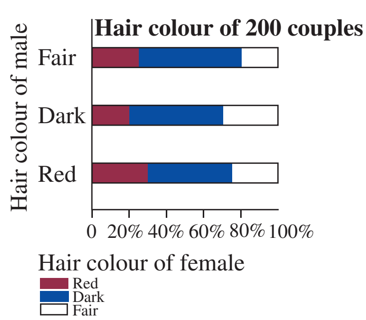


Although this graph displays the data so that comparisons are readily visible, the chart is difficult to read and figures can not be read accurately.

If we considered representing the data as a 2-dimensional **segmented bar chart**, this could be done in two ways.

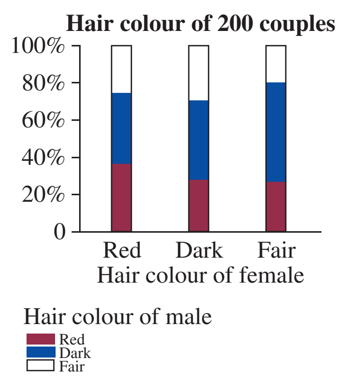
Splitting the data into categories based on the hair colour of the male and calculating percentages in each category would yield the following figures and segmented bar graph:

		Female			
		Red	Dark	Fair	Total
Male	Fair	24%	56%	20%	100%
	Dark	19%	52%	29%	100%
	Red	30%	47%	23%	100%



Splitting the data into categories based on the hair colour of the female and calculating percentages in each category would yield the following figures and segmented bar graph:

		Female		
		Red	Dark	Fair
Male	Fair	23%	24%	18%
	Dark	41%	50%	56%
	Red	36%	26%	26%
	Total	100%	100%	100%



It is obvious that the interpretation of the data depends on the reference basis. We may wish to interview those couples where the male is fair haired and the female dark haired. Note that this represents 25 couples. What if we talk about percentages? Comparing the percentages in the two tables, it can be seen that:

1. 56% of fair-haired males have female partners with dark hair
2. 24% of dark-haired females have male partners with fair hair.

These percentages have vastly different values, yet they both describe the same set of 25 couples of fair-haired males and dark-haired females. It is important, particularly when dealing with contingency tables, to consider the reference basis for percentages.

WORKED Example 4

A new test was designed to assess the reading ability of students entering high school.
The results were used to determine if the students' reading level was adequate to cope with high school. The students' results were then checked against existing records.
Of the 150 adequate readers who sat for the test, 147 of them passed.
Of the 50 inadequate readers who sat for the test, 9 of them passed.
Present this information in a contingency table.

THINK

Draw up the table showing the number of students whose reading was adequate and the number of students for whom the results of the new test were confirmed.

WRITE

	Test results		
	Passed	Did not pass	Total
Adequate readers	147	3	150
Inadequate readers	9	41	50
Total	156	44	

When information on a test is presented in a contingency table, conclusions can be made about the accuracy of the test.

WORKED Example 5

A batch of sniffer dogs is trained by customs to smell drugs in suitcases. Before they are used at airports they must pass a test. The results of that test are shown in the contingency table below.

	Test results		
	Detected	**Not detected**	**Total**
No of bags with drugs	24	1	25
No. of bags without drugs	11	164	175
Total	35	165	

a **How many bags did the sniffer dogs examine?**

b **In how many bags did the dogs detect drugs?**

c **In what percentage of bags without drugs did the dogs incorrectly detect drugs?**

d **Based on the above results, what percentage of the time will the dogs not detect a bag carrying drugs?**

THINK

a Add both total columns; they should give the same result.

b The total of the detected column.

c There were 175 bags without drugs but dogs incorrectly detected them in 11 bags. Write this as a percentage.

d Of 25 bags with drugs, 1 went undetected. Write this as a percentage.

WRITE

a 200 bags were examined.

b The dogs detected drugs in 35 bags.

c Percentage incorrectly detected

$$= \frac{11}{175} \cdot 100\%$$

$$= 6.3\%$$

d Percentage not detected $= \frac{1}{25} \cdot 100\%$

$$= 4\%$$

As a result of studying a contingency table, we should also be able to make judgements about the information given in the tables. In the previous worked example only one bag out of 25 with drugs went undetected. Although the dogs incorrectly detected drugs in 11 bags that did not have drugs, they still have an overall accuracy of 94% as shown by the calculation $[(24 + 164) | 200] \cdot 100\%$.

Many contingency tables will require you to make your own value judgements about the conclusions established. For example, the 94% overall accuracy recorded may be considered ‘very acceptable’.

WORKED Example 6

The contingency table at right shows the composition of the employees of a small law firm.

	Full-time	Part-time
Female	4	11
Male	30	5

a Extend the table to show totals in all categories and an overall total.

b Draw a table showing percentages with respect to type of employment (full or part-time).

c Redraw the table showing percentages based on the gender of the employee.

d What percentage of females work full time?

e What percentage of full-time workers are female?

f Explain why, in the workforce in general, it would be easier to estimate an answer to part d than it would to obtain an estimate for part e.

THINK / WRITE

a Add the numbers in the cells for all the rows and columns and enter the totals. Check that the overall total is consistent for the rows and columns.

a

	Full-time	Part-time	Total
Female	4	11	15
Male	30	5	35
Total	34	16	50

b Percentages are based on totals in columns. The totals in the columns are on the denominator when calculating percentages.

b

	Full-time	Part-time
Female	$\frac{4}{34} \cdot 100 = 12\%$	$\frac{11}{16} \cdot 100 = 69\%$
Male	$\frac{30}{34} \cdot 100 = 88\%$	$\frac{5}{16} \cdot 100 = 31\%$
Total	100%	100%

c Percentages are based on totals in rows. The totals in the rows are on the denominator when calculating percentages.

c

	Full-time	Part-time	Total
Female	$\frac{4}{15} \cdot 100 = 27\%$	$\frac{11}{15} \cdot 100 = 73\%$	100%
Male	$\frac{30}{35} \cdot 100 = 86\%$	$\frac{5}{35} \cdot 100 = 14\%$	100%

d 1 This is based on female totals in table **c**.

d $\frac{\text{full time}}{\text{female total}} \cdot 100 = \frac{4}{15} \cdot 100 = 27\%$

2 Write the answer.

Percentage of females who work full time = 27%.

e 1 This is based on full-time totals in table **b**.

e $\frac{\text{female}}{\text{full-time total}} \cdot 100 = \frac{4}{34} \cdot 100 = 12\%$

2 Write the answer.

Percentage of full-time workers who are female = 12%.

f An estimate is easier if the required sample is smaller.

f It would be easier to obtain an estimate for the percentage of females who work full time because the number of females is fewer than the number of full-time workers. This means that the sample size would be smaller.

investigation

Climatic influences in Queensland

For this activity we will investigate relationships between geographical features that influence our weather. We could pose questions such as:

What effect does latitude have on temperature?

What factor has the main influence on day length?

What part does elevation play in influencing temperature?

This investigation should be conducted using a spreadsheet. Data on Queensland towns from the Bureau of Meteorology's website have been collated and shown in two spreadsheet tables which follow. Graphs have been provided for stimulus when investigating relationships between the variables in the spreadsheet.

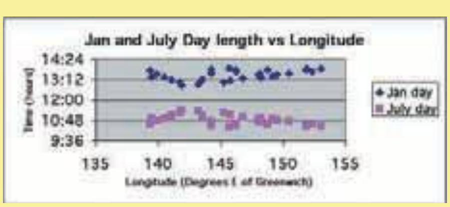
1 Use the **Locality by longitude** and **Locality by latitude** weblinks in your eBookPLUS and download the spreadsheets. (The longitude file does not contain the graphs displayed here.)

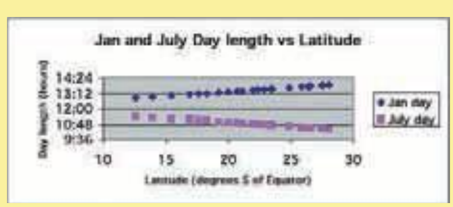
2 Experiment by graphing pairs of variables to determine whether a relationship exists between the pair. You may wish to sort the spreadsheet using a different classification.

3 Write a report on the geographical factors influencing daily temperatures and sunlight hours. Support your conclusions by providing graphical evidence.

4 Sites on the World Wide Web provide weather conditions for many places throughout the world. Conduct a search to collate data from locations around the globe. Investigate the geographical features which might have an influence on their weather.

	A	B	C	D	E	F	G	H	I
1	Locality	Latitude S	Longitude E	Jan		July		Jan day	July day
2				Sunrise	Sunset	Sunrise	Sunset		
3	Brisbane	27.48	153.03	4:56 AM	6:47 PM	6:39 AM	5:05 PM	13:51	10:26
4	Bundaberg	24.87	152.35	5:05 AM	6:44 PM	6:36 AM	5:13 PM	13:39	10:37
5	Toowoomba	27.58	151.93	5:00 AM	6:51 PM	6:43 AM	5:09 PM	13:51	10:26
6	Kingaroy	26.55	151.85	5:03 AM	6:49 PM	6:42 AM	5:11 PM	13:46	10:29
7	Rockhampton	23.4	150.5	5:15 AM	6:48 PM	6:40 AM	5:23 PM	13:33	10:43
8	St Lawrence	22.35	149.54	5:21 AM	6:50 PM	6:42 AM	5:29 PM	13:29	10:47
9	Mackay	21.15	149.18	5:25 AM	6:48 PM	6:41 AM	5:33 PM	13:23	10:52
10	Roma	26.55	148.78	5:15 AM	7:01 PM	6:54 AM	5:24 PM	13:46	10:30
11	Bowen	20	148.25	5:31 AM	6:50 PM	6:43 AM	5:39 PM	13:19	10:56
12	Emerald	23.5	148.15	5:24 AM	6:57 PM	6:50 AM	5:32 PM	13:33	10:42
13	Moranbah	21.99	148.03	5:28 AM	6:55 PM	6:47 AM	5:36 PM	13:27	10:49
14	Townsville	19.25	146.77	5:38 AM	6:54 PM	6:47 AM	5:46 PM	13:16	10:59
15	Tambo	24.88	146.26	5:29 AM	7:08 PM	7:00 AM	5:37 PM	13:39	10:37
16	Charleville	26.42	146.22	5:26 AM	7:11 PM	7:04 AM	5:34 PM	13:45	10:30
17	Cairns	16.93	145.78	5:47 AM	6:54 PM	6:47 AM	5:55 PM	13:07	11:08
18	Cunnamulla	28.05	145.7	5:24 AM	7:17 PM	7:09 AM	5:33 PM	13:53	10:24
19	Cooktown	15.46	145.25	5:52 AM	6:53 PM	6:46 AM	5:59 PM	13:01	11:13
20	Quilpie	26.58	144.27	5:33 AM	7:20 PM	7:12 AM	5:41 PM	13:47	10:29
21	Longreach	23.45	144.25	5:40 AM	7:13 PM	7:05 AM	5:48 PM	13:33	10:43
22	Georgetown	18.29	143.55	5:53 AM	7:05 PM	6:58 AM	6:01 PM	13:12	11:03
23	Coen	13.94	143.2	6:03 AM	6:59 PM	6:52 AM	6:10 PM	12:56	11:18
24	Weipa	12.63	141.88	6:10 AM	7:01 PM	6:54 AM	6:18 PM	12:51	11:24
25	Kowanyama	15.47	141.75	6:06 AM	7:07 PM	7:00 AM	6:13 PM	13:01	11:13
26	Normanton	17.65	141.07	6:04 AM	7:14 PM	7:07 AM	6:12 PM	13:10	11:05
27	Cloncurry	20.71	140.52	6:01 AM	7:22 PM	7:15 AM	6:09 PM	13:21	10:54
28	Boulia	22.88	139.9	5:58 AM	7:29 PM	7:22 AM	6:07 PM	13:31	10:45
29	Mt Isa	20.73	139.48	6:05 AM	7:26 PM	7:19 AM	6:13 PM	13:21	10:54
30	Birdsville	25.9	139.35	5:54 AM	7:38 PM	7:30 AM	6:03 PM	13:44	10:33
31									
32		Sorted by Longitude							





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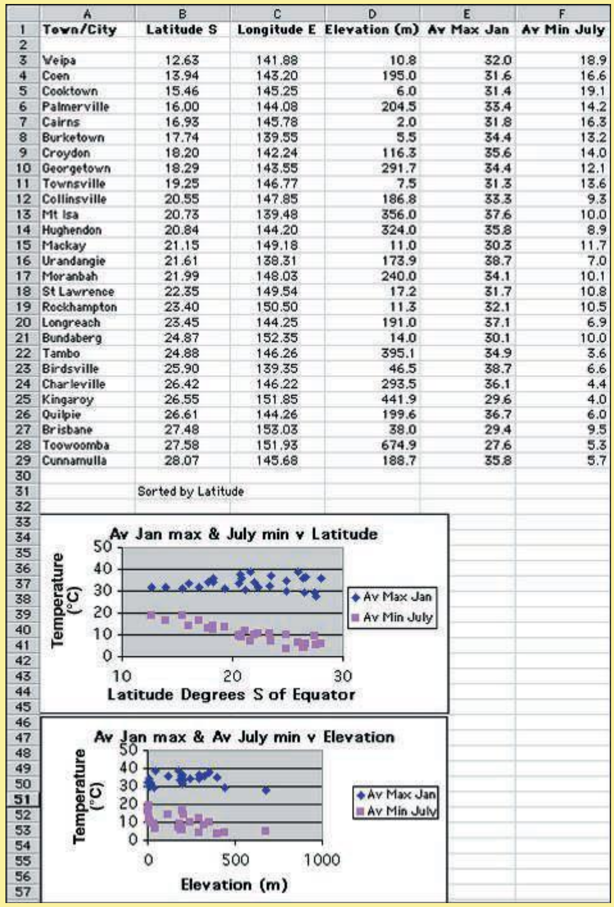


Town/City	Latitude S	Longitude E	Elevation (m)	Av Max Jan	Av Min July
Weipa	12.63	141.88	10.8	32.0	18.9
Coen	13.94	143.20	195.0	31.6	16.6
Cooktown	15.46	145.25	6.0	31.4	19.1
Palmerville	16.00	144.08	204.5	33.4	14.2
Cairns	16.93	145.78	2.0	31.8	16.3
Burketown	17.74	139.55	5.5	34.4	13.2
Croydon	18.20	142.24	116.3	35.6	14.0
Georgetown	18.29	143.55	291.7	34.4	12.1
Townsville	19.25	146.77	7.5	31.3	13.6
Collinsville	20.55	147.85	186.8	33.3	9.3
Mt Isa	20.73	139.48	356.0	37.6	10.0
Hughendon	20.84	144.20	324.0	35.8	8.9
Mackay	21.15	149.18	11.0	30.3	11.7
Urandangie	21.61	138.31	173.9	38.7	7.0
Moranbah	21.99	148.03	240.0	34.1	10.1
St Lawrence	22.35	149.54	17.2	31.7	10.8
Rockhampton	23.40	150.50	11.3	32.1	10.5
Longreach	23.45	144.25	191.0	37.1	6.9
Bundaberg	24.87	152.35	14.0	30.1	10.0
Tambo	24.88	146.26	395.1	34.9	3.6
Birdsville	25.90	139.35	46.5	38.7	6.6
Charleville	26.42	146.22	293.5	36.1	4.4
Kingaroy	26.55	151.85	441.9	29.6	4.0
Quilpie	26.61	144.26	199.6	36.7	6.0
Brisbane	27.48	153.03	38.0	29.4	9.5
Toowoomba	27.58	151.93	674.9	27.6	5.3
Cunnamulla	28.07	145.68	188.7	35.8	5.7

Sorted by Latitude

We are constantly bombarded with statistics, some of which are a valid interpretation of the data, and some of which are not. On occasions, the misuse of statistics may be unintentional or through ignorance, but there are occasions when misleading figures are quoted intentionally. If the raw data are available, it is wise to check the validity of any claims.

WORKED Example 7

This ABS data from the 2006 census shows the age groups and education details for residents of Brisbane.

Australian Bureau of Statistics
2006 Census of Population and Housing
Brisbane (SD 305) 5904.8 sq. Kms

B01 SELECTED PERSON CHARACTERISTICS BY SEX (FIRST RELEASE PROCESSING)
Count of persons

	Males	*Females*	*Persons*
Total persons	866 431	896 701	1 763 132
Age groups:			
0–4 years	60 495	57 123	117 618
5–14 years	123 323	117 807	241 130
15–19 years	63 978	61 948	125 926
20–24 years	68 038	68 593	136 631
25–34 years	126 312	130 313	256 625
35–44 years	129 361	136 460	265 821
45–54 years	115 978	121 807	237 785

	Males	Females	Persons
55–64 years	92 204	92 648	184 852
65–74 years	49 229	52 354	101 583
75–84 years	29 472	40 342	69 814
85 years and over	8 041	17 306	25 347
Age of persons attending an educational institution:			
0–4 years	8 738	8 213	16 951
5–14 years	110 354	105 767	216 121
15–19 years	40 549	42 704	83 253
20–24 years	19 677	23 509	43 186
25 years and over	24 103	35 135	59 238

When discussing the probability of female attendance of 15–19 year olds at an educational institution, it was said that 51% of the females in this age group attended an educational institute.

a **Construct a contingency table displaying the attendance/non-attendance of 15–19 year old males and females at an educational institution.**

b **Use your contingency table to discuss the validity of the claim.**

THINK / **WRITE**

a 1 Extract the attendance and total figures for 15–19 year old males and females from the table.

2 Form a contingency table with totals for rows and columns. Subtract the attendance figure from its relevant total to determine the non-attendance figure.

a

For 15–19 year olds

	Male	Female	Total
Attend an educational institution	40 549	42 704	83 253
Do not attend an educational institution	23 429	19 244	42 673
Total	63 978	61 948	125 926

b 1 Calculate the percentage of females who attend an educational institution; that is,

$$\frac{\text{no. females at education institute}}{\text{total number of females}} \cdot 100$$

b *P*(females who attend an educational institution)

$= \frac{42\,704}{61\,948} \cdot 100$

$= 69\%$

69% of females attend an educational institution.

2 Calculate the percentage of those in an educational institution who are female; that is,

$$\frac{\text{no. females at education institute}}{\text{total number in an educational institution}} \cdot 100$$

P(those in an educational institution who are female)

$= \frac{42\,704}{82\,253} \cdot 100$

$= 51\%$

51% of those in an educational institution are female.

3 Compare these probability figures with the statement above to determine if the statement is correct.

The statement is not correct. It should have said that 69% of 15–19 year old females attend an educational institution.

An error frequently occurs when statistics of the kind in the previous worked example are quoted. The reference basis for the probability percentage should be carefully noted.

investigation

Contingency tables from census data

The table below displays data collected from the 2006 census. It shows the numbers of males and females in various forms of employment in Australia for persons aged 15 years and over.

Cat. No. 2068.0 - 2006 Census Tables
2006 Census of Population and Housing
Australia (Australia)
INDUSTRY OF EMPLOYMENT - 2006 ANZSIC (DIVISION)(a) BY SEX
Count of employed persons aged 15 years and over
Based on the 2006 Australian and New Zealand Standard Industrial Classification (ANZSIC), Second Edition
Based on place of usual residence

	Males	Females	Persons
Accommodation and Food Services	247 685	327 419	575 104
Administrative and Support Services	137 241	149 379	286 620
Agriculture, Forestry and Fishing	195 001	85 911	280 912
Arts and Recreation Services	66 484	60 912	127 396
Construction	613 949	95 891	709 840
Education and Training	212 813	484 993	697 806
Electricity, Gas, Water and Waste Services	69 856	19 599	89 455
Financial and Insurance Services	158 750	189 842	348 592
Health Care and Social Assistance	204 501	751 644	956 145
Information Media and Telecommunications	101 921	74 906	176 827
Manufacturing	704 850	247 159	952 009
Mining	90 832	16 054	106 886
Other Services	191 113	147 100	338 213
Professional, Scientific and Technical Services	328 196	273 821	602 017
Public Administration and Safety	341 906	266 699	608 605
Rental, Hiring and Real Estate Services	75 848	78 064	153 912
Retail Trade	441 975	591 215	1 033 190
Transport, Postal and Warehousing	328 750	99 039	427 789
Wholesale Trade	258 850	137 516	396 366
Inadequately described	72 667	40 783	113 450
Not stated	67 947	55 103	123 050
Total	**4 911 135**	**4 193 049**	**9 104 184**

Using these data, we could form a contingency table to compare the proportion of males and females in, for example, the retail trade. (Confirm the figures in the table below.)

	Male	Female	Total
Retail trade	441 975	591 215	1 033 190
Non-retail trade	4 469 160	3 601 834	8 070 994
Total	4 911 135	4 193 049	9 104 184

1 Use this table to:
   a determine the percentage of male workers who are in the retail trade
   b calculate the percentage of retail workers who are male
   c explain why these two percentages are different
   d plan a strategy to survey the workforce for an estimate of the number of males in the retail trade.

2 Choose another category of the workforce from the census data. Construct a contingency table, then answer questions similar to those on the facing page.
3 Reports from early recordings of census data showed that more than 50% of Australians lived and worked on the land, providing food and clothing for our population. Most recent reports indicate that only 3% of Australians now work the land, providing for the remaining 97%. Use the data in the table to confirm that this is indeed true.
4 It is important for future planning that these changes are recorded and made known. Search the World Wide Web or reference books to obtain industry data from the 2006 census. Examine the figures, noting changing trends in industry employment. Report on your findings.

remember

1. Contingency tables can be used to display data that have been classified into different types.
2. The table displays 2 variables which have been split into categories in a horizontal and a vertical direction.
3. Calculations can be made with regard to a variety of reference bases.

EXERCISE 4C Contingency tables

WORKED Example 4

1 A test is developed to test for infection with the flu virus. To test the accuracy, the following 500 people are tested.
- Of the 100 people who are known to have the flu who are tested, the test returns 98 positive results.
- Of the 400 people who are known not to be infected with the virus who are tested, 12 false positives are returned.

Display this information in the contingency table below.

	Test results		Total
	Accurate	Not accurate	
With virus			
Without virus			
Total			

2 One thousand people take a lie detector test. Of 800 people known to be telling the truth, the lie detector indicates that 23 are lying. Of 200 people known to be lying, the lie detector indicates that 156 are lying.
Present this information in a contingency table.

WORKED Example 5

3 The contingency table shown below displays the information gained from a medical test screening for a virus. A positive test indicates that the patient has the virus.

	Test results		
	Accurate	Not accurate	Total
With virus	45	3	48
Without virus	922	30	952
Total	967	33	1000

eBook plus

Digital docs:
SkillSHEET 4.1 Converting a fraction to a percentage
SkillSHEET 4.2 Writing one quantity as a percentage of another

a How many patients were screened for the virus?

b How many positive tests were recorded? (That is, in how many tests was the virus detected?)

c What percentage of test results were accurate?

d Based on the medical results, if a positive test is recorded what is the percentage chance that you actually have the virus?

4 The contingency table below indicates the results of a radar surveillance system. If the system detects an intruder, an alarm is activated.

	Test results		
	Alarm activated	Not activated	Total
Intruders	40	8	48
No intruders	4	148	152
Total	44	156	200

a Over how many nights was the system tested?

b On how many occasions was the alarm activated?

c If the alarm is activated, what is the percentage chance that there actually is an intruder?

d If the alarm was not activated, what is the percentage chance that there was an intruder?

e What was the percentage of accurate results over the test period?

f Comment on the overall performance of the radar detection system.

The information below is to be used in questions 5 to 7.

A test for a medical disease does not always produce the correct result. A positive test indicates that the patient has the condition. The table indicates the results of a trial on a number of patients who were known to either have the disease or known not to have the disease.

	Test results		
	Accurate	Not accurate	Total
With disease	57	3	60
Without disease	486	54	540
Total	543	57	600

5 **multiple choice**

The overall accuracy of the test is:

A 9.5% **B** 90% **C** 90.5% **D** 92.5% **E** 95%

6 **multiple choice**

Based on the table, what is the probability that a patient who has the disease has it detected by the test?

A 9.5% **B** 90% **C** 90.5% **D** 92.5% **E** 95%

7 **multiple choice**

Which of the following statements is correct?

A The test has a greater accuracy with positive tests than with negative tests.
B The test has a greater accuracy with negative tests than with positive tests.
C The test is equally accurate with positive and negative test results.
D The test is equally inaccurate with positive and negative test results.
E There is insufficient information to compare positive and negative test results.

8 Airport scanning equipment is tested by scanning 200 pieces of luggage. Prohibited items were placed in 50 bags and the scanning equipment detected 48 of them. The equipment detected prohibited items in five bags that did not have any forbidden items in them.

a Use the above information to complete the contingency table below.

	Test results		Total
	Accurate	Not accurate	
Bags with prohibited items			
Bags with no prohibited items			
Total			

b Use the table to answer the following:

i What percentage of bags with prohibited items were detected?
ii What was the percentage of false positives among the bags that had no prohibited items?
iii What percentage of prohibited items pass through the scanning equipment undetected?
iv What is the overall percentage accuracy of the scanning equipment?

9 In some cases it is easier to count numbers in a particular category by considering a different population. In each of the following pairs of proportions, which one would be easier to determine?

a **i** Proportion of males who are left-handed.
ii Proportion of left-handers who are males.

b **i** Proportion of Mathematics A students in your school who are over 16.
ii Proportion of over 16 year olds in your school who study Mathematics A.

c **i** Proportion of state school students who live in Queensland.
ii Proportion of Queensland school students who attend a state school.

WORKED Example 6

10 Refer to the 2006 census data on industry of employment in Australia on page 200.

a Draw up a contingency table showing the males and females employed in education and training compared with those employed in other industries.

b Extend your table to show totals in all categories as well as an overall total.

c Draw up a table showing percentages with respect to gender.

d Redraw your table showing percentages based on industry.

e What percentage of females are employed in education?

f What percentage of those employed in education are female?

g At some period in between census times, if it were necessary to obtain an estimate of the number of females employed in education by surveying a sample, what approach would you recommend?

WORKED Example 7

Use the following data collected from the 2006 census for questions **11** and **12**. It details age groups and education details for residents of Townsville.

Australian Bureau of Statistics

2006 Census of Population and Housing
Townsville (C) (LGA 37 000) 1869.6 sq. Kms

B01 SELECTED PERSON CHARACTERISTICS BY SEX (FIRST RELEASE PROCESSING)
Count of persons
Based on place of usual residence

	Males	*Females*	*Persons*
Total persons	48 396	47 068	95 464
Age groups:			
0–4 years	3 003	2 850	5 853
5–14 years	6 415	5 978	12 393
15–19 years	3 749	3 819	7 568
20–24 years	4 783	4 266	9 049
25–34 years	7 306	6 900	14 206
35–44 years	6 873	6 846	13 719
45–54 years	6 630	6 295	12 925
55–64 years	4 919	4 422	9 341
65–74 years	2 723	2 797	5 520
75–84 years	1 576	2 073	3 649
85 years and over	419	822	1 241
Age of persons attending an educational institution			
0–4 years	357	344	701
5–14 years	5 662	5 293	10 955
15–19 years	2 171	2 623	4 794
20–24 years	1 054	1 487	2 541
25 years and over	1 351	2 219	3 570

11 a Construct a contingency table displaying the number of male and female 15–19 year olds in Townsville compared with all other age groups there. Show all totals.

b Is it correct to claim that:

i 50.5% of 15–19 year olds are female?

ii the percentage of males who are 15–19 years of age is greater than the percentage of females who are 15–19 years old?

Provide calculations to support your answers to each of these.

12 a Construct a contingency table displaying males and females of 15–19 years 'Attending an educational institution' and 'Not attending an educational institution'. Show all totals.

eBook plus

Digital doc: Worksheet 4.1

b From your contingency table calculate:

i the percentage of males in this age group in an educational institution

ii the percentage of those in an educational institution who are 15–19 years old and male.

c Would it be correct to say that only 55% of 15–19 year old females attend an educational institution?

10 Quick Questions 1

For questions 1 to 3, state whether a census or a survey has been used.

1 A school votes to elect a school captain.
2 Five hundred drivers complete a questionnaire on the state of a major highway.
3 All insurance customers complete a questionnaire when renewing their policies.
4 'The rich should pay more in tax to allow the poorer families to have access to better services, wouldn't you agree?' Explain why this question is biased.
5 Rewrite the above question to eliminate bias.
6 Explain why a telephone phone-in response to an issue aired on TV would almost certainly have a sampling bias.

This contingency table shows the test results of a radar surveillance system. If the system detects an intruder, an alarm is activated. Refer to it when answering questions 7 to 10.

	Alarm activated	Not activated	Total
Intruders	40	8	48
No intruders	4	148	152
Total	44	156	

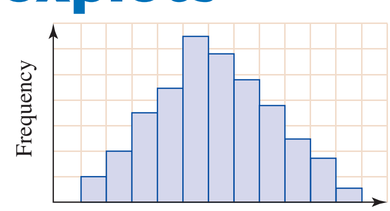
7 Over how many nights was the system tested?
8 On how many occasions was the alarm activated?
9 If the alarm was activated, what is the percentage probability that there actually was an intruder?
10 If the alarm was *not* activated, what is the percentage probability that there actually was an intruder?

Interpreting the shape of histograms, stem-and-leaf plots and boxplots

eBook*plus*

Interactivity: Boxplots and five figure summaries int-0802

Symmetric distributions



The data shown in the histogram at right can be described as **symmetric**.

There is a single peak and the data trail off on both sides of this peak in roughly the same fashion.

Similarly in the stem-and-leaf plot at right, the distribution of the data could be described as symmetric.

The single peak for these data occur at the stem 3. On either side of the peak, the number of observations reduces in approximately matching fashion.

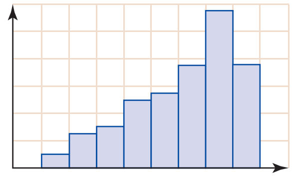
Stem	Leaf
0	7
1	2 3
2	2 4 5 7 9
3	0 2 3 6 8 8
4	4 7 8 9 9
5	2 7 8
6	1 3

Skewed distributions

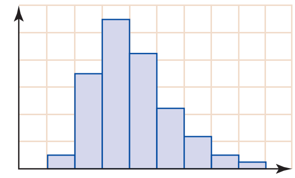
Both of the histograms shown below are examples of skewed distributions.

The figure below left shows data which are **negatively skewed**. The data in this case peak to the right and trail off to the left.

The figure below right shows **positively skewed** data. The data in this case peak to the left and trail off to the right.



Negatively skewed distribution



Positively skewed distribution

eBook*plus*

Tutorial:
Worked example 8
int-0426

WORKED Example 8

The ages of a group of people who were taking out their first home loan is shown below.

Key: 1|9 = 19 years

Stem	Leaf
1	9 9
2	1 2 4 6 7 8 8 9
3	0 1 1 2 3 4 7
4	1 3 5 6
5	2 3
6	7

Describe the shape of the distribution of these data.

THINK

Check whether the distribution is symmetric or skewed. The peak of the data occurs at the stem 2. The data trail off as the stems increase in value. This seems reasonable since most people would take out a home loan early in life to give themselves time to pay it off.

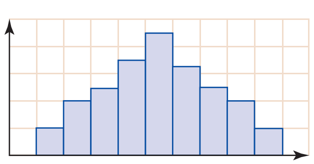
WRITE

The data are positively skewed.

It is useful to compare the corresponding boxplots of distributions with shapes which are symmetric, negatively skewed and positively skewed.

In the figure below a symmetric distribution is represented in the histogram and in the boxplot.

The characteristics of this boxplot are that the whiskers are about the same length and the median is located about halfway along the box.



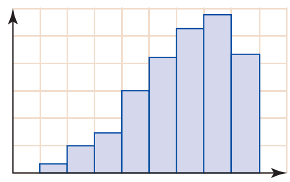
Symmetric histogram

Symmetric boxplot

The figure below shows a negatively skewed distribution. In such a distribution, the data peak to the right on the histogram and trail off to the left.

In corresponding fashion on the boxplot, the bunching of the data to the right means that the left-hand whisker is longer and the right-hand whisker is shorter; that is, the lower 25% of data are sparse and spread out whereas the top 25% of data are bunched up.

The median occurs further towards the right end of the box.



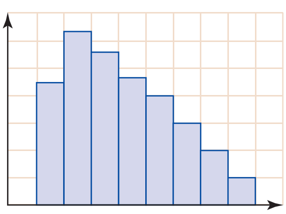
Negatively skewed histogram

Negatively skewed boxplot

In the figure below we have a positively skewed distribution. In such a distribution, the data peak to the left on the histogram and trail off to the right.

In corresponding fashion on the boxplot, the bunching of the data to the left means that the left-hand whisker is shorter and the right-hand whisker is longer; that is, the upper 25% of data are sparse and spread out whereas the lower 25% of data are bunched up.

The median occurs further towards the left end of the box.

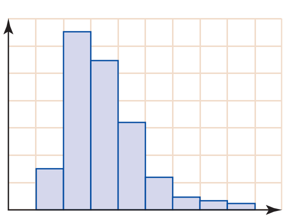


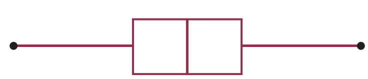
Positively skewed histogram

Positively skewed boxplot

WORKED Example 9

Explain whether or not the histogram and the boxplot shown below could represent the same data.





THINK

The histogram shows a distribution which is positively skewed.

The boxplot shows a distribution which is approximately symmetric.

WRITE

The histogram and the boxplot could not represent the same data since the histogram shows a distribution that is positively skewed and the boxplot shows a distribution that is approximately symmetric.

WORKED Example 10

eBook plus

Tutorial: Worked example 10 int-0427

The results (out of 20) of oral tests in a Year-12 Indonesian class are:

15 12 17 8 13 18 14 16 17 13 11 12

Display these data using a boxplot and discuss the shape obtained.

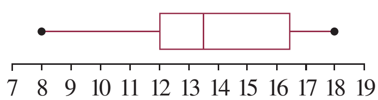
THINK

1. Find the lowest and highest scores, Q_1, the median (Q_2) and Q_3 by first ordering the data.

WRITE

8 11 12 12 13 13 14 15 16 17 17 18
The median score is 13.5.
The lower half of the scores are
8 11 12 12 13 13.
So, $Q_1 = 12$
The upper half of the scores are
14 15 16 17 17 18.
So, $Q_3 = 16.5$
The lowest score is 8.
The highest score is 18.

2. Using these five-figure summary statistics, draw the boxplot.

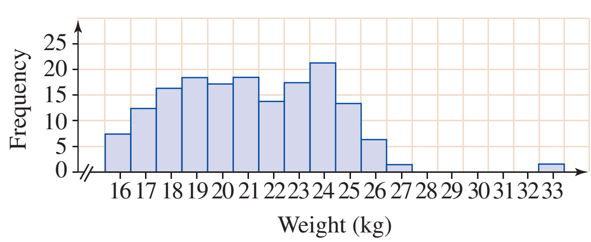


3. Consider the spread of each quarter of the data.

The scores are grouped around 12 and 13, as well as around 17 and 18 with 25% of the data in each section. The scores are more spread elsewhere.

Outliers

When one observation lies well away from other observations in a set, we call it an **outlier**. Sometimes an outlier occurs because data have been incorrectly obtained or misread. For example, below we see a histogram showing the weights of a group of 5-year-old boys.



The outlier, 33, may have occurred because a weight was incorrectly recorded as 33 rather than 23 or perhaps there was a boy in this group who, for some medical reason, weighed a lot more than his counterparts. When an outlier occurs, the reasons for its occurrence should be checked.

To identify possible outliers, we can apply a simple rule.

An outlier is a score, x, which lies outside the interval

$$Q_1 - 1.5 \cdot \text{IQR} \le x \le Q_3 + 1.5 \cdot \text{IQR}$$

An outlier is not included in the boxplot but simply plotted as a point beyond the end of the whisker.

WORKED Example 11

The times (in seconds) achieved by the 12 fastest runners in the 100-m sprint at a school athletics meeting are listed below.

11.2	**12.3**	**11.5**	**11.0**	**11.6**	**11.4**
11.9	**11.2**	**12.7**	**11.3**	**11.2**	**11.3**

a Draw a boxplot to represent the data, describe the shape of the distribution and comment on the existence of any outliers.

b Use a graphics calculator to draw a boxplot for the data.

THINK / **WRITE/DISPLAY**

a 1 Determine the five-figure summary statistics by first ordering the data and obtain the interquartile range.

a 11.0 11.2 11.2 11.2 11.3 11.3 11.4 11.5 11.6 11.9 12.3 12.7

lowest score = 11.0

highest score = 12.7

$Q_2 = 11.35$

$Q_1 = 11.2$

$Q_3 = 11.75$

IQR = 11.75 – 11.2 = 0.55

2 Identify any outliers by applying the outlier rule.

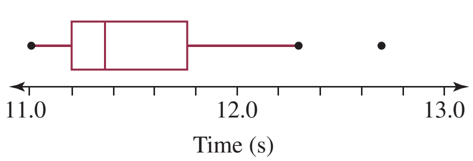
$Q_1 - 1.5 \cdot \text{IQR} = 11.2 - 1.5 \cdot 0.55 = 10.375$

The lowest score lies above 10.375, so there is no outlier below.

$Q_3 + 1.5 \cdot \text{IQR} = 11.75 + 1.5 \cdot 0.55 = 12.575$

The score 12.7 lies above 12.575, so it is an outlier and 12.3 becomes the end of the upper whisker.

3 Draw the boxplot with the outlier.



4 Describe the shape of the distribution. Data peak to the left and trail off to the right with one outlier.

The data are positively skewed with 12.7 seconds being an outlier. This may be due to incorrect timing or recording but more likely the 12th time was simply slower than the first eleven.

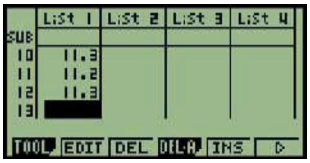
For the Casio *fx*-9860G AU

b 1 Enter data.

To enter the data, press:
- MENU
- 2: STAT.

Enter the data into List 1.

b



Continued over page

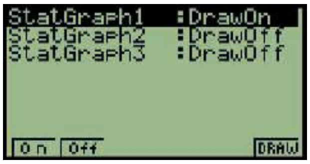
THINK | **WRITE/DISPLAY**

2 Draw the boxplot.

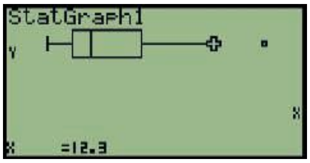
i To draw the boxplot press:
- F1 (GRPH)
- F6 (SET).

For StatGraph1, set the fields as shown.

ii To draw one boxplot, press F4 (SEL). Set StatGraph1 to DrawOn. Set StatGraphs 2 and 3 as DrawOff.



iii To see the boxplot, press F6 (DRAW). To find the five-number summary, press SHIFT F1 (TRCE). Note the position of the outlier.

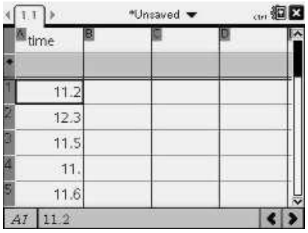


For the TI-Nspire CAS

1 Enter data.

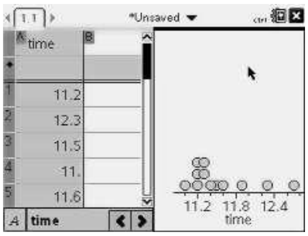
i To enter the data, open a Lists & Spreadsheet page.

ii Enter the list of scores in column A and label the column with the title 'time'.



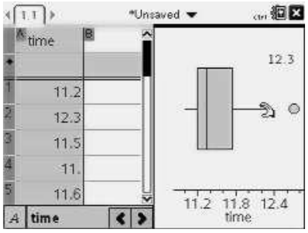
2 Draw the boxplot.

i To highlight the first column, press the up arrow ▲ in column A until it is highlighted. Press:
- MENU (menu)
- 3: Data (3)
- 6: Quick Graph (6).



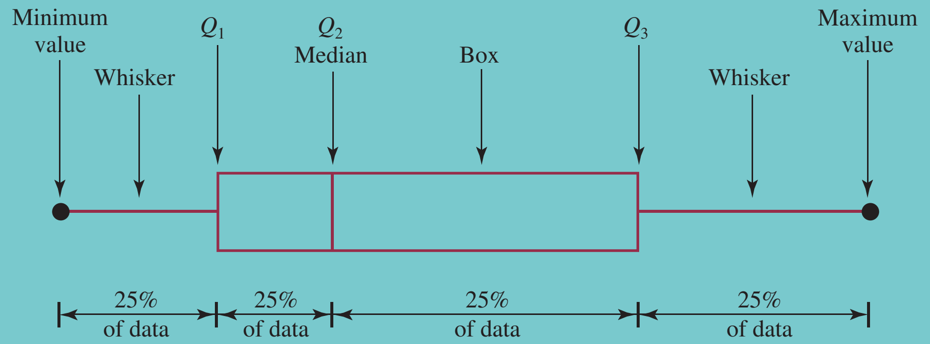
ii To draw one boxplot, press:
- MENU (menu)
- 1: Plot Type (1)
- 2: Box Plot (2).

Use the NavPad to navigate over the boxplot to show the five-number summary. Note the position of the outlier.



remember

1. When data are displayed in a histogram or a stem-and-leaf plot, we say that their distribution is:
   (a) symmetric if there is a single peak and the data trail off on either side of this peak in roughly the same fashion
   (b) negatively skewed if the data peak to the right and trail off to the left
   (c) positively skewed if the data peak to the left and trail off to the right.



A boxplot

2. When data are displayed in a boxplot we say that the distribution of the data is:
   (a) *symmetric* if the whiskers are about the same length and the median is about halfway along the box
   (b) *negatively skewed* if the left-hand whisker is longer than the right-hand whisker and the median occurs closer to the right-hand end of the box
   (c) *positively skewed* if the left-hand whisker is shorter than the right-hand whisker and the median occurs closer to the left-hand end of the box.
3. An outlier is a score, x, which lies outside the interval

$$Q_1 - 1.5 \cdot \text{IQR} \leq x \leq Q_3 + 1.5 \cdot \text{IQR}$$

EXERCISE 4D Interpreting the shape of histograms, stem-and-leaf plots and boxplots

WORKED Example 8

1 For each of the following stem-and-leaf plots, describe the shape of the distribution of the data.

a Key: 1|2 = 12

Stem	Leaf
0	1 3
1	2 4 7
2	3 4 4 7 8
3	2 5 7 9 9 9 9
4	1 3 6 7
5	0 4
6	4 7
7	1

b Key: 2|6 = 2.6

Stem	Leaf
1	3
2	6
3	3 8
4	2 6 8 8 9
5	4 7 7 7 8 9 9
6	0 2 2 4 5

c Key: 10|4 = 104

Stem	Leaf
2	3 5 5 6 7 8 9 9
3	0 2 2 3 4 6 6 7 8 8
4	2 2 4 5 6 6 6 7 9
5	0 3 3 5 6
6	2 4
7	5 9
8	2
9	7
10	

d Key: 2|4 = 24

Stem	Leaf
1	
1*	5
2	1 4
2*	5 7 8 8 9
3	1 2 2 3 3 3 4 4
3*	5 5 5 6
4	3 4
4*	

e Key: 4|3 = 0.43

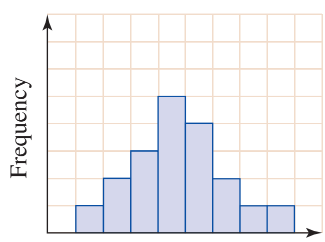
Stem	Leaf
3	
3	8 9
4	0 0 1 1 1
4	2 3 3 3 3 3
4	4 5 5 5
4	6 7
4	8

f Key: 62|3 = 623

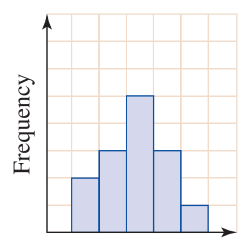
Stem	Leaf
60	2 5 8
61	1 3 3 6 7 8 9
62	0 1 2 4 6 7 8 8 9
63	2 2 4 5 7 8
64	3 6 7
65	4 5 8
66	3 5
67	4

2 For each of the following histograms, describe the shape of the distribution of the data and comment on the existence of any outliers.

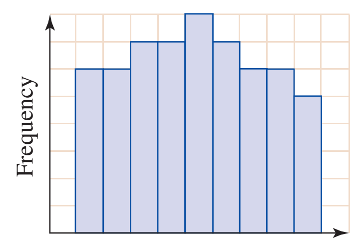
a



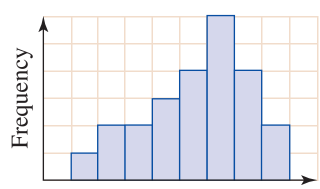
b



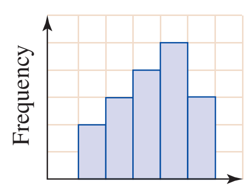
c



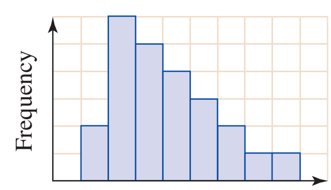
d



e



f



3 **multiple choice**

The distribution of the data shown in this stem-and-leaf plot could be described as:

A negatively skewed
B negatively skewed and symmetric
C positively skewed
D positively skewed and symmetric
E symmetric.

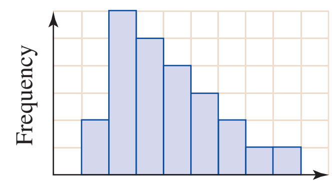
Key: 1|8 = 18

Stem	Leaf
0	1
0	2
0	4 4 5
0	6 6 6 7
0	8 8 8 8 9 9
1	0 0 0 1 1 1 1
1	2 2 2 3 3 3
1	4 4 5 5
1	6 7 7
1	8 9

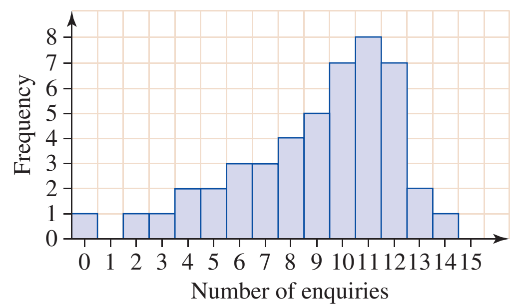
4 **multiple choice**

The distribution of the data shown in the histogram at right could be described as:

A negatively skewed
B negatively skewed and symmetric
C positively skewed
D positively skewed and symmetric
E symmetric.



5 The average number of product enquiries per day received by a group of small businesses who advertised in the Yellow Pages telephone directory is given at right. Describe the shape of the distribution of these data.



6 The number of nights per month spent interstate by a group of flight attendants is shown on the stem plot at right. Describe the shape of distribution of these data and explain what this tells us about the number of nights per month spent interstate by this group of flight attendants.

Key: 1|4 = 14 nights

Stem	Leaf
0	0 0 1 1
0	2 2 3 3 3 3 3 3 3 3
0	4 4 5 5 5 5 5
0	6 6 6 6 7
0	8 8 8 9
1	0 0 1
1	4 4
1	5 5
1	7

7 The mass (to the nearest kilogram) of each dog at a dog obedience school is shown on the stem plot at right.

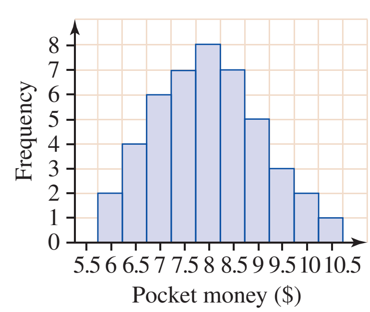
a Describe the shape of the distribution of these data.
b What does this information tell us about this group of dogs?

Key: 0|4 = 4 kg

Stem	Leaf
0	4
0*	5 7 9
1	1 2 4 4
1*	5 6 6 7 8 9
2	1 2 2 3
2*	6 7

8 The amount of pocket money (to the nearest 50 cents) received each week by students in a Year-6 class is illustrated in the histogram at right.

a Describe the shape of the distribution of these data.
b What conclusions can you reach about the amount of pocket money received weekly by this group of students?



9 The number of hours of exercise completed each week by a group of employees at a company is shown on the stem plot at right.

a Describe the shape of the distribution of these data.

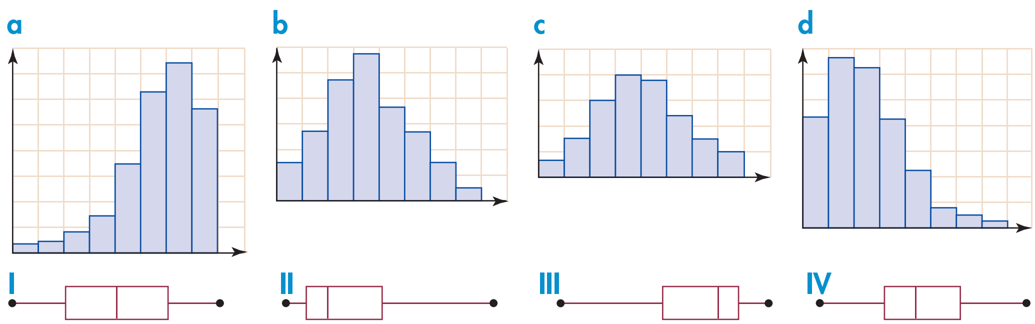
b What does this tell us about the number of hours of exercise completed weekly by the employees in this company?

Stem	Leaf
0	0 0 0 0 1 1
0	2 2 2 2 3 3 3
0	4 4 5
0	6 7
0	8

Key: 0|1 = 1 hour

WORKED Example 9

10 Each of the histograms shown below is labelled with a letter and each of the boxplots is labelled with a number. Match each histogram with a boxplot which could show the same distribution.



11 For each of the following sets of data, construct a boxplot.

a 3 5 6 8 8 9 12 14 17 18

b 3 4 4 5 5 6 7 7 7 8 8 9 9 10 10 12

c 4.3 4.5 4.7 4.9 5.1 5.3 5.5 5.6

d 11 13 15 15 16 18 20 21 22 21 18 19 20 16 18 20

e 0.4 0.5 0.7 0.8 0.8 0.9 1.0 1.1 1.2 1.0 1.3

WORKED Example 10

eBook *plus*

Digital doc: Spreadsheet 004 Boxplots

12 **multiple choice**

For the distribution shown in the boxplot below, it is true to say that:

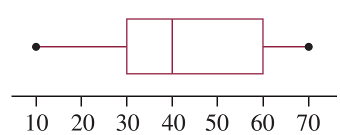
A the median is 30

B the median is 45

C the interquartile range is 10

D the interquartile range is 30

E the interquartile range is 60.



13 The number of clients seen each day over a 15-day period by a tax consultant is:

3 5 2 7 5 6 4 3 4 5 6 6 4 3 4

Represent these data on a boxplot.

14 The maximum daily temperatures (in °C) for the month of September in Brisbane are:

18 26 28 23 16 19 21 27 31 23 24 26 21 18 26 27
23 21 24 20 19 25 27 32 29 21 16 19 23 25 27

Represent these data on a boxplot.

WORKED Example 11

15 The number of rides that 16 children had during their time at the annual show are listed below.

8	5	9	4	9	0	8	7
9	2	8	7	9	6	7	8

a Draw a boxplot to represent the data, describe the shape of the distribution and comment on the existence of any outliers.
b Use a graphics calculator to draw a boxplot for these data.

16 Twenty Year 7 students were asked how many music CDs they owned. The results are presented below.

9	6	10	7	8	15	5	8	9	8
11	8	7	9	12	7	10	9	8	9

a Draw a boxplot to represent the data, describe the shape of the distribution and comment on the existence of any outliers.
b Use a graphics calculator to draw a boxplot for these data.

Interpreting data in practical situations

By exploring data collected from samples (provided the samples have been chosen carefully) we are able to estimate characteristics of the population. We can determine past trends and speculate on future trends. Through a series of investigations we will explore the application of statistics and probability to life-related situations.

Using histograms to estimate probabilities

Discrete data (the type where the scores can take only set values) can be represented as a frequency histogram.

Continuous data (the type where the scores may take any value, usually within a certain range) can also be represented in the form of a frequency or probability histogram. Let us construct a frequency histogram of continuous data from which we can then estimate probabilities.

WORKED Example 12

A battery company tested a random sample of a batch of their batteries to determine their lifetime. The results are shown below.

Lifetime (hours)	20–<25	25–<30	30–<35	35–<40	40–<45	45–<50
Frequency	6	25	70	61	30	8

a Represent the data as a frequency histogram.
b If you chose a battery from this batch, estimate the probability that the battery would last:
i at least 25 hours
ii less than 40 hours.
c In an advertising campaign, the battery manufacturer claims that they will replace the battery if it does not last at least 30 hours. Based on these results, what is the probability they will have to replace a battery?

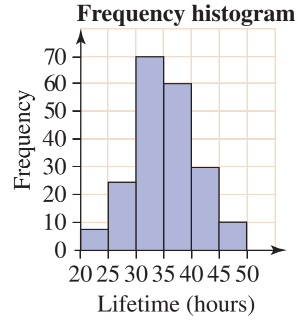
Continued over page

THINK

a Construct a frequency histogram with lifetime on the x-axis and frequency on the y-axis.

WRITE

a



THINK

b 1 Find the total number of scores.

2 The total area under the curve is 1, so each class interval represents a fraction of 1 in terms of area (and probability).

i 1 Find the total of frequencies with a score of at least 25 hours.

2 Estimated probability

$$= \frac{\text{total of frequencies at least 25 h}}{\text{total number of scores}} \cdot 1$$

3 Write the answer.

ii 1 Find the total frequencies with a score of less than 40 hours.

2 Apply the same rule as in part **i**.

3 Write the answer.

WRITE

b Total number of scores

$= 6 + 25 + 70 + 61 + 30 + 8$

$= 200$

i Total frequency at least 25 hours

$= 25 + 70 + 61 + 30 + 8$

$= 194$

$P(\geq 25\text{ h}) = \frac{194}{200} \cdot 1$

$= 0.97$

The probability that the battery would last for at least 25 hours is 0.97.

ii Total frequency less than 40 hours

$= 6 + 25 + 70 + 61$

$= 162$

$P(<40\text{ h}) = \frac{162}{200} \cdot 1$

$= 0.81$

The probability that the battery would last less than 40 hours is 0.81.

THINK

c 1 Find the total frequency for those batteries lasting less than 30 hours.

2 Apply the probability rule.

3 Write the answer.

WRITE

c Total frequency less than 30 hours

$= 6 + 25$

$= 31$

$P(<30\text{ h}) = \frac{31}{200} \cdot 1$

$= 0.155$

P(replacing battery) = 0.155

The probability that the manufacturer will have to replace the battery is 0.155.

It should be noted that, if we are not given a table of results (as we were in the previous worked example), but simply a frequency histogram, we would have to estimate frequencies from the histogram. In this case, the probability answers obtained would be estimates rather than exact values.

Interpreting histograms

The aim of this investigation is to highlight the pitfalls in interpreting the shape of histograms. The activity is more readily conducted using a graphics calculator.

1 Consider the percentages received by a class of 36 students in their end-of-semester test.
67, 90, 83, 85, 73, 80, 78, 79, 68, 71, 53, 65, 74, 64, 77, 56, 66, 63, 70, 49, 56, 71, 67, 58, 60, 72, 67, 57, 60, 90, 63, 88, 78, 46, 64, 81.

For the Casio *fx*-9860G AU

2 To enter the data as a list into a graphics calculator, press:
- MENU
- 2: STAT.

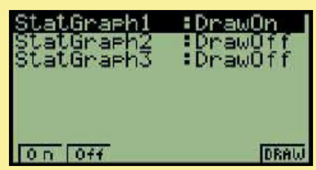
Enter the data into List 1.

3 **i** To graph the data as a histogram, press:
- F1 (GRPH)
- F6 (SET).

Adjust the settings as shown, press EXE.

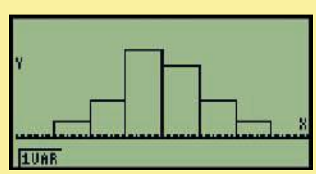
ii To set StatGraph1 DrawOn, press F4 (SEL).

Ensure StatGraphs 2 and 3 are set to DrawOff.

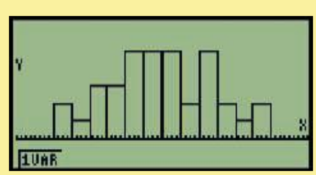


iii To set the histogram to start at 40 with a width of 10, press EXE and then adjust settings as shown.

iv To draw the histogram, press EXE.



4 To view the histogram with different widths, repeat **ii** to **iv** of part **3** to set the histogram to start at 46, with a width of 4.



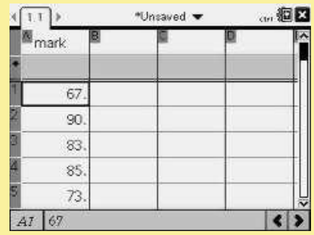
Note that, while the previous histogram appeared to have one modal class, this one appears multimodal.

(*Continued*)

For the TI-Nspire CAS

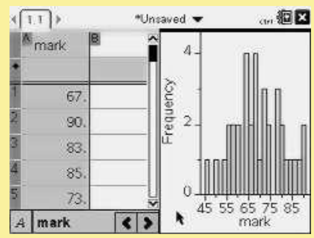
2 To enter the data as a list, open a Lists & Spreadsheet page.

Label column A as 'mark' and enter the data in the cells below.



3 Graph the data as a histogram starting at 40, with a class interval of 10.

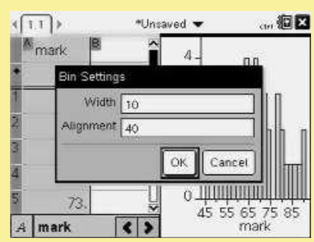
i To highlight column A place the cursor over the title, 'mark' and press the up arrow ▲. Then press:
- MENU (menu)
- 3: Data (3)
- 6: Quick Graph (6)
- MENU (menu)
- 1: Plot Type (1)
- 3: Histogram (3).



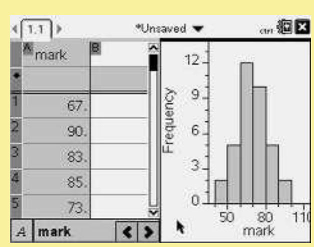
Note that this is a histogram of individual marks.

ii To adjust the class interval, called the *bin width,* press:
- MENU (menu)
- 2: Plot Properties (2)
- 2: Histogram Properties (2)
- 2: Bin Settings (2).

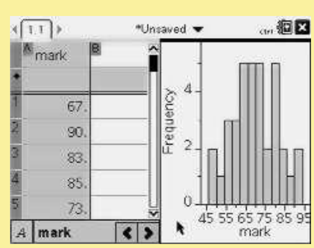
Set the width to 10 and the alignment to 40.



iii To redraw the histogram with the adjusted class interval, press:
- MENU (menu)
- 5: Window/Zoom (5)
- 2: Zoom – Data (2).



4 To view the histogram with different widths, repeat the previous steps to change the bin width to 4 and the alignment to 46.



Note that, while the previous histogram appeared to have one modal class, this one appears multimodal.

5 Use your calculator to investigate changing the class interval and the starting value of the percentages. What do you observe?

6 All these histograms are graphical representations of the same data. While they all indicate distributions with higher frequencies towards the middle, some suggest bimodal or multimodal distributions. What do you conclude from this investigation?

Using scatterplots to consider relationships between data sets

WORKED Example 13

Are tall mothers likely to produce tall sons?
The table below details the heights of 12 mothers and their adult sons.

Height of mother (cm)	185	152	168	166	173	172	159	154	168	148	162	171
Height of son (cm)	188	162	168	172	179	182	160	148	178	152	184	180

a Construct a scatterplot of the data.
b Draw the line of best fit.
c Estimate the height of a son born to a 180-cm tall mother.
d Discuss the relationship between the heights of mothers and their sons as shown by these data.
The solution to this problem will be shown using three methods.
1. Pen and paper 2. Graphics calculator 3. Spreadsheet
It should be noted that, when a line of best fit is drawn by eye, variations in answers will occur for those dependent on the position of the line.

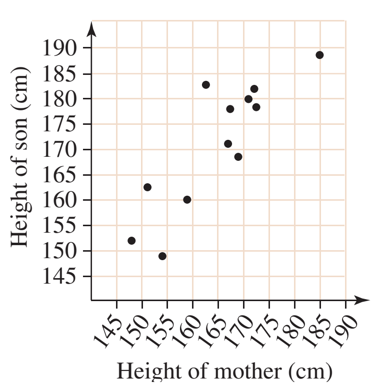
THINK

WRITE/DRAW

Method 1. Using pen and paper

a Plot points on a graph with height of mother on x-axis (the independent variable) and height of son on y-axis (the dependent variable). This results in a scatterplot.

a



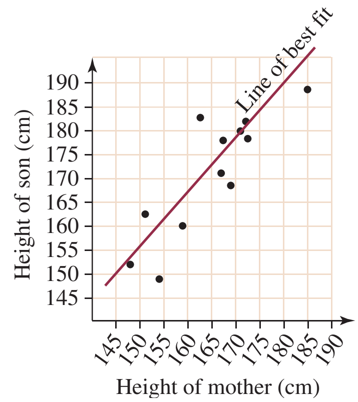
Continued over page

THINK

WRITE/DISPLAY

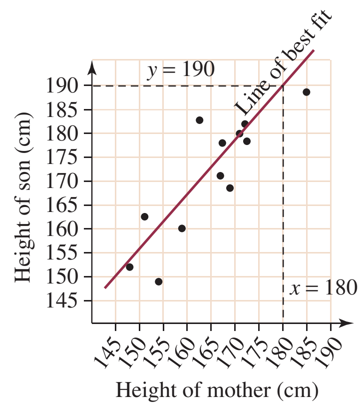
b Draw in the line of best fit. Balance an equal number of points either side of the line and as close to the line as possible.

b



c Draw a vertical line from the 180 cm point on the x-axis to the line of best fit. From this point on the line, draw a horizontal line to the y-axis. Read this y-value.

c



From the graph
when $x = 180$
$y = 190$
So, a 180-cm tall mother could produce a son approximately 190 cm tall.

d Look at the slope of the line and the proximity of the points to the line.

d The slope of the line of best fit is positive, indicating that, as one variable increases, the other also increases. The points lie fairly close to the line, so this indicates a fairly strong positive relationship between the two variables. This seems to support the view that tall mothers are likely to produce tall sons.

Method 2. Using a graphics calculator

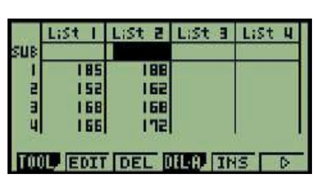
These instructions apply to the Casio *fx*-9860 G AU and TI-Nspire CAS graphics calculators.

For the Casio *fx*-9860G AU

a 1 To enter the data, press:
- MENU
- 2: STAT.

Clear any data in the lists. Enter the mother's height in List 1 and the son's height in List 2.

a

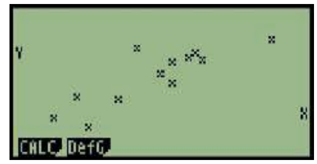


THINK

WRITE/DISPLAY

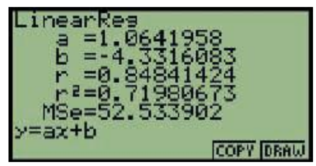
2 To draw a scatterplot, press:
- (F1) (GRPH)
- (F6) (SET).

Set StatGraph1 to be a scatterplot, XList to List 1 and YList to List 2. Press (EXE) to return to the previous screen. Press (F4) (SEL). Turn StatGraph1 on, making sure StatGraph2 and StatGraph3 are both turned off, then press (F6) (DRAW).



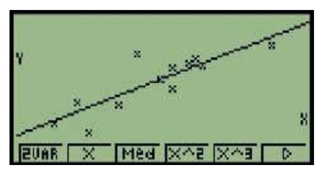
3 To calculate the equation of the regression line, press:
- (F1) (CALC)
- (F2) (X).

This shows the equation of the regression line (or line of best fit) to be $y = 1.064x - 4.332$.



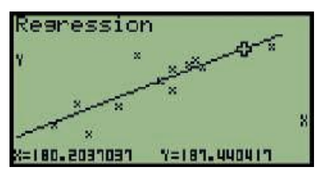
b To draw the regression line, press:
- (F5) (COPY)
- (EXE)
- (F6) (DRAW).

b



c To find the sons's height when the mother's height is 180 cm, press (SHIFT) (F1) (TRCE). Use the arrow keys to move the cursor along the line to $x = 180$.

c



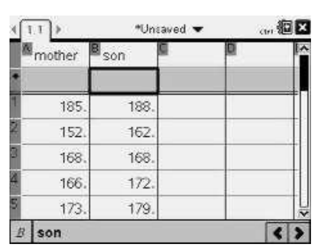
The sons's height is predicted to be about 187 cm.

For the TI-Nspire CAS

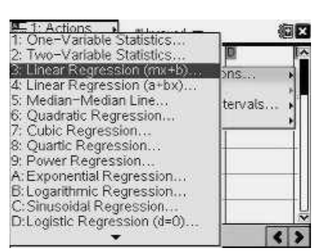
a 1 To enter the data into a spreadsheet, open a Lists & Spreadsheet page.

Enter the mother's height in column A and the son's height in column B, labelling both columns as shown.

a



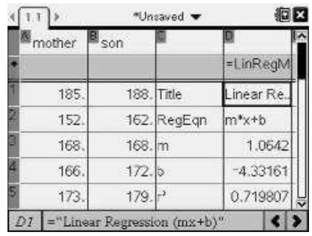
2 To calculate the least-squares regression line, highlight both columns and then press:
- MENU (menu)
- 4: Statistics (4)
- 1: Stat Calculations . . . (1)
- 3: Linear Regression (mx + b) (3).



Continued over page

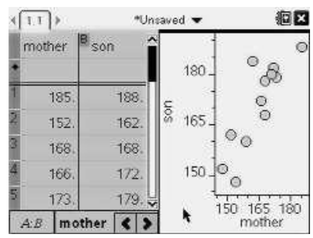
THINK | **WRITE/DISPLAY**

3. Ensure that X List: a[] and Y List: b[]. Press Tab (tab) to OK, then press ENTER (enter). This shows the regression line (or line of best fit) to have the equation $y = 1.064x - 4.332$.



4. To draw a scatterplot, highlight the first two columns then press:
   - MENU (menu)
   - 3: Data (3)
   - 6: Quick Graph (6).

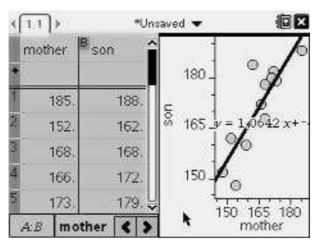
   A scatterplot appears.



b To graph the regression line, press:
- MENU (menu)
- 4: Analyse (4)
- 6: Regression (6)
- 1: Show Linear (mx + b) (1).

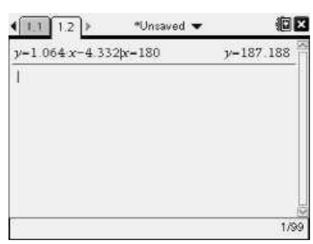
The regression line will be displayed on the scatterplot.

b



c To estimate the height of a son born to a 180-cm tall mother, open a Calculator page.
Complete the entry line as:
$y = 1.064x - 4.332 \mid x = 180$.
Then press ENTER (enter).

c



The son's height is predicted to be about 187 cm.

d Look at the angle of the straight line and the proximity of the points to the line.

d The line of best fit from both calculators predicts that a 180-cm tall mother could produce an adult son approximately 187 cm tall. The slope of the line of best fit is upwards, indicating that as one variable increases the other also increases. Most of the points lie close to the line, so it is reasonable to assume that the relationship between mother and son heights is quite strong. This supports the proposal that tall mothers are likely to produce tall sons.

THINK

Method 3. Using a spreadsheet

a 1 Open up a spreadsheet and enter the data for the mother's and son's heights in columns under headings.

2 Use the chart wizard to graph the data as a scatterplot.

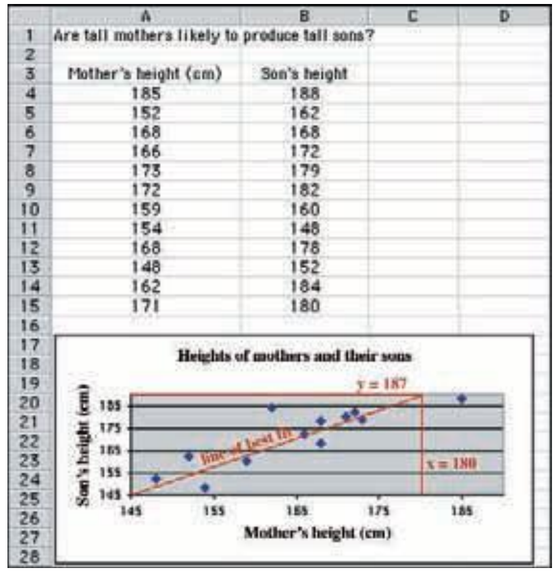
3 Label the axes and provide a title for the graph.

4 Adjust the range and scale on the x- and y-axes to more appropriate values if necessary (suggest 145 to 190 range with a scale of 5).

5 Print out a copy of the scatterplot.

WRITE/DISPLAY

a



	A	B	C	D
1	Are tall mothers likely to produce tall sons?			
2				
3	Mother's height (cm)	Son's height		
4	185	188		
5	152	162		
6	168	168		
7	166	172		
8	173	179		
9	172	182		
10	159	160		
11	154	148		
12	168	178		
13	148	152		
14	162	184		
15	171	180		

b Draw in the line of best fit. Balance an equal number of points either side of the line and as close to the line as possible.

b From the scatterplot of the data above, the line of best fit is shown on the scatterplot.

c From the graph, read the corresponding y-value for $x = 180$ cm.

c When $x = 180$, $y = 187$.
So a 180-cm tall mother would produce an adult son approximately 187 cm tall.

d Look at the slope of the line and the proximity of the points to the line.

d The slope of the line of best fit is positive, indicating that, as one variable increases, the other also increases. The points lie fairly close to the line, so this indicates a fairly strong positive relationship between the two variables. This seems to support the view that tall mothers are likely to produce tall sons.

Comparison of data sets

When multiple data displays are used to display similar sets of data, comparisons and conclusions can then be drawn about the data.

WORKED Example 14

A bank surveys the average morning and afternoon waiting time for customers. The figures were taken each Monday to Friday in the morning and afternoon for one month. The stem-and-leaf plot at right shows the results.

Key: 1 | 2 = 1.2 minutes

Morning	Stem	Afternoon
7	**0**	7 8 8
8 6 3 1 1	**1**	1 1 2 4 4 5 6 6 6 7
9 6 6 6 5 5 4 3 3 1	**2**	2 5 5 8
9 5 2	**3**	1 6
5	**4**	
	5	7

a Find the median morning waiting time and the median afternoon waiting time.

b Calculate the range for morning waiting times and the range for afternoon waiting times.

c What conclusions can be made from the display about the average waiting time at the bank in the morning compared with the afternoon?

THINK

a There are 20 scores in each set and so the median will be the average of the 10th and 11th scores.

b For each data set, subtract the lowest score from the highest score.

c Conclude that waiting time in the afternoon is generally less and more consistent except for one outlier.

WRITE

a Morning: Median = (2.4 + 2.5) | 2
= 2.45 minutes

Afternoon: Median = (1.6 + 1.6) | 2
= 1.6 minutes

b Morning: Range = 4.5 − 0.7
= 3.8 minutes

Afternoon: Range = 5.7 − 0.7
= 5 minutes

c The waiting time is generally shorter in the afternoon. There is one outlier in the afternoon scores which causes the range to be larger. However, apart from this outlier the afternoon scores are less spread.

Contingency tables can also be a meaningful way of displaying data. A contingency table allows for two variables to be compared.

WORKED Example 15

A survey of 25 000 people is taken. The sex of each respondant is noted and whether they are a smoker or non-smoker is also noted. The results are displayed in the two-way table below.

	Males	Females	Totals
Smokers	4125	4592	8717
Non-smokers	8436	7847	16 283
Totals	12 561	12 439	**25 000**

a What percentage of the females surveyed were smokers?

b What percentage of the smokers surveyed were female?

THINK

a Write 4592 as a percentage of 12 439.

b Write 4592 as a percentage of 8717.

WRITE

a Percentage of females who smoke

$= \frac{4592}{12\,439} \cdot 100\%$

$= 36.9\%$

b Percentage of smokers who are female

$= \frac{4592}{8717} \cdot 100\%$

$= 52.7\%$

The most common method, however, for comparing data sets is to compare the summary statistics from the data sets. The measures of location such as mean and median are used to compare the typical score in a data set. Measures of spread such as range, interquartile range and standard deviation are used to make assessments about the consistency of scores in the data set.

WORKED Example 16

Below are the scores for two students in eight mathematics tests throughout the year.

Jane: 45, 62, 64, 55, 58, 51, 59, 62

Pierre: 84, 37, 45, 80, 74, 44, 46, 50

a Use the statistics function on the calculator to find the mean and standard deviation for each student.

b Which student had the better overall performance on the eight tests?

c Which student was more consistent over the eight tests?

THINK

a Enter the statistics into your calculator to determine the mean and the standard deviation.

b The student with the higher mean performed better overall.

c The student with the lower standard deviation was more consistent.

WRITE

a Jane: $\bar{x} = 57$, SD = 6
Pierre: $\bar{x} = 57.5$, SD = 17.4

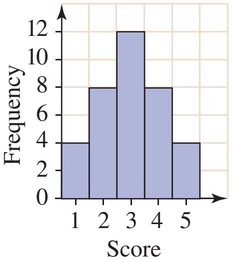
b Pierre performed slightly better overall, as his mean mark was higher than Jane's.

c Jane was the more consistent student, as her standard deviation was much lower than Pierre's.

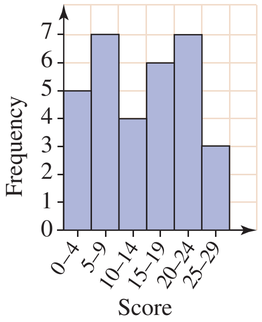
remember

1. Frequency histograms can be used to estimate probabilities in data sets.
2. Scatterplots display the relationship between two variables.
3. Scatterplots enable past and future trends to be considered.
4. When multiple displays are used for two or more sets of data, we can compare and contrast the data sets and determine if any relationship exists between them.
5. A multiple stem-and-leaf plot allows for a quick comparison of medians, ranges and interquartile ranges.
6. The summary statistics from two data sets can be compared quickly on a box-and-whisker plot.
7. Contingency tables can be used to make a comparison of data where two variables are involved.
8. The most commonly used comparisons are summary statistics to compare what is a typical score and what the spread of the data is.

EXERCISE 4E Interpreting data in practical situations



1 In the distribution on the right:
- **a** is the graph symmetrical?
- **b** what is/are the modal class(es)?
- **c** can the mean and median be seen from the graph and, if so, what are their values?
- **d** which score has the greatest probability of occurring?



2 For the distribution shown on the right:
- **a** are the data symmetrical?
- **b** what is/are the modal class(es)?
- **c** can the mean and median be seen from the graph and, if so, what are their values?
- **d** which classes have the same probability of occurring?
- **e** which class has the least probability of occurring?

3 The table on the right shows the number of goals scored by a hockey team throughout a season.
- **a** Show this information in a frequency histogram.
- **b** Are the data symmetrical?
- **c** What is/are the mode(s)?
- **d** Can the mean and median be seen for this distribution and, if so, what are their values?
- **e** The probability that the team will score 5 goals is the same as their probability of scoring what other number of goals?

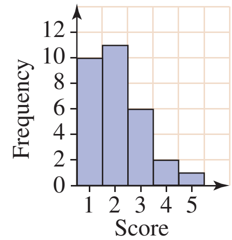
No. of goals	Frequency
0	6
1	4
2	4
3	4
4	4
5	6

4 For the distribution shown on the right:

a what is/are the modal score(s)?

b which score has the greatest probability of occurring?

c which score has the least probability of occurring?



WORKED Example 12

5 The table at right shows the number of goals scored by a basketball team throughout a season.

a Draw a frequency histogram of the data.

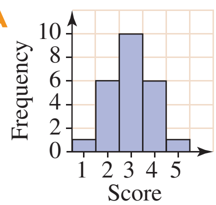
b What is the probability that the team will score more than 40 goals?

No. of goals	Frequency
11–20	3
21–30	6
31–40	7
41–50	23
51–60	21

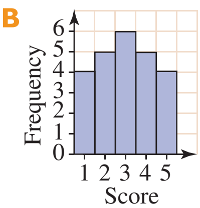
6 multiple choice

Which of the distributions below has the smallest standard deviation?

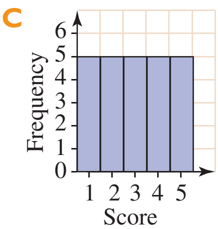
A



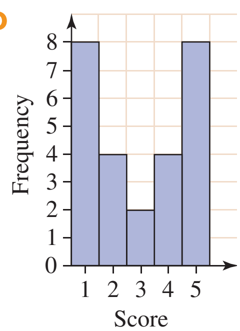
B



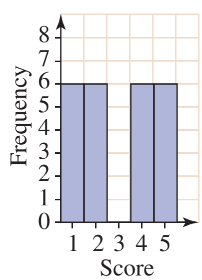
C



D



E



7 A movie is shown at a cinema 30 times during the week. The number of people attending each session of the movie is shown in the table at right.

a Present the data in a frequency histogram.

b Are the data symmetrical?

c What is/are the modal class(es)?

d What is the probability of more than 150 people attending a session?

e What is the probability of having up to 100 people at a session?

No. of people	Frequency
1–50	2
51–100	3
101–150	5
151–200	10
201–250	10

8 Year 12 at Wallarwella High School sit exams in chemistry and mathematics. The results are shown in the table below.

Mark	Chemistry	Maths
31–40	2	3
41–50	9	4
51–60	7	6
61–70	4	7
71–80	7	9
81–90	9	7
91–100	2	4

a Is either distribution symmetrical? **b** State the mode of each distribution.
c In which subject is the standard deviation greater? Explain your answer.
d The students were told that the probability of achieving more than 80% in chemistry was the same as it was in mathematics. Is this true? Explain.
e Is the probability of obtaining more than a pass mark (50%) greater in chemistry or mathematics?
f What is the propability of achieving over 90% in each subject?

Note that some answers in the following questions may vary depending on the position of the line of best fit. The use of a graphics calculator is recommended, if available.

WORKED Example 13

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Digital docs:
SkillSHEET 4.3 Using an equation to make predictions
Spreadsheet 057 Interpolation/ Extrapolation

9 A drug company wishes to test the effectiveness of a drug to increase red blood cell counts in people who have a low count. The following data were collected.

Day of experiment	4	5	6	7	8	9
Red blood cell count	210	240	230	260	260	290

Construct a scatterplot, then draw in the line of best fit to find the red blood cell count at the beginning of the experiment (that is, on day 0).

10 A wildlife exhibition is held over 6 weekends and features still and live displays. The number of live animals that are being exhibited varies each weekend. The number of animals participating, together with the number of visitors to the exhibition each weekend, is shown in the table which follows.

Number of animals	6	4	8	5	7	6
Number of visitors	311	220	413	280	379	334

Construct a scatterplot, then draw in the line of best fit to find the predicted number of visitors if there are no live animals.

11 A study of the dining-out habits of various income groups in a particular suburb produces the results shown in the table below.

Weekly income ($)	100	200	300	400	500	600	700	800	900	1 000
Number of restaurant visits per year	5.8	2.6	1.4	1.2	6	4.8	11.6	4.4	12.2	9

Use the data to predict:

a the number of visits per year by a person on a weekly income of \$680

b the number of visits per year by a person on a weekly income of \$2000.

12 The following table represents the costs for transporting a consignment of shoes from Brisbane factories. The cost is given in terms of distance from Brisbane. There are two factories which can be used. The data are summarised below.

Distance from Brisbane (km)	10	20	30	40	50	60	70	80
Factory 1 cost (\$)	70	70	90	100	110	120	150	180
Factory 2 cost (\$)	70	75	80	100	100	115	125	135

a Draw the line of best fit for each factory.

b Which factory is likely to have the lowest cost to transport to a shop in Brisbane?

c Which factory is likely to have the lowest cost to transport to Mytown, 115 kilometres from Brisbane?

d Which factory has more 'linear' transport rates?

WORKED Example 14

13 The stem-and-leaf plot drawn below shows the marks obtained by 20 students in both English and Maths.

Key: 7 | 1 = 71

English		Maths
	4	1 7
7 4 1 0	**5**	2 4 7 9 9
9 9 7 6 6 5 3 1 1 0	**6**	1 3 3 4 6 6
8 7 7 5 2	**7**	4 4 4 8
2	**8**	3 6
	9	4

a Calculate the median mark for both English and Maths.

b Calculate the range of marks for both English and Maths.

c Comment on the distribution of marks in each of the subjects.

14 Tracey measures the heights (in meters) of twenty Year 10 boys and twenty Year 10 girls and produces the following five-number summaries for each data set.

Boys: 1.47, 1.58, 1.64, 1.72, 1.81
Girls: 1.55, 1.59, 1.62, 1.66, 1.73

a Draw a box-and-whisker plot for both sets of data and display them on the same scale.

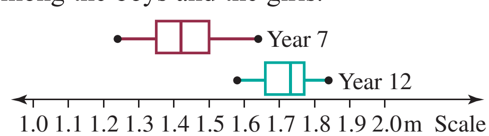
b What is the median of each distribution?

c What is the range of each distribution?

d What is the interquartile range for each distribution?

e Comment on the spread of the heights among the boys and the girls.

15 The box-and-whisker plots on the right show the heights of a sample of Year 7 boys and a similar-sized sample of Year 12 boys.



a Calculate the range of heights among both the Year 7 and Year 12 boys.

b Calculate the interquartile range of the heights among both the Year 7 and Year 12 boys.

c Comment on the relationship between the two data sets, both in terms of measures of location and measures of spread.

WORKED Example 15

16 The contingency table below shows the results of random breath testing by Brisbane police over one weekend. A driver is charged if they record a reading of over 0.05% blood alcohol content (BAC).

	Males	Females	Totals
Over 0.05 PCA	26	7	33
Below 0.05 PCA	962	743	1705
Totals	988	750	**1738**

a What percentage of the drivers tested were female?
b What percentage of the drivers tested had a PCA over 0.05?
c What percentage of female drivers had a PCA over 0.05?
d What percentage of male drivers had a PCA over 0.05?
e Based on the above results, can any conclusion be drawn concerning the prevalence of drink driving among males and females? Explain your answer.

17 Ashley is the star player of a football team. To analyse the importance of Ashley to the team, the coach prepares the contingency table below showing the results of games over three years both when Ashley is playing and not playing.

	Won	Lost	Totals
Ashley playing	38	4	42
Ashley not playing	10	8	18
Totals	48	12	**60**

a What percentage of games were won when Ashley played?
b What percentage of games were won when Ashley did not play?
c Do you think that Ashley has a significant impact on the performance of the team? Explain your answer.

WORKED Example 16

18 Calvin recorded his marks for each test that he did in Physics and Chemistry throughout the year.
Physics: 65, 74, 69, 66, 72, 64, 75, 60
Chemistry: 45, 85, 91, 42, 47, 72, 87, 85
a In which subject did Calvin achieve the better average mark?
b In which subject was Calvin more consistent? Explain your answer.

19 The police set up two radar speed checks in a country town. In both places the speed limit is 60 km/h. The results of the first 10 cars that have their speed checked are given below.
Point A: 60, 62, 58, 55, 59, 56, 65, 70, 61, 64
Point B: 55, 58, 59, 50, 40, 90, 54, 62, 60, 60
a Calculate the mean and standard deviation of the readings taken at each point.
b At which point are drivers generally driving faster?
c At which point is the spread of the readings taken greater? Explain your answer.

20 Aaron and Sunil open the batting for the local cricket team. The number of runs they have scored in each innings this season are listed below.
Aaron: 45, 43, 33, 56, 21, 38, 0, 29, 76, 40
Sunil: 5, 70, 12, 54, 68, 11, 8, 64, 32, 69

a Calculate the mean number of runs scored for each player.
b What is the range of runs scored by each player?
c What is the interquartile range of runs scored by each player?
d Which player would you consider to be the more consistent player? Explain your answer.

21 **multiple choice**

The figures below show the ages of the men's and women's champions at a tennis tournament.
Men's: 23, 24, 25, 26, 25, 25, 22, 23, 30, 24
Women's: 19, 27, 20, 26, 30, 18, 28, 25, 28, 22
Which of the following statements is correct?

A The mean age of the men's champions is greater than the mean age of the women's champions.
B The range is greater among the men's champions than among the women's champions.
C The interquartile range is greater among the men's champions than among the women's champions.
D The standard deviation is greater among the men's champions than among the women's champions.
E None of the above

22 A company producing matches advertise that there are 50 matches in each box. Two machines are used to distribute the matches into the boxes. The results from a sample taken from each machine are shown in the stem-and-leaf plot below.

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Digital doc: WorkSHEET 4.2

Key: 5 | 1 = 51 5* | 6 = 56

Machine A		Machine B
4	**4**	
9 9 8 7 7 6 6 5	**4***	5 7 8 9 9 9 9 9 9 9
4 3 2 2 2 2 1 1 1 0 0 0 0 0	**5**	0 0 0 0 0 1 1 1 1 1 2 2 3
5 5	**5***	9

a Display the data from both machines on a box-and-whisker plot.
b Calculate the mean and standard deviation of the number of matches distributed from both machines.
c Which machine is the more dependable? Explain your answer.

investigation

Year 2018 Commonwealth Games

The Gold Coast region has submitted a bid to host the 2018 Commonwealth Games. Obviously a great deal of planning is required for such an event. It should be held at a time which is not too hot or too cold. Since rain could interrupt proceedings, this must be considered when choosing an appropriate time of year. Humidity and wind speeds could also be factors to consider.

The table over the page (downloaded from the Bureau of Meteorology site) shows climatic data collected for the Gold Coast from 1987 to 2008.

(Continued)

Monthly Climate Statistics for 'GOLD COAST SEAWAY'

040764 GOLD COAST SEAWAY
Commenced: 1987
Last Record: 2008
Latitude: 27.94 Degrees South

Longitude: 153.43 Degrees East
Elevation: 3 m
State: QLD

Statistic Element	January	February	March	April	May	June	July	August	September	October	November	December	Annual
Mean max temp (ºC)	28.5	28.4	27.6	25.6	23.3	21.2	21.1	21.6	23.6	25.1	26.4	27.6	25
Highest temp (ºC)	38.5	40.5	36.3	33.3	29.4	27.1	26.8	28.5	33	36.8	35.5	39.4	40.5
Date of highest temp	12-Jan-02	22-Feb-04	1-Mar-93	4-Apr-06	4-May-07	3-Jun-04	4-Jul-07	10-Aug-07	10-Sep-03	8-Oct-04	28-Nov-95	26-Dec-01	22-Feb-04
Lowest max temp (ºC)	22.8	22.3	22.2	19.3	17.7	15.7	15.6	15	16.9	17.5	19.8	21.3	15
Date of lowest max temp	4-Jan-96	19-Feb-94	13-Mar-03	13-Apr-94	31-May-00	16-Jun-98	3-Jul-95	12-Aug-05	23-Sep-93	23-Oct-96	6-Nov-96	2-Dec-94	12-Aug-05
Mean min temp (ºC)	21.8	21.8	20.6	18.3	15.6	13.2	12	12.6	14.8	17	18.7	20.4	17.2
Lowest temp (ºC)	17.2	17.2	13.4	8.9	6.6	3.8	2.5	4.3	8.3	10	8.2	14.7	2.5
Date of lowest temp	8-Jan-03	1-Feb-04	31-Mar-08	29-Apr-08	29-May-04	21-Jun-04	19-Jul-07	1-Aug-03	1-Sep-00	11-Oct-03	17-Nov-06	19-Dec-05	19-Jul-07
Highest min temp (ºC)	26	27.3	26.2	23.7	21.8	19.1	21.8	18.3	20.7	22.2	24	24.6	27.3
Date of highest min temp	21-Jan-00	22-Feb-04	6-Mar-97	2-Apr-00	6-May-00	6-Jun-02	11-Jul-00	31-Aug-04	18-Sep-96	27-Oct-05	28-Nov-95	19-Dec-97	22-Feb-04
Mean rainfall (mm)	107.8	169.2	89.1	89.5	110.7	124.1	41.7	68.7	43.6	91.5	124.8	124.9	1190.2
Highest rainfall (mm)	193	471.2	210.2	226.2	339.6	455.8	172.2	204.2	163.6	255.4	379.4	197	1673.4
Year of highest rainfall	1996	2003	2001	1998	1996	2005	1999	1999	1998	1997	2004	1995	1999
Lowest rainfall (mm)	5.2	17.4	13	13.4	22.6	14.6	1	0	3.2	13.4	30.8	67.4	851.2
Year of lowest rainfall	2003	2005	1998	2007	2004	1998	2007	1997	2000	1998	1994	2006	2002
Highest daily rainfall (mm)	87	140	122	59	126	350.8	100.6	144	87	127	149.8	75.6	350.8
Date of highest daily rainfall	10-Jan-96	4-Feb-03	10-Mar-01	17-Apr-98	15-May-03	30-Jun-05	1-Jul-05	29-Aug-99	12-Sep-98	8-Oct-97	7-Nov-04	8-Dec-04	30-Jun-05
Mean number of days of rain	14.3	13.5	13.9	11.9	12.1	11.2	7.6	8.2	8.4	10.1	12.1	11.7	135
Mean 9am temp (ºC)	25.8	25.7	24.7	22.6	19.7	17	16.5	17.6	20.1	22.1	23.6	24.9	21.7
Mean 9am relative humidity (%)	71	72	71	68	69	68	65	62	62	64	65	67	67
Mean 9am wind speed (km/h) for years 1991 to 2008	18.4	17.3	18.6	16.9	14.3	13.7	13.1	15.4	16.7	17.4	18.3	17.8	16.5
Mean 3pm temp (ºC)	26.5	26.7	25.7	24	22	20	19.7	20.1	21.6	22.6	23.9	25.2	23.2
Mean 3pm relative humidity (%)	71	71	69	65	63	58	55	57	62	67	68	69	65
Mean 3pm wind speed (km/h) for years 1991 to 2008	25.2	23.7	25.1	22.7	19.5	18.8	19.1	22.5	24.2	24.8	24.4	24.8	22.9

Assume you work for an organisation which has a contract to prepare the submission for the 2018 Commonwealth Games on behalf of the Gold Coast. Your task is to:

1. decide on the best time of the year to hold this event
2. prepare a proposal on behalf of the Gold Coast council to host the 2018 Commonwealth Games
3. support your submission with statistical analysis and graphical displays
4. convince the Commonwealth Games organisers that the Gold Coast is the best place to hold the event that year.

Sampling text to predict population characteristics

Research shows that the letter 'e' is the most frequently used letter in the English language. This activity aims to determine the frequency of its occurrence on a page of English text using a sample selected from the page.

1 Choose a book with an extended section of continuous prose. Select 20 full lines of text from a page (ignore incomplete lines).
2 Draw up the table below.

Line number	Number of e's	Cumulative total
1		
2		
3		
20		

3 Count the number of e's per line and complete the second column.
4 Using the figures in the second column, complete the 'cumulative total' column.
5 Use a graphics calculator, spreadsheet or graph paper to draw a scatterplot of the number of lines against the accumulated total.
6 Draw the line of best fit on your scatterplot.
7 Count the number of lines of text on your page. Extrapolate your scatterplot to estimate the number of e's on the page you have chosen.
8 Compare your answer with those obtained from different texts by other members of your class. Comment on the similarities and variations in your answers. Factors that must be taken into account when making comparisons between the number of e's per page of printed material include page size, font size, line length, line spacing and so on.

Aside: The novel, *A void* by George Perec (Harper Collins 1994) is written entirely without the letter 'e'.

Comparing population characteristics

The following tables display data collated from the 2006 ABS Census relating to Queensland and Australia.

Australian Bureau of Statistics
2006 Census of Population and Housing
(Queensland (STE 3) 1740378.0 sq. Kms

B01 SELECTED PERSON CHARACTERISTICS BY SEX (FIRST RELEASE PROCESSING)
Count of persons
Based on place of usual residence

	Males	*Females*	*Persons*
Total persons	1 935 381	1 969 153	3 904 534
Age groups:			
0–4 years	132 171	124 909	257 080
5–14 years	281 714	267 742	549 456

(*Continued*)

Age groups	*Males*	*Females*	*Persons*
15–19 years	138 175	132 124	270 299
20–24 years	135 241	133 661	268 902
25–34 years	258 405	265 192	523 597
35–44 years	280 410	295 158	575 568
45–54 years	265 750	273 434	539 184
55–64 years	220 859	216 694	437 553
65–74 years	129 382	130 044	259 426
75–84 years	74 092	92 717	166 809
85 years and over	19 182	37 475	56 657
Counted on Census Night:			
At home (a)	1 825 872	1 883 519	3 709 391
Elsewhere in Australia(a)	109 509	85 634	195 143
Indigenous persons:			
Aboriginal	48 235	50 481	98 716
Torres Strait Islander	9 300	9 076	18 376
Both Aboriginal and Torres Strait Islander(b)	5 155	5 333	10 488
Total	*62 690*	*64 890*	*127 580*
Birthplace:			
Australia	1 451 360	1 483 900	2 935 260
Elsewhere(c)	342 660	356 788	699 448
Language spoken at home:			
English only	1 666 245	1 705 439	3 371 684
Other language(d)	144 061	159 035	303 096
Australian citizen	1 655 316	1 706 732	3 362 048

(a) Data are based on place of enumeration.
(b) Applicable to persons who are of both Aboriginal and Torres Strait Islander origin.
(c) Includes 'Australian External Territories', 'Inadequately described', 'At sea', and 'Not elsewhere classified'.
(d) Includes 'Inadequately described' and 'Non-verbal, so described'.

Australian Bureau of Statistics
2006 Census of Population and Housing
Australia (0) 7759538.2 sq. Kms

B01 SELECTED PERSON CHARACTERISTICS BY SEX (FIRST RELEASE PROCESSING)
Count of persons
Based on place of usual residence

	Males	*Females*	*Persons*
Total persons	9 799 250	10 056 038	19 855 288
Age groups:			
0–4 years	647 411	612 987	1 260 398
5–14 years	1 373 927	1 302 887	2 676 814
15–19 years	695 798	661 110	1 356 908
20–24 years	681 651	665 706	1 347 357
25–34 years	1 321 061	1 355 330	2 676 391
35–44 years	1 437 242	1 500 598	2 937 840
45–54 years	1 360 081	1 402 443	2 762 524
55–64 years	1 096 133	1 096 545	2 192 678
65–74 years	668 447	704 987	1 373 434
75–84 years	412 460	535 628	948 088
85 years and over	105 042	217 814	322 856
Counted on Census Night:			
At home(a)	9 298 488	9 631 062	18 929 550
Elsewhere in Australia(a)	500 758	424 980	925 738
Indigenous persons:			
Aboriginal	200 164	207 536	407 700
Torres Strait Islander	15 068	14 448	29 516
Both Aboriginal and Torres Strait Islander(b)	8 841	8 970	17 811
Total	*224 073*	*230 954*	*455 027*
Birthplace:			
Australia	6 931 756	7 141 194	14 072 950
Elsewhere(c)	2 163 339	2 252 693	4 416 032

Language spoken at home:			
English only	7 671 497	7 909 834	15 581 331
Other language(d)	1 522 742	1 623 453	3 146 195
Australian citizen	8 386 595	8 708 974	17 095 569

(a) Data are based on place of enumeration.
(b) Applicable to persons who are of both Aboriginal and Torres Strait Islander origin.
(c) Includes 'Australian External Territories', 'Inadequately described', 'At sea', and 'Not elsewhere classified'.
(d) Includes 'Inadequately described' and 'Non-verbal, so described'.

1 Examine the data carefully.
2 Prepare a report outlining the similarities and differences between the characteristics of Queenslanders and the overall Australian population. Support your statements by reference to figures.

Modelling Olympic Games times

The running time for the men's 100-m event in the Olympic Games broke the 10-second barrier for the first time in 1968 when Jim Hines of the United States clocked a time of 9.95 seconds. Knowing that records are broken over time, is it possible to predict a year when a runner could break the 9.5-second barrier?

We could model this situation by looking at past times for the event. These are shown in the table at right.

Note: The times have not decreased consistently over the years. In fact, after 1968, the 10-second barrier was not broken again until 1984 when Carl Lewis of the United States won with a time of 9.99 s. For this reason, along with other factors which accompany feats of human endurance, in modelling situations such as this, any resulting predictions can only be considered as estimates.

Year	Time (seconds)
1948	10.30
1952	10.40
1956	10.50
1960	10.20
1964	10.00
1968	9.95
1972	10.14
1976	10.06
1980	10.23
1984	9.99
1988	9.92
1992	9.96
1996	9.84
2000	9.87
2004	9.85
2008	9.69

1 Using a graphics calculator, spreadsheet or graph paper, draw a scatterplot of 'year' (on the x-axis) against 'time' (on the y-axis). This could be viewed as a time series as the years are at equal intervals.
2 Draw the line of best fit for your scatterplot.

(*Continued*)

3 Use your line of best fit to find the year in which the 9.5-second barrier could be broken in the 100-m sprint. This represents a situation when we are extrapolating data (that is, determining a value outside the range of data plotted). Note that the Olympic Games are only held every four years and that your answer may not be one of those years. It remains to be seen how accurate your prediction is!

4 Use some resources available to you (World Wide Web, reference books, almanacs and so on) to collect data on other sporting events (such as high jump heights, shot put distances and swimming times). Model the situation, enabling a prediction to be proposed.

EXTENSION: How fast could Usain Bolt have run? Use the **Usain Bolt** weblink in your eBookPLUS to find out.

investigation

Predicting test results

Over the year Sally had sat for nine mathematics tests, but had been sick at the time of the tenth test. She had achieved above average marks for each of her nine tests, so her teacher did not want to give her the class average for her last test. As sometimes happens, the class as a whole found this last test more difficult than the previous ones, so generally all marks were depressed. It would not then be fair to give Sally the average of her previous tests for this last one. How could her teacher give Sally an estimated mark based on her past performance compared with that of her fellow students?

Below is a table displaying the class average percentage for each test and Sally's percentage on the tests.

Test	1	2	3	4	5	6	7	8	9	10
Class %	64	59	60	55	62	66	58	63	65	49
Sally's %	72	70	77	65	75	80	71	75	75	

Because Sally's marks were consistently above the class average by 10% or more, we could explore a relationship between the class average and Sally's marks.

1 Using a graphics calculator, or otherwise, construct a scatterplot of the first nine tests, plotting the class average percentage on the x-axis and Sally's percentage on the y-axis.

2 Draw in the line of best fit.

3 Use your line of best fit and a value of 49 for the class average percentage to determine an estimate for Sally's performance had she sat for the tenth test.

4 Do you think this is a fair mark to award to Sally? Justify your answer.

This method can be used in a variety of situations to estimate values for missing results. The predictions become less accurate if there is a great deal of inconsistency in past performance.

investigation

The door game

Imagine you are a contestant in a game on television. The conditions and rules are as follows:

1. Three doors (door 1, door 2, and door 3) stand before you.
2. Behind one of these doors which face you and the audience lies the prize of your dreams (a trip to Wimbledon or a Ferrari car). (The organisers know where the prize lies.)
3. Behind each of the other two doors there is nothing.
4. You see the three closed doors before you and you have no hints.
5. With the benefit (or distraction) of audience participation, you are asked to select the door behind which you think the prize lies.
6. You tell the compere and audience which door you choose.
7. Before opening the door you have chosen, the compere tells you that first, one of the doors, different from the one you have chosen, will be opened; it will be a door that does not have the prize behind it.
8. There are two doors now unopened and you are given an offer to change your choice.
9. Of the two unopened doors, one is your original choice and the other not your choice.
10. Using the theory of probability, would you have a better chance of winning if you stayed with your original choice, or made the switch?

This was a game which was actually played on television for a lengthy period many years ago. Regular viewers of the program formed their own opinions regarding the better option by analysing the results of contestants' choices. The background to the game can be researched by a web search of the name Monty Hall. You may be interested to conduct a search at this stage, or leave your search until you have had time to formulate a logical reason for your choice.

Part I

This activity provides practical experience for the door game. The simulation could be undertaken as a class activity or in pairs.

1. Simulate three doors using books lying on a flat surface instead of doors.
2. Underneath one of them place a small piece of paper. (The contestant is to be unaware of the location of the paper.)
3. Ask the contestant to select one of the three books underneath which he or she thinks this piece of paper lies.
4. Turn over one book other than one chosen by the contestant and underneath which the object does not lie.
5. Ask the contestant whether he/she would like to choose a different book.
6. Turn over the book under which the paper lies.

(*Continued*)

7 Repeat the simulation ten times. Copy and complete the table below, recording your results after each turn shown in the table.

Game	Stay with choice	Change mind	Win/Lose
1	✔		✔
2		✔	✖
3		✔	✔
4			
5			
6			
7			
8			
9			
10			

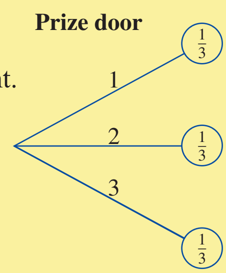
8 Combine your results with those of other members of your class so that your combined set consists of a large number of simulations.

9 What are your conclusions from your experiment? Is it wise to stay with your original choice or should you change your mind?

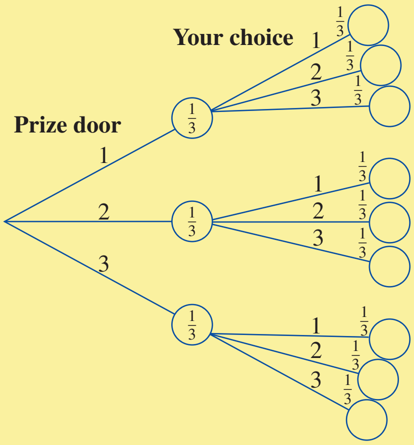
Part II

Let us consider the probabilities of the choices in the 'door game'.

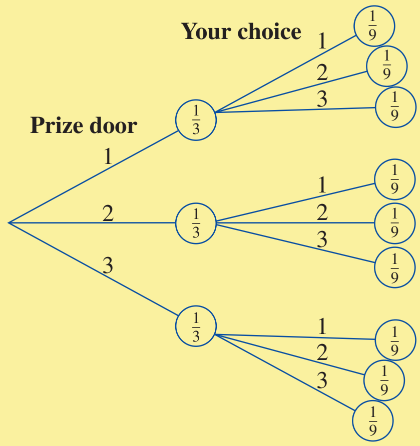
1. Since there are three doors, the probability of the prize being behind any of these doors is $\frac{1}{3}$.
   This enables us to start a tree diagram as shown at right.



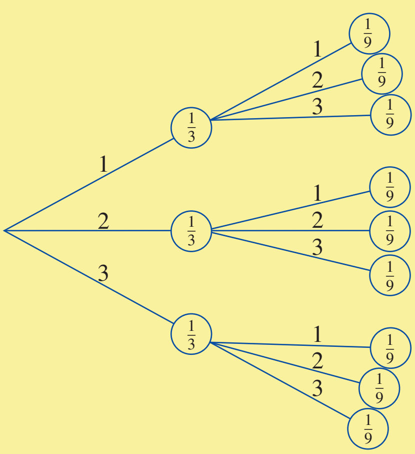
2. As you have three doors from which to select, the probability that your choice will be the correct selection is $\frac{1}{3}$.
   This realisation enables us to extend the branches of the tree as at right.



3. Multiplying the probabilities along the branches enables us to fill in the probability for each selection (shown at right).



4. The next stage is the crucial factor. Depending on which door is opened, will you win if you stay with your choice, or are you more likely to win if you change your mind? Follow the results completed for door 1. The combinations of opened doors have been completed below for the remainder of the tree. Complete the blank spaces.



Door opened	Win if I stay?	Win if I change my mind?
2 or 3	Yes ($P = \frac{1}{9}$)	No
3	No	Yes ($P = \frac{1}{9}$)
2	No	Yes ($P = \frac{1}{9}$)
3	____	____
1 or 3	____	____
1	____	____
2	____	____
1	____	____
1 or 2	____	____

5. Add the probabilities in the 'Win if I stay?' column. This gives the overall probability of your winning if you stay with your original choice.
6. Similarly, add the probabilities in the 'Win if I change mind?' column. This figure represents the overall probability of your winning if you change your mind and choose the other unopened door.

   Based on these probabilities, which choice should you make? How does this agree with your experimental results from your practical simulation?

If you have not yet visited websites by searching for Monty Hall, it would now be appropriate to do so. This game has created much discussion among great mathematicians and non-mathematicians over many years.

10 QUICK QUESTIONS 2

For the set of scores 23, 45, 24, 19, 22, 16, 16, 27, 20, 21, use your calculator to find:

1 the mean
2 the median
3 the mode
4 the range
5 the interquartile range
6 the standard deviation.
7 In the data set, is the distribution symmetrical?
8 Does the data set have an **outlier** (a score which is not typical of the data set)?
9 Which measure of **central tendency** is the best measure of location in this data set?
10 Explain why the interquartile range is a better measure of spread than the range.

summary

Populations and samples

- A statistical investigation can be done by either census or sample.
- A census is when an entire population takes part in the investigation.
- A sample is when a small group takes part in the investigation and the results are taken to be representative of the whole group.
- A random sample occurs when chance is the only factor in deciding who participates.

Bias

If the sample is poorly chosen the results of the investigation will be biased. This means that the results will be skewed towards one section of the population.

Contingency tables

- Display horizontal and vertical categories of two variables of a set of data
- Calculations with regard to a variety of reference bases can be made.

Interpreting the shape of histograms, stem-and-leaf plots and boxplots

- When data are displayed in a histogram or a stem-and-leaf plot, we say that the distribution of those data is:
  1. symmetric if there is a single peak and the data trail off on either side of this peak in roughly the same fashion
  2. negatively skewed if the data peak to the right and trail off to the left
  3. positively skewed if the data peak to the left and trail off to the right.

Interpreting data in practical situations

- Frequency histograms can be used to estimate probabilities in data sets.
- Scatterplots display the relationship between two variables and allow predictions to be made.
- The summary statistics from two data sets can be compared from a stem-and-leaf plot or box-and-whisker plot.
- Contingency tables are used to compare data where there are two variables involved.
- Data are most commonly compared using the mean and standard deviation.

Measures of central tendency and spread

- The mean, median and mode are measures of central tendency in a data set.
- The mean is calculated by adding the scores then dividing by the number of scores.
- The median is the middle score or average of two middle scores.
- The mode is the most frequently occurring score.
- The range, interquartile range and standard deviation are measures of spread.
- The range is the difference between the highest and lowest scores.
- The interquartile range is the difference between the upper and lower quartiles.
- The standard deviation is found using the population or sample functions of a calculator.
- An outlier is a score that is much less or much greater than the other scores.

CHAPTER review

4A

1 For each of the following statistical investigations, state whether a census or a survey has been used.

a The average price of petrol in Townsville was estimated by averaging the price at 40 petrol stations.

b The Australian Bureau of Statistics has every household in Australia complete an information form once every five years.

c The performance of a cricketer is measured by looking at his performance in every match he has played.

d Public opinion on an issue is sought by a telephone poll of 2000 homes.

4A

2 **multiple choice**

Which of the following is an example of a census?

A A newspaper conducts an opinion poll of 2000 people.

B A product survey of 1000 homes to determine what brand of washing powder is used

C Every 200th jar of Vegemite is tested to see if it is the correct mass.

D Information is collected to determine the number 1 hit single for the week.

E A federal election

4A 4B

3 Use your random number generator to select 10 numbers between 1 and 1000.

4 Bias can be introduced into statistics through:

a questionnaire design

b sample selection

c interpretation of statistical results.

Discuss how bias could be a result of techniques in the above three areas.

4C

5 A medical test screens 200 people for a virus. A positive test result indicates that the patient has the virus.

1. Of 50 people known to have the virus, the test produced 48 positive results.
2. Of the remainder who were known not to have the virus, the test produced one positive result.

Use the above information to complete the table below.

	Test results		Total
	Accurate	Not accurate	
With virus			
Without virus			
Total			

4C

6 The results of a lie detector test are given below.

1. Of 80 people who are known to be telling the truth, the lie detector indicates that three are lying.
2. Of 20 people known to be lying, the lie detector indicates that 17 are lying.

Display this information in a contingency table.

4C

7 Below are the results of a test screening for a disease. A positive test indicates that the patient has the disease.

	Test results		
	Accurate	Not accurate	Total
With disease	18	2	20
Without disease	108	12	120
Total	126	14	

a How many people were tested for the disease?
b How many positive test results were recorded?
c What percentage of those people with the disease were correctly diagnosed by the test?
d If a person without the disease is chosen at random, what percentage returned a positive test?

4C

8 A reading test for people with dyslexia is given and the results are shown in the contingency table below.

	Test results		
	Accurate	Not accurate	Total
With dyslexia	39	1	40
Without dyslexia	85	5	90
Total	124	6	

a How many people were tested?
b What percentage of people tested positive for dyslexia?
c Based on the above results, if a person with dyslexia takes the test, what is the percentage chance that they will be accurately diagnosed?

4C

9 **multiple choice**

The contingency table below shows the results of a trial on new metal detectors for aircraft. The metal detector scans a piece of hand luggage and lights up if metal is found.

	Test results		
	Accurate	Not accurate	Total
With metal	9	1	10
Without metal	87	3	90
Total	96	4	

Based on the above results, the chance of metal going undetected in a piece of hand luggage is:

A 1% B 10% C 25% D 75% E 90%

4C

10 A medical test for a disease does not always give the correct result. A positive test indicates that the patient has the disease. The contingency table below shows the results of a new screening test for the disease. It was tested on a group of people, some of whom were known to be suffering from the disease; some of whom were not.

	Test results		Total
	Accurate	Not accurate	
With disease	28	2	30
Without disease	164	6	170
Total	192	8	

a How many people were tested for the disease?
b What percentage of the results were accurate?
c How many patients tested positive to the disease?
d What percentage of patients with the disease were correctly diagnosed by the new test?
e Based on the above results, what is the probability that a patient with the disease will have the disease detected by this test?

4C

11 The contingency table below compares the number of men and women who are right- and left-handed.

	Men	Women	Totals
Right-handed	158	172	330
Left-handed	17	15	32
Totals	175	187	362

a What percentage of males are left-handed?
b What percentage of females are left-handed?
c Based on the above data, is there any significant difference between the percentage of male and female left-handers?

4C

12 **multiple choice**

The contingency table below shows the number of men and women who work in excess of 45 hours per week.

	Men	Women	Totals
≤45 hours	132	128	260
>45 hours	69	34	103
Totals	201	162	363

The percentage of men who work greater than 45 hours per week is closest to:

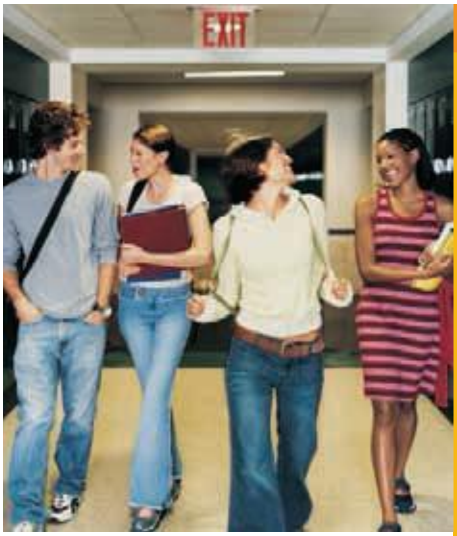
A 19% B 28% C 34% D 51% E 67%

4C

13 Sixty-seven primary and 47 secondary school students were asked their attitude to the number of school holidays which should be given. They were asked whether there should be more, fewer or the same number. Five primary students and 2 secondary students wanted fewer holidays, 29 primary and 9 secondary students thought that they had enough holidays (that is, they chose the same number) and the rest thought that they needed to be given more holidays.

a Form a contingency table with reference to primary and secondary school percentages.

b Use your table to compare the opinions of primary and secondary school students.



4D

14 **multiple choice**

The distribution of data shown in the stem-and-leaf plot at right could be described as:

A negatively skewed
B negatively skewed with one outlier
C positively skewed
D positively skewed with one outlier
E symmetric.

Key: 3|1 = 31

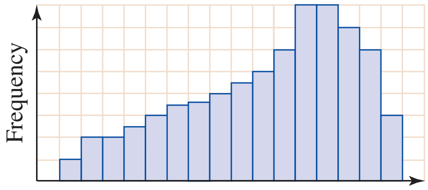
Stem	Leaf
2	3 4
2*	5 6 8
3	0 1 2 3 4 4
3*	5 5 7 9 9
4	0 1 3 3
4*	6 8 8
5	0 1
5*	6
6	9
6*	

4D

15 **multiple choice**

The distribution of the data shown in this histogram could be described as:

A negatively skewed
B negatively skewed with one outlier
C positively skewed
D positively skewed with one outlier
E symmetric.



4D

16 **a** The money raised (to the nearest whole dollar) by each student in a Year-3 class on the school walkathon is shown in the stem-and-leaf plot below.
Describe the shape of the distribution of these data.

Key: 0|8 = $8

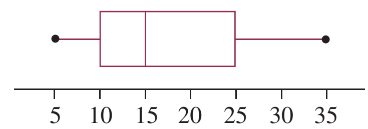
Stem	Leaf
0	8 9
1	2 3 4 7
2	1 2 2 3 5 7 9
3	0 1 4 5 8
4	3 5 6 7
5	1 3 5
6	4 6
7	6

b Describe how this distribution would need to change for it to become a symmetric one.

4D

17 **multiple choice**

For the distribution shown in this boxplot it is true to say that:



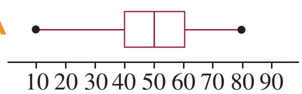
A the range is 35
B the interquartile range is 10
C the median is 20
D the interquartile range is 25
E the median is equal to the interquartile range.

4D

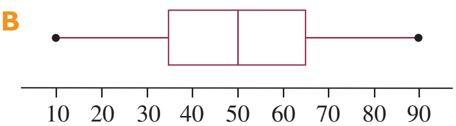
18 **multiple choice**

A distribution has a range of 80, an interquartile range of 30 and a median of 50. Which one of the following boxplots could represent this distribution?

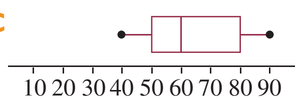
A



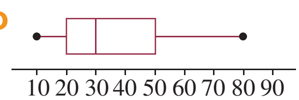
B



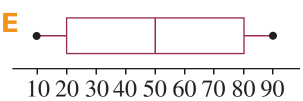
C



D



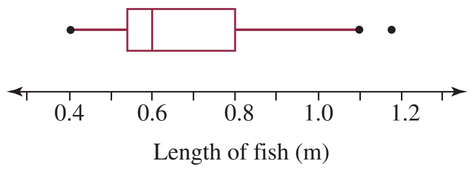
E



4D

19 **multiple choice**

The boxplot at right represents the lengths of barracuda caught by fishing boats during one day. Which one of the following statements is not true about these data?



A The data contain an outlier.
B The shortest length is 0.4 m.
C The median is 60 cm.
D The interquartile range is 0.2 m.
E The distribution is positively skewed.

4D

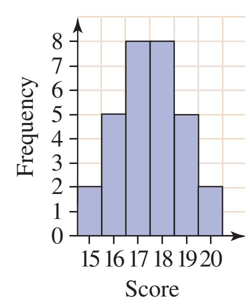
20 **a** For the set of data below, construct a boxplot to display the distribution.

2	5	4	6	3	7	9	8	5	3
1	4	6	8	7	5	2	9	5	6

b Describe the shape of the distribution.

4E

21 Consider the data set represented by the frequency histogram on the right.



a Are the data symmetrical?
b Can the mean and median of the data be seen?
c What is the mode of the data?
d Which score has the highest probability of occurring?

4E

22 Consider this data set which measures the sales figures for a new salesperson.

Day (d)	1	2	3	4	5	6	7	8
Units sold (s)	1	2	4	9	20	44	84	124

a Draw the line of best fit.
b Use your line to predict the sales figures for the tenth day.

23 The table below shows the number of attempts that 20 members of a Year 12 class took to obtain a driver's licence. 4E

Number of attempts	Frequency
1	11
2	4
3	2
4	2
5	0
6	1

a Show these data in a frequency histogram.
b Are the data symmetrical?
c What is the probability of a student of the class taking more than three attempts to obtain a driver's licence?

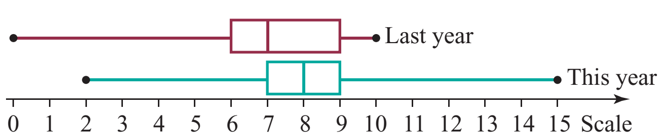
24 The stem-and-leaf plot below compares the crowds (correct to the nearest thousand) at a football team's home and away matches. 4E

Key: 2 | 5 = 25 000

Home		Away
8	**0**	6 7
7 3 2	**1**	0 1 1 6 8 9 9
6 6 3 2	**2**	4 5
5 5 2	**3**	

a Calculate the median of both data sets.
b Calculate the range of both data sets.
c Calculate the interquartile range of both data sets.
d Display both sets of data on a box-and-whisker plot.

25 The figure on the right shows a box-and-whisker plot showing the average number of weekly car sales made in last year and this year. 4E



a What was the median for each year?
b In which year was the range of sales greatest?
c In which year was the interquartile range of sales greatest?
d In which year did the car yard perform better? Explain your answer.

26 Hsiang compares her marks in 10 English exams and 10 Maths exams. 4E
English: 76, 74, 80, 77, 73, 70, 75, 37, 72, 76
Maths: 80, 56, 92, 84, 65, 58, 55, 62, 70, 71
a Calculate Hsiang's mean mark in each subject.
b Calculate the range of marks in each subject.
c Calculate the standard deviation of marks in each subject.
d Based on the above data, in which subject would you say that Hsiang performs more consistently?

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Chapter 4

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4A Populations and samples

Digital doc

- Extension: Investigate more about populations and samples *(page 187)*

4C Contingency tables

Digital docs

- SkillSHEET 4.1: Practise converting a fraction to a percentage *(page 202)*
- SkillSHEET 4.2: Practise writing one quantity as a percentage of another *(page 202)*
- WorkSHEET 4.1: Calculate measures of central tendency, five figure summary values, identify outliers and interpret tables and graphs *(page 204)*

4D Interpreting the shape of histograms, stem-and-leaf plots and boxplots

Digital doc

- Spreadsheet 004: Investigate boxplots *(page 214)*

Tutorials

- WE10 Int-0427: Watch how to construct a boxplot *(page 208)*
- WE11 Int-0426: Watch an example on how to determine the interquartile range *(page 209)*

Interactivity

- Boxplots and five figure summaries int-0802: Consolidate your understanding of constructing boxplots *(page 205)*

4E Interpreting data in practical situations

Digital docs

- SkillSHEET 4.3: Using an equation to make predictions *(page 228)*
- Spreadsheet 057: Investigate interpolation/extrapolation *(page 228)*

Chapter review

Digital doc

- Test Yourself: Take the end-of-chapter test to test your progress *(page 247)*.

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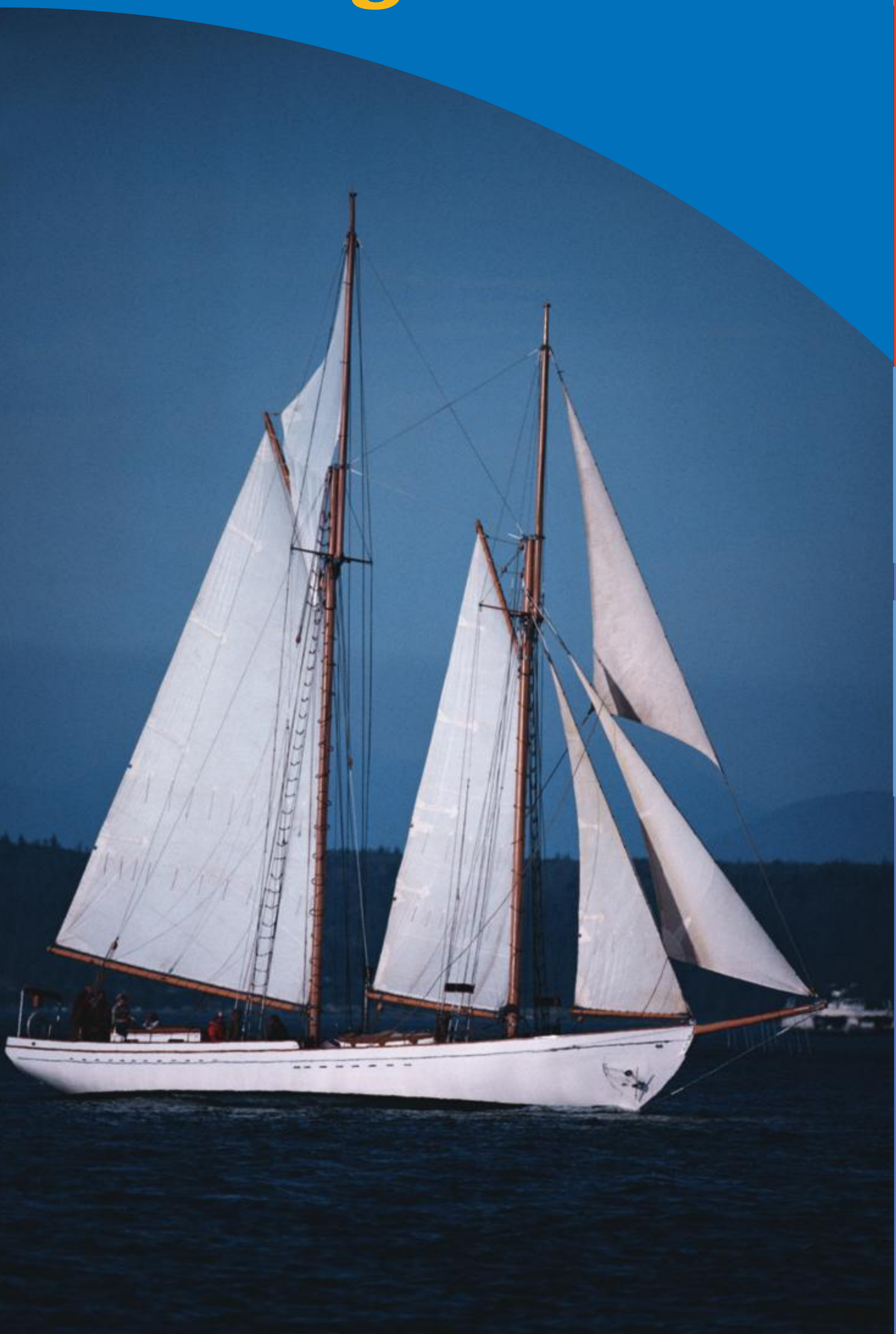
5

Navigation



syllabus reference

Elective topic

Maps and compasses — navigation

In this chapter

Introduction

'Mayday, Mayday, Mayday!

'This is *Kestrel*. We're sinking! There's a man overboard! *Kestrel* is a yacht, 10 metres, and we're a couple of nautical miles north of Great Keppel Island. We've hit a submerged object and the right side of the hull's damaged. We're taking water . . . electricals damaged . . . GPS isn't working . . . engine won't start. Rudder damage also. Impact threw one crew member overboard. Life jacket has been thrown to him.

'Repeat: man overboard. Male . . . Glen Smith . . . strong swimmer but he's drifting further away. Seas here are very rough. Strong currents are taking us out to sea. We require immediate assistance! We're taking a lot of water and the bilge pumps can't keep up!

'Two males still on board. Skipper, Sean Quinn speaking. Repeat: Mayday. We need urgent assistance. Vessel may sink within the hour. Rapidly taking water.'

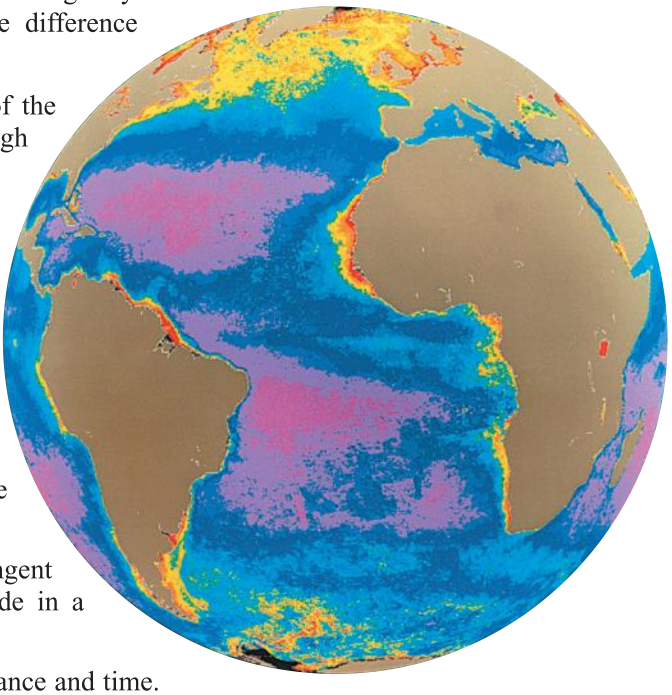
'*Kestrel*, this is Yeppoon Coast Guard. We've received your distress call. Rescue Shark Cat preparing to depart. *Kestrel*, can you accurately state your position? Over.'

Unfortunately, scenarios like the above are too common. The ability to accurately and rapidly describe the position of the sinking vessel could mean the difference between life and death. So what will Captain Quinn do in the next 60 seconds to determine his position at sea?

In this chapter, we shall investigate circumstances similar to those described above, and develop a working knowledge of techniques needed for coastal navigation. Some principles related to air navigation are also included. Related questions that will be answered include:

1. How can any position on the Earth's surface be accurately described?
2. How can the Earth, a globe, be represented using only two dimensions?
3. Why do navigators use the terms *nautical mile* and *knot* when most quantities these days are described in metric units?
4. What is a compass, why does it work and how can it be used in navigation?
5. How can charts assist us in navigation, and how can we fix position on a chart?
6. How can lighthouses be used to assist mariners?
7. How do GPS (Global Positioning System) devices work, and how can they be of assistance to navigators?

SKILLS CHECK



1 Lines of latitude and longitude are imaginary lines which circle the Earth. Explain the difference between the two.

2 What is the longitude, in degrees, of the prime meridian passing through Greenwich?

3 What is the latitude, in degrees, of the equator?

4 When stating the position of a point on the Earth's surface, which comes first, latitude or longitude?

5 Give the formula for the circumference of a circle.

6 What is the circumference of the Earth if the radius is 6371 km?

7 State the relationship between tangent ratio, opposite side and adjacent side in a right-angled triangle.

8 Give the formula relating speed, distance and time.

9 What does Greenwich Mean Time (GMT) refer to?

10 What is an isosceles triangle?

Review of Earth geometry

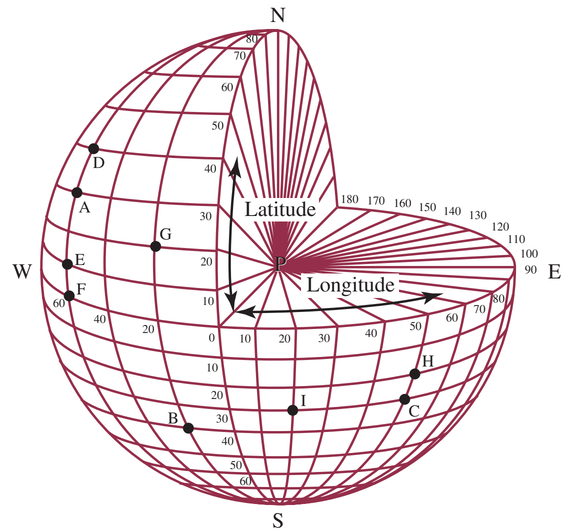
This section, including exercise 5A, revises concepts covered in *Maths A Year 11 2nd edition*, Chapter 6, *Earth geometry*.

Although our Earth is not quite spherical, it is useful, and acceptable, to assume that it is a sphere for the purposes of mapping. The Earth rotates every 24 hours about an axis joining the geographic North and South Poles. The line joining these poles and passing through the Earth's centre can be considered a diameter of the sphere.

Any plane that passes through the centre of the sphere intersects its surface to form circles known as **great circles**. The **equator** is an example of a great circle. Great circles that pass through the North and South Poles consist of two semicircles called **meridians**. As the radius of the Earth averages 6371 km, all great circles have this distance for their radii. A meridian is also a **line of longitude**. The line of longitude passing through Greenwich is called the **prime meridian**. By international agreement, the **longitude** of Greenwich is 0°. The location of all other meridians is measured by the number of degrees they lie east or west of this line.

Any circle traced out on the surface of the Earth with a radius less than 6371 km is known as a **small circle**. **Lines of latitude** are small circles on the surface of the Earth whose planes are parallel to the equator.

Using latitude and longitude, it is possible to fix accurately any point on the surface of the Earth. The cut-away globe in the figure at right shows a number of points on its surface. The point A has the coordinates (30°N, 60°W); B is (40°S, 20°W); C is (30°S, 50°E). Latitude is always stated first, followed by longitude. (Think alphabetically here — LAT, LONG.)



WORKED Example 1

Use the cut-away globe in the figure above, to give the coordinates of D.

THINK	WRITE
1 Recall LAT LONG, so latitude is required first.	
2 D is 40° above the equator; that is, 40°N.	
3 D is 60° west of the prime meridian; that is, 60°W.	The coordinates of D are (40°N, 60°W).

To find the distance separating two points on the same line of longitude (that is, meridian), the length of the arc of a circle of radius 6371 km is calculated. The distance from the South Pole to the North Pole would be:

$$\begin{aligned}\text{Distance} &= (\text{fraction of circle}) \times (2\pi r)\\ &= \frac{180}{360} \times 2 \times \pi \times 6371 \text{ km}\\ &= 20\,015 \text{ km.}\end{aligned}$$

If 180° along a line of longitude is equivalent to a distance of 20 015 km, then

$$\begin{aligned}1° &= \frac{1}{180} \times 20\,015 \text{ km}\\ &= 111.2 \text{ km.}\end{aligned}$$

WORKED Example 2

Using the figure above, find the distance from A (15°S, 120°E) to B (45°S, 120°E).

THINK	WRITE
1 A and B are on the same line of longitude; that is, 120°E.	
2 Calculate the angular distance AB.	Angular distance = 45° − 15° = 30°
3 Convert the angular distance to kilometres using 1° = 111.2 km.	Distance = 30 × 111.2 km = 3336 km

WORKED Example 3

Using the figure on the previous page, find the distance from A (50°N, 80°E) to B (82°S, 80°E) to the nearest kilometre.

THINK	WRITE
1 Points A and B are on opposite sides of the equator, and on the same meridian.	
2 Calculate the angular distance AB (latitude angles must be added).	Angular distance = 50° + 82° = 132°
3 Convert the angular distance to kilometres using 1° = 111.2 km.	Distance = 132 × 111.2 km = 14 678 km

remember

1. Any plane that passes through the centre of a sphere intersects the surface of the sphere to form circles known as *great circles*, for example the equator.
2. Great circles which pass through the North and South Poles consist of two semicircles called *meridians* or *lines of longitude*.
3. The line passing through Greenwich is the *prime meridian* (0° longitude).
4. *Lines of latitude* are circles on the surface of the Earth parallel to the equator; they are assigned a number depending on the number of degrees they are north or south of the equator.
5. To fix a position, state latitude then longitude.
6. Along a meridian line, 1° = 111.2 km.

EXERCISE 5A Review of Earth geometry

WORKED Example 1

eBook plus

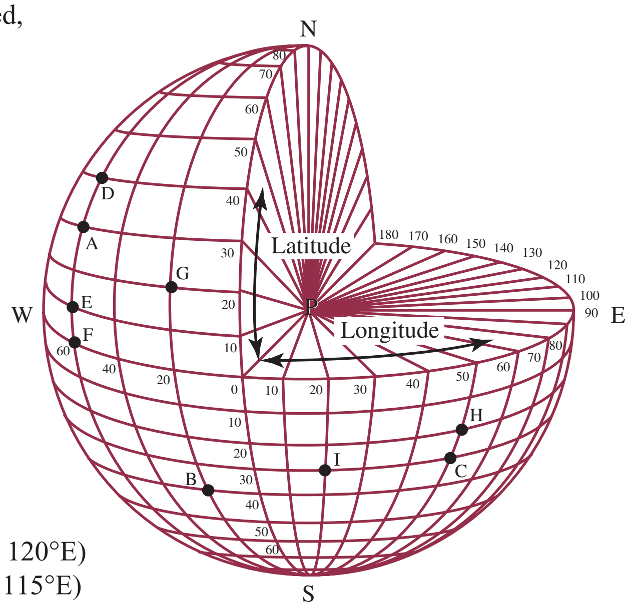
Digital doc: SkillSHEET 5.1 Angle measures in degrees and minutes

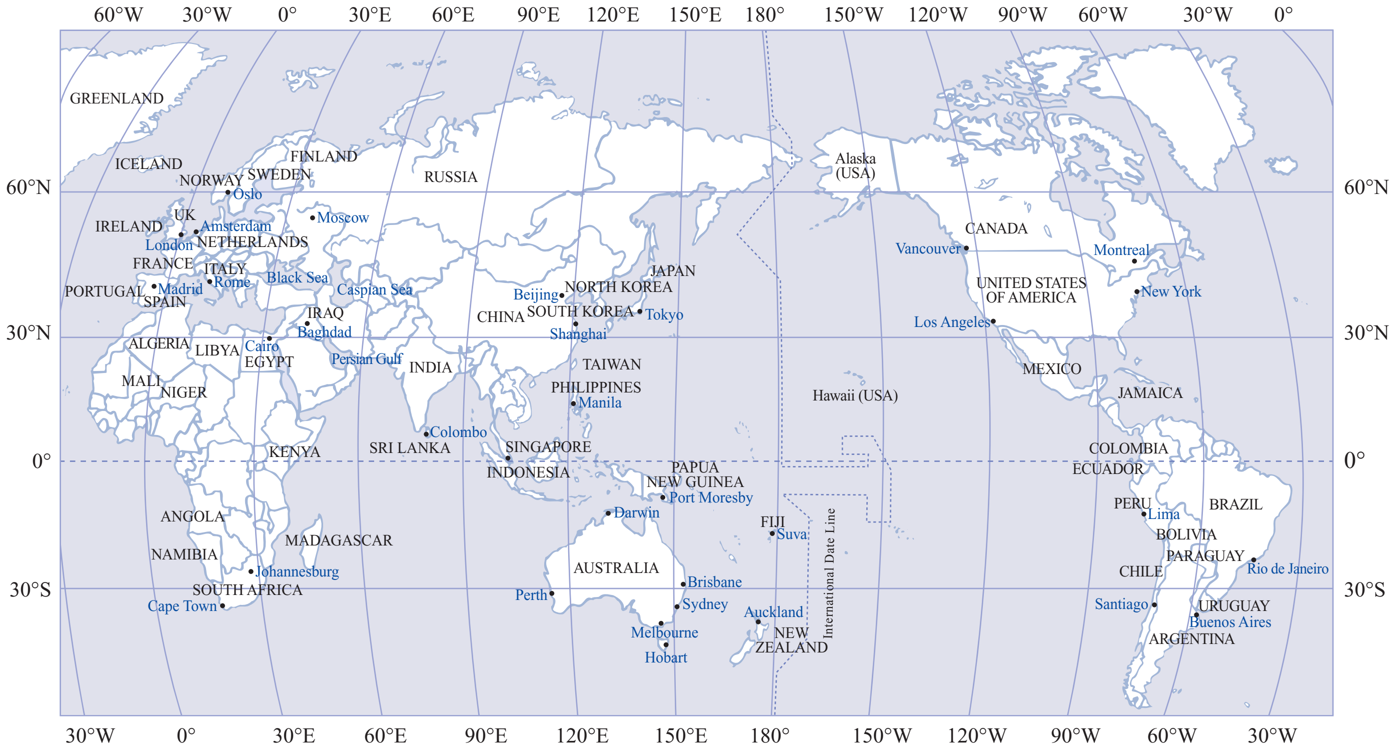
1 Use the cut-away globe provided, to give the coordinates of:
  a A b B
  c C d D
  e G f I.

2 Name two meridians in the figure in question 1 (for example, NGS is a meridian).

3 If P represents the centre of the Earth, state the value of:
  a ∠DPF b ∠DPE
  c ∠HPC d ∠BPG.

4 Using the map over the page, identify the major cities closest to the following locations.
  a (30°S, 30°E) b (30°N, 120°E)
  c (45°N, 75°W) d (32°S, 115°E)





5 Give the approximate coordinates of:
a Los Angeles **b** Cape Town **c** Singapore **d** Beijing.

WORKED Example 2

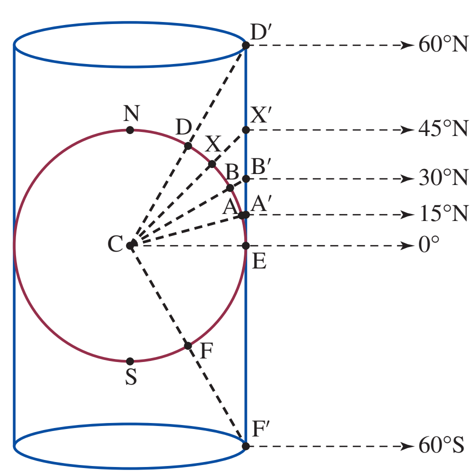
6 Find the shortest distance (to the nearest km) from:
a A (20°S, 110°E) to B (60°S, 110°E) **b** D (10°S, 140°E) to E (80°S, 140°E)
c T (15°N, 80°W) to R (75°N, 80°W) **d** E (20°N, 115°W) to F (86°N, 115°W).

WORKED Example 3

7 Find the distance (to the nearest km) from:
a X (20°S, 86°W) to Y (50°N, 86°W) **b** G (40°N, 135°E) to H (18°S, 135°E).

8 How far is Melbourne (38°S, 145°E) from the equator?

Representing the Earth in two dimensions



Maps play an extremely important role in our modern world. They are essential in order to travel safely and efficiently by sea, land and air. They are necessary for the subdivision of land, the construction of roads, dams, bridges and railways, and helping us out when we are lost.

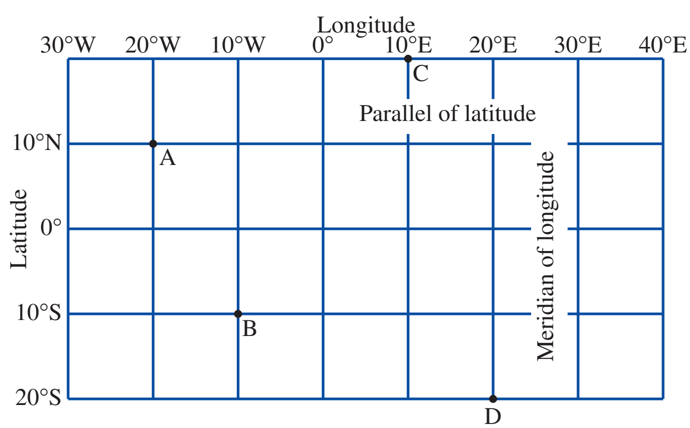
A map is a 2-dimensional representation of a 3-dimensional portion of the Earth's curved surface. There are difficulties in attempting to make a curved surface flat. Trying to flatten half a tennis ball gives us some idea of the problems arising. All maps, then, possess some form of distortion. However, since maps are used for navigation, the 2-dimensional representation must still be extremely accurate even if it contains distortion.

The charts most commonly used for shipping are drawn using a projection known as **Mercator's projection**. This is a cylindrical projection based on the following idea. Imagine a light source at the centre of the sphere. This sphere is then enclosed in a cylinder, and shadows from all land masses on the sphere are projected onto the walls of the cylinder. The curved wall of the cylinder is then opened up to give a 2-dimensional map (see the figure on the previous page).

A portion of the meridian NES is shown projected onto the cylinder wall. D is projected to D′, X to X′, B to B′, A to A′ and F to F′. The point E on the equator is touching the cylinder wall and maps onto itself. The curved section DEF of this meridian then is mapped onto a straight line D′EF′ on the cylinder wall. All meridians will likewise appear as straight vertical lines and parallel to one another.

The parallels of latitude 15°N, 30°N, 45°N, 60°N and 75°N are equidistant on the Earth's surface. This can be easily verified by examining a model globe of our planet. All parallels of latitude which pass through E, A, B, X and D are separated by a constant distance on the Earth's surface. When mapped onto the cylinder wall, these lines are still parallel, but are no longer equidistant. Note that DX = XB = BA = AE, but D′X′ (on the map) > X′B′ > B′A′ > A′E. Consequently, distortion in the higher latitudes results. Distortion in this region is also due to the fact that all meridians on our Earth become closer as we approach the poles, yet are drawn as parallel lines on the Mercator's projection. Regions near the poles, such as Greenland, appear much larger on a Mercator's map than they do on a globe.

For navigation purposes, Mercator's projection is useful between 60°N and 60°S. Despite the distortion as we move away from the equator, one advantage of this form of mapping is that meridians are equally spaced parallel lines and are perpendicular to the lines of latitude (see figure at right).



This simplifies the task of a navigator who can now work with straight lines rather than curves.

Another advantage is that distance can easily be determined from these charts by using the latitude scale (outlined later in this chapter).

investigation

Mercator's projection

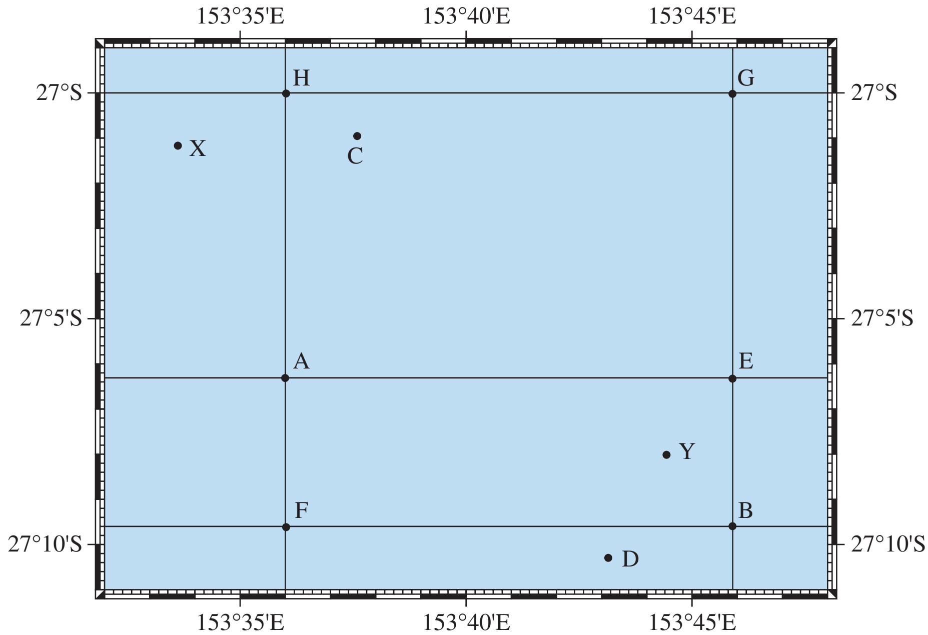
Mercator's projection is but one method of attempting to represent our 3-dimensional world on a flat page. By referring to an atlas or the internet, identify projections which preserve rather than distort land area.

Also, find examples of projections which preserve angles correctly.

Why is it that both area and angle cannot simultaneously be preserved on one 2-dimensional representation?

Accurate position description

Any point on the Earth's surface can be described by stating latitude, then longitude, in degrees. To describe a location accurately, the degree is further divided into 60 minutes. Navigation charts will often show latitude in degrees, minutes and tenths of minutes. The chart below shows a section off the coast of Brisbane. Point A has a latitude of 27°6.3′S and longitude of 153°36′E. Point B is at (27°9.6′S, 153°45.9′E). Note that in this chart, each minute of latitude and longitude has been divided into fifths. Alternative minute divisions have been shaded to assist with ease of reading.



WORKED Example 4

Use the chart above to give the position of point E.

THINK

1. Find latitude first. The scale is increasing as we go down the scale. Point E is somewhere between 27°6′S and 27°7′S. The minutes are divided into fifths. Point E is midway between 27°6.2′S and 27°6.4′S; that is, 27°6.3′S.
2. The longitude scale is increasing as we go across from left to right. Point E is greater than 153°45′E, but just less than 153°46′E. As it is midway between 45.8′ and 46.0′, use 45.9′.

WRITE

1. Point E is at latitude (27°6.3′S).
2. Point E is at (27°6.3′S, 153°45.9′E).

remember

1° = 60 minutes, written as 60′.

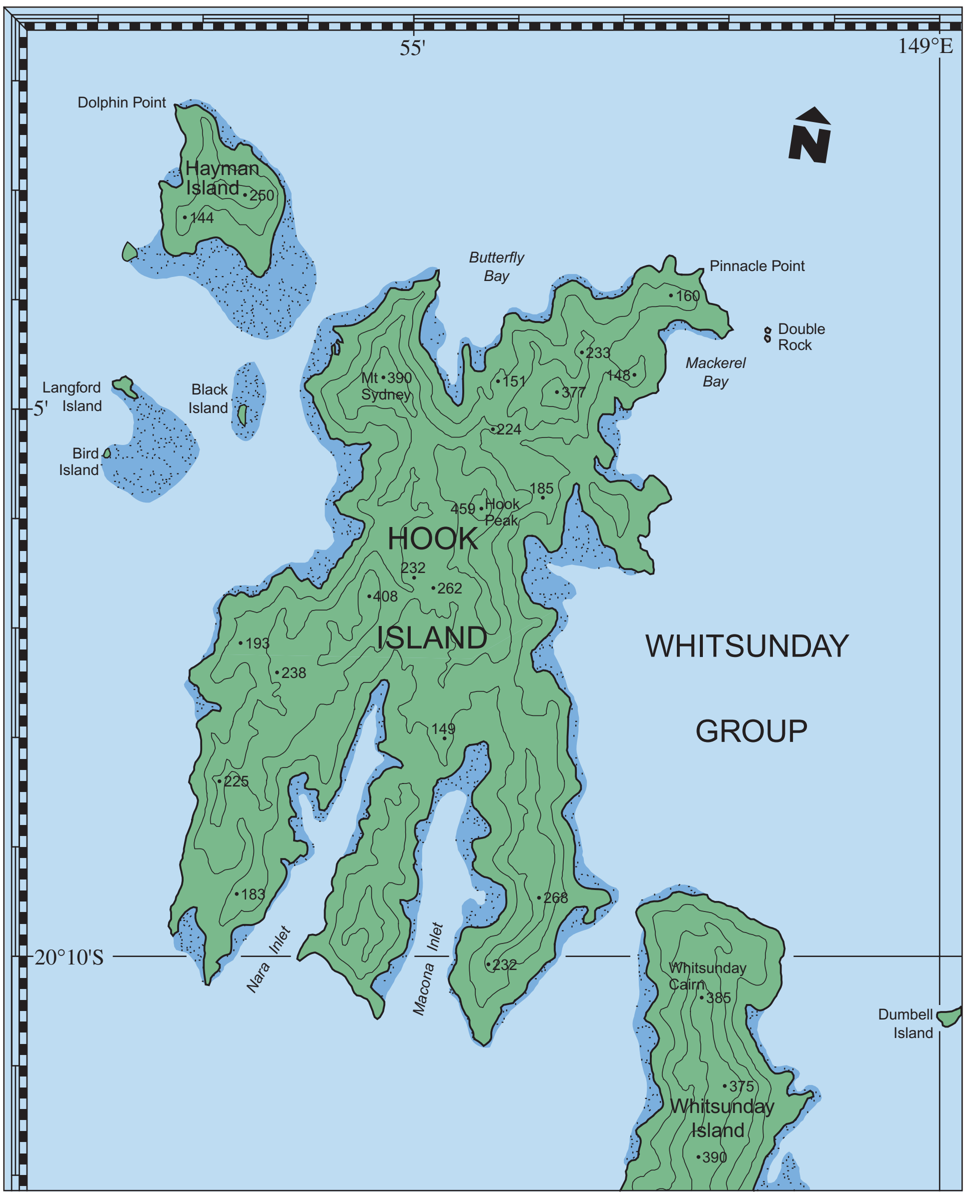
EXERCISE 5B Accurate position description

WORKED Example 4

1 Use the chart on page 256 to give the positions of points:
a F **b** G **c** H **d** C **e** X **f** Y.

2 Your teacher will supply a copy of the chart on page 256. Plot the following positions on this copy.
a (27°S, 153°45′E) **b** (27°6.3′S, 153°40′E)
c (27°7.4′S, 153°39.8′E) **d** (27°10′S, 153°42.7′E)

3 The figure below shows a portion of a map of Whitsunday Passage. Name the feature at:
a (20°5′S, 148°54.7′E) **b** (20°5.1′S, 148°53.4′E) **c** (20°3.8′S, 148°57.8′E).



4 Use the map of the Whitsunday Passage on page 257 to give the position of:
- **a** Dolphin Point on Hayman Island
- **b** Double Rock (off the north-east coast of Hook Island)
- **c** Langford Island (to the south of Hayman Island)
- **d** the entrance to Nara Inlet
- **e** the entrance to Macona Inlet.

investigation

Tidal variation

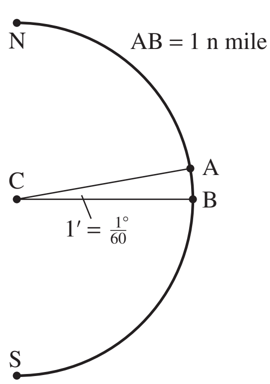
Use the **Seabreeze** weblink in your eBookPLUS, or a similar website, to investigate the tidal variation in your local area. This site displays the variation graphically. Now choose a different location in the state or country and observe the variation in that region. At what stages of the tide is the movement of water the least and most?

Why might information such as this be very useful to boat owners anchoring at island beaches for day trips?

Investigate the rule of twelfths and how it can be used to estimate tide height at any time if one knows the time and height of high and low tide.

The nautical mile and the knot

One **nautical mile** is defined as the length of the arc of a great circle which subtends an angle of 1 minute (1′) at the centre of the Earth; that is, it is the distance between two points on the same line of longitude or meridian with a difference in latitude of 1′. The nautical mile is an SI unit and is equivalent to 1852 metres.



The figure at left shows a cross-section of the Earth with centre C and North (N) and South (S) Poles. Points A and B lie on the line of longitude joining the two poles. Angle ACB is an angle of 1 minute $\left(\frac{1}{60}^\circ\right)$, so the distance AB on the Earth's surface is 1 nautical mile.

The distance between two locations on a map is measured using an instrument with two points called *dividers*. The dividers are spread so that the two points are directly over the two locations. The spread of the dividers is then transferred to the latitude scale. The number of minutes on this scale gives the distance in nautical miles.

Because some distortion is inherent in all charts, distance is always determined on the latitude scale in the region that is of similar latitude to where the distance is being determined.

When measuring distance, use a scale correct for that latitude.

The mechanical harpoon log was patented in 1802.

The **knot** is a unit of velocity and is defined as 1 nautical mile per hour. Wind speeds are often given in knots.

The term *knot* has its origins in the 16th century, when a piece of wood or 'log' was thrown from the stern of the ship to determine speed. The log was attached to a rope that had a number of knots tied in it at equal spacings. The number of knots passing the rear of the vessel in a fixed period of time, usually 30 seconds, gave a measure of speed.

Several devices have been invented that more accurately determine speed. The device pictured on the previous page, patented in 1802, was the first commercially available instrument designed for this task.

Study this device closely. How do you think it worked? Which pieces might move as it operates?

WORKED Example 5

Convert 12°45′ to minutes.

THINK	WRITE
1 1 degree = 60 minutes, so 12° is 12 × 60 or 720 minutes.	$12°45' = 12 \times 60' + 45'$
2 Now add the remaining 45 minutes to 720′.	$= 720' + 45'$ $= 765'$

WORKED Example 6

Add 13°37′ to 38°52′.

THINK	WRITE
1 Keep the minutes and degrees separate. Add the minutes first.	$37' + 52' = 89'$ or $1°29'$
2 Now add the degrees.	$13° + 38° = 51°$
3 Add the sum of minutes to the sum of degrees.	$13°37' + 38°52' = 51° + 1°29'$ $= 52°29'$

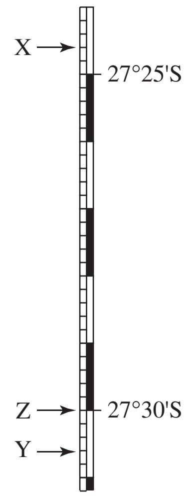
WORKED Example 7

Subtract 25°46′ from 47°13′.

THINK	WRITE
1 Convert each to minutes, then subtract.	$47°13' - 25°46' = (47 \times 60 + 13)' - (25 \times 60 + 46)'$ $= 2833' - 1546'$ $= 1287'$
2 Convert 1287′ to degrees by dividing by 60.	$= 21.45°$
3 Convert 0.45° to minutes by multiplying by 60.	$= 21°27'$

WORKED Example 8

The scale in the figure at right indicates latitude. Find distances a XY and b XZ.



THINK

a

1. (a) X has a latitude of 27°24.6′S.
   (b) Y has a latitude of 27°30.6′S.
   (c) The difference in these latitudes will give the distance in nautical miles.
2. Calculate the difference in latitude.

WRITE

a 27°30.6′S – 27°24.6′S = 6′
XY = 6 n mile

THINK

b

1. (a) X has a latitude of 27°24.6′S.
   (b) Z has a latitude of 27°30′S.
   (c) The difference in these gives the distance.
2. Calculate the difference in latitude.

WRITE

b 27°30′S – 27°24.6′S = 5.4′
XZ = 5.4 n mile

WORKED Example 9

Find the shortest distance in nautical miles and kilometres from A (40°N, 150°E) to B (30°S, 150°E).

THINK	WRITE
1. A and B are on the same meridian. Calculate the angular distance.	Angular distance = 40° + 30° = 70°
2. Convert 70° to minutes. (Recall that 1° = 60′.)	= 70 × 60′ = 4200′
3. Convert distance AB to n mile. Recall that 1′ = 1 n mile.	AB = 4200 n mile
4. Covert AB to kilometres. Recall that 1 n mile = 1.852 km.	AB = 4200 × 1.852 km = 7778 km

WORKED Example 10

A vessel picks up anchor off Sandy Cape, Fraser Island (24°30′S, 153°E) at 7.30 am, and travels north to Saumarex Reef (21°50′S, 153°E). The vessel averages 12 knots. Find the estimated time of arrival.

THINK / **WRITE**

1. Find change in latitude.

$$\begin{aligned}\text{Latitude change} &= 24°30' - 21°50' \\ &= (24 \times 60 + 30)' - (21 \times 60 + 50)' \\ &= 1470' - 1310' \\ &= 160'\end{aligned}$$

2. Convert minutes to n mile. Recall that 1′ = 1 n mile.

Distance = 160 n mile

3. Calculate time.
   (a) Recall that speed $= \dfrac{\text{distance}}{\text{time}}$.

$$\text{Speed} = \frac{\text{distance}}{\text{time}}$$

$$12 \text{ n mile/h} = \frac{160 \text{ n mile}}{\text{time}}$$

   (b) Rearrange to find time.

$$\begin{aligned}\text{Time} &= \tfrac{160}{12} \\ &= 13.33 \text{ hours}\end{aligned}$$

4. Convert to hours and minutes.

= 13 h 20 min

5. Calculate ETA: Boat departs at 7.30 am.

ETA is 13 hours 20 min after 7.30 am.
ETA = 8.50 pm on the same day.

remember

1. A nautical mile is the length of the arc of a great circle which subtends an angle of 1 minute (1′) at the centre of the Earth.
2. Distance is always determined on the latitude scale in the region that is of similar latitude to that at which the distance is being measured.
3. 1 degree = 60 minutes (1° = 60′).
4. 1 minute = $\frac{1}{60}$ degree.
5. 1 nautical mile (n mile) = 1852 metres.
6. 1 knot = 1 nautical mile per hour.
7. Speed = $\dfrac{\text{distance}}{\text{time}}$.

EXERCISE 5C The nautical mile and the knot

WORKED Example 5

1 Convert to minutes:
a 2° **b** 2.5° **c** 23°42′ **d** 47°51.7′.

WORKED Example 6

2 Add:
a 3°15′ and 6°28′ **b** 15°26′ and 23°42.7′.

WORKED Example 7, 8

3 Subtract:
a 11°28′ from 28°45′ **b** 17°6.4′ from 18°3.7′.

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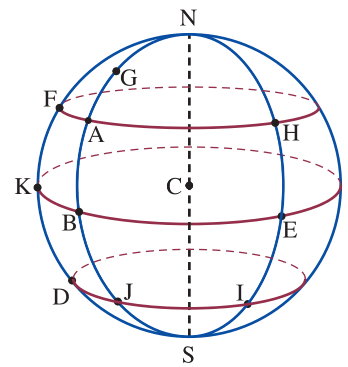
SkillSHEET 5.1
Angle measures in degrees and minutes

4 In the figure at right, C is the Earth's centre, E is a point on the equator, G is Greenwich, I is at (60°S, 80°E) and F is at (50°N, 30°W).



a Name two points on the same parallel of latitude as I.
b Name two points on the same parallel of latitude as F.
c Name two points on the same meridian as E.
d State the position of:
i H **ii** B **iii** J **iv** K **v** A.

WORKED Example 9

e Find the shortest distance across the Earth's surface in nautical miles from the North Pole to:
i F **ii** H **iii** E **iv** I **v** D.
f Find the shortest distance in nautical miles from:
i H to I **ii** A to J.
g Calculate the shortest distance in nautical miles from the equator to:
i D **ii** J **iii** H.

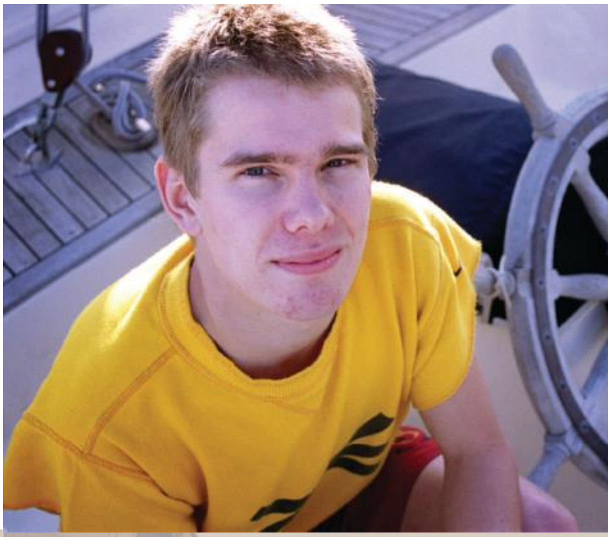
5 Calculate the distance in nautical miles from Brisbane (27.5°S, 153°E) to:
a the equator **b** the South Pole **c** the North Pole
d Woodlark Island (off the east coast of Papua New Guinea (9°S, 153°E).

6 The vessel *Blue Fin* covers 40 nautical miles in 5 hours. Find its speed in knots.

7 The *Trader Horn* leaves Wynnum Creek at 11 am and arrives at Dunwich at 3 pm. It has travelled 14 nautical miles. Calculate the speed in:
a knots **b** km/h.

8 Find the unknowns:

Speed (knots)	Distance (n mile)	Time (units given)
a	25	6.5 hours
b	1548	5 days 3 hours
17	**c**	17 hours
125	**d**	25 minutes
40	1200	**e** hours
72	12	**f** minutes



9 The youngest person to circumnavigate the Earth was an 18-year-old Australian, Jesse Martin. He sailed into Port Philip Bay, Victoria (and into history!) on 31 October, 1999, completing a continuous and solo 50 000-kilometre voyage in the 10-metre vessel, *Lionheart*. Jesse's journey, which included the rounding of the notorious Cape Horn, took 328 days.

- **a** How many hours did Jesse spend alone at sea?
- **b** Calculate the average speed of *Lionheart* in kilometres per hour and knots. Round your answers to the nearest 0.1.

Note: This amazing adventure is described in the book *Lionheart*. A video of the same title is also available.

10 A and B are two points on the Earth's surface on the 80°E meridian. Their latitudes differ by 60°.

- **a** Find the distance between them in nautical miles.
- **b** Another two locations, C and D, are likewise separated by 60° in latitude and lie on the 30°W meridian.
  - **i** Find the distance between C and D.
  - **ii** Why is this answer equal to that of **a**?
- **c** A ship travels from C to D with an average speed of 18 knots. Find the time taken for the journey.

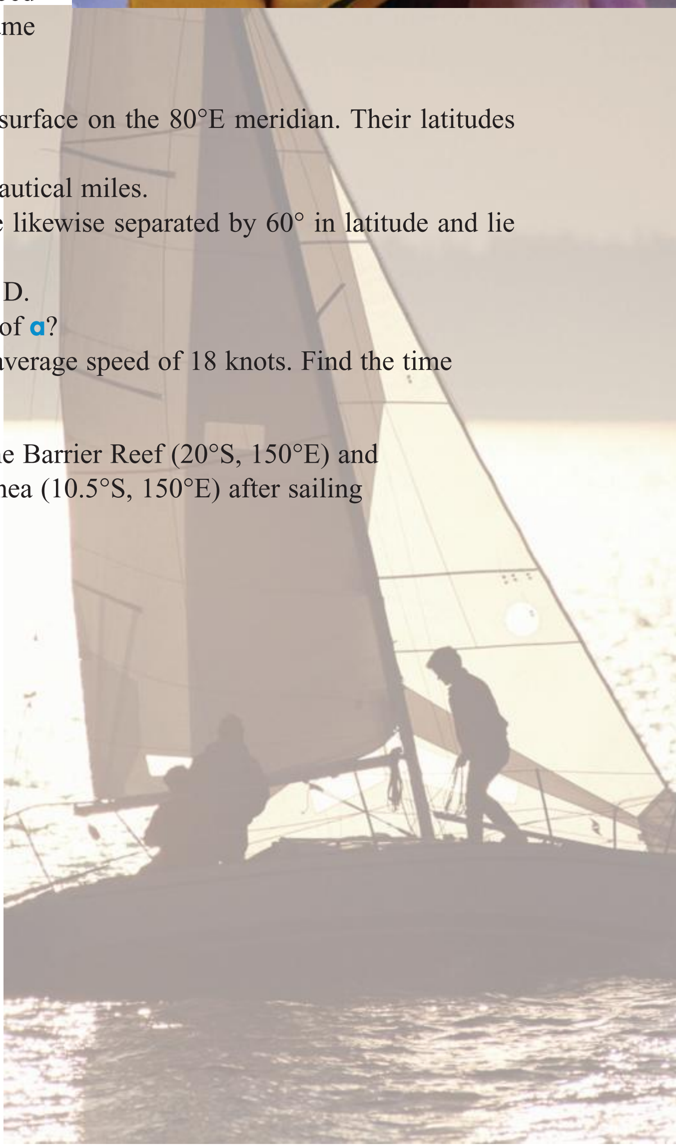
11 A sailing boat leaves from a point off the Barrier Reef (20°S, 150°E) and arrives at Orangie Bay, Papua New Guinea (10.5°S, 150°E) after sailing for 5 days 3 hours. Find:

- **a** the distance travelled
- **b** the average speed in knots.

WORKED Example 10

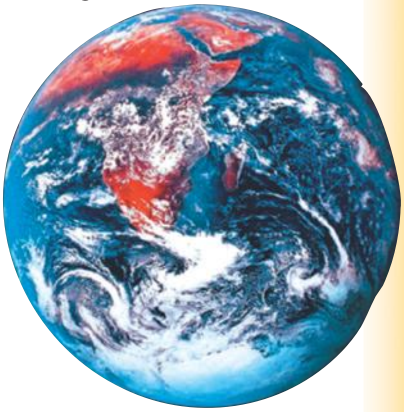
12 A vessel departs a point off Moreton Island (27°15′S, 153°40′E) at 1.30 pm and sails north to Fraser Island (24°45′S, 153°40′E). The vessel averages 11 knots. Find the ETA.

13 Why is it that the nautical mile could not be defined as the distance between two points on the same line of latitude with a difference in longitude of 1′?



10 Quick Questions 1

1 Which comes first when describing position: latitude or longitude?



2 Which scale on a map is used to determine distance in nautical miles: latitude or longitude?

3 How many minutes make 1 degree?

4 What distance in metres is equivalent to 1 nautical mile?

5 A vessel changes its latitude by 2°30′. Its longitude remains unchanged. How many nautical miles has it covered?

6 Give the rule relating speed, distance and time.

7 One nautical mile per hour is equal to which unit of speed?

8 A boat completes a 30 n mile trip in 5 hours. What is its speed?

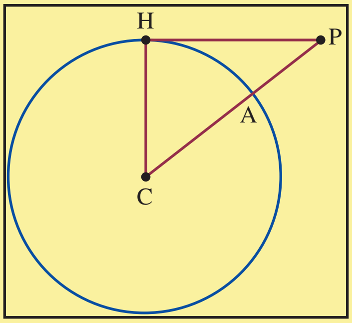
9 How far is the South Pole from the equator, in nautical miles?

10 A boat leaves port at 2 pm and travels at 15 knots to cover a distance of 45 n mile. What is its ETA?

investigation: Distance to the horizon

If we know the height of our position above sea level (that is, our altitude), the distance to the horizon can be determined.

In the figure at right, the circle represents the Earth with centre C. An observer is at P, a distance of AP above sea level and can see the horizon at H, a distance of HP from the observer.



1 What is the magnitude of ∠PHC? Why? You may wish to draw several lines from various positions outside the circle to 'horizons' (that is, lines just touching the circle) to see if this is always the case.

2 Can you recall a theorem relating the three sides CH, HP and PC? State the relationship between these sides. (*Hint:* Name a great Greek mathematician beginning with P.)

3 Is there a relationship between CH and AC? Can you supply a value for these lengths? (See 'Review of Earth geometry' section.)

4 Determine the distance to the horizon if the height above sea level is:
   a 50 m **b** 500 m **c** 1 km **d** 10 km.

5 As height above sea level increases, what happens to the distance to the horizon? If we lived on a flat Earth, would we expect the same results that we calculated in question 4?

Using the compass

The most important aid to a navigator is the **compass**. It allows the vessel to be steered on a predetermined course, and its accuracy is vital to the safety of all seafarers.

Types of compass

The **magnetic compass** has many forms. The simplest type of magnetic compass has a free-swinging needle that aligns itself with the Earth's magnetic north–south line. These are usually quite inexpensive and in the past have even been built into the soles of shoes! Most small boat owners would possess a **hand bearing compass** and many types of these are readily available (see figure **a** below). The needle is sometimes built into a **card** which is immersed in alcohol to protect the swinging card from impact or sudden movement. The card is graduated from 0° to 360°. Many boats are also fitted with a master compass that is permanently mounted in a location that is visible to the helmsman (see figure **b** below).

Compass error can cause many problems to air, sea and land navigators, and so compasses must be checked regularly for accuracy. Compass error can be caused by damage or by external magnetic influences. If a magnet or even a piece of steel is brought near a compass, the needle deviates. Also, the presence of electrical currents can cause the compass needle to deviate. Even the proximity of canned foods can cause a deviation. Fortunately, there are procedures that can easily and accurately determine the sum total of these errors, which are referred to as **errors of deviation**. Such procedures are beyond the scope of this course, and in all exercises in this text we will assume that these errors do not occur.

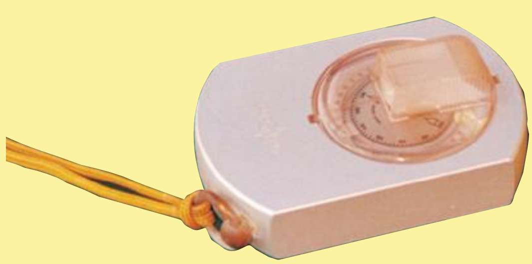


Figure a: Hand bearing compass

Figure b: Master compass

Another type of compass, the **gyro compass**, operates on a different principle from that of the magnetic compass. The gyro compass consists of a spinning gyroscope whose axis aligns with the *true* north–south axis of the Earth. This has obvious advantages as an aid to navigation because magnetic variation no longer needs to be considered. Also, it is unaffected by the influences that interfere with magnetic compasses. However, these compasses are expensive and extremely sensitive precision instruments which require reliable electricity sources.

Magnetic variation

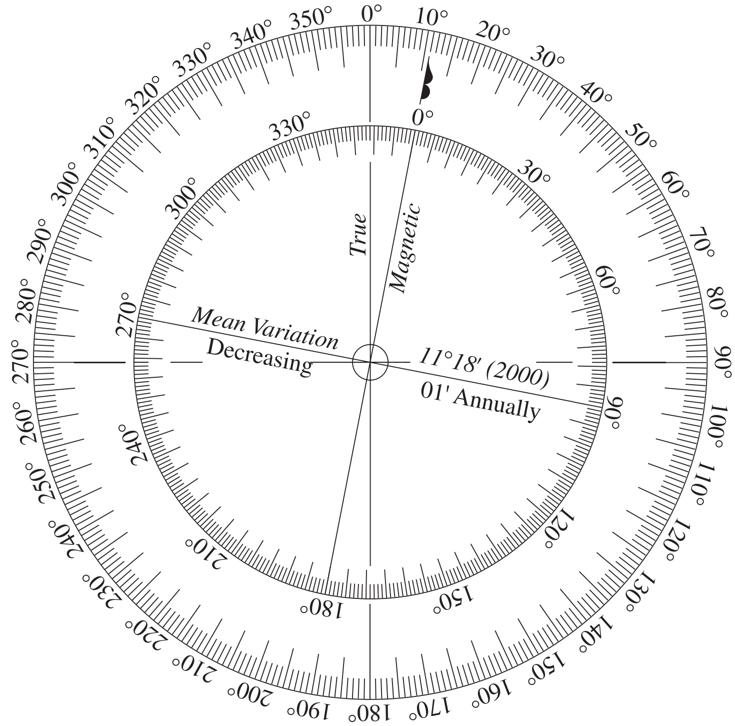
The Earth's magnetic north pole and true north are not in the same position on the Earth's surface. **True north** is the northern location of the Earth's axis of spin, which is the basis for lines of latitude and longitude. **Magnetic north** is the position on the Earth's surface of the north pole of the Earth's 'internal magnet' (really a magnetic field that extends out into space, and whose lines of force act like magnetic meridians). The axis of this internal magnet is not the same as the axis of spin, and the magnetic north pole lies about 1500 km away from the true North Pole, in the Arctic Ocean north of Canada.

Consequently, depending on its position on Earth, a compass does not usually point to true north. **Magnetic variation** is the angle between true north and the direction of north as determined by a compass in that region (see figure below). In some shipping channels magnetic variation can be as high as 30°. (Can it be greater than this? Has it a maximum? Where on the Earth's surface would a compass have to be located to achieve a maximum variation?)

Notes

1. Do not confuse magnetic variation (the *angle*) with other variations which *affect* the angle.
2. The Earth's magnetic field changes slowly, as though the internal magnet is moving.
3. Magnetic 'meridians' are related to lines of magnetic force. They are not straight lines, being bent by local magnetic influences such as mineral deposits. The magnetic variation will often be different in different parts of the one map.

Navigators must take this variation into account. It may deflect to the east or west of true north, but it is to the east in Queensland and in most parts of the Southern Hemisphere. This variation is well researched and documented and is nearly always shown on charts at the centre of what is called the **compass rose**, as is shown in the figure below.



This compass rose is taken from a chart of Moreton Bay. The magnetic variation at this point on the Earth's surface in the year 2000 was 11°18′E, and has since decreased by 1′ annually. Such variation, once known, can easily be taken into account. If a vessel is to follow a **bearing** of 45°T (45° true or 45° to the east of true north) and is in a region where the magnetic variation is 10°E, then the compass course to steer must be 35°C (the C indicating that this is the bearing with respect to the compass in that location).

If a vessel is to follow a bearing of 238°T in a region where magnetic variation is 4°W, then the course to steer must be 242°C.

The conversion from true (T) to compass (C) bearing can easily be recalled by the rhyme:

Variation east — compass least (subtract)

Variation west — compass best (add).

This means that if the magnetic variation is to the east, then the compass bearing will be the smaller of the two bearings.

If the magnetic variation is to the west, then the compass bearing will be the larger of the two bearings.

In Queensland, magnetic variation is to the east.

WORKED Example 11

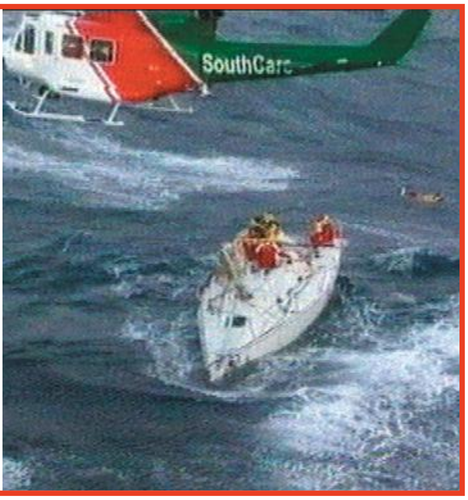
Convert the true course to a compass course for the given variation.

a 138°T, variation 11°30′E **b 225°50′T, variation 5°20′W**

THINK	WRITE
a 1 Recall: Variation east — compass least. This means that the compass course must be less than the true course.	**a**
2 Subtract 11°30′ from 138°.	Compass course = 138° − 11°30′ = 126°30′
b 1 Recall: Variation west — compass best. This means that the compass course must be more than the true course.	**b**
2 Add 5°20′ to 225°50′.	Compass course = 225°50′ + 5°20′ = 231°10′

WORKED Example 12

***Kestrel* follows a compass course of 129°. The skipper notes that this is identical to a true course of 140°. Find the magnetic variation in this region.**



THINK	WRITE
1 The compass course (129°) is less than the true course (140°). Recall: Variation east — compass least.	The variation must be to the east.
2 Subtract the compass course from the true.	Magnetic variation = 140° – 129° = 11°E

remember

To convert a true bearing to a compass bearing:

1. Subtract the magnetic variation from the true bearing if variation is to the east.
2. Add the magnetic variation to the true bearing if variation is to the west.
   Variation east — compass least
   Variation west — compass best

EXERCISE 5D Using the compass

WORKED Example 11

1 Find the compass course for the given variation and true bearings.
 a True course 135°, variation 7°E
 b True course 302°, variation 10°E
 c True course 189°, variation 4°W
 d True course 32°, variation 8°W

2 A skipper steers a vessel on a course of 280°C. The chart for this locality indicates the magnetic variation to be 11°E. Find the true course.

WORKED Example 12

3 A vessel follows a course of 115°C, which is identical to 121°T. Find the magnetic variation.

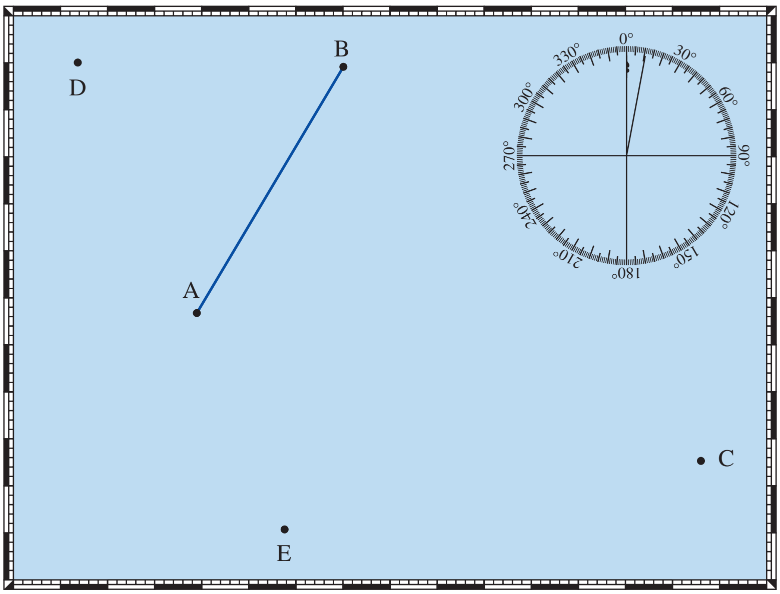
4 Find the unknowns in the table below.

True course	125°T	219°T	311°T	d
Variation	5°E	7°W	c	4°12′E
Compass course	a	b	315°C	253°17′C

Compass bearings and reverse bearings

True bearings

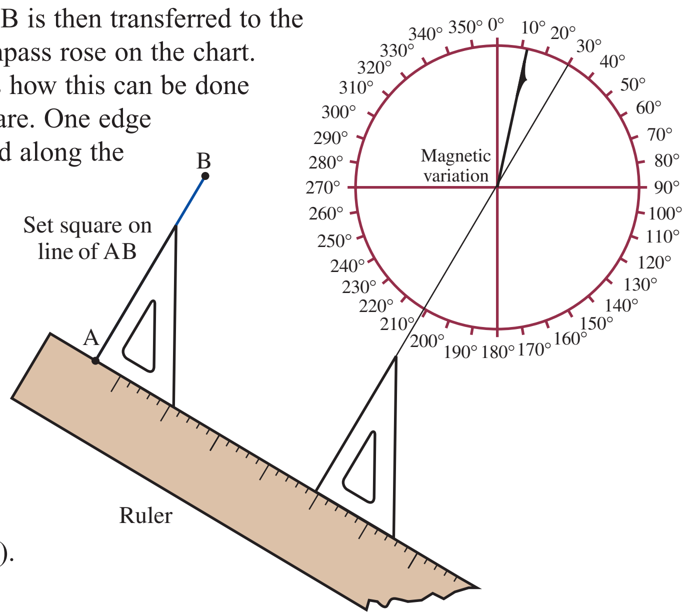
The figure below shows a section of a chart off the coast of Queensland. If a navigator is intending to sail from A to B, the true bearing of B from A can be determined by a variety of methods depending on the equipment available. Three methods are described: using a ruler, set square and pencil; using parallel rulers; and using a chart plotter.



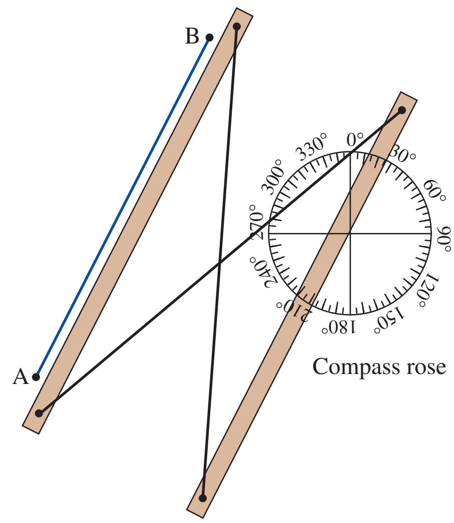
Method 1: Using a ruler, set square and pencil

The ruler, set square and pencil are used as follows.

1. Draw a line on the chart from A to B.
2. The exact direction A to B is then transferred to the centre of the nearest compass rose on the chart. The figure at right shows how this can be done using a ruler and set square. One edge of the set square is placed along the line AB, and a ruler held perpendicular to AB. The set square is moved until it is aligned with the centre of the compass rose. The true bearing can then be read, in this case 30°. (This could then be adjusted to a compass course by subtracting 11° variation E from 30°).



Method 2: Using parallel rulers

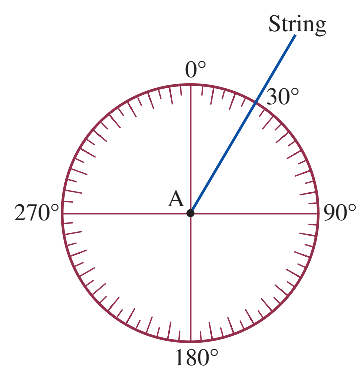


Although a set of parallel rulers is more expensive than a ruler and set square, it is a very simple device to operate and provides a high degree of accuracy. It is used as follows.

1. Place one edge of one of the parallel rulers on the line joining A to B.
2. Apply pressure to the ruler on the line AB and slide the other ruler across until it is aligned with the centre of the compass rose. The true bearing can then be read. Parallel rulers have this name because they are constructed with a mechanism which ensures that the two rulers always stay parallel.

You may have to 'walk' the rulers some distance to the nearest compass rose.

Method 3: Using a course plotter (chart plotter)



A course plotter (or chart plotter) is a flat, transparent plastic sheet with a grid and compass rose marked on it. There is a string tied to the centre of the compass rose. Chart plotters are very simple devices to use, and no lines need to be drawn on charts.

An inexpensive chart plotter can be made by simply drilling a small hole in the centre of a full circle protractor and passing a piece of string through it. A knot is tied at one end of the string to secure it.

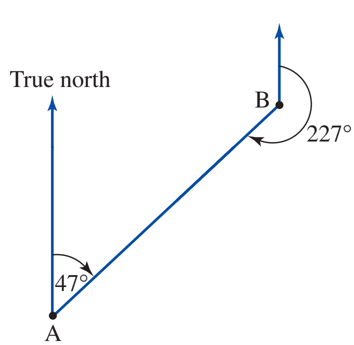
The chart plotter is used as follows.

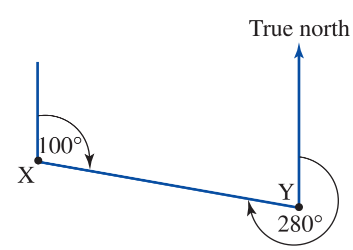
1. Place the centre of the compass rose on the chart plotter over point A on the chart, aligning the grid lines of both the chart plotter and the chart.
2. Extend the string to point B so that it is taut. The bearing can now be read from the compass rose.

Reverse bearings

If the bearing of B from A is 47°T, then the bearing of A from B must be 227° (47° + 180°; see figure below left).

Likewise, if the bearing of X from Y is 280°T, then the bearing of Y from X must be 100°T (see figure below right).





The bearing of B from A is known as the **reverse bearing** or **back bearing** of A from B. Note that these bearings differ by 180°. Likewise, the bearing of X from Y is the reverse bearing of Y from X (280°T and 100°T).

If the first bearing is less than 180°, add 180° to get the reverse bearing. If it is more than 180°, subtract 180°.

WORKED Example 13

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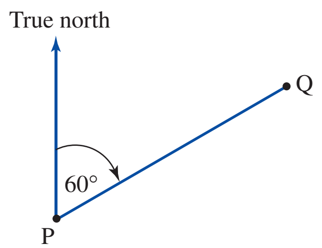
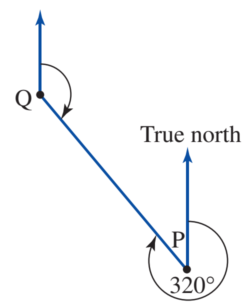
Find the bearing of P from Q in the two situations below.

a



b



THINK	WRITE
a 1 Recall that reverse bearings differ by 180°.	**a**
2 Add 180° to 60°.	Bearing of P from Q = 60° + 180° = 240°T
b 1 Recall that reverse bearings differ by 180°.	**b**
2 Subtract 180° from 320°.	Bearing of P from Q = 320° − 180° = 140°T

WORKED Example 14

Use the chart on page 269 to answer the following.

a Find the distance in nautical miles from A to B. (Shaded divisions on latitude scale each represent 1′.)

b A vessel goes from A to B in 20 minutes. Find its speed in knots.

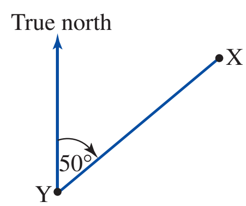
THINK	WRITE
a 1 Dividers are spread with one point on A and the other on B. (A ruler can also be used, or just a straight edge of paper can have the distance marked.)	**a**
2 This distance (6′) is transferred to the latitude scale.	AB = 6 n mile
b Calculate speed.	**b**
1 Recall that speed = $\frac{\text{distance}}{\text{time}}$.	Speed = $\frac{\text{distance}}{\text{time}}$
2 Calculate time in hours.	Time = 20 min = $\frac{20}{60}$ h = 0.333 h
3 Calculate speed.	Speed = $\frac{6 \text{ n mile}}{0.333 \text{ h}}$ = 18 knots

EXERCISE 5E Compass bearings and reverse bearings

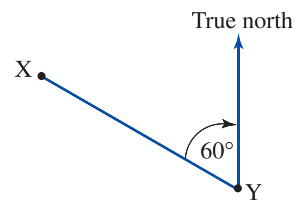
WORKED Example 13

1 Give the bearing of X from Y in each of the following.

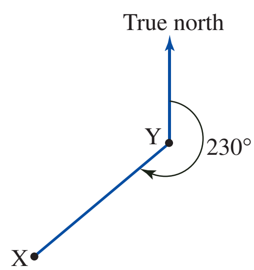
a



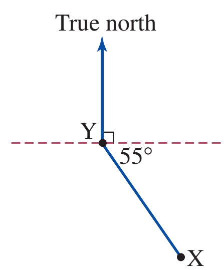
b



c



d



2 In each part of question **1**, give the bearing of Y from X.

3 Taking true north to be the top of the page, use a protractor, ruler and pencil to sketch a diagram so that the true bearing from X of:

a A is 30°T and AX = 4 cm
b B is 45°T and BX = 6 cm
c C is 80°T and CX = 7 cm
d D is 125°T and DX = 5 cm
e E is 210°T and EX = 3 cm
f F is 320°T and FX = 4 cm.

4 Use the diagram constructed in question **3** above to find the bearing of:

a F from A
b A from C
c B from D
d E from C
e F from D
f X from B.

Use the chart on page 269 to answer questions **5** to **7**.

WORKED Example 14

5 Find the following distances in n mile. (Recall that the latitude scale is to be used.)

a AD **b** AE **c** AC **d** CB **e** CD **f** ED

6 A vessel departs E and travels 4 n mile on a bearing of 302°T.

a Plot its final position.
b If it travels for 20 minutes, find its average speed.

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7 A vessel sails from B to E.

a On what true bearing does it sail?
b If the magnetic variation is 11°E, what compass course does it follow?
c The vessel is averaging 12 knots. Find the time taken.
d What is the compass bearing of B from E?

Reverse bearings

investigation

Students are to work in pairs for this activity. Each student has a compass and takes the compass bearing of the partner. The two bearings of each pair of students are recorded. Students then move to another position and again take bearings to one another. Repeat this process 10 times. What is the relationship between a bearing and its reverse bearing?

10 QUICK QUESTIONS 2

1 Give the name of the device which aligns itself with the Earth's magnetic north–south line.
2 The North Pole, which is the starting point for lines of longitude, is either true north or magnetic north. Which one?
3 When we are relating compass bearings to a map, we have to take account of the difference between true north and magnetic north (in degrees). What is the name we give to that difference?
4 If variation is to the east, which is least: true bearing or compass bearing?
5 If variation is to the west, which is least: true bearing or compass bearing?
6 Convert a compass course of 50° to a true bearing if variation is 10°E.
7 Convert a true bearing of 146° to a compass bearing if variation is 12°W.
8 By what angle do reverse bearings differ?
9 The distance between two points on the surface of the Earth can be determined by comparing their distance on a chart to which scale: latitude or longitude?
10 A vessel travels 56 nautical miles in 8 hours. What is its speed in knots?

Fixing position

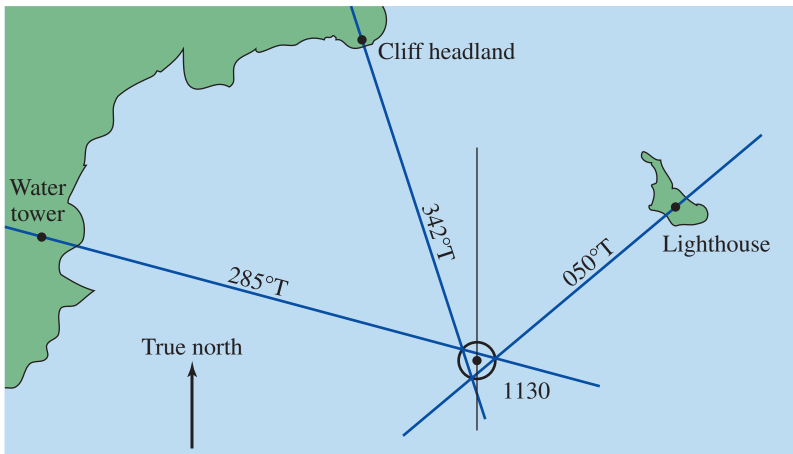
The ability of navigators to determine the position of their vessel at any time is an extremely important and often necessary skill. This is called **fixing position**. Coastal navigators look for objects that can easily be seen on land. (Mariners crossing the oceans rely on celestial navigation; that is, determining position by observing the location of certain heavenly bodies.)

Landmarks that are often used in coastal navigation include prominent features such as **lighthouses**, headlands, church steeples and small islands. Such useful landmarks will be shown on charts. These landmarks can be used to fix the vessel's position. Two or more position lines are marked on the chart and the intersection of these lines indicates the vessel's position. Several methods of determining a fix are available.

Fix by cross bearings

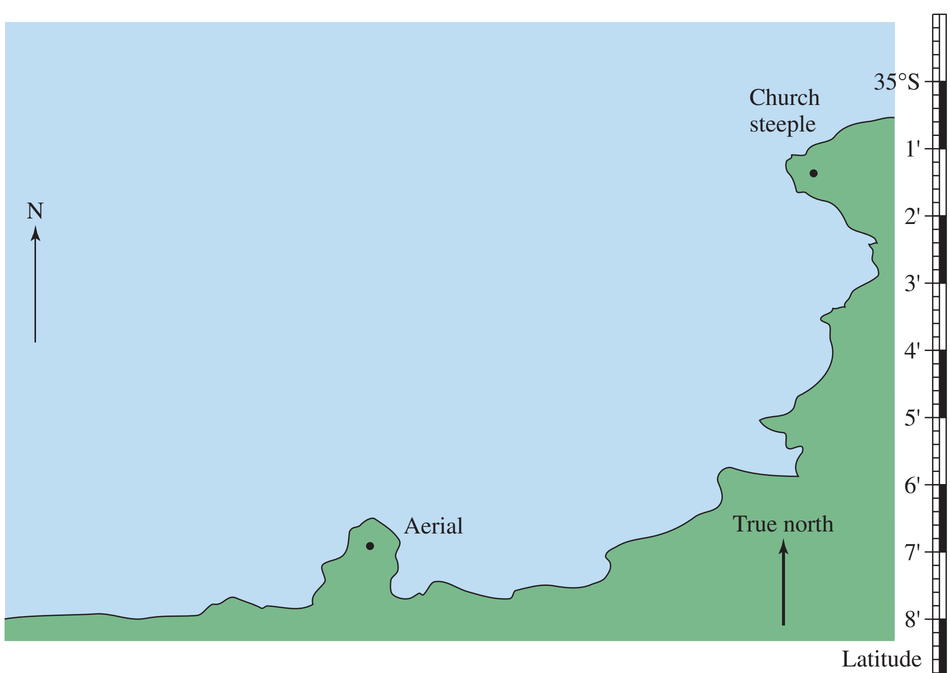
The **cross bearing fix** is the most commonly used, and most easily mastered method of establishing a fix. It requires at least two bearings to objects spaced ideally at approximately 90°. The figure below shows how a fix has been determined. A navigator on board the vessel notes the bearings of three prominent landmarks that are also marked on the chart. These bearings are converted to true bearings and then are marked on the chart as shown.

The intersection of the position lines of the lighthouse (050°T), cliff headland (342°T) and water tower (285°T) fix the position. Because of small errors, the intersections of the lines form a small triangle, known as a **cocked hat**. This is circled and the time the fix was taken is noted (in this case 1130 hours, or 11.30 am). This gives an excellent description of the vessel's position at that time.



WORKED Example 15

At 10.30 am, the yacht *Lady Jacqlin* records the bearing of a church steeple (070°T) and an aerial (105°T). At 11.15 am, the bearing of the church (095°T) and aerial (200°T) are noted.



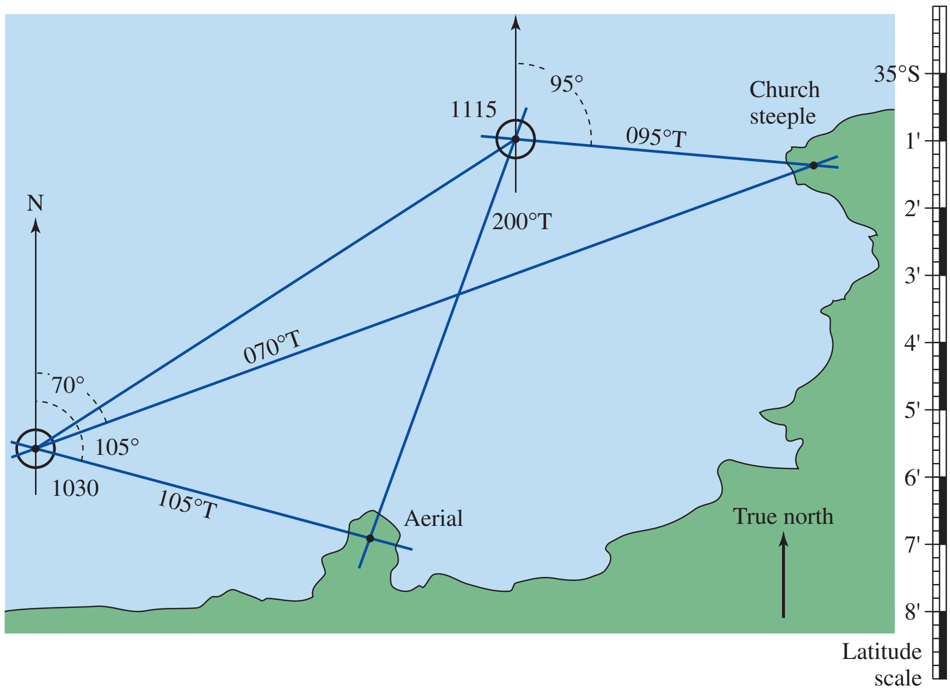
a **Use this information to fix the vessel's position at these times.**
b **How far has the vessel travelled in this time?**
c **Calculate the speed of the vessel in knots.**
d **On what true bearing is the vessel travelling?**
e **How far is the vessel from the church at 11.15 am?**

THINK	WRITE
a 1 For the position at 10.30 am: (a) Place the protractor centre over the church steeple.	**a**
(b) Measure an angle of 70° clockwise from true north.	
(c) Draw the 70° position line, extending it across the map.	Draw the 70° position line as shown on the chart on the following page.
(d) Move the protractor to the aerial.	
(e) Measure an angle of 105°T.	
(f) Draw this position line and extend it back until it intersects the 70° position line.	Draw the 105° position line.
(g) Draw a circle around this point of intersection and mark the time 1030.	Draw a circle, write at 1030 (see chart).
2 For the position at 11.15 am: (a) Draw the 95° and 200° position lines as at 11.15 am.	
(b) Extend both back to the point of intersection.	Draw the position lines.
(c) Draw a circle.	Draw a circle.
(d) Note the time.	Time is 1115.
b 1 Draw a line joining the positions at 1030 and 1115.	**b** Draw a line joining the points.
2 Using dividers or a ruler, transfer the distance between these points to the latitude scale and measure this distance. Recall that 1′ = 1 n mile.	

Continued over page

THINK | **WRITE**

3 Measure the distance in n mile. — Distance = 8.6 n mile



c Recall that speed = $\frac{\text{distance}}{\text{time}}$

The distance travelled is 8.6 n mile, and the time is from 1030 to 1115; that is, 45 minutes. Convert this to hours and determine the speed.

c Speed = $\frac{\text{distance}}{\text{time}}$

Time = 45 minutes

$= \frac{45}{60}$ hours

= 0.75 hour

Speed = $\frac{8.6 \text{ n mile}}{0.75 \text{ h}}$

= 11.5 n mile/h

= 11.5 knots

d Place the protractor at the 1030 position. Measure the angle clockwise from true north.

d True bearing (course) = 57°

e Using dividers or a ruler, transfer the distance between the 1115 position and the church steeple to the latitude scale.

e Distance = 4.5 n mile

WORKED Example 16

At 8.00 am, the vessel *Aqua Jet* is on a course of 270°T. Her navigator notes that the bearing of a tower is 0°T. At 8.30 am, the bearing of the same tower is 045°T. *Aqua Jet* is cruising at 18 knots.

a **Draw a sketch showing this information.**
b **How far has the vessel moved between the two sightings?**
c **How far is the vessel from the tower at 8.00 am?**
d **How far is the vessel from the tower at 8.30 am?**
e **If *Aqua Jet* continues on this course at the same speed, what bearing of the tower is expected at 9.00 am?**

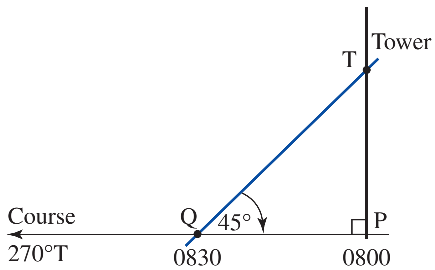
THINK | **WRITE**

a 1 The vessel is on a course of 270°T. Take true north to be towards the top of the page. Measuring 270° clockwise from north, the vessel's direction can be represented as a straight line from right to left.

2 Label a point on this line to represent the position at 8.00 am. Now draw the 0°T position line directly upwards from this point.

3 At a point to the left of the 0800 point, mark the position at 0830. (As only a sketch is required, there is no need to draw the diagram to scale. A scale diagram, though, could be drawn using, say, 1 cm to represent 1 n mile.) Now draw the 0830 position line at 45°. Extend it sufficiently so that it intersects with the 0800 position line.

a



b 1 To determine distance, recall that speed = $\frac{\text{distance}}{\text{time}}$.

2 Convert time to hours.

3 Calculate distance.

b

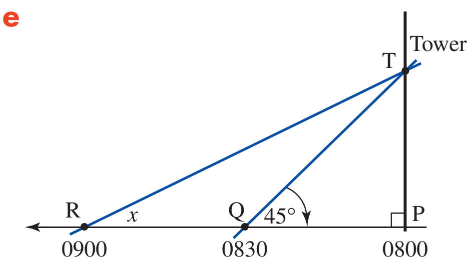
$$30 \text{ min} = \frac{30}{60} \text{ hours} = 0.5 \text{ h}$$

$$\text{Distance} = \text{speed} \times \text{time} = 18 \text{ n mile/h} \times 0.5 \text{ h} = 9 \text{ n mile}$$

c 1 The triangle has one angle of 90° and another of 45°, so the remaining angle must be 45°.

c $\angle TQP = 45°$
$\angle TPQ = 90°$
So $\angle QTP = 45°$

Continued over page

THINK	WRITE
2 Hence, the triangle is isosceles with two equal sides, TP and QP.	TP = QP = 9 n mile
d 1 The unknown distance is the hypotenuse of a right-angled triangle. We know the angle and adjacent side. Recall that cosine = $\frac{\text{adjacent}}{\text{hypotenuse}}$.	**d** $\cos 45° = \frac{9}{H}$
2 Rearrange to make H the subject.	$H = \frac{9}{\cos 45°}$
3 Calculate H.	The distance is 12.73 n mile.
e 1 Extend the line of travel to the left. After 1 hour the vessel will have covered 18 n mile. △ TRP is a right-angled triangle and we know the lengths of the adjacent and opposite sides; therefore, recall that tangent = $\frac{\text{opposite}}{\text{adjacent}}$.	**e**  $\tan x = \frac{TP}{RP}$
2 Calculate tan x.	$\tan x = \frac{9}{18}$ $= 0.5$
3 Now use the inverse function of your calculator to find x.	$x = 27°$
4 The bearing of the tower at 9.00 am from R is measured from true north clockwise down to the line TR.	Bearing = 90° − 27° = 63°T The bearing at 9.00 am is 63°T.

remember

1. The intersection of two or more position lines marked on a chart can be used to determine a vessel's position.
2. The intersection of three position lines forms a small triangle known as a *cocked hat*.

EXERCISE 5F Fixing position

WORKED Example 15

1 The bearings of a tower (090°T) and church (135°T) are recorded at 1400 hours (see figure at right). At 1430 hours, the bearings are 030°T and 090°T respectively.

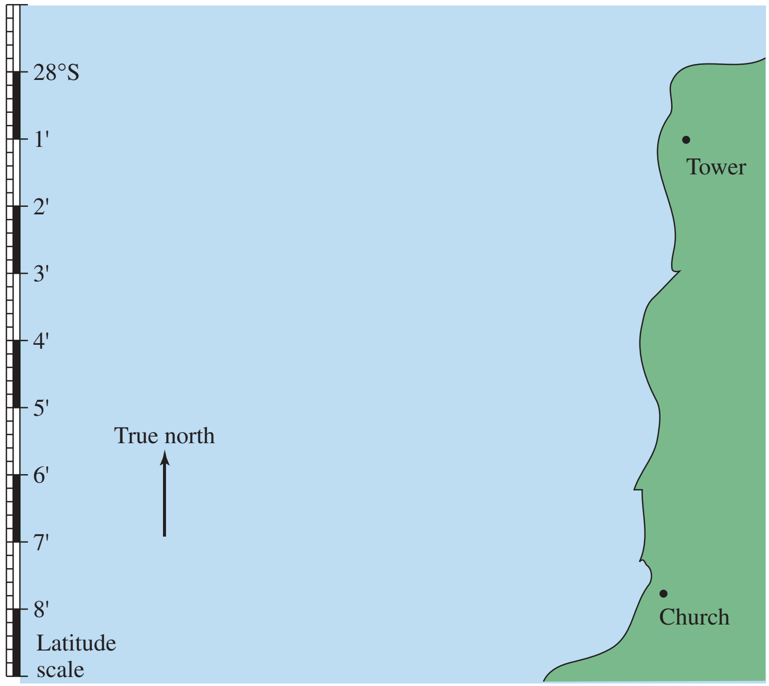
a Fix the position at these times on the figure.

b Plot the course of the vessel and determine its true bearing.

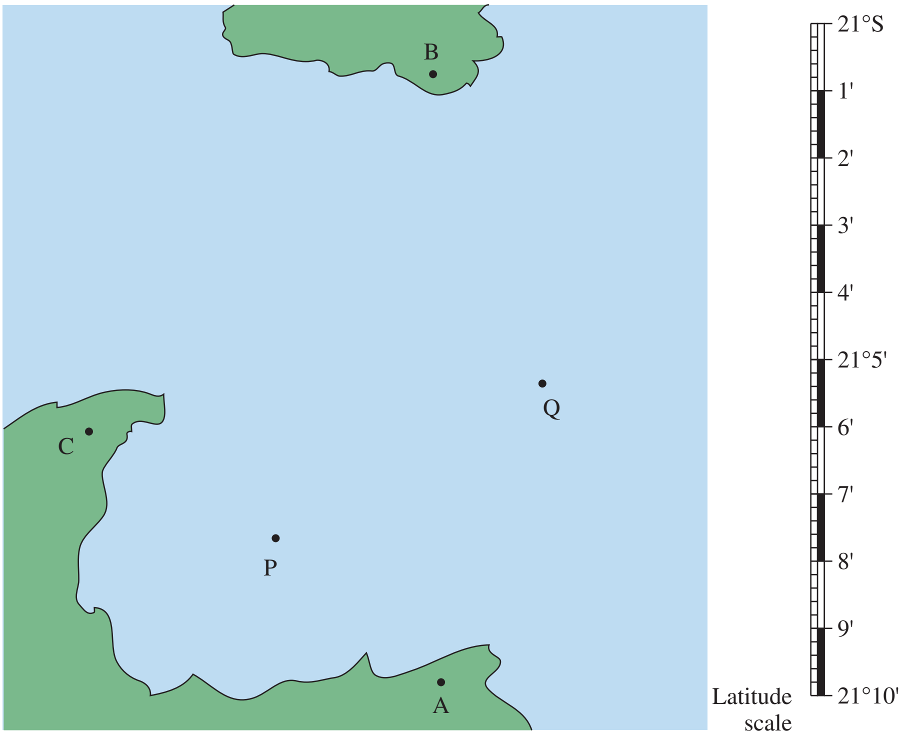
c How far has the vessel travelled?

d At what speed is the vessel travelling?

e At 1430 hours, how far is the vessel from the church?



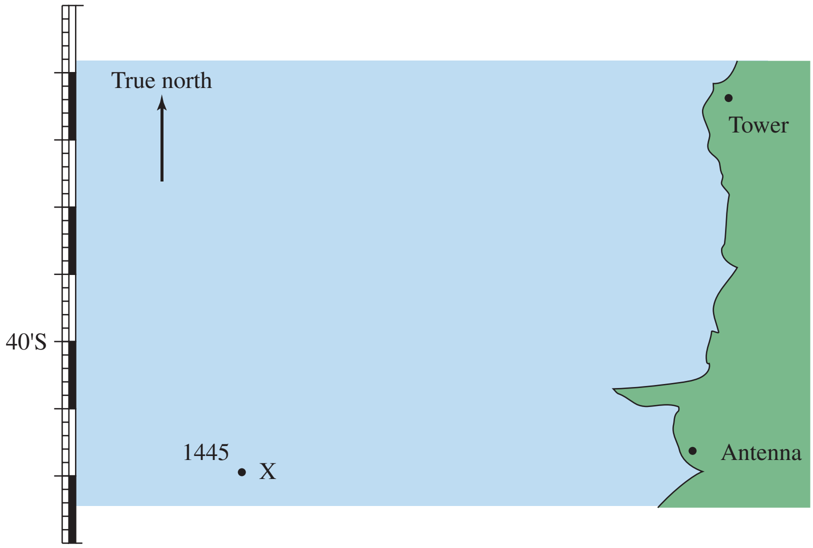
2 At 0830 hours the true bearings, in the figure below, of A (270°T) and B (330°T) are recorded. At 0900 hours, the true bearings are 200°T (A) and 270°T (B).



a Fix the vessel's position at these times.
b How far has the vessel travelled in this time?
c Calculate its speed.

3 A vessel is at Point P (in the figure in question **2**) at 6.00 am.
a What true bearings of A, B and C with respect to P are expected?
b The vessel travels directly to Q. What are the true bearings at Q of A, B and C?
c The time noted at Q is 6.25 am. Calculate the vessel's speed in knots.
d Estimate the vessel's position at 6.50 am by giving bearings of A, B and C. Assume it maintains its previous direction and speed.

4 The yacht *Cool Change* is at point X, in the figure below, at 2.45 pm.



a What true bearings of the tower and antenna are expected with respect to X?
b If *Cool Change* is on a course of 030°T and moving at 10 knots, plot the expected positions at 3.15 pm and 3.45 pm.

WORKED Example 16

5 At 10.15 am a schooner on a course 090°T sights a lighthouse on a bearing of 180°T. At 10.45 am the bearing of the lighthouse is 225°T. The speed of the vessel is 28 knots.
a Draw a sketch representing this information.
b How far has the vessel travelled in the time between the two sightings?
c How far is the vessel from the lighthouse at 10.15 am?
d How far is the vessel from the lighthouse at 10.45 am?
e If the schooner continues on this course, what bearing of the lighthouse is expected at:
  i 11.15 am?
  ii 11.45 am?

6 A ship departs A and sails 50 n mile on a bearing of 0°T to B. It then sails 60 n mile to C on a bearing of 120°.
a Sketch a scale diagram to represent this information.
b Find the distance from C to A.
c Give the bearing of A from C.

Navigation methods through the ages

For thousands of years, accurate navigation over land and sea has been extremely important for exploration, trade and survival.

Research the navigational methods used by indigenous groups, Captain Cook, Sir Francis Chichester, Amelia Earhart, Sir Charles Kingsford Smith and Kay Cottee.

How do inertial, radio and satellite navigational systems operate, and how are they helpful in everyday life?

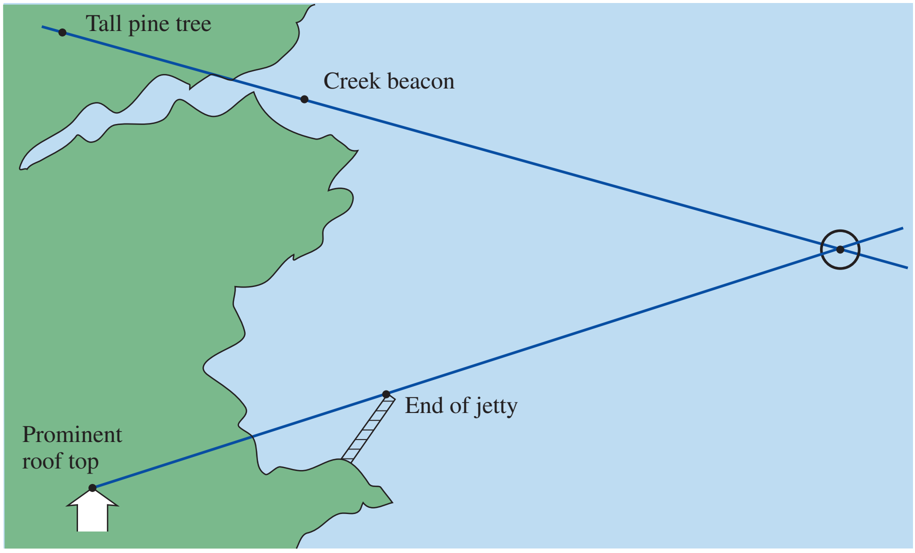
What is an EPIRB and why is it now mandatory for many vessels to carry this item? Someone in class may be able to bring one along and describe its operation. (Warning — do not activate it!)

How is the computerised navigational system on a ship or commercial fishing boat or passenger vessel used? You may be able to visit the bridge of a ship or wheelhouse of a vessel on a navigation excursion.

Transit fix

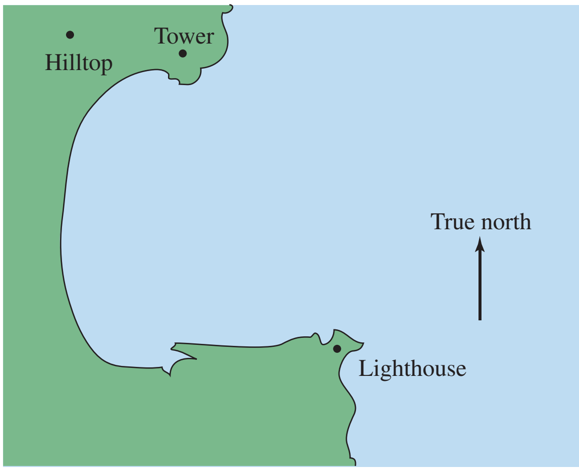
If a vessel notes that two prominent objects are in line, then the vessel must lie on the line joining these two objects. This line is known as the **transit line** and a fix arrived at by using this method is known as a **transit fix**.

The figure below shows that if a vessel has sighted the creek beacon and tall pine tree in line, then the vessel must be somewhere on this transit line. Further, if a prominent roof top and the jetty end are simultaneously in line, then this gives another transit line. The intersection of these transit lines gives the vessel's position. This is known as a **two-transit fix**. Such a fix is quite accurate. However, sufficient prominent landmarks are not always available. More often, a single transit line and a position line are used.



WORKED Example 17

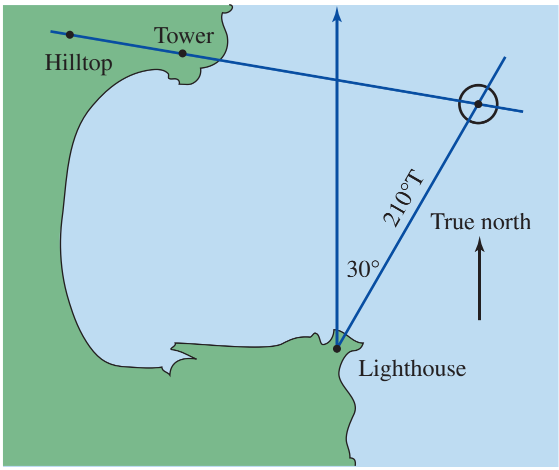
A fisherman is anchored on a particular spot where he is catching a lot of fish. He is keen to mark the location of this spot. He notices that a hilltop, and a tower to the front of it, are in line. He also records the bearing of a lighthouse as 210°T. Use this information to determine his position on the chart.



THINK

1. The tower and hilltop are in line, so draw a line joining these two points.
2. Place the protractor over the lighthouse and measure an angle of 210° clockwise from true north. Extend this position line until it intersects the transit line of the hilltop and tower. Circle this point of intersection. (Alternatively, find the reverse bearing of 210°; that is, 210° – 180° or 30°. Measure an angle of 30° at the lighthouse and extend this position line up to intersect the transit line.)

WRITE



remember

1. If an observer notes that two prominent shore objects are in line, then the observer must be on the line of sight connecting these two objects. This line is the *transit line*.
2. A position fix using transit lines is known as a *transit fix*.
3. A *two-transit fix* uses the intersection of two transit lines to determine a position.

EXERCISE 5G Transit fix

Use the figure below to answer questions **1** and **2**.

WORKED Example 17

1 A vessel is positioned such that points A and B are in line and points C and D are in line at 0730 hours.

a Use a two-transit fix to locate the vessel's position at this time.

b At this time, calculate the vessel's distance from:

- **i** A
- **ii** E
- **iii** C.

c The vessel heads true north until E is on a bearing of 270°T. The time is now 0745 hours. Fix the vessel's position at this time.

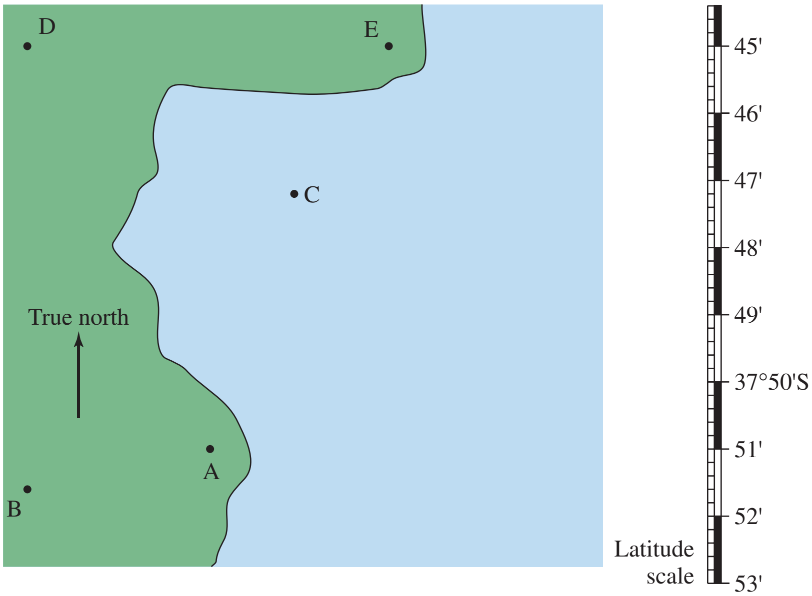
d Calculate the speed of the vessel.

2 A vessel is positioned such that E and D are in line. The bearing of A at the time, 2.30 pm, is 225°T.

a Locate the vessel at 2.30 pm.

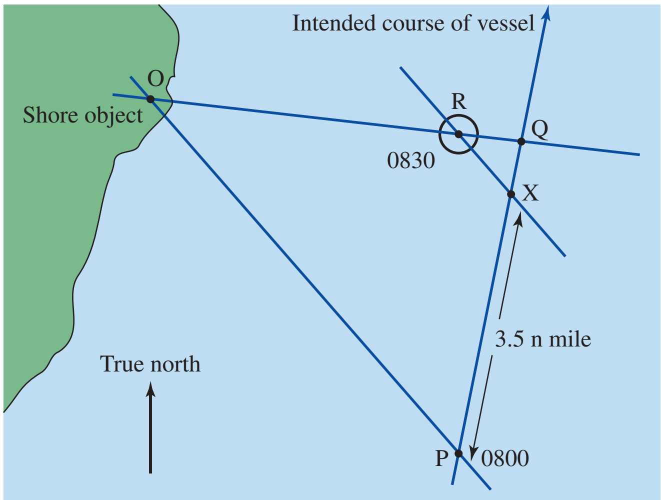
b The vessel sails on a bearing of 180°T at 10 knots. Plot its position at 2.50 pm.

c Give the expected bearings of B, D and E at 2.50 pm.



Running fix

The **running fix** is used where only one shore object is visible. This is often the case when navigating along the less-populated coastlines. Lighthouses that are separated by large distances are often the only means of determining a fix at night. The procedure involved in obtaining a running fix is as follows, and is shown in the figure at right.



1. Take the bearing of the shore object O and note the time (in the figure, 0800 hours).
2. Convert to a true bearing and then use the reverse bearing to plot the position line OP from the shore object. This position line intersects the vessel's intended course already plotted on the chart and will give an approximate position of the vessel at that time at point P. Any error in this position is to the left or right of P, along the line OP.
3. After some time, (at least 30° bearing change) the bearing of the same shore object is taken and converted to true. The reverse bearing is used to draw the second position line (OQ) which will intersect with the vessel's course at Q. The time of this bearing is again noted (0830 hours).
4. Given that the vessel has been travelling for some time, previous information kept in the vessel's log can give a reasonable estimate of the speed. Since times have been noted when the two bearings were taken, the distance travelled from point P can be calculated. This is then marked with an X on the chart.
5. Using parallel rules, or set square and ruler, transfer the first bearing through X so that the line RX is parallel to the line OP. The intersection of the second position line and the transferred bearing give a reasonable estimate of the vessel's position (point R). In this case, the estimated position earlier at P contained some error. The vessel was almost certainly closer to the shore at 0800 hours, along the line OP.

In this case, previous information indicated that the vessel was travelling at 7 knots, so X is marked by measuring a distance of 3.5 n mile from P (the distance covered by a vessel moving at 7 knots for 30 minutes). Note that the vessel has moved towards the coast. This could be due to currents or winds and the navigator must now take this into account. The fix at R is considered more reliable than position X since at 0830, the vessel was on the position line OQ.

EXERCISE 5H Running fix

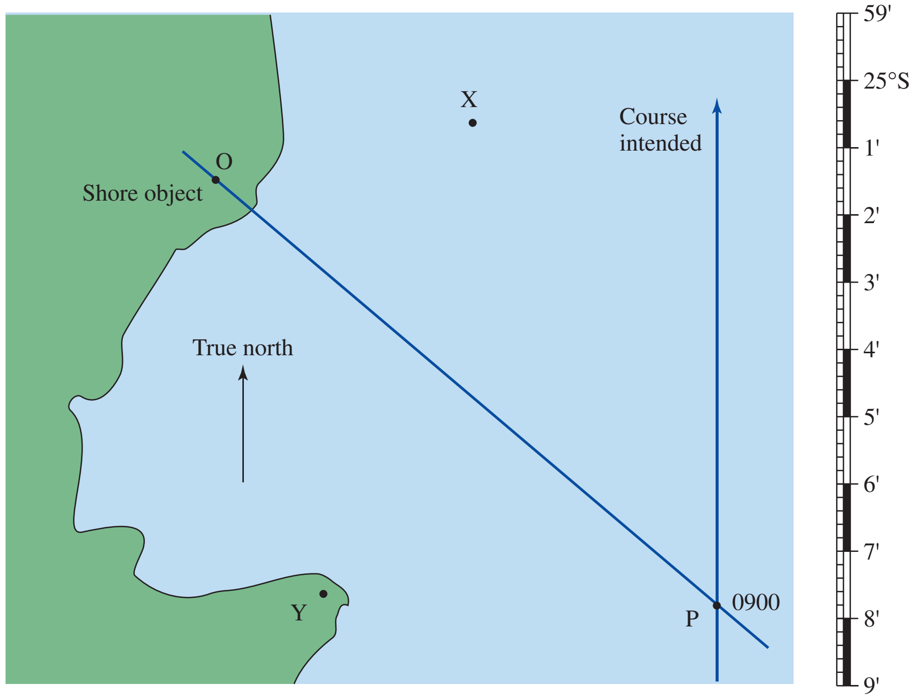
Use the diagram on the following page to answer questions **1** and **2**.

1 A vessel is at point P and has been travelling at 10 knots. The intended course is 0°T. The bearing of shore object O is taken at 0900 hours. At 0936 hours, the bearing of the shore object is 270°T.

a Draw the position line at 0936 hours using the 270°T bearing of O.

b Calculate the distance the vessel has covered from P, assuming that the 10 knots speed is maintained.

c Use the running fix method to estimate the vessel's position at 0936 hours.



2 A vessel moving at 11 knots is at X at 1010 hours, and intends to follow a course of 180°T. The bearing of Y is noted. At 1040 hours the bearing of Y is 260°T.

a Plot the intended course.

b What true bearing of Y from X is expected at 1010 hours?

c Plot the position line at 1040 hours.

d Use the running fix method to estimate the vessel's position at 1040 hours.

Doubling the angle on the bow

The front of a boat is known as the **bow**, and the rear is called the **stern**. To someone at the stern looking to the bow, the left-hand side is known as **port** and the right-hand side **starboard** as shown in the figure at right.

The **'doubling-the-angle-on-the-bow'** method is a reasonably accurate way of determining how far a vessel is from the coast at a particular time. It requires measuring the relative angle on the bow. Figure a (over page, left) shows the relative angle on the bow as 35° of a vessel at 8.00 pm (2000 hours) located at position P.

This is the angle between the ship's course and the bearing of a prominent landmark — in this case the lighthouse. This angle can be determined by either subtracting from or adding to the true bearing of the landmark. In figure a, the true bearing of the lighthouse is 055°T and the ship's course is 090°T. The relative angle on the bow then is 35°. The bearing of the landmark is taken at regular intervals until the angle on the bow has doubled. At this point, Q, the time of this sighting is again noted as shown in figure b (over page, right) as 8.30 pm or 2030 hours.

If previously gathered information indicated that the vessel was travelling at 12 knots, then the distance PQ is equal to 6 n mile (the vessel has travelled at

12 knots for half an hour). In △PQL, ∠PQL = 110° because the angles on the straight line are supplementary (add to 180°). Also, because angles of a triangle add to 180°, ∠QLP = 35°, △PQL is isosceles and so QL = PQ = 6 n mile. So the vessel at Q is 6 n mile from the lighthouse.

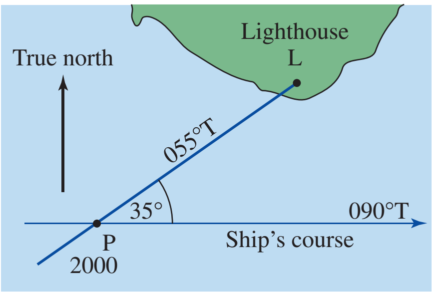


Figure a

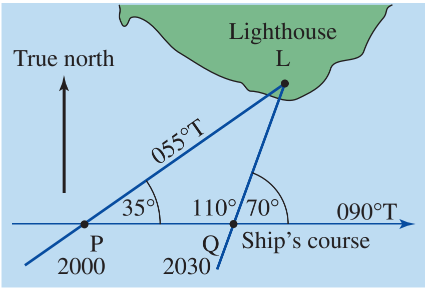
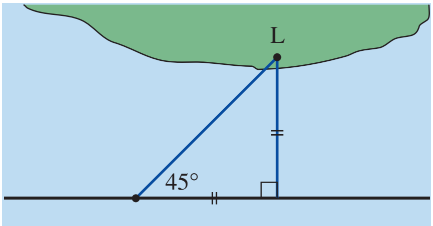


Figure b

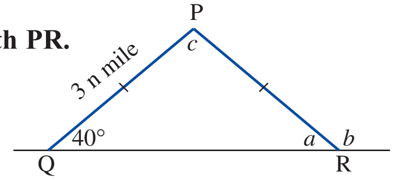
An adaptation of the doubling-the-angle-on-the-bow method is the **fix-by-four-point bearing**. This method is often used by navigators, and is a convenient way of easily determining if a vessel is on course. When the angle on the bow is 45° (commonly referred to as the **four-point bearing**) the time is noted. See the diagram at right.



Fittings previously set up on deck can assist in readily detecting a 45° angle on the bow. The vessel then continues on course until the angle on the bow is 90°, again an angle that is easily determined. The time of this sighting is again noted. The navigator can estimate the distance the vessel is at this time from the landmark. Previous chart-work would have predicted the expected distance from the landmark. Hence, the navigator can determine whether or not the vessel is on course.

WORKED Example 18

In the diagram at right, find angles *a*, *b* and *c* and length PR.



THINK	WRITE
1 The triangle is isosceles (given PQ = PR). Hence, $a = 40°$.	$a = 40°$
2 Calculate b. If $a = 40°$, then $b = 180° - 40°$ (since $a + b = 180°$).	$b = 180° - 40°$ $= 140°$
3 Calculate c. The sum of three angles in a triangle add to 180°.	$a + c + 40° = 180°$ But $a = 40°$ $\therefore c = 180° - 2 \times 40°$ $= 100°$
4 It is given that PQ = PR.	PR = 3 n mile

WORKED Example 19

A ship is on a course of 010°T. At 1100 hours, the navigator notes the bearing of a lighthouse, L, to be 340°T. The vessel is travelling at 24 knots. At 1120 hours the angle on the bow has doubled.

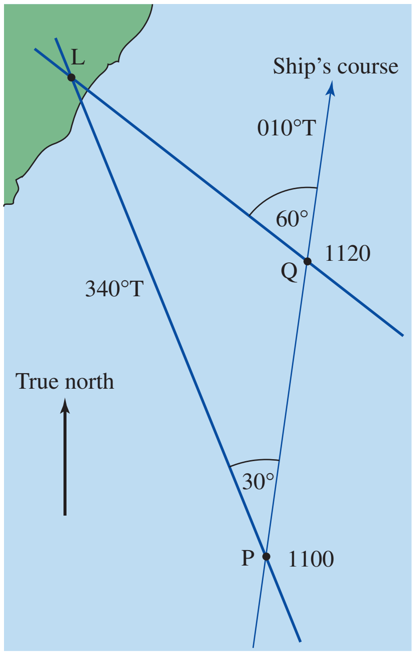
a Draw a neat diagram representing this information.
b Calculate the angle on the bow at 1100 hours and 1120 hours.
c How far has the vessel travelled between 1100 and 1120 hours?
d How far is the vessel from the lighthouse at 1120 hours?

THINK

a
1. Draw a line 10° clockwise from vertical (or true north) to mark the ship's course.
2. Towards the base of this line, mark a point, P, representing the ship's position at 1100. From this point, draw a position line at 340°T. Draw a point, L, on land, to represent the lighthouse. The angle between the two lines now drawn must be 30°.
3. At 1120 the angle on the bow has doubled, and therefore will now be 60°. Draw a position line through L and intersecting the ship's course at 60°. This will give the vessel's position, Q, at 1120.

WRITE

a



THINK

b
1. The angle on the bow at 1100 hours is the angle between the direction of the vessel and the bearing of the lighthouse; that is, the angle between 10°T and 340°T.
2. At 1120, the angle on the bow has now doubled to 60°.

WRITE

b At 1100, the angle on the bow is 30°.

At 1120, the angle on the bow is 60°.

THINK

c
1. Calculate distance. We know the speed and the time, therefore recall that $\text{speed} = \dfrac{\text{distance}}{\text{time}}$.
2. Convert time to hours.

WRITE

c $24 \text{ knots} = \dfrac{\text{distance}}{\text{time}}$

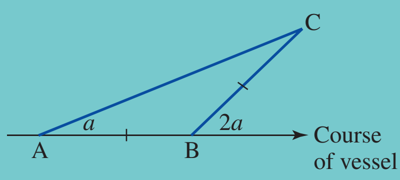
$$1120 - 1100 = 20 \text{ min} = \tfrac{20}{60} \text{ hours} = \tfrac{1}{3} \text{ h}$$

Continued over page

THINK	WRITE
3 Rearrange the equation to make distance the subject.	Distance = speed × time $= 24$ n mile/h $\times \frac{1}{3}$ h $= 8$ n mile
4 Find the distance.	The vessel has travelled 8 n mile.
d The triangle formed by the three position lines is isosceles. Hence, the distance between the 1100 and 1120 positions is equal to the distance between the 1120 position point and the lighthouse; that is, 8 n mile.	**d** Distance = 8 n mile

remember

1. The front of a boat is known as the *bow.*
2. The *angle on the bow* is the angle between the boat's course and the bearing of a prominent feature.
3. The *doubling-of-the-angle-on-the-bow* method uses the properties of isosceles triangles — triangles which have one pair of sides equal and base angles equivalent. When angle a has doubled to $2a$, the distance BC is equal to the distance AB.



EXERCISE 5I Doubling the angle on the bow

WORKED Example 18

eBook plus

Digital doc: SkillSHEET 5.2 Angle relationships

1 In each of the diagrams (**a** to **f**) below, find the following unknowns.

a i x ii y iii z iv AC

b i x ii y iii PQ

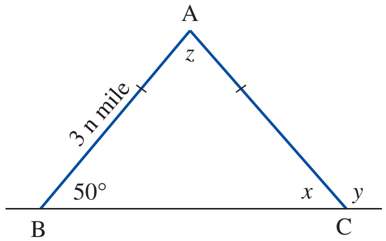
c i x ii y iii z iv AB

d i y ii x iii QL iv true bearing of L from P

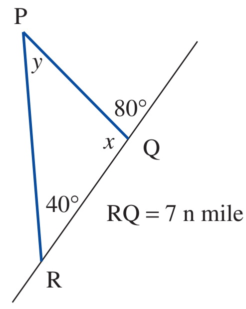
e i ∠LPR ii ∠PRL iii ∠PLR iv LR

f i angle on bow at 0800 hours, x ii ∠LRP iii ∠RLP iv RP v RL

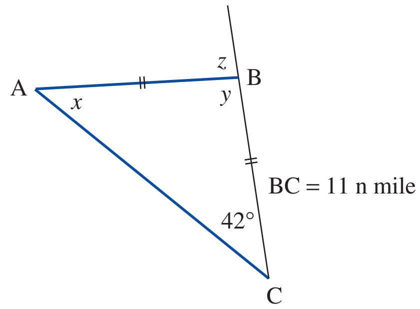
a



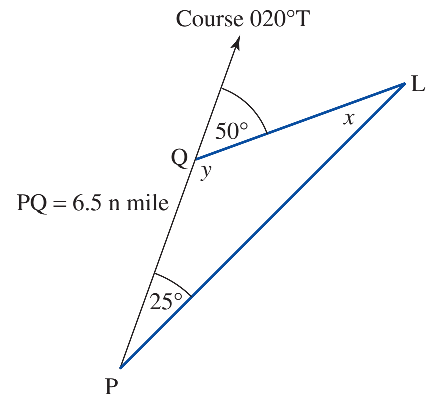
b



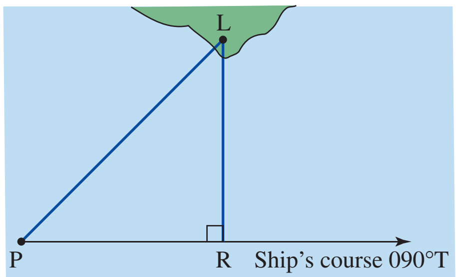
c



d

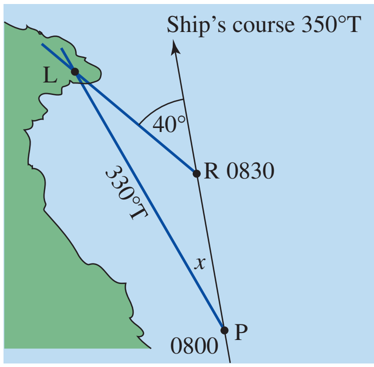


e



Given: Bearing of L at P is 045°T
PR = 10 n mile

f

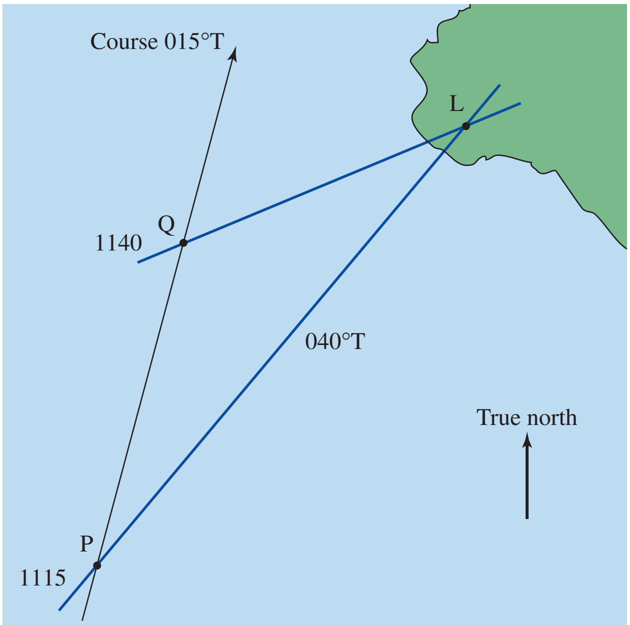


Given: Speed of vessel = 16 knots

2 The figure at right shows the course (015°T) of a vessel travelling at 12 knots. At 11.15 am the navigator observes a land feature, L, on a bearing of 040°T. At 11.40 am the angle on the bow has doubled.

a Determine the angle on the bow at 11.15 am.

b How far is the vessel from the feature L at 11.40 am?



WORKED Example 19

3 At 1300 hours, the navigator of a vessel on a course of 080°T observes the bearing of a landmark to be 050°T. At 1330 hours the angle on the bow has doubled. The speed of the vessel is 18 knots.

a Using a ruler and protractor, draw a neat diagram representing this information.

b Calculate the angle on the bow at 1300 and 1330 hours.

c How far has the vessel travelled between sightings of the landmark; that is, from 1.00 pm to 1.30 pm?
d How far is the vessel from the landmark at 1.30 pm?

4 A vessel is on a bearing of 250°T travelling at 24 knots. At 6.30 am, a tower is observed at a bearing of 225°T. At 6.50 am the angle on the bow has doubled.
a Use a ruler and protractor to represent this information on a diagram.
b How far is the vessel from the tower at 6.50 am?
c Find the shortest distance the vessel is from the tower as the vessel continues on its course. (*Hint:* What is the angle on the bow when the vessel is at this point?)
d At what time is the vessel at the point in **c**?
e What bearing of the tower would be expected at 7.30 am? (*Hint:* This can be answered by using an accurate scale diagram or trigonometric ratios.)

10 QUICK QUESTIONS 3

1 What is the minimum number of intersecting position lines required to fix the position of an object?
2 The intersection of three position lines forms a small triangle. What is the triangle called?
3 What name is given to the line of sight connecting two prominent objects?
4 Name the type of triangle that has two equal sides and two equal base angles.
5 The bow refers to which part of a vessel: front or rear?
6 What is the name of the angle between a boat's course and the bearing of a prominent feature?
7 A vessel covers 63 nautical miles from 10 am to 5 pm. What is its average speed in knots?
8 How long does it take a vessel to cover 40 n mile if it is travelling at 10 knots?
9 The distance between two points on a chart is transferred to the latitude scale using dividers. The spread of the dividers shows 1°24′. How far apart are the two points in nautical miles?
10 How many visible shore objects are required for a running fix?

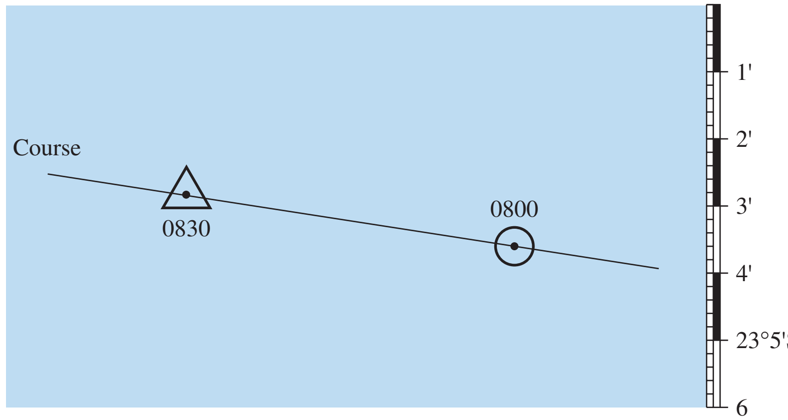
investigation

Magnetic variation

Obtain several adjacent 1 : 50 000 scale maps of your local area. Explore the relationships between true north, magnetic north and grid north on these maps. You may wish to peruse the section in Chapter 6 on topographic maps.

How has magnetic variation in your area changed over the past century? What is the cause of this?

Dead reckoning



Dead reckoning (or **deduced reckoning** or DR) refers to the method of calculating a vessel's position from previously collected information. It is an estimate only and does not involve taking a fix by sighting. It is often used by navigators when usual methods of fixing cannot be used; for example, when a vessel is unable to sight landmarks because of distance from the coastline. An estimated position arrived at by dead reckoning is indicated on the chart by a dot in the centre of a triangle, ⚠, to distinguish it from the more accurate fix position ⊙.

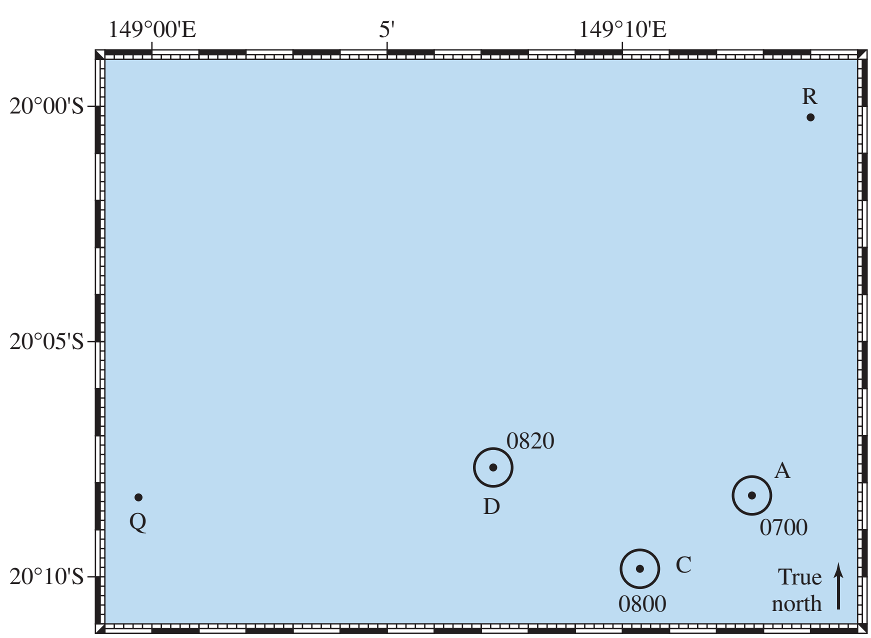
The simplest method of establishing the DR position is to plot on the chart the course and distance the vessel has followed since the last fix position. The DR position is only an estimate due to factors such as changing winds and currents. The diagram above shows a fix taken at 0800 hours and marked ⊙. From previous fixes, the speed of the vessel has been determined at 10 knots. Hence, the DR position at 0830 hours can be determined and marked ⚠. (Distance = time × speed = $\frac{1}{2}$ hour × 10 knots = 5 n mile.) The dividers are used to transfer 5 n mile from the latitude scale.

WORKED Example 20

The yacht *Escapade* is at point A at 0700 hours as shown in the chart, and is sailing in a true northerly direction at 8 knots. Locate its expected position at:

a 0730 hours and

b 0745 hours.



Continued over page

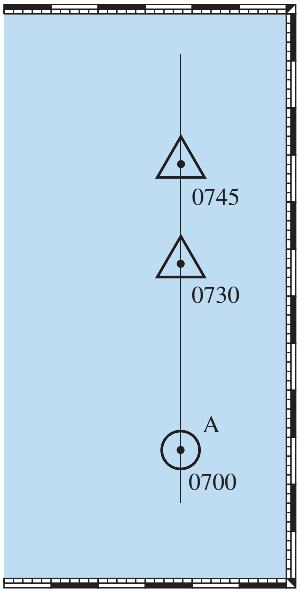
THINK

a 1. The course of the vessel is drawn as a line directly upwards from point A.
2. The vessel then moves a distance of 8×0.5 n mile (speed × time) or 4 n mile in the next 30 minutes. This position is then plotted by transferring 4 n mile from the latitude scale.
3. This expected position is marked in the triangle.

b 1. At 0745 hours, *Escapade* has travelled for $\frac{45}{60}$ hour or 0.75 hour.
2. At 8 knots, it has covered 8×0.75 n mile, or 6 n mile, since 0700 hours.
3. Again, the position is marked in a triangle.

WRITE

a, **b**



remember

1. Dead reckoning or deduced reckoning is a method of estimating the position of a vessel.
2. It does not involve a fix by sighting; rather, it uses previously collected information about speed and course (direction) of the vessel.

EXERCISE 5J Dead reckoning

Use the chart in Worked example 20 on page 291 to answer questions 1 to 3.

WORKED Example 20

1 A vessel is at Q at 0930 hours and is travelling at 6 knots on a true bearing of 060°.
  a Plot this course using a protractor.
  b State the latitude and longitude of Q.
  c Plot the estimated position of the vessel at:
    i 0950 hours   ii 1010 hours   iii 1030 hours.
  d State the position of the vessel for the times given in c above.

2 A vessel travelling at 7 knots is at R at 1300 hours. Plot and state its estimated position at 1400 hours if it follows a true bearing course of:
  a 180°   b 270°   c 225°   d 260°.

3 Use the fixes determined by a vessel at C and D to determine:
  a the distance travelled from 0800 to 0820 hours
  b the speed of the vessel
  c the DR position (latitude and longitude) of the vessel at:
    i 0840 hours **ii** 0900 hours.

The lighthouse and navigation

Throughout the world, all coastlines that can be navigated are lined with *lighthouses* and lights. There are three main categories of lights:

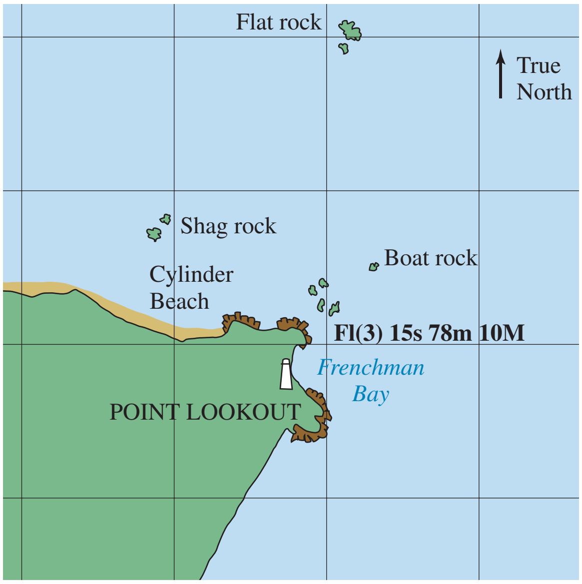
1. the long range **ocean lights**, commonly known as lighthouses
2. **coastal lights**, shorter range lights used for indicating harbours and rivers
3. **harbour lights**, including buoys and beacons, and indicating channels and hazards.

Most lighthouses have a lens that concentrates the light. This concentrated beam is then rotated at a set speed. The rotating beam can often be seen as a flare or 'loom' across the sky, even though the light source is below the horizon. A vessel that has been well off the coast must be able to distinguish one lighthouse from another to ascertain its correct position. Consequently, lighthouses flash at different rates.

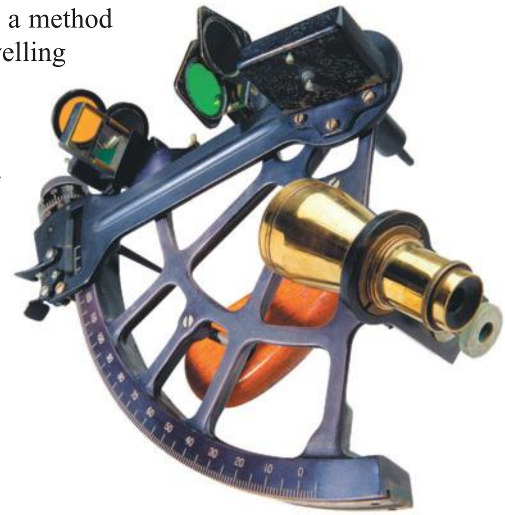
A light in which periods of light are shorter than dark periods is marked F1 (for **flashing**) on a chart. If the periods of light exceed the period of darkness, it is marked OCC (**occulting**).

A lighthouse emitting two short flashes of light followed by a long period of darkness is marked F1 (2). Three long flashes of light followed by a short dark period is marked OCC (3).

The length of the cycle in seconds is also given, followed by the height of the light above sea level and its range in nautical miles. The chart below shows the lighthouse at Point Lookout, North Stradbroke Island, marked on a chart as F1 (3) 15 s 78 m 10 M. This means that the light gives 3 short flashes and a long period of darkness in each 15-second cycle. It is 78 m above mean low water mark and has a range of 10 nautical miles. Because no light anywhere in the region flashes in a similar fashion, even an off-course navigator could readily establish position if charts were available.

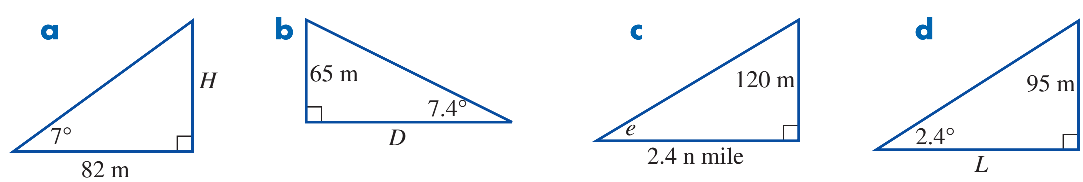


Lighthouses offer a very accurate fix by using a method known as the *extreme-range fix*. A navigator travelling along the coast will notice the 'loom' or flaring beam from the lighthouse well before the light itself is seen. As soon as the light is seen on the horizon, the time is noted and the bearing of the lighthouse is taken. Navigation tables can then be used to give the distance from the lighthouse. Trigonometric calculation using the tangent ratio can be used as the vessel nears the lighthouse. A sextant is used to determine the **angle of elevation** of the light above the horizon.



WORKED Example 21

Find the unknowns in a to d below.



THINK / **WRITE**

a
1. Recall that tangent $= \dfrac{\text{opposite}}{\text{adjacent}}$. H is opposite 7°, and 82 m is the adjacent side. So $\tan 7° = \dfrac{H}{82}$.
   $\tan 7° = \dfrac{H}{82}$
2. Rearrange to make H the subject.
   $H = 82 \times \tan 7°$
   $= 10.1$ m

b
1. The 65-m side is opposite 7.4° and D is the adjacent side.
   $\tan 7.4° = \dfrac{65}{D}$
2. Rearrange the equation to make D the subject.
   $D = \dfrac{65}{\tan 7.4°}$
   $= 500$ m

c
1. Note that the units of length are different. Convert 2.4 n mile to metres by multiplying by 1852.
   2.4 n mile $= 2.4 \times 1852$ m
   $= 4445$ m
2. Calculate $\tan e$; opposite is 120 m, adjacent is 4445 m.
   $\tan e = \frac{120}{4445}$
   $= 0.027$
3. Calculate e.
   $e = 1.55°$
4. Convert to degrees and minutes. Recall that $1° = 60'$, so $0.55° = 0.55 \times 60' = 33'$.
   $e = 1°33'$

THINK / **WRITE**

d 1. The 95-m side is opposite; L is adjacent, so $\tan 2.4° = \frac{95}{L}$.

d $\tan 2.4° = \frac{95}{L}$

2. Rearrange the equation to make L the subject.

$L = \frac{95}{\tan 2.4°}$

3. Calculate L.

$L = 2267$ m

4. Convert to nautical miles. Recall that 1 n mile = 1852 m so $L = \frac{2267}{1852}$ n mile.

$L = 1.22$ n mile

WORKED Example 22

A lighthouse is marked on a map as F1 (3) 15 s 78 m 10 M. A ship's navigator, using a sextant, measures its angle of elevation as 1°30′.

a Describe the flashing pattern expected.

b How high above sea level is the lighthouse, and what is the range of the light emitted by it?

c How far is the vessel from the lighthouse? Give your answer in metres and nautical miles.

d What angle of elevation is expected if the vessel is 0.5 n mile from the light?

THINK / **WRITE**

a F1 means short flashes of light then a long period of darkness. The symbol (3) gives the number of light flashes.

a There are 3 short flashes of light, then a long period of darkness.

b 1. 78 m refers to height above sea level in metres.

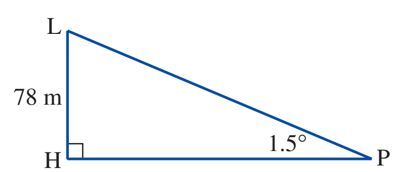
b Height = 78 m

2. 10 M gives the range of light in n mile.

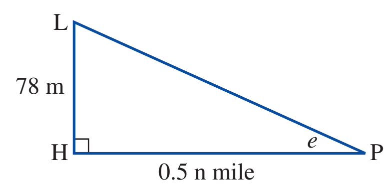
Range = 10 n mile

c 1. Draw triangle PLH.
(a) P represents the vessel's position.
(b) LH is the height of the light; that is, 78 m.
(c) The angle of elevation (∠HPL) = 1°30′ = 1.5°.
(d) HP is the adjacent side and 78 m is the opposite.

c



Continued over page

THINK	WRITE
2 The tangent ratio relates LH and PH.	$\tan 1.5° = \frac{78}{HP}$
3 Make HP the subject and calculate the distance.	$HP = \frac{78}{\tan 1.5°}$ $= 2979$ m
4 Now convert to n mile by dividing by 1852.	$HP = \frac{2979}{1852}$ HP = 1.61 n mile
d 1 Draw a diagram as shown. (a) Mark the unknown angle of elevation as *e*. (b) The opposite side is 78 m. (c) The adjacent side is 0.5 n mile (or 0.5 × 1852 = 926 m).	**d**  0.5 n mile = 0.5 × 1852 m = 926 m
2 The tangent ratio relates *e*, LH and PH.	$\tan e = \frac{78}{926}$
3 Find *e*.	$e = 4.8°$

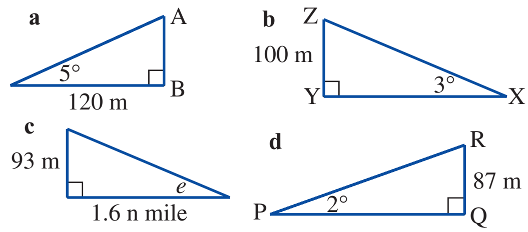
remember

1. Lighthouse lights are either flashing (F1) or occulting (OCC). A flashing light gives a number of short flashes of light followed by a long period of darkness. A lighthouse marked OCC refers to a light which gives long flashes of light followed by a short period of darkness.
2. F1 (2) 14 s 108 m 13 M describes a lighthouse that has 2 short flashes of light followed by a long period of darkness every 14 seconds. It is 108 metres above sea level and has a range of 13 nautical miles.
3. The tangent ratio of an angle = $\frac{\text{opposite side}}{\text{adjacent side}}$.
4. 1° = 60′; that is, 1 degree = 60 minutes.
5. The angle of elevation of a lighthouse is the angle measured from the horizontal upwards to the light.
6. 1 nautical mile = 1852 metres.

EXERCISE 5K The lighthouse and navigation

WORKED Example 21

1 Find the unknowns in the diagrams at right.
a AB in m
b XY in m
c e (in degrees and minutes)
d PQ in n mile



WORKED Example 22

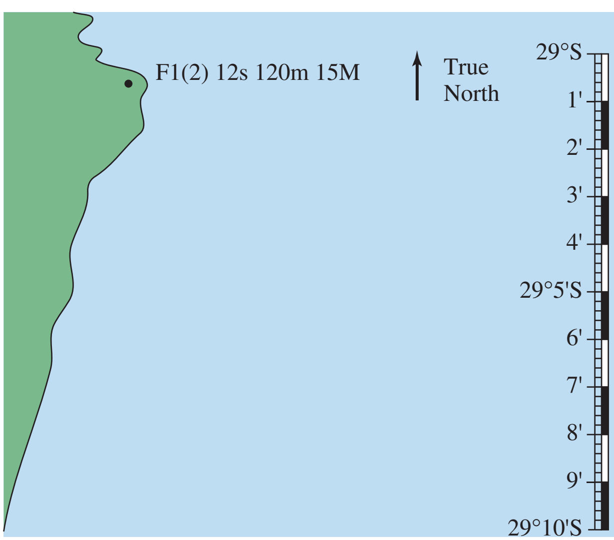
2 The Cape Moreton lighthouse is described on the chart as F1 (4) 20 s 122 m 27 M. From the trawler *Seaspray*, the angle of elevation of this lighthouse is 2°15′.
a What is the flashing pattern of this lighthouse?
b Draw a right-angled triangle showing the height of the lighthouse and the angle of elevation given.
c How far is the vessel from the lighthouse when the sighting is made?

eBook plus

Digital doc:
SkillSHEET 5.3
Sides of a right-angled triangle

3 At 2100 hours the navigator on a vessel heading south notes that the angle of elevation of a lighthouse is 1°20′ on a bearing of 340°T. The lighthouse is marked F1 (2) 12 s 120 m 15 M (see figure below).
a Describe the light pattern.
b Use a protractor to draw the position line from the lighthouse.
c Calculate the distance from the lighthouse.
d Fix the vessel's position at 2100 hours.
e If the vessel is moving at 10 knots, plot its expected position at 2130 hours.
f How far is the vessel from the lighthouse at 2130 hours?
g Predict the angle of elevation of the lighthouse at:
i 2130 hours ii 2148 hours.

4 At 1300 hours a vessel's navigator notes the bearing of the lighthouse in the figure shown to be 270°T. The vessel is on a course 180°T. The navigator notes the angle of elevation of the lighthouse to be 1°.
a Plot the line of sight to the lighthouse at 1300 hours.
b Calculate the distance from the lighthouse if it is 120 m above sea level. Plot the vessel's position at 1300 hours.



c If the vessel is moving at 8 knots, plot the position at 1330 hours.
d What is the expected angle of elevation of the lighthouse at 1330 hours?

Obtaining a speed boat licence

What steps are required to legally operate a speed boat in Queensland? Someone in your class may be able to speak to you about this.

A group may choose to obtain a licence as you study this unit. Many centres along our coast offer short courses.

Let’s go cruising

The ability to rapidly determine accurate position at sea is an essential skill required by seafarers. In this section, we shall explore position fixing using GPS devices and chart work.

Global Positioning System (GPS)

It is now possible to purchase a relatively inexpensive and portable device (hand-held if required) which can accurately determine one’s position in the air, on sea or on land. The techniques for fixing position that we have examined in earlier sections all depend upon the visibility of coastal objects; these new devices are superior as they operate in all weather conditions, 24 hours a day.

The **Global Positioning System** (or GPS), is a navigation system that is operated by the United States Government. Twenty-four satellites orbit the Earth every twelve hours. The satellites are positioned so that a GPS receiver is usually in range of six or more of them. The GPS receiver then determines the position on Earth by decoding information picked up from the satellites.

Until 1 May 2000, civilian users were exposed to an error that was deliberately added to all received signals. This error was known as selective availability and often caused position errors of approximately 50 metres. Position in terms of latitude and longitude can now be described with an accuracy of 2 to 5 metres on the more sophisticated devices. The illustrations on page 299 show a number of devices used for determining position. They can also provide altitude, absolute ground speed (unaffected by wind and currents) and deviation from a planned course. They are also capable of converting a true bearing to a corresponding magnetic bearing and vice versa. They are rapidly becoming an essential item for the safety conscious navigator.

These fish were caught on an offshore reef using a GPS.

GPS

This activity requires a small hand-held GPS and operating manual. These devices are now inexpensive. If your school does not have one, you may be able to have a student bring one to school and describe its operation to the class.

Finding your way back to a previous location

Move outside the classroom away from buildings and stop. Switch on the GPS and observe the screen as it depicts the satellites that it is locking on to.

1 Obtain your position. Note the degree of accuracy that is provided by the latitude and longitude description. Store this position in the GPS memory as Waypoint 1 (WP 001).

2 Walk approximately 50 metres from your first point, stop, and store this new position in the memory (WP 002). Continue moving to new locations and storing new positions as waypoints.

3 Now highlight your first stored position on the screen, Waypoint 1. Use the GPS to see if you can return to this (your original) position. Use the GPS device to return to your other stored positions.

Finding a hidden object

To conduct this activity, first break the class into two groups. One group hides an object and records its position in the GPS. The second group, using the stored waypoint, then attempts to locate the object. The groups swap roles, and repeat the activity. The times taken to recover the hidden object are recorded. This process can be repeated four or five times, and the group with the lowest average declared the winner.


Orienteering

Orienteering is a physically active and enjoyable pastime involving practical land navigation using a map and compass. A class activity is described in Chapter 6 on page 372.

Cruising your local area — practical navigation

The activities described below can form the basis of an exciting and educational excursion. Assessment items can be drafted around the activities conducted before as well as during the trip. Many charter boat operators are more than happy to assist with the teaching of practical navigation.

Materials

The materials that are required are:
1. a laminated chart of an area of coastline close to your school
2. a hand bearing compass
3. a parallel ruler
4. chart pencils
5. a portable GPS device (the vessel on which you are travelling may have one)
6. a camera — to record the dolphins, whales, seagulls and boats under sail!
7. a large charter boat to carry your Maths A class.

Before the trip

Study the chart of your local coastline closely.

1. What scale has been used to produce this chart?
2. Look at the compass rose in the area closest to where you will conduct your excursion. What is the magnetic variation for this year?
3. Study the shoreline closely. Which prominent features might be observed from a vessel cruising within a few nautical miles of your coastline? Give the position of each prominent feature by stating latitude and longitude.
4. Look closely at the river, harbour or ramp from which you will depart. Note any beacons present that you are likely to encounter. Use the legend at the base of the chart to decipher any unfamiliar symbols. What is the depth of water in the region of your planned excursion? Again the legend gives information regarding soundings.
5. Choose any two prominent locations on your chart. Estimate the distance between them in kilometres and nautical miles. Remember to use the latitude scale to determine distance. Repeat this process for other pairs of locations.
6. Determine the true and magnetic bearing of a number of prominent features from your departure point.

During the voyage

A trip of only 3 hours duration could include the following activities.
1. As you depart, note the presence of any beacons that you expected to see when you studied the chart.
2. Stop the vessel soon after departing the harbour.
   - a. Use the compass to determine north. Which way to east, west and south?
   - b. Locate the prominent features you expected to find prior to the trip. Use the hand bearing compass to give the true and magnetic bearing of each. How do these values compare with your expected bearings?

3 Cruise out for a couple of nautical miles and stop the vessel.
   a Quickly take the bearings of three prominent objects using the hand bearing compass. You could use towers, aerials or beacons — as long as the features you have chosen are shown on your chart. Convert these compass bearings to true bearings using the appropriate magnetic variation.
   b Use the parallel rule to pinpoint your position on the chart. A fairly small triangle ought to result. It is best to undertake this task while the vessel is still. (Ask the skipper to keep the boat steady until all groups have recorded the three bearings.) What would happen if the boat drifted between each of your sightings?
   c Now state your current position in terms of latitude and longitude. Check the accuracy of your work using either a portable GPS device or one that the skipper of the boat has at the wheel.
   d Just as you depart this position, note the time and write this on your chart beside the cocked hat.
4 Ask the skipper to now move directly at constant speed to another location, perhaps one or two nautical miles away. The skipper will later tell you the speed he was cruising at, using his instruments at the wheel, and the course he followed. You're going to calculate both first, and then check to see if the skipper is correct!
   a As soon as the boat stops, record the time.
   b Again, have the vessel still as you record the compass bearings of three features. Convert these bearings to true and determine your new location on the chart. Check your position with the GPS reading.
   c Now calculate the distance in nautical miles between these two locations and the time taken to travel between them. Determine the speed of the vessel in knots.
   d Use the chart to also find the true and compass course of the boat.
   e Ask the skipper for the actual speed and course followed to check your work. Good luck!

Some practical considerations

1 If your class does not have this equipment, you may be able to use materials used by the Marine Studies classes at your school or a nearby school.
2 Charts suitable for this excursion are produced by the Marine Division of Queensland Transport, Brisbane.
3 Charts, parallel rulers, chart pencils, course plotters and many books, videos and CDs related to navigation are available from Boat Books Australia. To contact Boat Books Australia or to find out more about their products, use the **Boat Book** weblink in your eBookPLUS. An extensive catalog is available free of charge.
4 Group sizes of 3 to 4 work well for the above activities. Rotate activities at each site so that each member uses the compass and plots a line of sight.

EXERCISE 5L Let's go cruising

This exercise uses the chart of a region of Moreton Bay, Brisbane on pages 304 and 305.

1 Study the compass rose on the chart. What is the magnetic variation in the region when this chart was produced:

a to the nearest minute?

b to the nearest degree?

2 State the position of:

a South West Rocks on Peel Island

b the wreck *Platypus* on Peel Island

c Lake Kounpee on North Stradbroke Island (at the right of the map)

d Cleveland Point on the mainland (base of lighthouse)

e the rocks to the north of Pott's Point on Macleay Island (at the bottom of the chart).

3 Name the feature at:

a (27°34′S, 153°20′E)

b (27°30′S, 153°22.4′E)

c (27°34.4′S, 153°20.3′E)

d (27°28.5′S, 153°24.3′E).

4 The *Kelly* is at Blaksley Anchorage on North Stradbroke Island (the land mass to the right of the map). (Use the anchor symbol on the map for this reference point.)

a What true bearing is expected of:

i South West Rocks on Peel Island?

ii The Bluff on Peel Island?

iii the township of Dunwich on North Stradbroke Island?

iv Pott's Point on Macleay Island (to the west, and about halfway to the mainland)?

b What compass bearing is expected of each of these locations?

c Estimate the distance in nautical miles from Blaksley Anchorage to each of these four locations.

5 The abbreviation F1 G 3 s describes a light, green in colour, which flashes every 3 seconds. R and Y indicate red and yellow lights.

a Describe the light indicated by:

i F1 Y 2.5 s   ii F1 R 4 s   iii F1 G 6 s.

b Why are the lights of different colours and flashing rates?

6 A boat departs Horseshoe Bay (on Peel Island) at 10.25 am and heads directly to Blaksley Anchorage on North Stradbroke Island.

a How far is this in nautical miles? (Use the anchor symbol at each location.)

b On what true and magnetic bearing would it travel?

c How long would this trip take at 9 knots?

d Calculate the ETA.

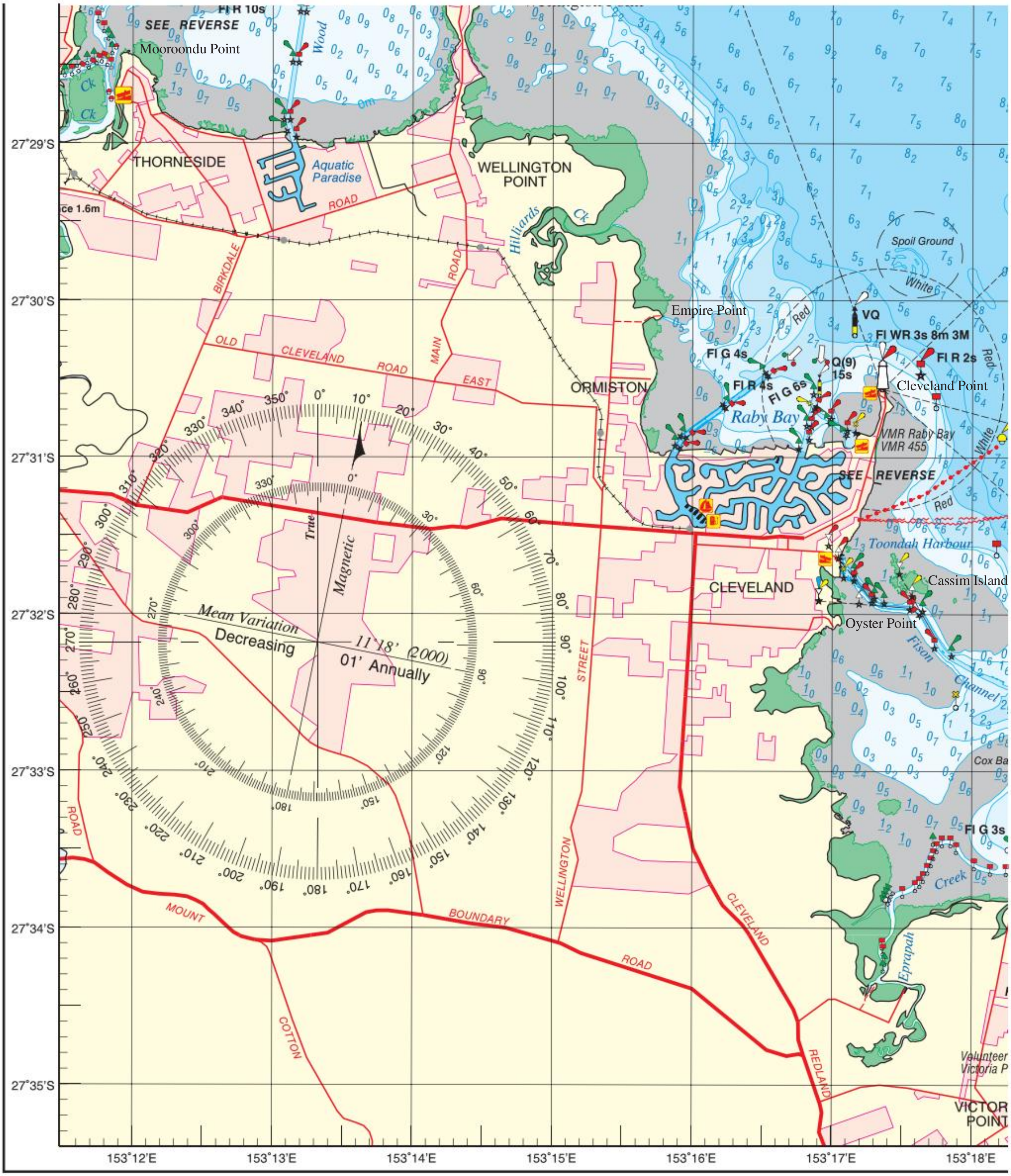
e On this trip, immediately after departure, rain reduces visibility considerably and a strong wind blows from the south-west. Can you suggest a possible hazard on this trip?

7 A large cruiser departs Norfolk Beach on Coochiemudlo Island and travels to Blaksley Anchorage, then to Dunwich. It then visits Horseshoe Bay on Peel Island before returning to Norfolk Beach. The vessel cruises at 8 knots. How much fuel is consumed if it uses 85 litres per hour at this speed?

8 Mary is fishing at a spot in Moreton Bay. She notes that the compass bearing of the light F1 G 2.5 s at South West Rocks on Peel Island is 340° and that of the centre of Coochiemudlo Island is 216°.

a Convert each bearing to a true bearing.

b Locate the position where Mary is fishing.



Section of Moreton Bay (reproduced from Queensland Transport (Maritime Division) Boating Safety Chart, Number MB8, published June 2000 — this reproduction not to be used for navigation).


Peel Island
Cucumber Point
Lazaret Gutter
Line of Leads 194° 46′
CAUTION: (Coral Reef)
Coral
The Spit
'The Corner'
Horseshoe Bay
'Platypus'
The Bluff
CAUTION: South West Rocks
(disused)
Douglas Light
Bird Island
CAUTION: Rocks
Goat Island
Polka Point
Hospital Point
One Mile
'Norman'
Nim Love Beacon
Myora Light
Volunteer Marine Rescue North Stradbroke Island VMR 449
DUNWICH
SEE REVERSE
Tower (conspic.)
Deanbilla Bay
Deanbilla
Wallen Wallen
Lake Kounpee
Sandy Island
Banana Bank
CAUTION: Submerged rocks
Coochiemudlo Island
Norfolk Beach
Main Beach
Point Halloran
Victoria Point
Potts Point (Coondooroopa)
rocks
Pelican Banks
Blaksley Anchorage
Canalpin
Fl Y 3s
Q (9) 15s
Dir Q Y
Dir Iso 2s
Fl G 4s
QG
Fl G 2.5s
Fl R 2.5s
Q(6) & L Fl 15s
QR
Fl Y 4s
Fl Y 2s
Fl R 3s
Fl G 3s
Fl R 4s
Fl R 6s
Fl Y 2.5s
Fl R 3s
Fl G 4s
Q
153°19′E
153°20′E
153°21′E
153°22′E
153°23′E
153°24′E
153°25′E

Air navigation

There are many similarities between air and sea navigation. Given that a pilot is certain of the starting position, and distance travelled is known, the position at any time can be easily determined.

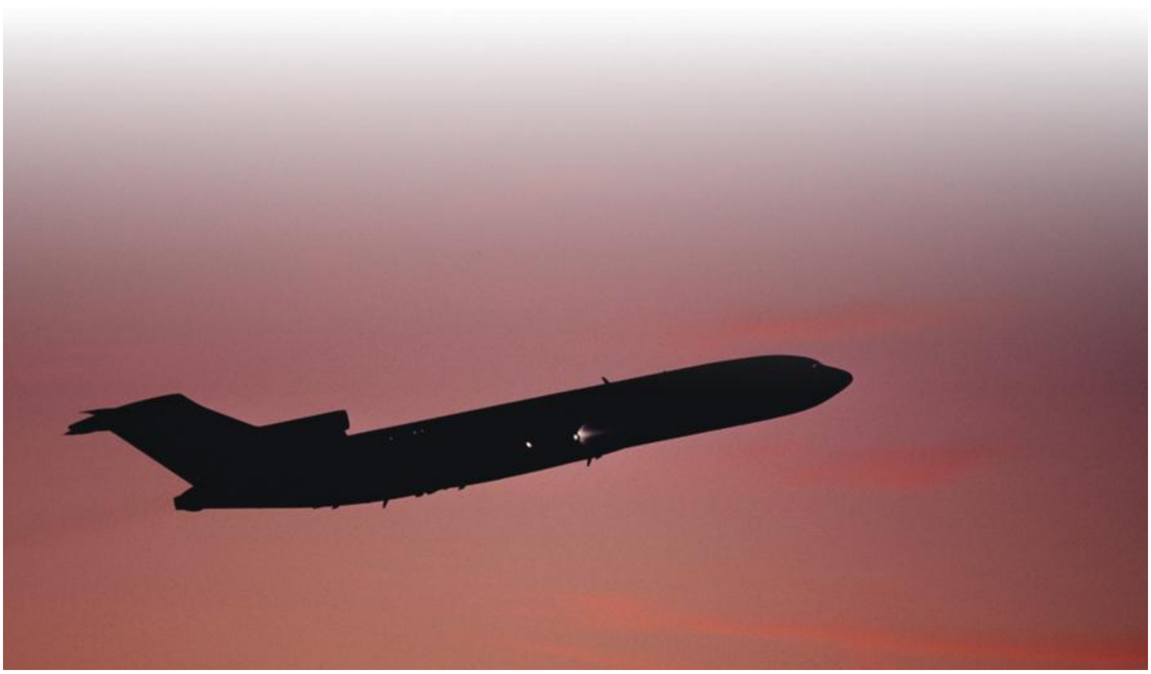
Latitude and longitude, as previously described, is the usual method a pilot uses to specify position when preparing a flight plan. In the air, however, the position of an aircraft could be specified by:

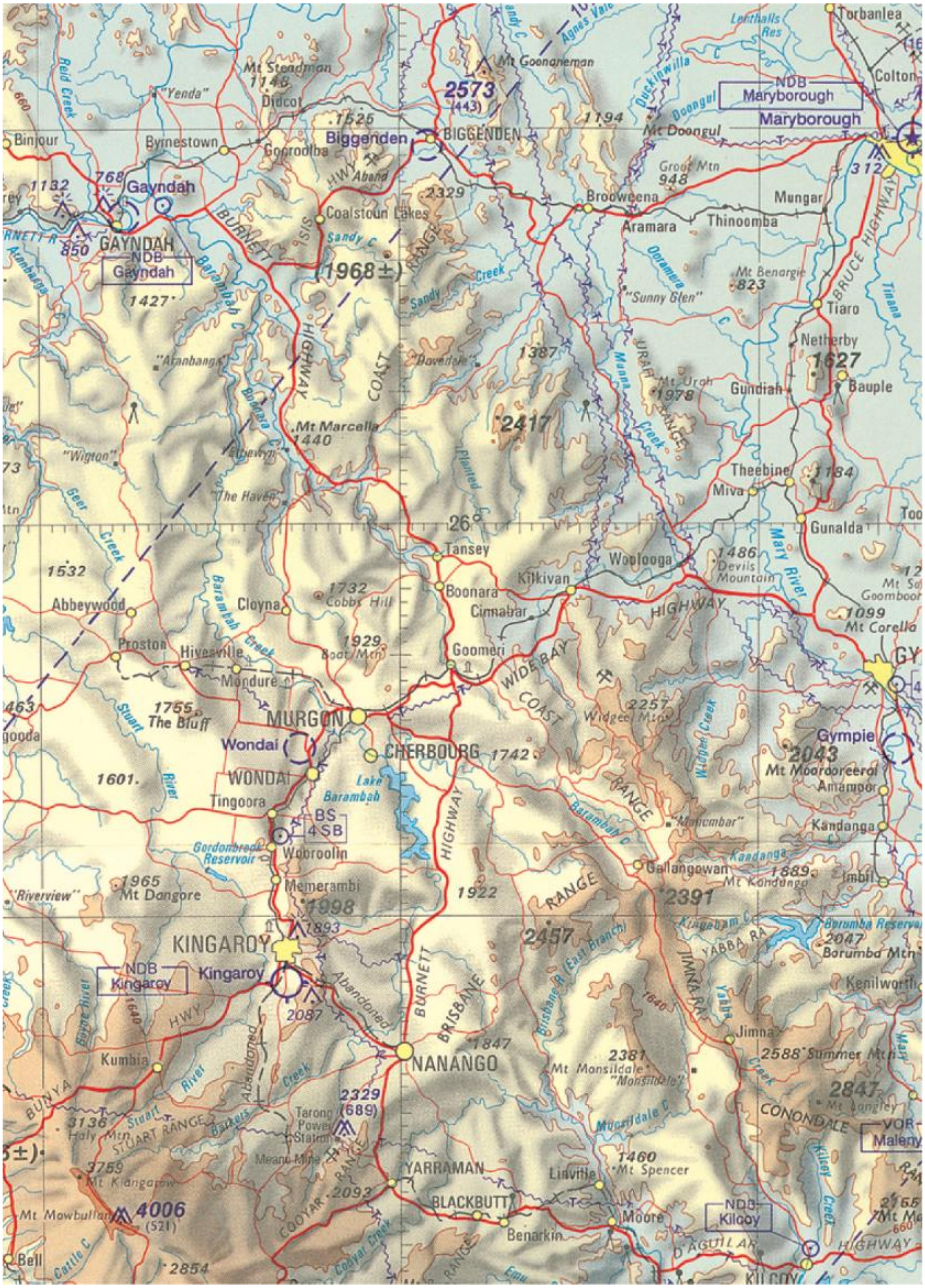
1. the description of a landmark or feature below (for example over the Gateway Bridge, or over the Goondoola Homestead)
2. by range (or distance) and bearing from a radio beacon or landmark (for example 10 nautical miles on a bearing of 310°T from St George).

Many maps drawn to varying scales are used by air navigators. The World Aeronautical Chart (WAC) Series are drawn to a 1 : 1 000 000 scale (see the map on page 307). These charts contain topographical data including rivers, lakes, mountains and coastlines; and cultural information such as towns, roads and railway lines. These maps are useful for visual en route navigation when the pilot is well away from busy aerodromes. (Aeronautical information concerning restricted airspace around major terminals is found in the Visual Terminal Chart (VTR) Series of scale 1 : 250 000.)

The map on page 307 shows a section of the Brisbane WAC. Note that shading is used to give a 3-dimensional relief effect, with the highest regions in the darkest shades. Spot elevations or heights above sea level are also given at many locations on the chart. Also, the highest point in each 30-minute by 30-minute square is marked in large bold type. For example Mt Kiangarow (3759 feet) is the highest point in the square containing Kingaroy, and Devil's Mountain (26°4′S, 152°26′S) is 1486 feet above sea level.

Isogonals (lines joining places of equal magnetic variation) are also shown on the WAC by dashed lines drawn across the maps. The map shows an isogonal with $10\frac{1}{2}°$ east variation.




25°30′
26°S
26°30′
151°30′
152°
152°30′
Mt Goonaneman
Mt Steadman
1148
Didcot
"Yenda"
Binjour
Byrnestown
Gooroolba
Biggenden
BIGGENDEN
2573
(443)
1525
1194
Mt Doongul
NDB
Maryborough
Maryborough
Torbanlea
Colton
768
Gayndah
GAYNDAH
NDB
Gayndah
1132
850
1427
Coalstoun Lakes
2329
Brooweena
Aramara
Thinoomba
Mungar
Groot Mtn
948
Mt Benargie
823
"Sunny Glen"
Tiaro
Netherby
1627
Bauple
Gundiah
(1968±)
1387
"Aranbanga"
"Wigton"
Mt Marcella
1440
2417
Mt Urah
1978
Theebine
1184
Miva
"The Haven"
26°
Tansey
Boonara
Kilkivan
Woolooga
Gunalda
1486
Devils
Mountain
1099
Mt Corella
1532
Abbeywood
Cloyna
1732
Cobbs Hill
Cinnabar
1929
Boat Mtn
Goomeri
Proston
Hivesville
Mondure
MURGON
1755
The Bluff
Wondai
WONDAI
CHERBOURG
1742
2257
Widgee Mtn
Gympie
2043
Mt Mooroocreerol
Amamoor
1601
Tingoora
Lake
Barambah
BS
4SB
Gordonbrook
Reservoir
Wooroolin
Memerambi
"Manumbar"
Kandanga
Gallangowan
1889
Mt Kandanga
Imbil
1965
Mt Dongore
"Riverview"
1998
1893
1922
2391
2457
KINGAROY
NDB
Kingaroy
Kingaroy
2087
Borumba Reservoir
2047
Borumba Mtn
Kenilworth
Jimna
2588 Summer Mtn
1847
NANANGO
2381
Mt Monsildale
"Monsildale"
2847
CONONDALE
VOR
Maleny
Kumbia
2329
(689)
Tarong
Power
Station
Meandu Mine
3136
Holy Mtn
STUART RANGE
3759
Mt Kiangarow
4006
(521)
Mt Mowbullan
2093
YARRAMAN
BLACKBUTT
Benarkin
Linville
1460
Mt Spencer
Moore
NDB
Kilcoy
KILCOY
D'AGUILAR
HIGHWAY
Bell
2854
BRUCE HIGHWAY
BURNETT HIGHWAY
WIDE BAY HIGHWAY
BUNYA HWY
COOYAR RANGE
CONONDALE RANGE
JIMNA RA
YABBA RA

EXERCISE 5M Air navigation

Use the map on page 307 to answer the following.

1 Give the latitude and longitude of:
 a Murgon b Nanango c Gympie d Kingaroy.

2 Name the town or feature at:
 a (26°3′S, 152°3′E) b (26°15′S, 151°43′E) c (26°07′S, 151°37′E).

3 Give the height of the highest point in the 30′ × 30′ square containing:
 a Murgon b Nanango c Gympie.

4 Aerodromes are marked if they have passenger facilities, or if they do not have passenger facilities but are frequently used.
 A pilot leaves Kingaroy airport. State the true and compass bearing expected if she is heading to:
 a Gayndah airport b Murgon airport.

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Digital doc:
WorkSHEET 5.2

5 The Flying Doctor departs Gayndah on a bearing of 160°T.
 a Convert this to a compass bearing.
 b Describe the features that would be observed by the pilot on this course.

investigation

Navigation

Consider the problems that could arise when attempting to navigate:
- a small plane by air
- across land
- at sea within sight of the coast
- at sea beyond sight of land.

Why is a GPS such an important aid to modern navigation?

summary

Earth geometry

- Any geometric plane passing through the centre of a sphere intersects the surface of the sphere to form circles known as *great circles*.
- The equator is a great circle.
- Great circles passing through the North and South Poles consist of two semicircles called *meridians* or *lines of longitude*.
- The line of longitude passing through Greenwich is the *prime meridian* (0° longitude).
- Lines of latitude are circles on the surface of the Earth parallel to the equator, assigned a number depending on the number of degrees north or south of the equator.
- To fix a position, state latitude then longitude (for example, 30°N, 50°W).
- 1 degree of movement along a meridian line = 111.2 km.
- 1 degree = 60 minutes.

Nautical mile and knot

- 1 nautical mile is the length of the arc of a great circle which subtends an angle of 1 minute (1′) at the centre of the Earth.
- 1 nautical mile = 1852 metres
- 1 knot = 1 nautical mile per hour
- $\text{Speed} = \dfrac{\text{distance}}{\text{time}}$

Compasses and bearings

- Magnetic compasses point to the magnetic north pole, the position of which varies slightly each year.
- The conversion from true bearing to compass bearing can be recalled by:
  (a) Variation east — compass least (subtract the magnetic variation from the true bearing to get the compass bearing).
  (b) Variation west — compass best (add the magnetic variation to the true bearing to get the compass bearing).
- The bearing of B from A is known as the *reverse bearing* of A from B. These bearings differ by 180°.

Determining position by navigation

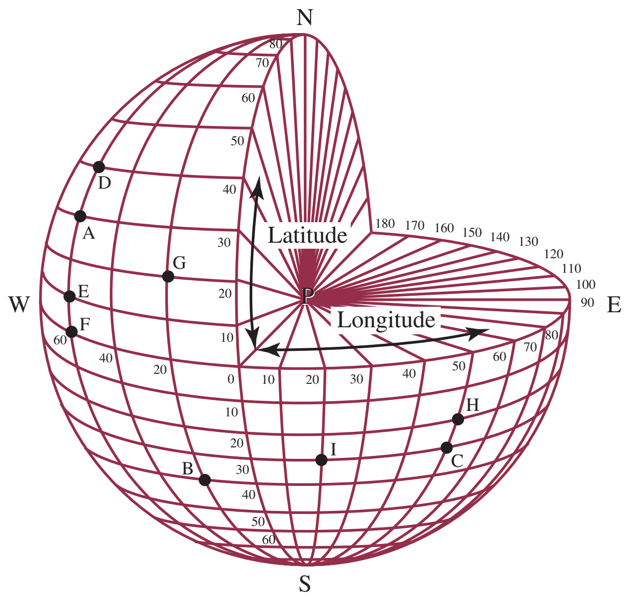
- The intersection of two or more position lines marked on a chart can be used to determine a vessel's position.
- The intersection of three position lines forms a small triangle known as a *cocked hat*.
- If an observer notes that two prominent shore objects are in line, then the observer must be on the line of sight connecting these two objects. This line is called the *transit line*.
- A position fix using transit lines is known as a *transit fix*.
- A *two-transit fix* uses the intersection of two transit lines to determine position.
- The front of the vessel is called the *bow*.

- The *angle on the bow* is the angle between the boat's course and the bearing of a prominent feature.
- The *doubling-of-the-angle-on-the-bow* method uses the properties of isosceles triangles — triangles which have one pair of sides equal and base angles equivalent.
- *Dead reckoning* or *deduced reckoning* (DR) is a method of estimating position. It does not involve a fix by sighting. Rather, it uses previously collected information about speed and course direction of the vessel.
- Lighthouses are either flashing (Fl) or occulting (OCC). A flashing light gives a number of short flashes of light followed by a long period of darkness. A lighthouse marked OCC refers to a light which gives long flashes of light followed by a short period of darkness.
- Tangent ratio of an angle $= \dfrac{\text{opposite side}}{\text{adjacent side}}$
- The angle of elevation of a lighthouse is the angle measured from the horizontal upwards to the light.

CHAPTER review

1 The diagram at right represents the Earth. 5A

a Give the position of A, B, C and D.
b Name 3 meridians.
c Name a point on the equator.
d If P is the centre of the Earth, give 4 radii.



2 Use the map of the Whitsunday Group on page 257 to give the position of: 5B

a Dolphin Point on Hayman Island
b the entrance to Nara Inlet.

3 Find the shortest distance in nautical miles from the North Pole to: 5C

a the equator
b the South Pole
c (20°N, 150°E)
d (42°S, 84°W).

4 Find the shortest distance in nautical miles from: 5C

a (2°N, 100°E) to (20°N, 100°E)
b (52°S, 170°W) to (37°N, 170°W).

5 Convert to minutes: 5C

a 6° **b** 18.5° **c** 28°15′ **d** 57°37.4′.

6 Find the distance from Woodlark Island (9°S, 153°E) to the equator in nautical miles and kilometres. 5C

7 Find the unknowns. 5C

Speed (knots)	Distance (n miles)	Time
a	18	3 hours
b	1453	4 days 13 hours
145	**c**	3.6 hours
140	**d**	1 hour 25 minutes
52	2600	**e** hours
96	24	**f** minutes

8 The *Red Devil* departs (27°22′S, 153°42′E) and heads north to (25°46′S, 153°42′E) averaging 15 knots. Find the time of the trip. 5C

5C

9 Sue departs her homestead in her ultralight, *Skyfox*, on a latitude of 27°52′S at 7.25 am, and flies due north to an airport on a latitude of 21°12′S. She refuels and has lunch at the airport, being grounded for a total of 1 hour and 20 minutes, before returning to the homestead. Her plane cruises at 80 knots.

a How far (in nautical miles) is the homestead from the airport?
b On what true bearing should Sue fly to return to the homestead?
c How long does it take Sue to fly from the homestead to the airport?
d What is her estimated time of return to the homestead?

5D

10 Find the unknown values in the table below.

True course	120°T	245°T	318°T	d
Variation	6°E	8°W	c	10°E
Compass course	a	b	324°C	196°C

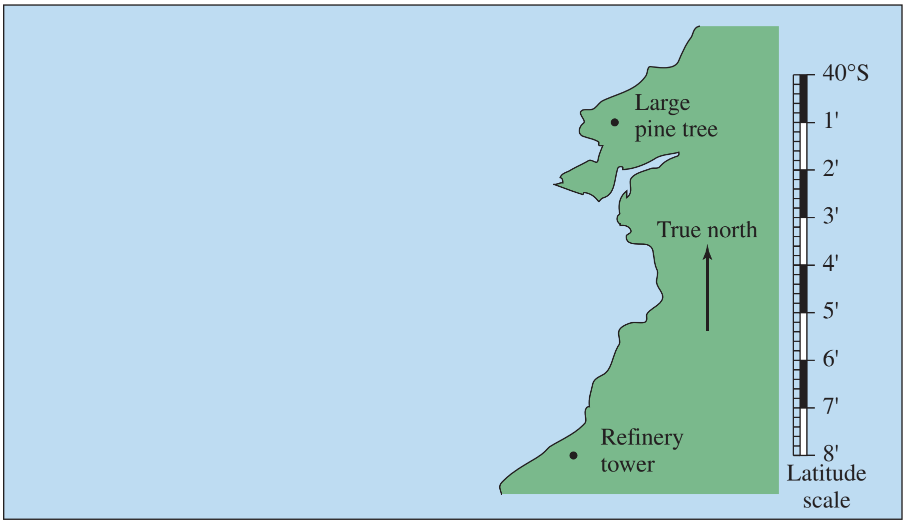
5E

11 The bearing of Skull Rock from Pirate Cove is 106°T. What is the bearing of Pirate Cove from Skull Rock? Include a sketch.

5E

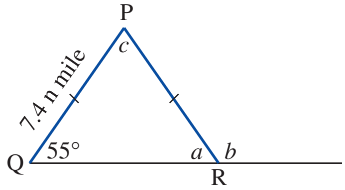
12 At 12.30 pm, the yacht *Shotover* records the bearings of a refinery tower (090°T) and a large pine tree (060°T). At 12.50 pm the bearings are tower (150°T) and pine tree (085°T) (see chart below).

a Use this information to fix the vessel's position at these times.
b How far has *Shotover* travelled in this time?
c Calculate the speed in knots.
d On what true bearing is *Shotover* travelling?

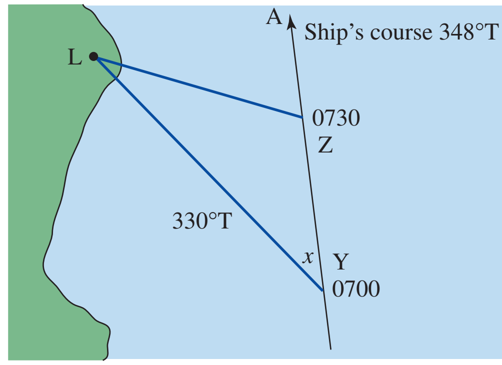


5J

13 In the diagram at right, find angles a, b, c and length PR.



14 The figure at right shows the path of a vessel on a course of 348°T. At 7.00 am, the vessel sights a large tower on a cliff on a bearing of 330°T.
At 7.30 am the angle on the bow has doubled. The vessel is travelling at 26 knots. Find: 5J

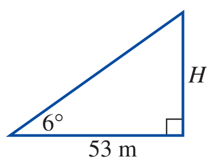


- **a** the angle on the bow at 7.00 am
- **b** ∠LZA
- **c** the distance travelled from 7.00 am to 7.30 am; that is, YZ
- **d** the distance from the vessel to the tower at 7.30 am.

15 At 0800 hours, the cruiser *Marlin King*, on a course 020°T, notes the bearing of Camel Rock to be 070°T. *Marlin King* is travelling at 18 knots. At 0840 hours the angle on the bow has doubled. 5J
- **a** Draw a neat diagram representing this information.
- **b** Calculate the angle on the bow at 0800 hours and 0840 hours.
- **c** How far has the *Marlin King* travelled between these two sightings of Camel Rock?
- **d** How far is the cruiser from Camel Rock at 0840 hours?

16 Find the unknowns in the triangles below. 5L

a



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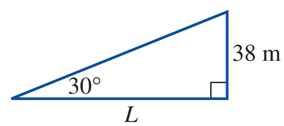
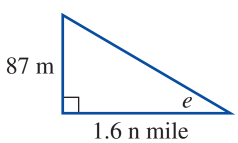
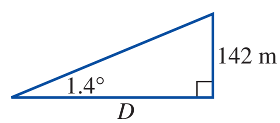


c



d



17 A ship sights a lighthouse marked on a chart as Fl (3) 16 s 130 m 16 M. 5L
- **a** Describe the flashing pattern expected.
- **b** How high is the lighthouse above sea level?
- **c** What is the range of the light?
- **d** A navigator observes from a porthole at sea level that the angle of elevation of this lighthouse is 3°. How far from the lighthouse is the ship?

18 Use the chart on page 291 to answer this question. A vessel is at point A at 7.00 am. 5K
- **a** How far is it from R?
- **b** On what true bearing should the vessel travel to arrive at R?
- **c** The vessel departs A at 7.00 am heading for R at 6 knots. Estimate its ETA at R.

19 In the chart on page 291 *Escapade* sails from R to Q at 18 knots, departing at 10.50 am. 5K
- **a** On what true bearing does it sail?
- **b** How far is R from Q?
- **c** Plot the position of *Escapade* at 11.20 am.
- **d** What is the ETA (estimated time of arrival) at point Q?

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Chapter 5

eBookplus ACTIVITIES

5A Review of Earth geometry

Digital doc

- SkillSHEET 5.1: Practise working with angle measures in degrees and minutes *(page 253)*

5C The nautical mile and the knot

Digital doc

- SkillSHEET 5.1: Practise working with angle measures in degrees and minutes *(page 262)*

5E Compass bearings and reverse bearings

Digital doc

- WorkSHEET 5.1: Calculate distances on great circles and write angles in degrees, minutes and seconds *(page 272)*

Tutorial

- WE13 Int-0472: Watch how to calculate compass and true bearings *(page 271)*

5I Doubling the angle on the bow

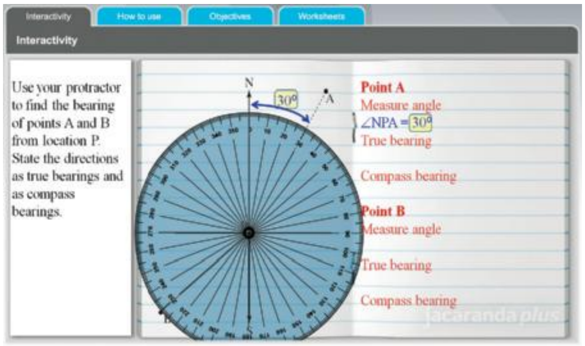
Digital doc

- SkillSHEET 5.2: Practise angle relationships in triangles and on straight lines *(page 288)*

5K The lighthouse and navigation

Digital doc

- SkillSHEET 5.3: Practise identifying sides of a right-angled triangle with respect to the given angle *(page 297)*



5M Air navigation

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- WorkSHEET 5.2: Calculate bearings and distances using trigonometry and speeds travelled *(page 308)*

Chapter review

Digital doc

- Test Yourself: Take the end-of-chapter test to test your progress *(page 313)*.

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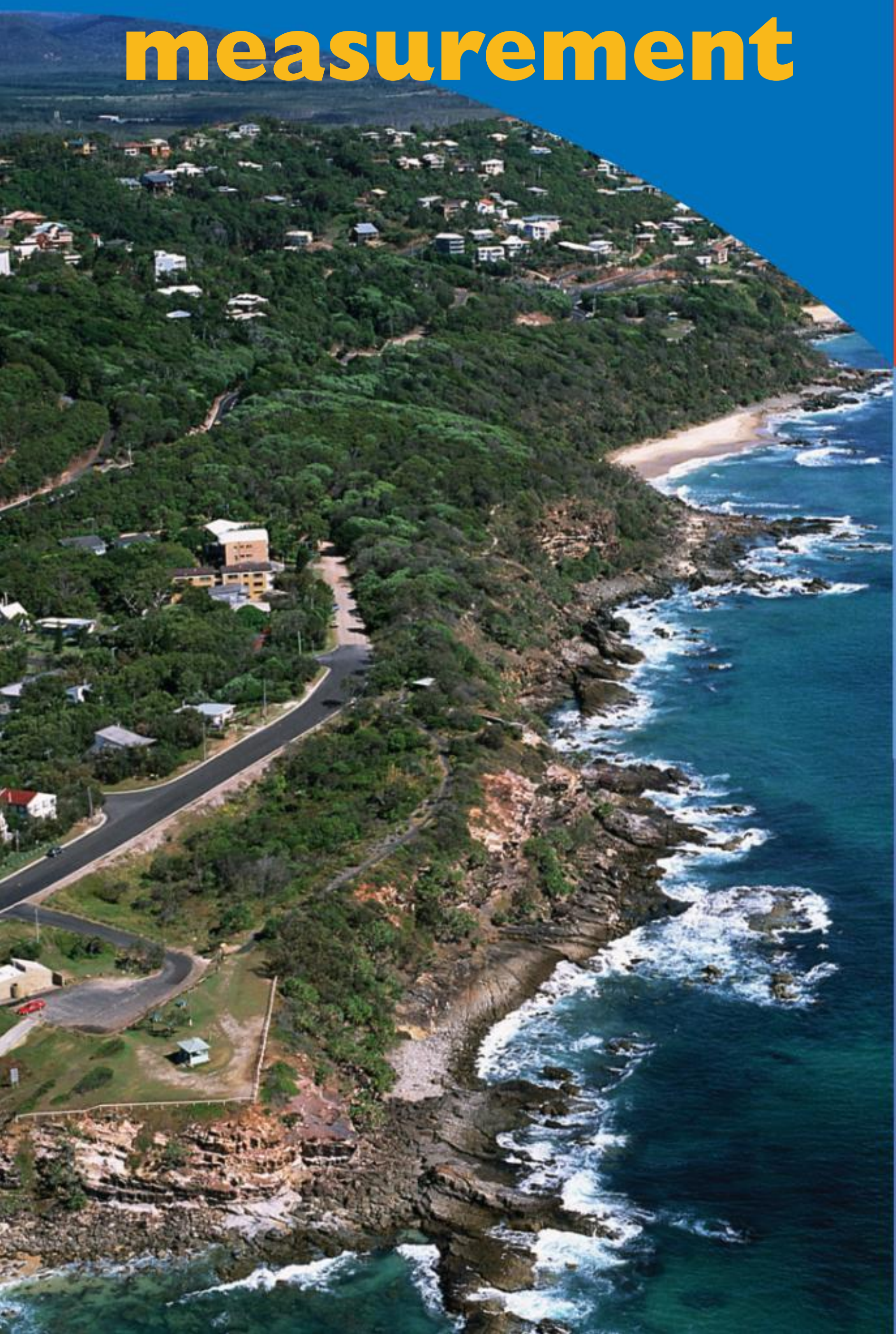
Land measurement

6



syllabus reference

Elective topic

Maps and compasses — land measurement

In this chapter

Introduction

Greg and Margaret have always wanted to live close to the city. They inspect a small, elevated, vacant block of land in an established area. It is between two quite old houses. Their ideal home plan is a spacious split-level dwelling that they expect will cover much of the block. They engage an architect, Sally, who meets Greg and Margaret on the block to discuss design features that would suit this allotment.

'Well, Sally, what do you think of this? Will we manage to fit our dream home on it?' Greg asks enthusiastically.

'Lovely breezes and views,' replies Sally as she approaches the old fence separating the block from the neighbours. 'Yes, what you want should fit in here just nicely . . . provided . . .'

'Provided what?' Greg looks startled.

'Well,' continues Sally, 'does this fence follow the exact boundary of your block? You'll need to know exactly what you're buying.'

'We hope so! How can we find out?' asks Margaret.

'Time for a surveyor!' replies Sally.

The above scenario is extremely common. Land boundaries must be clearly established. Areas of allotments must also be accurately known.

How can the area of an irregularly shaped lot of land be determined?

How can scale diagrams of large areas be drawn?

How do we interpret maps?

This unit illustrates a variety of simple methods for conducting surveys and land measurements that can be employed using readily available materials. It also includes many investigations that focus on the use of modern technology incorporating the operation of the theodolite, laser distance measuring devices, GPS, and perimeter and area calculation programs.

Scenarios like the one above are only part of a wide range of situations that require the measurement of boundaries and areas of land.

SKILLS CHECK

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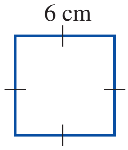
Spreadsheet
063 Perimeter and area

1 Give four metric units of length.

2 What name is given to the distance around the boundary of a shape?

3 Calculate the perimeter of the following shapes.

a


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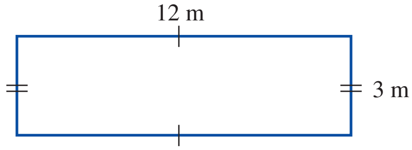

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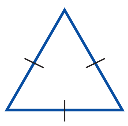

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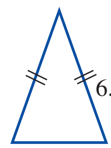

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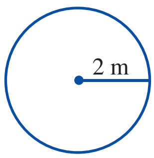

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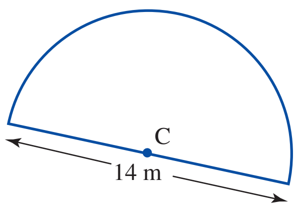
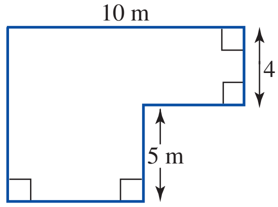


g


4 Give a formula for the area of:

a a square of side length l

b a rectangle of length l and width w

c a circle of radius r

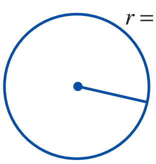
d a triangle of base b and perpendicular height h

e a parallelogram of base l and perpendicular height h.

5 Convert each of the following quantities to the units indicated.

a 15 mm → cm
b 18 cm → m
c 123m → cm
d 0.68 km → m
e 12.5 km → m

6 Calculate the area of the figures below.

a


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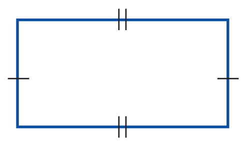

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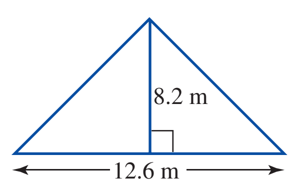

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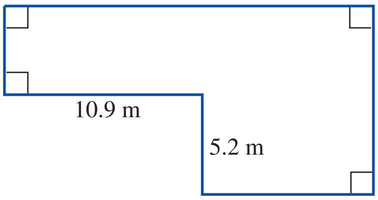

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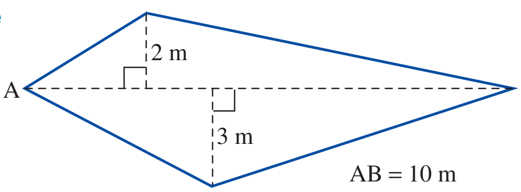

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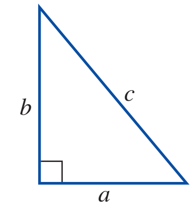
SkillSHEET 6.1
Pythagoras' theorem
Spreadsheet
054 Introducing the trig ratios

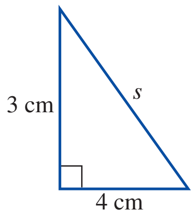
7 Evaluate, using your calculator:
a sin 70° **b** cos 15° **c** tan 18°.

8 State Pythagoras' theorem for the right-angled triangle shown at right.



9 Find the length of the remaining side in the triangles below.

a



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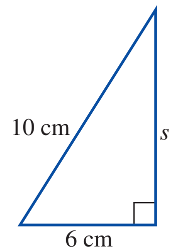

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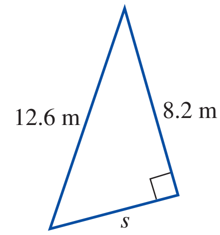
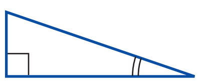


10 Copy the diagrams below and label the sides as opposite, adjacent or hypotenuse with respect to the marked angle.

a



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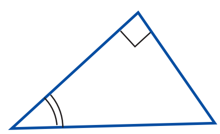

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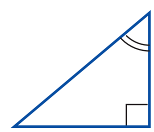
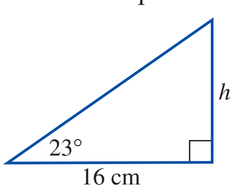


11 Define the sine, cosine and tangent ratios in terms of the hypotenuse, and the adjacent and opposite sides.

12 In each of the following, find the length of the side marked with a pronumeral, correct to one decimal place.

a



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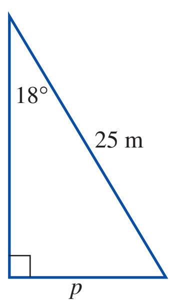

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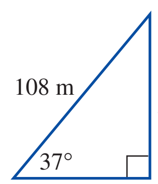


Perimeters and areas of triangles

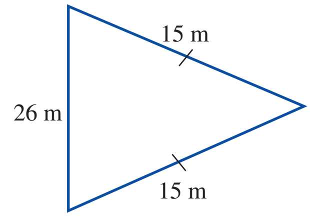
It is essential that surveyors accurately measure land area. The triangle is a figure regularly employed in surveying techniques and calculations. You are already familiar with determining the area of a triangle using its base and perpendicular height. Other methods will now be described.

Finding area using Pythagoras' theorem

The perpendicular height of a triangle can sometimes be determined using Pythagoras' theorem. This height can then be used to calculate the area of the triangle.

WORKED Example 1

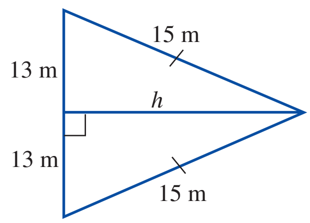
Find the area of the triangle shown.



THINK

WRITE

1. An altitude can be constructed which will bisect the base.



2. Pythagoras' theorem can be used to find the perpendicular height.

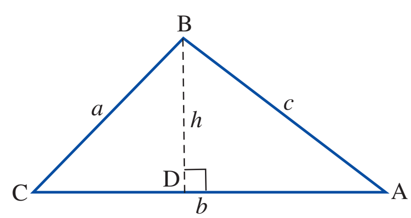
$15^2 = h^2 + 13^2$
$225 = h^2 + 169$
$h^2 = 225 - 169$
$h^2 = 56$
$h = \sqrt{56}$
$h = 7.48$ m

3. The area can now be determined using area = $\frac{1}{2}$ × base × perpendicular height.

Area = $\frac{1}{2}$ × base × perp. height
$= \frac{1}{2} \times 26 \times 7.48$
$= 97.3 \text{ m}^2$

Finding area using the sine ratio

The derivation of this rule is as follows:
The sides of the triangle opposite vertices A, B and C have been labelled a, b and c.



Area of $\triangle$ABC = $\frac{1}{2}$ base × perpendicular height
$= \frac{1}{2}\, b \times h$

In $\triangle$DBC, $\sin C = \frac{\text{opposite side}}{\text{hypotenuse}}$

where C is the angle at the vertex C.

$$\sin C = \frac{h}{a}$$

Therefore, $h = a \sin C$.

Substituting this value for h into the area formula above, we obtain:

$$\textbf{Area of } \triangle\textbf{ABC} = \frac{1}{2}\, a \times b \sin C$$

By constructing perpendiculars to the other two sides it is possible to show that:

$$\text{area } \triangle ABC = \tfrac{1}{2} bc \sin A$$

and

$$\text{area } \triangle ABC = \tfrac{1}{2} ac \sin B.$$

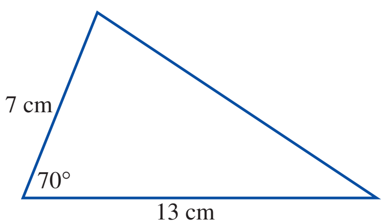
The rule can be recalled as: the area of any triangle is

$\frac{1}{2} \times$ side 1 $\times$ side 2 $\times$ sin (angle between sides 1 and 2).

WORKED Example 2

Find the area of the triangles given below.

a



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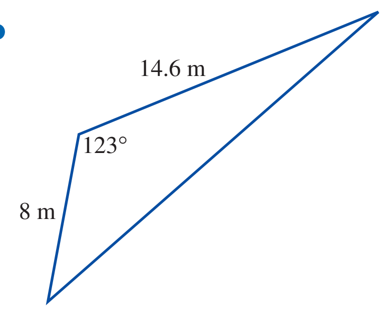


THINK

a 1 The area formula involving sine should be used as we have two sides and an included angle.

2 Substitute the values into the formula and solve.

b 1 The area formula involving sine should be used as we have two sides and an included angle.

2 Substitute values and solve.

WRITE

a Area $= \frac{1}{2} ab \sin C$

$= \frac{1}{2} \times 7 \text{ cm} \times 13 \text{ cm} \times \sin 70°$
$= 42.8 \text{ cm}^2$

b Area $= \frac{1}{2} ab \sin C$

$= \frac{1}{2} \times 8 \text{ m} \times 14.6 \text{ m} \times \sin 123°$
$= 49.0 \text{ m}^2$

Finding area using Heron's formula

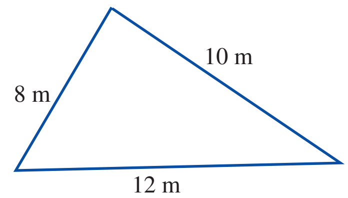
Heron's formula (sometimes also referred to as Hero's formula) is represented by the equation:

$$\text{Area} = \sqrt{S(S-a)(S-b)(S-c)}$$

where a, b and c are the lengths of the 3 sides of a triangle and S is the semi-perimeter and is calculated by $S = \frac{1}{2}(a + b + c)$.

WORKED Example 3

Find the area of the triangle with sides 8, 10 and 12 m.



THINK

1. Because three sides are given, use Heron's formula. The semi-perimeter is to be found first.
2. The values of S, a, b and c are substituted into Heron's formula.

WRITE

$S = \frac{1}{2}(8 + 10 + 12)$

$= \frac{1}{2} \times 30$ m

$= 15$ m

$\text{Area} = \sqrt{S(S-a)(S-b)(S-c)}$

$= \sqrt{15(15-8) \times (15-10) \times (15-12)}$

$= \sqrt{15 \times 7 \times 5 \times 3}$

$= \sqrt{1575}$

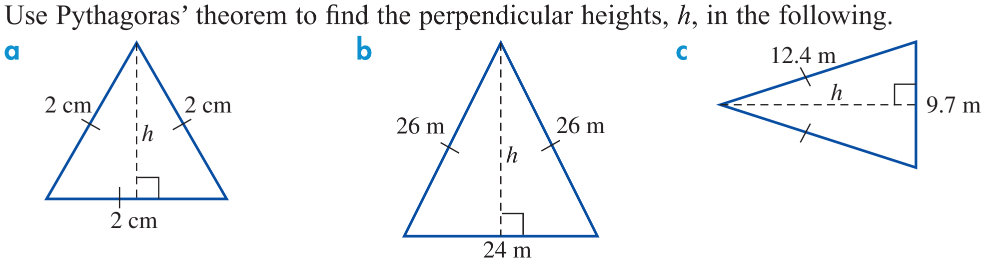
$= 39.7 \text{ m}^2$

remember

1. Area of a triangle $= \frac{1}{2} ab \sin C$
   where a, b are sides and $\angle C$ is the included angle.
2. Pythagoras' theorem relates the length of the hypotenuse, c, and the remaining sides, a and b, in a right-angled triangle:
   $$a^2 + b^2 = c^2$$
3. The area of a triangle of sides a, b and c can be found by Heron's formula:
   $$\text{Area} = \sqrt{S(S-a)(S-b)(S-c)}$$
   where S is the semi-perimeter. $S = \frac{1}{2}(a + b + c)$.

EXERCISE 6A Perimeters and areas of triangles

1 Use Pythagoras' theorem to find the perpendicular heights, h, in the following.

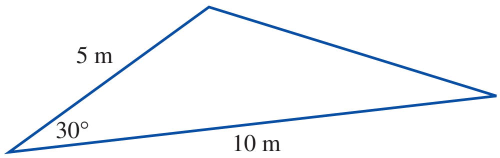


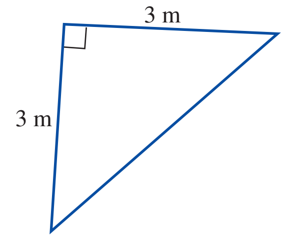
WORKED Example 1

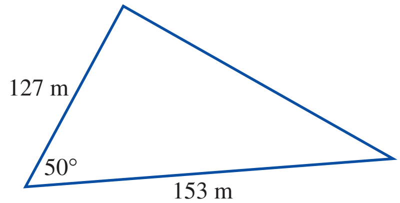
2 Find the areas of the triangles shown in question 1.

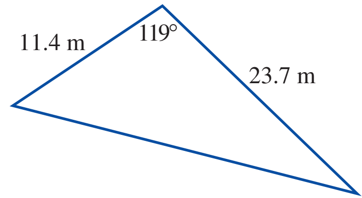
WORKED Example 2

3 Find the area of the following triangles.

a


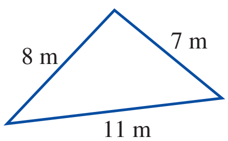
b


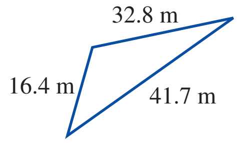
c


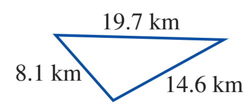
d


WORKED Example 3

4 Find the perimeter and semi-perimeter of the triangles below.

a


b


c


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SkillSHEET 6.2
Heron's formula and the sine rule

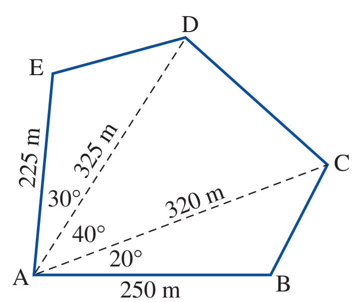
5 Use Heron's formula to find the area of the triangles in question 4.

Perimeters and areas of polygons

Surveyors are often engaged to determine the area of a portion of land that may have an irregular shape. Any **polygon** can be divided into a number of triangles. The area of the polygon is the sum of the areas of the individual triangles.

WORKED Example 4

Find the area of the field shown to the nearest square metre. (This sketch is not drawn to scale.)



THINK

1. The area of the polygon can be divided into three triangles.
2. Find the area of each of the three triangles.

WRITE

Area of ABCDE = area ABC + area ACD + area ADE

$$\text{Area ABC} = \tfrac{1}{2} \times 250 \text{ m} \times 320 \text{ m} \times \sin 20° = 13\,681 \text{ m}^2$$

$$\text{Area ACD} = \tfrac{1}{2} \times 320 \text{ m} \times 325 \text{ m} \times \sin 40° = 33\,425 \text{ m}^2$$

THINK | **WRITE**

Area ADE $= \frac{1}{2} \times 325 \text{ m} \times 225 \text{ m} \times \sin 30°$
$= 18\,281 \text{ m}^2$

3 Calculate the total area.

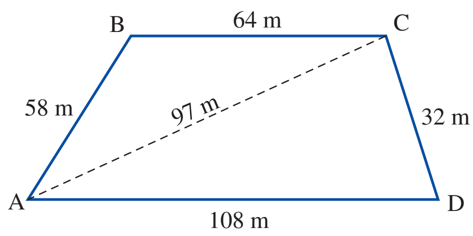
Area ABCDE $= 13\,681 + 33\,425 + 18\,281 \text{ m}^2$
$= 65\,387 \text{ m}^2$

Note: Realistically, the accuracy of the third figure is not certain, and $65\,400 \text{ m}^2$ would be as close as we could justify in this example (in fact, the answer lies somewhere between 65 200 and $65\,600 \text{ m}^2$). However, to illustrate the process in worked examples, all digits have been calculated. You should always be cautious about the accuracy of later digits in such calculations.

The area of land can also be expressed in hectares. Recall that 1 ha $= 10\,000 \text{ m}^2$. So the area of the field in Worked example 4 is 6.5387 ha or 7 ha to the nearest hectare.

WORKED Example 5

Find the area of the paddock shown to the nearest square metre. This sketch is not drawn to scale.



THINK | **WRITE**

1 The figure can be broken into two triangles, △ABC and △ACD.

2 Calculate the area of △ABC using Heron's formula.

For △ABC

$S = \frac{1}{2}(64 + 58 + 97)$
$= 109.5$

Area $= \sqrt{S(S-a)(S-b)(S-c)}$
$= \sqrt{109.5 \times (109.5 - 58) \times (109.5 - 64) \times (109.5 - 97)}$
$= \sqrt{109.5 \times 51.5 \times 45.5 \times 12.5}$
$= 1791 \text{ m}^2$

3 Calculate the area of △ACD.

For △ACD

$S = \frac{1}{2}(97 + 32 + 108)$
$= 118.5$

Area $= \sqrt{118.5 \times 21.5 \times 86.5 \times 10.5}$
$= 1521 \text{ m}^2$

4 Find the total area by adding the two areas.

Area of ABCD = area ABC + area ACD
$= 1791 + 1521 \text{ m}^2$
$= 3312 \text{ m}^2$

remember

1. The area of a triangle can be determined using the formula:

$$\text{Area of a triangle} = \tfrac{1}{2}\, ab \sin C$$

where a and b are two sides of a triangle and C is the angle between those sides.

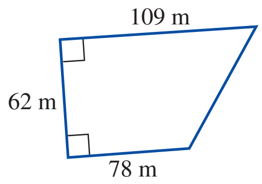
2. The area of polygons can be found by dividing the figure into a number of triangles and then finding the area of each triangle.

EXERCISE 6B Perimeters and areas of polygons

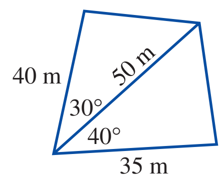
WORKED Example 4

1 Find the area of the regions shown, to the nearest square metre. Diagrams are not to scale.

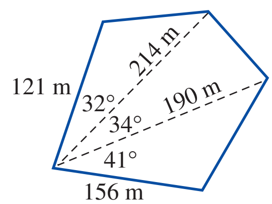
a



b



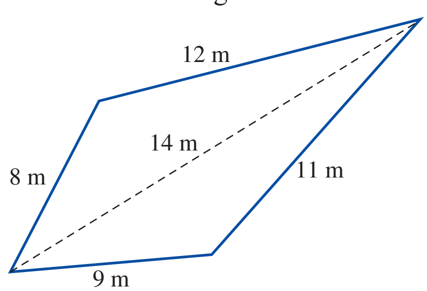
c



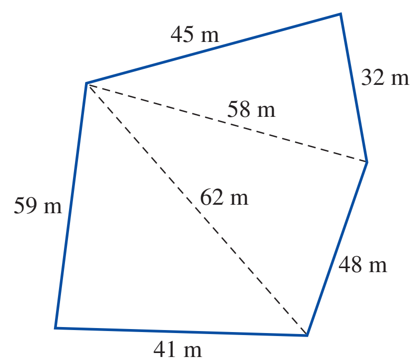
WORKED Example 5

2 Find the area of the figures shown.

a



b



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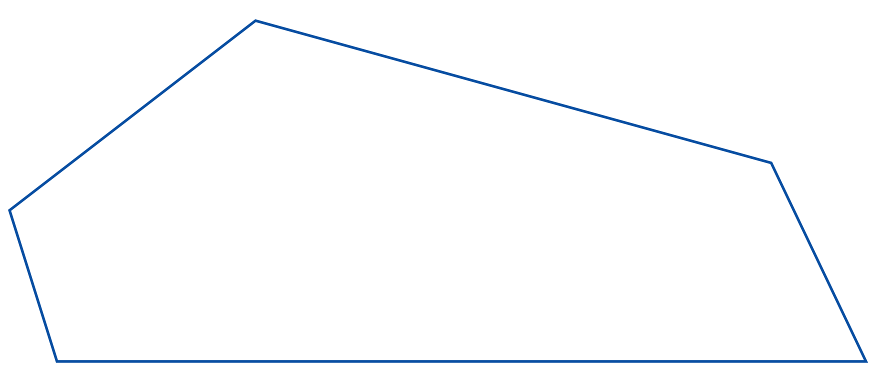
SkillSHEET 6.2 Trigonometric values and angles
Spreadsheet 063 Perimeter and area

3 Find the area of the paddock shown in the figure below, to the nearest hectare. The scale used is 1 : 10 000.



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Digital docs:
Spreadsheets
047 Map scales 1
048 Map scales 2

4 Draw scale diagrams of the regions shown in the figures in question **1** using a 1 : 1000 scale for **a** and **b**, and a 1 : 5000 scale for **c**.

Finding perimeter and area using pace length

Determining pace length

Materials required: tape measure or metre rule.

Mark out an accurately measured distance of 100 m. Record the number of normal walking paces required to cover this distance. Repeat this process.

Find the average number of paces required to cover 100 metres. This figure can now be used to estimate lengths. Each member of the group should record his or her number of paces required to cover 100 m. Now calculate the number of strides per 10 metres.

Estimating distance and area using pacing

Materials required: 5 marker domes marked 1 to 5, drawing compass.

This activity is best done on a large quadrangle or oval. Your teacher will set out 5 marker domes defining the perimeter of an irregular polygon.

Your tasks are:

1. Draw a neat scale diagram showing the location of the domes numbered 1 to 5. You are to use pacing to estimate distances, and you are to collect sufficient measurements to ensure that the position of the domes can be accurately located on your diagram. A drawing compass may assist in the sketching.
2. Determine the perimeter of the polygon defined by the 5 domes.
3. Calculate the area of the region enclosed by the perimeter of this polygon.

Finding perimeter and area using your computer

Visit the website of your local council. They allow access to information, maps and plans concerning the block of land where you live and your school grounds. Use this site to find the plan showing the boundaries of your school and/or home site. Now use a measuring tool such as the Measure facility in Red-e-Map (use the **Redland Shire Council** weblink in your eBookPLUS to access this facility) to determine the perimeter of your school and/or home site. You can also use this facility to accurately determine the area of your school grounds, home site or any part thereof.

Measuring a perimeter made simple

Many modern hand-held laser meters that are now available are quite inexpensive, and can be useful for simple and rapid distance measurement. These devices allow areas to be determined using the push of a button, and are now an invaluable aid to many trades people, as time involved in estimation and quoting is significantly

(*Continued*)

reduced. Obtain a laser distance meter and use it to determine the perimeter and area of your classroom and other areas around your school. These devices can be used to check the accuracy of your pace length in the previous investigation. A laser distance meter can also be employed to verify the accuracy of hand-held GPS devices. However, measurements made with hand-held GPS instruments are less accurate than those made with a laser distance meter.

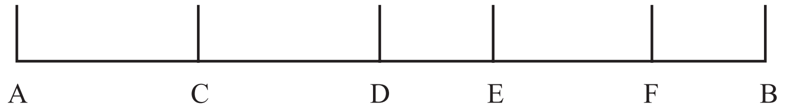
Surveying on level ground without obstacles

It is essential that the size and shape of the Earth's surface be accurately recorded. The task of the surveyor is to link the shapes and sizes of parts of our real three-dimensional world (with all its hills, gullies, ridges, boulders and trees) to the two-dimensional survey map drawn to scale. Initially, we shall investigate working on level ground without obstacles; later, we shall examine more difficult surveys, where physical obstacles get in the way.

A surveyor is interested in measuring distance between objects in a straight line. This is done by first setting out a straight line known as a **survey line**, which requires the use of a number of poles called **staffs**.

Julie is a surveyor who needs to determine the distance from A to B (which is too large to measure directly with a tape) over level ground.

How does she do it? She and an assistant set up staffs at A and B. The line AB is the survey line.



Julie moves directly behind staff A. Looking back to staff B, she has the assistant place the staffs C, D, E and F between A and B, ensuring they are all in a straight line. The smaller distances (AC, CD, DE etc.) can then be measured.

The example below shows how a surveyor would record the results of a survey of flat land in a field book. The locations of some features of the landscape are to be included.

A field is to be surveyed, and a field sketch is made, showing major features at 5 locations (see H, I, J, K and L in figure **a**). The area is first studied to select a suitable survey line. The line AB is selected in the diagram below because there are no obstructions along it.

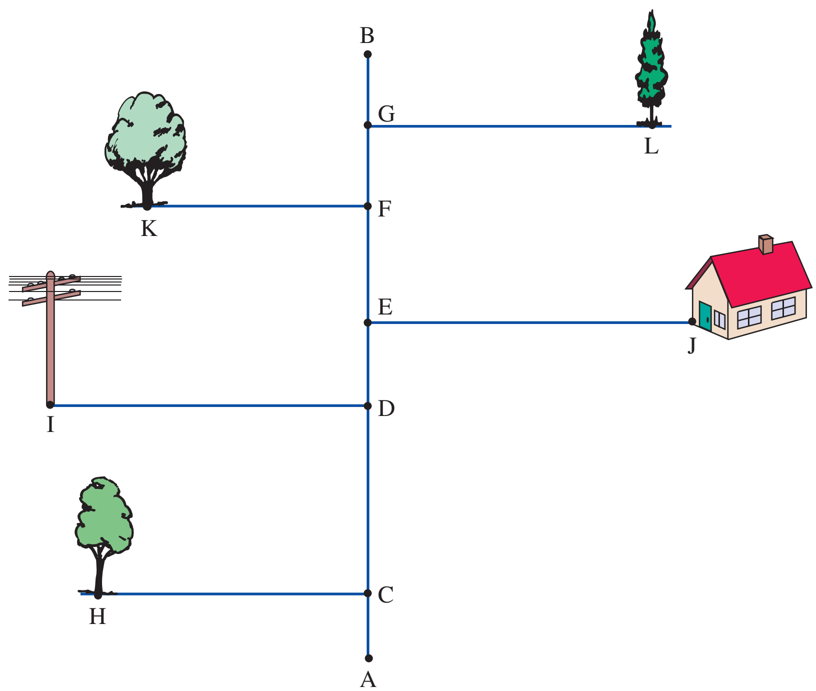


Figure a

Intermediate staffs are then placed at C, D, E, F and G opposite the features at H, I, J, K and L respectively. The staffs at C to G are placed so that CH, ID, EJ, KF and GL are at right angles to AB. These distances (CH to GL) are known as **offsets**.

Measurements are then recorded in a field book so that a scale diagram of this area can later be drawn.

Figure **b** shows the field book entry, as a surveyor would make it. The survey line AB is shown as a double line ruled vertically in the middle of the page.

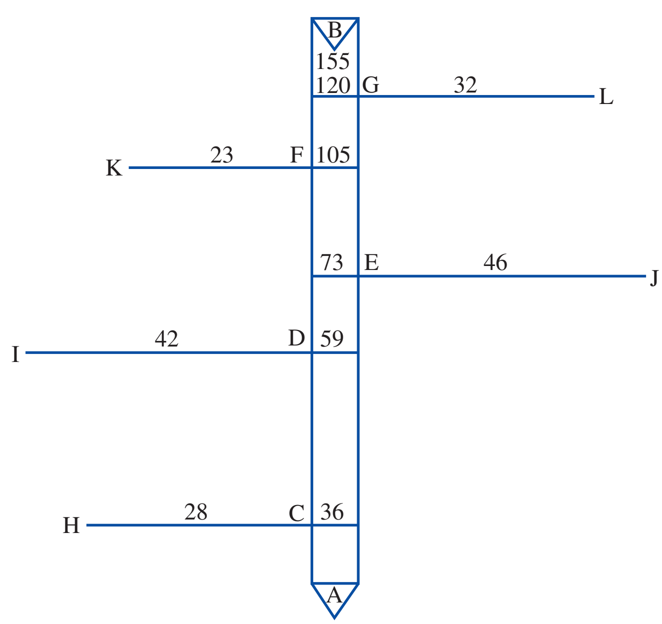


Figure b

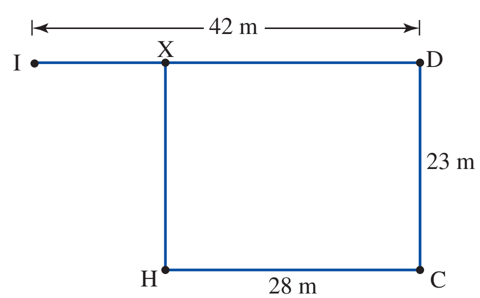
The length of the survey line is indicated at B (155 m). The lengths of all offset measurements are shown (for example CH = 28 m) and the position of intermediate staffs is given (for example D is 59 m from A). The field book entries are not drawn to scale.

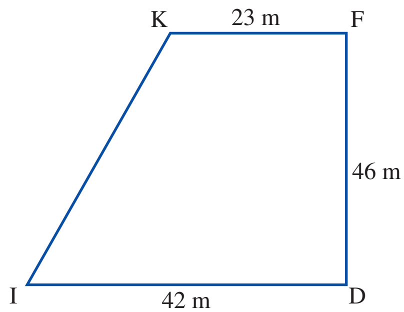
This field book entry can then be used to draw a scale diagram of the area surveyed. This scale diagram allows the surveyor to find distances between points that had not been directly measured in the field. However, some distances, angles and areas can be determined from the field book entries by calculation, as shown in the example below.

WORKED Example 6

Use the field book entry shown in figure b on page 327 to find:

a ID **b** GA **c** EJ
d CD **e** HI **f** the magnitude of ∠AHC
g the area of △JEA **h** the area of KFDI.

THINK	WRITE
a–c The field book entry gives the distances of ID, GA and EJ directly.	**a** ID = 42 m **b** GA = 120 m **c** EJ = 46 m
d The point C is 36 m from A and point D is 59 m from A, so CD = AD − CA.	**d** CD = AD − CA = 59 − 36 = 23 m
e 1 The points C, D, H and I are redrawn. 2 A line HX, perpendicular to ID, is constructed. 3 HX then must equal 23 m.	**e** 
4 HI can be determined using Pythagoras' theorem.	$HI^2 = IX^2 + HX^2$ $= 14^2 + 23^2$ $HI^2 = 725$ $HI \approx 27$ m
f 1 In △AHC, AC is the opposite side and HC the adjacent side for ∠AHC. Tangent ratio can now be used to find ∠AHC. 2 The inverse function on the calculator is now used.	**f** $\tan \angle AHC = \frac{\text{opposite}}{\text{adjacent}}$ $= \frac{36}{28}$ $= 1.286$ $\angle AHC = 52°$

THINK	WRITE
g The area of △JEA can be calculated using (base × perpendicular height) ÷ 2.	**g** Area $= \frac{1}{2}$ base × perpendicular height $= \frac{1}{2} \times 73 \times 46$ $= 1679 \text{ m}^2$
h 1 A diagram of KFDI is drawn with all dimensions shown.	**h** 
2 FD can be calculated as AF – AD.	FD = AF – AD = 105 – 59 = 46 m
3 The figure has one pair of parallel sides and hence it is a trapezium. The area of a trapezium rule can be used.	Area $= \frac{\text{sum of parallel sides}}{2} \times$ perpendicular height $= \frac{23 + 42}{2} \times 46$ $= 1495 \text{ m}^2$

remember

1. To produce an accurate scale diagram of an area, a survey line is chosen.
2. Perpendicular distances from a survey line to surrounding features are known as offsets.

EXERCISE 6C Surveying on level ground without obstacles

WORKED Example 6a–d

1 Use the field book entry (figure **b**) on page 327 to find the distance of:
a LG **b** HC **c** DA **d** AE **e** GE.

WORKED Example 6e

2 Use the field book entry (figure **b**) on page 327 and Pythagoras' theorem to find:
a AJ **b** AK **c** BL **d** LJ.

3 The sketch, figure **a**, on the next page (not to scale) shows a survey line AB and a number of features in an area being surveyed. Intermediate staffs at C to G are placed at locations where features are seen at right angles to the survey line. The distance AB is 180 m, AC = 40 m, AD = 80 m, AE = 95 m, AF = 110 m, AG = 150 m.

a Sketch the entry that a surveyor would make in a field book. Your sketch should resemble the field entry shown in figure **b** on page 327 and it does not need to be drawn to scale.

b Draw a map of this area using a 1 : 1000 scale. Label all features.

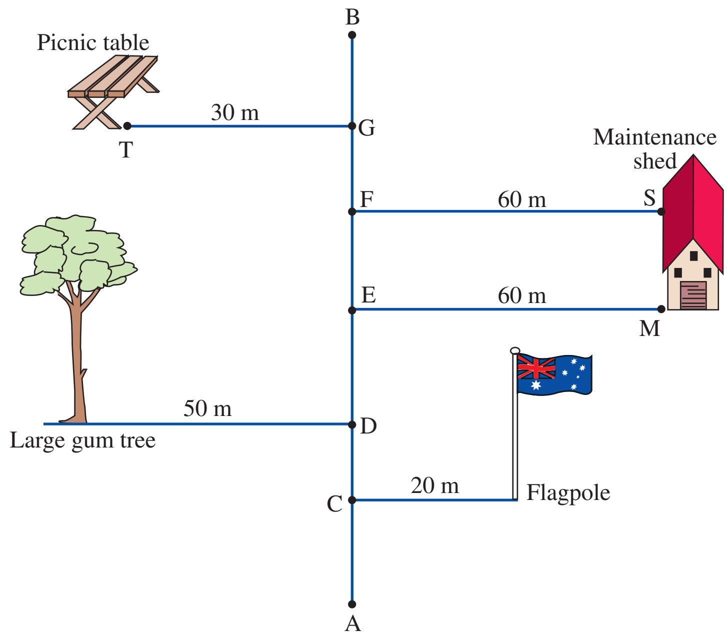


Figure a

4 A field book entry is shown at right.

a How long is the survey line?

b How many intermediate staffs were placed between A and B?

c What is the distance between:

- **i** D and A?
- **ii** D and F?
- **iii** E and G?
- **iv** Z and C?
- **v** W and F?

d Find the distance from:

- **i** V to A
- **ii** B to X
- **iii** Z to Y
- **iv** Y to W.

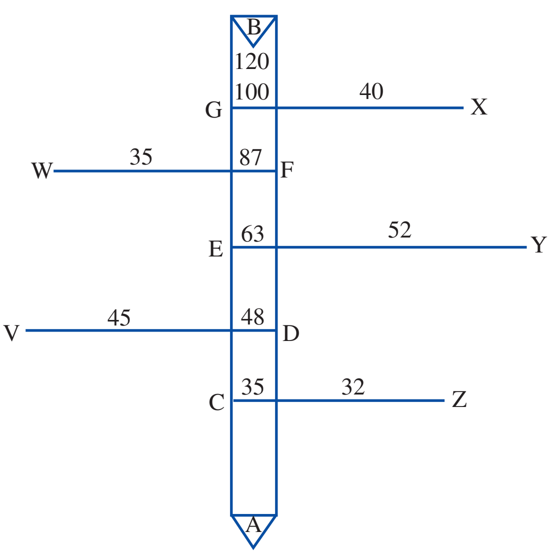


Figure b

WORKED Example 6f, g

e **i** What is the magnitude of ∠VAB?

ii Find the area of △DYF.

iii An observer is at feature Y, looking at V. Find the angle of intersection of the line of sight with survey line AB.

f Draw a 1 : 1000 sketch representing this information.

g Briefly describe the steps that would have occurred to produce this field map.

WORKED Example 6h

h Find the area of XZCG.

Drawing a field map by survey

A school quadrangle is ideal for this exercise. Groups of 3 to 4 students are required. Equipment needed: 10-m tape measures; several poles (metre rules can be used).

Select a relatively level piece of ground that has a number of large features on it; for example buildings, trees, flagpole. Choose a survey line AB so that there are no obstacles in the path of this line. An observer at one end is to direct assistants to move their staffs so they are directly on the line AB. The assistants are to position themselves so that they are perpendicular to features that the group wishes to include on the field map. (An estimate of 90° is sufficient for this activity although you may wish to employ a little help from Pythagoras.)

Record all information gathered on a field map using the method shown above. The length of the survey line and the distance from the survey line to features are to be measured. Use this field map to then draw a scale diagram of the area using a suitable scale. Determine the area of the figure defined by joining the ends of the survey line and the feature points. You can check the accuracy of your work using a laser distance meter described in an earlier investigation on pages 325–6.

Mapping using the GPS

This investigation involves the collection of data outdoors, using a hand-held GPS and then plotting points on graph paper back in the classroom. Prior knowledge needed for this investigation includes:

- plotting a position point using latitude and longitude (see Chapter 5 page 251)
- an understanding of the term **minute**. One minute = $1' = \frac{1}{60^\circ}$.

Choose a large outdoor area bounded by a number of significant features. Alternatively, you can work on an oval or large paddock with marker domes.

Use the GPS to store the position of each feature bounding the area.

Now use the GPS to determine the distance between any two features and record this figure.

You should notice that there is a small amount of error in the device. You can change your position slightly, and the latitude and longitude reading on the screen will not change.

A far more accurate measure of distance between any two features can be determined using a laser distance meter described in the investigation on pages 325–6, or the electronic theodolite (page 352).

You will notice that the GPS describes latitude and longitude to the nearest 0.001′.

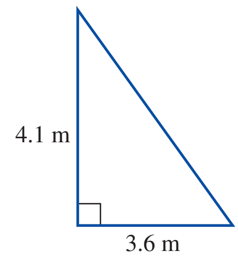
Back in the classroom, plot the position of each point on graph paper. Use the horizontal axis for longitude (°East) and the vertical axis for (°South). Use a scale that will include all points, well separated across your paper. You have now produced a simple map of the area including your features. Label the distance between the two features you recorded. You can now determine a scale for your map if you wish.

You can also plot these points on a graphics calculator.

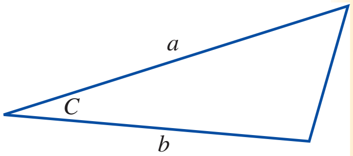
It is also possible for students to produce high quality maps using OCAD software. An investigation on mapping is included in the Orienteering section on page 372.

10 QUICK QUESTIONS 1

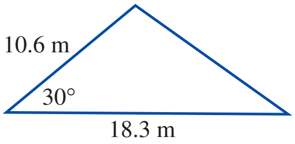
1 Find the length of the hypotenuse in this figure.

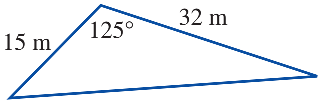


2 State the formula for the area of the triangle shown, in terms of sides *a* and *b*, and the angle *C*.



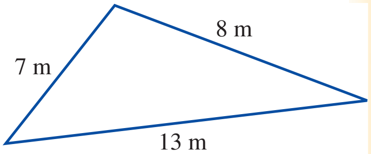
3 Find the area of each of the triangles below.



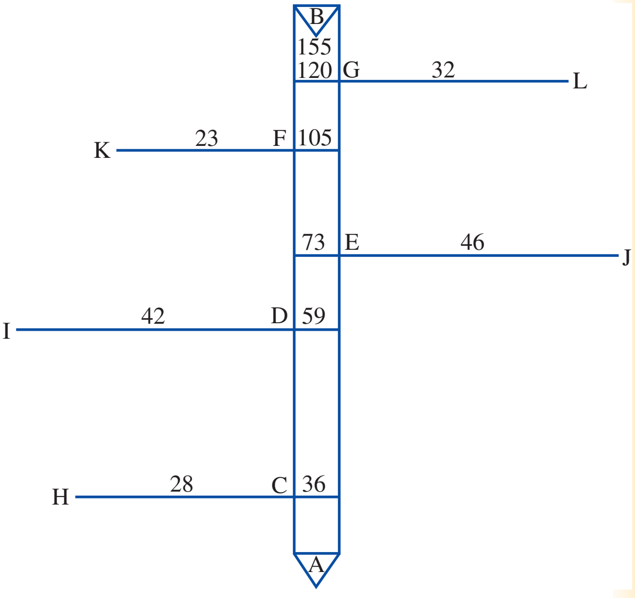


4 State Heron's formula for the area of a triangle of sides *a*, *b* and *c*.

5 Find the area of this figure, using Heron's formula.



Use the sketch at right, which shows a field book entry, to answer questions 6 to 10. All entries are in metres.



6 Which line is the survey line?

7 GL is an offset. Name four other offsets.

8 Give the lengths of AC, AD, AE, EF and GC.

9 Give the area of △GFL.

10 Calculate the magnitude of:

a ∠ECJ

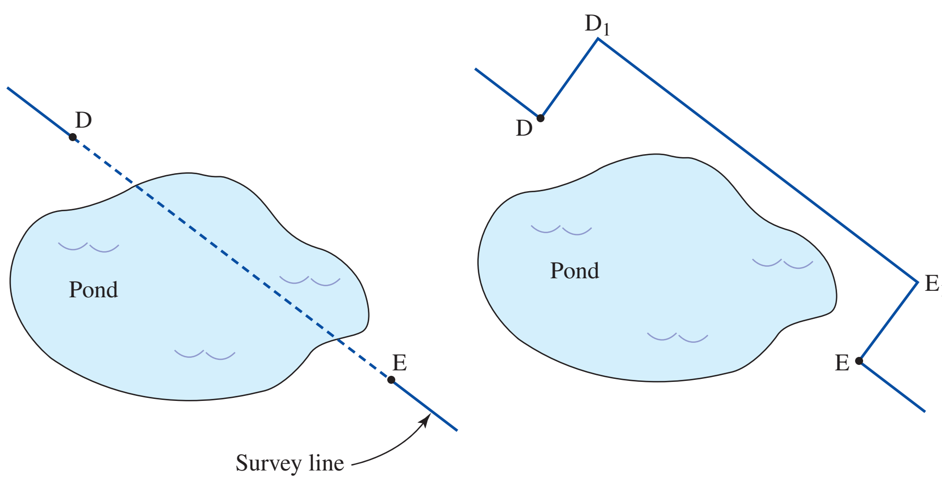
b CJ.

Surveying around obstacles

Quite often, obstacles such as rivers, hills or trees will prevent a surveyor from having uninterrupted survey lines. Two methods to overcome such obstacles are the **offset method** and **triangulation**.

Offset method

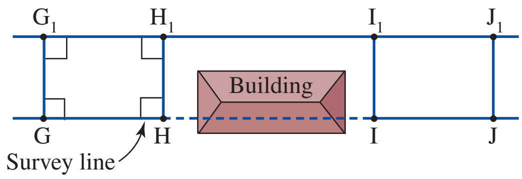
This method is useful when direct measurement is not possible (for example, due to the presence of a marsh or pond) but vision is not restricted. The diagram shows intermediate staffs located at D and E on the survey line on either side of the pond.



Equal offsets EE_1 and DD_1 are marked out perpendicular to the survey line. Points D_1 and E_1 are chosen so that they are clear of the pond.

Line D_1E_1 is parallel to the survey line and of equal length to DE. D_1E_1 is measured, which gives the length of DE.

The offset method can also be used when vision is prevented by tall objects. If a building is in the path of a survey line, the surveyor cannot measure directly along this path. The figure below shows how the offset method can be used to survey around a house. G and H are points on the main survey line.



Offsets GG_1 and HH_1 are set perpendicular to, and equidistant from, the main survey line. A second survey line $G_1H_1I_1J_1$ is then set up parallel to the main survey line. Equal offsets I_1I and J_1J are set off perpendicular to $G_1H_1I_1J_1$ so that IJ can now be continued. The length H_1I_1 is equal to the desired length HI.

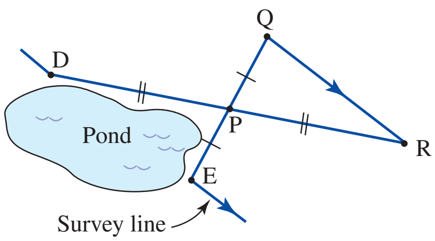
Triangulation method

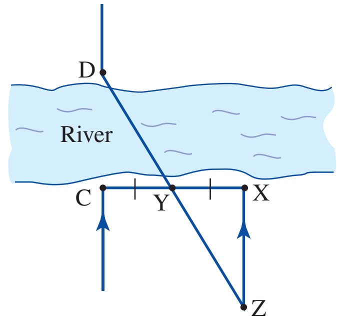
The figure below shows the triangulation method for surveying across the pond.

Staff P is placed so that lines EP and DP are free of obstructions. EP is continued to Q such that EP = PQ and DP is continued to R so that DP = PR. The triangles formed are congruent and so DE = QR. Once again we are able to measure QR and hence DE.

Triangulation can also be used across streams and rivers where vision is not interrupted. The figure below right shows intermediate staffs at C and D. A staff is placed at X so that CX is perpendicular to the survey line. Another staff is positioned at Y, the midpoint of CX.

A staff is then placed at Z so that D, Y and Z are in a straight line and ZX is parallel to the survey line. Triangles formed are congruent and so CD = ZX.





WORKED Example 7

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Worked example 7
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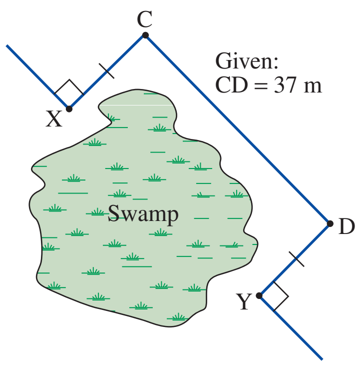
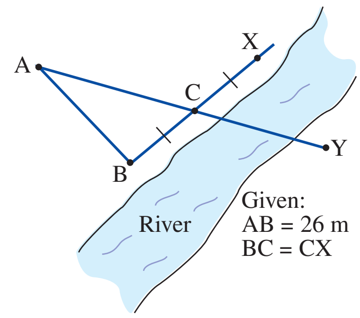
Find the unknown, XY, in each of the following.

a



b



THINK	WRITE
a Since the offsets are equal in length and perpendicular to the line XY, CDYX is a rectangle.	**a** CDXY is a rectangle. ∴ CD = XY ∴ XY = 37 m
b △ABC and △YXC are congruent.	**b** Triangles ABC and YXC are congruent. ∴ XY = BA ∴ XY = 26 m

remember

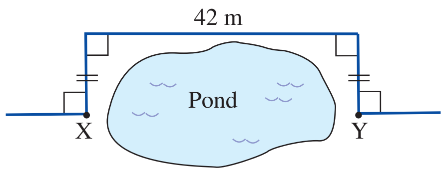
The presence of obstacles such as rivers, swamps, hills or dwellings can result in the survey line being interrupted. Obstacles can be bypassed using the offset and triangulation methods.

EXERCISE 6D Surveying around obstacles

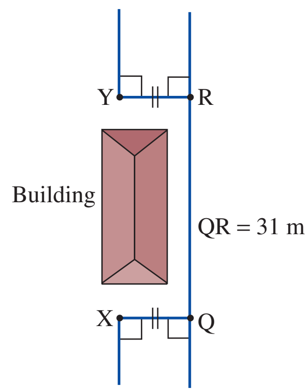
WORKED Example 7

1 Find the unknown side, XY, in the following diagrams.

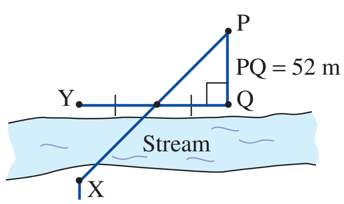
a



b

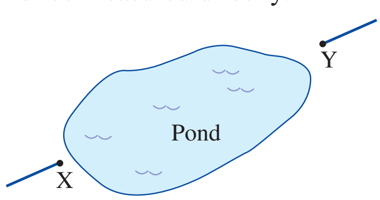


c

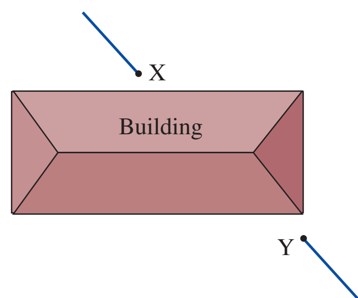


2 Describe with the aid of a diagram how the distance XY could be found in each of the following figures. All surfaces are level. X and Y are points on a main survey line. XY cannot be measured directly.

a



b



3 Sally and Jack are on opposite sides of a river and are attempting to determine the distance between them. Sally marks her position with a staff, and directs her assistant Peter, on her side of the river, to move 20 m away from her, and perpendicular to the line of sight across the river.

Peter marks this spot with a staff. Peter then moves another 20 m and places a second staff so that the two staffs that he has positioned, and Sally's staff, are in a straight line.

Peter then walks away from the river, at 90° to the line of staffs he has set up with Sally. He keeps walking until the first staff he positioned is in line with Jack. He then sets up a third staff.

a Show this information on a diagram.

b Indicate a distance Peter could measure that would be equal to the width of the river.

Bearings and reverse bearings

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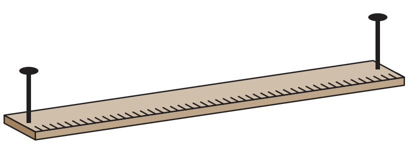
Some of the topics discussed in Chapter 5 on navigation are required for an understanding of land measurement. At this point, you should cover the sections 'Using the compass' and 'Compass bearings and reverse bearings' including exercises 5D and 5E (pages 268 to 272) if you have not already covered these topics.

Plane table surveying: intersection or triangulation

A **plane table survey** is a reasonably rapid method of mapping an area and can be done with relatively inexpensive and readily available equipment. However, the results obtained are not as accurate as those determined by some other methods. Equipment required includes:

1. a level table — a picnic table or school desk will suffice
2. a spirit level to check the level of the table
3. a large board — the type used by graphics students is ideal
4. a large piece of drawing paper attached to the board with pins or tape
5. an alidade — this is a sight rule (a rule with pins inserted at either end will suffice for this — see the figure below).



6. a plumb bob (may be used to improve the accuracy of results)
7. a marker peg and hammer (if the plane table is located on an earthy or grass surface) or chalk (if working on concrete or asphalt)
8. a compass (so that the direction of magnetic north can be marked on the map)
9. staffs or metre rulers.

The triangle is frequently used in surveying to fix the position of a landmark. This is because any 3-sided figure is a rigid shape.

The steps involved in the intersection or **triangulation** method using the plane table are as follows (see the sequence of figures a, b, c and d in this section).

Step 1 A base survey line, AB, is chosen within the area that is to be surveyed. This distance, AB, should be as large as is convenient. Rods are placed at points A and B. Points A and B should be chosen so that all features required on the map to be drawn are clearly visible from both of them. These points (A and B) are then marked with pegs hammered into the ground (if working on a grassy surface) or with chalk (if working on concrete or asphalt). The distance AB between these rods is carefully measured and noted. This distance is the only measurement required using this method.

The line AB is then drawn to a suitable scale, as line A′B′ on the paper fixed to the board. (The points A and B on the ground are represented by the points A′ and B′ on the paper.)

Step 2 Place the plane table over position A. Ensure that it is level using the spirit level. Move the board on the table so that A′ already marked on the paper is directly above point A marked on the ground below. You may use a plumb bob to assist with this, although a reasonably accurate estimation of position is possible.

Ensure that the line A′B′ drawn on the paper is in line with the actual line AB as shown in figure **a**. This is done by looking down the drawn line A′B′ to an assistant holding a pole at point B.

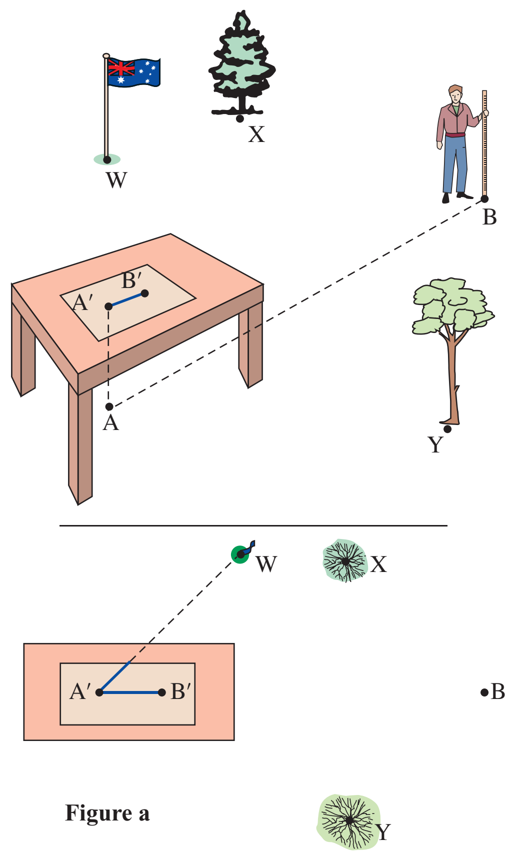


Figure a

Step 3 Use the alidade (sight rule) to align point A′ on the paper with feature W. Draw a faint line on the paper in pencil showing the direction from A′ to W. Check again that the board is in its correct position; that is, that the drawn line A′B′ is in line with the base survey line AB, and that the drawn position of A′ is directly over A.

Then mark on the paper the lines of sight from point A′ to all other features.

Figure **b** shows the lines of sight from point A′ to features W, X and Y (drawn as W′, X′ and Y′ respectively, near the edge of the paper).

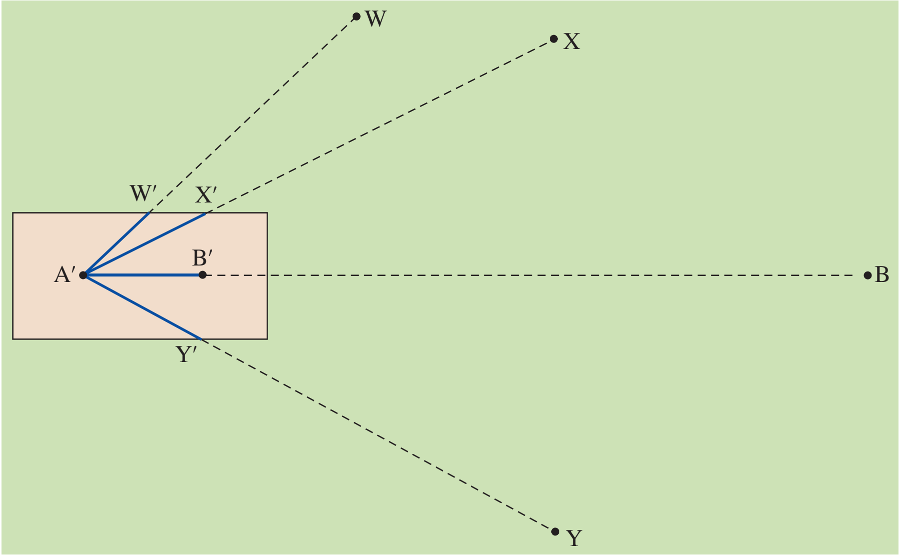


Figure b

Step 4 The table and board are then moved to point B so that B′ marked on the paper is directly over B on the ground. Again, ensure this is done as accurately as possible and that the table is level. Line A′B′ on the paper is again positioned in line with the base survey line AB (see figure **c**).

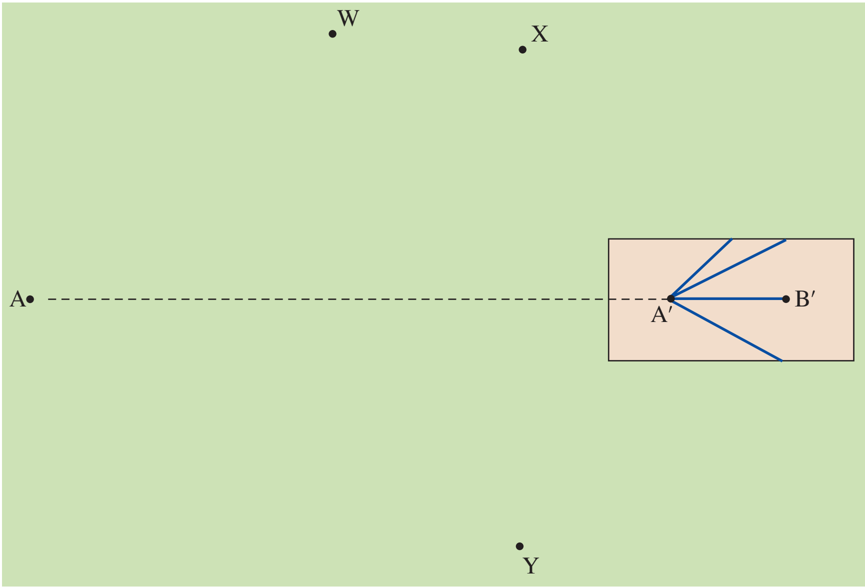


Figure c

Step 5 The sight rule is used to position the lines of sight to relevant features W, X and Y. These are marked on the paper as lines to cross the lines A′W′, A′X′ and A′Y′. The position of the board is regularly checked by ensuring B′ on the map is directly over B on the ground, and the line A′B′ on the map is aligned with the base survey line.

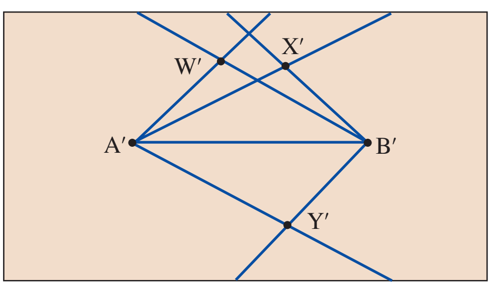
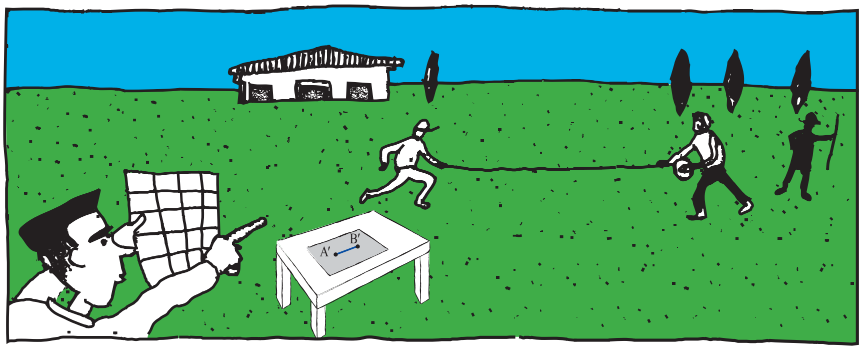


Figure d

The intersection of the marked lines of sight (figure **d**) will give the new, correct positions of W′, X′ and Y′, and thus the features W, X and Y.

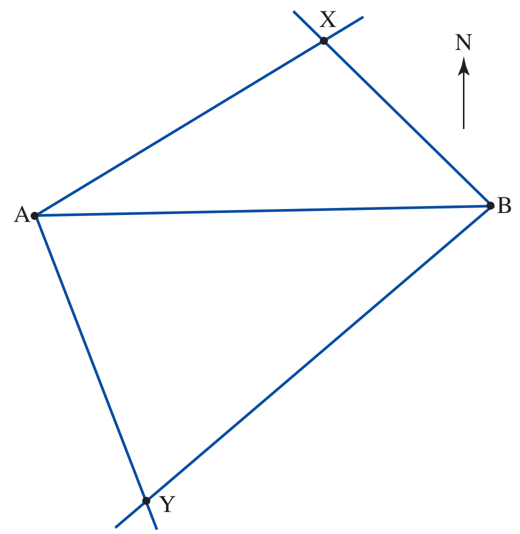
Step 6 The compass can be used to determine the direction of north and this, as well as the scale used, can be noted on the map.



WORKED Example 8

The figure at right shows a map resulting from a plane table survey using triangulation. A 1 : 2000 scale has been used and the direction of true north is shown.
Find:

a **the length of the survey line AB**
b **the bearing of**
 i **X from A**
 ii **Y from A**
 iii **B from A**
 iv **X from B**
 v **Y from B**
c **the perimeter of AXBY**
d **the area of AXBY to the nearest 0.01 hectare.**



THINK	WRITE
a 1 Measure AB on the figure and convert to distance on the ground.	**a** AB = 6 cm × 2000 = 12 000 cm = 120 m
b A protractor is used to measure each of the bearings.	**b** Bearings of: **i** X from A = 060° **ii** Y from A = 160° **iii** B from A = 090° **iv** X from B = 315° **v** Y from B = 230°
c 1 Determine the lengths of all sides.	**c** AX = 44.5 mm XB = 31 mm BY = 60 mm YA = 41 mm
2 To find the perimeter of the figure, add the lengths of AX, XB, BY and YA.	Total = 176.5 mm
3 Convert to the distance on the ground.	Perimeter = 176.5 mm × 2000 = 353 m
d 1 Divide the figure AXBY into two triangles, and calculate the area of each triangle using Heron's formula. $A = \sqrt{S(S-a)(S-b)(S-c)}$	**d** Area AXBY = Area of △AXB + Area of △AYB
2 Calculate the area of △AXB. (a) Calculate S.	$S = \frac{a+b+c}{2}$ $= \frac{89+62+120}{2}$ = 135.5 m

Continued over page

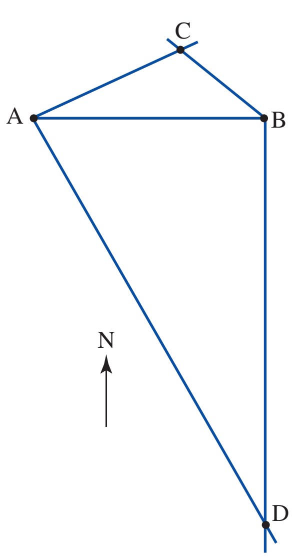
THINK	WRITE
(b) Calculate the area.	$\text{Area} = \sqrt{S(S-a)(S-b)(S-c)}$ $= \sqrt{135.5 \times 46.5 \times 73.5 \times 15.5}$ $= 2679 \text{ m}^2$
3 Calculate the area of $\triangle$AYB.	
(a) Calculate *S*.	$S = \dfrac{120 + 82 + 120}{2}$ $= 161$
(b) Calculate the area.	$\text{Area} = \sqrt{161 \times 41 \times 79 \times 41}$ $= 4624 \text{ m}^2$
4 The total area of the figure is found by adding areas of $\triangle$AXB and $\triangle$AYB.	Area AXBY = 2679 + 4624 $= 7303 \text{ m}^2$ $= 0.73$ ha (1 ha = 10 000 m^2)

remember

Plane table surveying makes use of a horizontal table. One of the key plane table surveying methods is called *intersection* or *triangulation*. Lines of sight to prominent features are taken from either end of the base survey line. The intersection of these lines on the survey map gives the positions of the features.

EXERCISE 6E Plane table surveying: intersection or triangulation

Note: Students should be able to measure lines accurately to the nearest 0.5 mm. However, some lines will be difficult to assess, and in marginal cases, discrepancies of 0.5 mm may be unavoidable. The focus should remain on mathematical processes, so when an answer depends on line measurement, allowance should be made for differences that may arise from such discrepancies.



WORKED Example 8

1 During an environmental study, a map is drawn from a plane table survey. This map is shown in the figure at right. (Scale 1 : 2000.) Find:

a the length of the survey line AB

b the lengths of:

i AC **ii** CB **iii** BD **iv** AD

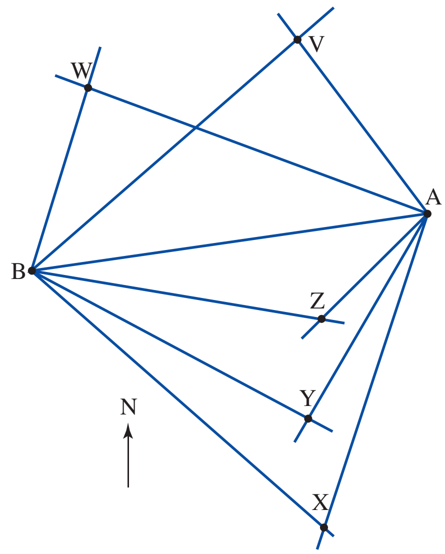
c the bearing of:

i C from A **ii** C from B **iii** D from B **iv** D from A

d the perimeter of ACBD

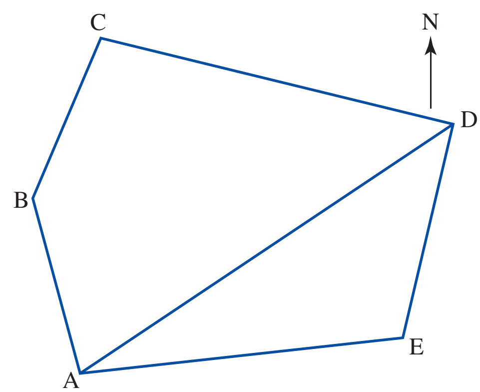
e the area of ACBD to 0.01 ha.

2 The figure at right shows a drawing resulting from a plane table survey using triangulation. The scale used is 1 : 1000. The direction of true north is marked.



- **a** Find the length of the base survey line AB.
- **b** Find the distance from:
  i B to W **ii** W to V **iii** V to A
  iv V to X **v** A to X.
- **c** By dividing the area into triangles, determine the area of figure BWVAX to the nearest 0.01 hectare.
- **d** An observer is at B. Give the bearings of points:
  i W **ii** A **iii** X.

3 The simplified results of a plane table survey are shown in the figure at right (scale 1 : 500). Find:



- **a** the perimeter of ABCDE
- **b** the cost of fencing this perimeter at $16 per metre
- **c** the area of ABCDE.

4 Jason is at point A. Peta is 100 m from Jason on a bearing 090° at point X. Jason observes a tower at point B on a bearing 050° and a tree at 150° at Y. Peta notes the bearing of the tower as 310° and tree as 240°.

- **a** Use this information to draw a 1 : 1000 sketch.
- **b** Use this diagram to find the length of:
  i AX **ii** BX **iii** AY **iv** YB **v** XY.
- **c** An observer is at X. Give the bearing of:
  i A **ii** B **iii** Y.
- **d** Calculate the area of ABXY to the nearest 100 m^2.

investigation

Drawing a scale diagram

Materials required: table (or desk, picnic or card table), spirit level, board, paper, tape or drawing pins, sight rule, wooden pegs and hammer or chalk, poles or metre rulers, compass.

In groups of 3, you are to draw a scale diagram of an area in your school grounds using the intersection method. Include the scale used and the direction of magnetic north. Include as many prominent features in this area as you can.

Less-prominent features can be included by recording them as offsets from the main lines of sight.

Determine:

1. the perimeter formed when outermost features on the map are joined
2. the area bounded by this perimeter.

You can check the accuracy of your work by using the laser distance meter described on pages 325–6.

Plane table surveying: radiation and traversing

We shall now consider two other methods of plane table surveying: the radiation survey (using the table at only one point) and the traversing survey (using the table at several points).

Radiation survey

The radiation method of surveying is suited to the mapping of small areas. It can be completed quickly because the table is set up at one point only, rather than at two points using the triangulation technique. The method is as follows:

1. A suitable spot is chosen (point A) located approximately at the centre of the region to be surveyed. All features to be plotted on the map should be visible. This point is then marked by a peg or with chalk.
2. The table is placed over point A. The table is levelled and the board with paper taped to it is positioned on the table. Point A′ is marked on the paper directly over point A on the ground. A plumb bob can be used to assist with this procedure.
3. Use the sight rule to draw the line of sight to feature B, as shown in figure **a** at right.
4. This is repeated so that lines of sight to C, D, E and F (off page to left) are drawn.
5. The distances AB, AC, AD, AE and AF are measured. A suitable scale is chosen and points B′, C′, D′, E′ and F′ are then marked on this map.
6. The direction of magnetic north is determined using a compass and marked on the map.

Figure **b** shows an example of a sketch drawn by the radiation method.

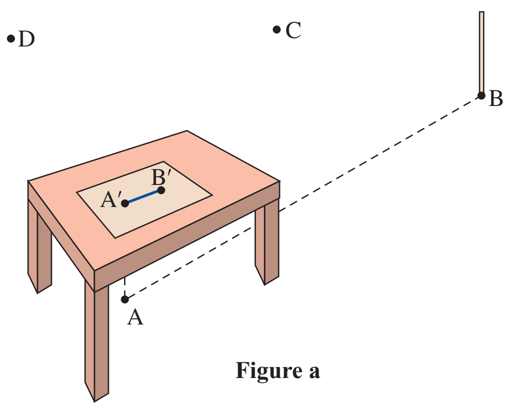


Figure a

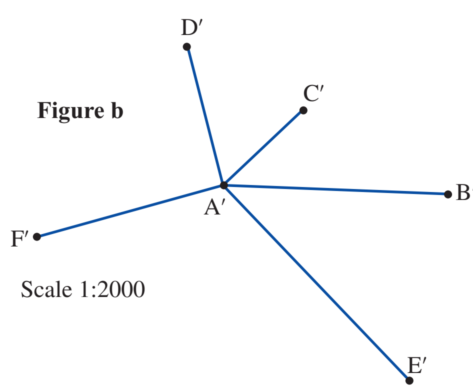


Figure b

WORKED Example 9

Use figure b above to answer the following.

a Find the length of: i AF ii AD iii AB iv DE v CB.

b If C is due north of A, give the bearing from A of: i B ii E iii D.

c Find the area of △ABC.

THINK	WRITE
a Lengths are measured and converted to metres using the scale 1 : 2000.	**a** AF = 51 m AD = 38 m AB = 60 m DE = 106 m CB = 44 m

THINK	WRITE
b A protractor is used to measure each of the angles clockwise from north.	**b** **i** B is 045°. **ii** E is 090°. **iii** D is 300°.
c 1 Three sides are known, so Heron's formula can be used.	**c** AB = 60 m; AC = 29 m; CB = 44 m
2 First find S, the semi-perimeter.	$S = \frac{60 + 29 + 44}{2}$ $= 66.5$
3 Calculate area.	Area $= \sqrt{66.5(66.5-60)(66.5-29)(66.5-44)}$ $= 604 \text{ m}^2$

Traversing survey

The traversing-plane-table method requires the plane table to be moved over each important feature that is to be identified on the map. Distances between each feature are to be measured. Steps involved are:

1. Examine the area to be surveyed and determine points A, B, C, D, etc. to be included on the map. These can be marked with pegs or chalk depending on the type of surface.
2. Position the plane table directly over point A. Ensure that it is level. Tape paper to the board and mark point A′ on the paper such that it is directly above A on the ground. Use the alidade (sight rule) to sight a pole held by an assistant at point B. The line of sight A′B is marked on the map. The line of sight A′D is then also marked as shown at right (figure **a**).

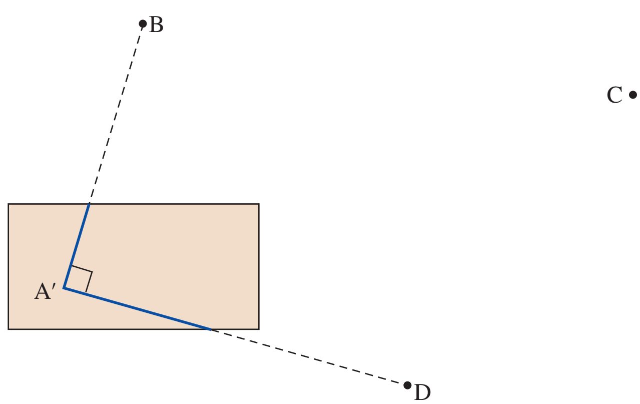


Figure a

3. The distances AB and AD are measured. Points B′ and D′ are then marked on the map using a suitable scale.
4. The table is moved to B and levelled. The board is moved so that B′ on the map is directly above B on the ground. Line B′A′ on the map is aligned with the line of sight to A. The line of sight to C can then be drawn as shown in figure **b**. Distance BC is measured and C′ is then plotted on the map.

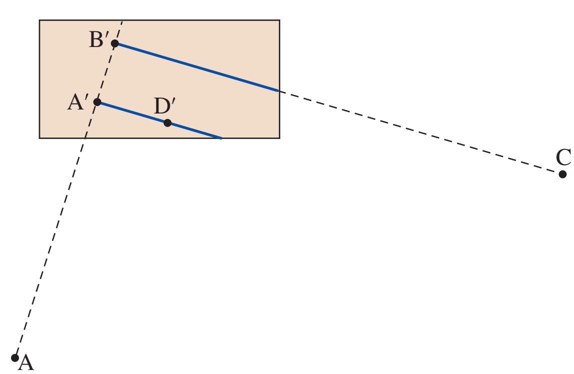


Figure b

5. The table is then moved directly above C. The board is positioned so that C′ on the map is above C on the ground. C′B′ on the map is aligned with the line of sight from C to B. The line of sight to D can now be drawn, and should match that predicted by the map positions of C′ and D′, The accuracy of the map can be checked by moving the table to D. From D, the lines of sight to A and C should match lines D′A′ and D′C′ on the map.

WORKED Example 10

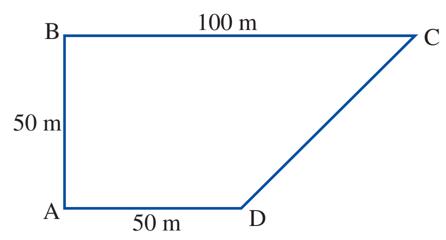
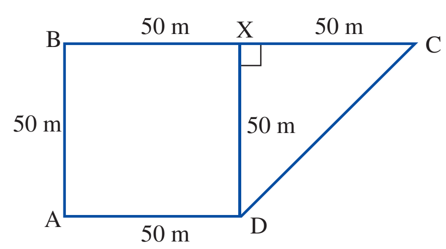
Bill places a table over point A and sights B, 50 metres away, on a bearing of 0°, and point D, 50 metres away, on a bearing of 090°. He moves the plane table directly over point B and now notes the position of point C as 100 metres from B on a bearing of 090°.

a What method of surveying is Bill using?

b Draw a neat sketch representing this information.

c What is the expected bearing of D from C?

d Find the area of ABCD.

THINK	WRITE
a The table has been moved over the top of a number of features, so the traversing method was used.	**a** Bill is using the traversing method.
b 1 Starting at A, draw a vertical line upwards to represent AB. Label AB as 50 m. 2 Draw AD at right angles to AB. Mark this as 50 m. 3 Draw BC at 90° to AB and parallel to AD.	**b** 
c 1 (a) If the diagram is accurately drawn to scale, angle BCD will be measured as 45°. (b) Alternatively, an isosceles triangle can be constructed by drawing line DX perpendicular to BC. Then, in △CXD, XC = XD.	**c**  CX = XD ∴ ∠XCD = 45°
2 Measuring clockwise from north, the bearing of D from C is 225°.	Bearing of D from C = 225°
d The area of ABCD is the sum of the areas of the square ABXD and the triangle CXD.	**d** Area $= 50^2 + \frac{1}{2} \times 50 \times 50$ $= 3750\ \text{m}^2$

remember

Two other plane table survey methods are:
1. Radiation: Lines of sight and distances to prominent features are marked from one point only.
2. Traversing: The plane table is moved over several prominent features and lines of sight are drawn at each location.

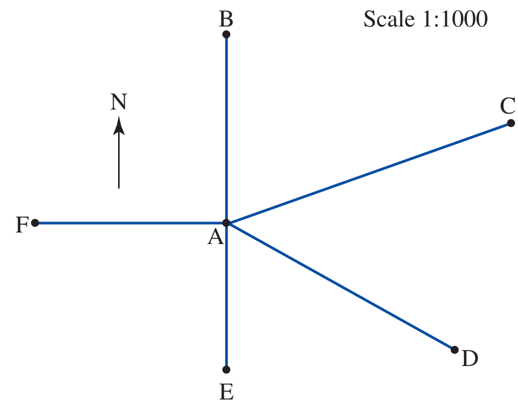
eBook*plus*

Digital docs:
Spreadsheets
047 Map scale 1
048 Map scale 2

EXERCISE 6F Plane table surveying: radiation and traversing

WORKED Example 9

1 The figure at right shows a map produced by the plane table radiation method.

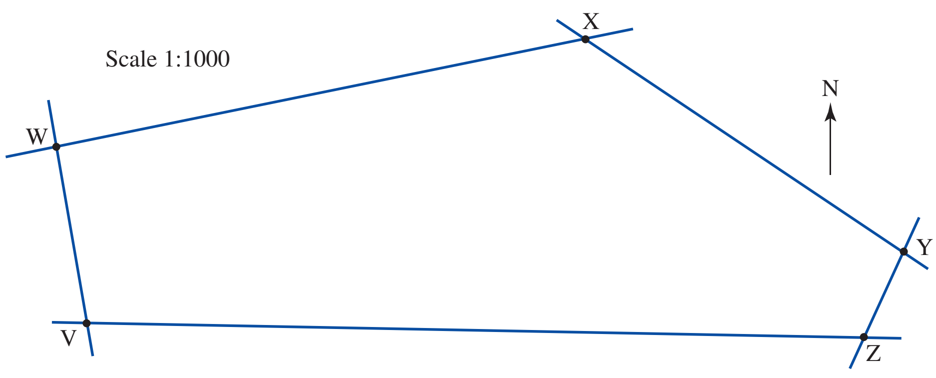


a Find the length of:
   i AB   ii AD
   iii DE   iv BD.

b What is the bearing from A of:
   i B?   ii C?   iii E?

c Find the area, to the nearest 100 m^2, of the figure FBCDE.

2 The figure below shows the results of a plane table survey using the traverse method.



a Give the lengths of:
   i VW   ii WX   iii XY
   iv YZ   v VZ   vi VY
   vii VX.

b Find the area enclosed by this figure, to the nearest 0.01 ha.

WORKED Example 10

3 Cameron sets up a plane table at O and from this point makes the following notes about the surrounding features A to E.

Feature	Distance to feature	Bearing of feature from O
A	25 m	045°
B	35 m	120°
C	20 m	180°
D	40 m	250°
E	65 m	325°

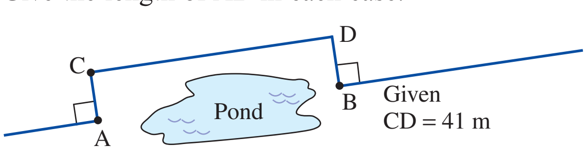
a What surveying method is Cameron using?
b Draw a neat sketch representing this information. Use a 1 : 1000 scale.
c An observer is at E. From E, what would you expect to be the bearings of A, B, C and D?
d Estimate the area enclosed by the perimeter linking features A to E. Express to the nearest 10 m^2.

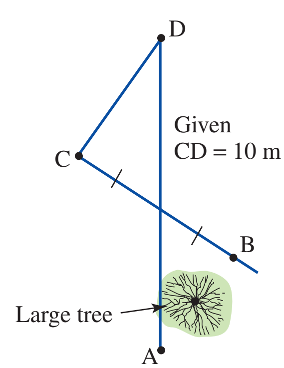
4 Margaret places a plane table directly over point A and sights B, 60 m from her on a bearing of 020°, and D, 50 m away on a bearing of 100°. She moves the plane table to B and notes the bearing of C, 70 m from her, to be 090°.
a Draw a neat 1 : 1000 sketch showing this information.
b What method of surveying is Margaret employing?
c Margaret then moves the plane table to C. What bearing is expected of:
  i D? **ii** B?
d What is the distance from C to:
  i A? **ii** D?
e Determine the area of ABCD to the nearest 100 m^2.

10 QUICK QUESTIONS 2

1 Name two methods that can be used to survey around obstacles.

2 Give the length of AB in each case.





3 Name three methods of plane table surveying.

4 Which of these three methods requires lines of sight taken from either end of a survey line?

5 Which method requires all lines of sight to be taken from only one point?

6 Which method requires the plane table to be moved over several points with lines of sight taken at each?

7 In the field, Janelle measures a distance of 57.5 metres. She is producing a survey map with a scale of 1 : 200. What length on the map (in mm) would represent this distance?

8 An irregularly shaped field has 6 sides. Daniel intends to determine its area by dividing it into triangles. Daniel will need to find the area of what minimum number of triangles?

9 A triangle has sides of 24, 27 and 41 metres. State Heron's formula and use it to find the area of the triangle.

10 A triangle has sides of 5 and 8 metres with the included angle (the angle between them) of 30°. Find the area of this triangle.

Levelling: vertical measurements in relation to a datum

The survey methods discussed so far assume that one is working on relatively level surfaces. This is often not the case: undulations within the perimeter of a portion of land will affect the area of this portion. Consequently, the above methods cannot be used on sloping surfaces unless height variations are taken into account.

Levelling is the surveying process of determining the relative heights of points on the Earth. It is concerned with finding the difference in height, or level, between two points. Clearly, if point X is 10 m above ground level and Y is 3 m above ground level, then the difference in level is 7 m.

Levelling is of great importance in the construction of buildings. Various levelling devices (the spirit level, string line and water level) and their principles of operation are described in *Maths Quest, Maths A Year 11 for Queensland 2nd edition,* (Chapter 7). The **dumpy level** is a levelling device that is often employed in preparing a site for construction. It derives its name from early models which were compact telescopes mounted on tripods. The dumpy level establishes a level line of sight known as a **line of collimation**.

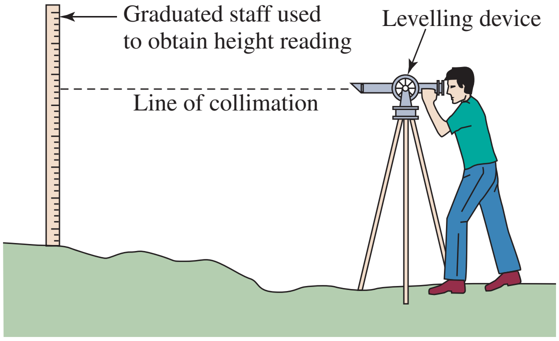


Figure a

The dumpy level must be level itself when it is being used. It therefore has a bubble which can be centred by adjusting the footscrews.

Laser levels emit horizontal rays of laser light. These can be detected either by sensors which slide up and down the graduated staff or by telescope.

A **theodolite** is a multi-purpose surveying instrument. It consists of a telescopic sight mounted on a tripod, again with a levelling mechanism. It allows measurements of angles in the horizontal plane and angles of depression and elevation in the vertical plane.

A level line or surface to which other heights or levels are referenced is known as a **datum line** or **datum surface**. Official datum levels are known as **ordnance levels**. A **benchmark** (BM) is a fixed point of known level with reference to an ordnance level. There are thousands of benchmarks used, and these are usually placed on features such as bridges and buildings. Any survey can be referenced by a benchmark, as their heights above mean sea level are accurately recorded.

A **temporary benchmark** (TBM) is a point selected by a surveyor for a particular task. A **profile** is a cross-sectional view of the surface being surveyed. The **profile level** is the survey undertaken by a surveyor when heights of various positions are taken and recorded. The diagram in figure **b** (below) shows a theodolite located on sloping ground between points A and B. The survey begins by placing a staff on point A which is either a benchmark or temporary benchmark. The theodolite is then positioned between A and B. A sighting back to point A is made first and the backsight is noted. The sighting to B is then made, and the foresight is noted. The backsight and foresight recordings can then be used to calculate the reduced levels of A and B. The **reduced level** (RL) is the height of a point above the datum line. Figure **c** shows a sketch of a profile level.

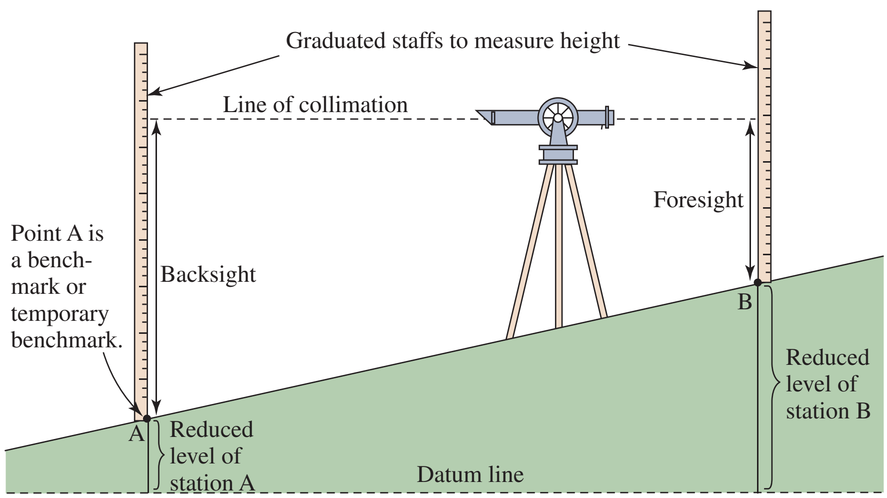


Figure b

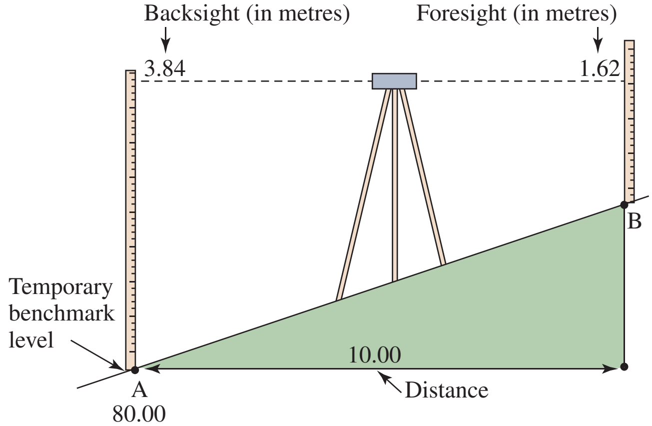


Figure c

The information contained in this sketch is usually recorded in a surveyor's book as follows:

Sta.	BS	IS	FS	HI	RL	Dist.	Notes
A	3.84			83.84	80.00	0	Temp. benchmark
B			1.62	83.84	82.22	10.00	

The abbreviations used in this table are Sta: station; BS: backsight; IS: intermediate sights; FS: foresights; HI: height of instrument (the height entered in the table is the height above the datum line); RL: reduced level (the heights of stations above the datum line); Dist: distance along the line containing the TBM parallel to the datum line.

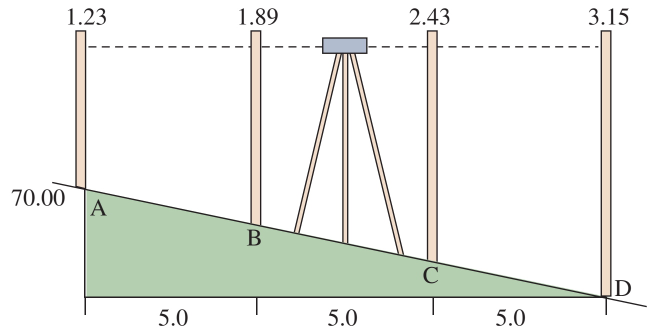


Figure d

Figure **d** shows profile levelling using intermediate sights. Information from this survey would be recorded in the table as shown.

Sta.	BS	IS	FS	HI	RL	Dist.	Notes
A	1.23			71.23	70.00	0.00	TBM 70.00
B		1.89		71.23	69.34	5.00	
C		2.43		71.23	68.80	10.00	
D			3.15	71.23	68.08	15.00	

The RL column (in red) shows the reduced levels of the four stations. Station A is a benchmark 70.00 m above the datum line. Station B has the RL of 69.34 m (or 71.23 – 1.89 m) and station C has the RL of 68.80 m (or 71.23 – 2.43 m).

WORKED Example 11

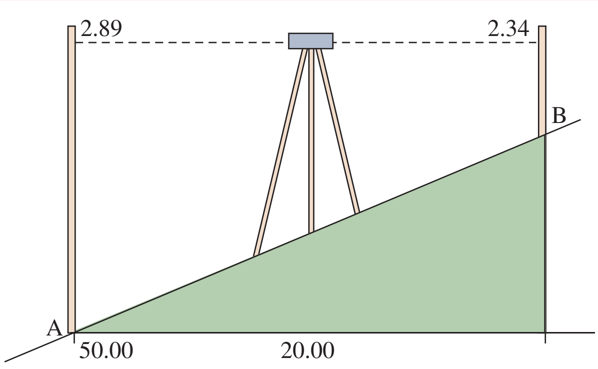
The sketch shows a profile level. All measurements are given in metres. Use this sketch to:

a state the RL (reduced level) of A

b state the foresight if the backsight is 2.89 metres

c find the height of the instrument

d complete the table below.



Sta.	BS	IS	FS	HI	RL	Dist.	Notes
A	2.89						TBM
B							

Continued over page

THINK

a A is marked as 50.00. This is the reduced level.

b Foresight is measurement in the direction opposite to backsight.

c Height of instrument
HI = RL of A + 2.89 m

d Complete the table.

WRITE

a RL of A = 50.00 m

b Foresight = 2.34 m

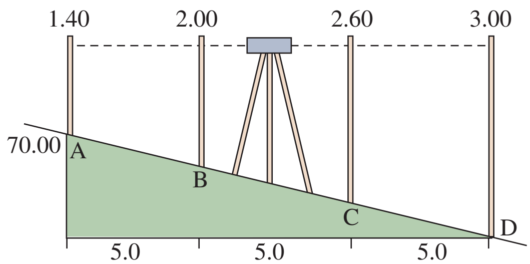
c HI = 50.00 + 2.89
= 52.89 m

d

Sta.	BS	IS	FS	HI	RL	Dist.	Notes
A	2.89			52.89	50.00	0.00	TBM
B			2.34	52.89	50.55	20.00	

WORKED Example 12

The sketch at right has been made by Sally, who is surveying from point A to D using intermediate sights. Use this sketch to complete the unknowns a to k in the table below.



Sta.	BS	IS	FS	HI	RL	Dist.	Notes
A	1.40			a	e	0.00	TBM
B		2.00		b	f	i	
C		2.60		c	g	j	
D			3.00	d	h	k	

THINK

1. Determine HI (unknowns a to d). Station A is a benchmark 70.00 m above the datum line; BS is 1.40; 70.00 + 1.40 = 71.40.
2. RL of A (unknown e) is benchmark 70.00 m.
3. RL of B (unknown f) is 71.40 – 2.00 = 69.4.
4. RL of C (unknown g) is 71.40 – 2.60 = 68.8 m.
5. RL of D (unknown h) is 71.40 – 3.00 = 68.4 m.
6. Distances parallel to the datum line from A to B, C and D (unknowns i, j and k) are at 5-m intervals.

WRITE

Sta.	BS	IS	FS	HI	RL	Dist.	Notes
A	1.40			71.4	70.00	0.00	TBM
B		2.00		71.4	69.4	5.00	
C		2.60		71.4	68.8	10.00	
D			3.00	71.4	68.4	15.00	

remember

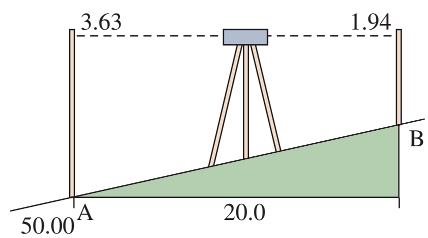
1. Levelling is the process of determining relative heights of points on the Earth.
2. This reduced level, RL, is the height of a point above the datum line.

EXERCISE 6G Levelling: vertical measurements in relation to a datum

WORKED Example 11

1 The figure at right shows a sketch of a profile level. All measurements are given in metres. Use this sketch to:

a state the RL of: **i** A **ii** B
b state the foresight if the backsight is 3.63 m
c find the height of the instrument
d complete the table below.



Sta.	BS	IS	FS	HI	RL	Dist.	Notes
A	3.63						TBM
B							

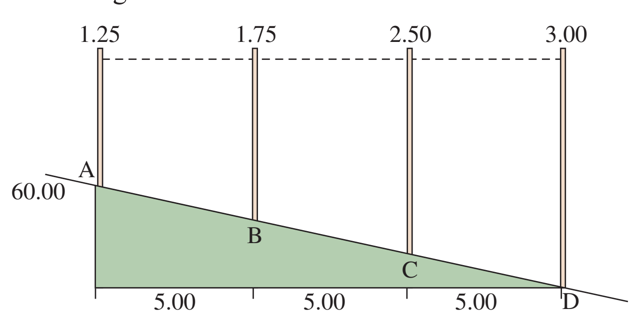
2 The table below shows the recordings made by a surveyor.

Sta.	BS	IS	FS	HI	RL	Dist.	Notes
A	3.60			53.60	50.00	20.00	TBM
B			2.80	53.60			

a State the:
i backsight **ii** foresight **iii** height of instrument **iv** reduced level of A.
b Calculate the RL of B.
c Sketch a profile showing points A and B, and the theodolite positioned between them. Include the backsight and foresight.

WORKED Example 12

3 The figure at right shows a sketch made by a surveyor working from point A to D using intermediate sights. Use this sketch to complete the unknowns **a** to **k** in the table below.



Sta.	BS	IS	FS	HI	RL	Dist.	Notes
A	1.25			**a**	**e**	0.00	TBM
B		1.75		**b**	**f**	**i**	
C		2.50		**c**	**g**	**j**	
D			3.00	**d**	**h**	**k**	

The theodolite

The theodolite is an extremely important device used in land measurement. Modern electronic theodolites give accurate measures of angles of elevation and depression as well as distances to surrounding objects.

Research the history of the development of the theodolite.

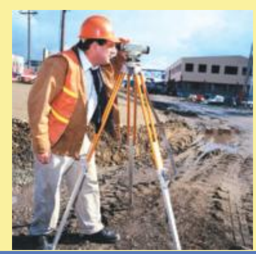
Obtain a non-electronic theodolite and investigate its use. These are still used by some builders and developers.

Investigate the operation of the modern theodolite. These are quite simple devices to use and should be operated by each student during the study of this unit. If you cannot have a demonstration done by a parent or surveyor in your region, these devices can also be hired from most large construction hire companies.

The work of the surveyor

Invite a surveyor to visit your class to discuss the variety of tasks performed in this role. To find out more about surveyors who work in your region use the **Spatial Industries Business Association** weblink in your eBookPLUS.

Ask your speaker in advance to demonstrate the operation of modern surveying devices.



Topographic maps

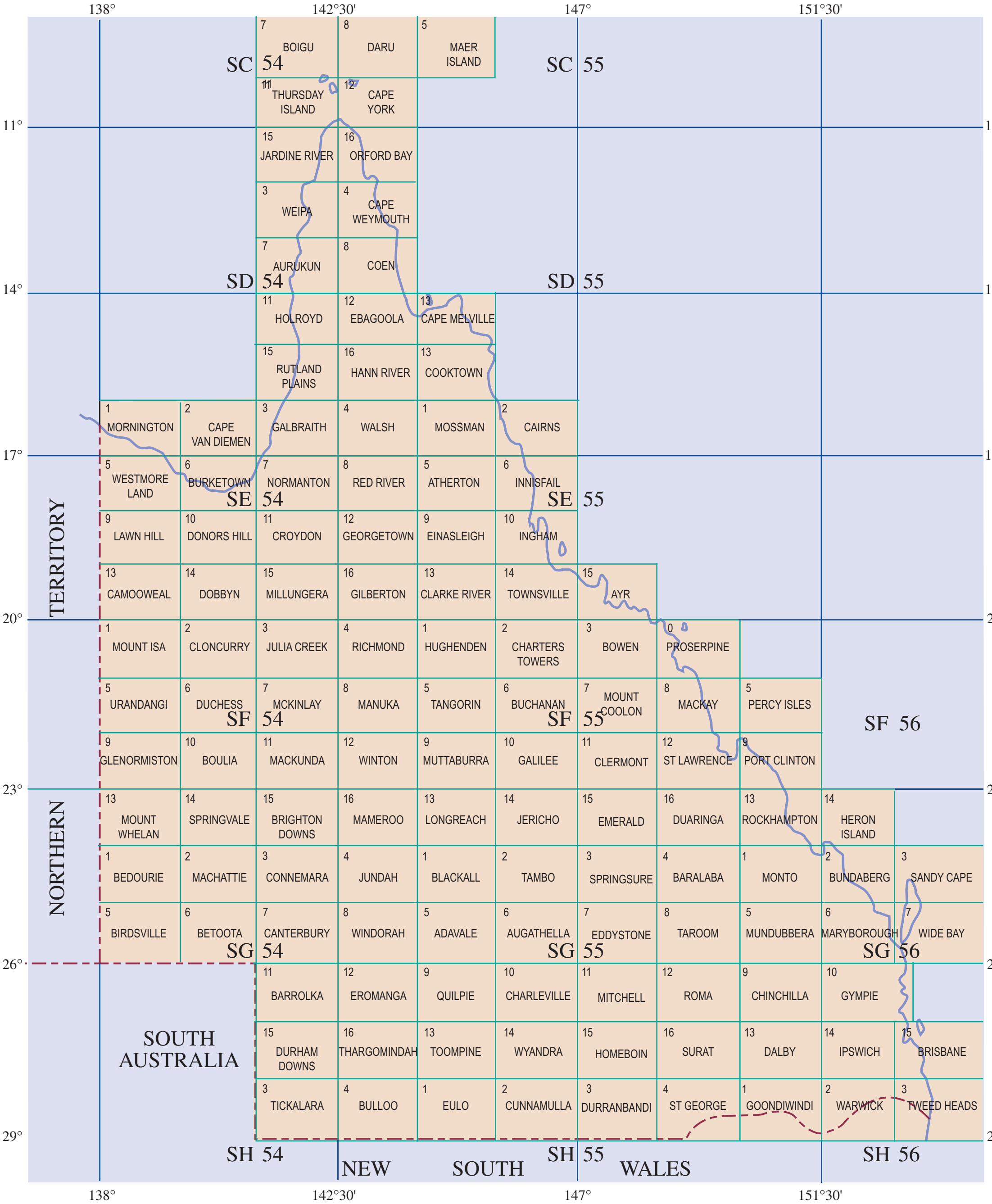
A **topographic map** is a detailed sketch of a portion of the surface of the Earth. These maps include natural features such as landforms, rivers and forests; and cultural features such as towns, railway lines and bridges.

Topographic maps are the most frequently employed of all maps, and are used by police, bush walkers, tourists, the armed forces and emergency teams. They are readily available from the Department of Natural Resources agencies throughout the State. The entire State has been divided into regions and all are shown on topographic maps. The map on page 352 shows the names of regions that have been mapped, drawn to a scale of 1 : 250 000. Other scales of maps also available are 1 : 10 000, 1 : 25 000, 1 : 31 680, 1 : 50 000, 1 : 100 000 and 1 : 1 000 000.

To successfully interpret a topographic map, the reader must have a knowledge of symbols, scales, contours, grid references, cross-sections and compass bearings.

An extract of a topographic map is shown on pages 356 and 357. This shows a section of the Mount Crosby region drawn to a scale of 1 : 25 000. A key is included to assist with the interpretation of this map (see legend on page 358). Most of the symbols shown can be grouped under the headings of vegetation, transportation, landform, urban area features, buildings and land use.

INDEX TO SHEETS OF
1:250 000 MAPPING



MAP DESCRIPTION

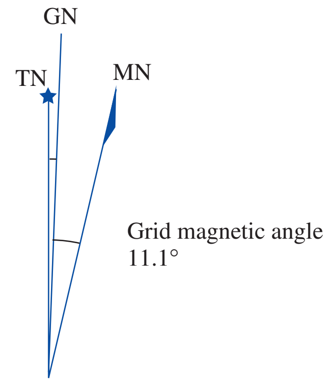
TOPOGRAPHIC MAPPING: Produced on the Australian Map Grid and showing 10 000 metre grid interval, these maps are compiled from aerial photographs which are coordinated to ground surveys. They are a multicoloured publication showing cultural detail, vegetation, drainage systems and relief. Relief is represented by means of 50 metres contours and spot heights. The sheet format is 1° of latitude by 1°30' longitude.

Source: Reproduced with permission of the Department of Lands, Queensland.

Mapping reproduced with permission of The Department of Natural Resources and Mines (1990 approx.)

A topographic map is an extremely useful aid in navigating our way over the Earth's surface. To determine the direction that we must travel to reach a certain feature, we need to understand compass bearings and north points.

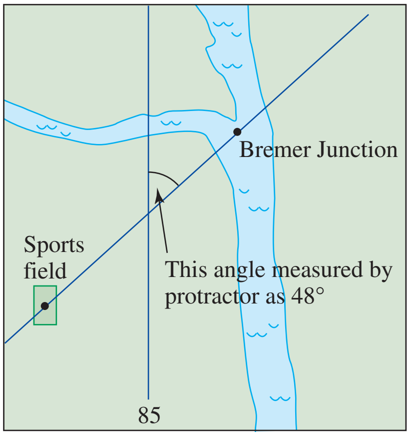
A topographic map shows three north points: true north, magnetic north and grid north. The figure at right shows how these three points are related for the Mount Crosby region.



True north is the direction that we would travel to reach the North Pole. A line in a true north direction links the North and South Poles.

Magnetic north is the direction in which a compass needle points because the Earth is a giant magnet. The direction of magnetic north varies annually and this variation will always be shown on the map. The legend from the map shows that magnetic north is 11.1° to the east of grid north and is gradually changing to the east at a rate of about 0.12° every 5 years. The angle between grid north and magnetic north is the *grid magnetic angle.*

Vertical grid lines on a topographic map, then, do not lie in a true north–south direction; they lie at an angle to true north. Bearings are used to describe the direction from grid north. Grid north has a bearing of 0° for these maps. The diagram at right shows how the bearing of Bremer Junction (the point where the two rivers join) from the sports field can be found. A line linking these two features makes an angle of 48° with the vertical grid line. This can be measured using a protractor.



The grid bearing of Bremer Junction from the sports field is 48° (often written as 048°).

More information on bearings, reverse bearings and compass use is included in Chapter 5, Navigation.

Grid references

Any position on a topographic map can be given by using **grid references**. Location is described by using a 4- or 6-figure grid reference, depending upon the degree of position accuracy required. Because grid references really refer to a square region rather than points on a map they are also referred to as *area references*.

Examine the map on pages 356 and 357.

To describe the grid reference of Finlay Island (just north of Bremer Junction, the point where the two rivers meet), the following steps are required.

Step 1 The bottom left-hand corner of the grid square containing Finlay Island is located.

Step 2 The easting (vertical line) forming the left-hand boundary of the grid square is noted as 85.

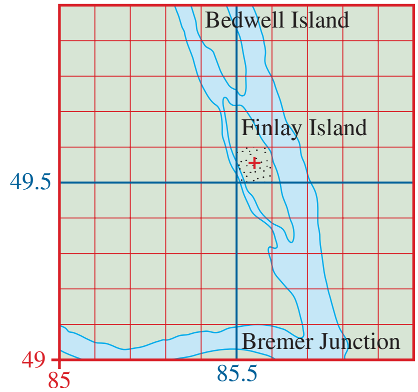
Step 3 The northing (horizontal line) forming the bottom boundary of the grid square is noted as 49.

Step 4 The grid reference for Finlay Island is written as GR 8549.

Note: The letters GR stand for *grid reference*. A 4-figure grid reference for the Bundamba Racecourse is GR 8045.

A 6-figure grid reference is a more accurate description of position giving one hundredth the area of a 4-figure reference. Now, the reference is given to the bottom left-hand corner of the new, smaller area in which the point lies. The centre of Finlay Island has the 6-figure grid reference determined as follows (see figure below). The bottom left-hand corner of grid square 8549 is again located. The easting is now described using three numbers rather than two. An estimate is made of the location to the nearest tenth of an easting, in this case 85.5. An estimate is then made of the northing, 49.5. The grid reference is written as easting followed by northing, omitting decimal points; so the reference is GR 855495. (Note that *every* point in this small square has the same six-figure grid reference.) The 6-figure grid reference of the Bremer Institute of TAFE (right-hand end of the building on the map on page 356) is GR 814465.

Hint: Use the 'tenths' scale at the base of the map on page 357. Copy the tags onto the edge of a piece of paper that you can lay on the map.

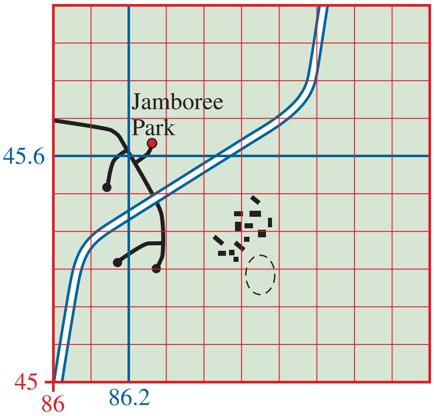


WORKED Example 13

On the map on page 357, what feature is at GR 862456?

THINK

The first three numbers give an easting of 86.2. The next three give a northing of 45.6. These lines intersect to give the area containing the point labelled Jamboree Park.

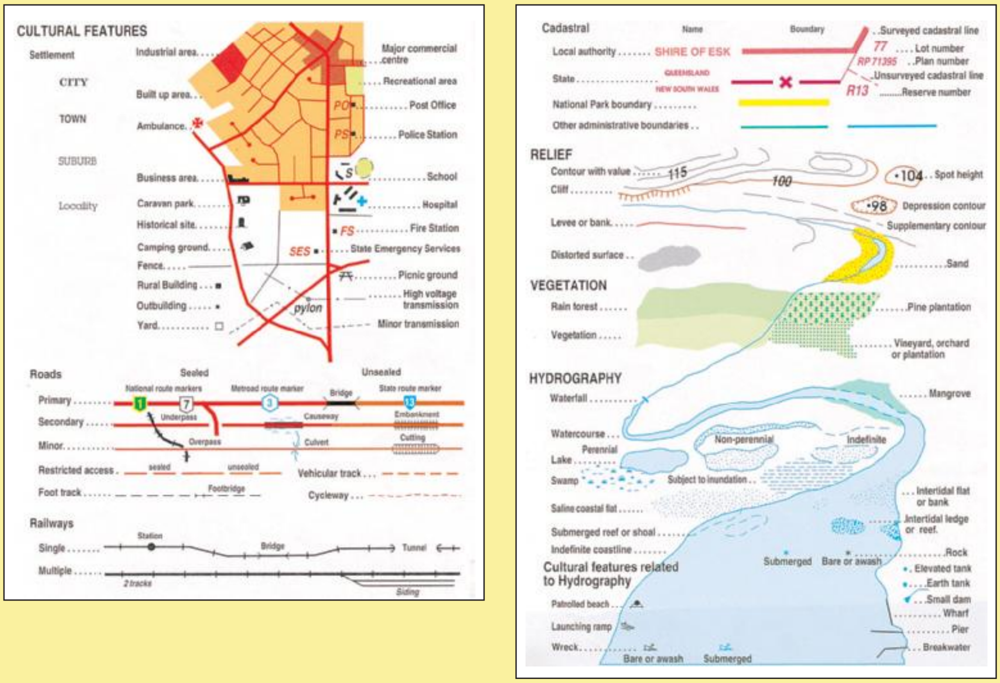


WRITE

The feature at GR 862456 is Jamboree Park.

CITY OF IPSWICH

GASCOYNE DRIVE
OXLEY
Springdale Park
PHILLIP CRES
FAWKNER CRES
STUART ST
JUNCTION
ROAD
MITCHELL ST
BENDEMEER ST
AVE
BARELLAN POINT
JUNCTION ROAD
ELANORA WAY
FIRST AVE
RIVERSIDE
FIFTH AVE
ISLANDVIEW ST
AVENUE
Bedwell Island
Finlay Island
Joseph Brady Park
Bremer Junction
HAWKESBURY
WITTY
ROAD
MOGGILL
MATFIELD
SUB-ARTERIAL
ROAD
BRISBANE GPO 22km
abattoir
quarry
sawmill
council depot
waste transfer station
boys home
Moggill Reach
Redbank Reach
BRISBANE RIVER
mines testing and research station
SMITH STREET
MONTGOMERY RD
RIVERVIEW
RIVERVIEW ROAD
sports field
retirement village
DINMORE
ALFRED ST
LINE
MOGGILL ROAD
ENDEAVOUR RD
Mc EWAN ST
vehicular ferry
silos
railway workshops
railway stores
REDBANK
industrial estate
St Peter Claver College
Frank Mitchell Park
Maculata Park
TESSMAN ST
DUNCAN STREET
CUNNINGHAM HIGHWAY
BRISBANE ROAD
MAIN ROAD
IPSWICH
JACOB ST
microwave tower
disused clay pits
school
oval
brick works
HIGHWAY
VALE
sports ground
shopping centre
CHALK ST
Creek
Goupong Park
NAMATJIRA DR
COLLINGWOOD DRIVE
slag heaps
DUNCAN STREET
Jamboree Park
Banjo Paterson Park
KRUGER PAR
Six Mile Creek
GIBBS AVE
open cut mine
NEW CHUM
refuse site
microwave tower
COLLINGWOOD PARK
GOSS DRIVE
LAWRIE DR
Goodna
HENDERSON ST
CUNNINGHAM
83
84
85
86
87
51
50
49
48
47
46
45
27°37'3
Tenths



WORKED Example 14

Using the map on page 357, give the grid bearing of the building near the vehicular ferry at GR 859475 from Jamboree Park (GR 862456).

THINK	WRITE
The two features are located on the map. A line joining them intersects with the vertical grid line at 350° (measured using a protractor).	The bearing of the building near the vehicular ferry from Jamboree Park is 350°.

remember

1. A topographic map is a detailed sketch of a portion of the surface of the Earth.
2. Location is described using 4- or 6-figure grid references, depending on the accuracy required.

EXERCISE 6H Topographic maps

eBook plus

Digital doc:
SkillSHEET 6.3
Trigonometric values and angles

1 A location has a 4-figure grid reference of GR 8446. State its easting and northing.

Use the map on pages 356 and 357 to answer questions 2 to 6.

WORKED Example 13

2 Name the main feature located at GR:
a 851471 **b** 819473 **c** 830486.

3 Give the 6-figure grid reference for:
  a the shopping centre south of the Cunningham Highway at Redbank (the far right-hand corner)
  b the boy's home at Bremer Junction (where the two rivers meet) (use the most north-easterly building)
  c the bushland native reserve on the Bremer River (use the position of the 'n' in 'native').

4 The map has been drawn to a scale of 1 : 25 000. Estimate the direct distance from:
  a the bridge over the Bremer River, just to the north-east of the bushland native reserve to the centre of the Bundamba Racecourse.
  b the quarry at GR 8248 to the centre of the bridge in **a** above.

WORKED Example 14

5 From the centre of the oval at GR 819473, give the grid bearing of:
  a the centre of the bridge near the bushland native reserve on the Bremer River
  b Redbank Industrial Estate (GR 8647).

6 a Ian is flying directly over Bremer Junction (GR 8549) on a grid bearing of 270°. Briefly describe what he could expect to see directly below as he continues on this bearing.
  b On another flight, he passes GR 803492 and later, GR 821452. On what grid bearing is he travelling? Could you answer this without a map?

investigation

Local features on topographic maps

Activity 1: Identifying local features

Materials required: sketchbook, magnetic compass.

From a point on top of a hill or a tall building in your school, sketch a map of your local area. Include as many prominent local features on this map as you can in a radius of about 5 kilometres. Include also on the map the direction of magnetic north.

Activity 2: Using a topographic map of your locality

Materials required: *Sunmap guide to topographic maps, Queensland, Australia*, and magnetic compass.

Obtain a copy of this free guide by contacting the Department of Natural Resources in Brisbane. The guide will be posted to you, or can be collected from agencies in Brisbane, Bundaberg, Cairns, Rockhampton, Toowoomba, Maryborough, Nambour, Roma and Townsville. Use the **NRW** weblink in your eBookPLUS to contact the Department of Natural Resources.

Study the section 'How to use this guide' and then select the appropriate large-scape map that has been drawn to describe the area in which your school is situated. A large-scale topographic map is one that covers a small area of land but includes many features. Maps of scales 1 : 25 000 or 1 : 31 680 will suffice for this exercise. Obtain this map from the Department of Natural Resources.

1 Use it to identify prominent local features in your area, giving the grid reference of each.
2 Compare the printed topographic map of your local area with the map you drew in activity 1 above.
3 Locate your sketch point on the topographic map.
4 Determine the distance from the sketch point to prominent features using the scale shown on the map.

(*Continued*)

5 Use the topographic map to determine the grid bearing of prominent features from the sketch point. Use the grid magnetic angle shown on the map to convert each grid bearing to a compass bearing. Use a compass to compare the bearing of each feature taken from the sketch point.
6 Calculate the area in square kilometres covered by the topographic map.
7 Use the topographic map to determine the highest point in your region, and the bearing of this point from your sketch position. Can you locate this point by eye?
8 Use the map to determine the steepest hill in the region. Calculate its gradient. (The section following, on contours, will be of assistance in this calculation.)

Note: You can now view maps of your area using the above website.

investigation

Surveying — Then and now

Use the **NRW** weblink in your eBookPLUS to visit the Queensland Government Department of Natural Resources and Water website. The Mapping and surveying section includes information showing how modern technology is employed to create a wide range of maps and images.

This site also includes a Virtual Museum which provides an overview of the historic development of surveying and mapping in Queensland.

What instruments were used by surveyors prior to the development of today's modern devices?

How different would the current lifestyle of a surveyor be compared to that of one at work in the mid 1880s in our state?

Contour maps

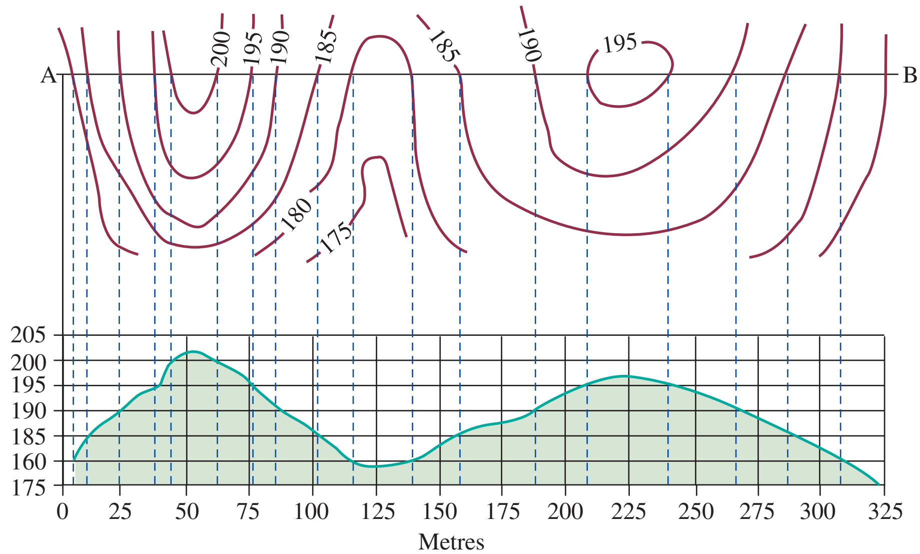
Although a map is a 2-dimensional representation of the 3-dimensional surfaces of the Earth, it can still be used to show landforms such as hills, valleys and cliffs. Major features such as hilltops will often show heights above sea level. **Contour lines** are imaginary lines that join points having the same height above sea level. These lines are drawn following accurate surveys by air and ground. Contour maps are extremely useful for hikers and emergency workers.

Engineers who are selecting routes for railway lines and roads require accurate contour maps. These maps are also necessary when an estimate of cut-and-fill requirements for roadworks or the capacity of a proposed dam is to be calculated.

Contour lines on a map show the height of imaginary lines above sea level. On any one map, the vertical difference in height between contour lines is constant and is known as the **vertical interval** or **contour interval**. The contour interval on the Mount Crosby topographic map shown on pages 356 and 357 is 5 metres. Contour lines that are close together indicate a steep slope and lines far apart indicate a gentle slope. Lines evenly spaced show a uniform slope.

Topographic maps also show the heights of highest hills in a region. A dot, with height shown in metres, is used to indicate this. The hill at Collingwood Park (GR 8544 in the map on page 357) has a height of 68 metres.

The figure below shows a section of a map with contour intervals of 5 metres. A hiker walking in a straight line from A to B as shown would go up a hill and down the other side and then across a comparatively less-steep hill and again down.



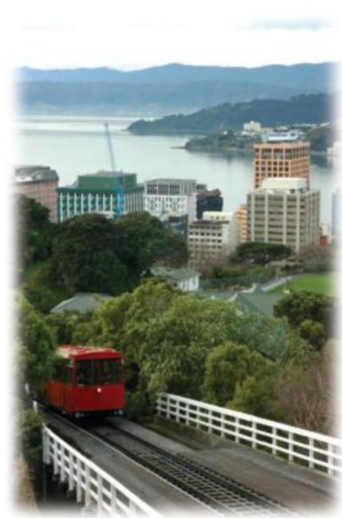
Profile of line AB

A *profile*, or cross-section, represents the surface of the ground. Profiles are drawn to show clearly the variation in height. Often, different scales are used on the axes, with the larger scale used on the vertical axis to exaggerate height changes. Vertical exaggeration is often necessary: if the height variation is only 200 metres and the profile length spans several kilometres then, without exaggeration, the profile would appear almost flat.

Vertical exaggeration (VE) is defined as:

$$\text{VE} = \frac{\text{vertical scale}}{\text{horizontal scale}}.$$

If the vertical scale used is 1 centimetre to 25 metres and the horizontal scale is 1 : 25 000 then:



$$\begin{aligned}\text{Vertical exaggeration} &= \frac{1\text{ cm} : 25\text{ m}}{1 : 25\,000} \\ &= \frac{1\text{ cm} : 2500\text{ cm}}{1\text{ cm} : 25\,000\text{ cm}} \\ &= \frac{\frac{1}{2500}}{\frac{1}{25\,000}} \\ &= \frac{1}{2500} \div \frac{1}{25\,000} \\ &= \frac{1}{2500} \times \frac{25\,000}{1} \\ \text{VE} &= 10\end{aligned}$$

A vertical exaggeration of 10 is often used. In mountainous areas, a VE of less than 10 may be adequate.

To construct a cross-section or profile between two points on a map, an edge of a piece of paper is placed on the line joining the points. The points where the contour lines are cut by the edge of the paper are marked on the paper. The paper edge is transferred to the horizontal scale of the profile. A suitable VE is chosen. At each point where a contour line crosses the paper, the height is plotted. The points are joined with a smooth line (as seen in the previous figure).

The gradient between two points is calculated by the formula:

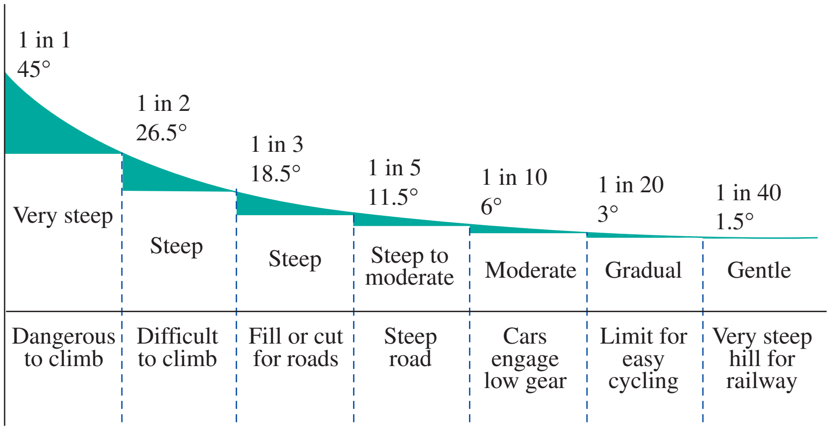
$$\text{Gradient} = \frac{\text{difference in heights}}{\text{horizontal distance}}.$$

Since this corresponds to the tangent ratio, the angle of fall or rise can be calculated. If point X is at 870 m and Y is at 720 m and these points are separated by 1600 m, then:

$$\text{Gradient A to B} = \frac{150}{1600}$$

$$= \frac{1}{10.66} \text{ (by dividing top and bottom by 150)}$$

Rounding the denominator to the nearest whole number gives a gradient of 1 in 11. Geographers often quote gradients in this manner. A gradient of $\frac{1}{10.66}$ or 0.093 75 is equivalent to an angle of 5.4° (using inverse or second function and tangent keys). The figure below shows the classification of gradients.



Classification of gradient

WORKED Example 15

Use the contour map and profile on page 361, to answer the following. (All contours are in metres).

a What contour interval is used to draw this map?
b What is the greatest height shown?
c What is the smallest height shown?
d A hiker walks from B to A. Briefly describe what the hiker would experience.
e Why is vertical exaggeration (VE) used to construct this profile?

THINK	WRITE
a The lines are marked 200, 195, 190 etc., so the contour interval must be 5 m.	**a** The contour interval is 5 m.
b The largest number shown on any line is 200.	**b** The greatest height is 200 m.
c The smallest number shown on any line is 175 m.	**c** The smallest height is 175 m.
d From B the profile rises and falls twice.	**d** The hiker would climb a hill, then come down the other side to encounter another rise and fall.
e Explain the need for vertical exaggeration.	**e** Without any exaggeration on the vertical scale, the profile would appear almost flat.

WORKED Example 16

A profile shows the vertical scale as 1 cm : 10 m and a horizontal scale of 1 : 10 000. What vertical exaggeration has been used?

THINK	WRITE
1 Recall the definition for vertical exaggeration.	$\text{VE} = \dfrac{\text{vertical scale}}{\text{horizontal scale}}$
2 Show the horizontal scale in cm.	$= \dfrac{1\text{ cm} : 10\text{ m}}{1\text{ cm} : 10\,000\text{ cm}}$

Continued over page

THINK | **WRITE**

3. Convert vertical scale to cm.

$$\text{VE} = \frac{1\text{ cm} : 1000\text{ cm}}{1\text{ cm} : 10\,000\text{ cm}}$$

4. Convert ratios to fractions and divide to obtain VE.

$$= \frac{1}{1000} \div \frac{1}{10\,000}$$
$$= 10$$

WORKED Example 17

Point A is at 1250 metres and point B is at 1160 metres further down the hill. The two points are separated by a horizontal distance of 900 metres.

a Calculate the gradient.

b Describe the nature of this gradient using the classification in the figure on page 362.

c Calculate the angle of this gradient.

THINK | **WRITE**

a 1. Recall the formula for gradient.

a $$\text{Gradient AB} = \frac{\text{difference in heights}}{\text{horizontal distance}}$$

2. Substitute values into the formula.

$$= \frac{1250 - 1160}{900}$$
$$= \frac{90}{900}$$
$$= \frac{1}{10}$$

b A gradient of 1 in 10 is described as 'moderate' (see figure on page 362).

b Gradient of 1 in 10 is moderate.

c To find the angle which has a tangent ratio, convert $\frac{1}{10}$ to a decimal (that is, 0.1) and use the inverse key.

c
$$\tan a = \frac{1}{10}$$
$$= 0.1$$
$$\therefore a = 5.7°$$

remember

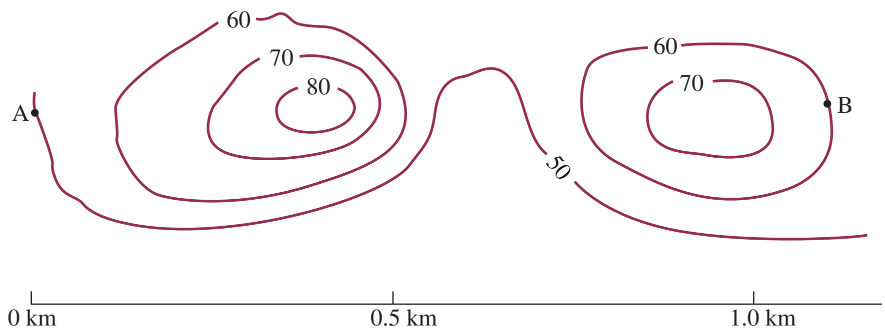
1. Contour lines are imaginary lines that join points having the same height above sea level.
2. Vertical exaggeration (VE) is defined as:

$$\text{VE} = \frac{\text{vertical scale}}{\text{horizontal scale}}.$$

EXERCISE 6I Contour maps

WORKED Example 15, 16

1 The figure below shows a contour map of a region. All contours are in metres.



a What contour interval is used to draw this map?
b What is the greatest height shown?
c What is the smallest height shown?
d A hiker walks from A to B. Briefly describe what the hiker would experience.
e Construct a profile of A to B. Use a horizontal scale of 10 cm : 1 km and a vertical scale of 1 cm : 20 m.
f Calculate the vertical exaggeration in **e**.

2 Use the topographic map on page 357 to state the height of the hill at GR:
a 8446 (the hill with the tower)
b 8544 (the hill at Collingwood Park).

3 The contour map on page 366 shows a region of the coastline. A river enters the sea. The river and the coastline do not have a contour figure on them.
a What contour interval is used on this map?
b What is the distance from X to Y?
c Use the scaled frame at the bottom of the figure to draw a cross-section from X to Y.
d Find the vertical exaggeration for this cross-section. (Note that 20 mm represents 100 m for the vertical scale.)
e Estimate the grid bearing of X from Y using a protractor.
f Do you think it would be possible to see X from Y?

WORKED Example 17

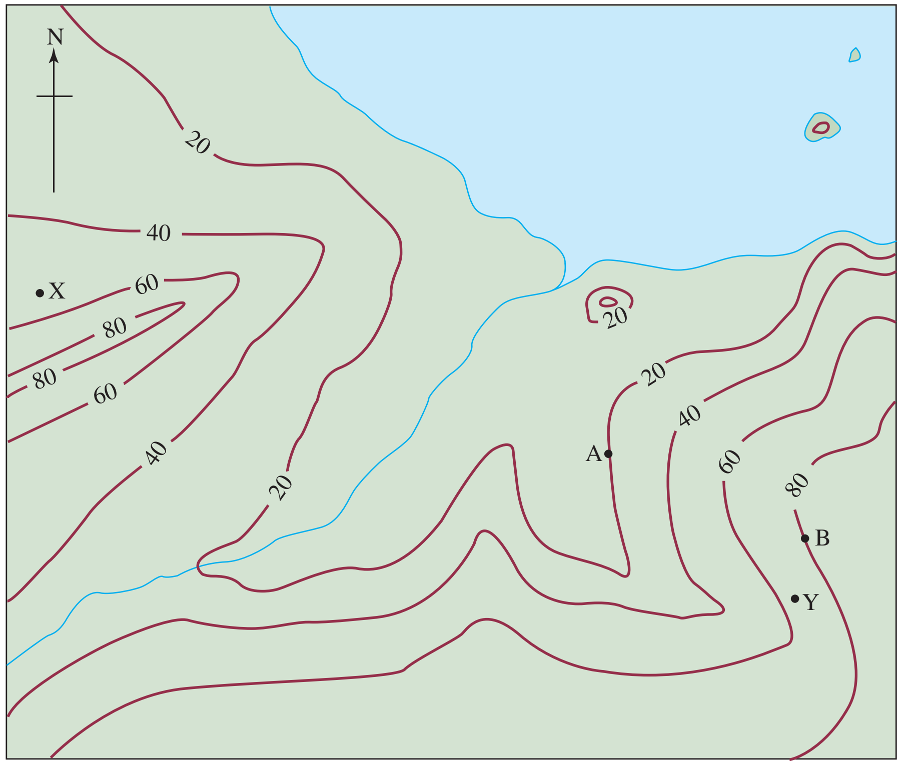
4 Point P is at 2147 m and Q is at 3527 m. They are separated by a horizontal distance of 4.7 kilometres. Calculate:
a the gradient
b the angle of this gradient.
How would you describe this gradient?

eBook plus

Digital doc: SkillSHEET 6.3 Trigonometric values and angles

5 Convert the following gradients to angles.
a 1 in 1
b 1 in 3
c 10 m in 1000 m
d $\frac{1}{50}$
e 85 m in 1640 m

6 Use the map on page 366 to determine the angle of rise from A to B.



SCALE 1:100 000

Contour interval: 20 metres

Local contours

Materials required: local topographic map.

1 Use the topographic map for your area to find:

a the highest point in the region

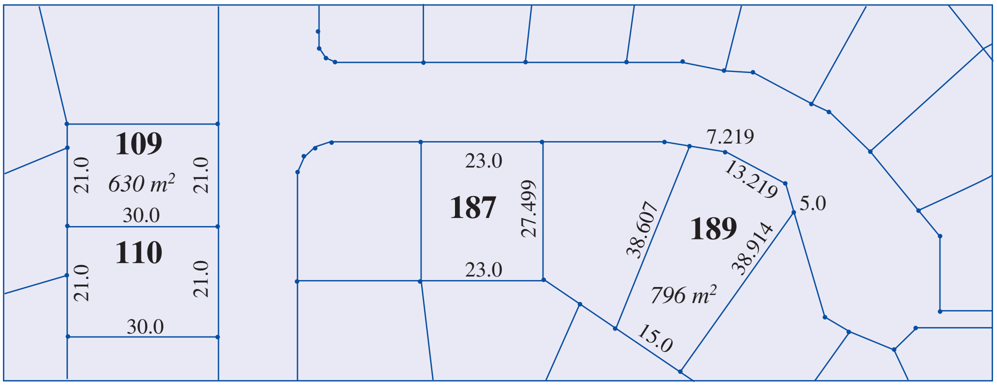
b the steepest hill in the region.

2 Use the topographic map for your local area to construct a cross-section between two prominent landmarks.

3 Use the topographic map to determine the gradient of the steepest hill at its steepest section.

Cadastral maps and site plans

A **cadastral map** is a map that shows the boundaries of blocks of land in a survey plan (see map below). The names of owners of allotments must be accurately recorded and each lot must be well defined. Queensland has been divided into counties and then parishes. Parishes are further subdivided into portions. Further subdivisions ultimately allocate a lot number to each block of land. Surveyor's plans must be extremely accurate for these maps.



A cadastral surveyor's equipment includes:

1. a theodolite for reading horizontal and vertical angles
2. a level to provide a horizontal reference plane to determine height differences (rotating laser beams are often used for this purpose)
3. steel tape to measure distances (these tapes are regularly checked for accuracy, and corrections are made to take temperature variations into account)
4. laser beam distance measuring equipment.

Data collected on-site can be processed by computer to produce extremely accurate maps.

Surveyors and engineers require key pieces of information in the initial stages of development of a new town or suburb. The height above sea level and latitude and longitude of a certain location are important starting points. Information such as this is retained by the Department of Natural Resources and is referred to as *survey control information*. Thousands of marks have been placed over Queensland by surveyors as permanent reference points. These **permanent survey marks** (PSMs) are most commonly circular bronze plaques, 10 cm in diameter and are set in concrete, level with the surrounding soil. The photograph above shows an example of a permanent survey mark. Alternatively, long rods and pickets are used in unstable soil and sand.

Each mark forms part of an expanding network of survey information. All three levels of government contribute to this from time to time. The Main Roads Department or Railways Department may place permanent survey marks when extending roads and railway lines. Local governments may place marks when installing drainage and sewerage facilities.

Recent technology allows points to be positioned on the Earth's surface using orbiting satellites. A repeated signal monitored from the passing satellite can give the height and position of a point to within a few metres.

WORKED Example 18

The figure on page 367 is a cadastral map and has a number of blocks of land that are labelled.

a Give the area of Lot 189.

b What are the dimensions of Lot 109?

c Is the described area for Lot 109 correct? Give a reason for your answer.

THINK	WRITE
a The area is indicated on the map.	**a** 796 m^2
b The dimensions are 21 m × 30 m.	**b** 21 × 30 m
c Lot 109 is a rectangle, and the area of a rectangle is length × width. 21 × 30 = 630 m^2.	**c** Yes. Area = 21 × 30 = 630 m^2

remember

A cadastral map shows boundaries of blocks of land in a survey plan.

EXERCISE 6J Cadastral maps and site plans

WORKED Example 18

1 The cadastral map on page 367 has several lots of land showing areas and dimensions in metres.

a Give the area of Lot 109.

b Give the dimensions (length and breadth) of Lot 187.

c The area of Lot 187 is not shown. Calculate its area.

d What scale has been used to produce this plan?

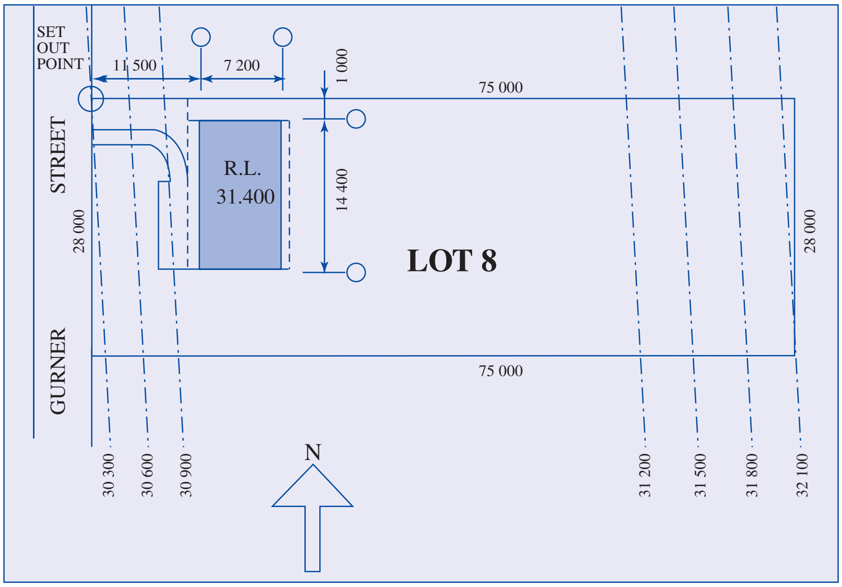
e Draw a 1 : 500 plan of Lot 109.

f Before the metric system was introduced, the area of house blocks was measured in perches (1 perch = 25.3 m^2).

- **i** A block of 42 perches is advertised for sale at $61 500. Convert the area to square metres and find the price per square metre.
- **ii** A lot is 850 m^2 and another is 28 perches. Which is the larger lot?

g Lot 109 is for sale at $59 850 and Lot 189 is for sale at $82 000.
  i Which has the lower cost per square metre?
  ii What features of a block of land might attract a purchaser even though its dollar value per square metre may be higher than surrounding blocks?
  (Comparing Lots 109 and 189 can assist in your answer, but include as many features as possible.)

2 The figure below shows the site plan for Lot 8 on Gurner Street. All dimensions given are in millimetres.



a Find the area of Lot 8 in square metres and perches.
b The shaded sketch shows the area of the proposed dwelling to be erected on this lot. What is the area of the proposed dwelling?
c What is the distance from the rear of the dwelling to the back boundary?
d What fraction of the land does the proposed dwelling occupy?
e The dashed lines are contour lines (lines of height). All points along the 31 800 line are 31 800 mm above sea level.
  i Is the block rising or falling as I walk from the Gurner Street entrance to the rear of the block?
  ii What is the difference in height from the front boundary to the rear?
  iii Calculate the angle of rise or fall from the front to the rear.

investigation

Survey maps: old and new

Compare survey maps

Obtain a number of survey maps from your area. It will be possible to find survey maps that were drawn well over 100 years ago, particularly if you live in a well-established town. Contact your local council and ask parents and relatives for maps. Many people keep copies of survey maps when purchasing property. Organise a wall display showing the survey maps in chronological order of production. Use a recent map to locate a piece of land in your area.

Compare features noted in past maps with present-day maps.

Cadastral maps and site plans

1 Obtain a Permanent Mark Sketch Plan from the Department of Natural Resources for your area. This shows the exact location of individual permanent survey marks. Use this plan to locate several of these features on the ground.

2 Contact your local government authority and obtain a flood-risk map for your area. Note local regions that are subject to flooding.

3 Obtain the site and drainage plans for the school from either the school office or the local authority. Locate features marked on these plans. Check the plans for accuracy by comparing positions of objects drawn with their actual position.

Mapping from air and space

Orthophoto maps

Land feature or orthophoto maps show the Earth's natural and man-made features and are based on an aerial photograph overlaid with contour information and major road and site names. They are of great use in flood control and water supply.

Investigate the method of production and applications of these maps. A brochure describing these maps is available from the Department of Natural Resources. Obtain an orthophoto map of your area and identify key features.

Remotely sensed imagery

Remotely sensed imagery means that an image has been collected remotely; that is, at a distance. The Landsat Satellite orbits Earth at an altitude of 705 km and records scenes that measure 185 km by 185 km.

1 What information is collected by Landsat?

2 How is this information interpreted?

3 How is the imagery from Landsat used?

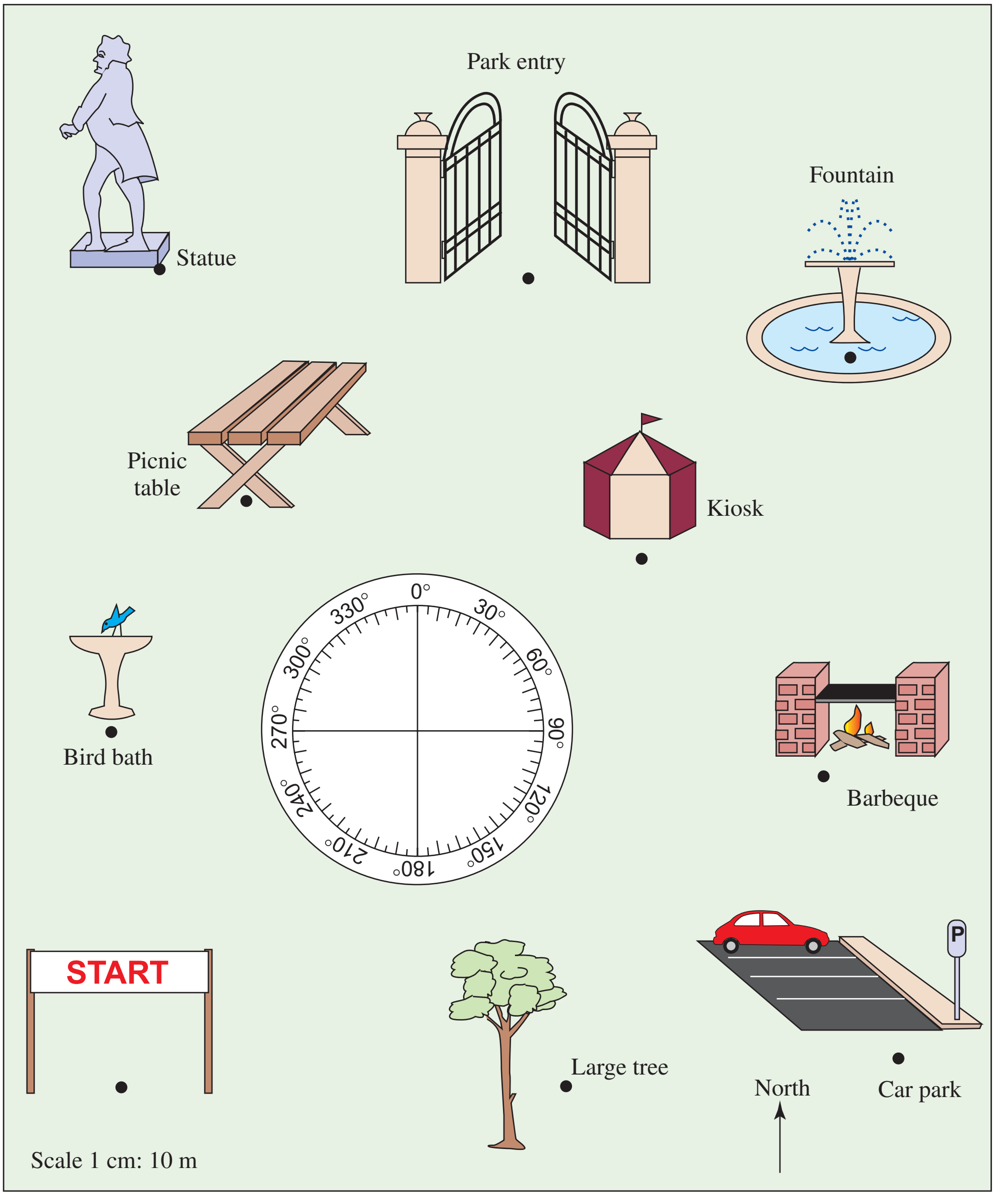
Brochures and posters showing examples of remotely sensed imagery can be obtained from the Department of Natural Resources.

Orienteering

Orienteering is a physically active and enjoyable pastime that requires map and compass reading skills and an ability to estimate distances. The aim of orienteering is to complete a given course that has been planned and mapped. There are many checkpoints in an orienteering course with competitors visiting each.

The following exercise is an example of a simple orienteering course conducted in a park. The directions that are listed below could be either:

1. given to each participant prior to commencing the course, or
2. left at each checkpoint.



Use the compass rose or a protractor and your ruler to follow this course in the figure shown above.

1. Find the point in the park labelled 'Start'.
2. Proceed on a bearing of 090° for 60 m.
3. Follow 020° for 103 m.
4. Now move 94 m on 277°.
5. Proceed along 186° for 62 m.
6. Now follow 094° for 95 m.

If you followed the course correctly you should be at the barbeque!

The bearings used in orienteering exercises are magnetic to save competitors the time of converting between magnetic and true.

remember

The aim of orienteering is to complete a given course that has been mapped.

EXERCISE 6K Orienteering

1 Use the figure on the previous page to find the bearing of:
   a the kiosk from the large tree
   b the car park from the statue
   c the bird bath from the park entry
   d the fountain from the start.

2 Find the distance separating the landmarks listed in questions a to d above.

3 Devise a set of 8 instructions, each a distance and a bearing, that would have you start and finish at the bird bath. Swap your set of instructions with another student and follow each other's course.

eBook plus
Digital doc: WorkSHEET 6.2

investigation — Planning an orienteering course

Materials: compass, paper, pen.
In your school yard or nearby park, design an orienteering course for other class members to follow. The instructions can be either:

1 listed as a complete series of instructions (distance to travel and bearing) and handed to the competitors before starting, or

2 posted at each checkpoint along the course.

To add interest, a letter of a word could be at each checkpoint and competitors could be required to unjumble the word on finishing the course.

Notes

1. The Queensland Orienteering Association welcomes new members and conducts regular events. To find out more about the Queensland Orienteering Association use the **Orienteering** weblink in your eBookPLUS.

2. You can now produce your own high quality colour orienteering map of any area where you would like to set up a course using the latest version of OCAD.

summary

Area of triangle

- The triangle is a figure regularly used in surveying techniques. Its area can be found by:
  1. using base and perpendicular height
     Area = $\frac{1}{2}$ × base × perpendicular height
  2. using 2 side lengths and the angle between them
     Area = $\frac{1}{2}$ × side 1 × side 2 × sin (angle between sides 1 and 2)
  3. Heron's formula
     Area = $\sqrt{S(S-a)(S-b)(S-c)}$
     where a, b and c are the sides and the semi-perimeter, S, is given by
     $$S = \frac{1}{2}(a + b + c)$$
- Any polygon can be divided into a number of triangles.

Surveying

- Surveyors often set up a survey line when surveying an area. Perpendicular distances to surrounding features are known as *offsets*.
- The presence of obstacles such as rivers, swamps, hills or dwellings can result in the survey line being interrupted. Obstacles can be bypassed using the offset and triangulation methods.
- Plane table surveying makes use of a horizontal table. Three plane table surveying methods are:
  1. intersection or triangulation
  2. radiation, and
  3. traversing.

Topographic maps

- A topographic map is a detailed sketch of a portion of the surface of the Earth and includes natural and artificial features.
- These maps have grid lines showing the direction of grid north.
- True north is the direction of the North Pole and magnetic north is the direction in which a compass points.
- Magnetic north varies slightly annually and its relationship to grid and true north is shown on topographic maps.
- Contour lines are imaginary lines joining points having the same height above sea level.
- A profile or cross-section shows the surface of the Earth between two points.
- Vertical exaggeration is often used when drawing profiles.

$$\text{Vertical exaggeration} = \frac{\text{vertical scale}}{\text{horizontal scale}}$$

$$\text{The gradient between two points} = \frac{\text{difference in heights}}{\text{horizontal difference}}$$

This corresponds to the tangent ratio and can be given as a ratio or an angle.

Cadastral maps

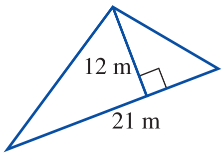
- A cadastral map shows the boundaries of blocks of land.

CHAPTER review

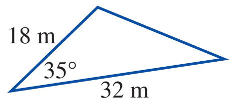
6A

1 Find the area of the diagrams below, to the nearest square metre.

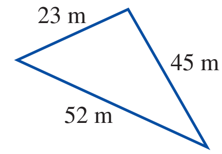
a



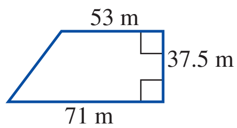
b



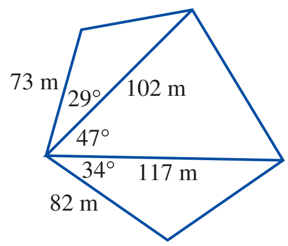
c



d

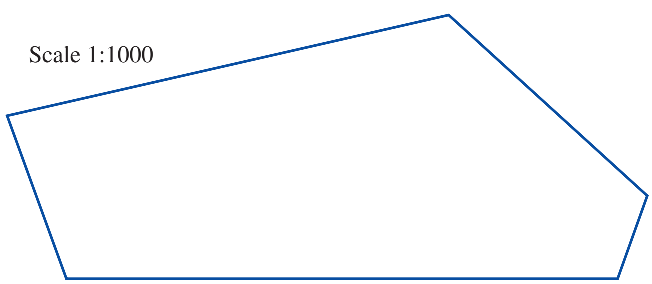


e



6B

2 The figure below shows a paddock drawn to a scale of 1 : 1000. Find the area of the paddock, to the nearest 0.1 hectare.



6C

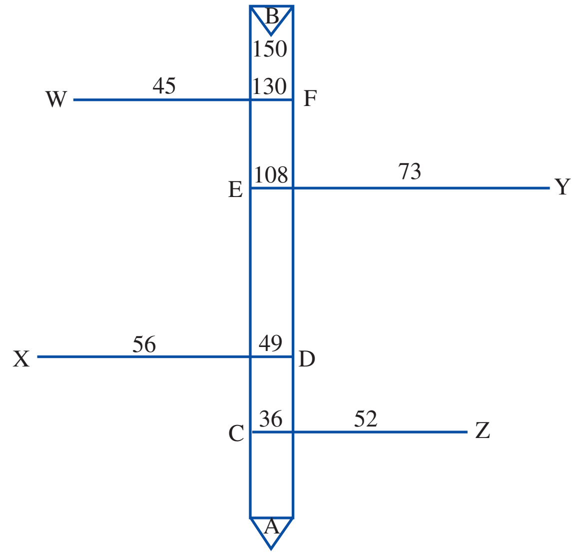
3 A field sketch is shown at right, not to scale. All figures are in metres.

a Find the distance:

i AB ii CZ iii AZ
iv CD v ZY vi XY.

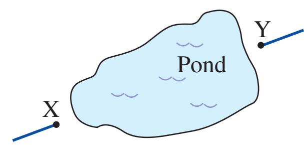
b Find the area of:

i △ACZ ii △BEY
iii figure EYZC iv figure AXWB.

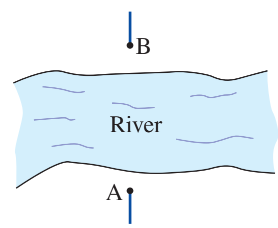


6D

4 a Show diagramatically how XY in the figure at right could be found using the offset method.

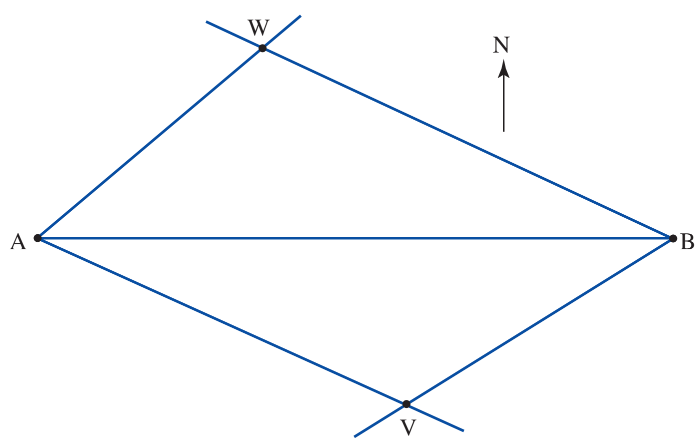


b In the figure at right, show how AB, the distance across a river, could be determined using triangulation.



6E

5 The following diagram shows a map resulting from a plane table survey using triangulation (1 : 1000 scale).



Find:

a the length of survey line AB

b the bearing of:

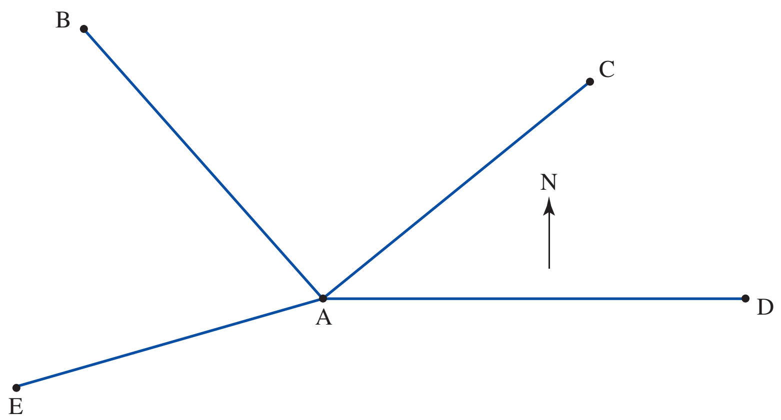
i W from A ii V from A iii W from B iv V from B v B from A

c the area of quadrilateral AWBV to the nearest 100 m^2

d the perimeter of AWBV (to the nearest 10 metres).

6F

6 The figure below shows a sketch drawn by the plane table radiation method drawn to a scale of 1 : 1000.



Find:

a the length of:
 i AE **ii** AB **iii** AC **iv** AD **v** EC

b the bearing of:
 i C from A **ii** D from A **iii** E from A

c the area of △ABC to the nearest 100 m^2.

6G

7 The table shows the recordings made by a surveyor.

Sta.	BS	IS	FS	HI	RL	Dist.	Notes
A	3.90			53.60	50.00	20.00	TBM
B			2.70	53.60			

a State the:
 i backsight **ii** foresight **iii** height of instrument **iv** reduced level of A.

b Calculate the RL of B.

c Sketch a profile showing points A and B, and the theodolite positioned between them. Include the backsight and foresight.

6H

8 **a** With reference to the Mount Crosby topographic map (see pages 356 and 357), what feature is located at:
 i GR 868473? **ii** GR 855495?

b Use the scale shown to estimate the distance from Bremer Junction to the centre of the shopping centre south of the Cunningham Highway (near Redbank). Scale is 1 : 25 000.

c A plane flies directly over the route in **b** above. On what grid bearing does it fly?

6I

9 Point A is at 1620 m and Point B is at 1870 m. They are separated by a horizontal distance 1250 m. Calculate:

a the height difference

b the gradient

c the angle of this gradient. Include a description using the classification on page 362.

6I

10 Convert each of the following gradients to angles.

a 1 in 1 **b** 1 in 2

c 10 m in 500 m **d** $\frac{1}{20}$

e 80 m in 630 m

6I

11 A profile shows the vertical scale as 1 cm : 10 m and a horizontal scale of 1 : 10 000. What vertical exaggeration has been used?

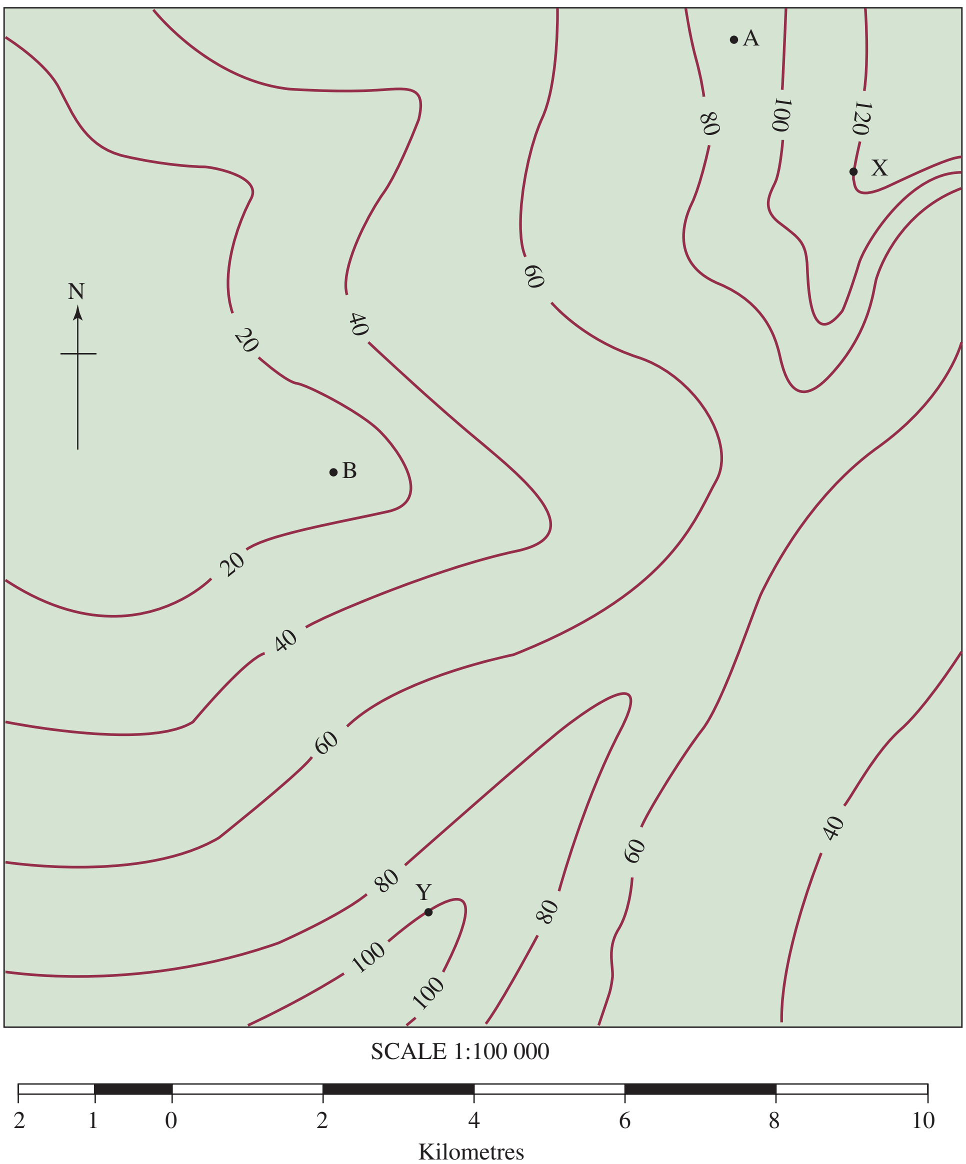
6I

12 Refer to the contour map below.

a Estimate the height of A.

b What is the contour interval?

c Draw a profile from Y to X. Use a vertical exaggeration of 20.



6J

13 Use the figure on page 367 to:

a find the area of Lot 110

b draw a 1 : 500 plan of Lot 110.

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Digital doc: Test Yourself Chapter 6

eBook*plus* ACTIVITIES

6A Perimeters and areas of triangles

Digital docs

- Spreadsheet 063: Investigate perimeter and area *(page 317)*
- SkillSHEET 6.1: Practise using Pythagoras' theorem *(page 318)*
- Spreadsheet 054: Investigate trigonometric ratios *(page 318)*
- SkillSHEET 6.2: Practise calculating areas of triangles using Heron's formula and the sine rule *(page 322)*

6B Perimeters and areas of polygons

Digital docs

- SkillSHEET 6.2: Practise calculating areas of triangles using Heron's formula and the sine rule *(page 324)*
- Spreadsheet 063: Investigate perimeter and area *(page 324)*
- Spreadsheet 047: Investigate map scale 1 *(page 324)*
- Spreadsheet 048: Investigate map scale 2 *(page 324)*

6D Surveying around obstacles

Digital doc

- WorkSHEET 6.1: Calculate perimeter, area of triangles and composite shapes using different methods, use Pythagoras' theorem and determine bearings *(page 336)*

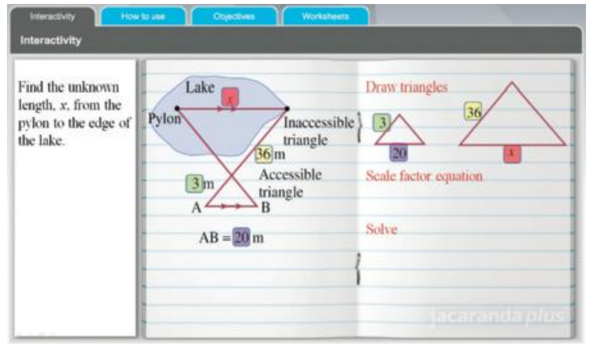
Tutorial

- WE7 Int-0474: Watch how to make triangulation and similarity calculations *(page 334)*

6F Plane table surveying: radiation and traversing

Digital docs

- Spreadsheet 047: Investigate map scale 1 *(page 345)*
- Spreadsheet 048: Investigate map scale 2 *(page 345)*



6H Topographic maps

Digital doc

- SkillSHEET 6.3: Practise finding trigonometric values and angles *(page 358)*

6I Contour maps

Digital doc

- SkillSHEET 6.3: Practise finding trigonometric values and angles *(page 365)*

6K Orienteering

Digital doc

- WorkSHEET 6.2: Make land measurement calculations including survey measurements, using scales, calculating and converting gradients to angles *(page 372)*

Chapter review

Digital doc

- Test Yourself: Take the end-of-chapter test to test your progress *(page 377)*.

To access eBookPLUS activities, log on to

www.jacplus.com.au

7

Linear programming

syllabus reference

Elective topic

Operations research — linear programming

In this chapter

What is linear programming?

Linear programming was developed during the late 1940s to assist in the organisation of military supplies. Today it is extensively used in fields such as business, engineering, industry and social sciences.

Linear programming is a mathematical technique used to solve real-life situations in which a particular quantity is to be maximised or minimised — for example time, money, profit — subject to given constraints or restrictions. A specific example would be maximising the profit a company makes, subject to:

1. the number of employees
2. the number of hours they can work
3. the cost of producing goods

and so on.

Linear implies that the restrictions and the quantity to be maximised or minimised follow linear patterns. **Programming** means that it follows a systematic plan.

The problem at the bottom of this page is an example of a linear programming problem.

SKILLS CHECK

1 Consider the equation $y = 4x - 6$. Calculate the value of y when:
a $x = 2$ **b** $x = 0$ **c** $x = -5$.

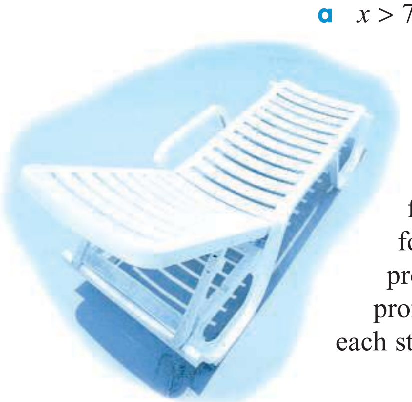
2 Consider the equation $2x - 3y + 12 = 0$.
a Determine the value of x when $y = 0$.
b Determine the value of y when $x = 0$.

3 Rearrange the following equations to make y the subject of the formula.
a $5x - y = 6$ **b** $6x + 5y = 10$ **c** $7x + 2y + 5 = 0$

4 Describe the graph of each of the following lines.
a $x = 3$ **b** $y = 6$ **c** $x = y$

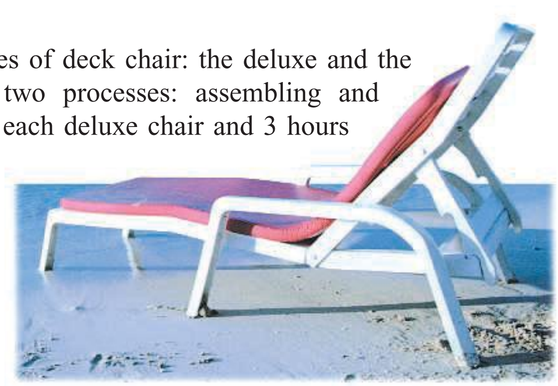
5 Give the equation of:
a the x-axis **b** the y-axis.

6 Give possible integer values which would satisfy each of the following relationships.
a $x > 7$ **b** $x \le 4$ **c** $3 \le x < 10$



Problem

A local manufacturer makes two types of deck chair: the deluxe and the standard. Each chair goes through two processes: assembling and finishing. It takes 4 hours to assemble each deluxe chair and 3 hours for each standard chair. The finishing process takes 2 hours for each chair. The profit on each deluxe chair is $15 and on each standard chair, $12. If employees spend



at most 48 hours on assembling and 28 hours on finishing each week, how many chairs of each type should be made weekly to maximise the manufacturer's profit?

Mathematical formulation of the problem

1. Define the variables:
   (a) Let d represent the number of deluxe chairs.
   (b) Let s represent the number of standard chairs.
   (c) Let P represent the profit.
2. Write the constraints as inequations in terms of the variables:
   $4d + 3s \leq 48$ time available for assembling chairs
   $2d + 2s \leq 28$ time available for finishing chairs
   $d \geq 0$, $s \geq 0$ } Since the number of chairs can not be negative, the variables d and s must be positive or zero.
3. Determine what must be maximised: the maximum profit needs to be obtained.
4. Write the function that needs to be maximised in terms of the variables:

$$P = 15d + 12s$$

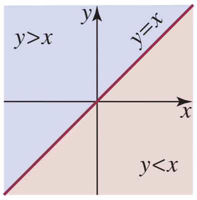
From this point the problem can be solved by graphical methods, which we use later in the chapter.

To solve linear programming problems, it is important to be able to sketch graphs of inequations and determine points of intersection between graphs. Once we understand this, we can then begin to maximise and minimise linear functions and start to solve linear programming problems like this one.

Graphs of linear inequations

When a straight line is drawn on a plane, it divides the plane into three sets of points: the points above the line, on the line and below the line.

The diagram on the right shows the graph of $y = x$. The points on the line itself satisfy the equation $y = x$. The region above the line represents the points that satisfy the inequation $y > x$, while the region below the line represents the points that satisfy the inequation $y < x$.



To indicate which part of the plane we want, shading is used. *Note:* Throughout this chapter the following format will be used:
1. The area that is required will be shaded.
2. The area that is not required will remain unshaded.

To sketch the graph of an inequation, the following algorithm can be used:
1. For the boundary, replace the inequality sign with an equals sign and sketch the graph of the equation thus formed.
   (a) If the inequality sign was < or >, use a dotted line (to indicate that the points on the line itself are not included).
   (b) If the inequality sign was ≤ or ≥, use a solid line (to indicate that the points on the line are included in the region required).
2. To determine the side of the line where the required region lies:
   (a) Choose any point on one side of the line (a test point).
   (b) Substitute the coordinates of the test point into the inequation.

(c) If the result is a true statement, the point that was chosen belongs to the required region.
(d) If the result is not a true statement, the selected point belongs to the region that is not required.

3. Shade the region that is required.
4. Add the legend to your graph: ▭ Region required

The following worked examples illustrate this concept.

WORKED Example 1

Sketch the graph of:

a $y \geq 0$ **b** $x < 6$

and shade the required region.

THINK

a
1. Rule a labelled set of axes including the origin.
2. For the boundary, replace $\geq$ with = and sketch the equation $y = 0$ (this is a horizontal line which coincides with the x-axis). Since the inequation contains $\geq$, a solid line must be drawn.
3. Choose a test point on the y-axis, say $y = 6$.
4. Substitute 6 into $y \geq 0$ to see if it satisfies the inequation; that is, is $6 \geq 0$? (Correct.)
5. Since the inequation is correct, shade the side of the line that contains the point.
6. Add the legend to the graph, indicating that the region required is shaded.

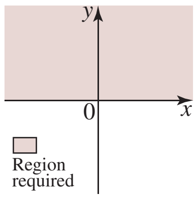
WRITE

a

Boundary equation is $y = 0$.

Test point: (0, 6)

Is $6 \geq 0$? Yes



THINK

b
1. Rule a labelled set of axes, including the origin.
2. Sketch the equation $x = 6$ (a vertical line that passes through 6 on the x-axis). Since the inequation contains <, a broken line must be drawn.
3. Choose a test point on the x-axis, say $x = 0$.
4. Substitute 0 into $x < 6$ to see if it satisfies the inequation; that is, is $0 < 6$? (Correct.)
5. Since the inequation is correct, shade the side of the line that contains the point.
6. Add the legend.

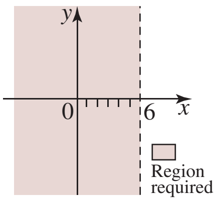
WRITE

b

Boundary equation is $x = 6$.

Test point: (0, 0)

Is $0 < 6$? Yes



The graph in part **a** in the previous worked example is a *closed half-plane* defined by $y \geq 0$. It is called 'closed' since it includes the equation $y = 0$. It includes the points on or above the line $y = 0$.

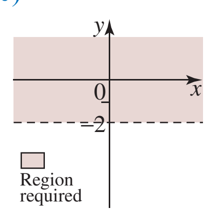
The graph in part **b** is an *open half-plane* defined by $x < 6$ ('open' since it excludes the equation $x = 6$). It is the set of points to the left of the line $x = 6$.

Sometimes the inequation has to be transposed before sketching to make x (or y) the subject. While transposing, keep in mind that multiplying or dividing both sides of an inequation by a negative number changes the direction of the sign of inequality to its opposite.

WORKED Example 2

Sketch the graph of $-y + 3 < 5$ and shade the required region.

THINK	WRITE
1 To make y the subject, subtract 3 from both sides of the inequation.	$-y + 3 < 5$ $-y + 3 - 3 < 5 - 3$ $-y < 2$
2 Multiply both sides of the inequation by −1.	$-1 \cdot -y < -1 \cdot 2$ $y > -2$
3 Change the direction of the inequality sign.	
4 Rule a labelled set of axes, including the origin.	
5 For the boundary, sketch the equation $y = -2$ (a horizontal line, passing through −2 on the y-axis). Since the inequation contains >, a broken line must be drawn.	Boundary equation is $y = -2$.
6 Choose a point on the y-axis, say $y = 0$.	Test point: (0, 0)
7 Substitute 0 into $-y + 3 < 5$ to see if it satisfies the inequation; that is, is $0 > -2$? (Correct.)	Is $0 > -2$? Yes
8 Since the inequation is correct, shade the side of the line that contains the point.	
9 Add the legend.	



In the following worked example, we consider the graphing of linear inequations with two variables, x and y.

WORKED Example 3

Sketch the graph of the inequation $y - 4x \leq 8$ and shade the required region.

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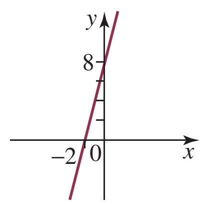
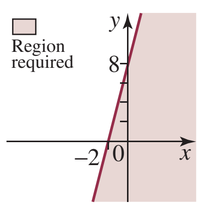
Tutorial: Worked example 3 int-0482

THINK	WRITE
1 Replace ≤ with = to find the x- and y-intercepts of the boundary.	Boundary equation is $y - 4x = 8$.

Continued over page

THINK	WRITE
2 To determine the x-intercept, let $y = 0$.	x-intercept: $y = 0$ $0 - 4x = 8$ $-4x = 8$ $x = -2$ $(-2, 0)$
3 To determine the y-intercept, let $x = 0$.	y-intercept: $x = 0$ $y - 4 \cdot 0 = 8$ $y = 8$ $(0, 8)$
4 Rule a labelled set of axes, including the origin.	
5 To sketch the graph of the equation $y - 4x = 8$, mark the x- and y-intercepts on the set of axes and join them with the straight line. Since the inequation contains ≤, a solid line must be drawn.	
6 To determine the region required, choose a test point on one side of the line, say (0, 0).	Test point: (0, 0) Substitute into $y - 4x \le 8$ $0 - 4 \cdot 0 \le 8$? Is $0 \le 8$? Yes
7 Substitute the coordinates of the test point into the inequation to see if (0, 0) satisfies it; that is, is $0 \le 8$?	
8 Since the inequation is correct, shade the side of the line that contains the point.	
9 Add the legend.	

Graphics Calculator tip! Graphing linear inequations

Linear inequations can be sketched with the aid of a graphics calculator. For instance, the graph of the inequation in Worked example 3 can be graphed as follows.

First transpose the inequation to make y the subject: $y \le 4x + 8$.

For the Casio *fx*-9860G AU

1. To graph an inequation, press:
   - MENU
   - 5: GRAPH
   - F3 (TYPE)
   - F6 (▶).
   - F4 (Y ≤).

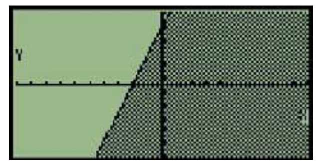
Complete the entry line as:
Y1 ≤ 4 · X + 8.

Then press EXE.

2. To alter the viewing window, press
   - SHIFT
   - F3 (V-WIN).

Enter the settings as shown, and then press EXE.

3. To draw the graph, press F6 (DRAW). The graph appears, with the required region shaded.



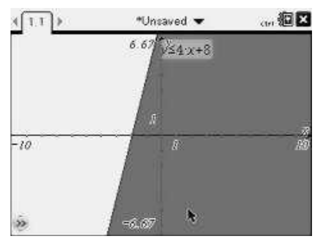
For the TI-Nspire CAS

To graph an inequation, open a Graphs page.

In the function area at the bottom of the screen, press the Clear button to delete = .

Complete the entry line as:
$y \leq 4 \cdot \; x + 8$.

Then press ENTER.
The graph appears, with the required region shaded.



remember

1. The graph of an inequation containing ≤ or ≥ is a closed half-plane; ≤ or ≥ indicates that a solid line is drawn and it is included in the required region.
2. The graph of an inequation containing < or > is the open half-plane; < or > indicates that a dotted line is drawn and it is excluded from the region required.
3. The required region is shaded; this is stated by adding the legend to the graph:
   ▭ Region required
4. Before sketching the inequation, it must be transposed to make the pronumeral the subject.
5. Multiplying or dividing both sides of an inequation by a negative number changes the direction of the sign of inequality to its opposite.
6. The origin (0, 0) is the most convenient test point to use when determining the region required, unless the straight line passes through it.

EXERCISE 7A Graphs of linear inequations

WORKED Example 1

1 Sketch graphs to represent the following inequations. Shade the required region.

a $y \geq 2$ b $y \leq 0$ c $y > -1$ d $y < 6$
e $y > 0$ f $y \leq -6$ g $x \geq 4$ h $x \leq 0$
i $x > -2$ j $x < 1$ k $x > 0$ l $x \leq -1$

WORKED Example 2

2 Sketch graphs to represent the following inequations. Shade the required region.

a $-x < 7$ b $-y \geq 2$ c $-x > 3$ d $-x \geq -5$
e $-y > -4$ f $-y \leq 3$ g $x + 2 > 0$ h $y - 3 \leq 0$
i $x - 8 < 0$ j $-x + 4 \geq 0$ k $y - 2 < 3$ l $-y - 6 \leq 2$

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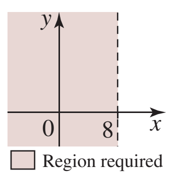
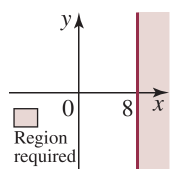
Digital doc:
SkillSHEET 7.1
Solving linear inequations

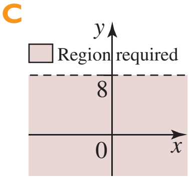
Interactivity:
Sketching inequations
int-0083

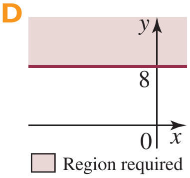
3 **multiple choice**

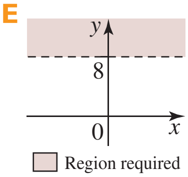
The expression $y > 8$ is best represented by the following graph:

A


B


C


D


E


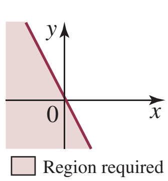
WORKED Example 3

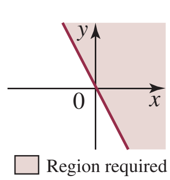
4 Sketch graphs to represent these inequations. Shade the required region.

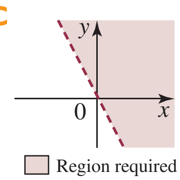
a $y - 2x \leq 4$ b $y + x \geq -1$ c $4x + 4y \leq 16$
d $y - x \leq 0$ e $y \geq -x + 2$ f $y > 2x - 14$
g $y < 6x - 24$ h $y \geq -7x + 21$ i $x + y > 0$
j $y \geq x + 7$ k $x > y - 2$ l $y > 12x - 24$
m $3y \leq x + 12$ n $-2y \geq 4x + 6$ o $x - y < 10$
p $y < x - 4$ q $2y > 4x - 8$ r $4x - 2y \leq 8$
s $2x - y > -1$ t $y - x - 4 < 0$ u $y + 2x - 6 \geq 0$
v $2y + 8x + 4 \leq 0$ w $9x + 9y + 9 \geq 0$ x $5x + 2y - 10 > 0$

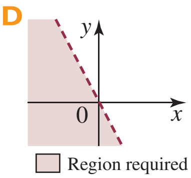
5 **multiple choice**

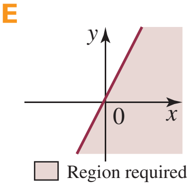
The expression $y \geq -2x$ is best represented by the following graph:

A


B


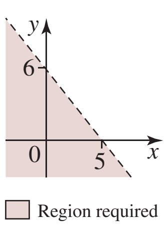
C


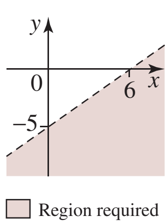
D


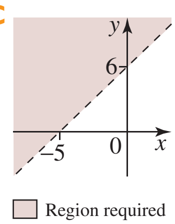
E


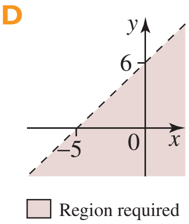
6 **multiple choice**

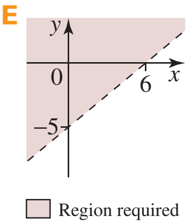
The expression $-12x + 10y - 60 < 0$ is best represented by the following graph:

A


B


C


D


E


Solutions of simultaneous linear equations

In this section we revise how to solve simultaneous linear equations.

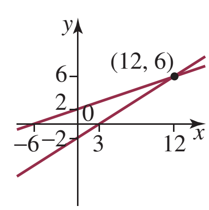
Recall the following facts:

1. Equations that are valid at the same time are called *simultaneous equations*.
2. The graphical solution to the pair of simultaneous linear equations is given by the point of intersection of the straight lines representing these equations.
3. The coordinates of the point of intersection can be read from the graph. The accuracy of the solution depends to a very large extent on the accuracy of the graph. Hence, the solution should be verified using algebraic methods.
4. To solve simultaneous equations algebraically, either elimination or substitution methods are used.

WORKED Example 4

Sketch the graphs of, and solve, the following pair of simultaneous linear equations.

$$2x - 3y = 6$$
$$x = 3y - 6$$

THINK	WRITE	
1 Find the intercepts of $2x - 3y = 6$.		
(a) Write the equation.	$2x - 3y = 6$	
(b) To determine the x-intercept, let $y = 0$.	x-intercept: $y = 0$ $2x - 0 = 6$ $2x = 6$ $x = 3$	(3, 0)
(c) To determine the y-intercept, let $x = 0$.	y-intercept: $x = 0$ $0 - 3y = 6$ $-3y = 6$ $y = -2$	(0, −2)
2 Find the intercepts of $x = 3y - 6$.		
(a) Write the equation.	$x = 3y - 6$	
(b) To determine the x-intercept, let $y = 0$.	x-intercept: $y = 0$ $x = 0 - 6$ $x = -6$	(−6, 0)
(c) To determine the y-intercept, let $x = 0$.	y-intercept: $x = 0$ $0 = 3y - 6$ $6 = 3y$ $y = 2$	(0, 2)
3 Sketch the graphs of $2x - 3y = 6$ and $x = 3y - 6$. The solution is the point at which the two graphs intersect.		
4 Verify the graphical solution obtained algebraically by substituting $x = 3y - 6$ into $2x - 3y = 6$.	$2(3y - 6) - 3y = 6$	

Continued over page

THINK	WRITE
5 Solve for y.	$6y - 12 - 3y = 6$ $3y - 12 = 6$ $3y = 6 + 12$ $= 18$ $y = 6$
6 Substitute $y = 6$ into either of the two equations to find the value of x. Choose $x = 3y - 6$.	$x = 3 \cdot 6 - 6$ $= 18 - 6$ $x = 12$
7 The solution set to the simultaneous equations $2x - 3y = 6$ and $x = 3y - 6$ is (12, 6).	The solution is (12, 6).

Graphics Calculator tip!

Solving simultaneous linear equations

A calculator can be used to find a solution for Worked example 4.

For the Casio *fx*-9860G AU

The Casio can be used to show a graphical solution.

1. To solve the simultaneous equations graphically press:
   - MENU
   - 5: GRAPH.

Rearrange the equations to make y the subject of the formula first, then complete the entry lines as:

Y1 = (2 ÷ 3)X − 2

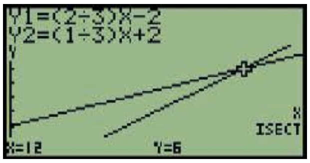
Y2 = (1 ÷ 3)X + 2

Press EXE after each entry.

2. To adjust the window to accommodate the graphs, press:
   - SHIFT
   - F3 (V-WIN).

Set the fields as shown and then press EXE.

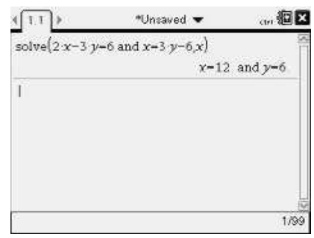
3. To find the coordinates of the point of intersection of these two lines, press:
   - F6 (DRAW)
   - SHIFT F5 (G-SLV)
   - F5 (ISCT).



The coordinates of the point of intersection will be displayed.

For TI-Nspire CAS

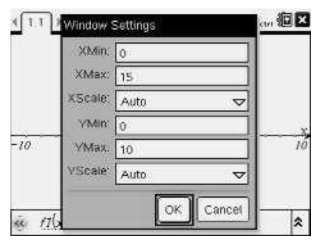
This calculator can be used to solve simultaneous equations both algebraically and graphically.



1. To solve the simultaneous equations algebraically, open a Calculator page. Press:
   - MENU (menu)
   - 3: Algebra (3)
   - 1: Solve (1).

Complete the entry line as:
solve($2x - 3y = 6$ and $x = 3y - 6$, x).
Then press ENTER (enter).

2. To solve the simultaneous equations graphically, open a Graphs page.



3. To adjust the window settings to accommodate the graphs, press:
   - MENU (menu)
   - 4: Window/Zoom (4)
   - 1: Window Settings (1).

Set the fields as shown and then select OK.

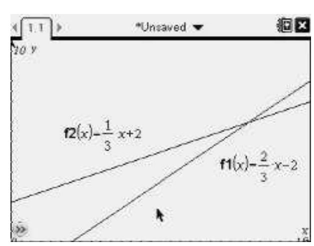
4. Rearrange the equations to make y the subject and then complete the entry lines as:

$$f1(x) = \frac{2}{3}x - 2$$

$$f2(x) = \frac{1}{3}x + 2 .$$

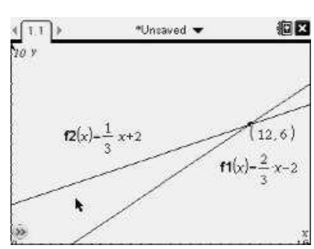
   Press ENTER (enter) after each entry.

The two graphs will appear on the screen.



5. To find the coordinates of the point of intersection of these two lines, press:
   - MENU (menu)
   - 7: Points & Lines (7)
   - 3: Intersection Point(s) (3).

Use the NavPad to move the cursor and press CLICK (click) once on each graph. The coordinates of this point will appear as shown.



remember

1. The graphical solution to a pair of simultaneous linear equations is given by the point of intersection of the two lines representing the equations.
2. The coordinates of the point of intersection can be obtained from the graph, but should always be verified using algebra.
3. To verify the graphical solution, substitute the coordinates of the point of intersection into the equations and check whether the substitution will result in the true statement.
4. Alternatively, solve the equations algebraically by using the substitution or the elimination method.

EXERCISE 7B Solutions of simultaneous linear equations

WORKED Example 4

1 Sketch the graphs, and solve the following pairs of simultaneous equations. (Use an appropriate method to verify your solutions algebraically.) You may wish to use a graphics calculator.

a $x = 4$, $x + y = 2$ **b** $y = 1$, $x - y = 1$ **c** $y = 1$, $x - y = -5$

d $x + y = 7$, $y = 3x - 1$ **e** $2x + 2y = 12$, $y = 2x - 3$ **f** $x - y = 10$, $y = -3x + 2$

g $2x + y = 22$, $x = 3y - 3$ **h** $-x + 3y = 15$, $x = 7y + 1$ **i** $3x - y = 9$, $x = -y + 1$

j $x - y = 1$, $x + y = -1$ **k** $x + y = 8$, $x - y = 1$ **l** $2x + y = 15$, $3x + y = 17$

m $5x - 2y = 28$, $6x + 2y = 16$ **n** $2x - 4y = 4$, $x + 3y = 6$ **o** $x + y = 7$, $5x + 2y = 7$

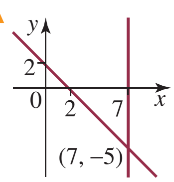
p $2x + y = 1$, $4x + 3y = 0$ **q** $3x + y = 5$, $12x + 8y = 7$ **r** $2x + 3y = -7$, $6x + 21y = 3$

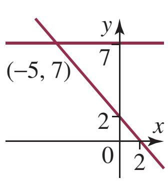
2 **multiple choice**

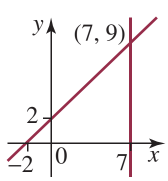
For the simultaneous equations

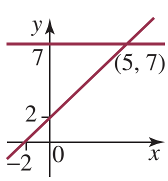
$$x = 7$$
$$x + y = 2$$

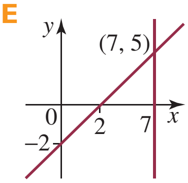
the figure showing the correct graphs and solution is:

A


B


C


D


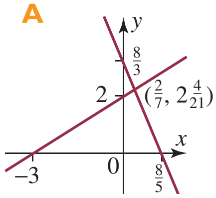
E


3 multiple choice

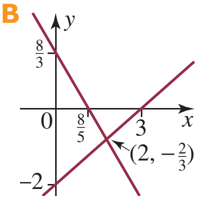
For the simultaneous equations $2x - 3y = 6$
$5x + 3y = 8$

the figure showing the correct graphs and solution is:

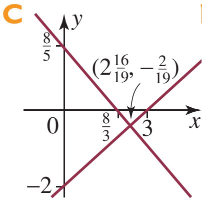
A



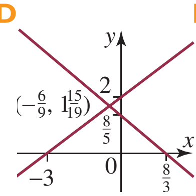
B



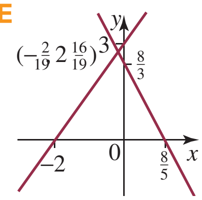
C



D



E



Graphs of simultaneous linear inequations

The graph of a linear equation is a straight line and the solution to a pair of simultaneous linear equations is a point of intersection of the two lines.

The graph of a linear inequation is a half plane and the solution to a pair of simultaneous linear inequations is the area that is common to both half planes; that is, the area of their intersection.

To find the graphical solution to a pair of simultaneous linear inequations, the following algorithm can be used:

1. On the same set of axes sketch each of the inequations, shading the regions that are required.
2. The solution is represented by the area that is shaded by both graphs, so specify this by adding a legend to the graph (that is, ▭ Region required).

WORKED Example 5

Sketch the following pair of simultaneous linear inequations, determine the point of intersection and shade the required region (that is, the solution).

$$x \geq 2, \quad x + 2y \leq 0$$

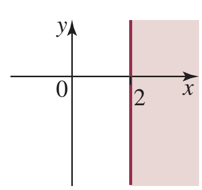
THINK

1. Sketch the graph of $x \geq 2$.
   (a) The graph of the equation $x = 2$ is a vertical line which intersects the x-axis at 2 (there is no y-intercept). Since the inequation contains $\geq$, draw a solid line.
   (b) Substitute $x = 3$ into $x \geq 2$ to see if it satisfies the inequation; that is, is $3 \geq 2$?
   (c) Since the inequation is correct, shade the side of the line that contains the point.

WRITE

Boundary equation for $x \geq 2$ is
$x = 2$.

Test point: (3, 0)
Is $3 \geq 2$? Yes



2. Sketch the graph of $x + 2y \leq 0$.
   (a) To sketch the graph of $x + 2y = 0$, first determine the x-intercept by letting $y = 0$.

Boundary equation for $x + 2y \leq 0$ is
$x + 2y = 0$.
x-intercept: $y = 0$
$x + 0 = 0$
$x = 0$ (0, 0)

Continued over page

THINK | **WRITE**

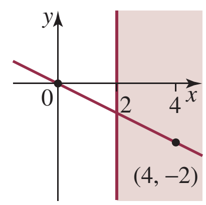
(b) Determine the y-intercept by letting $x = 0$.

y-intercept: $x = 0$
$0 + 2y = 0$
$y = 0$ $(0, 0)$

(c) Since x- and y-intercepts coincide, an alternative point must be chosen. To obtain the alternative point, let x or y equal any number other than 0. Say, let $x = 4$.

If $x = 4$,
$4 + 2y = 0$
$2y = -4$
$y = -2$ $(4, -2)$

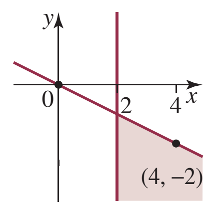
(d) To sketch the graph of $x + 2y = 0$, mark the points (0, 0) and (4, −2) on the set of axes and join them with a straight line. (Use a solid line, since the inequation contains the ≤ sign.)



(e) Select any point on one side of the line, say (1, 1) and substitute its coordinates into $x + 2y \le 0$ to see if it satisfies the inequation; that is, is $3 \le 0$?

(f) Since the inequation is false, shade the side of the line that does not contain the point.

Test point: (1, 1)
$1 + 2 \cdot 1 \le 0$
Is $3 \le 0$? No



3 Find the coordinates of the point of intersection of the two lines.

(a) Write the given inequations as equations and label them [1] and [2].

$x = 2$ [1]
$x + 2y = 0$ [2]

(b) Substitute $x = 2$ into equation [2].

Substituting [1] into [2]:
$2 + 2y = 0$

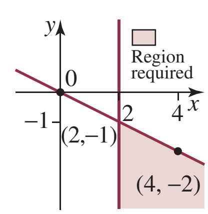
(c) Solve for y.

$2y = -2$
$y = -1$

(d) State the coordinates of the point of intersection.

The solution set is (2, −1).

4 Write the coordinates of the point of intersection on the graph and add the legend.



WORKED Example 6

Sketch the following pair of simultaneous linear inequations, determine the point of intersection and shade the required region.

$$2x + 3y \le 6$$
$$x - y \ge 3$$

THINK

1 Sketch the graph of $2x + 3y \le 6$.
(a) To sketch the graph of $2x + 3y = 6$, first determine the x-intercept by letting $y = 0$.

WRITE

Boundary of $2x + 3y \le 6$ is
$2x + 3y = 6$.
x-intercept: $y = 0$
$2x + 0 = 6$
$x = 3$ (3, 0)

(b) Determine the y-intercept by letting $x = 0$.

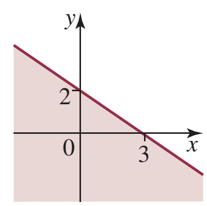
y-intercept: $x = 0$
$0 + 3y = 6$
$y = 2$ (0, 2)

(c) On the set of axes mark x- and y-intercepts and join them with a solid straight line (since the inequation contains the ≤ sign).

(d) Substitute the coordinates of the point (0, 0) into $2x + 3y \le 6$ to see if it satisfies the inequation; that is, is $0 \le 6$?

Test point: (0, 0)
Is $2 \cdot 0 + 3 \cdot 0 \le 6$? Yes

(e) Since the inequation is correct, shade the side of the line that contains the point.



2 Sketch the graph of $x - y \ge 3$.
(a) To sketch $x - y = 3$, first determine the x-intercept by letting $y = 0$.

Boundary of $x - y \ge 3$ is
$x - y = 3$.
x-intercept: $y = 0$
$x - 0 = 3$
$x = 3$ (3, 0)

(b) Determine the y-intercept by letting $x = 0$.

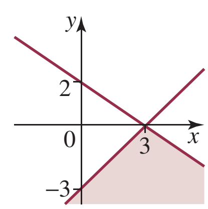
y-intercept: $x = 0$
$0 - y = 3$
$y = -3$ (0, −3)

(c) Mark the x- and y-intercepts and join them with a solid straight line.

Test point: (0, 0)
Is $0 - 0 \ge 3$? No

(d) Substitute the coordinates of the point (0, 0) into $x - y \ge 3$ to see if it satisfies the inequation; that is, is $0 \ge 3$?

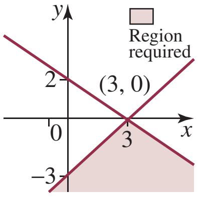
(e) Since the inequation is false, shade the side of the line that does not contain the point.



3 Find the point of intersection of the two lines:
(a) Write the given inequations as equations and label them [1] and [2].

$2x + 3y = 6$ [1]
$x - y = 3$ [2]

Continued over page

THINK	WRITE
(b) To equate the coefficients of y, multiply equation [2] by 3 and label the resulting equation [3].	[2] · 3: $3x - 3y = 9$ [3]
(c) Eliminate y by adding equations [1] and [3].	[1] + [3]: $2x + 3y + 3x - 3y = 6 + 9$
(d) Solve for x.	$5x = 15$ $x = 3$
(e) Substitute $x = 3$ into either equation, say [2], and solve for y.	Substituting $x = 3$ into [2]: $3 - y = 3$ $y = 0$
(f) State the coordinates of the point of intersection.	Solution set is (3, 0).
4 Add to the graph the coordinates of the point of intersection and the legend.	

Graphics Calculator tip! Solutions to simultaneous linear inequations

Graphs of simultaneous linear inequations can be obtained using a graphics calculator. For instance, the problem discussed in Worked example 6 can be solved as shown below.

First transpose both inequations to make y the subject:

$$2x + 3y \leq 6 \qquad \text{and} \qquad x - y \geq 3$$

$$y \leq -\tfrac{2}{3}x + 2 \qquad\qquad y \leq x - 3$$

For the Casio *fx*-9860G AU

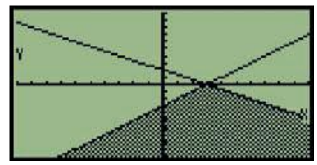
1. To find the graphical solution of the simultaneous linear inequations, press:
   - MENU
   - 5: GRAPH
   - F3 (TYPE)
   - F6 (▶)
   - F4 (Y ≤).

Complete the entry lines as:
Y1 ≤ (−2 ÷ 3) · X + 2
Y2 ≤ X − 3.
Press EXE after each line.

2. To set the viewing screen, press:
   - SHIFT
   - F3 (V-WIN).

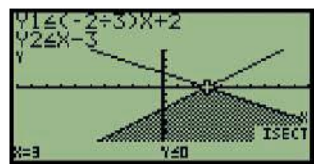
Set the fields as shown and then press EXE.

3. To display the required region, press F6 (DRAW).
   The shaded area shows the required region.



4. To find the point of intersection, press:
   - SHIFT
   - F5 (G-SLV)
   - F5 (ISCT).

The coordinates of the intersection point will be displayed.



For the TI-Nspire CAS

1. To find the graphical solution of the simultaneous linear inequations, open a Graphs page.

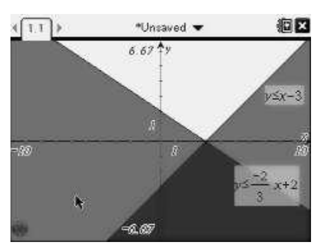
In the function area at the bottom of the screen, press Clear to delete = .
Complete the entry lines as:

$$y \le \frac{-2}{3} \cdot\ x + 2$$

$$y \le x - 3.$$

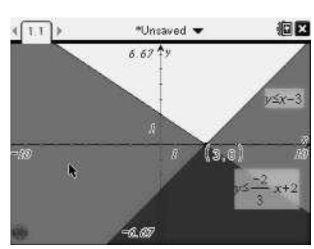
Press ENTER after each line.

The required region is the section where the two shadings overlap.



2. To find the intersection point, press:
   - MENU
   - 7: Points & Lines
   - 3: Intersection Point(s).

Press CLICK once on each graph and the coordinates of the intersection point will be displayed.



remember

1. The graphical solution to a pair of simultaneous linear inequations is the region common to both inequations.
2. To find the graphical solution of simultaneous linear inequations:
   (a) Sketch each of the inequations on the same set of axes.
   (b) Find the coordinates of the point of intersection of the two lines that form the boundaries and add it to the graph.
   (c) Add the legend to the graph to indicate that the solution (region required) is the area which is shaded.

EXERCISE 7C Graphs of simultaneous linear inequations

WORKED Example 5

1 Sketch the graphs of the following pairs of simultaneous inequations, determine the point of intersection and shade the required region.

a $x \geq 0$, $x + 4y \leq 0$ **b** $y \geq 2$, $x + y \leq 1$ **c** $x \leq 1$, $x - y \leq 2$

d $y \leq 2x$, $y \leq -3x$ **e** $y \leq 6$, $5x + 10y \leq 20$ **f** $y \geq -3$, $3x - 4y \leq -24$

eBook*plus*

Digital docs:
SkillSHEET 7.2 Simultaneous equations
WorkSHEET 7.1

WORKED Example 6

2 Sketch the graphs of the following pairs of simultaneous inequations, determine the point of intersection and shade the required region.

a $2x - 3y \leq 0$, $x + 2y \leq 0$ **b** $2x + 4y \geq 8$, $3x + y \leq 3$ **c** $4x + 3y \leq 12$, $x + 4y \geq 4$

d $x + y \geq 10$, $x - y \leq 10$ **e** $5x + 4y \geq 20$, $x - y \leq 5$ **f** $3x + 2y \geq 6$, $3x - 2y \geq 6$

g $5x + 2y \geq 15$, $3x + 6y \leq 18$ **h** $4x - 6y \leq 12$, $2x + 2y \leq 10$ **i** $7x - y \leq 14$, $3x + 4y \geq 9$

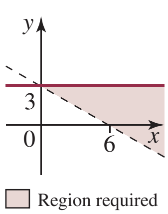
j $4x - y \geq 8$, $14x + 2y \geq 14$ **k** $2x + 2y \geq 6$, $x - y \leq -4$ **l** $-6x + y \geq 12$, $6x - 3y \geq 6$

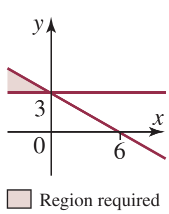
m $4x - y \leq 2$, $4x + y \leq 2$ **n** $3x + 3y \leq 3$, $2x + y \geq -1$ **o** $x - 5y \geq 10$, $4x + 2y \leq 12$

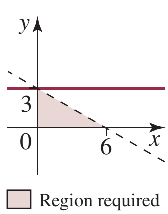
p $6x - 3y \leq -3$, $3x + 4y \geq 4$ **q** $8x + 4y \leq -8$, $x - \frac{y}{2} \leq 1$ **r** $\frac{x}{2} - \frac{y}{3} \geq -2$, $\frac{x}{3} + \frac{y}{2} \leq 2$

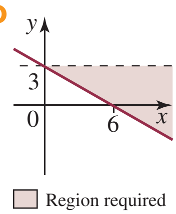
3 **multiple choice**

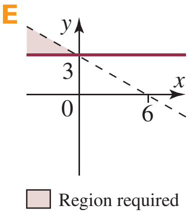
a For the simultaneous inequations $y \geq 3$ and $x + 2y \leq 6$, the figure showing the correct graphs and required region (shaded) is:

A


B


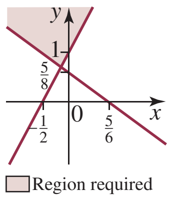
C


D


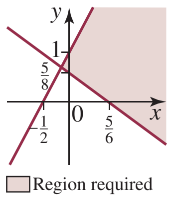
E


b For the simultaneous inequations $24x - 12y \le -12$ and $12x + 16y \ge 10$, the figure showing the correct graphs and required region (shaded) is:

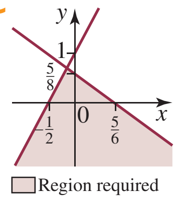
A



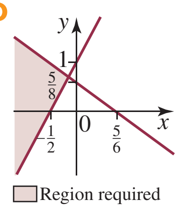
B



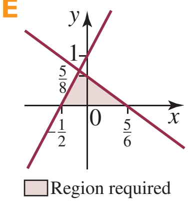
C



D



E



Graphs of systems of linear inequations

In the previous section we discussed how to sketch the graphs of pairs of simultaneous linear inequations. The number of inequations to be graphed simultaneously can be extended. We refer to the groups that contain more than two inequations as *systems*.

The solution to a system of linear inequations is the area, common to all half-planes, representing those inequations. Graphically, the solution is given by the region where all the shadings overlap.

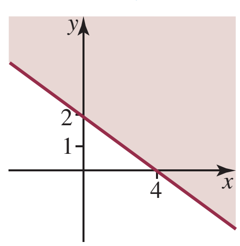
WORKED Example 7

eBook*plus*

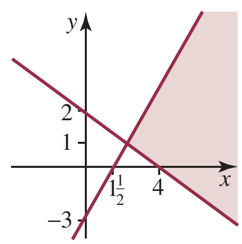
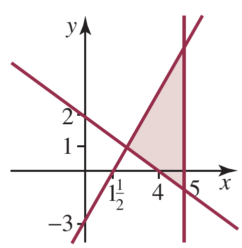
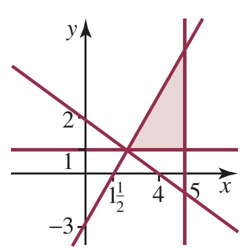
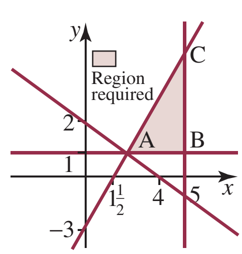
Tutorial: Worked example 7 int-0483

Sketch the following system of linear inequations and shade the required region. (Do not calculate the coordinates of the points of intersection of the straight lines.)

$x + 2y \ge 4$ **[1]** $\quad 2x - y \ge 3$ **[2]** $\quad x \le 5$ **[3]** $\quad y \ge 1$ **[4]**

THINK	WRITE
1 Sketch the graph of $x + 2y \ge 4$. (a) To sketch $x + 2y = 4$, first determine the x-intercept by letting $y = 0$.	Boundary equation for $x + 2y \ge 4$ is $x + 2y = 4$. x-intercept: $y = 0$ $x + 2 \cdot 0 = 4$ $x = 4$ $\quad$ (4, 0)
(b) Determine the y-intercept by letting $x = 0$.	y-intercept: $x = 0$ $1 \cdot 0 + 2y = 4$ $2y = 4$ $y = 2$ $\quad$ (0, 2)
(c) On the set of axes mark the x- and y-intercepts and join them with a solid straight line (since the inequation contains a ≥ sign). (d) Substitute the coordinates of the point (0, 0) into $x + 2y \ge 4$ to see if it satisfies the inequation; that is, is $0 \ge 4$? (e) Since the inequation is false, shade the side of the line that does not contain the point.	 Test point: (0, 0) Is $0 \ge 4$? No

Continued over page

THINK	WRITE
2 Sketch the graph of $2x - y \geq 3$.	Boundary equation for $2x - y \geq 3$ is $2x - y = 3$.
(a) To sketch $2x - y = 3$, first determine the x-intercept by letting $y = 0$.	x-intercept: $y = 0$ $2x - 0 = 3$ $2x = 3$ $x = 1.5$ (1.5, 0)
(b) Determine the y-intercept by letting $x = 0$.	y-intercept: $x = 0$ $0 - y = 3$ $y = -3$ (0, −3)
(c) Mark the x- and y-intercepts on the set of axes and join them with a solid straight line (since the inequation contains a $\geq$ sign).	 Test point: (0, 0)
(d) Substitute the coordinates of the point (0, 0) into $2x - y \geq 3$ to see if it satisfies the inequation; that is, is $0 \geq 3$?	Is $0 \geq 3$? No
(e) Since the inequation is false, shade the side of the line that does not contain the point.	
3 Sketch the graph of $x \leq 5$.	Boundary equation for $x \leq 5$ is $x = 5$.
(a) Sketch the graph of $x = 5$ (a vertical straight line passing through 5 on the x-axis). Use a solid line, as the inequation contains a $\leq$ sign.	 Test point: (0, 0)
(b) Substitute $x = 0$ into $x \leq 5$ to see if it satisfies the inequation; that is, is $0 \leq 5$?	Is $0 \leq 5$? Yes
(c) Since the inequation is correct, shade the side of the line that contains the point.	
4 Sketch the graph of $y \geq 1$.	Boundary equation for $y \geq 1$ is $y = 1$.
(a) Sketch the graph of $y = 1$ (a horizontal line, passing through 1 on the y-axis). Use a solid line, as the inequation contains the $\geq$ sign.	 Test point: (0, 0)
(b) Substitute $y = 0$ into $y \geq 1$ to see if it satisfies the inequation; that is, is $0 \geq 1$?	Is $0 \geq 1$? No
(c) Since the inequation is false, shade the side of the line that does not contain the point.	
5 The polygon which represents the overlapping of all the shaded areas is the region required. Label the vertices of the polygon A, B and C and add the legend to the graph.	

In the previous example a boundary polygon, ABC, has been formed. The terminology below will be used in the sections that follow.

1. The shaded region and the polygon are together called the **feasible region**. They represent all the points that satisfy the system of linear inequations.
2. Points A, B and C are *vertices* of the feasible region and can be determined by finding the points of intersection of the relevant lines (that is, solving simultaneous equations where necessary). The vertices are also referred to as *feasible points* or *corner points*.

remember

1. A group of simultaneous linear inequations is referred to as a *system*.
2. The graphical solution to the system of linear inequations is the area, common to all half-planes, representing those inequalities. It is the region that is the overlap of all the shaded areas.
3. The shaded area together with the boundary polygon, formed as a result of sketching the system of simultaneous linear inequations, is called a *feasible region*.
4. The vertices of the polygon are called *feasible* (or *corner*) *points*.

EXERCISE 7D Graphs of systems of linear inequations

WORKED Example 7

1 Sketch the graphs of the following systems of inequations and shade the required region. (Do not calculate the coordinates of the points of intersection of the straight lines.)

a $x \geq 0$, $y \geq 0$, $x \geq 3$, $y \leq 4$

b $x \geq 0$, $y \geq 0$, $x \geq 2$, $y \geq 6$

c $x \leq 0$, $y \leq 0$, $x \leq 5$, $y \geq -7$

d $2x + 3y \leq 3$, $x \geq 2$, $y \leq 2$

e $6x + 5y \leq 30$, $x \geq -1$, $y \geq -3$

f $-2x - 4y \geq 8$, $x \leq 4$, $y \leq -2$

g $-4x + y \leq 4$, $2x + y \leq 4$, $y \geq 0$

h $4x + 6y \geq 12$, $3x + y \leq 9$, $x \geq 0$

i $-8x - 4y \leq 4$, $y \geq x$, $x \geq 0$, $y \geq 6$

j $x + y \geq 1$, $x - y \geq 1$, $x \geq 2$, $y \geq 2$

k $2x - y \leq 2$, $x + 2y \geq 1$, $x \leq 3$, $y \leq 1$

l $x - 3y \leq 1$, $2x \geq y$, $x \geq \frac{1}{2}$, $y \geq 0$

m $4x + 3y \geq 12$, $2x + 5y \leq 10$, $x \geq 1$, $y \leq 1$

n $2x - 3y \leq 0$, $x + 2y \leq 0$, $x \leq 5$, $y \leq 0$

o $3x + 2y \geq 6$, $3x - 2y \leq 6$, $x \geq 0$, $y \geq 0$

p $4x - 2y \leq 2$, $4x + y \geq 2$, $x \leq 2$, $y \geq 2$

q $3x + 3y \leq 3$, $2x + y \geq -1$, $x \leq 1$, $y \geq 1$

r $6x + 3y \geq 12$, $-4x + 2y \leq 16$, $x \geq 0$, $y \geq 0$

s $5x + 10y \leq 15$, $6x - 2y \leq 9$, $x \geq 0$, $y \geq 0$

t $x + 2y \leq 16$, $2x + 5y \geq 15$, $x \leq 5$, $x \geq 0$, $y \geq 0$

u $3x + 5y \geq 15$
$x + y \leq 8$
$x \geq 5$
$x \geq 0$
$y \geq 0$

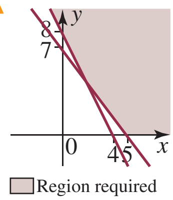
v $6x + 3y \geq 18$
$3x - y \geq -6$
$x \leq 4$
$x \geq 0$
$y \geq 0$

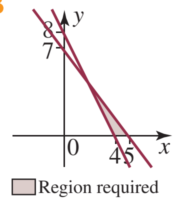
w $x + y \leq 9$
$8x - 3y \leq 24$
$y \leq \frac{1}{2}x$
$x \geq 0$
$y \geq 0$

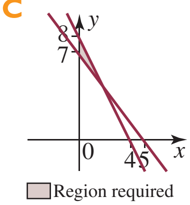
x $-7x + 3y \geq 21$
$y \geq -3x$
$y \leq 7$
$x \leq 0$
$y \leq 0$

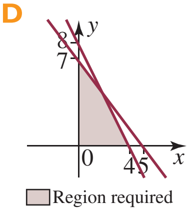
2 multiple choice

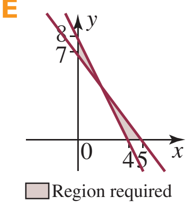
The required region for the system of inequations $x \geq 0$, $y \geq 0$, $7x + 5y \leq 35$ and $2x + y \geq 8$ is:

A


B


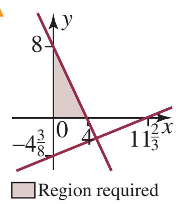
C


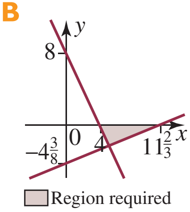
D


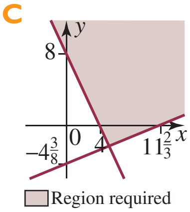
E


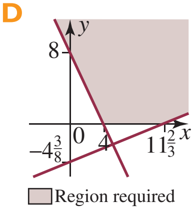
3 multiple choice

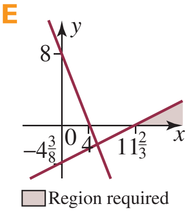
The required region for the system of inequations $x \geq 0$, $y \geq 0$, $3x - 8y \geq 35$ and $x + \frac{1}{2}y \geq 4$ is:

A


B


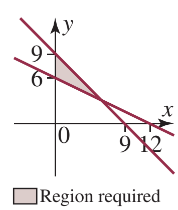
C


D


E


4 multiple choice

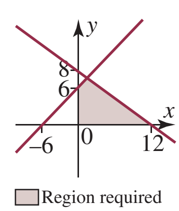
The system of inequations which best describes the graph at right is:



A $x \geq 0, y \geq 0, x + y \geq 9, x + 2y \leq 12$
B $x \leq 0, y \geq 0, x + y \geq 9, x + 2y \geq 12$
C $x \geq 0, y \leq 0, x + y \geq 12, 2x + y \geq 9$
D $x \geq 0, y \geq 0, x + y \leq 9, x + 2y \geq 12$
E $x \geq 0, y \geq 0, x + y \leq 9, 2x + y \geq 12$

5 multiple choice

The system of inequations which best describes the graph at right is:



A $x \geq 0, y \geq 0, 2x + 3y \geq 24, 2x - 2y \geq -12$
B $x \geq 0, y \geq 0, 2x + 3y \leq 24, 2x - 2y \geq -12$
C $x \geq 0, y \geq 0, 2x + 3y \leq 24, 2x - 2y \leq -12$
D $x \leq 0, y \leq 0, 2x + 3y \leq 24, 2x - 2y \geq -12$
E $x \geq 0, y \geq 0, 3x + 2y \leq 24, 2x - 2y \geq -12$

Maximising and minimising linear functions

Linear programming is employed to maximise or minimise linear functions subject to the constraints given by a system of linear inequations.

In this section we learn to maximise and minimise linear functions using two methods: the *sliding-line method* and the *corner point method*.

Sliding-line method

To maximise/minimise linear functions using the sliding-line method:

1. Sketch the feasible region.
2. Determine the coordinates of all corner points.
3. Graph the linear function to be maximised or minimised.
4. (a) To maximise the linear function, slide the line up and find the last point the line touches in the feasible region.
   (b) To minimise the linear function, slide the line down and find the last point the line touches in the feasible region.

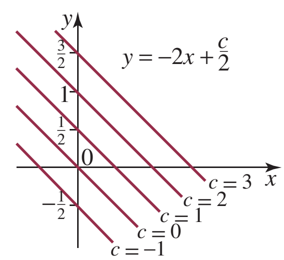
While maximising/minimising a linear function, you will at one stage need to graph it. The function is usually expressed in terms of x and y and needs to be transposed first to make y the subject.

Consider the function $c = 4x + 2y$.

When transposed, it will give: $y = -2x + \frac{c}{2}$.

The linear function has a gradient of -2 and a y-intercept of $\frac{c}{2}$.

If values of $-1, 0, 1, 2, 3, \ldots$ are assigned to c, a series of parallel lines will be formed (that is, lines with the same gradient, but different y-intercepts), as shown in the diagram at right.



For maximising/minimising linear functions using the sliding-line method, any one of the lines shown in the diagram at right can be selected to be 'the sliding line'. The following example illustrates the concept.

WORKED Example 8

a **Sketch the system of linear inequations given by:**

$6x + 8y \le 24, \quad x \le 2, \quad x \ge 0, \quad y \ge 0$

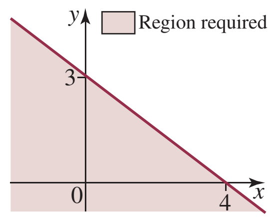
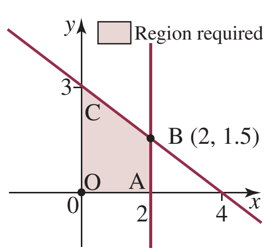
and shade the required region.

b **Determine the coordinates of the vertices of the feasible region.**

c **Determine the maximum and minimum values of $R = 2x + 2y$ subject to the constraints above, using the sliding-line method.**

THINK	WRITE
a 1 Sketch the graph of $6x + 8y \le 24$. (a) To sketch the graph of $6x + 8y = 24$, first determine the x-intercept by letting $y = 0$.	**a** Boundary equation of $6x + 8y \le 24$ is $6x + 8y = 24$. x-intercept: $y = 0$ $6x + 8 \cdot 0 = 24$ $6x = 24$ $x = 4$ (4, 0)

Continued over page

THINK	**WRITE**
(b) Determine the y-intercept by letting $x = 0$.	y-intercept: $x = 0$ $6 \cdot 0 + 8y = 24$ $8y = 24$ $y = 3$ (0, 3)
(c) Mark the x- and y-intercepts on the set of axes and join them with a straight line. Since the inequation contains the ≤ sign, a solid line must be drawn.	Test point: (0, 0) Is $0 \le 24$? Yes
(d) Substitute the coordinates of the point (0, 0) into $6x + 8y \le 24$ to see if it satisfies the inequation; that is, is $0 \le 24$?	
(e) Since the inequation is correct, shade the side of the line that contains the point.	
2 Sketch the graph of $x \le 2$.	
(a) Sketch the graph of $x = 2$ (a vertical straight line, passing through the 2 on the x-axis). Use a solid line, since the inequation contains a ≤ sign.	Boundary equation of $x \le 2$ is $x = 2$.
(b) Substitute $x = 0$ into $x \le 2$ to see if it satisfies the inequation; that is, is $0 \le 2$?	Test point: (0, 0) Is $0 \le 2$? Yes
(c) Since the inequation is correct, shade the side of the line that contains the point.	
3 Sketch the graph of $x \ge 0$.	
(a) Sketch $x = 0$ (which is actually the y-axis), using a solid line, since the inequation contains a ≥ sign.	Boundary equation for $x \ge 0$ is $x = 0$.
(b) Choose a point on the x-axis, say $x = 1$, and check that it satisfies the inequation $x \ge 0$. That is, is $1 \ge 0$?	Test point: (1, 0) Is $1 \ge 0$? Yes
(c) Since the inequation is correct, shade the side of the line that contains the point.	
4 Sketch the graph of $y \ge 0$.	Boundary equation of $y \ge 0$ is $y = 0$.
(a) Sketch $y = 0$ (which is actually the x-axis), using a solid line, as the inequation contains a ≥ sign.	Test point: (0, 1) Is $1 \ge 0$? Yes
(b) Choose a point on the y-axis, say $y = 1$, and check that it satisfies $y \ge 0$; that is, is $1 \ge 0$?	
(c) Since the inequation is correct, shade the side of the line that contains the point.	
5 Transfer all of the above information onto a graph. Label the vertices of the feasible region A, B, C and O.	

THINK / **WRITE**

b

1. Determine the coordinates of the corner points: read the coordinates of points O, A and C from the graph.

b O (0, 0), A (2, 0) and C (0, 3)

2. The coordinates of point B can be determined by finding the point of intersection of the lines $6x + 8y = 24$ and $x = 2$. Substitute $x = 2$ into $6x + 8y = 24$ and solve for y.

$6x + 8y = 24$ [1]
$x = 2$ [2]

Substituting [2] into [1]:

$$\begin{aligned} 6 \cdot 2 + 8y &= 24 \\ 12 + 8y &= 24 \\ 8y &= 12 \\ y &= 1.5 \end{aligned}$$

3. State the coordinates of point B.

The solution set is B (2, 1.5).

c

1. Transpose the equation that needs to be maximised or minimised to make y the subject.

c

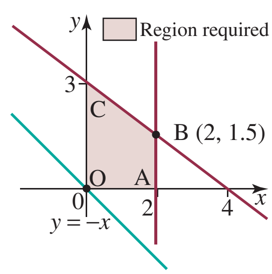
$$\begin{aligned} R &= 2x + 2y \\ 2y &= -2x + R \\ y &= -x + \frac{R}{2} \end{aligned}$$

2. Select any value of R, say, 0.

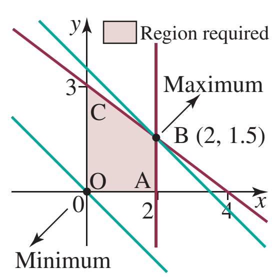
Let $R = 0$.

$$y = -x + \frac{0}{2}$$
$$y = -x$$

3. Sketch the linear function $y = -x$.



4. Treat the linear function as a sliding line.
   (a) To maximise the linear function, *slide* the line up and find the last corner point the line touches in the feasible region. (This can be done easily by placing a ruler along the line and sliding it up, parallel to the line, until it touches the last corner point.)
   (b) To minimise the linear function, *slide* the line down and find the last corner point the line touches in the feasible region.



5. Observe from the graph at which point the maximum and minimum intercepts occur within the feasible region.

Maximum occurs at B (2, 1.5).
Minimum occurs at O (0, 0).

6. Substitute the coordinates of point B into equation $R = 2x + 2y$ to determine the maximum value of R.

$$\begin{aligned} R(\text{max.}) &= 2 \cdot 2 + 2 \cdot 1.5 \\ &= 4 + 3 \\ &= 7 \end{aligned}$$

7. Substitute the coordinates of point O into $R = 2x + 2y$ to find the minimum value of R.

$$\begin{aligned} R(\text{min.}) &= 2 \cdot 0 + 2 \cdot 0 \\ &= 0 + 0 \\ &= 0 \end{aligned}$$

Corner point method

To maximise/minimise linear functions using the corner point method:

1. Sketch the feasible region.
2. Determine the coordinates of all corner points.
3. Apply the corner point method by substituting coordinates of each corner point into the linear function which is to be maximised or minimised.
4. Select maximum and minimum values.

Using Worked example 8, substitute the values of each corner point into the equation $R = 2x + 2y$, and then select the maximum and minimum values.

O (0, 0)	$R = 2 \cdot 0 + 2 \cdot 0 = 0$	(Minimum)
A (2, 0)	$R = 2 \cdot 2 + 2 \cdot 0 = 4$	
B (2, 1.5)	$R = 2 \cdot 2 + 2 \cdot 1.5 = 7$	(Maximum)
C (0, 3)	$R = 2 \cdot 0 + 2 \cdot 3 = 6$	

WORKED Example 9

a **Sketch the following system of linear inequations and shade the required region.**

$$x + y \le 10,\ y \ge x - 4,\ y \le 2x + 1,\ x \ge 0,\ y \ge 0$$

b **Determine the coordinates of the vertices of the feasible region.**

c **Determine the maximum and minimum values of $z = 3x - y$ subject to the above constraints, using the corner point method.**

THINK

a 1 For the boundary equations, replace the inequality signs with an = sign in each inequation and label the resulting equations [1], [2], [3], [4] and [5].

2 Determine the intercepts for [1], [2] and [3].

3 Sketch all graphs.

4 Find the required region for each inequation by using a test point.

5 Transfer all of the information onto the graph. Label the vertices of the feasible region A, B, C, D and O.

b 1 Read the coordinates of points O, A and D from the graph.

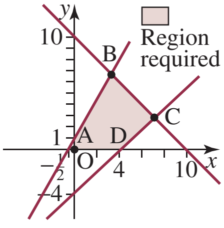
WRITE

a $x + y = 10$ [1]
$y = x - 4$ [2]
$y = 2x + 1$ [3]
$x = 0$ [4]
$y = 0$ [5]

$x + y = 10$
x-intercept: $y = 0, x = 10$ (10, 0)
y-intercept: $x = 0, y = 10$ (0, 10)

$y = x - 4$
x-intercept: $y = 0, x = 4$ (4, 0)
y-intercept: $x = 0, y = -4$ (0, –4)

$y = 2x + 1$
x-intercept: $y = 0, x = -0.5$ (–0.5, 0)
y-intercept: $x = 0, y = 1$ (0, 1)



b O (0, 0), A (0, 1) and D (4, 0)

THINK	WRITE
2 Find point B.	
(a) The coordinates of point B can be determined by solving equations [1] and [3] simultaneously: substitute [3] into [1].	$x + y = 10$ [1] $y = 2x + 1$ [3] Substituting [3] into [1]: $x + 2x + 1 = 10$
(b) Solve for x.	$3x + 1 = 10$ $3x = 9$ $x = 3$
(c) Substitute $x = 3$ into equation [3] to find the value of y.	Substituting $x = 3$ into [3]: $y = 2 \cdot 3 + 1$ $y = 7$
(d) State the coordinates of the point B.	The solution set is B (3, 7).
3 Find point C.	
(a) To obtain the coordinates of the point C, solve equations [1] and [2] simultaneously.	$x + y = 10$ [1] $y = x - 4$ [2]
(b) Substitute [2] into [1].	Substituting [2] into [1]: $x + x - 4 = 10$
(c) Solve for x.	$2x - 4 = 10$ $2x = 14$ $x = 7$
(d) Substitute $x = 7$ into equation [2] to find the value of y.	Substituting $x = 7$ into [2]: $y = 7 - 4$ $y = 3$
(e) State the coordinates of the point C.	The solution set is C (7, 3).
c **1** Substitute the coordinates of each point into the linear function, z, to determine its maximum and minimum values.	**c** $z = 3x - y$ O (0, 0) $z = 3 \cdot 0 - 0 = 0$ A (0, 1) $z = 3 \cdot 0 - 1 = -1$ B (3, 7) $z = 3 \cdot 3 - 7 = 2$ C (7, 3) $z = 3 \cdot 7 - 3 = 18$ D (4, 0) $z = 3 \cdot 4 - 0 = 12$
2 Select maximum and minimum values of z.	$z_{min.} = -1$ (vertex A) $z_{max.} = 18$ (vertex C)

The linear function to be maximised or minimised is often referred to as the *objective function*. The maximum or minimum values of the objective function always occur at a corner point (vertex) of the feasible region.

remember

1. To maximise/minimise a linear (objective) function:
   (a) Sketch the feasible region.
   (b) Determine the coordinates of all corner points.
   (c) Apply the sliding-rule method by drawing a graph of the objective function and moving a ruler up or down along it. The last corner point of the feasible region to be passed gives the maximum or minimum value of the objective function.
2. Alternatively, apply the corner point method by substituting the coordinates of each corner point into the objective function and selecting minimum and/or maximum value(s).

EXERCISE 7E Maximising and minimising linear functions

WORKED Example 8

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Digital docs:
SkillSHEET 7.3 Vertices of feasible regions
SkillSHEET 7.4 Sliding line method
SkillSHEET 7.5 Corner point method
Spreadsheet 068 Linear programming

1 For each of the following systems of inequations:
  i Sketch the system of inequations and shade the required region.
  ii Determine the coordinates of the vertices (corner points) of the feasible region.
  iii Determine the maximum or minimum value (as specified) of the objective function for the given constraints, using the sliding-line method.

a Maximise $z = x - y$ subject to $x \ge 0$, $y \ge 0$, $x \le 4$, $y \le 6$.
b Minimise $z = x + 3y$ subject to $x \ge 0$, $y \ge 0$, $y \ge x$, $y \le 7$.
c Maximise $z = x + 2y$ subject to $x \ge 0$, $y \ge 0$, $y \le x$, $x \le 5$.
d Maximise $z = 4x + 6y$ subject to $x \ge 0$, $y \ge 0$, $x + y \le 4$, $3x + 8y \le 24$.
e Minimise $z = 3x - 6y$ subject to $x \ge 0$, $y \ge 0$, $2x + 2y \le 8$, $6x + 8y \le 30$.
f Maximise $z = 0.8x + 1.2y$ subject to $x \ge 0$, $y \ge 0$, $x - 4y \le 10$, $2x + 7y \le 28$.
g Minimise $z = 9x + 3y$ subject to $x \ge 0$, $y \ge 0$, $2x + 3y \ge 18$, $3x + 4y \le 30$.
h Maximise $z = -3x + y$ subject to $x \ge 0$, $y \ge 0$, $5x + 2y \ge 20$, $x \le 3$, $y \le 9$.
i Minimise $z = x + y$ subject to $x \ge 0$, $y \ge 0$, $-2x + y \le 3$, $6x - 3y \le 12$, $3x + 3y \le 15$.
j Maximise $z = 3x + 4y$ subject to $x \ge 0$, $y \ge 0$, $-2x + y \le 9$, $3x - 5y \le 12$, $x \le 6$, $y \le 10$.

WORKED Example 9

2 For each of the following systems of inequations:
  i Sketch the system of inequations and shade the required region.
  ii Determine the coordinates of the vertices (corner points) of the feasible region.
  iii Determine the maximum or minimum value (as specified) of the objective function for the given constraints, using the corner point method.

a Minimise $z = 2x + y$ subject to $x \ge 0$, $y \ge 0$, $-x + y \le 3$, $4x + 7y \le 28$, $2x + 8y \ge 0$, $x \le 6$.
b Maximise $z = 2x + y$ subject to $x \ge 0$, $y \ge 0$, $x + y \le 8$, $y \le 2$.
c Minimise $z = 5x - y$ subject to $x \ge 0$, $y \ge 0$, $x \le 6$, $x - y \ge -8$.
d Minimise $z = 3x + 4y$ subject to $x \ge 0$, $y \ge 0$, $x + y \ge 4$, $x - y \ge -4$, $x \le 8$.
e Maximise $z = 1.8x + 2.2y$ subject to $x \ge 0$, $y \ge 0$, $x \le 10$, $y \le 7$, $\frac{1}{2}x + y \ge 8$.
f Minimise $z = 0.7x - 0.3y$ subject to $x \ge 0$, $y \ge 0$, $2x - y \ge -4$, $3x + 4y \le 36$, $y \ge 5$.
g Maximise $z = 1.5x + 2.7y$ subject to $x \ge 0$, $y \ge 0$, $5x - 6y \ge -30$, $x + y \le 10$, $y \ge 6$.
h Minimise $z = 3.2x - 1.4y$ subject to $x \ge 0$, $y \ge 0$, $-7x + 4y \le 28$, $4x + 2y \ge 16$, $x + y \le 14$.
i Maximise $z = 7x - 3y$ subject to $x \ge 0$, $y \ge 0$, $2x + y \le 9$, $-2x + 6y \le 18$, $x - y \le 3$.
j Minimise $z = 9.2x - 5.1y$ subject to $x \ge 0$, $y \ge 0$, $15x + 6y \ge 30$, $-6x - 4y \ge -36$, $y \le 7$, $x \le 4$.

3 **multiple choice**

a The minimum value of the equation $z = x - y$ subject to $x \ge 0$, $y \ge 0$, $x \le 7$, $y \le 8$ is:
  A −8 B −1 C 0 D 1 E 7
b The maximum value of the equation $z = 2x + 5y$ subject to $x \ge 0$, $y \ge 0$, $y \le 4$, $2x + 4y \le 24$ is:
  A 0 B 16 C 20 D 28 E 40
c The maximum value of the equation $z = 5x - 7y$ subject to $x \ge 0$, $y \ge 0$, $x + y \le 4$, $3x + 9y \le 24$ is:
  A −28 B −18.9 C −4 D 20 E 32
d The minimum value of the equation $z = 10x - 4y$ subject to $x \ge 0$, $y \ge 0$, $-7x + 4y \le 14$, $-8x - 8y \le -16$, $x \le 3$ is:
  A −28 B −20 C −14 D −5 E 20

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Digital doc:
WorkSHEET 7.2

Solving linear programming problems

The techniques developed earlier in this chapter can now be employed to solve linear programming problems. For the purpose of this course, all problems will involve exactly two variables. It is important to note that linear programming problems involve only ≤ and ≥ signs; therefore, when graphing inequalities, only solid lines will be used.

To solve linear programming problems, use the following algorithm:

1. Define the variables.
2. Write the inequations (constraints) in terms of the variables.
3. Determine what must be maximised or minimised. This is called the *objective function*.
4. Write the objective function in terms of the variables.
5. Draw the graph of each constraint and obtain the feasible region (shaded region).
6. Obtain the coordinates of the corner points.
7. Employ the sliding-line method or the corner point method to obtain the maximum or minimum value of the objective function.

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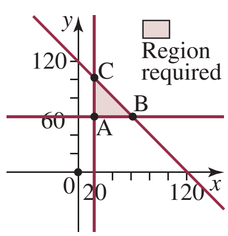
Tutorial: Worked example 10 int-0484

WORKED Example 10

Bright Spark Enterprises Pty Ltd produces two types of computer games, A and B. The company is contracted to produce at least 20 type-A games and at least 60 type-B games each week. The factory can produce a maximum of 120 games per week. The profit on type-A games is \$10 and the profit on type-B games is \$15. How many of each game should be produced each week so as to make the greatest weekly profit, assuming all games produced can be sold?

THINK	WRITE
1 Define the variables.	Let x = number of type-A computer games produced. Let y = number of type-B computer games produced.
2 Write the inequations (constraints) in terms of the variables:	
(a) The number of type-A games produced should be 20 or more.	$x \geq 20$
(b) The number of type-B games produced should be 60 or more.	$y \geq 60$
(c) The total production can not exceed 120 games.	$x + y \leq 120$
3 Determine what must be maximised or minimised.	Maximum profit (P) required.
4 Express the objective function in terms of the variables x and y.	$P = 10x + 15y$

Continued over page

THINK	WRITE
5 Find the required region for $x + y \le 120$. (a) To sketch the graph of $x + y = 120$, determine the x-intercept and the y-intercept.	Boundary for $x + y \le 120$ is $x + y = 120$. x-intercept: $y = 0$, $x = 120$ (120, 0) y-intercept: $x = 0$, $y = 120$ (0, 120)
(b) Substitute the coordinates of the point (0, 0) into $x + y \le 120$ to see if it satisfies the inequation; that is, is $0 \le 120$?	Test point: (0, 0) Is $0 \le 120$? Yes
6 Find the required region for $x \ge 20$.	Test point for $x \ge 20$: (0, 0) Is $0 \ge 20$? No
7 Find the required region for $y \ge 60$.	Test point for $y \ge 60$: (0, 0) Is $0 \ge 60$? No
8 Transfer all of the above information onto the graph. Label the corner points.	
9 Determine the points of the feasible region. (a) Point A: Read the coordinates of the point of intersection of the horizontal and vertical lines from the graph.	A (20, 60)
(b) Point B: Solve $x + y = 120$ and $y = 60$ simultaneously; substitute $y = 60$ into $x + y = 120$ and solve for x.	$x + y = 120$ [1] $y = 60$ [2] Substituting [2] into [1]: $x + 60 = 120$ $x = 60$
(c) State the coordinates of point B.	B (60, 60)
(d) Point C: Solve $x + y = 120$ and $x = 20$ simultaneously; substitute $x = 20$ into $x + y = 120$ and solve for y.	$x + y = 120$ [3] $x = 20$ [4] Substituting [4] into [3]: $20 + y = 120$ $y = 100$
(e) State the coordinates of point C.	C (20, 100)
10 Employing the corner point method, substitute the coordinates of the corner points into the objective function P.	$P = 10x + 15y$ At A (20, 60) $P = 10 \cdot 20 + 15 \cdot 60 = 1100$ At B (60, 60) $P = 10 \cdot 60 + 15 \cdot 60 = 1500$ At C (20, 100) $P = 10 \cdot 20 + 15 \cdot 100 = 1700$
11 Select the maximum value of P.	$P_{max.} = 1700$ at C (20, 100)
12 Relate the answer to the original question.	A maximum profit of $1700 will be obtained when 20 type-A and 100 type-B games are produced.

remember

To solve linear programming problems:
1. define the variables
2. write the constraints in terms of the variables
3. define the objective function
4. draw the graphs of the constraints to obtain the feasible region (use solid lines only)
5. find the coordinates of the corner points
6. use either the sliding line or the corner point method to find the maximum or minimum value of the objective function.

EXERCISE 7F Solving linear programming problems

WORKED Example 10

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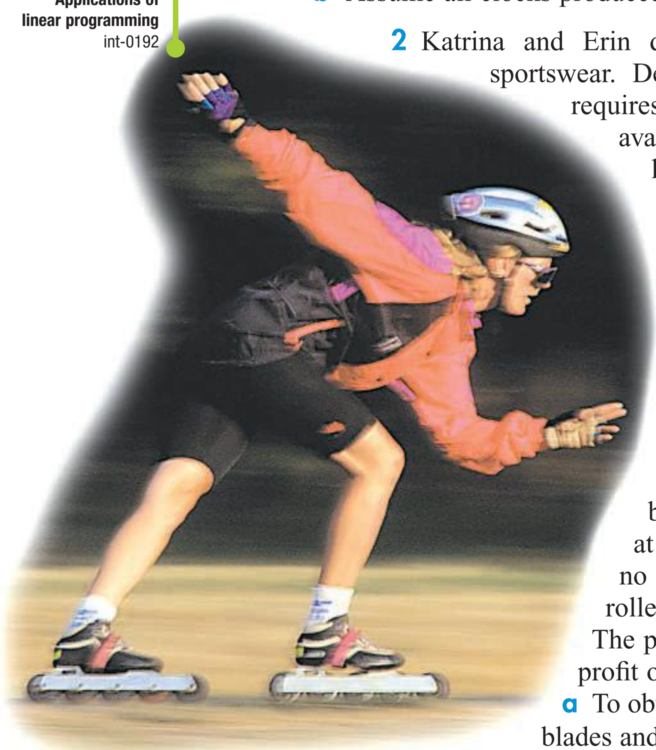
Digital doc: Spreadsheet 068 Linear programming

Interactivity: Applications of linear programming int-0192

1 For a semester project, Cathy's Business Management Team produces two styles of sundial clock. Each clock is made from an old vinyl record. The team is able to produce up to 24 clocks weekly. A minimum of 5 style-A and 3 style-B clocks are ordered each week. The profit on style-A clocks is $2 and the profit on style-B clocks is $3.

a To obtain a maximum weekly profit, how many of each style of clock should be produced each week?

b Assume all clocks produced can be sold. What is the maximum weekly profit?



2 Katrina and Erin design two types of tracksuit for Right-on-Track sportswear. Design A requires 2 m of material while design B requires 3 m of material. The total amount of material available each day is 60 m. At least 3 of design-A and at least 4 of design-B tracksuits must be produced each day to satisfy orders. Design-A tracksuits are sold at a profit of $5 while design-B tracksuits are sold at a profit of $6.50.

a To obtain a maximum daily profit, how many tracksuits of each design should be produced each day?

b Assume all tracksuits produced can be sold. What is the maximum daily profit?

3 Active-8 Enterprises hires out rollerblades and bicycles along the bay. Each day the company supplies at least 15 pairs of rollerblades, and a minimum of 5 but no more than 25 bicycles. No more than 40 pairs of rollerblades and bicycles are hired on any particular day. The profit on hiring out a pair of rollerblades is $4 and the profit on hiring out a bicycle is $3.

a To obtain a maximum daily profit, how many pairs of roller-blades and how many bicycles should be hired out each day?

b What is the maximum daily profit?

4 Squeaky Clean soap manufacturers produce two brands of liquid soap. To meet demand, at least 20 litres of brand A and at least 14 litres of brand B must be produced. Due to other factors the manufacturers are able to produce a maximum of 80 litres. The profit on brand A is $20 and the profit on brand B is $24.
  a To obtain a maximum profit, how much of brand A and B should be produced?
  b What is the maximum profit?

5 The Sweat It Out Gymnasium offers its participants aerobic classes and circuit classes. At least 25 aerobic classes and at least 9 circuit classes must be held each week. The gym is able to offer a maximum of 45 classes per week. Aerobic classes produce a profit of $6 while circuit classes produce a profit of $4.
  a In order to obtain a maximum profit, how many aerobic and circuit classes should be held each week?
  b What is the maximum profit?

6 Jillaroo's Adventures factory manufactures two types of tent: a 2-person tent and a 3-person tent. To meet demand, the factory manufactures at least eighteen 2-person and at least eighteen 3-person tents each week. The 2-person tent takes $1\frac{1}{2}$ hours and the 3-person tent takes $2\frac{1}{2}$ hours to make. The equipment needed to produce the tents can be used for a maximum of 75 hours per week. Two-person tents return a profit of $24 while 3-person tents return a profit of $28.
  a How many of each type of tent should be produced weekly to obtain the maximum profit?
  b What is the maximum profit?

7 It-Will-Print manufactures bubble jet and laser printers. To meet demand, the company must produce a minimum of 5 laser printers, and the total of printers must be, at most, 25 each week. A bubble jet printer takes 2 hours to make and a laser printer takes 3 hours. Due to power restrictions the manufacturing plant can operate for only 60 hours per week. Bubble jet printers return a profit of $12 while laser printers return a profit of $15.
  a How many of each type of printer should be produced weekly to obtain the maximum profit?
  b What is the maximum profit?

8 A farmer decides to divide his land into two sections and plant corn and peas. He has 80 hectares of land available and must devote at least 10 hectares to peas and at least 10 hectares to corn. At harvest time it takes 1 hour per hectare to collect the corn and 3 hours per hectare to collect the peas. The maximum time available for collecting the crops is 120 hours. He can make a profit of $180 per hectare of corn and $160 per hectare of peas.
  a How much of each crop should be sown to obtain a maximum profit?
  b What is the maximum profit?

9 Peter's task as the new assistant manager at Sureway supermarket is to decide how to achieve maximum profit from sales on two brands of fruit juice. He notes that, on any given day, fruit juice sales are greater than 70 litres but less than 90 litres. Peter also notices that Nature's Own fruit juice sales are equal to or greater than generic brand sales, and that at least 10 litres of generic brand juice is sold daily.

a How many litres of each type of juice should be sold for maximum profit if the profit on Nature's Own fruit juice is $2.50/L while the profit on the generic brand is $1.20/L?

b What is the maximum profit?

10 Sandra and Loreta's Shantai resort has been redesigned specifically to cater for tourists and people attending business conventions. It is able to cater for a minimum of 360 guests and a maximum of 510 on a monthly basis. The number of tourists is always greater than the number of people at the conventions but never double the people attending the conventions. The profit made per tourist per month is $15 while the profit made per businessperson per month is $18.

a How many tourists and businesspeople should the hotel cater for each month to make a maximum profit?

b What is the maximum profit?

11 Mathematically Minded Limited produces 3-D puzzles and logic games. It takes 3 hours to produce the parts of a batch of twenty 3-D puzzles and 2 hours to produce the parts of a batch of twenty logic games. The minimum time available for production of these items is 12 hours. A batch of 3-D puzzles takes 1 hour to assemble while a batch of logic puzzles takes 2 hours to assemble. A maximum of 10 hours is allocated to assembling the games. The packaging of a batch of 3-D puzzles and a batch of logic games requires 1 and 2 hours respectively. The minimum time allocated for the packing of these items is 8 hours.

a How many of each item should be made to minimise costs if the overhead cost of each 3-D puzzle is $1.80 while the overhead cost of each logic game is $1.25?

b What is the minimum cost?

12 A refrigerator manufacturer makes two models of refrigerator: Arctic Snow and Cool Breeze. The manufacturers are able to produce up to 40 Arctic Snow and 50 Cool Breeze models per fortnight, and production must not exceed 80 models (per fortnight). Each Arctic Snow model requires 12 hours to make and each Cool Breeze model takes 10 hours. The factory is able to operate for a maximum of 840 production hours per fortnight. Arctic Snow is able to generate a profit of $200 and Cool Breeze is able to generate a profit of $110.

a How many of each model should be manufactured to obtain a maximum profit?

b What is the maximum profit?

c If the profit generated by Arctic Snow was $140 (and the profit on Cool Breeze remained the same), would the number of each model made still produce the maximum profit?

13 Zorko Industries has produced two new cement products: CP_1 and CP_2. Each 50-kg bag of the cement products consists of specific amounts of substances *a*, *b* and *c* (in units per bag) according to the table shown below:

Product	*a*	*b*	*c*
CP_1	20	20	15
CP_2	25	15	10

The amounts of substances a, b and c available are 400, 300 and 210 units respectively. Each 50-kg bag of CP_1 yields a profit of \$45 and each 50-kg bag of CP_2 yields a profit of \$50.

a Let x represent the number of bags of CP_1 and let y represent the number of bags of CP_2. Explain why.

b Given that $20x + 25y \leq 400$, write two similar constraints on x and y.

c There are two other constraints. What are they?

d What is the objective function?

e What is the maximum profit (to the nearest dollar)?

14 **multiple choice**

Elio is commissioned to paint still-lifes and abstracts for the local gallery. He must produce a minimum of 20 pieces for an upcoming exhibition. It takes him on average 10 hours to paint a still-life and 6 hours to paint an abstract, and the maximum time he has to spend on his paintings is 180 hours.

If s represents still-lifes and a represents abstracts, the inequations for this information are:

A $a \geq 0, s \geq 0, s + a \geq 20, 10s + 6a \leq 180$

B $a \geq 0, s \geq 0, s + a \geq 20, s \geq 10, a \geq 6, s + a \leq 180$

C $a \geq 0, s \geq 0, s + a \leq 180, 10s + 6a \geq 20$

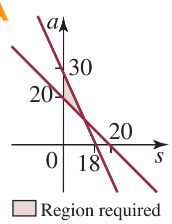
D $s \geq 10, a \geq 6, s + a \leq 180, 10s + 6a \geq 20$

E $s \geq 10, a \geq 6, s + a \geq 20, 10s + 6a \leq 180$

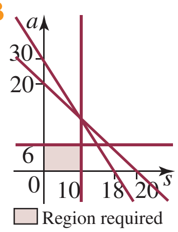
15 **multiple choice**

The feasible region (shaded region) for the previous problem can best be defined by the graph:

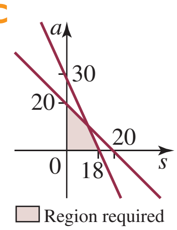
A



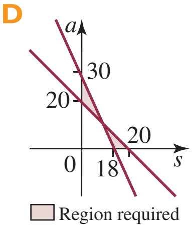
B



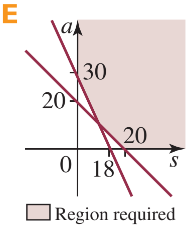
C



D



E



investigation

How many in the research team?

Sonic Boom Sound Systems has developed a new product and needs to organise a research team to run a series of tests.

The team is to comprise experienced engineers (e) and training technicians (t). The team is to consist of no more than 8 people, and at least 2 engineers but no more than 6 engineers, and at least 1 but no more than 5 technicians. The number of engineers must be greater than the number of technicians.

A minimum of 18 tests need to be conducted on the product in a week. Engineers are able to conduct 5 tests per week and technicians 3. Engineers are paid \$1600 per week while technicians are paid \$800.

1 How many engineers and technicians should be chosen to keep wages to a minimum?

2 What is the minimum weekly wage bill?

Further applications of linear programming

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Tutorial:
Worked example 11
int-0485

In this section we consider more-complex linear programming problems. Although you might find it harder to write the constraints and to define the objective function, the technique of solving these problems is exactly the same as the one discussed in the previous section.

WORKED Example 11

The dietitian of the local football club purchases two types of powdered food products for her team. The nutritional contents of the two products per 250 g are listed in the table below:

Component	Product A	Product B
Carbohydrates	25 g	30 g
Fat	2 g	4 g
Protein	15 g	10 g

Continued over page

The team's minimum daily requirements of carbohydrates, fat and protein are 30 g, 4 g and 15 g respectively. If product A costs 50 cents per 250 g and product B costs 60 cents per 250 g, how much of each type should be used to supply the team's daily nutritional requirements at the least cost?

THINK	WRITE
1 Define the variables.	Let x = the amount of product A (in units of 250 g) y = the amount of product B (in units of 250 g)
2 Write the inequations (constraints) in terms of the variables.	$25x + 30y \geq 30$ $2x + 4y \geq 4$ $15x + 10y \geq 15$ $x \geq 0$ $y \geq 0$
3 Determine what must be maximised or minimised.	Minimum cost required.
4 Express the objective function in terms of the variables.	$C = 0.50x + 0.60y$
5 For boundary equations, replace the $\geq$ sign in the constraints with the = sign and label the resulting equations [1] to [5].	Boundary equations: $25x + 30y = 30$ [1] $2x + 4y = 4$ [2] $15x + 10y = 15$ [3] $x = 0$ [4] $y = 0$ [5]
6 Determine the intercepts for [1], [2] and [3].	$25x + 30y = 30$ x-intercept: $y = 0, x = 1.2$ (1.2, 0) y-intercept: $x = 0, y = 1$ (0, 1) $2x + 4y = 4$ x-intercept: $y = 0, x = 2$ (2, 0) y-intercept: $x = 0, y = 1$ (0, 1) $15x + 10y = 15$ x-intercept: $y = 0, x = 1$ (1, 0) y-intercept: $x = 0, y = 1.5$ (0, 1.5)
7 Sketch all graphs. (a) Sketch the graphs of the equations [1], [2] and [3] by marking their respective x- and y-intercepts on the set of axes and joining them with a straight line. (b) Sketch the graphs of the equations [4] and [5] — these are the x- and y-axes.	
8 Find the required region for $25x + 30y \geq 30$. (a) Substitute the coordinates of the point (0, 0) into $25x + 30y \geq 30$ to see if it satisfies the inequation; that is, is $0 \geq 30$? (b) Since the inequation is incorrect, shade the side of the line that does not contain the point.	For $25x + 30y \geq 30$, Test point: (0, 0) Is $0 \geq 30$? No

THINK | **WRITE**

9 Find the required region for $2x + 4y \geq 4$.
(a) Substitute the coordinates of the point (0, 0) into $2x + 4y \geq 4$ to see if it satisfies the inequation; that is, is $0 \geq 4$?
(b) Since the inequation is incorrect, shade the side of the line that does not contain the point.

For $2x + 4y \geq 4$,
Test point: (0, 0)
Is $0 \geq 4$? No

10 Find the required region for $15x + 10y \geq 15$.
(a) Substitute the coordinates of the point (0, 0) into $15x + 10y \geq 15$ to see if it satisfies the inequation; that is, is $0 \geq 15$?
(b) Since the inequation is incorrect, shade the side of the line that does not contain the point.

For $15x + 10y \geq 15$,
Test point: (0, 0)
Is $0 \geq 15$? No

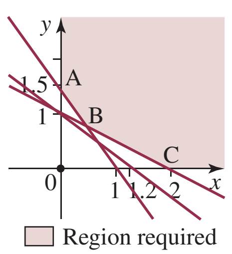
11 Find the required region for $x \geq 0$.
(a) Choose a point on the x-axis, say $x = 1$, and check that it satisfies the inequation $x \geq 0$. That is, is $1 \geq 0$?
(b) Since the inequation is correct, shade the side of the line that contains the point.

For $x \geq 0$,
Test point: (1, 0)
Is $1 \geq 0$? Yes

12 Find the required region for $y \geq 0$.
(a) Choose a point on the y-axis, say $y = 1$, and check that it satisfies the inequation $y \geq 0$. That is, is $1 \geq 0$?
(b) Since the inequation is correct, shade the side of the line that contains the point.

For $y \geq 0$,
Test point: (0, 1)
Is $1 \geq 0$? Yes

13 Transfer all of the above information onto a graph. Label the corner points.



14 Determine the coordinates of the corner points of the feasible region.
(a) Read the coordinates of the points A and C from the graph.

A (0, 1.5) and C (2, 0)

(b) The coordinates of point B can be determined by solving equations [2] and [3] simultaneously. Write equations [2] and [3].

$2x + 4y = 4$ [2]
$15x + 10y = 15$ [3]

Continued over page

THINK	WRITE
(c) Multiply equation [2] by 2.5 and call the resultant equation [6].	[2] · 2.5: $(2x + 4y = 4) \cdot 2.5$ $5x + 10y = 10$ [6]
(d) Subtract equation [6] from [3] and solve for x.	[3] − [6]: $15x + 10y - (5x + 10y) = 15 - 10$ $10x = 5$ $x = 0.5$
(e) Substitute $x = 0.5$ into equation [2] and solve for y.	Substituting $x = 0.5$ into [2]: $2 \cdot 0.5 + 4y = 4$ $1 + 4y = 4$ $4y = 3$ $y = 0.75$
Write the coordinates of point B.	B (0.5, 0.75)
15 Employing the corner point method, substitute the coordinates of the corner points into the objective function C.	$C = 0.50x + 0.60y$ At A (0, 1.5) $C = 0.50 \cdot 0 + 0.60 \cdot 1.5$ $= 0.90$ At B (0.5, 0.75) $C = 0.50 \cdot 0.5 + 0.60 \cdot 0.75$ $= 0.70$ At C (2, 0) $C = 0.50 \cdot 2 + 0.60 \cdot 0$ $= 1.00$
16 Select the minimum value of C.	$C_{min.} = 0.70$ at B (0.5, 0.75)
17 Relate the answer to the original question:	
(a) We used x to denote the amount of product A in 250 g units. Find the amount of product A in grams.	Amount of product A $= 0.5 \cdot 250$ $= 125$ g
(b) We used y to denote the amount of product B in 250 g units. Find the amount of product B in grams.	Amount of product B $= 0.75 \cdot 250$ $= 187.5$ g
(c) Write the answer to the problem in words.	A minimum cost of $0.70 will be spent by using 125 grams of product A and 187.5 grams of product B.

remember

In practical problems involving time, number of items and so on, variables cannot take negative values. Therefore, in linear programming, variables such as x and y are always positive or 0; that is, $x \geq 0$ and $y \geq 0$.

EXERCISE 7G Further applications of linear programming

Solve the following linear programming problems, applying skills learned in the previous exercises.

WORKED Example 11

eBook plus

Digital doc: Spreadsheet 068 Linear programming

1 A rug cleaning service has designed a revolutionary treatment which involves two chemicals, A and B. Each of these chemicals contains (among other components) different amounts (in units per kg) of substances a, b and c as shown in the table below.

Substance	Number of units (per kg)	
	Chemical A	Chemical B
a	10	5
b	1	2
c	2	12

The minimum amounts of substances a, b and c required are 20, 6 and 18 units respectively. One kilogram of chemical A costs the company \$16, while 1 kg of chemical B costs the company \$22. What is the minimum possible cost (to the nearest dollar) of the treatment if both chemicals must be used?

2 Smelter's Steel Works manufactures two types of steel rod: type A and type B. Steel rod A takes 2 hours to make while steel rod B takes 4 hours to make. For optimal plant utilisation, the machine press used to make the rods must operate for a minimum of 56 hours over a 1-week period. At least 6 of each type of rod must be made weekly but no more than 16 of steel rod A and no more than 10 of steel rod B can be made per week.

a If the profit on steel rod A is \$300 and the profit on steel rod B is \$900, how many rods of each type must be manufactured to obtain a maximum profit?

b What will be the maximum profit?

3 A clothing manufacturer makes two styles of uniform: style A and style B. Each uniform needs to be sewn, pressed and packaged. Each style-A uniform requires 5 minutes for sewing, 6 minutes for pressing and 3 minutes for packaging. Each style-B uniform requires 8 minutes for sewing, 12 minutes for pressing and 3 minutes for packaging. The profit on each style-A uniform is \$7 and \$12 on each style-B uniform. The times required for the sewing, pressing and packaging, at most, are 480, 600 and 450 minutes respectively.

a Specify the variables.
b Write the 5 constraints.
c Specify the objective function.
d Determine how many uniforms of each style should be made each day to maximise the manufacturer's daily profit.
e What is the maximum daily profit?

4 The members of a local football team require a diet which provides them with the daily minimum requirements of essential vitamins A, B and C. The daily minimum requirements are 36 units of A, 12 units of B and 8 units of C (per kilogram). These requirements could be met if two products, Zest and Boom, were combined. The amounts of vitamins (in units per kilogram) are shown in the table below:

Product	A	B	C
Zest	12	3	1
Boom	6	4	8

a How many kilograms of products Zest and Boom should be used to keep costs to a minimum, given that product Zest costs $4.80 per kilogram and product Boom costs $3.50 per kilogram?
b What is the minimum cost?

5 Luxurious Limousine Services offer two types of chauffeured limousine package: the Gold Pass and the Classic. The number of Gold Pass packages ranges from 140 to 200 while the number of Classic packages ranges from 80 to 120. Gold Pass packages are at least twice as popular as Classic packages. A profit of $20 is made on each Gold Pass ride while a profit of $10 is made on each Classic ride.
a Which combination of chauffeur-driven rides will yield a maximum profit?
b What is the maximum profit?

6 Let It Grow industries have been developing a new type of fertiliser in their two production plants. The fertiliser requires 3 ingredients: I_1, I_2 and I_3. The amounts of these ingredients (in units per tonne) available at each plant are provided in the table below:

Production plant	I_1	I_2	I_3
Plant A	5	9	5
Plant B	3	3	10

A minimum amount of 15 units of I_1 is available while a maximum of 27 and 50 units of I_2 and I_3 respectively are available. Plant A yields a profit of $270 per day and plant B yields a profit of $500.
a How many tonnes of fertiliser should be produced daily at each plant to yield a maximum profit?
b What is the maximum profit?

investigation

Deck chairs

For the situation presented on pages 380 and 381 of this chapter, use linear programming to solve the problem given. That is, how many chairs of each type should be made weekly to maximise the manufacturer's profit?

summary

Graphs of linear inequations

- The graph of an inequation containing a ≤ or ≥ sign is a closed half-plane; a solid line indicates that the points on the line are included in the region required.
- The graph of an inequation containing a < or > sign is an open half-plane. The points on the line are not included in the region required. This is indicated by using a dotted line.
- To find which side of the line contains points that make the inequation a true statement, a test point is used.
- The required region is shaded.

Graphs of simultaneous linear equations

- The graphical solution to the pair of simultaneous linear equations is given by the point of intersection of the two lines representing those equations.
- The coordinates of the point of intersection can be read from the graph, or found using algebra.
- To solve simultaneous linear equations algebraically, either the elimination or the substitution method is used.

Graphs of simultaneous linear inequations

- The graphical solution to a pair of simultaneous linear inequations is given by the intersection of the two half-planes which represent those inequations.

Graphs of systems of linear inequations

- A group of simultaneous linear inequations is called a system.
- The graphical solution to the systems of linear inequations is given by the area common to all half-planes representing those inequations.
- The shaded area (region required) together with the boundary polygon is called the feasible region.
- The vertices of the polygon (feasible region) are referred to as feasible points or corner points.

Solving linear programming problems

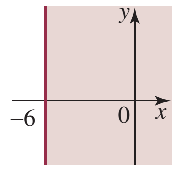
- To solve linear programming problems:
  1. Define the variables.
  2. Write the constraints (inequations) in terms of the variables.
  3. Define the objective function (the linear function to be minimised or maximised).
  4. Obtain the feasible region by sketching graphs of the constraints.
  5. Find the coordinates of the corner points.
  6. Use either the sliding-line or corner point method to find the maximum/minimum value of the objective function.
- Only solid lines are used when sketching the constraints.
- In linear programming, variables always take positive values or 0.

CHAPTER review

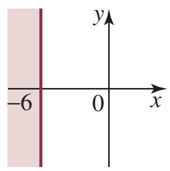
7A

1 multiple choice

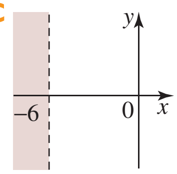
The region required (shaded region) for the inequation $x > -6$ is represented by:

A


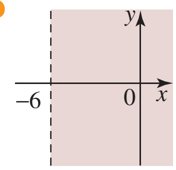
Region required

B


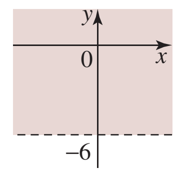
Region required

C


Region required

D


Region required

E


Region required

7A

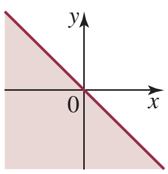
2 Sketch the graphs to represent the following inequations. Shade the required region.

a $3 - x \geq 0$ **b** $y + 1 < -2$ **c** $3x \geq 2y - 6$

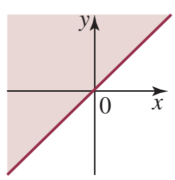
7A

3 multiple choice

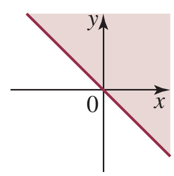
The region required (shaded region) for the inequation $y < -x$ is represented by:

A


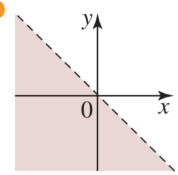
Region required

B


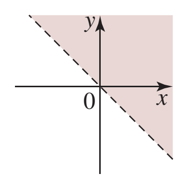
Region required

C


Region required

D


Region required

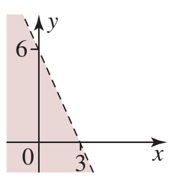
E


Region required

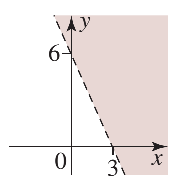
7A

4 multiple choice

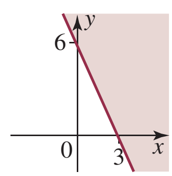
The region required (shaded region) for the inequation $2x + y \leq 6$ is represented by:

A


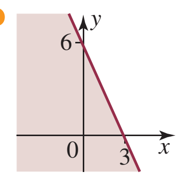
Region required

B


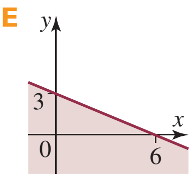
Region required

C


Region required

D


Region required

E


Region required

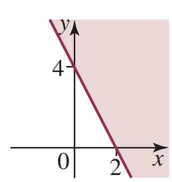
7A

5 multiple choice

The region required (shaded region) for the graph at right can be defined by the inequation:

A $y < 2x + 4$ **B** $y \leq -2x + 4$ **C** $y > 2x + 4$

D $y > -2x + 4$ **E** $y \geq -2x + 4$



Region required

7B

6 Sketch the graphs of, and solve, the following pair of simultaneous equations. Use an appropriate method to verify your solution algebraically.

$$y + 3x = 9$$
$$3y + 4x = 12$$

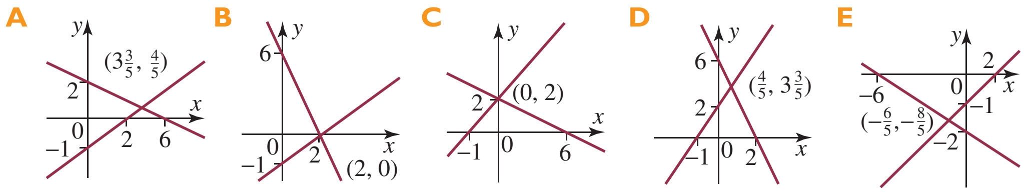
7B

7 **multiple choice**

For the simultaneous equations

$$4x - 8y = 8$$
$$2x + 6y = 12$$

the figure showing the correct graphs and solution is:



7C

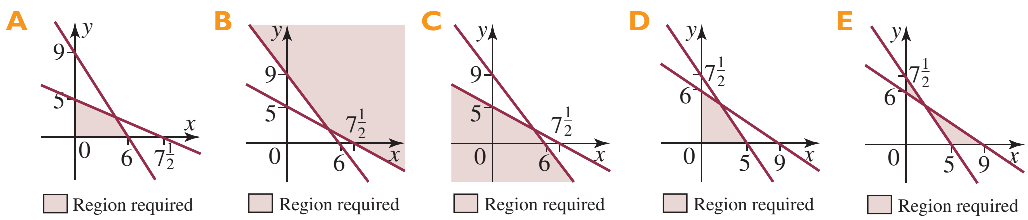
8 Sketch the graphs of the following pair of simultaneous inequations and shade the required region.

$$2y + 5x \geq 6$$
$$x - \tfrac{1}{2}y \leq 4$$

7C

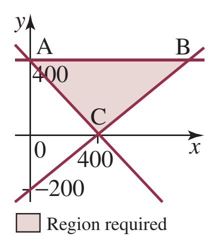
9 **multiple choice**

The graph which best displays the required (shaded) region of the simultaneous inequations $3x + 2y \geq 18$, $4x + 6y \geq 30$ is shown in:



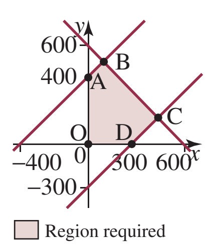
7D

10 The shaded region ABC on the graph at right is the solution set for a system of 3 simultaneous linear inequations. Find the 3 inequations.



7D

11 The shaded region OABCD on the graph at right is the solution set for a system of 5 simultaneous linear inequations. Find the 5 inequations.



7D

12 Graph the region required (shaded region) for the following systems of inequations and specify all corner points.

a $x \geq 0, y \geq 0, 3y - 2x \leq 7$ and $-2y + 5x \leq 10$

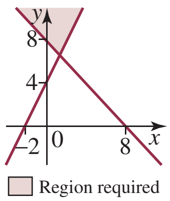
b $x \geq 0, y \geq 0, 6x + 8y \geq 24$ and $8x + 12y \leq 48$

7D

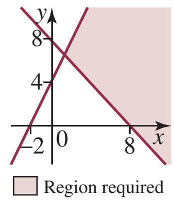
13 **multiple choice**

The shaded region which best represents the system of inequations $x \geq 0$, $y \geq 0$, $x + y \leq 8$ and $y - 2x \leq 4$ is shown in:

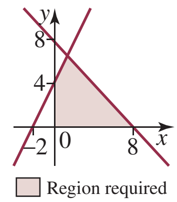
A



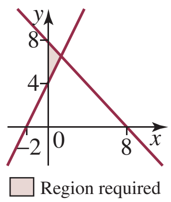
B



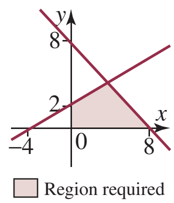
C



D



E



7D,E

14 Graph the region required (shaded region) for the following system of inequations: $x \geq 0$, $y \geq 0$, $2y - 6x \leq 12$, $2y + 4x \leq 14$ and $2y - 2x \geq -2$.

a Specify each of the corner points.

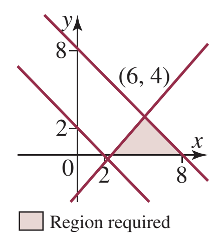
b If the profit is given by $P = 5x + 4y$, determine the maximum profit subject to the above constraints.

7E

15 **multiple choice**

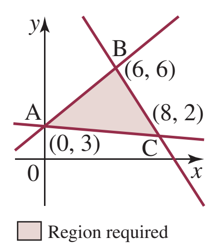
The region required for a system of inequations is given by the graph at right. If the revenue (in dollars) is given by $R = 4x + 3y$, the maximum revenue will be:

A \$8 **B** \$32 **C** \$34 **D** \$36 **E** \$38

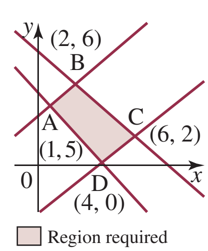


7E

16 **a** Maximise the objective function $D = 4x + 5y$, subject to the system of inequations shown in the graph at right.



b Minimise the objective function $C = 6x - 2y$, subject to the system of inequations shown in the graph at right.

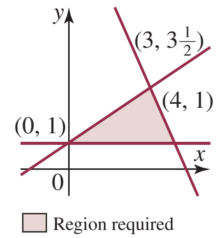


7E

17 **multiple choice**

The maximum value of the objective function S, where $S = 5y - x$, for the feasible region at right is:

A 2 **B** 9 **C** 11 **D** 14.5 **E** 27



7E

18 For which values of x and y will the objective function $P = 10x + 12y$ be a minimum, subject to:

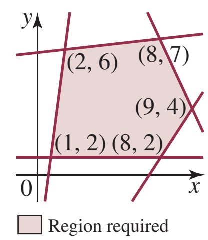
$$2x + 6y \le 24$$
$$2x + 4y \le 18$$
$$x \ge 0$$
$$y \ge 0?$$

7E

19 **multiple choice**

The minimum value of the objective function C, where $C = 3x - 2y$, for the feasible region at right is:

A −6 **B** −5 **C** 2 **D** 4.5 **E** 10



7E

20 For which values of x and y will the objective function $C = 100x + 1200y$ be a maximum, subject to:

$$7x + 5y \le 35$$
$$5x + 10y \le 34$$
$$x \le 2$$
$$x \ge 0$$
$$y \ge 0?$$

Questions **21**–**23** refer to the following information. A drink manufacturer produces two types of sports drink. Each month, at least 300 litres of type A sports drink and 500 litres of type B sports drink must be produced to meet demand. The factory must produce at least 900 litres of sports drink but no more than 1400 litres.

7F

21 **multiple choice**

If x represents the amount in litres of sports drink A, and y represents the amount in litres of sports drink B, then the system of inequations for the above information is:

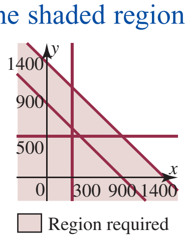
A $x \le 300$, $y \le 500$, $x + y \ge 900$, $x + y \le 1400$
B $x \ge 300$, $y \ge 500$, $x + y \ge 900$, $x + y \le 1400$
C $x \ge 300$, $y \ge 500$, $x + y \le 900$, $x + y \ge 1400$
D $x \ge 300$, $y \ge 500$, $x + y \le 900$, $x + y \le 1400$
E $x \ge 300$, $y \ge 500$, $x + y \ge 900$, $x + y \ge 1400$

7F

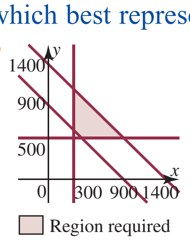
22 **multiple choice**

The shaded region which best represents the system of inequations is shown in:

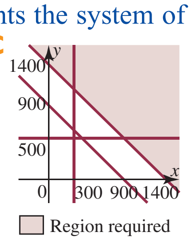
A



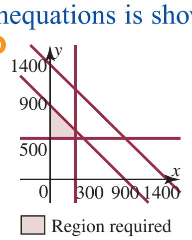
B



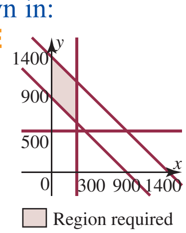
C



D



E



7G

23 **multiple choice**

Given that the profit on a litre of sports drink A is $1.20 and the profit on a litre of sports drink B is $1.00, the maximum profit possible is:

A $960 **B** $1460 **C** $1500 **D** $1580 **E** $1620

7F,G

24 A local factory produces runners and walking shoes. It is able to produce a minimum of 400 pairs of runners and 350 pairs of walking shoes, and must meet the weekly demand of up to 900 pairs of shoes altogether. The profit on a pair of runners is $12.50 and on a pair of walking shoes, $10.

a Specify the variables.

b Write the three constraints.

c If we need to maximise profit, what is the objective function?

25 Nick intends to sow o hectares of oats and w hectares of wheat on his farm. He has 35 hectares of land available on which to sow crops. Oats requires 3 hours of labour and wheat requires 4 hours of labour per hectare, and a total of 120 hours of labour is available.

a If the profit on oats per hectare is $200 and on wheat $240, how much of each crop must be sown to obtain the maximum profit?

b What is the maximum profit Nick can make?

c Would this change if the profit on oats per hectare became $250 and the profit on wheat remained unchanged?

26 A company manufactures two products, A and B. Each product must undergo three chemical processes for the number of hours specified in the table at right.

The minimum amount of time required for processes 1, 2 and 3 is 70, 28 and 36 hours respectively.

Process	A	B
P1	10	5
P2	2	4
P3	3	3

a Using this information, construct the three constraints.
Let x represent the number of items of product A produced.
Let y represent the number of items of product B produced.

b Are any other constraints assumed in this situation?

c Sketch the 5 constraints, shading the region required (feasible region).
The costs associated with the chemical processes on product A and product B are $300 and $200 per item respectively.

d If we want to minimise cost, what is the objective function?

e Find the minimum cost of the chemical processes associated with manufacturing the products in question.

27 Vicki's Vitamin Company sells two Vitamin C products, C_1 and C_2, each boosted with a certain amount of macronutrients , and (units per kilogram) as shown in the table at right.

Scott Scurvy has been advised by his dietitian to combine the 2 products so that the resulting mixture provides at least 50 units of , 28 units of and 60 units of .

Macronutrients	C_1	C_2
	10	5
	4	4
	6	15

If C_1 costs $3.50 per kilogram and C_2 costs $5.00 per kilogram:

a find how Scott should combine the products to achieve minimum cost, and

b state this minimum.

28 Sklo, a glass and crystalware company, is about to launch a new line of products called Spring Blooms. All products in this line (cake platters, cheese platters and fruit bowls) feature the same flower design. The company plans to produce a trial batch of 2100 items. Based on previous experience, they know that the number of cake platters should be at least double the number of cheese platters and that the demand for fruit bowls will not exceed 600. They also need to produce at least 200 cheese platters and no more than 1600 cake platters.

a Let x be the number of cake platters and let y be the number of cheese platters produced. Write the constraints to represent the information.

b Sketch the constraints on a set of axes. Shade the feasible region.

c Find the coordinates of the corner points of the feasible region.

Assuming that every item produced will be sold, the company can make a profit of $24 on every cake platter, $18 on every cheese platter and $21.50 on every fruit bowl sold.

d If the Sklo company wishes to maximise their profit, write the objective function.

e Find the quantities of each type of merchandise that need to be produced and sold in order to maximise the profit.

f State the maximum profit.

29 Chantelle, a discerning cat, likes two types of dry food produced by the Superior Cat Food company: Chicken Bites and Fish Bites. Each of these two products contains three main nutrients (A, B and C), essential for healthy teeth and shiny fur. Chicken Bites contain 3% of nutrient A, 5% of nutrient B and 5% of nutrient C; Fish Bites contain 5% of nutrient A, 8% of nutrient B and 2.5% of nutrient C. Chantelle needs at least 80 g of dry food every day, and her minimum daily requirements of the A, B and C nutrients are 2.5 g, 4.5 g and 3 g respectively.

a Let x represent the quantity (in grams) of Chicken Bites and let y represent the quantity (in grams) of Fish Bites fed to Chantelle on a daily basis. Write 6 constraints to represent the information.

b Sketch the constraints, shading the feasible region.

c Find the coordinates of the corner points of the feasible region.

Alan, the local vet, sells both of Chantelle's favourite Superior Cat Food products at the following prices: $12 for a 1-kg packet of Chicken Bites and $16 for a 1-kg packet of Fish Bites.

d Elena, Chantelle's owner, wants to minimise the cost of her cat's food. Write the objective function that can be used to help Elena achieve her goal.

e Use the objective function from part **d** to find the daily quantities of each type of food that Chantelle needs to be fed in order to minimise the cost.

f What is the minimum cost?

During Chantelle's annual health-check, Alan told Elena that Fish Bites are now available in 1.5-kg packets and can be purchased from him at $21 per packet. Elena quickly calculated that it would be cheaper to buy these new, larger packets of Fish Bites.

g Construct a new function for the cost of pet food.

h Calculate the new amounts of each type of food needed to minimise the cost.

i Find the new minimum daily cost of Chantelle's food.

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Test Yourself
Chapter 7

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ACTIVITIES

7A Graphs of linear inequations

Digital doc

- SkillSHEET 7.1: Solving linear inequations *(page 386)*

Tutorial

- WE3 Int-0482: Watch how to graph an inequation *(page 383)*

Interactivity

- Sketching inequations int-0083: Consolidate your understanding of sketching inequations *(page 386)*

7C Graphs of simultaneous linear inequations

Digital docs

- SkillSHEET 7.2: Practise solving simultaneous equations *(page 396)*
- WorkSHEET 7.1: Graphs of simultaneous linear inequations *(page 396)*

7D Graphs of systems of linear inequations

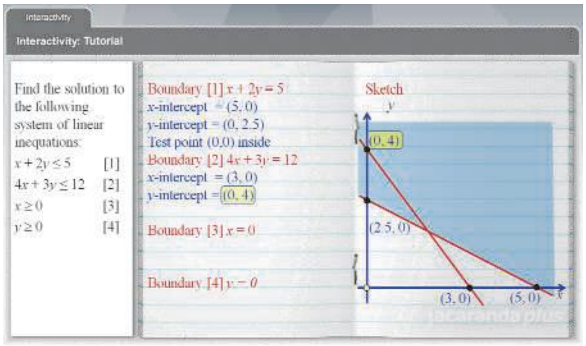
Tutorial

- WE7 Int-0483: Watch how to solve simultaneous linear inequations *(page 397)*

7E Maximising and minimising linear functions

Digital docs

- SkillSHEET 7.3: Practise using vertices of feasible regions *(page 406)*
- SkillSHEET 7.4: Practise using the sliding line method *(page 406)*
- SkillSHEET 7.5: Practise using the corner point method *(page 406)*
- Spreadsheet 068: Investigate linear programming *(page 406)*
- WorkSHEET 7.2: Maximise and minimise linear functions *(page 406)*



7F Solving linear programming problems

Digital doc

- Spreadsheet 068: Investigate linear programming *(page 409)*

Tutorial

- WE10 Int-0484: Watch how to maximise the objective function *(page 407)*

Interactivity

- Applications of linear programming int-0192: Consolidate your understanding of linear programming *(page 409)*

7G Further applications of linear programming

Digital doc

- Spreadsheet 068: Investigate linear programming *(page 417)*

Tutorial

- WE11 Int-0485: Watch how to solve a linear programming problem *(page 413)*

Chapter review

Digital doc

- Test Yourself: Take the end-of-chapter test to test your progress *(page 425)*.

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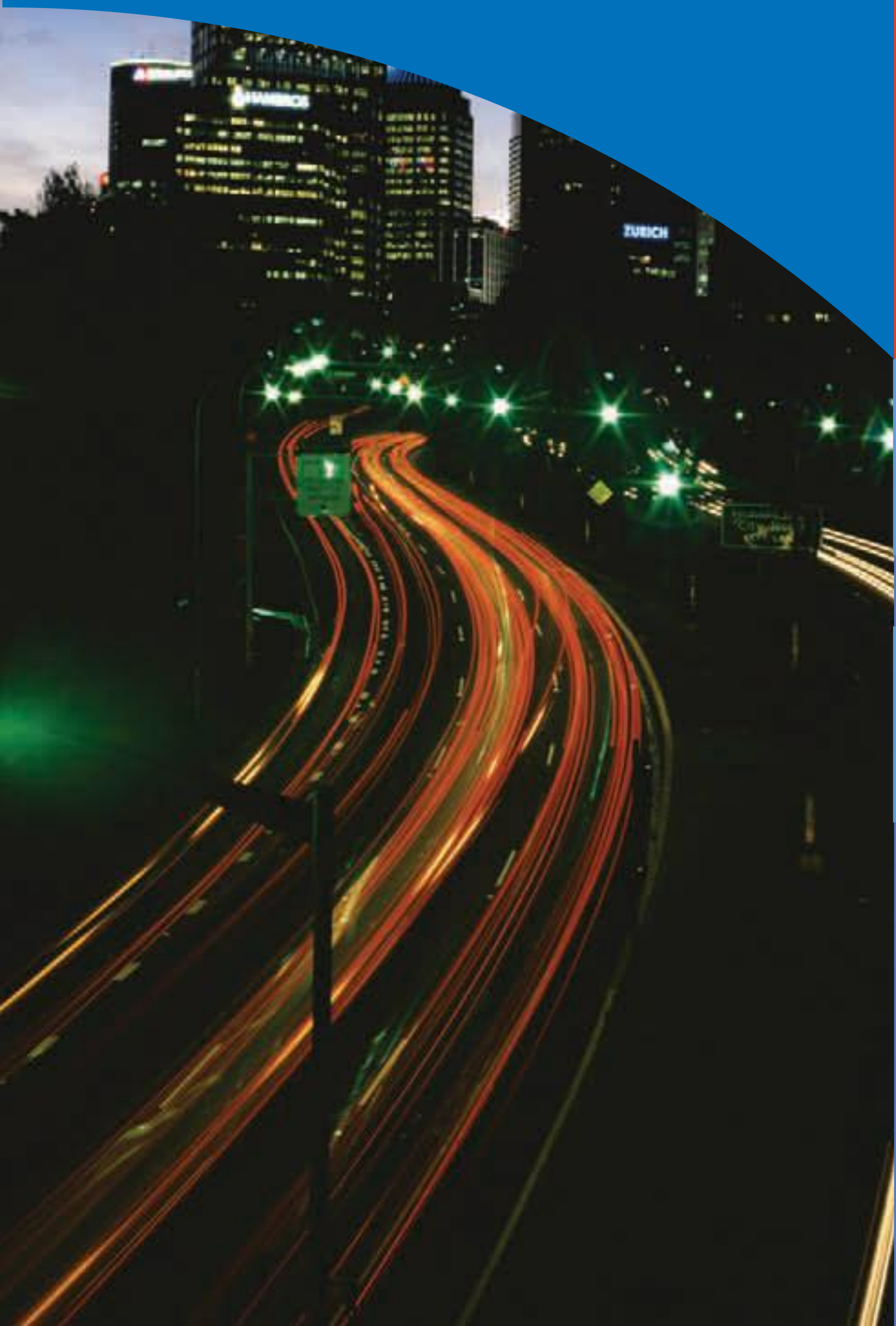
8

Networks



syllabus reference

Elective topic

Operations research — networks and queuing

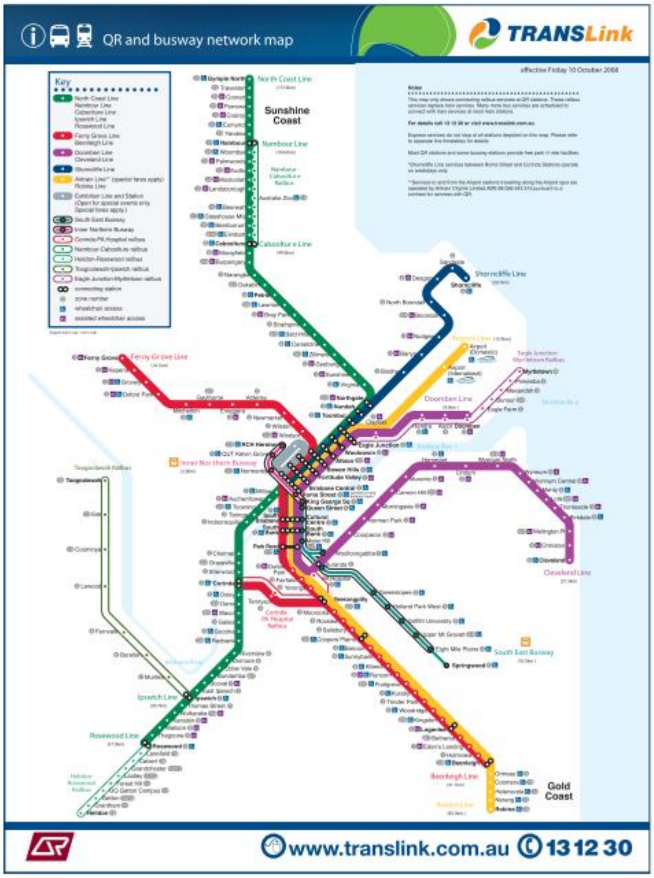
In this chapter

Introduction to networks

Mathematical models may be computer programs, drawings, a system of equations or a combination of these. Through models, people attempt to understand real situations. A postman plans the shortest delivery route or a builder schedules jobs on a large construction project so that the formwork is done as soon as the foundations are completed and the plasterers do not arrive before the walls have gone up.

Models allow these people to think about and plan tasks before actually doing them.

In particular *operations research* is the science of planning and executing an operation to make the most economical use of available resources.



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Networks, nodes and arcs

Networks are maps that can represent an amazing variety of different things: simplified maps, relationships between people, sub-tasks in a building project, computer terminals or the flow of traffic through a city. In each case the network provides a means of studying real-life situations so that decisions can be made. When drawing a network, irrelevant information, such as bends in the roads of a map, is ignored.

1. **A network is a collection of objects connected to each other in some way.**
2. **Networks are made up of *nodes* joined by *arcs*. If nodes are connected they are joined by an arc.**
3. **When the arcs have arrows they are called *directed networks* and travel is possible only in the direction of the arrows.**

There are many examples where networks can be used to model a situation. The first worked example uses a network to plan a drive that takes the shortest possible path or distance.

The **network** can be drawn and each **node** labelled. A path is a specific set of **arcs** connecting nodes and can be represented by the letters in the nodes, as we will see.

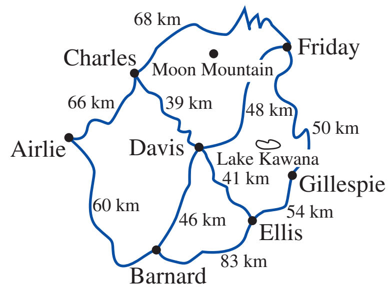
WORKED Example 1

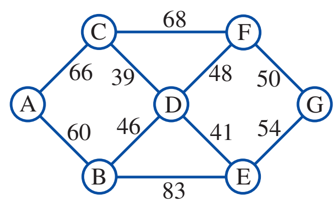
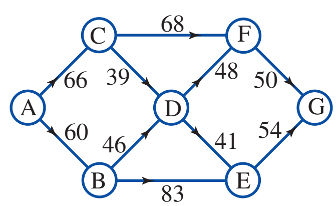
George and Effie want to drive from Airlie to Gillespie using the map at right.

a Draw a network which represents the map.

b Given that each road taken must bring them closer to Gillespie, list the number of ways from Airlie to Gillespie. How many ways are possible?

c Identify the shortest path from the possible routes in b.

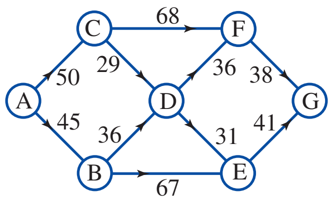


THINK	WRITE/DRAW
a 1 Represent towns with circles, called nodes, labelled with the first letter of the town. 2 Ignore the bends in the roads and use straight lines to represent roads connecting the towns. 3 Check that towns not connected by roads on the map are not joined with an arc.	**a** 
b 1 Each road taken from Airlie must go towards Gillespie. Indicate the direction on each arc with an arrow.	**b** 
2 Use the network to list the number of ways from A to G.	A–B–D–E–G A–B–D–F–G A–B–E–G A–C–D–E–G A–C–D–F–G A–C–F–G
3 Answer the question.	There are 6 ways to go from Airlie to Gillespie.
c 1 Add the lengths of the nodes to calculate the distances of the 6 routes in part **b**.	**c** A–B–D–E–G (60 + 46 + 41 + 54) 201 km A–B–D–F–G 204 km A–B–E–G 197 km A–C–D–E–G 200 km A–C–D–F–G 203 km A–C–F–G 184 km
2 Answer the question.	The shortest path is A–C–F–G: Airlie to Charles to Friday to Gillespie.

If Effie and George were more concerned with time, rather than distance, they might have consulted their travel adviser about the times for each of these stages and redrawn the network with the arcs representing average times for travelling on each connecting road. This network would help them to find the shortest time.

WORKED Example 2

Effie consults the local travel adviser about the travel times for the stages in the journey planned in Worked example 1. She then redraws the network with the average time (in minutes) taken to drive between the towns as shown at right. Which path would take the least time and what is that time?



THINK

1. List all the feasible paths and the times they will take.

WRITE

A-B-D-E-G	153 min
A-B-D-F-G	155 min
A-B-E-G	153 min
A-C-D-E-G	151 min
A-C-D-F-G	153 min
A-C-F-G	156 min

2. The path of least time is ACDEG. — The path ACDEG takes 151 min, the least time.

So, Effie and George would plan different routes depending on whether they were interested in shortest distance or shortest time. In addition to distances and times, arcs may also represent other relationships between nodes. In the following worked example we look at cost relationships between nodes.

WORKED Example 3

The costs of connecting various locations on a university campus with computer cable are given in the table below. A blank space indicates no direct connection.

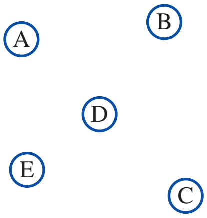
	A	B	C	D	E
A	——	4000		5000	3000
B	——	——	1500	2200	4500
C	——	——	——	2200	1500
D	——	——	——	——	2500

Draw a network to represent this situation, showing the cost of connection along each arc.

THINK

1. There are 5 nodes. Draw them as labelled circles. Because A and C have 3 connections, put them on the outside.

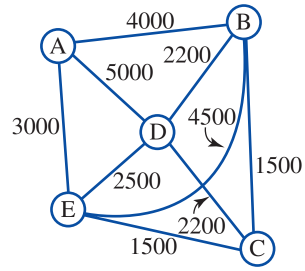
WRITE/DRAW



THINK

2 From the table insert, in a systematic way, draw each arc and label each arc with its cost.

WRITE/DRAW



remember

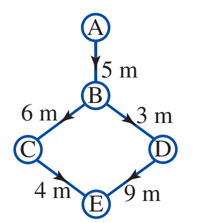
1. A network is a collection of objects connected to each other in some specific way.
2. A network consists of nodes which may be connected by arcs.
3. In a directed network, the arcs will have a direction indicated by arrows.
4. Networks can be used to model situations and calculate shortest paths.

Note: Branch is an alternative term for 'arc'.

EXERCISE 8A Networks, nodes and arcs

WORKED Example 1c

1 Examine the network at right. (All the lengths are in metres.)
   a Which is the longest path?
   b Which is the shortest path?

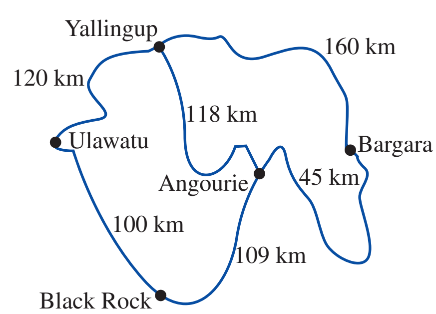


WORKED Example 1

2 A traveller plans a journey from Ulawatu to Bargara (shown on the road map at right).
   a Draw a network to represent this situation.
   b Calculate the longest path if no road is travelled twice.
   c Calculate the shortest path.
   d The travelling times between each town are:

Ulawatu–Yallingup	85 min
Ulawatu–Black Rock	75 min
Yallingup–Angourie	80 min
Black Rock–Angourie	82 min
Yallingup–Bargara	120 min
Angourie–Bargara	34 min.

   i Draw a network of this situation showing the time taken to travel between towns on each arc of the network.
   ii Calculate the longest time taken to travel from Ulawatu to Bargara, assuming you don't return to the same place twice.



iii Calculate the shortest time taken to travel from Ulawatu to Bargara.

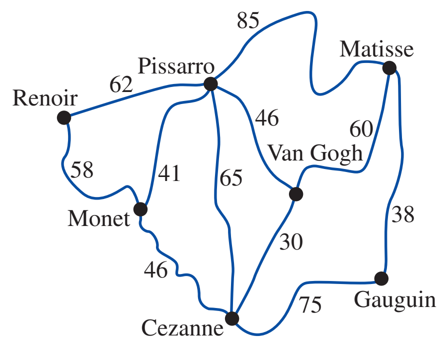
iv Complete the table showing the shortest distance between each of the towns.

	Ulawatu	Yallingup	Black Rock	Angourie	Bargara
Ulawatu	——	120	100	209	
Yallingup	——	——	220		
Black Rock	——	——	——		
Angourie	——	——	——	——	

v Produce a similar table showing the travelling times between each of the towns shown on the map.

WORKED Example 2

3 A traveller plans a journey from Renoir to Gauguin. The distances between various nearby towns are shown on the map at right.



a Calculate the shortest path.

b The travelling times between the following towns are:

Renoir–Pissarro	47 min
Renoir–Monet	44 min
Monet–Cezanne	40 min
Pissarro–Cezanne	45 min
Pissarro–Van Gogh	34 min
Pissarro–Matisse	75 min
Pissarro–Monet	25 min
Cezanne–Van Gogh	20 min
Van Gogh–Matisse	38 min
Cezanne–Gauguin	59 min
Matisse–Gauguin	28 min

i Draw a network of this situation showing the time taken to travel between towns on each arc of the network.

ii Calculate the longest time to travel from Renoir to Gauguin, without returning to the same town twice.

iii Calculate the shortest time to travel from Renoir to Gauguin.

c Complete the table below showing the shortest distance between each of the towns.

	Renoir	Pissarro	Monet	Cezanne	Van Gogh	Matisse	Gauguin
Renoir	——						179
Pissarro	——	——	41				123
Monet	——	——	——				
Cezanne	——	——	——	——			
Van Gogh	——	——	——	——	——		
Matisse	——	——	——	——	——	——	

d Produce a similar table showing the travelling times between each of the towns shown on the map.

WORKED Example 3

4 The cost of trips on McFlaherty's Bus service are given in the table below.

	Port St	Land St	Tork Rd	Bell St	Key St
Port St	——	2.40			1.80
Land St	——	——		2.40	1.50
Tork Rd	——	——	——	1.80	1.50
Bell St	——	——	——	——	2.00

a Draw a network representing this information.
b What is the minimum cost of travelling from Port St to Tork Rd?
c What is the minimum cost of travelling from Bell St to Port St?

5 **multiple choice**

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Digital doc: SkillSHEET 8.1 Constructing networks

The distances, in kilometres, between towns in a region are given in the table below. *Note:* Where a blank appears no direct link between the towns exists.

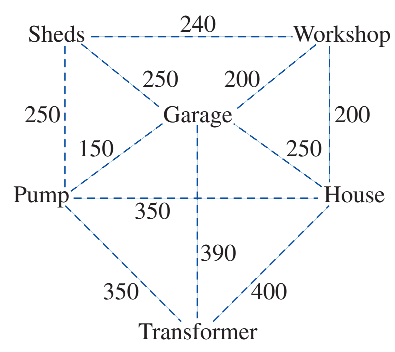
	Grantha	Tamwor	Armida	Beech	Kianga
Grantha	——		85	104	122
Tamwor	——	——	43		100
Armida	——	——	——	85	

In a big storm the bridge on the Armida to Beech road was washed out. How far is the journey from Beech to Armida now?

A 163 km **B** 128 km **C** 189 km **D** 154 km

Minimal spanning trees

The diagram at right represents a farm complex. Each site needs to be connected directly or indirectly to the transformer so that it can get electrical power. For example, the garage can get its power directly from the transformer or indirectly from the house, if the house is connected. The numbers represent the distance between each site. How should the connections be arranged so that the minimum length of cabling is used?



To answer this question in a systematic way we consider the following aspects of networks.

A **tree** is a series of connections in a network that does not contain a loop.

A **spanning tree** in a network is a tree that contains each node.

To identify a minimal spanning tree, we use the minimal spanning tree algorithm which has the following steps.

Step 1 **Choose any node at random and connect it to its closest neighbour.**

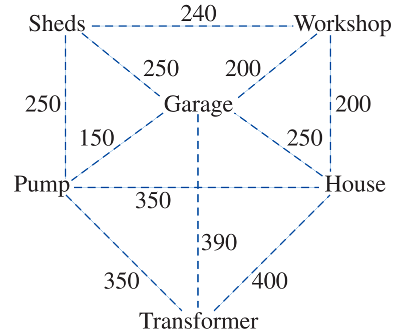
Step 2 **Choose an unconnected node which is the closest to any connected node. Connect this node to the nearest connected node. (If two or more nodes are nearest; that is have the same value, just select any one.)**

Step 3 **Repeat Step 2 until all the nodes are connected.**

The **minimal spanning tree algorithm** can be used to determine the least length of cable needed to connect each building of the farm complex considered above.

WORKED Example 4

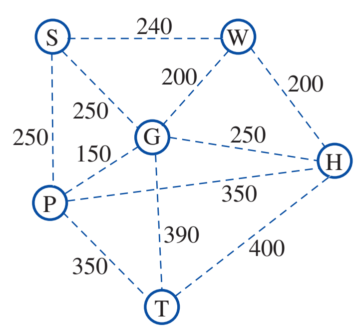
Find the minimal spanning tree to determine the minimum amount of cable needed to connect all the buildings in this farm complex to the transformer. Distances between locations are shown in this plan and are in metres.

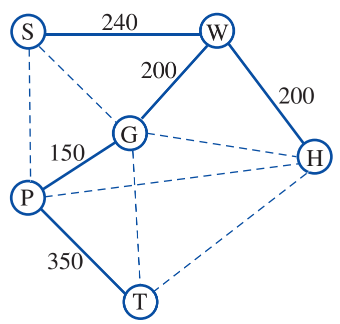


THINK

1. Draw a network with nodes using the first letter of each building.
2. Use dotted lines for the arcs and label each arc with distances between the nodes.
3. Start with the transformer and find the shortest arc. The unconnected node closest to T is P, so join T to P with an arc.
4. Find the unconnected node closest to P or T. It is G. Connect P and G with an arc.
5. Find the unconnected node closest to P, T or G. It is W. Connect G and W with an arc.
6. Find the unconnected node closest to P, T, G or W. It is H. Connect W and H.
7. The sheds, S, are still not connected. Find the node closest to P, T, G, W or H which is closest to the unconnected node S. Connect W and S with an arc.
8. Add up the lengths in the minimal spanning tree.
9. Answer the question.

WRITE/DRAW





350 + 150 + 200 + 200 + 240 = 1140

The minimal length of cable to connect the buildings is 1140 m.

For the minimal spanning tree in the previous worked example, it does not matter which node was used as the starting point. The same spanning tree would have resulted. However, suppose that the distance between the sheds and the pump had been 240 m — the same distance from the sheds to the workshop. Then we could have chosen the final arc as either SW or SP but not both. However, the total length of the minimal spanning tree would have been the same.

History of mathematics

JOHN FORBES NASH (1928–)

When the movie *A Beautiful Mind* won an Oscar for best film in 2002, John Nash was in the audience. The movie, based on a book by the same name, is his story.

John Nash was born in Bluefield, West Virginia in the United States. His schoolteachers did not recognise his brilliance and they focussed on his lack of social skills.

His mother was a schoolteacher who encouraged his love of books and experiments. One of his chemistry experiments with explosives caused the death of a school friend. He enjoyed *Compton's Pictured Encyclopedia*, and the book, *Men of Mathematics* by E T Bell, first excited him about mathematics. He succeeded in proving difficult mathematical problems such as Fermat's Theorem for himself.

He entered Carnegie Technical College in Pittsburgh to follow his father's footsteps in engineering. He moved to chemistry to avoid the rigidity of mechanical drawing. Then, encouraged by the mathematics faculty, he moved from chemistry to major in mathematics, realising that it was possible to make a good career in America as a mathematician.

He excelled in mathematics and graduated with an MS as well as a BS because of his advanced mathematical knowledge. On graduation from Carnegie, where an elective course in international economics influenced his mathematical ideas, he was offered fellowships at both Harvard and Princeton.

In 1948, he chose Princeton where he was closer to his family in Bluefield. He avoided lectures and studied on his own, and was full of mathematical ideas. His interest in game theory grew and he developed the mathematics of equilibrium strategies to predict behaviour. In two papers *Equilibrium Points in n-person Games* and *Non-cooperative Games*, Nash proved the existence of a strategic equilibrium for non-cooperative games, the Nash equilibrium, and suggested approaching the study of cooperative games by their reduction to non-cooperative form. In his two papers on bargaining theory, he proved the existence of the Nash bargaining solution and provided the first execution of the Nash program.

He was awarded the Nobel Prize in Economic Science in 1994, for this work on game theory 45 years earlier.

In the movie, *A Beautiful Mind*, we see a version of how his ideas were stimulated by thinking about non-predictable strategies in a bar scene. In another scene we see him

(*Continued*)

mapping the interactions between pigeons and saying that he is developing an algorithm to predict their behaviour. An algorithm is a step-by-step procedure for a particular mathematical problem and is the idea that lies at the heart of all the computer programming and the code which drives digital computers.

After obtaining his degree in 1950, he worked as an instructor at Princeton but moved to the mathematics faculty of Massachusetts Institute of Technology (MIT) where he met his wife, Alicia, a physics graduate. In 1958 he was described as *the most promising mathematician in the world*. He became mentally disturbed in 1959 when Alicia was pregnant.

Nash attributes his recovery from mental illness to a determined effort to think rationally, aided by light mathematical work. He rejected his delusions and in his acceptance speech for the Nobel Prize in 1994 said, '*I am still making the effort and it is conceivable that with the gap period of about 25 years of partially deluded thinking providing a sort of vacation, my situation may be atypical. Thus I have hopes of being able to achieve something of value through my current studies or with any new ideas that come in the future.*'

In 1999 John Nash was also awarded the Leroy P Steele Prize by the American Mathematical Society for contributions to research.

Questions

1. Which book first stimulated John Nash's interest in mathematics?
2. Which two prizes did John Nash receive?
3. What is an algorithm?

Research

1. Find out about game theory.
2. What opportunities are there to study mathematics after finishing school?

WORKED Example 5

The cost, in dollars, of connecting 7 offices with a computer network is given in the table.

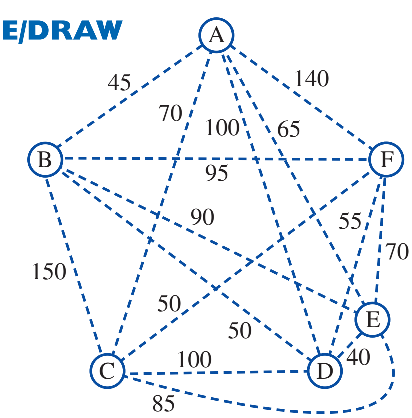
	A	B	C	D	E	F
A	——	45	70	100	65	140
B	——	——	150	50	90	95
C	——	——	——	100	85	50
D	——	——	——	——	40	55
E	——	——	——	——	——	70

Use the minimal spanning tree algorithm to calculate the minimum cost of connecting the offices.

THINK

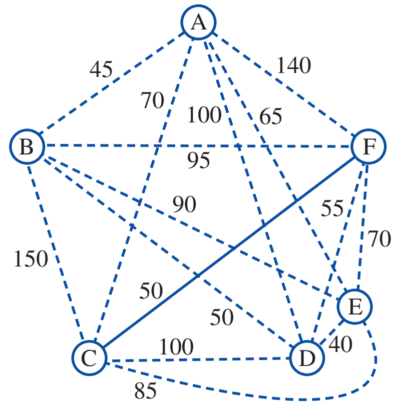
1. Draw a network to represent the information given in the table.
2. Select any starting point, say C.

WRITE/DRAW

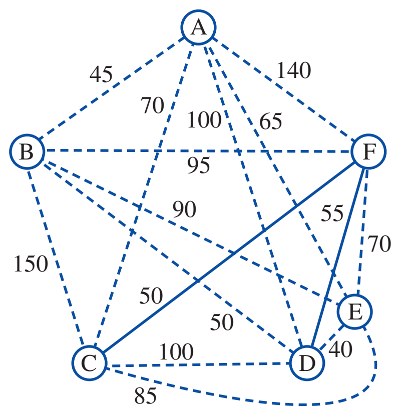


THINK | **WRITE/DRAW**

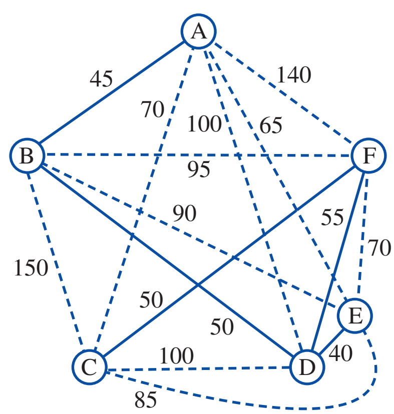
3 Identify the shortest arc connected to C. This is arc CF.



4 Identify the shortest arc connected to C or F to an unconnected node. This is arc FD.



5 Continue, using the minimal spanning algorithm to get the figure opposite.



6 Use the minimal spanning tree to answer the question.

The minimum cost of linking the offices is $\$45 + \$50 + \$50 + \$55 + \$40 = \$240$.

remember

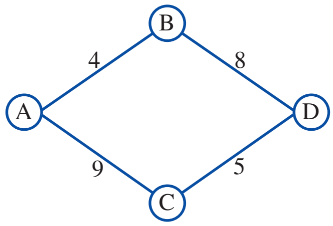
1. A spanning tree connects all nodes in the network and does not contain any loops.
2. A minimal spanning tree is the smallest spanning tree.
3. To find the minimal spanning tree use the minimal spanning tree algorithm.
   - Step 1 Choose any node at random and connect it to its closest neighbour.
   - Step 2 Choose an unconnected node which is the closest to any connected node. Connect this node to the nearest connected node.
   - Step 3 Repeat Step 2 until all nodes are connected.

EXERCISE 8B Minimal spanning trees

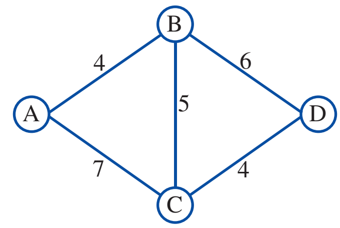
WORKED Example 4

1 Find the minimal spanning tree for each of the following networks.

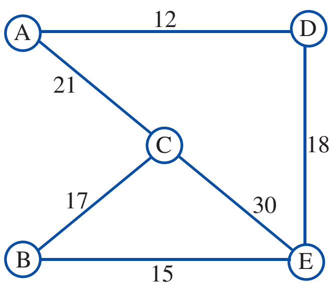
a



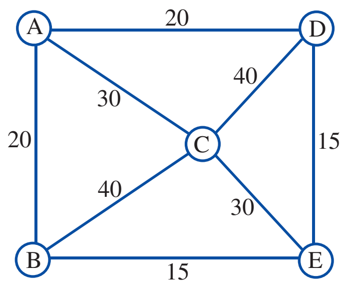
b



c



d



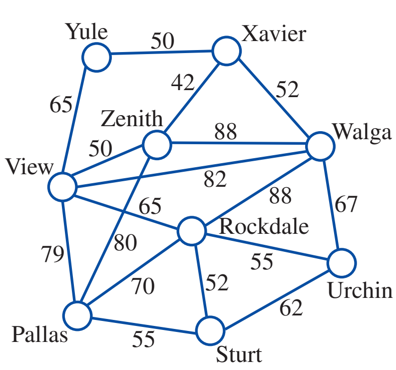
2 The rail authority plans to connect the country centres shown with a rail network (distances are in kilometres). What is the minimum length of track required to achieve this? Use a minimal spanning tree algorithm as follows.

a Begin at Pallas and connect it to its nearest neighbour. Which town is this?

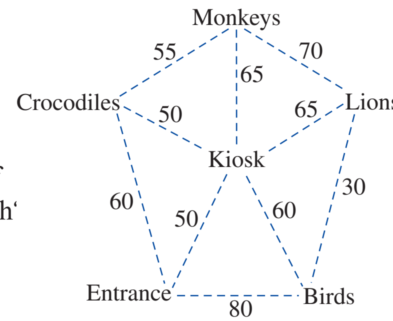
b Which unconnected town is closest to Pallas or to the town selected in **a**?

c Connect this town to the existing link in the shortest way possible.

d Continue by connecting the closest unconnected nodes to any connected ones, one at a time, until all nodes are connected.



3 The paths between the various cages at the Nolonger Park Zoo are dirt and when it rains they become muddy. The figure at right shows all paths, with distances in metres. Management has decided to put in concrete paths.



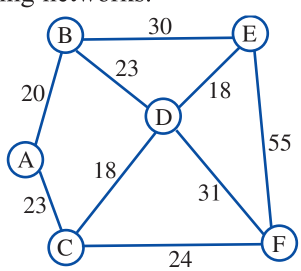
a What total length of path would be required if each dotted line was to become a concrete path‘

b Use the minimal spanning tree algorithm to find the minimum length of concrete path that is required so that patrons could see each exhibit and visit the kiosk without walking on a dirt path.

c Repeat the minimal spanning tree algorithm using a different starting point and show that it does not matter where you start.

4 Use the minimal spanning tree algorithm to find the minimal spanning tree for the following networks.

a



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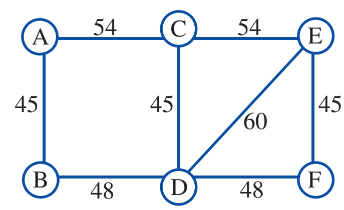
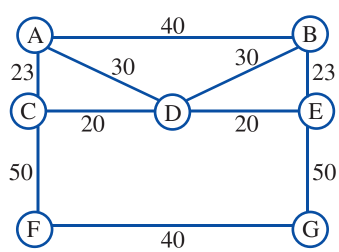
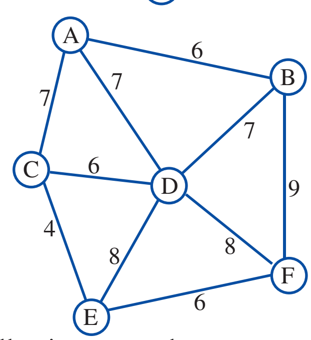


c

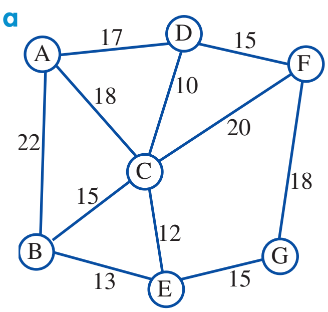


d

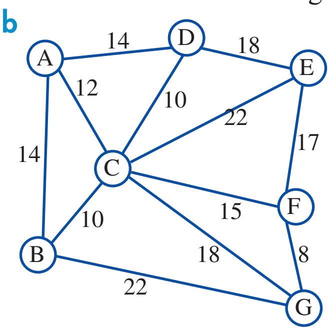


5 Find the minimal spanning tree for each of the following networks.

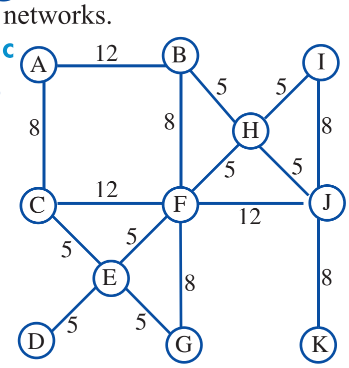
a



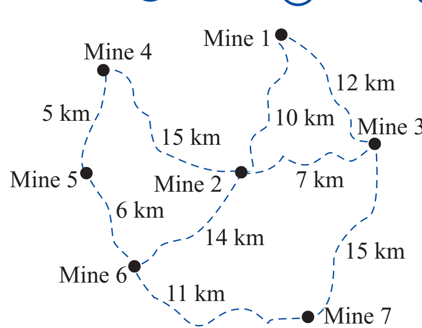
b



c



6 A number of small, private mines have opened up in Waller Flats and the local shire council wants to link them by bitumen roads as shown in the figure at right. What is the minimum length of road that is needed? (Assume the only connections that can be made are those marked on the map of Waller Flats at right.)



7 In question **6** the dotted lines connecting the mines represent dirt roads. If an inspector wants to visit all the mines and is willing to travel on dirt roads, what is the shortest distance he or she needs to travel to visit each of them, starting from Mine 1?

WORKED Example 5

8 A gas pipeline is to be connected between 5 towns so that each town has at least one connection to the system. The gas pipeline costs $25 000 per kilometre. The distance (in km) between the towns is given in this table.

	A	B	C	D	E
A	——	16	23	10	43
B	——	——	32	17	19
C	——	——	——	35	43
D	——	——	——	——	38

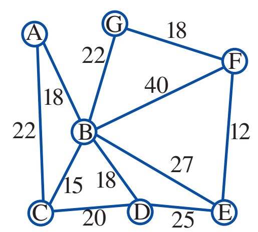
a Find the length of the network connecting these towns in the shortest way.
b What is the cost of this connection?

9 An office computer system requires the linking of 8 terminals. Each terminal has to have at least one connection with the system. The cost (in dollars) of connecting each terminal with another is given in the table.

	A	B	C	D	E	F	G	H
A	——	35	50	75	50	100	65	105
B	——	——	100	40	65	70	90	105
C	——	——	——	70	60	40	55	15
D	——	——	——	——	30	40	105	100
E	——	——	——	——	——	55	40	30
F	——	——	——	——	——	——	25	50
G	——	——	——	——	——	——	——	75

a What is the smallest possible cost for linking the computer terminals if each terminal has at least one connection with the system?
b If each terminal is connected to every other terminal, what is the cost of the linking?

Use the network at right to answer questions **10** and **11**. The dimensions are in km.



10 multiple choice

Which of the following arcs are *not* in the minimal spanning tree?

A AB **B** AC **C** BC **D** BG

11 multiple choice

What is the length of the minimal spanning tree?

A 120 km **B** 105 km **C** 98 km **D** 103 km

Shortest paths

Given a network representing the distance between towns, consider the question, ‘How far is it from town A to town X?’.

In earlier sections we have approached such a question using a trial and error method. However, when networks become more complex, a systematic method is required. The method used is called the **shortest path algorithm**.

Shortest path algorithm

To find the shortest path between A and X in a network, follow these steps.

Step 1 For all nodes that are one step away from A, write the shortest distance from A inside the circle representing the closest node.

Step 2 For all nodes which are two steps away from A, write the shortest distance from A inside the circle representing the closest node two steps away.

Step 3 Continue in this way until X is reached.

Step 4 The shortest path can be identified by starting at X and moving back to the node from which the minimum value at X was obtained, then continuing this process until A is reached. This will be explored in the next worked example.

WORKED Example 6

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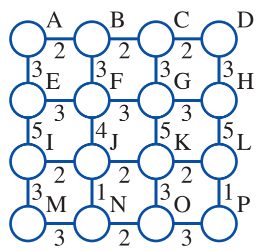
Tutorial: Worked example 6 int-0504

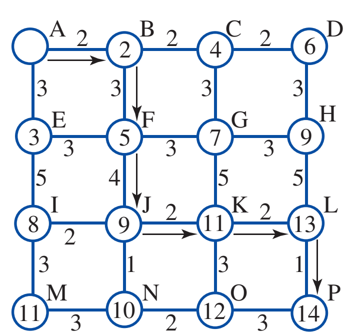
Find the shortest path from A to P in the network at right. The units are in minutes and represent time taken.
Note: **We have placed the labels outside the nodes so that the times can be placed inside the circles.**



THINK

1. Beginning at A write inside the nodes at B and E the shortest time taken to get to them.
2. Then write in the shortest time for all nodes which are two steps away from A. That is, C = 4, F = 5 and I = 8.
3. Continue in this way until P is reached. For example, at node J, the time from I would be 10, so the shorter time, 9, from F is put in the node.
4. Now back-track from P moving from node to node along the arcs which produced the minimum values. Check to see if this is the shortest path.
5. This is the shortest path. Put arrows on this path.
6. Write the answer.

WRITE/DRAW



The shortest path from A to P is A–B–F–J–K–L–P and is 14 minutes long.

remember

To find the shortest path from A to X in a network:

1. For all nodes one step away from A, write the shortest distance.
2. For all nodes two steps away from A, write the shortest distance.
3. Continue until X is reached.

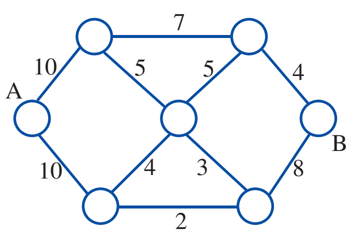
The shortest path is located by starting at X and working backwards to A.

EXERCISE 8C Shortest paths

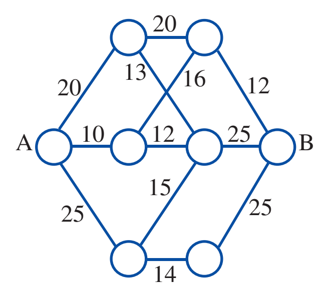
WORKED Example 6

1 Find the length of the shortest path from A to B in each of the following networks.

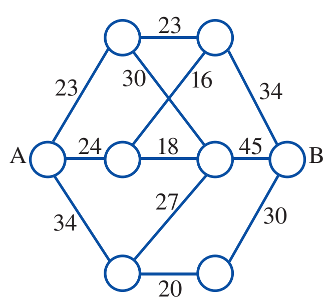
a



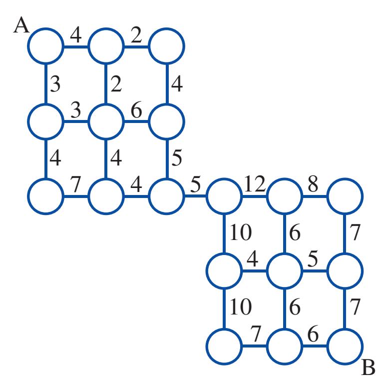
b



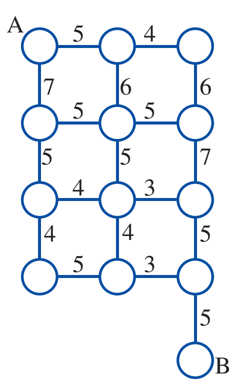
c



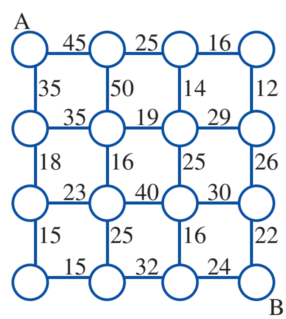
d



e

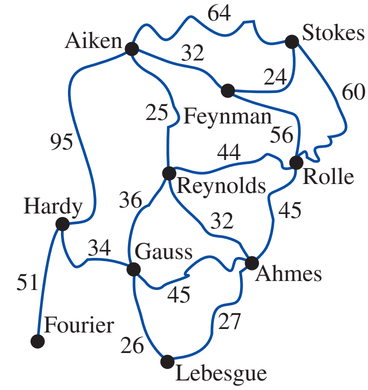


f



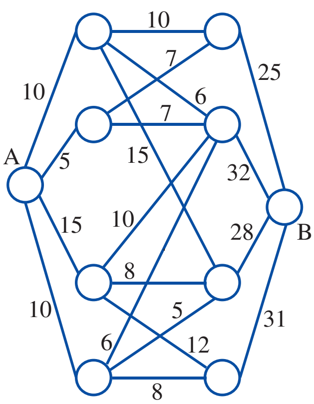
2 From the map at right, where the units are km, answer the following questions.

- **a** What is the shortest distance from Fourier to Rolle?
- **b** What is the shortest distance from Fourier to Stokes?
- **c** What is the shortest distance from Fourier to Stokes travelling through Reynolds?

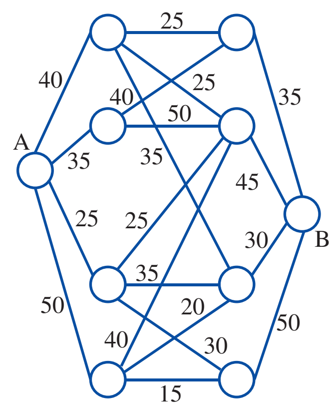


3 For each of the following networks, find the shortest path from A to B.

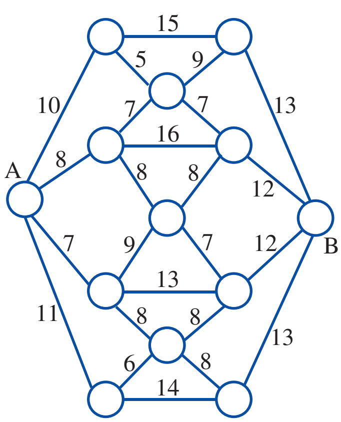
a



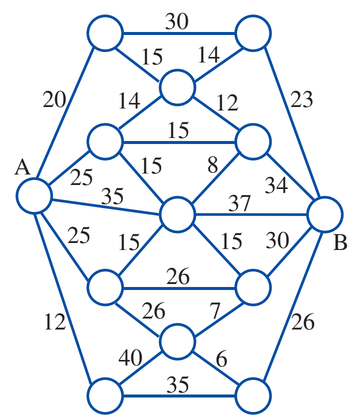
b



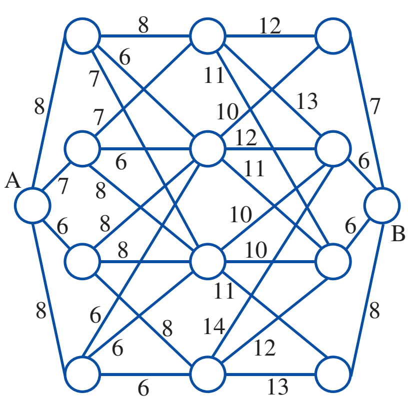
c



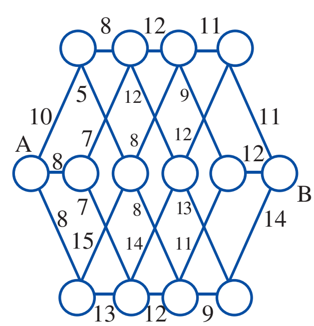
d



e



f



4 This table shows the travelling times in minutes between towns which are connected directly to each other. *Note:* The line indicates that towns are not connected directly to each other.

	Addisba	Bundong	Callop	Dilger	Eric
Addisba	0	50	20	25	—
Bundong	50	0	25	30	30
Callop	20	25	0	—	60
Dilger	25	30	—	0	70
Eric	—	30	60	70	0

a Draw a network to show the connection of the towns by these roads.
b Find the shortest travelling time between Addisba and Eric.

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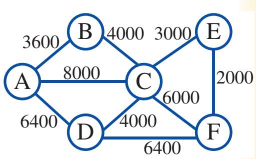
Digital doc:
WorkSHEET 8.1

10 QUICK QUESTIONS 1

This network at right represents the potential cost of a covered walkway between various locations on a campus.



1 How many nodes are there in this network?

2 How many arcs are there in this network?

3 Which node/s have more than 4 arcs meeting?

The cost of the walkway is to be kept to a minimum but it should be possible to go from any location to any other via a covered walkway.

4 Find the minimal spanning tree.

5 What arcs are not included in the minimal spanning tree?

6 What is the minimum cost of such an arrangement of walkways?

7 If one is to travel from D to F under cover, what path should be taken?

It is found that there was an error in the estimate for the walkway connecting A to C. The correct value should be $3600.

8 Find the new minimal spanning tree.

9 What is the new minimum cost for a suitable arrangement of walkways?

10 If one were to travel from B to C under cover, what path should be taken?

Network flow

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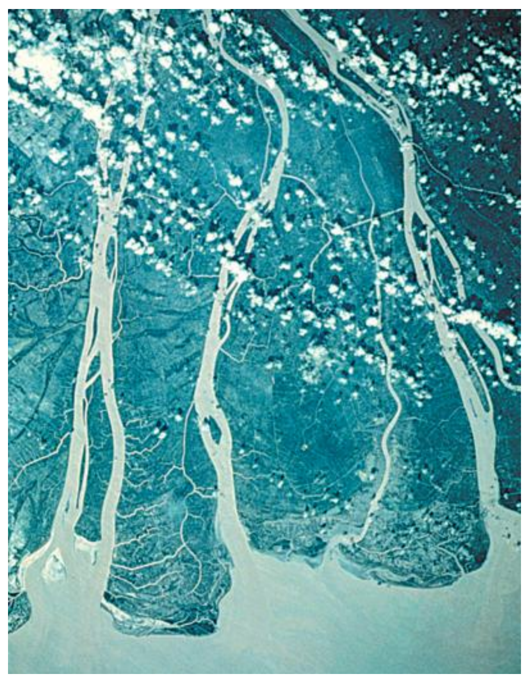
Interactivity:
Maximum flow
int-0196

An application of networks used to analyse flow of traffic or water is *network flow*. These usually involve directed networks where arrows show the direction of flow. An example is described below.

A driver starts for work in the city at 7.30 am each morning. He lives in an outer suburb and as he travels from his driveway through a few streets in his local neighbourhood, there is not much traffic on the roads. As he joins the road that connects his suburb to the next suburb, he notices an increase in the volume of the traffic. As this two-lane road joins the four-lane freeway into the city, the flow of traffic becomes immense. Cars are following bumper to bumper, with drivers changing lanes to drive in the fastest lane. The costs involved, financial and otherwise, for those who participate in the morning rush are significant.

It is in everyone's best interest that the traffic flow smoothly and that traffic jams be avoided at all costs. Engineers use mathematical models of network flow to ensure smooth flow of traffic.

Flow capacities and maximum flow



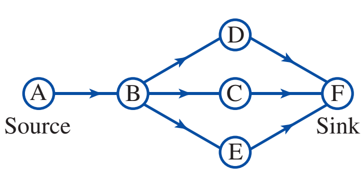
The network's starting node(s) is called the *source*. This is where all flows commence. The flow goes through the network to the end node(s) which is called the *sink*.

The **flow capacity** (capacity) of an arc is the amount of flow that an arc can allow through if it is not connected to any other arcs.

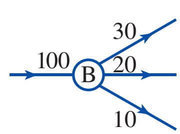
The **inflow** of a node is the total of the flows of all arcs leading into the node.

The **outflow** of a node is the *minimum* value obtained when one compares the inflow to the sum of the capacities of all the arcs leaving the node.

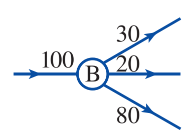
Consider the following figures.



All flow commences at A. It is therefore the source. All flow converges on F indicating it is the sink.



B has an inflow of 100. The flow capacity of the arcs leaving B is 30 + 20 + 10 = 60. The outflow is the minimum of 100 and 60, which is 60.



B still has an inflow of 100 but now the capacity of the arcs leaving B is 80 + 20 + 30 = 130. The outflow from B is now 100.

The flow capacity of the network is the total flow possible through the entire network.

WORKED Example 7

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Tutorial: Worked example 7 int-0508

Consider the information presented in the following table.

From		To		Quantity (kilolitres per minute)	Demand (E)
Rockybank Reservoir	(R)	Marginal Dam		1000	—
Marginal Dam	(M)	Freerange	(F)	200	200
Marginal Dam	(M)	Waterlogged	(W)	200	200
Marginal Dam	(M)	Dervishville	(D)	300	300

a **Convert the information to a network diagram, clearly indicating the direction and quantity of the flow.**

b **Determine the flow capacity of the network.**

c **Determine whether the flow through the network is sufficient to meet the demand of all the towns.**

THINK

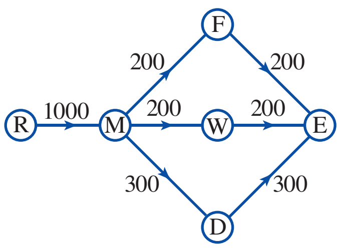
a Construct and label the required number of nodes. The nodes are labelled with the names of the source of the flow and the corresponding quantities are recorded on the arcs.

b 1 Examine the flow into and out of the Marginal Dam node. Record the smaller of the two at the node. This is the maximum flow through this point in the network.

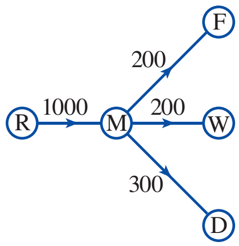
2 In this case the maximum flow through Marginal Dam is also the maximum flow of the entire network.

WRITE

a

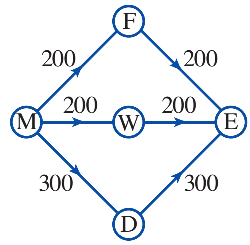


b



Even though it is possible for the reservoir to send 1000 kL/min (in theory), the maximum flow that the dam can pass on is 700 kL/min (the minimum of the inflow and the sum of the capacities of the arcs leaving the dam).

Maximum flow is 700 kL/min.

THINK	WRITE
c 1 Determine that the maximum flow through Marginal Dam meets the total flow demanded by the towns.	**c**  Flow through Marginal Dam = 700 kL/min Flow demanded = 200 + 300 + 200 = 700 kL/min
2 If the requirements of step 1 are able to be met, then determine that the flow into each town is equal to the flow demanded by them.	By inspection of the table, all town inflows equal town demands (capacity of arcs leaving the town nodes).

Consider what would happen to the system if Rockybank Reservoir continually discharged 1000 kL/min into Marginal Dam while its output remained at 700 kL/min.

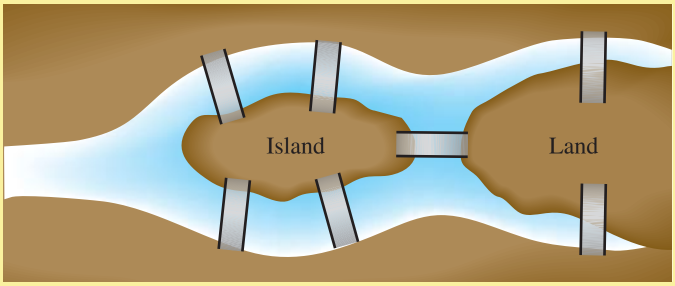
Such flow networks enable future planning. Future demand may change, the population may grow or a new industry that requires more water may come to one of the towns. The next worked example will examine such a case.

Excess flow capacity is the surplus of the capacity of an arc less the flow into the arc.

investigation

The seven bridges of Königsberg

On the River Pregel in the European town of Königsberg, there were 7 bridges arranged as below.



People wondered if it was possible to cross all 7 bridges without crossing any bridge more than once.

Can you see if it can be done?

WORKED Example 8

Use the information contained in Worked example 7 for this worked example. A new dairy factory, Creamydale (C), is to be set up on the outskirts of Dervishville. The factory will require 250 kL/min of water.

a **Determine whether the original flow to Dervishville is sufficient.**

b **If the answer to part a is no, is there sufficient flow capacity into Marginal Dam to allow for a new pipeline to be constructed directly to the factory to meet their demand?**

c **Determine the maximum flow through the network if the new pipeline was constructed.**

THINK

a 1 Add the demand of the new factory to Dervishville's original flow requirements. If this value exceeds the flow into Dervishville then the new demand cannot be met.

2 The new requirements exceed the flow.

b 1 Reconstruct the network including a new arc for the factory after Marginal Dam.

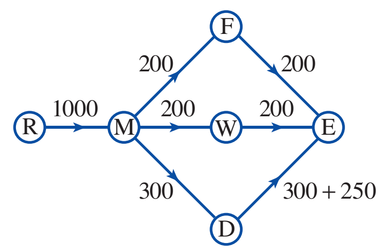
2 Repeat step 1 from Worked example 7 to find the outflow of node M.

3 Determine if the flow is sufficient for a new pipeline to be constructed.

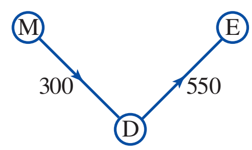
c This answer can be gained from part **b** step 2 above.

WRITE

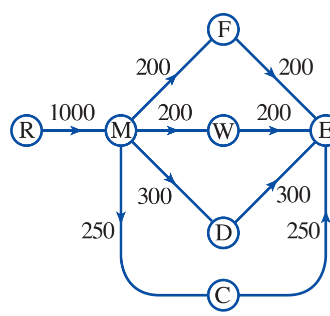
a



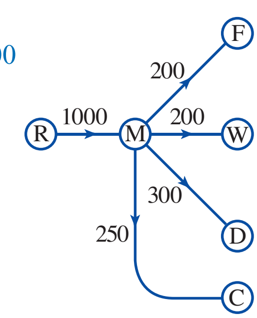
The present network is not capable of meeting the new demands.



b



Marginal Dam inflow = 1000
Marginal Dam outflow
= 200 + 200 + 300 + 250
= 950



There is excess flow capacity of 300 into Marginal Dam which is greater than the 250 demanded by the new factory. The existing flow capacity to Marginal Dam is sufficient.

c The maximum flow through the new network is 950 kL/min.

The maximum flow through most simple networks can be determined using these methods, but more complex networks require different methods to be used.

remember

1. In a network flow diagram, the *arcs have quantities* that indicate rates of flow; for example, litres per minute, cars per second, people per hour and so on.
2. The starting node(s) from which all flows commence is called the *source*.
3. The flow goes through the network to the end node(s) which is called the *sink*.
4. The flow capacity (or capacity) of an arc is the amount of flow that an arc would allow if it were not connected to any other arcs.
5. The *flow capacity* of the network is the total flow possible through the network.
6. The *inflow of a node* is the total of the flows of all arcs leading into the node.
7. The *outflow of a node* is the minimum of either the inflow or the sum of the capacities of all the arcs leaving the node.
8. Excess flow capacity of an arc equals the flow capacity of an arc minus the flow into the arc.

EXERCISE 8D Network flow

WORKED Example 7a

1 Convert the following flow tables into network diagrams, clearly indicating the direction and quantity of the flow.

a

From	To	Flow capacity
A	B	100
A	C	200
B	C	50
C	D	250
D	E	300

b

From	To	Flow capacity
R	S	250
S	T	200
T	U	100
T	E	100
U	E	50

c

From	To	Flow capacity
M	N	20
M	Q	20
N	O	15
N	R	5
Q	R	10
O	E	12
R	E	12

d

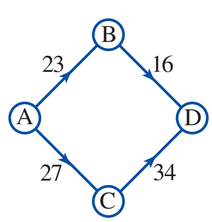
From	To	Flow capacity
D	F	8
D	G	8
G	H	5
G	J	3
F	H	2
F	J	6
J	E	8
H	E	8

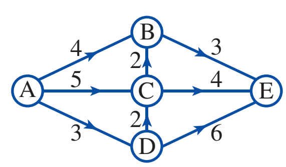
2 For node B in the network at right, state:

a the inflow at B

b the arc capacities flowing out of B

c the outflow from B.





3 Repeat question **2** for the network at right.

WORKED Example 7b, c

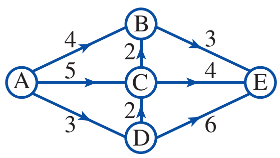
4 For each of the networks in question 1, determine:

i the flow capacity

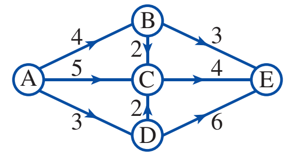
ii whether the flow through the network is sufficient to meet the demand.

5 Convert the following flow diagrams to tables as in question 1.

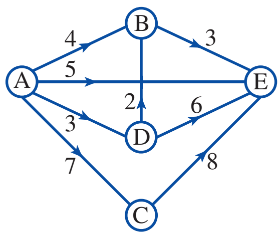
a



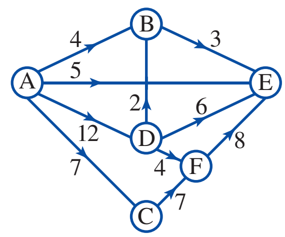
b



c



d



6 Calculate the capacity of each of the networks in question 5.

WORKED Example 8

7 i Introduce new arcs, from the information which follows, to each of the network diagrams produced in question 1.

ii Calculate the new network flow capacities.

a

From	To	Flow capacity
A	B	100
A	C	200
B	C	50
C	D	250
D	E	300
B	E	100

b

From	To	Flow capacity
R	S	250
S	T	200
T	U	100
T	E	100
U	E	50
S	T	100

c

From	To	Flow capacity
M	N	20
M	Q	20
N	O	15
N	R	5
Q	R	10
O	E	12
R	E	12
N	E	5

d

From	To	Flow capacity
D	F	8
D	G	8
G	H	5
G	J	3
F	H	2
F	J	6
J	E	8
H	E	8
D	E	10

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Digital doc: WorkSHEET 8.2

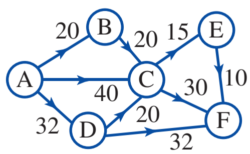
8 multiple choice

In question 7c the outflow from N is:

A 5 B 20 C 15 D 25

10 QUICK QUESTIONS 2

Questions 1 to 4 refer to the network at right. The network represents the distance between towns in kilometres.



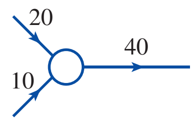
1 What is the shortest path from A to F?

2 Give the length of the shortest path from A to F.

3 Give the shortest path from B to F.

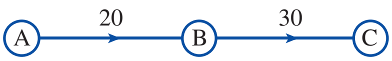
4 What is the length of the shortest path from B to F?

5 In the network at right, what is the inflow at the node?



6 In the same network as question 5, what is the outflow at the node?

7 What is the excess flow capacity of arc BC in the network at right?



Questions 8 to 10 also refer to the first network above. This network shows the capacity of irrigation pipes in kilolitres per hour.

8 What is the inflow at C?

9 What is the outflow at C?

10 What is the maximum flow in the network?

Traffic research

Traffic engineers often use road features and rules to slow traffic or to make it travel more quickly.

List features that are used to:
- slow traffic down
- speed traffic up.

If there is a set of traffic lights near where you live or go to school:
- draw a map showing the flow through the intersection
- time the light phases during peak and off-peak times.

Submit an analysis of the operation of the traffic lights to the local council.

summary

Networks, nodes and arcs

- A network consists of a number of nodes connected by arcs.
- When the arcs have arrows the network is called a directed network and travel is possible only in the direction of the arrows.

Minimal spanning tree

- A tree is a series of connections in a network that does not contain a loop.
- A spanning tree in a network is a tree that contains each node of the network.
- A minimal spanning tree is the arrangement of arcs in which every node is connected to at least one other node in such a way as to minimise the total length of these arcs.
- To find the minimal spanning tree use the minimal spanning tree algorithm:
  Step 1 Choose any node at random and connect it to its closest neighbour.
  Step 2 Choose an unconnected node which is the closest to any connected node. Connect this node to the nearest connected node.
  Step 3 Repeat Step 2 until all the nodes are connected.
- A path is a series of nodes connected by arcs.

Shortest path

- The shortest path is the shortest distance from a given starting point to a given end point.
- To find the shortest path between A and X:
  1. For all nodes that are one step away from A, write the shortest distance from A inside the circle representing the node.
  2. For all nodes which are two steps away from A, write the shortest distance from A inside the circle representing the node.
  3. Continue in this way until X is reached.
  4. The shortest path can be identified by starting at X and moving back to the node from which the minimum value at X was obtained, then continuing this process until A is reached.

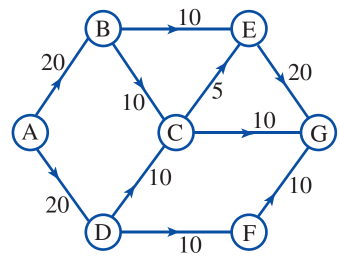
Network flow

- A network can be used to represent the network flow of quantities such as water, traffic or telephone calls.
- Arcs indicate rates of flow.
- The inflow of a node is the total of the flows of all arcs leading into the node.
- The outflow of a node is the minimum of either the inflow or the sum of the capacities of all the arcs leaving the node.
- In a network flow diagram, the arcs have quantities that indicate rates of flow, for example, litres per minute, cars per second people per hour and so on.
- The starting node(s) is called the source, from which all flows commence.
- The flow goes through the network to the end node(s) which is called the sink.
- The flow capacity (or capacity) of an arc is the amount of flow that an arc would allow if it were not connected to any other arcs.
- The flow capacity of the network is the total flow possible through the entire network.
- Excess flow capacity equals the flow capacity of an arc minus the flow into the arc.

CHAPTER review

8A

1 For the network at right, write down:
a the number of nodes
b the number of arcs.



8A

2 The following table represents the cost, in tens of thousands of dollars, of resurfacing roads connecting various locations in a district. Draw a network representing this situation.

	A	B	C	D	E
A	——	5		11	12
B	——	——	4		7
C	——	——	——	8	
D	——	——	——	——	
E	——	——	——	——	——

8B

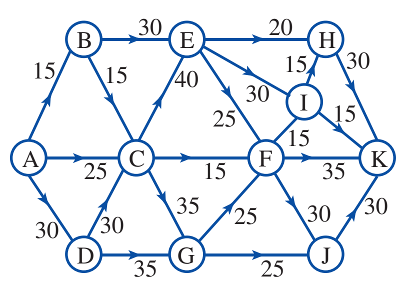
3 Describe an algorithm used to identify the minimal spanning tree.

8B

4 Give the minimal spanning tree for the network in question 1.

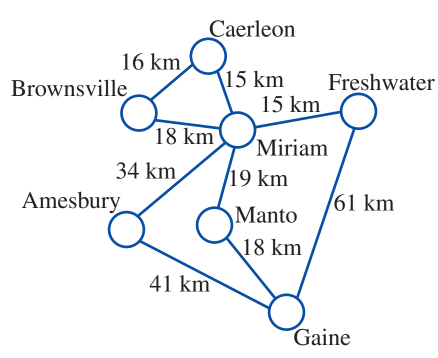
8B

5 Determine the minimal spanning tree for the figure at right.



8B

6 It is planned to join the towns shown on the map at right by a rail link. Use a minimal spanning algorithm to find the shortest length of track needed to connect each town by rail.



8C 7 Identify the shortest path from A to G in question 1. What is the length of this path?

8C 8 Identify the shortest path from A to K in question 5. What is the length of this path?

8D 9 If the arcs in the network in question 1 represent capacity for flow, calculate the following:
a inflow at C
b outflow at C
c the maximum flow.

8D 10 If the arcs in question 5 represent capacity for flow, calculate each of the following:
a inflow at C
b outflow at C
c maximum flow from A to K.

8D 11 From the table at right produce a network flow diagram.

From	To	Flow quantity
A	B	13
A	C	6
B	C	10
B	D	4
C	D	3
C	E	14
D	F	10
E	F	15

8D 12 Draw the network flow diagram for the table at right.

From	To	Flow quantity
A	B	13
A	C	6
A	G	16
B	C	10
B	D	4
B	G	2
C	D	3
C	E	14
D	F	10
E	F	15
G	D	3
G	H	10
H	F	13

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eBookplus ACTIVITIES

8A Networks nodes and arcs

Digital doc

- SkillSHEET 8.1: Practise constructing networks *(page 433)*

8C Shortest paths

Digital doc

- WorkSHEET 8.1: Perform forward and backward scanning, determine earliest completion time, critical paths and float times *(page 444)*

Tutorial

- WE6 Int-0504: Watch how to determine the minimum spanning tree and the shortest path *(page 441)*

8D Network flow

Digital doc

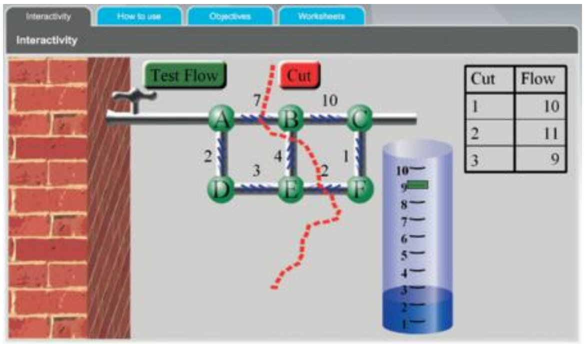
- WorkSHEET 8.2: Critical path analysis and queuing *(page 450)*

Tutorial

- WE7 Int-0508: Watch how to determine the maximum flow of a network *(page 446)*

Interactivity

- Maximum flow int-0196: Consolidate your understanding of maximum flow in a directed network *(page 445)*



Chapter review

Digital doc

- Test Yourself: Take the end-of-chapter test to test your progress *(page 455)*.

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9 Critical path analysis and queuing

syllabus reference

Elective topic

Operations research — networks and queuing

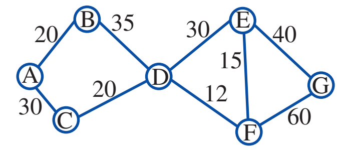
In this chapter

SKILLS CHECK

1 The table at right represents the time taken, in minutes, to travel between various stops on a bus route.
Draw a directed network to represent this information.

	A	B	C	D
A	—	20	30	—
B	15	—	35	45
C	—	30	—	50
D	—	40	—	—

2 The distances between towns in a region are shown in the network at right.
a Find the shortest path from A to G.
b Find the longest path from A to G if the path cannot contain a loop.



Critical path analysis

Activity charts and networks

No matter what we do in our lives, there are many tasks that we must fit into our daily schedule. If the daily tasks are not organised, we tend to run out of time or double-book ourselves. Similarly, operations such as major construction tasks must be efficiently planned so that the right people and materials are at the right place at the right time. If one of these components is wrong, then time, and therefore money, is wasted. To demonstrate the advantages of planning, we will use a simple example.

Cameron organises the things he *has* to do, to allow time for the activities he *wants* to do. (These activities include getting up early for half an hour of tai chi, and taking his dog for a run.)

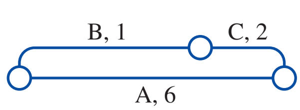
While getting ready for school in the morning, Cameron is faced with the following problem. He has three tasks to do: downloading his email from the computer, reading the email and eating his breakfast. The first two tasks take 1 minute and 2 minutes respectively, while the last takes 6 minutes in all. Cameron needs to complete all these tasks in 7 minutes. How might he accomplish this?

Clearly he needs to be able to do some tasks simultaneously. Although this seems like a simple problem, let us look at what might happen each minute.

Time	Activity	Activity
1st minute		Download email
2nd minute	Eat breakfast	
3rd minute	Eat breakfast	
4th minute	Eat breakfast	
5th minute	Eat breakfast	Read email
6th minute	Eat breakfast	Read email
7th minute	Eat breakfast	

More complex activities require a much greater amount of planning and analysis. A network diagram can be used to represent the 'flow' of activities. Let us construct such a diagram for the problem above.

In the figure at right, the *arcs* of our network represent the three activities of downloading, reading and eating. The left *node* represents the start of all activity, the right node the end of all activity and the middle node indicates that the downloading must occur before the reading starts. In other words, downloading (B) is the *immediate predecessor* of reading (C).



Another way of representing this information is in an activity chart.

Activity letter	Activity	Predecessor	Time (min)
A	Eat breakfast	—	6
B	Download email	—	1
C	Read email	B	2

This chart also shows that activity B (downloading) is the immediate predecessor of activity C (reading), and that the other two activities have no predecessors.

Let us now extend the activity chart to a more complex set of activities for Cameron's morning routine.

WORKED Example 1

From the activity chart below, prepare a network diagram of Cameron's morning schedule.

Activity letter	Activity	Predecessor	Time (min)
A	Prepare breakfast	—	4
B	Cook breakfast	A	2
C	Eat breakfast	B, E, G	6
D	Have shower	A	4
E	Get dressed	D	4
F	Brush teeth	C, H	2
G	Download email	A	1
H	Read email	B, E, G	2
		Total time	25

THINK / **WRITE/DRAW**

1. Begin the diagram by drawing the starting node.

2. (a) Examine the table looking for activities that have no predecessors. There must be at least one of these. Why?

   Activity A has no predecessors.

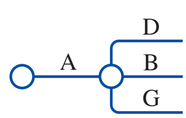
   (b) This activity becomes the first arc and is labelled with its activity letter.

3. (a) List all activities for which A is the immediate predecessor.

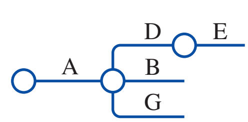
   Activity B has A as an immediate predecessor.
   Activity D has A as an immediate predecessor.
   Activity G has A as an immediate predecessor.

   (b) Add a node to the end of the arc for activity A.

   (c) Create one arc from this node for each of the listed activities. Label these arcs.
   Note: The end node for each of these activities is not drawn until either you are certain that it is not the immediate predecessor of any later activities, or all activities have been completed.



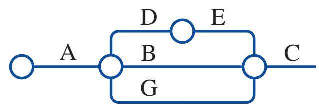
4. Repeat step 3 for activity D. Since it is the *only* immediate predecessor of activity E, this can be added to the diagram. Otherwise, activity E could not be added yet.



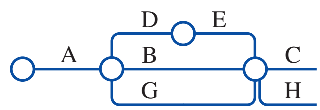
THINK

WRITE/DRAW

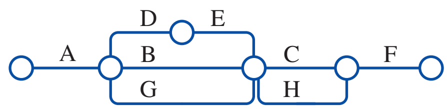
5 (a) Repeat step 3 for activities B and G. They have no activities for which they are the only predecessors. Since activity C is preceded by all of B, G and E, join all the arcs at a single node.
(b) Add activity C after this joining node. Note that activity H is also preceded by all of B, G and H but *not* by activity C.



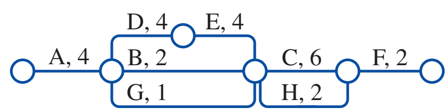
6 Determine whether activity C and H are independent of each other.
Since they are independent, activity H starts from the same node as activity C.



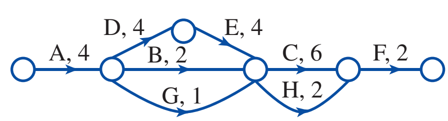
7 The last activity is F, which has C and H as its immediate predecessors. Therefore join C and H with a node, then add an arc for F.
Since F is the final activity, also add the end node.



8 Add the time required for each activity next to its letter.



An alternative network diagram is shown at right. This diagram also indicates a direction: we start at task A, then perform tasks B, D and G . . . and end up at task F. Because of the implied direction, these networks are called **directed graphs** or **directed networks**.



Now that the tasks have been reduced to a network diagram, we can use the diagram to help Cameron reduce the total time spent on all these tasks. If all the tasks were spread out in a straight line so that no tasks were completed at the same time then his morning routine would take 25 minutes (see the activity chart). The diagram shows that some of Cameron's tasks can take place at the same time. Let us investigate the time savings available.

Forward scanning

By forward scanning through a network we can calculate the earliest start times for each activity and the earliest completion time for the whole project. The **earliest start time** (EST) is the earliest that any activity can be started after all prior activities have been completed. The EST is determined by looking at all the previous activities, starting with the immediate predecessors, and working back to the start of the project. An activity can start no earlier than the *completion* of such predecessors. Obviously, the EST for the first activity is 0.

To determine the time saving, first determine the earliest start time for each activity. For simplicity we will return to the initial three tasks with which Cameron was faced: downloading and reading his email and eating his breakfast.

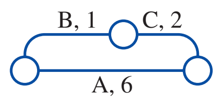
WORKED Example 2

Use forward scanning to determine the earliest completion time for Cameron's initial three tasks from Worked example 1.

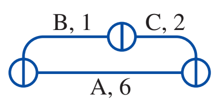
THINK

WRITE/DRAW

1 Begin with the network diagram.

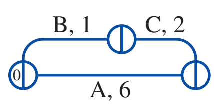


2 Separate nodes into two halves.

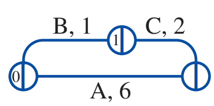


3 The earliest start time for each node is entered in the left-hand side of the node. Nodes with no immediate predecessors are given the value of zero.

As activities B and A have no immediate predecessor then their earliest start time is zero.

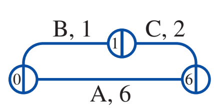


4 Move to another node and enter the earliest start time in the left-hand side. In the case of activity C it must wait one minute while its immediate predecessor, B, is completed.



5 The last node's earliest start time is entered. When more than one arc joins at a node then the earliest start time is the largest value of the paths to this node. This is because all tasks along these paths must be completed before the job is finished.
There are two paths converging at the final node. The top path takes 3 minutes to complete and the bottom path, 6 minutes. The larger value is entered in the node.

Path B–C = 1 + 2 = 3 minutes
Path A = 6 minutes



6 The earliest completion time is the value in the node.

All tasks can be completed in 6 minutes.

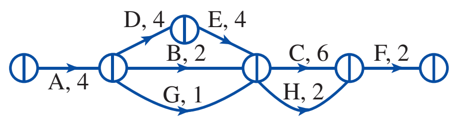
It is important for anybody planning many tasks to know which tasks can be delayed and which tasks must be completed immediately. In the worked example above, the eating must be commenced immediately if the six-minute time is to be attained, whereas downloading the email could be delayed three minutes and still allow enough time for it to be read while Cameron is eating.

WORKED Example 3

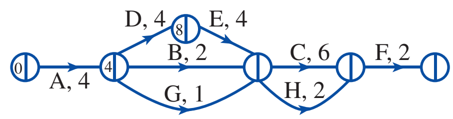
Using all the activities listed in Cameron's morning routine in Worked example 1 on page 460, find the earliest completion time and hence identify those tasks that may be delayed without extending the completion time.

THINK | **WRITE/DRAW**

1. Draw a directed network with split circles at each node.

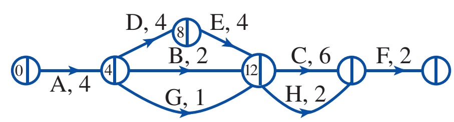


2. Begin forward scanning. The earliest start time for the first three nodes in the path can be entered immediately.



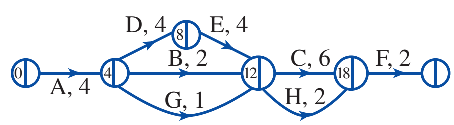
3. Calculate the time values for the paths to the fourth node. Enter the largest value into the left-hand side of the node.

A–D–E = 4 + 4 + 4
= 12 minutes
A–B = 4 + 2
= 6 minutes
A–G = 4 + 1
= 5 minutes



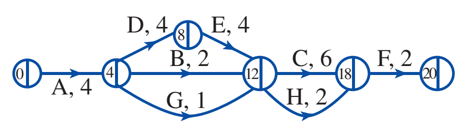
4. Repeat step 3 for the next node. Note that calculations begin by using the time from the previous node (12 minutes).

A–E–C = 12 + 6
= 18 minutes
A–E–H = 12 + 2
= 14 minutes



5. There is only one path to the last activity (F). Add its time requirement to that of the previous node (18 minutes).

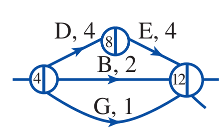
A–C–F = 18 + 2
= 20 minutes
Earliest completion time is 20 minutes.



6. The time in the last node indicates the earliest completion time.

Earliest completion time = 20 minutes

7. Identify sections of the network where there was a choice of paths. There are two such sections of the network. Examine the first one (the 4th node).

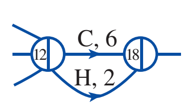


8. List and total the time for each path through this section of the network. The largest value indicates the path that cannot be delayed.

D–E = 4 + 4 = 8 minutes
B = 2 minutes
G = 1 minute
Paths B and G can be delayed.

9. Repeat step 8 for the next section identified in step 7.

C = 6 minutes
H = 2 minutes
H can be delayed.



The path through the network which follows those activities that cannot be delayed without causing the entire project to be delayed is called the *critical path*.

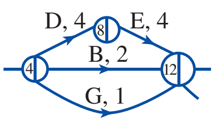
Therefore the **critical path** for the activities listed in Cameron's morning routine would be A–D–E–C–F. It is easily seen that this path takes the longest time (20 minutes).

Float time and latest start time

Float time is the maximum time an activity can be deferred without delaying the entire project. The **latest start time** for such activities is defined as the latest time they may be started without delaying the project.

WORKED Example 4

Work out the float time for activities B and G in Worked example 3, and hence identify the latest starting time for these activities.



THINK	WRITE
1 List the alternative paths for the section containing activities B and G and the times for these alternatives.	D–E = 4 + 4 = 8 minutes B = 2 minutes G = 1 minute
2 Subtract the smaller times separately from the maximum time.	Float time for activity B = 8 − 2 = 6 minutes Float time for activity G = 8 − 1 = 7 minutes
3 Look up the **earliest completion time** for the activity on the critical path and subtract the activities' times.	D–E is on the critical path. Earliest completion time = 12 minutes Latest start time for activity B = 12 − 2 = 10 minutes Latest start time for activity G = 12 − 1 = 11 minutes

The float times indicate the amount of time for which these activities can be deferred without delaying the completion of all tasks. Furthermore, activity B could begin up to 6 minutes (10 − 4) after the start of the critical activity (D), while G could begin up to 7 minutes (11 − 4) after the same critical activity (D). There will be a more formal treatment of float time in the next section.

remember

1. Directed networks have arcs with associated quantities which imply direction.
2. The arcs represent activities with associated quantities such as activity time, the time it takes to complete the task. Directional arrows on the arcs indicate the sequence of activities.
3. Nodes represent the end of one activity and the start of subsequent activities.
4. The immediate predecessor(s) of an activity is an activity (activities) that must be completed immediately before the next activity can commence.
5. The earliest start time (EST) is the earliest that any activity can be started after all prior activities have been completed.
6. The earliest completion time is the earliest time in which all activities in the network can be finished after taking into account all activities that can run simultaneously.
7. Float time is the maximum time an activity can be deferred without delaying the entire project.
8. The latest start time for an activity is the latest time the activity may begin without delaying the entire project.
9. Forward scanning through a network allows for the calculation of earliest start times for each activity and the earliest completion time for the entire project.
10. The critical path is the path through the network along activities that cannot be delayed without delaying the entire project.

EXERCISE 9A Critical path analysis

WORKED Example 1

1 From each of the activity charts below, prepare a network diagram.

a

Activity	Immediate predecessor
A	—
B	—
C	A

b

Activity	Immediate predecessor
D	—
E	D
F	D
G	E, F

c

Activity	Immediate predecessor
A	—
B	A
C	A
D	C
E	B
F	B
G	F
H	D, E, G
J	D, E, G
I	J, H

d

Activity	Immediate predecessor
N	—
O	N
P	O, T
Q	P
R	—
S	N
T	S, Y
U	O, T
V	O, T
W	V
X	Y
Y	R
Z	U, X

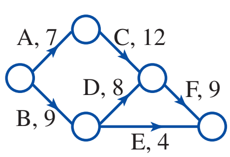
2 When a personal computer is being assembled the following processes must be performed.

Activity letter	Activity	Predecessor	Time (min)
A	Install memory board	—	2
B	Test hard drive	A	20
C	Install hard drive	B, E	4
D	Install I/O ports	A	5
E	Install CD-ROM	D	3
F	Test CD-ROM	E	5
G	Install operating system	C, F	10
H	Test assembled computer	G	12
	Total time		61

a Construct a network diagram.
b Determine the minimum time in which all tasks could be completed.

3 **multiple choice**

Consider the network diagram at right. Times shown are in minutes.



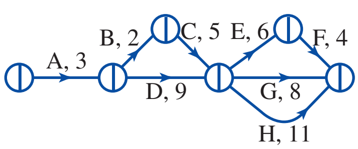
a Which of the following statements is true?
A Activity A is an immediate predecessor of F.
B Activity D is an immediate predecessor of F.
C Activity F must be done before activity D.
D Activity F must be done before activity E.

b The minimum time taken to complete all activities is:
A 19 minutes **B** 21 minutes **C** 23 minutes **D** 28 minutes

WORKED Example 2

WORKED Example 3

4 Refer to the diagram at right.

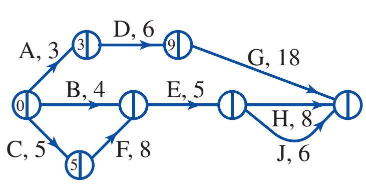


a Use forward scanning to determine the earliest completion time.
b Identify tasks that may be delayed without increasing the earliest completion time.

5 Determine the critical path for the network in question **3**.

6 **multiple choice**

Refer to the network diagram at right.



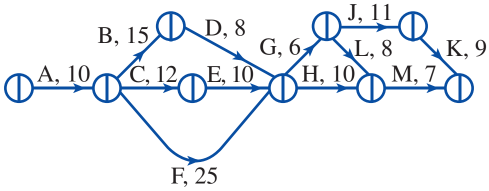
a The number required in the left-hand side of the node after activities B and F is:
A 0 **B** 13 **C** 5 **D** 8

b The number required in the left-hand side of the node after activity E is:
A 5 **B** 9 **C** 10 **D** 18

c The earliest completion time for all tasks is:
A 27 **B** 24 **C** 21 **D** 18

7 Identify the critical path of the network in question **6**.

8 **a** Find the earliest start time for each node in the network shown at right.
b Hence, find the earliest completion time for the project.



9 From the network diagram in question **3**, produce an activity chart.

10 From the network diagram in question **6**, produce an activity chart.

11 From the network in question **8**, produce an activity chart.

WORKED Example 4

12 For the network in question **6**:
a find the critical path
b determine which activities have float time and hence calculate their float times
c determine the latest start time for all non-critical activities.

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13 For the network in question **8**:
a find the critical path
b determine which activities have float time.

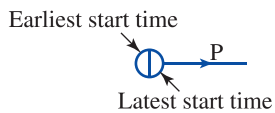
Critical path analysis with backward scanning

With more-complex projects requiring the coordination of many activities, it is necessary to record more information on the network diagrams and to display the information using charts.

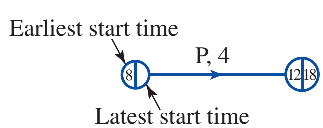
In the previous section the float times and the critical path were worked out using somewhat informal methods. In this section a more formal method will be shown to enable float times to be calculated and the critical path to be determined. This method involves backward scanning.

Backward scanning

To complete critical path analysis, a procedure called *backward scanning* must be performed. In forward scanning, we record the earliest start time for an activity in the left-hand side of each node; in backward scanning, we record the *latest* start time in the *right-hand side* of each node — that is, the latest time that this activity can start without delaying the project.

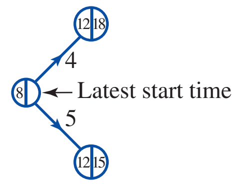


Backward scanning starts at the end node and moves backward through the network, subtracting the time of each arc from the earliest start time of each succeeding node. When two or more paths are followed back to the same node the smallest such difference is recorded.

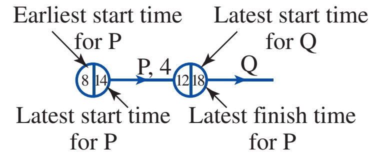


$$\text{Latest start time} = 18 - 4 = 14$$

The results of each backward scanning step yield the latest start time for each activity. Latest start time is the latest time an activity can start without delaying the project.



Latest start time = 15 − 5 (because it is smaller than 18 − 4)
= 10



Latest finish time for an activity is equal to the latest start time of the following activity.

Float time is the maximum time that an activity can be delayed without delaying a subsequent activity on the critical path and thus affecting the earliest completion time.

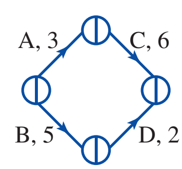
From the above it can be seen that there is a relationship between float time and the other quantities, namely:

Float time = latest finish time − earliest start time − activity time

The technique of backward scanning is best explained with an example.

WORKED Example 5

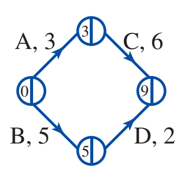
The network diagram at right has been constructed for a project manager. Use forward and backward scanning to clearly display the critical path and to list any float times.



THINK

1. Forward scan through the network and record the earliest start time for each activity in the left-hand side of the appropriate node.

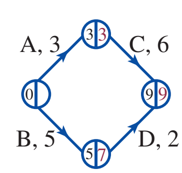
WRITE/DRAW

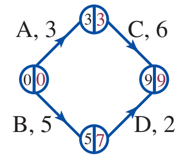
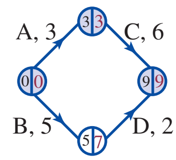


2. Begin backward scanning.
   (a) Start at the end node and trace backwards along all paths from this node.
   (b) Subtract the times of the activities along each path from the earliest completion time (9) and record the value in the right-hand side of the previous node. These values are the latest start times for the activities along the path.

Along path C: 9 − 6 = 3
Along path D: 9 − 2 = 7

Latest start time for activity C = 3
Latest start time for activity D = 7



THINK	**WRITE/DRAW**
3 Repeat the process backwards through the diagram. Where two (or more) paths come together (activities A and B), record the *smaller* value in the right-hand side of the node.	Along path A: 3 − 3 = 0 Along path B: 7 − 5 = 2 Smallest value = 0. 
4 The critical path can now be clearly identified. It is the path that has the same numbers in both the left and right sides of any node. Remember to include *all* such nodes in the critical path.	Critical path shown in pink. 
5 (a) Float times are calculated now. Construct a table with the headings shown. (b) Record the times from the left-hand side of the nodes in the earliest start times (EST) column, the times in the right-hand side of the nodes in the latest finish times (LFT) column as well as the activity times (T). Calculate float times using the equation: Float = LFT − EST − T In this example the float times are also the differences between the corresponding times in the nodes. This is not the rule in the general case.	See table and calculations below.

Activity	Activity time	Earliest start time	Latest finish time	Float time
A	3	0	3	0
B	5	0	7	2
C	6	3	9	0
D	2	5	9	2

For activity D: Float = 9 − 5 − 2 = 2
For activity C: Float = 3 − 0 − 3 = 0
For activity B: Float = 7 − 0 − 5 = 2
For activity A: Float = 3 − 0 − 3 = 0

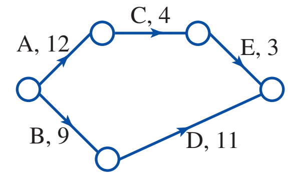
remember

1. The *latest start time* is the time by which an activity must be started to avoid delaying the entire project.
2. *Float time* is the maximum time an activity can be deferred without delaying the entire project.
3. *Backward scanning* starts at the end node and moves backward through the network, subtracting the time for each arc from the earliest start time of each succeeding node.
4. The *latest finish time* for an activity is equal to the latest start time of the following activity.
5. Float time = latest finish time − earliest start time − activity time
6. The *critical path* is identified by finding nodes in a critical path diagram where the number in the left-hand side of the node equals the number in the right-hand side of the node.

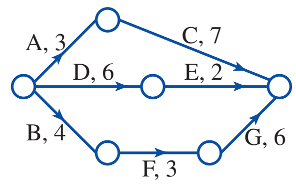
EXERCISE 9B Critical path analysis with backward scanning

WORKED Example 5

1 For the network diagram shown, use forward and backward scanning to clearly display the critical path and to list any float times. Times are in minutes.



2 For the network diagram shown, use forward and backward scanning to clearly display the critical path and to list any float times for non-critical activities. Times are in hours.



3 **multiple choice**

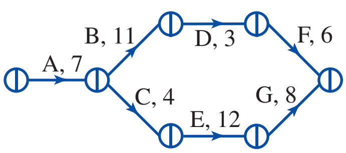
The earliest completion times for questions **1** and **2** respectively, are:

A 13 min; 19 h **B** 17 min; 10 h
C 20 min; 12 h **D** 20 min; 13 h

4 Complete the figure at right by forward and backward scanning and hence:

a determine the earliest completion time
b indicate the critical path.

Note: Times are in days.



5 **multiple choice**

The float time for activity D in question **4** is:

A 1 day **B** 2 days **C** 3 days **D** 4 days

6 **multiple choice**

The latest start time for activity D in question **4** is:

A 18 days **B** 21 days **C** 22 days **D** 25 days

7 The manufacturing of bicycles can be considered as a 7-step process:

A — Collect all the parts — 12 minutes
B — Paint frame — 35 minutes (requires A to be completed first)
C — Assemble brakes — 16 minutes (requires A to be completed first)
D — Assemble gears — 20 minutes (requires B to be completed first)
E — Install brakes — 12 minutes (requires C to be completed first)
F — Install seat — 5 minutes (requires C to be completed first)
G — Final assembly — 18 minutes (requires D and E to be completed first)

a Construct an activity chart.
b Construct a network diagram.
c Determine the earliest completion time using forward and backward scanning.
d Determine the critical path.

8 **multiple choice**

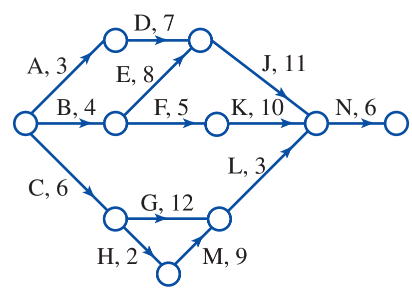
In the bicycle manufacturing system described in question **7**, activities with float time are:

A A, B, C, D, E, F, G
B A, B, D, G
C C, E, F
D C only

9 In question **7**, determine the amount of time saved, as a percentage, using the critical path approach versus completing each task sequentially.

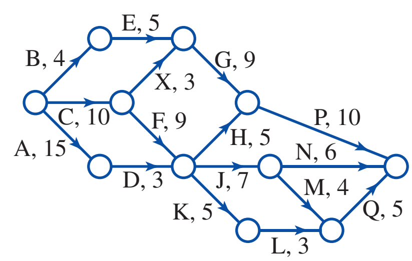
10 From the network diagram at right:

a complete a forward scan and hence determine the earliest completion time
b complete a backward scan and hence determine the critical path.



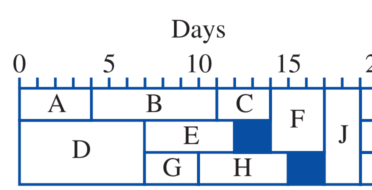
11 From the network diagram at right:

a forward scan to determine the earliest completion time
b backward scan to determine the critical path
c determine the float time for activity X.



12 A method often used in business to display the critical path is a critical path chart, as shown.

The chart indicates that the activities A–B–C–F–J are the critical path. The chart works as follows. Activities immediately to the left are immediate predecessors. For example, A is the immediate predecessor of B, while D is the immediate predecessor of E and G.

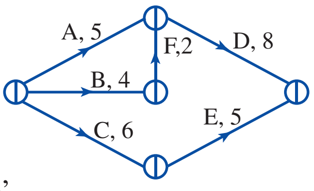


The length of activity is read off the scale (days) at the top. For example, activity C is 3 days long.

a Construct a network diagram.
b Determine the float times for each non-critical activity. (*Hint:* This can be determined directly from the critical path chart.)

10 Quick Questions 1

All questions refer to the network at right.



1. Which activity is an immediate predecessor to E?
2. In one sentence explain the term 'earliest starting time'.
3. In one sentence explain the term 'latest starting time'.
4. Give the formula for the float time of an activity.
5. Re-draw this network to show all earliest starting times and latest finishing times.
6. State the earliest finishing time for activity C.
7. State the earliest finishing time for activity D.
8. State the latest starting time for activity E.
9. State the float time for activity C.
10. Which activities lie on the critical path?

Queues: one service point

Terms used in queuing

Queues come in all types and sizes: long, short, fast, slow, orderly and chaotic. They are formed by humans at an automatic bank teller machine or at the football, by cars at the traffic lights, by planes on the ground or in the air, by telephone calls at the exchange and by pieces of paper in an office.

Queues can be classified as FIFO or LIFO.

FIFO stands for 'first in first out' and most of the queues formed by people take this format. The people who are served first are those who arrived first. (Customers have been known to become upset when this principle is violated.)

LIFO (last in first out) queues may operate with the distribution of items such as nuts and bolts. New stock is deposited in a bin on top of old stock and taken from the top as required. Thus the latest addition to this queue is used first.

Next time you are at a shopping centre, observe people in queues and try to identify the following behaviours. *Baulking* occurs when a customer looks at a queue, decides it is too long and leaves the store. *Reneging* describes the behaviour of the customer who tires of waiting in the queue, gives up and leaves. When a number of queues are available, customers can be seen to be *jockeying* as they move from queue to queue trying to find the shortest. These behaviours have been illustrated using people at a shopping centre but may equally apply to many other situations.

For a business or service it is crucial that queues associated with their activity be at an optimum length. If they are too long, customers will become disgruntled and leave and/or complain. If they are too short then the service provided is not cost effective.

Through modelling, the mathematician is able to plan effective methods for serving customers in queues.

A bank manager receives a phone call from a dissatisfied customer who complains about always waiting for service at the bank. She decides to investigate. Experience has shown that a customer arrives every 2 minutes and that the average time taken to service customers is 3 minutes.

The time between customers arriving is called the *inter-arrival time* and the time taken to serve them is called the *service time*.

It is clear that if the service time is longer than the **inter-arrival time**, then the queue will grow longer if there is only one service point.

The next worked example analyses the growth of the queue.

WORKED Example 6

Customers arrive at the bank with an inter-arrival time of 2 minutes. The service time for each customer is 3 minutes. There is one bank officer serving customers.

a **When the bank opens at 9.00 am customer A has just arrived. When does the next customer, B, arrive?**

b **When is customer A finished?**

c **When is customer B finished?**

d **Continue the table from 9.09 am to show the arrival, queuing and service of customers until 9.15 am.**

Time	Customer served	Arrivals	People in queue	Length of queue
9.00 am	A	—		0
9.01 am	A	—		0
9.02 am	A	B	B	1
9.03 am	B	—		0
9.04 am	B	C	C	1
9.05 am	B	—	C	1
9.06 am	C	D	D	1
9.07 am	C	—	D, E	2
9.08 am	C	E	E	1
9.09 am	D	—	E	1

e **Using the table, calculate the length of time customer F waited in the queue.**

Continued over page

THINK / **WRITE**

a The inter-arrival time is 2 minutes. — **a** Customer B arrives at 9.02.

b The service time is 3 minutes. — **b** Customer A was finished at 9.03.

c Customer B takes 3 minutes to be served at 9.03. — **c** Customer B was finished at 9.06.

d

1. At 9.10, F arrived 2 minutes after E.
2. At 9.12, D is finished, E is served and G arrived.
3. At 9.14, H arrived.
4. At 9.15, E is finished, F is served.

d

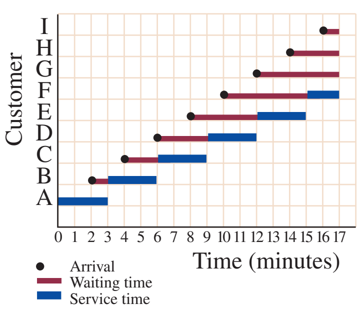
Time	Customer served	Arrivals	People in queue	Length of queue
9.09 am	D	—	E	1
9.10 am	D	F	E, F	2
9.11 am	D	—	E, F	2
9.12 am	E	G	F, G	2
9.13 am	E	—	F, G	2
9.14 am	E	H	F, G, H	3
9.15 am	F	—	G, H	2

e F arrived at 9.10. F was served at 9.15. — **e** Customer F waited in the queue for 5 minutes.

The passage of customers through queues and service points can also be represented graphically as is illustrated in the following example.

WORKED Example 7

Use the information given in the graph to answer the following questions.
a When did customer E arrive?
b How long did E wait in the queue?
c How many people were in the queue at 11 minutes?
d Were customers E and G in the queue at the same time?



THINK / **WRITE**

a The arrival is indicated by a dot. — **a** Customer E arrived at 8 minutes.

b The waiting time is indicated by the thin line. — **b** Customer E waited in the queue for 4 minutes.

c Reading up from $t = 11$ we find thin lines associated with E and F. — **c** Two customers, E and F, were waiting in the queue at 11 minutes.

d E was in the queue from $t = 8$ until $t = 12$. G arrived at $t = 12$. — **d** No, customers E and G did not wait in the queue at the same time.

remember

1. The inter-arrival time is the time between the arrival of one customer and the next.
2. Service time is the time taken to service a customer.
3. A queue will grow if the service time is greater than the inter-arrival time.
4. Use capital letters to represent customers; for example, A, B, C, …

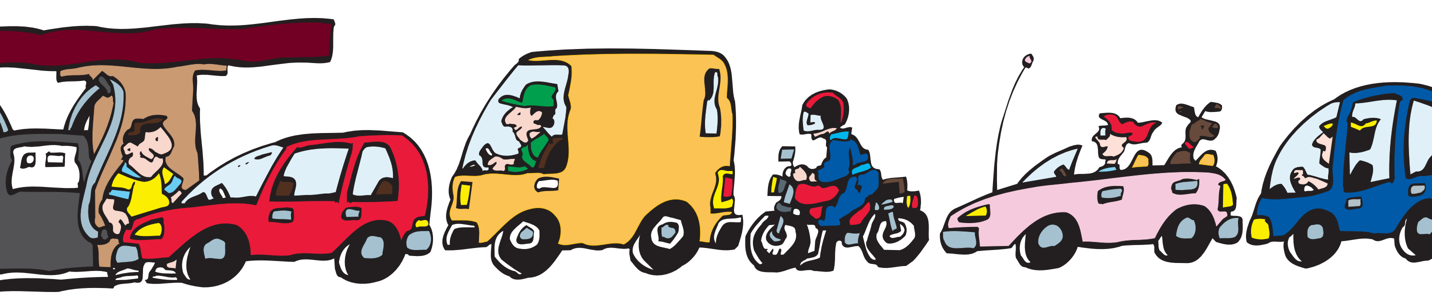
EXERCISE 9C Queues: one service point

WORKED Example 6

1 Consider this table which describes the service of customers at a petrol station.

Time (min)	Customer served	Arrivals	Length of queue	People in queue
0	A	B, C	2	B, C
1	A	—	2	B, C
2	B	—	1	C
3	B	D	2	C, D
4	B	E	3	C, D, E
5	C	—	2	D, E
6	C	F	3	D, E, F
7	D	G, H	4	E, F, G, H
8	D	—		
9	D	I		
10	E	—		
11	E	J		
12	F	K		
13	G	—		
14	G	—		
15	G	L		
16	H	—		

a Copy and complete the table showing the length of the queue for the times after 7 minutes.
b How long did customer C wait in the queue?
c How long did customer H wait to get served?
d How long did it take to serve customer C? (This does not include waiting time.)
e What was the average service time?
f What was the average inter-arrival time? (Add the total time between arrivals and divide by the number of arrivals. If two people arrive at the same time their inter-arrival time is 0.)



2 The time-plot table shown below describes the service of customers at a pet shop. Time is in minutes.

Time	Customer served	Arrivals	Length of queue	People in queue
0	A	—	0	
1	A	B	1	B
2	A	—	1	B
3	A	C	2	B, C
4	B	D	2	C, D
5	B	—	2	C, D
6	C	E	2	D, E
7	D	F, G	3	E, F, G
8	D	—	3	E, F, G
9	D	H	4	E, F, G, H
10	D	—		
11	E	—		
12	F	—		
13	G	—		
14	G	—		
15	G	—		
16	H	—		

a Copy and complete the table showing the length of the queue for the times after 9 minutes.
b How long did customer C wait in the queue?
c How long did customer H wait to get served?
d How long did it take to serve customer C? (This does not include waiting time.)

3 **multiple choice**

The inter-arrival time for customers at the hardware store is 45 s. The number of customers expected between 12.05 and 12.41 is:

A 27 **B** 36 **C** 48 **D** 54

4 **multiple choice**

The time taken to serve a customer at Famished Fred's Fastburgers is 75 seconds. If Nigelli arrives and notices there are 4 people in the queue ahead of her — not including the person who has just begun to be served, how long will it be before she has her burgers?

A 6 min **B** 7.5 min **C** 4.5 min **D** 4 min

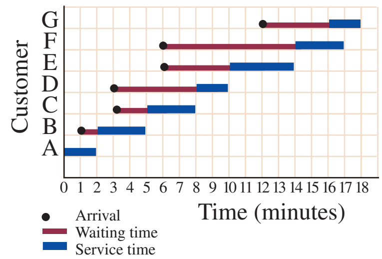
5 At the Fine Foods Delicatessen the inter-arrival time for customers is 2 minutes. The proprietors pride themselves on their friendliness, and the average time spent serving a customer is 3 minutes.
a Construct a time-plot table for the service of customers in this shop. Start at time = 0 with customer A just arriving. (Represent each customer with a capital letter.)
b How long will the queue be when the fifth customer arrives?
c How long will this customer have to wait?

6 A government department has only one person rostered to service customer enquiries. Customers arrive at a rate of about 20 per hour. It takes about 6 minutes to serve each customer.

- **a** What is the inter-arrival time for customers?
- **b** Do you expect a queue to form?
- **c** Construct a time-plot table for the service of customers in this department. Use 3-min intervals.

WORKED Example 7

7 Examine the graph at right.
From this graph deduce the following information.



- **a** How long did it take to serve customer A?
- **b** When did customer C arrive and leave?
- **c** How long was customer C in the queue?
- **d** What was the waiting time of customer E?
- **e** How long was the queue at time = 7 minutes?
- **f** How long was the queue at time = 11 minutes?
- **g** Which customer waited the longest and for how long?
- **h** At what time did two customers arrive at the same time?
- **i** From this graph construct a time-plot table using the following headings.

Time	Customer served	Arrivals	People in queue	Length of queue

8 Draw a graph to represent the behaviour of the queue in question **1**.

9 Draw a graph to represent the behaviour of the queue in question **2**.

10 The passenger ferry to Holiday Island doesn't leave until all passengers have boarded and there are none waiting. It takes each passenger 30 seconds to buy a ticket and board the ferry. When boarding started there were 25 passengers waiting and more passengers were arriving at the rate of 1 every 2 minutes. How long is it before the ferry leaves?

10 Quick Questions 2

1 If customers arrive at the rate of one every 12 seconds, how many customers would you expect in one hour?

2 If a service point can deal with 40 clients each hour, what inter-arrival time is needed to ensure that a queue does not continue to grow?

The table below shows the progress of customers at a service point (in this case, people entering a football game). Use the information to answer questions **3** to **10**.

Time (min)	Customer served	Arrivals	People in queue
0	A	B, C	B, C
1	B	D, E	C, D, E
2	B	F	C, D, E, F
3	C	—	D, E, F
4	D	G, H	E, F, G, H
5	D	I	E, F, G, H, I
6	E	—	F, G, H, I

3 How long did it take to serve customer B?

4 How long did it take to serve customer C?

5 How long did customer C wait to be served?

6 How long did customer E wait to be served?

7 What was the greatest length of the queue?

8 Who was in the queue while C was being served?

9 Who was in the queue while E was being served?

10 How many customers had been completely served in the first four minutes?

Queues: multiple service points

At times it is necessary to have more than one service point to prevent queues growing too long.

WORKED Example 8

At a fast food outlet there are 20 customers waiting when the outlet opens for business. It takes, on average, 80 seconds to serve each customer.

a If no more customers arrive and there are 5 service points operating, how long will it take to serve these people?

b If the inter-arrival time for customers is generally 15 seconds, how many service points are needed to prevent the length of the queue growing?

THINK

a Total time needed is 20 × 80 seconds. So, divide by the number of service points, 5.

b 1 Need the service time to be less than the inter-arrival time.
2 Find the service time with 5 service points.
3 Find the service time with 6 service points.
4 For a queue not to grow, the service time must be less than the inter-arrival time.

WRITE

a Time to serve customers $= \dfrac{20 \times 80}{5}$
$= 320$ s
$= 5$ min 20 s

b Service time = 80 ÷ no. of service points

Service time for 5 points = 16 seconds
Service time for 6 points = 13.7 seconds
This is less than the inter-arrival time.
Therefore, 6 service points are needed.

WORKED Example 9

Five people are waiting in a queue being served by one teller. On average, it takes three minutes to serve one customer and every two minutes another customer arrives. When the second service point opens the situation looks like this:

Time (min)	Point 1 customer	Point 2 customer	Arrivals	People in queue	Length of queue
0	A (2 min to go)	B	F	C, D, E, F	4

a At what time will customer C be served?
b When will customer D be served?
c Complete the table showing the progress of the queue in the first 10 minutes.
d Use the table to determine for how long customer G waited in the queue.
***Note:* Use the convention of assigning A to the first customer, B to the next customer etc.**

THINK

a Customer C will be served by the first service point. Customer A has only two minutes to go.

b Customer D will follow customer B.

WRITE

a Customer C will be served after 2 minutes.

b Customer D will be served after 3 minutes.

Continued over page

THINK **WRITE**

c

1. At $t = 1$, there is no change.
2. At $t = 2$, A finishes and G arrives.
3. At $t = 3$, B finishes.
4. At $t = 4$, H arrives.
5. At $t = 5$, C finishes.
6. At $t = 6$, D finishes and I arrives.
7. At $t = 8$, E finishes and J arrives.
8. At $t = 9$, F finishes.
9. At $t = 10$, K arrives.

c

Time (min)	Point 1 customer	Point 2 customer	Arrivals	People in queue	Length of queue
0	A (2 to go)	B	F	C, D, E, F	4
1	A	B		C, D, E, F	4
2	C	B	G	D, E, F, G	4
3	C	D		E, F, G	3
4	C	D	H	E, F, G, H	4
5	E	D		F, G, H	3
6	E	F	I	G, H, I	3
7	E	F		G, H, I	3
8	G	F	J	H, I, J	3
9	G	H		I, J	2
10	G	H	K	I, J, K	3

d G arrived at $t = 2$.
G was served at $t = 8$.
Answer the question by subtracting.

d Customer G waited 6 minutes in the queue.

investigation

Role play

In this activity the class uses the time-plot table from the previous worked example as a script for acting out the behaviour of a queue. Students are assigned the following roles:

1. 2 tellers
2. 11 customers (A to K)
3. 1 timekeeper

Each of the tellers sets a service point at a desk in the classroom. A point for the head of the queue is designated. The timekeeper begins by calling the time, beginning at 0. As soon as the timekeeper is assured that the action for a particular time is complete, he or she will call the next time. Those not directly involved should observe to see if the customers, tellers and timekeepers are following the script accurately.

remember

1. The first customer in a queue is served at the first available service point.
2. Capital letters represent customers.

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Spreadsheet
052 Generating random numbers

EXERCISE 9D Queues: multiple service points

WORKED Example 8

1 At Loony Larry's Discount Store it takes, on average, 65 seconds to serve a customer. At lunch time there were 18 customers waiting in the queue.

a If no more customers arrive and there are 5 service points operating, how long will it take to serve these people?

b If the inter-arrival time for customers is generally 25 seconds, how many service points are needed to prevent the length of the queue growing?

2 At Mike's Mechanics, a mechanic takes, on average, 2.5 hours to give a vehicle a service. Each mechanic works for only 8 hours a day, 5 days a week. In a particular week 53 customers had booked in to have a service for their vehicles. How many mechanics need to be on duty that week to service this need?

3 At a bank's ATM, customers arrive, on average, at a rate of one every minute and it takes 90 seconds to serve each customer. The bank is considering introducing another ATM. Would another machine prevent the queue growing?

4

Time (min)	Customer served by teller 1	Customer served by teller 2	Arrivals	Length of queue	People in queue
0	A	B	—	4	C, D, E, F
1	A	B	—	4	C, D, E, F
2	A	B	G	5	C, D, E, F, G
3	C	D	—	3	E, F, G
4	C	D	H	4	E, F, G, H
5	C	D	—	4	E, F, G, H
6	E	F	I	3	G, H, I
7	E	F	—	3	G, H, I
8	E	F	J	4	G, H, I, J
9	G	H	—	2	I, J
10	G	H	K	3	I, J, K
11	G	H	—	3	I, J, K
12	I	J	L	2	K, L
13	I	J	—	2	K, L
14	I	J	M	3	K, L, M
15	K	L	—	1	M
16	K	L	N	2	M, N
17	K	L	—	2	M, N

Use the above table to determine the following:

a the number of customers served from $t = 0$ to $t = 6$

b when customer G arrived

c when customer G was served

d the time customer G waited in the queue

e the time customer L waited in the queue.

5 At the First National Bank, 2 tellers are serving customers. A time-plot table for the queue is shown at right.

a Copy and complete the table, showing the length of the queue for times after 8 minutes.

b How long did customer C wait in the queue?

c How long did customer H wait to get served?

d How long did it take to serve customer C? (This does not include waiting time.)

Time (min)	Customer served by teller 1	Customer served by teller 2	Arrivals	Length of queue
0	A	B	C, D, E, F	4
1	A	B	—	4
2	C	B	G	4
3	C	D	—	3
4	C	D	H, I	5
5	C	E	—	4
6	F	E	—	3
7	F	E	—	3
8	G	E	—	2
9	G	H	J, K	
10	G	H	L	
11	G	H	—	
12	G	H	M	
13	I	J	—	
14	I	J	N	
15	K	L	—	
16	K	L	O	
17	K	L	—	

WORKED Example 9

6 Research on customer behaviour at the Second National Bank has shown that the customer inter-arrival time is 2 minutes and service time is 3 minutes.
Consider the following situation where service is provided by two service points:

Time (min)	Point 1 customer	Point 2 customer	Arrivals	People in queue	Length of queue
0	A (1 min to go)	B (2 min to go)	E	C, D, E	3

a At what time will customer D be served?
b When will customer F be served?
c Complete the table, showing the progress of the queue in the first 5 minutes.
d Use the table to determine how long customer F waited in the queue.

7 multiple choice

At the circus there are 8 service points serving 340 customers. Each customer requires, on average, 30 seconds to serve. The show will not start until all customers have been served. How long will it be before the show can start?

A Approximately 21 minutes
B Approximately 15 minutes
C Approximately 17 minutes
D Approximately 32 minutes

8 multiple choice

At the Freewill Games Athletics, spectators arrive at a rate of 250 per minute. If it takes 25 seconds to check each person's ticket, how many service points are needed if queues are not to grow?

A 55
B 105
C 25
D 80

9 It takes a nurse 2 minutes to administer a Hepatitis B vaccine. At a workplace where all employees are being innoculated, the inter-arrival time of patients is 45 seconds. If two nurses are giving the injections, construct a time-plot table to show the progress of the queue that forms. Begin by assuming that patients A and B are being attended and there is no waiting. Use 45-second and 2-minute intervals for the first 8 minutes.

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investigation — Call centres

1 What is a call centre? Distinguish between the purposes of inbound and outbound calls.
2 What is a virtual call centre?
3 Discuss the role of temporary agents.
4 Conduct a class debate:
Call centres — an economic marvel or a social curse.

summary

Introduction to critical path analysis

- Directed networks have arcs whose associated quantity also has an implied direction.
- The arcs represent activities whose associated quantity, activity time, is the time it takes to complete the task. Directional arrows on the arcs indicate the sequence of activities.
- Nodes represent the end of one activity and the start of subsequent activities.
- The immediate predecessor(s) of an activity must be completed immediately before the next activity can commence.
- The earliest start time (EST) is the earliest that any activity can be started after all prior activities have been completed.
- The earliest completion time is the earliest time in which all activities in the network can be finished after taking into account all activities that can run simultaneously.
- Forward scanning through a network allows the calculation of earliest start times for each activity and the earliest completion time for the entire project.
- The critical path is the path through the network along activities that cannot be delayed without delaying the entire project.
- In a critical path diagram, the critical path can be identified where there is the same number in the left- and right-hand sides of any node.

Critical path analysis with backward scanning

- The latest start time is the time by which an activity must be started to avoid delaying the entire project.
- Float time is the maximum amount of time an activity can be deferred without delaying the entire project.
- Backward scanning starts at the end node and moves backward through the network, subtracting the time for each arc from the earliest start time of each succeeding node.
- The latest finish time for an activity is equal to the latest start time of the following activity.
- Float time = latest finish time − earliest start time − activity time
- The critical path is the path through the network formed by nodes for which the left-hand side value equals the right-hand side value.

Queues

- Queues can be classified as FIFO (First in first out) or LIFO (Last in first out).
- Types of queue behaviour:
  - Baulking — a customer refuses to join a queue because it is too long.
  - Reneging — a customer leaves a queue after waiting too long.
  - Jockeying — a customer moves from queue to queue to pick the shortest.
- Inter-arrival time is the time between the arrival of successive customers at a queue.
- Capital letters are used to represent customers arriving at a queue with the convention that the first customer is A, the second is B and so on.
- Service time is the time taken to serve a customer.
- A queue will increase in length if the inter-arrival time is less than the service time.

CHAPTER review

Questions **1** to **4** refer to the following table:

Activity	Time	Immediate predecessor
A	12	—
B	11	A
C	13	—
D	24	C
E	11	D, B
F	21	E

1 **multiple choice** 9A

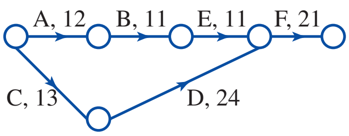
Using the table above, the activities that come before activity E are:

A D and B
B A, D and B
C A, B, C and D
D A, C and D
E F

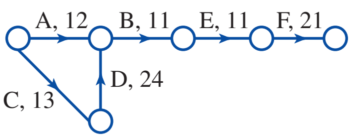
2 **multiple choice** 9A

The correct diagram for the table above is:

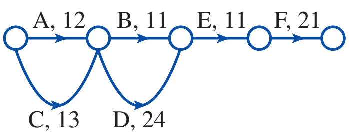
A



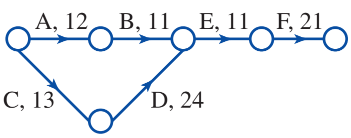
B



C



D



E None of the above.

9A

3 **multiple choice**

The earliest start time for activity E is:

A 23 B 24 C 36 D 37 E 30

9A

4 **multiple choice**

The earliest completion time for the network is:

A 56 B 55 C 68 D 69 E 62

The precedence table below has been provided to control the movement of stock within a store during a refit. The junior manager has been given the task of planning the operation so that the total time taken by the project is kept to a minimum, and that staff are used most efficiently. All time is given in hours. This table will be needed for questions 5 to 8.

Task	Time (min)	Immediate predecessor
A	5	—
B	5	—
C	4	B
D	8	A
E	1	D, G
F	1	E
G	2	C
H	3	D
J	4	H
K	3	F, J

9A

5 Produce a network diagram from the information in the table.

9A

6 Determine, by forward scanning, the earliest completion time for the refit.

9B

7 Perform a backward scan and clearly show the critical path.

9B

8 Copy the table, adding columns for earliest start, earliest finish and float times. Complete the table.

9C

9 Consider the time-plot table which describes the service of customers at a 24–7 store.

Time (min)	Customer served	Arrivals	People in queue	Length of queue
0	A	B		
1	A			
2	B	C, D		
3	C	E, F		
4	C			
5	C	G		
6	D	H, I		
7	E			
8	F	J		
9	F	K		

a Copy and complete this table showing the people in the queue and the length of the queue.
b How long did customer F wait to get served?
c How long did it take to serve customer C?
d What was the average service time?
e What was the average inter-arrival time?

9C

10 Draw a graph to represent the information in question **5**.

9C

11 At Ticketex, people can book concert tickets. It takes 2 minutes to serve each customer and customers arrive at the rate of 1 per minute. If there is only one service point, construct a table to show the growth of the queue over the first 8 minutes. Assume customer A has just arrived at $t = 0$.

9D

12 Calls arrive at a call centre at the rate of 80 per minute and on average each call takes 30 seconds to deal with.
a What is the inter-arrival time of calls in seconds?
b How many service points are needed to prevent the length of the queue of customers growing?

9D

13 Cars arrive at a toll booth at the rate of 1000 per hour. It takes 10 seconds, on average, to serve each car.
a Show that a queue would grow if there is only one service point.
b How many service points would handle this volume without a queue growing?

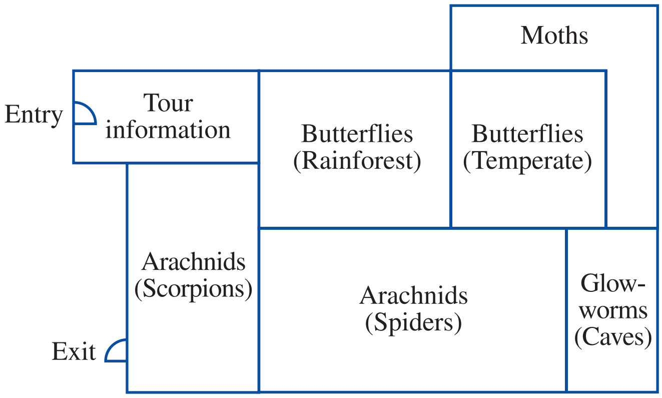
9D

14 The following table describes the behaviour of a queue where there are two service points and customers move to the next available service point.

Time (min)	Point 1 customer	Point 2 customer	Arrivals	People in queue	Length of queue
0	A	B	C, D, E	C, D, E	
1	A	C	F, G		
2	A	D	H		
3	E	D	I, J		
4	E	F	K		
5	G	H	L		
6	G	H	M, N		
7	G	I	O		

a Complete the table showing the people in the queue and the length of the queue.
b How long does customer C wait to be served?
c Calculate the average inter-arrival time for customers.
d For customers B to I, calculate the average time spent waiting in the queue.

15 A Lepidoptera and Arachnid building is to be set up at the zoo. The floor plan is shown at right. The building is to be designed so that the people can flow through in one direction only. Each doorway will open only one way and is designed to ensure that there is no mixing of the exhibits.
a Draw the doors leading from one section to the next, clearly indicating in which direction they open. (The entry and exit doors have been completed for you.)



An analysis of the the flow of visitors through various exhibits at similar zoos in other cities has provided the following table.

Section from	Section to	Arrival rate Number of people per minute
Entry	Tour information	12
Tour information	Rainforest butterflies	13
Rainforest butterflies	Temperate butterflies	12
Temperate butterflies	Moths	2
Temperate butterflies	Glow-worms	4
Temperate butterflies	Arachnids	2
Moths	Glow-worms	4
Glow-worms	Spiders	6
Spiders	Scorpions	5
Scorpions	Exit	12

b If the doors can be represented by an arc (A) with a capacity of 12, convert the information given in the table and plan into a network flow diagram using letters A–J.

c Analyse the inflows, capacities and outflows, and then describe what would happen to the number of people in the rainforest butterflies' room.

d At what rate should people be admitted so that they can flow smoothly through the building?

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Chapter 9

eBook*plus* ACTIVITIES

9A Critical path analysis

Digital doc

- WorkSHEET 9.1: Perform forward and backward scanning, determine earliest completion time, critical paths and float times *(page 467)*

9D Queues: multiple service points

Digital docs

- Spreadsheet 052: Investigate generating random numbers *(page 481)*
- WorkSHEET 9.2: Critical path analysis and queuing *(page 483)*

Chapter review

Digital doc

- Test Yourself: Take the end-of-chapter test to test your progress *(page 489)*.

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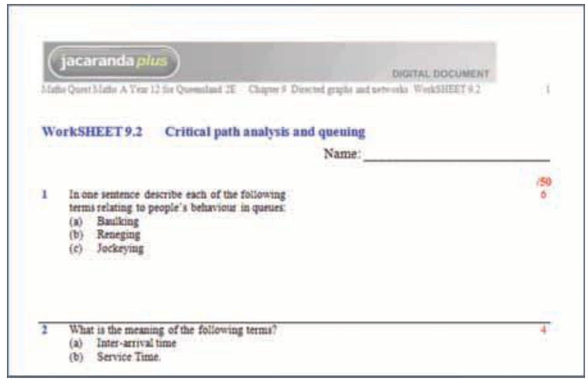
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Maths Quest Maths A Year 12 for Queensland 2E – Chapter 9 Directed graphs and networks WorkSHEET 9.2 1

WorkSHEET 9.2 Critical path analysis and queuing

Name: ____________________

/50

1 In one sentence describe each of the following terms relating to people's behaviour in queues: 6
(a) Baulking
(b) Reneging
(c) Jockeying

2 What is the meaning of the following terms? 4
(a) Inter-arrival time
(b) Service Time.

Probability and the binomial distribution

10

syllabus reference

Elective topic
Introduction to models for data

In this chapter

Introduction

Daily, we hear and use words and phrases that refer to probabilities — 8 out of 10 people use , thunderstorms are likely

A **probability** becomes more complex when the outcome depends on the result of two or more factors. For example, does a take-away restaurant sell more chicken kebabs with coffee or beef kebabs with soft drink? In rolling two dice, what are my chances of getting a total of 7? If I roll a 2 on the first die, what are my chances of obtaining a total of 7 on rolling the second die? On rolling a die 20 times, what chance do I have that the outcome will show all sixes?

To find answers to questions like these, we need an understanding of the dependence (or independence) of one event on another.

This chapter considers questions such as:

1. Does the outcome of a particular event affect the outcome of a second event?
2. Are there any common elements in two or more events?
3. Even though there may be many outcomes to an event, can we group the outcomes to consider only two categories?

SKILLS CHECK

1 State the number of outcomes possible in each of the following trials:

a rolling a die
b tossing a coin
c choosing a card from a deck of cards
d rolling two dice.

2 Calculate each of the following.

a $\frac{1}{2} \cdot \frac{1}{6}$ **b** $\frac{1}{2} + \frac{1}{6}$ **c** $1 - \frac{1}{6}$
d $(\frac{1}{6})^2$ **e** $4 \cdot (\frac{1}{6})^3$ **f** $10 \cdot (\frac{1}{2})^3 \cdot (\frac{1}{2})^2$

3 Evaluate each of the following.

a $(0.2)^2$ **b** $0.3 \cdot 0.1$ **c** $1 - 0.46$
d $3 \cdot 0.4 \cdot 0.6$ **e** $1 - [0.3 + 0.03]$ **f** $0.5 \cdot 0.5 \cdot 0.5$

4 Convert each of the following to decimals (correct to 3 decimal places).

a $\frac{1}{8}$ **b** $\frac{3}{16}$ **c** $\frac{75}{100}$
d $4 \cdot (0.6)^1 \cdot (0.4)^3$ **e** $1 - (\frac{3}{4})^4$ **f** $35 \cdot (\frac{1}{4})^4 \cdot (\frac{3}{4})^3$

5 State, in words, the complement of:

a at least **b** less than **c** no more than.

Compound events — independent events

There are cases when more than one activity is being considered at the same time, such as tossing a coin *and* rolling a die together. It may be interesting to find the probability of (for example) tossing a Tail on a coin and rolling a 5 on a die. For these **compound events** it is necessary to understand first the concept of *independence*.

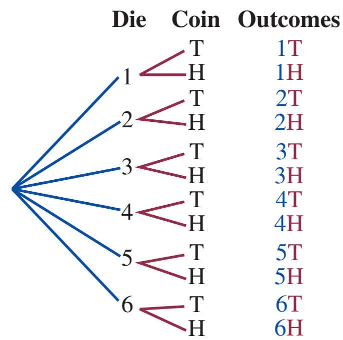
Independent events

When studying compound events we must establish if the events (or outcomes) are **independent** of each other or not. Two events, A and B, are independent if the occurrence of event A has no effect on the probability of event B occurring. For example, when we toss a coin and a die at the same time, the result on the coin does *not* depend on the result on the die. In this case we say that the events are *independent* of each other.

However, consider tossing two dice. Suppose that event A is a 2 appearing on the first die and event B is the sum of the numbers on the two dice being less than 4. Then event B is *dependent* on event A, because the number on the first die is part of the sum of numbers which make up event B.

The multiplication rule for independent events

Consider the first case outlined above. To find the probability of obtaining a Tail on the coin *and* a 5 on the die, list all possible outcomes from a **tree diagram**:



{(1, T), (1, H), (2, T), (2, H), (3, T), (3, H),
(4, T), (4, H), (**5, T**), (5, H), (6, T), (6, H)}

In this experiment there is only one 'successful' outcome, (5, T); so, assuming all outcomes are equally likely, the probability is:

$$\frac{\text{Number of favourable outcomes}}{\text{Total number of possible outcomes}} = \frac{1}{12}.$$

It is tedious to draw tree diagrams (particularly if there are a large number of outcomes) so we will look at a method to shorten the work involved. The problem can also be solved by using the multiplication rule for independent events:

The multiplication rule states that if two (or more) events are *independent* of each other, then the probability of *both* (or all) of them happening is the *product* (multiplication) of the individual probabilities.

$$\mathbf{P(A \text{ and } B \text{ both occurring}) = P(A) \cdot P(B)}$$

In our example, the probability of a Tail is $\frac{1}{2}$ and the probability of a 5 is $\frac{1}{6}$, so the probability of a Tail on the coin and a 5 on the die is $\frac{1}{2} \cdot \frac{1}{6} = \frac{1}{12}$.

Note that this method works for 2, 3 or as many independent events as we wish. It is worth emphasising the fact that the events must all be independent.

WORKED Example 1

Two dice are tossed at once. Find the probability of a double 6 appearing.

THINK

1. Assume that each die toss is independent of the other. State the individual probabilities.
2. Apply the multiplication rule for independent events.

WRITE

Die 1: $P(6) = \frac{1}{6}$
Die 2: $P(6) = \frac{1}{6}$

$P(\text{double } 6) = \frac{1}{6} \cdot \frac{1}{6}$
$= \frac{1}{36}$

Complementary events

Complementary events were discussed in Chapter 12 of the Year 11 text. We shall now consider a more mathematical approach to writing these concepts.

Let A be the specific event we are interested in. Then $P(A)$ is the probability that A happens. What is the probability that A does not happen? Mathematically, this probability is denoted by $P(A')$.

An event either happens or does not happen, so $P(A) + P(A') = 1$.

For example, if $P(A) = 0.4$, then $P(A$ doesn't happen$) = 0.6$ and this is obtained from $1 - P(A)$.

Mathematically this is written as $P(A') = 1 - P(A)$.

A and A' are said to be *complementary events* (one is the complement of the other).

WORKED Example 2

A die is rolled.

a What is the probability of a 6 appearing uppermost?

b What is the probability of a number that is not a 6 appearing uppermost?

THINK

a
1. List the elements in the event space and find the number of possible outcomes.
2. List the successful outcomes and find how many there are.
3. Find the probability of a 6 appearing uppermost.

b
1. Write the formula for an event not occurring.
2. Find the probability of a number that is not a 6 appearing.

WRITE

a $S = \{1, 2, 3, 4, 5, 6\}$
Number of possible outcomes = 6
If A is the event of a 6 appearing, $A = \{6\}$ and number of successful outcomes = 1.

$P(A) = \frac{\text{number of successful outcomes}}{\text{total number of outcomes}}$
$= \frac{1}{6}$

b $P(A') = 1 - P(A)$

$P(A') = 1 - \frac{1}{6}$
$= \frac{5}{6}$

WORKED Example 3

On any day in September the probability of rain in Sydney is 0.18, while in Hong Kong it is 0.39 and in London it is 0.41. Assuming the independence of these events, find:

a **the probability that it is raining in all three cities on the same day in September**

b **the probability that it is not raining in all three cities on the same day in September**

c **the probability that it is raining in Sydney and Hong Kong but not in London on the same day in September.**

THINK	WRITE
a 1 State the individual probabilities.	**a** P(rain in Sydney) = 0.18 P(rain in Hong Kong) = 0.39 P(rain in London) = 0.41
2 Apply the multiplication rule for independent events.	P(rain in all 3 cities) $= 0.18 \cdot 0.39 \cdot 0.41$ $= 0.0288$
b 1 The probabilities of *no* rain in each city is 1 − P(rain in that city).	**b** P(no rain in Sydney) $= 1 - 0.18$ $= 0.82$ P(no rain in Hong Kong) $= 1 - 0.39$ $= 0.61$ P(no rain in London) $= 1 - 0.41$ $= 0.59$
2 Apply the multiplication rule for independent events.	P(no rain in all 3 cities) $= 0.82 \cdot 0.61 \cdot 0.59$ $= 0.2951$
c 1 In this case we need a mix of rain and no-rain probabilities.	**c** P(rain in Sydney) = 0.18 P(rain in Hong Kong) = 0.39 P(no rain in London) = 0.59
2 Apply the multiplication rule for independent events.	P(rain, rain, no rain) $= 0.18 \cdot 0.39 \cdot 0.59$ $= 0.0414$

Are there any other combinations possible for the three cities? Not all possible combinations of rain–no rain for the three cities are listed;
for example P(rain in Sydney) · P(no rain in Hong Kong) · P(no rain in London).

The following section provides a method of discovering all combinations: the use of tree diagrams.

Using tree diagrams for independent events

While the arithmetic approach can be used for computing probabilities for independent events, a tree diagram gives us a *visual* method for calculating probabilities. Tree diagrams also help us to enumerate (list) all possible combinations of outcomes.

WORKED Example 4

Two coins are tossed.

a Find all the possible outcomes, namely 'Head–Head', 'Head–Tail' . . . and so on.

b Find the probability of:

i no (0) Heads turning up **ii exactly 1 Head turning up**

iii at least one Head turning up **iv exactly 2 Heads turning up.**

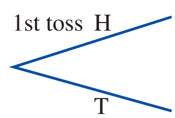
THINK / **WRITE**

a 1 (a) It is reasonable to assume that the outcome of the first coin toss is independent of the second toss.

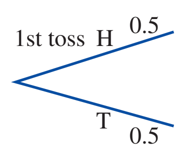
(b) Display the possible outcomes for the first event, namely the first coin toss.
Note: The two lines are called *branches*, one for each possible event (H or T) in the event space {H, T}.

(c) Label one branch H for Heads, the other one T for Tails.

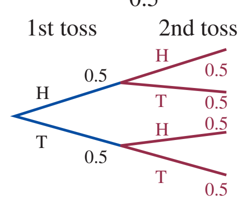
a



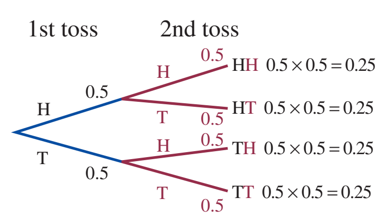
2 Show the probabilities for each branch. In this case they are both 0.5.



3 Repeat for the second coin toss.
Notes
1. If a Head turns up on the 1st toss, either a Head or a Tail turns up on the second toss.
2. Observe the pattern of H above T. Try to follow this pattern.



4 Apply the multiplication rule for independent events for each pathway. The first pathway is 'HH'. There are 4 pathways in all. Note that the sum of all 4 final probabilities = 1 because when two coins are tossed one of the 4 outcomes must occur. The probability of a certainty occurring is 1.



b Use the tree diagram to answer the question.
Note: There is only one way of turning up 2 Heads. There is only one way of turning up 2 Tails. There are two ways of turning up 1 Head and 1 Tail.
P(1 Head) = P(HT) + P(TH).

b **i** P(TT) = 0.25

ii P(1 Head) = P(HT) + P(TH)
= 0.5

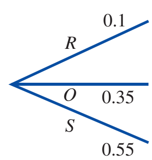
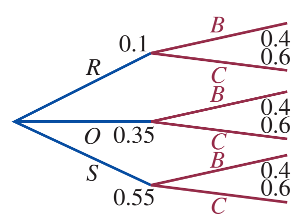
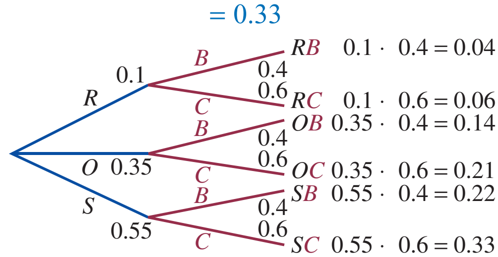
iii P(at least 1 Head)
= P(HT) + P(TH) + P(HH)
= 0.25 + 0.25 + 0.25
= 0.75

iv P(HH) = 0.25

It is not necessary for each activity to have only two branches, or for the probabilities to be equal. For example, a tree diagram representing a die toss would have 6 branches, while a tree diagram representing months of the year would have 12 branches.

WORKED Example 5

Mary's father can drive her to school in his car (*C*) on Tuesdays, Wednesdays and Fridays only. On the other two days Mary must ride her bicycle (*B*). In the month of August, the weather is wet, overcast or sunny. The probability that it rains (*R*) is 0.1 and the probability that it is overcast (*O*) is 0.35; the rest of the days are sunny (*S*). Using a tree diagram, list the possible outcomes and probabilities, and find the probability that Mary gets wet when riding her bicycle.

THINK	WRITE
1 (a) Assume that weather is *independent* of mode of transport.	
(b) Find P(sunny). Remember that the weather is either sunny or it is not sunny.	P(sunny) = 1 − P(not sunny) = 1 − [P(rain) + P(overcast)] = 1 − (0.1 + 0.35) = 1 − 0.45 = 0.55
(c) Set up the weather branches.	
2 For each of the possible weather branches, find P(car) and P(bicycle). Set up the second set of branches for these probabilities.	P(car) = $\frac{3}{5}$ = 0.6 P(bicycle) = $\frac{2}{5}$ = 0.4 
3 Apply the multiplication rule for independent events and work out each possible combination of weather and mode of transport.	P(rain and bike) = $0.1 \cdot 0.4$ = 0.04 P(rain and car) = $0.1 \cdot 0.6$ = 0.06 P(overcast and bike) = $0.35 \cdot 0.4$ = 0.14 P(overcast and car) = $0.35 \cdot 0.6$ = 0.21 P(sunny and bike) = $0.55 \cdot 0.4$ = 0.22 P(sunny and car) = $0.55 \cdot 0.6$ = 0.33 

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THINK	WRITE
4 The only outcome where Mary gets wet is (rain, bicycle). Change this probability into a fraction and interpret it.	P(rain, bicycle) = 0.1 · 0.4 = 0.04 P(rain, bicycle) = $\frac{4}{100}$ = $\frac{1}{25}$ (about 1 day in 25)
5 To verify calculations, add up all 6 final probabilities to ensure that the total is 1. This ensures that you have included all possible outcomes.	0.04 + 0.06 + 0.14 + 0.21 + 0.22 + 0.33 = 1

It is also worth noting that the sum of the probabilities of the first set of branches (R, O, S) also equals 1 because the assumption is that the weather must be rainy, sunny or overcast. This may also be useful in verifying your work. In the previous example, 0.1 + 0.35 + 0.55 = 1.

Advantages and disadvantages of tree diagrams

The main advantage of a tree diagram is that it provides a graphical layout for compound probability problems. If done correctly, it includes all possible branches. However, for displaying three or more independent events (such as outcomes from dice, coins and days of the week), there could be many (6 · 2 · 7 = 84) branches and the tree diagram could become extremely large and hard to use. Thus, they are best used for simpler problems with fewer branches.

remember

1. Compound events occur when more than one event is being considered at the same time.
2. Two events, A and B, are independent if the occurrence of event A has no effect on the probability of the occurrence of event B.
3. The formula for independent events states that if two (or more) events are *independent* of each other, then the probability of *both* (or all) of them happening is the *product* (multiplication) of the individual probabilities: P(A and B both occurring) = P(A) · P(B) where P(A) is the probability of event A occurring, and P(B) is the probability of event B occurring.
4. P(A) is the probability of an event A happening, and P(A') is the probability of the event A not happening. Then:

$$P(A) + P(A') = 1$$
$$P(A') = 1 - P(A)$$

5. Tree diagrams can be used as a graphical method of displaying the outcomes of, and computing the probabilities for, independent events.

EXERCISE 10A Compound events — independent events

WORKED Example 1

1 A coin and a die are tossed at once. Find the probability of a Head appearing on the coin and an even number appearing on the die. (Assume independence.)

WORKED Example 2

2 A die is rolled.
a What is the probability of an odd number appearing uppermost?
b What is the probability of an even number (a number which is not odd) appearing uppermost?

3 List all possible outcomes for tossing two dice. Find the probability that the total on the two dice equals 9.

4 When we toss a pair of dice, the sum of the two numbers appearing uppermost can be 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 or 12. Using the same list of possible outcomes as in question **3**, find which sum is most likely. What is its probability?

WORKED Example 3

5 In any week the probability of the Wallabies winning a rugby union match is 0.49, while the probability of the Brisbane Lions winning an AFL match is 0.9, and the probability of the Brisbane Broncos winning a rugby league match is 0.6. Find:
a the probability that all three teams will win in that week
b the probability that none of these teams win in that week
c the probability that the Wallabies and the Brisbane Lions win that week, but the Brisbane Broncos do not.

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SkillSHEET 10.1
Probability skills
Spreadsheet
069 Tree diagrams

WORKED Example 4

6 Three coins are tossed. List all possible outcomes, namely 'Head–Head–Head', 'Head–Head–Tail' . . . and so on. Find the probability that:
a no (0) Heads turn up
b exactly 1 Head turns up
c at least 1 Head turns up
d exactly 2 Heads turn up.

7 **multiple choice**

The probability of getting 10 Heads in a row is closest to:
A 0.1 B 0.01 C 0.001 D 0.000 98 E 0.000 000 001

8 **multiple choice**

The events that result from tossing a red die, tossing a blue die and tossing a green die are:
A never independent
B always independent
C sometimes independent
D independent if they are tossed at the same time only
E independent only if they are tossed one after the other.

9 Find the probability of getting triple-6 when tossing 3 dice.

WORKED Example 5

10 The Stock Exchange index goes either up or down. The probability that the New York Stock Exchange index goes up is 0.7; the Tokyo exchange index has a probability of only 0.45 of rising; the Australian Exchange index has a probability of rising of 0.5.

Assume the three events are independent. (Is this a realistic assumption?) List the possible outcomes and probabilities using a tree diagram and find the probability that all three exchanges go up.

11 Using the same data as in question **10**, find the probability that all three exchanges go down.

12 An unfair coin has a probability of Heads of 0.55. Make a tree diagram representing two tosses of this coin. What is the probability of getting two Tails?

13 Make a tree diagram for the tossing of three fair coins.
- **a** How many different paths are there?
- **b** Calculate the probability of each path and use your results to check the answer to question **6**.

14 **multiple choice**

If two events are independent, then the probability of *neither* occurring is:
- **A** the product of their individual probabilities
- **B** the sum of their individual probabilities
- **C** 1 – the product of their individual probabilities
- **D** 1 – the sum of their individual probabilities
- **E** (1 – the probability of one event)(1 – the probability of the other event).

15 In the local supermarket, 60% of the customers are women. It is also observed that 20% of the female customers buy bread, and that 20% of the male customers buy bread. Make a tree diagram and find:
- **a** the probability that a customer is a woman and buys bread
- **b** the probability that a customer is a man and does not buy bread.

16 Which of the following are likely to be independent events? (*Note:* There may be more than one correct answer.)
- **a** Potatoes sell for more than \$1/kg in Queensland and potatoes sell for more than \$1/kg in New South Wales.
- **b** The Liberals win the next election and the Brisbane Lions win the next grand final.
- **c** It rains today and it rains tomorrow.
- **d** It rains today and it rains 6 months from today.
- **e** It rains today and the Stock Exchange goes up.

17 A restaurant serves a set two-course meal and records the probabilities that customers make various choices, according to this table.

Make a tree diagram of this situation and find the probability that a customer has food from the ocean for both courses.

First course	Second course
P(prawns) = 0.4	P(beef) = 0.3
P(salad) = 0.25	P(fish) = 0.25
P(tomato soup) = 0.35	P(lamb) = 0.45

18 A medical researcher is testing a new drug as a cure for cancer. She has 40 patients and uses the following testing scheme.

Patient groups	Recover	Don't recover
Do not receive the drug	4	10
Receive a small dose of the drug	6	10
Receive a large dose of the drug	8	2

Using these data:

a for each of the three groups, find the probability that it will include any one patient

b make a tree diagram with the result from part **a** forming the first set of branches

c continue from **b**, with the second branch being the result ('recover' or 'don't recover')

d calculate the probabilities for all possible combinations

e discuss whether the drug is effective or not.

Compound events — mutually exclusive events

In the previous section we looked at one kind of compound event, namely two events that were independent of each other. In that case we were able to multiply probabilities. In this section we look at different kinds of compound events — events where there are two (or more) outcomes from a single event space. We are interested in the probability that *any* of these outcomes happen, but we must first check to see if there is some *overlap* between the outcomes.

Mutually exclusive events

Mutually exclusive events are events which cannot happen at the same time.

WORKED Example 6

For each of the following activities, determine which outcome pairs are mutually exclusive.

a A card is drawn from a pack of cards.
  i The card is a king. **ii The card is a queen.**

b A card is drawn from a pack of cards.
  i The card is a king. **ii The card is a spade.**

c A die is thrown.
  i The upper face shows a 1. **ii The upper face shows an even number.**

d A die is thrown.
  i The upper face shows a 1. **ii The upper face shows an odd number.**

e A shopper makes purchases at a supermarket.
  i The shopper buys eggs. **ii The shopper buys milk.**

f A car buyer purchases a car.
  i The car is red. **ii The car is green.**

Continued over page

THINK

a A card can't be a king and queen at the same time.

b It is possible for a card to be both a king and a spade (the king of spades).

c Since 1 is an odd number, it cannot be an even number.

d 1 is an odd number.

e A shopper can buy both milk and eggs.

f A buyer cannot buy both a red and green car (only 1 car is bought!)

WRITE

a Drawing a king and drawing a queen *are* mutually exclusive events.

b Drawing a king and drawing a spade *are not* mutually exclusive.

c Tossing a 1 and an even number *are* mutually exclusive events because 1 is not an even number.

d Tossing a 1 and an odd number *are not* mutually exclusive events because 1 is an odd number.

e Buying milk and eggs are *are not* mutually exclusive.

f Buying a red car and buying a green car *are* mutually exclusive events.

To decide whether events are mutually exclusive, look for an *overlap* between them. If there appears to be no possible way that both events could happen *together*, then the events are mutually exclusive.

Calculating probabilities for mutually exclusive events

WORKED Example 7

A card is drawn at random from a deck of 52 cards.
What is the probability that it is a king or queen?

THINK

1. Determine if the events are mutually exclusive.
2. Find the individual probabilities.
3. Calculate the probability that the card is a king or a queen.
   Since there is no overlap, there are 8 cards that are either a king or a queen.

WRITE

1. The card cannot be both a king and a queen at the same time, so the events are mutually exclusive.
2. Since there are 4 kings, $P(K) = \frac{4}{52}$ and since there are 4 queens, $P(Q) = \frac{4}{52}$.
3. $P(\text{king or queen}) = \frac{8}{52}$

This probability could have been found also by adding individual probabilities.

$$\begin{aligned} P(\text{king or queen}) &= P(\text{king}) + P(\text{queen}) \\ &= \frac{4}{52} + \frac{4}{52} \\ &= \frac{8}{52} \end{aligned}$$

To find the probability of mutually exclusive events occurring, add the individual probabilities. In mathematical notation, we say that if A and B are mutually exclusive events, then $P(A \text{ or } B) = P(A) + P(B)$. This is called the **addition rule of probability**. It works for two or more mutually exclusive events.

WORKED Example 8

Students entering Corvard University can enrol in one of the following degree programs: Architecture, Arts, Engineering, Law, Medicine or Science. Only Arts and Law students are exempted from doing compulsory First Year Mathematics. Furthermore, Architecture and Engineering students must study Strength of Materials. The enrolments of students in the six faculties for the previous year are shown in the table which follows.

Faculty	Architecture	Arts	Engineering	Law	Medicine	Science
Students	105	420	345	620	155	420

Use the information provided to find:

a the probability that a first-year student chosen at random is doing First Year Mathematics

b the probability that a first-year student chosen at random is doing Strength of Materials.

THINK

1 Find the total number of students enrolled.

2 Since students enrol in one faculty only, each faculty is *mutually exclusive*, so for each faculty, find the probability that a student is studying Mathematics.

a Apply the addition rule for First Year Mathematics (FYM).

b Apply the addition rule for Strength of Materials (SM).

WRITE

Total number of students enrolled

$= 105 + 420 + 345 + 620 + 155 + 420$

$= 2065$

$P(\text{Architecture}) = \frac{105}{2065} = 0.051$

$P(\text{Engineering}) = \frac{345}{2065} = 0.167$

$P(\text{Medicine}) = \frac{155}{2065} = 0.075$

$P(\text{Science}) = \frac{420}{2065} = 0.203$

a $P(\text{FYM}) = P(\text{Architecture}) + P(\text{Engineering}) + P(\text{Medicine}) + P(\text{Science})$

$= 0.051 + 0.167 + 0.075 + 0.203$

$= 0.496$

b $P(\text{SM}) = P(\text{Architecture}) + P(\text{Engineering})$

$= 0.051 + 0.167$

$= 0.218$

remember

1. Mutually exclusive events are events which cannot happen at the same time.
2. Let P(A), P(B) be the probabilities of events A and B occurring, respectively. If the events are mutually exclusive then:

$$P(A \text{ or } B) = P(A) + P(B).$$

This is called the *addition rule of probability*.

EXERCISE 10B Compound events — mutually exclusive events

WORKED Example 6

1 For each of the following activities, determine which of the events are likely to be mutually exclusive.

a A card is drawn at random from a pack of cards.
  i The card is a spade. ii The card is a club.

b A card is drawn at random from a pack of cards.
  i The card is a spade.
  ii The card is a picture card (J, Q, K).

c A shopper buys items from the supermarket.
  i He buys bacon.
  ii He buys eggs.

d A shopper buys a stove.
  i It is a gas stove. ii It is an electric stove.

e During the day, a woman watches the weather.
  i It rains. ii It doesn't rain.

f During the day, a woman watches the weather.
  i It rains. ii It is cold.

g A Member of Parliament belongs to a Party.
  i She belongs to the Labor Party. ii She belongs to the Liberal Party.

WORKED Example 7

2 A card is drawn at random from a deck of 52 playing cards. Find the probability that it is:

a either a black jack or a red ace
b either an even-numbered card or a king
c either a heart or a black jack.

3 In a bag of coloured marbles there were 7 red marbles, 8 green marbles, 9 yellow marbles, 10 white marbles and 11 purple marbles. If one marble is drawn from the bag, find:

a the probability that the marble is either red or yellow
b the probability that the marble is not white
c the probability that the marble is either white, green or purple
d the probability that the marble is a colour which has the letter 'w' in it
e the probability that the marble is a colour which has the letter 'e' in it.

4 Over the years, Liverpool and Manchester United football clubs have played 180 games. Liverpool has won 86, 16 were drawn and 2 were cancelled. Teams get points for either a win or a draw. Find the probability that Manchester United gets points when playing Liverpool in their next game.

5 **multiple choice**

When tossing a pair of dice, the pair of events which is mutually exclusive is:

A a 'double' or a total of 10
B a 'double' or a total greater than 4
C a 'double' or a total of 2
D a 'double' or a total of an even number
E none of these.

WORKED Example 8

6 In a parliament there are 62 Liberal, 59 Labor, 14 Greens and 13 independent members. If the Liberals propose a change to the tax laws, the Labor and independent members will oppose it.

a What is the probability that an MP, chosen at random, opposes the bill?
b What are the chances of the bill passing?

7 When tossing a pair of dice find the probability that:

a the total is either 7 or a number greater than 9
b the total is an even number or 7
c the total is a prime number.

8 A professional cricketer has recorded his results in great detail over the years. Specifically he has recorded the method by which he was dismissed.

Method of dismissal	Number of times
Leg-before-wicket	17
Caught	13
Run out	6
Bowled by fast bowler	29
Bowled by spin bowler	11
Hit wicket	1
Not out	12

a Find the probability that he was dismissed either by leg-before-wicket or by being run out.
b Find the probability that he was dismissed by being bowled.
c Find the probability that he was dismissed by *any* method.

9 On a bookshelf there are several types of book (shown in the table at right). A book is chosen at random from the shelf. Giving your answer to 3 decimal places, find the probability that it was:

a a poetry book
b a novel or a short-story book
c a non-fiction book
d a fiction book.

Type of book	Number
Novel	12
Biography	5
History	8
Poetry	2
Computer manual	11
Short story	4
Mathematics text	12

10 In a recent survey of 100 people it was found that 23 read just the morning newspaper, 37 read just the evening newspaper, and 9 read both papers. Find the probability that a person chosen at random:

a reads the morning paper **b** reads the evening paper **c** reads no paper.

11 In the game of poker there are many possible hands. Ranking from highest to lowest, their probabilities are given in the table at right.

a A poker machine pays if you get one pair or better. What is the probability of winning money? (Give your answer to 4 decimal places.)
b In a real game of poker, you have a full house. What is the probability that you win when playing against one other opponent? Assume that all full houses are equal. (Give your answer to 4 decimal places.)
c About how many games would you have to play to get a royal flush?

Type of hand	Probability
Royal flush	0.000 001 539
Straight flush	0.000 013 857
Four-of-a-kind	0.000 24
Full house	0.001 44
Flush	0.001 97
Straight	0.003 92
Three-of-a-kind	0.0211
Two pairs	0.0476
One pair	0.4236

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Compound events — Venn diagrams

In the previous section on mutually exclusive events, we had to ensure that the two (or more) events making up the compound event had no overlap before we could proceed. What do we do when there is an overlap?

Finding the overlap

Consider the following example.

WORKED Example 9

In a room of 10 students, 7 can speak English, 4 can speak French and none speak neither. How many students:

a **speak both languages?** **b** **speak English only?** **c** **speak French only?**

THINK	WRITE
a 1 Add up the *frequencies* of each of the simple events that make up the compound event.	**a** There are 7 English + 4 French students = 11 students.
2 Establish that there are none who belong to *neither* group.	Everyone speaks either English and/or French.
3 Subtract the total number of students from the total found in step 1.	11 − 10 = 1 Thus there must be 1 student who speaks *both*.
b 1 Find how many speak English and how many speak both.	**b** 7 speak English and 1 speaks both.
2 Subtract the number who speak both from the number who speak English.	7 − 1 = 6 There are 6 students who speak English only.
c 1 Find how many speak French and how many speak both.	**c** 4 speak French and 1 speaks both.
2 Subtract the number who speak both from the number who speak French.	4 − 1 = 3 There are 3 students who speak French only.

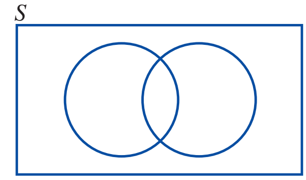
Thus we can divide our students into 3 groups:
Group 1 those who speak English only (6)
Group 2 those who speak French only (3)
Group 3 those who speak *both* English and French (1).
In this example there is an overlap of 1 student. This overlap is called the *intersection*. This situation can be displayed graphically using a *Venn diagram*.

Venn diagrams

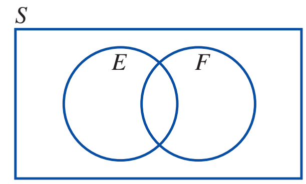
A **Venn diagram** is made up of a rectangle that represents all possible outcomes with the other activities represented by circles drawn within the rectangle.

Let us construct a Venn diagram from the information in Worked example 9.

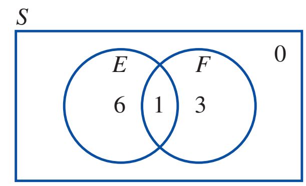
Step 1 Construct a rectangle with 2 circles inside. Since there is an overlap, show the two circles overlapping. The rectangle represents *all* the students in the room.



Step 2 Add labels to indicate that the rectangle represents all the students in the room (the **sample space** S), one circle represents students who speak English (E) and the other circle represents French (F) speaking students.



Step 3 Enter the number of students in all possible regions to represent the students in those categories. The 0 outside the circles indicates that no students speak neither French nor English.



From the values in the four regions, it is a simple task to work out probabilities. Note that the sum of all the numbers in the Venn diagram equals the *total* number of students.

$$\text{P(English only)} = \frac{6}{10} = \frac{3}{5}$$

$$\text{P(French only)} = \frac{3}{10}$$

$$\text{P(both French and English)} = \frac{1}{10}$$

WORKED Example 10

In a group of 85 Year-11 students at Heartbeat High, it was found that 47 study Mathematics A, 48 study Geography and 6 study neither. Construct a Venn diagram and find the probability that a student, chosen at random:

a studies both Mathematics A and Geography **b studies Mathematics A only.**

THINK	WRITE
a 1 State the number of students who are in Year 11.	**a** There are 85 Year-11 students.
2 State the number of students who do not study Mathematics A or Geography.	There are 6 students who do not study Mathematics A or Geography.
3 Hence, find the number who study Mathematics A or Geography.	$85 - 6 = 79$ There are 79 students who study Mathematics A or Geography.
4 Find the total enrolments in Mathematics A and Geography.	$47 + 48 = 95$ enrolments
5 Subtract the total in step 2 from the total in step 4 to find the intersection.	Intersection $= 95 - 79$ $= 16$ There are 16 students who study both Mathematics A and Geography.

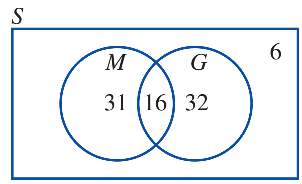
THINK

6. Work out the remaining regions by subtracting this intersection from the number studying Mathematics A and Geography only.
7. Construct the Venn diagram, then calculate the required probabilities. Verify that the sum of all parts is equal to the total number of students.
8. Answer the question.

b Answer the question.

WRITE

Mathematics A only: 47 − 16 = 31
Geography only: 48 − 16 = 32



31 + 16 + 32 + 6 = 85

$$\text{P(both)} = \frac{16}{85}$$
$$= 0.188 \text{ (about 19\%)}$$

b $\text{P(Mathematics A only)} = \frac{31}{85}$
$$= 0.365$$

We can apply this Venn diagram technique to probabilities directly.

WORKED Example 11

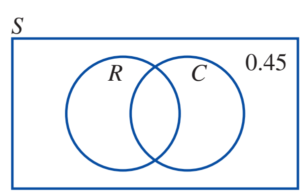
Based on previous weather data, the probability that it is rainy *and* cold is 0.15. The probability that it is *neither* rainy *nor* cold (that is, fine and warm) is 0.45. Also, it is known that the probability it will rain is 0.35. Construct a Venn diagram and determine the probability that it will be cold but will not rain.

THINK

1. Find the probability that neither event occurs and put this number in the upper right corner of the Venn diagram.

WRITE

P(not rainy nor cold) = 0.45

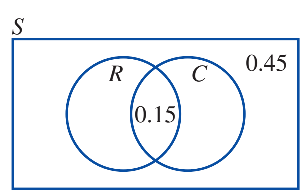


THINK

2. Find the probability of the intersection.

WRITE

P(rainy and cold) = 0.15



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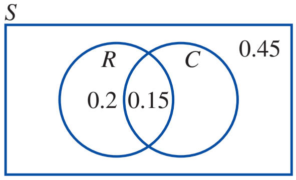
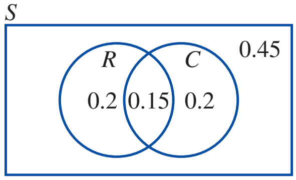
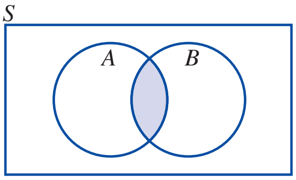
THINK	**WRITE**
3 Find P(rainy only) from the given information. The *total* P(rainy) = 0.35, P(rainy and cold) = 0.15.	P(rainy only) = P(rain) − P(rainy and cold) P(rainy only) = 0.35 − 0.15 = 0.2
	
4 There is only one region left; work out how much probability is left, given that total = 1.	Total so far = 0.45 + 0.15 + 0.2 = 0.8 So, P(cold but not rainy) = 0.2
5 Complete the Venn diagram.	

The General Addition Rule for Probability

From the results of a probability problem such as Worked example 11, we can generalise a rule for the relationship between the various components of compound events that are not mutually exclusive. It is clear that there is an *arithmetic* (specifically an addition) relationship between the probabilities of the overlapping part and the non-overlapping parts of a Venn diagram.

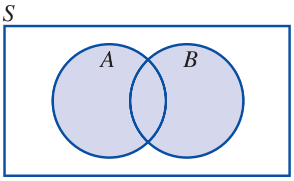
But first, we need to develop some new symbolism or notation.

Let A and B be two overlapping events, with corresponding probabilities P(A) and P(B). Let the intersection, as shown at right, be denoted as P(both A and B).



This gives us the probability of the overlap.

Let the *entire* region where either A or B can happen, as shown at right, be denoted by P(either A or B).

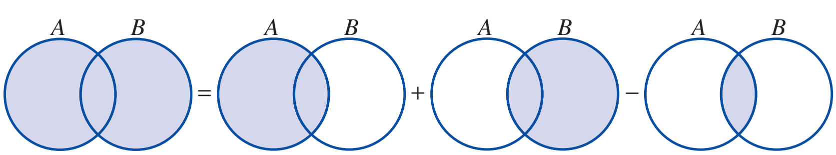


P(either A or B) ≠ P(A) + P(B), because we end up counting P(both A and B) twice.

Thus we can correct the formula by subtracting P(both A and B) once, to give us the General Addition Rule for Probability:

P(either A or B) = P(A) + P(B) − P(both A and B)

This can be shown, graphically, as:

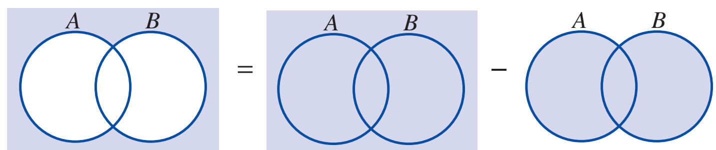


For mutually exclusive events, P(either A or B) = P(A) + P(B) because P(both A and B) = 0.

This formula can be used to solve compound probability problems without using Venn diagrams. Provided we know 3 of the 4 quantities in the formula, we can find the fourth. Note that this formula does not take into account the region outside both circles. This region can be calculated using the formula:

P(neither A nor B) = 1 − P(either A or B)

This can be shown, graphically, as:



WORKED Example 12

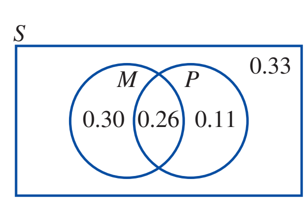
It was found that 56% of students study Mathematics, 37% study Physics and 26% study both Mathematics and Physics. Find:

a the percentage of students who study either Mathematics or Physics

b the percentage who study neither Mathematics nor Physics.

THINK	WRITE
a 1 Identify the known probabilities.	**a** 56% = 0.56 so P(Mathematics) = 0.56 37% = 0.37 so P(Physics) = 0.37 26% = 0.26 so P(both Mathematics and Physics) = 0.26
2 State the General Addition Rule for Probability and apply it to find the percentage of students who study either Mathematics or Physics.	P(either A or B) = P(A) + P(B) − P(both A and B) P(either Mathematics or Physics) = 0.56 + 0.37 − 0.26 = 0.67 Therefore 67% study either Mathematics or Physics.
b Students study either Mathematics or Physics or they study neither. So the sum of these probabilities is 1.	**b** 1 − 0.67 = 0.33 Therefore, P(neither Mathematics nor Physics) = 0.33. So 33% study neither Mathematics nor Physics.

A Venn diagram for this problem would be:



remember

1. For any compound event, P(both A and B) is the overlap, or *intersection*, between 2 events.
2. For any compound event, P(either A or B) is the *union* of the 2 events.
3. The addition rule for probability states that:
$$P(\text{either } A \text{ or } B) = P(A) + P(B) - P(\text{both } A \text{ and } B)$$
4. For mutually exclusive events:
$$P(\text{either } A \text{ or } B) = P(A) + P(B)$$

EXERCISE 10C Compound events — Venn diagrams

WORKED Example 9

1 In a group of 17 female athletes there are 9 who play netball and 12 who play soccer and none who play neither. Find how many females:
a play both sports **b** play netball only **c** play soccer only.

WORKED Example 10

2 In a survey of 200 television viewers, it was found that 135 watched the 7 o'clock news, 89 watched the 11 o'clock news and 31 watched neither program. Construct a Venn diagram and find the probability that a viewer, chosen at random:
a watched both news programs **b** watched the 7 o'clock news only.

3 In a group of 76 students, 31 study Information Processing and Technology, 29 study Technology Studies and 16 study neither subject. By constructing a Venn diagram, show that the number of students who study both subjects is 0. Therefore, what can you say about the events 'study IPT' and 'study TS'?

WORKED Example 11

4 A survey of adults between the ages of 18 and 25 found that the probability that someone smokes is 0.26, while the probability that someone consumes alcohol is 0.41. It also found that there is a probability of 0.51 that young adults neither drink nor smoke. Construct a Venn diagram and find the probability that they both drink and smoke.

5 In researching two lotteries, it was found that the probability of winning Lottery A was 0.03, the probability of winning Lottery B was 0.07 and the probability of winning both was 0.0009. Construct a Venn diagram and find the probability of winning neither.

6 **multiple choice**

The managers of a large motel found that 17% of their customers stayed on a Friday night, 14% of their customers stayed on a Saturday night and 71% of their customers stayed on neither of these two nights. Construct a Venn diagram and show that the probability that a customer stayed on both nights was:
A 2 **B** 0.2 **C** 0.02 **D** 0.002 **E** 0.0002

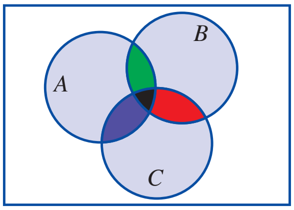
WORKED Example 12

7 The same motel managers also found that 16% of their customers ordered room service for dinner, 26% ordered room service for breakfast and 5% ordered it for both dinner and breakfast. Find:
a the percentage who ordered either dinner or breakfast
b the percentage who ordered neither dinner nor breakfast.

8 A professional gambler plays both blackjack and roulette. During a period of a month, he recorded his results as follows. The probability that he ended the night losing money was 0.32. The probability that he ended the night winning money if he just played blackjack was 0.13, and if he just played roulette the probability of winning money was 0.28.

By constructing a Venn diagram, or otherwise, find the probability of a winning night if he played both games. Would you say he was a better roulette player or blackjack player?

9 It is possible to construct a Venn diagram for three overlapping events as shown. The black area represents the probability of all three, while the green area represents the probability of events A and B, but not C, and so on. Let A be the number of students studying Mathematics B, B be the number of students who study Mathematics C and C be the number of students who study Chemistry. It is known that 6 students study all three subjects, 10 study both Mathematics subjects, 9 study Mathematics C and Chemistry and 8 study Mathematics B and Chemistry. Also, there are 26 students in the Mathematics B class, 20 students in the Mathematics C class and only 11 students in the Chemistry class. How many students are there altogether?



10 In a group of sports people, 25 played soccer, 25 played netball and 25 played cricket. Altogether there were 47 people in the group. If 8 played all three sports, 4 played netball and soccer but not cricket, and 3 played soccer and cricket but not netball, how many played just cricket?

10 QUICK QUESTIONS 1

1 On the tossing of two dice, what is the probability that the sum of the numbers will be 7?

2 What is the probability of rolling a double on two rolls of a die?

3 A biased coin results in a head 60% of the time. On two tosses of the coin, what is the chance that two Heads will result?

4 For the coin in question **3**, what is the chance that two Tails will result?

5 For the coin in question **3**, what is the chance that the two tosses will result in different faces?

6 A card is drawn from a deck of 52 playing cards. Find the probability that it is a red queen.

7 What would be the probability that the card in question **6** is red or a queen?

8 In a class of 20 students, there are 5 students who wear glasses and 18 students who are right-handed. How many right-handed students also wear glasses, if it is known that there is only one left-hander who doesn't wear glasses?

9 In question **8**, how many right-handers don't wear glasses?

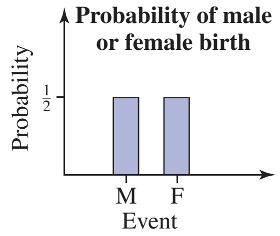
10 What is the probability that a person chosen from the class in question **8** is left-handed and wears glasses?

The binomial distribution using Pascal's triangle

Probability distributions

In the birth of a child, the **outcome** can be either a male (M) or a female (F). This means that $P(M) = \frac{1}{2}$ and $P(F) = \frac{1}{2}$.

A graph of such a probability distribution where each outcome has an equal chance of occurring produces a rectangular shape.

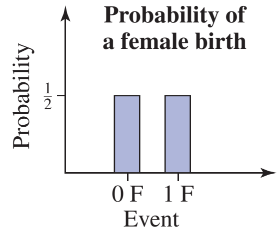


The variables here (male/female) are able to be counted, so we call this type of distribution a discrete **rectangular distribution** or a discrete **uniform distribution**.

The results of the birth of a child could be recorded differently. We could look at the birth in terms of the number of females born. For every birth, the outcome would be 0 females or 1 female.

$$P(0\text{ F}) = P(M) = \tfrac{1}{2}$$

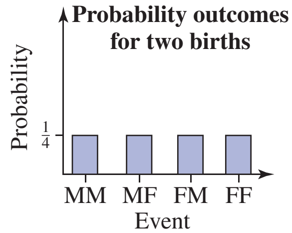
$$P(1\text{ F}) = P(F) = \tfrac{1}{2}$$



This also produces a discrete uniform distribution with the same rectangular graphical shape shown top right.

We could extend this concept to consider the distribution resulting from the birth of two children. As we have noted before, the possibilities are as follows.

Male, male	(MM)	Female, male	(FM)
Male, female	(MF)	Female, female	(FF)

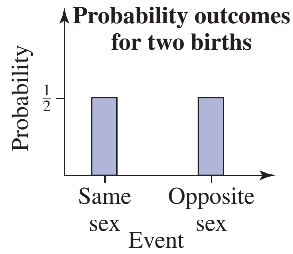


Each of these outcomes has a probability of $\frac{1}{4}$. Graphing this distribution we would observe the familiar rectangular shape.

Looking at these outcomes from a different perspective, we could record whether the two children were of the same sex or of opposite sex.

$$P(\text{same sex}) = P(\text{MM or FF}) = \tfrac{2}{4} = \tfrac{1}{2}$$

$$P(\text{opposite sex}) = P(\text{MF or FM}) = \tfrac{2}{4} = \tfrac{1}{2}$$



A graphical representation of these recordings would also produce a rectangular shape.

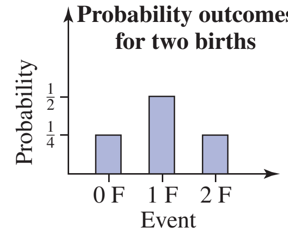
Another way of recording these outcomes would be to look at the possible number of females in the births of two children. This is obviously 0, 1 or 2. The probabilities of each of these outcomes are not all the same.

$$P(0\text{ F}) = P(\text{MM}) = \tfrac{1}{4}$$

$$P(1\text{ F}) = P(\text{MF or FM}) = \tfrac{2}{4} = \tfrac{1}{2}$$

$$P(2\text{ F}) = P(\text{FF}) = \tfrac{1}{4}$$

A graphical representation of these outcomes would not produce a uniform, rectangular distribution. This type of distribution is known as a **binomial distribution**.



The binomial distribution

A binomial distribution is the result of repeated trials which have only two outcomes. In rolling a die, we may be interested in rolling a six, or not rolling a six. In tossing a coin we may be concerned with getting a Head, or not getting a Head. The two possible outcomes are labelled a '**success**' and a '**failure**'. The probability of a success is represented by 'p' while the probability of a failure is represented by 'q'. Since these two events are complementary

$$p + q = 1 \qquad \text{that is,} \qquad \text{P(success)} + \text{P(failure)} = 1.$$

The number of trials in the experiment is represented by 'n'.

Let us extend our previous discussions on child births to consider up to four births. If we are interested in the number of females born, we could represent the birth of a female as a 'success' and the birth of a male as a 'failure'. We could have considered the male birth to be a 'success' and the female birth to be a 'failure'. Whichever outcome we choose to allocate as a success or a failure will not affect the results.

Let p = P(sucess) = P(F)

then q = P(failure) = P(M).

The outcomes resulting from the births of 1 to 4 children could be summarised in the table which follows.

n	Outcome	Successes	Failures	Probability
1	F	1	0	p
	M	0	1	q
2	FF	2	0	$pp = p^2$
	FM	1	1	$\left.\begin{matrix} pq \\ pq \end{matrix}\right\} = 2pq$
	MF	1	1	
	MM	0	2	$qq = q^2$
3	FFF	3	0	$ppp = p^3$
	FFM	2	1	$\left.\begin{matrix} ppq \\ pqp \\ qpp \end{matrix}\right\} = 3p^2q^1$
	FMF	2	1	
	MFF	2	1	
	FMM	1	2	$\left.\begin{matrix} pqq \\ qpq \\ qqp \end{matrix}\right\} = 3p^1q^2$
	MFM	1	2	
	MMF	1	2	
	MMM	0	3	$qqq = q^3$

(*continued*)

n	Outcome	Successes	Failures	Probability
4	FFFF	4	0	$pppp = p^4$
	FFFM	3	1	$\left.\begin{matrix} pppq \\ ppqp \\ pqpp \\ qppp \end{matrix}\right\} = 4p^3q^1$
	FFMF	3	1	
	FMFF	3	1	
	MFFF	3	1	
	FFMM	2	2	$\left.\begin{matrix} ppqq \\ pqpq \\ pqqp \\ qppq \\ qpqp \\ qqpp \end{matrix}\right\} = 6p^2q^2$
	FMFM	2	2	
	FMMF	2	2	
	MFFM	2	2	
	MFMF	2	2	
	MMFF	2	2	
	FMMM	1	3	$\left.\begin{matrix} pqqq \\ qpqq \\ qqpq \\ qqqp \end{matrix}\right\} = 4p^1q^3$
	MFMM	1	3	
	MMFM	1	3	
	MMMF	1	3	
	MMMM	0	4	$qqqq = q^4$

Collating the results of the table we can notice the following patterns.

Number of trials n	Probability				
	4 successes	3 successes	2 successes	1 success	0 successes
1				p^1	q^1
2			p^2	$2p^1q^1$	q^2
3		p^3	$3p^2q^1$	$3p^1q^2$	q^3
4	p^4	$4p^3q^1$	$6p^2q^2$	$4p^1q^3$	q^4

The power of p is the same as the number of successes in the trial.
The sum of the p and q powers in each term is the same as the number of trials.
The coefficient of each term can be obtained from Pascal's triangle.

							1							
						1		1						
					1		2		1					
				1		3		3		1				
			1		4		6		4		1			
		1		5		10		10		5		1		
	1		6		15		20		15		6		1	
1		7		21		35		35		21		7		1

Pascal's triangle

The arrangement of numbers in **Pascal's triangle** begins by writing 1s along the outside of the triangle. On the inside, numbers are obtained by adding adjacent numbers in the same row and writing this number below and between this pair of numbers. For example:

1   3
  4

For reasons that will become clear later, the elements in Pascal's triangle are labelled as follows:

The first row   1   is called **row 0**

the next row   1   1   is called **row 1**

and so on.

Row 5 is   1   5   10   10   5   1

Note that, in the table,

when $n = 1$, the coefficients are   1   1

when $n = 2$, the coefficients are   1   2   1

when $n = 3$, the coefficients are   1   3   3   1   and so on.

So, when we are looking for the term which represents the probability of 2 successes in, say, 5 trials, we need the term with p to the power of 2. Since the powers of p and q add to give the value of n (5 in this case), the power of q is 3; that is, the term p^2q^3 (2 successes and 3 failures).

To get the coefficient of this term we go to row 5 in Pascal's triangle.

1   5   10   10   5   1

The first number represents the number of outcomes with 5 successes. The second number represents the number of outcomes with 4 successes, and so on.

So the coefficient of the term representing 2 successes is 10. This means that the probability of 2 successes in 5 trials is represented by the value of $10p^2q^3$ when values are substituted for p and q.

Remember that, in these experiments, we are concerned with only two possible outcomes. We could use this technique if, for example, a die was rolled repeatedly and the number of 6s resulting was observed. We would not use this technique if we wanted to observe the number of 1s, 2s, 3s, 4s, 5s or 6s resulting.

WORKED Example 13

Calculate the probability of tossing 3 Heads in 4 tosses of a coin.

THINK	WRITE
1 This is a binomial experiment as we have only two outcomes (Head or Tail).	
2 Let a success occur when a Head results, and a failure when a Tail results. Assign values to these two outcomes.	Let p = P(success) = P(H) $= \frac{1}{2}$ = 0.5 q = P(failure) = P(T) $= \frac{1}{2}$ = 0.5
3 Specify the number of trials (n).	n = 4 trials
4 Locate the coefficient of the term from row 4 of Pascal's triangle.	1 4 6 4 1 is row 4 of Pascal's triangle. Coefficient required = 4 (3 successes)
5 Write the term with p to the power 3 and q to the power 1: $(3 + 1 = n = 4)$.	P(3 Heads in 4 tosses) $= 4p^3q^1$
6 Substitute values for p and q.	$= 4 \cdot (0.5)^3 \cdot 0.5$ $= 0.25$
7 Calculate the probability.	The probability of 3 Heads in 4 tosses is 0.25.

WORKED Example 14

A coin is biased so that the probability of it showing a Tail is 0.4. If the coin is tossed 7 times, what is the chance that 4 Heads will appear?

THINK	WRITE
1 Since we are wanting Heads to appear, let a success be represented by the appearance of a Head.	
2 Allocate values to the probability of a success p and the probability of a failure q. (Remember that $p + q = 1$.)	Let p = P(success) = P(H) = 0.6 q = P(failure) = P(T) = 0.4
3 State the number of trials.	$n = 7$
4 Locate the seventh row of Pascal's triangle. Find the coefficient representing 4 successes.	1 7 21 35 35 21 7 1 is row 7 of Pascal's triangle. Coefficient required = 35 (4 successes)
5 The term representing 4 sucesses and 3 failures in 7 trials is p^4q^3.	
6 Write the probability for 4 successes in 7 trials.	P(4 successes in 7 trials) $= 35p^4q^3$
7 Substitute values for p and q.	$= 35 \cdot (0.6)^4 \cdot (0.4)^3$
8 Calculate the probability.	$= 0.29$
9 Write the answer.	So, the probability of obtaining 4 Heads in 7 tosses of the coin is 0.29.

WORKED Example 15

In the previous worked example, what would be the chance that no more than 4 Heads would appear in the 7 tosses of the coin?

THINK	WRITE
1 Define a success and a failure, allocating values to their probabilities.	Let p = P(success) = P(H) = 0.6 q = P(failure) = P(T) = 0.4
2 State the number of trials.	$n = 7$
3 No more than 4 Heads means: 0 Heads, 1 Head, 2 Heads, 3 Heads or 4 Heads. We must find each of these probabilities and add them.	P(≤ 4 H) = P(0 H) + P(1 H) + P(2 H) + P(3 H) + P(4 H)
4 State the individual terms, locating their coefficients in Pascal's triangle.	$= q^7 + 7pq^6 + 21p^2q^5 + 35p^3q^4 + 35p^4q^3$
5 Substitute values for p and q.	$= (0.4)^7 + 7 \cdot 0.6 \cdot (0.4)^6 + 21 \cdot (0.6)^2 \cdot (0.4)^5 + 35 \cdot (0.6)^3 \cdot (0.4)^4 + 35 \cdot (0.6)^4 \cdot (0.4)^3$
6 Calculate the answer.	= 0.002 + 0.017 + 0.077 + 0.194 + 0.29 = 0.58
7 Write the answer.	The probability that no more than 4 Heads would appear in 7 tosses of the coin is 0.58.

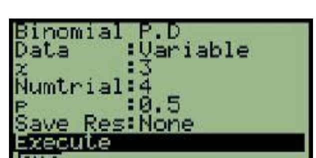
Graphics Calculator tip! Calculating binomial probabilities

Graphics calculators can greatly simplify calculating binomial probabilities. Here are the steps required for solutions to Worked examples 13, 14 and 15.

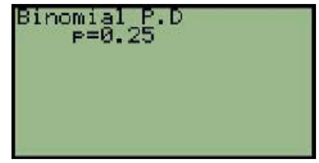
For the Casio *fx*-9860G AU

1. To calculate binomial probabilities, press:
   - MENU
   - 2: STAT
   - F5 (DIST)
   - F5 (BINM)
   - F1 (Bpd).

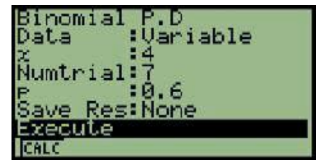
2. For Worked example 13, enter the data as shown.



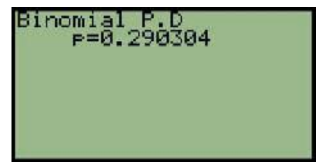
3. Press F1 (CALC) to see the answer.



4. Repeat the steps above for Worked example 14.



5. Press F1 (CALC) to see the answer.



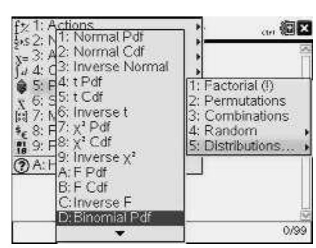
6. For Worked example 15, press:
   - MENU
   - 2: STAT
   - F5 (DIST)
   - F5 (BINM)
   - F2 (Bcd).

Enter the data as shown.

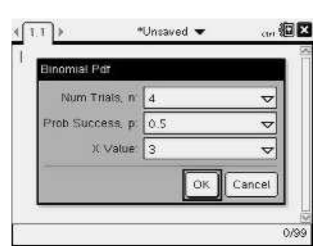
7. Press F1 (CALC) to see the answer.

For the TI-Nspire CAS

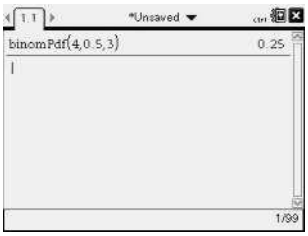
1. To calculate binomial probabilities, open a Calculator page.
   Press:
   - MENU (menu)
   - 5: Probability (5)
   - 5: Distributions . . . (5)
   - D: Binomial Pdf (D).



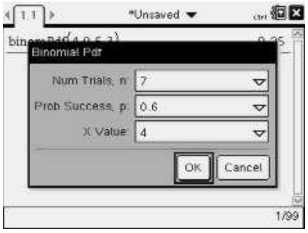
2. For Worked example 13, enter the data as shown.
   Note: Press Tab (tab) to move between fields.



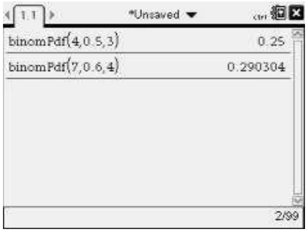
3. Select OK, then press ENTER (enter).



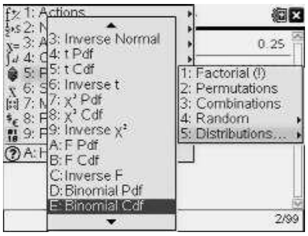
4. Repeat the steps above for Worked example 14.



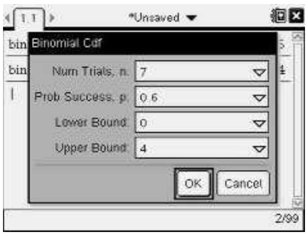
5. Select OK, then press ENTER (enter).



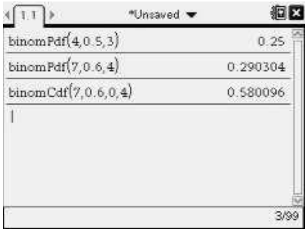
6. For Worked example 15, open a Calculator page.
   Press:
   - MENU (menu)
   - 5: Probability (5)
   - 5: Distributions . . . (5)
   - E: Binomial Cdf (E).



7. Enter the data as shown.



8. Select OK, then press ENTER (enter).



remember

1. A discrete uniform distribution is produced by an experiment where each outcome has an equal chance of occurring.
2. The graph of such a distribution is rectangular in shape.
3. A binomial distribution results from an experiment which consists of only two outcomes, called a 'success' and a 'failure'.
4. Pascal's triangle can be used in calculating the probability of obtaining particular successes in repeated trials.

EXERCISE 10D The binomial distribution using Pascal's triangle

Note: Use the table on page 526 for these questions. Alternatively, you may wish to use a graphics calculator.

1 Complete the entries in Pascal's triangle in the row *following* each of these rows.
a 1 6 15 20 15 6 1 **b** 1 8 28 56 70 56 28 8 1

2 Deduce the entries in Pascal's triangle in the row *before* each of these rows.
a 1 6 15 20 15 6 1 **b** 1 8 28 56 70 56 28 8 1

3 Use Pascal's triangle to determine the number of outcomes in the following experiments.
a 3 trials, 0 successes **b** 4 trials, 1 success
c 8 trials, 4 successes **d** 110 trials, 110 successes
e 110 trials, 109 successes

4 Use the power button on your calculator to evaluate the following.
a $(0.4)^4$ **b** $(0.7)^6$ **c** $(0.5)^3$
d $(0.3)^3(0.7)^2$ **e** $(0.7)^4(0.3)^1$ **f** $(0.5)^2(0.5)^4$

5 Calculate the value of each of the following.
a $15(0.7)^2(0.3)^5$ **b** $5(0.7)^1(0.3)^4$ **c** $6(0.4)^2(0.6)^2$

The values in Pascal's triangle to row 15 are shown in the table on page 526. You should consult the table for the necessary coefficient values for questions **6** to **13**.

WORKED Example 13

6 A fair coin is tossed 6 times. What is the probability that:
a a Tail appears exactly 4 times? **b** a Head appears exactly 3 times?

WORKED Example 14, 15

7 A coin is biased with the probability of tossing a Head being 0.6. What is the probability that when the coin is tossed 8 times a Head appears:
a exactly 4 times?
b less than 7 times? (*Hint:* P(less than 7 times) = 1 − P(7 or 8 times).)

8 A die is rolled 5 times. What is the probability that a 6 will appear:
a exactly 1 time? **b** less than 2 times?

9 On a roulette wheel we may assume that all numbers are either red or black. The chance of the ball coming to rest on a red number is close to 50%. If the wheel is spun 7 times, what is the probability that:
a the first number is a red number?
b exactly 3 of the numbers are black?

eBook*plus*

Digital docs:
SkillSHEET 10.2 Listing possibilities
Spreadsheets
052 Generating random integers
043 Pascal's triangle

10 Lopac is attempting to get his driver's licence. He has to pass a written true-or-false test. There are 10 questions in the test. If he guesses the answer to all the questions, what is the probability that he gets:

a a particular question correct?
b all 10 questions correct?

11 Kelly is running out of time on her aptitude test and she decides to guess the last 6 multiple-choice questions. For each question she must circle A, B, C or D. What is the probability that she guesses:

a any particular question correctly?
b no correct answers?
c exactly 4 correct answers?

12 Gabrielle and Mary-Jo have played many tennis matches against each other. The statistics show that in any particular game the probability that Gabrielle wins is 0.6. Note that a tennis match consists of many games! What is the probability that:

a out of 8 games Gabrielle wins exactly 6?
b out of 8 games Mary-Jo wins exactly 6?

13 Sean is the local golf professional. Every hole he plays he expects to get par or better with a probability of 0.8. If he plays 9 holes, what is the probability that he will score par or better on every hole?

investigation

Rectangular and binomial distributions

Task 1

1 Set your calculator to generate random integers in the range 1 to 5 inclusive.

2 Copy and complete the table below to record the result of generating 100 random integers. Calculate the probability of the occurrence of each number by dividing the frequency of each number by 100 (the total number of trials), expressing your answer as a decimal to 2 decimal places.

Number	Tally	Frequency	Probability
1			
2			
3			
4			
5			
Total		100	1

(*continued*)

3 Draw a histogram of your distribution with the number value on the *x*-axis and probability on the *y*-axis.

4 Describe the shape of your distribution.

Task 2

1 Again set your calculator to generate random integers in the range 1 to 5 inclusive.

2 Consider performing 20 trials, each trial consisting of generating a set of five random integers. For each trial, record the number of 2s which appear. Copy and complete the table at right.

Trial number	Number of 2s
1	
2	
3	
4	
20	

3 Count the number of trials where no 2s result, one 2 results, two 2s result and so on. Copy and complete the table below. Calculate the probability for each category to 2 decimal places.

Number of 2s resulting	Frequency	Probability
0		
1		
2		
3		
4		
5		
Total	100	1

4 Draw a histogram of your distribution with the number of resulting 2s on the *x*-axis and probability on the *y*-axis.

5 Comment on the shape of your distribution.

Task 3

1 Repeat Task 2, counting the occurrence of a number other than 2 (1, 3, 4 or 5) in the random integer generation.

2 Compare your graph with the one obtained in Task 2.

Conclusion

What do you conclude from this experiment? Write about a page comparing the results of your three tasks. Support any comments by referring to data collected, figures calculated and graphs drawn.

Pascal's triangle

There are many features of Pascal's triangle which at first are not obvious. Let us investigate.

1. Complete Pascal's triangle to row 10.
2. Comment on the symmetry of the triangle.
3. What is the relationship between the number of entries in a row and the row number?
4. What do you notice about the number of entries in each odd-numbered row? What about the number in each even-numbered row?
5. Compare middle numbers with odd and even numbers of trials.
6. Investigate to find a relationship between the sum of the numbers in a row and that row number.
7. Look at the entries in each row. Consider row 1 to be a two-digit number, row 2 to be a three-digit number, and so on.
   Note that in row 1, $11^1 = 11$
   in row 2, $11^2 = 121$
   Does this pattern continue?
8. The six numbers forming a circle around another number are said to form a ring. If we multiply these six numbers together, this product has a special property. What is it?
9. **Fibonacci's sequence** is the set of numbers 1, 1, 2, 3, 5, 8, 13, 21, Each new number is found by adding the two previous numbers. Where could you find the Fibonacci sequence in Pascal's triangle?
10. The **square numbers** 4, 9, 16, 25, can be found by adding two adjacent numbers. Where does this pattern occur?
11. The **triangular numbers** 1, 3, 6, 10, can be located easily. Where do they occur?
12. The first diagonal consists of the numbers 1, 2, 3, 4, Locate the position of the first four numbers of this diagonal. Where does the sum of the first four numbers of this diagonal lie? Does this pattern continue? Does this pattern also follow for the sum of the numbers in the second diagonal (1, 3, 6, 10,)?
13. Find the sum of all the elements in Pascal's triangle down to and including the first six rows. Copy and complete the table below.

Row	0	1	2	3	4	5	6
Sum	1	3	7				

   What pattern would enable you to determine the sum down to any row without actually adding all the elements?
14. Many visual patterns are also apparent in Pascal's triangle if multiples of numbers are highlighted. To obtain the full effect of these patterns it would be necessary to extend your triangle beyond the tenth row.
   a. Try highlighting all the multiples of 2.
   b. On a new triangle, highlight the multiples of 3.
15. There are many more patterns which lie hidden in Pascal's triangle. Investigate to discover different ones for yourself. Templates can be found by logging into www.jacplus.com.au and locating the weblinks for this chapter.
16. Create a poster or write a summary of the properties of Pascal's triangle. This could be prepared as an oral presentation to your class.

Pascal's triangle binomial coefficients

Trial *n*	Number of successes															
	15	**14**	**13**	**12**	**11**	**10**	**9**	**8**	**7**	**6**	**5**	**4**	**3**	**2**	**1**	**0**
1															1	1
2														1	2	1
3													1	3	3	1
4												1	4	6	4	1
5											1	5	10	10	5	1
6										1	6	15	20	15	6	1
7									1	7	21	35	35	21	7	1
8								1	8	28	56	70	56	28	8	1
9							1	9	36	84	126	126	84	36	9	1
10						1	10	45	120	210	252	210	120	45	10	1
11					1	11	55	165	330	462	462	330	165	55	11	1
12				1	12	66	220	495	792	924	792	495	220	66	12	1
13			1	13	78	286	715	1287	1716	1716	1287	715	286	78	13	1
14		1	14	91	364	1001	2002	3003	3432	3003	2002	1001	364	91	14	1
15	1	15	105	455	1365	3003	5005	6435	6435	5005	3003	1365	455	105	15	1

History of mathematics

BLAISE PASCAL (1623–62)

During his life . . .
The Taj Mahal is started.
Rembrandt does most of his paintings.
Oliver Cromwell governs England.

Blaise Pascal was a French mathematician and physicist who studied combinatorics and developed the theory of probability.

He was born in the town of Clermont in France. His father was an important taxation officer. His mother died when he was only 4. Pascal was a sickly child and so was not sent to school initially but was educated at home by his father. Because he was not healthy his father would not let him study mathematics. It took about 5 years before Blaise could convince his father to let him try.

When Blaise was 16, his father was in trouble with the courts because he would not set any more taxes. He had to leave Paris, and the family moved to Rouen.

Blaise Pascal discovered and proved a major theorem of geometry when he was only 16 years old. This theorem was about the intersections of points on a conic plane.

When he was 18 he became very ill. He eventually recovered, after being temporarily paralysed and close to death. After this scare he became very religious and started to study philosophy and religion. His research into mathematics and science often conflicted with his religious beliefs.

At age 19, Pascal invented a calculating machine that could do simple addition and subtraction. He sold many of these machines and they were so well made that some still exist today.

He demonstrated that air pressure decreases with height by taking accurate measurements at various levels on the side of the Puy de Dôme mountain. He persuaded his brother to climb the mountain and take measurements using a heavy barometer.

Like many mathematicians, Blaise Pascal had arguments with other mathematicians, including René Descartes, who came to visit him. Descartes did not believe that Pascal was capable of such difficult mathematics and claimed that Pascal had stolen some of his ideas from Descartes himself. Blaise Pascal developed the pattern of numbers now known as *Pascal's triangle* that is used in probability, permutations and combinations.

When Blaise Pascal's father died, his sister went into a monastery, and he was left to live free of family and spiritual conflicts. His health improved and he took up an active social life including gambling and driving a fast, horse-drawn carriage!

In late 1654 he was involved in an accident. His horses went over the edge of a bridge and were killed, but he survived. Pascal was shaken up by this and again saw the event as a message from God. In 1655 he moved in with his married sister. Later that year, Pascal became ill and eventually died from the effects of a brain tumor and stomach ulcer in 1662.

The computer language 'Pascal' is named after him.

Questions

1. How old was Pascal when he proved his theorem on conics?
2. What did he develop at age 19 that earned him a lot of money?
3. Upon which mountain was his work on air pressure done, and who did the real work?
4. What is 'Pascal's triangle' used for?
5. What did he die from?

Binomial probabilities through tables

The use of Pascal's triangle to help in calculating binomial probabilities makes the task easier. However, this in turn becomes difficult when questions such as the following are posed: *What is the probability of passing (getting 50% or better) through pure guess-work on a true-or-false test of 20 questions?*

To solve this, one needs to add the probability of scoring exactly $\frac{10}{20}$ to the probability of scoring exactly $\frac{11}{20}$ and so on. Tables have been constructed that make the task much simpler. These tables are found in the binomial distribution tables on pages 532–3. The values in the table give the probability for x or less successes in n trials with the probability of success, p. Their use is demonstrated in the following example. Note that the probability values are given to 4 decimal places.

WORKED Example 16

A fair coin is tossed 10 times. What is the probability that:
a **4 Heads or fewer appear?**
b **more than 4 Heads appear?**
c **at least 4 Heads appear?**

THINK	WRITE
a 1 State values for n and p.	**a** $n = 10$ $p = \text{P(success)}$ $= \text{P(H)}$ $= 0.5$
2 Use the binomial cumulative distribution table for $n = 10$ and $p = 0.5$.	Using the binomial cumulative distribution table for $n = 10$ and $p = 0.5$, $\text{P}(\leq 4\ \text{H}) = 0.3770$.
3 Write the answer.	The probability of 4 Heads or fewer is 0.377.
b 1 Use tables for $n = 10$ and $p = 0.5$. 2 Use the complementary event.	**b** $\text{P}(> 4\ \text{H}) = 1 - \text{P}(\leq 4\ \text{H})$ $= 1 - 0.3770$ $= 0.6230$
3 Write the answer.	The probability of more than 4 Heads is 0.623.
c 1 Use the same table and consider the complementary event.	**c** $\text{P}(\geq 4\ \text{H}) = 1 - \text{P}(\leq 3\ \text{H})$ $= 1 - 0.1719$ $= 0.8281$
2 Write the answer.	The probability of at least 4 Heads is 0.8281.

WORKED Example 17

George attempts to guess the answers to a multiple-choice test of 20 questions. Each question has 4 options: A, B, C and D. What is the probability that, through guesswork, George scores 10 or better?

THINK	WRITE
1 Locate the binomial cumulative distribution tables on pages 532–3.	
2 State the values for n and p.	$n = 20$ $p = \text{P(success)}$ $= \text{P(correct answer)}$ $= \frac{1}{4}$ $= 0.25$

THINK	WRITE
3 We need P(≥ 10 correct) so use the complementary event. Look up the binomial cumulative distribution table for $n = 20$, $p = 0.25$ and $x = 9$.	P(≥ 10 correct) = 1 − P(≤ 9 correct) = 1 − 0.9861 = 0.0139
4 Write the answer.	So, the probability that George scores 10 or better through guessing is 0.0139.

Note: A graphics calculator can also be used to answer these questions.

remember

Binomial distribution tables are useful in determining binomial probabilities in repeated trials. A graphics calculator is also useful.

EXERCISE 10E Binomial probabilities through tables

1 Use the binomial cumulative distribution tables on pages 532–3 to calculate the probability of these events. (Here n stands for the number of trials, p the probability of success and x the number of successes.) A graphics calculator could also be used.

a $P(x \le 6)$ $p = 0.8$ $n = 15$
b $P(x \le 5)$ $p = 0.3$ $n = 10$
c $P(x \le 20)$ $p = 0.8$ $n = 25$
d $P(x \le 10)$ $p = 0.4$ $n = 15$

2 Use the binomial cumulative distribution tables on pages 532–3 or a graphics calculator to calculate the probability of these events.

a $P(x \ge 4)$ $p = 0.5$ $n = 10$
b $P(x \ge 10)$ $p = 0.8$ $n = 15$
c $P(x \ge 15)$ $p = 0.9$ $n = 20$
d $P(x \ge 17)$ $p = 0.8$ $n = 25$

3 Use the binomial cumulative distribution tables on pages 532–3 or a graphics calculator to calculate the probability of these events.

a $P(x \ge 8)$ $p = 0.5$ $n = 15$
b $P(x \le 8)$ $p = 0.5$ $n = 15$
c $P(x > 10)$ $p = 0.4$ $n = 25$
d $P(x < 8)$ $p = 0.9$ $n = 25$

WORKED Example 16, 17

4 A fair coin is tossed 20 times. What is the probability that it will show Heads at least 8 times?

5 A die is rolled 25 times. What is the probability that:

a an even number will show at least 12 times?
b a 6 will turn up at most 5 times?

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Digital docs:
SkillSHEET 10.3
Multiple probabilities
WorkSHEET 10.2

6 What is the probability of achieving at least 50% on the following tests due to pure guesswork only?
  a A true-or-false test of 10 questions.
  b A true-or-false test of 20 questions.
  c A true-or-false test of 25 questions.
  d A multiple-choice test of 10 questions each with options A, B, C and D.
  e A multiple-choice test of 20 questions each with options A, B, C and D.
  f A multiple-choice test of 25 questions each with options A, B, C and D.

7 Though Fred is unaware of it, he has a (fictitious) disease called pyronia. The disease is contagious and people who have been in contact with him have a 40% chance of contracting it. If 10 people come into contact with Fred, what is the probability that:
  a exactly 1 person will catch the disease?
  b at least 2 people will catch the disease?
  c at least 1 person will catch the disease?

8 A scientist estimates that in a certain lake there are 1200 striped trout and of these 240 are tagged.
  a If a striped trout is caught, what is the probability that it is tagged?
  b A sample of 10 striped trout are caught. What is the probability that:
    i exactly 3 of these are tagged?
    ii at least 3 fish are tagged?
    iii none of the fish are tagged?
  c What conclusion could be made by the scientist if 8 of the 10 fish were tagged?

9 A political poll revealed that 60% of the population favoured the Green Party candidate while the remainder said that they would vote for the Yellow Party candidate. What is the probability that in a group of 20 people there would be a majority in favour of the Yellow Party?

10 Paul finds that he knows none of the answers in his Chinese test and has to guess them. Each of the 10 questions is multiple-choice with 3 possible answers. What is the probability that he guesses:
  a a particular answer correctly?
  b all answers correctly?
  c he passes; that is, scores at least 5 out of 10?

investigation

The birthday problem

In a large group of people, you might expect to find two with the same birthday. In a smaller group, you might think that the probability of that occurring is almost negligible.

Statistically, it has been calculated that, in a group of only about 60 people, the chance of finding two people with the same birthday is almost 100%. The term ‘with the same birthday’ in this case means celebrating a birthday on the same date and month — not taking into account the year of birth.

1 Before undertaking this investigation, record the birthday of each student in your class. Did you find an incidence of two people with the same birthday? If you have quite small numbers, you may need to extend the group to your whole year level. Conduct a survey on a group of about 60 people and test the statistic.
2 There are many web sites dealing with this problem, also providing a detailed solution. Access one of the sites and follow the solution and logic provided.

3 You might like to use the ideas in this solution to pose alternative problems that you could test in your classroom environment. Here is a suggestion to get you started.

Consider a group of two students. Calculate the theoretical chance of them both having a birthday on the same day of the week this year. Add a third, fourth . . . person to the group. What is the theoretical minimum number of students required to be assured of an almost 100% chance that two of the students would share a birthday on the same day of the week this year? Of course, for a group of 8 students, the probability would be 100%, but it may be very close to 100% for a much smaller group. Test your theory by randomly selecting two students from your class. Add a third, fourth . . . and test your theoretical calculations. Repeat the experiment with another group of students. What are your conclusions?

10 QUICK QUESTIONS 2

1 In a family of 3 children, what is the probability that they will all be the same sex?

2 Write row 4 of Pascal's triangle.

3 A die is rolled 4 times and the number of times a 6 results is noted. What would be the values for 'p', 'q' and 'n' in this experiment?

4 For the experiment in question 3, write the term which would represent the probability of obtaining three 6s.

5 Calculate the probability of obtaining three 6s in four rolls of a die.

6 A coin is biased so that it results in a Tail 70% of the time. What is the probability of obtaining 4 Tails in 5 tosses of the coin?

7 Use the binomial cumulative distribution tables on pages 532–3 to determine the probability of obtaining at most 10 Heads in 15 tosses of an unbiased coin.

8 What would be the probability of obtaining more than 10 Heads in the experiment in question 7?

9 A multiple-choice test consists of 20 questions each with a choice A, B, C or D. What would be the chance of passing the test by purely guessing the answers?

10 If the test in question 9 had five choices A, B, C, D or E, what chance would there be of passing the test by merely guessing?

The binomial cumulative distribution tables

The values in the tables give the probability for x or fewer successes in n trials with the probability of success, p.

$n = 5$ (p)

x	.01	.05	.10	.20	.25	.30	.40	.50	.60	.70	.75	.80	.90	.95	.99
0	.9510	.7738	.5905	.3277	.2373	.1681	.0778	.0313	.0102	.0024	.0010	.0003	.0000		
1	.9990	.9774	.9185	.7373	.6328	.5282	.3370	.1875	.0870	.0308	.0156	.0067	.0005	.0000	
2	1.0000	.9988	.9914	.9421	.8965	.8369	.6826	.5000	.3174	.1631	.1035	.0579	.0086	.0012	.0000
3		1.0000	.9995	.9933	.9844	.9692	.9130	.8125	.6630	.4718	.3672	.2627	.0815	.0226	.0010
4			1.0000	.9997	.9990	.9976	.9898	.9688	.9222	.8319	.7627	.6723	.4095	.2262	.0490

$n = 10$ (p)

x	.01	.05	.10	.20	.25	.30	.40	.50	.60	.70	.75	.80	.90	.95	.99
0	.9044	.5987	.3487	.1074	.0563	.0282	.0060	.0010	.0001	.0000					
1	.9957	.9139	.7361	.3758	.2440	.1493	.0464	.0107	.0017	.0001	.0000				
2	.9999	.9885	.9298	.6778	.5256	.3828	.1673	.0547	.0123	.0016	.0004	.0001	.0000		
3	1.0000	.9990	.9872	.8791	.7759	.6496	.3823	.1719	.0548	.0106	.0035	.0009	.0000		
4		.9999	.9984	.9672	.9219	.8497	.6331	.3770	.1662	.0473	.0197	.0064	.0001	.0000	
5		1.0000	.9999	.9936	.9803	.9527	.8338	.6230	.3669	.1503	.0781	.0328	.0016	.0001	
6			1.0000	.9991	.9965	.9894	.9452	.8281	.6177	.3504	.2241	.1209	.0128	.0010	.0000
7				.9999	.9996	.9984	.9877	.9453	.8327	.6172	.4744	.3222	.0702	.0115	.0001
8					1.0000	.9999	.9983	.9893	.9536	.8507	.7560	.6242	.2639	.0861	.0043
9						1.0000	.9999	.9990	.9940	.9718	.9437	.8926	.6513	.4013	.0956

$n = 15$ (p)

x	.01	.05	.10	.20	.25	.30	.40	.50	.60	.70	.75	.80	.90	.95	.99
0	.8601	.4633	.2059	.0352	.0134	.0047	.0005	.0000							
1	.9904	.8290	.5490	.1671	.0802	.0353	.0052	.0005	.0000						
2	.9996	.9638	.8159	.3980	.2361	.1268	.0271	.0037	.0003	.0000					
3	1.0000	.9945	.9444	.6482	.4613	.2969	.0905	.0176	.0019	.0001	.0000				
4		.9994	.9873	.8358	.6865	.5155	.2173	.0592	.0093	.0007	.0001	.0000			
5		.9999	.9978	.9389	.8516	.7216	.4032	.1509	.0338	.0037	.0008	.0001			
6		1.0000	.9997	.9819	.9434	.8689	.6096	.3036	.0950	.0152	.0042	.0008			
7			1.0000	.9958	.9827	.9500	.7869	.5000	.2131	.0500	.0173	.0042	.0000		
8				.9992	.9958	.9848	.9050	.6964	.3902	.1311	.0566	.0181	.0003	.0000	
9				.9999	.9992	.9963	.9662	.8491	.5968	.2784	.1484	.0611	.0022	.0001	
10				1.0000	.9999	.9993	.9907	.9408	.7827	.4845	.3135	.1642	.0127	.0006	
11					1.0000	.9999	.9981	.9824	.9095	.7031	.5387	.3518	.0556	.0055	.0000
12						1.0000	.9997	.9963	.9729	.8732	.7639	.6020	.1841	.0362	.0004
13							1.0000	.9995	.9948	.9647	.9198	.8329	.4510	.1710	.0096
14								1.0000	9995	.9953	.9866	.9648	.7941	.5367	.1399

$n = 20$								p							
x	.01	.05	.10	.20	.25	.30	.40	.50	.60	.70	.75	.80	.90	.95	.99
0	.8179	.3585	.1216	.0115	.0032	.0008	.0000								
1	.9831	.7358	.3917	.0692	.0234	.0076	.0005	.0000							
2	.9990	.9245	.6769	.2061	.0913	.0355	.0036	.0002							
3	1.0000	.9841	.8670	.4114	.2252	.1071	.0160	.0013	.0000						
4		.9974	.9568	.6296	.4148	.2375	.0510	.0059	.0003						
5		.9997	.9887	.8042	.6172	.4164	.1256	.0207	.0016	.0000					
6		1.0000	.9976	.9133	.7858	.6080	.2500	.0577	.0065	.0003	.0000				
7			.9996	.9679	.8982	.7723	.4159	.1316	.0210	.0013	.0002	.0000			
8			.9999	.9900	.9591	.8867	.5956	.2517	.0565	.0051	.0009	.0001			
9			1.0000	.9974	.9861	.9520	.7553	.4119	.1275	.0171	.0039	.0006			
10				.9994	.9961	.9829	.8725	.5881	.2447	.0480	.0139	.0026	.0000		
11				.9999	.9991	.9949	.9435	.7483	.4044	.1133	.0409	.0100	.0001		
12				1.0000	.9998	.9987	.9790	.8684	.5841	.2277	.1018	.0321	.0004		
13					1.0000	.9997	.9935	.9423	.7500	.3920	.2142	.0867	.0024	.0000	
14						1.0000	.9984	.9793	.8744	.5836	.3828	.1958	.0113	.0003	
15							.9997	.9941	.9490	.7625	.5852	.3704	.0432	.0026	
16							1.0000	.9987	.9840	.8929	.7748	.5886	.1330	.0159	.0000
17								.9998	.9964	.9645	.9087	.7939	.3231	.0755	.0010
18								1.0000	.9995	.9924	.9757	.9308	.6083	.2642	.0169
19									1.0000	.9992	.9968	.9885	.8784	.6415	.1821

$n = 25$								p							
x	.01	.05	.10	.20	.25	.30	.40	.50	.60	.70	.75	.80	.90	.95	.99
0	.7778	.2774	.0718	.0038	.0008	.0001	.0000								
1	.9742	.6424	.2712	.0274	.0070	.0016	.0001								
2	.9980	.8729	.5371	.0982	.0321	.0090	.0004	.0000							
3	.9999	.9659	.7636	.2340	.0962	.0332	.0024	.0001							
4	1.0000	.9928	.9020	.4207	.2137	.0905	.0095	.0005	.0000						
5		.9988	.9666	.6167	.3783	.1935	.0294	.0020	.0001						
6		.9998	.9905	.7800	.5611	.3407	.0736	.0073	.0003						
7		1.0000	.9977	.8909	.7265	.5118	.1536	.0216	.0012	.0000					
8			.9995	.9532	.8506	.6769	.2735	.0539	.0043	.0001					
9			.9999	.9827	.9287	.8106	.4246	.1148	.0132	.0005	.0000				
10			1.0000	.9944	.9703	.9022	.5858	.2122	.0344	.0018	.0002	.0000			
11				.9985	.9893	.9558	.7323	.3450	.0778	.0060	.0009	.0001			
12				.9996	.9966	.9825	.8462	.5000	.1538	.0175	.0034	.0004			
13				.9999	.9991	.9940	.9222	.6550	.2677	.0442	.0107	.0015			
14				1.0000	.9998	.9982	.9656	.7878	.4142	.0978	.0297	.0056	.0000		
15					1.0000	.9995	.9868	.8852	.5754	.1894	.0713	.0173	.0001		
16						.9999	.9957	.9461	.7265	.3231	.1494	.0468	.0005		
17						1.0000	.9988	.9784	.8464	.4882	.2735	.1091	.0023	.0000	
18							.9997	.9927	.9264	.6593	.4389	.2200	.0095	.0002	
19							.9999	.9980	.9706	.8065	.6217	.3833	.0334	.0012	
20							1.0000	.9995	.9905	.9095	.7863	.5793	.0980	.0072	.0000
21								.9999	.9976	.9668	.9038	.7660	.2364	.0341	.0001
22								1.0000	.9996	.9910	.9679	.9018	.4629	.1271	.0020
23									.9999	.9984	.9930	.9726	.7288	.3576	.0258
24									1.0000	.9999	.9992	.9962	.9282	.7226	.2222

summary

Independent events

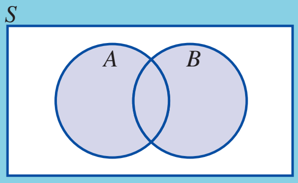
- The probability of an event, A, is symbolised by P(A) and the probability of the event not happening, A', is given by P(A').
- $P(A) + P(A') = 1$ and $P(A') = 1 - P(A)$.
- If two compound events have absolutely no influence upon each other, they are called *independent* events.
- Let P(A) be the probability of event A, and P(B) the probability of event B. If the events are independent then $P(A \text{ and } B) = P(A) \cdot P(B)$. This is called the *multiplication rule*.
- Tree diagrams can be used as a graphical method of displaying the outcomes and computing the probabilities for independent events.

Mutually exclusive events

- In a compound event from a single event space, if two events cannot happen at the same time they are said to be *mutually exclusive*.

Venn diagrams

- Venn diagrams are a graphical method of showing the relationship between sets.
- Let P(A), P(B) be the probabilities of events A and B, respectively. If the events are mutually exclusive then $P(\text{either } A \text{ or } B) = P(A) + P(B)$. This is called the *addition rule*.
- For any compound event, P(both A and B) is the overlap, or *intersection*, between two events.
- The addition rule for probability states that $P(\text{either } A \text{ or } B) = P(A) + P(B) - P(\text{both } A \text{ and } B)$.



Probability distributions

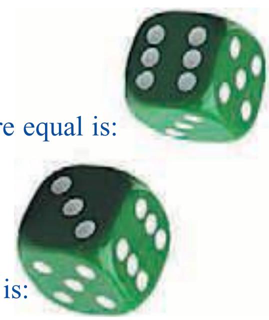
- A rectangular distribution results from trials where each outcome has an equal chance of occurring.
- A binomial experiment is concerned with only two outcomes, labelled a success and a failure.
- Pascal's triangle and binomial cumulative distribution tables are useful in determining binomial probabilities in repeated trials.
- A graphics calculator is useful in calculating binomial probabilities.

CHAPTER review

1 **multiple choice** 10A

When two coins are tossed, the probability of getting at *least* one Head is:

A 0 **B** 0.25 **C** 0.5 **D** 0.75 **E** 1



2 **multiple choice** 10A

Two dice are tossed. The probability that the numbers on each die are equal is:

A 0 **B** $\frac{1}{36}$ **C** $\frac{1}{18}$ **D** $\frac{1}{6}$ **E** $\frac{1}{2}$

3 **multiple choice** 10A

When tossing a pair of dice, the probability of obtaining a total of 4 is:

A $\frac{1}{3}$ **B** $\frac{1}{4}$ **C** $\frac{1}{9}$ **D** $\frac{1}{12}$ **E** $\frac{1}{36}$

4 Two 6-sided dice are rolled and the sum on their faces is noted. 10A

a Copy the table below into your books and complete it, giving the probability of each outcome.

Sum	2	3	4	5	6	7	8	9	10	11	12
Probability	$\frac{1}{36}$	$\frac{1}{18}$									

b What is the probability that the sum is an even number?

c To win the game of Craps on the first throw of the dice the player must roll a 7 or an 11. What is the probability of winning on the first throw?

d To lose the game of Craps on the first throw the player must roll a 2, 3 or 12. What is the probability of losing on the first throw?

5 If the probability of a baby being born a girl is 0.5, calculate the probability of 3 children in a family being: 10A

a a boy, girl, boy in that order

b two boys and a girl in any order.

6 In a survey of voters, it was found that 3240 supported the Labor Party, 3670 supported the Liberal Party and 1240 supported neither party. Find the probability that a voter, chosen at random, is a Labor Party voter. Give your answer correct to 4 decimal places. 10A

7 Australia and South Africa play two rugby matches. Based on previous meetings the probability that Australia wins is 0.56. Construct a tree diagram and find the probability that the teams win one game each. 10A

8 A factory is protected by three independent alarm systems. The probability that an alarm fails to work is 0.05. What is the probability that at least one alarm is working? 10A

10A,B

9 When tossing two dice, find the probability that the total is:
a an odd number
b a number in the range 2 to 6
c either an even number or a prime number.

10A,B

10 **multiple choice**

If the probability of event A depends upon the outcome of event B then the events are:
A mutually exclusive
B independent
C not independent
D complementary
E supplementary

10B

11 **multiple choice**

If two events are mutually exclusive, then:
A their probabilities must be equal
B their probabilities must not be equal
C the probability of either happening is the product of their individual probabilities
D the probability of either happening is the sum of their individual probabilities
E the probability of one event is conditional upon the other.

10B

12 The probability that Brian wins a set of tennis against John is 0.75. To win a match a player must win 2 sets. What is the probability that:
a Brian wins the first 2 sets?
b the match lasts for 3 sets?
c Brian wins (in 2 or 3 sets)?

10B

13 **multiple choice**

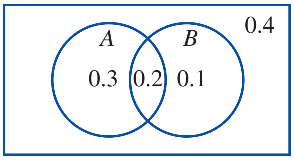
In an urn there are 17 green balls, 12 white balls and 23 red balls. The probability that a randomly selected ball is not green is closest to:
A 0.231 B 0.326 C 0.558 D 0.660 E 0.673

10B

14 A container holds 7 balls — 5 green and 2 yellow. A ball is drawn, its colour noted and then replaced, and then a second ball is drawn.
a What is the probability that:
i both are green? ii they are of different colours?
b What are the probabilities for i and ii if the balls are not replaced?

10C

15 **multiple choice**



From the data in the figure at right, P(both A and B) is equal to:
A 0.1 B 0.2 C 0.3 D 0.4 E 0.6

10C

16 In a group of 20 students, 10 have neither blond hair nor blue eyes, 5 have both and 9 have blue eyes. Construct a Venn diagram. How many have blond hair?

10C

17 Let $P(A) = 0.3$, $P(B) = 0.5$ and P(either A or B) = 0.55. By sketching a Venn diagram of this situation, find the intersection; that is, P(both A and B).

10C

18 **multiple choice**

The students attending a secondary school in Wallaby Plains are encouraged to play either football or tennis or both. Of the 50 students in the school, 30 decide to play football and 15 decide to play both football and tennis. The number who play tennis is:
A 5 B 15 C 20 D 25 E 35

19 At a meeting of the Annerley Junior Rugby League Supporters Club a vote was taken on a key proposal. Of the 43 people at the meeting, 30 were in favour of the proposal and 25 were against it. Some had voted both ways. 10C

a Draw a Venn diagram illustrating this situation. (Assume everyone at the meeting voted at least once.)

b What is the probability that a person selected at random from the meeting voted in favour of the proposal?

20 **multiple choice** 10D

Which of the following could be used to simulate coin tossing?

A Tossing two dice and letting an even total represent a Head.

B Using a calculator's random-number generator and letting any number < 0.5 represent a Head.

C Opening a book at a random page and if the page number is odd, letting that represent a Head.

D Either **A** or **B**.

E Either **A** or **B** or **C**.

21 Provide each of the following: 10D

a two conditions for an event to be classed as binomial

b an example of an event which is binomial

c an example of an event which is not binomial.

22 A die is rolled 5 times. What is the probability that a 6 will appear exactly: 10D

a once?

b 3 times?

23 A multiple-choice test has 10 questions each with 4 options. What is the probability of scoring exactly 6 out of 10 by pure guesswork? 10D

24 A manufacturing process produces on average 9% defective items. In a group of 10 items, what is the probability that there will be: 10D

a no defective items?

b 1 defective item?

c no more than 1 defective item?

25 Use the binomial cumulative distribution table from pages 532–3 or a graphics calculator to calculate: 10E

a $P(x \geq 10)$ for $n = 20$ and $p = 0.6$

b $P(x \geq 15)$ for $n = 20$ and $p = 0.8$.

26 A multiple-choice test has 20 questions each with 5 options. 10E

a What is the probability of guessing any one answer?

b Using the binomial cumulative distribution tables on pages 532–3 or a graphics calculator, calculate the probability of scoring at least 5 out of 20 by pure guesswork.

27 Find the probability that in a binomial experiment where P(success) = 0.5: 10E

a exactly 12 successes occur in 20 trials

b at least 12 successes occur in 20 trials.

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Digital doc:
Test Yourself
Chapter 10

eBookplus ACTIVITIES

10A Compound events – independent events

Digital docs

- SkillSHEET 10.1: Review probability skills *(page 499)*
- Spreadsheet 069: Investigate tree diagrams *(page 499)*

10B Compound events – mutually exclusive events

Digital doc

- WorkSHEET 10.1: Introduction to models for data using tree diagrams *(page 506)*

10D The binomial distribution using Pascal's triangle

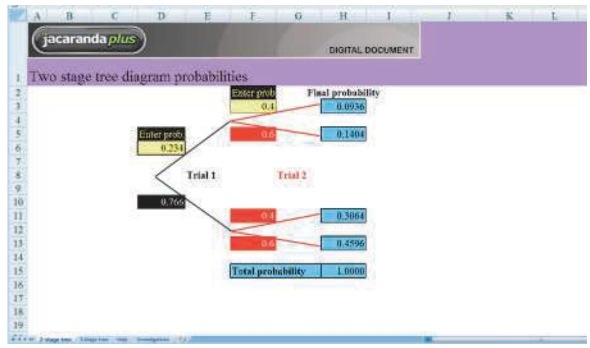
Digital docs

- SkillSHEET 10.2: Practise listing probabilities *(page 523)*
- Spreadsheet 052: Investigate generating random integers *(page 523)*
- Spreadsheet 043: Investigate Pascal's triangle *(page 523)*

10E Binomial probabilties through tables

Digital docs

- SkillSHEET 10.3: Practise calculating multiple probabilities *(page 529)*
- WorkSHEET 10.2: Calculate probabilities using the binomial distribution and Pascal's triangle *(page 529)*



Chapter review

Digital doc

- Test Yourself: Take the end-of-chapter test to test your progress *(page 537)*.

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11

The normal distribution and games of chance

syllabus reference

Elective topic:
Introduction to models for data

In this chapter

Introduction

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Digital docs:
Spreadsheets
081 Finding the mode
082 Finding the mode — DIY
079 Finding the median
080 Finding the median — DIY
072 Bar graphs — DIY

As we have seen, frequency distributions can be developed by **direct measurement**. In many circumstances, however, statisticians are able to calculate frequency distributions from bulk data, without taking direct measurements at all. They would, for example, be able to evaluate the proportion of a given population whose height fell between 175 cm and 185 cm. They can do this because data are frequently distributed in special patterns that can be examined mathematically.

One of the most important frequency distributions is the **normal distribution**. In this chapter we shall see how the normal distribution can be used to model many different situations: the scores of a group of students taking a test; physical characteristics such as height, weight and strength; the odds of winning in games of chance; and the quality of manufactured products.

The normal distribution is widely used in research and industry. Those who are responsible for quality control can take samples and test whether (for example) the cables they make are strong enough or whether their cereal boxes contain enough of their product. Consequently, they can determine if there are problems with their manufacturing equipment or its settings.

SKILLS CHECK

1 Use your calculator to generate 50 random integers in the range 1 to 5 inclusive. Draw a histogram to display your resulting distribution.

2 a Find the mean value of the following scores.
4, 6, 8, 10, 5, 9, 6, 9, 2, 8
b Use your calculator to determine the standard deviation.
c A score of 2 would be how far from the mean?

3 Which of the following two distributions has the scores spread more tightly around the mean?
a mean 50, standard deviation 10
b mean 50, standard deviation 5

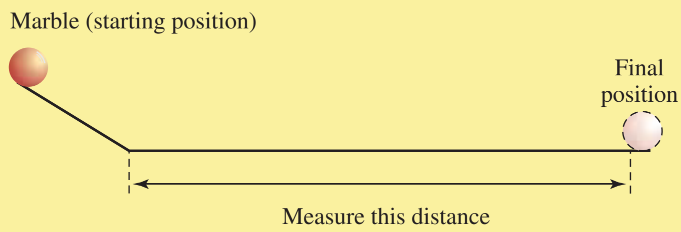
4 Calculate the range represented by 50 ± 5.

5 Explain each of the following.
a $x > 40$
b $x \leq 40$
c $20 < x < 30$

investigation

Rolling marbles

To develop an intuitive feel for the normal distribution, collect and collate data through the following activity.



1 Roll a marble down an incline as shown in the diagram above. Ensure the marble is released from exactly the same point in the same way each time, and measure the distance that the marble takes to stop.

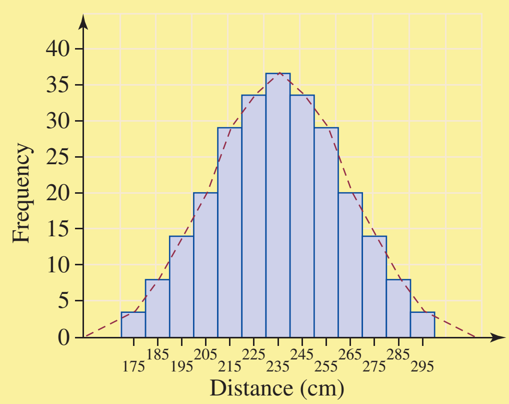
2 Repeat the experiment 60 times. For each of these 60 trials, record:

a trial number

b horizontal distance travelled.

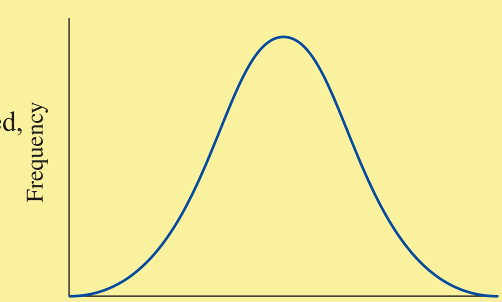
3 Collate the data into approximately 12 equally spaced intervals, and draw a **histogram** for the data.

The distribution of your data should be similar to that shown at right. These data were obtained from 250 trials.



You can see from the shape of the frequency polygon that if the number of points were increased and the interval width were reduced, a curve like that at right would result.

This curve is called a **normal curve**.



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Digital docs:
Spreadsheets
074 Histograms and frequency polygons
075 Histograms and frequency polygons — DIY

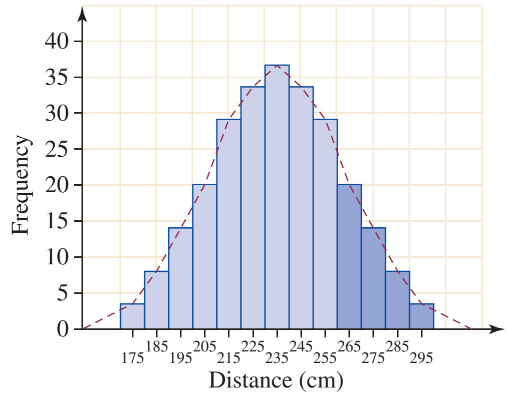
The normal (or Gaussian) distribution is one of the most important in statistical theory. It is named after Carl Friedrich Gauss, one of the great mathematicians in history.

Probability and the normal curve

Consider the histogram at right, representing the distance taken for the rolling marble to come to rest.

What is the probability that one of the trials selected at random has a stopping distance greater than or equal to 260 cm?

There are two ways of answering this question.



Method 1: Count the number of trials in the appropriate categories:

261–270 → 20
271–280 → 13
281–290 → 8
291–300 → 4
Total 45

$$\text{P(stopping distance} > 260) = \frac{45}{250} = 0.18$$

Method 2: The second method may seem similar in this context but has a key difference that will be useful later.

$$\text{P(stopping distance} > 260) = \frac{\text{area under the histogram to the right of 260}}{\text{total area}}$$

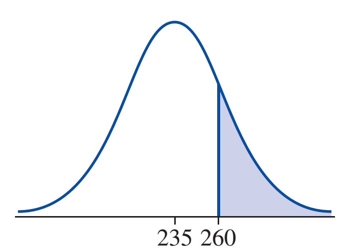
Each of the rectangles in the histogram has a base whose length is 10.

$$\text{P(stopping distance} > 260) = \frac{10 \cdot 20 + 10 \cdot 13 + 10 \cdot 8 + 10 \cdot 4}{10 \cdot 4 + 10 \cdot 8 + 10 \cdot 13 + \quad + 10 \cdot 8 + 10 \cdot 4}$$

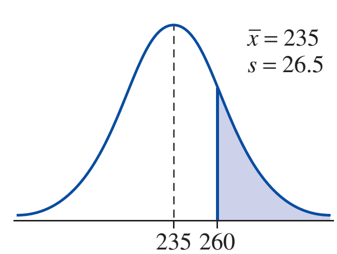
$$= \frac{10 \cdot 45}{10 \cdot 250}$$

$$= 0.18$$

We now consider the probability distribution for a very large number of trials. This discussion will use the terms **mean** and **standard deviation**. As the number of trials increases and the measurements on the x-axis become finer, the histogram becomes a smooth curve called the *normal curve*. Because of its shape, it is sometimes described as a *bell curve*.



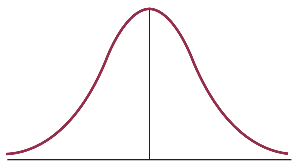
To answer a question such as, 'What percentage of stopping distances is greater than 260 cm?', we would need to calculate the shaded area and divide by the total area under the curve.



We shall return to this problem after practising easier calculations of this type.

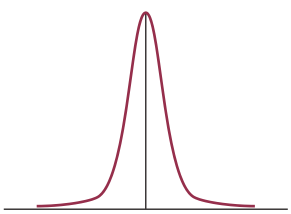
z-scores

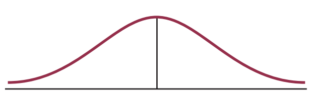
A normal distribution is a statistical representation of data, where a set of scores is symmetrically distributed about the mean. Most continuous variables in a population — such as height, mass and time — are normally distributed. In a normal distribution, the frequency histogram is symmetrical and begins to take on a bell shape as shown by the following figure.



The normal distribution is symmetrical about the mean, which has the same value as the median and mode in this distribution. The graph of a normal distribution will extend symmetrically in both directions and will always remain above the x-axis.

The spread of the normal distribution will depend on the standard deviation. The lower the standard deviation, the more clustered the scores will be around the mean. The figure below left shows a normal distribution with a low standard deviation, while the figure below right shows a normal distribution with a much greater standard deviation.





To gain a comparison between a particular score and the rest of the population we use the ***z*-score**. The z-score (or **standardised score**) indicates the position of a particular score in relation to the mean. A z-score is a very important statistical measure, and later in the chapter some of its uses will be explained.

A z-score of 0 indicates that the score obtained is equal to the mean; a negative z-score indicates that the score is below the mean; a positive z-score indicates a score above the mean.

The z-score measures the distance from the mean in terms of the standard deviation. A score that is exactly one standard deviation above the mean has a z-score of 1. A score that is exactly one standard deviation below the mean has a z-score of −1.

To calculate a z-score we use the formula:

$$z = \frac{x - \bar{x}}{s}$$

where x = the score, $\bar{x}$ = the mean and s = the standard deviation.

WORKED Example 1

In an IQ test the mean IQ is 100 and the standard deviation is 15. Dale's test results give an IQ of 130. Calculate this as a *z*-score.

THINK	WRITE
1 Write the formula.	$z = \frac{x - \bar{x}}{s}$
2 Substitute for x, $\bar{x}$ and s.	$= \frac{130 - 100}{15}$
3 Calculate the *z*-score.	$= 2$

Dale's *z*-score is 2, meaning that his IQ is exactly two standard deviations above the mean.

Not all *z*-scores will be whole numbers; in fact most will not be whole numbers. A whole number indicates only that the score is an exact number of standard deviations above or below the mean.

WORKED Example 2

A sample of professional basketball players gives the mean height as 192 cm with a standard deviation of 12 cm. Dieter is 183 cm tall. Calculate Dieter's height as a *z*-score.

THINK	WRITE
1 Write the formula.	$z = \frac{x - \bar{x}}{s}$
2 Substitute for x, $\bar{x}$ and s.	$= \frac{183 - 192}{12}$
3 Calculate the *z*-score.	$= -0.75$

The negative *z*-score in Worked example 2 indicates that Dieter's height is below the mean but, in this case, by less than one standard deviation.

When examining *z*-scores, care must be taken to use the appropriate value for the standard deviation. If examining a population, the population standard deviation σ_x or $x\sigma_n$ should be used and if a sample has been taken, the sample standard deviation S_x or $x\sigma_{n-1}$ should be used.

WORKED Example 3

To obtain the average number of hours study done by Year 12 students per week, Kate surveys 20 students and obtains the following results.

12	**18**	**15**	**14**	**9**	**10**	**13**	**12**	**18**	**25**
15	**10**	**3**	**21**	**11**	**12**	**14**	**16**	**17**	**20**

a **Calculate the mean and standard deviation (correct to 2 decimal places).**

b **Robert does 16 hours of study each week. Express this as a *z*-score based on the above results. (Give your answer correct to 2 decimal places.)**

THINK	WRITE
a 1 Enter the data into your calculator.	**a**
2 Obtain the mean from your calculator.	$\bar{x} = 14.25$
3 Obtain the standard deviation from your calculator using the sample standard deviation.	$s = 4.88$
b 1 Write the formula.	**b** $z = \frac{x - \bar{x}}{s}$
2 Substitute for x, $\bar{x}$ and s.	$= \frac{16 - 14.25}{4.88}$
3 Calculate the z-score.	$= 0.36$

remember

1. A data set is *normally* distributed if it is symmetrical about the mean.
2. The graph of a normally distributed data set is a bell-shaped curve that is symmetrical about the mean. In such a distribution the mean, median and mode are equal.
3. A z-score is used to measure the position of a score in a data set relative to the mean.
4. The formula used to calculate a z-score is $z = \frac{x - \bar{x}}{s}$, where x = the score, $\bar{x}$ = the mean and s = the standard deviation.

EXERCISE 11A z-scores

1 In a mathematics exam the mean score is 60 and the standard deviation is 12. Chifune's mark is 96. Calculate her mark as a z-score.

2 In an English test the mean score was 55 with a standard deviation of 5. Adrian scored 45 on the English test. Calculate Adrian's mark on the test as a z-score.

3 Tracy is a nurse, and samples the mass of 50 newborn babies born in the hospital in which she works. She finds that the mean mass is 3.5 kg, with a standard deviation of 0.4 kg. What would be the standardised score of a baby whose birth mass was:

a 3.5 kg? **b** 3.9 kg?
c 2.7 kg? **d** 4.7 kg?
e 3.1 kg?

4 Ricky finds that the mean number of hours spent watching television each week by Year 12 students is 10.5 hours, with a standard deviation of 3.2 hours. How many hours of television are watched by a person who has a standardised score of:

a 0? **b** 1? **c** 2? **d** 1? **e** 3 ?

WORKED Example 2

5 Intelligence (IQ) tests have a mean of 100 and a standard deviation of 15. Calculate the z-score for a person with an IQ of 96. (Give your answer correct to 2 decimal places.)

6 The mean time taken for a racehorse to run 1 km is 57.69 s, with a standard deviation of 0.36 s. Calculate the z-score of a racehorse that runs 1 km in 58.23 s.

7 In a major exam every subject has a mean score of 60 and a standard deviation of 12.5. Clarissa obtains the following marks on her exams. Express each as a z-score.

a English 54 **b** Maths A 78 **c** Biology 61
d Geography 32 **e** Art 95

8 The mean time for athletes over 100 m is 10.3 s, with a standard deviation of 0.14 s. What time would correspond to a z-score of:

a 0? **b** 2? **c** 0.5?
d 3? **e** 0.35? **f** 1.6?

WORKED Example 3

9 The length of bolts being produced by a machine needs to be measured. To do this, a sample of 20 bolts are taken and measured. The results (in mm) are given below.

20	19	18	21	20	17	19	21	22	21
17	17	21	20	17	19	18	22	22	20

a Calculate the mean and standard deviation of the distribution.
b A bolt produced by the machine is 22.5 mm long. Express this result as a z-score. (Give your answer correct to 2 decimal places.)

10 A garage has 50 customers who have credit accounts with them. The amount spent by each credit account customer each week is shown in the table below.

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Spreadsheets
007 Calculating the mean from a frequency table
078 Calculating the mean from a frequency table — DIY

Amount ($)	Class centre	Frequency
0–<20		2
20–<40		8
40–<60		19
60–<80		15
80–<100		6

a Copy and complete the table.
b Calculate the mean and standard deviation.
c Calculate the standardised score that corresponds to a customer's weekly account of:
i $50 **ii** $100 **iii** $15.40.

11 **multiple choice**

In a normal distribution, the mean is 21.7 and the standard deviation is 1.9. A score of 20.75 corresponds to a z-score of:

A 1 **B** 0.95 **C** –0.5 **D** 0.5 **E** 1

12 **multiple choice**

In a normal distribution the mean is 58. A score of 70 corresponds to a standardised score of 1.5. The standard deviation of the distribution is:

A 6 **B** 8 **C** 10 **D** 12 **E** 18

13 **multiple choice**

In a normal distribution, a score of 4.6 corresponds to a z-score of –2.4. It is known that the standard deviation of the distribution is 0.8. The mean of the distribution is:

A 1.76 **B** 2.2 **C** 2.68 **D** 6.52 **E** 6.8

14 The results of 24 students sitting a mathematics exam are listed below.

95 63 45 48 78 75 80 66 60 58 59 62
52 57 64 75 81 60 65 70 65 63 62 49

a Calculate the mean and standard deviation of the exam marks.

b Calculate the standardised score of the highest score and the lowest score, correct to 2 decimal places.

15 The results of Luke's exams are shown in the table below.

Subject	Luke's mark	Mean	Standard deviation
English	72	60	12
Mathematics	72	55	13
Biology	76	64	8
Legal studies	60	70	5
Drama	60	50	15
Music	50	58	10

Convert each of Luke's results to a standardised score.

Comparison of scores

An important use of z-scores is to compare scores from different data sets. Suppose that in your mathematics exam your result was 74 and in English your result was 63. In which subject did you achieve the better result?

It may appear, at first glance, that the mathematics result is better, but this does not take into account the difficulty of the test. A mark of 63 on a difficult English test may in fact be a better result than 74 if it was an easy maths test.

The only way that we can fairly compare the results is by comparing each result with its mean and standard deviation. This is done by converting each result to a z-score.

If for mathematics, $\bar{x} = 60$ and $s = 12$, then

$$\begin{aligned} z &= \frac{x - \bar{x}}{s} \\ &= \frac{74 - 60}{12} \\ &= 1.167 \end{aligned}$$

If for English, $\bar{x} = 50$ and $s = 8$, then $z = \dfrac{x - \bar{x}}{s}$

$$= \frac{63 - 50}{8}$$

$$= 1.625$$

The English result is better because the higher z-score shows that the 63 is higher in comparison to the mean of each subject.

WORKED Example 4

Janine scored 82 in her Physics exam and 78 in her Chemistry exam. In Physics, the mean was 62 and the standard deviation 10, while in Chemistry, the mean was 66 and the standard deviation 5.

a Write both results as a standardised score.

b Which is the better result? Explain your answer.

THINK	WRITE
a 1 Write the formula for each subject.	**a** Physics: $z = \dfrac{x - \bar{x}}{s}$ Chemistry: $z = \dfrac{x - \bar{x}}{s}$
2 Substitute for x, $\bar{x}$ and s.	$= \dfrac{82 - 62}{10}$ $= \dfrac{78 - 66}{5}$
3 Calculate each z-score.	$= 2$ $= 2.4$
b Explain that the subject with the highest z-score is the better result.	**b** The Chemistry result is better because of the higher z-score.

In each example the circumstances must be read carefully to see whether a higher or lower z-score is better. For example, if we were comparing times for runners over different distances, the lower z-score would be the better one.

WORKED Example 5

In international swimming the mean time for the men's 100-m freestyle is 50.46 s with a standard deviation of 0.6 s. For the 200-m freestyle, the mean time is 1 min 51.4 s with a standard deviation of 1.4 s.

Sam's best time is 49.92 s for 100 m and 1 min 49.3 s for 200 m. At a competition Sam can enter only one of these events. Which event should he enter?

THINK	WRITE
1 Write the formula for both events.	100 m: $z = \dfrac{x - \bar{x}}{s}$ 200 m: $z = \dfrac{x - \bar{x}}{s}$
2 Substitute for x, $\bar{x}$ and s. (For 200 m convert time to seconds.)	$= \dfrac{49.92 - 50.46}{0.6}$ $= \dfrac{109.3 - 111.4}{1.4}$
3 Calculate the z-scores.	$= -0.9$ $= -1.5$
4 The best event is the one with the lower z-score.	The z-score for 200 m is lower, indicating that Sam's time is further below the mean and that this is the event that he should enter.

remember

1. Scores can be compared by their z-scores because z-scores compare the score with the mean and the standard deviation.
2. Read each question carefully to see if a higher or lower z-score is a better outcome.

EXERCISE 11B Comparison of scores

WORKED Example 4

1 Ken's English mark was 75 and his Mathematics mark was 72. In English, the mean was 65 with a standard deviation of 8, while in Mathematics the mean mark was 56 with a standard deviation of 12.

a Convert the mark in each subject to a z-score.

b In which subject did Ken perform better? Explain your answer.

2 In the first Mathematics test of the year the mean mark was 60 and the standard deviation was 12. In the second test the mean was 55 and the standard deviation was 15. Barbara scored 54 in the first test and 50 in the second test. In which test did Barbara do better? Explain your answer.

3 **multiple choice**

The table at right shows the mean and standard deviation in four subjects. Kelly's marks were: English 67.5, Mathematics 70, Biology 62 and Geography 55. In which subject did Kelly achieve her best result?

Subject	Mean	Standard deviation
English	60	12
Mathematics	65	8
Biology	50	16
Geography	52	7.5

A English **B** Mathematics
C Biology **D** Geography
E Both English and Mathematics

4 **multiple choice**

The table below shows the mean and standard deviation of unit prices in four Australian cities. The table also shows the cost of building a similar one-bedroom unit in each of the cities.

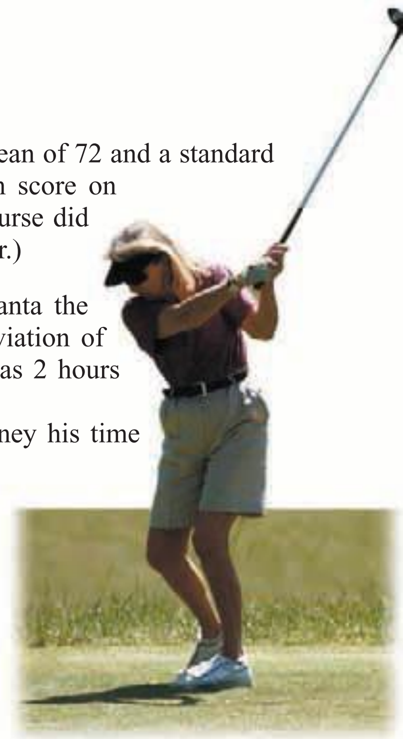
City	Mean	Standard deviation	Cost
Sydney	$230 000	$30 000	$215 000
Melbourne	$215 000	$28 000	$201 000
Adelaide	$185 000	$25 000	$160 000
Brisbane	$190 000	$20 000	$165 000

In which city is the standardised cost of building the unit least?

A Sydney **B** Melbourne **C** Both Sydney and Melbourne
D Adelaide **E** Brisbane

WORKED Example 5

5 Karrie is a golfer who scored 70 on course A, which has a mean of 72 and a standard deviation of 2.5. On course B, Karrie scores 69. The mean score on course B is 72 and the standard deviation is 4. On which course did Karrie play the better round? (In golf the lower score is better.)



6 Steve is a marathon runner. On the Olympic course in Atlanta the mean time was 2 hours and 15 minutes with a standard deviation of 4.5 minutes. On Sydney's Olympic course the mean time was 2 hours and 16 minutes with a standard deviation of 3 minutes.
In Atlanta Steve's time was 2 hours 17 minutes and in Sydney his time was 2 hours 19 minutes.
a Write both times as a standardised score.
b Which was the better performance? Explain your answer.

7 **multiple choice**

The table below shows the mean and standard deviation of times in the 100-m by the same group of athletes on five different days. It also shows Matt's time on each of these days.

Day	Mean	Standard deviation	Matt's time
1 Jan.	10.25	0.14	10.18
8 Jan.	10.21	0.15	10.12
15 Jan.	10.48	0.28	10.30
22 Jan.	10.14	0.09	10.05
29 Jan.	10.22	0.12	10.11

On what day did Matt give his best performance?
A 1 Jan. B 8 Jan. C 15 Jan.
D 22 Jan. E 29 Jan.

8 **multiple choice**

In which of the following subjects did Alyssa achieve her best standardised result?

Subject	Alyssa's mark	Mean	Standard deviation
English	54	60	12
Mathematics	50	55	15
Biology	60	65	8
Music	53	62	9
Accounting	56	64	10

A English B Mathematics C Biology
D Music E Accounting

9 Shun Mei received a mark of 64 on her Mathematics exam and 63 on her Chemistry exam. To determine how well she actually did on the exams, Shun Mei sampled 10 people who sat for the same exams and the results are shown below.

Mathematics:

56 45 82 90 41 32 65 60 55 69

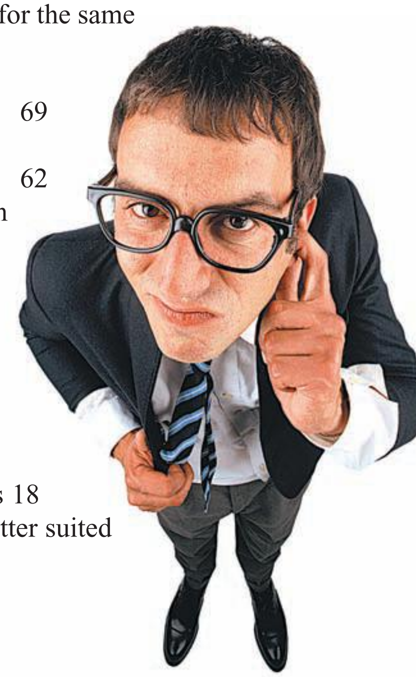
Chemistry:

55 63 39 92 84 46 47 50 58 62

a Calculate the mean and standard deviation for Shun Mei's sample in each subject.

b By converting each of Shun Mei's marks to z-scores, state the subject in which she performed better.

10 Ricardo scored 85 on an entrance test for a job. The test has a mean score of 78 and a standard deviation of 8. Kory sits a similar exam and scores 27. In this exam the mean is 18 and the standard deviation is 6. Who is the better suited candidate for the job? Explain your answer.



10 QUICK QUESTIONS 1

1 In a normal distribution the mean is 32 and the standard deviation 6. Convert a score of 44 to a z-score.

2 In a normal distribution the mean is 1.2 and the standard deviation is 0.3. Convert a score of 0.6 to a z-score.

3 The mean of a distribution is 254 and the standard deviation is 39. Write a score of 214 as a standardised score, correct to 2 decimal places.

4 The mean mark on an exam is 62 and the standard deviation is 9.5. Convert a mark of 90 to a z-score. (Give your answer correct to 2 decimal places.)

5 Explain what is meant by a z-score of 1.

6 Explain what is meant by a z-score of –2.

7 In a distribution, the mean is 50 and the standard deviation is 10. What score corresponds to a z-score of 0?

8 In a distribution the mean score is 60. If a mark of 76 corresponds to a standardised score of 2, what is the standard deviation?

9 Cynthia scored a mark of 65 in English where the mean was 55 and the standard deviation is 8. In Mathematics Cynthia scored 66 where the mean was 52 and the standard deviation 10. Convert the mark in each subject to a z-score.

10 In which subject did Cynthia achieve the better result?

investigation

Comparison of subjects

1 List all the subjects that you study. Arrange the subjects in the order that you feel is from your strongest subject to your weakest.

2 List your most recent examination results in each subject.

3 From your teachers, find out the mean and standard deviation of the results in each subject.

4 Convert each of your marks to a standardised score.

5 List your subjects from best to worst based on the standardised score and see how this list compares with the initial list that you wrote.

Distribution of scores

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Normal distributions
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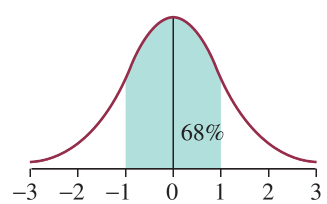
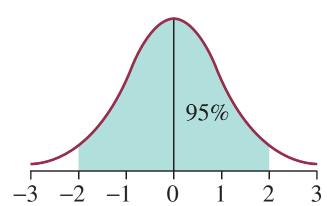
In any normal distribution, the percentage of scores that lie within a certain number of standard deviations of the mean is always the same, provided that the sample is large enough. This is true irrespective of the values of the mean and standard deviation.

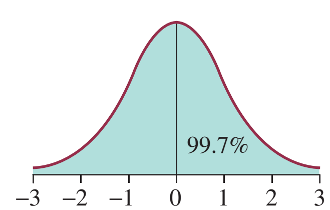
In any normal distribution, approximately 68% of the values will lie within one standard deviation of the mean. This means 68% of scores will have a z-score between 1 and 1. This is shown on the normal curve at right:



Approximately 95% of the values lie within 2 standard deviations, or have a z-score of between 2 and 2.



Approximately 99.7% of scores lie within 3 standard deviations, or have a z-score that lies between 3 and 3.



If we know that a random variable is approximately normally distributed, and we know its mean and standard deviation, then we can use this rule to quickly make some important statements about the way in which the data values are distributed.

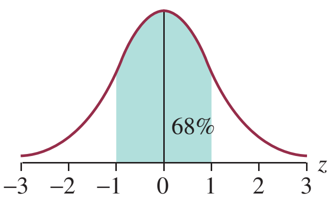
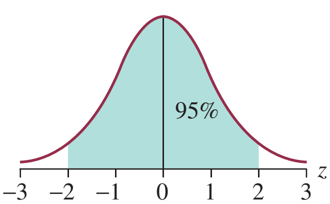
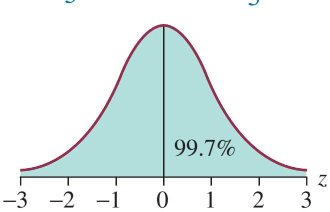
WORKED Example 6

Experience has shown that the scores obtained on a commonly used IQ test can be assumed to be normally distributed with a mean of 100 and a standard deviation of 15. Draw a curve to illustrate each of the following and find approximately what percentage of the distribution lies:

a **between 85 and 115?**

b **between 70 and 130?**

c **between 55 and 145?**

THINK	WRITE
a 1 Calculate the z-scores for 85 and 115.	**a** $z = \frac{85 - 100}{15} = -1$ $z = \frac{115 - 100}{15} = 1$
2 Draw a diagram.	
3 68% of scores have a z-score between 1 and 1.	68% of the scores will lie between 85 and 115.
b 1 Calculate the z-scores for 70 and 130.	**b** $z = \frac{70 - 100}{15} = -2$ $z = \frac{130 - 100}{15} = 2$
2 Draw a diagram.	
3 95% of scores have a z-score between 2 and 2.	95% of the scores will lie between 70 and 130.
c 1 Calculate the z-scores for 55 and 145.	**c** $z = \frac{55 - 100}{15} = -3$ $z = \frac{145 - 100}{15} = 3$
2 Draw a diagram.	
3 99.7% of scores have a z-score between 3 and 3.	99.7% of the scores will lie between 55 and 145.

Graphics Calculator tip! Distribution of scores

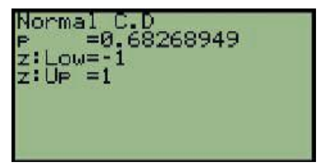
A graphics calculator can be used to readily determine the percentage of a distribution which lies between certain values. This can be illustrated with Worked example 6.

For the Casio *fx*-9860G AU

1. To calculate the cumulative probability between two scores, press:
   - MENU
   - 2: STAT
   - F5 (DIST)
   - F1 (NORM)
   - F2 (Ncd).

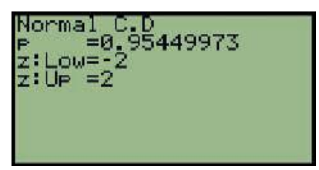
Enter the data as shown for part **a**.

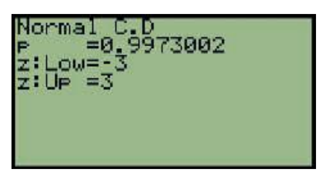
2. To calculate the probability, press F1 (CALC).



This shows that 68% of the scores lie between 85 and 115. It also shows that the z-score of 85 is –1, and the z-score of 115 is 1.

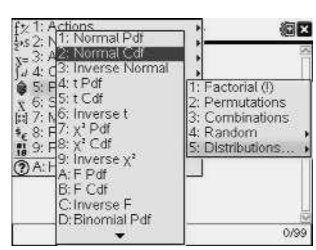
3. Repeat this procedure for parts **b** and **c**.



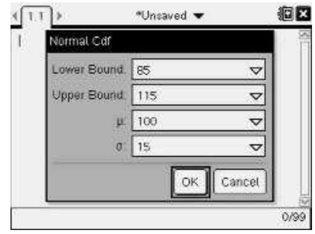


For the TI-Nspire CAS

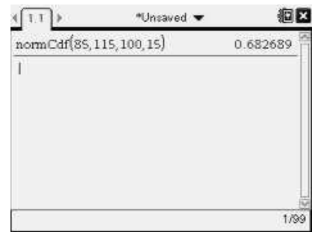
1. To calculate the cumulative probability between two scores, open a Calculator page.
   Press:
   - MENU (menu)
   - 5: Probability (5)
   - 5: Distributions . . . (5)
   - 2: Normal Cdf (2).



2. Enter the data as shown for part **a**, pressing Tab (tab) to advance to the next field.

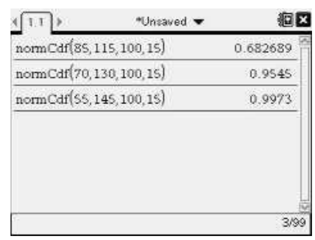


3. To calculate the probability select OK and then press ENTER (enter).



This shows that 68% of the scores obtained on a commonly used IQ test lie between 85 and 115.

4. Repeat the steps above for parts **b** and **c**.



We can also make statements about the percentage of scores that lie in the tails of the distribution by using the symmetry of the distribution and remembering that 50% of scores will have a z-score of greater than 0 and 50% will have a z-score less than 0.

WORKED Example 7

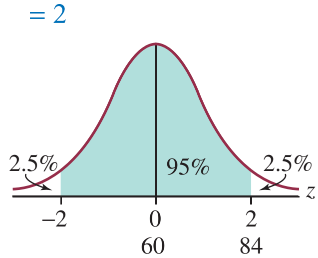
When the results of an examination were analysed, the mean was found to be 60 and the standard deviation, 12. What percentage of candidates in the examination scored above 84?

THINK | **WRITE**

1. Calculate 84 as a z-score using $\bar{x} = 60$ and $s = 12$.

$$z = \frac{x - \bar{x}}{s}$$
$$= \frac{84 - 60}{12}$$
$$= 2$$

2. Draw a sketch showing 95% of z-scores lie between –2 and 2.
3. 5% of z-scores therefore lie outside this range. Half of these scores lie below –2 and half are above 2.



4. Give a written answer.

2.5% of scores are greater than 84.

Some important terminology is used in connection with this rule. We can say that if 95% of scores have a z-score between −2 and 2, then if one member of the population is chosen, that member will *very probably* have a z-score between −2 and 2.

If 99.7% of the population has a z-score between −3 and 3, then if one member of that population is chosen, that member will *almost certainly* have a z-score between −3 and 3.

WORKED Example 8

A machine produces tyres that have a mean thickness of 12 mm, with a standard deviation of 1 mm. If one tyre that has been produced is chosen at random, within what limits will the thickness of the tyre:

a very probably lie?

b almost certainly lie?

THINK	WRITE
a 1 A score will very probably have a z-score between −2 and 2. 2 A z-score of −2 corresponds to a tyre of 10 mm thickness.	**a** If $z = -2$, $x = \bar{x} - 2s = 12 - 2 \cdot 1 = 10$; If $z = 2$, $x = \bar{x} + 2s = 12 + 2 \cdot 1 = 14$
3 A z-score of 2 corresponds to a tyre of 14 mm thickness.	A tyre chosen will very probably have a thickness of between 10 and 14 mm.
b 1 A score will almost certainly have a z-score between −3 and 3. 2 A z-score of −3 corresponds to a tyre of 9 mm thickness.	**b** If $z = -3$, $x = \bar{x} - 3s = 12 - 3 \cdot 1 = 9$; If $z = 3$, $x = \bar{x} + 3s = 12 + 3 \cdot 1 = 15$
3 A z-score of 3 corresponds to a tyre of 15 mm thickness.	A tyre chosen will almost certainly have a thickness of between 9 and 15 mm.

Because it is almost certain that all members of the data set will lie within three standard deviations of the mean, if a possible member of the data set is found to be outside this range, one should suspect a problem.

For example, if a machine is set to deposit 200 mL of liquid into a bottle, with a standard deviation of 5 mL, and then a bottle is found to have contents of 220 mL, one would expect there to be a problem with the settings on the machine because a figure of 220 mL is four standard deviations above the mean.

This knowledge of z-scores is then used in industry by the quality control department. In the previous example, a sample of bottles would be tested and the z-scores recorded. The percentage of z-scores between −1 and 1 should be close to 68%, between −2 and 2 close to 95% and between −3 and 3 close to 99.7%. If these percentages are not correct, the machinery needs to be checked for faults.

remember

1. In a normal distribution:
   (a) 68% of scores will have a z-score between −1 and 1
   (b) 95% of scores will have a z-score between −2 and 2
   (c) 99.7% of scores will have a z-score between −3 and 3.
2. The symmetry of the normal distribution allows us to make calculations about the percentage of scores lying within certain limits.
3. If a member of a normally distributed population is chosen, it will:
   (a) very probably have a z-score between −2 and 2
   (b) almost certainly have a z-score between −3 and 3.
4. Any score further than three standard deviations from the mean indicates that there may be a problem with the data set.

EXERCISE 11C Distribution of scores

WORKED Example 6

1 The temperature on a January day in a city is normally distributed with a mean of 26° and a standard deviation of 3°. What percentage of January days lie between:
a 23° and 29°? **b** 20° and 32°? **c** 17° and 35°?

2 The marks of students sitting for a major exam are normally distributed with $\bar{x} = 57$ and a standard deviation of 13. What percentage of marks on the exam were between:
a 44 and 70? **b** 31 and 83? **c** 18 and 96?

3 The mean thickness of bolts produced by a machine is 2.3 mm, with a standard deviation of 0.04 mm. What percentage of bolts will have a thickness between 2.22 mm and 2.38 mm?

WORKED Example 7

4 Experience has shown that the scores obtained on a commonly used IQ test can be assumed to be normally distributed with a mean of 100 and a standard deviation of 15. What percentage of scores lie above 115?

5 The heights of young women are normally distributed with a mean $\bar{x} = 160$ cm and a standard deviation of 8 cm. What percentage of the women would you expect to have heights:
a between 152 and 168 cm?
b greater than 168 cm?
c less than 136 cm?

6 The age at which women give birth to their first child is normally distributed with $\bar{x} = 27.5$ years and a standard deviation of 3.2 years. From these data we can conclude that about 95% of women have their first child between what ages?

7 Fill in the blanks in the following statements. For any normal distribution:
 a 68% of the values have a z-score between ___ and ___
 b ___% of the values have a z-score between –2 and 2
 c ___% of the values have a z-score between ___ and ___.

8 **multiple choice**

Medical tests indicate that the amount of an antibiotic needed to destroy a bacterial infection in a patient is normally distributed with $\bar{x} = 120$ mg and a standard deviation of 15 mg. The percentage of patients who would require more than 150 mg to clear the infection is:

A 0.15% **B** 2.5% **C** 5% **D** 95% **E** 97.5%

9 **multiple choice**

The mean mark on a test is 55, with a standard deviation of 10. The percentage of students who achieved a mark between 65 and 75 is:

A 13.5% **B** 22.5% **C** 34% **D** 63.5% **E** 95%

10 In a factory, soft drink is poured into cans such that the mean amount of soft drink is 500 mL with a standard deviation of 2 mL. Cans with less than 494 mL of soft drink are rejected and not sold to the public. What percentage of cans are rejected?

11 The distribution of IQ scores for the inmates of a certain prison is approximately normal with a mean of 85 and a standard deviation of 15.
 a What percentage of this prison population have an IQ of 100 or higher?
 b If someone with an IQ of 70 or less can be classified as mentally disabled, what percentage of the prison population could be classified as mentally disabled?

12 The distribution of blood pressures (systolic) among women of similar ages is normal with a mean of 120 (mm of mercury) and a standard deviation of 10 (mm of mercury). Determine the percentage of women with a systolic blood pressure:
 a between 100 and 140
 b greater than 130
 c between 120 and 130
 d between 90 and 110
 e between 110 and 150.

WORKED Example 8

13 The mass of packets of chips is normally distributed with $\bar{x} = 100$ g and a standard deviation of 2.5 g. If I purchase a packet of these chips, between what limits will the mass of the packet:
 a very probably lie? b almost certainly lie?

14 The heights of army recruits are normally distributed about a mean of 172 cm and a standard deviation of 4.5 cm. A volunteer is chosen from the recruits. The height of the volunteer will very probably lie between what limits?

15 A machine is set to deposit a mean of 500 g of washing powder into boxes with a standard deviation of 10 g. When a box is checked, it is found to have a mass of 550 g. What conclusion can be drawn from this?

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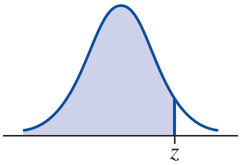
16 The average mass of babies is normally distributed with a mean of 3.8 kg and a standard deviation of 0.4 kg. A newborn baby will almost certainly have a mass between what limits?

Examining a normal distribution

Complete a sample of the heights or masses of 50 people.

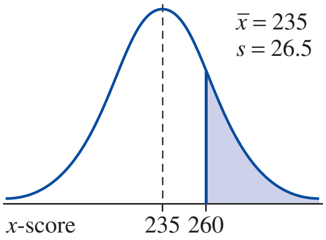
1. Calculate the mean and the standard deviation of your sample.
2. Calculate the percentage of people whose height or mass has a standardised score of between −1 and 1.
3. Calculate the percentage of people whose height or mass has a standardised score of between −2 and 2.
4. Calculate the percentage of people whose height or mass has a standardised score of between −3 and 3.
5. Compare the percentage found in **2**, **3** and **4** with those you would expect if the group of 50 people is normally distributed. Can you think of reasons why your distribution is the same as, or different from, a normal distribution?
6. Write up your investigation, presenting your data, together with graphs. Draw conclusions from the results of your experiment.

Standard normal tables



Obviously, not all z-scores lie exactly one, two or three standard deviations either side of the mean. To deal with situations such as these, we consult a set of **standard normal tables**. The tables have been computed to give the area under the curve to the left of a particular z-value. The total area under the curve is 1. An area to the right of a particular z-score can be calculated by subtracting the area to the left from 1. The standard normal tables are shown on the following page.

Let us return to our rolling marble problem introduced at the beginning of the chapter. The graph of the x-scores is the same as the graph shown at right.



The mean rolling distance was 235 cm and the standard deviation 26.5 cm. As z-scores, the mean represents 0 and the standard deviation represents 1. The question, 'What percentage of stopping distances is greater than 260 cm?' requires us to convert the x-score of 260 cm into a z-score.

$$z = \frac{x - \bar{x}}{s}$$

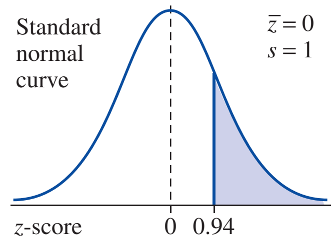
$$= \frac{260 - 235}{26.5}$$

$$= 0.94$$

We can now draw the standard normal curve in terms of z-scores as shown at right.

This problem requires us to determine the area under the curve to the right of the z-score, 0.94.

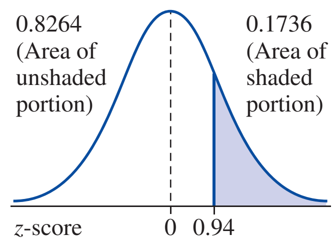
Consulting the standard normal tables which follow tells us that the area under the curve to the left of a z-score of 0.94 is 0.8264.



The area we require is the shaded area shown in the diagram at right.

$$\text{Total area under curve} = 1$$
$$\therefore \text{Area to the right of } 0.94 = 1 - 0.8264$$
$$= 0.1736$$

So the answer to our question is that 0.1736 · 100; that is, 17.36% of balls will have a stopping distance greater than 260 cm.



Standard normal tables: area under the standard normal curve

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

WORKED Example 9

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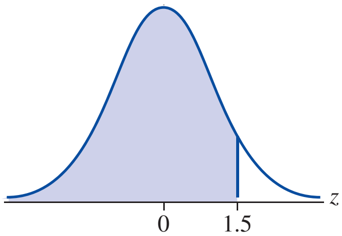
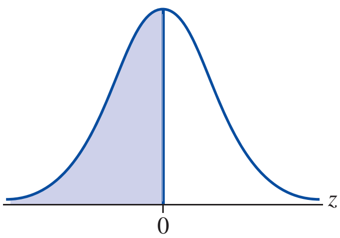
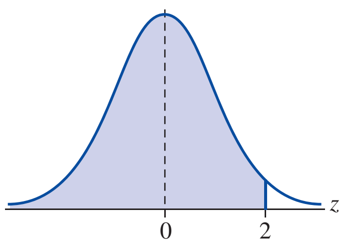
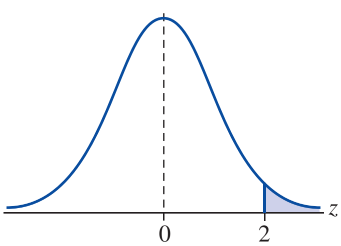
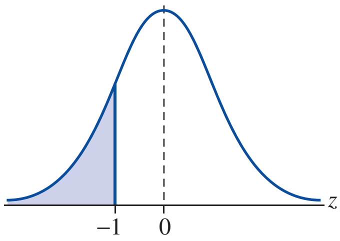
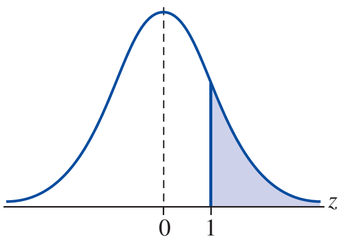
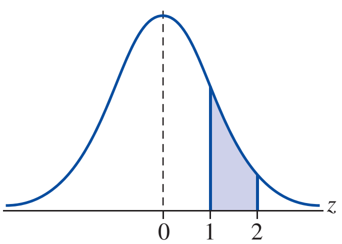
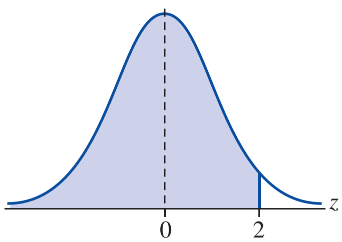
Tutorial:
Worked example 9
int-0429

Use the standard normal tables on page 560 to find values for each of the following.

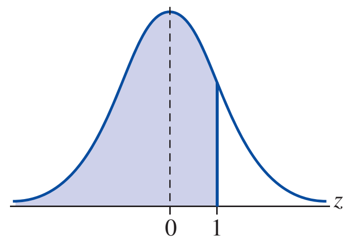
a $P(z < 1.5)$
b $P(z < 0)$
c $P(z < 2)$
d $P(z > 2)$
e $P(z < -1)$
f $P(1 < z < 2)$

THINK	WRITE
a 1 Draw a diagram and shade in the required area. 2 Use the tables to read off a z-value <1.5.	**a** 
3 Write the answer showing correct nomenclature.	$P(z < 1.5) = 0.9332$
b Repeat the steps in part **a**.	**b**  $P(z < 0) = 0.5$
c Repeat the steps in part **a**.	**c**  $P(z < 2) = 0.9772$
d 1 Repeat steps 1 and 2 in part **a**. 2 Use complement to find required area.	**d** 

Continued over page

THINK	WRITE
3 Write the answer showing the correct nomenclature.	$P(z < 2) = 0.9772$ $P(z > 2) = 1 - 0.9772$ $= 0.0228$
e 1 Draw a diagram and shade in the required area. 2 Use the symmetry property of the curve.	**e**  Because the curve is symmetrical, the area would be the same as shown below. 
3 Use the complement to find the required area. 4 Write the answer showing correct nomenclature.	$P(z < -1) = 1 - P(z < 1)$ $= 1 - 0.8413$ $= 0.1587$
f 1 Draw a diagram and shade in the required area. 2 Consider the two z-scores separately.	**f**  This is equivalent to the z-score area for 1 taken from the z-score area for 2. 

THINK | **WRITE**



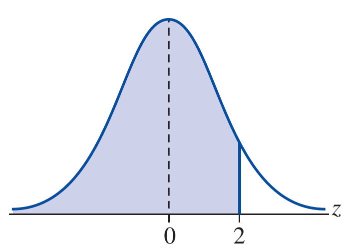
3. Subtract the required areas.
   $P(z < 2) = 0.9772$
   $P(z < 1) = 0.8413$
4. Write the answer showing the correct nomenclature.
   $P(1 < z < 2) = 0.9772 - 0.8413$
   $= 0.1359$

WORKED Example 10

For a group of students attempting an entrance examination to Fullsome University it was found that the scores were normally distributed with a mean of 57 and a standard deviation of 12. What is the probability that an entrant selected at random scored less than 81? (Assume the marks vary continuously; that is, marks of 62.6 are possible.)

THINK | **WRITE**

1. Define the variables.
   $\bar{x} = 57$
   $s = 12$
   $x = 81$
2. Convert the x-score to a z-score.
   $z = \frac{x - \bar{x}}{s}$
   $= \frac{81 - 57}{12}$
   $= \frac{24}{12}$
   $= 2$
3. Sketch the standard normal curve, shading in the required area.



4. Use the tables to determine the area.
   P(entrant scores less than 81)
   $= P(x < 81)$
   $= P(z < 2)$
   $= 0.9772$
5. Write the answer using correct nomenclature.

WORKED Example 11

A normal distribution has a mean of 41 and a standard deviation of 6. If x is a value selected at random from this distribution, calculate the following.

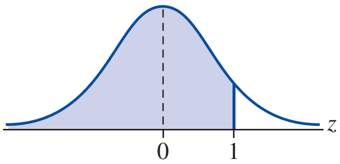
a P(x < 47) **b** P(x < 29)

THINK / **WRITE**

a 1 Convert the x-score to a z-score.

$$z = \frac{x - \bar{x}}{s} = \frac{47 - 41}{6} = \frac{6}{6} = 1$$

2 Draw the standard normal curve and shade the required area.



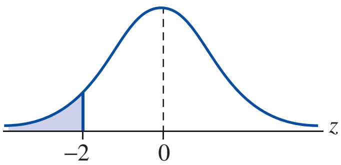
3 Use tables to determine the area and write using correct nomenclature.

P(x < 47) = P(z < 1)
= 0.8413

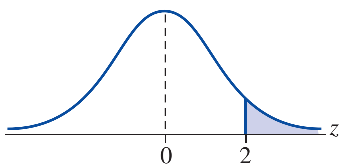
b 1 Convert the x-score to a z-score.

$$z = \frac{x - \bar{x}}{s} = \frac{29 - 41}{6} = \frac{-12}{6} = -2$$

2 Draw the standard normal curve and shade the required area.



3 Consider the negative z-score in terms of the equivalent positive z-score.



P(x < 29) = P(z < –2)
= P(z > 2)
= 1 – P(z < 2)
= 1 – 0.9772
= 0.0228

remember

1. Standard normal tables give the area under the curve to the left of a particular z-value.
2. The total area under the curve is 1.
3. The area to the right of a z-score can be calculated by subtracting the area to the left from 1.

EXERCISE 11D Standard normal tables

WORKED Example 9

1 Use the standard normal tables on page 560 to find the value of each of the following.

a $P(z < 1)$ **b** $P(z < 1.4)$ **c** $P(z < 1.8)$
d $P(z > 1)$ **e** $P(z < -1.7)$ **f** $P(0.5 < z < 1.5)$
g $P(-1 < z < 1)$ **h** $P(-2 < z < 2)$ **i** $P(-3 < z < 3)$
j $P(-2 < z < -1)$ **k** $P(2 < z < 3)$ **l** $P(-1.5 < z < 1.5)$

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SkillSHEET 11.1
Cumulative normal distribution table

2 If a normal distribution has a mean of 34 and a standard deviation of 4, find z-values for the following scores.

a $x = 34$ **b** $x = 31$ **c** $x = 30$ **d** $x = 40$

3 If a normal distribution has a mean of 4 and a standard deviation of 0.1, find z-values for the following scores.

a $x = 4.05$ **b** $x = 3.95$ **c** $x = 3.87$ **d** $x = 4.12$

4 If a normal distribution has a mean of 5 and a standard deviation of 1, calculate each of the following.

a $P(x < 6)$ **b** $P(x < 6.6)$ **c** $P(x < 5)$
d $P(x < 2)$ **e** $P(4 < x < 5)$ **f** $P(3 < x < 6)$

5 If a normal distribution has a mean of 165 and a standard deviation of 14, calculate each of the following.

a $P(x < 170)$ **b** $P(x < 180)$ **c** $P(x < 165)$
d $P(x < 160)$ **e** $P(160 < x < 170)$ **f** $P(150 < x < 175)$
g $P(158 < x < 160)$ **h** $P(180 < x < 184)$

Note: Round z-values to 2 decimal places.

WORKED Example 10

6 A machine manufactures components with a mean lifetime of 45 h with a standard deviation of 4.5 h. If we assume that the variation in the lifetime of the components is normally distributed, calculate the probability that a component will last at least:

a 45 h **b** 50 h **c** 53 h 30 min **d** 40 h.

WORKED Example 11

7 The heights of the Year 12 students at Echo Beach State High School are normally distributed with a mean of 160 cm and a standard deviation of 15 cm. What is the probability that a student's height will be:

a less than 170 cm? **b** less than 180 cm?
c greater than 170 cm? **d** between 140 cm and 170 cm?

8 Assume that the time taken for a group of 60 competitors to complete an obstacle course was normally distributed with a mean of 26 min and a standard deviation of 6 min.

a What percentage of competitors would take less than 30 min to finish the course?
b How many of the 60 competitors would take less than 28 min to finish the course?
c How many competitors would still be going after 22 min?

9 A machine is designed to manufacture sheets of metal, each 24.0 cm in length. A sample of the metal sheets shows that their lengths are normally distributed with a mean of 24.2 cm and a standard deviation of 0.2 cm.

a What is the probability that the length of a sheet of metal is:

i less than 24.5 cm?

ii greater than 24.0 cm?

iii 24.0 cm long?

(*Hint:* A measure of 24.0 cm would represent the interval from 23.95 cm to 24.05 cm.)

b If sheets of metal are rejected when they are less than 24.0 cm or greater than 24.5 cm, calculate the percentage of metal sheets that are rejected.

10 The diameters of 4-year-old Woop pine trees are normally distributed with a mean of 31 cm and a standard deviation of 2.5 cm. What is the probability that one of these a trees has a diameter which:

a is less than 33 cm?

b is less than 30 cm?

c is greater than 34 cm?

d is greater than 29 cm?

e lies between 30 cm and 34 cm?

11 The lifetime of Larson's Light Bulbs is normally distributed with a mean of 55 h and a standard deviation of 3 h. The company advertises that the bulbs should last 50 h. In what percentage of cases would you expect this claim to be false?

12 Packets of Watto's Wheat Flakes are supposed to contain 500 g of cereal. In a sample the mean mass was 508 g with a standard deviation of 3 g. What percentage of packets of Watto's Wheat Flakes are underweight?

(What assumption have you made in answering this question?)

investigation

Standardised scores

This investigation gives you an insight into part of the process involved in calculating an OP (Overall Position). For each OP-eligible subject studied, each student receives an SAI (Student Assessment Index). This index ranges from 200 to 400, the lowest student in each subject receiving a score of 200 and the top student receiving a score of 400. The relative gaps between the students' scores are an indication of the difference in performance between the students. So that all students can be compared, the SAIs for each subject are converted to standardised scores (z-scores).

Consider a school that has a total of 20 Maths A students. The school has assigned the following SAIs to their 20 students in order from top student to bottom student (let's call them student A to Student T).

Student	A	B	C	D	E	F	G	H	I	J
SAI	400	390	387	385	380	360	325	319	300	298

Student	K	L	M	N	O	P	Q	R	S	T
SAI	292	290	270	240	239	233	230	215	202	200

1 Enter the scores into your calculator to calculate the mean and standard deviation of the SAI scores.

2 Convert each of the SAI scores to a z-score. Copy and complete the table below.

Student	A	B	C	D	E	F	G	H	I	J
SAI	400	390	387	385	380	360	325	319	300	298
z-score										

Student	K	L	M	N	O	P	Q	R	S	T
SAI	292	290	270	240	239	233	230	215	202	200
z-score										

3 What do you notice about the sum of the z-scores?

4 Analyse the z-scores to determine the percentage of students with a standardised score of between 1 and 1, 2 and 2, 3 and 3. Are these scores normally distributed? Because we have considered only a small number of scores, you may not find that this results in a normal distribution.

5 You may be able to obtain an actual set of SAIs from your school. If so, you could use those figures in your investigation.

Odds

Gamblers — whether at the racetrack, at the casino or in the comfort of their living rooms — think about probability. However, they do not usually think in terms of a probability of 1 in 5, 20% or 0.2. Gamblers usually think in terms of **odds**. A probability of 1 in 5 produces odds of 4 to 1.

Odds can be thought of as a ratio of the number of ways of losing to the number of ways of winning. When a die is rolled there are 5 ways it can come up not-6 and 1 way it can come up 6. The fair odds against rolling a 6 are 5 to 1. Some common odds and their probabilities are given in the table on the next page.

Odds for events with a probability greater than $\frac{1}{2}$ are given by expressing the ratio 'winning ways to losing ways' as **on**. For example, 4 : 6 is written '6 : 4 on'.

Odds
4 : 6
4 losing ways : 6 winning ways
10 outcomes in total
So, the probability of winning is P(winning) = $\frac{6}{10}$

Odds	Probability of winning
1:1 (evens)	$\frac{1}{2}$
4:1	$\frac{1}{5}$
6:4	$\frac{4}{10}$
4:6 (6:4 on)	$\frac{6}{10}$

In the betting arena, payouts on wins are calculated according to the following formula.

$$\text{Winning payout} = \frac{\text{chances of losing}}{\text{chances of winning}} \cdot \text{bet} + \text{bet returned}$$

WORKED Example 12

A gambler bets $20 on a horse at 5 to 1. If the horse wins:

a what amount does this bet win?

b what return does the gambler receive?

THINK

a Odds are 5:1, so any winning bet will win 5 times the bet.

WRITE

a $\text{Amount won} = \frac{\text{chances of losing}}{\text{chances of winning}} \cdot \text{bet}$

$= \frac{5}{1} \cdot \$20$

$= \$100$

THINK

b The bet is also returned on a win.

WRITE

b Return to gambler = win + bet

= \$100 + \$20

= \$120

It is sometimes necesssary to convert an expression in terms of *odds* to one in terms of *probability* and vice versa. The following worked example demonstrates this technique.

WORKED Example 13

Convert each of the following:

a odds of 3 to 1 on, to a probability

b a probability of 0.16 to fair odds.

THINK	WRITE
a 1 Translate what '3 to 1 on' means as a ratio; '3 to 1 on' means 1 to 3.	**a** '3 to 1 on' means 1:3.
2 This is loss:win.	This represents 1 chance of losing to 3 of winning.
3 Find the total number of chances.	So out of the 4 chances there are 3 of winning.
4 Calculate P(winning).	P(winning) = $\frac{3}{4}$
b 1 Probability of 0.16 means there is 0.16 chance of winning in 1 trial.	**b** P(winning) = 0.16 $= \frac{0.16}{1}$
2 Convert this to a ratio with denominator 100.	$= \frac{16}{100}$
3 Calculate the chances of losing.	So out of every 100 trials, there are 16 chances of winning. This means there are 84 chances of losing.
4 Represent the odds as losing to winning.	So odds = losing chances:winning chances = 84:16
5 Simplify this ratio.	= 21:4

remember

1. The odds represent the ratio of the number of ways of losing to the number of ways of winning.
2. Odds for events with a probability greater than $\frac{1}{2}$ are given by expressing 'winning ways : losing ways' as 'on'.
3. The winning payout is calculated as follows.

$$\text{Winning payout} = \frac{\text{chances of losing}}{\text{chances of winning}} \cdot \text{bet} + \text{bet returned}$$

EXERCISE 11E Odds

WORKED Example 12a

1 What amount would a punter expect to win on the following wagers?
a $35 at 3 to 1
b $70 at 6 to 4
c $78 at 11 to 2
d $120 at 5 to 2
e $45 at 3 to 1 on
f $50 at 6 to 4 on
g $150 at 9 to 4 on

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Digital doc: Spreadsheet 073 Converting odds

WORKED Example 12b

2 What return could the punter expect on each of the wagers in question 1?

WORKED Example 13a

3 Convert each of the following odds to a probability.
a 4 to 1 on
b 3 to 1
c 3 to 2 on
d 5 to 2
e 7 to 3
f 2 to 1 on

WORKED Example 13b

4 What fair odds are equivalent to these probabilities?
a $\frac{1}{3}$
b $\frac{1}{5}$
c $\frac{3}{5}$
d $\frac{2}{7}$
e $\frac{5}{12}$
f $\frac{2}{3}$
g 0.4
h 0.45

5 What fair odds should be offered on the following events?
a Roll an even number with one die
b Roll a score of 7 with a pair of dice
c Draw a heart from a pack of 52 cards
d Draw an ace from a pack of 52 cards
e Toss 2 Heads with 2 coins

6 Calculate the odds obtained by a person who bet $50 and who collected:
a $150
b $200
c $100
d $120
e $75
f $70.

7 Many people have a bet on one particular horse, Slipper, with a bookmaker, Tom. Tom could lose a lot of money if Slipper wins. In this situation, Tom may ‘lay off’ some of these bets. This means that Tom bets on this same horse with another bookmaker. Suppose that Tom accepts bets totalling $7200 at 5 to 1, on Slipper to win.
a What could he lose if Slipper wins?
b If Tom takes $3000 of this money and bets on Slipper with another bookmaker at 9 to 2:
i what amount does Tom win if Slipper wins?
ii what are his net losses on the race if the horse wins?

8 The odds offered by a bookmaker are not static, but fluctuate with the amount of money being wagered on various horses in the field. If large amounts are bet on a particular horse its odds will ‘shorten’ while other odds may ‘lengthen’. For this reason many punters shop around for the best odds.

How much extra is won if $320 is invested on a winner at:
a 5 to 2 rather than 2 to 1?
b 6 to 4 on rather than 9 to 4 on?
c 11 to 4 rather than 5 to 2?

Two-up

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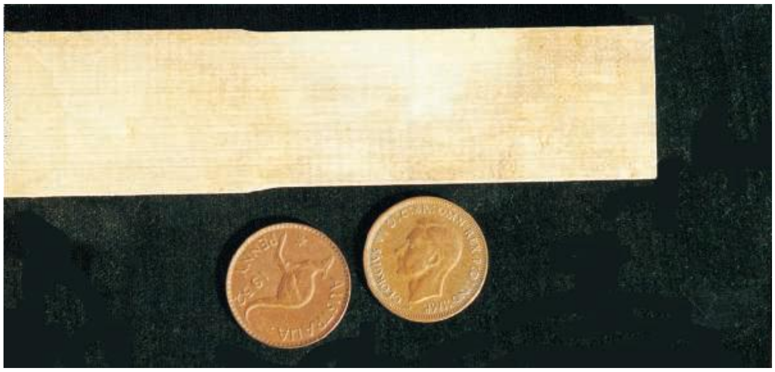
There are many variations of **two-up** played around Australia. The simplest of these involves tossing 2 coins and betting on odds (a Head and a Tail) or evens (2 Heads or 2 Tails).



A version commonly played in casinos has the following rules of operation:

A person, called 'the spinner', tosses 2 coins.

Players can bet on either HH or TT. The odds for these bets are even money.

If the spinner throws HT, he or she continues until a HH or a TT is thrown. A game is finished when this occurs.

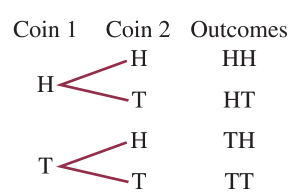
However, if the spinner throws HT 5 times in succession, all bets lose and the game is finished. Thus a game must finish on or before the fifth toss of the coins.

WORKED Example 14

What is the probability of the spinner tossing a HT then a TT?

THINK | **WRITE/DRAW**

1 Draw a tree diagram showing the outcomes of tossing two coins.



2 Calculate the probability of a HT and TT.

From the tree diagram it can be seen that

$$P(HT) = \frac{2}{4} = \frac{1}{2}$$

$$P(TT) = \frac{1}{4}$$

3 Find the probability of one outcome followed by the other.

Since the two tosses are independent of each other

$$P(\text{HT then TT}) = P(HT) \cdot P(TT) = \frac{2}{4} \cdot \frac{1}{4} = \frac{1}{8}$$

remember

1. The game of two-up involves tossing two coins. The rules of probability apply to the outcomes.
2. Players can bet on either HH or TT. If a HT is thrown, tosses continue until a HH or a TT results.
3. If 5 successive tosses result in HT, all bets lose and the game is finished.

EXERCISE 11F Two-up

Answer the following questions for the game of two-up.

WORKED Example 14

1 What is the probability of tossing HH?

2 What is the probability of tossing TT?

3 What is the probability of tossing TH or HT?

4 What is the probability that the game finishes on the first toss?

5 The game can finish on the second toss through the following sequence.

Throw 1	Throw 2
HT	HH

What is the probability of this outcome?

6 The game can finish on the third toss through the following sequence.

Throw 1	Throw 2	Throw 3
HT	HT	HH

What is the probability of this outcome?

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069 Simulating coin tosses — DIY
067 Coin toss lister

7 If the game is undecided after the 4th toss:
- **a** what sequence has occurred?
- **b** what is the probability of this outcome?

8 What is the probability that the bank will take all bets?

9 If one bets on TT, what is the probability of:
- **a** winning on the 2nd toss?
- **b** winning on the 3rd toss?
- **c** winning on the 4th toss?
- **d** winning on the 5th toss?
- **e** winning overall?

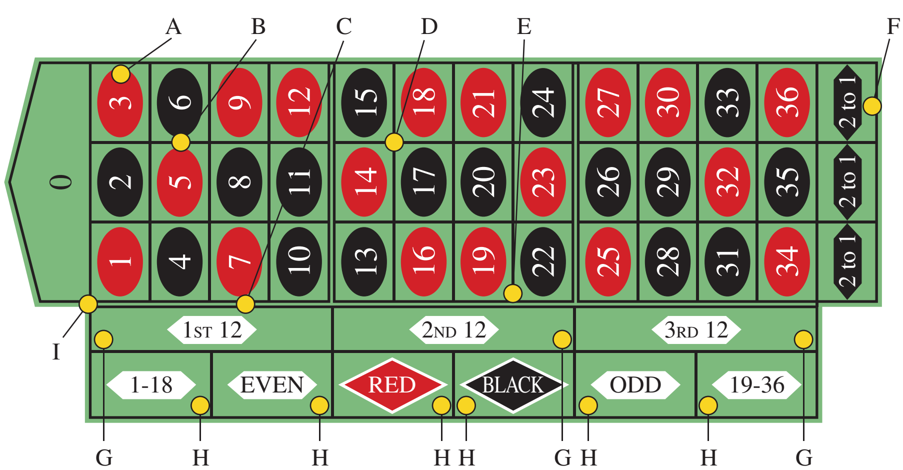
10 The odds offered for betting on TT are even money. Are these odds fair?

Roulette

Roulette is a game of chance in which a ball is placed into a spinning wheel that has numbered slots. Gamblers eagerly await the final resting place of the ball to see whether they have won or lost. There are 37 slots on the roulette wheel numbered from 0 to 36, and by placing your chips strategically on the table there are many ways to bet on the outcome. If the ball lands on 0, the bank takes all money (except for those bets that are 'straight up' on 0 — see below).

The diagram below shows where to place your bet and at what odds you can win.

A A 'straight up' on any single number (including 0). Odds 36 to 1.
B A 'split' covers any one of two numbers. Odds 17 to 1.
C A 'street' covers any one of three numbers. Odds 11 to 1.
D A 'corner' covers any one of four numbers. Odds 8 to 1.
E A 'six line' covers any one of 6 numbers. Odds 5 to 1.
F A 'column' covers any of the 3 vertical columns. Odds 2 to 1.
G A 'dozen' covers any of the series of twelve. Odds 2 to 1.
H The 'even chances'. Odds even money.
I Cover 0, 1, 2 and 3. Odds 8 to 1.



remember

1. In the game of roulette, a ball is placed into a wheel with 37 numbered slots (0 to 36).
2. Many betting options are available to the gambler.

EXERCISE 11G Roulette

1 What is the probability that the ball will land in:
  a an even number?
  b a 'six line'?

2 a, b Calculate the fair odds for the events in question 1.
  c Are the odds offered by the casino 'fair' in the mathematical sense? In a short paragraph write a justification of the fairness of these odds from the casino's point of view.

3 What amount would a punter win or lose (in total) on each of these rolls of the wheel if the following wagers were made?
  a $20 on red and $10 on the 20-21-23-24 corner. The winner was black 24.
  b $10 on 12 and $20 on odd and $10 on black. The winner was red 25.
  c $20 on 1st 12 and $20 on the 13-14-15 street. The winner was black 15.
  d $50 on red and $50 on even and $5 on 0. The winner was 0.

4 If a roulette player bets $20 on the black and $20 on the red, what is going to happen most of the time? What is the problem with this strategy?

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Digital docs: **Spreadsheet** 203 Simulating random numbers

This system can be applied to many different forms of gambling. To illustrate it in a simple context we choose roulette. The probability that an odd number comes up is $\frac{18}{37}$. It pays odds of 'even money', or 1 to 1. The system operates in this fashion.

Bet $5 on an odd number.
  If it wins, take the $5 and leave — Result: Win $5
  If it loses, then:
bet $10 on an odd number.
  If it wins, take the $10 and leave — Result: Win $5
  If it loses, then:
bet $20 on an odd number.
  If it wins, take the $20 and leave — Result: Win $5
  If it loses, then:
bet $40 on an odd number.
  If it wins, take the $40 and leave — Result: Win $5
  If it loses, then:
bet $80 on an odd number.
  . . . and so on.

In theory, an odd number will come up sooner or later and when it does you will win $5. Thus in theory this system can never lose.

If you have a roulette wheel play this system and see if it works. If you don't have a roulette wheel you can devise a system using the random number generator on a calculator or spreadsheet to model the situation. Alternatively, you could use a pack of cards.

The *fundamental* question is to determine the flaw in this method. If it works all the time wouldn't the casinos be bankrupt?

Common fallacies in probability

A misunderstanding of the nature of **independence** leads to numerous fallacies in probability. If a coin has just come up Heads 5 times in a row people feel strongly that in the next throw it will come up Tails; or if a roulette wheel has landed on 20 then it has a smaller chance of landing on 20 the next time around. Clearly, however, each of these events has just as good a chance of occurring as any other. The coin and the roulette wheel have no capacity for 'remembering' what happened last time and so operate independently of previous outcomes.

If two events A and B are independent, then their probabilities are multiplied.

$$P(A \text{ and } B) = P(A) \cdot P(B)$$

WORKED Example 15

The probability that a person has black hair is $\frac{1}{4}$.

The probability that a person has a black moustache is $\frac{1}{4}$.

What is the probability that a person has black hair and a black moustache?

THINK	WRITE
1 These are independent events so multiply the probabilities.	P(black hair and black moustache) $= P(\text{black hair}) \cdot P(\text{black moustache})$ $= \frac{1}{4} \cdot \frac{1}{4}$ $= \frac{1}{16}$
2 Interpret your answer. Are these events really independent?	These two events are biologically linked so cannot be multiplied. The events are not independent so this answer is not correct.

remember

1. When two or more events are independent, the outcome of each event has no effect on the outcome of the others.
2. On each toss of a fair coin, a Head has the same chance of occurring as a Tail. If a Tail has resulted each time in four tosses of the coin, the chance of a Tail occurring on the fifth toss is still fifty-fifty.
3. A common fallacy in games of chance is that if one particular outcome has occurred repeatedly in a number of trials, then it is less likely to occur in the next trial.

EXERCISE 11H Common fallacies in probability

WORKED Example 15

1 A coin is tossed. What is the probability of:
  a getting 4 Heads in a row?
  b getting 5 Heads in a row?
  c getting 1 more Head if you have just thrown 4 in a row?

2 a Juanita is at the State tennis championships and she estimates that she has a 0.75 chance of winning each match that she has to play. What is the probability that she wins:
    i 4 matches in a row?
    ii 5 matches in a row?
  b She wins her first 4 matches and her coach says to her, 'You can't keep winning like this. The chance of winning 5 in a row is 0.24, so your chances of winning the 5th match are not good.' How should Juanita reply to this lack of confidence?

3 'Lightning never strikes the same place twice.' This old saying is yet another example of misunderstanding independence in probability. Can you think of any other examples?

4 Shane was attempting a question in probability. The question was:

> In Runaway Bay the population is 18 000 and of these people, 3200 are aged 12 to 18 years. In the town there are 1900 people who own a surfboard. What is the probability that a person selected at random in Runaway Bay is aged 12 to 18 and owns a surfboard?

Shane's solution was:

$$P(12 \le \text{age} \le 18) = \frac{3200}{18\,000} = 0.18$$

$$P(\text{own surfboard}) = \frac{1900}{18\,000} = 0.11$$

Thus

$$P(\text{own surfboard and } 12 \le \text{age} \le 18) = 0.18 \cdot 0.11 = 0.02$$

  a What is the error in Shane's thinking?
  b What extra information would you need before the problem can be solved?

Mathematical expectation

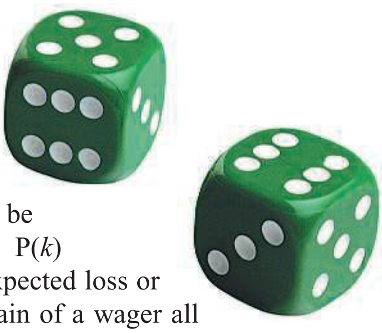
A die is rolled for a large number of times and the number on the uppermost face is noted. What value could be expected for the average of these numbers? A little common sense would suggest that the average would be $(1 + 2 + 3 + 4 + 5 + 6) \mid 6 = 3.5$. Now suppose the die was biased as shown in the table at right.

Event (X)	P(X)
1	0.1
2	0.1
3	0.2
4	0.2
5	0.2
6	0.2

If this die were to be rolled a large number of times, what average could be expected?

In this case the **expected value** would be

$$1 \cdot 0.1 + 2 \cdot 0.1 + 3 \cdot 0.2 + 4 \cdot 0.2 + 5 \cdot 0.2 + 6 \cdot 0.2 = 3.6$$



In general, if an experiment has outcomes $a, b, c, \ldots k$ then the average of the outcomes is expected to be

$$\text{expected value} = a \cdot P(a) + b \cdot P(b) + \ldots + k \cdot P(k)$$

A variation on the idea of expected value is the expected loss or gain of a wager. To calculate the expected loss or gain of a wager all possible outcomes are listed. A loss is counted as a negative gain as seen in the following example.

WORKED Example 16

A lottery sells 1200 tickets at $5 each and offers prize money of $4500. What is the expected gain or loss by a person who buys one ticket?

THINK

1. Calculate the gain · P(gain) for a win.
2. Calculate the gain · P(gain) for a loss. *Note:* This will be a negative value.

WRITE

Outcome	Gain	Probability	Gain · P
Ticket wins	$4500	$\frac{1}{1200}$	$\frac{4500}{1200}$
Ticket loses	–$5	$\frac{1199}{1200}$	$-5 \cdot \frac{1199}{1200}$

3. Add these two to give overall gain — a negative sign is seen as a loss.

$$\text{Expected gain} = \frac{4500}{1200} + -5 \cdot \frac{1199}{1200}$$
$$= -\frac{1495}{1200}$$
$$= -1.25$$

A loss of $1.25.

remember

If an event has numerical outcomes a, b, c, k, then the expected outcome, or the 'average' outcome, for this event will be:

$$a \cdot P(a) + b \cdot P(b) + c \cdot P(c) + \quad + k \cdot P(k).$$

EXERCISE 11I Mathematical expectation

1 A die is biased as shown in the table at right. What average would you expect for a large number of rolls of this die?

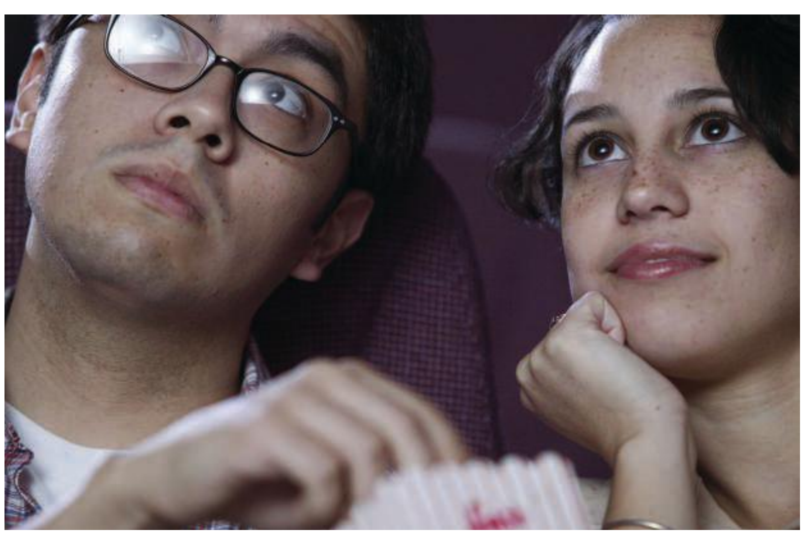
Number	Probability
1	0.2
2	0.1
3	0.1
4	0.2
5	0.3
6	0.1

2 A die is biased as shown in the table at right. What is the expected value for a roll of this die?

Number	Probability
1	0.2
2	0.2
3	0.2
4	0.2
5	0.1
6	0.1

3 A group of people attended a showing of Rocky 25. The distribution of their ages is shown in the table at right.
If a person is selected at random from this group, what is the expected value of his or her age?

Age	Proportion (%)
14	18
15	26
16	40
17	11
18	5



WORKED Example 16

4 A lottery sells 5000 tickets at $2 each. If a first prize of $8000 is offered, what is the expected loss or gain for buying this ticket?

5 The following bet is suggested to you:
Roll the die (6-sided) and if it:
1. shows an even number, you get $10
2. shows a 5, you pay $30
3. shows a 3, you pay $15
4. shows a 1, nothing happens.
What is the expected loss or gain for this wager?

6 In my pocket I have 5 coins: a $2 coin, a 50c coin and three 20c coins. If I take one coin from my pocket, what is the mathematical expectation of a random selection?

7 On a roulette wheel there are 37 numbers: 0 to 36. If a 0 turns up you lose. If you bet $5 on the odd numbers you receive $10. What is the expected return on the $5 bet?

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Digital doc: WorkSHEET 11.2

8 If you are given one of 250 tickets in a raffle which has a prize of $400, what is the value of this ticket?

9 What is the expected value for the sum of the uppermost faces when a pair of dice are rolled?

10 QUICK QUESTIONS 2

1 Is it 'very probable' or 'almost certain' that a member of a population will lie within a z-score range of -2 to $+2$?

2 In a test where the mean was 62% and the standard deviation 12%, what percentage of the candidates scored above 86%?

3 A machine fills 1-litre drink bottles. The standard deviation of the machine is 10 mL. What is the least volume acceptable in the bottle?

4 Use the standard normal tables (on page 560) to determine $P(z < 1.65)$.

5 Hence determine $P(z > 1.65)$.

6 What is $P(z < -1.65)$?

7 Calculate $P(-1.65 < z < 1.65)$.

8 If the odds are 3:2, what is the probability of winning?

9 If I placed a $100 bet on a horse at 4:1, how much would I receive if the horse wins?

10 At a school fete there are 1000 $2-tickets in a raffle with a prize money of $1000. What is the expected gain or loss by a person who buys one ticket?

investigation

Keno

Keno is a popular game in the large clubs around Australia. In one version of the game, each round a machine randomly generates 15 numbers from 1 to 50. In one entry you can select 1 or 2 or up to 15 numbers. The return on a bet is given in the table below.

Keno prizes — example

Numbers selected	Numbers matched		$1 bet	$2 bet	$5 bet
1	MATCH	1	3	6	15
2	MATCH	2	12	24	60
3	MATCH	2	1	2	5
		3	43	86	215
4	MATCH	2	1	2	5
		3	4	8	20
		4	112	224	560
5	MATCH	3	2	4	10
		4	14	28	70
		5	610	1 220	3 050
6	MATCH	3	1	2	5
		4	5	10	25
		5	85	170	425
		6	1 500	3 000	7 500
7	MATCH	3	1	2	5
		4	2	4	10
		5	14	28	70
		6	147	294	735
		7	JACKPOT[1]	JACKPOT[1]	JACKPOT[1]
8	MATCH	4	2	4	10
		5	7	14	35
		6	50	100	250
		7	835	1 670	4 175
		8	JACKPOT[2]	JACKPOT[2]	JACKPOT[2]
9	MATCH	4	1	2	5
		5	5	10	25
		6	30	60	150
		7	220	440	1 100
		8	3 500	7 000	17 500
		9	50 000	100 000	250 000
10	MATCH	5	2	4	10
		6	10	20	50
		7	80	160	400
		8	820	1 640	4 100
		9	10 000	20 000	50 000
		10	JACKPOT[3]	JACKPOT[3]	JACKPOT[3]
11	MATCH	0	3	6	15
		1	1	2	5
		5	1	2	5
		6	5	10	25
		7	35	70	175
		8	220	440	1 100
		9	2 500	5 000	12 500
		10	22 000	44 000	110 000
		11	130 000	250 000	250 000
12	MATCH	0	4	8	20
		1	1	2	5
		5	1	2	5
		6	4	8	20
		7	15	30	75
		8	80	160	400
		9	600	1 200	3 000
		10	7 600	15 200	38 000
		11	56 000	112 000	250 000
		12	160 000	250 000	250 000
13	MATCH	0	5	10	25
		1	1	2	5
		5	1	2	5
		6	2	4	10
		7	8	16	40
		8	45	90	225
		9	350	700	1 750
		10	2 000	4 000	10 000
		11	9 000	18 000	45 000
		12	80 000	160 000	250 000
		13	190 000	250 000	250 000
14	MATCH	0	7	14	35
		1	1	2	5
		6	1	2	5
		7	7	14	35
		8	35	70	175
		9	220	440	1 100
		10	1 000	2 000	5 000
		11	8 500	17 000	42 500
		12	25 000	50 000	125 000
		13	100 000	200 000	250 000
		14	225 000	250 000	250 000
15	MATCH	0	15	30	75
		1	2	4	10
		6	1	2	5
		7	5	10	25
		8	15	30	75
		9	50	100	250
		10	330	660	1 650
		11	2 600	5 200	13 000
		12	20 000	40 000	100 000
		13	60 000	120 000	250 000
		14	110 000	220 000	250 000
		15	250 000	250 000	250 000

1. Minimum jackpot: $5000 2. Minimum jackpot: $20 000 3. Minimum jackpot: $1 000 000

Analyse the returns given for each of the bets. How are the odds calculated? Are they fair? If you select 15 numbers, the payout for getting 0 right is larger than the payout for getting 1 right. Why is this?

Rock, paper, scissors


Scissors


Paper

Rock

The game *Rock, paper, scissors* is played all over the world, not just for fun but also as a way of settling a disagreement.

The game uses the three different hand signs shown left and right.

Simultaneously, two players 'pound' the fist of one hand into the air three times. On the third time each player displays one of the hand signs. Possible results are shown below.

Paper covers rock

Paper wins

Rock breaks scissors

Rock wins

Scissors cut paper

Scissors win

1 Play 20 rounds of *Rock, paper, scissors* with a partner. After each round, record each player's choice and the result in a table like the one shown below. (Use R for rock, P for paper and S for scissors.)

Round number	Player 1	Player 2	Result
1	P	R	Player 1 wins
2	S	R	Player 2 wins
3	S	S	Tie

2 Based on the results of your 20 rounds, what is the experimental probability of:
  a you winning? b your partner winning? c a tie?

3 Do you think playing *Rock, paper, scissors* is a fair way to settle a disagreement? Explain.

Two person, zero sum is an adaptation of this game with only two optional hand shapes; Paper, P, or Scissors, S. Through the toss of a coin, players decide who will be A and who will be B.

Players win or lose according to these rules:

Player A makes scissors and player B makes paper: Player A loses $5 and player B gains $5

Player A makes paper and player B makes scissors: Player A loses $3 and player B gains $3

Both players make paper: Player A gains $5 and player B loses $5

Both players make scissors: Player A gains $3 and player B loses $3.

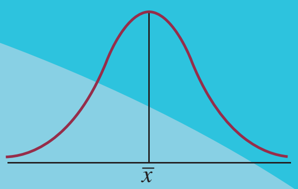
Players start with $100 and the winner of this game is the leader after 30 rounds.

4 Record the results after 30 rounds in a table.

5 What would happen if this game is complicated by introducing some predictability into one of the contestant's actions?
Suppose player A chooses between paper and scissors randomly but with a 0.6 bias in favour of paper. Now player A uses the random number generator on a calculator to choose between scissors and paper. If the first digit of the random number is 0 to 5, choose paper; otherwise choose scissors.

6 Is there a strategy which can now be adopted to maximise the chances of player B winning?

summary



z-scores

- A data set is normally distributed if it is symmetrical about the mean.
- A z-score measures the position of a score relative to the mean and standard deviation.
- A z-score is found using the formula $z = \frac{x - \bar{x}}{s}$, where x = the score, $\bar{x}$ = the mean and s = the standard deviation.

Comparison of scores

- Standardising both scores best compares scores from different data sets.
- When comparing exam marks, the highest z-score is the best result.

Distribution of scores

- A data set that is normally distributed will be symmetrical about the mean.
- 68% of scores will have a z-score of between −1 and 1.
- 95% of scores will have a z-score between −2 and 2. A score chosen from this data set will very probably lie in this range.
- 99.7% of scores will have a z-score of between −3 and 3. A score chosen from the data set will almost certainly lie within this range.

Normal distribution

- The normal distribution is used to describe quantities such as test scores, physical characteristics such as height and the distribution of errors.
- The standard normal curve has a mean of 0 and a standard deviation of 1.
- The normal variable x can be scaled to a z-score on the standard normal curve by the formula

$$z = \frac{x - \bar{x}}{s}.$$

Odds

- If an outcome has p ways of success and q ways of failure, then the odds against the event occurring are q to p. If an event has a probability of $\frac{1}{6}$ of occurring, then the fair odds offered for this event are 5 to 1.

Casino games

- The techniques in probability that have been developed can be used to analyse a number of casino games.
- *Two-up* is played by tossing 2 coins and noting the results. If the spinner tosses HH 3 times in a row before tossing TT, he or she wins at odds of 7.5 to 1. All players lose if the spinner tosses TH 5 times in a row.
- *Roulette* offers the gambler a large variety of betting opportunities as the ball rolls around the 37 black and red numbers.
- *Keno* is a popular game in the large clubs around Australia. In each round a machine randomly generates 15 numbers from 1 to 50. In one entry you can select 1 or 2 or up to 15 numbers.

Mathematical expectation

- If an event has numerical outcomes, $a, b, c, \ldots k$, then the expected outcome, or the 'average' outcome, for this event will be:

$$a \cdot \text{P}(a) + b \cdot \text{P}(b) + c \cdot \text{P}(c) + \ldots + k \cdot \text{P}(k).$$

CHAPTER review

1 Measurements of the amount of acid in a certain chemical are made. The results are normally distributed such that the mean is 6.25% and the standard deviation is 0.25%. Harlan gets a reading of 5.75%. What is Harlan's reading as a z-score? **11A**

2 A set of scores is normally distributed such that the mean is 15.3 and the standard deviation 5.2. Convert each of the following members of the distribution to z-scores. **11A**
a 15.3 **b** 20.5 **c** 4.9 **d** 30.9 **e** 10.1

3 On an exam the results are normally distributed with a mean of 58 and a standard deviation of 7.5. Jennifer scored a mark of 72 on the exam. Convert Jennifer's mark to a z-score, giving your answer correct to 2 decimal places. **11A**

4 A set of scores is normally distributed with a mean of 2.8 and a standard deviation of 0.6. Convert each of the following members of the data set to z-scores, correct to 2 decimal places. **11A**
a 2.9 **b** 3.9 **c** 1 **d** 1.75 **e** 1.6

5 The table at right shows the length of time for which a sample of 100 light bulbs will burn. **11A**

Length of time (hours)	Class centre	Frequency
0–<500		3
500–<1000		28
1000–<1500		59
1500–<2000		10

a Find the mean and standard deviation for the data set.
b A further sample of five light bulbs are chosen. The length of time for which each light bulb burnt is given below. Convert each of the following to a standardised score.
i 1000 hours **ii** 1814 hours **iii** 256 hours **iv** 751 hours **v** 2156 hours

6 Anji conducts a survey on the water temperature at her local beach each day for a month. The results (in °C) are shown below. **11A**

20	21	19	22	21	18	17
23	17	16	22	20	20	20
21	20	21	18	22	17	16
20	20	22	19	21	22	23
24	20					

a Find the mean and standard deviation of the scores.
b Find the highest and lowest temperatures in the data set and express each as a z-score.

11B
7 Betty sat examinations in both Physics and Chemistry. In Physics the examination results showed a mean of 48 and a standard deviation of 12, while in Chemistry the mean was 62 with a standard deviation of 9.
 a Betty scored 66 in Physics. Convert this result to a z-score.
 b Betty scored 71 in Chemistry. Convert this result to a z-score.
 c In which subject did Betty achieve the better result? Explain your answer.

11B
8 In Geography, Carlos scored a mark of 56 while in Business Studies he scored 58. In Geography $\bar{x} = 64$ with a standard deviation of 10. For Business Studies $\bar{x} = 66$ with a standard deviation of 15.
 a Convert each mark to a standardised score.
 b In which subject did Carlos achieve the better result?

11B
9 A psychologist records the number of errors made on a series of tests. On a literacy test the mean number of errors is 15.2 and the standard deviation is 4.3. On the numeracy test the mean number of errors is 11.7 with a standard deviation of 3.1. Barry does both tests and makes 11 errors on the literacy test and 8 errors on the numeracy test. In which test did Barry do better? Explain your answer.

11C
10 A data set is normally distributed with a mean of 40 and a standard deviation of 8. What percentage of scores will lie in the range:
 a 32 to 48? b 24 to 56? c 16 to 64?

11C
11 The value of sales made on weekdays at a take-away store appears to be normally distributed with a mean of \$1560 and a standard deviation of \$115. On what percentage of days will the days sales lie between:
 a \$1445 and \$1675? b \$1330 and \$1790? c \$1215 and \$1905?

11C
12 A data set is normally distributed with a mean of 56 and a standard deviation of 8. What percentage of scores will:
 a lie between 56 and 64? b lie between 40 and 56? c be less than 40?
 d be greater than 80? e lie between 40 and 80?

11C
13 A machine is set to produce bolts with a mean diameter of 5 mm with a standard deviation of 0.1 mm. A bolt is chosen and it is found to have a diameter of 4.5 mm. What conclusion can be drawn about the settings of the machine?

11D
14 Use the normal tables on page 560 to find the value of:
 a $P(z < 1.3)$ b $P(z < 2.4)$ c $P(z > 1)$
 d $P(z < -1.5)$ e $P(0.6 < z < 1.5)$.

11D
15 If a normal distribution has a mean of 45 and a standard deviation of 6, find z-values for the following scores.
 a $x = 45$ b $x = 51$ c $x = 40$ d $x = 77$

11D
16 If a normal distribution has a mean of 25 and a standard deviation of 3, calculate:
 a $P(x < 25)$ b $P(x < 28)$ c $P(x < 22)$
 d $P(x < 20)$ e $P(24 < x < 25)$ f $P(23 < x < 26)$.

11D
17 A machine manufactures components with a mean mass of 215 grams and a standard deviation of 8 grams. If we assume that the variation in the mass of the components is normally distributed, calculate the probability that a component will have a mass:
 a less than 215 grams b less than 223 grams
 c more than 210 grams d between 220 and 230 grams.

11E

18 A bookmaker takes the following bets on two separate races. Which horses, if they win, will result in a loss for the bookmaker?

a

Horse	Odds	Bets
1	8 to 1	$280
2	14 to 1	$175
3	6 to 4 on	$1250
4	evens	$870
5	8 to 1	$420
6	25 to 1	$250

b

Horse	Odds	Bets
1	15 to 1	$780
2	40 to 1	$275
3	6 to 4 on	$4250
4	evens	$670
5	15 to 1	$820
6	2 to 1	$2250

11F

19 The casino offers odds of 7.5 to 1 if the spinner can toss HH 3 times in a row before tossing TT or TH 5 times in a row. Are these fair odds? Answer this question by simulation; that is, toss 2 coins many times and record the number of wins and losses.

11G

20 Which of the following is better in a roulette game?

a Place $5 on each even number (except 0) or place $90 on evens.

b Place $5 on each of the numbers 20, 21, 23 and 24 or place $20 on the corner 20-21-23-24.

11H

21 At Rockaway College there are 435 girls and 450 boys. The school's policy is that only girls can play netball and in fact 221 girls play netball.

a What is the probability that a student selected at random:

- **i** is a girl?
- **ii** plays netball?
- **iii** is a girl who plays netball?

b Are the events P(girl) and P(plays netball) independent?

11I

22 A game is played where you win $10 for rolling a double using 2 dice.

a What is the expected value of your winnings?

b How much would you expect to pay, per throw, to play this game, if the operator of the game wanted to make a 20% profit (calculated on total bets placed) in the long run?

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11A z-scores

Digital docs

- Spreadsheet 081: Investigate finding the mode *(page 540)*
- Spreadsheet 082: Investigate finding the mode — DIY *(page 540)*
- Spreadsheet 079: Investigate finding the median *(page 540)*
- Spreadsheet 080: Investigate finding the median — DIY *(page 540)*
- Spreadsheet 072: Investigate finding bar graphs — DIY *(page 540)*
- Spreadsheet 074: Investigate histograms and frequency polygons *(page 541)*
- Spreadsheet 075: Investigate histograms and frequency polygons — DIY *(page 541)*
- Spreadsheet 007: Investigate calculating the mean from a frequency table *(page 546)*
- Spreadsheet 078: Investigate calculating the mean from a frequency table — DIY *(page 546)*

11C Distribution of scores

Digital doc

- WorkSHEET 11.1: Use the 68%, 95% and 99.7% rules to make calculation for normal distributions *(page 558)*

Interactivity

- Normal distributions int-0182: Consolidate your understanding of normal distributions *(page 552)*

11D Standard normal tables

Digital doc

- SkillSHEET 11.1: Practise using the cumulative normal distribution table *(page 565)*

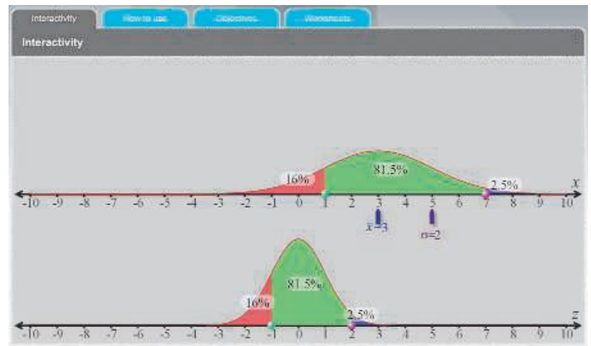
Tutorial

- WE9 Int-0429: Watch how to determine probabilities for a normal distribution *(page 561)*

11E Odds

Digital doc

- Spreadsheet 073: Investigate converting odds *(page 570)*



11F Two-up

Digital docs

- Spreadsheet 069: Investigate simulating coin tosses — DIY *(page 572)*
- Spreadsheet 067: Investigate the coin toss lister *(page 572)*

Interactivity

- Random numbers int-0089: Consolidate your understanding of random number generation *(page 571)*

11G Roulette

Digital doc

- Spreadsheet 203: Investigate simulating random numbers *(page 574)*

11I Mathematical expectation

Digital doc

- WorkSHEET 11.2: Calculate probabilities for normal distributions, expected values and game probabilities *(page 579)*

Chapter review

Digital doc

- Test Yourself: Take the end-of-chapter test to test your progress. *(page 585)*

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Appendix

Instructions for the TI-89 Titanium graphics calculator

Chapter 7 — Linear programming

Chapter 10 — Probability and the binomial distribution

Chapter 11 — The normal distribution and games of chance

Note: This set of graphics calculator tips for the TI-89 Titanium uses the CellSheet application. If this application has not already been downloaded to your calculator, use the **TI-89 Titanium** weblink in your eBookPLUS and follow the instructions.

Chapter 1 page 6

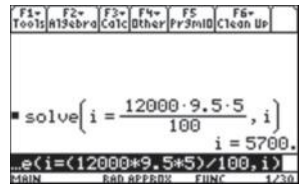
Graphics Calculator tip!

Calculating simple interest and amount

A graphics calculator can be used to solve Worked example 1.

1. Press:
   - HOME
   - CATALOG.

   Scroll down to choose the solve command and complete the entry line as:



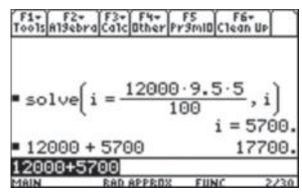
   $$\text{solve}\left(i = \frac{12\,000 \times 9.5 \times 5}{100}, i\right).$$

   Then press ENTER.

2. $A = P + I$

   $= 12\,000 + 5700$

   $= \$17\,700$



   Verify the answers in part **b** of Worked example 1 using your calculator.

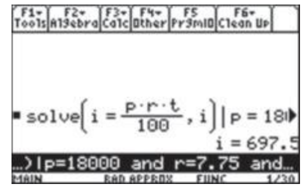
Chapter 1 page 8

Graphics Calculator tip! Simple interest

The following instructions can be used for Worked example 2.

1. Press:
   - HOME
   - CATALOG.

   Scroll down to select the solve command. Complete the entry line as:

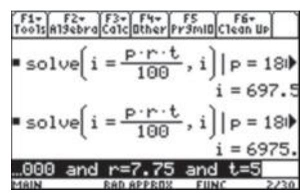
   $$\text{solve}\left(i = \frac{p \times r \times t}{100}, i\right) \mid p = 18\ 000 \text{ and } r = 7.75$$

   and t = 0.5.

   Then press ENTER.

   Note: The word ‘and’ can be entered by using ALPHA and the letter keys or selecting ‘and’ from the CATALOG menu.

2. To obtain the answer to part **b**, use the arrow keys to highlight the previous equation and paste it into the entry line. Use the arrow keys to edit the equation as shown.

   $$\text{solve}\left(i = \frac{p \times r \times t}{100}, i\right) \mid p = 18\ 000 \text{ and } r = 7.75$$

   and t = 5

   Then press ENTER.

Chapter 1 page 13

WORKED Example 4

THINK

1. Press:
   - HOME
   - CATALOG.

   Then scroll down to choose the solve command.
   Complete the entry line as:

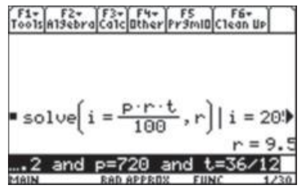
   $$\text{solve}\left(i = \frac{p \times r \times t}{100}, r\right) \mid i = 205.2 \text{ and}$$

   $$p = 720 \text{ and } t = \frac{36}{12}.$$

   Then press ENTER.

2. Write your answer.

WRITE/DISPLAY



Solve $I = \frac{P \times R \times T}{100}$, for R, given $I = 205.20$,

$P = 720$ and $T = \frac{36}{12} = 3$.

The interest rate offered was 9.5% per annum.

Chapter 1 page 14

WORKED Example 5

THINK

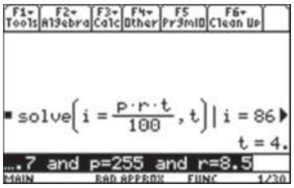
1. Press:
   - HOME
   - CATALOG.

   Then scroll down to choose the solve command.
   Complete the entry line as:

   $$\text{solve}\left(i = \frac{p \times r \times t}{100}, t\right) \mid i = 86.7 \text{ and}$$

   p = 255 and r = 8.5.
   Then press ENTER.

WRITE



2. Write your answer.

Solve $I = \frac{P \times R \times T}{100}$, for T,
given $I = 205.20$, $P = 720$ and $R = 8.5$.
The period of the investment was 4 years.

Chapter 1 page 19

WORKED Example 8

Use a graphics calculator to answer the question in Worked example 7.

THINK | **WRITE/DISPLAY**

a 1 To enter the data in a CellSheet, press (APPS).
Select CellSheet and press:
- 3: (New)
- (ENTER).

Enter the values 1 to 5 in column A. These values represent years.

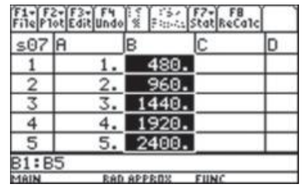
In column B move the cursor to cell B1. Press:
- (F3) (Edit)
- 3: (Fill Range).

Complete the table as:

Initial formula: = 12 000 × 4 × a1 ÷ 100

Range: B1: B5.
Then press (ENTER).

a



2 In column C the interest rate becomes 5%.
Repeat the above steps completing the table as:

Initial formula: = 12 000 × 5 × a1 ÷ 100

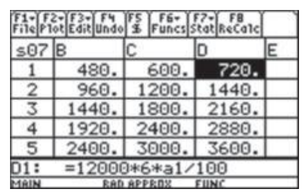
Range: C1: C5.

Then press (ENTER).

In column D the interest rate becomes 6%.
Again repeat these steps completing the table as:

Initial formula: = 12 000 × 6 × a1 ÷ 100

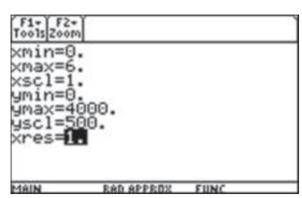
Range: D1: D5.
Then press (ENTER).



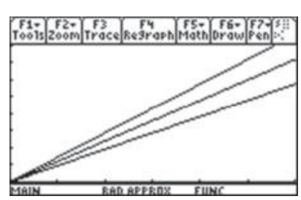
b 1 To show this information in graph form, press:
- (APPS)
- (♦) [GRAPH].

To set appropriate window settings for the graph, press (♦) [WINDOW].
Adjust the window settings as shown.

b



2 Press (♦) [Y=] and enter the following functions:
y1(x) = 12 000 × 4 × x ÷ 100
y2(x) = 12 000 × 5 × x ÷ 100
y3(x) = 12 000 × 6 × x ÷ 100.
Press (♦) [GRAPH] to graph the functions.



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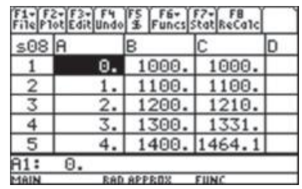
Graphics Calculator tip! Comparing investments

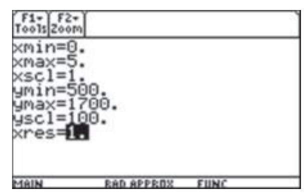
The data found on page 25 may be entered into a graphics calculator and used to plot each of the graphs.

1. To enter the data in a CellSheet, press (APPS).
   Select CellSheet and press:
   - 3: (New)
   - (ENTER).

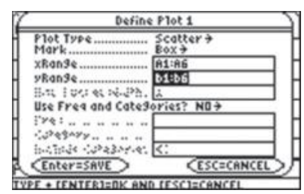
   In column A enter the years, in column B enter the simple interest and in column C enter the compound interest.



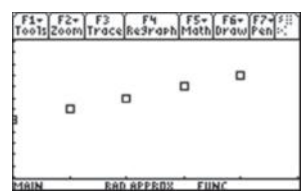
2. To adjust the window settings, press (♦) [WINDOW] and enter the settings shown.



3. Return to the CellSheet and press (F2) (Plots). Enter the settings shown.

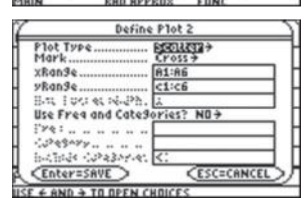


4. To draw the graph of the function, press (♦) [GRAPH].

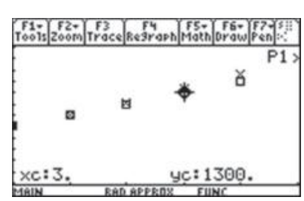


5. Return to the CellSheet to draw the second scatter plot. Press:
   - (F2) (Plot)
   - 1: (Plot Setup).

   Use the arrow keys to move to Plot 2 and press (F1) (Define). Enter the settings shown.



6. To draw both scatterplots, press (♦) [GRAPH]. Press (F3) (Trace) to explore the points on the graph.



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WORKED Example 11

THINK

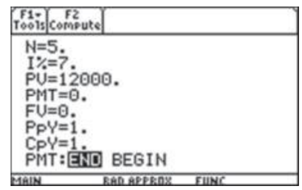
1. Enter the following data in the Finance application on the TI-89 Titanium graphics calculator.

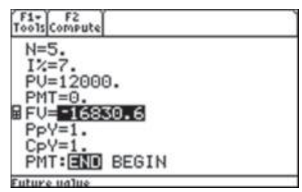
$n = 5$	(Interest is calculated for 5 years.)
$I\% = 7$	(Annual interest rate of 7% p.a.)
$PV = 12\ 000$	(Present value is \$12 000.)
$PMT = 0$	(No regular payments are made.)
$FV = 0$	(This value will be calculated.)
P/Y or $PpY = 1$	(One instalment period per year.)
C/Y or $CpY = 1$	(Interest is calculated once a year.)

2. Use the arrow keys to move to the *FV* line and press ENTER.

3. Write the answer.

WRITE/DISPLAY





Future value is \$16 830.60.

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WORKED Example 12

THINK

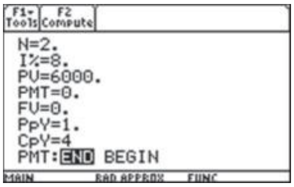
1. Enter the following data:

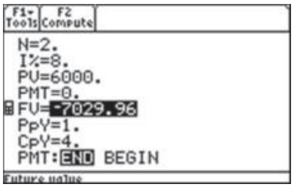
$n = 2$	(Interest is calculated for 2 years.)
$I\% = 8$	(Annual interest rate of 8% p.a.)
$PV = 6000$	(Present value is \$6000.)
$PMT = 0$	(No regular payments are made.)
$FV = 0$	(This value will be calculated.)
P/Y or $PpY = 1$	(One instalment period per year.)
C/Y or $CpY = 4$	(Interest is calculated 4 times a year.)

2. Use the arrow keys to move to FV line and press ENTER.

3. Write the answer.

WRITE/DISPLAY





Future value would be \$7030.

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WORKED Example 16

THINK

a 1 To enter the data in a CellSheet press (APPS) and select CellSheet. Press:

- 3: (New)
- (ENTER).

Enter the values 0 to 5 in column A to represent the years.

Move the cursor to cell B1 and press:

- (F3) (Edit)
- 3: (Fill Range).

Complete the table as:
Initial Formula: = 2000(1 + 5 ÷ 100) ^ a1
Range: B1: B6.
Then press ENTER.

2 Repeat the steps for column C, completing the table as:
Initial Formula: = 2000(1 + 6 ÷ 100) ^ a1
Range: C1: C6.
Then press (ENTER).

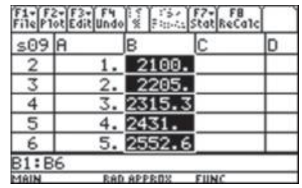
Again repeat these steps for column D, completing the table as:
Initial Formula: = 2000(1 + 7 ÷ 100) ^ a1
Range: D1: D6.
Then press (ENTER).

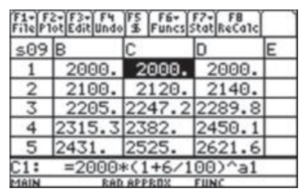
3 To adjust the window settings, press (♦) [WINDOW] and enter the settings shown.

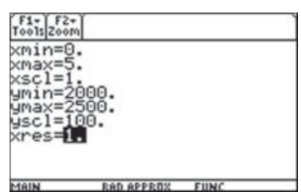
4 Return to the CellSheet, press (F2) (Plots) and enter the settings shown to graph the first series.

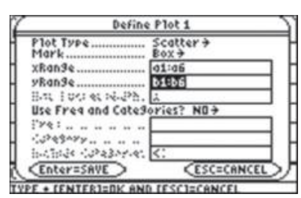
WRITE/DISPLAY

a





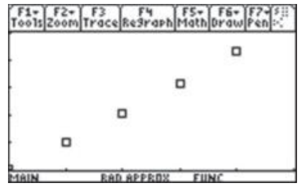




THINK | **WRITE/DISPLAY**

b ① To draw the graph of the function press ♦ [GRAPH].

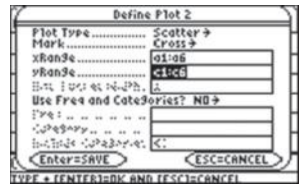
b



② To draw the second scatter plot, return to the CellSheet. Press:

- F2 (Plot)
- 1: (Plot Setup).

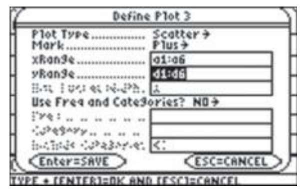
Use the arrow keys to move to Plot 2 and press F1 (Define). Enter the settings shown.



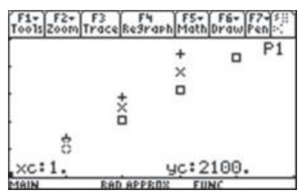
③ Return to the CellSheet to draw the second scatter plot. Press:

- F2 (Plot)
- 1: (Plot Setup).

Use the arrow keys to move to Plot 3 and press F1 (Define). Enter the settings shown.



④ To draw all three scatterplots together press ♦ [GRAPH]. To explore points on the graph press F3 (Trace).



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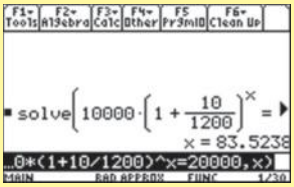
Doubling your money

Task 3

1 We can use a graphics calculator to determine the time required for a sum of money to double in value.

Let's consider our original investment of $10 000 at 10% p.a. compounding monthly.

The formula is:



$$A = P\left(1 + \frac{R}{100 \times n}\right)^{n \times T}.$$

Where:

$A = 20\,000$

$P = 10\,000$

$R = 10$

$n = 12$.

To find T, complete the entry line as:

$$\text{solve}\left(10\,000 \times \left(1 + \frac{10}{1200}\right)^{x} = 20\,000, x\right).$$

Then press ENTER.

The time taken to double the value of the investment is 84 months (7 years).

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Comparing simple and compound interest functions using graphics calculators

Task 1

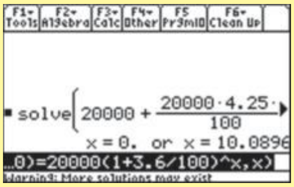
Your grandfather left you \$20 000 in his will. You have no need to use the money at this stage, so you are looking at investing it for approximately 12 years. Your research has narrowed down your options to 4.25% p.a. simple interest or 3.6% p.a. interest compounding yearly. At this stage, you do not anticipate having to withdraw your money in the short term; however, it may be necessary to do so.

Let us investigate to determine which would be the better option if you were forced to withdraw your money at any period of time within 12 years. When using graphics calculators, there are often different ways of approaching a problem, as we often find when solving problems using pen and paper. The TI-89 Titanium graphics calculator allows us to show two different methods that could be used to solve this problem.

1 To find an algebraic solution, it is necessary to find the intersection of the two simultaneous equations:

$FV = 20\,000 + \dfrac{20\,000 \times 4.25 \times x}{100}$ for simple interest

$FV = 20\,000\left(1 + \dfrac{3.6}{100}\right)^x$ for compound interest.



On a calculator screen, complete the entry line as:

$$\text{solve}\left(20\,000 + \frac{20\,000 \times 4.25 \times \text{x}}{100}\right) = 20\,000\left(1 + \frac{3.6}{100}\right)^{\text{x}}, \text{x}.$$

Then press ENTER.

The simple and compound interest funds will have the same value after approximately 10.1 years. At this stage it is unknown which of the two funds provide the better option prior to this time. Looking at their graphs can determine this.

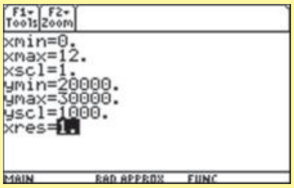
2 To draw the graphs, press **APPS** and select Y = editor.

Complete the entry lines as:

y1(x) = 20 000 + (20 000 + 4.25 × x ÷ 100)

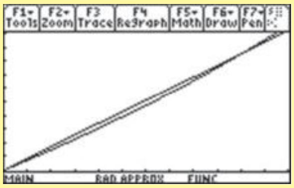
$$y2(x) = 20\,000\left(1 + \frac{3.6}{100}\right)^x.$$

Adjust the window settings as shown.



3 To draw the two graphs, press ◆ [GRAPH].

The graph shows that the simple interest investment (the straight line) is initially better than the compound interest investment (curved line). The two intersect at a point, after which the compound interest investment is better.



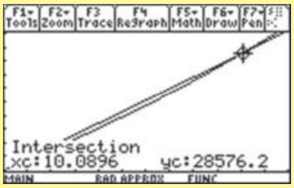
4 To find the point of intersection, press:

- **F5** (Math)
- 5: (Intersection).

Press **ENTER** on any point on the first curve then at any point on the second curve.

To set the bounds between which the calculator will look for the intersection, select points before and after the intersection.

The point of intersection will appear.



After approximately 10.1 years, the two investments both amount to $28 576.20.

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Graphics Calculator tip! Calculating effective interest rates

The effective interest rate can be calculated quickly using a graphics calculator. From Worked example 17 take $I\% = 3.5\%$ and $n = 12$.

1. To calculate the effective interest rate press:
   - HOME
   - CATALOG
   - F3 (Flash Apps).

Scroll down and select the Eff(function.

Complete the entry line as:

TIFinance.Eff(3.5, 12).

This gives an effective interest rate of 3.56% p.a. (to 2 decimal places).

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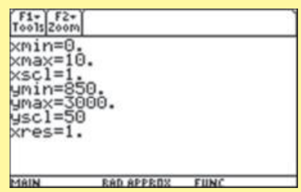
Modelling appreciation with the aid of a graphics calculator

You have purchased a rare coin that the coin dealer told you should appreciate by 15% each year. You paid $850 for the coin and hope that its value will treble within the next 10 years. The coin dealer is not sure whether this is the case, so you offer to produce a graph for him displaying the value of the coin over the next 10 years.

Using a graphics calculator greatly simplifies the calculations and will produce a graph that can be used to determine the value of the coin at any period of time. The following screens displayed and instructions supplied are those of the TI-89 Titanium graphics calculator.

1 To draw a graph, press (APPS) and select the Graphs option. To set an appropriate scale press (♦) [WINDOW].

Complete the fields as shown and press (ENTER).

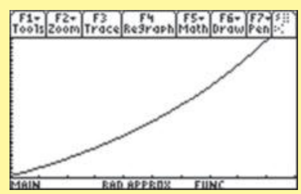


2 Press (♦) [Y=] and complete the entry line as:

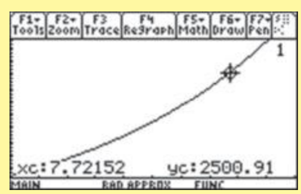
y1(x) = 850 × (1 + 15 ÷ 100)^ x.

Then press (ENTER).

To draw the graph press (♦) [GRAPH].



3 To trace the values on the graph, press (F3) (Trace). Then use the arrow keys to move along the curve and find a value as close as possible to $850 × 3.



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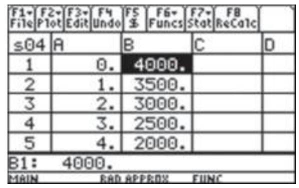
WORKED Example 6

With the aid of a graphics calculator, produce a graph showing the relationship between the age and value of the computer in Worked example 4.

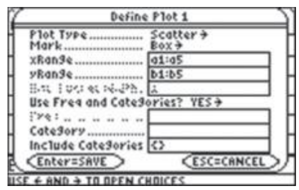
THINK

1. To draw a scatterplot of the data, press (APPS) and select CellSheet.
Enter the year data in column A and the values in column B.

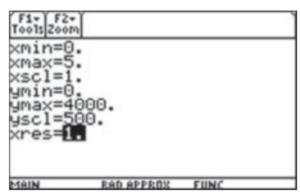


2. Press:
   - (F2) (Plot)
   - 1: (Plot Setup)
   - (F1) (Define).

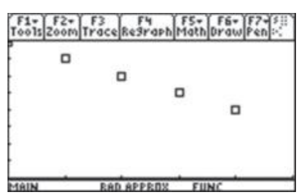
   Complete the table as shown.
   Note that the xRange is the years (A1:A5) and the yRange is the values (B1:B5).



3. Press (♦) [WINDOW] and enter the settings shown at right.

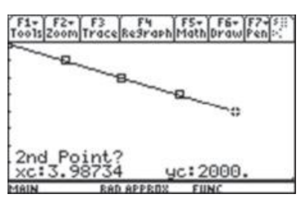


4. To draw the scatterplot, press (♦) [GRAPH].
Note: Ensure any previous entries in the Y= editor have been cleared.



5. To connect the points, press:
   - (F7)
   - 3: (Line).

   Move the cursor to the first point and press (ENTER), then move it to the last point and press (ENTER) again.



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WORKED Example 9

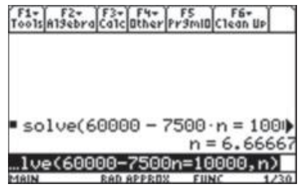
Use a graphics calculator for the problem in Worked example 8.

THINK

1. To solve an equation, press:
   - HOME
   - CATALOG.

   Then use the arrow keys to select the solve command.
   Complete the entry line as:
   solve(60 000 – 7500n = 10 000, n).
2. Write the solution.

WRITE/DISPLAY



Solve $S = V_o - Dn$, for n,
given $S = 10\ 000$, $V_o = 60\ 000$ and $D = 7500$.
The equipment should be replaced after 7 years.

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WORKED Example 15

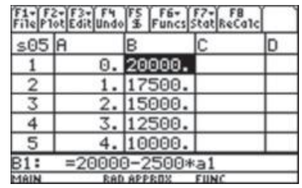
Use a graphics calculator to solve Worked example 14.

THINK / **WRITE/DISPLAY**

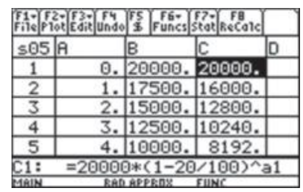
a 1 To create a table of values using a spreadsheet, press (APPS) and select CellSheet.
Enter the year data 0 to 8 in column A.
Move the cursor to cell B1. Press:
- (F3)
- 3: (Fill Range).

Complete the table as:
Initial Formula: = 20 000 – 2500 × a1
Range: B1: B9.
Then press (ENTER).

a

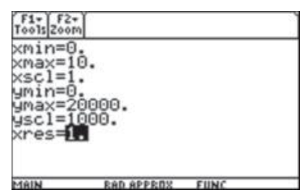


2 Repeat the above steps for column C entering the formula as:
= 20 000 × (1 – 20 ÷ 100)^a1.

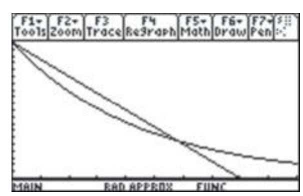


b 1 To draw graphs of the Straight Line Depreciation and the Diminishing Value Depreciation, press (APPS) and select Graph. Clear any existing function, select (♦) [WINDOW], enter the values as shown and press (ENTER).

b



2 To enter the functions press (♦) [Y=].
Complete the entry lines as:
$y1 = 20\,000 - 2500 \times x$
$y2 = 20\,000(1 - 20 \div 100)$^$x$.
Pressing (ENTER) after each line.
Press (♦) [GRAPH] to draw the graphs.



THINK

WRITE/DISPLAY

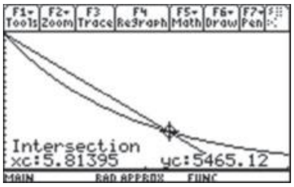
3 To find the intersection point of these two graphs press:
- F5 (Math)
- 5: (Intersection).

Move the cursor to any point on the first graph and press ENTER.

Move the cursor to any point on the second graph and press ENTER.

Move the cursor to any point before the intersection to set the lower bound and press ENTER.

Move the cursor to any point after the intersection to set the upper bound and press ENTER.



4 Write the solution.

The straight line depreciation becomes less than the diminishing value depreciation after 6 years.

Chapter 2 page 89

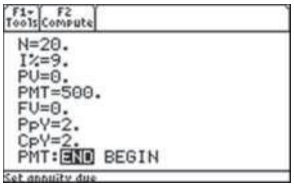
WORKED Example 19

Christina invests \$500 into a fund every 6 months at 9% p.a. interest, compounding six monthly for 10 years. Calculate the future value of the annuity after 10 years.

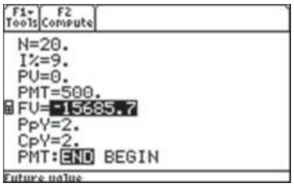
THINK / **WRITE/DISPLAY**

1. Press (APPS) and select the Finance option.
2. Enter the data as shown.

$n = 10 \times 2$	(Interest is calculated twice a year for 10 years.)
$I\% = 9$	(Interest rate is 9% p.a.)
$PV = 0$	(No deposit is made initially — only regular 6-monthly payments.)
$PMT = 500$	(Regular \$500 payments are made.)
$FV = 0$	(This value will be calculated.)
PpY or $P/Y = 2$	(Regular payments are made twice a year.)
CpY or $C/Y = 2$	(Interest is calculated twice a year.)



3. Move the cursor to the FV field and press (F2) (Compute) to calculate its value.



4. Write the answer.

The future value of the annuity after 10 years is \$15 685.71.

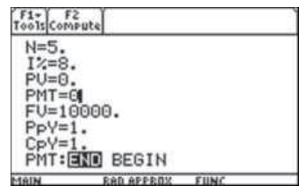
Chapter 2 page 90

WORKED Example 20

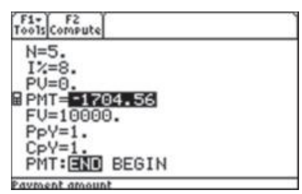
Vicky has the goal of saving $10 000 in the next 5 years. The best interest rate that she can obtain is 8% p.a., with interest compounded annually. Calculate the amount of the annual contribution that Vicky must make.

THINK | **WRITE/DISPLAY**

1. Press (APPS) and select the Finance option.
   Enter the values in the fields as shown.
   $n = 5 \times 1$ (Interest is calculated once a year for 5 years.)
   $I\% = 8$ (Interest rate is 8% p.a.)
   $PV = 0$ (No deposit is made initially.)
   $PMT = 0$ (This is the unknown value which will be calculated.)
   $FV = 10\ 000$ (The future value is $10 000.)
   PpY or $P/Y = 1$ (Vicky makes one payment per year.)
   CpY or $C/Y = 1$ (Interest is calculated yearly.)



2. Use the arrow keys to highlight the PMT line and press (ENTER).



3. Write the answer.

A payment of $1704.56 is required as the annual contribution.

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Loan repayments

Using the TI-89 Titanium graphics calculator

The features used in this investigation are available in the finance section of the TI-89 Titanium graphics calculator. You are advised to consult with your manual before commencing, so you are familiar with the variables and format of the functions.

The finance section enables us to examine aspects of periodic payments on loans or investment accounts.

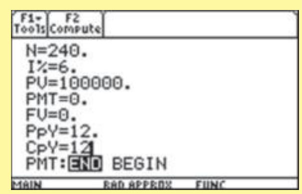
The amortisation section provides the facility to display:

- the interest **Σ**Int(and principal **Σ**Prn(portion of each instalment
- the balance bal(of the principal after any instalment
- the total interest **Σ**Int(paid on the loan to any particular point
- a graph showing the progress of the loan.

Imagine you obtain a $100 000 housing loan. The interest rate on the loan is 6% p.a. on a monthly reducing balance. You agree to repay the loan plus interest in equal monthly instalments over a period of 20 years.

Task 1

To calculate the monthly repayment required to pay off the loan of $100 000 in 20 years, press (APPS) and select the Finance option.



From the information given, enter the data as shown.

n = 12 × 20 = 240 (Monthly payments for 20 years)

I% = 6%

PV (present value) = 100 000

PMT = 0 (this is the value required)

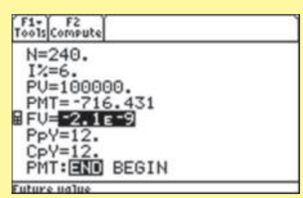
FV (future value) = 0 (loan paid off)

*P*p*Y* = 12 (Payments are monthly — 12 per year)

*C*p*Y* = 12 (Interest is calculated monthly)

PMT: END (Interest calculated at the end of the month)

Move to the *PMT* field and then press (F2) (Compute). The screen displays a monthly payment of $716.43. Move to the *FV* field and then press (F2) (Compute). The future value is $2.1 × 10^{-9} ($0.000 000 002 1 — a value very close to zero), meaning that the loan has been paid off.



Task 2

1 Looking at the interest and principal portion of each instalment.

To calculate the interest paid on the first payment, press HOME and then press CATALOG. Press (F3) (Flash Apps) and select the **Σ**Int function.

Complete the entry line as:

TIFnance.**Σ**Int(1,1).

Then press (ENTER) to display the interest for the first month.

The interest paid on the first payment is $500.

To see the principal component of Instalment 1, press:
- HOME
- CATALOG
- (F3) (Flash Apps).

Then select the **Σ**Prn function.
Complete the entry line as:

TIFnance.**Σ**Prn (1,1).

Press (ENTER) to display the principal component for the first month. This shows only $216.43 of the $716.43 (30%) goes towards paying off the principal; the remainder is interest.
To display the interest of payment 2, repeat the previous procedure, and complete the entry line as:

TIFnance.**Σ**Int(2, 2).

Then press (ENTER).

To display the principal component of payment 2, repeat the previous procedure, complete the entry line as:

TIFnance.**Σ**Prn(2, 2).

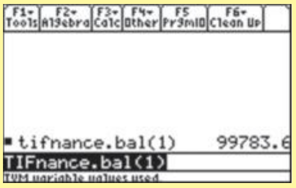
Then press (ENTER).

Describe what you notice.

Investigate the change in the interest and principal components as the loan progresses from Payment 1 to Payment 240 (the last instalment).

2 Looking at the balance of the principal after any instalment

To calculate the balance of the loan after the first payment, press:
- HOME
- CATALOG
- (F3) (Flash Apps).



Select the bal(function.

Complete the entry line as:

TIFnance.bal(1).

Then press (ENTER).

This will display the balance at the end of the first month.

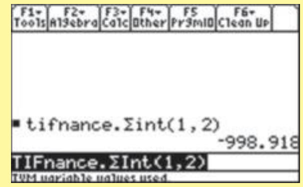
The balance on the loan after the first payment is $99 783.60. Change the payment number and investigate what happens over the duration of the loan?

3 Looking at the total interest paid on the loan to any particular point.

To find the total interest paid on Instalments 1 and 2, repeat the previous procedures however complete the entry line as:

TIFnance.ΣInt(1, 2).

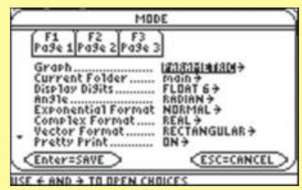
The total interest paid on instalments 1 and 2 is $998.92.



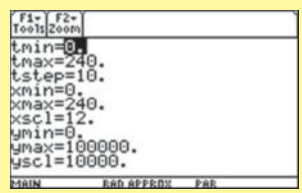
Investigate the total interest payable over other time periods. Compare the interest paid in the first two instalments with that paid in the last two instalments. What do you notice?

4 Looking at a graph showing the progress of the loan.

1. To graph the progress of the loan, press (MODE) and change the graph type to parametric.

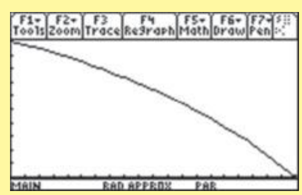


2. To change the window settings to suitable values, press (♦) [WINDOW] and enter the settings shown at right.

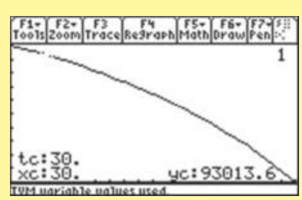


3. Press (♦) [Y=] and complete the entry lines as:
   xt1 = t
   yt1 = tifinancebal(240).
   Pressing (ENTER) after each line.

   To enter this *y*-function, press CATALOG, then (F3) (Flash Apps) and select the balance function. Press (♦) [GRAPH] to graph this function.



4. To trace the balance over the life of the loan, press (F3) (Trace). Move the cursor along the curve to trace the balance of the loan as time progresses.



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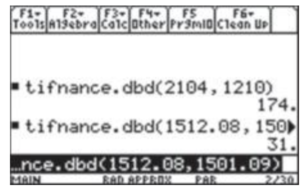
Graphics Calculator tip!

Calculating the number of days between dates

The TI-89 graphics calculator has a function which will calculate the number of days between dates. This can be helpful in interest calculations involving daily balances.



To find the number of days between:

(a) 21 April and 12 October

(b) 15 December 2008 and 15 Jan 2009.

Press:

- HOME
- CATALOG
- F3 (Flash Apps).

Select the dbd(function.

Complete the entry lines as:

(a) dbd(2104,1210)

(b) dbd(1512.08,1501.09).

Dates are expressed in the form DDMM (if no year is mentioned) or DDMM.YY (if a year is specified).

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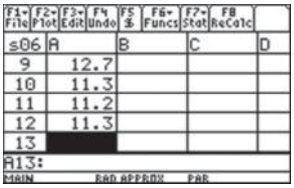
WORKED Example 11

THINK

WRITE/DISPLAY

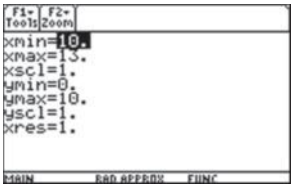
1. Enter data.
   - i To enter the data press APPS and select CellSheet. Then press: 3: (New).
   - ii Enter the list of scores in column A.



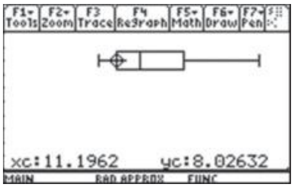
2. Draw the boxplot.
   - i Press:
     - F2 (Plot)
     - 1: (Plot Setup)
     - F1 (Define).

     Enter the settings shown at right.

   - ii Press ◆ [WINDOW] and enter the settings shown at right.



   - iii To draw the Boxplot, press ◆ [GRAPH]. Use F3 (Trace) and navigate over the boxplot to show the five-number summary.



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Interpreting histograms

The aim of this investigation is to highlight the pitfalls in interpreting the shape of histograms. The activity is more readily conducted using a graphics calculator.

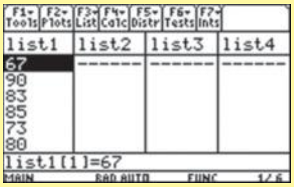
1 Consider the percentages received by a class of 36 students in their end-of-semester test.

67, 90, 83, 85, 73, 80, 78, 79, 68, 71, 53, 65, 74, 64, 77, 56, 66, 63,
70, 49, 56, 71, 67, 58, 60, 72, 67, 57, 60, 90, 63, 88, 78, 46, 64, 81.

2 To enter the data as a list, press:

- (APPS)
- Stats/List Editor.

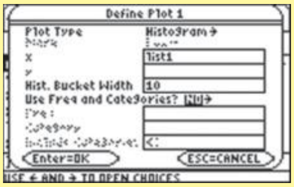
Enter the data in list 1.



3 **i** To graph the data as a histogram starting at 40, with a class interval of 10, press:

- (F2) (Plots)
- 1: (Plot Setup)
- (F1) (Define).

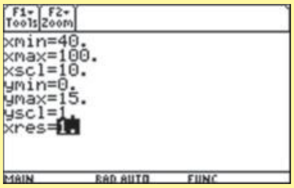
Complete the table as shown above right.



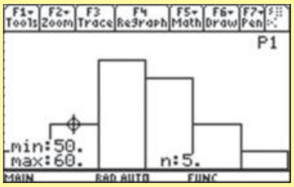
ii Then adjust the window settings by pressing:

- ENTER
- (F5) (Zoom Data)
- (♦) [WINDOW].

Complete the settings as shown at right and then press (♦) [GRAPH] to create the histogram.



iii To see the values of the histogram, press (F3) (Trace).



4 To view the histogram with different widths, repeat the previous steps changing the values that have been set.

Note: The bucket width must match the xscl.

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WORKED Example 13

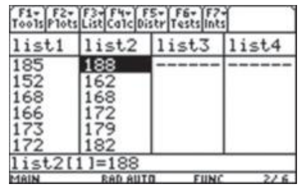
Using scatterplots to consider relationships between data sets

THINK | **WRITE/DISPLAY**

Method 2 Using a graphics calculator

a 1 To enter the data into a spreadsheet, press APPS and select Stats/List Editor.

Enter the mother's height in list 1 and the son's height in list 2.

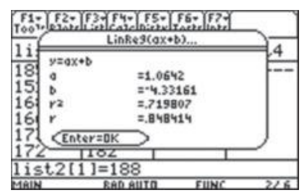


2 To calculate the least-squares regression line, press:
- F4 (Calc)
- 3: (Regressions)
- 2: (LinReg (ax + b)).

Complete the table as shown and press ENTER.

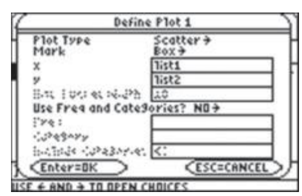
Note the equation of the least-squares regression line will be stored in y1(x).

3 This shows that the equation of the regression line (or line of best fit) is $y = 1.064x - 4.332$.

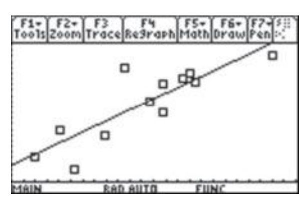


4 To draw a scatterplot, press:
- ENTER
- F2 (Plots)
- 1: (Plot Setup)
- F1 (Define).

Complete the table as shown and then press:
- ENTER
- F5 (Zoom Data).



b The regression line will be displayed on the scatterplot.

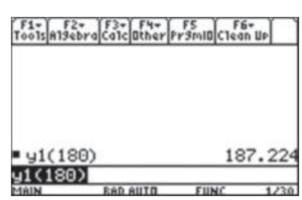


c To estimate the height of a son born to a 180 cm tall mother, press HOME.

Complete the entry line as:

y1(180).

Then press ENTER.



The son's height is predicted to be about 187 cm.

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Graphics Calculator tip! Graphing linear inequations

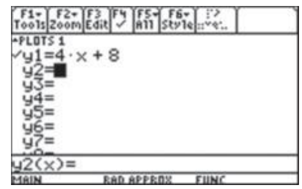
Linear inequations can be sketched with the aid of a graphics calculator. For instance, the graph of the inequation in Worked example 3 can be graphed as follows.

First transpose the inequation to make y the subject: $y \leq 4x + 8$.

1. To graph an inequation, press APPS and select Graphs. Press ♦ [Y =] and complete the entry line as:

   y1 = 4x + 8.

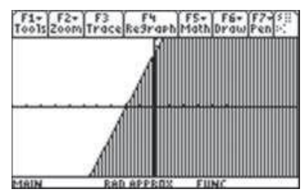
   Then press ENTER.



2. To determine if the shaded region should be above or below the line, test the point (0, 0). Since $0 < 4(0) + 8$ we must shade the area below containing the point (0, 0).

3. In the Y = editor highlight the function y1 and press:
   - F6 (Style)
   - 8: (Below).

   Then press ♦ [GRAPH] to graph the inequality.



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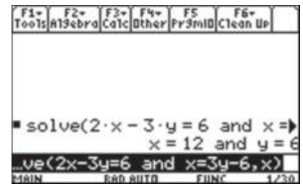
Graphics Calculator tip!

Solving simultaneous linear equations

A calculator can be used to find a solution for Worked example 4

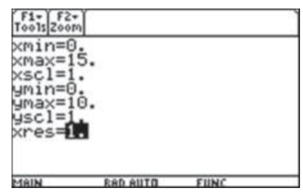
The TI-89 graphics calculator can be used to solve simultaneous equations both algebraically and graphically.

To solve the simultaneous equations algebraically, press (APPS) and select the Home screen.



Complete the entry line as:
solve(2x – 3y = 6 and x = 3y – 6, x).
Then press (ENTER).

1. To solve the simultaneous equations graphically, press (APPS) and select the Graph application.
2. To adjust the window settings to accommodate the graphs press (♦) [WINDOW] and enter the settings shown at right.



3. Press (♦) [Y =] and rearrange the equations to make *y* the subject.

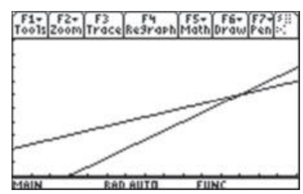
   Complete the entry lines as:

   $$y1 = \frac{2}{3}x - 2$$

   $$y2 = \frac{1}{3}x + 2.$$

   Pressing (ENTER) after each line.

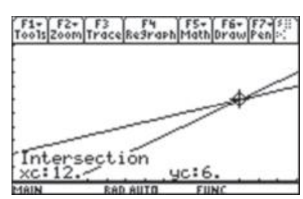
   Press (♦) [GRAPH] and the two graphs will appear on the screen.



4. To find the coordinates of the point of intersection of these two lines, press:
   - (F5) (Math)
   - 5: (Intersection).

   Press (ENTER) at a point on the first curve and again at a point on the second curve.

   To select the upper and lower bound, move the cursor to a point near the intersection and press (ENTER) at a point just before and just after the point of intersection. The point of intersection will be displayed.



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Graphics Calculator **tip!**

Solutions to simultaneous linear inequations

1. To find the graphical solution of the simultaneous linear inequations, press (APPS) and select the Graph application.

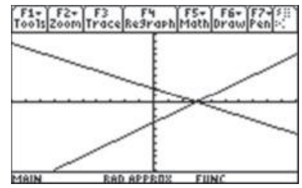
   Press (♦) [Y =] and complete the entry lines as:

   $y1 = \frac{-2}{3} \times x + 2$

   $y2 = x - 3$

   Press (ENTER) after each line.

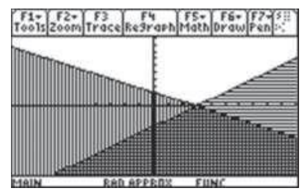
   Press (♦) [GRAPH] to draw the graph of the two functions.



2. To shade the required region, in the Y = editor, use a test point.
   Substituting the point (0, 0) into the inequality $y1 \leq \frac{-2}{3} \times x + 2$ gives 0 < 2, so the region below the line, which contains (0, 0) should be shaded. Substituting the point (0, 0) into the inequality $y2 \leq x - 3$ gives $0 \leq -3$ and so the region below the line, which does not contain (0, 0) should be shaded.

   Highlight the equation $y1 \leq \frac{-2}{3} \times x + 2$ and press:
   - (F6) (Style)
   - 8: (Below).

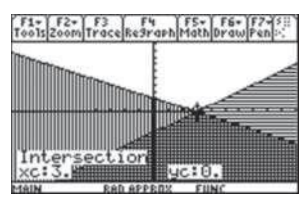
   Repeat for the equation $y2 = x - 3$
   Then press (♦) [GRAPH].



3. To find the intersection point, press:
   - (F5) (Math)
   - 5: (Intersection).

   Press (ENTER) at a point on the first curve and again at a point on the second curve.

   Move the cursor to a point near the intersection and press (ENTER) at a point just before and just after the point of intersection to select the upper and lower bound. The point of intersection will be displayed.



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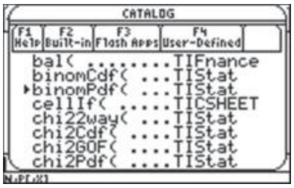
Graphics Calculator tip! Calculating binomial probabilities

Graphics calculators can greatly simplify calculating binomial probabilities. Here are the steps required for solutions to Worked examples 13, 14 and 15.

1. To calculate binomial probabilities, press:
   - HOME
   - CATALOG
   - F3 (Flash Apps).

   Select the binomPdf function.



2. For Worked example 13, there were 4 trials with a 0.5 probability of success, with an X value (number of required heads) of 3.

   Complete the entry line as:

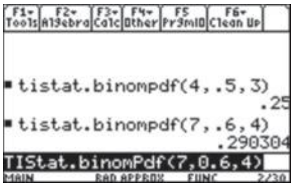
   TIStat.binomPdf(4,0.5,3).

   Then press ENTER.

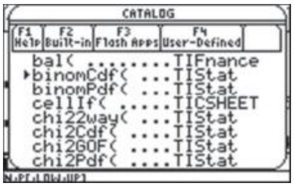
3. Repeat the steps above for Worked example 14 and complete the entry line as:

   TIStat.binomPdf(7, 0.6, 4).

   Then press ENTER.



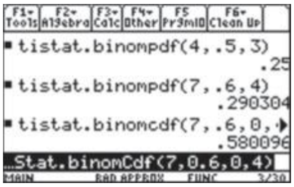
4. For Worked example 15 repeat the previous steps, however in this case select the binomCdf function.



5. Complete the entry line as:

   TIStat.binomCdf(7, 0.6, 4).

   Then press ENTER.



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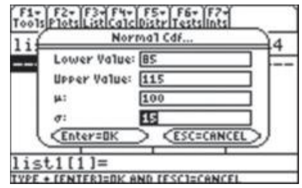
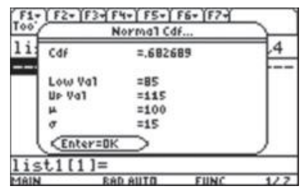
Graphics Calculator tip! Distribution of scores

A graphics calculator can be used to readily determine the percentage of a distribution which lies between certain values. This can be illustrated with Worked example 6.

1. To calculate the cumulative probability between two scores, press (APPS) and select Stat/List Editor then press:
   - (F5) (Distr)
   - 4: (Normal Cdf).

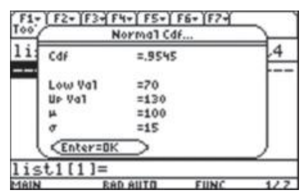


2. Complete the table as shown for part **a** and press (ENTER).

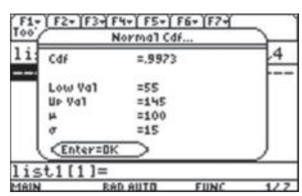


This shows that 68% of the scores obtained on a commonly used IQ test lie between 85 and 115.

3. Repeat the steps above for parts **b**.



4. Repeat the steps above for parts **c**.



Glossary

Adjacent — The side next to the angle used for reference in a right-angled triangle.

Allowable deduction — A deduction from taxable income permitted by the Australian Taxation Office. Allowable deductions include expenditure incurred in earning income.

Allowance — An extra payment made to a worker for working in unfavourable conditions.

Angle of depression — The angle through which you must look down from the horizontal to sight an object.

Angle of elevation — The angle through which you must look up from the horizontal to sight an object.

Annual leave — A period of time that each permanent employee is allowed each year for holidays.

Annual leave loading — An extra payment of $17\frac{1}{2}$% of the gross pay made to employees when they take their annual leave.

Annulus — The area between two circles that have the same centre (concentric).

Area — The amount of space within the boundary of a closed figure.

Bar graph — A graph where categorical data are displayed in horizontal bars, with the categories on a vertical axis and quantity on the horizontal axis.

Bimodal — A set of scores for which two scores occur most often.

Bivariate data — Sets of data containing two variables.

Bottom plate — Timber or metal strip at the bottom of a frame.

Box-and-whisker-plot — A method of graphically displaying a five-number summary. The plot is drawn to scale with the box representing the interquartile range and the whiskers representing the range. Within the box the median is also shown.

Brace — Sheets of timber, or strips of timber or metal used to provide strength to a frame and help the frame to retain its shape.

Budget — A list of a person's income and expenses. A personal budget is made to try to avoid spending more than is earned. A balanced budget is where income equals expenditure.

Building square — A device used by builders to check whether corners of buildings are square.

Capacity — The quantity of solid, liquid or gas that a 3-D object could hold.

Casual rate — A higher rate of pay to compensate casual workers for the lack of holiday and sick pay.

Categorical data — Data which are not numerical and are put into categories such as types of car.

Causality — When the occurrence of one variable causes another. For example there is a strong positive relationship between a person's shirt size and shoe size but one does not cause the other. On the other hand, there is a strong positive relationship between the amount of a Lottery jackpot and the number of tickets sold. In this case, it would seem that one does cause the other.

Census — Data gathered from the entire population.

Central tendency — A method for describing a typical score in a data set. There are three measures of central tendency — mean, median and mode.

Closed question — One that must be answered within given categories.

Column graph — Similar to a bar graph, but the data are displayed in vertical columns.

Commission — Payment made to a salesperson. A commission is usually paid as a percentage of sales.

Complementary events — Two events that cover all possible outcomes to a probability experiment. The sum of the probabilities to complementary events is 1.

Continuous data — Data which can take any value within a given range.

Cosine ratio— The ratio of the adjacent side and hypotenuse in a right-angled triangle.

Course — A 'course of bricks' is an alternative expression for a 'row of bricks'.

Cumulative frequency — A progressive total of the frequencies.

Curing — Allowing concrete to dry slowly to increase its strength.

Cyclic trends — Trends that fluctuate up and down but not according to season.

Data — Information before it is organised.

Database — An organised set of data on a population.

Deduction — A sum of money that is deducted from an employee's gross pay before receiving net pay.

Dependent variable — A variable whose value responds to changes in the independent variable.

Discrete data — Discrete data are where the data can take only certain values, usually whole numbers.

Double time — A penalty rate which pays the employee twice the normal hourly rate.

Drop — A vertical length. The term is commonly used in wallpapering and curtaining.

Effective width — The width of a sheet, taking into account overlap between adjacent sheets.

Elevation — A scale drawing of what a building will look like from one side.

Enlargement — A figure is drawn similar to, but larger than the original. The corresponding sides will be in equal ratio and all corresponding angles will be equal.

Equally likely outcomes — These occur when each element of the sample space for a probability experiment is equally likely to occur.

Event — An occurrence that is being examined in a probability experiment.

Exchange rate — The rate at which currencies can be interchanged. Buying rate refers to the rate at which banks will buy the currency from you. Selling rate refers to the rate banks will sell you a currency.

Extrapolate — To extend a graph so as to make predictions about future trends.

Favourable outcomes — Elements from the sample space that meet the requirement for an event to occur.

Feasible region — All the points that satisfy a system of linear inequations.

Five-number summary — A summary of a data set consisting of the lower extreme, lower quartile, median, upper quartile and upper extreme.

Floor plan — A plan showing the floor dimensions of a structure and detailed dimensions of features duch as doors, windows, wall thicknesses and stairs.

Footings — Trenches (in the shape of rectangular prisms) dug around the perimeter of a slab, and sometimes within the slab as a support for internal walls.

Frequency — The number of times an event occurs.

Frequency histogram — A graph suitable for statistical (quantitative) data. It is a column graph drawn with scores or class centres on the horizontal axis and frequency on the vertical axis. A $\frac{1}{2}$ unit (half column width) space is drawn before the first column with no other gaps between columns.

Frequency polygon — A line graph often drawn on the same axes as a frequency histogram. The line is drawn from the corner of the axes to the centre of each column.

Frequency table — A table displaying statistical data. For ungrouped data the table will have columns for score, tally, frequency and possibly cumulative frequency. For grouped data the score column will be replaced with a class column and a class centre column.

Fundamental counting principle — The number of elements of the sample space for a multi-stage probability experiment is found by multiplying the number of ways each stage can occur. This is the fundamental counting principle.

Gable roof — In the shape of an inverted 'V' with two rectangular surfaces.

Goods and Services Tax — A tax that is levied on the price of all items other than fresh food. The GST is levied at a rate of 10%.

GPS (Global Positioning System) — A satellite navigation system accessed by users on land, sea, or in the air, operated by the US Department of Defense.

Gradient — The rate of increase (or decrease) in the dependent variable per one unit increase in the independent variable.

Great circle — A circle of the greatest possible diameter that can be drawn on the surface of a sphere.

Greenwich Mean Time (GMT) — The standard time in Greenwich which is used as the basis for calculating the time in all other parts of the world.

Greenwich Meridian — The meridian of longitude from which angular distances in the east–west direction are measured. Using the longitude calculated from the Greenwich Meridian, time in different places on the Earth's surface is calculated.

Gross pay — A person's earnings before any deductions are taken out.

Group certificate — *see* PAYG Payment Summary Statement.

Grouped data — A data set tabulated in small groups rather than as individual scores.

Grout — A mixture rubbed between tiles to provide separation and to bind them together.

Hip roof — Generally consists of two trapezium-shaped surfaces and two triangular surfaces.

Histogram — A column graph which displays the frequency for a set of scores.

Horizontal — Level, flat and parallel to the horizon or the ground.

Hypotenuse — The longest side of a right-angled triangle. The hypotenuse is opposite the right angle.

Income — Money received by a person that is taxable and is usually in exchange for labour or the result of an investment.

Income tax — Tax that is paid on all income received.

Independent variable — A variable whose value does not depend on the value of another variable.

Indirect tax — Any tax that is not paid directly to the government by the taxpayer. For example, the GST is an indirect tax because it is paid to the retailer who then passes it on to the government.

International Date Line — The meridian of longitude opposite to the Greenwich Meridian. The International Date Line is, however, bent for convenience. When crossing the International Date Line, the date changes.

Interpolate — Drawing a graph using data found at the end points.

Interquartile range — A number that represents the spread of a data set. The interquartile range is calculated by subtracting the lower quartile from the upper quartile.

King post — Vertical post from the horizontal tie beam of a truss to the apex of the truss.

Latitude — The angular distance of a point on the Earth's surface either north or south of the equator.

Linear metre — Length expressed in metres.

Linear programming — A mathematical technique used to solve real-life problems in which a particular quantity is to be maximised or minimised.

Line of best fit — A line drawn on a scatterplot that passes through or is close to as many points as possible.

Lintel — A timber or metal strip above a door or window.

Lower extreme — The lowest score in the data set.

Lower quartile — The lowest 25% of scores in a data set.

Mean — The average of a data set, found by totalling all the scores then dividing by the number of scores.

Median — The middle score or the average of the two middle scores in a data set.

Medicare levy — A payment made as part of our tax system that covers the cost of basic health care services. The basic levy is 1.5% of gross income; however, low income earners pay the levy at a reduced rate.

Meridian of longitude — A line on the Earth's surface that runs from the North Pole to the South Pole. Each meridian of longitude is measured by the number of degrees east or west it is of the Greenwich Meridian.

Mode — The score in a data set with the highest frequency.

Mortar — A mixture of cement and sand used to bind bricks together (and keep them a fixed distance apart).

Multi-stage event — This occurs when there is more than one part to a probability experiment. For example, tossing two coins can be considered as tossing one coin then tossing another, therefore there are two parts to this experiment.

Net pay — The amount of money actually received by the employee after all deductions have been subtracted from the gross pay.

Nogging — Horizontal separators between studs in a frame.

Nominal data — Categorical data which have no order associated with them.

Non-compliant response — A response that does not fit within the expected responses or categories provided in a questionnaire.

Numerical data — Data which involve numbers or measurements.

Open question — One that has no guidelines within which to answer.

Opposite — The side opposite to the angle used for reference in a right-angled triangle.

Ordinal data — Categorical data that are associated with some qualitative scale.

Ordinary rate — The normal hourly rate for a wage earner.

Outcome — A possible result to a probability experiment.

Overtime — This is when a person earns more than the regular hours each week.

Parallel of latitude — A line on the Earth's surface parallel to the equator. Each parallel of latitude is measured in terms of the angular distance either north or south of the equator.

PAYG — Pay As You Go. The method usually applied to the collection of tax.

PAYG Payment Summary Statement— A statement of gross income and the PAYG tax deducted from that income throughout the financial year. It is given to the employee by the employer at the end of each financial year.

Payment by piece (Piecework) — Payment for the amount of work completed.

Penalty rate — A higher rate of pay made to a person who is working overtime.

Per annum — per year.

Percentage chance — The probability of an event expressed as a percentage.

Perimeter — The distance around the boundary of a figure.

Piecework — see Payment by piece.

Pitch — The angle the roof makes with the horizontal.

Pitch ratio — The pitch of a roof expressed as a tangent ratio in the form $1 : x$.

Plumb bob — A device consisting of a length of string with a weight attached at one end. It is used to test whether a surface is vertical.

Polygon — A line graph displaying the frequency for a set of scores.

Population — An entire group of people or objects to which a statistical inquiry is applied.

Prism — A solid shape with a constant cross-section.

Probability — A number between 0 and 1 that describes the chance of an event occurring.

Pyramid — A solid shape with a plane shape as its base and triangular sides meeting at an apex.

Pythagoras' theorem — In a right-angled triangle, the square on the hypotenuse is equal to the sum of the squares on the other two sides.

Pythagorean triad or Pythagorean triple — Sets of three numbers which satisfy Pythagoras' theorem.

Qualitative data — Data which do not involve numbers or measurement.

Quantitative data — Data which can be measured. A numerical value can be assigned to them.

Quartile — 25% of the data set. The upper quartile is the top 25% of the data set and the lower quartile is the bottom 25% of the data set.

Questionnaire — A set of questions completed for a statistical investigation.

Rafter — Beam at the top of a truss to which the roof cladding is attached.

Random trends — Those trends which occur randomly, caused by external events such as wars, floods.

Range — A number which represents the spread of a data set. The range is calculated by subtracting the smallest score from the largest score.

Recording error — Where data have been incorrectly recorded.

Reduction — A similar figure, drawn smaller in size than the original.

Regression line — A line of best fit that is extrapolated to make predictions about data.

Reinforcing mesh — Steel mesh laid in the concrete in the footings and slab to provide structural strength.

Relative frequency — A number between 0 and 1, usually a decimal, which describes how often an event has occurred. The relative frequency is found by dividing the number of times an event has occurred by the total number of trials.

Retainer — A fixed payment usually paid to someone receiving commission. They receive the retainer regardless of the number of sales made.

Roof truss — Frame providing shape and strength for a roof.

Royalty — A royalty is a payment made to the owner of a copyright such as a musician or author. The royalty is usually a percentage of sales.

Salary — A form of payment where a person is paid a fixed amount to do their job. A salary is usually based on an annual amount divided into weekly or fortnightly instalments.

Sample — When data are gathered from a portion of the population, that is taken to be representative of the whole population.

Sample space — A list of all possible outcomes to a probability experiment.

Scale factor — A number by which the side lengths on the first of two similar figures is multiplied by to obtain the measurements on the second of the figures.

Scatterplot — A graph that shows two variables, one on each axis, and their relationship by plotting the points generated by each data pair.

Score — Each piece of quantitative data is a score.

Seasonal trend — A trend that fluctuates with the changing seasons.

Sector — The area between any two radii of a circle.

Sector graph — A graph where a circle is cut into sectors. Each sector then represents a section of the data set. Each sector is the same proportion of the circle as the part of the data set it represents.

Secular trend — A trend that appears to either increase or decrease steadily over time, with no major changes of direction.

Sill — Timber of metal strip below a window.

Similar (figures) — Two or more figures with corresponding angles equal and corresponding sides in the same ratio.

Sine ratio — The ratio of the opposite side and the hypotenuse in a right-angled triangle.

Site plan — A plan showing the boundaries of a block of land and the position of the structure on the lot.

Slab — Foundations for a structure.

Small circle — A circle that is drawn on the surface of a sphere that is of a smaller diameter than a great circle.

Sphere — A closed surface consisting of points in space that are a fixed distance, the radius, from a given point, the centre.

Spirit level — A device, usually constructed of aluminium, containing a vial of liquid with an air bubble. It can be used to determine whether a surface is horizontal and whether a surface is vertical.

Standard deviation — A measure of the spread of a data set. The standard deviation is found on a calculator using either the population standard deviation or the sample standard deviation.

Statistics — Numerical facts compiled to describe a data set.

Stem-and-leaf-plot — A method of displaying a data set where the first part of a number is written in the stem and the second part of the number is written in the leaves.

Studs — Vertical strips of timber or metal in a frame.

Summary statistic — A number such as the mean, median or mode which describes a data set.

Survey plan — A plan showing all boundaries of blocks of land and the position of roadways.

Tangent ratio — The ratio of the opposite side and the adjacent side in a right-angled triangle.

Taxable income — The amount of income upon which the amount of tax due is calculated. Taxable income is calculated by subtracting any allowable tax deductions from the total gross income.

Three-dimensional — Can be described using three measurements (for example a length, a width and a height).

Tie beam — Beam at the base of a truss.

Time and a half — A penalty rate where the employee is paid $1\frac{1}{2}$ times the normal hourly rate.

Time series — Bivariate data where one of the variables is time.

Top plate — Timber or metal strip at the top of a frame.

Tree diagram — A method of listing the sample space to a multi-stage probability experiment. The diagram branches once for each stage of the experiment at each level showing all possible outcomes to each stage.

Trend line — A straight line used to represent a time series.

Trial — The number of times a probability experiment has been conducted.

Trigonometry — A branch of mathematics in which sides and angles of triangles are calculated.

Two-dimensional — Can be described using two measurements (for example a length and a width).

Upper extreme — The highest score in a data set.

Upper quartile — The highest 25% of scores in a data set.

Value Added Tax — Similar to the GST, a VAT is levied in many countries on the cost of goods and services. The rate of VAT varies from country to country.

Volume — The amount of space contained in, or occupied by, a 3-D object.

Wage — A form of payment that is based on an hourly rate.

Water level — A device consisting of a length of clear hose filled with water. It can be used to establish levels between two poins which are separated by some distance.

***y*-intercept** — The value of y when a function crosses the vertical axis.

Answers

CHAPTER 1 Simple and compound interest

Skills check

1 **a** \$1 **b** \$8.75 **c** \$1.50
d \$0.25 **e** \$4 **f** \$0.25

2 **a** 7.25 **b** 0.0725 **c** 0.2
d 0.002 **e** 0.125 **f** 0.001

3 **a** $\frac{1}{4}$ year **b** $\frac{1}{6}$ year **c** $\frac{2}{3}$ year
d $2\frac{1}{12}$ years **e** $4\frac{5}{12}$ years **f** $2\frac{1}{2}$ years

4 **a** 450 **b** 525 **c** 21 000
d 1.157 625 **e** 1.083 **f** 1.877

Exercise 1A — Simple interest

1 **a** \$136.00 **b** \$56.70 **c** \$145.25
d \$110.40 **e** \$255 **f** \$336.89
g \$178.57 **h** \$43.88 **i** \$11.76
j \$229.68 **k** \$544.05

2 **a** \$103.50 **b** \$2700 **c** \$325
d \$131.25

3 **a** \$360 **b** \$1020 **c** \$27 700
d \$17.70 **e** \$13.67

4 B 5 C 6 B 7 A
8 B 9 D 10 D 11 B
12 A 13 \$465.50 14 \$25.50 15 \$2418.75
16 \$584.50

17 **a** The Big-4 Bank offers the better rate.
b The Big-4 Bank charges $11\frac{1}{3}$% p.a. for a loan while The Friendly Building Society charges 12% (=12 × 1% per month).

18 **a** \$627.13 **b** \$12 542.50

19 **a** **i** \$1540.63 **ii** \$6162.50
b Yes

20 **a** \$2247 **b** \$15 729 **c** $7\frac{1}{2}$ years

Exercise 1B — Finding *P*, *R* and *T*

1 **a** \$3070 **b** \$4400 **c** \$5425
d \$236.36 **e** \$2500

2 **a** 10% **b** 6.25% **c** 80%
d 2.125% or $2\frac{1}{8}$% **e** 3.36%

3 **a** 1 year **b** 18 months **c** 3 months
d 7 years **e** 1 month

4 \$1515.79 5 \$2133.33 6 \$352
7 24 months 8 3 years 9 C
10 C 11 D 12 E 13 A

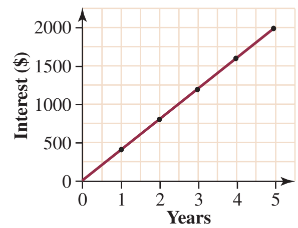
14 **a** Yes (\$1112.50) **b** No
c Yes (\$1600 in 23 months) **d** Yes (\$1281.60)

Exercise 1C — Graphing simple interest functions

1 **a**

No. of years	1	2	3	4	5
Interest	\$400	\$800	\$1200	\$1600	\$2000

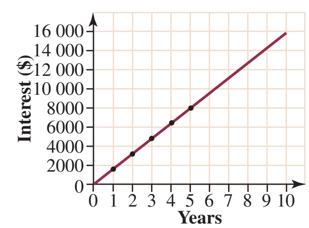
b



2 **a**

No. of years	1	2	3	4	5
Interest	\$1600	\$3200	\$4800	\$6400	\$8000

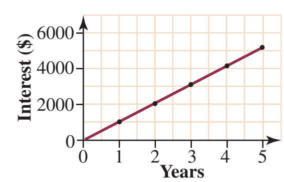
b



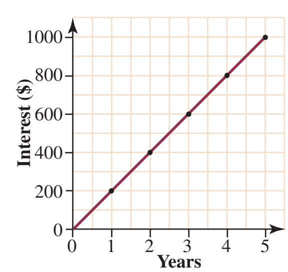
c 1600

d \$16 000

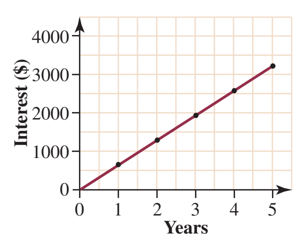
3 **a**



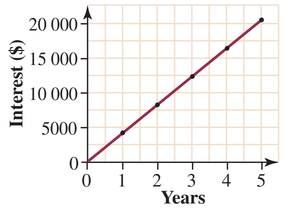
b



c

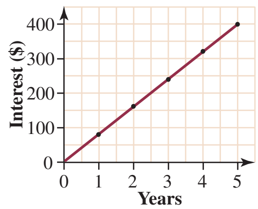


d

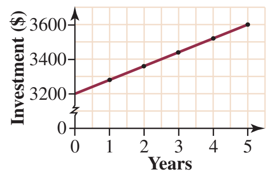


4 288

5 a



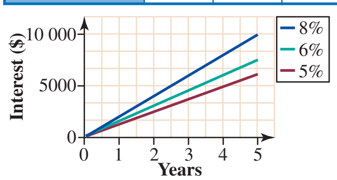
b



6 a

No. of years	1	2	3	4	5
Interest (5%)	$1250	$2500	$3750	$5000	$6250
Interest (6%)	$1500	$3000	$4500	$6000	$7500
Interest (8%)	$2000	$4000	$6000	$8000	$10 000

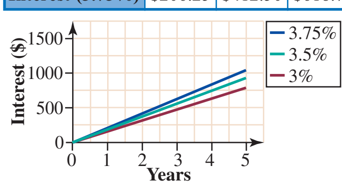
b



7 a

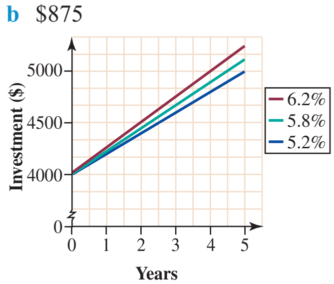
No. of years	1	2	3	4	5
Interest (3%)	$165.00	$330.00	$495.00	$660.00	$825.00
Interest (3.5%)	$192.50	$385.00	$577.50	$770.00	$962.50
Interest (3.75%)	$206.25	$412.50	$618.75	$825.00	$1031.25

b



8 a $4500, $5000, $5200
b $875

9



10 Quick Questions 1

1 $800 **2** $1260 **3** $2700 **4** $1.90
5 $7000 **6** $138 **7** $215.44 **8** $1998.75
9 53c **10** $173.58

Exercise 1D — Calculation of compound interest

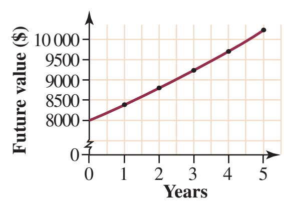
1 $6655 **2** $17 173.50 **3** $2938.66
4 a $4630.50 **b** $9274.19 **c** $24 488.80
d $13 503.78 **e** $12 588.72
5 $70 555.25 **6** $502
7 $14 059.91 **8** $31 850.33
9 a $1003.69 **b** $9111.56 **c** $181 402.12
d $20 039.67 **e** $1 264 568.95
10 B **11** C **12** E **13** D
14 a $15 746.40 **b** $15 793.09
c $15 817.40 **d** $15 833.99
15 a 0.0219% **b** $108 320.72
c $8320.72 **d** $320.72
16 a $4720 **b** $4726.24
c Compounding interest

Exercise 1E — Graphing compound interest functions

1 a

No. of years	1	2	3	4	5
Future value	$8400	$8820	$9261	$9724	$10 210

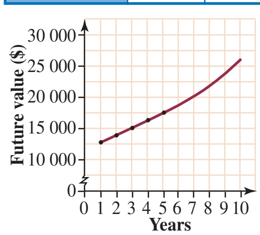
b



2 a

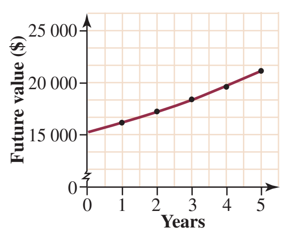
No. of years	1	2	3	4	5
Future value	$12 960	$13 997	$15 117	$16 326	$17 632

b

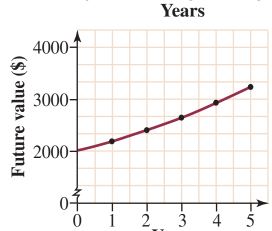


c $25 900

3 a



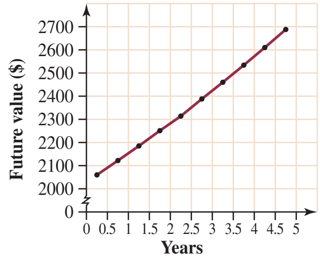
b



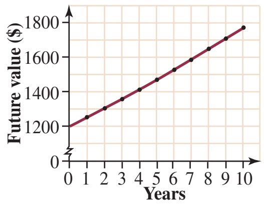
4 a

Years	0.5	1	1.5	2	2.5	3	3.5	4	4.5	5
FV	$2060	$2122	$2185	$2251	$2314	$2388	$2460	$2534	$2610	$2688

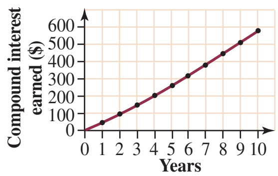
b



5 a



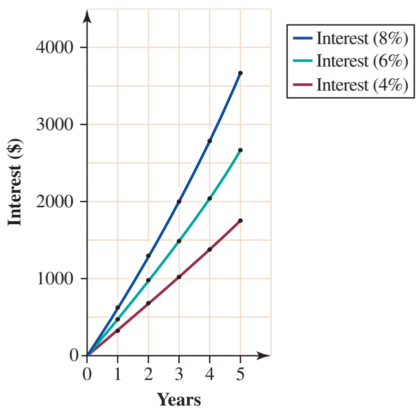
b



6 a

No. of years	1	2	3	4	5
Interest (4%)	$320	$653	$999	$1359	$1733
Interest (6%)	$480	$989	$1528	$2100	$2706
Interest (8%)	$640	$1331	$2078	$2884	$3755

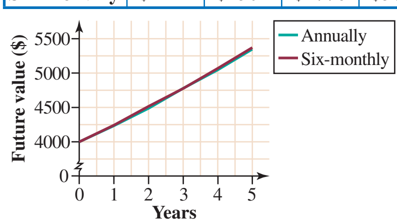
b



7 a

No. of years	1	2	3	4	5
Annually	$4240	$4494	$4764	$5050	$5353
Six-monthly	$4244	$4502	$4776	$5067	$5376

b



10 Quick Questions 2

1 $2051.28 **2** $17 253 **3** $3437.50
4 $429.69 **5** $2315.25 **6** $315.25
7 $6792.58 **8** $6805.66 **9** $6812.41
10 $14 700.69

Exercise 1F — Nominal and effective interest rates

1 4.06% p.a. **2** 4.08% p.a. **3** 3.79% p.a.
4 a 4.27% p.a. effective rate
b 4.26% p.a. effective rate
So 4.2% p.a. compounding quarterly is better.
5 5.095% effective rate, 5.12% effective rate
Choose 5% p.a. compounding monthly.

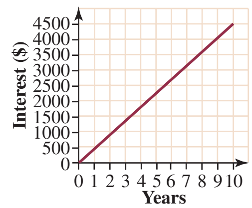
Chapter review

1 $1000
2 a $1296 **b** $2820 **c** $42
d $4.05 **e** $7617.58
3 a $7280 **b** $39 780 **c** $455
4 B **5** D **6** B
7 6.5% p.a. **8** 15 months **9** D
10 E **11** B

12 a

No. of years	1	2	3	4	5
Interest	$450	$900	$1350	$1800	$2250

b

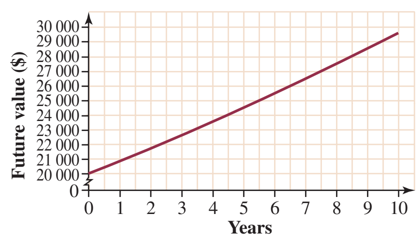


c 450 **d** $4500
13 $2778.30 **14** $5700.47
15 a $3932.39 **b** $12 596.90 **c** $14 457.96
d $23 851 **e** $5334.67
16 $756.94
17 a $12 024.02 **b** $12 052.04
c Compounding interest

18 a

No. of years	1	2	3	4	5
Future value	$20 800	$21 632	$22 497	$23 397	$24 333

b



c $29 600
19 3.94% p.a.
20 4.18% p.a., 4.08% p.a.
Choose 4.1% p.a. compounding monthly.
21 a 3.95% p.a. effective **b** 3.97% p.a. effective
c 3.96% p.a. effective
Choose 3.895% p.a. compounding monthly.

CHAPTER 2 Appreciation and depreciation

Skills check

1 a 1.1 **b** 1.01 **c** 1.0175
d 0.9625 **e** 0.955 **f** 0.9975
2 a $4 **b** $5 **c** $3.95
d 41c **e** $1.10 **f** $4.37
3 a $44 **b** $3 **c** $7.64
d $18 **e** $135.94 **f** $3940
4 a 40% **b** 87.5% **c** 64.5%
d 4.5% **e** 51.4% **f** 1.56%
5 a 1.06 **b** 0.94 **c** 1.04
d 0.97 **e** 1.03 **f** 0.97

Exercise 2A — Inflation and appreciation

1 $20 800

2 a $618 b $48.15 c $3.71
d $579.60 e $932.40

3 a $878.05 b $901.76

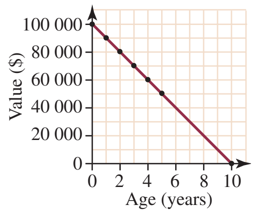
4 $117.90 5 $619 6 $2.52

7 $2.35 8 $122.80 9 E

10 $500 11 $2350 12 $2460

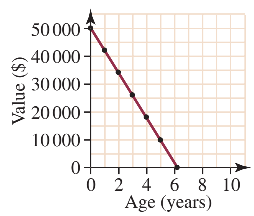
Exercise 2B — Modelling depreciation

1 a



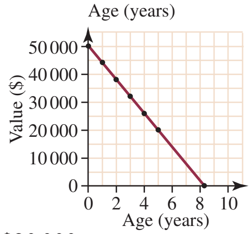
b $V = 100\,000 - 10\,000t$

2



$V = 50\,000 - 8000t$

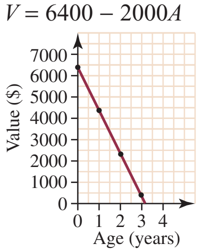
3 a



b $20 000 c 9 years

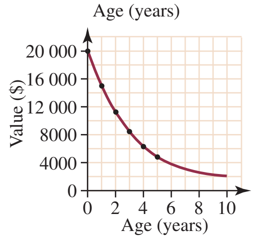
4 a $V = 6400 - 2000A$

b



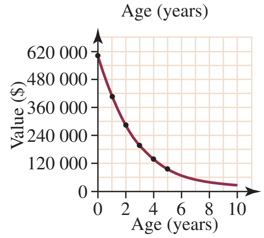
c 4

5 a



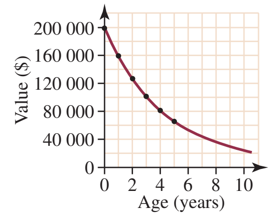
b $2000

6 a



b $17 000 c 7

7 a i $160 000 ii $128 000 iii $102 400
iv $81 920 b



8 B

9 a

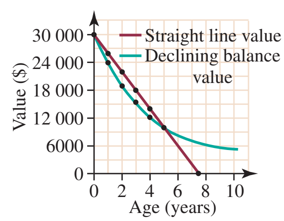
Age (years)	Value ($)
New (0)	30 000
1	26 000
2	22 000
3	18 000
4	14 000
5	10 000

b See part d.

c

Age (years)	Value ($)
New (0)	30 000
1	24 000
2	19 200
3	15 360
4	12 228
5	9 830

d



e 6 years

Exercise 2C — Straight line depreciation

1 $20 000

2 a $1000 b $10 300 c $270 000
d $145 e $32 000

3 a $7 125 000 b $3 750 000

4 $10 600

5 8 years

6 a 6 years b 5 years
c 8 years d 7 years

7 $2500/year

8 a $4000/year b $12 500/year c $14 500/year

9 $900/year

10 $25 000

11 a $110 000 b $26 500 c $1450

12 $78 000

Exercise 2D — Declining balance or diminishing value method of depreciation

1 $20 480

2 a $2220 b i $750 ii $390

3 7 years

4 $383 000

5 a $5900 b $68 100 c $1200
d $62 100 e $3900

6 $6174

7 $676 000

8 a $14 600 b $20 400

9 A

10 C

11 a $5360 b $2640
c $3591 d $1769

12 5 years

10 Quick Questions 1

1 $650 2 $2350/year

3 7 years 4 $21 000

5 $S = V_0\left(1 - \frac{R}{100}\right)^T$ 6 $7250

7 $11 000 8 $389 000

9 $37 500 10 16 years

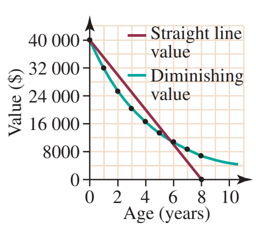
Exercise 2E — Depreciation tables

1 a $1638.50 b $9537.50 c $34 870

2 a

Age of car (years)	Straight line value ($)	Diminishing value ($)
New (0)	40 000	40 000
1	35 000	32 000
2	30 000	25 600
3	25 000	20 500
4	20 000	16 400
5	15 000	13 100
6	10 000	10 500
7	5000	8400
8	0	6700

b

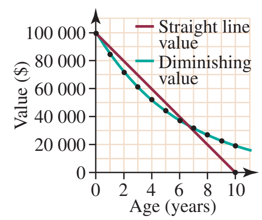


c After 6 years

3 a

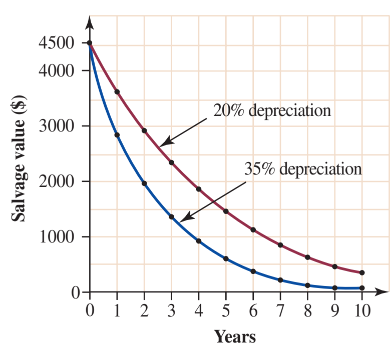
Age of equipment (years)	Straight line value ($)	Diminishing value ($)
New (0)	100 000	100 000
1	90 000	85 000
2	80 000	72 250
3	70 000	61 400
4	60 000	52 200
5	50 000	44 350
6	40 000	37 700
7	30 000	32 050
8	20 000	27 250
9	10 000	23 150
10	0	19 700

b



4

Age of computer (years)	Salvage value at 20% ($)	Salvage value at 35% ($)
1	3520.00	2860.00
2	2816.00	1859.00
3	2252.80	1208.35
4	1802.24	785.43
5	1441.79	570.53
6	1153.43	331.84
7	922.75	215.70
8	738.20	140.21
9	590.56	91.14
10	472.45	59.24



5

Years	Salvage value ($)	Tax deduction ($)
1	4355	2145
2	2918	1437
3	1955	963
4	1310	645
5	878	432
6	0	878

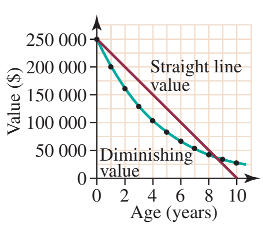
6

Years	Salvage value ($)	Tax deduction ($)
1	33 750	11 250
2	25 313	8438
3	18 985	6328
4	14 239	4746
5	10 679	3560
6	8009	2670
7	6007	2002
8	4505	1502

7 a

Age of truck (years)	Straight line value ($)	Diminishing value ($)
New (0)	250 000	250 000
1	225 000	200 000
2	200 000	160 000
3	175 000	128 000
4	150 000	102 400
5	125 000	81 920
6	100 000	65 536
7	75 000	52 429
8	50 000	41 943
9	25 000	33 554
10	0	26 844

b



c

Age of truck (years)	Salvage value — straight line ($)	Tax deduction ($)
1	225 000	25 000
2	200 000	25 000
3	175 000	25 000
4	150 000	25 000
5	125 000	25 000
6	100 000	25 000
7	75 000	25 000
8	50 000	25 000
9	25 000	25 000
10	0	25 000

Age of truck (years)	Salvage value — declining balance ($)	Tax deduction ($)
1	200 000	50 000
2	160 000	40 000
3	128 000	32 000
4	102 400	25 600
5	81 920	20 480
6	65 536	16 384
7	52 429	13 107
8	41 943	10 486
9	33 554	8389
10	26 843	6711

8 a $10 000 b $\frac{1}{3}$ c $2500
9 a $3000 b $75 c $1600 d $750

Exercise 2F — Future and present value of an annuity

1 $7049.37
2 a $6691.13 b $16 859.14
c $6158.56 d $3974.56
e $17 713.21 f $3530.21
3 $4472.93 4 $90 237.49
5 a $20 326.23 b $24 297.37
c $45 881.32 d $69 770.03
6 D 7 $4067.23
8 a $4524.37 b $7068.59 c $1930.08
9 $4787.76
10 a $1324.00 b $23 932.35
c $7503.81 d $62 953.50
11 a 4% b 10 c $6003.05
12 a $4103.92 b $5335.38 c $7546.74
13 5% for 6 years. $1 will grow to $6.8019 but at 6% for 5 years it will grow to $5.6371.
14 E 15 $6918.50
16 a $1845.09 b $12 289.20
c $4455.79 d $16 604.40

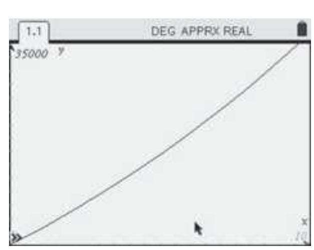
10 Quick Questions 2

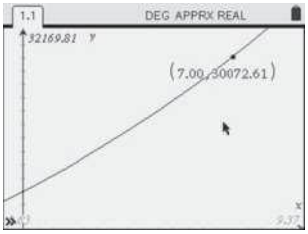
1 $15 937.42 2 $15 937.40
3 $13 537.79 4 $1435.91
5 $5084.04 6 $19 277.16
7 $22 094.93 8 $8513.56
9 $10.64 10 $13 300

Investigation — A growing investment

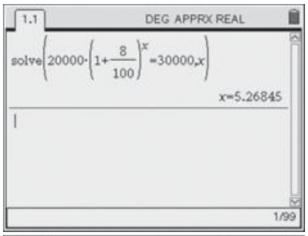
1 $A = 20\,000\left(1 + \frac{6}{100}\right)^{T}$

2



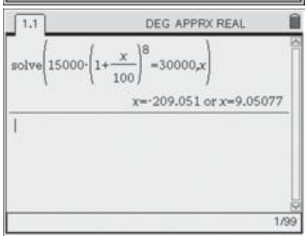
3 

The investment will grow to $30 000 in 7 years.

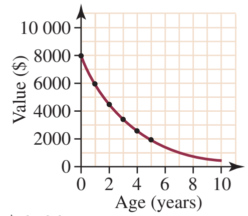
4 

The investment grows to $30 000 in 6 years; that is, a year earlier.

The solved function shows that $30 000 is actually reached in 5.3 years.

5 

Investing $15 000 at 9.05% p.a. will reach $30 000 in 8 years.

8 

9 $6500

10 **a** $1300 **b** $15 000 **c** $235 000

11 12 years **12** $250/year **13** After 6 years

14 $20 880

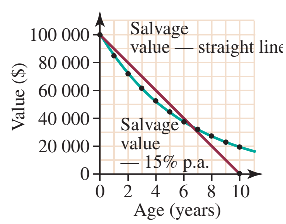
15 $474 000

16 **a** $23 620 **b** $1000 **c** $24 290 **d** $27 210 **e** $49 380

17 **a** $167 100 **b** $432 900

18 **a**

Age (years)	Salvage value — straight line ($)	Salvage value — 15% p.a. ($)
New (0)	100 000	100 000
1	90 000	85 000
2	80 000	72 250
3	70 000	61 400
4	60 000	52 200
5	50 000	44 350
6	40 000	37 700
7	30 000	32 050
8	20 000	27 250
9	10 000	23 150
10	0	19 700

b 

19

Year	Salvage value ($)	Tax deduction ($)
1	3015	1485
2	2020	995
3	1353	667
4	907	446
5	0	907

20 $41 039.20

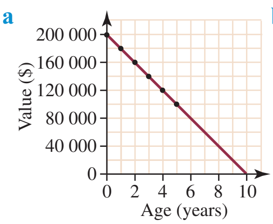
21 **a** $4399.95 **b** $34 641.25 **c** $1842.84 **d** $51 014.25

22 $2242.95

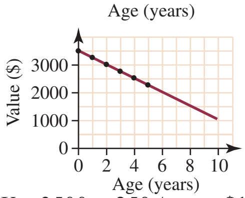
23 **a** $1516.32 **b** $14 047.20 **c** $4055.45 **d** $11 177.64

Chapter review

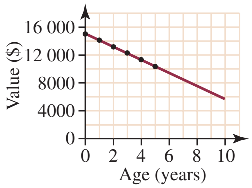
1 $83.44 **2** $149.90 **3** $7900

4 **a** 

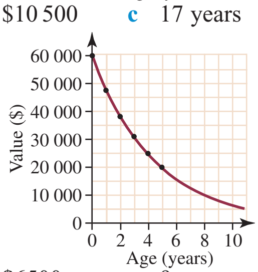
b $V = 200\,000 - 20\,000A$

5 **a** 

b $V = 3500 - 250A$ **c** $1250

6 **a** 

b $10 500 **c** 17 years

7 **a** 

b $6500 **c** 9 years

CHAPTER 3 Consumer credit and investments

Skills check

1 a $\frac{1}{12}$ b $1\frac{1}{2}$ c $\frac{1}{365}$
d $2\frac{1}{2}$ e $\frac{1}{52}$ f $\frac{1}{26}$
2 a 0.1 b 0.01 c 0.025
d 0.0375 e 0.005 f 0.0125
3 a 18 b 33 c 8
d 2 e 69 f 15
4 a i 30 ii 30 iii 31
b i 45 ii 33 iii 56
c i 25 Feb. ii 24 Aug. iii 13 Apr.
5 a $440 b $20 c 37.5%

Exercise 3A — Flat rate interest

1 a $700 b $1200
c $7500 d $2850 e $390
2 $1584
3 $5000
4 a $4060 b $21 330
c $1803.75 d $308.25
e $275 000
5 a $1650 b $3850
c $693 d $6193
6 a $1600 b $6600
c $137.50
7 a $800 b $2800
c $53.85
8 a $2000 b $6000
c $2160 d $8160
e $226.67
9 $43.33
10 B 11 D 12 8% p.a.
13 a $2400 b $9600
c $16 319.88 d 15% p.a.
14 15% p.a.

Exercise 3B — Home loans

1 a $800 b $79 950
2 a $312.50, $49 848.99 b $151.01
c $311.56, $49 697.04 d $151.95
3

Month	Principal ($)	Interest ($)	Balance owing ($)
1	150 000.00	1200.00	149 791.99
2	149 791.99	1198.34	149 582.32
3	149 582.32	1196.66	149 370.97
4	149 370.97	1194.97	149 157.93
5	149 157.93	1193.26	148 943.18
6	148 943.18	1191.55	148 726.72
7	148 726.72	1189.81	148 508.52
8	148 508.52	1188.07	148 288.58
9	148 288.58	1186.31	148 066.88
10	148 066.88	1184.54	147 843.41

4 a

Month	Principal ($)	Interest ($)	Balance owing ($)
1	85 000.00	637.50	84 872.73
2	84 872.73	636.55	84 744.51
3	84 744.51	635.58	84 615.32
4	84 615.32	634.61	84 485.16
5	84 485.16	633.64	84 354.03
6	84 354.03	632.66	84 221.92
7	84 221.92	631.66	84 088.81
8	84 088.81	630.67	83 954.71
9	83 954.71	629.66	83 819.60
10	83 819.60	628.65	83 683.48
11	83 683.48	627.63	83 546.34
12	83 546.34	626.60	83 408.17

b

Month	Principal ($)	Interest ($)	Balance owing ($)
1	85 000.00	637.50	84 837.50
2	84 837.50	636.28	84 673.78
3	84 673.78	635.05	84 508.83
4	84 508.83	633.82	84 342.65
5	84 342.65	632.57	84 175.22
6	84 175.22	631.31	84 006.53
7	84 006.53	630.05	83 836.58
8	83 836.58	628.77	83 665.35
9	83 665.35	627.49	83 492.84
10	83 492.84	626.20	83 319.04
11	83 319.04	624.89	83 143.93
12	83 143.93	623.58	82 967.51

c $440.66
5 $243 123
6 a $302 308.80 b $241 500 c $60 808.80
7 A
8 B
9 a $112 034 b $86 072
c $61 789.40 d $39 329.60
10 a Smith – $6000, Jones – $9000
b i

	Smith family		
Month	Principal ($)	Interest ($)	Balance owing ($)
1	50 000.00	395.83	49 895.83
2	49 895.83	395.01	49 790.84
3	49 790.84	394.18	49 685.02
4	49 685.02	393.34	49 578.36
5	49 578.36	392.50	49 470.86
6	49 470.86	391.64	49 362.50
7	49 362.50	390.79	49 253.29
8	49 253.29	389.92	49 143.21
9	49 143.21	389.05	49 032.26
10	49 032.26	388.17	48 920.43
11	48 920.43	387.29	48 807.72
12	48 807.72	386.39	48 694.11

b ii

	Jones family		
	Principal ($)	Interest ($)	Balance owing ($)
1	50 000.00	395.83	49 645.83
2	49 645.83	393.03	49 288.86
3	49 288.86	390.20	48 929.06
4	48 929.06	387.36	48 566.42
5	48 566.42	384.48	48 200.90
6	48 200.90	381.59	47 832.49
7	47 832.49	378.67	47 461.16
8	47 461.16	375.73	47 086.89
9	47 086.89	372.77	46 709.66
10	46 709.66	369.78	46 329.44
11	46 329.44	366.77	45 946.21
12	45 946.21	363.74	45 559.95

c $3134.16

10 Quick Questions 1

1 $420 **2** $1135.20
3 $7025 **4** $975
5 $5525 **6** $2817.75
7 $8342.75 **8** $231.74
9 $1640 **10** $265 577

Exercise 3C — The cost of a loan

1 11.6%
2 a 11.6% **b** 8.32% **c** 15.2%
d 10.6% **e** 12.2%
3 a 8.32% **b** 8.66% **c** 9.01%
d 9.39% **e** 11.6% **f** 18.3%
4 a $213 996 **b** $128 996 **c** 6.0704%
5 9.01%
6 Loan 1
7 a $231 546 **b** $200 745.60
c $145 593.60
8 Loan 2 – they will save $6041.
9 C
10 a $341 376 **b** $337 578
11 D
12 a $562 279.20 **b** 6.25% **c** 5.8%

Exercise 3D — Loan repayments

1 $674.25
2 a $90.46 **b** $341.25 **c** $819.84
d $1101.00 **e** $1515.54
3 a $400 **b** $3600 **c** $123.05
4 They will not need to increase their repayments.
5 a $1510.20 **b** $1620.14
6 Yes. The repayment is $744 and the most he can afford is $750.
7 a $7000 **b** $1750 **c** $178 000
8 a $733.40 **b** $174.80

Exercise 3E — Bonds, debentures and term deposits

1 $3200 **2** $315 **3** $472.50 **4** $1500
5 $1800 **6** $612.50 **7** B **8** A
9 A **10** D **11** B **12** D
13 a $22.15 **b** $84.99 **c** $297
14 a i $1406.25 **ii** $1350 **iii** $1321.88
b No difference

Exercise 3F — Bank savings accounts

1 a $2066.10 **b** $9.47
2 a 8 cents **b** 12 cents
3 9%
4

Date	Transaction	Debit	Credit	Balance
1 May	Balance B/F			2132.20
3 May	Cheq 4217	460.27		1671.93
7 May	Deposit		230.16	1902.09
17 May	Cheq 4218	891.20		1010.89
26 May	Wages		1740.60	2751.49
31 May	Interest		5.69	2757.18
2 June	Deposit		415.10	3172.28
8 June	Cheq 4220	2217.00		955.28
19 June	Cheq 4219	428.50		526.78
21 June	Cheq 4222	16.80		509.98
23 June	Wages		1740.60	2250.58
30 June	Interest		2.87	2253.45
1 July	Deposit		22.80	2276.25
4 July	Cheq 4221	36.72		2239.53
18 July	Cheq 4223	280.96		1958.57
26 July	Wages		1740.60	3699.17
31 July	Interest		11.02	3710.19

5 a $33.95 **b** $14.37
6

Date	Transaction	Debit	Credit	Balance
4 Aug	Salary		1410.20	1410.20
5 Aug	Health fund	327.60		1082.60
10 Aug	Health fund		68.20	1150.80
15 Aug	Electricity a/c	150.26		1000.54
18 Aug	Salary		1410.20	2410.74
20 Aug	Rent	620.80		1789.94
30 Aug	Visa	180.00		1609.94
31 Aug	Interest		6.25	1616.19
1 Sept	Salary		1410.20	3026.39
2 Sept	Telephone a/c	180.64		2845.75
5 Sept	Tax refund		461.27	3307.02
5 Sept	Health fund	327.60		2979.42
15 Sept	Salary		1410.20	4389.62
20 Sept	Rent	620.80		3768.82
29 Sept	Salary		1410.20	5179.02
30 Sept	Interest		17.79	5196.81

7 a i $6.25 **ii** $15.06 **iii** $8.81
b i $4.79 **ii** $4.77 **iii** –$0.02
c i $10.94 **ii** $16.86 **iii** $5.92

10 Quick Questions 2

1 $2835 **2** 160% p.a.
3 $299 191.20 **4** 10.3%
5 6.0% **6** $45.23
7 $264.40 **8** $375
9 $5825 **10** $2.23

Exercise 3G — Investing in real estate

1 a $2448.75 b $3656.25 c $12 000
2 a $244.88 b $365.63 c $1200
3 a $77 256.37 b $124 228.12 c $448 800
4 $80 750
5 a $1873 b $2693 c $10 249
6 a $189 123 b $271 933 c $552 369
7 a $266 942.50 b $482 875 c $15 932.50
8 $17 995
9 $127 500
10 $385 000
11 $620 000

Exercise 3H — Investing in the stock market

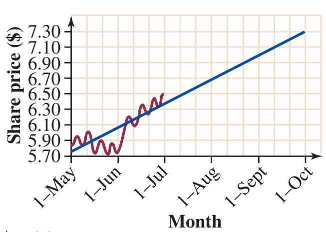
1 $18 808.13
2 $2479.95
3 $5170.05
4 $15 950.40
5 25 c/share
6 $1.50/share
7 6 c/share
8 29.27 c/share
9 a $1.224 million b $2.176 million
c 43.52 c/share
10 a $9.28 million b $6.08 million
c 76 c/share
11 $3.276 million
12 4.57%
13

Dividend	Share price	Dividend yield
$0.56	$8.40	6.7%
$0.78	$7.40	10.5%
$1.20	$23.40	5.1%
$1.09	$15.76	6.9%
$0.04	$0.76	5.3%

14 2.91%
15 B
16 $364
17 a 6.6% b $1.06/share
18 a $1.44 b $7.488 million
19 8.5%
20 a 0.59% b $10.64
c 6.4 c/share d 0.6%
21 a $77.50 b 1.2% c 82.6
22 a $60 b 1.2% c 83.3
23 a 5000 b 0.75% c 133.3
24 23.3

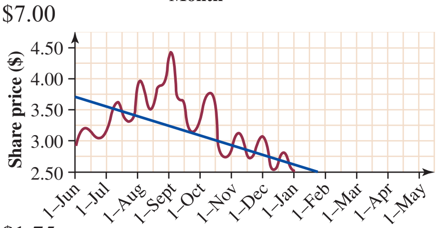
Exercise 3I — Graphing share performance

1 a



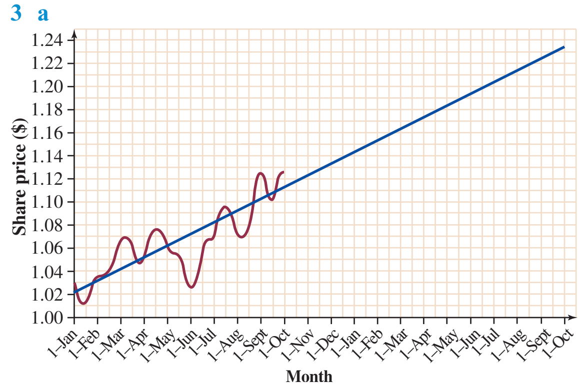
b $7.00

2 a



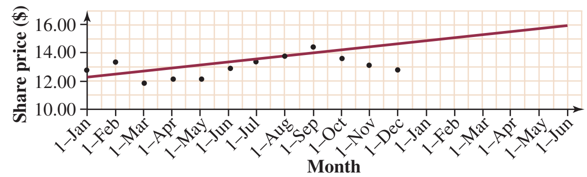
b $1.75

3 a



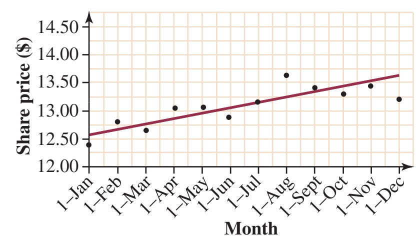
b $1.20

4 a



b $16.00

5 a



b $14.50

History of mathematics — The Wall Street Crash

1 The Wall Street Crash was caused mainly by panic, which caused there to be more sellers and virtually no buyers in the market.
2 Share prices declined rapidly.
3 The crash caused the collapse of the American and other economies throughout the world, paralysing businesses and causing widespread poverty.

History of mathematics — The Dow Jones Industrial Average

1 Wall Street Journal journalists Charles Dow and Eddie Jones
2 30
3 Sum of 30 stock prices divided by 0.2252
4 Technology, telecommunications

Chapter review

1 a $1120 b $7187.50 c $1281.60
d $39.60 e $12 285.00
2 $6760
3 $191.02
4 6.15%
5 a $1250 b $124 873.64

6 a

Month	Principal ($)	Interest ($)	Balance owing ($)
1	130 000.00	866.67	129 779.30
2	129 779.30	865.20	129 557.13
3	129 557.13	863.71	129 333.47
4	129 333.47	862.22	129 108.32
5	129 108.32	860.72	128 881.67
6	128 881.67	859.21	128 653.51
7	128 653.51	857.69	128 423.83
8	128 423.83	856.16	128 192.62
9	128 192.62	854.62	127 959.87
10	127 959.87	853.07	127 725.57
11	127 725.57	851.50	127 489.70
12	127 489.70	849.93	127 252.26

b

Month	Principal ($)	Interest ($)	Balance owing ($)
1	130 000.00	866.67	129 366.67
2	129 366.67	862.44	128 729.11
3	128 729.11	858.19	128 087.30
4	128 087.30	853.92	127 441.22
5	127 441.22	849.61	126 790.83
6	126 790.83	845.27	126 136.10
7	126 136.10	840.91	125 477.01
8	125 477.01	836.51	124 813.52
9	124 813.52	832.09	124 145.61
10	124 145.61	827.64	123 473.25
11	123 473.25	823.16	122 796.41
12	122 796.41	818.64	122 115.05

c $5137.21

7 a $596 844 b $18 884

8 a 7.25% b 13.70%
c 25.65% d 14.11%

9 a $18 223.20 b $4723.20 c 7%

10 Loan 2

11 $21.15

12 a $316.75 b $599.40
c $2110.45 d $5100

13 a $2453.49 b $2618.06

14 C 15 E 16 A

17 $2700 18 $694.17 19 $2520

20 $5000 21 A

22 a $1.08 b $1.15

23 a $3.33 b $3.64

24 $10 575 25 $901.63 26 $236 425.45

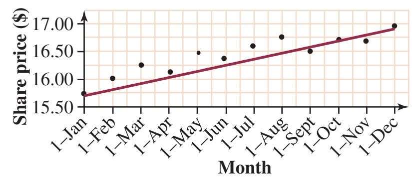
27 $475 166 28 $15 548.05 29 $47 352.75

30 $1.93/share 31 $14.74/share 32 5.22%

33 1.6% 34 81.8 c/share

35 a $260 b 2.5% c 40.4

36 a



b $18.00

CHAPTER 4 Exploring and understanding data

Skills check

1 a 0.375 b 0.083 c 0.813 d 0.590

2 a 75% b 12.5% c 42.5% d 4%

3 Answers will vary.

4 a 4 b 4 c 3 d 7 e 1

5 a $a = 8$ b $b = 9$ c $c = 22.5$
d $d = 17.5$ e $e = 10.5$

6 Scale on axes, omitting certain values, giving a 3D visual impression, using a non-linear scale on the axes

7 a 73 b 7.3 c 7 d 6
e 6 f 8 g 6 h 2

Investigation — Australia's population and housing census

1 This is a statistical collection of data to determine the number of people in Australia on Census Night, the characteristics of these people and the dwellings in which they live.

2 All people in Australia on Census Night take part.

3 It is compulsory.

4 Questions asked include: age, marital status, birthplace, income, type of dwelling, type of job… The questions have changed over the years to take into account changing social conditions of the population; such as language spoken at home, computer usage…

5 A census can provide information necessary for future planning.

6 The ABS has access to the information and details of individuals are protected by the Privacy Act.

7 All dwellings are issued with census booklets, which are delivered and collected by ABS workers. The booklets are completed by all individuals on the same night.

Exercise 4A — Populations and samples

1 Census, sample

2 Census — every member of the population participates

3 Survey

4 a Survey b Survey c Census
d Census e Survey

5 a Survey b Census c Census
d Sample

6 Survey

7 80, 84, 71, 85, 79, 54, 56, 51, 81, 67

8 Range of answers

9 Range of answers

10 Should be three different sets of numbers

Exercise 4B — Bias

1 Check with your teacher.
2 Check with your teacher.
3 a Sample does not represent characteristics of population
b No control over responses
c Unrepresentative sample
d Abnormal conditions
e Only extreme groups in sample
4 The decrease in the value of the Australian dollar compared with the American dollar is accentuated by the large scale on the y-axis. The decrease is actually only 2 cents. The scale on the x-axis is not uniform (9 May, 11 May, 12 May).
5 What type of university tests? What do the terms 'consistently', 'majority', 'more effective', 'most other' mean? No hard evidence has been provided to support the claim.
6 a There would be many more student drivers in Year 12 than in Year 11 — perhaps also some in Year 10.
b Students with part-time jobs are in lower year levels as well.
c Residents not at the neighbourhood watch meeting have been ignored.
d Other music students who play instruments and don't belong to the choir have been excluded.
e The composition of cars in a shopping centre car park is not representative of the cars on the road.
f Females have been excluded.
g Users of the local library would not reflect the views of teenagers.

Investigation — Contingency tables from census data

1 a 9.0% b 42.8%
c Part (a) compares the number of males in the retail trade with the total number of male workers, while in part (b) the comparison is with the total number in the retail trade.
d It would be easier to survey those in the retail trade rather than surveying the male population, as the former number is smaller than the second.
2 Choose another category to survey.
3 Percentage of persons in agriculture, forestry and fishing

$$= \frac{280\,912}{9\,104\,184} \times 100\%$$
$$= 3.1\%$$

4 Analyse data from the 2001 census.

Exercise 4C — Contingency tables

1

	Test results		
	Accurate	Not accurate	Total
With virus	98	2	100
Without virus	388	12	400
Total	486	14	500

2

	Test results		
	Accurate	Not accurate	Total
Telling truth	777	23	800
Telling lies	156	44	200
Total	933	67	1000

3 a 1000 b 75 c 96.7% d 60%
4 a 200 b 44 c 90.9% d 5.1%
e 94% f Check with your teacher.
5 C 6 E 7 A
8 a

	Test results		
	Accurate	Not accurate	Total
Bags with prohibited items	48	2	50
Bags with no prohibited items	145	5	150
Total	193	7	200

b i 96% ii 3.3% iii 4% iv 96.5%
9 a ii b ii c ii
10 a

	Male	Female
Education	212 813	484 993
Other	4 698 322	3 708 056

b

	Male	Female	Total
Education	212 813	484 993	697 806
Other	4 698 322	3 708 056	8 406 378
Total	4 911 135	4 193 049	9 104 184

c

	Male	Female
Education	4.3%	11.6%
Other	95.7%	88.4%
Total	100%	100%

d

	Male	Female	Total
Education	30.5%	69.5%	100%
Other	55.9%	44.1%	100%

e 11.6%
f 69.5%
g Survey a sample of those in education rather than conducting a survey on a sample of females as the total education group is fewer in number than the total female group.

11 a

	Male	Female	Total
15–19 years	3 749	3 819	7 568
All other ages	44 647	43 249	87 896
Total	48 396	47 068	95 464

b **i** Yes

ii No — 7.7% for males and 8.1% for females

12 a

	Male	Female	Total
Attending an educational institution	2171	2623	4794
Not attending an educational institution	1578	1196	2774
Total	3749	3819	7568

b **i** 58% **ii** 45%

c No — 69% attend

10 Quick Questions 1

1 Census 2 Survey 3 Census

4 The question leads the responder to an expected answer of 'yes' by using emotional words and ideas.

5 Should the tax rates for upper-income earners be raised?

6 Those supporting one side of the issue may be more motivated to call in; certain groups of the population may be misrepresented among viewers of the program; it is costly to make the call and this might deter some.

7 200 nights 8 44 times 9 90.9%

10 5.1%

Exercise 4D — Interpreting the shape of histograms, stem-and-leaf plots and boxplots

1 **a** Symmetric
b Negatively skewed
c Positively skewed
d Symmetric
e Symmetric
f Positively skewed

2 **a** Symmetric
b Symmetric
c Symmetric
d Negatively skewed
e Negatively skewed
f Positively skewed

3 E 4 C

5 Negatively skewed

6 Positively skewed. This tells us that most of the flight attendants in this group spend a similar number of nights interstate per month. A few stay away more than this and a very few stay away a lot more.

7 **a** Symmetric
b This tells us that there are few low-weight dogs and few heavy dogs but most dogs have a weight in the teens (in kg).

8 **a** Symmetric
b Most students receive about $8 (give or take $2).

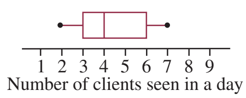
9 **a** Positively skewed
b This indicates that most workers do up to 3 hours of exercise per week. Very few do more and the most time spent is 8 hours.

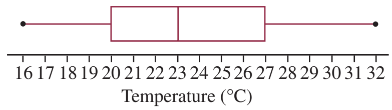
10 **a** III **b** IV **c** I **d** II

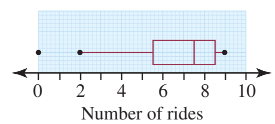
11 The boxplots should show the following:

	Minimum value	Q_1	Median	Q_3	Maximum value
a	3	6	8.5	14	18
b	3	5	7	9	12
c	4.3	4.6	5	5.4	5.6
d	11	15.5	18	20	22
e	0.4	0.7	0.9	1.1	1.3

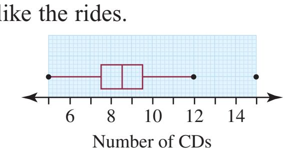
12 D

13


14


15


The data are negatively skewed with an outlier on the lower end. The reason for the outlier may be that the person wasn't at the show for long or possibly didn't like the rides.

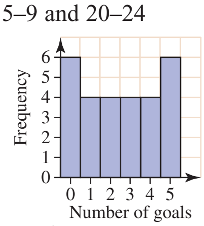
16


The data are symmetric, although there is an outlier on the upper end. The reason for the outlier may be that the person loves music CDs or is able to get CDs more easily.

Exercise 4E — Interpreting data in practical situations

1 **a** Yes **b** 3 **c** Yes, both equal 3 **d** 3

2 **a** No **b** 5–9 and 20–24 **c** No
d 5–9 and 20–24 **e** 25–29

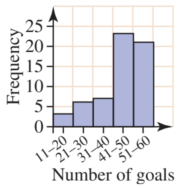
3 **a**


b Yes

c 0 and 5 **d** Yes, both equal 2.5
e 0

4 **a** 2 **b** 2 **c** 5

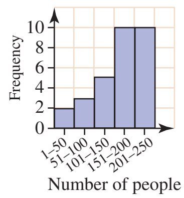
5 **a**



b 0.73

6 A

7 **a**



b No

c 151–200 and 201–250
d 0.67
e 0.17

8 **a** Chemistry is symmetrical.
Maths is not symmetrical.
b Chemistry: mode = 41–50 and 81–90,
Maths: mode = 71–80
c Maths, because there are more scores further away from the centre of the distribution.
d Yes, both 0.275
e Mathematics
f $P(>90\%$ Chem$) = 0.05$
$P(>90\%$ Maths$) = 0.1$

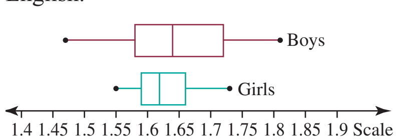
9 157 **10** About 32 visitors

11 **a** 7 **b** 18

12 **a** Lines vary.
b Factory 1 is cheaper at \$43.21 (compared to Factory 2 at \$56.61).
c Factory 2 is cheaper at \$167 (compared to Factory 1 at \$217).
d Factory 2 is marginally more linear.

13 **a** English 66, Maths 63.5
b English 32, Maths 53
c The marks are more spread in Maths than in English.

14 **a**



b Boys 1.64 m, girls 1.62 m
c Boys 0.34 m, girls 0.18 m
d Boys 0.14 m, girls 0.07 m
e The spread of heights is much greater among boys than among girls.

15 **a** Year 7: range = 0.4, Year 12: range = 0.26
b Year 7: interquartile range = 0.15,
Year 12: interquartile range = 0.11
c The range of heights is greater in Year 7 as shown by the range and the IQR. The heights become less spread by the time they get to Year 12.

16 **a** 43.2% **b** 1.9%
c 0.9% **d** 2.6%
e More evident in males with three times the number of drivers over the limit

17 **a** 90.5% **b** 55.6%
c Yes, as a much greater percentage of games are won with Ashley playing.

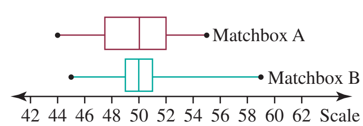
18 **a** Chemistry, 69.25
b Physics, because of the lower standard deviation

19 **a** Point A: $\bar{x} = 61$, SD = 4.5,
Point B: $\bar{x} = 58.8$, SD = 12.7
b Point A because of the higher mean
c Point B because of the greater standard deviation

20 **a** Aaron: $\bar{x} = 38.1$, Sunil: $\bar{x} = 39.3$
b Aaron: range = 76, Sunil: range = 65
c Aaron: interquartile range = 16,
Sunil: interquartile range = 57
d Aaron is more consistent because although he has a larger range this is caused by one outlier. Aaron's interquartile range is much less, showing his consistency.

21 A

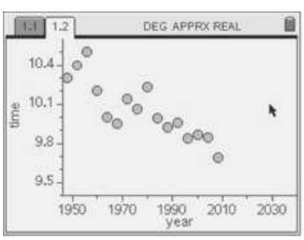
22 **a**



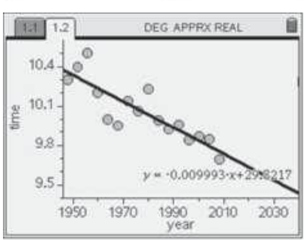
b Machine A: $\bar{x} = 49.96$, SD = 2.96,
Machine B: $\bar{x} = 50.12$, SD = 2.49
c Machine B has a lower standard deviation and so is more dependable.

Investigation — Modelling Olympic Games times

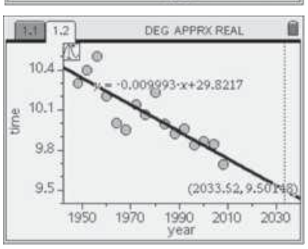
1 Scatterplot



2 Line of best fit



3 Prediction



The line of best fit predicts a time of 9.5 seconds in the year 2034. The Olympic Games closest to this year is 2032 or 2036.

Investigation — The door game

Part II

P(winning if stay) = $\frac{1}{3}$

P(winning if change mind) = $\frac{2}{3}$

If you change your mind you will double your chance of winning from 1 in 3 to 2 in 3.

10 Quick Questions 2

1 23.3
2 21.5
3 16
4 29
5 5
6 7.93
7 No
8 Yes, 45 is an outlier.
9 Median, because the outlier inflates the mean
10 The outlier makes the range very large.

Chapter review

1 **a** Survey **b** Census **c** Census **d** Survey

2 E

3 Check with your teacher.

4 Check with your teacher.

5

	Test results		
	Accurate	**Not accurate**	**Total**
With virus	48	2	50
Without virus	149	1	150
Total	197	3	200

6

	Test results		
	Accurate	**Not accurate**	**Total**
Telling truth	77	3	80
Telling lies	17	3	20
Total	94	6	100

7 **a** 140 **b** 30 **c** 90% **d** 10%

8 **a** 130 **b** 33.8% **c** 97.5%

9 B

10 **a** 200 **b** 96% **c** 34 **d** $93\frac{1}{3}$% **e** 93%

11 **a** 9.7% **b** 8.0% **c** No significant difference

12 C

13 **a**

Attitude	**Primary**	**Secondary**
Fewer	7.5%	4.3%
Same	43.3%	19.1%
More	49.2%	76.6%
Total	100%	100%

b Secondary students were much keener on having more holidays than were primary students.

14 C

15 A

16 **a** Positively skewed

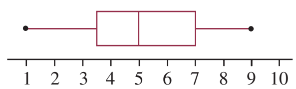
b There would need to be a shift of some of the amounts in the twenties to the thirties and forties.

17 E

18 B

19 D

20 **a**

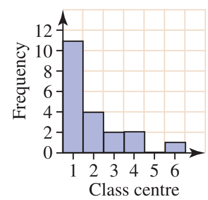


b Approximately symmetric

21 **a** Yes **b** Both are 17.5. **c** 17 and 18 **d** 17 and 18

22 **a** A variety of answers **b** About 130

23 **a**



b No

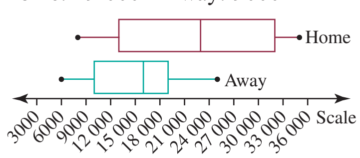
c 0.15

24 **a** Home: 23 000 Away: 16 000

b Home: 27 000 Away: 19 000

c Home: 19 000 Away: 9000

d



25 **a** Last year: median = 7, this year: median = 8

b This year

c Last year

d This year — Higher median, lower limit, lower quartile and upper limit

26 **a** English: $\bar{x}$ = 71, Maths: $\bar{x}$ = 69.3

b English: range = 43, Maths: range = 37

c English: SD = 11.6, Maths: SD = 12.0

d English, because of the lower standard deviation

CHAPTER 5 Navigation

Skills check

1 Lines of latitude run parallel to the equator. Lines of longitude run from one pole to the other and are east or west of Greenwich.

2 0° 3 0° 4 Latitude 5 $C = 2\pi r$

6 40 030 km 7 Tangent = $\frac{\text{opposite}}{\text{adjacent}}$

8 Speed = $\frac{\text{distance}}{\text{time}}$

9 The time at the prime meridian (0° longitude)

10 A triangle which has 2 sides congruent, and base angles congruent

Exercise 5A — Review of Earth geometry

1 a (30°N, 60°W) b (40°S, 20°W)
c (30°S, 50°E) d (40°N, 60°W)
e (20°N, 20°W) f (30°S, 20°E)
2 Any 2 meridians; for example, NDS, NGS; or any line of longitude; for example, 20°W
3 a 40° b 30°
c 10° d 60°
4 a Johannesburg b Shanghai
c Montreal d Perth
5 a (35°N, 118°W) b (35°S, 20°E)
c (0°, 100°E) d (38°N, 115°E)
6 a 4448 km b 7784 km
c 6672 km d 7339 km
7 a 7784 km b 6450 km
8 4226 km

Exercise 5B — Accurate position description

1 a 27°9.6′S, 153°36′E b 27°S, 153°45.9′E
c 27°S, 153°36′E d 27°0.9′S, 153°37.6′E
e 27°1.1′S, 153°33.6′E f 27°8′S, 153°44.5′E
2 Sketch
3 a Mt Sydney b Black Island
c Pinnacle Point
4 a 20°2.2′S, 148°52.7′E b 20°4.3′S, 148°58.3′E
c 20°4.8′S, 148°52.2′E d 20°10′S, 148°53.6′E
e 20°10.5′S, 148°55′E

Exercise 5C — The nautical mile and the knot

1 a 120′ b 150′ c 1422′ d 2871.7′
2 a 9°43′ b 39°8.7′
3 a 17°17′ b 57.3′
4 a J, D b A, H c H, I
d i 50°N, 80°E ii 0°, 0° iii 60°S, 0°
iv 0°, 30°W v 50°N, 0°
e i 2400 n mile ii 2400 n mile iii 5400 n mile
iv 9000 n mile v 9000 n mile
f i 6600 n mile ii 6600 n mile
g i 3600 n mile ii 3600 n mile iii 3000 n mile
5 a 1650 n mile b 3750 n mile
c 7050 n mile d 1110 n mile
6 8 knots
7 a 3.5 knots b 6.5 km/h
8 a 3.85 knots b 12.6 knots
c 289 n mile d 52.1 n mile
e 30 hours f 10 minutes
9 a 7872 hours
b 6.4 km/h, 3.4 knots
10 a 3600 n mile
b i 3600 n mile
ii The Earth is a sphere and any arc joining 2 points on its surface subtending an angle of 60° must be separated by the same distance.
c 200 hours
11 a 570 n mile b 4.63 knots
12 3.08 am
13 A separation of 1′ near the equator on a line of latitude is greater than that further from the equator.

10 Quick Questions 1

1 Latitude 2 Latitude 3 60′
4 1852 metres 5 150 n mile
6 Speed = $\frac{\text{distance}}{\text{time}}$ 7 The knot
8 6 knots 9 5400 n mile 10 5 pm

Investigation — Distance to the horizon

1 Angle PHC = 90° (PH is a tangent to the circle, so CH is perpendicular to PH.)
2 $PC^2 = CH^2 + HP^2$ (by Pythagoras' theorem)
3 CH = AC (Both are radii of the Earth; both = 6371 km.)
4 a 25.2 km b 79.8 km c 112.9 km d 357.1 km
5 As height increases, distance also increases. (On a flat Earth, distance to horizon would be greater.)

Exercise 5D — Using the compass

1 a 128°C b 292°C c 193°C d 40°C
2 291°T 3 6°E
4 a 120°C b 226°C c 4°W d 257°29′T

Exercise 5E — Compass bearings and reverse bearings

1 a 50°T b 300°T c 230°T d 145°T
2 a 230°T b 120°T c 50°T d 325°T
5 a 6 n mile b 5 n mile c 11.2 n mile
d 11.4 n mile e 15.7 n mile f 10.9 n mile
6 b 12 knots
7 a 187°T b 176°C c 50 min d 356°C

10 Quick Questions 2

1 Compass 2 True north
3 Magnetic variation 4 Compass
5 True 6 60°T 7 158°C 8 180°
9 Latitude 10 7 knots

Exercise 5F — Fixing position

1 a Check with your teacher.
b 155°T c 7.5 n mile
d 15 knots e 3.5 n mile
2 a Check with your teacher.
b 9.3 n mile c 18.6 knots
3 a 131°T, 18°T, 299°T b 198°T, 340°T, 265°T
c 11 knots d A 219°, B 293°, C 254°
4 a Tower 53°, Antenna 88°
5 b 14 n mile c 14 n mile d 19.8 n mile
e i 243° ii 252°
6 b 56 n mile c 250°

Exercise 5G — Transit fix

1 b i 6 n mile ii 5.6 n mile iii 5.3 n mile
c Plot d 20 knots
2 c B 250°, D 291°, E 316°

Exercise 5H — Running fix

1 b 6 n mile
2 b 197°

Exercise 5I — Doubling the angle on the bow

1 a i 50° ii 130° iii 80° iv 3 n mile
b i 100° ii 40° iii 7 n mile
c i 42° ii 96° iii 84° iv 11 n mile
d i 130° ii 25° iii 6.5 n mile iv 45°T
e i 45° ii 90° iii 45° iv 10 n mile
f i 20° ii 140° iii 20° iv 8 n mile
v 8 n mile

2 a 25° b 5 n mile

3 b At 1300, 30°; at 1330, 60°
c 9 n mile d 9 n mile

4 b 8 n mile c 6.2 n mile (from sketch)
d 7.03 am e 100°

10 Quick Questions 3

1 Two 2 Cocked hat 3 Transit line
4 Isosceles 5 Front
6 Angle on the bow
7 9 knots 8 4 hours
9 84 n miles 10 One

Exercise 5J — Dead reckoning

1 a Check with your teacher.
b 20°08.3′S, 148°59.7′E
c Check with your teacher.
d i 20°07.3′S, 149°01.5′E
ii 20°06.3′S, 149°03.3′E
iii 20°05.3′S, 149°04.9′E

2 a 20°07.3′S, 149°14′E
b 20°00.2′S, 149°07′E
c 20°05.2′S, 149°09.2′E
d 20°01.4′S, 149°07.3′E

3 a 3.8 n mile b 11.4 knots
c i 20°05.4′S, 149°04.2′E
ii 20°03.3′S, 149°01′E

Exercise 5K — The lighthouse and navigation

1 a AB = 10.5 m b 1908 m
c 1°48′ d 1.345 n mile

2 a 4 short flashes of light followed by a long period of darkness every 20 seconds
c 3105 m

3 a 2 flashes, then darkness every 12 seconds
c 5156 m f 7.6 n mile
g i 0.48° ii 0.36°

4 b 6875 m d 0.68°

Exercise 5L — Let's go cruising

1 a 11°18′ east b 11° east

2 a 27°30.9′S, 153°20.7′E
b 27°30.1′S, 153°22.4′E
c 27°32.7′S, 153°25.2′E
d 27°30.6′S, 153°17.3′E
e 27°34.8′S, 153°21.6′E

3 a Coochiemudlo Island b The Bluff
c Submerged rocks d Myora Light

4 a i 308° ii 338° iii 0° iv 266°
b i 297° ii 327° iii 349° iv 255°
c i 5.1 n mile ii 5.1 n mile
iii 4.8 n mile iv 2.6 n mile

5 a i Yellow light flashes every 2.5 seconds
ii Red every 4 seconds
iii Green every 6 seconds
b So that they can be readily identified as different from neighbouring lights

6 a 5 n miles b 150°T, 139°C
c 33 minutes d 10.58 am
e A southwest wind could push the vessel towards the rocks near Goat Island.

7 Approx. 15 n mile, so approx. 160 litres

8 a 351°T, 227°T b 27°32.8′S, 153°21.6′E

Exercise 5M — Air navigation

1 a 26°15′S, 151°56′E b 26°40′S, 152°00′E
c 26°17′S, 152°41′E d 26°33′S, 151°51′E

2 a Tansey b The Bluff c Abbeywood

3 a 1998 b 2457 c 2043

4 a 350°T, 339.5°C b 05°T, 355°C

5 a 149°30′C
b Barambah Ck, Clonya, Murgon, Nanango

Chapter review

1 a A, 30°N, 60°W; B, 40°S, 20°W; C, 30°S, 50°E; D, 40°N, 60°W
b NDS, NGS, NHS or any line of longitude (for example, 40°W)
c F
d PG, PN, PH, PC etc.

2 a 20°2.2′S, 148°52.7′E
b 20°10′S, 148°53.7′E

3 a 5400 n mile b 10 800 n mile
c 4200 n mile d 7920 n mile

4 a 1080 n mile b 5340 n mile

5 a 360′ b 1110′ c 1695′ d 3457.4′

6 540 n mile, 1000 km

7 a 6 knots b 13.3 knots c 522 n mile
d 198.3 n mile e 50 hours f 15 minutes

8 6 hours 24 minutes

9 a 400 n mile b 180°T
c 5 hours d 6.45 pm

10 a 114°C b 253°C
c 6°W d 206°T

11 286°T

12 b 9.8 n mile c 29.4 knots d 48°T

13 $a = 55°$, $b = 125°$, $c = 70°$, PR = 7.4 n mile

14 a 18° b 36°
c 13 n mile d 13 n mile

15 b 50°, 100° c 12 n mile d 12 n mile

16 a 5.57 m b 65.8 m
c 1.68° d 5810 m

17 a 3 short flashes then long period of darkness every 16 seconds
b 130 m c 16 n mile d 2480 m

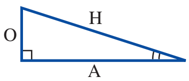
18 a 8.1 n mile b 8°T
c Approx. 1 h 20 min trip, ETA 8.20 am

19 a 240°T b 16.5 n mile
c Plot d 11.45 am

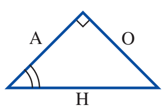
CHAPTER 6 Land measurement

Skills check

1 Millimetre, centimetre, metre, kilometre
2 Perimeter
3 a 24 cm b 30 m c 15.6 cm d 16.8 m e 12.6 m f 36.0 m g 38 m
4 a l^2 b $l \times w$ c πr^2 d $b \times h \times \frac{1}{2}$ e $l \times h$
5 a 1.5 cm b 0.18 m c 12 300 cm d 680 m e 12 500 m
6 a 40.7 m^2 b 435.8 m^2 c 51.7 m^2 d 177 m^2 e 25 m^2
7 a 0.9397 b 0.9659 c 0.3249
8 $c^2 = b^2 + a^2$
9 a 5 cm b 8 cm c 9.6 m
10 a



b



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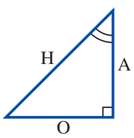


11 Sine = $\frac{\text{opposite}}{\text{hypotenuse}}$ Cosine = $\frac{\text{adjacent}}{\text{hypotenuse}}$

Tangent = $\frac{\text{opposite}}{\text{adjacent}}$

12 a 6.8 cm b 7.7 m c 65.0 m

Exercise 6A — Perimeters and areas of triangles

1 a 1.73 cm b 23.1 m c 11.4 m
2 a 1.73 cm^2 b 277 m^2 c 55.3 m^2
3 a 12.5 m^2 b 4.5 m^2 c 7443 m^2 d 118.2 m^2
4 a 26 m, 13 m b 90.9 m, 45.45 m c 42.4 km, 21.2 km
5 a 27.9 m^2 b 250 m^2 c 52.4 km^2

Exercise 6B — Perimeters and areas of polygons

1 a 5797 m^2 b 1062 m^2 c 27 952 m^2
2 a 97.4 m^2 b 3195 m^2
3 Approx. 36 ha

Exercise 6C — Surveying on level ground without obstacles

1 a 32 m b 28 m c 59 m d 73 m e 47 m
2 a 86.3 m b 107.5 m c 47.4 m d 49.0 m
3 Sketch
4 a 120 m b 5
c i 48 m ii 39 m iii 37 m iv 32 m v 35 m
d i 65.8 m ii 44.7 m iii 34.4 m iv 90.2 m
e i 43.15° ii 1014 m^2 iii 81.2°
f Sketch
g AB survey line established and measured. Staffs at features Z and C, measurements taken. Staffs at V and D, measurements taken.
h 2340 m^2

10 Quick Questions 1

1 5.47 m
2 $\frac{1}{2} \times a \times b \times \sin C$
3 48.5 m^2, 196.6 m^2
4 $\sqrt{S(S-a)(S-b)(S-c)}$
5 24.2 m^2
6 AB
7 KF, JE, ID, HC
8 36 m, 59 m, 73 m, 32 m, 84 m
9 240 m^2
10 a 51.2° b 59 m

Exercise 6D — Surveying around obstacles

1 a 42 m b 31 m c 52 m
2 Sketch
3 a Sketch
b The distance between the second and third staffs placed by Peter

Exercise 6E — Plane table surveying: intersection or triangulation

1 a 61 m
b i 43 m ii 28 m iii 106 m iv 124 m
c i 065° ii 310° iii 180° iv 150°
d 301 m e 0.38 ha
2 a 59 m
b i 28.5 m ii 31.5 m iii 32 m iv 73 m v 49 m
c 0.25 ha
d i 15° ii 81° iii 151°
3 a 83 m b $1328 c 450 m^2
4 a Sketch
b i 100 m ii 66 m iii 50 m iv 90 m v 86 m
c i 270° ii 310° iii 240°
d 4300 m^2

Exercise 6F — Plane table surveying: radiation and traversing

1 a i 25 m ii 35 m iii 30.5 m iv 51.5 m
b i 0° ii 070° iii 180°
c 1900 m^2
2 a i 23 m ii 72 m iii 51 m iv 12.5 m v 104 m vi 109 m vii 75 m
b 0.31 ha
3 a Radiation b Sketch
c A, 123°; B, 136°; C, 152°; D, 180°
d 3160 m^2
4 a Sketch b Traversing
c i 212° ii 270°
d i 107 m ii 77 m e 3800 m^2

10 Quick Questions 2

1 Offset and triangulation
2 41 m, 10 m
3 Intersection (or triangulation), radiation, traversing
4 Intersection
5 Radiation

6 Traversing
7 287.5 mm
8 4 triangles
9 Area = $\sqrt{S(S-a)(S-b)(S-c)}$ = 310 m^2
10 10 m^2

Exercise 6G — Levelling: vertical measurements in relation to a datum

1 a i 50.00 m ii 51.69 m
b 1.94 m c 53.63 m
d

Sta.	BS	IS	FS	HI	RL	Dist.	Notes
A	3.63			53.63	50.00	0.00	TBM
B			1.94	53.63	51.69	20.00	

2 a i 3.60 m ii 2.80 m iii 53.60 m iv 50.00 m
b 50.80 m
3 a 61.25 m b 61.25 m c 61.25 m
d 61.25 m e 60.00 m f 59.50 m
g 58.75 m h 58.25 m i 5.00 m
j 10.00 m k 15.00 m

Exercise 6H — Topographic maps

1 Easting 84, northing 46
2 a Maculata Park b Oval
c Building at quarry
3 a GR 871464 b GR 854487
c GR 813488
4 a 3350 m b 1250 m
5 a 352° b 090°
6 a Abattoirs, bridge over river on Warrego Highway, then along river and over slag heaps
b 155°. Yes. A scale diagram could be sketched and trigonometry used to calculate angles.

Exercise 6I — Contour maps

1 a 10 m b 80 m c 50 m
d Up a hill then down a steep descent, then up and down another smaller hill
e Sketch f 5
2 a 93 m b 68 m
3 a 20 m b 10.3 km c Sketch
d 20 e 293°
f No, not if X and Y are at the surface.
4 a $\frac{1}{3.41}$ b 16.4°, steep
5 a 45° b 18.4° c 0.57°
d 1.15° e 2.97°
6 a 1.27°

Exercise 6J — Cadastral maps and site plans

1 a 630 m^2 b 23.0 × 27.499 m
c 632.477 m^2 d 1 : 1500
e Rectangle of length 60 mm and width 42 mm
f i \$57.88/m^2 ii 850 m^2
g i Lot 109
ii Location, elevation, road frontage size, views
2 a 2100 m^2, 83 perches
b 103.68 m^2 c 56.3 m
d 0.049 or approx. $\frac{1}{20}$
e i Rising ii 1800 mm iii 1.375°

Exercise 6K — Orienteering

1 a 8° b 137° c 222° d 45°
2 a 67 m b 136 m c 77 m d 130 m
3 Any suitable set of 8 instructions

Chapter review

1 a 126 m^2 b 165 m^2 c 516 m^2
d 2325 m^2 e 8850 m^2
2 0.2 ha
3 a i 150 m ii 52 m iii 63.2 m iv 13 m
v 75 m vi 141.9 m
b i 936 m^2 ii 1533 m^2 iii 4500 m^2
iv 5912.5 m^2
4 Sketch
5 a 84 m
b i 050° ii 115° iii 295° iv 238° v 090°
c 2000 m^2 d 190 m
6 a i 43 m ii 48 m iii 46 m iv 56 m v 86 m
b i 051° ii 090° iii 253°
c 3200 m^2
7 a i 3.90 m ii 2.70 m iii 53.60 m iv 50.00 m
b 50.90 c Sketch
8 a i Industrial Estate ii Finlay Island
b 2.5 km c 153°
9 a 250 b 1 in 5
c 11.3°, steep to moderate
10 a 45° b 26.6° c 1.1°
d 2.9° e 7.2°
11 10
12 a 90 m b 20 m c Sketch
13 a 630 m^2 b Sketch

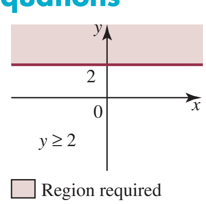
CHAPTER 7 Linear programming

Skills check

1 a 2 b −6 c −26
2 a −6 b 4
3 a $y = 5x - 6$ b $y = \frac{-6}{5}x + 2$ c $y = \frac{-7}{2}x - \frac{5}{2}$
4 a A vertical line passing through (3, 0)
b A horizontal line passing through (0, 6)
c A line at an angle of 45° in the positive direction of the x-axis
5 a $y = 0$
b $x = 0$
6 a {8, 9, 10, 11, 12, . . .}
b {. . . −3, −2, −1, 0, 1, 2, 3, 4}
c {3, 4, 5, 6, 7, 8, 9}

Exercise 7A — Graphs of linear inequations

1 a



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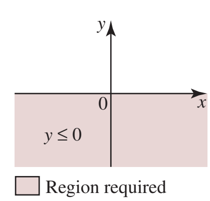

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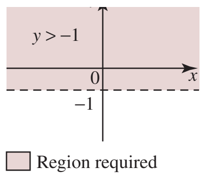

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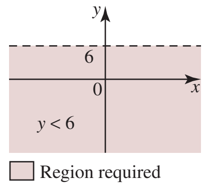

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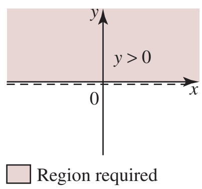
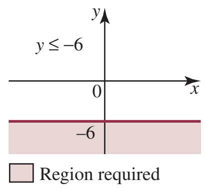


f


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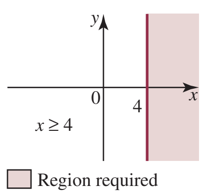

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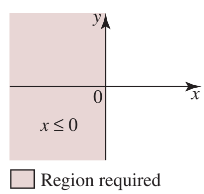

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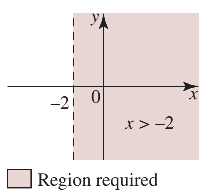

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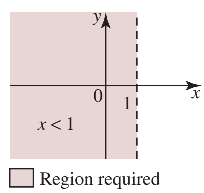

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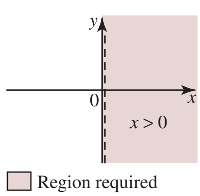

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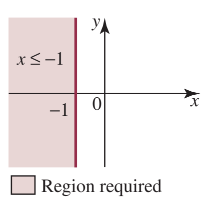

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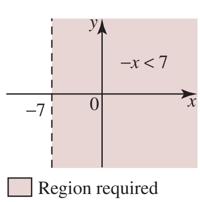

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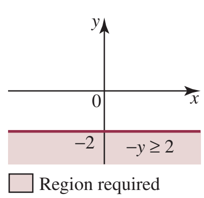

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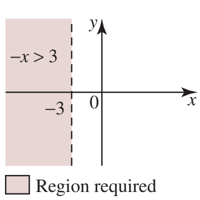

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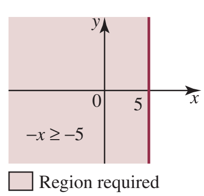

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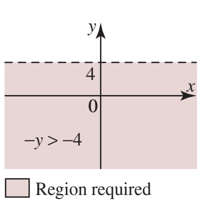

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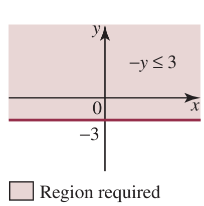

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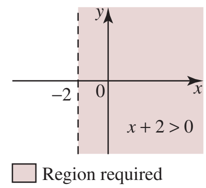

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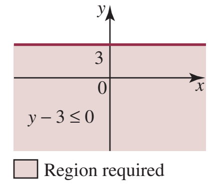

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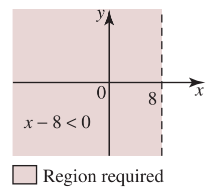

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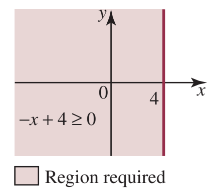

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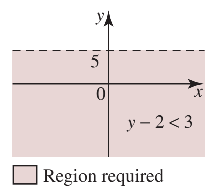

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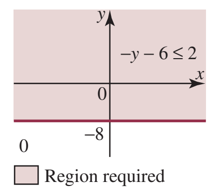

3 E

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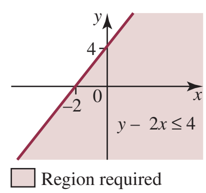

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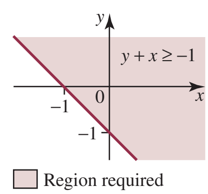

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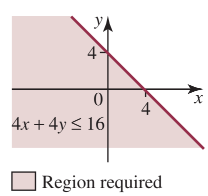

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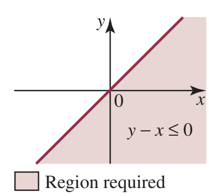

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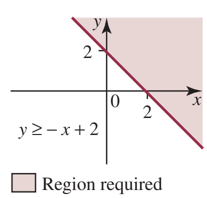

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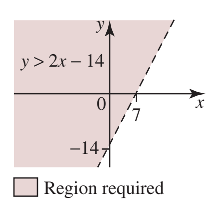

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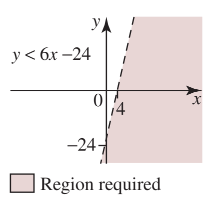

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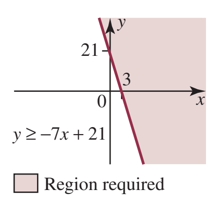

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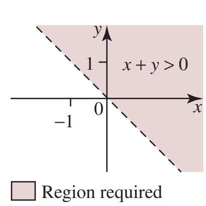

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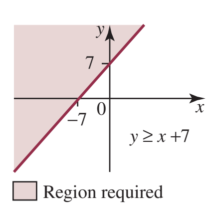

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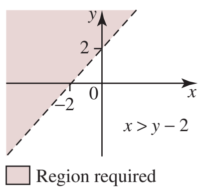

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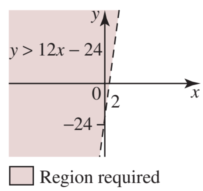

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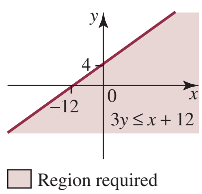

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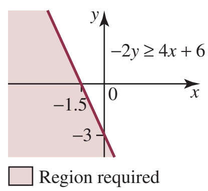

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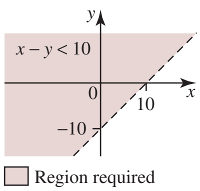

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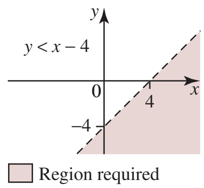

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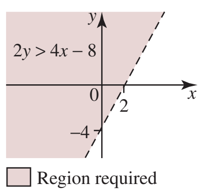

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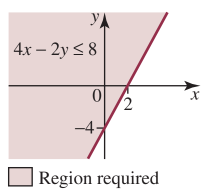

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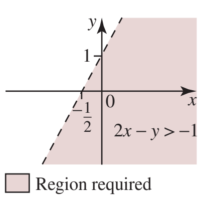

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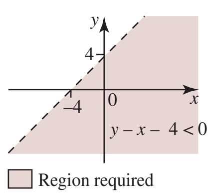

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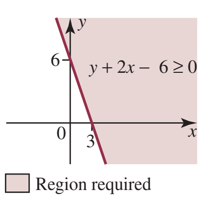

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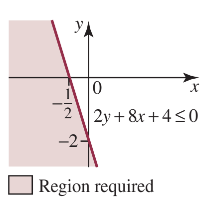

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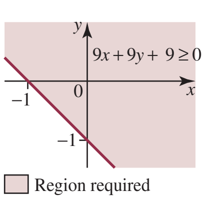

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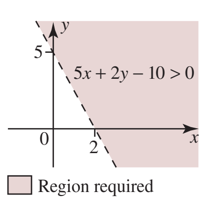

5 B

6 D

Exercise 7B — Solutions of simultaneous linear equations

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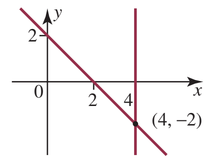

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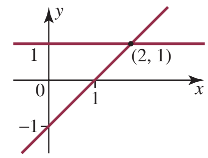

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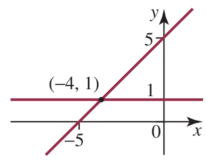

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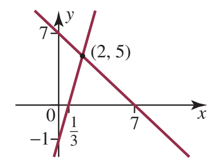

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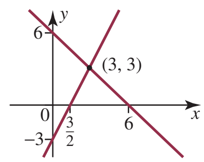

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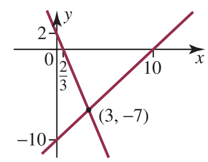

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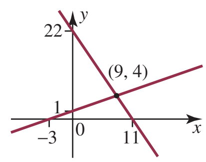

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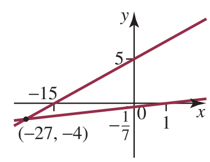

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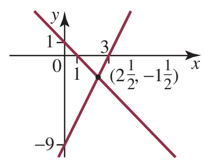

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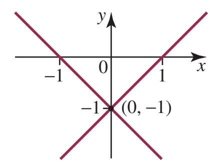

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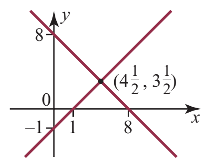

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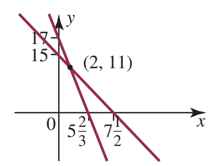

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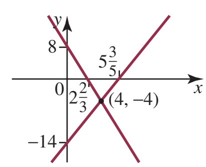

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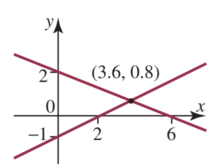

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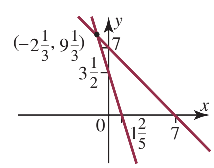

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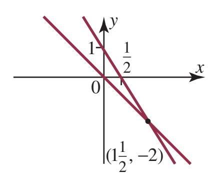

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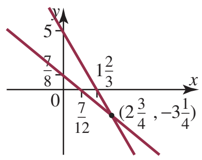

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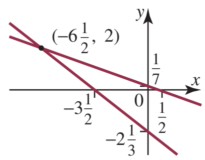

2 A

3 B

Exercise 7C — Graphs of simultaneous linear inequations

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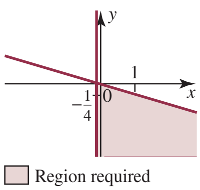

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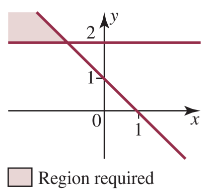

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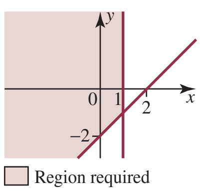

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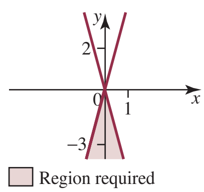

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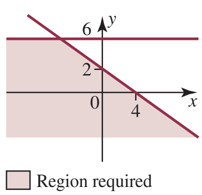

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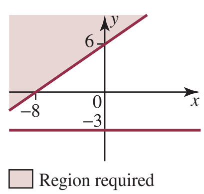

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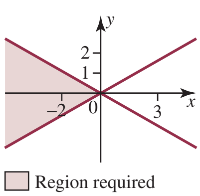

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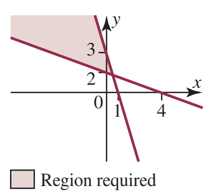

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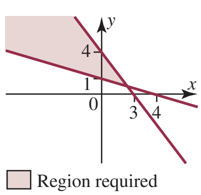

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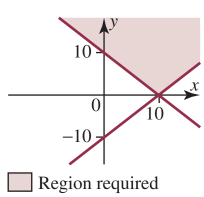

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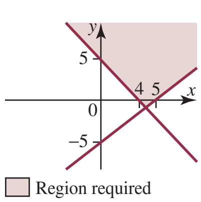

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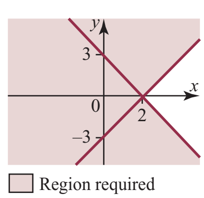

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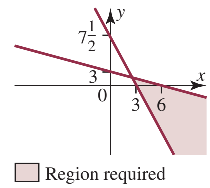

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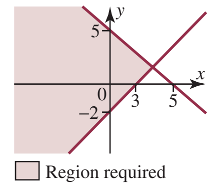
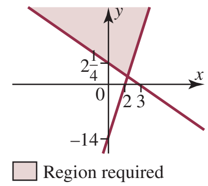


i



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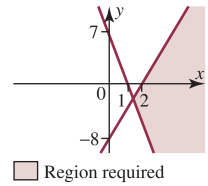

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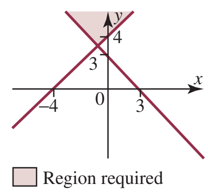

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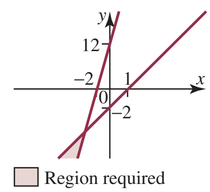

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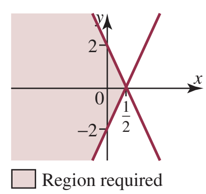

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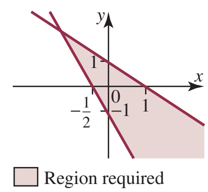
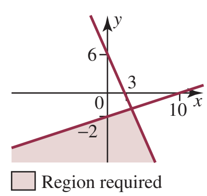


o



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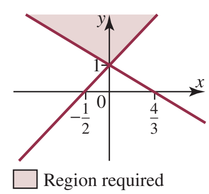

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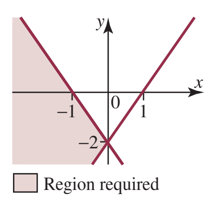

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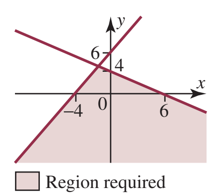
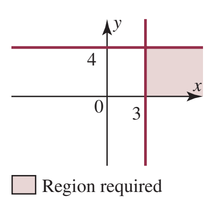


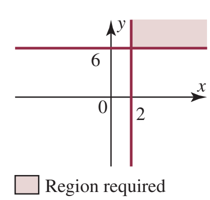
3 a B **b** A

Exercise 7D — Graphs of systems of linear inequations

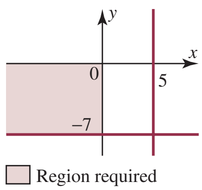
1 a



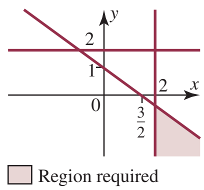
b



c



d



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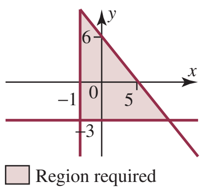

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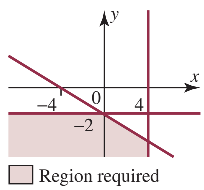

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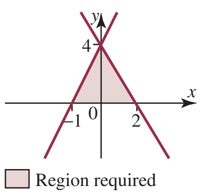

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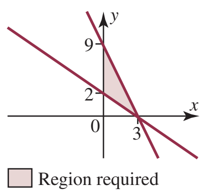

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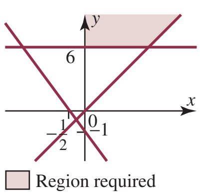

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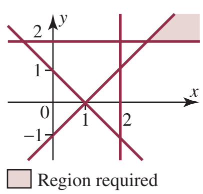

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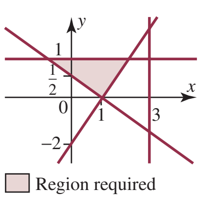

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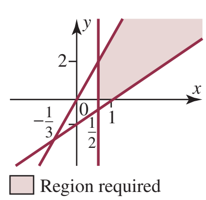

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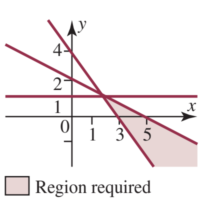

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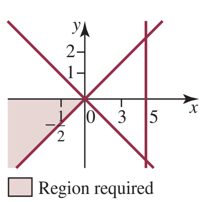

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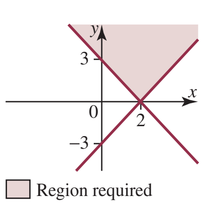

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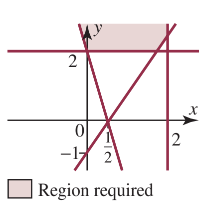

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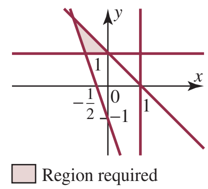

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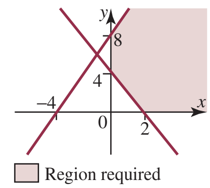

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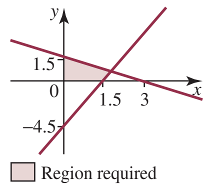

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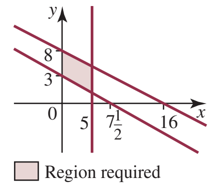

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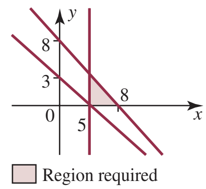

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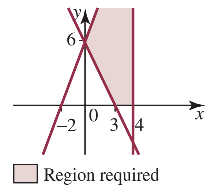

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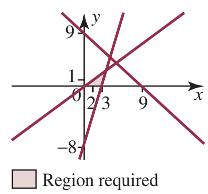

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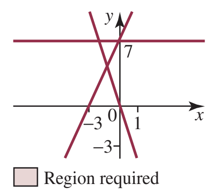
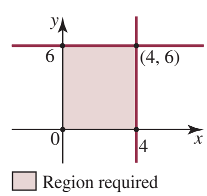


2 B **3** E **4** D **5** B

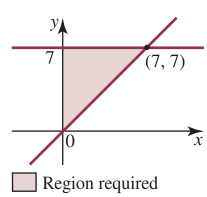
Exercise 7E — Maximising and minimising linear functions

1 a i, ii



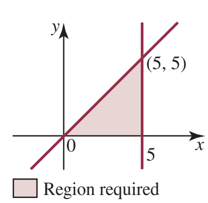
iii Maximum value 4

b i, ii



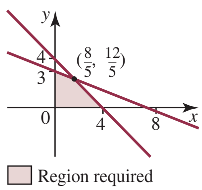
iii Minimum value 0

c i, ii



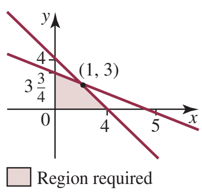
iii Maximum value 15

d i, ii



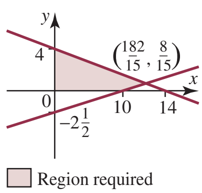
iii Maximum value $20\frac{4}{5}$

e i, ii



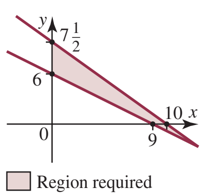
iii Minimum value $-22\frac{1}{2}$

f i, ii



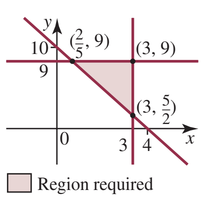
iii Maximum value 10.35

g i, ii



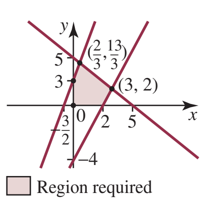
iii Minimum value 18

h i, ii



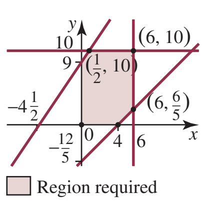
iii Maximum value $7\frac{4}{5}$

i i, ii



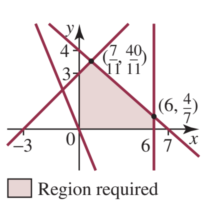
iii Minimum value 0

j i, ii



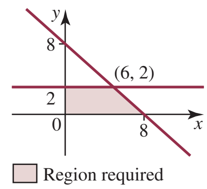
iii Maximum value 58

2 a i, ii



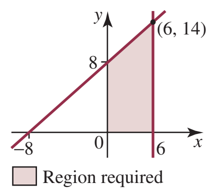
iii Minimum value 0

b i, ii



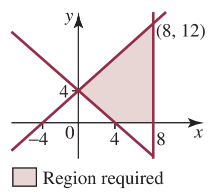
iii Maximum value 16

c i, ii



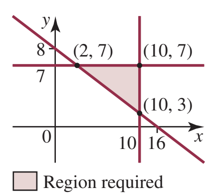
iii Minimum value −8

d i, ii



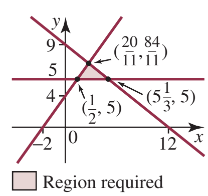
iii Minimum value 12

e i, ii



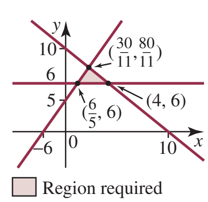
iii Maximum value 33.4

f i, ii



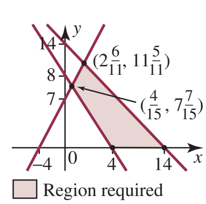
iii Minimum value −1.15

g i, ii



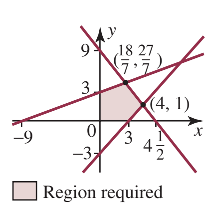
iii Maximum value 23.72

h i, ii



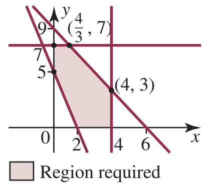
iii Minimum value −9.6

i i, ii



iii Maximum value 25

j i, ii



iii Minimum value −35.7

3 a A **b** D **c** D **d** C

Exercise 7F — Solving linear programming problems

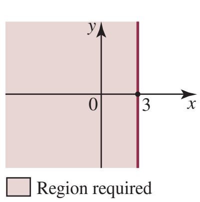
1 a Style A = 5, Style B = 19 **b** $67
2 a Design A = 24, Design B = 4 **b** $146
3 a 35 rollerblades, 5 bicycles **b** $155
4 a Brand A = 20, Brand B = 260 **b** $1840
5 a 36 aerobic classes, 9 circuit classes **b** $252
6 a 20 2-person tents, 18 3-person tents **b** $984
7 a 15 bubble jet printers, 10 laser printers **b** $330
8 a 70 hectares of corn, 10 hectares of peas
b $14 200
9 a Nature's Own = 80 litres, generic brand = 10 litres **b** $212
10 a 255 tourists, 255 businesspeople **b** $8415
11 a 2 batches of 3-D puzzles, 3 batches of logic games **b** $147
12 a 40 Arctic and 40 Cool Breeze
b $11 960 **c** Yes ($9560)
13 a Variables need to be defined in order to determine how many bags of CP_1 and CP_2 will produce maximum profit.
b $20x + 15y \le 300$, $15x + 10y \le 210$
c $x \ge 0$, $y \ge 0$ **d** $P = 45x + 50y$ **e** $836
14 A **15** A

Exercise 7G — Further applications of linear programming

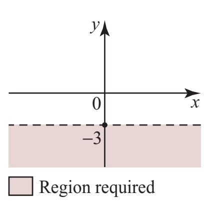
1 $69
2 a Rod A = 16, rod B = 10 **b** $13 800
3 a x = the number of type A uniforms produced; y = the number of type B uniforms produced.
b $5x + 10y \le 480$, $6x + 12y \le 600$, $3x + 3y \le 450$, $x \ge 0$, $y \ge 0$
c $P = 7x + 12y$
d 80 type A uniforms and 10 type B uniforms
e $588
4 a $2\frac{2}{5}$ kg of Zest, $1\frac{1}{5}$ kg of Boom
b $15.72 (rounded down $15.70)
5 a 200 Gold Pass packages, 100 Classic packages
b $5000
6 a 1.6 tonnes at plant A, 4.2 tonnes at plant B
b $2532

Chapter review

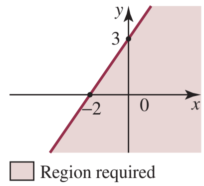
1 D
2 a



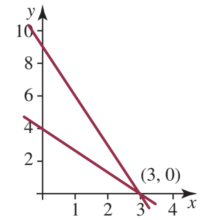
b



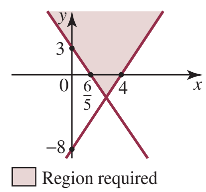
c



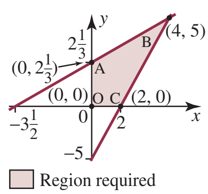
3 D **4** D **5** E
6



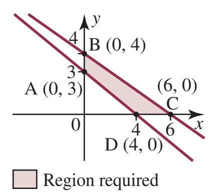
7 A
8



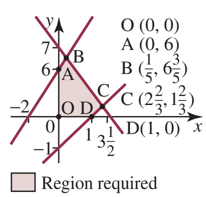
9 B
10 $x + y \ge 400$, $y \le 400$, $x - 2y \le 400$
11 $x \ge 0$, $y \ge 0$, $-x + y \ge -300$, $x + y \le 600$, $x - y \ge -400$
12 a



b

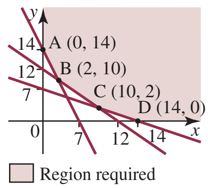


13 C
14 a



b $27.40
15 D
16 a 54 **b** −4
17 D
18 $x = 0$, $y = 0$, minimum = 0
19 A
20 $x = 0$, $y = 3.4$, maximum = 4080
21 B **22** B **23** D
24 a x = the number of runners produced, y = the number of walking shoes produced
b $x \ge 400$, $y \ge 350$, $x + y \le 900$
c $P = 12.5x + 10y$
25 a 20 hectares of oats, 15 hectares of wheat
b $7600 **c** Yes
26 a $10x + 5y \ge 70$, $2x + 4y \ge 28$, $3x + 3y \ge 36$
b Yes, $x \ge 0$ and $y \ge 0$
We are dealing with items produced; therefore $x \ge 0$ and $y \ge 0$.

c

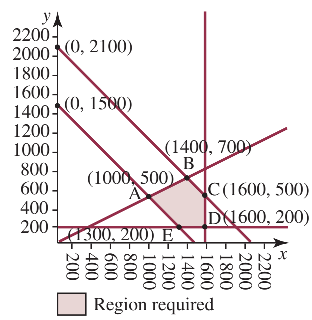


d $C = 300x + 200y$ **e** \$2600

27 a 5 kg C_1, 2 kg C_2 **b** \$27.50

28 a $x \geq 0; y \geq 0; x \leq 1600; y \geq 200; x \geq 2y;$
$x + y \geq 1500; x + y \leq 2100$

b



c A(1000, 500) B(1400, 700) C(1600, 500)
D(1600, 200) E(1300, 200)

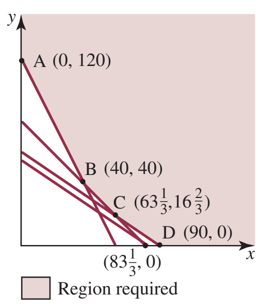
d $P = 2.5x - 3.5y + 45\,150$

e 1600 cake platters, 200 cheese platters and 300 fruit bowls

f \$48 450

29 a $x \geq 0; y \geq 0; x + y \geq 80; 0.03x + 0.05y \geq 2.5;$
$0.05x + 0.08y \geq 4.5; 0.05x + 0.025y \geq 3$

b



c A(0, 120) B(40, 40) $C(63\frac{1}{3}, 16\frac{2}{3})$ D(90, 0)

d $C = 0.012x + 0.016y$

e $63\frac{1}{3}$ g of Chicken Bites and $16\frac{2}{3}$ g of Fish Bites per day

f \$1.03 per day **g** $C = 0.012x + 0.014y$

h $63\frac{1}{3}$ g of Chicken Bites and $16\frac{2}{3}$ g of Fish Bites per day

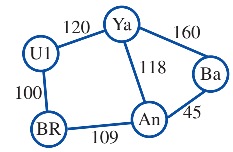
i \$0.99 per day

CHAPTER 8 Networks

Exercise 8A — Networks, nodes and arcs

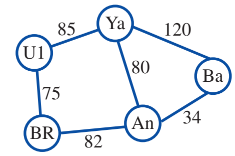
1 a ABDE **b** ABCE

2 a



b 487 km **c** 254 km

d i



ii 357 min **iii** 191 min

iv

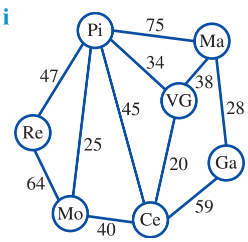
	Ulawatu	Yallingup	Black Rock	Angourie	Bargara
Ulawatu	0	120	100	209	254
Yallingup	120	0	220	118	160
Black Rock	100	220	0	109	154
Angourie	209	118	109	0	45
Bargara	254	160	154	45	0

v

	Ulawatu	Yallingup	Black Rock	Angourie	Bargara
Ulawatu	0	85	75	157	191
Yallingup	85	0	160	80	114
Black Rock	75	160	0	82	116
Angourie	157	80	82	0	34
Bargara	191	114	116	34	0

3 a 179 km

b i



ii 277 min **iii** 143 min

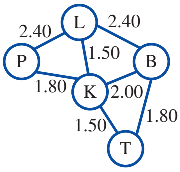
c

	Re	Pi	Mo	Ce	VG	Ma	Ga
Re	0	62	58	104	108	147	179
Pi	62	0	41	65	46	85	123
Mo	58	41	0	46	76	126	121
Ce	104	65	46	0	30	90	75
VG	108	46	76	30	0	60	98
Ma	147	85	126	90	60	0	38
Ga	179	123	121	75	98	38	0

d

	Re	Pi	Mo	Ce	VG	Ma	Ga
Re	0	47	44	84	81	119	143
Pi	47	0	25	45	34	75	100
Mo	44	25	0	40	59	97	99
Ce	84	45	40	0	20	58	59
VG	81	34	60	20	0	38	66
Ma	119	72	97	58	38	0	28
Ga	143	104	99	59	28	28	0

4 a



b \$3.30 **c** \$3.80

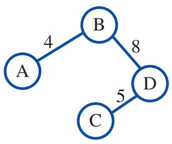
5 C

History of mathematics

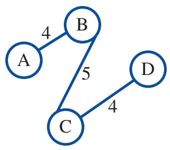
1 *Men of Mathematics* by E. T. Bell
2 The Nobel Prize and the Leroy P. Steele Prize
3 An algorithm is a procedure for solving a problem by a number of steps.

Exercise 8B — Minimal spanning trees

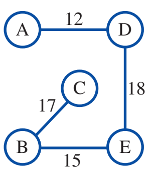
1 a



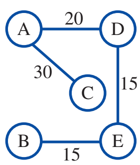
b



c

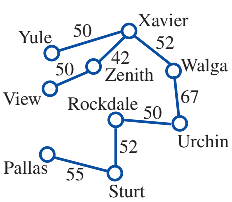


d



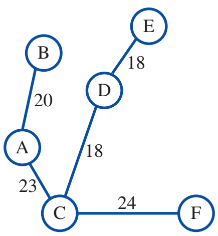
2 a Sturt **b** Rockdale **c** To Sturt

d

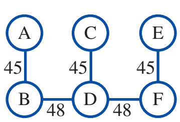


3 a 585 m **b** 245 m
c Check with your teacher.

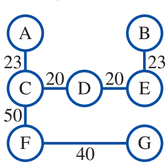
4 a



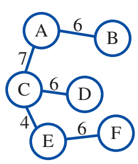
b



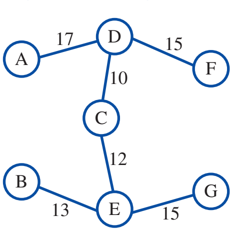
c



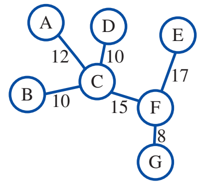
d



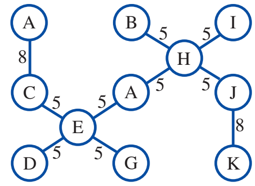
5 a



b



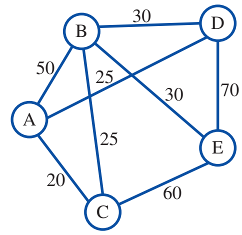
c



6 53 km
7 53 km
8 a 68 km **b** \$1.7 million
9 a \$215 **b** \$1740
10 B **11** D

Exercise 8C — Shortest paths

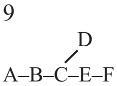
1 a 20 **b** 38 **c** 74
d 45 **e** 28 **f** 139
2 a 165 km **b** 210 km **c** 210 km
3 a 37 **b** 90 **c** 32
d 72 **e** 30 **f** 44
4 a



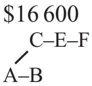
b 80 min

10 Quick Questions 1

1 6 **2** 9
3 1 **4**



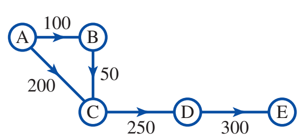
5 AC, AD, DF, CF **6** \$16 600
7 D–C–E–F **8**



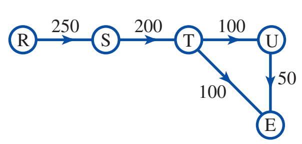
9 \$16 200 **10** B–A–C

Exercise 8D — Network flow

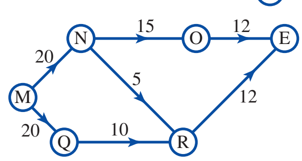
1 a



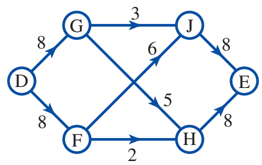
b



c



d



2 a 23 b 16 c 16

3 a 6 b 3 c 3

4 a i 250 ii No b i 150 ii Yes
c i 24 ii Yes d i 15 ii No

5 a

From	To	Flow capacity
A	B	4
A	C	5
A	D	3
B	E	3
C	B	2
C	E	4
D	C	2
D	E	6

b

From	To	Flow capacity
A	B	4
A	C	5
A	D	3
B	E	3
B	C	2
C	E	4
D	C	2
D	E	6

c

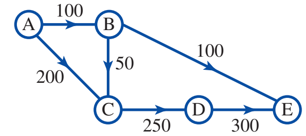
From	To	Flow capacity
A	B	4
A	C	7
A	D	3
A	E	5
B	E	3
C	E	8
D	B	2
D	E	6

d

From	To	Flow capacity
A	B	4
A	C	7
A	D	12
A	E	5
C	F	7
D	B	2
D	E	6
D	F	4
F	E	8

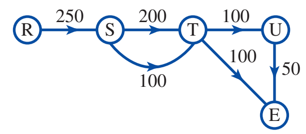
6 a 10 b 10 c 18 d 22

7 a i



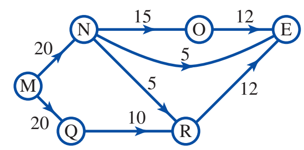
ii 300

b i



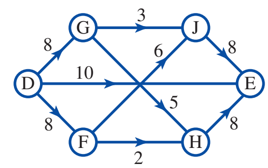
ii 150

c i



ii 29

d i



ii 25

8 B

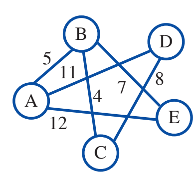
10 Quick Questions 2

1 A–D–F
2 64 km
3 B–C–E–F
4 45 km
5 30
6 30
7 10
8 80
9 45
10 72

Chapter review

1 a 7 b 10

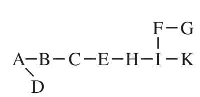
2



3 Step 1 Choose any node at random and connect it to its closest neighbour.
Step 2 Choose any unconnected node which is closest to any connected node. Connect this node to the nearest connected node.
Step 3 Repeat Step 2 until all nodes are connected.

4 A–B–E–C–D–F–G

5



6 117 km

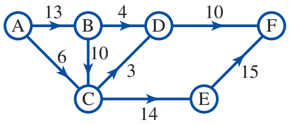
7 ADFG, 40

8 ACFIK, 70

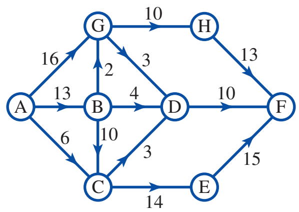
9 a 20 b 15 c 35

10 a 70 b 70 c 70

11



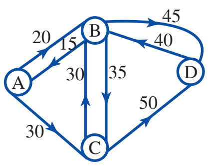
12



CHAPTER 9 Critical path analysis and queuing

Skills check

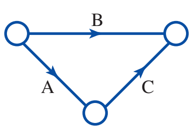
1



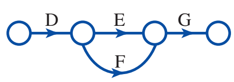
2 a 117 km b 160 km

Exercise 9A — Critical path analysis

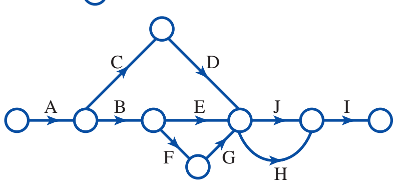
1 a



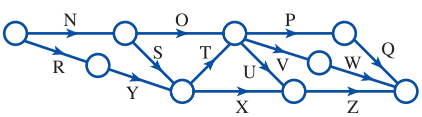
b



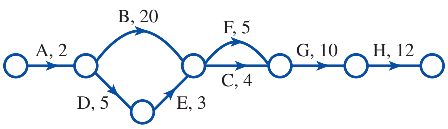
c



d



2 a



b 49 minutes

3 a B b D

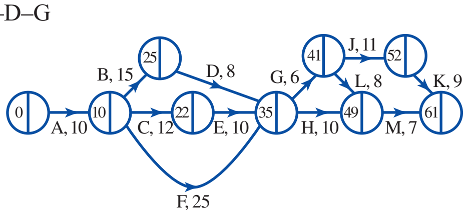
4 a 23 minutes b B, C, E, F, G

5 A–C–F

6 a B b D c A

7 A–D–G

8 a



b 61 minutes

9

Activity letter	Immediate predecessor	Time
A	—	7
B	—	9
C	A	12
D	B	8
E	B	4
F	C, D	9

10

Activity letter	Immediate predecessor	Time
A	—	3
B	—	4
C	—	5
D	A	6
E	B, F	5
F	C	8
G	D	18
H	E	8
J	E	6

11

Activity letter	Immediate predecessor	Time
A	—	10
B	A	15
C	A	12
D	B	8
E	C	10
F	A	25
G	D, E, F	6
H	D, E, F	10
J	G	11
K	J	9
L	G	8
M	H, L	7

12 a A–D–G

b Float (H) = 1, Float (J) = 3, Float (E) = 1, Float (B) = 10, Float (C) = 1, Float (F) = 1

c Activity B can be delayed 10 minutes, activity C can be delayed 1 minute, activity E can be delayed 1 minute, activity F can be delayed 1 minute, activity H can be delayed 1 minute, activity J can be delayed 3 minutes.

13 a A–F–G–J–K

b M, L, H, C, E, B, D

Exercise 9B — Critical path analysis with backward scanning

1 Critical path = B–D; Float (E) = 1 min, Float (C) = 1 min, Float (A) = 1 min

2 Critical path = B–F–G; Float (C) = 3 h, Float (E) = 5 h, Float (A) = 3 h, Float (D) = 5 h

3 D

4 a 31 days b Critical path = A–C–E–G

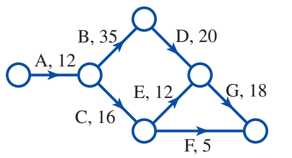
5 D

6 C

7 a

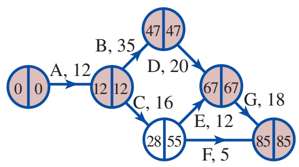
Activity letter	Activity	Immediate predecessor	Time
A	Collect parts	—	12
B	Paint frame	A	35
C	Assemble brakes	A	16
D	Assemble gears	B	20
E	Install brakes	C	12
F	Install seat	C	5
G	Final assembly	D, E	18

b



c 85 minutes

d



A–B–D–G

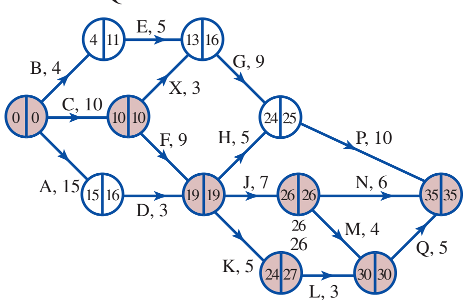
8 C

9 28%

10 a 29 **b** B–E–J–N

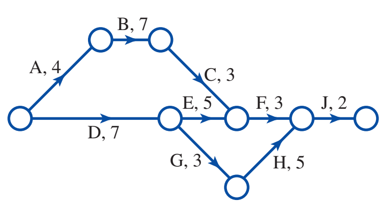
11 a 35

b C–F–J–M–Q



c 3

12 a

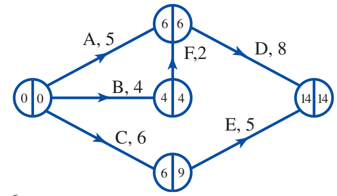


b Float (D), Float (E) = 2, Float (G), Float (H) = 2

10 Quick Questions 1

1. C
2. The earliest starting time of an activity is the earliest time by which all prior activities can be completed.
3. The latest start time of an activity is the latest time the activity can start if the project is not to be delayed.
4. Float time = latest finish time − earliest start time − activity time

5



6 6

7 14

8 9

9 3

10 B, F, D

Exercise 9C — Queues: one service point

1 a

Time	Customer served	Arrivals	Length of queue	People in queue
8	D	—	4	E F G H
9	D	I	5	E F G H I
10	E	—	4	F G H I
11	E	J	5	F G H I J
12	F	K	5	G H I J K
13	G	—	4	H I J K
14	G	—	4	H I J K
15	G	L	5	H I J K L
16	H	—	4	I J K L

b 5 min **c** 9 min **d** 2 min
e 2.28 min **f** 1.36 min

2 a

Time	Customer served	Arrivals	Length of queue	People in queue
10	D	—	4	E F G H
11	E	—	3	F G H
12	F	—	2	G H
13	G	—	1	H
14	G	—	1	H
15	G	—	1	H
16	H	—	0	

b 3 min **c** 7 min **d** 1 min

3 C **4** B

5 a

Time	Customer served	Arrivals	Length of queue	People in queue
0	A	—	0	—
1	A	—	0	—
2	A	B	1	B
3	B	—	0	—
4	B	C	1	C
5	B	—	1	C
6	C	D	1	D
7	C	—	1	D
8	C	E	2	D E
9	D	—	1	E
10	D	F	2	E F
11	D	—	2	E F
12	E	G	2	F G

b 2 **c** 4 min

6 **a** 3 min **b** Yes

c

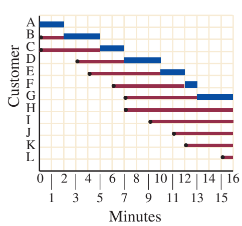
Time	Customer served	Arrivals	Length of queue	People in queue
0	A	—	0	—
3	A	B	1	B
6	B	C	1	C
9	B	D	2	C D
12	C	E	2	D E
15	C	F	3	D E F
18	D	G	3	E F G

7 **a** 2 min **b** 3, 10 **c** 2 min
d 4 min **e** 3 **f** 1
g F **h** 3 min and 6 min

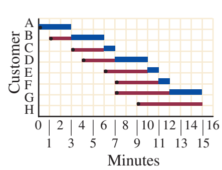
i

Time	Customer served	Arrivals	Length of queue	People in queue
0	A	—	0	—
1	A	B	1	B
2	B	—	0	—
3	B	C D	2	C D
4	B	—	2	C D
5	C	—	1	D
6	C	E F	3	D E F
7	C	—	3	D E F
8	D	—	2	E F
9	D	—	2	E F
10	E	—	1	F
11	E	—	1	F
12	E	G	2	F G
13	E	—	2	F G
14	F	—	1	G
15	F	—	1	G
16	F	—	1	G
17	G	—	0	—

8



9



10 17 min

10 Quick Questions 2

1 300
2 Greater than 1.5 minutes
3 2 min
4 1 min
5 3 min
6 5 min
7 5
8 D, E, F
9 F, G, H, I
10 3

Exercise 9D — Queues: multiple service points

1 **a** 234 s (3 mins 54 s)
b 3
2 4
3 Yes
4 **a** 4 had been completely served.
b 2 **c** 9 **d** 7 **e** 3

5 **a**

Time	Teller 1	Teller 2	Arrivals	Length of queue
9	G	H	J, K	3
10	G	H	L	4
11	G	H		4
12	G	H	M	5
13	I	J		3
14	I	J	N	4
15	K	L		2
16	K	L	O	3
17	K	L		3

b 2 min **c** 5 min **d** 4 min

6 **a** 2 min
b 5 min

c

Time	Service 1	Service 2	Arrivals	People in queue	Length of queue
0	A	B	E	C, D, E	3
1	C	B		D, E	2
2	C	D	F	E, F	2
3	C	D		E, F	2
4	E	D	G	F, G	2
5	E	F		G	

d 2 min

7 A **8** B

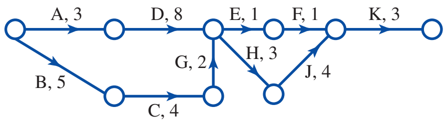
9

Time	Nurse 1	Nurse 2	Arrivals	People in queue	Length of queue
0.00	A	B			
0.45	A	B	C	C	1
1.30	A	B	D	C, D	2
2.00	C	D			0
2.15	C	D	E	E	1
3.00	C	D	F	E, F	2
3.45	C	D	G	E, F, G	3
4.00	E	F		G	1
4.30	E	F	H	G, H	2
5.15	E	F	I	G, H, I	3
6.00	G	H	J	I, J	2
6.45	G	H	K	I, J, K	3
7.30	G	H	L	I, J, K, L	4
8.00	I	J		K, L	2

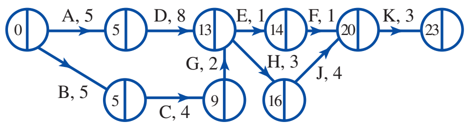
Chapter review

1 C **2** D **3** C **4** D

5

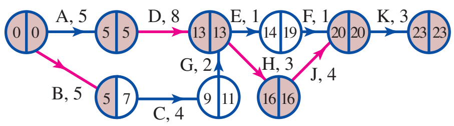


6



The refit can be completed in 23 hours.

7



8

Activity letter	Time	Earliest start time	Earliest finish time	Float time	Immediate predecessor
A	5	0	5	0	—
B	5	0	5	0	—
C	4	5	11	2	B
D	8	5	13	0	A
E	1	13	19	5	D, G
F	1	14	20	5	E
G	2	9	13	2	C
H	3	13	16	0	D
J	4	16	20	0	H
K	3	20	23	0	F, J

9 a

Time	Customer	Arrivals	People in queue	Length of queue
0	A	B	B	1
1	A		B	1
2	B	C, D	C, D	2
3	C	E, F	D, E, F	3
4	C		D, E, F	3
5	C	G	D, E, F, G	4
6	D	H, I	E, F, G, H, I	5
7	E		F. G. H. I	4
8	F	J	F, G, H, I, J	4
9	F	K	G, H, I, J, K	5

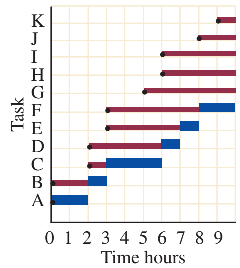
b 5 min

c 3 min

d $(2 + 1 + 3 + 1 + 1 + 2) \div 6 = 1.67$ min

e $(0 + 2 + 0 + 1 + 0 + 2 + 1 + 0 + 2 + 1) \div 10$
$= 0.9$ min

10



11

Time	Customer served	Arrivals	People in queue	Length of queue
0	A	—	—	—
1	A	B	B	1
2	B	C	C	1
3	B	D	C, D	2
4	C	E	D, E	2
5	C	F	D, E, F	3
6	D	G	E, F, G	3
7	D	H	E, F, G, H	4
8	E	I	F, G, H, I	4

12 a Inter-arrival time = 0.75 s

b Number of service points = 40

13 a Inter-arrival time = $\frac{3600}{1000}$ s = 3.6 s (less than service time)

b 3

14 a

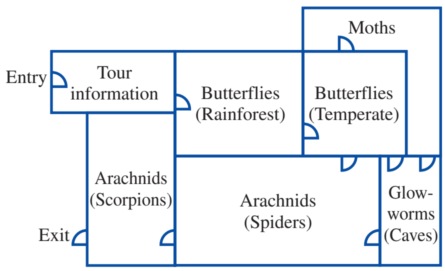
Time	Point 1	Point 2	Arrivals	People in queue	Length of queue
0	A	B	C, D, E	C, D, E	3
1	A	C	F, G	D, E, F, G	4
2	A	D	H	E, F, G, H	4
3	E	D	I, J	F, G, H, I, J	5
4	E	F	K	G, H, I, J	4
5	G	H	L	I, J, L	3
6	G	H	M, N	I, J, L, M, N	5
7	G	I	O	J, L, M, N, O	5

b 1 min

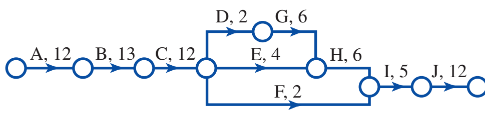
c $(0 + 0 + 0 + 1 + 0 + 1 + 1 + 0 + 1 + 1 + 1 + 0 + 1) \div 13 = 0.54$ min

d $(0 + 1 + 2 + 3 + 3 + 4 + 3 + 4) \div 8 = 2.5$ min

15 a



b



c The room would quickly become crowded.
d Five people per minute

CHAPTER 10 Probability and the binomial distribution

Skills check

1 a 6 **b** 2 **c** 52 **d** 36

2 a $\frac{1}{12}$ **b** $\frac{2}{3}$ **c** $\frac{5}{6}$
d $\frac{1}{36}$ **e** $\frac{1}{54}$ **f** $\frac{5}{16}$

3 a 0.04 **b** 0.03 **c** 0.54
d 0.72 **e** 0.67 **f** 0.125

4 a 0.125 **b** 0.188 **c** 0.75
d 0.154 **e** 0.684 **f** 0.058

5 a Less than
b Greater than or equal to
c Greater than

Exercise 10A — Compound events — independent events

1 $\frac{1}{4}$

2 a $\frac{1}{2}$ **b** $\frac{1}{2}$

3 $S = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6)$
$(2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6)$
$(3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6)$
$(4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6)$
$(5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6)$
$(6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)\}$
$P(9) = \frac{4}{36} = \frac{1}{9}$

4 7 is mostly likely, $P(7) = \frac{6}{36} = \frac{1}{6}$

5 a 0.2646 **b** 0.0204 **c** 0.1764

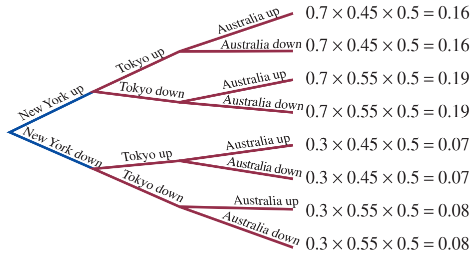
6 {(H, H, H), (H, H, T), (H, T, H), (H, T, T), (T, H, H), (T, H, T), (T, T, H), (T, T, T)}
a $\frac{1}{8}$ **b** $\frac{3}{8}$ **c** $\frac{7}{8}$ **d** $\frac{3}{8}$

7 D

8 B

9 $\frac{1}{6} \times \frac{1}{6} \times \frac{1}{6} = \frac{1}{216}$

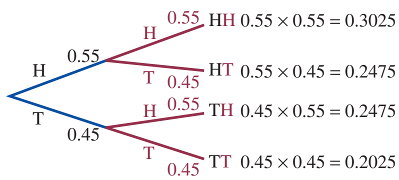
10



$0.7 \times 0.45 \times 0.5 = 0.1575$

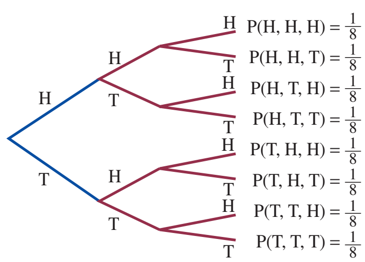
11 $0.3 \times 0.55 \times 0.5 = 0.0825$

12



P(2 Tails) = 0.2025

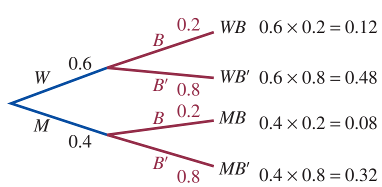
13



a There are 8 paths.

14 E

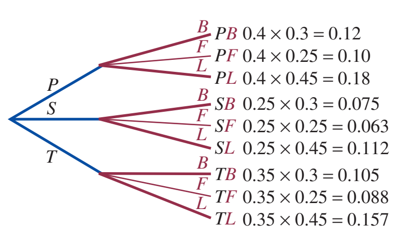
15



a 0.12 = 12% **b** 0.32 = 32%

16 **b**, **d**, **e**

17



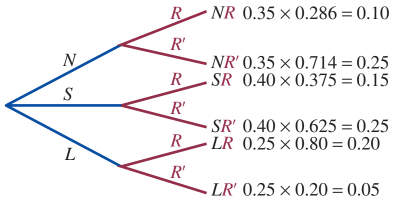
$P(P - F) = 0.4 \times 0.25 = 0.1$

18 a P(no drug) $= \frac{14}{40} = 0.35$

P(small dose) $= \frac{16}{40} = 0.40$

P(large dose) $= \frac{10}{40} = 0.25$

b, c, d



e Although few patients were tested, it appears that a greater percentage (80%) of those given a large dose of the drug recovered, whereas a much smaller percentage (29%) of those not given the drug recovered. 20% of people tested were given a large dose of the drug and recovered, 15% of people tested were given a small dose and recovered, whereas only 10% of people were not given the drug and recovered. So it could be said that a patient is more likely to recover if the drug is taken.

Exercise 10B — Compound events — mutually exclusive events

1 a, d, e, g

2 a $\frac{4}{52} = \frac{1}{13}$ **b** $\frac{6}{13}$ **c** $\frac{15}{52}$

3 a $\frac{16}{45}$ **b** $\frac{35}{45} = \frac{7}{9}$ **c** $\frac{29}{45}$ **d** $\frac{19}{45}$ **e** 1

4 $\frac{16+76}{178} = 0.517$

5 E

6 a $\frac{59+13}{148} = \frac{72}{148} = \frac{18}{37}$ **b** $\frac{19}{37}$

7 a $\frac{12}{36} = \frac{1}{3}$ **b** $\frac{24}{36} = \frac{2}{3}$ **c** $\frac{15}{36} = \frac{5}{12}$

8 a 0.258 **b** 0.449 **c** 0.865

9 a 0.037 **b** 0.296 **c** 0.667 **d** 0.333

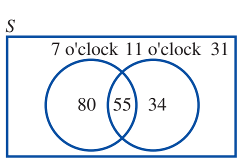
10 a 0.32 **b** 0.46 **c** 0.31

11 a 0.4999 **b** 0.9997 **c** 649 773

Exercise 10C — Compound events — Venn diagrams

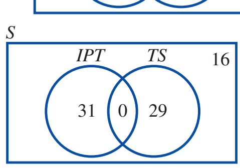
1 a 4 **b** 5 **c** 8

2 a



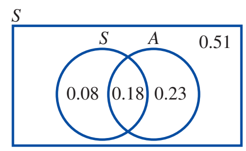
$\frac{55}{200} = 0.275$ **b** 0.4

3



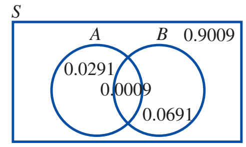
The events are mutually exclusive and the Venn diagram could have been drawn as two circles which did not overlap.

4



0.18

5



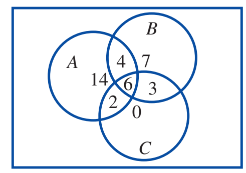
0.9009

6 C

7 a 37% **b** 63%

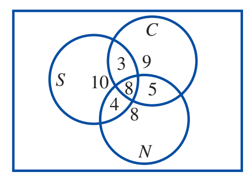
8 0.27, much higher probabilities of winning with roulette

9



36

10



9

10 Quick Questions 1

1 $\frac{1}{6}$ **2** $\frac{1}{6}$ **3** 0.36 or 36%

4 0.16 or 16% **5** 0.48 or 48% **6** $\frac{1}{26}$

7 $\frac{7}{13}$ **8** 4 **9** 14

10 $\frac{1}{20}$

Exercise 10D — The binomial distribution using Pascal's triangle

1 a 1 7 21 35 35 21 7 1
b 1 9 36 84 126 126 84 36 9 1

2 a 1 5 10 10 5 1
b 1 7 21 35 35 21 7 1

3 a 1 **b** 4 **c** 70 **d** 1 **e** 110

4 a 0.0256 **b** 0.1176 **c** 0.125
d 0.0132 **e** 0.0720 **f** 0.0156

5 a 0.0179 **b** 0.0284 **c** 0.3456

6 a 0.2344 **b** 0.3125

7 a 0.2322 **b** 0.8936

8 a 0.4019 **b** 0.8038

9 a 0.5 **b** 0.2734

10 a 0.5 **b** $(0.5)^{10}$

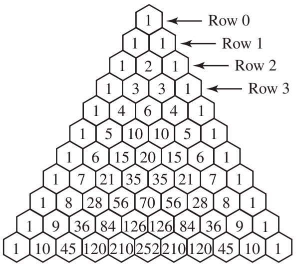
11 a 0.25 **b** $(0.75)^6$ **c** 0.0330

12 a 0.2090 **b** 0.0413

13 0.1342

Investigation — Pascal's triangle

1



2 The triangle is symmetrical about a vertical line through the centre.

3 Number of entries in row = row number + 1

4 Odd-numbered rows have an even number of entries. Even-numbered rows have an odd number of entries.

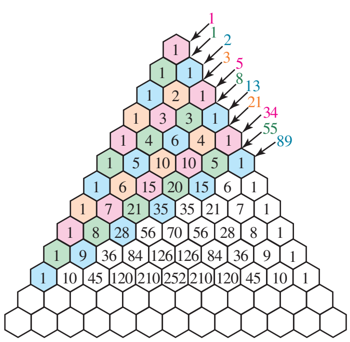
5 An odd number of trials has two middle numbers of the same value.
An even number of trials has one middle number.

6 Sum of numbers in row = $2^{\text{row number}}$ (2 to the power of the row number)

7 Yes. $11^3 = 1331$
$11^4 = 14\,641$

8 They are square numbers.

9 Fibonacci's sequence



10 Use the diagonal

1
3 $\quad 1 + 3 = 4 = 2^2$
6 $\quad 3 + 6 = 9 = 3^2$
10 $\quad 6 + 10 = 16 = 4^2$
15

11 The triangular numbers are located in the diagonal shown in question 10.

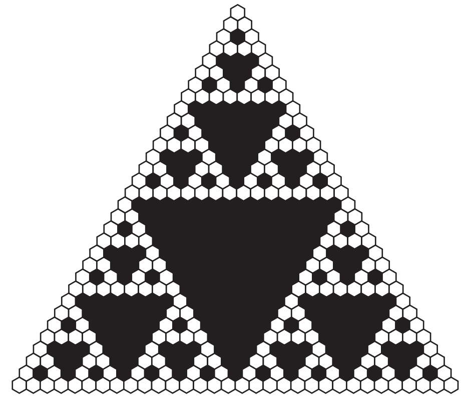
12 Sum of 10 lies in the position below the 4 to the right. This pattern continues. The same pattern continues for the numbers in the second diagonal.

13

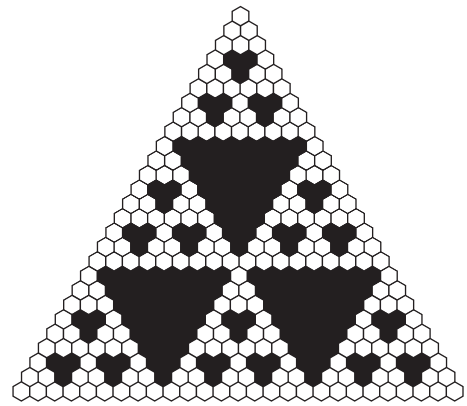
Row (r)	0	1	2	3	4	5	6
Sum (s)	1	3	7	15	31	63	127

$s = 2^{r+1} - 1$

14 a Multiples of 2



b Multiples of 3



History of mathematics — Blaise Pascal

1 16 years old
2 A calculating machine
3 The Puy de Dôme mountain
4 For probability, permutations and combinations
5 A brain tumour and stomach ulcer

Exercise 10E — Binomial probabilities through tables

1 a 0.0008 b 0.9527 c 0.5793 d 0.9907
2 a 0.8281 b 0.9389 c 0.9887 d 0.9532
3 a 0.5000 b 0.6964 c 0.4142 d 0.0000
4 0.8684
5 a 0.6550 b Between 0.6167 and 0.9666
6 a 0.6230 b 0.5881 c 0.5000 d 0.0781
e 0.0139 f 0.0034
7 a 0.0404 b 0.9536 c 0.9940
8 a 0.2
b i 0.2013 ii 0.3222 iii 0.1074
c The probability of this is very slight; there may be some error in the sampling technique.
9 0.1275
10 a $\frac{1}{3}$ b $\left(\frac{1}{3}\right)^{10}$ c Approximately 0.2

10 Quick Questions 2

1 $\frac{1}{4}$ 2 1 4 6 4 1

3 $p = \frac{1}{6}, q = \frac{5}{6}, n = 4$ 4 $4p^3q$

5 0.0154 6 0.3602 or 36% 7 0.9408

8 0.0592 9 0.0139 10 0.0026

Chapter review

1 D 2 D 3 D

4 a

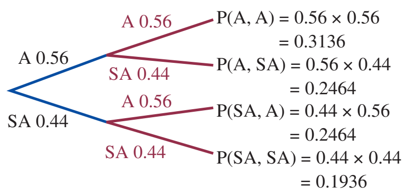
Sum	2	3	4	5	6	7	8	9	10	11	12
Prob.	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

b $\frac{1}{2}$ c $\frac{2}{9}$ d $\frac{1}{9}$

5 a $\frac{1}{8}$ b $\frac{3}{8}$

6 0.3975

7 0.4928



8 0.9999

9 a 0.5 b $\frac{15}{36}$ c $\frac{32}{36}$

10 C 11 D

12 a 0.5625 b 0.375 c 0.843 75

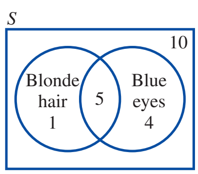
13 E

14 a i $\frac{25}{49}$ ii $\frac{20}{49}$

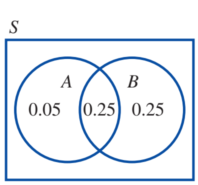
b i $\frac{20}{42}$ ii $\frac{20}{42}$

15 B

16 6

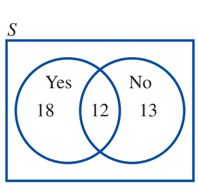


17 P(both A and B) = 0.25



18 E

19 a



b $\frac{30}{43}$

20 E

21 a Two outcomes, success or failure, same event repeated
b Tossing a coin 6 times
c Rolling a die and noting the uppermost face

22 a 0.402 b 0.032

23 0.016

24 a 0.39 b 0.39 c 0.78

25 a 0.8725 b 0.8042

26 a 0.2 b 0.3704

27 a 0.12 b 0.2517

CHAPTER 11 The normal distribution and games of chance

Skills check

1 Answers will vary. Check with your teacher.

2 a 6.7 b 2.4 c 4.7 below the mean

3 Distribution **b**

4 45 to 55

5 a The value of x is larger than 40.
b The value of x is less than or equal to 40.
c The value of x is larger than 20 and less than 30.

Exercise 11A — *z*-scores

1 3

2 −2

3 a 0 b 1 c −2 d 3 e −1

4 a 10.5 b 13.7 c 16.9 d 7.3 e 0.9

5 −0.27

6 1.5

7 a −0.48 b 1.44 c 0.08 d −2.24 e 2.8

8 a 10.3 s b 10.58 s c 10.37 s
d 9.88 s e 10.251 s f 10.524 s

9 a $\bar{x}$ = 19.55, SD = 1.76 b 1.68

10 a

Amount ($)	Class centre	Frequency
0–<20	$10	2
20–<40	$30	8
40–<60	$50	19
60–<80	$70	15
80–<100	$90	6

b $\bar{x}$ = 56, SD = 20.1
c i −0.3 ii 2.2 iii −2.0

11 C

12 B

13 D

14 a $\bar{x}$ = 64.7, SD = 11.4
b Highest score z = 2.66, Lowest score z = −1.73

15 English 1, Mathematics 1.31, Biology 1.5, Legal studies −2, Drama 0.67, Music −0.8

Exercise 11B — Comparison of scores

1 a English 1.25, Maths 1.33
b Maths mark is better as it has a higher z-score.
2 2nd test, Barbara's z-score was –0.33 compared to –0.5 in the first test.
3 C
4 E
5 Course A, z-score of –0.8 compared to –0.75 on course B
6 a Atlanta 0.44, Sydney 1
b In Atlanta because of the lower z-score
7 D
8 B
9 a Mathematics $\bar{x}$ = 59.5, SD = 17.9
Chemistry $\bar{x}$ = 59.6, SD = 16.8
b Mathematics 0.25, Chemistry 0.20. So Mathematics is the better result.
10 Kory is the better candidate with a z-score of 1.5 compared with 0.875 for Ricardo.

10 Quick Questions 1

1 2
2 –2
3 –1.03
4 2.95
5 One standard deviation above the mean
6 Two standard deviations below the mean
7 50
8 8
9 English 1.25, Maths 1.4
10 Maths

Exercise 11C — Distribution of scores

1 a 68% b 95% c 99.7%
2 a 68% b 95% c 99.7%
3 95%
4 16%
5 a 68% b 16% c 0.15%
6 21.1 and 33.9 years
7 a 68% of the values have a z-score between –1 and 1.
b 95% of the values have a z-score between –2 and 2.
c 99.7% of the values have a z-score between –3 and 3.
8 B
9 A
10 0.15%
11 a 16% b 16%
12 a 95% b 16% c 34%
d 15.85% e 83.85%
13 a 95 g to 105 g b 92.5 g to 107.5 g
14 163 cm – 181 cm
15 Faulty, as the one chosen has a z-score greater than 3.
16 2.6 kg – 5 kg

Exercise 11D — Standard normal tables

1 a 0.8413 b 0.9192 c 0.9641
d 0.1587 e 0.0446 f 0.2417
g 0.6826 h 0.9544 i 0.9974
j 0.1359 k 0.0215 l 0.8664
2 a 0 b –0.75 c –1 d 1.5
3 a 0.5 b –0.5 c –1.3 d 1.2
4 a 0.8413 b 0.9452 c 0.5
d 0.0013 e 0.3413 f 0.8185
5 a 0.6406 b 0.8577 c 0.5 d 0.3594
e 0.2812 f 0.6188 g 0.0509 h 0.0554
6 a 0.5000 b 0.1335 c 0.0294 d 0.8665
7 a 0.7486 b 0.9082 c 0.2514 d 0.6568
8 a 75% b 37 or 38 c 45
9 a i 0.9332 ii 0.8413 iii 0.1210
b 22.6%
10 a 0.7881 b 0.3446 c 0.1151
d 0.7881 e 0.5403
11 4.75%
12 0.38%, assuming mass to be normally distributed

Exercise 11E — Odds

1 a $105 b $105 c $429 d $300
e $15 f $33.33 g $66.67
2 a $140 b $175 c $507 d $420
e $60 f $83.33 g $216.67
3 a $\frac{4}{5}$ b $\frac{1}{4}$ c $\frac{3}{5}$
d $\frac{2}{7}$ e $\frac{3}{10}$ f $\frac{2}{3}$
4 a 2:1 b 4:1 c 3:2 on d 5:2
e 7:5 f 2:1 on g 6:4 h 11:9
5 a Evens b 5:1 c 3:1
d 12:1 e 3:1
6 a 2:1 b 3:1 c Evens
d 7:5 e 2:1 on f 5:2 on
7 a $36 000 b i $13 500 ii $22 500
8 a $160 b $71.11 c $80

Exercise 11F — Two-up

1 $\frac{1}{4}$ 2 $\frac{1}{4}$ 3 $\frac{1}{2}$
4 $\frac{1}{2}$ 5 $\frac{1}{8}$ 6 $\frac{1}{16}$
7 a TH, TH, TH, TH b $\frac{1}{16}$
8 $\frac{1}{32}$
9 a $\frac{1}{8}$ b $\frac{1}{16}$ c $\frac{1}{32}$ d $\frac{1}{64}$ e 0.48
10 Not quite, probability of winning = 0.48

Exercise 11G — Roulette

1 a $\frac{18}{37}$ b $\frac{6}{37}$
2 a 19:18 b 31:6
c No, slightly lower
3 a $60 b 0 c $200 d $80
4 Nothing happens except when the ball lands on 0, then he loses both bets.

Exercise 11H — Common fallacies in probability

1 a $\frac{1}{16}$ b $\frac{1}{32}$ c $\frac{1}{2}$
2 a i 0.32 ii 0.24
b Her chances of winning any match remain 0.75.
4 a Events are not independent.
b How many people in this group own surfboards?

Exercise 11I — Mathematical expectation

1 3.6
2 3.1
3 15.6 years
4 –40 cents
5 –\$2.50
6 \$0.62
7 –13.5 cents
8 \$1.60
9 7

10 Quick Questions 2

1 Very probable
2 2.5%
3 970 mL
4 0.9505
5 0.0495
6 0.0495
7 0.9010
8 $\frac{2}{5}$
9 \$500
10 \$1 loss

Chapter review

1 –2
2 a 0 b 1 c –2 d 3 e –1
3 1.87
4 a 0.17 b 1.83 c –3 d –1.75 e –2
5 a $\bar{x}$ = 1130, SD = 334.2
b i –0.39 ii 2.05 iii –2.62 iv –1.13 v 3.07
6 a $\bar{x}$ = 20.1, SD = 2.1
b Highest = 1.86, Lowest = –1.95
7 a 1.5 b 1 c Physics, higher z-score
8 a Geography: –0.8, Business studies: –0.53
b Business studies: higher z-score
9 Numeracy: lower z-score
10 a 68% b 95% c 99.7%
11 a 68% b 95% c 99.7%
12 a 34% b 47.5% c 2.5%
d 0.15% e 97.35%
13 Faulty, as it is more than three standard deviations from the mean.
14 a 0.9032 b 0.9918 c 0.1587
d 0.0668 e 0.2075
15 a 0 b 1 c –0.83 d 5.33
16 a 0.5 b 0.8413 c 0.1587
d 0.0475 e 0.1293 f 0.3779
17 a 0.5 b 0.8413 c 0.7357 d 0.2342
18 a Horses 5 and 6
b Horses 1, 2 and 5
19 Test this by simulation on a computer.
20 a Same b Same
21 a i 0.49 ii 0.25 iii 0.25
b No
22 a \$1.67
b About \$2.65 (if the operator's percentage is based on this figure)

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