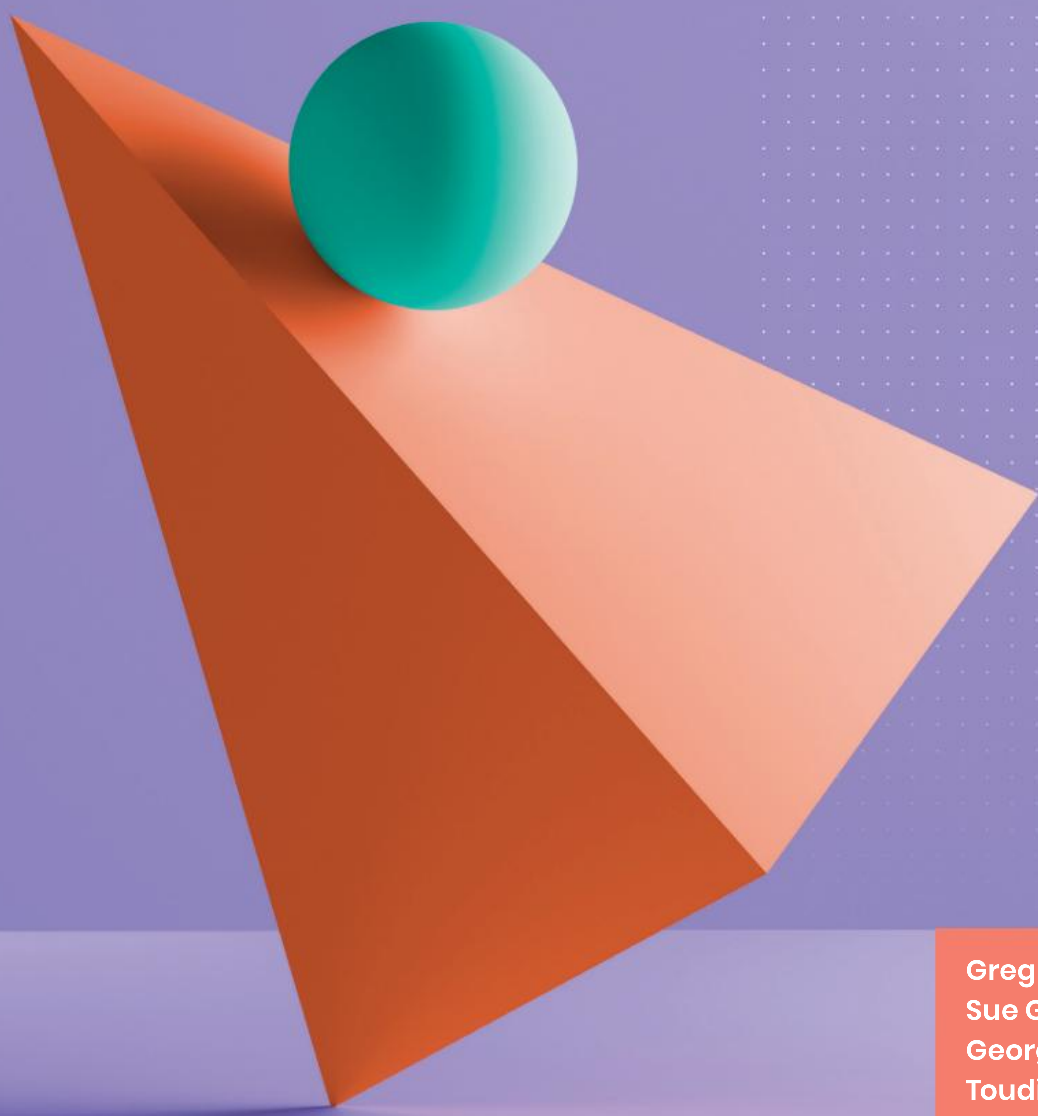
NELSON

VICmaths

VCE UNITS 1 + 2



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mathematical methods 11

Nelson VICmaths Mathematical Methods 11
1st Edition
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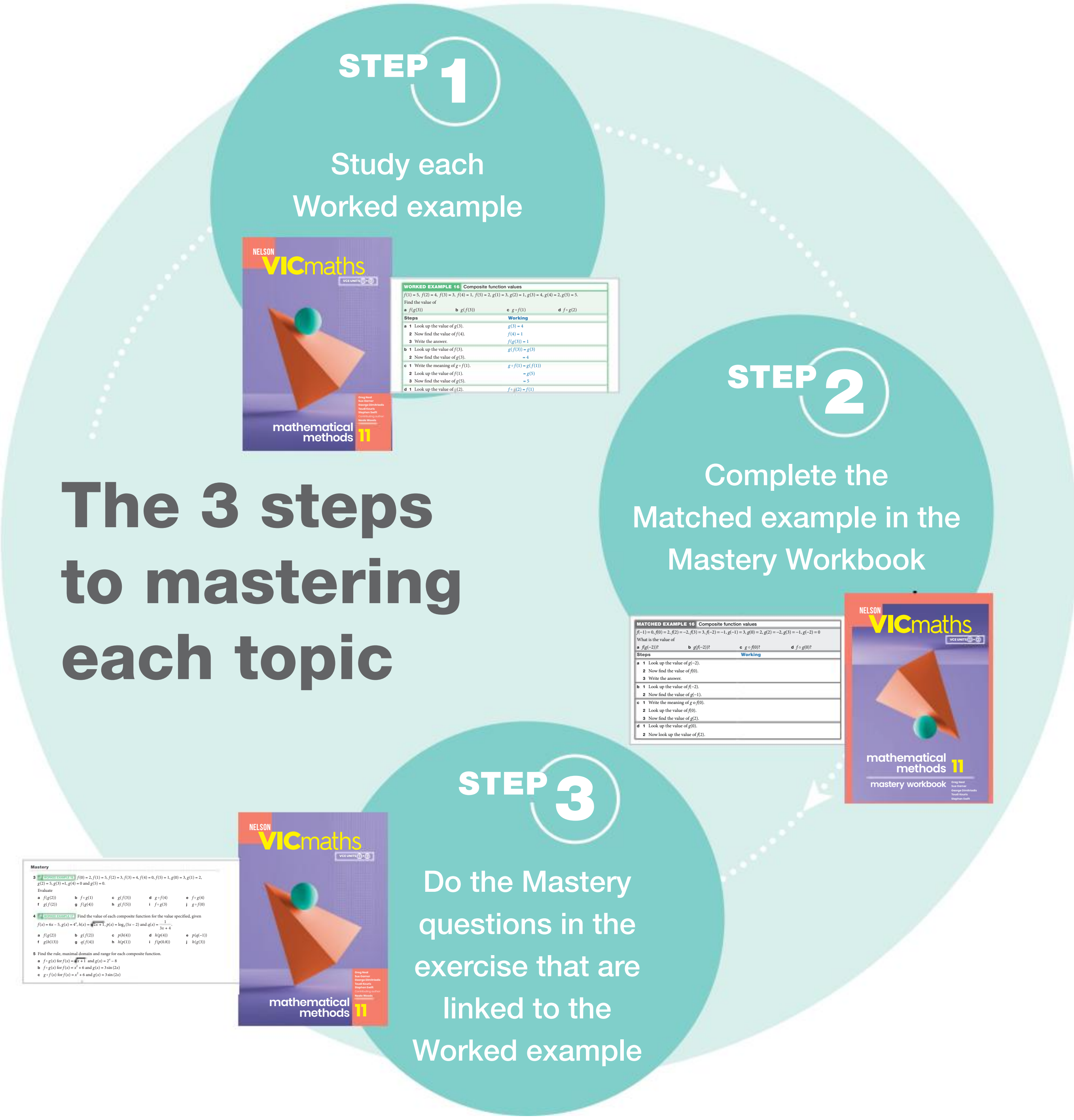
Contents

UNIT 2

To the student

Nelson VICmaths is your best friend when it comes to studying Mathematical Methods in Year 11. It has been written to help you maximise your learning and success this year. Every explanation, every exam hack and every worked example has been written with the exams in mind.



To the teacher

Now there's a better way to VCE maths mastery.

Nelson VICmaths 11–12 is a new VCE mathematics series that is backed by research into the science of learning. The design and structure of the series have been informed by teacher advice and evidence-based pedagogy, with the focus on preparing VCE students for their exams and maximising their learning achievement.

- Using **backwards learning design**, this series has been built by analysing past VCE exam questions and ensuring that all theory and examples are precisely mapped to the VCE Study Design.
- To reduce the **cognitive load** for learners, explanations are clear and concise, using the technique of **chunking** text with accompanying diagrams and infographics.
- The student book and workbook combination has been designed for **mastery** of the learning content.
- The exercise structure of **Recap**, **Mastery** and **Exam practice** leads students from procedural fluency to **higher-order thinking** using the learning technique of **interleaving**.
- **Exam practice** includes exam-style questions and graded past VCE exam questions with success percentages based on **VCAA performance data**.
- The cumulative structure of Exercise **Recaps** and chapter-based **Cumulative examinations** is built on the learning and memory techniques of **spacing** and **retrieval**.

About the authors

Greg Neal has taught for over 40 years and was Head of Mathematics at Ballarat High School. He has been a VCAA examination assessor for Mathematical Methods for 20 years and has expertise with CAS technology. Greg has written a range of Nelson resources for Further Mathematics, Mathematical Methods and Specialist Mathematics.

Sue Garner has taught senior mathematics at Ballarat Grammar School for 30 years. She has been a VCAA examination assessor for Mathematical Methods and Specialist Mathematics for many years and works for the VCAA in the role of Curriculum Manager for VCE Coursework. Sue has expertise with ClassPad technology and regularly presents workshops on CAS.

George Dimitriadis teaches at Montmorency Secondary College and has taught for 40 years. He is a VCAA Examination Development panellist and examination assessor for Specialist Mathematics. George has extensive experience in writing textbooks for Years 7 to 12 and contributing to mathematics enrichment blogs.

Toudi Kouris is Head of Mathematics at Strathcona Girls Grammar School, Canterbury and has taught for over 30 years. She has led mathematics at several independent schools and has written a range of student resources for VCE mathematics. Toudi has been a VCAA examination assessor and moderator for Mathematical Methods.

Stephen Swift is the lead author of *Nelson Senior Maths* (Australian Curriculum) and *Nelson QMaths* (QLD) for Mathematical Methods and Specialist Mathematics. He has taught all levels of mathematics in several states for 40 years and was Head of Mathematics at Wellington Point State High School in Brisbane.

Contributing author

Neale Woods coordinated the *Using CAS* content. He is a CAS specialist who often presents at conferences and conducts technology workshops for teachers and students. Neale has been teaching mathematics for over 45 years, including for Virtual Schools Victoria.

Darren Smyth (maffsguru.com) created the VCE question analysis videos.

Study Design grid

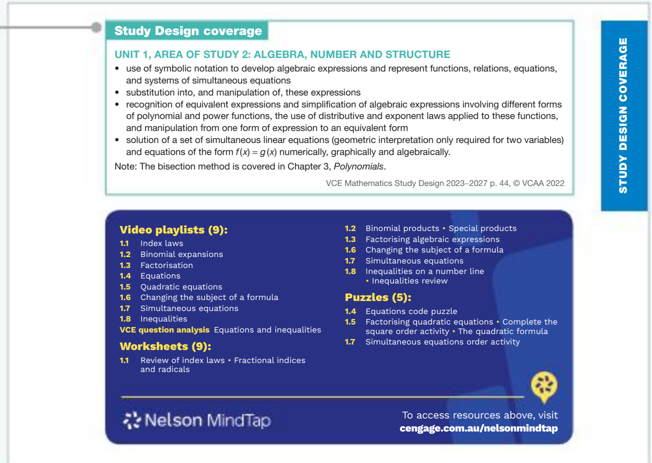
Area of study	*Nelson VICmaths Mathematical Methods 11* chapter	
UNIT 1		
1 Functions, relations and graphs	2	Functions and graphs
	3	Polynomials
	4	Power and inverse functions
2 Algebra, number and structure	1	Equations and inequalities
	3	Polynomials
	4	Power and inverse functions
3 Calculus	6	Rates of change
4 Data analysis, probability and statistics	5	Probability and counting
UNIT 2		
1 Functions, relations and graphs	7	The exponential and logarithmic functions
	9	The circular functions
2 Algebra, number and structure	7	The exponential and logarithmic functions
	11	Trigonometric and exponential equations
3 Calculus	6	Rates of change
	10	Differentiation
	12	Applications of differentiation
4 Data analysis, probability and statistics	8	Probability theory

9780170448260

About this book

In each chapter

Study Design coverage and extracts are shown at the front of the chapter, along with a listing of **Nelson MindTap** chapter resources.



Study Design coverage

UNIT 1, AREA OF STUDY 2: ALGEBRA, NUMBER AND STRUCTURE

- use of symbolic notation to develop algebraic expressions and represent functions, relations, equations, and systems of simultaneous equations
- substitution into, and manipulation of, these expressions
- recognition of equivalent expressions and simplification of algebraic expressions involving different forms of polynomial and power functions, the use of distributive and exponent laws applied to these functions, and manipulation from one form of expression to an equivalent form
- solution of a set of simultaneous linear equations (geometric interpretation only required for two variables) and equations of the form $f(x) = g(x)$ numerically, graphically and algebraically.

Note: The bisection method is covered in Chapter 3, *Polynomials*.

VCE Mathematics Study Design 2023–2027 p. 44, © VCAA 2022

STUDY DESIGN COVERAGE

Video playlists (9):
1.1 Index laws
1.2 Binomial expansions
1.3 Factorisation
1.4 Equations
1.5 Quadratic equations
1.6 Changing the subject of a formula
1.7 Simultaneous equations
1.8 Inequalities
VCE question analysis Equations and inequalities

Worksheets (9):
1.1 Review of index laws • Fractional indices and radicals
1.2 Binomial products • Special products
1.3 Factorising algebraic expressions
1.6 Changing the subject of a formula
1.7 Simultaneous equations
1.8 Inequalities on a number line • Inequalities review

Puzzles (5):
1.4 Equations code puzzle
1.5 Factorising quadratic equations • Complete the square order activity • The quadratic formula
1.7 Simultaneous equations order activity

Nelson MindTap

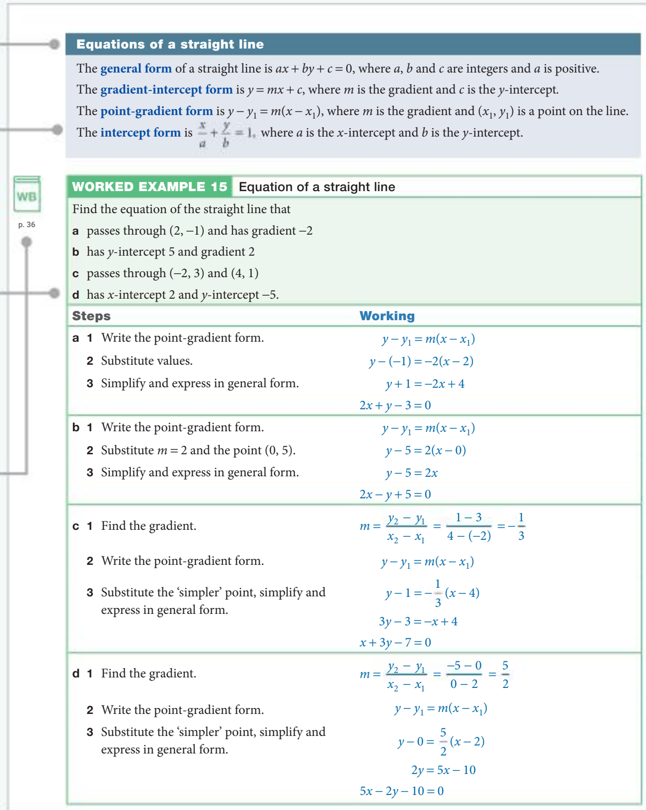
To access resources above, visit **cengage.com.au/nelsonmindtap**

Important facts and formulas are highlighted in a shaded box.

Important words and phrases are printed in blue and listed in the **Glossary and index** at the back of the book.

Worked examples are explained clearly step-by-step, with the mathematical working shown on the right-hand side.

Links to scaffolded matched examples in the Mastery Workbook (WB).



Equations of a straight line

The **general form** of a straight line is $ax + by + c = 0$, where a, b and c are integers and a is positive.

The **gradient-intercept form** is $y = mx + c$, where m is the gradient and c is the y-intercept.

The **point-gradient form** is $y - y_1 = m(x - x_1)$, where m is the gradient and (x_1, y_1) is a point on the line.

The **intercept form** is $\frac{x}{a} + \frac{y}{b} = 1$, where a is the x-intercept and b is the y-intercept.

WB p. 36

WORKED EXAMPLE 15 Equation of a straight line

Find the equation of the straight line that

a passes through $(2, -1)$ and has gradient -2

b has y-intercept 5 and gradient 2

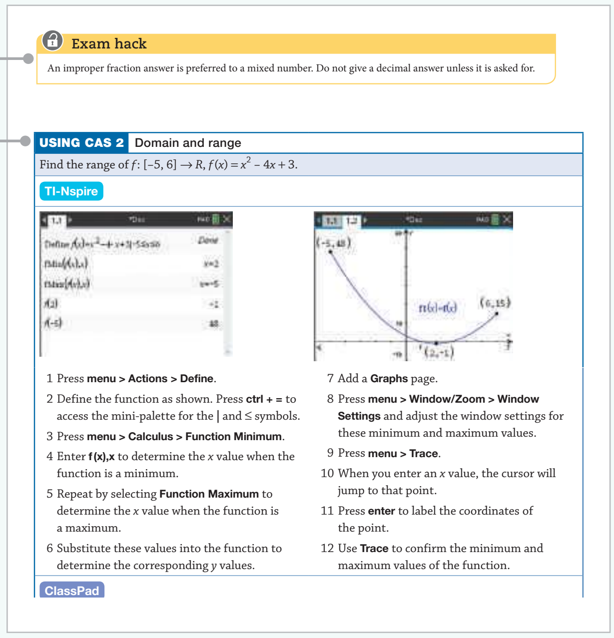
c passes through $(-2, 3)$ and $(4, 1)$

d has x-intercept 2 and y-intercept -5.

Steps	Working
a 1 Write the point-gradient form.	$y - y_1 = m(x - x_1)$
2 Substitute values.	$y - (-1) = -2(x - 2)$
3 Simplify and express in general form.	$y + 1 = -2x + 4$ $2x + y - 3 = 0$
b 1 Write the point-gradient form.	$y - y_1 = m(x - x_1)$
2 Substitute $m = 2$ and the point $(0, 5)$.	$y - 5 = 2(x - 0)$
3 Simplify and express in general form.	$y - 5 = 2x$ $2x - y + 5 = 0$
c 1 Find the gradient.	$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1 - 3}{4 - (-2)} = -\frac{1}{3}$
2 Write the point-gradient form.	$y - y_1 = m(x - x_1)$
3 Substitute the 'simpler' point, simplify and express in general form.	$y - 1 = -\frac{1}{3}(x - 4)$ $3y - 3 = -x + 4$ $x + 3y - 7 = 0$
d 1 Find the gradient.	$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-5 - 0}{0 - 2} = \frac{5}{2}$
2 Write the point-gradient form.	$y - y_1 = m(x - x_1)$
3 Substitute the 'simpler' point, simplify and express in general form.	$y - 0 = \frac{5}{2}(x - 2)$ $2y = 5x - 10$ $5x - 2y - 10 = 0$

Exam hacks highlight valuable exam hints and common student errors.

Using CAS provides clear instructions for TI-Nspire and Casio ClassPad calculators.


Exam hack

An improper fraction answer is preferred to a mixed number. Do not give a decimal answer unless it is asked for.

USING CAS 2 Domain and range

Find the range of $f: [-5, 6] \to R, f(x) = x^2 - 4x + 3$.

TI-Nspire

1 Press **menu > Actions > Define**.
2 Define the function as shown. Press **ctrl + =** to access the mini-palette for the | and ≤ symbols.
3 Press **menu > Calculus > Function Minimum**.
4 Enter **f(x),x** to determine the x value when the function is a minimum.
5 Repeat by selecting **Function Maximum** to determine the x value when the function is a maximum.
6 Substitute these values into the function to determine the corresponding y values.
7 Add a **Graphs** page.
8 Press **menu > Window/Zoom > Window Settings** and adjust the window settings for these minimum and maximum values.
9 Press **menu > Trace**.
10 When you enter an x value, the cursor will jump to that point.
11 Press **enter** to label the coordinates of the point.
12 Use **Trace** to confirm the minimum and maximum values of the function.

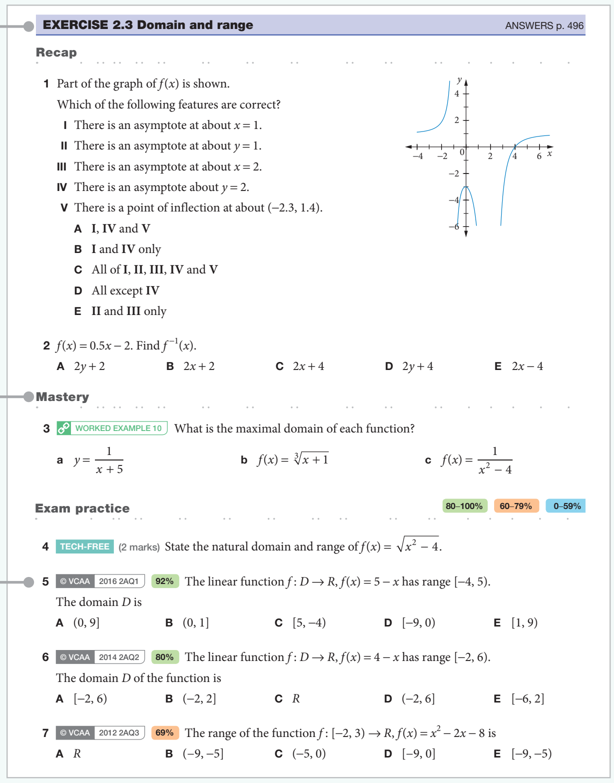
ClassPad

Graded exercises, including **Recap**, **Mastery** and **Exam Practice** questions, are linked to worked examples and Using CAS, and include past VCE exam and exam-style questions.

Recap questions revise skills from the previous exercise, **Mastery** questions provide skill practice, while **Exam practice** applies learned skills to VCE exam and exam-style problems.

Past VCE exam questions are clearly tagged and graded by colour-coded success percentages based on VCAA data, presented in the order Exam 1, Exam 2A, Exam 2B.

© VCAA 2013 1Q5b 76%


EXERCISE 2.3 Domain and range ANSWERS p. 496

Recap

1 Part of the graph of $f(x)$ is shown.
Which of the following features are correct?
I There is an asymptote at about $x = 1$.
II There is an asymptote at about $y = 1$.
III There is an asymptote at about $x = 2$.
IV There is an asymptote about $y = 2$.
V There is a point of inflection at about $(-2.3, 1.4)$.
A I, IV and V
B I and IV only
C All of I, II, III, IV and V
D All except IV
E II and III only

2 $f(x) = 0.5x - 2$. Find $f^{-1}(x)$.
A $2y + 2$ B $2x + 2$ C $2x + 4$ D $2y + 4$ E $2x - 4$

Mastery

3 WORKED EXAMPLE 10 What is the maximal domain of each function?
a $y = \frac{1}{x+5}$ b $f(x) = \sqrt[3]{x+1}$ c $f(x) = \frac{1}{x^2-4}$

Exam practice 80–100% 60–79% 0–59%

4 TECH-FREE (2 marks) State the natural domain and range of $f(x) = \sqrt{x^2 - 4}$.

5 © VCAA 2016 2AQ1 92% The linear function $f: D \to R, f(x) = 5 - x$ has range $[-4, 5)$.
The domain D is
A $(0, 9]$ B $(0, 1]$ C $[5, -4)$ D $[-9, 0)$ E $[1, 9)$

6 © VCAA 2014 2AQ2 80% The linear function $f: D \to R, f(x) = 4 - x$ has range $[-2, 6)$.
The domain D of the function is
A $[-2, 6)$ B $(-2, 2]$ C R D $(-2, 6]$ E $[-6, 2]$

7 © VCAA 2012 2AQ3 69% The range of the function $f: [-2, 3) \to R, f(x) = x^2 - 2x - 8$ is
A R B $(-9, -5]$ C $(-5, 0)$ D $[-9, 0]$ E $[-9, -5)$

KEY

2021	2021 exam year
<year>N	Northern hemisphere exam
<year>S	VCAA sample exam
TECH-FREE	Technology not permitted
1	Exam 1 (tech-free)
2A	Exam 2, Section A (multiple-choice)
2B	Exam 2, Section B

For multiple-choice questions and other 1-mark questions, the success percentage is the percentage of students who answered correctly. For questions worth more than 1 mark, the success percentage is the mean percentage result scored by students.

Success percentage range	Example
80–100% (straightforward)	83%
60–79% (standard)	69%
0–59% (complex)	45%

Note: Questions from Northern hemisphere and VCAA sample exams do not have success percentages.

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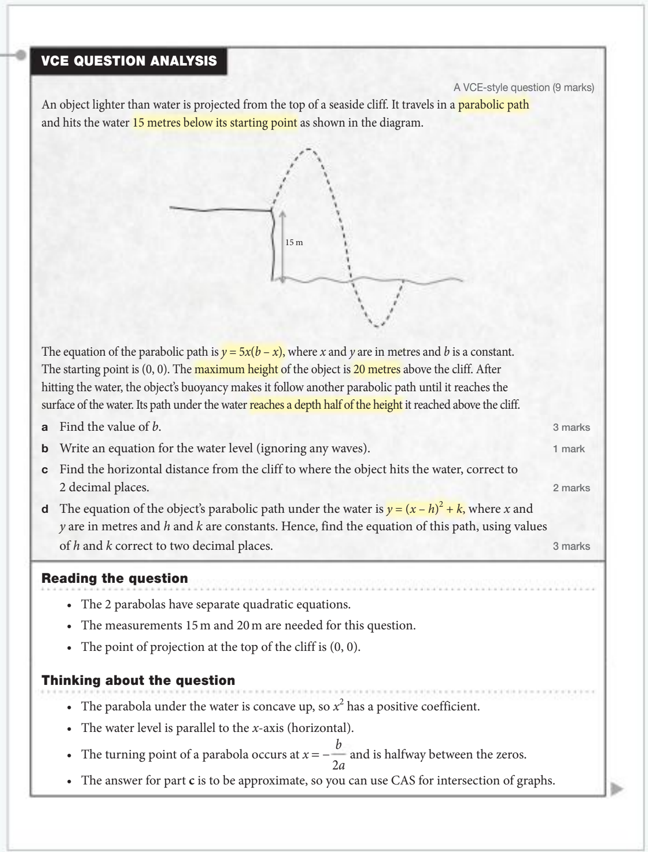
At the end of each chapter

VCE question analysis leads students through a VCE-style exam question that exemplifies the chapter, discussing how to approach the question, providing advice on interpreting the question and common student errors.



VCE QUESTION ANALYSIS

A VCE-style question (9 marks)

An object lighter than water is projected from the top of a seaside cliff. It travels in a parabolic path and hits the water 15 metres below its starting point as shown in the diagram.

The equation of the parabolic path is $y = 5x(b - x)$, where x and y are in metres and b is a constant. The starting point is (0, 0). The maximum height of the object is 20 metres above the cliff. After hitting the water, the object's buoyancy makes it follow another parabolic path until it reaches the surface of the water. Its path under the water reaches a depth half of the height it reached above the cliff.

a Find the value of b. 3 marks

b Write an equation for the water level (ignoring any waves). 1 mark

c Find the horizontal distance from the cliff to where the object hits the water, correct to 2 decimal places. 2 marks

d The equation of the object's parabolic path under the water is $y = (x - h)^2 + k$, where x and y are in metres and h and k are constants. Hence, find the equation of this path, using values of h and k correct to two decimal places. 3 marks

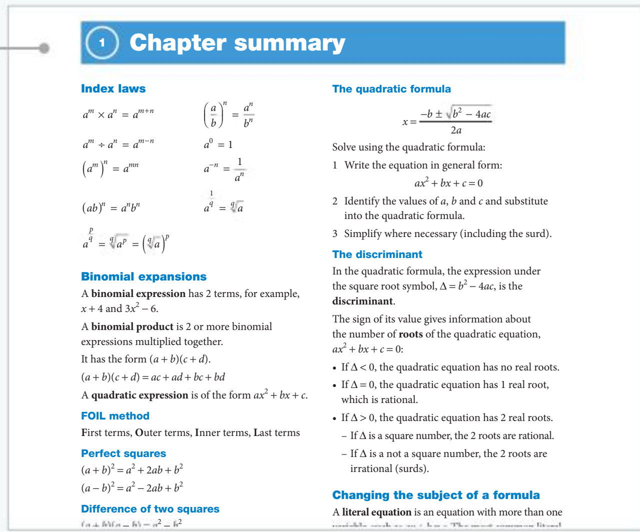
Reading the question

- The 2 parabolas have separate quadratic equations.
- The measurements 15 m and 20 m are needed for this question.
- The point of projection at the top of the cliff is (0, 0).

Thinking about the question

- The parabola under the water is concave up, so x^2 has a positive coefficient.
- The water level is parallel to the x-axis (horizontal).
- The turning point of a parabola occurs at $x = -\frac{b}{2a}$ and is halfway between the zeros.
- The answer for part **c** is to be approximate, so you can use CAS for intersection of graphs.

Chapter summary for easy reference.



1 Chapter summary

Index laws

$a^m \times a^n = a^{m+n}$ $\quad \left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$

$a^m \div a^n = a^{m-n}$ $\quad a^0 = 1$

$(a^m)^n = a^{mn}$ $\quad a^{-n} = \frac{1}{a^n}$

$(ab)^n = a^n b^n$ $\quad a^{\frac{1}{q}} = \sqrt[q]{a}$

$a^{\frac{p}{q}} = \sqrt[q]{a^p} = \left(\sqrt[q]{a}\right)^p$

Binomial expansions

A **binomial expression** has 2 terms, for example, $x + 4$ and $3x^2 - 6$.

A **binomial product** is 2 or more binomial expressions multiplied together.

It has the form $(a + b)(c + d)$.

$(a + b)(c + d) = ac + ad + bc + bd$

A **quadratic expression** is of the form $ax^2 + bx + c$.

FOIL method

First terms, Outer terms, Inner terms, Last terms

Perfect squares

$(a + b)^2 = a^2 + 2ab + b^2$

$(a - b)^2 = a^2 - 2ab + b^2$

Difference of two squares

The quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Solve using the quadratic formula:

1 Write the equation in general form: $ax^2 + bx + c = 0$

2 Identify the values of a, b and c and substitute into the quadratic formula.

3 Simplify where necessary (including the surd).

The discriminant

In the quadratic formula, the expression under the square root symbol, $\Delta = b^2 - 4ac$, is the **discriminant**.

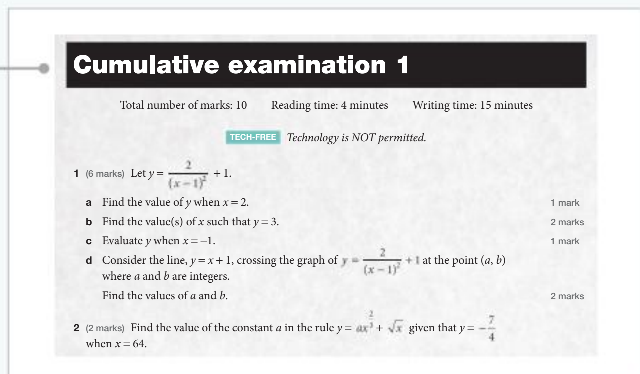
The sign of its value gives information about the number of **roots** of the quadratic equation, $ax^2 + bx + c = 0$:

- If $\Delta < 0$, the quadratic equation has no real roots.
- If $\Delta = 0$, the quadratic equation has 1 real root, which is rational.
- If $\Delta > 0$, the quadratic equation has 2 real roots.
  - If Δ is a square number, the 2 roots are rational.
  - If Δ is a not a square number, the 2 roots are irrational (surds).

Changing the subject of a formula

A **literal equation** is an equation with more than one

Cumulative examinations 1 and **2** are mini-exams based on the format of the VCE examinations 1 and 2, with around 50% of questions focusing on the chapter in which they appear.



Cumulative examination 1

Total number of marks: 10 Reading time: 4 minutes Writing time: 15 minutes

TECH-FREE *Technology is NOT permitted.*

1 (6 marks) Let $y = \frac{2}{(x-1)^2} + 1$.

a Find the value of y when $x = 2$. 1 mark

b Find the value(s) of x such that $y = 3$. 2 marks

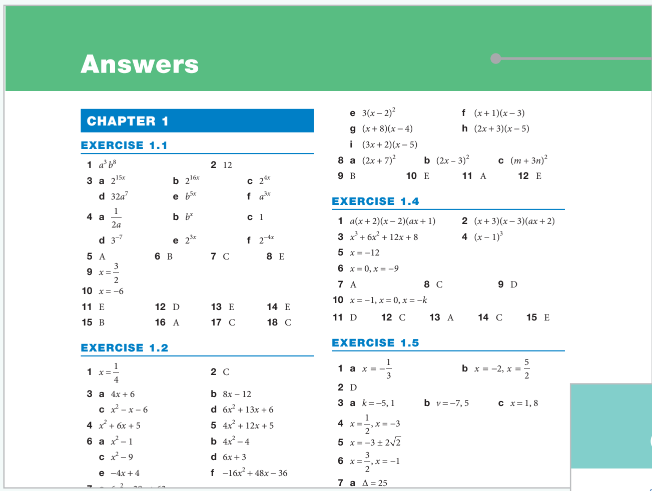
c Evaluate y when $x = -1$. 1 mark

d Consider the line, $y = x + 1$, crossing the graph of $y = \frac{2}{(x-1)^2} + 1$ at the point (a, b) where a and b are integers.

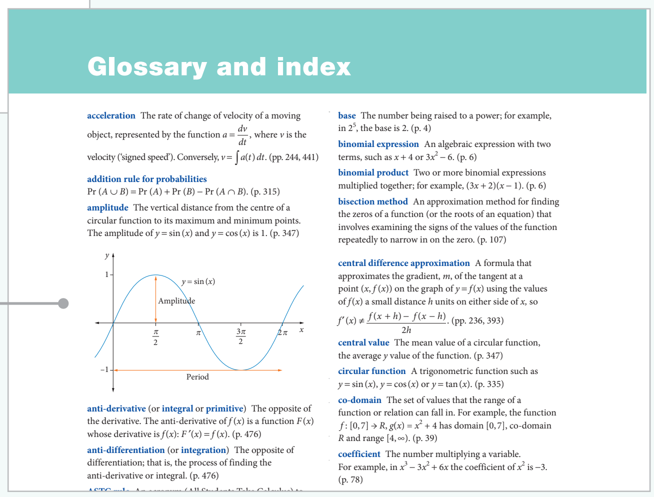
Find the values of a and b. 2 marks

2 (2 marks) Find the value of the constant a in the rule $y = ax^{\frac{2}{3}} + \sqrt{x}$ given that $y = -\frac{7}{4}$ when $x = 64$.

At the end of the book



Answers (with worked solutions provided on Nelson MindTap for teachers to allocate to students).



A combined **Glossary and index**.

Nelson MindTap

Video playlists

Worksheets

Skillsheets

Nelson MindTap is an online learning space that provides students with tailored learning experiences. Access tools and content that make learning simpler yet smarter to help you achieve VCE maths mastery. **Nelson MindTap** includes an eText with integrated interactives and online assessment. Margin links in the student book signpost multimedia student resources found on MindTap.

Nelson MindTap for students:

- **Watch** video tutorials featuring expert teacher advice to unpack new concepts and develop your understanding.
- **Revise** using quizzes, worksheets and skillsheets to practise your skills and build your confidence.
- **Navigate** your own path, accessing the content, analytics and support as you need it.

Nelson MindTap for teachers*:

- Tailor content to different learning needs – assign directly to the student, or the whole class.
- Monitor progress using assessment tools like Gradebook and Reports.
- Integrate content and assessment directly within your school's LMS for ease of access.
- Access topic tests, teaching plans and worked solutions to each exercise set.

*Complimentary access to these resources is only available to teachers who use this book as part of a class set, book hire or booklist. Contact your Cengage Education Consultant for information about access and conditions.

Nelson VICmaths 11–12 series

Companion resources

Mastery Workbook

- Step-by-step scaffolding to guide students towards mastery of the course content.
- Write-in support that encourages students to show working.
- Matched examples for *every* worked example in the student book.
- Full integration between student book and workbook.
- Answers (worked solutions for teachers on Nelson MindTap).

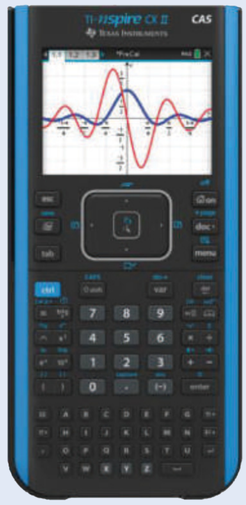
Exam*plus*

- Create and simulate exam-like conditions in minutes.
- Save time with an extensive bank of filterable and difficulty-graded questions for Year 12.
- Over 1000 past VCAA and unseen exam-style questions and solutions all in the one place.
- Extensively researched and user-tested.

TI-Nspire CAS introduction

The latest TI-Nspire model is TI-Nspire CX II CAS. When purchasing a new handheld, you also gain access to the student software. If you purchase a used handheld, then you can pay an additional fee for the student software.

Alternatively, you can connect your handheld to your computer using the TI-Nspire™ CX II Connect web-based app, which allows you to perform a variety of functions such as screen captures, file transfers and operating system updates. Note that TI-Nspire non-CAS technology is also available. It is **vital** that you use the CAS technology.



TI-Nspire CX II CAS

Student book instructions

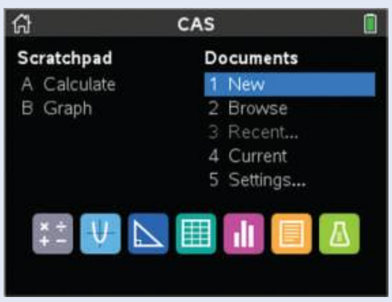
The instructions in this student book use words instead of symbols. Most keys on the keypad are clearly labelled with a word or an abbreviation. Four words that are used to represent less obvious keys are:

Word	Key
home	
catalog	
template	
Scratchpad	

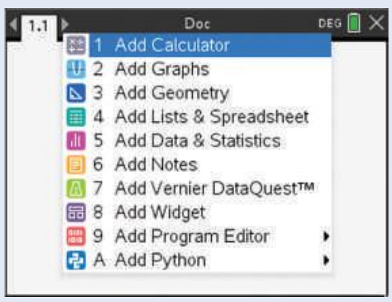
Several keys have a second function highlighted in blue above the key. For example, press **ctrl + x²** to access the square root function $\sqrt{\ }$.

Applications

The applications available are outlined below.



Press **home** to view the home page. The **Scratchpad** options on the left are available for quick calculations and graphing. The **Documents** options on the right are used for navigation. The seven icons on the bottom are the main applications.

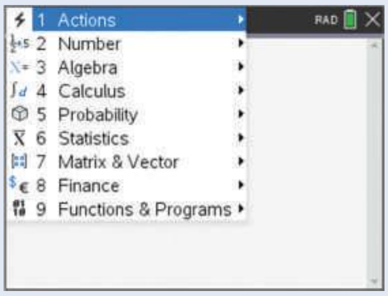


When you select **Documents > New** from the home page, a list of the seven applications plus three additional menu options will be displayed. From any application, press **ctrl + I** (for insert) to display this list and add a new page to the document.

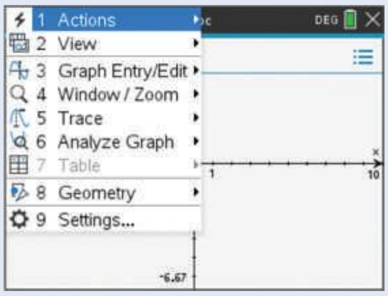
Menus

The instructions in this student book primarily use the **Calculator**, **Graphs**, **Lists & Spreadsheet** and **Data & Statistics** applications (see **Hints** on page xviii). The following figures show the initial menu options for these four applications. These menu options link to submenu options, which are not shown below. On the handheld and software, the applications are referred to as **Documents**. In the student book instructions, the applications are referred to as **pages** of the document (e.g. Add a **Graphs** page).

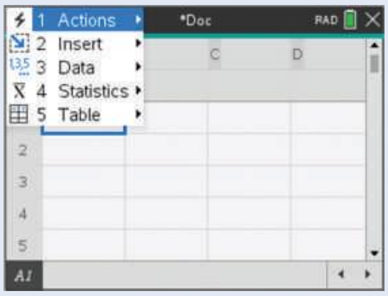
Calculator



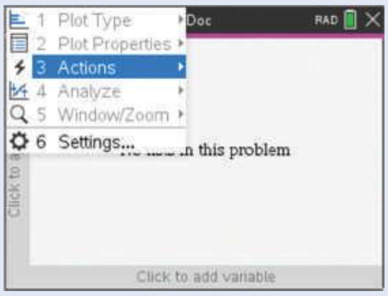
Graphs



Lists & Spreadsheet



Data & Statistics



If your document contains more than one page, move among them by clicking on the numbered tabs at the top of the screen. Alternatively, press **ctrl + left arrow** or **ctrl + right arrow**. To view all the pages of a document, press **ctrl + up arrow**.

All menu and submenu options include numbers. The student book instructions do not include numbers. For example, the instruction in a **Calculator** page for clearing all calculations is 'press **menu > Actions > Clear History**'. The shortcut is 'press **menu > 1 > 5**'. For efficiency, you are encouraged to learn the sequence of numbers for frequently used commands.

Document Settings

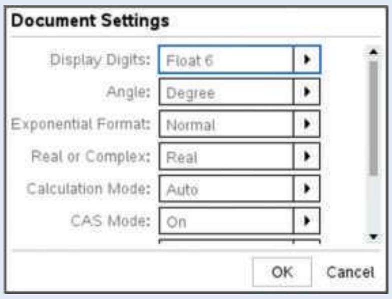
The document settings can be accessed in the following ways:

1 From the **home** page, press **Settings > Document Settings**.

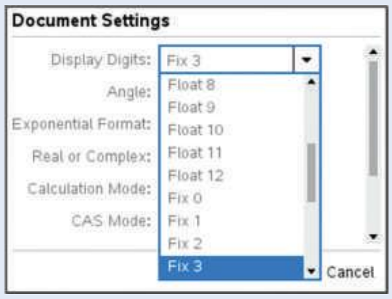
2 From a document page, press the **doc** key or click on **Doc** at the top of the page, then select **Settings & Status > Document Settings**.

3 Click on the **battery icon** in the top right-hand corner of the page, then select **Document Settings**.

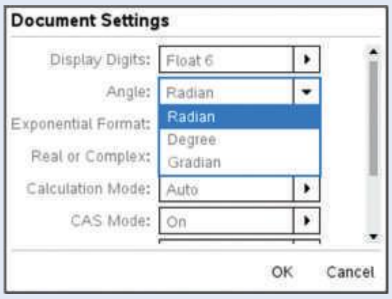
The document settings shown below are the primary ones you will be using.



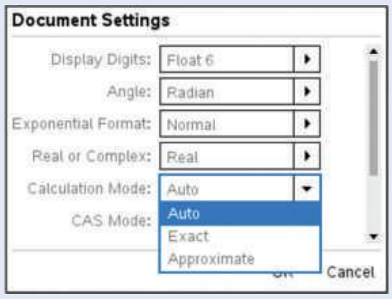
The screen above shows the default document settings. The **Display Digits** field is set to **Float 6**, which means up to six significant figures will be displayed.



Click in any field to display the options. The screen above shows the **Display Digits** options. Scroll down to select a specific number of decimal places.



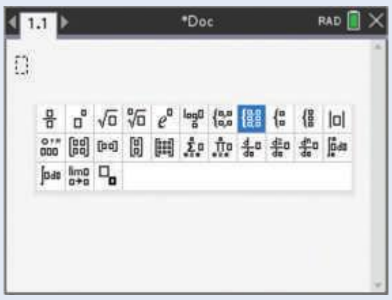
There are three **Angle** options shown above. Select either **Radian** or **Degree**. These two angle settings can be toggled at any time by clicking on **DEG** or **RAD** in the top right-hand corner of the screen.



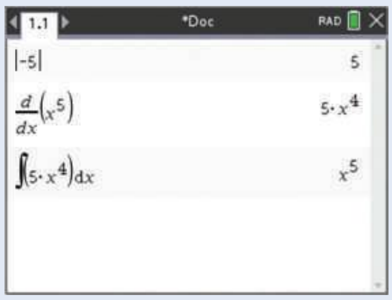
There are three **Calculation Mode** options shown above. It is recommended that you keep the default setting of **Auto**. If you require an approximate or decimal answer, press **ctrl + enter** or include a decimal point in your calculation.

Templates

The **template** key is located to the right of the **9** key.



Press **template** to view the template options. Most templates will be directly inserted, but for some, you will be prompted with a dialogue box. For example, after selecting the **3 piecewise function** template, you will be prompted for the number of function pieces.

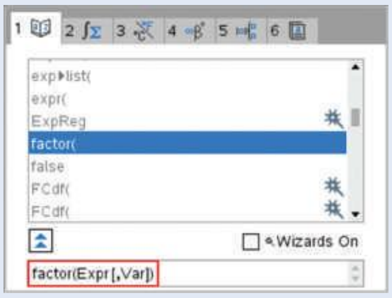


The screen above shows three examples using templates: an **absolute value**, a **derivative** and an **integral**. Many of the templates can be accessed using keys on the keyboard; for example, fraction, square root.

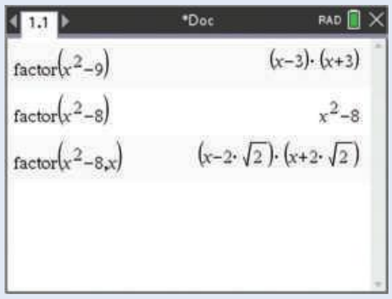
Catalog

The **catalog** key is located to the right of the **template** key. In addition to using the menus and submenus, you can access all the commands using the catalog. The advantage of using the catalog is that it shows the parameters required for each command at the bottom of the screen. Optional parameters appear in square brackets.

For example:



Press **catalog** and ensure tab **1** is selected. Press **F** to jump to the commands starting with F. Scroll down to **factor(**. The parameters for the **factor** command appear at the bottom of the screen (see **red** rectangle). **Expr** is a required parameter. The square brackets around **Var** means it is an optional parameter.



When you **factor** the expression $x^2 - 9$, the solution is $(x - 3)(x + 3)$. When you **factor** $x^2 - 8$, the expression remains unchanged. However, if you include the optional parameter **x**, the expression will factorise to $\left(x - 2\sqrt{2}\right)\left(x + 2\sqrt{2}\right)$

Symbols

All symbols are listed at the end of the **catalog**. To access the symbol palette, press **ctrl + catalog.** Frequently used symbols are available in mini-palettes by pressing the keys shown below.

sin	cos	tan	csc	sec	cot
$\sin^{-1}$	$\cos^{-1}$	$\tan^{-1}$	$\csc^{-1}$	$\sec^{-1}$	$\cot^{-1}$

Press **trig** to access the trigonometry functions.

>	<	≠
≥	≤	|

Press **ctrl + =** to access the inequalities and the **constraint** symbol.

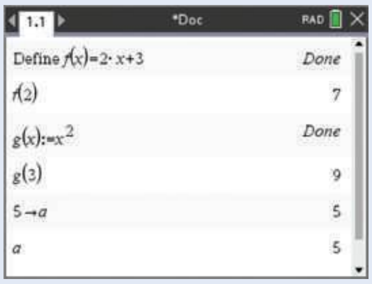
π	*i*	∞	*e*	θ
°	r	g	'	

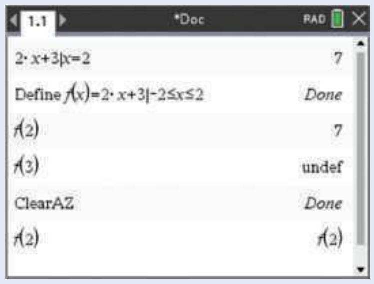
Press **π** to access commonly used symbols.

Hints

The instructions below provide a few hints to assist with using the 4 main applications.

Calculator





There are a few options available to define or store functions and variables.

1 Press **menu > Actions > Define**.

2 Press **ctrl + template** for the **:=** symbol.

3 Press **ctrl + var** for the **store** symbol.

Press **var** to view the list of defined and stored variables.

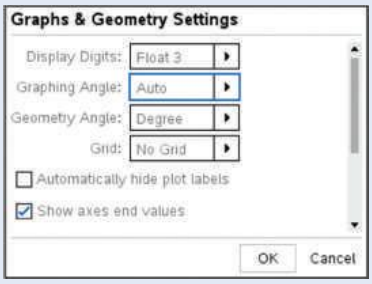
Press **ctrl + =** to access the **constraint** symbol (|).

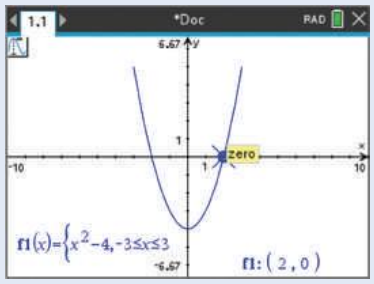
1 Use **constraint** for substitution.

2 Use **constraint** to limit the domain of a function.

Press **menu > Actions > Clear AZ** to delete defined variables.

Graphs

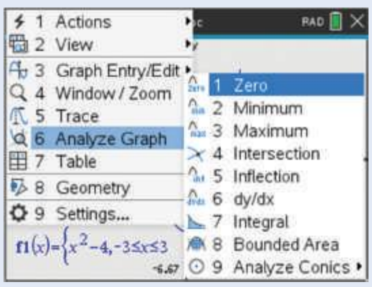


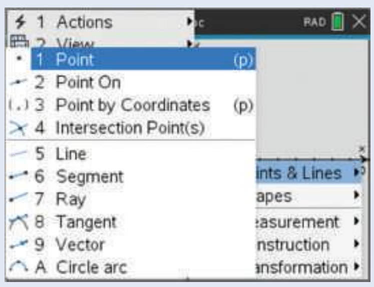


Press **menu > Settings** to view and/or change the settings for the **Graphs** and **Geometry** applications. The default setting for **Display Digits** is **Float 3** so change this if you need greater accuracy.
The **Graphing Angle** is set to **Auto** but can be changed to **Degree** or **Radian**.

The **constraint** command can also be used in a **Graphs** page to specify a domain.

Press **menu > Trace > Graph Trace** to identify points on a graph.

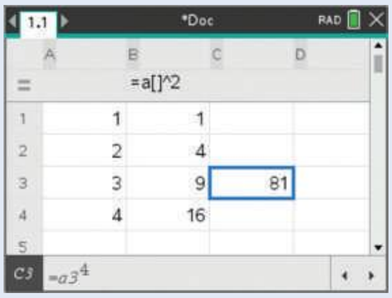


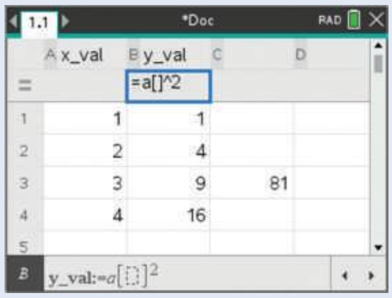


A second option for identifying key points on a graph is to press **menu > Analyze Graph**. These options prompt you for a **lower bound** and **upper bound** to locate the point.

A third option is to press **menu > Geometry > Points & Lines**. Locate points of intersection or place points on the graph and move them along the graph. Click on a coordinate to manually change a value.

Lists & Spreadsheet

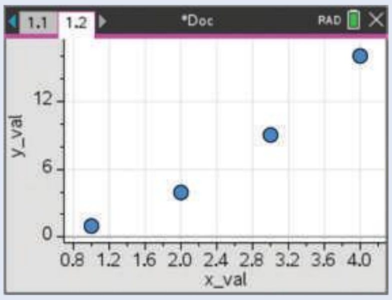


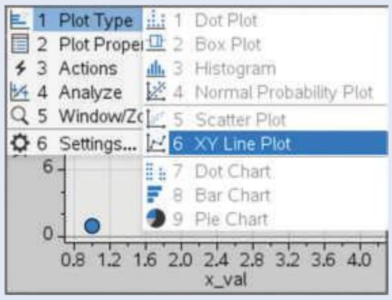


The columns in the **Lists & Spreadsheet** application can be used for lists. Above, the values in the list in column **B** are the squares of the values of the list in column **A**. It can also be used as a spreadsheet. The value in cell **C3** is the value in cell **A3** raised to the power of 4.

Calculations using lists can be completed using the column headings **A**, **B**, **C** etc. Lists need to be labelled to be used in other applications. The list **A** above has been labelled **x_val** and list **B** labelled **y_val**. List names cannot have spaces so spaces are replaced by asterisks. Alternatively, press **ctrl + space bar** to insert the **underscore** character instead of a space.

Data & Statistics

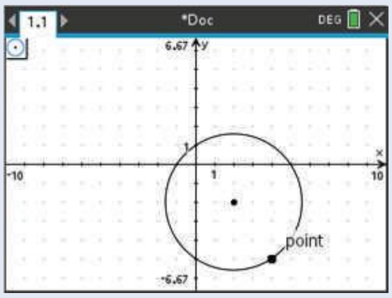




The **Data & Statistics** application is reliant on lists generated in other applications. It is designed for ungrouped data. The plot above displays the **x_val** and **y_val** lists from the **Lists & Spreadsheet** application.

Press **menu > Plot Type** to view the various graphing options. Options 1 to 4 are for univariate data (one list). Options 5 and 6 are for bivariate data (two lists). Options 7 to 9 are for categorical data (one list).

Geometry

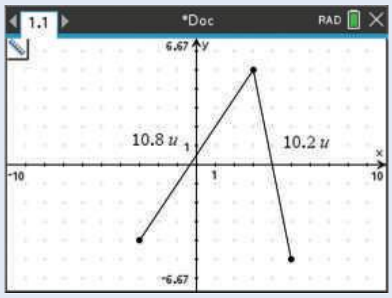


Add a **Graphs** page and press **esc** to hide the **Graph Entry Line**.

Press **menu > View > Grid > Dot Grid**.

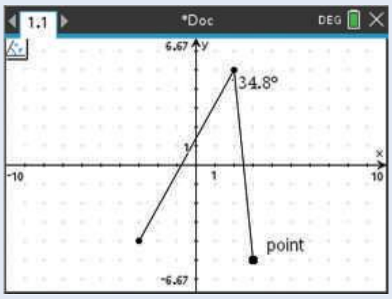
Press **menu > Geometry > Shapes** to view the submenu options. Select **Circle**.

Click on a centre and another point to draw a circle.



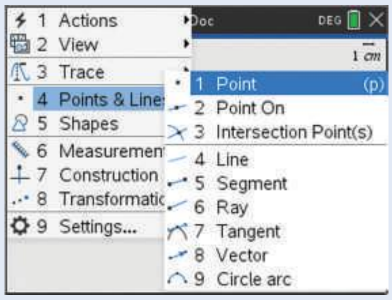
Press **menu > Geometry > Points & Lines** to view the submenu options. Select **Segment** then draw the segments.

Press **menu > Geometry > Measurement > Length** then click on the segments to measure them.



Press **menu > Geometry > Measurement > Angle** then click on the three points to measure the angle.

The tool being used is shown in the top left-hand corner of the screen. Press **esc** to remove a tool, which will then allow you to move objects around the screen.



Another option available is the **Geometry** page, which is restricted to plane geometry. Some of the menu options are shown above.

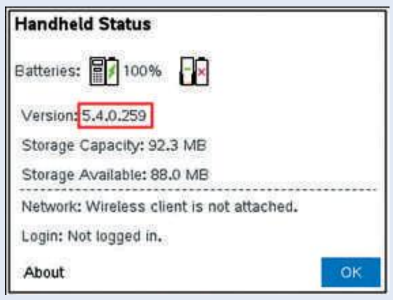
There is no grid or axes available on a Geometry page but an analytical Geometry page can be inserted.

Operating systems

Ensure the latest operating system is installed on your handheld and software.

Installing the latest operating system is relatively straightforward. Using the USB cable provided, connect the handheld to a computer with the student or computer link software installed. Select **Help > Check for OS Updates**. If you see a message that a new OS is available, follow the links to install it. Alternatively, go to the TI website at https://education.ti.com/ to download the latest operating system. Select **Tools > Install OS** then select the downloaded file.

To determine the version of your operating system, press **home > Settings > Status**. At the time of publication, the operating system for the CX II is version **5.4.0.259**.



Casio ClassPad introduction

The latest model of the Casio ClassPad is the fx-CP400. The connectivity software Screen Receiver, Share Assistant and Program Link Software can be downloaded for free. The ClassPad Manager software emulator is a separate program available at an additional cost.

Casio ClassPad

Student book instructions

The instructions in this student book use words instead of symbols. ClassPad tools are located at the top of the screen. These tools vary with each application. Initially, these instructions will show a tool enclosed in a red rectangle with the corresponding word highlighted in red. Examples are shown here:

Word	Tool
Graph	
View Window	
Table	
Table Input	

Applications

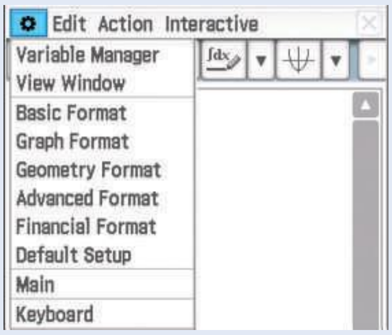
The applications available are outlined below.

Tap **Menu** to view the applications.

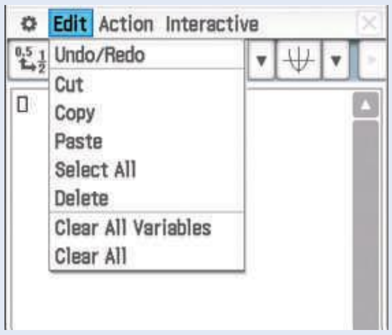
Slide the scroll bar at the bottom of the screen to access the full list.

Menus

The instructions in this student book will primarily use the **Main**, **Statistics**, **Geometry**, **Graph&Table** and **Conics** and applications (see **Applications** on page xxiv). All of these applications have the ⚙ (system) and **Edit** menus available at the top left of the screen.

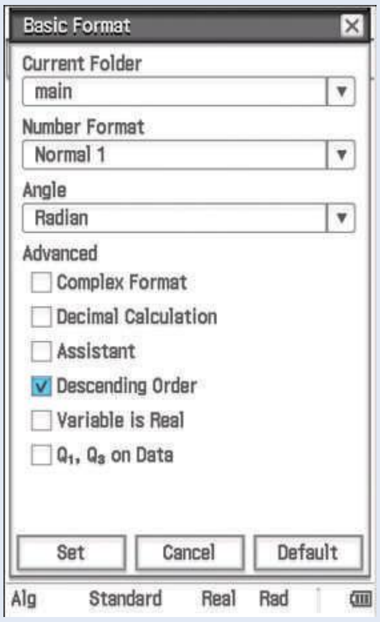


Tap ⚙ to view the system menu. The menu options allow you to manage variables and format applications.

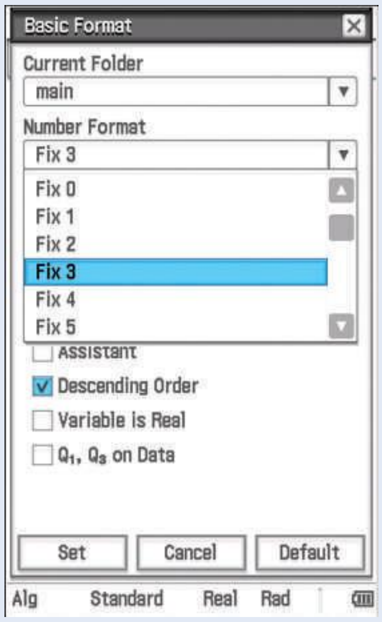


Tap **Edit** to view the edit menu. The menu options allow you to cut, copy, paste and delete screen content and clear variables. The **Edit** menu varies with each application.

Document Settings



Tap ⚙ > **Basic Format**. The screen above shows the default document settings. The **Number Format** default field is set to **Normal 1**.

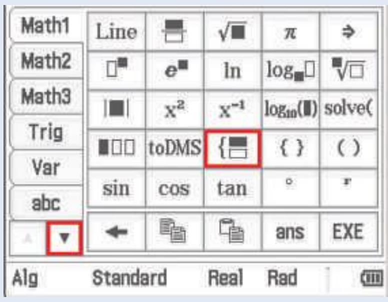


Tap the **Number Format** field to display the options. The screen above shows the **Display Digits** options. Scroll down to fix a specific number of decimal places.

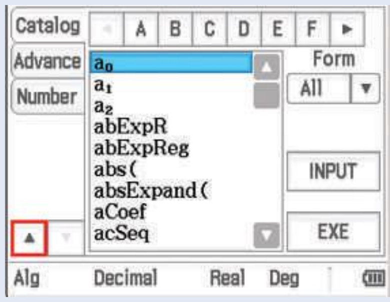
The settings **Standard/Decimal**, **Real/Cplx** and **Rad/Deg/Gra** can be toggled at the bottom of the screen. The recommended settings are **Standard**, **Real** and **Rad**.

Keyboard

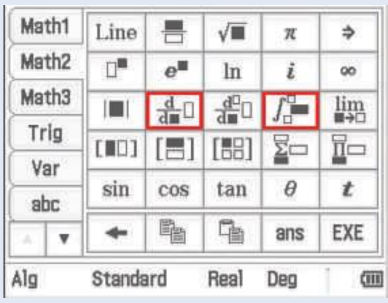
There are 9 soft keyboards available.



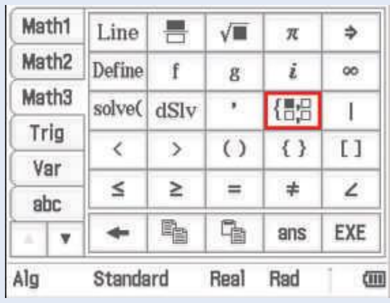
Press **Keyboard** to view the soft keyboards. The **Math1** soft keyboard is shown above with the **simultaneous equation** template highlighted in red. Tap the left tabs to access the other keyboards. Press the **down arrow** to view the second screen.



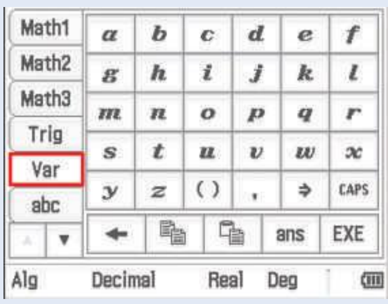
All functions can be accessed from **Catalog**. Tap on the letters at the top to jump through the list. Press the **up arrow** to return to the first screen.



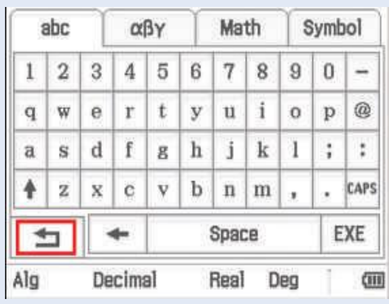
Tap **Math2** to display the soft keyboard. The **derivative** and **integral** templates are highlighted in red.



The **Math3** template is shown above with the **piecewise** template highlighted in red. Use this keyboard for inequalities.



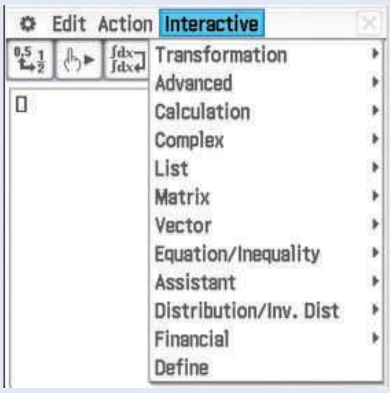
Tap **Var** to access the variables. Use variables, not letters, in your algebraic calculations.

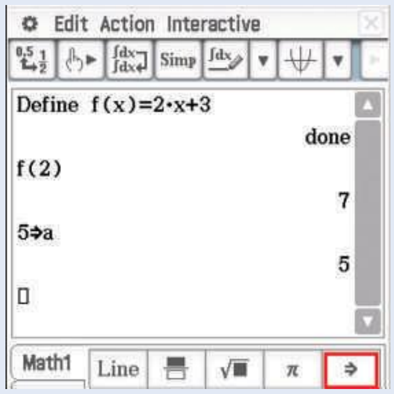


Tap **abc** to access letters and symbols. Use letters to name functions and matrices etc. Tap the tabs at the top of the screen to access the range of symbols. Press **back** to return to the main screen.

Applications

Main





In **Main**, there are two menus available to enter functions: **Active** and **Interactive**. With the **Interactive** menu, users are prompted with a dialogue box to enter the parameters. With few exceptions, the instructions in this student book are written using the **Interactive** menu.

For **derivative** and **integral**, it is easier to use **Interactive > Calculation > diff or summa (∫)** rather than the icons in **Math2**.

To define a function, enter and highlight the expression then tap **Interactive > Define**. The **store** arrow is available from the **Math** and **Trig** soft keyboards. **Store** can be used to store values.

Use the **abc** soft keyboard to label values.

All algebraic working uses **Var** and not **abc.**

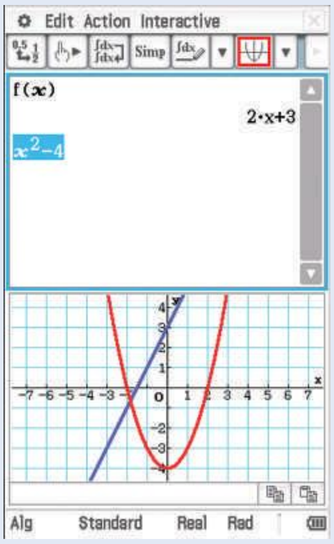
Define a function will be over-saved with a new defined function.

In contrast, store variables need to be cleared using **Variable Manager**.

9780170448260

Graphing

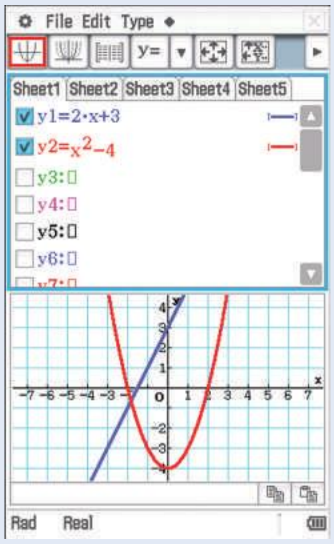
There are two main options for graphing functions and relations.



In **Main**, tap **Graph** to open the graph window. Enter and highlight the function or expression and drag it into the **Graph** window from either the right or left side.

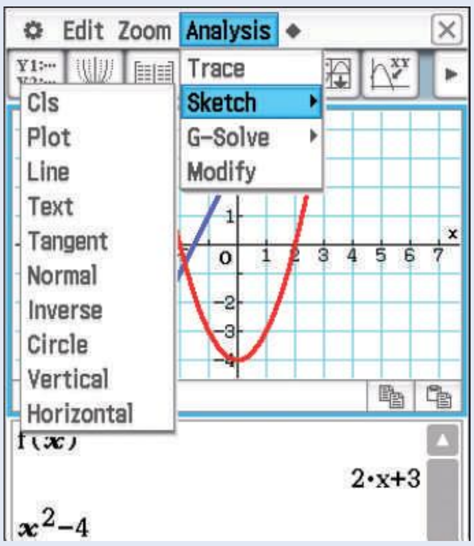
If the function has a restricted domain, it must be dragged from the left side of the screen.

The significance of **Main** is that the algebra is on the top and matching graphs are on the bottom of the twin screens.

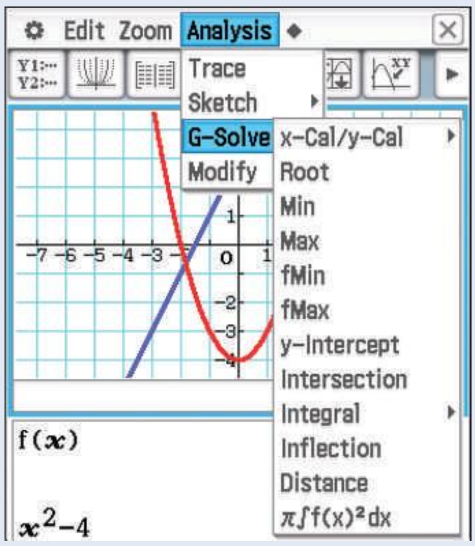


Graph&Table is used much less frequently because it requires rearrangement in terms of **y = …**

Tap **Menu > Graph&Table**. Enter the function then tap **Graph**. The instructions in this student book use the **Main** option in preference to the **Graph&Table** option.

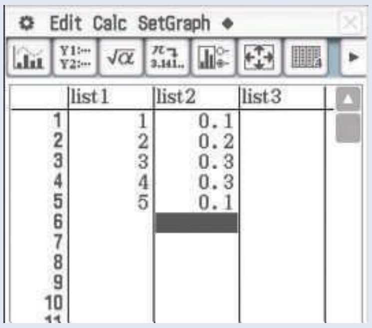


There are several menu options available to analyse graphs. Tap **Analysis > Trace** and press the **arrow** keys to move along the graph. Tap **Analysis > Sketch** for the options shown above.

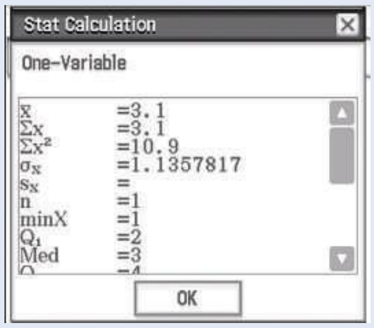


Tap **Analysis > G-Solve** to identify key features of a graph. For example, select **Root** to find the x-intercepts and select **Intersection** to locate points of intersection of two graphs. **G-Solve** gives decimal values. Use **Main** to calculate exact values.

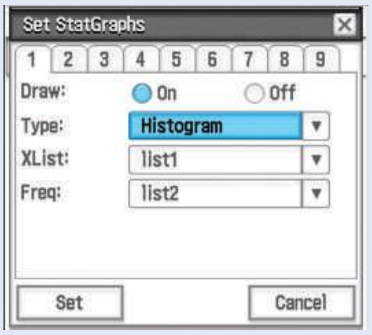
Statistics



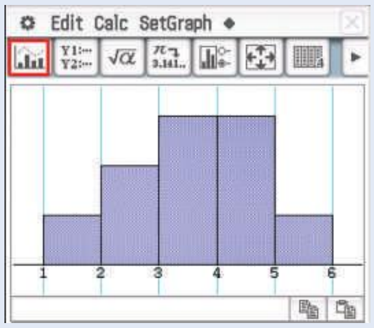
Tap **Menu > Statistics** to open the Statistics application. The default settings show **list1**, **list2**, **list3**, etc. If required, tap on the list name to enter a new heading.



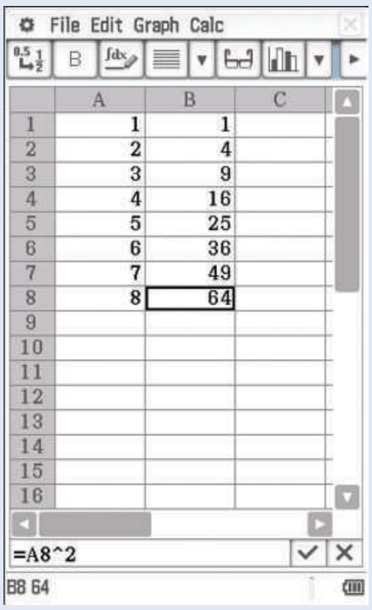
Tap **Calc** to view the Calculation menu options. Tap **One-Variable** for the dialogue box used to calculate statistical analysis of a list. The screen above displays the one variable statistics for **list1** for the frequencies in **list2**.



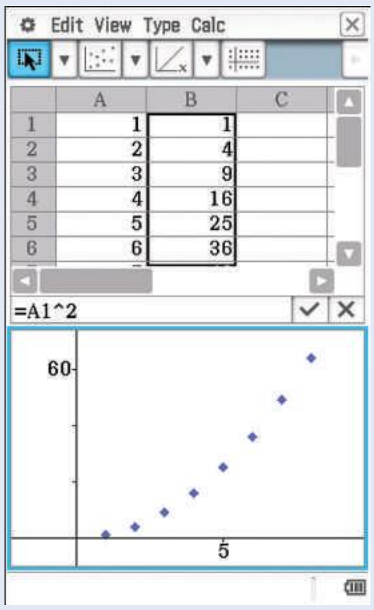
Tap **SetGraph** to set up the statistics graphs. Tap **Type:** to view the graphing options for statistical graphs. When finished, tap **Set**.



Tap **Graph** to display the statistical graph. The graph above displays a histogram of the **list1** values for the frequencies in **list2**.



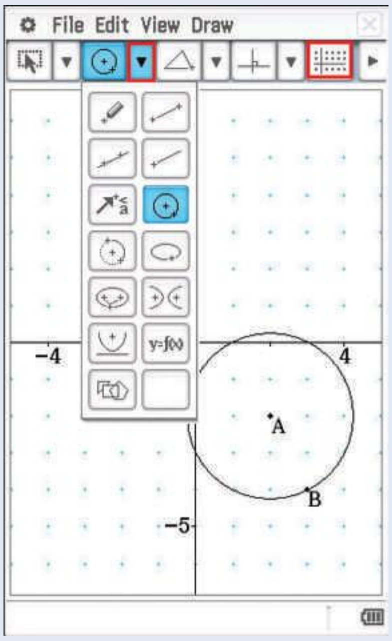
Tap **Menu > Spreadsheet** to open the Spreadsheet application. Values and formulas are entered using standard spreadsheet techniques. Column **A** shows the numbers 1 to 8 and Column **B** shows the square of the numbers.



Highlight the required data and tap **Calc** to access the range of statistical functions.

Highlight the data and tap **Graph** to access the range of graphs available. The graph above shows a **Scatterplot** of the squared values.

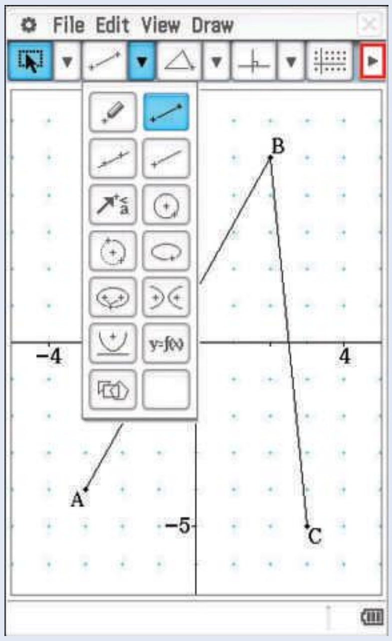
Geometry



Tap the **Grid** tool to toggle through the various grid options. The other tools at the top of the screen can also be accessed by tapping on the **Draw** menu.

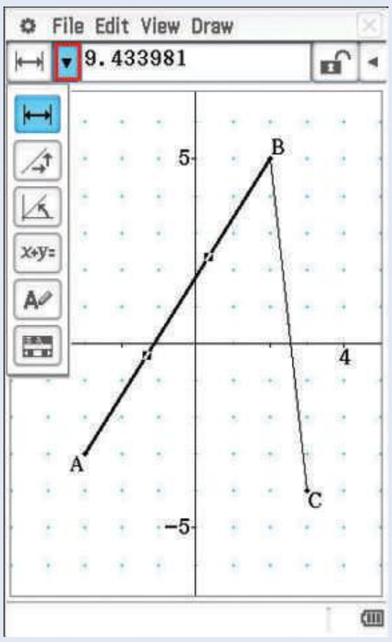
Tap the basic object **down arrow** to view the objects.

Select the circle object. Tap a centre and another point to draw a circle.



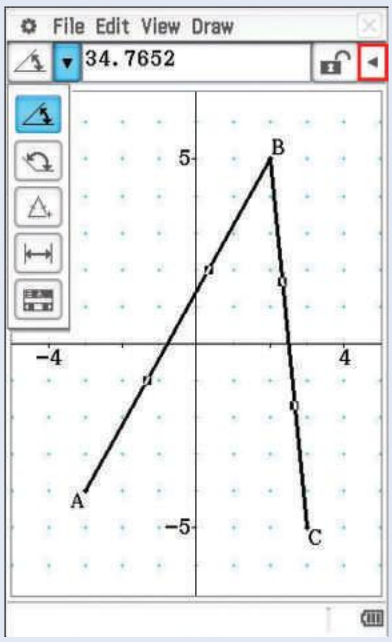
Select the line segment object then draw segments.

To measure or find equations of objects, tap the **right arrow** to view additional tools.



Tap to highlight an object then tap the **down arrow** to view the measurement box options.

The length of the line is shown at the top of the screen.



Tapping two objects gives different measurement box objects.

The angle between the line segments is shown at the top of the screen.

Tap the **left arrow** to return to the first screen.

Operating systems

Ensure you have the latest operating system installed on your handheld.

Tap **Settings** (located in the bottom left corner of the screen) then select **Version**. At the time of publication, the latest version is **02.01.7001**.

To download the latest operating system, go to the Australian Shriro website at http://www.casio.edu.shriro.com.au/classpad.php. Using the USB cable provided, connect the handheld to a computer. Start the installation program and follow the prompts. Some of these prompts will be on the computer and others will be on the handheld.

Algorithms and pseudocode

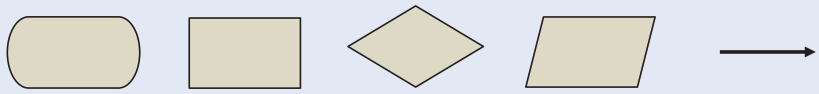
Algorithms

An algorithm is a set of instructions for successfully completing a task. The success of activities such as baking a cake, finding your way through a maze and mowing the lawn relies on executing a defined sequence of steps that produce the desired outcomes. If the algorithm is efficient, the cake will be delicious, you will find your way out of the maze and the lawn will be cut evenly.

An algorithm can be a set of instructions using everyday language, a **flowchart or pseudocode**.

Flowcharts

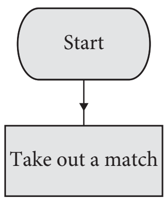
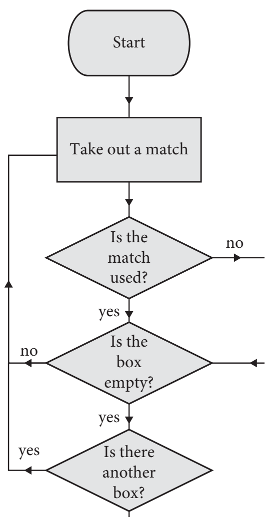
A flowchart uses a prescribed set of symbols to display the instructions of an algorithm.



WORKED EXAMPLE 1 Flowchart design for lighting a match

Draw a flowchart to show the following steps in the algorithm to light a match.

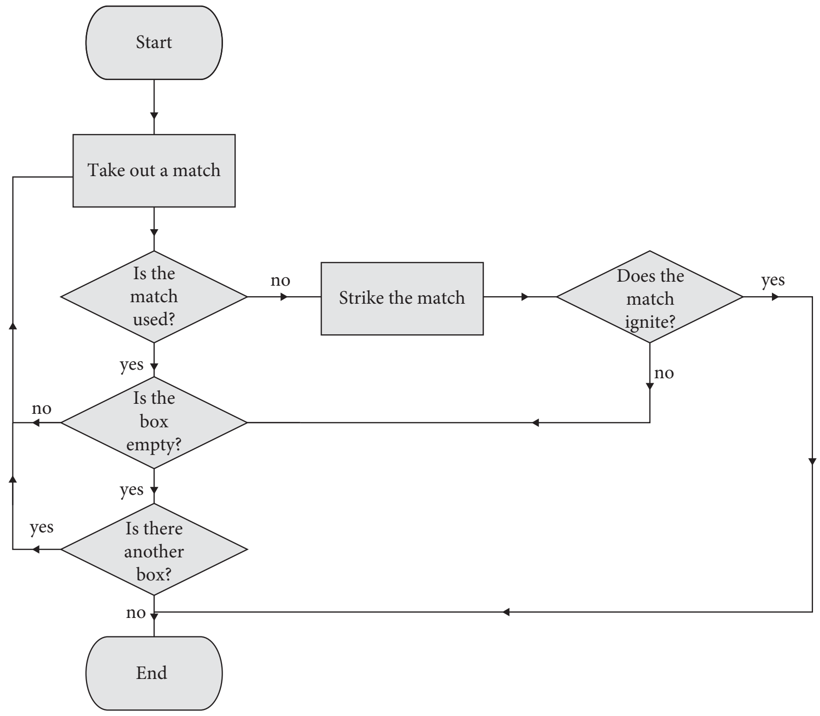
- Open the match box.
- Take out a match.
- If the match is used, take out another match.
- Scratch the match against the striking surface.
- If the match doesn't light and there is another match, repeat the method above.

Steps	Working
1 Use a start/end symbol and perform the first instruction.	Use a process symbol to 'take out a match' 
2 Show the decisions to be made after a match is taken out.	Use decision boxes to decide what to do if the match is used, the matchbox is empty or if there is no other matchbox. 

3 Complete the flowchart.

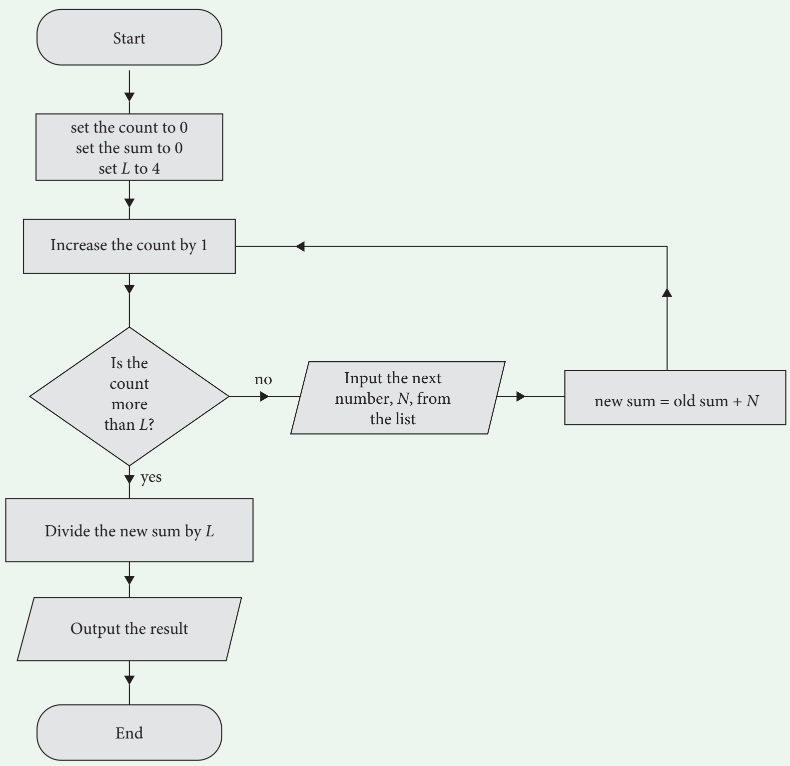
Show what happens after the match is struck.

Ensure that lines with arrows point in the right direction and connect to the appropriate sections of the flowchart.



WORKED EXAMPLE 2 Interpret a flowchart

The flowchart below describes an algorithm.



a State the values for count, sum and new sum when $N = 5, 6, 1, 8$.

b Describe what the algorithm is calculating.

Steps	Working
a **1** State the initial values.	count = 0 sum = 0 $L = 4$
Apply the flowchart instructions for the first value of count.	**count** is increased by 1, so count = 0 + 1 = 1.
2 Follow the correct path for the decision.	The count (1) is not more than L (4), so the first number from the list (5) is used. This value is added to the value of the sum, so the new sum is 0 + 5 = 5.
3 Repeat the loop until the count value is greater than the N value.	count = 1 + 1 = 2 $N = 6$, sum = 5 + 6 = 11 count = 2 + 1 = 3 $N = 1$, sum = 11 + 1 = 12 count = 3 + 1 = 4 $N = 8$, sum = 12 + 8 = 20
4 Follow the final set of instructions prior to terminating the program.	count = 4 + 1 = 5 count is more than L, so follow the 'yes' path at the decision box. final sum = sum ÷ L = 20 ÷ 4 = 5 The output is 5
b **1** Describe the situation by looking at what happens in the process symbols.	For each value of the count from 1 to 4, a number, N, is taken from the list and added to the previous sum. In the final step, the sum is divided by how many numbers were used.
2 State the conclusion.	The algorithm finds the arithmetic mean of a given set of 4 numbers.

Pseudocode

In Worked example **1**, a set of easily understood instructions were used to create a flowchart describing how to light a match. In a similar way, pseudocode consists of a combination of technical terms and commonly used words that provide an informal but effective way of depicting the flow and operation of a task. Think of pseudocode as the 'planning' stage that guides how the actual computer program will be developed. Because it is not tied to any specific programming language, pseudocode allows flexibility in choosing words and phrases. However, by adopting a number of conventions, a better understanding of the basic operations required to accomplish the task can be achieved.

The following are commonly accepted practices.

- Keywords and commands are capitalised.
- One statement is given in each line.
- Indent sections that are used to convey a specific task.

Below are the main commands that will be used with pseudocode.

INPUT
read data
 	
OUTPUT/PRINT
print data
	INTEGER Name[size]
	Reserves size + 1 locations to store values.
	A ← B (A is assigned the value of B)
		IF (a given statement is true)
		THEN (perform action 1)
		ELSE (perform action 2)
		ENDIF

DO
<procedure>
WHILE (condition is true)
ENDWHILE
	FOR variable = first value TO last value
	<procedure>
	ENDFOR
		REPEAT
		<procedure>
		UNTIL (condition is true)

+ addition – subtraction / division * multiplication ^ power	< less than > greater than ≤ less than or equal to ≥ greater than or equal to	

WORKED EXAMPLE 3 Interpreting pseudocode

a Describe what the program below is doing.

```
INTEGER array[10]
count ← 0
FOR loop = 1 TO 10
   IF loop MOD 3 = 1
THEN
count ← count + 1
array[count] ← loop
ENDIF
number[count] ← number
ENDFOR
PRINT "The sum of the 10 numbers is"
FOR numeral = 1 TO count
PRINT number[numeral]
ENDFOR
```

b State the output for each cycle of the PRINT command.

Steps	Working
a Describe what is happening in each section.	An array to store up to 11 numbers is initialised. (line 1) A counter is set to 0. (line 2) A loop consisting of the numbers 1 to 10 is used. (line 3, 10) Each of the numbers 1 to 10 is divided by 3 and if the remainder is 1, the counter is increased by 1 and the number is stored in the array. (lines 4 – 9) A descriptive heading for the output is printed. (line 11) The numbers which have been stored are printed. (lines 12 – 14)
b **1** Show the output from the first PRINT statement. **2** Write the output from the second PRINT command.	Line 11 prints the output heading, *The sum of the 10 numbers is* From line 13, the numbers to print are numbers from 1 to 10 which have a remainder of 1 when divided by 3. These are 1, 4, 7 and 10.

WORKED EXAMPLE 4 Pseudocode involving addition of numbers

Write pseudocode that finds the sum of 10 numbers and prints the result.

Steps	Working
1 Initialise the sum of the numbers and set up a loop to read the numbers.	There are 10 numbers to read, one at a time. current sum ← 0 FOR count = 1 TO 10 INPUT next number from list procedure ENDFOR

2 The procedure keeps a running total of the numbers.	current sum ← 0
	FOR count = 1 TO 10
	    INPUT next number from list
	    current sum ← current sum + next number from list
	ENDFOR

3 Print the result.	PRINT "The sum of the 10 numbers is" PRINT current sum

WORKED EXAMPLE 5	**Pseudocode using random numbers, modulus, product**

The function *A* MOD *B* returns the remainder after *A* is divided by *B*, for example, 8 MOD 3 = 2.

The function RAND(*A*) generates a random whole number from 1 to *A*, for example, RAND(5) = 3.

Write pseudocode that generates 3 random even numbers from 2 to 10 and then finds their product.

Steps	**Working**
1 Initialise the product of the numbers and create a loop for the random numbers.	There are 3 random numbers to generate. current product ← 1 FOR count = 1 TO 3 procedure ENDFOR

2 Check to see if the random number is even. Print the even number.	A number is even if the remainder after division by 2 is 0.
	current product ← 1
	FOR count = 1 TO 3
	REPEAT
	    random number ← RAND(10)
	    remainder ← random number MOD 2
	UNTIL remainder = 0
	PRINT random number
	ENDFOR

3 Update the product and output the result.	current product ← 1
	FOR count = 1 TO 3
	    REPEAT
	    random number ← RAND(10)
	    remainder ← random number MOD 2
	    UNTIL remainder = 0
	PRINT random number
	current product ← current product * random number
	ENDFOR
	PRINT current product

CHAPTER

1 EQUATIONS AND INEQUALITIES

Study Design coverage

UNIT 1, AREA OF STUDY 2: ALGEBRA, NUMBER AND STRUCTURE

- use of symbolic notation to develop algebraic expressions and represent functions, relations, equations, and systems of simultaneous equations
- substitution into, and manipulation of, these expressions
- recognition of equivalent expressions and simplification of algebraic expressions involving different forms of polynomial and power functions, the use of distributive and exponent laws applied to these functions, and manipulation from one form of expression to an equivalent form
- solution of a set of simultaneous linear equations (geometric interpretation only required for two variables) and equations of the form $f(x) = g(x)$ numerically, graphically and algebraically.

Note: The bisection method is covered in Chapter 3, *Polynomials*.

STUDY DESIGN COVERAGE

Video playlists (9):

1.1 Index laws
1.2 Binomial expansions
1.3 Factorisation
1.4 Equations
1.5 Quadratic equations
1.6 Changing the subject of a formula
1.7 Simultaneous equations
1.8 Inequalities
VCE question analysis Equations and inequalities

Worksheets (9):

1.1 Review of index laws • Fractional indices and radicals
1.2 Binomial products • Special products
1.3 Factorising algebraic expressions
1.6 Changing the subject of a formula
1.7 Simultaneous equations
1.8 Inequalities on a number line • Inequalities review

Puzzles (5):

1.4 Equations code puzzle
1.5 Factorising quadratic equations • Complete the square order activity • The quadratic formula
1.7 Simultaneous equations order activity

Nelson MindTap

To access resources above, visit **cengage.com.au/nelsonmindtap**

1.1 Index laws

In the term a^n, a is the base and n is called the **index** or the **exponent**. Index laws for simplifying index expressions are listed below. These laws are easy to understand if we use the definition

$$a^n = a \times a \times a \times a \times \ldots \text{ (to } n \text{ factors)}$$

For example, $2^5 = 2 \times 2 \times 2 \times 2 \times 2 = 32$.

Video playlist Index laws

Worksheets Review of index laws

Fractional indices and radicals

The index laws

$a^m \times a^n = a^{m+n}$	$a^0 = 1$
$a^m \div a^n = a^{m-n}$	$a^{-n} = \dfrac{1}{a^n}$
$\left(a^m\right)^n = a^{mn}$	$a^{\frac{1}{q}} = \sqrt[q]{a}$
$(ab)^n = a^n b^n$	$a^{\frac{p}{q}} = \sqrt[q]{a^p} = \left(\sqrt[q]{a}\right)^p$
$\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$	

Using the index law $(ab)^n = a^n b^n$, the expression $(3xy)^2 = 9x^2y^2$.

Using the index law $a^m \div a^n = a^{m-n}$, the expression $x^{a+1} \div x^{2a} = x^{-a+1}$.

Using the index law $a^{\frac{p}{q}} = \sqrt[q]{a^p} = \left(\sqrt[q]{a}\right)^p$, the expression $27^{\frac{2}{3}}$ becomes $\sqrt[3]{27^2}$ or $\left(\sqrt[3]{27}\right)^2 = 9$.

WB

p. 1

WORKED EXAMPLE 1 Index laws 1

Simplify the expression $a^6 \times a^8 \times b^8 \times b^{-9} \div a^2$.

Steps	Working
1 Write the expression and group the terms with the same base.	$a^6 \times a^8 \times b^8 \times b^{-9} \div a^2$ $= a^6 \times a^8 \div a^2 \times b^8 \times b^{-9}$
2 Use index laws to simplify.	$= a^{6+8-2} \times b^{8-9}$
3 Simplify indices.	$= a^{12}b^{-1}$

Exam hack

It is not necessary to express your answer with positive indices unless specifically asked in the question.

WB

p. 2

WORKED EXAMPLE 2 Index laws 2

Simplify the expression $8^{\frac{1}{3}} \times 16^{\frac{3}{4}} \times 5^{\frac{2}{3}} \div 5^{\frac{1}{3}}$.

Steps	Working
1 Write the expression and simplify the bases to prime numbers if possible.	$8^{\frac{1}{3}} \times 16^{\frac{3}{4}} \times 5^{\frac{2}{3}} \div 5^{\frac{1}{3}}$ $= (2^3)^{\frac{1}{3}} \times (2^4)^{\frac{3}{4}} \times 5^{\frac{2}{3}} \div 5^{\frac{1}{3}}$ Write 8 and 16 as powers of 2 to simplify.
2 Use index laws to simplify.	$= 2^{3\times\frac{1}{3}} \times 2^{4\times\frac{3}{4}} \times 5^{\frac{2}{3}-\frac{1}{3}}$ $= 2^1 \times 2^3 \times 5^{\frac{1}{3}}$ $= 2^4 \times 5^{\frac{1}{3}}$

Exponential equations

An **exponential equation** is an equation in which the variable is in the **exponent** (power). Exponential equations can be solved by using index laws to express all terms with a common **base**. The base chosen is usually a prime number.

WORKED EXAMPLE 3 Exponential equations

Solve the equation $3^{2x} \times 3^{1-3x} = 9^x$ for x.

Steps	Working
1 Express each term with the base 3 and simplify using index laws.	$3^{2x} \times 3^{1-3x} = 9^x$ $3^{2x+1-3x} = (3^2)^x$ Write 9 as a power of 3. $3^{1-x} = 3^{2x}$
2 Equate powers.	$1 - x = 2x$
3 Solve for x.	$1 = 3x$ $x = \frac{1}{3}$

EXERCISE 1.1 Index laws

ANSWERS p. 491

1.1

WB p. 3

Mastery

1 WORKED EXAMPLE 1 TECH-FREE Simplify the expression $a^6 \times b^8 \div a^3$.

2 WORKED EXAMPLE 2 TECH-FREE Simplify the expression $27^{\frac{1}{3}} \times 16^{\frac{3}{4}} \div 8^{\frac{1}{4}} \div 2^{\frac{1}{4}}$.

3 TECH-FREE Simplify each expression.

a $32^{2x} \times 2^x \times 16^x$ **b** $32^x \times 8^x \times 16^{2x}$ **c** $2^{2x} \times 8^{2x} \times 16^{-x}$

d $a^2 \times 2a^3 \times 16a^2$ **e** $b^{2x} \times b^x \times b^{2x}$ **f** $a^{2x} \times a^{2x} \times a^{-x}$

4 TECH-FREE Simplify each expression.

a $a^2 \div 2a^3$ **b** $b^{2x} \div b^x$ **c** $a^{2x} \div a^{2x}$

d $3^2 \div 27^3$ **e** $2^{2x} \div 2^{-x}$ **f** $4^{2x} \div 16^{2x}$

5 WORKED EXAMPLE 3 The solution to the equation $2^{2x} \times 2^{1+3x} = 4^x$ is

A $x = -\frac{1}{3}$ **B** $x = -\frac{1}{2}$ **C** $x = \frac{1}{3}$ **D** $x = 2$ **E** $x = 4$

6 The solution to the equation $3^{x+1} \times 9^{2x} = 27^x$ is

A $x = -1$ **B** $x = -\frac{1}{2}$ **C** $x = 0$ **D** $x = \frac{1}{2}$ **E** $x = 1$

7 The value of $9^{\frac{1}{2}} \times 81^{\frac{1}{2}}$ is

A 3 **B** 9 **C** 27 **D** 81 **E** 729

8 The expression $ab \times (4ab)^{\frac{1}{2}} \div (ab)^2$ is equivalent to

A $(ab)^{\frac{1}{2}}$ **B** $2(ab)^{\frac{3}{2}}$ **C** $\frac{\sqrt{ab}}{ab}$ **D** $\frac{1}{\sqrt{ab}}$ **E** $\frac{2}{\sqrt{ab}}$

Exam practice

80–100% 60–79% 0–59%

9 © VCAA 2014 1Q4 85% TECH-FREE (2 marks) Solve the equation $2^{3x-3} = 8^{2-x}$ for x.

10 © VCAA 2013 1Q5b 76% TECH-FREE (2 marks) Solve the equation $3^{-4x} = 9^{6-x}$ for x.

11 The expression $b^6 \times (2b)^6 \div b^2$ can be simplified to

A $2b^{10}$ **B** $2b^{14}$ **C** 2^6b^{34} **D** $2^{10}b^{10}$ **E** 2^6b^{10}

12 The expression $6^{\frac{1}{3}} \times 3^{\frac{2}{3}} \times 2^{\frac{2}{5}}$ can be simplified to

A $6 \times 2^{\frac{2}{5}}$ **B** $3 \times 2^{\frac{2}{5}}$ **C** $2^{\frac{7}{5}}$ **D** $3 \times 2^{\frac{11}{15}}$ **E** $2^{\frac{11}{15}}$

13 The solution, for x, of the equation $9^{2x} \times 3^{3x} = 27^{2x+2}$ is

A $x = 0$ **B** $x = \frac{2}{3}$ **C** $x = 1$ **D** $x = 3$ **E** $x = 6$

14 The expression $ab^3 \times (2a)^3$ is equivalent to

A $2a^4b^3$ **B** a^4b^3 **C** 2^3b^3 **D** 2^3ba^6 **E** $8a^4b^3$

15 The expression $8^{\frac{1}{3}} \times 8^{\frac{2}{3}} + 32^{\frac{2}{5}}$ is equivalent to

A $3^1 \times 2^{\frac{2}{5}}$ **B** 12 **C** $8 + 2^{\frac{2}{5}}$ **D** 32 **E** $8^3 \times 2^2$

16 The solution, for x, of the equation $3^{2x} \times 3^x = 27$ is

A $x = 1$ **B** $3^x = 1$ **C** $x = \frac{2}{3}$ **D** $x = 3$ **E** $3^{2x} = 1$

17 The expression $b^6 \times a^8 \div a^3$ can be simplified to

A a^{11} **B** a^{17} **C** $b^6 a^5$ **D** $a^6 b^5$ **E** b^{11}

18 The expression $81^{\frac{1}{4}} \times 9^{\frac{3}{4}} \div 3^{\frac{1}{4}}$ can be simplified to

A 3 **B** $3^{\frac{1}{4}}$ **C** $3^{\frac{9}{4}}$ **D** 3^6 **E** 3^8

Video playlist
Binomial expansions

Worksheets
Binomial products

Special products

1.2 Binomial expansions

Binomial products

A **binomial expression** is an expression with 2 terms, for example, $x + 4$ and $3x^2 - 6$.

A **binomial product** is the product of 2 or more binomial expressions, such as $(3x + 2)(x - 1)$.

It has the form $(a + b)(c + d)$.

To expand $(a + b)(c + d)$, we can use the following distributive law:

$$(a \times c) + (a \times d) + (b \times c) + (b \times d) = ac + ad + bc + bd$$

This method can also be remembered using the letters of the word FOIL.

FOIL method

First terms, **O**uter terms, **I**nner terms, **L**ast terms

F **O**

$$(a + b)(c + d) = ac + ad + bc + bd$$

I **L**

If done correctly, the O and I terms are like terms that can be collected.

When a binomial product such as $(3x + 2)(x - 1)$ is expanded, the simplified answer usually has 3 terms, such as $3x^2 - x - 2$.

This is called a **quadratic expression**, where the highest power of x is 2.

WORKED EXAMPLE 4 Binomial expansions

Expand and simplify the expression $(x-5)(2x+1)$.

Steps	Working
Use the distributive law to expand $(x-5)(2x+1)$.	$(x-5)(2x+1) = x(2x+1) - 5(2x+1)$ $= 2x^2 + x - 10x - 5$ $= 2x^2 - 9x - 5$
Or use the **FOIL method**	
First: $x \times 2x = 2x^2$	$(x-5)(2x+1) = 2x^2 + x - 10x - 5$
Outer: $x \times 1 = x$	$= 2x^2 - 9x - 5$
Inner: $-5 \times 2x = -10x$	
Last: $-5 \times 1 = -5$	

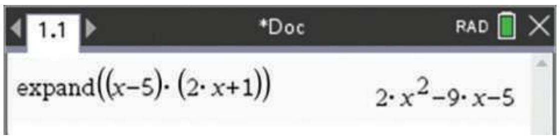
The O and I terms can be collected to give the middle term: $x - 10x = -9x$.

WB p. 4

USING CAS 1 Binomial products

Expand the expression $(x-5)(2x+1)$.

TI-Nspire



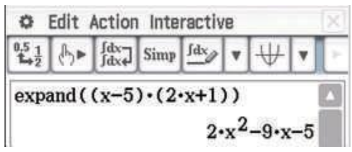
1 Add a **Calculator** page.

2 Press **menu > Algebra > Expand**.

3 Enter the expression **(x – 5)(2x + 1)**.

4 Press **enter**.

ClassPad



1 In **Main**, enter the expression **(x – 5)(2x + 1)**.

2 Highlight the expression and tap **Interactive > Transformation > Expand**.

3 Tap **OK**.

Perfect squares

$(a+b)^2$ and $(a-b)^2$ are special types of binomial expansions called **perfect squares**, which have their own formulas.

Perfect squares

$(a+b)^2 = a^2 + 2ab + b^2$

$(a-b)^2 = a^2 - 2ab + b^2$

WB p. 5

WORKED EXAMPLE 5 Perfect squares

Expand the expression $(x+5)^2$.

Steps	Working
Use the distributive law or the FOIL method to expand $(x+5)^2$.	$(x+5)^2 = (x+5)(x+5)$ $= x^2 + 5x + 5x + 25$ $= x^2 + 10x + 25$
OR use the formula $(a+b)^2 = a^2 + 2ab + b^2$.	$(x+5)^2 = x^2 + 2 \times x \times 5 + 5^2$ $= x^2 + 10x + 25$

Note the pattern:
- 1st term squared
- last term squared
- twice the 1st multiplied by the last

Difference of two squares

$(a + b)(a - b)$ is another special type of binomial expansion involving the sum of 2 terms multiplied by the difference between the same 2 terms. This type is called **difference of two squares**, because the expanded form is the difference between 2 square numbers a^2 and b^2.

Difference of two squares
$(a + b)(a - b) = a^2 - b^2$

This is also called 'difference of perfect squares', abbreviated to DOPS.

WB
p. 6

WORKED EXAMPLE 6 Difference of two squares

Expand the expression $(2x + 5)(2x - 5)$.

Steps	Working
1 Use FOIL method to expand $(2x + 5)(2x - 5)$.	$(2x + 5)(2x - 5) = 4x^2 - 10x + 10x - 25$ $= 4x^2 - 25$
2 Alternatively, use the difference of two squares formula $(a + b)(a - b) = a^2 - b^2$.	$(2x + 5)(2x - 5) = (2x)^2 - 5^2$ $= 4x^2 - 25$

EXERCISE 1.2 Binomial expansions ANSWERS p. 491

Recap

1 TECH-FREE Solve the equation $2^{2x} \times 2^{3x} = 2^{x+1}$ for x.

2 The solution, for x, in the equation $a^x \times a^{x+1} = a^4$ is

A $x = a^4$ **B** $x = \frac{2}{3}$ **C** $x = \frac{3}{2}$ **D** $x = 3$ **E** $x = 4$

Mastery

3 TECH-FREE Expand each expression.

a $2(2x + 3)$ **b** $4(2x - 3)$
c $(x + 2)(x - 3)$ **d** $(3x + 2)(2x + 3)$

4 WORKED EXAMPLE 4 TECH-FREE Expand and simplify the expression $(x + 5)(x + 1)$.

5 TECH-FREE Expand and simplify the expression $(2x + 5)(2x + 1)$.

6 TECH-FREE Expand and simplify each expression.

a $(x + 1)(x - 1)$ **b** $4(x + 1)(x - 1)$ **c** $(x + 3)(x - 3)$
d $3(2x + 1)$ **e** $4(-x + 1)$ **f** $-4(2x - 3)^2$

7 TECH-FREE Expand each expression.

a $(2x - 7)(3x - 9)$ **b** $(2x - 3)(3x + 8)$ **c** $(m - 5n)^2$
d $(7p + 4)(7p - 4)$ **e** $(x + 2)(2x - 3y + 7)$

8 Using CAS 1 The expression $(2x-5)(2x-1)$ can be simplified to

A $4x^2+12x+5$ **B** $4x^2-12x-5$ **C** $4x^2-12x+5$

D $4x^2+12x-5$ **E** $2x^2-10x+5$

9 WORKED EXAMPLE 5 The expression $(2x+3)^2$ can be expanded as

A $4x^2+9$ **B** $4x^2-9$ **C** $2x^2+3$

D $4x^2+12x+9$ **E** $4x^2+6x+9$

10 WORKED EXAMPLE 6 The expression $(4x+3)(4x-3)$ can be expanded as

A $4x^2+9$ **B** $4x^2-9$ **C** $16x^2-9$

D $4x^2+12x-9$ **E** $16x^2-24x+9$

Exam practice

11 The expression $(3x-5)(2x+7)$ can be expanded as

A $(3x-5)(2x+7)$ **B** $3x(2x+7)$ **C** $x^2+11x+35$

D $6x^2+11x-35$ **E** $3x^2+11x+35$

12 The expression $(3x-5)^2$ can be expanded as

A $(3x-5)(3x+5)$ **B** $9x^2-30x+25$ **C** x^2-25

D $9x^2-25$ **E** $9x^2+30x+25$

1.3 Factorisation

Video playlist Factorisation

Worksheet Factorising algebraic expressions

Factorising is the opposite of **expanding**. While expanding involves removing brackets and multiplying terms, factorising involves inserting brackets and 'taking out' common factors.

Examples of factorisation are:

- $3x+12=3(x+4)$
- $3x^2+12x=3x(x+4)$
- $6x^2-12x=6x(x-2)$
- $4x^2+12x+16=4(x^2+3x+4)$

Perfect squares

WB p. 7

WORKED EXAMPLE 7 Factorising perfect squares

Factorise the perfect square $4x^2-20x+25$.

Steps	Working
Use the formula $(a-b)^2=a^2-2ab+b^2$ in reverse, where $a=2x$, $b=5$.	$4x^2-20x+25=(2x-5)^2$ Note the pattern: • square root of the 1st term • square root of the last term Then check that the middle term is twice the 1st multiplied by the last.

Difference of two squares

WB

p. 8

WORKED EXAMPLE 8	Factorising the difference of squares
Factorise the expression $4x^2 - 25$.	
Steps	**Working**
Use the difference of two squares formula $(a+b)(a-b) = a^2 - b^2$ In the expression $4x^2 - 25$, $a = 2x$ and $b = 5$.	$4x^2 - 25 = (2x)^2 - 5^2$ $= (2x+5)(2x-5)$

Grouping in pairs

If there are 4 terms to factorise, then consider grouping in pairs. That is, factorise the first 2 terms, then the last 2 terms. Sometimes, a common factor will appear.

WB

p. 9

WORKED EXAMPLE 9 Factorising by grouping in pairs

Factorise each expression.

a $x^3 - 4x^2 - 4x + 16$

b $2x^2 - 4x + x - 2$

This method will be handy for factorising non-monic quadratic expressions introduced in Worked example 13 on page 15.

Steps	**Working**
a **1** Factorise the first 2 terms, then the last 2 terms.	$x^3 - 4x^2 - 4x + 16 = x^2(x-4) - 4(x-4)$
2 Factorise again using the common factor $(x-4)$.	$= (x-4)(x^2-4)$
3 Factorise $x^2 - 4$.	$= (x-4)(x+2)(x-2)$
b **1** Factorise the first 2 terms, then the last 2 terms.	$2x^2 - 4x + x - 2 = 2x(x-2) + 1(x-2)$
2 Factorise again using the common factor $(x-2)$.	$= (x-2)(2x+1)$

EXERCISE 1.3 Factorisation

ANSWERS p. 491

Recap

1 TECH-FREE Expand each expression.

a $(2x+3)(x+1)$

b $4(2x-3)(x+1)$

c $(x+2)(x+3)$

d $(3x+2)(2x-3)$

2 The expression $(x+2)(x+4)$ can be expanded as

A $x^2 - 6x - 8$ **B** $x^2 + 6x + 8$ **C** $x^2 + 6x - 8$ **D** $x^2 - 6x + 8$ **E** $x^2 + 8$

Mastery

3 TECH-FREE Expand the expression $(x+3)^2$.

4 WORKED EXAMPLE 7 TECH-FREE Factorise the perfect square $x^2 + 10x + 25$.

5 The expression $(2x-3)(2x+3)$ can be expanded as

A $4x^2 + 9$ **B** $4x^2 - 9$ **C** $4x^2 - 12x + 9$

D $4x^2 + 12x - 9$ **E** $2x^2 - 12x - 9$

6 WORKED EXAMPLE 8 TECH-FREE Factorise each expression.

a $3x^2 + 3x$
b $4x^2 - 4x$
c $-3x^2 - 9x$
d $-3x^2 - 9$
e $3x^2 + 6x + 3$
f $4x^2 - 16x + 16$
g $x^2 - 9$
h $x^2 - 16$
i $x^2 - y^2$
j $3x^2 - 12$
k $4x^3 - 16x^2$
l $2x^2 - 18a^2$
m $x^2 - 2x + 1$
n $x^2 + 8x + 16$
o $2x^2 + 4x + 2$

7 WORKED EXAMPLE 9 TECH-FREE Factorise each expression by grouping in pairs.

a $-3x^2 - 9x + 3x + 9$
b $3x^3 + 6x^2 + 3x + 6$
c $4x^3 - 16x^2 + 4x - 16$
d $a^2x^3 - 2ax^2 - 9a^2x + 18a$
e $3x^2 - 6x - 6x + 12$
f $x^2 + x - 3x - 3$
g $x^2 + 8x - 4x - 32$
h $2x^2 + 3x - 10x - 15$
i $3x^2 + 2x - 15x - 10$

8 TECH-FREE Factorise each perfect square.

a $4x^2 + 28x + 49$
b $4x^2 - 12x + 9$
c $m^2 + 6mn + 9n^2$

Exam practice

9 The expression $(3x - 5)^2$ can be expanded as

A $(3x - 5)(3x + 5)$
B $9x^2 - 30x + 25$
C $x^2 - 25$
D $9x^2 - 25$
E $9x^2 + 30x + 25$

10 The expansion of the expression $(ax - 1)(ax + 1)$ is

A $(ax - 1)^2$
B $(ax + 1)^2$
C $a^2x^2 + 1$
D $ax^2 - 2$
E $a^2x^2 - 1$

11 The factorisation of the expression $9m^2 - 4$ is

A $(3m - 2)(3m + 2)$
B $(3m + 2)(3m + 2)$
C $(3m - 2)(3m - 2)$
D $(3m - 4)(3m + 4)$
E $(3m - 2)^2$

12 The factorisation of the expression $9m^2 - 6m + 1$ is

A $(3m - 2)(3m + 2)$
B $(3m + 1)(3m + 1)$
C $(3m - 2)(3m - 2)$
D $(3m - 3)(3m + 3)$
E $(3m - 1)^2$

1.4 Equations

The solution to an equation is also called the **root** of the equation. To solve an equation, rearrange it using inverse operations to get the variable by itself on the left-hand side (LHS).

Video playlist Equations

Puzzle Equations code puzzle

WB p. 10

WORKED EXAMPLE 10	Solving linear equations
Solve $6(x + 2) = 4x - 1$ for x.	
Steps	**Working**
1 Expand the brackets.	$6(x + 2) = 4x - 1$ $6x + 12 = 4x - 1$
2 Subtract 12 from both sides.	$6x + 12 - 12 = 4x - 1 - 12$ $6x = 4x - 13$

3 Subtract $4x$ from both sides.	$6x - 4x = 4x - 4x - 13$ $2x = -13$
4 Divide both sides by 2.	$x = -\frac{13}{2}$

Exam hack

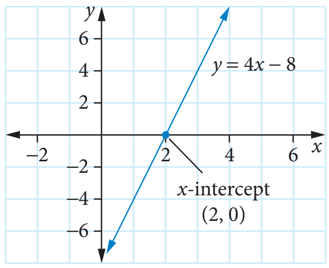
An improper fraction answer is preferred to a mixed number. Do not give a decimal answer unless it is asked for.

Factors, roots and *x*-intercepts

The **roots** of an equation are also the ***x*-intercepts** of its graph.

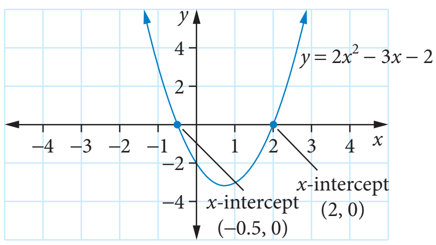
For example, consider the graph of $y = 4x - 8$.

- Solve $4x - 8 = 0$ to get $x = 2$.
- The x-intercept of the graph of $y = 4x - 8$ will be $x = 2$.



For example, consider the graph of $y = 2x^2 - 3x - 2$.

- Factorise to get $y = (x - 2)(2x + 1)$.
- Solve the equation $(x - 2)(2x + 1) = 0$ to get $x = 2$ or $x = -\frac{1}{2}$.
- The x-intercepts of the graph of $y = 2x^2 - 3x - 2$ will be $x = 2$ and $x = -\frac{1}{2}$.



WB

p. 11

WORKED EXAMPLE 11 Roots of an equation

Find the roots of $y = -2x^2 - 32x$.

Steps	Working
1 Factorise the expression $-2x^2 - 32x$.	$-2x^2 - 32x = -2x(x + 16)$
2 Solve the equation $-2x^2 - 32x = 0$.	$-2x(x + 16) = 0$ $\therefore\ x = 0,\ x = -16$

EXERCISE 1.4 Equations

ANSWERS p. 491

Recap

1 TECH-FREE Factorise $a^2x^3 + ax^2 - 4a^2x - 4a$.

2 TECH-FREE Factorise $ax^3 + 2x^2 - 9ax - 18$.

Mastery

3 TECH-FREE Expand the expression $(x + 2)^3$.

4 TECH-FREE Factorise the expression $x^3 - 3x^2 + 3x - 1$.

5 WORKED EXAMPLE 10 TECH-FREE Solve $3(x + 3) = 2x - 3$ for x.

6 WORKED EXAMPLE 11 TECH-FREE Find the roots of $y = x^2 + 9x$.

7 The solution to the equation $-2(x + 2) = 2x + 5$ is

A $x = -\frac{9}{4}$ **B** $x = -2$ **C** $x = \frac{4}{9}$

D $x = \pm\frac{9}{4}$ **E** $x = 5$

8 The x-intercepts of the graph of $y = 3x^2 - 9x$ are

A (0, 0) **B** (–3, 0) **C** (0, 0) and (3, 0)

D (0, 0) and (–3, 0) **E** (0, 3) and (3, 0)

9 The root(s) of $y = -2x^2 + 8x$ are

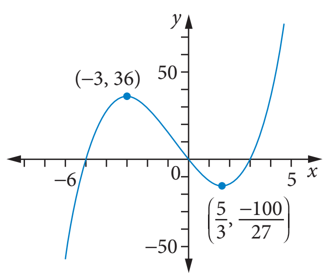
A $x = -4$ **B** $x = 0$ **C** $x = \pm 4$

D $x = 0, x = 4$ **E** $x = 0, x = \pm 4$

Exam practice

10 © VCAA 2010S 1Q1 TECH-FREE (3 marks) Solve $x^3 + (k + 1)x^2 + kx = 0$ for x.

11 Part of the graph of a cubic polynomial function f and the coordinates of its stationary points are shown below.

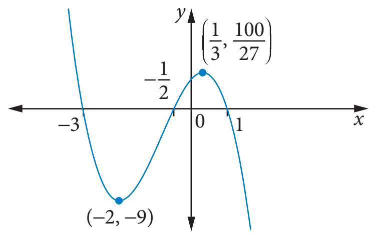


The roots of the graph are

A $x = 0, x = 3$ **B** $x = 0, x = 3, x = \frac{5}{3}$ **C** $x = 0, x = -3$

D $x = 0, x = 3, x = -5$ **E** $x = 0, x = -3, x = 5$

12 Part of the graph of the polynomial function f is shown.



The x-intercepts of the graph are

A $x = 1, x = \frac{1}{3}$ **B** $x = 1, x = \pm 3$ **C** $x = 1, x = -3, x = -\frac{1}{2}$

D $x = 1, x = 3, x = \frac{1}{2}$ **E** $x = -3, x = 1, x = \frac{1}{3}$

13 The solution(s) to the equation $\frac{3}{x+2} = \frac{5}{x}$ is

A $x = -5$ **B** $x = -3$ **C** $x = -\frac{5}{2}$ **D** $x = 3, x = 5$ **E** $x = 5$

14 When $5x^2 - 2x - 7$ is factorised, it becomes $(5x - 7)(x + 1)$.

The root(s) of the equation $5x^2 - 2x - 7 = 0$ are

A $x = -1$ **B** $x = -\frac{7}{5}, x = 1$ **C** $x = \frac{7}{5}, x = -1$ **D** $x = 1$ **E** $x = 5$

15 The solution to the equation $3(y + 7) = 2(2y - 9)$ is

A $y = -39$ **B** $y = -7$ **C** $y = 3$ **D** $y = \frac{9}{2}$ **E** $y = 39$

Video playlist Quadratic equations

Puzzles Factorising quadratic equations

Complete the square order activity

The quadratic formula

1.5 Quadratic equations

A **quadratic equation** is an equation where the highest power of the variable is 2, for example, $2x^2 - 9x - 5 = 0$. It has the general form $ax^2 + bx + c = 0$, and can be solved by **factorising**, **completing the square** or using the **quadratic formula**.

Factorising quadratic expressions

Quadratic equations that can be factorised are solved using the **null factor law**.

Null factor law

If $ab = 0$, then either $a = 0$, $b = 0$ or $a = b = 0$.

For example, solve the equation $2x^2 - 9x - 5 = 0$ for x.

$(x - 5)(2x + 1) = 0$

$x - 5 = 0$ or $2x + 1 = 0$

$x = 5$ or $2x = -1$

$x = 5$ or $x = -\frac{1}{2}$

1.5

A **monic** quadratic expression is one where the coefficient of x^2 is 1. We can factorise it using a 'sum and product' method. First, notice the pattern when we expand the binomial product $(x+a)(x+b)$.

$$\begin{aligned}(x+a)(x+b) &= x^2+bx+ax+ab\\ &= x^2+(a+b)x+ab\\ &= x^2+(\text{SUM})x+(\text{PRODUCT})\end{aligned}$$

WORKED EXAMPLE 12 Factorising a monic quadratic expression

Factorise the expression x^2+3x+2.

Hence, solve the quadratic equation $x^2+3x+2=0$.

Steps	Working
1 Write 2 pairs of brackets beginning with 'x'.	$x^2+3x+2=(x\quad)(x\quad)$
2 Find 2 numbers with a sum of +3 and a product of +2: they are 1 and 2. Look for factors of +2, positive and negative. $1+2=3,\ 1\times 2=2$	$x^2+3x+2=(x+1)(x+2)$ **Exam hack** You can always check your answer by expanding it. You may have been taught a different method of factorising. Choose the method that suits you.
3 Solve the equation.	$x^2+3x+2=0$ $(x+1)(x+2)=0$ $x+1=0$ or $x+2=0$ $x=-1$ or $x=-2$

A **non-monic** quadratic expression is one where the coefficient of x^2 is not 1. We can also factorise it using a different 'sum and product' method.

WORKED EXAMPLE 13 Factorising a non-monic quadratic expression

Factorise the expression $2x^2+x-3$.

Hence, solve $2x^2+x-3=0$.

Steps	Working
1 Find 2 numbers with a sum of +1 and a product of $2\times(-3)=-6$: they are +3 and −2. $+3-2=+1,\ +3\times(-2)=-6$ Use these numbers to split the middle term $(+x)$.	$2x^2+1x-3=2x^2+3x-2x-3$
2 Factorise in pairs.	$=x(2x+3)-1(2x+3)$
3 Take out the common factor $(2x+3)$.	$=(2x+3)(x-1)$
4 Solve the equation.	$2x^2+x-3=0$ $(2x+3)(x-1)=0$ $2x+3=0$ or $x-1=0$ $2x=-3$ or $x=1$ $x=-\frac{3}{2}$ or $x=1$

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WB p. 13

Completing the square

If a quadratic equation cannot be factorised, the technique of **completing the square** can be used. This involves writing the equation in the form $a^2 + 2ab + b^2 = (a + b)^2$ or $a^2 - 2ab + b^2 = (a - b)^2$.

To solve equations by completing the square:

1. Write the equation in the general form: $ax^2 + bx + c = 0$.
2. Take out the common factor of a.
3. Halve the coefficient of x and then square this number.
4. Add and then subtract (to compensate) the number found in the previous step.
5. Recognise the perfect square and factorise.
6. Solve for zero to find the roots of the equation.

WB
p. 14

WORKED EXAMPLE 14 Completing the square

Solve the equation $2x^2 + 12x = -1$ by completing the square.

Steps	Working
1 Write the equation in general form: $ax^2 + bx + c = 0$.	$2x^2 + 12x = -1$ $2x^2 + 12x + 1 = 0$
2 Take out the common factor of a.	$2\left(x^2 + 6x + \frac{1}{2}\right) = 0$
3 Halve the coefficient of x and then square this number.	Coefficient of x is 6. $\frac{1}{2}$ of $6 = 3$ and $3^2 = 9$
4 Add and then subtract the number found in the previous step.	$2\left(x^2 + 6x + 9 - 9 + \frac{1}{2}\right) = 0$
5 Recognise the perfect square and simplify.	$2\left[(x + 3)^2 - 9 + \frac{1}{2}\right] = 0$ $2\left[(x + 3)^2 - \frac{17}{2}\right] = 0$
6 Divide both sides of the equation by 2 and solve for x.	$(x + 3)^2 - \frac{17}{2} = 0$ $(x + 3)^2 = \frac{17}{2}$ $x + 3 = \pm\sqrt{\frac{17}{2}}$ $\therefore\ x = -3 \pm\sqrt{\frac{17}{2}}$

The quadratic formula

The quadratic formula

For any quadratic equation $ax^2 + bx + c = 0$, the solutions are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

To solve equations using the quadratic formula:

1 Write the equation in general form: $ax^2 + bx + c = 0$.
2 Identify the values of a, b and c and substitute into the quadratic formula.
3 Simplify where necessary (including the surd).

Exam hack

In most cases, the quadratic formula is easier to use than completing the square.

WORKED EXAMPLE 15 Using the quadratic formula

Solve the equation $2x^2 = 2x + 3$ using the quadratic formula.

Steps	Working
1 Write in general form: $ax^2 + bx + c = 0$.	$2x^2 = 2x + 3$ $2x^2 - 2x - 3 = 0$
2 Identify a, b, and c.	$a = 2,\ b = -2,\ c = -3$
3 Substitute into the quadratic formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ and simplify. **Exam hack** Students often make mistakes in this last cancelling step.	$x = \dfrac{-(-2) \pm \sqrt{(-2)^2 - 4(2)(-3)}}{2(2)}$ $= \dfrac{2 \pm \sqrt{28}}{4}$ $= \dfrac{2 \pm 2\sqrt{7}}{4}$ $= \dfrac{1 \pm \sqrt{7}}{2}$

1.5

WB p. 15

The discriminant

In the quadratic formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, the expression under the square root symbol, $b^2 - 4ac$, has a special name: the **discriminant**, Δ.

Δ is the Greek letter 'delta'.

The discriminant, Δ, gives information about the number of roots of the quadratic equation, $ax^2 + bx + c = 0$.

- When Δ is positive, there are 2 roots.
- When Δ is negative, there are no roots because we cannot find the square root of a negative number.
- When Δ is 0, there is only 1 root because $\sqrt{0} = 0$, so the root is $x = \dfrac{-b}{2a}$.

The discriminant

The discriminant for any quadratic equation $ax^2 + bx + c = 0$ (where a, b and c are rational) is $\Delta = b^2 - 4ac$.

- If $\Delta < 0$, the quadratic equation has no real roots.
- If $\Delta = 0$, the quadratic equation has 1 real root, which is rational.
- If $\Delta > 0$, the quadratic equation has 2 real roots.
  - If Δ is a square number, the 2 roots are rational.
  - If Δ is not a square number, the 2 roots are irrational (surds).

For the quadratic equation in the previous example, $2x^2 - 2x - 3 = 0$

$$\Delta = b^2 - 4ac = (-2)^2 - 4 \times 2 \times (-3) = 28$$

In this case, $\Delta > 0$ and is not a square number, so the equation has **2 irrational roots**.

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WORKED EXAMPLE 16 The discriminant

a Show that the quadratic equation $2x^2 + x + 4 = 0$ has no real roots.

b Find the value(s) of k for which the equation $5x^2 - 2x + k = 0$ has real roots.

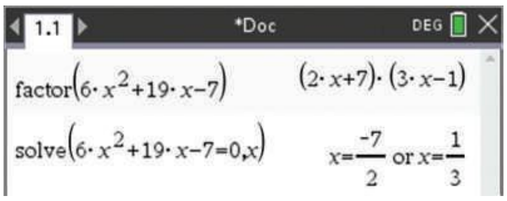
Steps	Working
a 1 Write the discriminant, $\Delta = b^2 - 4ac$, and substitute $a = 2$, $b = 1$, $c = 4$.	$\Delta = b^2 - 4ac$ $= 1^2 - 4 \times 2 \times 4$ $= -31$
2 Use the sign of the discriminant to explain why the quadratic equation has no real roots.	Since $\Delta < 0$, $2x^2 + x + 4 = 0$ has no real roots.
b 1 Write the discriminant and simplify.	$\Delta = b^2 - 4ac$ $= (-2)^2 - 4 \times 5 \times k$ $= 4 - 20k$
2 Make $\Delta \geq 0$ for real roots and solve for k.	$4 - 20k \geq 0$ $\therefore k \leq \frac{1}{5}$

USING CAS 2 Solving and factorising quadratics

a Factorise the expression $6x^2 + 19x - 7$.

b Solve the equation $6x^2 + 19x - 7 = 0$.

TI-Nspire



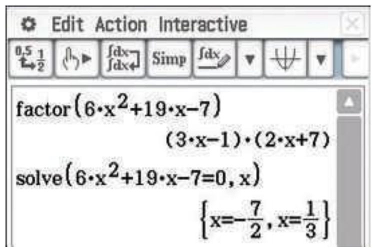
1 In a **Calculator** page, press **menu > Algebra > Factor**.

2 Enter the expression **6x²** + **19x** – **7** then press **enter**.

3 Press **menu > Algebra > Solve**.

4 Enter the equation **6x²** + **19x** – **7** = **0,x** then press **enter**. Be sure to include ',**x**' after the equation.

ClassPad



1 In **Main**, enter the expression **6x²** + **19x** – **7**.

2 Highlight the expression and tap **Interactive > Transformation > Factor**.

3 Tap **OK**.

4 Enter the equation **6x²** + **19x** – **7** = **0**.

5 Highlight the equation and tap **Interactive > Equation/Inequality > solve**.

6 Tap **OK**.

a $(2x + 7)(3x - 1)$

b $x = -\frac{7}{2}$ or $x = \frac{1}{3}$

EXERCISE 1.5 Quadratic equations

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1.5

Recap

1 TECH-FREE Solve each equation.

a $2(x+5) = -4(x-2)$ **b** $2(x^2-4) = x+2$

2 The x-intercepts of the graph of $y = x^2 - 4x$ are

A 0 **B** 0 and 2 **C** 2 **D** 0 and 4 **E** 4

Mastery

3 WORKED EXAMPLE 12 TECH-FREE Solve each equation by factorising.

a $k^2 + 4k - 5 = 0$ **b** $v^2 + 2v - 35 = 0$ **c** $x^2 = 9x - 8$

4 WORKED EXAMPLE 13 TECH-FREE Find the roots of the graph of $y = 2x^2 + 5x - 3$.

5 WORKED EXAMPLE 14 TECH-FREE Solve the equation $x^2 + 6x = -1$ by completing the square.

6 WORKED EXAMPLE 15 TECH-FREE Solve the equation $2x^2 = x + 3$ using the quadratic formula.

7 WORKED EXAMPLE 16 TECH-FREE

a Find the discriminant of the equation $2x^2 = x + 3$.

b Use the discriminant of the equation $2x^2 = x + 3$ to find the number and type of solutions to the equation.

8 TECH-FREE Show that the quadratic equation $2x^2 + 3x + 8 = 0$ has no real roots.

9 TECH-FREE Find the value(s) of k for which the equation $3x^2 - 2kx + 2k = 0$ has 1 real root.

10 To complete the square in the equation $x^2 + 6x + 1 = 0$, we need to add and subtract

A 1 **B** 3 **C** 6 **D** 9 **E** 36

11 When solving the equation $2x^2 + 3x + 7 = 0$ using the quadratic formula, the values of a, b and c are

A $a = 2, b = -3, c = 7$ **B** $a = 2, b = 3, c = 7$ **C** $a = -2, b = -3, c = -7$

D $a = 4, b = 3, c = 7$ **E** $a = 2, b = 3, c = 28$

12 The discriminant of $2x^2 + 3x - 7$ is

A -47 **B** 23 **C** 59 **D** 65 **E** 137

13 The equation $x^2(x^4 - 81) = 0$ has the solution

A $x = 0, x = -3$ **B** $x = 0, x = 9$ **C** $x = 0, x = 3$ **D** $x = 3$ only **E** $x = 0, x = \pm 3$

14 Using CAS 2 The equation $x(x^2 - 81) = 0$ has the solutions

A $x = 0, x = -3$ **B** $x = 0, x = \pm 9$ **C** $x = 0, x = 9$ **D** $x = 9$ only **E** $x = 0, x = \pm 3$

Exam practice

80–100% 60–79% 0–59%

15 TECH-FREE (6 marks) Solve each equation.

a $3x^2 + 5x - 12 = 0$ **b** $2x^2 - 24x + 22 = 0$ **c** $3x^2 + 15x + 12 = 0$

16 TECH-FREE (6 marks) Solve each equation.

a $x^2 + x - 4 = 0$ **b** $3x^2 - 5x + 1 = 0$ **c** $q^2 - 4q - 3 = 0$

17 The expansion of the expression $(3x-5)^2$ is

A $(3x-5)(3x+5)$ **B** $(9x-25)(9x+25)$ **C** $9x^2-30x+25$

D $6x^2-16x+25$ **E** $3x^2+15x+25$

18 The factorisation of the expression $3x^2-25$ is

A $(3x-5)(3x+5)$ **B** $\left(\sqrt{3}x-5\right)\left(\sqrt{3}x+5\right)$ **C** $\left(\sqrt{3}x-25\right)\left(\sqrt{3}x+25\right)$

D $\left(\sqrt{3}x-5\right)\left(\sqrt{3}x-5\right)$ **E** $(3x-5)(3x-5)$

19 The factorisation of the expression $4x^2-20x+25$ is

A $(3x-5)(3x+5)$ **B** $(4x-25)^2$ **C** x^2-25

D $2x^2-5$ **E** $(2x-5)^2$

20 The solutions to the equation $3x^2-x-2=0$ are

A $x=-1, \frac{2}{3}$ **B** $x=1, -\frac{2}{3}$ **C** $(x-1)(3x+2)$

D $3(x-1)(x+2)$ **E** $x=-1, 2$

21 For the quadratic equation $x^2+2x+p=0$, the values of p for which the function has 2 real roots are

A $p<1$ **B** $p>1$ **C** $p<-1$

D $p>-1$ **E** $p>0$

22 © VCAA 2019 2AQ2 MODIFIED 59% The values of k for which $x^2+2x-k=0$ has two real solutions is

A $k=-1$ or 1 **B** $k>-1$ **C** $k<-1$ **D** $k=-1$ **E** $k\geq-1$

23 The values of k for which $y=x^2-2kx+8$ has no real solutions are

A $-2<k<2$ **B** $-2\sqrt{2}<k<2\sqrt{2}$ **C** $-2\sqrt{2}\leq k\leq 2\sqrt{2}$

D $-2\leq k\leq 2$ **E** $k<2\sqrt{2}$

24 The values of m such that the equation $y=(m+1)x^2-mx+1$ has 2 real solutions are

A $m<2\sqrt{2}+2,\ m>2\sqrt{2}+2$ **B** $-2\sqrt{2}+2<m<2\sqrt{2}+2$ **C** $m\leq 2\sqrt{2}+2,\ m\geq 2\sqrt{2}+2$

D $m\leq 2, m\geq 2$ **E** $2<m<2$

Video playlist Changing the subject of a formula

Worksheet Changing the subject of a formula

1.6 Changing the subject of a formula

A **literal equation** is an equation with more than one variable, such as $ax+b=c$. The most common literal equation is a **formula**. We can rearrange a formula to make one of the variables the **subject** of the formula.

The formula for converting Fahrenheit temperatures to Celsius is $C=\frac{5}{9}(F-32)$. To make F the subject, rearrange the formula so that F is on the left-hand side (LHS):

$$C=\frac{5}{9}(F-32)$$

$$\frac{9}{5}C=F-32$$

$$\frac{9}{5}C+32=F$$

$$\therefore F=\frac{9}{5}C+32$$

To make x the subject of a literal equation such $ax + b = c$, solve for x in the same way that a standard equation, such as $2x + 4 = 12$, is solved for x. The only difference is that the final answer is a formula instead of a value.

For $ax + b = c$

$$ax = c - b$$

$$\therefore\ x = \frac{c-b}{a}$$

WB p. 17

WORKED EXAMPLE 17 Literal equations

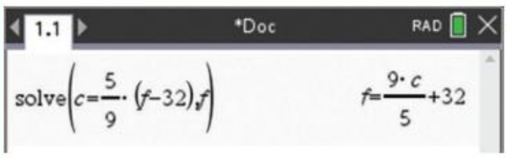
Make h the subject of the formula $V = \frac{1}{3}\pi r^2 h$.

Steps	Working
Solve the equation for h. Rearrange using inverse operations.	$V = \frac{1}{3}\pi r^2 h$ $3V = \pi r^2 h$ $\therefore\ h = \frac{3V}{\pi r^2}$

USING CAS 3 Transposing formula

Make f the subject of the formula $c = \frac{5}{9}(f - 32)$.

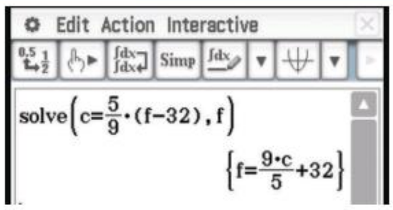
TI-Nspire



1 In a **Calculator** page, press **menu > Algebra > Solve**.

2 Enter the equation $\boldsymbol{c} = \frac{5}{9}(\boldsymbol{f} - 32), \boldsymbol{f}$ then press **enter**.

ClassPad



1 In **Main**, enter the equation $\boldsymbol{c} = \frac{5}{9}(\boldsymbol{f} - 32)$.

2 Highlight the equation. Tap **Interactive > > Equation/Inequality > solve**.

3 Make sure you change the variable to $\boldsymbol{f}$.

4 Tap **OK**.

EXERCISE 1.6 Changing the subject of a formula

ANSWERS p. 491

Recap

1 TECH-FREE

a What is the discriminant of the expression $m^2 - 6m + 9$?

b What does the discriminant of the equation $m^2 - 6m + 9 = 0$ show?

2 The solution for the equation $4x^2 - 6x + 2 = 0$ is

A $x = -\frac{1}{2}$ **B** $x = 0, -\frac{1}{2}$ **C** $x = \frac{1}{2}$ **D** $x = 1$ **E** $x = 1, \frac{1}{2}$

Mastery

3 WORKED EXAMPLE 17 TECH-FREE Make r the subject of the formula $S = 4\pi rh^2 + 2$.

4 TECH-FREE Change the subject of the formula to the variable given in brackets.

a $V = \frac{1}{3}Ah \quad (A)$

b $S = \frac{n}{2}[2a + (n-1)d] \quad (a)$

c $v^2 = u^2 + 2as \quad (s)$

d Given $b \geq 0$, $c^2 = a^2 + b^2 \quad (b)$

5 Make m the subject of the formula $E = mc^2$.

A $c^2 = mE$ **B** $m = E$ **C** $c = mE$ **D** $m = \frac{E}{c^2}$ **E** $m = \frac{E}{\sqrt{c}}$

6 The volume of a sphere is given by the formula $V = \frac{4}{3}\pi r^3$. An expression for r is

A $r^3 = \frac{0.75}{\pi}V$ **B** $r = \sqrt[3]{\frac{3V}{4\pi}}$ **C** $r = \pm\sqrt[3]{\frac{3V}{4\pi}}$ **D** $r = \sqrt[3]{\frac{3\pi V}{4}}$ **E** $r = \sqrt[3]{\frac{3\pi}{4}}V$

7 Using CAS 3 Make c the subject in the formula $E = mc^2$, given c is positive.

A $c^2 = \frac{E}{m}$ **B** $c = \frac{m}{E}$ **C** $c = \frac{E}{m}$ **D** $c = \sqrt{\frac{E}{m}}$ **E** $c = \frac{E}{\sqrt{m}}$

Video playlist Simultaneous equations

Worksheet Simultaneous equations

Puzzle Simultaneous equations order activity

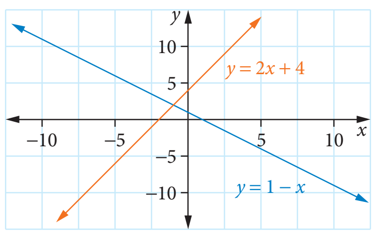
1.7 Simultaneous equations

Simultaneous equations are 2 or more equations with more than one variable. These are solved together to give a set of values that satisfy all equations. Simultaneous equations can be solved algebraically or graphically. Graphically, the solution is the point(s) of intersection of the graphs of the equations.

Graphical solution

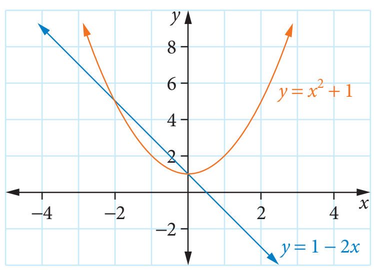
Consider the linear graphs of $y = 1 - x$ and $y = 2x + 4$.

The graphs have a point of intersection at (–1, 2).



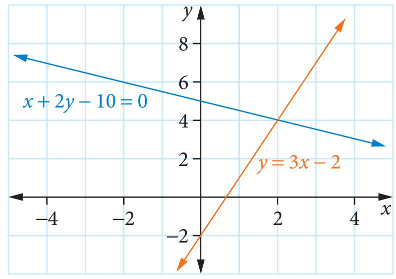
The graphs don't have to be linear. Consider the graphs of a linear equation $y = 1 - 2x$ and quadratic equation $y = x^2 + 1$.

They have 2 points of intersection, at (0, 1) and (–2, 5).



WORKED EXAMPLE 18 Simultaneous equations using graphs

Use graphs to solve the simultaneous equations $x + 2y - 10 = 0$ and $y = 3x - 2$.

Steps	Working
1 Graph $x + 2y - 10 = 0$ and $y = 3x - 2$.	
Exam hack You can always check your solution by substituting x and y into the 2 equations.	
2 Read the solution point from the graph.	The point of intersection is (2, 4). So the solution is $x = 2$, $y = 4$.

WB p. 18

Algebraic solution

There are 2 algebraic methods for solving simultaneous equations.

The **elimination method** combines the equations to eliminate x or y.

The **substitution method** substitutes an expression from one equation into the other.

WORKED EXAMPLE 19 Simultaneous equations using algebra

WB p. 19

Solve the simultaneous equations $x + 2y - 10 = 0$ and $y = 3x - 2$.

Steps	Working
Elimination method	$x + 2y = 10 \quad \ldots[1]$
1 Rearrange the equations to get matching terms and add or subtract to eliminate x or y.	$3x - y = 2 \quad \ldots[2]$ These are the same equations as in the previous example. To match y terms, multiply [2] by 2. $x + 2y = 10 \quad \ldots[1]$ $6x - 2y = 4 \quad \ldots[3]$ Add [1] and [3]: $7x = 14$ $\therefore x = 2$
2 Substitute $x = 2$ into the easiest equation, say [1], to find y.	$x + 2y = 10$ $2 + 2y = 10$ $y = 4$ The solution is $x = 2$, $y = 4$.
Substitution method	
1 Substitute $y = 3x - 2$ into $x + 2y - 10 = 0$ and solve to find x.	$x + 2(3x - 2) - 10 = 0$ $7x - 14 = 0$ $\therefore x = 2$
2 Substitute $x = 2$ into $y = 3x - 2$ to find y.	$y = 3(2) - 2$ $= 4$
You can always check your solutions by substituting x and y back into the 2 equations.	The solutions are $x = 2$, $y = 4$.

For simultaneous equations involving **non-linear equations**, substitution is the only method.

Consider the equations $y = 2x + 2$ and $3x^2 + y = 3$ and find where their graphs intersect.

Substitute the easier equation $y = 2x + 2$ into $3x^2 + y = 3$.

$$3x^2 + (2x + 2) = 3$$
$$3x^2 + 2x - 1 = 0$$
$$(3x - 1)(x + 1) = 0$$
$$x = \frac{1}{3} \text{ and } x = -1$$

Substitute x values into the easier equation $y = 2x + 2$ to get matching y values.

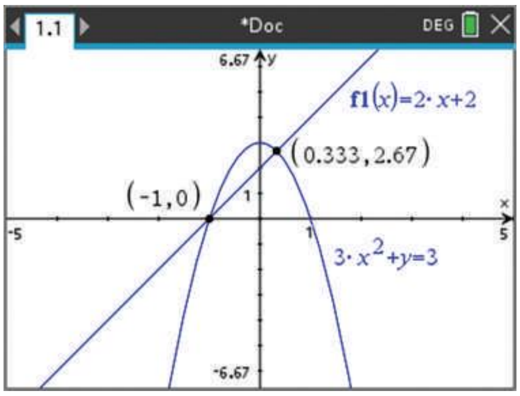
$$y = 2\left(\frac{1}{3}\right) + 2 = \frac{8}{3} \qquad \text{and} \qquad y = 2(-1) + 2 = 0$$

The graphs of the equations intersect at $\left(\frac{1}{3}, \frac{8}{3}\right)$ and $(-1, 0)$.

USING CAS 4 Simultaneous equations

Sketch the graphs of $y = 2x + 2$ and $3x^2 + y = 3$ and find their point(s) of intersection.

TI-Nspire



1 Add a **Graphs** page.

2 Enter **2x + 2** then press **enter** to display the first graph.

3 Press **menu > Graph Entry/Edit > Relation**.

4 Enter **3x² + y = 3** then press **enter** to display the second graph.

5 Press **menu > Analyze Graph > Intersection**.

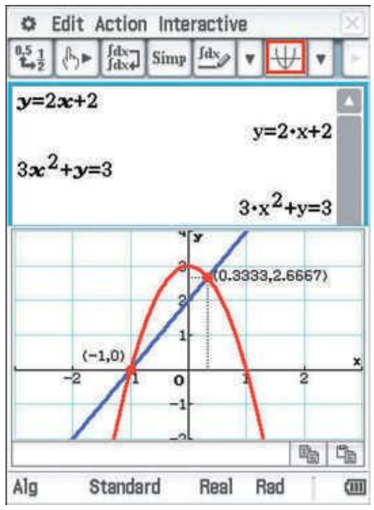
6 When prompted for the **lower bound?**, move the cursor to the left of the first point of intersection and press **enter**.

7 When prompted for the **upper bound?**, move the cursor to the right of the first point of intersection and press **enter**.

8 Repeat steps 6 and 7 to for the second point of intersection.

9 Adjust the window settings to suit.

ClassPad



1 In the **Main** menu, enter the equations ***y* = 2*x* + 2** and **3*x*² + *y* = 3**.

Using Main menu is preferable to Graph&Table as we don't have to make y the subject before we enter the equation.

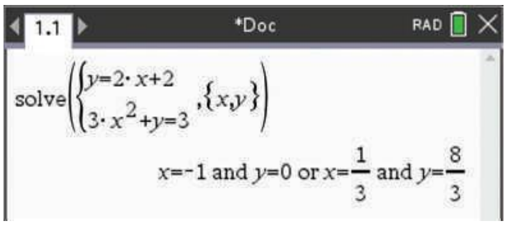
2 Tap **Graph** to open the Graph window.

3 Highlight and drag each equation into the Graph window below.

4 Adjust the viewing window for a suitable view.

5 In the lower screen, tap **Analysis > G-Solve > Intersection**, pressing the right arrow to move to the next point.

Algebraic solution



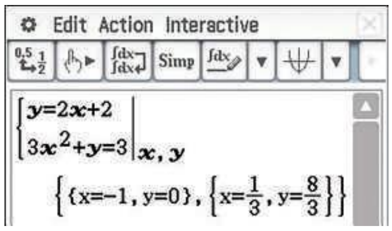
1 Add a **Calculator** page.

2 Press **menu > Algebra > Solve System of Equations > Solve System of Equations**.

3 Keep the default settings and select **OK**.

4 In the template, enter the equations as shown above and press **enter**.

Algebraic solution



1 In **Main**, open the **Keyboard** and select the **Math1** soft keyboard.

2 Tap on the **simultaneous equations** template.

3 Enter the equations and variables as shown above, then press **EXE**.

The points of intersection are (–1, 0) and $\left(\frac{1}{3}, \frac{8}{3}\right)$.

USING CAS 5 Solutions up to four unknowns

Solve the simultaneous equations for the 4 unknowns: x, a, b and c.

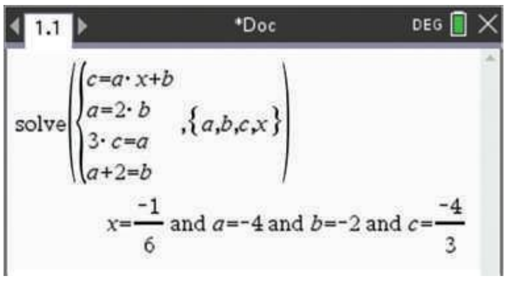
$c = ax + b$

$a = 2b$

$3c = a$

$a + 2 = b$

TI-Nspire



1 Add a **Calculator** page.

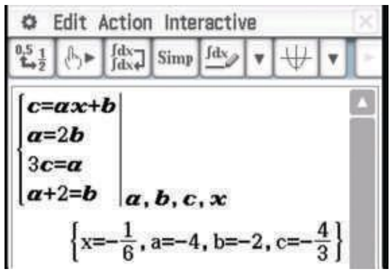
2 Press **menu > Algebra > Solve System of Equations > Solve System of Equations**.

3 On the next screen, change the **Number of equations:** to **4** and change the **Variables:** to ***a***, ***b***, ***c***, ***x***.

4 Select **OK**.

5 In the simultaneous equation template, enter the equations as shown above then press **enter**.

ClassPad



1 In **Main**, tap the **simultaneous equations** template in **Math1**, then tap twice more to allow space for 4 equations.

2 Enter each equation as shown above with symbols |***a***, ***b***, ***c***, ***x***.

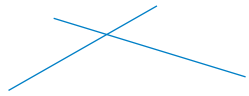
3 Press **EXE**.

The solutions are $x = -\frac{1}{6}$, $a = -4$, $b = -2$, $c = -\frac{4}{3}$.

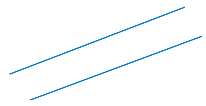
The solution to simultaneous equations allows us to analyse how 2 straight lines intersect.

There are 3 possibilities:

- one unique solution (different gradients)



- no solutions (parallel lines: same gradient, different y-intercepts)



- infinite solutions (the same line: same gradient, same y-intercepts)

WB p. 20

WORKED EXAMPLE 20 Finding a constant in simultaneous equations

Find the value(s) of m such that the simultaneous equations $mx + 2y = 20$ and $(m + 1)x - 2y = 10$ have no solution, where m is a real constant.

Steps	Working
1 Find the gradient of both linear equations by rearranging the equations in the form $y = mx + c$.	$mx + 2y = 20$ $2y = -mx + 20$ $y = -\frac{m}{2}x + 10$ gradient $= -\frac{m}{2}$ $(m + 1)x - 2y = 10$ $(m + 1)x - 10 = 2y$ $y = \left(\frac{m+1}{2}\right)x - 5$ gradient $= \frac{m+1}{2}$
2 If the equations have no solution, then the lines must be parallel, so their gradients must be equal but their y-intercepts must be different (so that they are not the same line).	$-\frac{m}{2} = \frac{m+1}{2}$ $-m = m + 1$ $-2m = 1$ $m = -\frac{1}{2}$
3 Substitute $m = -\frac{1}{2}$ to check that they are NOT the same line.	$y = -\frac{m}{2}x + 10$ $y = \frac{1}{4}x + 10$ ...[1] $y = \left(\frac{m+1}{2}\right)x - 5$ $y = \frac{1}{4}x - 5$ ...[2] Different y-intercepts (10 and −5), so parallel lines.
4 Write the answer.	$m = -\frac{1}{2}$

EXERCISE 1.7 Simultaneous equations

ANSWERS p. 492

1.7

Recap

1 TECH-FREE Make a the subject of the formula $am + 5 = -4(m - 2)$.

2 TECH-FREE Solve this equation for p:

$$pv = \frac{1}{2}(p + v)$$

Mastery

3 WORKED EXAMPLE 18 TECH-FREE Use graphs to solve the simultaneous equations $x + 2y = 8$ and $y = x - 2$.

4 TECH-FREE Use graphs to solve each pair of simultaneous equations.

a $y = x + 4$ and $y = -x - 6$

b $y = 2x + 1$ and $y = -x - 2$

5 WORKED EXAMPLE 19 TECH-FREE Solve each pair of simultaneous equations algebraically.

a $3a + 2b = 12$
$4a - 2b = 2$

b $3x - 4y = 8$
$2x + y = 9$

6 TECH-FREE Solve the simultaneous equations $y = x + 1$ and $x^2 - y = 1$.

7 TECH-FREE Solve each pair of simultaneous equations.

a $x + y = 10$
$2x - 2y = 5$

b $2x + y = 5$
$x - 2y = 5$

c $x + 3y = 20$
$2x + y = 1$

d $x + y = 1$
$2x + y = 5$

e $x - y = 5$
$4x - 2y = 10$

f $4x - y = 4$
$x + y = 1$

8 Using CAS 5 Solve the following simultaneous equations for x, y, a and b.

$y = ax + b$
$x = a + 2b$
$a = b$
$x = 2y - 1$

Exam practice

9 The solutions or factors to the equation $3x^2 - x - 2 = 0$ are

A $x = -1, \frac{2}{3}$ **B** $x = -1, -\frac{2}{3}$ **C** $(x - 1)(3x + 2)$ **D** $(3x - 1)(x + 2)$ **E** $x = 1, \frac{2}{3}$

10 WORKED EXAMPLE 20 © VCAA 2017N 2AQ3 MODIFIED The simultaneous linear equations $mx + 7y = 12$ and $7x + my = m$ have a unique solution only for

A $m = 7$ or $m = -7$

B $m = 12$ or $m = 3$

C any value of m except -7 or 7

D $m = 4$ or $m = 3$

E any value of m except 12 or 1

11 © VCAA 2019N 2AQ8 The simultaneous linear equations $2y + (m - 1)x = 2$ and $my + 3x = k$ have infinitely many solutions for

A $m = 3$ and $k = -2$
B $m = 3$ and $k = 2$
C $m = 3$ and $k = 4$
D $m = -2$ and $k = -2$
E $m = -2$ and $k = 3$

12 Using CAS 4 (3 marks) Let $f(x) = x^4 - 4x - 8$.

a Factorise the expression $x^4 - 4x - 8$. 1 mark

b Find the points of intersection between the graph of f and the line $y = 4x - 8$. 2 marks

Video playlist Inequalities

Worksheets Inequalities on a number line

Inequalities review

1.8 Inequalities

An **inequality** is a mathematical statement that contains an inequality symbol rather than an '=' sign. Inequality symbols are $>$, $<$, $\geq$ and $\leq$.

Inequalities can be solved like equations using inverse operations, except for the following rules:

Solving inequalities

- When multiplying or dividing both sides of an inequality by a negative number, reverse the inequality sign.
- When finding the reciprocal of both sides of an inequality, reverse the inequality sign.

Consider the true inequality $-7 < 5$.

Let's make each term the negative of itself to get $7 > -5$. Notice the change of $<$ to $>$ to make it true.

Consider the true inequality $7 > 5$.

Let's 'flip' each term upside down to get $\frac{1}{7} < \frac{1}{5}$. Notice the change of $>$ to $<$ to make it true.

Consider the inequality $2 - x > 4$.

So $-x > 2$

$\therefore\ x < -2$ Notice the change of $>$ to $<$ to make it true.

WB p. 21

WORKED EXAMPLE 21 Inequalities

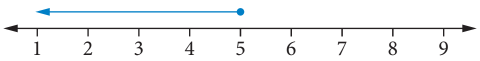
Solve for x the inequality $\frac{2x + 5}{6} \leq 2x + 1$.

Steps	Working
Solve using inverse operations, cross-multiplying first.	$\frac{2x+5}{6} \leq 2x + 1$ $2x + 5 \leq 6(2x + 1)$ $2x + 5 \leq 12x + 6$ $-10x \leq 1$ $x \geq -\frac{1}{10}$ When dividing both sides by (-10), reverse the inequality sign: $\leq$ changes to $\geq$.

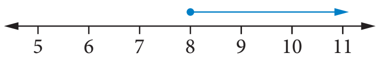
Graphing inequalities

Inequalities can be graphed on a number line as an arrow with a closed or open circle.

A **closed** circle means the value on the number line is **included**, and is used for inequalities involving $\geq$ or $\leq$.

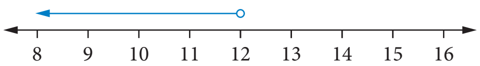


means $x \leq 5$ (5 included)

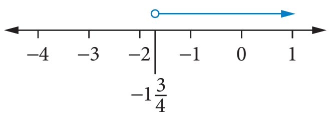


means $x \geq 8$ (8 included)

An **open** circle means the value on the number line is **not included**, and is used for inequalities involving > or <.



means $x < 12$ (12 not included)

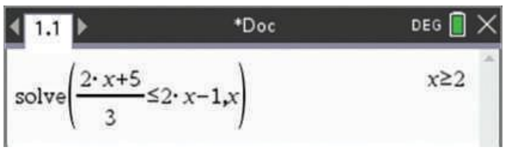


means $x > -1\frac{3}{4}$ $\left(-1\frac{3}{4} \text{ not included}\right)$

USING CAS 6 Solving inequalities

Solve the inequality $\dfrac{2x+5}{3} \leq 2x - 1$.

TI-Nspire

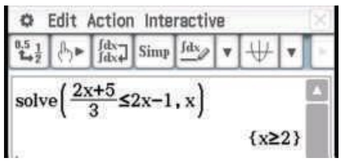


1 In a **Calculator** page, press **menu > Algebra > Solve**.

2 Enter the inequality followed by, ***x*** as shown above. Press **ctrl + =** to access the mini-palette for the **≤** symbol.

3 Press **enter**.

ClassPad



1 In **Main**, enter and highlight the inequality.

2 Tap **Interactive > Equation/Inequality > solve.** Tap **Keyboard > Math3** to access the **≤** symbol.

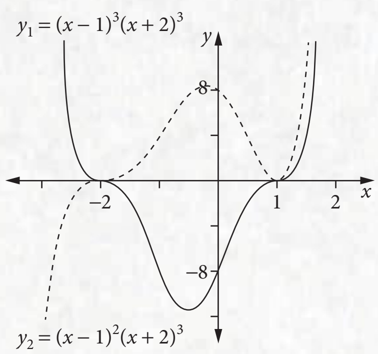
3 Tap **OK**.

VCE QUESTION ANALYSIS

© VCAA 2019N 2BQ1 MODIFIED 2019 Northern Hemisphere Examination 2 Section B Question 1 MODIFIED (8 marks)

Video
VCE question analysis: Equations and inequalities

The graphs of $y_1 = (x-1)^3(x+2)^3$ and $y_2 = (x-1)^2(x+2)^3$ are shown on the axes below.



The two graphs intersect at three points (–2, 0), (1, 0) and (c, d). The point (c, d) is not shown in the diagram above.

a Find the values of c and d. 2 marks

b Find the values of x such that $y_1 > y_2$. 1 mark

c **i** State the values of x where the graph $y_1 = (x-1)^3(x+2)^3$ gives $y = 8$. 1 mark

ii State the values of y where the graph $y_1 = (x-1)^3(x+2)^3$ gives $x = -1$. 1 mark

d State the values of x for which

i $y_1 = (x-1)^3(x+2)^3 > 0$ 1 mark

ii $y_2 = (x-1)^2(x+2)^3 > 0$ 1 mark

e Show that the value of y_1 is the same for $x = 1 + m$ and $x = -2 - m$, for all m. 1 mark

Reading the question

- The question is centred around the concept of graphs intersecting and being higher or lower.
- The first question asks where the graphs are equal.
- The second question asks where the graph of y_1 is higher than y_2.
- The question then moves into substitution of numbers and variables representing constants.

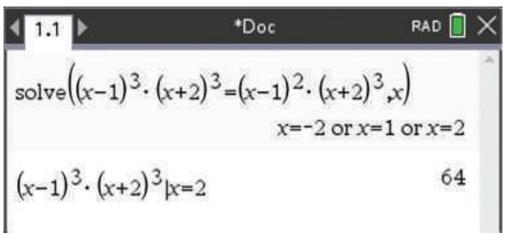
Thinking about the question

- As this question is technology active, the questions are much easier to solve than it looks on initial reading. Make sure your CAS is on 'exact' (TI) or 'Standard' (ClassPad) to get exact values, even if they are surds.
- If we want to find where $y = 8$ on the graph y_1, we just solve the equation $y_1 = 8$ for x and the answer will be given.
- Part **e** is a 'show that' question, so we must show a certain number of steps.

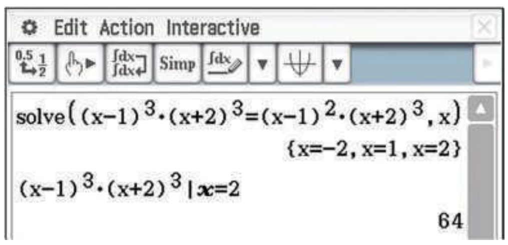
Worked solution (✓ = 1 mark)

a Equate y_1 and y_2 to give $\boldsymbol{x = -1, x = 1, x = 2}$ ✓

TI-Nspire



ClassPad



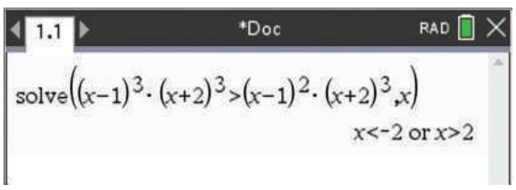
The 2 graphs intersect at 3 points (–2, 0), (1, 0) and (c, d), so $c = 2$.

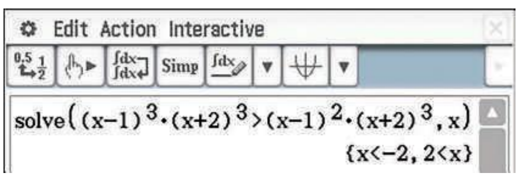
Substitute $x = 2$ into either equation to get $y = 64$.

$\boldsymbol{c = 2, d = 64}$ ✓

b We need to find x values where $y_1 > y_2$.

Using CAS we get $\boldsymbol{x < -2}$ **and** $\boldsymbol{x > 2}$. ✓



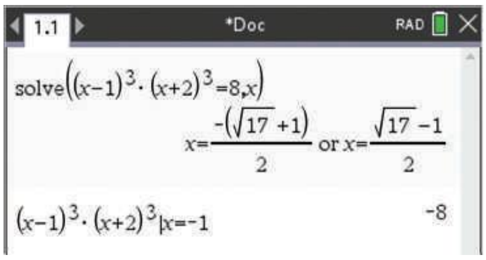


1.8

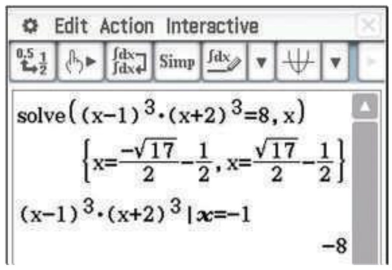
c i State the values of x where the graph $y_1 = (x-1)^3(x+2)^3$ gives $y = 8$.

This means to solve $(x-1)^3(x+2)^3 = 8$.

TI-Nspire



ClassPad



$\boldsymbol{x = \frac{-1 \pm \sqrt{17}}{2}}$ ✓

ii State the values of y where the graph $y_1 = (x-1)^3(x+2)^3$ gives $x = -1$.

This means to find y_1 when $x = -1$. Use CAS.

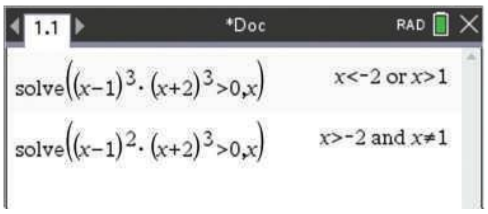
Refer to the graph to identify where these points are to confirm your answers.

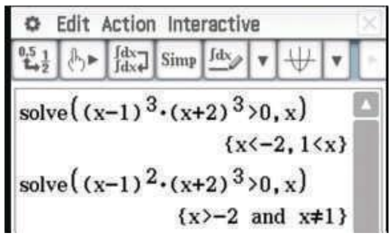
$\boldsymbol{y = -8}$ ✓

d i $y_1 > 0$

This means to find x where the graph y_1 is above the x-axis.

Refer to the graph to identify where these points are to confirm your answers.





$\boldsymbol{x < -2 \text{ and } x > 1}$ ✓

ii $y_2 > 0$

This means to find x where the graph y_2 is above the x-axis.

$\boldsymbol{x > -2 \text{ and } x \neq 1}$ (consider what $x \neq 1$ means) ✓

e Show that $x = 1 + m$ and $x = -2 - m$ have the same value of y_1 for all m.

CAS can show this, but this question is a 'show that' question.

Given $y_1 = (x-1)^3(x+2)^3$:

When $x = 1 + m$

$y_1 = (1 + m - 1)^3(1 + m + 2)^3$

$\quad = m^3(m+3)^3$

When $x = -2 - m$

$y_1 = (-2 - m - 1)^3(-2 - m + 2)^3$

$\quad = (-m-3)^3(-m^3)$

$\quad = [-1(m+3)]^3[(-1)m]^3$

$\quad = (-1)(m+3)^3(-1)m^3$

$\quad = \boldsymbol{(m+3)^3 m^3}$ as required ✓

EXERCISE 1.8 Inequalities

ANSWERS p. 492

Recap

1 TECH-FREE Solve the simultaneous equations $2x + y = 5$ and $x - 2y = 10$.

2 TECH-FREE Solve the simultaneous equations $y = x^2 + 1$ and $2x + y = 6$.

Mastery

3 WORKED EXAMPLE 21 TECH-FREE Solve for x in the inequality $\dfrac{2x+5}{3} \leq 2x + 1$.

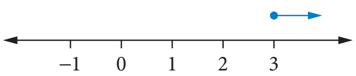
4 TECH-FREE Solve for x in the inequality $\dfrac{-2x-5}{5} > x - 1$.

5 The solution to the inequality $x + 7 > 2x + 4$ is

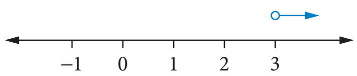
A $x > 3$ **B** $x < 3$ **C** $x \leq 3$ **D** $x \geq 3$ **E** $x = 3$

6 The number line representing the solution to the inequality $x + 7 > 2x + 4$ is

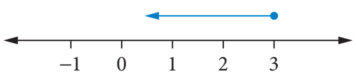
A



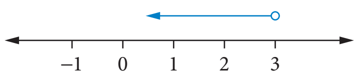
B



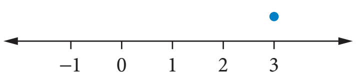
C



D



E



7 Using CAS 6 The solution to the inequality $\dfrac{x+5}{2} \geq \dfrac{3x-1}{4}$ is

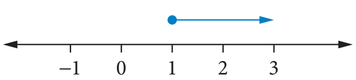
A $x > 11$ **B** $x < 11$ **C** $x \leq 11$ **D** $x \geq 11$ **E** $x = 11$

8 The solution to the inequality $\dfrac{x+5}{2} \geq \dfrac{3x-1}{4} + 1$ is

A $x > 7$ **B** $x < 7$ **C** $x \leq 7$ **D** $x \geq 7$ **E** $x \neq 7$

9 The number line representing the solution to the inequality $\dfrac{x+5}{2} \geq \dfrac{3x-1}{2} + 2$ is

A



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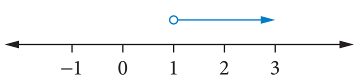

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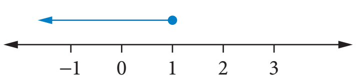

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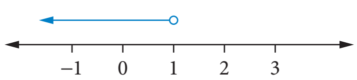
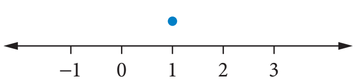


E



10 The graphs of the equations $x + 2y = 8$ and $3x - 4y = 2$ intersect at the point

A $\left(\dfrac{1}{2}, \dfrac{3}{4}\right)$ **B** $\left(\dfrac{1}{5}, \dfrac{3}{5}\right)$ **C** $\left(\dfrac{18}{5}, \dfrac{11}{5}\right)$ **D** $\left(\dfrac{18}{5}, -\dfrac{11}{5}\right)$ **E** $(-14, 11)$

1 Chapter summary

CHAPTER SUMMARY 1

Index laws

$a^m \times a^n = a^{m+n}$

$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$

$a^m \div a^n = a^{m-n}$

$a^0 = 1$

$\left(a^m\right)^n = a^{mn}$

$a^{-n} = \frac{1}{a^n}$

$(ab)^n = a^n b^n$

$a^{\frac{1}{q}} = \sqrt[q]{a}$

$a^{\frac{p}{q}} = \sqrt[q]{a^p} = \left(\sqrt[q]{a}\right)^p$

Binomial expansions

A **binomial expression** has 2 terms, for example, $x + 4$ and $3x^2 - 6$.

A **binomial product** is 2 or more binomial expressions multiplied together.

It has the form $(a + b)(c + d)$.

$(a + b)(c + d) = ac + ad + bc + bd$

A **quadratic expression** is of the form $ax^2 + bx + c$.

FOIL method

First terms, Outer terms, Inner terms, Last terms

Perfect squares

$(a + b)^2 = a^2 + 2ab + b^2$

$(a - b)^2 = a^2 - 2ab + b^2$

Difference of two squares

$(a + b)(a - b) = a^2 - b^2$

Quadratic equations

A **quadratic equation** is an equation where the highest power of the variable is 2. It has the general form $ax^2 + bx + c = 0$, and can be solved by **factorising**, **completing the square** or using the **quadratic formula**.

Completing the square

Solve by completing the square:

1 Write the equation in general form: $ax^2 + bx + c = 0$.

2 Take out the common factor of a.

3 Halve the coefficient of x and then square this number.

4 Add and then subtract (to compensate) the number found in the previous step.

5 Recognise the perfect square and factorise.

6 Solve for zero to find the roots of the equation.

The quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Solve using the quadratic formula:

1 Write the equation in general form:

$$ax^2 + bx + c = 0$$

2 Identify the values of a, b and c and substitute into the quadratic formula.

3 Simplify where necessary (including the surd).

The discriminant

In the quadratic formula, the expression under the square root symbol, $\Delta = b^2 - 4ac$, is the **discriminant**.

The sign of its value gives information about the number of **roots** of the quadratic equation, $ax^2 + bx + c = 0$:

- If $\Delta < 0$, the quadratic equation has no real roots.
- If $\Delta = 0$, the quadratic equation has 1 real root, which is rational.
- If $\Delta > 0$, the quadratic equation has 2 real roots.
  - If Δ is a square number, the 2 roots are rational.
  - If Δ is a not a square number, the 2 roots are irrational (surds).

Changing the subject of a formula

A **literal equation** is an equation with more than one variable, such as $ax + b = c$. The most common literal equation is a **formula**. We can rearrange a formula to make one of the variables the **subject** of the formula.

Simultaneous equations

Simultaneous equations are 2 or more equations with more than one variable. These are solved together to give a set of values that satisfy all equations. Simultaneous equations can be solved algebraically or graphically. Graphically, the solution is the point(s) of intersection of the graphs of the equations.

Algebraic solution

The **elimination method** combines the equations to eliminate x or y.

The **substitution method** substitutes an expression from one equation into the other.

Inequalities

Inequality symbols are $>$, $<$, $\geq$ and $\leq$.

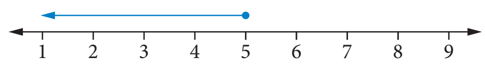
Inequalities can be solved like equations using inverse operations, except for the following rules:

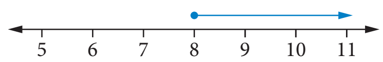
- When multiplying or dividing both sides of an inequality by a negative number, reverse the inequality sign.
- When finding the reciprocal of both sides of an equality, reverse the inequality sign.

Graphing inequalities

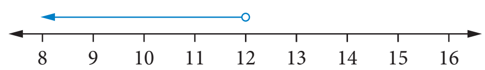
Inequalities can be graphed on a number line as an arrow with a closed or open circle.

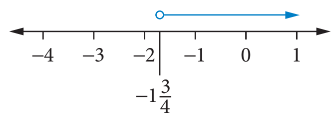
A **closed** circle means the value on the number line is **included**, and is used for inequalities involving $\geq$ or $\leq$.





An **open** circle means the value on the number line is **not included**, and is used for inequalities involving $>$ or $<$.





9780170448260

Cumulative examination 1

Total number of marks: 10 Reading time: 4 minutes Writing time: 15 minutes

TECH-FREE *Technology is NOT permitted.*

1 (6 marks) Let $y = \frac{2}{(x-1)^2} + 1$.

- **a** Find the value of y when $x = 2$. 1 mark
- **b** Find the value(s) of x such that $y = 3$. 2 marks
- **c** Evaluate y when $x = -1$. 1 mark
- **d** Consider the line, $y = x + 1$, crossing the graph of $y = \frac{2}{(x-1)^2} + 1$ at the point (a, b) where a and b are integers.

  Find the values of a and b. 2 marks

2 (2 marks) Find the value of the constant a in the rule $y = ax^{\frac{2}{3}} + \sqrt{x}$ given that $y = -\frac{7}{4}$ when $x = 64$.

3 © VCAA 2019N 1Q5a MODIFIED (2 marks) Let $h = \sqrt{2x+3} - 2$. Find the value(s) of x such that $h^2 = 1$.

Cumulative examination 2

Total number of marks: 19 Reading time: 4 minutes Writing time: 30 minutes

Approved technology is permitted.

Section A 5 multiple-choice questions

5 marks

1 The graph of $y = 4 - 2x$ has an x-intercept of

A -4 **B** -2 **C** 0 **D** 2 **E** 4

2 The graph of $y = x^2 + 2x - 3$ has x-intercepts at

A 3 only **B** 3 and 1 **C** 3 and -1 **D** -3 and 1 **E** -3 and -1

3 © VCAA 2017 2AQ7 The equation $(p - 1)x^2 + 4x = 5 - p$ has no real roots when

A $p^2 - 6p + 6 < 0$ **B** $p^2 - 6p + 1 > 0$ **C** $p^2 - 6p - 6 < 0$

D $p^2 - 6p + 1 < 0$ **E** $p^2 - 6p + 6 > 0$

4 If $x + a$ is a factor of $x^2 + 3x + a$, where $a \neq 0$, then the value of a is

A -4 **B** 0 or -4 **C** 0 **D** 0 or 2 **E** 2

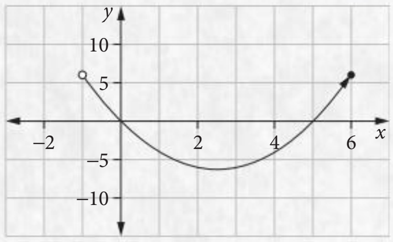
5 © VCAA 2019N 2AQ6 MODIFIED The equation $(x^2 + 1)^2 + 1 = \frac{185}{16}$ has real solution(s)

A $x = \pm\frac{\sqrt{13}}{4}$ **B** $x = \frac{\sqrt{13}}{4}$ **C** $x = \pm\frac{\sqrt{13}}{2}$ **D** $x = \frac{3}{2}$ **E** $x = \pm\frac{3}{2}$

Section B 1 question

14 marks

1 (14 marks) Let $y = x^2 - 5x$ for $-1 < x \leq 6$. The graph of y is shown.



a State the x-intercepts. 2 marks

b Find the coordinates of the turning point. 2 marks

c $A(-1, a)$ and $B(1, b)$ are two points on the graph.

i Find the equation of the straight line through A and B. 2 marks

ii Find the distance AB. 1 mark

d Let $C(-1, c)$ and $D(1, d)$ be two points on the graph of $y = x^2 - kx$, where k is a constant.

i Find the distance CD in terms of k. 2 marks

ii Find the values of k such that the distance CD is equal to $2k + 1$. 1 mark

e Find the point(s) of intersection between the line $y = x$ and the graph of $y = x^2 - 5x$. 2 marks

f Find the point(s) of intersection between the line $y = x$ and the graph of $y = x^2 - kx$ in terms of k. 2 marks

CHAPTER

2 FUNCTIONS AND GRAPHS

Study Design coverage

UNIT 1, AREA OF STUDY 1: FUNCTIONS, RELATIONS AND GRAPHS

- functions and function notation, domain, co-domain and range, representation of a function by rule, graph and table, inverse functions and their graphs
- qualitative interpretation of features of graphs of functions, including those of real data not explicitly represented by a rule, with approximate location of any intercepts, stationary points and points of inflection.

STUDY DESIGN COVERAGE

Video playlists (6):

2.1 Functions and relations
2.2 Functions and graphs
2.3 Domain and range
2.4 Linear functions
2.5 Quadratic functions
VCE question analysis Functions and graphs

Worksheets (17):

2.1 Functions and relations • Function notation

2.4 Graphing linear functions • x- and y-intercepts • Equations of lines • Linear functions code puzzle • Finding the equation of a line • $y = mx + c$ • Parallel and perpendicular lines • Linear functions

2.5 Sketching quadratic functions • Graphing quadratic functions • Quadratic functions • Graphing quadratics • Features of a parabola • Matching parabolas • The discriminant

Nelson MindTap

To access resources above, visit **cengage.com.au/nelsonmindtap**

2.1 Functions and relations

Video playlist Functions and relations

Worksheet Functions and relations

Relations

A **relation** R is a set of ordered pairs (x, y), where $x \in X$ and $y \in Y$. X and Y are usually both the set of real numbers. We can show that an ordered pair belongs to a relation R by writing $(x, y) \in R$ or $y = R(x)$. The set of values of the **independent variable** x is called the **domain** and is often the set X. The set Y is called the **co-domain** and the set of values of the **dependent variable** y is called the **range**.

We can show a relation as a rule (formula), a list of ordered pairs, a table, a mapping or a graph.

A **mapping** shows ordered pairs connected by arrows from the x values to the y values.

WB p. 22

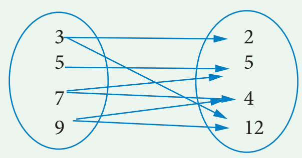
WORKED EXAMPLE 1 Mappings and ordered pairs

The mapping shows the relation M.

a Write M as a set of ordered pairs.

b Show M as a table.

c State the domain and range of M.



Steps	Working
a Write the ordered pairs.	$M = \{(3, 2), (3, 12), (5, 5), (7, 4), (7, 5), (9, 4), (9, 12)\}$
b Write the ordered pairs in a table.	see table below
c 1 The domain is the set of first numbers.	domain $= \{3, 5, 7, 9\}$
2 The range is the set of second numbers.	range $= \{2, 4, 5, 12\}$

x	3	3	5	7	7	9	9
y	2	12	5	4	5	4	12

Write the domain and range in order.

Functions

Worksheet
Function notation

Functions

A **function** $y = f(x)$ is a relation such that each x value in the domain has only one ordered pair (x, y). In other words, each x value matches with only one y value.

A function may be shown by a rule like $f(x) = 5x - 1$.

'$f(x)$', pronounced 'f of x', is called **function notation**, and means the rule for operating on the independent variable, x.

The value of $f(x)$ at $x = c$ is written as $f(c)$.

For example, $f(2)$ is the value of $f(x) = 5x - 1$ when $x = 2$.

$f(2) = 5 \times 2 - 1 = 9$

More formally, a function can be written like this: $f: R \to R$, $y = x^2 + 3$, whose domain and co-domain are the real numbers.

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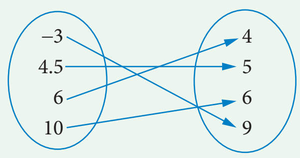
WORKED EXAMPLE 2 Relations and functions

State whether each relation is a function.

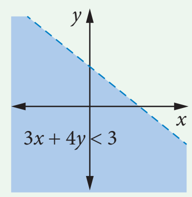
a



b $x^2 = y^2$

c $y = \sqrt{x - 2}$

d

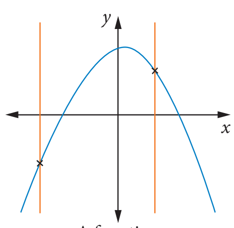


	Steps	Working
a	**1** Write the ordered pairs.	Relation is $(-3, 9)$, $(4.5, 5)$, $(6, 4)$, $(10, 6)$.
	2 Write the conclusion.	It is a function because each x value occurs only once.
b	**1** Do any ordered pairs have the same x value?	$(2, -2)$ and $(2, 2)$ have the same x value. $x = 2$ has 2 matching y values.
	2 Write the conclusion.	It is not a function as there are 2 ordered pairs with the same x value.
c	**1** Do any ordered pairs have the same x value?	$\sqrt{x - 2}$ means the positive value so each ordered pair (x, y) is unique.
	2 Write the conclusion.	It is a function because each x value has one y value.
d	**1** Do any ordered pairs have the same x value?	$(0, 0.5)$, $(0, 0)$, $(0, -0.2)$, $(0, -1)$ … have the same x value.
	2 Write the conclusion.	It is not a function as there are an infinite number of ordered pairs with the same x value.

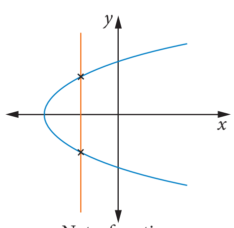
We can usually determine from a graph if a relation is a function. A function will show any value on the x-axis with (at most) one value for y. A relation may have 2 or more y values for one x value: there will be 2 or more parts of the graph directly above each other.

If any **vertical line** intersects the graph of a relation at most once, then it is a function.

If a vertical line intersects the graph of a relation in more than one place, then it is not a function.



A function

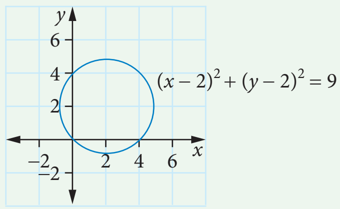


Not a function

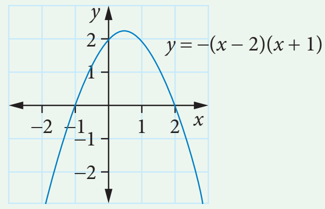
WORKED EXAMPLE 3 Relation and function graphs

State whether each of the following shows the graph of a function.

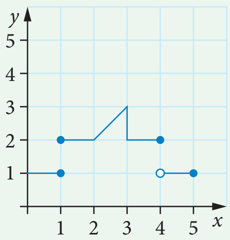
a



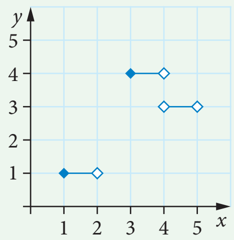
b



c



d

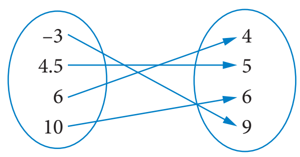


Steps		Working
a	**1** Do any points have the same x value?	There are 2 points at $x = 2$: (2, 5) and (2, −1).
	2 Write the conclusion.	$(x-2)^2+(y-2)^2=9$ is not a function.
b	**1** Do any points have the same x value?	There are no points with the same x value (but some with the same y value).
	2 Write the conclusion.	$y=-(x-2)(x+1)$ is a function.
c	**1** Do any points have the same x value?	(3, 2), (3, 2.1) … (3, 3) have the same x value. (1, 1) and (1, 2) have the same x value.
	2 Write the conclusion.	This graph does not show a function.
d	**1** Do any points have the same x value?	There are no points with the same x value. At $x = 4$, there is no value, so the relation is undefined there.
	2 Write the conclusion.	This graph does show a function.

Most relations with rules like 'y = algebraic rule involving x' are functions.

For example, $y=\dfrac{\sqrt{x-3}}{x-5}+\dfrac{1}{\sqrt{7-x}}$ is a function. Each value of x in its domain has only one corresponding value of y.

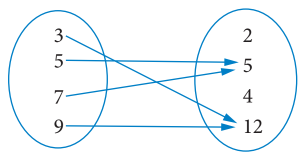
Consider this function mapping:



Each x value in the domain is matched with a unique y value. In the pairings, there aren't any repeated x or y values. This is called a **one-to-one function**.

The function {(1, 3), (2, 1), (3, 5), (4, −1)} is also one-to-one: every x or y value is in only one ordered pair.

Now consider this function mapping:



Each x value in the domain is matched with one y value. But notice that for both $x = 5$ and $x = 7$, $y = 5$. And for both $x = 3$ and $x = 9$, $y = 12$. In the pairings, there are repeated y values. This is called a **many-to-one function**.

The function $y = x^2$ is also many-to-one: there are many x values that match with the same y value, for example, {(7, 49), (−7, 49), (2, 4), (−2, 4), (1, 1), (−1, 1)}.

2.1

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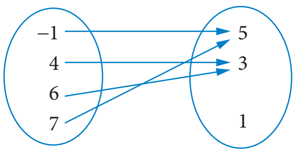
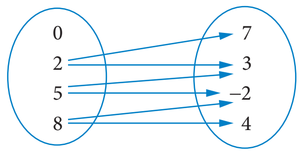
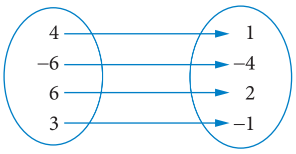
EXERCISE 2.1 Functions and relations
ANSWERS p. 492

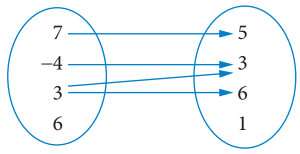
Mastery

1 WORKED EXAMPLES 1, 2 Write each mapping as ordered pairs. State the domain and range and whether the mapping is a function or not. If it is a function, state whether it is one-to-one or many-to-one.

a


b


c


d


2 State whether each set of ordered pairs is a function or not. If it is a function, state whether it is one-to-one or many-to-one.

a {(3, 5), (4, 7), (5, 7), (6, 7), (8, 9), (9, 5), (12, 5)}

b {(4, 7), (5, 6), (5, 4), (7, 3), (8, 2), (9, 1), (10, 0)}

c {(7, 3), (3, 7), (9, 2), (2, 9), (4, 8), (8, 4), (5, 7), (7, 5)}

d {(0.1, 3), (0.3, 9), (0.5, 15), (0.7, 21)}

3 State whether each relation is a function.

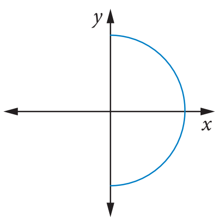
a $\dfrac{x-3}{5} - \dfrac{y+2}{3} = 1$

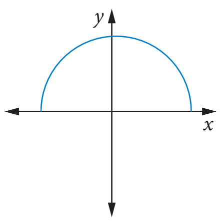
b $\dfrac{(x-3)^2}{25} - \dfrac{(y+2)^2}{9} = 1$

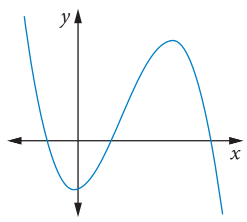
c $12x - 3y^2 = 5$

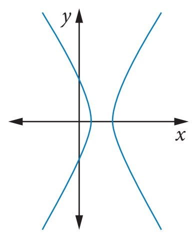
d $15x - 3y^3 = 7$

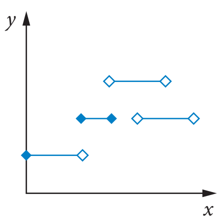
4 WORKED EXAMPLE 3 State whether each graph represents a function.

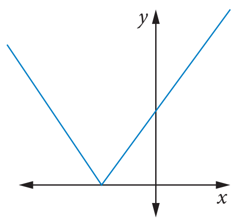
a


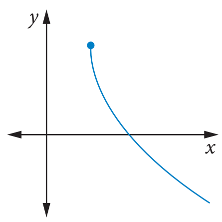
b


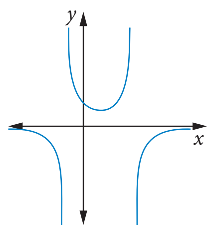
c


d


e


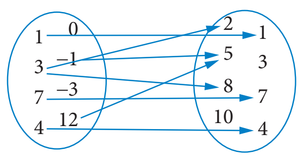
f


g


h


Exam practice

5 The relation P is shown by the mapping below.



The domain and range are

A domain: {0, –1, 1, –3, 3, 4, 7, 12}, range: {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}

B domain: {0, –1, 1, –3, 3, 4, 7, 12}, range: {1, 2, 3, 4, 5, 7, 8, 9, 10}

C domain: {0, –1, 1, –3, 3, 4, 7, 12}, range: {1, 2, 3, 4, 5, 7, 8, 10}

D domain: {0, –1, 1, –3, 3, 4, 7, 12}, range: {1, 2, 4, 5, 7, 8}

E domain: {–1, 1, 3, 4, 7, 12}, range: {1, 2, 4, 5, 7, 8}

6 Which relation is a function?

A $y = x^2 - 5$ **B** $y^2 + 5 = x$ **C** $y^2 = x + 5$

D $\frac{1}{x} = \frac{1}{y^2 + 1}$ **E** $x^2 - y^2 = 5$

7 Which relation is not a function?

A $y = \frac{1}{x + 3}$ **B** $y = x^2 - 5^2$ **C** $\frac{1}{x} = \frac{1}{y}$

D $x^2 = y^2 + 9$ **E** $\sqrt{y} = x^2 + 4$

2.2

2.2 Functions and graphs

Video playlist
Functions and graphs

WB
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WORKED EXAMPLE 4 Straight lines

A Chemistry student lubricated the plunger of a 30 mL syringe with silicone grease, pushed the plunger down to the 20 mL mark and put a cap on the end. She put the syringe into an ice-water bath and stirred the water while slowly heating it. The temperature and volume readings were as shown below.

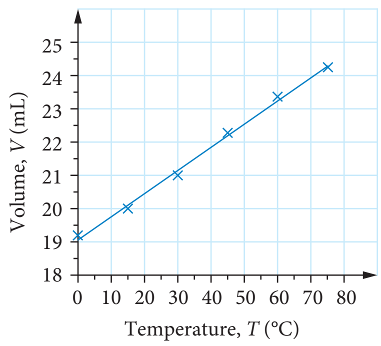
Draw a graph and determine the relationship between the temperature and volume of air in the syringe.

Temperature (°C)	0	15	30	45	60	75
Volume (mL)	19.2	20.0	21.0	22.3	23.4	24.3

Steps | **Working**

1 Plot the points on suitable graph paper.

The points appear to be in a line, so draw a line of best fit.



2 Choose 2 points on the line. (0, 19) and (70, 24) are on the line.

3 Calculate the gradient of the line. $m = \frac{24 - 19}{70 - 0} = \frac{1}{14}$

4 Write the equation using $y = mx + c$. $V = \frac{1}{14}T + 19$

The T-intercept of $V = \frac{1}{14}T + 19$ is easily calculated. When $V = 0$, $T = -266$. It suggests that the volume of the gas would be 0 mL at a temperature of −266°C. The lowest possible temperature (*absolute zero*) is now thought to be −273.15°C.

In many cases, we cannot draw a straight line. Points may also be widely scattered.

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WORKED EXAMPLE 5 Plotting a linear function

The total cost of running a sausage sizzle at a school sports day is $150.

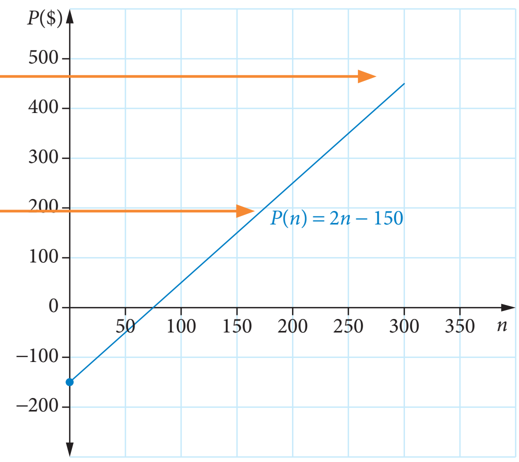
300 sausages are to be sold for $2 each.

a Write the function for the profit made when n sausages are sold.

b How many sausages must be sold to make a profit?

c Draw a graph of the profit function.

d How much profit is made if all 300 sausages are sold?

Steps	Working
a Profit = sales − cost	$P(n) = 2n - 150$
b **1** Find $P(n) > 0$.	$2n - 150 > 0$
	$n > 75$
2 Write the answer.	A profit is made when more than 75 sausages are sold.
c **1** Choose some values of n and calculate $P(n)$.	$P(0) = -150$, $P(100) = 50$, $P(200) = 250$
2 Draw the graph.	
The answer to part **d** agrees with the graph at this point.	
Label the graph with its equation.	
d **1** Find $P(300)$.	$P(300) = 450$
2 Write the answer.	If all the sausages are sold, there is a profit of $450.

From the graph or the function, the intercepts on the above graph are $(0, -150)$ and $(75, 0)$. For a linear function, we only need to plot 2 points to draw the line, but check using a third point.

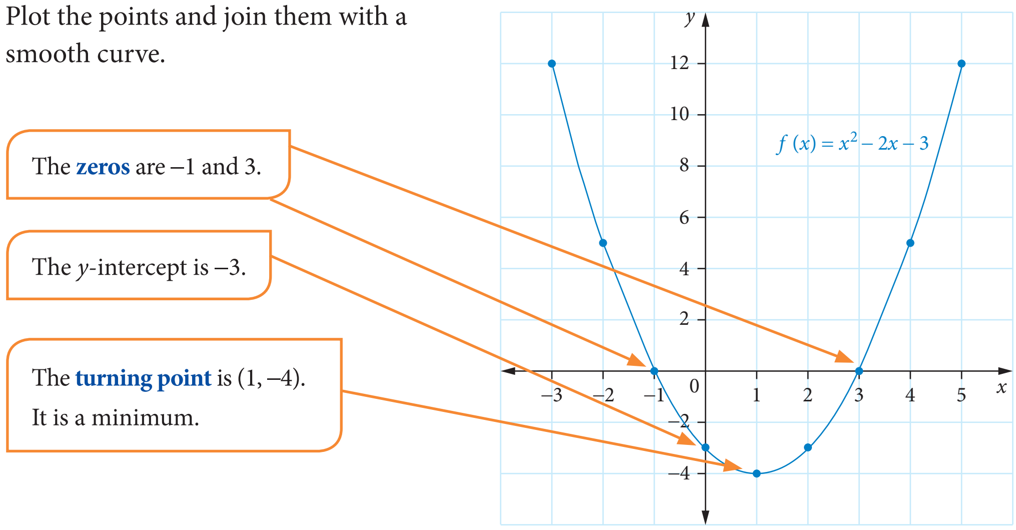
Quadratic functions have an x^2 term with a possible x term and constant.

WORKED EXAMPLE 6 Graphing a quadratic function

Graph $f(x) = x^2 - 2x - 3$ from $x = -3$ to $x = 5$.

Steps	Working
1 Make a table of values.	see table below
2 Plot the points and join them with a smooth curve.	see graph below

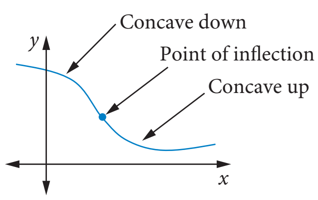
x	−3	−2	−1	0	1	2	3	4	5
y	12	5	0	−3	−4	−3	0	5	12



The curve above slopes down, then flattens out as it approaches the **minimum point**. Then it slopes upward. The gradient is negative for $x < 1$, zero at $x = 1$ and positive for $x > 1$. It is **concave up**. You can remember this as being *cup*-shaped.

A **stationary point** is a point on a graph where the gradient is zero. Any minimum or maximum on a continuous curve must be a stationary point.

A **point of inflection** is a point where a graph changes from concave up to **concave down**, or from concave down to concave up. (Inflection means 'bend'.)



2.2

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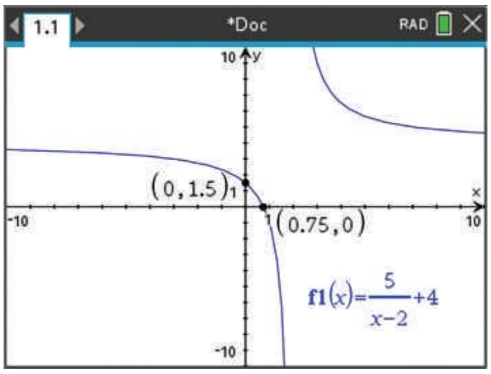
USING CAS 1 Sketching graphs

Sketch the graph of $f(x) = \dfrac{5}{x-2} + 4$.

TI-Nspire



1 Add a **Graphs** page.

2 Enter the function $\boldsymbol{f1(x) = \frac{5}{x-2} + 4}$.

3 Press **menu > Window/Zoom > Window Settings**.

4 Keep the default X window settings and change the Y settings to **YMin: -10, YMax: 10, Scale: 1**.

5 Press **menu > Analyze Graph > Zero**.

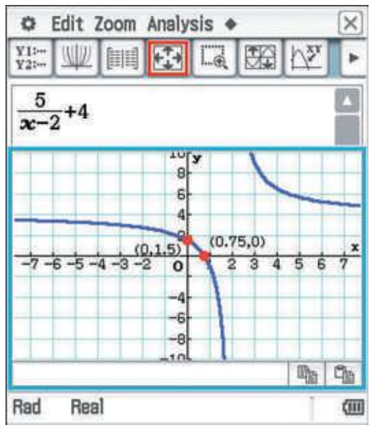
6 When prompted for the **lower bound?**, click to the left of the x-intercept.

7 When prompted for the **upper bound?**, click to the right of the x-intercept to display the coordinates.

8 Press **menu > Geometry > Points and Lines > Point On**.

9 Click on the graph and then on the y-intercept to display the coordinates.

ClassPad



1 In **Main**, enter the expression $\dfrac{5}{x-2} + 4$.

2 Tap and open the **Graph** window.

3 Highlight the expression and drag it down into the **Graph** window.

4 Tap **View Window**.

5 Tap **Default** and change the Y settings to:

YMin: -10 YMax: 10 Scale: 2.

6 Tap **Analysis > G-Solve > Root**.

7 Press **EXE** to label the coordinates.

8 Tap **Analysis > G-Solve > y-Intercept**.

9 Press **EXE** to label the coordinates.

The function is not defined at $x = 2$, and nearby the magnitude of $f(x)$ becomes very large.

As $x \to -\infty$, $f(x) < 4$ and $f(x) \to 4$.

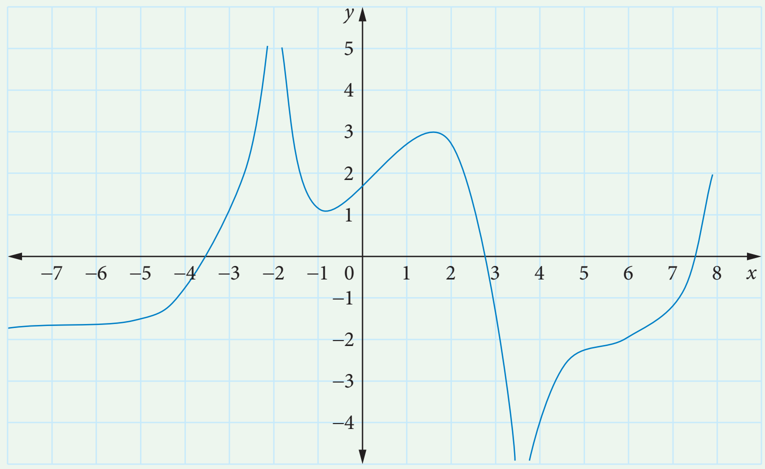
As $x \to \infty$, $f(x) > 4$ and $f(x) \to 4$.

The line $y = 4$ is called a horizontal **asymptote**. This is a value that the function approaches as $x \to -\infty$ or $x \to \infty$. In this case, the function approaches 4 for both cases.

The line $x = 2$ is called a vertical asymptote. As x approaches 2, the function becomes very large in magnitude. Functions are **undefined** at vertical asymptotes.

WORKED EXAMPLE 7 Features of a graph with no specific rule

The graph of a function is shown below. Identify and approximately locate its significant features.



Steps	Working
1 Where does the graph cross the y-axis?	The y-intercept is about (0, 1.5).
2 Where does it cross the x-axis?	The zeros are about −3.5, 2.8 and 7.5.
3 Are there any maximum or minimum points?	There is a minimum at about (−0.8, 1) and a maximum at about (1.6, 3).
4 Are there any asymptotes?	There may be a horizontal asymptote about $y = -2$ and vertical asymptotes at $x = -2$ and $x = 3.7$.
5 Are there any points of inflection?	Points of inflection occur at about (0.3, 1.9) and (6, −2).

2.2

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Inverse functions and graphs

A relation R is a set of coordinate pairs (x, y). The **inverse relation** R^{-1} is the relation formed by swapping all the ordered pairs to (y, x).

If the inverse relation of a function is also a function, then it is called the **inverse function** and is written as $f^{-1}(x)$. This function is the opposite to $y = f(x)$, and links the values of y in $y = f(x)$ back to x. For example, if $f(3) = -5$, then $f^{-1}(-5) = 3$.

A function has only one y value for each x value in its domain. This also applies to the inverse function. Its x values are the y values of the original function, so each y value can have only one x value. So this means that for a function to have an inverse function, it must be **one-to-one**: each x value has only one y value and each y value has only one x value. Both the x and y values in the ordered pairs of the function are unique.

The function $y = 3x - 5$ is one-to-one so it has an inverse.

The function $y = x^2 - 3$ is not one-to-one so it does not have an inverse. However, we can restrict its domain to make it one-to-one so it does have an inverse, for example, restrict it to $[0, \infty)$.

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WORKED EXAMPLE 8 Existence of an inverse function

State whether each function has an inverse function. If not, suggest how the domain could be restricted so that it does.

a (3, 7), (4, 9), (5, 9), (7, 11), (9, 17)

b $y = x^3 + 4$

c $f(x) = (x + 4)(x - 3)$

Steps	Working
a **1** Check the y values for duplicates.	(4, 9) and (5, 9) have the same y value.
2 State the conclusion.	The function is not one-to-one, so the inverse is not a function.
3 Suggest a restriction.	Restrict the domain to either {3, 4, 7, 9} or {3, 5, 7, 9} so it is one-to-one.
b **1** Check y values for duplicates.	Every y value is unique.
2 State the conclusion.	The function is one-to-one, so the inverse is a function.
c **1** Check y values for duplicates.	$f(-4) = 0$ and $f(3) = 0$, so it is not one-to-one and the inverse is not a function.
2 Find the turning point.	There is a minimum $(-0.5, -12.3)$, so the values above -0.5 duplicate those below.
3 Suggest a restriction.	Make the domain $x \le -0.5$ or $x \ge -0.5$ so it is one-to-one and the inverse is a function.

The inverse of a function has the x and y values swapped. To find the rule for the inverse of a function, swap x and y and make y the subject of the equation. The graphs of an inverse function and the original function are mirror images of each other in the line $y = x$.

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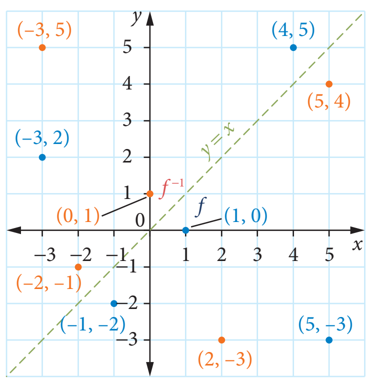
WORKED EXAMPLE 9 Function and inverse function graphs

Find the inverse function of each function and sketch both, together with the line $y = x$.

a f: (1, 0), (−3, 2), (4, 5), (5, −3), (−1, −2)

b $f(x) = 2x + 6$

Steps	Working
a **1** Write the inverse function.	f^{-1}: (0, 1), (2, −3), (5, 4), (−3, 5), (−2, −1)
2 Plot the graphs in different colours. Draw $y = x$.	

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b **1** Swap x and y. $x = 2y + 6$

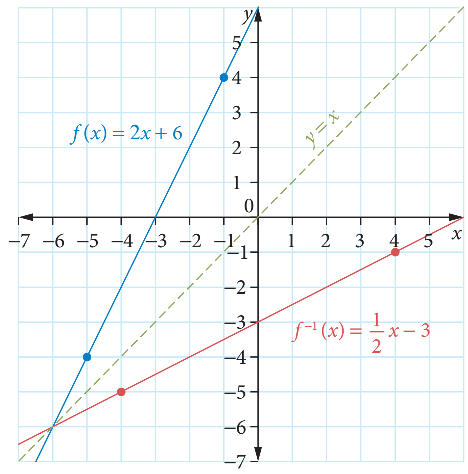
2 Make the new y the subject. $2y = x - 6$

$y = \frac{1}{2}x - 3$

3 Write the inverse function as $f^{-1}(x)$. $f^{-1}(x) = \frac{1}{2}x - 3$

4 Find some points on the graphs. $f(-1) = 4, f(-5) = -4,$
$f^{-1}(-4) = -5, f^{-1}(4) = -1$

5 Draw the graphs in different colours.



The graph of an inverse function is the graph of the original function reflected **in the line $y = x$**.

EXERCISE 2.2 Functions and graphs

ANSWERS p. 492

Recap

1 Consider the 3 relations X = {(3, 5), (−2, 3), (0, 4), (2, 5), (5, 3)}, Y = {(6, 2), (2, 5), (3, 7), (6, 3), (7, 3)} and Z = {(2, 2), (1, 4), (3, 7), (0, 0), (5, 8)}.

Which of the following is true?

A None of them are functions.
B Only Z is a function.
C Y and Z are functions.
D X and Z are functions.
E All of them functions.

2 Consider the 3 relations

$f: 3y^3 - 4x^2 = 1 \qquad g: 5x^3 - 4y^2 = 1 \qquad h: \frac{y-5}{x+4} = 3$

Which of the following is true?

A Only h is a function.
B f and h are functions.
C They are all functions.
D g and h are functions.
E None of them are functions.

Mastery

3 WORKED EXAMPLE 4 A call-out fee for a tradesperson normally covers some repair time. A whitegoods repairer's call-out fee covers the first half-hour and subsequent time is charged in half-hour lots.

Some recent jobs cost \$256, \$415, \$626 and \$312, exclusive of GST. They took 2 h, 1 h, 3 h and $1\frac{1}{2}$ h respectively. The cost of parts were \$35, \$242, \$357 and \$115 in the same order.

a Draw a graph of the call costs (less parts) against time.

b Find the equation of the line, where T is the total cost of repair and t is the time taken for the repair.

c What is the hourly rate?

d What is the call-out fee?

e What is the cost for a job taking $2\frac{1}{2}$ h with parts costing \$153, exclusive of GST?

f What is the cost including GST?

4 WORKED EXAMPLE 5 A group of students used the school kitchens to make 102 sushi rolls to sell at a school function. They were selling the sushi rolls for \$2.50 each.

One of the students worked out they would have to sell over 47 sushi rolls to make a profit.

a What is the gradient of the graph showing the profit against the number of sushi rolls sold (n)?

b Write the profit function in terms of the number of sushi rolls sold.

c By 4 pm, they had sold 87 sushi rolls. How much profit did they make?

d After 4 pm, they sold the rest at \$1.50 each. Draw a graph showing the profit against the number sold from $n = 0$ to $n = 102$.

e What was the profit as a percentage of costs for the full day? Round your answer to the nearest percentage.

5 WORKED EXAMPLE 6 Sketch the graph of each function for the specified domain.

a $f(x) = (x - 2)^2$ for $-1 \le x \le 5$

b $g(x) = (x - 2)^2 - 3$ for $-1 \le x \le 5$

c $h(x) = (x + 2)(4 - x)$ for $-4 \le x \le 6$

d $f(x) = (x + 2)(4 - x) + 3$ for $-4 \le x \le 6$

e $g(x) = 6 - 4x - 2x^2$ for $-5 \le x \le 3$

6 Using CAS 1 Sketch the graph of each function, comment and approximately locate its significant features.

a $f(x) = 9x - x^3$

b $g(x) = (x - 1)(x + 2)(x - 3)$

c $y = \dfrac{1}{x - 2}$

d $f(x) = \dfrac{x - 3}{(x + 1)^2(x - 4)}$

e $y = 3^x$

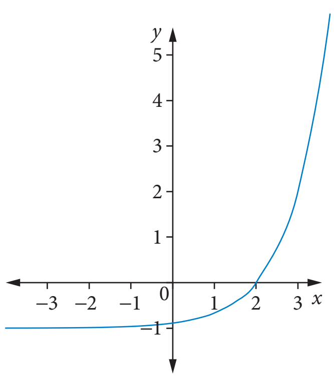
f $y = 3^x - 2$

g $f(x) = 2^{-(x+1)}$

h $g(x) = 3 - 4^{x-2}$

7 WORKED EXAMPLE 7 Identify and approximately locate significant features of each graph.

a



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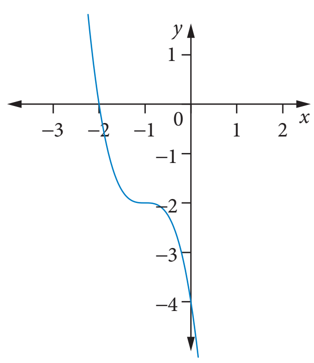

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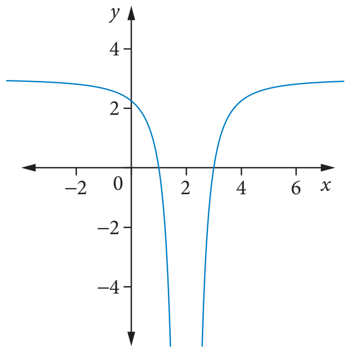

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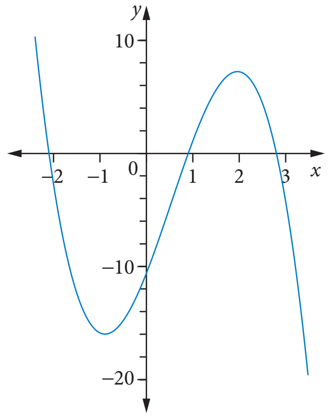

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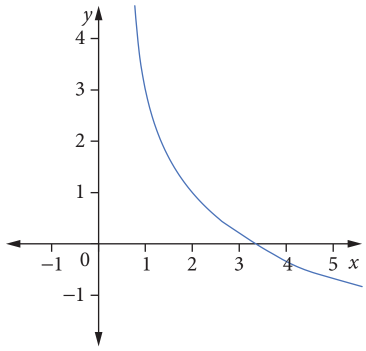
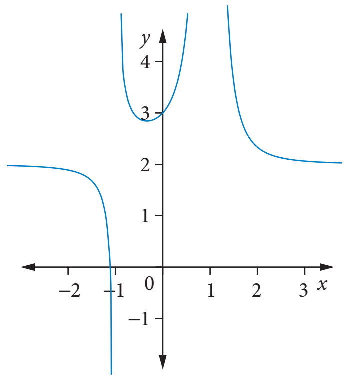
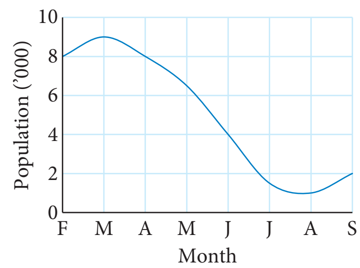


f



8 The population of mice in a grassed paddock over a period of 7 months is on the right.

Describe the characteristics of the graph and state possible reasons for the shape.

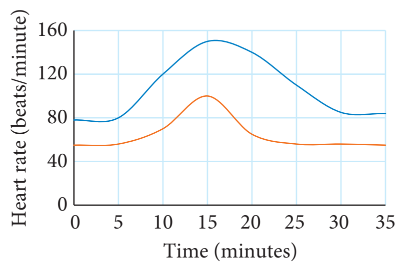


9 The graph on the right shows the heart rates of two people, one fit and the other unfit, before, during and after a period of 10 minutes exercise.

a Describe the characteristics of the graphs.

b When did they start exercising?

c Comment on their recovery times.



10 WORKED EXAMPLE 8 State whether each function has an inverse function. For those that do not, suggest how the domain could be restricted so that it does have an inverse function.

a $\{(-3, 5), (-1, 6), (0, 4), (1, 3), (4, -6), (3, 1)\}$

b $\{(-4, 2), (-2, -1), (1, 4), (3, 4), (5, 5), (7, 5), (8, -4)\}$

c $f(x) = \frac{1}{3}x - 2$

d $f(x) = 1 - x^2 + 4x$

e $f(x) = \frac{1}{x-3} + 2$

11 WORKED EXAMPLE 9 Find the inverse function of each function and sketch both, together with the line $y = x$.

a $f: \{(-1, -3), (-2, -2), (1, 3), (2, 4), (3, 6), (5, -4)\}$

b $f(x) = \dfrac{1}{4}x + 2$

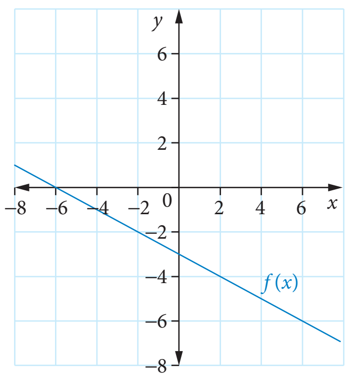
c $f(x) = 3x - 4$

d $f(x) = \dfrac{1}{x} + 2$

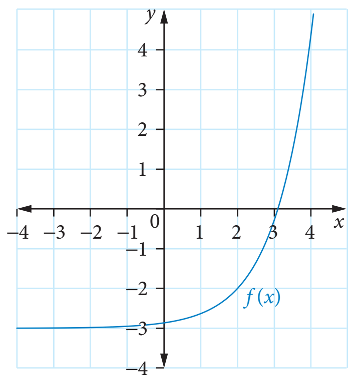
e $f(x) = (x - 2)^3 + 1$

12 Copy each graph and sketch its inverse function in a different colour.

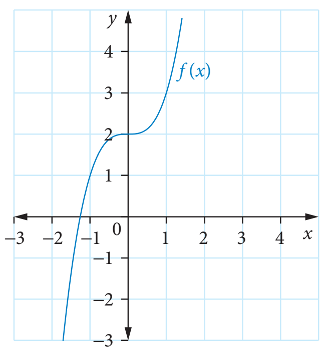
a



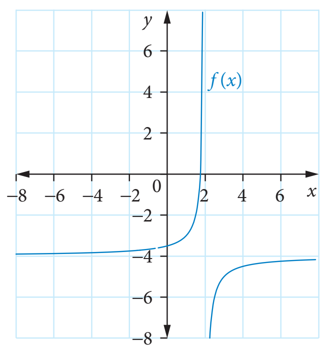
b



c



d



Exam practice

13 TECH-FREE (4 marks) Two quadratic functions are given by $f(x) = 2x^2 - 4x + c$ and $g(x) = bx - 2x^2 + 5$.

a What are the values of b and c if they intersect at (5, 5)? 1 mark

b What is the value if $b = c$ and they intersect when $x = 3$? 1 mark

c If $b = c$, write a relationship between the coordinates of the point of intersection (f, g). 2 marks

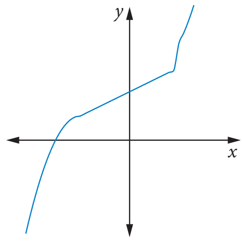
14 TECH-FREE (2 marks) Find the inverse function of $f: R^+ \to R, f(x) = 3 - \dfrac{\sqrt{x}}{2}$.

15 Consider the 3 functions $f(x) = 5x^3 - 3$, $g(x) = 5x^2 + 2$ and $h(x) = \dfrac{1}{x - 3} + 2$. Which functions are one-to-one?

A f, g and h **B** Only f **C** Only f and h

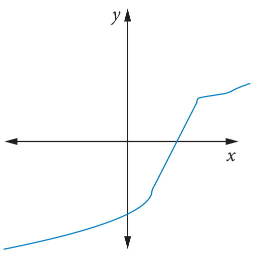
D Only h **E** None of them

16 Part of the graph of the function $f(x)$ is shown. The axes are shown at the same scale.

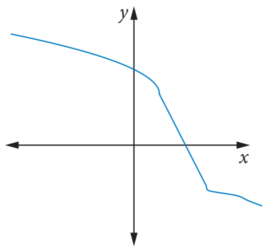


The graph of $f^{-1}(x)$ is best represented by

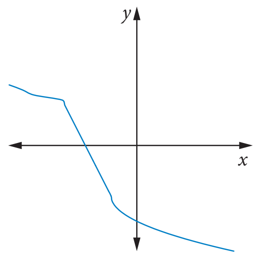
A



B



C



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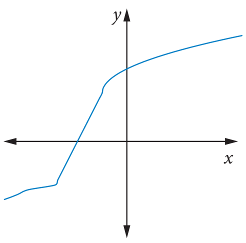

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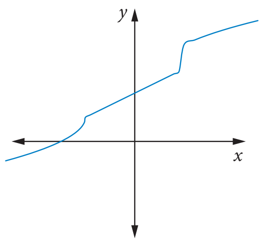


2.3

2.3 Domain and range

The range of a function is the set of y values that the domain produces. The y values are the images of the x values, so the range is the image of the domain.

The **implied domain** of a function such as $f(x) = \sqrt{x - 3}$ is the set of all real values of x for which the function is defined. This is also called the **maximal domain** or the **natural domain**.

We can use **interval notation** to specify the domain or range of a function.

[3, 5] means the real numbers from 3 to 5, including 3 and 5.

(3, 5] means the real numbers from 3 to 5, excluding 3 but including 5.

[4, ∞) means all the real numbers above 4, including 4.

(−∞, 6) means all the real numbers below 6, but not including 6.

Interval notation

- An interval is a set of all the real numbers from a lower to an upper boundary.
- The numbers are separated by a comma and enclosed in round or square brackets.
- A square bracket means the boundary is included. A round bracket means the boundary is excluded.
- The interval $[a, b)$ is the set of real numbers from a to b, including a but not b.
- The infinity symbol and a round bracket ', ∞)' is used for a set with no upper bound.
- Negative infinity '(−∞,' is used for a set with no lower bound.

Video playlist
Domain and range

We can also use set notation to specify the domain or range of a function. For example, $[3, 5] = \{x \in R: 3 \le x \le 5\}$, $(3, 5] = \{x \in R: 3 < x \le 5\}$, $[4, \infty) = \{x \in R: x \ge 4\}$ and $(-\infty, 6) = \{x \in R: x < 6\}$. The set exclusion symbol '\' is useful if only a few elements are excluded. $R\backslash\{0\}$ means all the real numbers except 0. $R\backslash\{-1, 0, 1\}$ means the real numbers excluding −1, 0 and 1. The set symbols make the value(s) a set.

Interval notation is commonly used to define or restrict the domain of a function.

$f: (0, 5] \to R, f(x) = \dfrac{5-x}{x}$ means the domain is restricted to the real numbers from 0 to 5, excluding 0, but including 5 and the co-domain is R.

WB

p. 31

WORKED EXAMPLE 10 Maximal domain of a function

What is the maximal domain of each of these functions?

a $y = \sqrt{x-3}$ **b** $h(x) = \dfrac{1}{x-4}$

Steps	Working
a **1** When is $\sqrt{x-3}$ defined?	$x - 3$ cannot be negative. $x - 3 \ge 0$ $x \ge 3$
2 Write the answer.	The maximal domain of $y = \sqrt{x-3}$ is $[3, \infty)$.
b **1** When is $\dfrac{1}{x-4}$ defined?	x cannot be 4, but can be any other value.
2 Write the answer.	The maximal domain of $h(x)$ is $R\backslash\{4\}$.

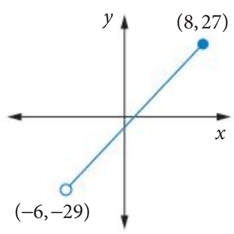
In Worked example 10 we could write the maximal domain of $h(x)$ as $\{x \in R: x \ne 4\}$ or $(-\infty, 4) \cup (4, \infty)$.

WB

p. 32

WORKED EXAMPLE 11 Range of a linear function with a restricted domain

What is the range of $g: (-6, 8] \to R, g(x) = 4x - 5$?

Steps	Working
1 $g(x)$ is a line between $x = -6$ and $x = 8$, so find the y values at the endpoints.	$g(-6) = -29$ and $g(8) = 27$ 
2 Write the range.	The range of g is $(-29, 27]$.

We need to check the maximum or minimum of quadratic functions to find the range.

2.3

WB p. 33

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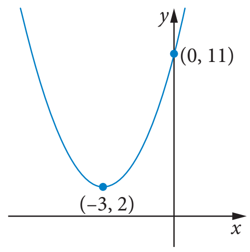
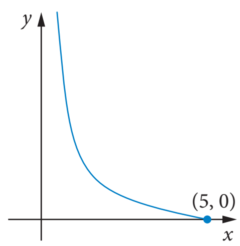
WORKED EXAMPLE 12 Range of a quadratic function

Find the range of the function given by $f: R \to R, f(x) = x^2 + 6x + 11$.

Steps	Working
1 Complete the square.	$f(x) = x^2 + 6x + 11$ $= x^2 + 6x + 9 + 2$ $= (x+3)^2 + 2$
2 State the turning point.	There is a minimum at $(-3, 2)$.
3 Sketch the graph.	
4 Write the range.	The range of f is $[2, \infty)$.

WORKED EXAMPLE 13 Range of a restricted function

Find the range of the function given by $f: (0, 5] \to R, f(x) = \dfrac{5-x}{x}$.

Steps	Working
1 Check values close to 0.	$f(0.1) = \dfrac{4.9}{0.1} = 49$ $f(0.001) = \dfrac{4.999}{0.001} = 4999$
You could write 'As $x \to 0, f(x) \to \infty$'.	As $x \to 0, f(x)$ increases without bound.
2 Check values close to and equal to 5.	$f(4.9) = \dfrac{0.1}{4.9} = 0.0204\ldots$ $f(4.999) = \dfrac{0.001}{4.999} = 0.000\,20\ldots$ $f(5) = \dfrac{0}{5} = 0$
3 Check other values.	$5 - x$ and x are both positive for $0 < x < 5$, so $f(x)$ is defined and positive for $0 < x < 5$.
4 Sketch the graph.	
5 Write the range.	The range is $[0, \infty)$.

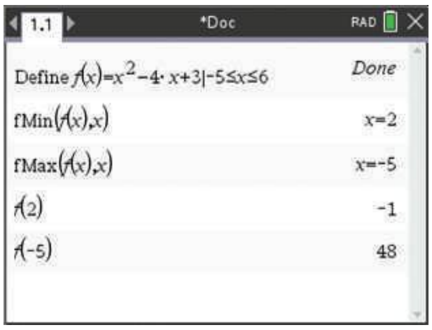
The function $f: (0, 5] \to R, f(x) = \dfrac{8-x}{x}$ has the range $\left[\dfrac{3}{5}, \infty\right)$ instead of $[0, \infty)$.

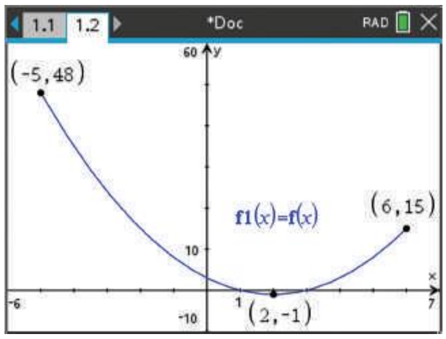
We need to consider possible higher and lower values between the endpoints for non-linear functions such as $f: [-5, 6] \to R, f(x) = x^2 - 4x + 3$.

USING CAS 2 Domain and range

Find the range of $f: [-5, 6] \to R, f(x) = x^2 - 4x + 3$.

TI-Nspire





1 Press **menu > Actions > Define**.

2 Define the function as shown. Press **ctrl + =** to access the mini-palette for the | and ≤ symbols.

3 Press **menu > Calculus > Function Minimum**.

4 Enter **f (x),x** to determine the *x* value when the function is a minimum.

5 Repeat by selecting **Function Maximum** to determine the *x* value when the function is a maximum.

6 Substitute these values into the function to determine the corresponding *y* values.

7 Add a **Graphs** page.

8 Press **menu > Window/Zoom > Window Settings** and adjust the window settings for these minimum and maximum values.

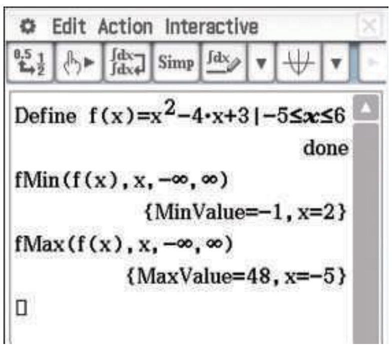
9 Press **menu > Trace**.

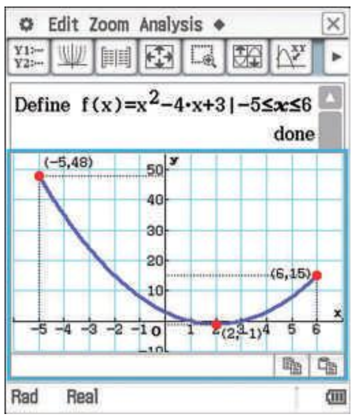
10 When you enter an *x* value, the cursor will jump to that point.

11 Press **enter** to label the coordinates of the point.

12 Use **Trace** to confirm the minimum and maximum values of the function.

ClassPad





1 Enter and highlight the expression $\boldsymbol{x^2 - 4x + 3}$.

2 Tap **Interactive > Define** to get $f(x) = x^2 - 4x + 3$.

3 Open the **Keyboard > Math3** and access the | and ≤ symbol to include the domain.

4 Copy and paste $f(x)$ to the next line, then tap **Interactive > Calculation > fMin/fMax > fMin**.

5 The minimum *y* value and corresponding *x* value will be displayed.

6 Repeat by selecting **fMax** to determine the maximum *y* and *x* values.

7 Open the **Graph** window and drag down from the **left side of screen** with the stated domain.

8 Tap **View Window** and adjust the window settings for the minimum and maximum values.

9 Tap **Analysis > G-Solve > Min** to find the local minimum.

10 Tap **Analysis > G-Solve > x-Cal/y-Cal** or **Analysis > G-Solve > fMax** to identify endpoints at $x = -5$ and $x = 6$.

11 Press **EXE** to label the coordinates of the points on the graph.

The range of the function is $[-1, 48]$.

We can work backwards to find the domain of a function when we are given the range.

WORKED EXAMPLE 14	**Domain and range**
What is the domain D of $h(x) = 5 - 4x$ if the range is $[3, 29)$?	
Steps	**Working**
1 Find the endpoint on the line when $y = 3$.	$3 = 5 - 4x$ $x = 0.5$ endpoint $(0.5, 3)$
2 Find the endpoint on the line when $y = 29$ (but excluded from domain).	$29 = 5 - 4x$ $x = -6$ other endpoint $(-6, 29)$
3 Write the domain.	Line goes from points $(0.5, 3)$ included, to $(-6, 29)$ excluded, so the domain is $(-6, 0.5]$.

EXERCISE 2.3 Domain and range

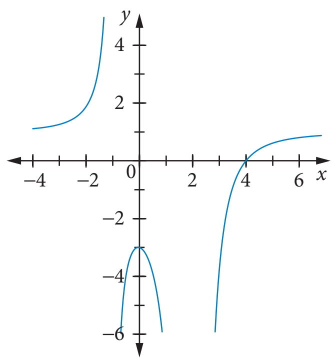
ANSWERS p. 496

2.3

WB p. 35

Recap

1 Part of the graph of $f(x)$ is shown.



Which of the following features are correct?

I There is an asymptote at about $x = 1$.

II There is an asymptote at about $y = 1$.

III There is an asymptote at about $x = 2$.

IV There is an asymptote about $y = 2$.

V There is a point of inflection at about $(-2.3, 1.4)$.

A **I**, **IV** and **V**

B **I** and **IV** only

C All of **I**, **II**, **III**, **IV** and **V**

D All except **IV**

E **II** and **III** only

2 $f(x) = 0.5x - 2$. Find $f^{-1}(x)$.

A $2y + 2$ **B** $2x + 2$ **C** $2x + 4$ **D** $2y + 4$ **E** $2x - 4$

Mastery

3 WORKED EXAMPLE 10 What is the maximal domain of each function?

a $y = \dfrac{1}{x+5}$ **b** $f(x) = \sqrt[3]{x+1}$ **c** $f(x) = \dfrac{1}{x^2 - 4}$

d $g(x) = \dfrac{1}{4x^2 + 1}$ **e** $y = \sqrt{x^2 - 16}$ **f** $g(x) = x^2\sqrt{x}$

g $h(x) = \sqrt{(x+2)(x-3)}$ **h** $y = \sqrt{(3+x)(5-x)}$ **i** $f(x) = \dfrac{1}{\sqrt{7-x}}$

4 WORKED EXAMPLES 11–13 What is the range of each function?

a $f: R \to R, y = 25 - x^2$

b $f: R \to R, y = x^2 - 4x + 3$

c $f: R \to R, y = 8 + 2x - x^2$

d $f: R \to R, y = \dfrac{1}{x + 2}$

e $f: R \to R, y = \dfrac{1}{(x - 4)^2}$

f $g: [3, 7] \to R, y = 2x - 10$

g $f: (-4, 4) \to R, y = 5x + 3$

h $h: [-4, 9) \to R, y = 6 - 3x$

i $g: [0, \infty) \to R, y = x^3 + 4$

j $h: [-4, 4) \to R, y = (x - 6)(x + 2)$

5 Using CAS 2 Find the range of each function.

a $f: [-4, 4] \to R, y = 6 - x - x^2$

b $f: (3, 10) \to R, x^2 - 7x - 8$

c $g: [-2, 4] \to R, y = 2^x + 5$

d $h: (-4, 5] \to R, \dfrac{4}{x^2 - 36}$

e $f: (-3, 3] \to R, y = \log_{10}(x + 5)$

f $g: (3, 10) \to R, y = x^3 - 12x^2 + 41x - 42$

6 WORKED EXAMPLE 14 Given the range of each function, find its domain.

a $f(x) = 5x - 7$, range: $(-17, 3]$

b $g(x) = 7 - 2x$, range: $[-5, 5)$

c $h(x) = 4 - 3x$, range: $(3, 8]$

d $k(x) = 4x - 5$, range: $[-3, 6)$

e $m(x) = \dfrac{1}{x}$, range: $(0.2, 10]$

f $p(x) = \dfrac{1}{2 - x}$, range: $[-4, -0.5) \cup [0.25, 1)$

7 What is the implied domain of $y = \sqrt{x + 3}$?

8 What is the natural domain of $f(x) = \dfrac{3}{\sqrt{3 - x}}$?

Exam practice

80–100% 60–79% 0–59%

9 TECH-FREE (2 marks) State the natural domain and range of $f(x) = \sqrt{x^2 - 4}$.

10 © VCAA 2016 2AQ1 92% The linear function $f: D \to R, f(x) = 5 - x$ has range $[-4, 5)$.

The domain D is

A $(0, 9]$ **B** $(0, 1]$ **C** $[5, -4)$ **D** $[-9, 0)$ **E** $[1, 9)$

11 © VCAA 2014 2AQ2 80% The linear function $f: D \to R, f(x) = 4 - x$ has range $[-2, 6)$.

The domain D of the function is

A $[-2, 6)$ **B** $(-2, 2]$ **C** R **D** $(-2, 6]$ **E** $[-6, 2]$

12 © VCAA 2012 2AQ3 69% The range of the function $f: [-2, 3) \to R, f(x) = x^2 - 2x - 8$ is

A R **B** $(-9, -5]$ **C** $(-5, 0)$ **D** $[-9, 0]$ **E** $[-9, -5)$

9780170448260

13 © VCAA 2015 2AQ7 56% The range of the function $f: (-1, 2] \to R, f(x) = -x^2 + 2x - 3$ is

A R **B** $(-6, -3]$ **C** $(-6, -2]$ **D** $[-6, -3]$ **E** $[-6, -2]$

14 Consider the function $f: (a, b] \to R, f(x) = \dfrac{b - x}{a - x}$.

The range of f is

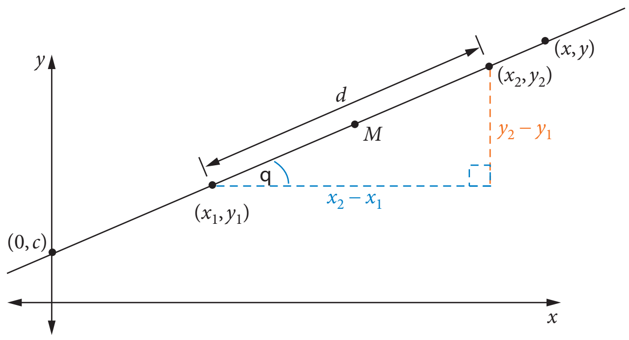
A $(-\infty, 0]$ **B** $(-\infty, 0)$ **C** $[0, \infty)$ **D** $(0, \infty)$ **E** $\left(-\infty, \dfrac{b}{a}\right]$

2.4

2.4 Linear functions

Equations of a line

A **linear function** has the form $f(x) = mx + c$. Its graph is a straight line with gradient m and y-intercept c.



The **distance** d between any 2 points (x_1, y_1) and (x_2, y_2) is given by Pythagoras' theorem, so $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

The **midpoint** of any 2 given points (x_1, y_1) and (x_2, y_2) is $M\left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$.

For any 2 points (x_1, y_1) and (x_2, y_2), the **gradient** is the ratio of the vertical and horizontal distances between the points, the **rise** and **run**.

$$m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1} = \tan(\theta)$$

where θ is the **inclination**, the angle between the line and the positive x direction.

For any general point (x, y) and any other point (x_1, y_1) on the line, $m = \dfrac{y - y_1}{x - x_1}$, so $y - y_1 = m(x - x_1)$.

For the ***y*-intercept** $(0, c)$, this becomes $y - c = mx$ or $y = mx + c$.

Equations of a straight line

The **general form** of a straight line is $ax + by + c = 0$, where a, b and c are integers and a is positive.

The **gradient-intercept form** is $y = mx + c$, where m is the gradient and c is the y-intercept.

The **point-gradient form** is $y - y_1 = m(x - x_1)$, where m is the gradient and (x_1, y_1) is a point on the line.

The **intercept form** is $\dfrac{x}{a} + \dfrac{y}{b} = 1$, where a is the x-intercept and b is the y-intercept.

Video playlist
Linear functions

Worksheets
Graphing linear functions

x- and y-intercepts

Equations of lines

Linear functions code puzzle

Finding the equation of a line

$y = mx + c$

WB

p. 36

WORKED EXAMPLE 15 Equation of a straight line

Find the equation of the straight line that

a passes through (2, –1) and has gradient –2

b has y-intercept 5 and gradient 2

c passes through (–2, 3) and (4, 1)

d has x-intercept 2 and y-intercept –5.

Steps	Working
a **1** Write the point-gradient form.	$y - y_1 = m(x - x_1)$
2 Substitute values.	$y - (-1) = -2(x - 2)$
3 Simplify and express in general form.	$y + 1 = -2x + 4$ $2x + y - 3 = 0$
b **1** Write the point-gradient form.	$y - y_1 = m(x - x_1)$
2 Substitute $m = 2$ and the point (0, 5).	$y - 5 = 2(x - 0)$
3 Simplify and express in general form.	$y - 5 = 2x$ $2x - y + 5 = 0$
c **1** Find the gradient.	$m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{1 - 3}{4 - (-2)} = -\dfrac{1}{3}$
2 Write the point-gradient form.	$y - y_1 = m(x - x_1)$
3 Substitute the 'simpler' point, simplify and express in general form.	$y - 1 = -\dfrac{1}{3}(x - 4)$ $3y - 3 = -x + 4$ $x + 3y - 7 = 0$
d **1** Find the gradient.	$m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{-5 - 0}{0 - 2} = \dfrac{5}{2}$
2 Write the point-gradient form.	$y - y_1 = m(x - x_1)$
3 Substitute the 'simpler' point, simplify and express in general form.	$y - 0 = \dfrac{5}{2}(x - 2)$ $2y = 5x - 10$ $5x - 2y - 10 = 0$

2.4

Worksheets
Parallel and perpendicular lines

Linear functions

WB
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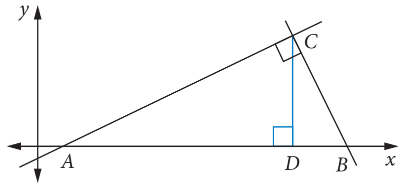
Parallel and perpendicular lines

Parallel lines are in the same direction, so they have the same inclination.

Since $m = \tan(\theta)$, **parallel lines** have the **same gradient**, so $m_1 = m_2$.

Consider 2 **perpendicular lines** AC and BC on the number plane, as shown on the right. It can be shown that $\triangle ACD$ and $\triangle DBC$ have corresponding equal angles, so the triangles are similar.



The gradient of line AC, $m_1 = \dfrac{CD}{AD}$ and the gradient of CB, $m_2 = -\dfrac{CD}{BD}$.

But in similar triangles $\triangle DBC$ and $\triangle ACD$, the ratios of the lengths of corresponding sides are equal, so $\dfrac{CD}{BD} = \dfrac{AD}{CD}$.

$$m_2 = -\frac{CD}{BD} = -\frac{AD}{CD}$$

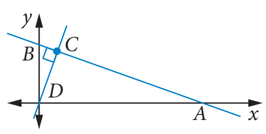
$$\therefore m_1 m_2 = \frac{CD}{AD} \times \left(-\frac{AD}{CD}\right) = -1$$

Hence, for perpendicular lines, $m_1 m_2 = -1$, or $m_2 = -\dfrac{1}{m_1}$.

We will usually use perpendicular or parallel lines incidentally in problems.

WORKED EXAMPLE 16 Using perpendicular lines

The linear function $x + 2y - 10 = 0$ has x and y intercepts A and B respectively. C is the point that divides AB in the ratio 4 : 1. Show that the line perpendicular to AB through C also passes through the origin $O(0, 0)$.

Steps	Working
1 Sketch a diagram.	
2 Find A and B.	At $y = 0$, $x = 10$, so $A = (10, 0)$. At $x = 0$, $y = 5$, so $B = (0, 5)$.
3 Find C.	C is $\frac{4}{5}$ of the way from A to B, so $C = A + \frac{4}{5}(-10, 5) = (10, 0) + (-8, 4) = (2, 4)$
4 Find the gradient of AB.	$m_{AB} = -\frac{5}{10} = -\frac{1}{2}$
5 Find the gradient of the perpendicular line, CD.	$m_{AB} \times m_{CD} = -1$ so $m_{CD} = 2$
6 Find the equation of CD.	$y - y_1 = m(x - x_1)$ $y - 4 = 2(x - 2)$ $y - 4 = 2x - 4$ $2x - y = 0$
7 Substitute 0 in the LHS.	LHS $= 2 \times 0 - 0 = 0 =$ RHS
8 State the conclusion.	The perpendicular line through C passes through the origin.

Lines **parallel to the x-axis** are of the form $y = b$, where (a, b) is on the line.

Lines **parallel to the y-axis** are of the form $x = a$, where (a, b) is on the line.

EXERCISE 2.4 Linear functions

ANSWERS p. 496

Recap

1 The natural domain of $f(x) = \sqrt{9 - x^2}$ is

A $(-\infty, 9]$ **B** $[-9, 9]$ **C** $(-3, 3)$ **D** $[-3, 3]$ **E** $[-3, 0) \cup (0, 3]$

2 The range of $f: R \to R, f(x) = 5 - 3x$ is $(-4, 8]$. The domain is

A $[1, 3)$ **B** $[-1, 3)$ **C** $(-1, 3]$ **D** $(-3, 1]$ **E** $[-3, 1)$

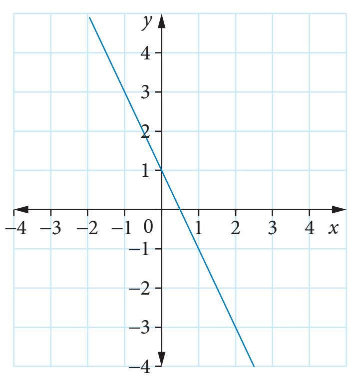
Mastery

3 WORKED EXAMPLE 15 Find the equation of the line

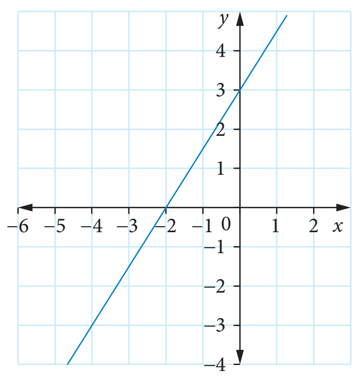
- **a** with gradient 2 and y-intercept 1
- **b** through $(-2, 5)$, with gradient $-\frac{3}{4}$
- **c** through $(3, 6)$ and $(-2, 7)$
- **d** with x-intercept 2 and y-intercept -4
- **e** through $(3, 5)$ and parallel to the x-axis
- **f** with gradient -1 and through $(3, -4)$
- **g** with x-intercept -3 and y-intercept -1
- **h** parallel to the y-axis through $(-5, 4)$
- **i** with gradient $-\frac{3}{4}$ and y-intercept -3
- **j** through $(1, -2)$ and $(-4, 2)$
- **k** with gradient $\frac{2}{3}$ and y-intercept $-\frac{1}{4}$
- **l** with gradient $\frac{1}{3}$ and through $(-1, -2)$
- **m** through $(-6, 4)$ and $(9, -1)$
- **n** with x-intercept -6 and y-intercept 2
- **o** through $(-4, -7)$ and $(-3, -5)$
- **p** with x-intercept 3 and y-intercept 3
- **q** through $(1, -7)$ and parallel to the y-axis
- **r** through $(5, 3)$, with gradient $\frac{5}{2}$
- **s** with gradient -3 and y-intercept 2
- **t** parallel to the x-axis through $(-2, -6)$

4 Write the equation of each line in the general form, $ax + by + c = 0$.

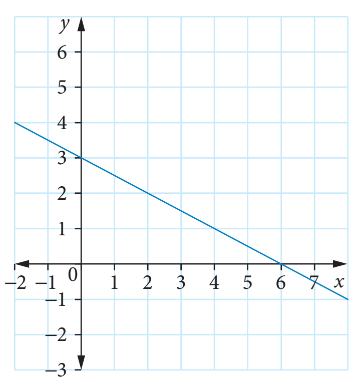
a



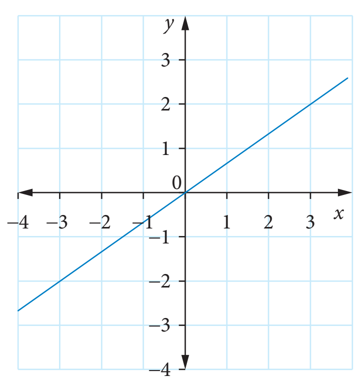
b



c



d



9780170448260

5 Find, correct to the nearest degree, the positive angle each line makes with the positive direction of the x-axis.

a $x - y + 3 = 0$ **b** $2x + y - 7 = 0$ **c** $x - \sqrt{3}\,y + 5 = 0$

d $x - 3y - 1 = 0$ **e** $x + y + 6 = 0$ **f** $\sqrt{3}\,x + 2y - 10 = 0$

2.4

6 Find the midpoint of each pair of points.

a (8, 9) and (2, –5) **b** (–6, –3) and (5, –4) **c** (–5, –2) and (8, 8)

d (–7, 9) and (–3, –1) **e** (2, 4) and (5, 9) **f** (–1, –3) and (–1, –9)

7 The midpoint of (9, –7) and A is (6, –5). Find the coordinates of A.

8 Find the distance between each pair of points.

a (–1, 5) and (0, 2) **b** (–6, 7) and (–2, 5) **c** (2, 5) and (–2, 2)

d (–7, 4) and (3, –6) **e** (–4, 10) and (4, –5) **f** (3, 7) and (7, 13)

9 The distance from (a, –2) to (5, 0.5) is 6.5 units. Find the possible values of a. Give your answer to the nearest whole number.

10 The distance between (a, 5) and (7, b) is 10 units. Find the relationship between a and b.

11 WORKED EXAMPLE 16 Find the lines perpendicular to

a $3x + 5y - 7 = 0$ through (1, –4) **b** $2x - y + 4 = 0$ through (–2, –2)

c $4x + 3y + 5 = 0$ through (–3, 4) **d** $x - 3y - 1 = 0$ through (2, 5)

12 Find the lines through

a (2, 5) parallel to $2x + 4y + 1 = 0$ **b** (3, –1) parallel to $x - 3y - 2 = 0$

c (–2, –6) parallel to $3x - 2y + 4 = 0$ **d** (–5, 2) parallel to $5x + 3y - 5 = 0$

13 The equations of 6 lines are as follows

L_1: $2x + 3y - 5 = 0$ L_2: $3x + 2y + 6 = 0$ L_3: $5x - 3y + 2 = 0$

L_4: $3x + 2y - 1 = 0$ L_5: $2x + 4y - 3 = 0$ L_6: $3x + 5y - 7 = 0$

a State which, if any, of the lines are parallel.

b State which, if any, of the lines are perpendicular.

Exam practice

80–100% 60–79% 0–59%

14 TECH-FREE (2 marks) $A = (6, 5)$, $B = (1, 2)$, $C = (3, 6)$ and $D = (8, 9)$. Show that $ABCD$ is a parallelogram.

15 TECH-FREE (2 marks) $ABCD$ is a rhombus, where $A = (-8, -7)$, $B = (-3, 5)$ and $C = (9, 10)$. Find the coordinates of D.

16 © VCAA 2013 2AQ2 74% The midpoint of the line segment that joins (1, –5) to (d, 2) is

A $\left(\frac{d+1}{2}, -\frac{3}{2}\right)$ **B** $\left(\frac{1-d}{2}, -\frac{7}{2}\right)$ **C** $\left(\frac{d-4}{2}, 0\right)$ **D** $\left(0, \frac{1-d}{3}\right)$ **E** $\left(\frac{5+d}{2}, 2\right)$

17 The equation of the straight line through (2, –4) with gradient $-\frac{4}{5}$ is

A $4x + 5y + 12 = 0$ **B** $4x + 5y - 28 = 0$ **C** $4x + 5y - 12 = 0$

D $4x - 5y - 12 = 0$ **E** $4x - 5y - 4 = 0$

18 The equation of the line that passes through (2, 5) and (5, −3) is

A $8x+3y+1=0$ **B** $8x+3y-31=0$ **C** $8x-3y-1=0$

D $8x-3y-31=0$ **E** $8x+3y+31=0$

19 © VCAA 2011 2AQ2 56% The gradient of a line perpendicular to the line which passes through (−2, 0) and (0, −4) is

A $\frac{1}{2}$ **B** -2 **C** $-\frac{1}{2}$ **D** 4 **E** 2

Video playlist
Quadratic functions

Worksheets
Sketching quadratic functions
Graphing quadratic functions
Quadratic functions
Graphing quadratics
Features of a parabola
Matching parabolas
The discriminant

2.5 Quadratic functions

A **quadratic function** has the general form $f(x)=ax^2+bx+c$. The shape of its graph is a **parabola**. It has a vertical **axis of symmetry** through the turning point.

The process of changing to the form $f(x)=a(x-h)^2+k$ is called **completing the square**.

Completing the square

1 Separate the constant and take the coefficient of x^2 as a common factor of the others.
2 Add and subtract the constant term that completes the square of the x^2 and x terms.
3 Simplify the constant terms.
4 Then $f(x)=a(x-h)^2+k$ gives the turning point (h, k).

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WORKED EXAMPLE 17 Completing the square

Find the turning point of $f(x)=3x^2-5x-4$.

Steps	Working
1 Separate the constant and take the coefficient of x^2 as a factor.	$f(x)=3x^2-5x-4$ $=3\left(x^2-\frac{5}{3}x\right)-4$
2 Complete the square.	$=3\left(x^2-\frac{5}{3}x+\left(\frac{5}{6}\right)^2-\left(\frac{5}{6}\right)^2\right)-4$
3 Simplify the constant terms.	$=3\left(x-\frac{5}{6}\right)^2-3\left(\frac{5}{6}\right)^2-4$ $=3\left(x-\frac{5}{6}\right)^2-\frac{25}{12}-4$ $=3\left(x-\frac{5}{6}\right)^2-\frac{73}{12}$
4 State the turning point.	$f(x)$ has a minimum value at $x=\frac{5}{6}$. The turning point is $\left(\frac{5}{6},-\frac{73}{12}\right)$.

For the general form $f(x)=ax^2+bx+c$, completing the square gives $f(x)=a\left(x+\frac{b}{2a}\right)^2-\frac{b^2-4ac}{4a}$.
The solutions of $f(x)=0$ are given by the **quadratic formula**

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

2.5

Quadratic functions

For the quadratic function $f(x) = ax^2 + bx + c$:

- If a is positive, then $f(x)$ is **concave upwards**, like ∪, and has a **minimum**.
- If a is negative it is **concave downwards**, like ∩, and has a **maximum**.
- The maximum or minimum is called the **turning point** or **vertex**.
- The turning point occurs at $x = -\frac{b}{2a}$.
- The x-intercepts, if they exist, are called **zeros** or **roots**.
- The zeros are given by the **quadratic formula** $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.
- The **discriminant**, $\Delta = b^2 - 4ac$, determines the nature of the solutions of $f(x) = 0$.
  - If $\Delta < 0$, there are no zeros and no real roots of the equation.
  - If $\Delta = 0$, there is 1 zero and 1 real root of the equation.
  - If $\Delta > 0$, there are 2 zeros and 2 real roots of the equation.
  - If Δ is a perfect square, the zeros are rational.

Compared to $y = x^2$, the graph of $y = a(x - h)^2 + k$:

- is stretched by the factor a from the x-axis
- has its turning point at (h, k).

WORKED EXAMPLE 18 Graphing $y = a(x - h)^2 + k$

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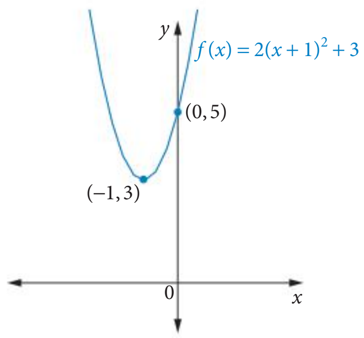
Sketch the graph of $f(x) = 2(x + 1)^2 + 3$.

Steps	Working
1 Compare to $y = x^2$.	$f(x)$ is twice as steep as $y = x^2$. The turning point is $(-1, 3)$.
2 State the shape.	It is concave upwards, so the turning point is a minimum.
3 State the y-intercept.	$f(0) = 2 + 3 = 5$
4 Sketch the graph, giving the coordinates of all significant points.	

Exam hack

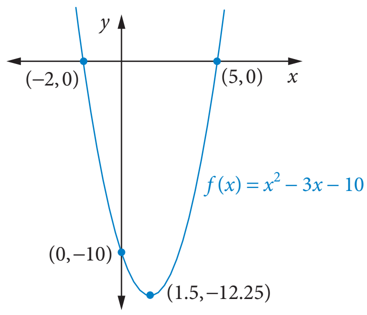
The position of the turning point and y-intercept tell you whether there are x-intercepts.

WB

p. 40

WORKED EXAMPLE 19 Sketching a factorable quadratic function

Sketch the graph of $f(x) = x^2 - 3x - 10$.

Steps	**Working**
1 Factorise if possible and state the zeros.	$f(x) = (x+2)(x-5)$ There are zeros at -2 and 5.
2 The zeros are symmetrical about the turning point.	The turning point is at $x = \frac{-2+5}{2} = \frac{3}{2}$.
3 State the turning point.	$f\left(\frac{3}{2}\right) = -\frac{49}{4}$, so the turning point is $\left(\frac{3}{2}, -\frac{49}{4}\right)$.
4 State the shape.	$a = 1$, so the graph is concave up.
5 State the y-intercept.	$f(0) = -10$
6 Sketch the graph.	

If we can't factorise a quadratic, we may need to complete the square.

WB

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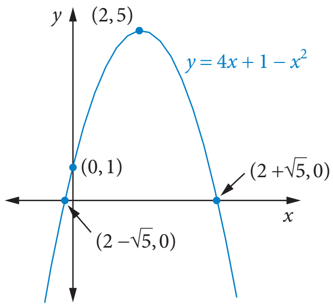
WORKED EXAMPLE 20 Sketching a non-factorable quadratic function

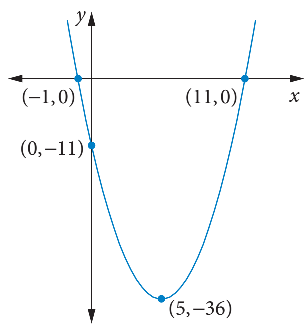
Sketch the graph of $y = 4x + 1 - x^2$.

Steps	**Working**
1 The quadratic won't factorise, so complete the square.	$y = -(x^2 - 4x + 4 - 4) + 1$ $= -(x^2 - 4x + 4) + 4 + 1$ $= -(x-2)^2 + 5$
2 Compare to $y = x^2$.	It has the same steepness as $y = x^2$. The turning point is (2, 5).
3 State the shape.	It is concave down, so the turning point is a maximum.
4 Find the zeros. The approximate value will help sketch the function, but label the graph with the exact values.	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ $= \frac{-4 \pm \sqrt{4^2 - 4 \times -1 \times 1}}{2 \times -1}$ $= \frac{-4 \pm \sqrt{20}}{-2}$ $= \frac{-(4 \mp 2\sqrt{5})}{-2}$ $= 2 \pm \sqrt{5}$
5 State the y-intercept.	At $x = 0$, $y = 1$.

6 Sketch the graph.	

WORKED EXAMPLE 21 Values of a quadratic coefficient

Find the values of b such that $f(x) = x^2 + bx - 5$ and $y = 3x + b$ have no intersections.

Steps	Working
1 Write the equation for intersections.	$f(x) = y$ $x^2 + bx - 5 = 3x + b$ $x^2 + (b - 3)x - 5 - b = 0$
2 Find the discriminant.	$\Delta = (b - 3)^2 + 4 \times 1 \times -(5 + b)$ $= b^2 - 6b + 9 - 20 - 4b$ $= b^2 - 10b - 11$
3 Use the condition for no intersections.	$\Delta < 0$ Δ has roots at $b = \dfrac{10 \pm \sqrt{100 + 44}}{2}$ $= \dfrac{10 \pm 12}{2}$ $= -1, 11$
4 Find other criteria of the graph of Δ.	Δ is concave upwards. Δ has a minimum at $x = 5$. At $b = 5$, $\Delta = -36$. The y-intercept is -11.
5 Sketch the graph of Δ.	
6 State the values of b such that $\Delta < 0$.	$\Delta < 0$ for $-1 < b < 11$.
7 State the answer.	$f(x) = x^2 + bx - 5$ and $y = 3x + b$ have no intersections for $-1 < b < 11$.

2.5

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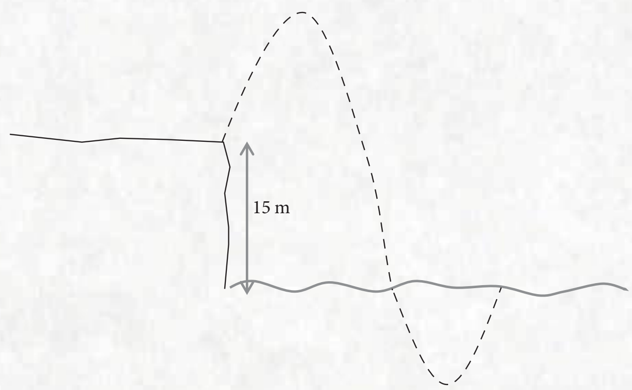
Video
VCE question analysis: Functions and graphs

VCE QUESTION ANALYSIS

A VCE-style question (9 marks)

An object lighter than water is projected from the top of a seaside cliff. It travels in a parabolic path and hits the water 15 metres below its starting point as shown in the diagram.



The equation of the parabolic path is $y = 5x(b - x)$, where x and y are in metres and b is a constant. The starting point is (0, 0). The maximum height of the object is 20 metres above the cliff. After hitting the water, the object's buoyancy makes it follow another parabolic path until it reaches the surface of the water. Its path under the water reaches a depth half of the height it reached above the cliff.

a Find the value of b. 3 marks

b Write an equation for the water level (ignoring any waves). 1 mark

c Find the horizontal distance from the cliff to where the object hits the water, correct to 2 decimal places. 2 marks

d The equation of the object's parabolic path under the water is $y = (x - h)^2 + k$, where x and y are in metres and h and k are constants. Hence, find the equation of this path, using values of h and k correct to two decimal places. 3 marks

Reading the question

- The 2 parabolas have separate quadratic equations.
- The measurements 15 m and 20 m are needed for this question.
- The point of projection at the top of the cliff is (0, 0).

Thinking about the question

- The parabola under the water is concave up, so x^2 has a positive coefficient.
- The water level is parallel to the x-axis (horizontal).
- The turning point of a parabola occurs at $x = -\dfrac{b}{2a}$ and is halfway between the zeros.
- The answer for part **c** is to be approximate, so you can use CAS for intersection of graphs.

Worked solution (✓ = 1 mark)

a $y = 5x(b - x)$ has zeros at $x = 0$ and $x = b$, so the x value of its turning point is $x = \frac{0+b}{2} = \frac{b}{2}$.

or

$y = 5x(b - x) = -5x^2 + 5bx$, so the x value of its turning point is $x = \frac{-5b}{2(-5)} = \frac{-5b}{-10} = \frac{b}{2}$.

$y = 20$ at $x = \frac{b}{2}$ ✓

$$5\left(\frac{b}{2}\right)\left(b - \frac{b}{2}\right) = 20$$

$$\frac{5b^2}{4} = 20 \text{ ✓}$$

$$b^2 = 16$$

$b > 0$ because the other zero must be on the right (positive) side of the starting point.

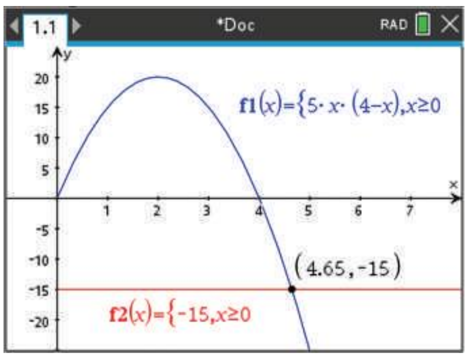
$\boldsymbol{b = 4}$ ✓

b The water level is a horizontal line 15 m below the origin, so its equation is

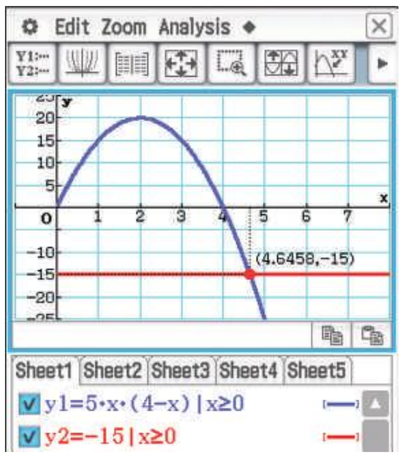
$\boldsymbol{y = -15}$ ✓

c The point where the object hits the water is the intersection of $y = -15$ and $y = 5x(4 - x)$.

TI-Nspire



ClassPad



✓

or $5x(4 - x) = -15$

$$x(4 - x) = -3$$

$$-x^2 + 4x = -3$$

$$x^2 - 4x - 3 = 0$$

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(-3)}}{2(1)}$$

$$= \frac{4 \pm \sqrt{28}}{2}$$

$$= \frac{4 \pm 2\sqrt{7}}{2}$$

$$= 2 \pm \sqrt{7}$$

But $x > 4$, so $x = 2 + \sqrt{7} \approx 4.65$.

It hits the water **4.65 m** horizontally from the cliff. ✓

d The object reaches a depth of $\frac{1}{2} \times 20 = 10$ m under the water.

The parabola passes through the point approximately $(4.65, -15)$ and has minimum $-15 - 10 = -25$.

The equation is of the form $y = (x - h)^2 + k$.

The turning point is (h, k) so $k = -25$. ✓

At $x \approx 4.65$, $y = -15$:

$$-15 = (4.65 - h)^2 - 25 \checkmark$$
$$(4.65 - h)^2 = 10$$
$$h = 4.65 \pm \sqrt{10}$$

The lowest point is at $x > 4.65$, so

$$h = 4.65 + \sqrt{10}$$
$$\approx 7.81$$

The equation is $\boldsymbol{y \approx (x - 7.81)^2 - 25}$ ✓

EXERCISE 2.5 Quadratic functions

ANSWERS p. 496

Recap

1 The equation of the line through $(-2, 3)$ and $(2, -3)$ is

A $3x - 2y + 12 = 0$ **B** $2x - 3y + 13 = 0$ **C** $2x + 3y + 5 = 0$

D $3x + 2y - 12 = 0$ **E** $3x + 2y = 0$

2 The gradient of a line perpendicular to $5x - 3y + 4 = 0$ is

A $-\frac{5}{3}$ **B** $-\frac{4}{3}$ **C** $-\frac{3}{5}$ **D** $\frac{3}{5}$ **E** $\frac{5}{3}$

Mastery

3 WORKED EXAMPLE 17 Find the turning point of each quadratic function.

a $y = x^2 + 4x + 6$ **b** $f(x) = 6x - x^2 - 12$ **c** $y = x^2 - 2x - 4$

d $f(x) = 2 - x^2 - 4x$ **e** $y = 3x^2 + 18x + 5$ **f** $f(x) = 12x - 2x^2 + 3$

4 WORKED EXAMPLE 18 Sketch each quadratic function.

a $f(x) = (x + 2)^2 + 2$ **b** $f(x) = -(x - 2)^2 - 3$ **c** $y = 3(x - 2)^2 - 3$

d $y = 4 - 2(x + 1)^2$ **e** $f(x) = 2\left(x - \frac{3}{2}\right)^2 + 5$ **f** $y = 5 - 0.5(x - 3)^2$

5 WORKED EXAMPLE 19 Sketch each quadratic function.

a $f(x) = 2x + 3 - x^2$ **b** $y = x^2 + 2x - 15$ **c** $f(x) = 3x + 10 - x^2$

d $y = 3x^2 - 5x - 2$ **e** $f(x) = 6x^2 + 11x + 4$ **f** $y = 2x^2 + 5x + 2$

6 WORKED EXAMPLE 20 Sketch each quadratic function.

a $y = x^2 + 4x + 1$ **b** $f(x) = x^2 + 5x - 4$ **c** $y = 1 - 2x^2 - 2x$

d $f(x) = 4x - x^2 - 1$ **e** $y = 4x^2 + 4x - 1$ **f** $f(x) = 5 - 4x^2 - 4x$

7 Sketch each quadratic function.

a $y = x^2 + 5x - 1$ **b** $f(x) = x - x^2 + 1$ **c** $y = x^2 - 2x - 1$

d $f(x) = 2x - x^2 - 1$ **e** $y = 3x^2 + 3x + 1$ **f** $f(x) = 2x - x^2 - 2$

8 WORKED EXAMPLE 21 Find the values of k such that $f(x) = 3x^2 - 4x + 5$ and $g(x) = kx^2 + 2x + 4$ have two points of intersection.

9 Find the values of c so that $y = 3 - x^2 + cx$ and $y = 3x + c$ have 2 points of intersection.

2.5

Exam practice

80–100% 60–79% 0–59%

10 TECH-FREE (3 marks) Sketch the graph of $f(x) = 6x - 2x^2 + 8$, showing all intercepts and other important points.

11 © VCAA 2015 2AQ20 61% If $f(x-1) = x^2 - 2x + 3$, then $f(x)$ is equal to

A $x^2 - 2$ **B** $x^2 + 2$ **C** $x^2 - 2x + 2$ **D** $x^2 - 2x + 4$ **E** $x^2 - 4x + 6$

12 © VCAA 2015 2AQ21 37% The graphs of $y = mx + c$ and $y = ax^2$ will have no points of intersection for all values of m, c and a such that

A $a > 0$ and $c > 0$

B $a > 0$ and $c < 0$

C $a > 0$ and $c > -\dfrac{m^2}{4a}$

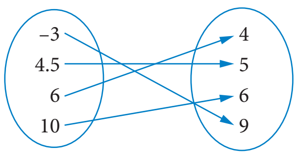
D $a < 0$ and $c > -\dfrac{m^2}{4a}$

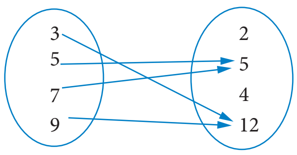
E $m > 0$ and $c > 0$

2 Chapter summary

Functions and relations

- A **relation** R is a set of ordered pairs (x, y), where $x \in X$ and $y \in Y$.
- The set of values of the **independent variable** x is called the **domain** and is often the set X. The set Y is called the **co-domain** and the set of values of the **dependent variable** y is called the **range**.
- A relation can be shown as a rule, a list of ordered pairs, a table, a mapping or a graph.
- A **function** f is a relation $f(x)$ such that each x value in the domain matches with only one y value.
- The **value of a function** at $x = c$ is written as $f(c)$.
- A function is either **one-to-one** or **many-to-one**.
- In a **one-to-one** function, each x value in the domain is matched with a unique y value. In the pairings, there aren't any repeated x or y values.

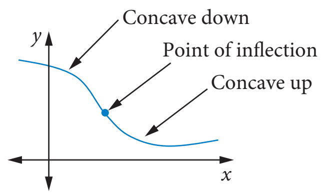




- The function $\{(1, 3), (2, 1), (3, 5), (4, -1)\}$ is one-to-one: every x or y value is in only one ordered pair.
- In a **many-to-one** function, 2 or more x values in the domain may be matched with the same y value. In the pairings, there are repeated y values.
- The function $y = x^2$ is many-to-one: there are many x values that match with the same y value, for example, $\{(7, 49), (-7, 49), (2, 4), (-2, 4), (1, 1), (-1, 1)\}$.
- A function written as $f: D \to R, y = f(x)$ is the function whose domain is the set D, co-domain the set R.

Features of graphs

- A **stationary point** is a point on a graph where the gradient is zero. Any minimum or maximum on a continuous curve must be a stationary point.
- A **point of inflection** is a point where a graph changes from concave up to concave down, or concave down to concave up as shown on the graph.



- A **horizontal asymptote** is a value that the function approaches as $x \to -\infty$ or $x \to \infty$.
- A **vertical asymptote** is a value where, as x approaches, the function becomes arbitrarily large in magnitude. Functions are undefined at vertical asymptotes.

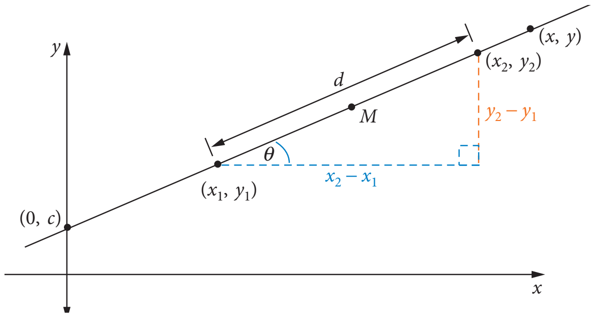
Inverse functions and relations

- The **inverse relation** R^{-1} of a relation R is the set of coordinate pairs with the coordinates in the reverse order (y, x).
- If the inverse relation of a function $f(x)$ is also a function, then it is called the **inverse function** $f^{-1}(x)$.
- For a function to have an inverse it must be **one-to-one**.
- The graph of an inverse function is the reflection of the graph of the original function in the line $y = x$.

Domain and range

- The **range** is the set of y values that the domain produces.
- The **implied domain** of a function is the set of all real values of x for which the function is defined. This is also called the **maximal domain** or the **natural domain**.
- We can use **interval notation** to specify the domain or range of a function.
- A square bracket, [or], means that the end of the interval is included.
- A round bracket, (or), means that the end of the interval is not included.
- The infinity symbol and a round bracket ', ∞)' is used for a set with no upper bound.
- Negative infinity '(−∞,' is used for a set with no lower bound.
- The set exclusion symbol ' \' is used to show excluded elements, such as $R\backslash\{0, 1\}$, which means the set of real numbers except 0 and 1.

Linear functions



- The distance d between (x_1, y_1) and (x_2, y_2) is $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.
- The midpoint is $M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.
- The gradient is $m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1} = \tan(\theta)$, where θ is the **inclination**, the angle between the line and the positive x direction.
- The **general form** of a straight line is $ax + by + c = 0$.
- The **gradient-intercept** form is $y = mx + c$, where c is the **y-intercept**.
- The **point-gradient** form is $y - y_1 = m(x - x_1)$.
- The **intercept** form is $\frac{x}{a} + \frac{y}{b} = 1$.
- Parallel lines have the same gradient, so $m_1 = m_2$.
- For **perpendicular lines**, $\boldsymbol{m_1 m_2 = -1}$, or $m_2 = -\frac{1}{m_1}$.
- Lines **parallel to the x-axis** are of the form $y = b$, where (a, b) is on the line.
- Lines **parallel to the y-axis** are of the form $x = a$, where (a, b) is on the line.

Quadratic functions

- A **quadratic function** has the general form $f(x) = ax^2 + bx + c$.
- The shape of its graph is a **parabola**. It has a vertical axis of symmetry through the centre.
- The process of changing to the form $f(x) = a(x - h)^2 + k$ is called **completing the square**.
- The solutions of $f(x) = 0$ are given by the **quadratic formula** $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.
- If a is positive the graph of $ax^2 + bx + c$ is **concave up**, like ∪, and has a **minimum**.
- If a is negative it is **concave down**, like ∩, and has a **maximum**.
- The maximum or minimum is called the **turning point** or **vertex**.
- The turning point occurs at $x = -\frac{b}{2a}$.
- The x-intercepts, if they exist, are called **zeros** or **roots**.
- The **discriminant** $\Delta = b^2 - 4ac$ determines the nature of the solutions of $f(x) = 0$.
  – If $\Delta < 0$, no zeros and no real roots
  – If $\Delta = 0$, 1 zero and 1 real root
  – If $\Delta > 0$, 2 zeros and 2 real roots
  – If Δ is a perfect square, the zeros are rational
- Compared to $y = x^2$, the graph of $y = a(x - h)^2 + k$:
  – is stretched by the factor a from the x-axis
  – has turning point (h, k), h from the y-axis and k from the x-axis.

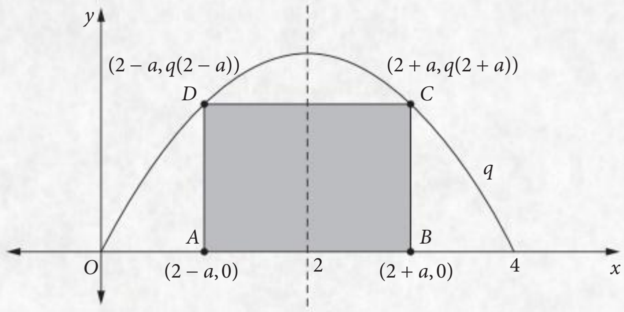
Cumulative examination 1

Total number of marks: 10 Reading time: 4 minutes Writing time: 15 minutes

TECH-FREE *Technology is NOT permitted.*

1 © VCAA 2021N 1Q7a (2 marks) Let $q : [0, 4] \to R$, $q(x) = x(4 - x)$.

A rectangle *ABCD* is inscribed between the graph of the function q and the x-axis. Its vertices are a units, where $a > 0$, from the axis of symmetry, $x = 2$, as shown below.



Find the value of a when the rectangle is a square. Give your answer in the form $b + \sqrt{c}$, where b is an integer and c is a positive integer.

2 (3 marks) Solve the simultaneous equations $2x + 3y = 5$ and $5x + 2y = -4$.

3 (2 marks) State the maximal domain of $f(x) = \dfrac{1}{\sqrt{x^2 - 16}}$.

4 (3 marks) Sketch the graph of $f(x) = 4x - 4x^2 + 1$, labelling the coordinates of all intercepts and other significant points.

Cumulative examination 2

Total number of marks: 20 Reading time: 4 minutes Writing time: 30 minutes

Approved technology is permitted.

Section A 5 multiple-choice questions

5 marks

1 The solution to the equation $4^{x+5} = 8^{2x+3}$ is

A -4 **B** $-\frac{1}{4}$ **C** $\frac{1}{4}$ **D** 4 **E** 9

2 Solve $3x - 2 \geq \frac{2x-3}{4}$.

A $x \geq -1$ **B** $x \leq -1$ **C** $x \leq \frac{1}{2}$ **D** $x \leq -\frac{1}{2}$ **E** $x \geq \frac{1}{2}$

3 $f(x+2) = x^2 + x - 7, f(x) =$

A $x^2 + 3x + 5$ **B** $x^2 - 7x + 5$ **C** $x^2 - 3x - 5$

D $x^2 + 5x - 1$ **E** $x^2 - 5x - 5$

4 © VCAA 2014 2AQ18 The graph of $y = kx - 4$ intersects the graph of $y = x^2 + 2x$ at two distinct points for

A $k = 6$ **B** $k > 6$ or $k < -2$ **C** $-2 \leq k \leq 6$

D $6 - 2\sqrt{3} \leq k \leq 6 + 2\sqrt{3}$ **E** $k = -2$

5 The range of $f: D \to R, f(x) = 3 - 2x$ is $(-\infty, -9] \cup (5, \infty)$. The domain D is

A $[-1, 6)$ **B** $(-6, 1]$ **C** $(-\infty, -1) \cup [6, \infty)$

D $(-\infty, -1] \cup (6, \infty)$ **E** $(-\infty, -6) \cup [1, \infty)$

Section B 2 questions

15 marks

1 (8 marks)

a Solve the simultaneous equations $2x - 3y = 5$ and $3x + y = 8$. 2 marks

b Solve the equations $2x + ay = 5$ and $3x + y = b$, where $a, b \in R$. 2 marks

c For what values of a and b are there no solutions? 2 marks

d For what values of a and b are there infinitely many solutions? 2 marks

2 (7 marks) Consider the functions $f(x) = ax - 2$ and $g(x) = \dfrac{1}{2x + b}$ on their maximal domains, where $a, b \in R$.

a What are the ranges of f and g? 2 marks

b Solve $f(x) = g(x)$ to show that any points of intersection between the graphs of $f(x)$ and $g(x)$ have coordinates

$$x = \frac{4 - ab \pm \sqrt{a^2b^2 + 8ab + 8a + 16}}{4a},\ y = \frac{-4 - ab \pm \sqrt{a^2b^2 + 8ab + 8a + 16}}{4}.$$

2 marks

c Find the relationship between a and b such that there is one point of intersection between the graphs of $f(x)$ and $g(x)$. 3 marks

CHAPTER

3 POLYNOMIALS

Study Design coverage

UNIT 1, AREA OF STUDY 1: FUNCTIONS, RELATIONS AND GRAPHS

- graphs of polynomial functions of low degree, and interpretation of key features of these graphs.

UNIT 1, AREA OF STUDY 2: ALGEBRA, NUMBER AND STRUCTURE

- the connection between the roots of a polynomial function, its factors and the horizontal axis intercepts of its graph, including the remainder, factor and rational root theorems
- solution of polynomial equations of low degree, numerically, graphically and algebraically, including numerical approximation of roots of simple polynomial functions using the bisection method algorithm.

Video playlists (9):

3.1 Polynomials
3.2 Equating coefficients
3.3 Dividing polynomials
3.4 The remainder and factor theorems
3.5 The rational root theorem
3.6 Solving polynomial equations
3.7 The bisection method
3.8 Graphing polynomial functions
VCE question analysis Polynomials

Worksheets (6):

3.4 The remainder theorem • Factorising polynomials
3.8 Graphing cubics • Polynomial graphs • Polynomials review • Sketching curves

Nelson MindTap

To access resources above, visit **cengage.com.au/nelsonmindtap**

Video playlist Polynomials

3.1 Polynomials

Terminology

A **polynomial** is an expression involving a sum of terms.

Polynomial comes from the Greek words poly- (meaning 'many') and -nomial (meaning 'term') so it translates to 'many terms'.

A polynomial in x is of the form:

$$P(x) = a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_1x^1 + a_0$$

where n is a positive whole number, a_0, a_1, a_2 … a_{n-1}, a_n are real numbers and $a_n \neq 0$.

- The numbers multiplying the different powers of x are called the **coefficients**.
  The coefficients are a_0, a_1, a_2 … a_{n-1}, a_n.
- The constant is a_0 and is also known as the **constant** term.
- The highest power of x is called the **degree of a polynomial**. The degree of $P(x) = a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_1x^1 + a_0$ is n.
- The term that contains the highest power is called the **leading term**, a_nx^n, and the coefficient of the leading term, a_n, is called the **leading coefficient**.
- When the leading coefficient, a_n, is 1, the polynomial is called a **monic polynomial**.

Some examples of polynomials are:

5	A polynomial of degree 0 (it could be written as $5x^0$)
$2x-3$	A polynomial of degree 1, called a **linear polynomial**
$4x^2+3\sqrt{2}x+9$	A polynomial of degree 2, called a **quadratic polynomial**
x^3-4x^2+x	A polynomial of degree 3, called a **cubic polynomial**
$\frac{2}{3}x^4+2x^2-5x+0.28$	A polynomial of degree 4, called a **quartic polynomial**

A polynomial with 2 terms is also called a **binomial expression**.

A polynomial with 3 terms is called a **trinomial expression**.

For example:

x^2-5	Degree 2 with 2 terms: a **quadratic binomial**
$7x^2-2x+1$	Degree 2 with 3 terms: a **quadratic trinomial**
$4x^3+25$	Degree 3 with 2 terms: a **cubic binomial**

3.1

WORKED EXAMPLE 1 Polynomial expressions and their features

WB p. 43

Determine whether each expression is a polynomial, giving reasons if it is not. State the degree, leading term and coefficients of any polynomial found.

a $4x-3x^2+2$ **b** $x^4+\frac{x^2}{4}-\frac{3}{x}+7x^3$

Steps	Working
a 1 Write the expression with descending powers of x. All of the powers are positive whole numbers, so it is a polynomial.	$-3x^2+4x+2$ This is a polynomial.
2 Identify the degree, leading term and coefficients.	degree = 2 leading term = $-3x^2$ coefficients = $-3, 4, 2$
b Write the expression with descending powers. Not all of the powers are positive whole numbers, so it is not a polynomial.	$x^4+7x^3+\frac{1}{4}x^2-3x^{-1}$ This is not a polynomial as $3x^{-1}$ has a negative power.

Adding and subtracting polynomials

We can add and subtract polynomial functions by collecting like terms and simplifying.

WB
p. 44

WORKED EXAMPLE 2 Simplifying polynomials

If $P(x) = 2x^4 - 3x^2 + 2x - 4$ and $Q(x) = 2x^4 - x^2 + 3$, find $P(x) + Q(x)$ and state its degree.

Steps	Working
1 Substitute for $P(x)$ and $Q(x)$.	$P(x) + Q(x) = (2x^4 - 3x^2 + 2x - 4) + (2x^4 - x^2 + 3)$
2 Add or subtract like terms to simplify.	$= 4x^4 - 4x^2 + 2x - 1$
	This is a polynomial function of degree 4.

WB
p. 45

WORKED EXAMPLE 3 Substituting values into polynomials

a Given that $P(x) = 5x^2 + 4x - 2$, find

i $P(-3)$ **ii** $P(2a - 1)$

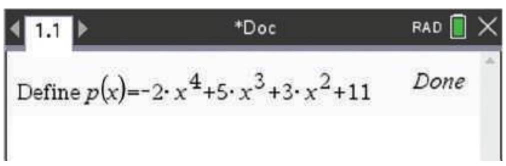
b x when $P(x) = 7$

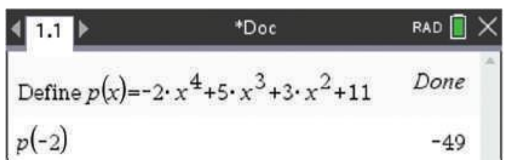
Steps	Working
a i Substitute $x = -3$ into $P(x)$ and evaluate.	$P(-3) = 5(-3)^2 + 4(-3) - 2$ $= 31$
ii Substitute $x = 2a - 1$ into $P(x)$ and simplify.	$P(2a - 1) = 5(2a - 1)^2 + 4(2a - 1) - 2$ $= 5(4a^2 - 4a + 1) + 8a - 4 - 2$ $= 20a^2 - 20a + 5 + 8a - 4 - 2$ $= 20a^2 - 12a - 1$
b Substitute $P(x) = 7$ and solve for x.	$5x^2 + 4x - 2 = 7$ $5x^2 + 4x - 9 = 0$ $(5x + 9)(x - 1) = 0$ $x = -\frac{9}{5}$ or $x = 1$

USING CAS 1 Defining and evaluating polynomials

Given $P(x) = -2x^4 + 5x^3 + 3x^2 + 11$, find $P(-2)$.

TI-Nspire

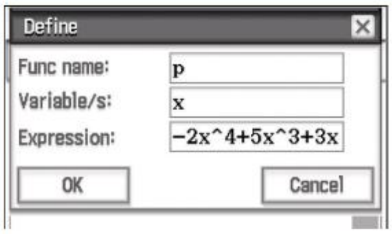


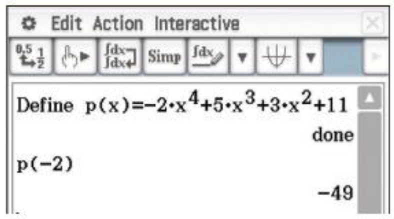


1 Add a **Calculator** page.
2 Press **menu > Actions > Define**.
3 Enter the function as shown above.
4 Press **enter**.
5 Enter **p(–2)**.
6 Press **enter**.
7 The solution **–49** will be displayed.

$P(-2) = -49$

ClassPad





1 Tap **Main** and clear all calculations.
2 Enter and highlight the polynomial.
3 Tap **Interactive > Define**.
4 Change the **Func name:** to **p** (use the letter **p**, not the variable ***p***).
5 The polynomial will appear in the **Expression:** field, as shown above.
6 Tap **OK**.
7 The defined function will be displayed.
8 Enter **p(–2)**.
9 Press **EXE**.
10 The solution –49 will be displayed.

$P(-2) = -49$

WORKED EXAMPLE 4 Finding unknown variables

WB p. 46

If $P(x) = 5x^3 + 4x^2 - mx + 7$ and $P(-2) = 14$, then find the value of m.

Steps	Working
1 Substitute $(-2, 14)$ into $P(x)$.	$5(-2)^3 + 4(-2)^2 - m(-2) + 7 = 14$
2 Solve for m.	$5(-8) + 4(4) + 2m + 7 = 14$ $-17 + 2m = 14$ $2m = 31$ $m = \frac{31}{2}$

WORKED EXAMPLE 5 Modelling with polynomials

WB p. 47

A ball is thrown upwards from the roof of a building that is 30 metres above the ground. The ball's height above the ground after t seconds is modelled by the equation $h(t) = -4.9t^2 + 14t + 30$.

a Find its height above the ground after 2 seconds.

b When will the ball hit the ground? Give your answer correct to 2 decimal places.

Steps	Working
a Substitute $t = 2$ into $h(t)$.	$h(2) = -4.9(2)^2 + 14(2) + 30$ $= 38.4$ The height of the ball after 2 seconds is 38.4 m.
b **1** This will happen when the height is 0. Solve the equation $h(t) = 0$.	$-4.9t^2 + 14t + 30 = 0$ $49t^2 - 140t - 300 = 0$ $(7t + 10)(7t - 30) = 0$ $t = -\frac{10}{7}$ or $t = \frac{30}{7}$
2 Check the feasibility of the solutions.	Since $t > 0$, $t = \frac{30}{7} = 4.2857\ldots \approx 4.29$ s The ball will hit the ground approximately 4.29 seconds after being thrown.

EXERCISE 3.1 Polynomials

ANSWERS p. 499

Mastery

1 WORKED EXAMPLE 1 State which expressions are not polynomials.

a $5x^4 - 3x^2 + x + \frac{1}{x}$
b $x^2 + 3^x$
c $x^2 + 3x - 7$
d $3x + 5$
e $6x^3 + 4\sqrt{x} - 8$
f $3x^2 - \frac{1}{2}x + 1$

2 For the polynomial $P(x) = x^5 - 3x^4 - 5x + 4$, find

a the degree of $P(x)$
b the constant term
c the coefficient of x^4
d the coefficient of x^2

3 $P(x) = (a + 1)x^3 + (b - 7)x^2 + c + 5$. Find values for a, b or c if

a $P(x)$ is monic
b the coefficient of x^2 is 3
c the constant is -1
d $P(x)$ has degree 2
e the leading term has a coefficient of 5

4 WORKED EXAMPLE 2 TECH-FREE Let $f(x) = -3x^3 - 4x^2 + 5$ and $g(x) = x^3 + 4x^2$.

a Find

i $f(x) + g(x)$
ii $f(x) - 2g(x)$

b Does $f(x) + f(x) = 2f(x)$? Give supporting evidence for your answer.

5 WORKED EXAMPLE 3 TECH-FREE

a Given $f(x) = x + 3$, find

i $f(1)$
ii $f(-3)$

b If $h(x) = x^2 - 2$, find

i $h(0)$
ii $h(2)$
iii $h(-4)$

c If $f(x) = 2x - 5$, find x when $f(x) = 13$.

d Given $P(x) = x^2 + 3$, find any values of x for which $P(x) = 28$.

e If $f(x) = 2x - 9$, find $f(x + h)$.

f Find $g(x - 1)$ when $g(x) = x^2 + 2x + 3$.

g Given $f(t) = t^2 + 2t + 1$, find t when $f(t) = 0$.

h Given $f(t) = t^2 + 2t + 1$, find t when $f(t) = 9$.

6 Using CAS 1

a Given $f(x) = x^2 - 5x + 2$, find

i $f(0)$
ii $f(-1)$

b If $h(x) = x^3 - 2x^2 - x + 6$, find

i $h(4)$
ii $h(-2)$

7 WORKED EXAMPLE 4 If $P(x) = 2x^2 + kx - 4$ and $P(-2) = 0$, find k.

8 If $f(x) = 4x^3 + bx^2 - 3x$ and $f(-2) = 3f(1)$, find b.

9 WORKED EXAMPLE 5 The height, h metres, above ground level of a falling stone after t seconds, $t \geq 0$, is given by $h(t) = -5t^2 + 30$.

a Find the stone's initial height above ground level.

b How far above ground level is the stone after it has fallen for 2 seconds?

c When, correct to one decimal place, will the stone hit the ground?

10 The area, A m^2, of a rectangular yard of width x metres is given by $A(x) = 40x - x^2$.

a If the area of the rectangular yard is 300 m^2, find its dimensions.

b Use a table of values to find the dimensions of the yard that would give the maximum area.

Exam practice 80–100% 60–79% 0–59%

11 © VCAA 2009 2AQ5 53% Let $f: R \to R, f(x) = x^2$.

Which one of the following is **not** true?

A $f(xy) = f(x)f(y)$ **B** $f(x) - f(-x) = 0$ **C** $f(2x) = 4f(x)$

D $f(x - y) = f(x) - f(y)$ **E** $f(x + y) + f(x - y) = 2(f(x) + f(y))$

12 © VCAA 2015 2AQ20 61% If $f(x - 1) = x^2 - 2x + 3$, then $f(x)$ is equal to

A $x^2 - 2$ **B** $x^2 + 2$ **C** $x^2 - 2x + 2$

D $x^2 - 2x + 4$ **E** $x^2 - 4x + 6$

3.2 Equating coefficients

Video playlist Equating coefficients

If 2 polynomials are equal for all values of x, then they are identical and the coefficients of the same powers of x are equal. This allows us to **equate the coefficients** of each corresponding term on either side of the '=' sign.

Equating coefficients of identical polynomials

If $P(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_1 x^1 + a_0$ and

$Q(x) = b_n x^n + b_{n-1}x^{n-1} + b_{n-2}x^{n-2} + \ldots + b_1 x^1 + b_0$

are equal for all values of x, then $a_n = b_n, a_{n-1} = b_{n-1}, a_{n-2} = b_{n-2}$ and so on up to $a_0 = b_0$.

WORKED EXAMPLE 6 Equating coefficients

WB p. 48

If $x^2 + 4x + 2 = a(x + 2)^2 + b$ for all x, find the values of a and b.

Steps	Working
1 Expand the RHS.	$a(x + 2)^2 + b = a(x^2 + 4x + 4) + b$ $= ax^2 + 4ax + 4a + b$
2 Let LHS = RHS. LHS = left-hand side RHS = right-hand side	So $x^2 + 4x + 2 = ax^2 + 4ax + 4a + b$.

3 Equate coefficients.	Coefficients of x^2: $1 = a$ Coefficients of x: $4 = 4a$, therefore, $a = 1$. Constant term: $2 = 4a + b$ Substituting $a = 1$: $2 = 4 + b$ $b = -2$ So $a = 1$, $b = -2$. (That is, $x^2 + 4x + 2 = (x + 2)^2 - 2$.)

EXERCISE 3.2 Equating coefficients

ANSWERS p. 499

Recap

1 Given that $P(x) = 3x^3 - 2x^2 + x - 2$, find $P(-2)$.

A -204 **B** -36 **C** -20 **D** -12 **E** 0

2 Which of the following expressions is a polynomial?

I $3x^2 - 2x^3 + \sqrt{7x^2}$ **II** $x^5 - \dfrac{3}{x^2} + 7x^2$ **III** $2^x + x^2$ **IV** $\left(x^4\right)^{\frac{1}{2}} - \dfrac{x^2}{3x}$

A I **B** II **C** I, IV **D** II, III **E** I, II, IV

Mastery

3 Find the values of a and b such that $a(x + 2) + b(x + 3) = 18x + 8$ for all values of x.

4 Find the values of A, B and C if $6x^2 - 24x + 14 = A(x + B)^2 + C$ for $x \in R$.

5 Find the values of m, n and p if $3x^3 - 18x^2 + 36x - 11 = m(x + n)^3 + p$ for $x \in R$.

6 WORKED EXAMPLE 6 $V(x) = 6x^3 + 2x^2 - x + p$ and $W(x) = 5x^3 - px^2 - 6x + 7k$.

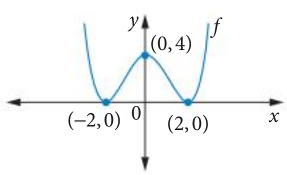
a Show that $2V(x) - 3W(x) = -3x^3 + (4 + 3p)x^2 + 16x + 2p - 21k$.

b If $2V(x) - 3W(x) = -3x^3 - 2x^2 + 16x - 25$, find the values of p and k for $x \in R$.

Exam practice

80–100% 60–79% 0–59%

7 © VCAA 2020 2BQ1 (2 marks) Let $f: R \to R$, $f(x) = a(x + 2)^2(x - 2)^2$, where $a \in R$. Part of the graph of f is shown below.



a 83% Show that $a = \dfrac{1}{4}$. 1 mark

b 80% Express $f(x) = \dfrac{1}{4}(x + 2)^2(x - 2)^2$ in the form $f(x) = \dfrac{1}{4}x^4 + bx^2 + c$, where b and c are integers. 1 mark

8 © VCAA 2014 2BQ5 (5 marks) Let $f: R \to R, f(x) = (x-3)(x-1)(x^2+3)$ and $g: R \to R, g(x) = x^4 - 8x$.

a 58% Express $x^4 - 8x$ in the form $x(x-a)\left((x+b)^2 + c\right)$. 2 marks

b 37% Describe the translation that maps the graph of $y = f(x)$ onto the graph of $y = g(x)$. 1 mark

c Find the values of d such that the graph of $y = f(x+d)$ has

i 7% one positive x-axis intercept. 1 mark

ii 19% two positive x-axis intercepts. 1 mark

3.3

3.3 Dividing polynomials

There are several methods for performing the division of one polynomial by another of equal or lesser degree. Some methods are quicker than others and some methods are more efficient than others. We will be looking at 3 methods:

- long division
- synthetic division
- division by inspection.

Video playlist Dividing polynomials

Long division

When evaluating $9515 \div 7$ using **long division**, we actually separate the division into place values, so in the thousands column we divide 9 by 7 first, getting 1, remainder 2.

To the 2 left over we bring down the 5, to get 25. Then we divide the 25 by 7 and repeat the process, giving 3, remainder 4. The process continues until the final answer is 1359, remainder 2, or $1359\frac{2}{7}$.

Polynomial division is done the same way, but we divide by separating the polynomial into descending powers instead of descending place values.

This is the 1 thousand from dividing

```
      1 3 5 9
  7 ) 9 5 1 5
      7          ← This is the 7 thousands
      2 5        ← This is the 25 hundreds
      2 1
        4 1
        3 5
          6 5
          6 3
            2
```

Division statement

In general, if $P(x)$ is divided by $x - a$, the quotient is denoted by $Q(x)$ and the remainder is R.

$$\frac{P(x)}{x-a} = Q(x) + \frac{R}{(x-a)}$$

It is usual to represent a polynomial division using this division statement:

$$P(x) = (x-a)Q(x) + R$$

WB
p. 49

WORKED EXAMPLE 7 Division of a polynomial by a linear divisor

Divide $P(x) = 3x^3 - 8x^2 + 7x - 1$ by $d(x) = x - 2$ and hence express $P(x)$ in the form $d(x) \times Q(x) + R$.

Steps | **Working**

1 Divide the highest power of x from $(x - 2)$ into the highest power of $(3x^3 - 8x^2 + 7x - 1)$.

This is $3x^3 \div x$, which gives $3x^2$.

Put this up the top in the x^2 place.

Work out how much of $(3x^3 - 8x^2)$ has been accounted for by doing $(x - 2) \times 3x^2 = 3x^3 - 6x^2$.

Put this under the $(3x^3 - 8x^2)$ and subtract to work out how many x^2 is left $(-2x^2)$.

Then bring down the $+ 7x$.

$$\begin{array}{r@{}l}
 & 3x^2 \\
x - 2 & \overline{)\,3x^3 - 8x^2 + 7x - 1} \\
 & \underline{3x^3 - 6x^2} \\
 & \; -2x^2 + 7x
\end{array}$$

Make sure each column has the same power of x.

2 x into $-2x^2$ goes $-2x$ times. Put the $-2x$ in the x place, above the $+7x$.

$(x - 2) \times (-2x) = -2x^2 + 4x$. Put this underneath the $-2x^2 + 7x$ and subtract to get $3x$.

Bring down the -1.

$$\begin{array}{r@{}l}
 & 3x^2 - 2x \\
x - 2 & \overline{)\,3x^3 - 8x^2 + 7x - 1} \\
 & \underline{3x^3 - 6x^2} \\
 & \; -2x^2 + 7x \\
 & \; \underline{-2x^2 + 4x} \\
 & \; 3x - 1
\end{array}$$

3 x into $3x$ goes 3 times. Put the 3 above the -1.

$(x - 2) \times 3 = 3x - 6$. Put this underneath the $3x - 1$ and subtract to get 5.

$$\begin{array}{r@{}l}
 & 3x^2 - 2x + 3 \\
x - 2 & \overline{)\,3x^3 - 8x^2 + 7x - 1} \\
 & \underline{3x^3 - 6x^2} \\
 & \; -2x^2 + 7x \\
 & \; \underline{-2x^2 + 4x} \\
 & \; 3x - 1 \\
 & \; \underline{3x - 6} \\
 & \; 5
\end{array}$$

4 Write the answer in the form $P(x) = d(x) \times Q(x) + R$.

$(3x^3 - 8x^2 + 7x - 1) \div (x - 2) = 3x^2 - 2x + 3$,

remainder $= 5$

$(3x^3 - 8x^2 + 7x - 1) = (x - 2)(3x^2 - 2x + 3) + 5$

3.3

WB

p. 50

WORKED EXAMPLE 8 Division of a polynomial by a non-linear divisor

Divide $P(x) = 4x^3 - 5x^2 + 2x - 6$ by $d(x) = 2x^2 - 1$ to find the quotient and remainder.

Steps	Working
1 Divide the highest power of x from $(2x^2 - 1)$ into the highest power of $(4x^3 - 5x^2 + 2x - 6)$. This is $4x^3 \div 2x^2$, which gives $2x$. Put this up the top in the x place. Work out how much of $(4x^3 - 5x^2)$ has been accounted for by doing $(2x^2 - 1) \times 2x = 4x^3 - 2x$. Put this under the $(4x^3 - 5x^2)$ and subtract to work out how many x^2 is left. Then bring down the -6.	$\begin{array}{r l} & \quad\quad\quad\quad\quad\; 2x \\ 2x^2 - 1 & \overline{)\,4x^3 - 5x^2 + 2x - 6} \\ & \;\; \underline{4x^3 \quad\quad\;\; - 2x} \\ & \quad\quad -5x^2 + 4x \end{array}$ Make sure each column has the same power of x.
2 $2x^2$ into $-5x^2$ goes $-\frac{5}{2}$ times. Put the $-\frac{5}{2}$ in the units place, above the -6. $\left(2x^2 - 1\right) \times \left(-\frac{5}{2}\right) = -5x^2 + \frac{5}{2}$. Put this underneath the $-5x^2 + 4x - 6$ and subtract to get $4x - \frac{17}{2}$.	$\begin{array}{r l} & \quad\quad\quad\quad\quad\; 2x - \frac{5}{2} \\ 2x^2 - 1 & \overline{)\,4x^3 - 5x^2 + 2x - 6} \\ & \;\; \underline{4x^3 \quad\quad\;\; - 2x} \\ & \quad\quad -5x^2 + 4x - 6 \\ & \quad\quad \underline{-5x^2 \quad\quad\;\; + \frac{5}{2}} \\ & \quad\quad\quad\quad\quad\quad\; -\frac{5}{2} \end{array}$
3 Write the answer.	$Q(x) = 2x - \frac{5}{2}$, remainder $= 4x - \frac{17}{2}$

Exam hack

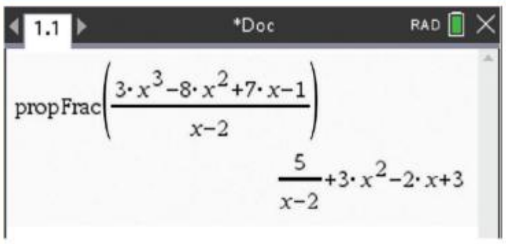
Make sure the **polynomial** is written in descending order. If any terms are missing, use a zero to fill in the missing term (this will help with the spacing). For instance, if $P(x) = 3x^3 - 2x^2 + 4$ is divided by $d(x) = x - 2$ then the long division would be set up like this:

$$x - 2\overline{)\,3x^3 - 2x^2 + 0x + 4}$$

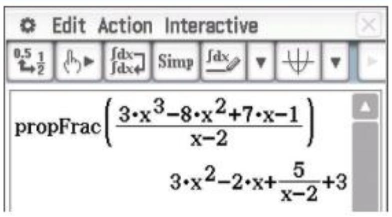
USING CAS 2 Division of polynomials

Given $P(x) = 3x^3 - 8x^2 + 7x - 1$ and $d(x) = x - 2$, express $P(x)$ in the form $d(x) \times Q(x) + R$.

TI-Nspire



ClassPad



1 In a **Calculator** page, press **menu > Algebra > Fraction Tools > Proper Fraction**.
2 Press **ctrl + +** to insert the fraction template.
3 Enter the polynomial division as shown above.
4 Press **enter**.
5 The quotient will be displayed.

1 In **Main**, enter and highlight the polynomial division.
2 Tap **Interactive > Transformation > Fraction > propFrac**.
3 In the dialogue box, tap **OK**.
4 The quotient will be displayed.

The $\frac{5}{x-2}$ term means that after dividing the polynomial by $x - 2$, the remainder is 5.

Therefore, $P(x) = (x-2)\left(3x^2 - 2x + 3\right) + 5$.

Synthetic division

Synthetic division is an alternative and shorter version of long division for dividing by a linear factor $(x - a)$. It uses a grid of numbers where the top row lists the coefficients of $P(x)$.

Here is Worked example 7 again, using synthetic division.

Remember, the question is to divide $P(x) = 3x^3 - 8x^2 + 7x - 1$ by $d(x) = x - 2$.

Steps	**Working**
1 Set $d(x) = 0$ and solve for x.	$x - 2 = 0$ $x = 2$
2 Place the solution to $d(x) = 0$ (2) in the top left-hand corner of the first row, to the left of the separation line. Place the coefficients of the terms in $P(x)$ in the top row to the right of the separation line, in order of their decreasing degree.	$\begin{array}{c\|cccc} 2 & 3 & -8 & 7 & -1 \\ & & & & \\ \hline \end{array}$
3 Bring the first coefficient, 3, down. $2 \times 3 = 6$ and place this underneath -8, and add to get $\mathbf{-2}$.	$\begin{array}{c\|cccc} 2 & 3 & -8 & 7 & -1 \\ & & 6 & -4 & 6 \\ \hline & 3 & -2 & 3 & 5 \end{array}$
4 The rest of the values in the second row are found by multiplying each value in the bottom row by the 2 from $(x - 2)$. The values in the bottom row are column totals, which make up the coefficients of the quotient, the last value being the remainder.	$\begin{array}{c\|cccc} 2 & 3 & -8 & 7 & -1 \\ & & 6 & -4 & 6 \\ \hline & 3 & -2 & 3 & 5 \end{array}$
5 Write the answer in the form $P(x) = d(x) \times Q(x) + R$.	$(3x^3 - 8x^2 + 7x - 1)$ $= (x - 2)(3x^2 - 2x + 3) + 5$

WORKED EXAMPLE 9 Division of a polynomial using synthetic division

Using synthetic division, divide $P(x) = 5x^3 - 2x^2 + 3x - 6$ by $d(x) = 2x - 1$, and hence express $P(x)$ in the form $d(x) \times Q(x) + R$.

Steps / **Working**

1 Set $d(x) = 0$ and solve for x.

$$\begin{aligned} 2x - 1 &= 0 \\ x &= \frac{1}{2} \end{aligned}$$

2 Place the solution to $d(x) = 0$, $\frac{1}{2}$, in the top left-hand corner of the first row, to the left of the separation line. Place the coefficients of the terms in $P(x)$ in the top row to the right of the separation line, in order of their decreasing power.

$$\begin{array}{c|cccc} \frac{1}{2} & 5 & -2 & 3 & -6 \\ & & & & \\ \hline \end{array}$$

3 Bring the first coefficient, 5, down.

$\frac{1}{2} \times 5 = \frac{5}{2}$ and place this underneath -2, and add to get $\frac{1}{2}$.

$$\begin{array}{c|cccc} \frac{1}{2} & 5 & -2 & 3 & -6 \\ & & \frac{5}{2} & & \\ \hline & 5 & \frac{1}{2} & & \end{array}$$

4 The rest of the values in the second row are found by multiplying each value in the bottom row by the 3 from $(x - 3)$. The values in the bottom row are column totals, which make up the coefficients of the quotient, the last value being the remainder.

$$\begin{array}{c|cccc} \frac{1}{2} & 5 & -2 & 3 & -6 \\ & & \frac{5}{2} & \frac{1}{4} & \frac{13}{8} \\ \hline & 5 & \frac{1}{2} & \frac{13}{4} & -\frac{35}{8} \end{array}$$

5 Express the answer in the form $P(x) = d(x) \times Q(x) + R$. In order to do this, $\left(x - \frac{1}{2}\right)$ must be factorised by taking out a factor of $\frac{1}{2}$ to get $\frac{1}{2}(2x - 1)$. This factor of $\frac{1}{2}$ is then multiplied to the second bracket.

$$\begin{aligned} &(5x^3 - 2x^2 + 3x - 6) \div (2x - 1) \\ &= \left(x - \frac{1}{2}\right)\left(5x^2 + \frac{1}{2}x + \frac{13}{4}\right) - \frac{35}{8} \\ &= \frac{1}{2}(2x - 1)\left(5x^2 + \frac{1}{2}x + \frac{13}{4}\right) - \frac{35}{8} \\ &= (2x - 1)\left(\frac{5}{2}x^2 + \frac{1}{4}x + \frac{13}{8}\right) - \frac{35}{8} \end{aligned}$$

6 Write the answer in the form $P(x) = d(x) \times Q(x) + R$.

$$(2x - 1)\left(\frac{5}{2}x^2 + \frac{1}{4}x + \frac{13}{8}\right) - \frac{35}{8}$$

3.3

WB

p. 51

Division by inspection

Division by inspection is useful to divide one polynomial by another polynomial of equal or lesser degree by expressing the numerator in terms of the denominator.

WB

p. 52

WORKED EXAMPLE 10 Division by inspection

Given $P(x) = 2x - 3$ and $d(x) = x + 1$, use division by inspection to express $P(x)$ in the form $Q(x) + \frac{R}{d(x)}$.

Steps	Working
1 Write the numerator $2x - 3$ as $2(x + 1) - 2 - 3$, which simplifies to $2(x + 1) - 5$.	$\frac{2(x+1)-5}{x+1}$
2 Split the expression into 2 partial fractions.	$\frac{2(x+1)}{x+1} - \frac{5}{x+1}$
3 Simplify.	$2 - \frac{5}{x+1}$

Exam hack

CAS presents the remainder (5) as a fraction over $(x - 2)$ and in different positions. So when writing your answer, remember to change the order and write the remainder last.

EXERCISE 3.3 Dividing polynomials ANSWERS p. 499

Recap

1 Find a and b given $x^2 + 4x + 10 = a(x + 2)^2 + b$.

2 Find a and b given $2x^2 + 8x + 10 = a(x + 2)^2 + b$.

Mastery

3 WORKED EXAMPLE 7 TECH-FREE Using long division, divide $P(x)$ by $d(x)$ and express $P(x)$ in the form $d(x) \times Q(x) + R$.

- **a** $P(x) = x^2 + 2x - 15, d(x) = x + 5$
- **b** $P(x) = 6x^2 - 17x + 12, d(x) = 3x - 4$
- **c** $P(x) = x^3 - 5x^2 + 3x - 2, d(x) = x - 2$
- **d** $P(x) = 2x^3 - x^2 - 13x + 9, d(x) = x + 2$
- **e** $P(x) = 6x^3 - 13x^2 - 4x + 35, d(x) = 3x + 4$
- **f** $P(x) = 5x^3 + 3x^2 + 8x - 8, d(x) = 5x - 2$
- **g** $P(x) = 4x^3 + 5x^2 + 10, d(x) = 2x - 1$
- **h** $P(x) = x^3 - 8, d(x) = x - 2$

4 WORKED EXAMPLE 8 TECH-FREE Divide the first polynomial by the second, stating the quotient $Q(x)$ and the remainder R in each case.

- **a** $x^2 + 4x - 14,\ x^2 - 3$
- **b** $3x^2 - 6x - 16,\ x^2 - 1$
- **c** $4x^3 - 2x^2 + 5x - 1, x^2 - 2$
- **d** $2x^3 - 6x^2 - 5x + 11, 2x^2 + 1$
- **e** $6x^3 + 4x^2 - 2x + 3, 3x^2 + 4$
- **f** $3x^3 + 5x^2 + 6x - 6, 6x^2 - 2$
- **g** $2x^3 - 5x^2 + x - 10, 4x^2 - 3$

5 Using CAS 2 Divide the first polynomial by the second, stating the quotient $Q(x)$ and the remainder R in each case.

- **a** $2x^5 - 5x^4 + 7x^3 + 4x^2 - 10x + 11,\ x^3 + 2$
- **b** $10x^5 + x^3 + 5x^2 - 2x - 2,\ 5x^2 - 2$

6 Using synthetic division, divide $P(x)$ by $d(x)$ and express $P(x)$ in the form $d(x) \times Q(x) + R$.

a $P(x) = x^2 + 3x - 1,\ d(x) = x - 2$

b $P(x) = x^3 - 4x^2 + 5x - 1, d(x) = x + 1$

c $P(x) = 2x^3 - 4x^2 + 3x - 5, d(x) = x + 3$

d $P(x) = 5x^3 + 6x^2 + 3x - 2, d(x) = x - 4$

7 WORKED EXAMPLE 9 Using synthetic division, divide $P(x)$ by $d(x)$ and express $P(x)$ in the form $d(x) \times Q(x) + R$.

a $P(x) = 2x^2 - 4x + 1, d(x) = 2x - 1$

b $P(x) = 6x^3 + 13x^2 - 2x + 35, d(x) = 3x + 1$

c $P(x) = 4x^3 - 7x^2 + 5x - 11, d(x) = 2x - 3$

d $P(x) = 4x^3 + 6x^2 - 10, d(x) = 2x + 5$

8 WORKED EXAMPLE 10 Use division by inspection to express $\frac{P(x)}{d(x)}$ in the form $Q(x) + \frac{R}{d(x)}$.

a $\frac{x+1}{x-2}$ b $\frac{x+3}{x+1}$ c $\frac{2x-5}{2x-3}$ d $\frac{3x-7}{3x+5}$

3.4

3.4 The remainder and factor theorems

Video playlist The remainder and factor theorems

Worksheet The remainder theorem

The remainder theorem

The remainder theorem

If a polynomial $P(x)$ is divided by $x - a$, the remainder is $P(a)$.

Also, if $P(x)$ is divided by $ax + b$, the remainder is $P\left(-\frac{b}{a}\right)$.

We can prove this by substituting $x = a$ in the division statement $P(x) = (x - a)Q(x) + R$ and $x = -\frac{b}{a}$ into $P(x) = (ax + b)Q(x) + R$.

$$\begin{aligned} P(a) &= (a - a)Q(a) + R \\ &= R \end{aligned}$$

$$\begin{aligned} P\left(-\frac{b}{a}\right) &= \left(a\left[-\frac{b}{a}\right] + b\right)Q\left(-\frac{b}{a}\right) \\ &= (-b + b)Q\left(-\frac{b}{a}\right) + R \\ &= R \end{aligned}$$

where R is the **remainder**.

From Worked example 7, we know that when $P(x) = 3x^3 - 8x^2 + 7x - 1$ is divided by $x - 2$, the remainder is 5. We can use the **remainder theorem** to give the same result.

$$\begin{aligned} P(x) &= 3x^3 - 8x^2 + 7x - 1 \\ P(2) &= 3(2)^3 - 8(2)^2 + 7(2) - 1 \\ &= 3 \times 8 - 8 \times 4 + 7 \times 2 - 1 \\ &= 5 \end{aligned}$$

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WORKED EXAMPLE 11 Using the remainder theorem

Find the remainder when $2x^3 - 3x^2 + 4x - 1$ is divided by $2x - 3$.

Steps	Working
1 Write down $P(x)$.	$P(x) = 2x^3 - 3x^2 + 4x - 1$
2 The remainder when $P(x)$ is divided by $2x - 3$ is $P\left(\frac{3}{2}\right)$. Evaluate $P\left(\frac{3}{2}\right)$.	$P\left(\frac{3}{2}\right) = 2\left(\frac{3}{2}\right)^3 - 3\left(\frac{3}{2}\right)^2 + 4\left(\frac{3}{2}\right) - 1$ $= 2 \times \frac{27}{8} - 3 \times \frac{9}{4} + 6 - 1$ $= 5$
3 Write the answer.	The remainder is 5.

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WORKED EXAMPLE 12 Using the remainder theorem to find a coefficient

Evaluate m if the remainder is 4 when dividing $2x^4 + mx + 5$ by $x + 3$.

Steps	Working
1 Write down $P(x)$.	$P(x) = 2x^4 + mx + 5$
2 The remainder when $P(x)$ is divided by $x + 3$ is $P(-3)$.	$P(-3) = 2(-3)^4 + m(-3) + 5$ $= 2 \times 81 - 3m + 5$ $= 167 - 3m$
3 The remainder is 4, so $P(-3) = 4$.	$167 - 3m = 4$
4 Solve the equation for m.	$-3m = -163$ $m = \frac{-163}{-3}$ $= 54\frac{1}{3}$

WB

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WORKED EXAMPLE 13 Using the remainder theorem to find coefficients

$f(x) = x^3 + ax^2 + bx + 3$, where a, b are constants.

a Given that when $f(x)$ is divided by $(x + 2)$ the remainder is 7, show that $2a - b = 6$.

b Given also that when $f(x)$ is divided by $(x - 1)$ the remainder is 4, find the values of a and b and hence determine the polynomial $f(x)$.

Steps	Working
a **1** Write down $f(x)$.	$f(x) = x^3 + ax^2 + bx + 3$
2 The remainder when $f(x)$ is divided by $x + 2$ is $f(-2)$.	$f(-2) = (-2)^3 + a(-2)^2 + b(-2) + 3$ $= -8 + 4a - 2b + 3$ $= 4a - 2b - 5$
3 The remainder is 7, so $f(-2) = 7$.	$4a - 2b - 5 = 7$
4 Simplify the equation and show the result given.	$4a - 2b = 12$ $2a - b = 6$
b **1** The remainder when $f(x)$ is divided by $x - 1$ is $f(1)$.	$f(1) = (1)^3 + a(1)^2 + b(1) + 3$ $= 1 + a + b + 3$ $= a + b + 4$
2 The remainder is 4, so $f(1) = 4$.	$a + b + 4 = 4$ $a + b = 0$

3 Set up a pair of simultaneous equations.	$2a - b = 6 \quad \ldots[1]$ $a + b = 0 \quad \ldots[2]$
4 Solve for a and b.	From [2]: $a = -b \quad \ldots[3]$ Substitute into [1]: $2(-b) - b = 6$ $-3b = 6$ $b = -2$ Substitute into [3] to find a: $a = -(-2) = 2$
5 Write the polynomial $f(x)$ using a and b.	$f(x) = x^3 + 2x^2 - 2x + 3$

3.4

USING CAS 3 Finding the remainder of polynomials

Find the remainder when $-2x^3 + 8x^2 + 9x - 15$ is divided by $x + 4$.

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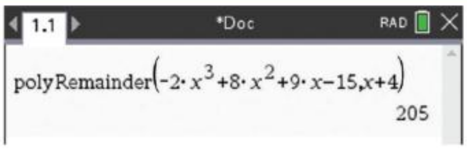
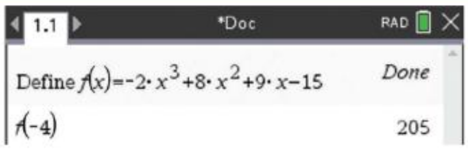




1 In a **Calculator** page, press **menu > Algebra > Polynomial Tools > Remainder of Polynomial**.

2 Enter the polynomials separated by a comma as shown above.

3 Press **enter**.

4 The remainder 205 will be displayed.

To confirm the answer:

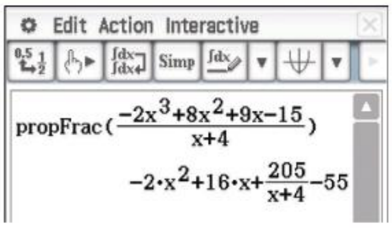
1 Define the function ***f*(*x*)**.

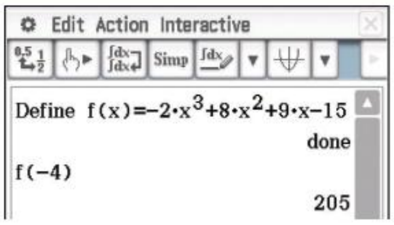
2 Enter ***f*(–4)**, which is the value of x when $x + 4 = 0$.

3 Press **enter**.

4 The remainder 205 will be displayed.

ClassPad





1 In **Main**, enter the 2 expressions as a polynomial division as shown above.

2 Tap **Interactive > Transformation > Fraction > propFrac**.

3 In the dialogue box, tap **OK**.

4 The remainder is the numerator of the fractional term, which is 205.

To confirm the answer:

1 **Define** the function ***f*(*x*)**.

2 Enter ***f*(–4)**, which is the value of x when $x + 4 = 0$.

3 Press **EXE**.

4 The remainder is 205.

The remainder is 205.

The factor theorem

We know 8 goes into 24 evenly leaving no remainder. For this reason, we call 8 a factor of 24. Similarly, if the division of a polynomial $P(x)$ by $(x - a)$ leaves no remainder, then the divisor $(x - a)$ must be a factor of the polynomial $P(x)$.

The **factor theorem** is a direct result of the remainder theorem. If a polynomial is divided by one of its factors, then there is no remainder.

> **The factor theorem**
>
> - For a polynomial $P(x)$, if $P(a) = 0$, then $(x - a)$ is a factor of the polynomial.
> - a is called a **zero** of the polynomial $P(x)$ or a **root** of the polynomial *equation* $P(x) = 0$.
> - Also, if $P\left(-\frac{b}{a}\right) = 0$, then the $(ax + b)$ is a factor of the polynomial.
>
> The converse of this theorem is also true.
>
> - If $(x - a)$ is a factor of $P(x)$, then $P(a) = 0$.
> - If $(ax + b)$ is a factor of $P(x)$, then $P\left(-\frac{b}{a}\right) = 0$.

Factorising polynomials

Worksheet Factorising polynomials

Long division

The factor theorem can be used to factorise a polynomial. If $x - a$ is a factor of the polynomial, then a is a factor of the **constant term** of the polynomial. This can help us to narrow down the search for factors. It is always worth trying 1 and −1.

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WORKED EXAMPLE 14 Factorising polynomials using long division 1

a Show that $x - 1$ is a factor of $P(x) = 2x^3 - x^2 - 7x + 6$.

b Using long division, divide $P(x)$ by $x - 1$.

c Factorise $P(x)$ completely.

Steps	Working
a **1** Write down $P(x)$.	$P(x) = 2x^3 - x^2 - 7x + 6$
2 To show that $(x - 1)$ is a factor of $P(x)$, we must prove $P(1) = 0$.	$P(1) = 2(1)^3 - (1)^2 - 7(1) + 6$ $= 2 - 1 - 7 + 6$ $= 0$
3 State your answer.	$\therefore\ x - 1$ is a factor of $P(x)$.
b **1** Use the process of polynomial long division to divide $x - 1$ into $P(x)$.	$\begin{array}{r} 2x^2 + x - 6 \\ x-1\,\overline{)\,2x^3 - x^2 - 7x + 6} \\ \underline{2x^3 - 2x^2} \qquad\qquad \\ x^2 - 7x \qquad \\ \underline{x^2 - x} \qquad \\ -6x + 6 \\ \underline{-6x + 6} \\ 0 \end{array}$
2 State division in the form $P(x) = (x - a)Q(x)$.	$2x^3 - x^2 - 7x + 6 = (x - 1)(2x^2 + x - 6)$
c **1** Factorise $2x^2 + x - 6$.	$2x^2 + x - 6 = (2x - 3)(x + 2)$
2 Write the answer.	$2x^3 - x^2 - 7x + 6 = (x - 1)(2x - 3)(x + 2)$

WORKED EXAMPLE 15 Factorising polynomials using long division 2

Factorise $f(x) = x^3 + 3x^2 - 4x - 12$.

Steps	Working
1 Write the polynomial.	$f(x) = x^3 + 3x^2 - 4x - 12$
2 Try factors of 12 (i.e. ±1, ±2, ±3, ±4, ±6, ±12) until the remainder of 0 is found.	$f(1) = (1)^3 + 3(1)^2 - 4(1) - 12$ $= -12$ $\neq 0$ So $(x - 1)$ is not a factor. $f(-1) = (-1)^3 + 3(-1)^2 - 4(-1) - 12$ $= -6$ $\neq 0$ So $(x + 1)$ is not a factor. $f(2) = (2)^3 + 3(2)^2 - 4(2) - 12$ $= 0$ So $(x - 2)$ is a factor.
3 Use the process of polynomial long division to divide $x - 2$ into $f(x)$.	$\begin{array}{r} x^2 + 5x + 6 \\ x-2\overline{) x^3 + 3x^2 - 4x - 12} \\ \underline{x^3 - 2x^2} \qquad\qquad \\ 5x^2 - 4x \qquad \\ \underline{5x^2 - 10x} \qquad \\ 6x - 12 \\ \underline{6x - 12} \\ 0 \end{array}$
4 Factorise $x^2 + 5x + 6$.	$x^2 + 5x + 6 = (x + 2)(x + 3)$
5 State your answer.	$x^3 + 3x^2 - 4x - 12 = (x - 2)(x + 2)(x + 3)$

Synthetic division

Factorising a polynomial using long division can be time-consuming. Factorising by synthetic division can be quicker.

WORKED EXAMPLE 16 Factorising polynomials using synthetic division

Factorise $f(x) = x^4 - x^3 - 7x^2 + x + 6$.

Steps	Working
1 Write the polynomial.	$f(x) = x^4 - x^3 - 7x^2 + x + 6$
2 Try factors of 6 (i.e. ±1, ±2, ±3, ±6) until the remainder of 0 is found.	$f(1) = (1)^4 - (1)^3 - 7(1)^2 + (1) + 6$ $= 1 - 1 - 7 + 1 + 6$ $= 0$ So $(x - 1)$ is a factor.
3 Place the 1 in the top left-hand corner of the first row, to the left of the separation line. Bring the first coefficient, 1, down. Place the coefficients of the terms in $f(x)$ in the top row to the right of the separation line, in order of their decreasing power.	$\begin{array}{r\|rrrrr} 1 & 1 & -1 & -7 & 1 & 6 \\ & & 1 & 0 & -7 & -6 \\ \hline & 1 & 0 & -7 & -6 & 0 \end{array}$ Note: The remainder is 0, which verifies that $(x - 1)$ is a factor.

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Steps	Working
The values in the second row are found by multiplying each value in the bottom row by the 1 from $(x-1)$. The values in the bottom row are column totals, which make up the coefficients of the quotient, the last value being the remainder.	
4 Write the answer in the form $f(x) = d(x) \times Q(x) + R$.	$(x-1)(x^3 + 0x^2 - 7x - 6)$
5 Write down the quotient.	$Q(x) = x^3 + 0x^2 - 7x - 6$
6 To factorise the quotient, repeat the process by first finding a linear factor of $Q(x)$. Try factors of 6 (i.e. ±1, ±2, ±3, ±6) until the remainder of 0 is found.	$Q(-1) = (-1)^3 - 0(-1)^2 - 7(-1) - 6$ $= -1 + 7 - 6$ $= 0$
The rest of the values in the second row are found by multiplying each value in the bottom row by the -1 from $(x+1)$. The values in the bottom row are column totals, which make up the coefficients of the quotient, the last value being the remainder.	So $(x+1)$ is a factor. $\begin{array}{r\|rrrr} -1 & 1 & 0 & -7 & -6 \\ & & -1 & 1 & 6 \\ \hline & 1 & -1 & -6 & 0 \end{array}$
7 Write down the factorised form of $f(x)$.	$f(x) = (x-1)(x+1)(x^2 - x - 6)$
8 Continue factorising by factorising the quadratic polynomial.	$f(x) = (x-1)(x+1)(x-3)(x+2)$

Equating coefficients

Factorising a polynomial using the method of **equating coefficients** can also be quicker than factorising by long division.

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WORKED EXAMPLE 17 Factorising polynomials by equating coefficients

Using the method of equating coefficients, factorise $f(x) = 3x^3 + 4x^2 - 13x + 6$.

Steps	Working
1 Write the polynomial.	$f(x) = 3x^3 + 4x^2 - 13x + 6$
2 Try factors of 6 (i.e. ±1, ±2, ±3, ±6).	$f(1) = 3(1)^3 + 4(1)^2 - 13(1) + 6$ $= 0$ So $x - 1$ is a factor.
3 Write $f(x)$ in the form of $(x-a)Q(x)$, where $Q(x)$ is a quadratic written in general form, where a, b, and c are constant coefficients.	$3x^3 + 4x^2 - 13x + 6 = (x-1)(ax^2 + bx + c)$
4 Find a by equating coefficients of x^3.	$3x^3 + 4x^2 - 13x + 6 = (x-1)(ax^2 + bx + c)$ Since $x \times ax^2 = 3x^3$, therefore $a = 3$.
5 Find c by equating the constant terms.	$3x^3 + 4x^2 - 13x + 6 = (x-1)(3x^2 + bx + c)$ Since $-1 \times c = 6$, therefore $c = -6$.
6 Find b by equating coefficients of x^2. Note: b can also be found by equating coefficients of x.	$3x^3 + 4x^2 - 13x + 6 = (x-1)(3x^2 + bx - 6)$ Since $-1 \times 3x^2 + x \times bx = -3x^2 + bx^2 = 4x^2$, therefore $-3 + b = 4$, so $b = 7$.
7 State your answer in the form $P(x) = (x-a)Q(x)$.	$3x^3 + 4x^2 - 13x + 6 = (x-1)(3x^2 + 7x - 6)$
8 Factorise completely.	$= (x-1)(3x-2)(x+3)$

There are several ways to factorise a polynomial, including those shown in this chapter: namely long division, synthetic division and the method of equating coefficients. Despite this, you will **not** be expected to factorise a polynomial using a specific technique. You may choose to factorise a polynomial the way that suits you.

USING CAS 4 Finding factors of polynomials

Factorise $P(x) = x^3 - 8x^2 + 19x - 12$.

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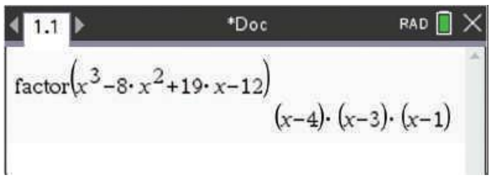
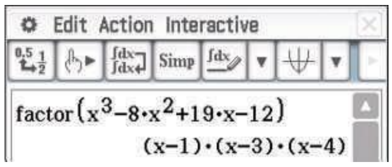


1. In a **Calculator** page, press **menu > Algebra > Factor**.
2. Enter the polynomial as shown above.
3. Press **enter**.
4. The factors will be displayed.

ClassPad



1. In **Main,** enter and highlight the polynomial.
2. Tap **Interactive > Transformation > factor > factor**. Note that **rfactor** is used for RootFactor using surds.
3. In the dialogue box, tap **OK**.
4. The factors will be displayed.

The factors are $(x - 4)$, $(x - 3)$ and $(x - 1)$.

Sum and difference of two cubes

We already know the difference of two squares: $a^2 - b^2 = (a - b)(a + b)$. Now we will extend our factorising skills to the **difference** and the **sum** of two **cubes**. The **sum of two cubes** and the **difference of two cubes** formulas can be used to factorise some cubic polynomials.

Sum of two cubes

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$

Difference of two cubes

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

Exam hack

When trying to remember these patterns, remember that the first binomial term in the factorised form in each pattern keeps the same sign as the sign between the perfect cubes. The sign separating the first and second terms of the trinomial is opposite the sign between the perfect cubes, and the last sign in the trinomial is always positive.

$$a^3 \pm b^3 = (a[\textit{same sign}]b)(a^2[\textit{opposite sign}]ab + b^2)$$

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WORKED EXAMPLE 18 Factorising using the sum and differences of two cubes

Factorise each expression.

a $x^3 - 8y^3$

b $9(x+2)^3 + 9$

Steps	Working
a **1** Recognise the difference of two cubes.	$x^3 - 8y^3 = x^3 - (2y)^3$
2 Use the formula $a^3 - b^3 = (a-b)(a^2+ab+b^2)$, where $a = x$ and $b = 2y$.	$(x-2y)(x^2+2xy+4y^2)$
b **1** Take out the highest common factor of 9.	$9(x+2)^3 + 9 = 9[(x+2)^3 + 1]$
2 Recognise the sum of two cubes.	$= 9[(x+2)^3 + 1^3]$
3 Use the formula $a^3 + b^3 = (a+b)(a^2-ab+b^2)$, where $a = x+2$ and $b = 1$.	$= 9(x+2+1)[(x+2)^2 - 1(x+2) + 1^3]$
4 Simplify.	$= 9(x+3)(x^2+4x+4-x-2+1)$ $= 9(x+3)(x^2+3x+3)$

Exam hack

When we factorise using the sum or difference of two cubes, the second bracket doesn't factorise further. This can be shown by working out the discriminant and seeing that it is always less than 0. In Worked example 12b, the second bracket is $x^2 + 3x + 3$.

$$\begin{aligned}\Delta &= b^2 - 4ac\\ &= (3)^2 - 4(1)(3)\\ &= -3\\ &< 0\end{aligned}$$

EXERCISE 3.4 The remainder and factor theorems

ANSWERS p. 500

Recap

1 Given that $P(x) = 4x^3 - 2x^2 + 3x - 1$, find the quotient when it is divided by $(x+1)$.

2 Divide $P(x) = 3x^3 - 2x^2 + x + 1$ by $d(x) = 3x + 1$ and state the remainder.

Mastery

3 WORKED EXAMPLE 11 TECH-FREE Find the remainder for each division.

a $(x^3 - 2x^2 + x + 5) \div (x - 4)$

b $(x^2 + 5x + 3) \div (x + 2)$

c $(2x^3 - 4x - 1) \div (x + 3)$

d $(3x^5 + 2x^2 - x + 4) \div (x - 5)$

e $(5x^3 + 2x^2 + 2x - 9) \div (x - 1)$

f $(x^4 - x^3 + 3x^2 - x - 1) \div (x + 2)$

4 Using CAS 3 Find the remainder for each division.

a $(2x^2 + 7x - 2) \div (x + 7)$

b $(x^7 + 5x^3 - 1) \div (x - 3)$

c $(2x^6 - 3x^2 + x + 4) \div (x + 5)$

d $(3x^4 - x^3 - x^2 - x - 7) \div (x + 1)$

5 WORKED EXAMPLE 12 TECH-FREE Find the value of k if

a the remainder is 3 when $5x^2 - 10x + k$ is divided by $x - 1$

b the remainder is -4 when $x^3 - (k - 1)x^2 + 5kx + 4$ is divided by $x + 2$

c the remainder is 0 when $2x^5 + 7x^2 + 1 + k$ is divided by $x + 6$

d $2x^4 - kx^3 + 3x^2 + x$ is divisible by $x - 3$

e the remainder is 25 when $2x^4 - 3x^2 + 5$ is divided by $x - k$.

6 When $f(x) = ax^2 - 3x + 1$ and $g(x) = x^3 - 3x^2 + 2$ are divided by $x + 1$, they leave the same remainder. Find the value of a.

7 WORKED EXAMPLE 13 $f(x) = 2x^3 + ax^2 + bx - 20$, where a, b are constants.

a Given that when $f(x)$ is divided by $(x + 3)$ the remainder is 7, show that $9a - 3b = 81$.

b Given also that when $f(x)$ is exactly divided by $(x - 2)$, find the value of a and b.

8 The remainder is 5 when $P(x) = ax^3 - 4bx^2 + x - 4$ is divided by $x - 3$, and the remainder is 2 when $P(x)$ is divided by $x + 1$. Find the values of a and b.

9 WORKED EXAMPLE 14

a Find the remainder when $f(x) = x^3 - 4x^2 + x + 6$ is divided by $x - 2$.

b Is $x - 2$ a factor of $f(x)$?

c Using long division, divide $x^3 - 4x^2 + x + 6$ by $x - 2$.

d Factorise $f(x)$ fully and write $f(x)$ as a product of its factors.

10 WORKED EXAMPLE 15 TECH-FREE Factorise each polynomial.

a $f(x) = x^3 + 4x^2 - 7x - 10$

b $f(x) = x^3 + 2x^2 - 41x - 42$

c $f(x) = x^3 + 2x^2 - 4x - 8$

d $f(x) = 2x^3 + 7x^2 - 10x - 24$

11 WORKED EXAMPLE 16 TECH-FREE Factorise each polynomial using long division or synthetic division.

a $f(x) = 2x^3 + x^2 - 22x + 24$

b $f(x) = 2x^3 - 11x^2 - x + 30$

c $f(x) = x^3 - 5x^2 - x + 5$

d $f(x) = x^3 - 9x^2 + 26x - 24$

e $f(x) = 2x^3 + x^2 - 13x + 6$

f $f(x) = x^3 + 2x^2 - x - 2$

12 WORKED EXAMPLE 17 By equating coefficients, factorise each polynomial.

a $f(x) = 3x^3 - 16x^2 - 37x + 14$

b $f(x) = 2x^3 - 3x^2 - 11x + 6$

c $f(x) = x^3 - 7x - 6$

d $f(x) = -4x^3 - x^2 + 27x - 18$

e $f(x) = -5x^3 + 26x^2 - 35x + 6$

f $f(x) = -30 + x - x^3 + 6x^2$

13 Using CAS 4 Factorise each polynomial.

a $f(x) = x^3 - 7x^2 + 4x + 12$

b $f(x) = x^3 + 2x^2 - 9x - 18$

c $f(x) = 2x^3 - 5x^2 - 23x - 10$

d $f(x) = 2x^3 + 3x^2 - 11x - 6$

14 WORKED EXAMPLE 18 TECH-FREE Factorise each polynomial.

a $x^3 + 27$

b $1 - x^3$

c $1000 + 8t^3$

d $\dfrac{x^3}{8} - 27$

e $y^3 + (3 + x)^3$

3.4

Exam practice

80–100% 60–79% 0–59%

15 © VCAA 2015 2AQ6 91% For the polynomial $P(x) = x^3 - ax^2 - 4x + 4$, $P(3) = 10$, the value of a is

A -3 **B** -1 **C** 1 **D** 3 **E** 10

16 © VCAA 2011 2AQ3 64% If $x + a$ is a factor of $4x^3 - 13x^2 - ax$, where $a \in R\backslash\{0\}$, then the value of a is

A -4 **B** -3 **C** -1 **D** 1 **E** 2

17 © VCAA 2015 2AQ17 60% A graph with rule $f(x) = x^3 - 3x^2 + c$, where c is a real number, has three distinct x-intercepts. The set of all possible values of c is

A R **B** R^+ **C** $\{0, 4\}$ **D** $(0, 4)$ **E** $(-\infty, 4)$

18 © VCAA 2013 2AQ3 59% If $x + a$ is a factor of $7x^3 + 9x^2 - 5ax$, where $a \in R\backslash\{0\}$, then the value of a is

A -4 **B** -2 **C** -1 **D** 1 **E** 2

19 © VCAA 2019N 2AQ3 If $x + a$ is a factor of $8x^3 - 14x^2 - a^2x$, where $a \in R\backslash\{0\}$, then the value of a is

A 7 **B** 4 **C** 1 **D** -2 **E** -1

20 © VCAA 2004 1AQ9 52% Which one of the following is a **complete** set of **linear** factors of the third degree polynomial $ax^3 - bx$, where a and b are positive real numbers?

A $x,\ ax^2 - b$

B $x,\ ax - b,\ ax + b$

C $x,\ \sqrt{ax - b}$

D $x,\ \sqrt{a}x - b,\ \sqrt{a}x + b$

E $x,\ \sqrt{a}x - \sqrt{b},\ \sqrt{a}x + \sqrt{b}$

Video playlist
The rational root theorem

3.5 The rational root theorem

The **rational root theorem** states the properties of the rational roots of a polynomial equation $P(x) = 0$.

Suppose we have to factorise $P(x) = 4x^3 - 9x^2 + 6x - 6$, whose roots are rational and of the form $x = \frac{p}{q}$ where p and q are integers. So that means $\left(x - \frac{p}{q}\right)$ is a factor of $P(x)$ and therefore so is $\frac{1}{q}(qx - p)$ and so is $(qx - p)$.

So $4x^3 - 9x^2 + 6x - 6 = (qx - p)(ax^2 + bx + c)$.

On the LHS, the constant term is -6. If we expand the RHS, the constant term is $-pc$.

Therefore, p must be an integer factor of -6, so it could be ±1, ±2, ±3 *or* ±6.

On the LHS, the leading coefficient is 4. If we expand the RHS, the leading coefficient is qa.

Therefore, q must be an integer factor of 4, so it could be ±1, ±2, *or* ±4.

The rational root theorem

If $P(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x^1 + a_0$ has integer coefficients, then every rational root of $P(x)$ has the form

$$\frac{p}{q} = \pm\frac{\text{factor of the constant term, } a_0}{\text{factor of the leading coefficient, } a_n}$$

where p and q are integers.

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The rational root theorem does not identify any irrational roots that the polynomial may have.

WORKED EXAMPLE 19 The rational root theorem

Find the possible rational roots of $P(x) = 4x^3 - 9x^2 + 6x - 6 = 0$.

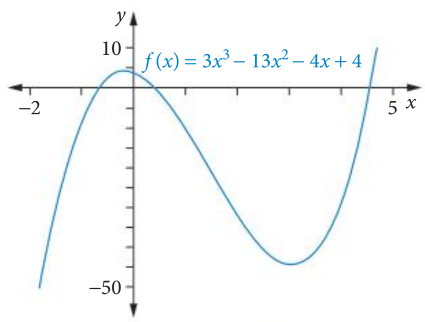
Steps	Working
1 Find the factors of the constant term, 6.	Factors of 6 are 1, 2, 3, 6.
2 Find the factors of the leading coefficient, 4.	Factors of 4 are 1, 2, 4.
3 Write the factors of the constant term over the factors of the leading coefficient.	$\frac{p}{q} = \pm\frac{\text{factors of 6}}{\text{factors of 4}}$ $\frac{p}{q} = \pm\frac{1,\ 2,\ 3,\ 6}{1,\ 2,\ 4}$
4 Write all possible rational roots separately (plus or minus).	$\pm 1, \pm\frac{1}{2}, \pm\frac{1}{4}, \pm 2, \pm 3, \pm\frac{3}{2}, \pm\frac{3}{4}, \pm 6$

WB p. 61

WB p. 62

WORKED EXAMPLE 20 Factorising using the rational root theorem

$f(x) = 3x^3 - 13x^2 - 4x + 4$ has rational zeros. Factorise $f(x)$.

Steps	Working
1 Find the factors of the constant term, 4.	Factors of 4 are 1, 2, 4.
2 Find the factors of the leading coefficient, 3.	Factors of 3 are 1, 3.
3 Write the factors of the constant term over the factors of the leading coefficient.	$\frac{p}{q} = \pm\frac{\text{factors of 4}}{\text{factors of 3}}$ $\frac{p}{q} = \pm\frac{1, 2, 4}{1, 3}$
4 Write all possible rational zeros separately.	$\pm 1, \pm\frac{1}{3}, \pm 2, \pm\frac{2}{3}, \pm 4, \pm\frac{4}{3}$
5 Instead of checking each possibility, we can narrow the choices by graphing $f(x)$ to identify possible zeros (x-intercepts).	 From the graph, $x = -\frac{2}{3}$ could be a zero.
6 Use the factor theorem to test.	$f\left(-\frac{2}{3}\right) = 3\left(-\frac{2}{3}\right)^3 - 13\left(-\frac{2}{3}\right)^2 - 4\left(-\frac{2}{3}\right) + 4$ $= 0$ Therefore, $x = -\frac{2}{3}$ is a zero and $\left(x + \frac{2}{3}\right)$ is a factor.
7 Factorise $f(x)$ further.	$f(x) = \left(x + \frac{2}{3}\right)(3x^2 - 15x + 6)$

8	Take out the common factor.	$= 3\left(x + \frac{2}{3}\right)(x^2 - 5x + 2)$
9	Remove the fraction.	$= (3x + 2)(x^2 - 5x + 2)$
10	Factorise $(x^2 - 5x + 2)$ by completing the square.	$x^2 - 5x + 2$ $= \left(x^2 - 5x + \frac{25}{4}\right) - \frac{25}{4} + 2$ $= \left(x - \frac{5}{2}\right)^2 - \frac{17}{4}$ $= \left(x - \frac{5}{2} - \frac{\sqrt{17}}{2}\right)\left(x - \frac{5}{2} + \frac{\sqrt{17}}{2}\right)$ $= \left(x - \frac{5 + \sqrt{17}}{2}\right)\left(x - \frac{5 - \sqrt{17}}{2}\right)$
11	Write the answer.	$f(x) = (3x + 2)\left(x - \frac{5 + \sqrt{17}}{2}\right)\left(x - \frac{5 - \sqrt{17}}{2}\right)$

Note from the answer that the other 2 zeros are irrational, $x = \frac{5 \pm \sqrt{17}}{2} \approx 4.56, -0.44$, as seen on the graph.

EXERCISE 3.5 The rational root theorem

ANSWERS p. 500

Recap

1 TECH-FREE Use the factor theorem to test whether $2x - 3$ is a factor of $P(x) = 5x^3 - 7x^2 + 5x + 3$.

2 TECH-FREE If $P(x) = 5x^3 - x^2 + 3x - 1$ and $Q(x) = 3x^3 - 8x^2 + 3x$, find $Q(x) - P(x)$.

Mastery

3 WORKED EXAMPLE 19 TECH-FREE Find the possible rational zeros for each polynomial.

a $3x^2 + 2x - 1$ b $x^2 + 8x + 10$ c $5x^3 - 2x^2 + 20x - 8$

d $x^3 - 27$ e $2x^4 - 9x^2 + 7$ f $5x^4 + 7x^3 + 32x^2 - 21$

4 WORKED EXAMPLE 20 TECH-FREE For each function

i find the possible rational roots

ii factorise the function.

a $x^3 + x^2 - 5x + 3$ b $x^3 - 13x^2 + 23x - 11$ c $4x^3 - 9x^2 + 6x - 1$

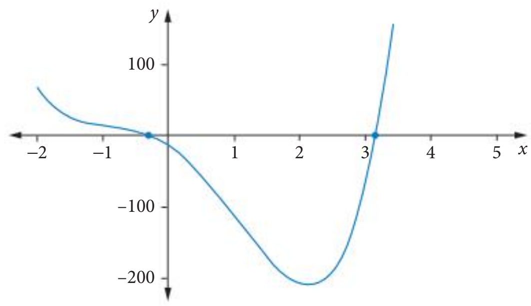
d $3x^3 + 11x^2 + 5x - 3$ e $3x^3 + 9x^2 + 4x + 12$ f $2x^3 + 9x^2 + 19x + 15$

g $5x^4 - 46x^3 + 84x^2 - 50x + 7$ h $3x^4 - 10x^3 - 24x^2 - 6x + 5$

5 Suppose that $\frac{7}{5}$ is a root of a polynomial equation with integer coefficients. What can you say about the leading coefficient and the constant term of the polynomial?

6 Does the rational root theorem apply to finding roots of $Q(x) = 10x^4 - 4\sqrt{2}x + 9$? Explain your response.

7 The graph of a polynomial function with equation $R(x) = 7x^4 - 2x^3 - 42x^2 - 61x - 14$ is shown.



a List all possible rational zeros of R.

b Which possible rational zeros of R are actual zeros?

Exam practice

8 (3 marks) Consider the function f, where $f(n) = 2n^4 + 5n^3 + 12$.

a List all possible rational zeros of this function. 2 marks

b Graph f and explain why it has no rational zeros. 1 mark

3.6 Solving polynomial equations

Video playlist Solving polynomial equations

An equation such as $x^3 + 3x + 5 = 0$ (cubic, degree 3) is not easily solved. In fact, only a small proportion of cubic and higher polynomial equations can be solved easily, and usually by using the factor theorem.

We use the **null factor law** to solve cubic and quartic equations, as we do with quadratic equations.

The null factor law for cubic equations

If $abc = 0$, then at least one of $a = 0$, $b = 0$, $c = 0$ is true.

If $P(x) = (x - a)(x - b)(x - c) = 0$, then the solutions are $x = a$, $x = b$, $x = c$.

This can be extended to any product of factors.

Solving cubic equations

A cubic equation should be written in **standard form**, $ax^3 + bx^2 + cx + d = 0$. The equation $x^2 + 4x - 1 = \frac{6}{x}$ is a cubic, though it is not written in the standard form. We need to multiply through by x and then subtract 6 from both sides, giving us $x^3 + 4x^2 - x - 6 = 0$. This is now in standard form.

Exam hack

Always look for a common factor, grouping in pairs, a perfect cube, sum or difference of cubes before trying other factorisation methods for cubic equations.

WORKED EXAMPLE 21 Solving equations involving perfect cubes

WB p. 63

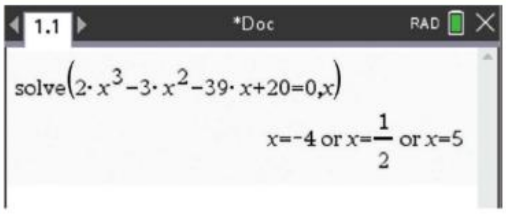
Solve $2(x - 1)^3 - 250 = 0$.

Steps	Working
1 Add 250 to both sides.	$2(x - 1)^3 = 250$
2 Divide both sides by 2.	$(x - 1)^3 = 125$
3 Take the cube root of both sides.	$x - 1 = 5$
4 Add 1 to both sides.	$x = 6$

USING CAS 5 Solving polynomial equations

Solve $2x^3 - 3x^2 - 39x + 20 = 0$.

TI-Nspire



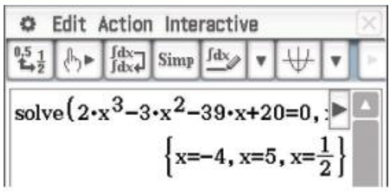
1 In a **Calculator** page, press **menu > Algebra > Solve**.

2 Enter the equation followed by **,x** as shown above.

3 Press **enter**.

4 The solutions will be displayed.

ClassPad



1 In **Main**, set the mode to **Standard**.

2 Enter and highlight the equation.

3 Tap **Interactive > Equation/Inequality > solve**.

4 In the dialogue box, tap **OK**.

5 The solutions will be displayed.

The solutions are $x = -4$, $x = \dfrac{1}{2}$ and $x = 5$.

Solving quartic equations

WB

p. 64

WORKED EXAMPLE 22 Solving quartic equations

Solve $6x^4 + 15x^3 - 12x^2 - 9x = 0$.

Steps	Working
1 Take $3x$ out as a common factor.	$3x(2x^3 + 5x^2 - 4x - 3) = 0$
2 Use the cubic polynomial and try factors of 3 (i.e. $\pm1, \pm2, \pm3$).	$P(x) = 2x^3 + 5x^2 - 4x - 3$ $P(1) = 2(1)^3 + 5(1)^2 - 4(1) - 3$ $= 0$ $x - 1$ is a factor of $P(x)$.
3 **Method 1: Using long division** Divide $P(x)$ by $x - 1$ by using long division.	$$\begin{array}{r} 2x^2 + 7x + 3 \\ x-1\overline{)\,2x^3 + 5x^2 - 4x - 3} \\ \underline{2x^3 - 2x^2} \\ 7x^2 - 4x \\ \underline{7x^2 - 7x} \\ 3x - 3 \\ \underline{3x - 3} \\ 0 \end{array}$$
Method 2: Using synthetic division Divide $P(x)$ by $x - 1$ by using synthetic division.	$$\begin{array}{r\|rrrr} 1 & 2 & 5 & -4 & -3 \\ & & 2 & 7 & 3 \\ \hline & 2 & 7 & 3 & 0 \end{array}$$ Note: The remainder is 0, which verifies that $(x - 1)$ is a factor.

3.6

Method 3: By equating coefficients Divide $P(x)$ by $x-1$ by equating coefficients. Write $f(x)$ in the form of $(x-a)Q(x)$, where $Q(x)$ is a quadratic written in general form ax^2+bx+c, where a, b and c are constant coefficients.	$2x^3+5x^2-4x-3=(x-1)(ax^2+bx+c)$ $a=2 \quad b-a=5 \quad -c=-3$ $b-2=5 \quad c=3$ $b=7$ $a=2, b=7, c=3$
4 State the polynomial as a product of the factors found so far.	$3x(2x^3+5x^2-4x-3)=0$ $3x(x-1)(2x^2+7x+3)=0$
5 Factorise the quadratic.	$3x(x-1)(2x+1)(x+3)=0$
6 Use the null factor law.	$3x=0$ or $x-1=0$ or $2x+1=0$ or $x+3=0$
7 Solve the equation.	$x=0$ or $x=1$ or $x=-\frac{1}{2}$ or $x=-3$

EXERCISE 3.6 Solving polynomial equations

ANSWERS p. 501

Recap

1 TECH-FREE Use the rational root theorem to find the possible rational zeros for $8x^3-1$.

2 TECH-FREE Factorise x^3+x^2-5x+3.

Mastery

3 WORKED EXAMPLE 21 TECH-FREE Solve each cubic equation.

a $3(x-4)^3-24=0$ **b** $3(x+5)^3+81=0$

c $-2(x-2)^3-2=0$ **d** $(3x-2)^3-8=0$

4 WORKED EXAMPLE 22 TECH-FREE Solve each cubic equation.

a $2x^3+x^2-15x=0$ **b** $x^3+3x^2-10x-24=0$

c $x^3+x^2-9x-9=0$ **d** $12-19x+8x^2-x^3=0$

5 Using CAS 5 Solve each cubic equation.

a $x^3-13x+12=0$ **b** $-x^3+2x^2+9x-18=0$

c $x^3+2x^2-4x-8=0$ **d** $x^3-5x^2+8x-4=0$

6 TECH-FREE Solve each quartic equation.

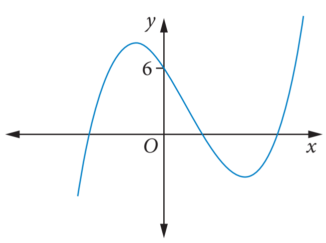
a $x(x+2)(x-3)(x-9)=0$ **b** $x^4+3x^3+2x^2=0$ **c** $-x^4+2x^3+3x^2=0$

d $x^4-3x^3-4x^2+12x=0$ **e** $4x^4+8x^3-x^2-2x=0$

Exam practice

80–100% 60–79% 0–59%

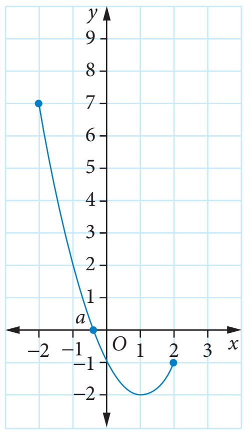
7 © VCAA 2005 non-CAS 1IIQ6 TECH-FREE (3 marks) Part of the graph of the curve with equation $y=x^3-2x^2-5x+6$ is shown.



a 73% Write the equation in the form $y=(x+2)(x^2+bx+c)$. 1 mark

b 72% Hence, by factorising, find the **exact** values of the x-intercepts. 2 marks

8 © VCAA 2013 1Q9a 50% TECH-FREE (1 mark) The graph of $f(x) = (x - 1)^2 - 2$, $x \in [-2, 2]$, is shown. The graph intersects the x-axis where $x = a$. Find the value of a.



9 © VCAA 2003 1Q20 61% Let $p(x) = (x^2 + a)(x + b)(x - c)$, where a, b and c are three distinct positive real numbers.

The number of real solutions to the equation $p(x) = 0$ is exactly

A 0 **B** 1 **C** 2 **D** 3 **E** 4

10 © VCAA 2011 2AQ18 52% The equation $x^3 - 9x^2 + 15x + w = 0$ has only one solution for x when

A $-7 < w < 25$ **B** $w \leq -7$ **C** $w \geq 25$

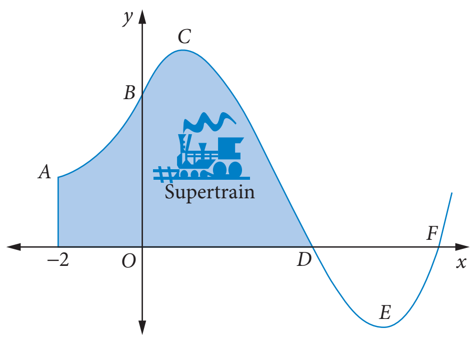
D $w < -7$ or $w > 25$ **E** $w > 1$

11 © VCAA 2018N 2AQ5 A set of three numbers that could be the solutions of $x^3 + ax^2 + 16x + 84 = 0$ is

A $\{3, 4, 7\}$ **B** $\{-4, -3, 7\}$ **C** $\{-2, -1, 21\}$

D $\{-2, 6, 7\}$ **E** $\{2, 6, 7\}$

12 © VCAA 2006 2BQ4aic (3 marks) A part of the track for Tim's model train follows the curve passing through A, B, C, D, E and F as shown. Tim has designed it by putting axes on the drawing as shown. The track is made up of two curves, one to the left of the y-axis and the other to the right.



B is the point $(0, 7)$.

The curve from B to F is part of the graph of $f(x) = px^3 + qx^2 + rx + s$ where p, q, r and s are constants.

a 86% Show that $s = 7$. 1 mark

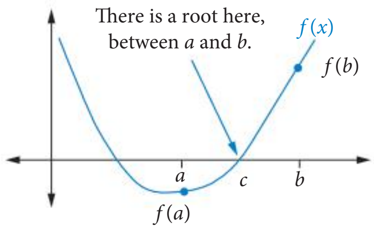
b 64% Assume $f(x) = 0.25x^3 - 2.5x^2 + 4.25x + 7$.

Find the exact coordinates of D and F. 2 marks

3.7 The bisection method

Polynomial equations are either difficult or impossible to solve algebraically. For this reason, we need to use numerical approximation methods that start with a good guess and then repeat a procedure to keep improving on the guesses. These are called 'iterative methods', where iterative means 'repeated'. The **bisection method** is one such method.

Consider a continuous (smooth) function $y = f(x)$ that has a zero at $x = c$ such that $f(c) = 0$. So $x = c$ is a root of the equation $f(x) = 0$ and the graph of $y = f(x)$ has an x-intercept at c.



Sign of $f(a) \neq$ sign of $f(b)$

Video playlist
The bisection method

The bisection method tests values of $f(x)$ until it finds where $f(x)$ is positive for one value of x and negative for another value of x, then finds an approximation for the root that is somewhere between these 2 values.

Suppose $f(x)$ is negative when $x = a$ and positive when $x = b$, as shown on the diagram.

Then, for a value of x between a and b, $f(x)$ must be equal to 0. The root of $f(x) = 0$ is in the interval $[a, b]$. The bisection method halves the interval (bisect means 'to halve') to find a smaller interval in which the sign changes. We repeat this to get closer and closer to the root.

WORKED EXAMPLE 23 The bisection method

WB p. 65

Use the bisection method to find an approximate root of $x^3 - 6x^2 - 33 = 0$ in the interval [4, 8]. Answer correct to 2 decimal places.

Steps	Working
1 Let $f(x) = x^3 - 6x^2 - 33$ and evaluate $f(4)$ and $f(8)$ to check that they are opposite in sign.	$f(4) = 4^3 - 6(4^2) - 33 = -65$ negative $f(8) = 8^3 - 6(8^2) - 33 = 95$ positive So the root lies in [4, 8].
2 Halve the interval and test the function at the halfway point.	$f\left(\frac{4+8}{2}\right) = f(6) = -33$ negative So the root lies in [6, 8] because $f(8)$ is positive.
3 Halve the interval again and test the function at the halfway point.	$f\left(\frac{6+8}{2}\right) = f(7) = 16$ positive So the root lies in [6, 7] because $f(6)$ is negative.

4 Repeat. Notice that we are getting closer to the root because $f(x)$ is getting closer to 0.

*Round approximations to 2 decimal places.

$f\left(\frac{6+7}{2}\right) = f(6.5) = -11.875$

So the root lies in [6.5, 7] because $f(7)$ is positive.

$f\left(\frac{6.5+7}{2}\right) = f(6.75) \approx 1.172$

So the root lies in [6.5, 6.75] because $f(6.5)$ is negative.

$f\left(\frac{6.5+6.75}{2}\right) = f(6.63)^* \approx -5.307$

So the root lies in [6.63, 6.75] because $f(6.75)$ is positive.

$f\left(\frac{6.63+6.75}{2}\right) = f(6.69) \approx -2.118$

So the root lies in [6.69, 6.75] because $f(6.75)$ is positive.

$f\left(\frac{6.69+6.75}{2}\right) = f(6.72) \approx -0.486$

So the root lies in [6.72, 6.75] because $f(6.75)$ is positive.

$f\left(\frac{6.72+6.75}{2}\right) = f(6.74)^* \approx 0.616$

So the root lies in [6.72, 6.74] because $f(6.72)$ is negative.

$f\left(\frac{6.72+6.74}{2}\right) = f(6.73) \approx 0.064$

So the root lies in [6.72, 6.73] because $f(6.73)$ is positive.

5 Of all the guesses, $f(6.73)$ is closest to 0 (correct to 2 decimal places).

$x \approx 6.73$ is a root of the equation $x^3 - 6x^2 - 33 = 0$ in the interval [4, 8].

Note that the bisection method requires a lot of effort and time because it requires repeated steps, so it is more efficiently implemented using technology such as calculators, spreadsheets or computers. Furthermore, this method has been used to find one root *within a given interval*. There may be *other* roots of the equation in a different interval or even within the same interval.

The bisection method for finding approximate roots of $f(x) = 0$

- Find a and b where $f(a)$ and $f(b)$ are opposite in sign: this means that a root is in the interval $[a, b]$.
- Bisect the interval by testing the function at the average of a and b: evaluate $f\left(\frac{a+b}{2}\right)$ and examine its sign.
- Check which half-interval contains the root.
- Repeat bisecting the interval until it is small enough to approximate the root.

EXERCISE 3.7 The bisection method

ANSWERS p. 501

Recap

1 TECH-FREE Solve $2(x-5)^3 - 16 = 0$.

2 TECH-FREE Solve $x^4 + 3x^3 + 2x^2 = 0$.

Mastery

3 WORKED EXAMPLE 23 Use the bisection method to approximate, correct to one decimal place, a root of each equation in the given interval.

a $x^3 - 4x^2 - 11x + 2 = 0$, $[0, 4]$

b $2x^4 + x^2 - 15 = 0$, $[0, 2]$

c $x^5 - 2x^3 - 12 = 0$, $[1, 5]$

d $4x^3 - 2x + 7 = 0$, $[-3, 1]$

e $3x^3 - 14x^2 - 27x + 126 = 0$, $[4, 6]$

3.8 Graphing polynomial functions

Graphing cubic functions

A **cubic function** is a polynomial of degree 3, whose equation has the general form $y = ax^3 + bx^2 + cx + d$. The simplest cubic function is $y = x^3$, shown here. The centre of the graph at (0, 0) is flat and is a special turning point called a **point of inflection**, where the graph changes from concave up to concave down, or from concave down to concave up. (Inflection means 'bend'.)

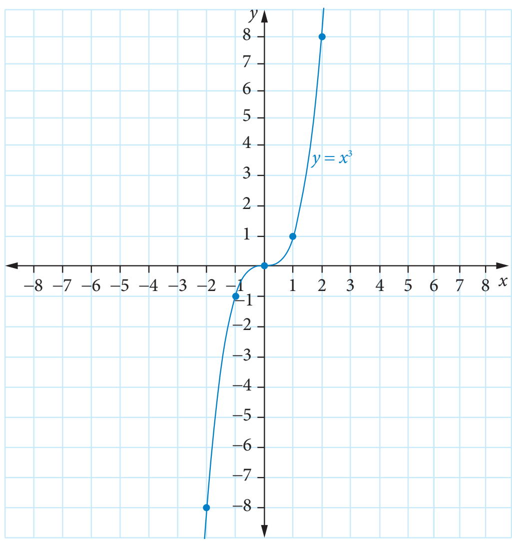
Video playlist Graphing polynomial functions

Worksheets Graphing cubics

Polynomial graphs

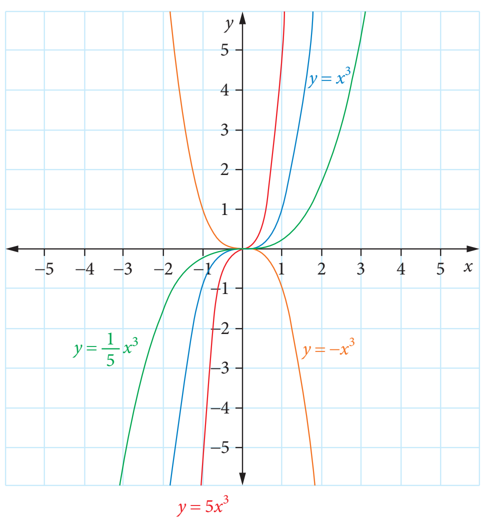
Polynomials review

Sketching curves



The graph of $y = ax^3$

The graphs of $y = ax^3$ are shown, where the value of the coefficient a makes the graph more steep or flat. If a is negative, such as in $y = -x^3$, then the graph is reflected in the x-axis, pointing in the opposite directions.



Compared to the graph of $y = x^3$, the effect of the value of a is to dilate ('stretch' or 'squash') $y = ax^3$ in the y-direction.

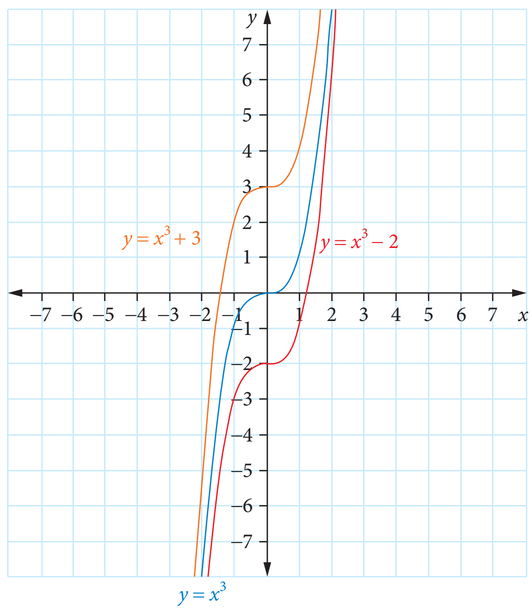
If a is large ($y = 5x^3$, for example), the cubic is **steeper**.

If a is small ($y = \frac{1}{5}x^3$, for example), the cubic is **flatter**.

If a is negative ($y = -x^3$, for example), the cubic is **reflected in the *x*-axis** (flipped upside-down).

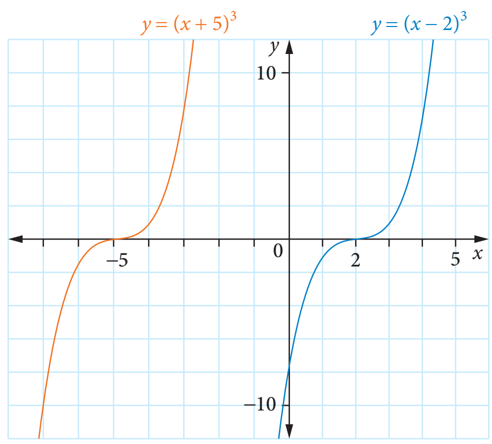
The graph of $y = x^3 + k$

The graphs of $y = x^3 + k$ are shown, where the value of the constant k moves the graph of $y = x^3$, k units up (or down if k is negative), so that k is the y-intercept and $(0, k)$ is the point of inflection.



The graph of $y = (x - h)^3$

The graphs of $y = (x - h)^3$ are shown, where the value of the constant h moves the graph of $y = x^3$, h units right (or left if h is negative), so that h is the x-intercept and $(h, 0)$ is the point of inflection.



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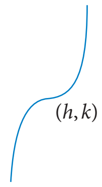
The graph of $y = a(x - h)^3 + k$

In Chapter 2 we learned that compared to the graph $y = x^2$, the graph of $y = a(x - h)^2 + k$ is a parabola that is stretched by a factor a from the x-axis and has turning point at (h, k).

The same is true for the cubic curve $y = a(x - h)^3 + k$, which is the graph of $y = ax^3$ that is stretched by a factor a from the x-axis and has point of inflection at (h, k).



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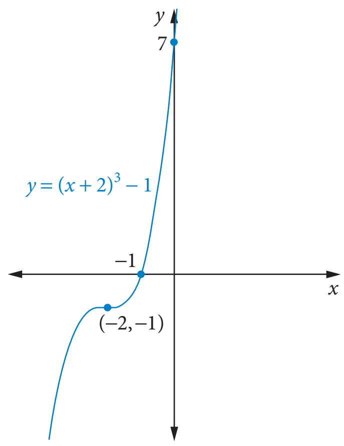
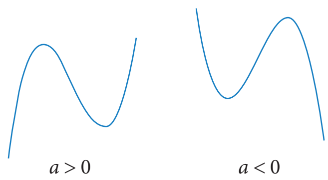
WORKED EXAMPLE 24 Graphing $y = (x - h)^3 + k$

Sketch the graph of $y = (x + 2)^3 - 1$.

Steps	Working
1 Compare $y = (x + 2)^3 - 1$ with $y = x^3$.	$y = (x + 2)^3 - 1$ is $y = x^3$ translated 2 units left and 1 unit down. Its new centre (point of inflection) is $(-2, -1)$.
2 Find its x-intercept(s).	Substitute $y = 0$: $0 = (x + 2)^3 - 1$ $1 = (x + 2)^3$ $\sqrt[3]{1} = x + 2$ $x + 2 = 1$ $x = -1$
3 Find its y-intercept.	Substitute $x = 0$: $y = (0 + 2)^3 - 1$ $= 7$
4 Sketch the graph. It will have the same shape as $y = x^3$.	

The graph of $y = ax^3 + bx^2 + cx + d$

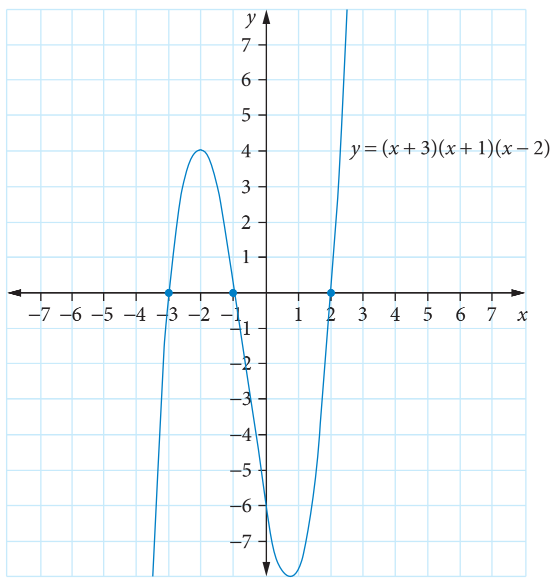
The graph of the general cubic function $y = ax^3 + bx^2 + cx + d$ does not have a flat point of inflection. Instead, it has 2 turning points or vertices: a maximum point and a minimum point. The shape of the cubic depends on the sign of the leading coefficient of the function, a:



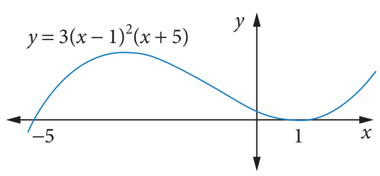
- If a is positive, the cubic curve starts at the bottom and finishes at the top: that is, $f(x) \to -\infty$ as $x \to -\infty$, and $f(x) \to +\infty$ as $x \to +\infty$.
- If a is negative, the cubic curve starts at the top and finishes at the bottom: that is, $f(x) \to +\infty$ as $x \to -\infty$, and $f(x) \to -\infty$ as $x \to +\infty$.

The graph of $y = a(x - b)(x - c)(x - d)$

If a cubic function can be factorised using the factor theorem so that it is of the form $\boldsymbol{y = a(x - b)(x - c)(x - d)}$, then b, c and d are its x-intercepts. For example, $y = (x + 3)(x + 1)(x - 2)$ is shown, with x-intercepts at -3, -1 and 2.

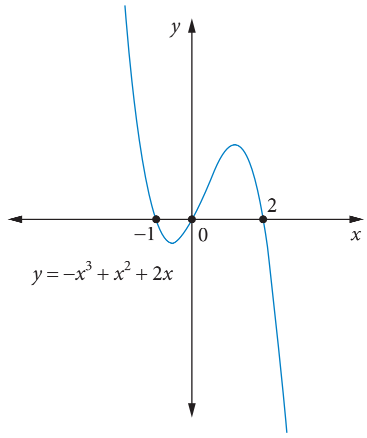


If one of the factors is squared, then the x-intercept 'touches' (rather than crosses) the x-axis and is a vertex. For example, $y = 3(x - 1)^2(x + 5)$ 'touches' the x-axis at $x = 1$. $(x - 1)^2$ is called a repeated factor or a double factor.



WORKED EXAMPLE 25 Graphing $y = ax^3 + bx^2 + cx + d$

Sketch the graph of $y = -x^3 + x^2 + 2x$, showing all intercepts.

Steps	Working
1 Factorise $y = -x^3 + x^2 + 2x$.	$y = -x^3 + x^2 + 2x$ $= -x(x^2 - x - 2)$ $= -x(x - 2)(x + 1)$
2 Find the x-intercepts.	Substitute $y = 0$: $0 = -x(x - 2)(x + 1)$ $x = 0, 2, -1$
3 Find the y-intercept.	Substitute $x = 0$: $y = -0(0 - 2)(0 + 1) = 0$
4 Determine the direction of the graph.	For $y = -x^3 + x^2 + 2x$, the leading coefficient $a = -1 < 0$, so the graph points down at the right.
5 Sketch the graph.	

3.8

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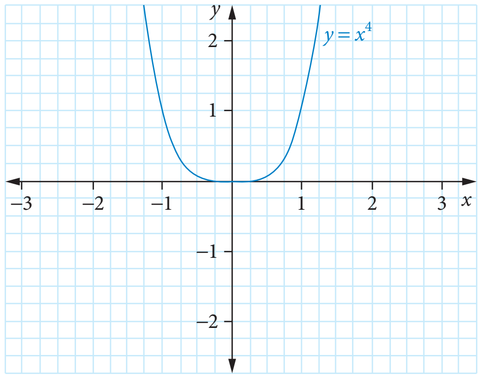
Graphing cubic functions

1. Determine the shape of the cubic curve from its equation and leading coefficient, a: it can have up to one maximum point and one minimum point.
2. Factorise the function if possible and note any repeated factors.
3. Find the x- and y-intercepts: it can have up to three x-intercepts but only one y-intercept. The y-intercept of $y = ax^3 + bx^2 + cx + d$ is d.

Graphing quartic functions

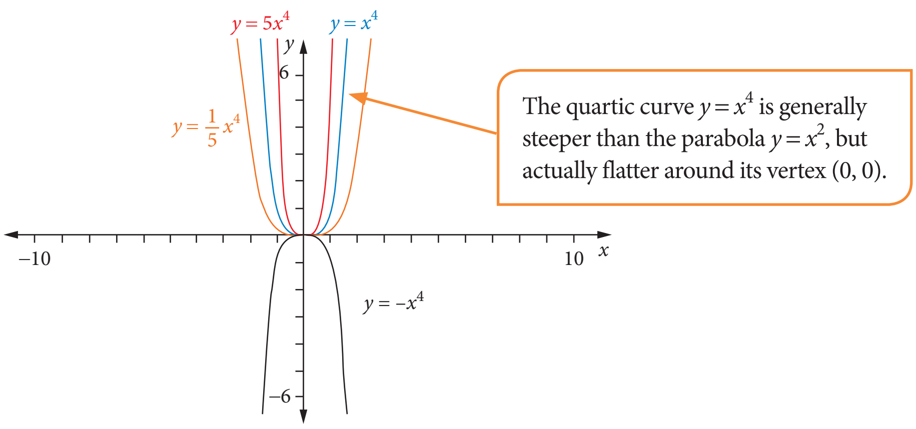
A **quartic function** is a polynomial of degree 4, whose equation has the general form $y = ax^4 + bx^3 + cx^2 + dx + e$.

The simplest cubic function is $y = x^4$, shown here. The centre of the graph at (0, 0) is its vertex or turning point.



The graph of $y = ax^4$

The graphs of $y = ax^4$ are shown, where the value of the coefficient a makes the graph more steep or flat. If a is negative, such as in $y = -x^4$, then the graph is reflected in the x-axis, pointing in the opposite directions.



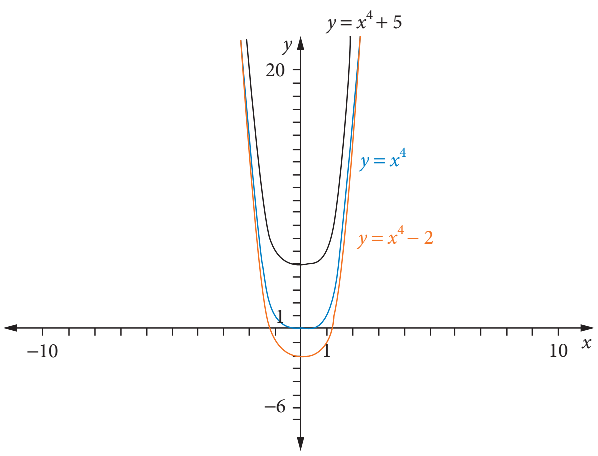
If a is large ($y = 5x^4$, for example), the quartic is **steeper**.

If a is small ($y = \frac{1}{5}x^4$, for example), the quartic is **flatter**.

If a is negative ($y = -x^4$, for example), the quartic is **reflected in the *x*-axis** (flipped upside-down).

The graph of $y = x^4 + k$

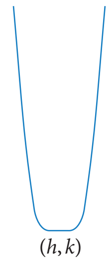
The graphs of $y = x^4 + k$ are shown, where the value of the constant k moves the graph of $y = x^4$, k units up (or down if k is negative), so that k is the y-intercept and $(0, k)$ is the vertex.



The graph of $y = a(x - h)^4 + k$

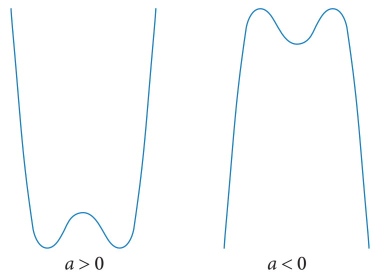
The graph of $y = (x - h)^4$ is the graph of $y = x^3$ moved h units right (left if h is negative), so that h is the x-intercept and $(h, 0)$ is the vertex.

So the quartic curve $y = a(x - h)^4 + k$ is the graph of $y = x^3$ that is stretched by a factor a from the x-axis and has a vertex at (h, k).



The graph of $y = ax^4 + bx^3 + cx^2 + dx + e$

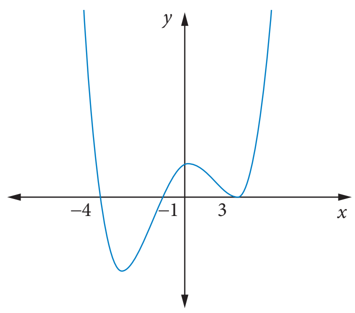
The graph of the general quartic function $y = ax^4 + bx^3 + cx^2 + dx + e$ has a W or M shape. It has 3 turning points. The shape of the quartic depends on the sign of the leading coefficient of the function, a:



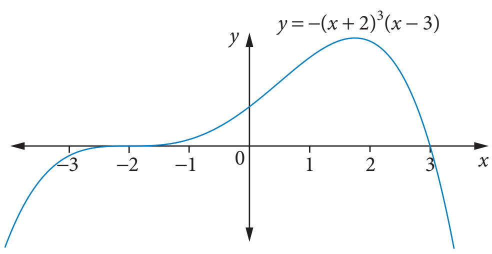
The graph of $y = a(x - b)(x - c)(x - d)(x - e)$

If a quartic function of the form $\boldsymbol{y = a(x - b)(x - c)(x - d)(x - e)}$ can be factorised, then b, c, d and e are its x-intercepts.

If one of the factors is squared, then the x-intercept 'touches' (rather than crosses) the x-axis and is a vertex. For example, $y = (x + 4)(x + 1)(x - 3)^2$ is shown, with x-intercepts at -4, -1 and 3. The quartic curve 'touches' the x-axis at $x = 3$, and $(x - 3)^2$ is a double factor.



If one of the factors is cubed, then the x-intercept is a flat point of inflection. For example, $y = -(x + 2)^3(x - 3)$ is shown, with x-intercepts at -2 and 3. The quartic curve has a point of inflection at $x = -2$, and $(x + 2)^3$ is a triple factor.



Graphing quartic functions

1 Determine the shape of the quartic curve from its equation and leading coefficient, a: it can have up to 3 turning points, made up of maximum and minimum points.

2 Factorise the function if possible and note any repeated factors.

3 Find the x- and y-intercepts: it can have up to four x-intercepts but only one y-intercept. The y-intercept of $y = ax^4 + bx^3 + cx^2 + dx + e$ is e.

WB

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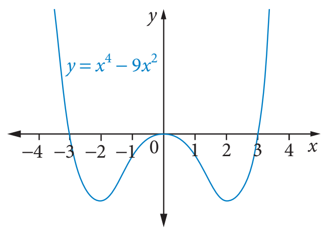
WORKED EXAMPLE 26 Graphing $y = ax^4 + bx^3 + cx^2 + dx + e$

Sketch the graph of $y = x^4 - 9x^2$, showing all intercepts.

Steps	Working
1 Factorise $y = x^4 - 9x^2$.	$y = x^4 - 9x^2$ $= x^2(x^2 - 9)$ $= x^2(x - 3)(x + 3)$
2 Find the x-intercepts.	Substitute $y = 0$: $0 = x^2(x - 3)(x + 3)$ $x = 0, 3, -3$
3 Find the y-intercept.	Substitute $x = 0$: $y = 0$
4 Determine the direction of the graph.	For $y = x^4 - 9x^2$, the leading coefficient $a = 1 > 0$, so the graph points up. x^2 is a double factor, so the graph will touch the x-axis at $x = 0$.
5 Sketch the graph.	

WB

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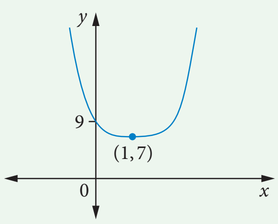
WORKED EXAMPLE 27 Determining the rule of a polynomial function from a graph

The function for the graph shown is of the form $y = a(x - b)^4 + c$. Find the equation of the function.



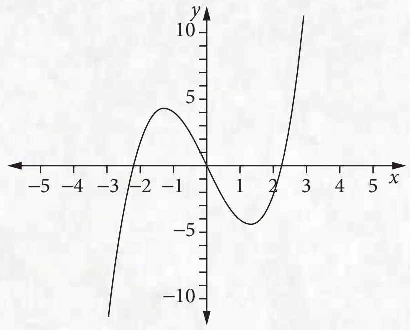
Steps	Working
a 1 Using the turning point, find the value of b and c.	Turning point is $(1, 7)$. Therefore, $b = 1$ and $c = 7$. So $y = a(x - 1)^4 + 7$.
2 Substitute the y-intercept $(0, 9)$ into the equation.	$9 = a(0 - 1)^4 + 7$ $9 = a(1) + 7$ $2 = a$
3 Write the equation of the function.	$y = 2(x - 1)^4 + 7$

VCE QUESTION ANALYSIS

© VCAA 2017 2BQ1 2017 Examination 2 Section B Question 1 (10 marks)

3.8

Let $f: R \to R, f(x) = x^3 - 5x$. Part of the graph of f is shown below.



a © VCAA MODIFIED Find the x-intercepts. 2 marks

b © VCAA MODIFIED The solutions of the quadratic equation $3x^2 - 5 = 0$ are the x-coordinates of the two turning points. Find the coordinates of the turning points. 2 marks

c $A(-1, f(-1))$ and $B(1, f(1))$ are two points on the graph of f.

- **i** Find the equation of the straight line through A and B. 2 marks
- **ii** Find the distance AB. 1 mark

Let $g: R \to R, g(x) = x^3 - kx$.

d Let $C(-1, g(-1))$ and $D(1, g(1))$ be two points on the graph of g.

- **i** Find the distance CD in terms of k. 2 marks
- **ii** Find the values of k such that the distance CD is equal to $k + 1$. 1 mark

Reading the question

- Take note of words in plural such as *intercepts, coordinates* and *values.*
- Take note of the formulas required, which questions will require working to be shown, whether answers are to be exact or rounded, and whether they will be numerical or algebraic.
- Appreciate, in the context of the question, the significance of key words and phrases such as **coordinates**, **in terms of *k***, **equation** and **distance**.

Thinking about the question

- Determine which algebraic or CAS techniques you are going to use.
- Define any given function on CAS at the start of the question. You can use this defined function to calculate values, solve equations and graph the functions.
- Many of the questions are 2-mark questions. This suggests that there are 2 steps in the solution and 2 things you need to show in your answers.
- Even if you cannot answer some parts of the question, there are often other sections of the question that are not directly linked and you may be able to do these sections.

Video
VCE question analysis: Polynomials

Worked solution (✓ = 1 mark)

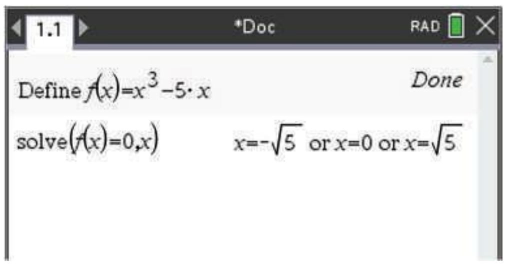
a For x-intercepts, solve $f(x) = 0$.

$x^3 - 5x = 0$

$x(x^2 - 5) = 0$ ✓

$\boldsymbol{x = 0, \sqrt{5}, -\sqrt{5}}$ ✓

TI-Nspire



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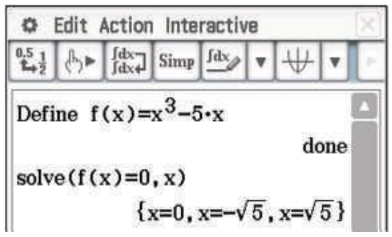


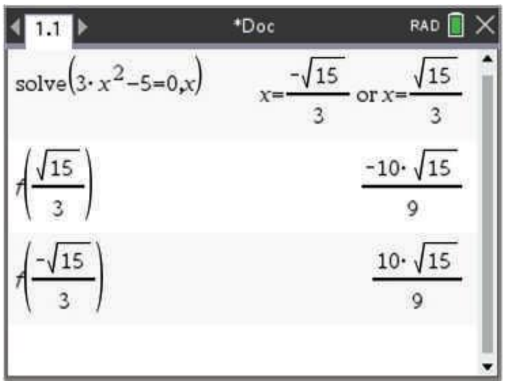
b $3x^2 - 5 = 0$

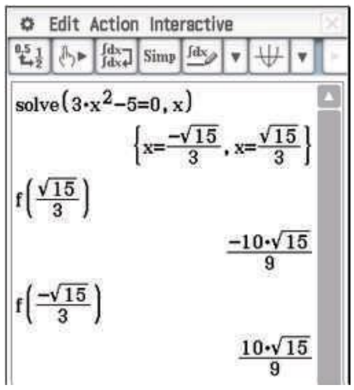
$3x^2 = 5$

$$\boldsymbol{x = \pm\sqrt{\frac{5}{3}} = \pm\frac{\sqrt{15}}{3}}$$

$$f\left(\frac{\sqrt{15}}{3}\right) = \left(\frac{\sqrt{15}}{3}\right)^3 - 5\left(\frac{\sqrt{15}}{3}\right) = \boldsymbol{-\frac{10\sqrt{15}}{9}} \checkmark$$

$$f\left(-\frac{\sqrt{15}}{3}\right) = \left(-\frac{\sqrt{15}}{3}\right)^3 - 5\left(-\frac{\sqrt{15}}{3}\right) = \boldsymbol{\frac{10\sqrt{15}}{9}} \checkmark$$





The turning points are at $\left(-\frac{\sqrt{15}}{3}, \frac{10\sqrt{15}}{9}\right)$ and $\left(\frac{\sqrt{15}}{3}, -\frac{10\sqrt{15}}{9}\right)$.

c **i** $f(1) = 1^3 - 5(1) = -4$ $(1, -4)$

$f(-1) = (-1)^3 - 5(-1) = 4$ $(-1, 4)$

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{-4 - 4}{1 - (-1)}$$

$$= \frac{-8}{2}$$

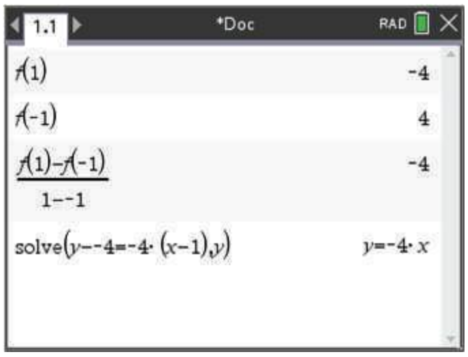
$\boldsymbol{= -4}$ ✓

9780170448260

Use $y - y_1 = m(x - x_1)$ with $(1, -4)$:

$$y - (-4) = -4(x - 1)$$
$$y + 4 = -4x + 4$$
$$\boldsymbol{y = -4x} \checkmark$$

TI-Nspire



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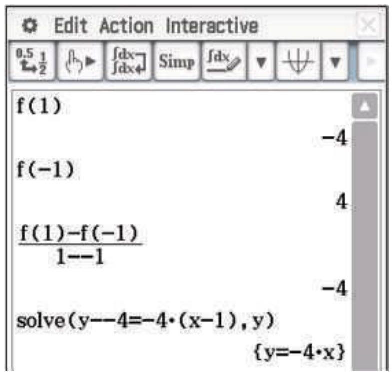


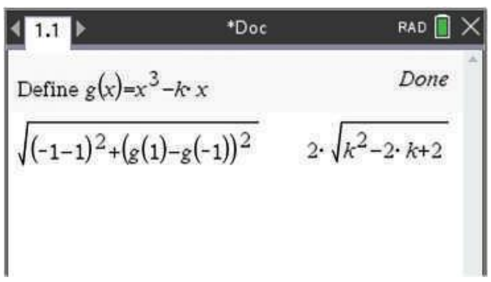
ii Use the distance formula.

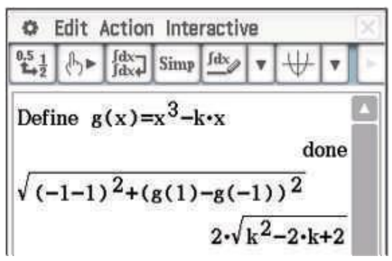
$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$= \sqrt{(-1 - 1)^2 + (4 - (-4))^2}$$
$$= \sqrt{68}$$
$$= \boldsymbol{2\sqrt{17}} \checkmark$$

d **i** $g(1) = 1^3 - k(1) = 1 - k$ $\quad (1, 1 - k)$

$g(-1) = (-1)^3 - k(-1) = -1 + k$ $\quad (-1, -1 + k) \checkmark$

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$= \sqrt{(-1 - 1)^2 + (-1 + k - (1 - k))^2}$$
$$= \sqrt{4 + (-2 + 2k)^2}$$
$$= \sqrt{4 + 4k^2 - 8k + 4}$$
$$= \sqrt{4k^2 - 8k + 8}$$
$$= \boldsymbol{2\sqrt{k^2 - 2k + 2}} \checkmark$$





ii $2\sqrt{k^2 - 2k + 2} = k + 1$

$4(k^2 - 2k + 2) = (k + 1)^2$

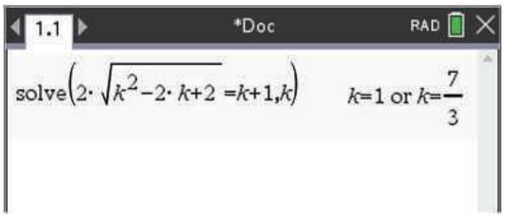
$4k^2 - 8k + 8 = k^2 + 2k + 1$

$3k^2 - 10k + 7 = 0$

$(k - 1)(3k - 7) = 0$

$\boldsymbol{k = 1, \frac{7}{3}}$ ✓

TI-Nspire



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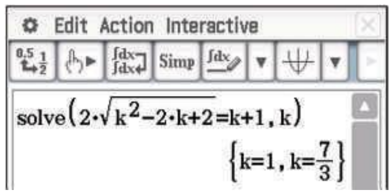


Student performance

80–100% 60–79% 0–59%

b 85% Well done, but some students scored one mark for giving rounded decimal answers when exact (surd) values were required. Some gave only the x-coordinates while others mixed up the signs.

c **i** 80% Also answered well. A common error was $y = 4x$, even though you could tell from the graph that the gradient should be negative. Some students made calculation errors when finding the gradient or equation of the line.

ii 78% Some students used an incorrect distance formula or made calculation errors, such as $1 - (-1) = 0$, so a common error was an answer of $\sqrt{64}$. This can be avoided by using CAS carefully.

d **i** 74% As above, some students did their solutions by hand and made arithmetic errors, especially with positive and negative signs. Use CAS. The answer was meant to be in terms of k only, but some students included x. Another common error: simplifying $\sqrt{2^2 + (2 - 2k)^2}$ as $2 + 2 - 2k$.

ii 63% Generally, students were able to solve the equation correctly. However, some gave only one solution rather than two.

EXERCISE 3.8 Graphing polynomial functions

ANSWERS p. 501

Recap

1 Use the bisection method to approximate, correct to one decimal place, a root of each equation in the given interval.

a $-x^3 + 2x^2 + 5x - 9 = 0$, $[-4, 0]$ **b** $x^4 - 3x^3 + 17x - 10 = 0$, $[0, 2]$

Mastery

2 WORKED EXAMPLE 24 TECH-FREE Graph each cubic function.

a $y = x^3 - 3$ **b** $y = -x^3 + 2$ **c** $y = -x^3 - 1$

d $y = x^3 + 1$ **e** $y = (x - 1)^3 + 3$ **f** $y = 4 - (x + 2)^3$

g $y = 3(x - 2)^3 + 1$ **h** $y = (2x - 1)^3 + 3$ **i** $y = (3x + 5)^3 - 2$

j $y = 4 - 2(3x - 6)^3$

3 WORKED EXAMPLE 25 TECH-FREE Graph each cubic function.

a $y = (x + 1)(x - 2)(x - 3)$ b $y = x(x + 4)(x - 2)$ c $y = x(x + 2)^2$

d $y = (5 - x)(x + 2)(x + 5)$ e $y = -x^3 - 4x^2 + 5x$ f $y = 2x^3 + x^2 - 15x$

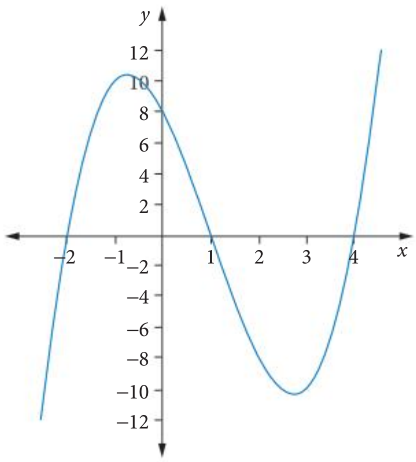
4 WORKED EXAMPLE 26 TECH-FREE Graph each quartic function.

a $y = (x + 1)(x - 2)(x - 3)(x + 2)$ b $y = -(x - 1)(x - 2)(x + 1)(x + 2)$

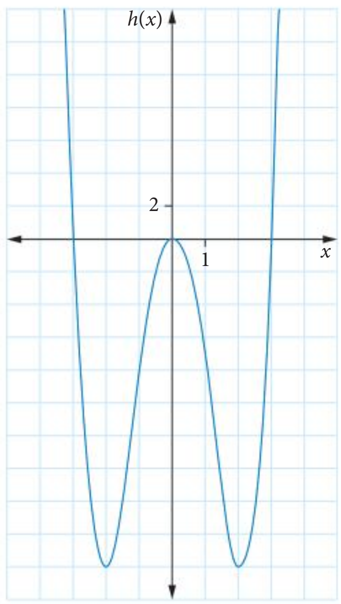
c $y = (x + 1)^2(x - 4)(x - 3)$ d $y = (x - 1)^2(x - 5)(x + 2)$

e $y = -(2x - 1)^3(3x + 5)$ f $y = -x^4 + 81x^2$

5 WORKED EXAMPLE 27 Find the rule for the function with the graph shown.



6 Find the rule for the function with the graph shown.



7 Kim wants to determine the length and height of her bookcase. The function $f(x) = x^3 + 14x^2 + 57x + 72$ represents the volume of the bookcase, where the width is $x + 3$ units. What are the possible dimensions for the length and height of the bookcase?

8 A specimen of bacteria is smeared on a round agar plate of diameter 12 cm and left for testing. The area (in cm^2) of the bacterial colony is given by the function $A = -t^3 + 6t^2 + 16t + 0.5$ where t is the number of weeks after the culture is started.

a What is the initial size of the colony?

b Sketch the graph for the first 10 weeks.

c When does the model of growth become inadequate?

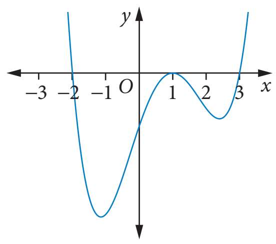
d What percentage of the plate is covered after 4 weeks?

e What is the maximum percentage of the plate that is covered?

Exam practice

80–100% 60–79% 0–59%

9 © VCAA 2005 1MQ16 95% Part of the graph of the function f is shown.



The rule for f is most likely to be

A $f(x) = (x+2)(x-1)^2(x-3)$

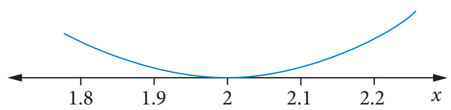
B $f(x) = (x-2)(x+1)^2(x+3)$

C $f(x) = (2-x)(x+1)^2(x+3)$

D $f(x) = (x-2)(x-1)^2(x-3)$

E $f(x) = (x+2)(x-1)^2(3-x)$

10 © VCAA 2003 1Q1 80% A polynomial function p has degree 3. A part of its graph, near the point on the graph with coordinates (2, 0), is shown.



Which one of the following could be the rule for the third degree polynomial p?

A $p(x) = -x(x-2)^2$

B $p(x) = (x-1)(x-2)^2$

C $p(x) = (x-2)^3$

D $p(x) = x^2(x-2)$

E $p(x) = x(x+2)^2$

11 © VCAA 2004 1MQ14 67% Let f be a polynomial function of degree 3. The graph of the curve with rule $y = f(x)$ either intersects or touches the x-axis at exactly two points, $(a, 0)$ and $(b, 0)$.

A possible rule for f could be

A $f(x) = (x-a)(x-b)$

B $f(x) = (x-a)(x+b)^2$

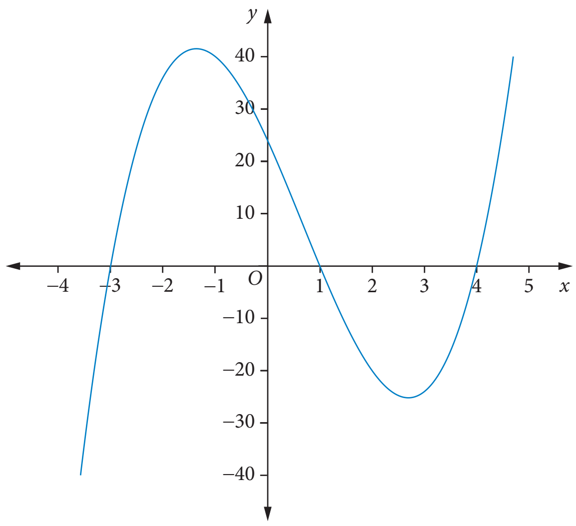
C $f(x) = (x-a)(x-b)^2$

D $f(x) = (x+a)^2(x-b)$

E $f(x) = (x+a)^2(x+b)$

9780170448260

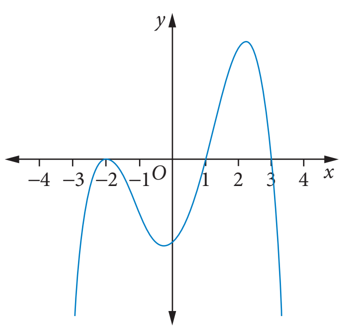
12 © VCAA 2006S 2AQ7 Part of the graph of $y = ax^3 + bx^2 + cx + d$ is shown.



The values of a, b, c and d could be

	a	b	c	d
A	1	−2	−11	12
B	2	−4	−22	24
C	1	2	−22	12
D	−2	−6	−2	24
E	3	−1	1	24

13 © VCAA 2018N 2AQ2 The diagram shows part of the graph of a polynomial function.



A possible rule for this function is

A $y = (x + 2)(x - 1)(x - 3)$

B $y = (x + 2)^2(x - 1)(x - 3)$

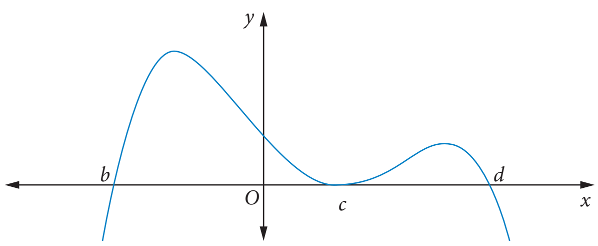
C $y = (x + 2)^2(x - 1)(3 - x)$

D $y = -(x - 2)^2(x - 1)(3 - x)$

E $y = -(x + 2)(x - 1)(x - 3)$

14 © VCAA 2015 2AQ3 20%

The rule for a function with the graph on the right could be



A $y = -2(x - b)(x + c)^2(x + d)$

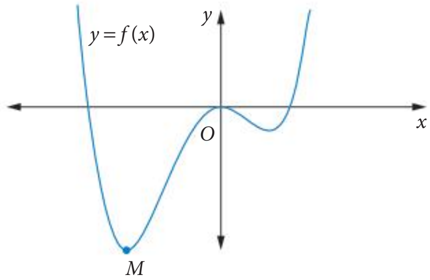
B $y = 2(x - b)(x - c)(x - d)$

C $y = 2(x + b)(x - c)^2(x - d)$

D $y = -2(x + b)(x - c)^2(x - d)$

E $y = -2(x - b)(x - c)^2(x - d)$

15 © VCAA 2018 2BQ1 (2 marks) Consider the quartic $f: R \to R, f(x) = 3x^4 + 4x^3 - 12x^2$ and part of the graph of $y = f(x)$ below.



a 95% Find the coordinates of the point M, at which the minimum value of the function f occurs. 1 mark

b 65% State the values of $b \in R$ for which the graph of $y = f(x) + b$ has no x-intercepts. 1 mark

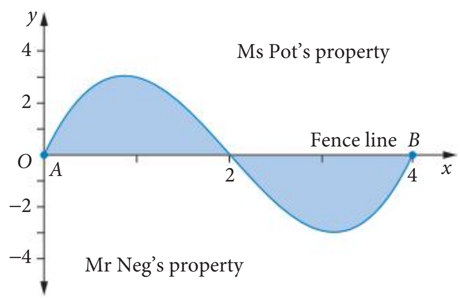
16 © VCAA 2019N 2BQ4 (4 marks) A mining company has found deposits of gold between two points, A and B, that are located on a straight fence line that separates Ms Pot's property and Mr Neg's property. The distance between A and B is 4 units.

The mining company believes that the gold could be found on both Ms Pot's property and Mr Neg's property.

The mining company initially models the boundary of its proposed mining area using the fence line and the graph of

$f: [0, 4] \to R, f(x) = x(x-2)(x-4)$

where x is the number of units from point A in the direction of point B and y is the number of units perpendicular to the fence line, with the positive direction towards Ms Pot's property. The mining company will only mine from the boundary curve to the fence line, as indicated by the shaded area below.



a Determine the total number of square units that will be mined according to this model. 2 marks

The mining company offers to pay Mr Neg \$100 000 per square unit of his land mined and Ms Pot \$120 000 per square unit of her land mined.

b Determine the total amount of money that the mining company offers to pay. 1 mark

The mining company reviews its model to use the fence line and the graph of

$$p: [0, 4] \to R,\ p(x) = x\left(x - 4 + \frac{4}{1+a}\right)(x-4)$$

where $a > 0$.

c Find the value of a for which $p(x) = f(x)$ for all x. 1 mark

3 Chapter summary

Polynomials

- A **polynomial** is an expression involving a sum of terms of the form:
  $P(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_1 x^1 + a_0$,
  where n is a positive whole number,
  $a_0, a_1, a_2 \ldots a_{n-1}, a_n$ are real numbers and $a_n \neq 0$.
- The **degree** of a polynomial is the highest power of x.
- The **leading term** is the term that contains the highest power of x.
- The constant term is a_0.
- The leading coefficient is a_n.
- A **monic polynomial** is a polynomial whose leading coefficient is 1.

Equating coefficients

- If $P(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_1 x^1 + a_0$ and $Q(x) = b_n x^n + b_{n-1}x^{n-1} + b_{n-2}x^{n-2} + \ldots + b_1 x^1 + b_0$ are equal for all values of x, then $a_n = b_n$, $a_{n-1} = b_{n-1}$, $a_{n-2} = b_{n-2}$ and so on up to $a_0 = b_0$.

Division of polynomials

$$
\begin{array}{r l}
 & 2x + 3 \leftarrow \text{Quotient} \\
x + 2 \,\big)\!\!\! & 2x^2 + 7x + 7 \\
 & \underline{2x^2 + 4x} \\
 & \quad\;\; 3x + 7 \\
 & \quad\;\; \underline{3x + 6} \\
 & \qquad\quad 1 \leftarrow \text{Remainder}
\end{array}
\qquad
\begin{array}{r|rrr}
-2 & 2 & 7 & 7 \\
 & \downarrow & -4 & -6 \\
\hline
 & 2 & 3 & 1
\end{array}
$$

Division statement of polynomials

- If $P(x)$ is divided by $x - a$, the quotient is denoted by $Q(x)$ and the remainder is R. It is usual to represent a polynomial division using the division statements:
  $P(x) = (x - a)Q(x) + R$
  $$\frac{P(x)}{x - a} = Q(x) + \frac{R}{(x - a)}$$

The remainder theorem

If a polynomial $P(x)$ is divided by $x - a$, the remainder is $P(a)$.

The factor theorem

- For a polynomial $P(x)$, if $P(a) = 0$, then $x - a$ is a factor of the polynomial.

Sum and difference of two cubes

- $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$
- $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

The rational root theorem

- If $P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x^1 + a_0$ has integer coefficients, then every rational root of $P(x)$ has the form:
  $$\frac{p}{q} = \pm\frac{\text{factor of the constant term, } a_0}{\text{factor of the leading coefficient, } a_n}$$

Solving polynomial equations

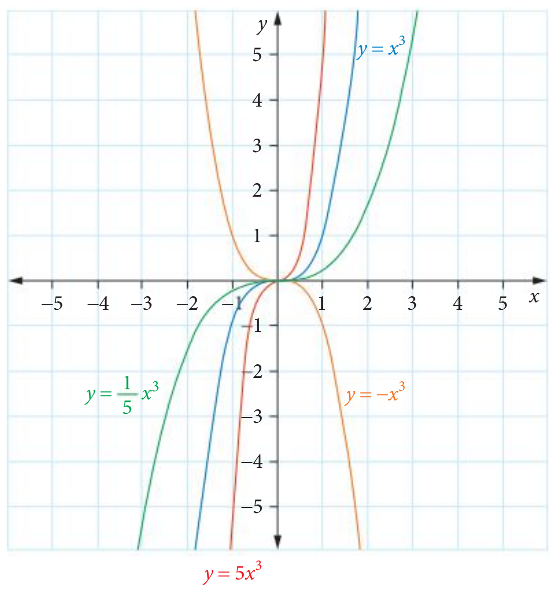
- If $P(x) = (x - a)(x - b)(x - c) = 0$, then the solutions are $x = a$, $x = b$, $x = c$.
  This can be extended to any number of products.

The bisection method for finding approximate roots of $f(x) = 0$

- Find a and b where $f(a)$ and $f(b)$ are opposite in sign: this means that a root is in the interval $[a, b]$.
- Bisect the interval by testing the function at the average of a and b: evaluate $f\left(\frac{a+b}{2}\right)$ and examine its sign.
- Check which half-interval contains the root.
- Repeat bisecting the interval until it is small enough to approximate the root.

Graphing cubic functions

A **cubic** function is a polynomial of degree 3, whose equation has the general form $y = ax^3 + bx^2 + cx + d$.



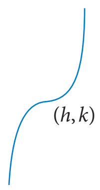
Compared to the graph of $y = x^3$, the effect of the value of a is to dilate ('stretch' or 'squash') $y = ax^3$ in the y-direction.

If a is large ($y = 5x^3$, for example), the cubic is **steeper**.

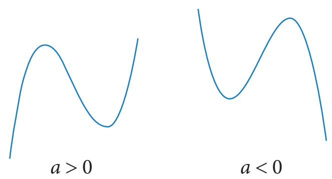
If a is small ($y = \frac{1}{5}x^3$, for example), the cubic is **flatter**.

If a is negative ($y = -x^3$, for example), the cubic is **reflected in the x-axis** (flipped upside-down).

The cubic curve $y = a(x - h)^3 + k$ is the graph of $y = ax^3$ that is stretched by a factor a from the x-axis and has point of inflection at (h, k).

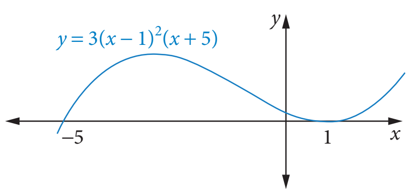


The graph of the general cubic function $y = ax^3 + bx^2 + cx + d$ has 2 turning points: a maximum point and a minimum point. The shape of the cubic depends on the sign of the leading coefficient of the function, a:



If a cubic function can be factorised to the form $\boldsymbol{y = a(x - b)(x - c)(x - d)}$, then b, c and d are its x-intercepts. For example, $y = (x + 3)(x + 1)(x - 2)$ has x-intercepts at -3, -1 and 2.

If one of the factors is squared, then the x-intercept 'touches' the x-axis and is a vertex. For example, $y = 3(x - 1)^2(x + 5)$ 'touches' the x-axis at $x = 1$. $(x - 1)^2$ is called a double factor.



Graphing cubic functions

1 Determine the shape of the cubic curve from its equation and leading coefficient, a: it can have up to one maximum point and one minimum point.

2 Factorise the function if possible and note any repeated factors.

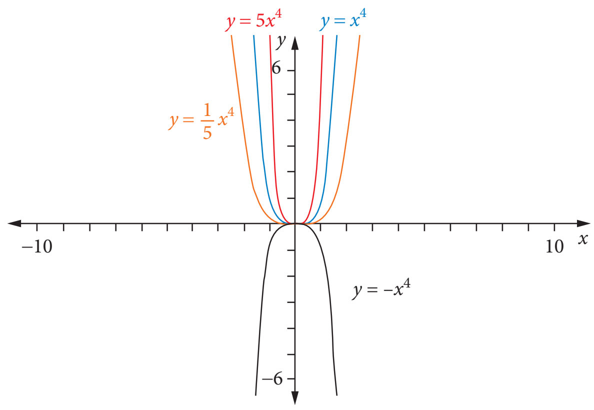
3 Find the x- and y-intercepts: it can have up to three x-intercepts but only one y-intercept.

   The y-intercept of $y = ax^3 + bx^2 + cx + d$ is d.

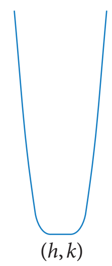
Graphing quartic functions

A **quartic function** is a polynomial of degree 4, whose equation has the general form $y = ax^4 + bx^3 + cx^2 + dx + e$.

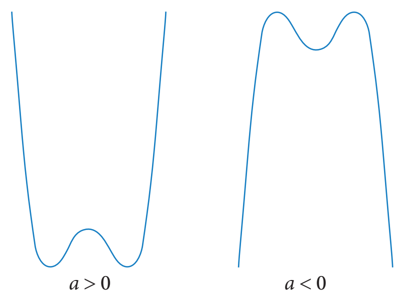
The graphs of $y = ax^4$ are shown, where the value of the coefficient a makes the graph more steep or flat. If a is negative, such as in $y = -x^4$, then the graph is reflected in the x-axis, pointing in the opposite directions.



The quartic curve $y = a(x - h)^4 + k$ is the graph of $y = x^3$ that is stretched by a factor a from the x-axis and has a vertex at (h, k).



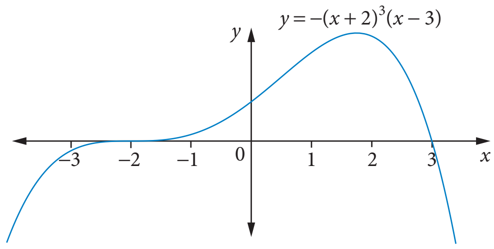
The graph of the general quartic function $y = ax^4 + bx^3 + cx^2 + dx + e$ has a W or M shape. It has 3 turning points. The shape of the quartic depends on the sign of the leading coefficient of the function, a:



If a quartic function can be factorised so that it is of the form $\boldsymbol{y = a(x - b)(x - c)(x - d)(x - e)}$, then b, c, d and e are its x-intercepts.

If one of the factors is squared, then the x-intercept 'touches' the x-axis and is a vertex. For example, $y = (x + 4)(x + 1)(x - 3)^2$ has x-intercepts at -4, -1 and 3. The curve 'touches' the x-axis at $x = 3$, and $(x - 3)^2$ is a double factor.

If one of the factors is cubed, then the x-intercept is a flat point of inflection. For example, $y = -(x + 2)^3 (x - 3)$ has x-intercepts at -2 and 3. The curve has a point of inflection at $x = -2$, and $(x + 2)^3$ is a triple factor.



Graphing quartic functions

1. Determine the shape of the quartic curve from its equation and leading coefficient, a: it can have up to 3 turning points, made up of maximum and minimum points.
2. Factorise the function if possible and note any repeated factors.
3. Find the x- and y-intercepts: it can have up to 4 x-intercepts but only 1 y-intercept.
   The y-intercept of $y = ax^4 + bx^3 + cx^2 + dx + e$ is e.

3 CHAPTER SUMMARY

Cumulative examination 1

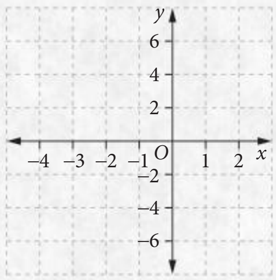
Total number of marks: 10 Reading time: 4 minutes Writing time: 15 minutes

TECH-FREE *Technology is NOT permitted.*

1 © VCAA 2017 1Q3 (4 marks) Let $f : [-3, 0] \to R, f(x) = (x + 2)^2(x - 1)$.

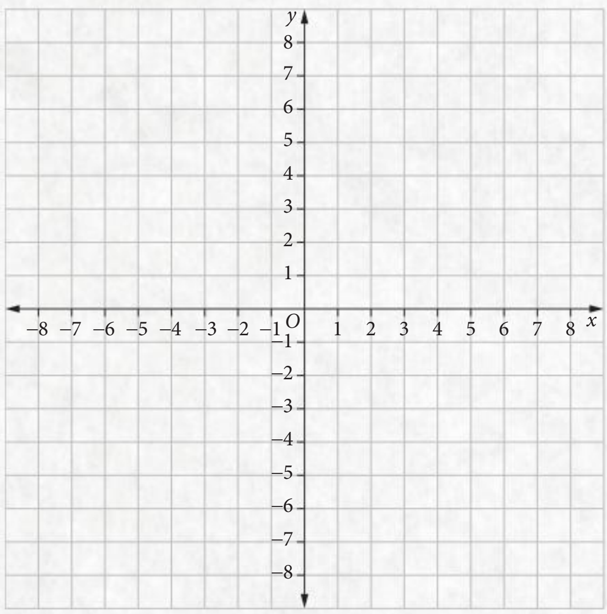
a Show that $(x + 2)^2(x - 1) = x^3 + 3x^2 - 4$. 1 mark

b Copy the axes below and on them sketch the graph of f. Label the axis intercepts and any turning points with their coordinates. 3 marks

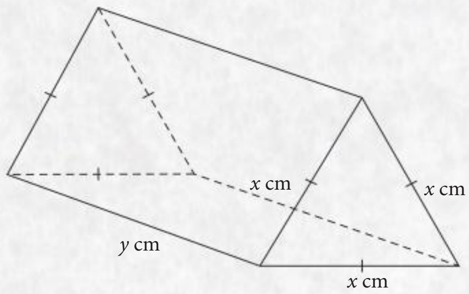


2 © VCAA 2015 1Q4b (2 marks) $f(x) = \frac{1}{2}(x - 1)(x + 2)^2$

Copy the axes below and on them sketch the graph of f, clearly indicating axis intercepts and turning points. Label the endpoints with their coordinates.



3 © VCAA 2008 1Q9 (4 marks) A plastic brick is made in the shape of a right triangular prism. The triangular end is an equilateral triangle with side length x cm and the length of the brick is y cm.



The volume of the brick is 1000 cm^3.

a Find an expression for y in terms of x. 2 marks

b Show that the total surface area, A cm^2, of the brick is given by $A = \dfrac{4000\sqrt{3}}{x} + \dfrac{\sqrt{3}x^2}{2}$. 2 marks

Cumulative examination 2

Total number of marks: 20 Reading time: 4 minutes Writing time: 30 minutes

Approved technology is permitted.

Section A 6 multiple-choice questions

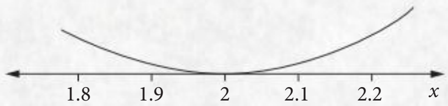
6 marks

1 © VCAA 2006 2AQ1 The graph with equation $y = x^2$ is translated 3 units down and 2 units to the right. The resulting graph has equation

A $y = (x - 3)^2 + 2$ **B** $y = (x - 2)^2 + 3$ **C** $y = (x - 2)^2 - 3$

D $y = (x + 2)^2 - 3$ **E** $y = (x + 2)^2 + 3$

2 © VCAA 2003 1Q1 A polynomial function p has degree 3. A part of its graph, near the point on the graph with coordinates (2, 0), is shown.



Which one of the following could be the rule for the third degree polynomial p?

A $p(x) = x(x + 2)^2$ **B** $p(x) = (x - 2)^3$ **C** $p(x) = x^2(x - 2)$

D $p(x) = (x - 1)(x - 2)^2$ **E** $p(x) = -x(x - 2)^2$

3 © VCAA 2013 2AQ2 The midpoint of the line segment that joins (1, −5) to (d, 2) is

A $\left(\dfrac{d+1}{2}, -\dfrac{3}{2}\right)$ **B** $\left(\dfrac{1-d}{2}, -\dfrac{7}{2}\right)$ **C** $\left(\dfrac{d-4}{2}, 0\right)$

D $\left(0, \dfrac{1-d}{3}\right)$ **E** $\left(\dfrac{5+d}{2}, 2\right)$

4 © VCAA 2015 2AQ20 If $f(x - 1) = x^2 - 2x + 3$, then $f(x)$ is equal to

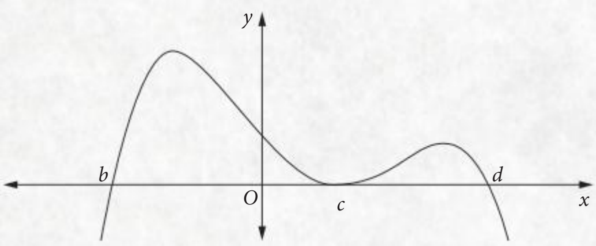
A $x^2 - 2$ **B** $x^2 + 2$ **C** $x^2 - 2x + 2$

D $x^2 - 2x + 4$ **E** $x^2 - 4x + 6$

5 © VCAA 2011 2AQ2 The gradient of the line **perpendicular** to the line which passes through (− 2, 0) and (0, −4) is

A $\dfrac{1}{2}$ **B** -2 **C** $-\dfrac{1}{2}$ **D** 4 **E** 2

6 © VCAA 2015 2AQ3



The rule for a function with the graph above could be

A $y = -2(x + b)(x - c)^2(x - d)$ **B** $y = 2(x + b)(x - c)^2(x - d)$

C $y = -2(x - b)(x - c)^2(x - d)$ **D** $y = 2(x - b)(x - c)(x - d)$

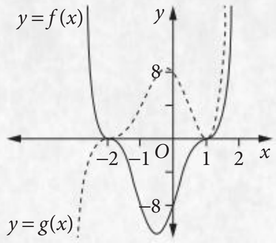
E $y = -2(x - b)(x + c)^2(x + d)$

Section B 2 questions

14 marks

1 © VCAA 2019N 2BQ1 (7 marks) Parts of the graphs of $f(x) = (x - 1)^3(x + 2)^3$ and $g(x) = (x - 1)^2(x + 2)^3$ are shown on the axes below.

The two graphs intersect at three points, $(-2, 0)$, $(1, 0)$ and (c, d). The point (c, d) is not shown in the diagram.



a Find the values of c and d. 2 marks

b Find the values of x such that $f(x) > g(x)$. 1 mark

c Show that $f(1 + m) = f(-2 - m)$ for all m. 1 mark

d Find the values of h such that $g(x + h) = 0$ has exactly one negative solution. 2 marks

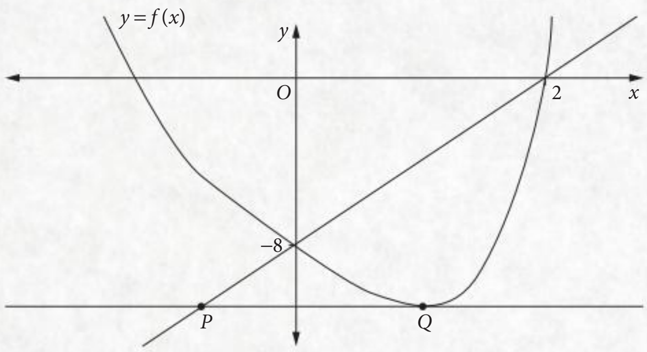
e Find the values of k such that $f(x) + k = 0$ has no solutions. 1 mark

2 © VCAA 2018N 2BQ1 (7 marks) Let $f: R \to R, f(x) = x^4 - 4x - 8$.

a Given $f(x) = (x - 2)(x^3 + ax^2 + bx + c)$, find a, b and c. 1 mark

b Find two consecutive integers m and n such that a solution to $f(x) = 0$ is in the interval (m, n), where $m < n < 0$. 2 marks

The diagram shows part of the graph of f and a straight line drawn through the points $(0, -8)$ and $(2, 0)$. A second straight line is drawn parallel to the horizontal axis and it touches the graph of f at the point Q. The two straight lines intersect at the point P.



c **i** Find the equation of the line through $(0, -8)$ and $(2, 0)$. 1 mark

ii State the equation of the line through the points P and Q. 1 mark

iii State the coordinates of the points P and Q. 2 marks

CHAPTER

4 POWER AND INVERSE FUNCTIONS

*According to the 2022 Study Design, matrix representation may be used but is not required.

Study Design coverage

UNIT 1, AREA OF STUDY 1: FUNCTIONS, RELATIONS AND GRAPHS

- graphs of power functions $f(x) = x^n$ for $n \in \{-2, -1, \frac{1}{3}, \frac{1}{2}, 1, 2, 3, 4\}$, and transformations of these graphs to the form $y = a(x + b)^n + c$ where $a, b, c \in R$ and $a \neq 0$
- inverse functions and their graphs.

UNIT 1, AREA OF STUDY 2: ALGEBRA, NUMBER AND STRUCTURE

- use of parameters to represent families of functions and determination of rules of simple functions and relations from given information
- transformations of the plane and application to basic functions and relations by simple combinations of dilations (students should be familiar with both 'parallel to an axis' and 'from an axis' descriptions), reflections in an axis and translations (matrix representation may be used but is not required).

STUDY DESIGN COVERAGE

Video playlists (6):

- **4.1** Power functions
- **4.2** Transformations of power functions
- **4.3** Inverse functions
- **4.4** Matrices
- **4.5** Matrices and transformations of functions
- **VCE question analysis** Power and inverse functions

Worksheets (8)

- **4.1** Cubic functions • Graphing cubics • Graphing cubics 2
- **4.2** Graphing translations of functions
- **4.3** Inverse functions • Inverse functions code puzzle
- **4.4** Addition and subtraction of matrices • Multiplying matrices
- **4.5** Graphing translations of functions

Nelson MindTap

To access resources above, visit **cengage.com.au/nelsonmindtap**

4.1 Power functions

Power functions are of the form $y = x^n$.

Video playlist Power functions

Worksheets Cubic functions

Graphing cubics

Graphing cubics 2

Cubic functions

Cubic functions

The basic cubic function is $y = x^3$.

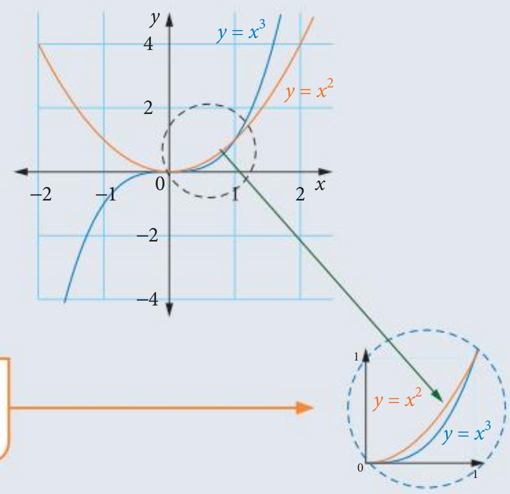
It is shown on the right, with $y = x^2$.

$y = x^3$ is steeper than $y = x^2$ for $x > 1$ and $x < -1$.

It is flatter than $y = x^2$ for $-1 < x < 1$.

There is a point of inflection at the central point (0, 0).

It changes from concave downwards to concave upwards at this point.

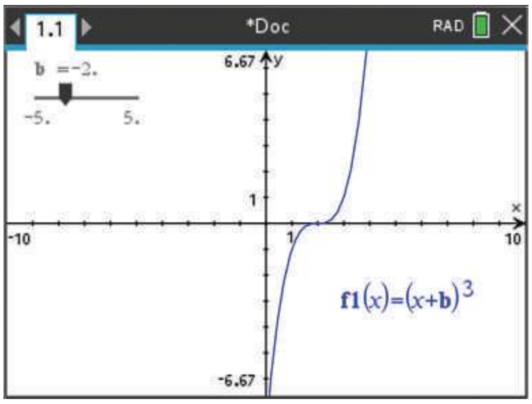


This is a close-up of the graphs of $y = x^2$ and $y = x^3$ more clearly between $x = 0$ and $x = 1$.

USING CAS 1 A family of cubic functions

Use CAS to examine the families of cubic functions given by $y = (x + b)^3$.

TI-Nspire



1 Add a **Graphs** page.

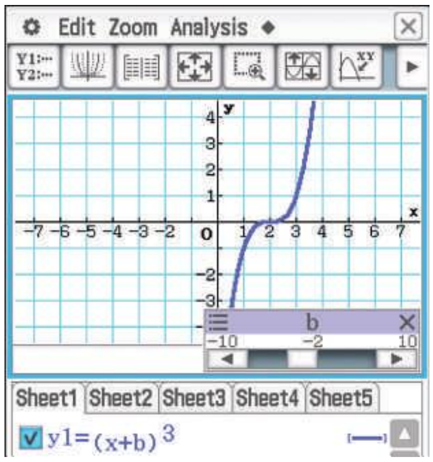
2 Enter **f1(*x*) = (*x* + *b*)^3** and press **enter**.

3 When the **Create Sliders** dialogue box appears, press **enter** to create a slider for ***b***.

4 Move the slider to a suitable location on the screen and click outside the slider to place it.

5 Adjust the slider to see the effect different values of b have on the shape of the graph.

ClassPad



1 Open the **Graph&Table** application to enable **Dynamic Graphing**.

2 Enter **y1 = (*x* + *b*)^3**.

3 Tap ◆ **> Dynamic Graph** to graph the function and insert a slider for ***b***.

4 Adjust the slider to see the effect different values of b have on the shape of the graph.

Note: the windows have been swapped.

The screenshots show the graph for $b = -2$.

A **family** of functions is a set of functions, each of which differs from the others by the value of a particular **parameter**. The family of parabolas $y = x^2 + c$ differ according to the value of c.

Different families of cubic functions are given by the parameters a, b and c in the cubic functions $y = ax^3$, $y = (x + b)^3$ and $y = x^3 + c$. Use CAS to examine each family of cubics.

Families of cubic functions

The family $y = ax^3$

- $y = ax^3$ is flipped upside-down (reflected in the x-axis) for $a < 0$.
- For $a < -1$ or $a > 1$, it is stretched away from the x-axis, steeper than $y = x^3$.
- For $-1 < a < 1$, it is shrunk towards the x-axis, less steep than $y = x^3$.

The family $y = (x + b)^3$

- $y = (x + b)^3$ is moved parallel to the x-axis.
- For $b > 0$ it is moved to the left of $y = x^3$.
- For $b < 0$ it is moved to the right of $y = x^3$.

The family $y = x^3 + c$

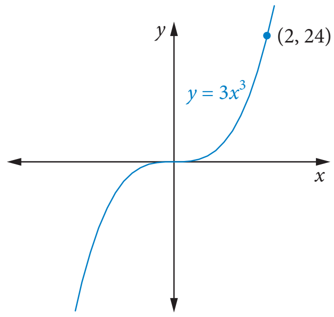
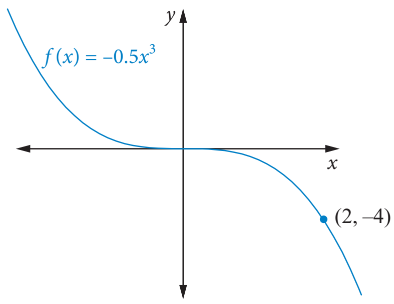
- $y = x^3 + c$ is moved parallel to the y-axis.
- For $c > 0$ it is moved upwards, higher than $y = x^3$.
- For $c < 0$ it is moved downwards, lower than $y = x^3$.

If the y-intercept is 0, we need to use another reference point in a sketch.

WORKED EXAMPLE 1 Sketching $y = ax^3$

Sketch the functions

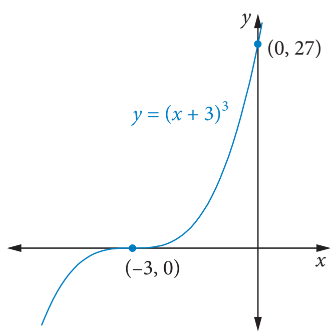
a $y = 3x^3$ **b** $f(x) = -0.5x^3$

Steps	Working
a **1** Write the comparison with $y = x^3$.	$y = 3x^3$ is 3 times as steep as $y = x^3$.
2 State the central point.	The point of inflection is still (0, 0).
3 State another reference point.	The curve passes through (2, 24).
4 Sketch the graph.	
b **1** Write the comparison with $y = x^3$.	$f(x) = -0.5x^3$ is less steep and reflected in the x-axis.
2 State the central point.	The point of inflection is still (0, 0).
3 State another reference point.	The curve passes through (2, −4).
4 Sketch the graph.	

It helps to understand the direction that the family of functions $y = (x + b)^3$ is moved compared to $y = x^3$. Use the y-intercept as a reference point wherever possible.

WORKED EXAMPLE 2 Sketching $y = (x + b)^3$

Sketch the graph of $y = (x + 3)^3$.

Steps	Working
1 Write the comparison with $y = x^3$.	$y = (x + 3)^3$ is moved 3 to the left from $y = x^3$.
2 State the central point.	The point of inflection is (−3, 0).
3 State another reference point.	The y-intercept is (0, 27).
4 Sketch the graph. Find any zeros if possible.	

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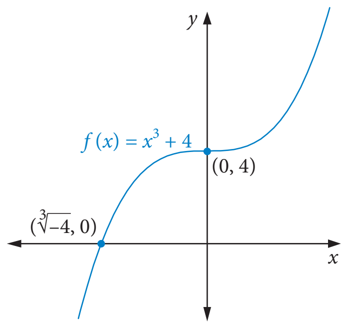
WORKED EXAMPLE 3 Sketching $y = x^3 + c$

Sketch the graph of $f(x) = x^3 + 4$.

Steps	Working
1 Write the comparison with $y = x^3$.	$y = x^3 + 4$ is moved 4 up from $y = x^3$.
2 State the central point.	The point of inflection is $(0, 4)$.
3 State another reference point.	The zero is given by $x^3 + 4 = 0$. The zero is $(\sqrt[3]{-4}, 0) \approx (-1.59, 0)$.
4 Sketch the graph.	

Quartic functions

The basic quartic function is $y = x^4$. Like $y = x^2$, it is always positive or 0 so it is also U-shaped. It is even steeper than a cubic for $x > 1$ and $x < -1$, and flatter for $-1 < x < 1$.

The **turning point** of $y = x^4$ is a **minimum**, while that of $y = -x^4$ is a **maximum**. The families of quartic functions, $y = ax^4$, $y = (x + b)^4$ and $y = x^4 + c$ are related to $y = x^4$ similarly to the families of $y = x^3$. Use CAS to investigate the families. Generally, the families of functions $y = af(x)$, $y = f(x + b)$ and $y = f(x) + c$ have the same relationships to $f(x)$ for any function $f(x)$.

Families of functions

The family $y = af(x)$

- $y = af(x)$ is flipped upside-down (reflected in the x-axis) for $a < 0$, compared to $f(x)$.
- For $a < -1$ or $a > 1$, it is stretched away from the x-axis, steeper than $y = f(x)$.
- For $-1 < a < 1$, it is shrunk towards the x-axis, less steep than $y = f(x)$.

The family $y = f(x + b)$

- $y = f(x + b)$ is moved parallel to the x-axis.
- For $b > 0$ it is moved to the left of $y = f(x)$.
- For $b < 0$ it is moved to the right of $y = f(x)$.

The family $y = f(x) + c$

- $y = x^3 + c$ is moved parallel to the y-axis.
- For $c > 0$, it is moved upwards, higher than $y = f(x)$.
- For $c < 0$, it is moved downwards, lower than $y = f(x)$.

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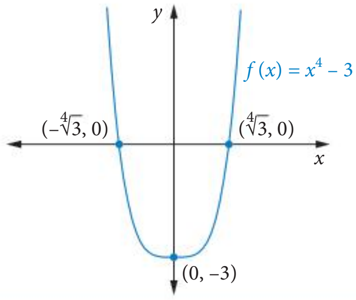
WORKED EXAMPLE 4 Sketching $y = x^4 + c$

Sketch the graph of $f(x) = x^4 - 3$.

Steps	Working
1 Write the comparison with $y = x^4$.	$y = x^4 - 3$ is moved 3 down from $y = x^4$.
2 State the turning point.	The minimum is $(0, -3)$.

3 Find the zeros if possible.	The zeros are given by $x^4 - 3 = 0$. The zeros are $(-\sqrt[4]{3}, 0)$ and $(\sqrt[4]{3}, 0)$.
4 Sketch the graph.	

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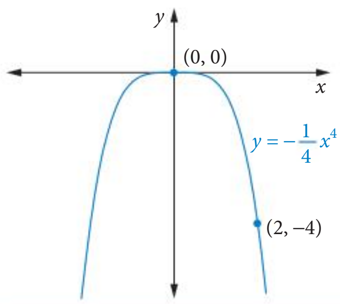
WORKED EXAMPLE 5 Sketching $y = ax^4$

Sketch the graph of $y = -\frac{1}{4}x^4$.

Steps	**Working**
1 Write the comparison with $y = x^4$.	$y = -\frac{1}{4}x^4$ is reflected in the x-axis and dilated by the factor $\frac{1}{4}$ from the x-axis from $y = x^4$.
2 State the turning point.	There is a maximum at $(0, 0)$.
3 State another reference point.	The point $(2, -4)$ is on the curve.
4 Sketch the graph.	

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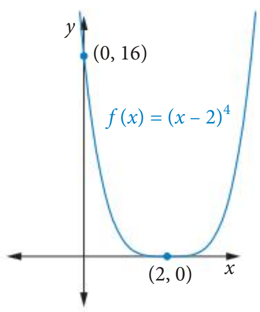
WORKED EXAMPLE 6 Sketching $y = (x + b)^4$

Sketch the graph of $f(x) = (x - 2)^4$.

Steps	**Working**
1 Write the comparison with $y = x^4$.	$y = (x - 2)^4$ is moved 2 to the right of $y = x^4$.
2 State the turning point.	The minimum is $(2, 0)$.
3 Find the y-intercept.	The y-intercept is $(0, 16)$.
4 Sketch the graph.	

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The hyperbola

The simple **hyperbola** is the graph of the function $f(x) = x^{-1} = \frac{1}{x}$.

As $x \to \pm\infty$, $\frac{1}{x} \to 0$.

The function is undefined at $x = 0$.

$f(0.1) = 10, f(0.001) = 1000, f(10^{-6}) = 10^6$ etc., so as $x \to 0$ from above $y \to \infty$.

$f(-0.1) = -10, f(-0.001) = 1000$ etc., so as $x \to 0$ from below $y \to -\infty$.

We can also plot points or use CAS to find the general shape of $f(x)$.

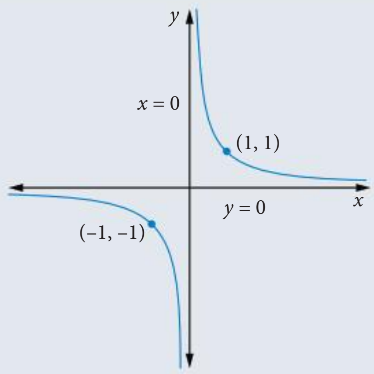
The hyperbola

There is a **discontinuity** at $x = 0$ (there is a break in the graph at $x = 0$). So the graph of $y = \frac{1}{x}$ is separated into 2 branches.

The x-axis ($y = 0$) is a **horizontal asymptote**: as $x \to \pm\infty$ (as x approaches positive or negative infinity), the curve approaches $y = 0$, but never reaches it.

The y-axis ($x = 0$) is a **vertical asymptote**: as $x \to 0$, $y \to \pm\infty$ (y approaches positive or negative infinity) so the curve approaches $x = 0$, but never reaches it.

Each of the points $(-1, -1)$ and $(1, 1)$ is a **vertex**.



The families of hyperbolas given by $f(x) = ax^{-1} = \frac{a}{x}$, $f(x) = (x + b)^{-1} = \frac{1}{x + b}$ and $f(x) = x^{-1} + c = \frac{1}{x} + c$ have the same relationship to the basic hyperbola as the families of other functions. The horizontal and vertical asymptotes are moved by the parameters b and c in the same way as the branches.

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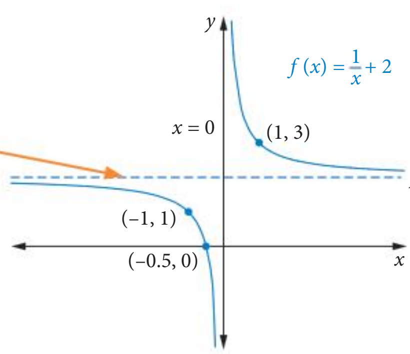
WORKED EXAMPLE 7 Sketching $f(x) = x^{-1} + c$

Sketch the graph of $f(x) = \frac{1}{x} + 2$.

Steps	Working
1 Write the comparison with $y = x^{-1}$.	$f(x) = \frac{1}{x} + 2$ is moved 2 up from $y = \frac{1}{x}$.
2 State the asymptotes and vertices.	The asymptotes are $x = 0$ and $y = 2$ and the vertices are $(1, 3)$ and $(-1, 1)$.
3 Find the zero if possible.	$\frac{1}{x} + 2 = 0 \Rightarrow x = -\frac{1}{2}$
4 Sketch the graph.	

Show the horizontal asymptote as a dashed line, with its equation.



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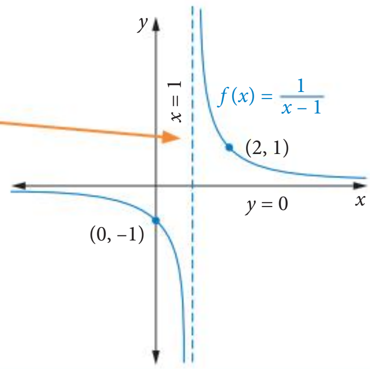
WORKED EXAMPLE 8 Sketching $f(x) = (x + b)^{-1}$

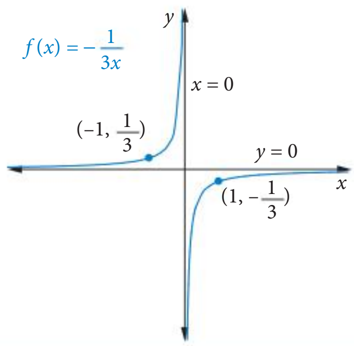
Sketch the graph of $f(x) = \dfrac{1}{x-1}$.

Steps	Working
1 Write the comparison with $y = x^{-1}$.	$f(x) = \dfrac{1}{x-1}$ is moved 1 to the right from $\dfrac{1}{x}$.
2 State the asymptotes.	The asymptotes are $x = 1$ and $y = 0$.
3 State the y-intercept.	$f(0) = -1$
4 Sketch the graph.	

Show the vertical asymptote as a dashed line, with its equation.

WORKED EXAMPLE 9 Sketching $f(x) = ax^{-1}$

Sketch the graph of $f(x) = -\dfrac{1}{3x}$.

Steps	Working
1 Write the comparison with $y = x^{-1}$.	$f(x) = -\dfrac{1}{3x} = -\dfrac{1}{3}x^{-1}$ is $y = x^{-1}$ reflected in the x-axis and dilated by a factor $\dfrac{1}{3}$ from the x-axis from $\dfrac{1}{x}$.
2 State the asymptotes.	The asymptotes are still $x = 0$ and $y = 0$.
3 State the vertices.	The vertices are $\left(1, -\dfrac{1}{3}\right)$ and $\left(-1, \dfrac{1}{3}\right)$.
4 Sketch the graph.	

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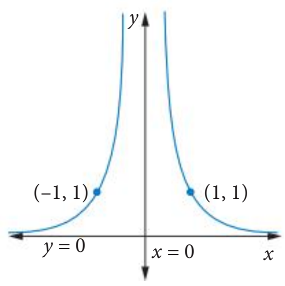
The truncus

The basic **truncus** is the power function $f(x) = x^{-2} = \dfrac{1}{x^2}$. It is similar to the hyperbola, but is flatter for $-1 < x < 1$ and steeper outside this interval.

The truncus is discontinuous at $x = 0$ and has 2 branches. The axes form horizontal and vertical asymptotes. It is positive on both sides of the y-axis.

The families of truncus's $y = ax^{-2}$, $y = (x + b)^{-2}$ and $y = x^{-2} + c$ bear the same relation to $y = x^{-2}$ as other functions.



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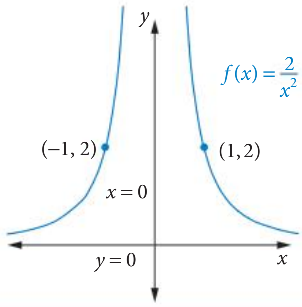
WORKED EXAMPLE 10 Sketching $f(x) = ax^{-2}$

Sketch the graph of $f(x) = \frac{2}{x^2}$.

Steps	Working
1 Write the comparison with $y = x^{-2}$.	$f(x) = \frac{2}{x^2} = 2x^{-2}$ is dilated by the factor 2 from the x-axis.
2 State the asymptotes.	The asymptotes are still $x = 0$ and $y = 0$.
3 State other reference points.	The curve passes through $(-1, 2)$ and $(1, 2)$.
4 Sketch the graph.	

We can sketch a member of a function family with more than one parameter value. We need to proceed from the innermost to the outermost parameter to make changes to the basic function correctly. We follow the normal order of operations on x.

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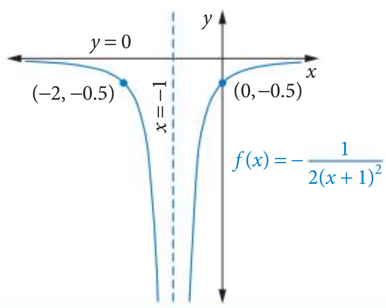
WORKED EXAMPLE 11 Sketching a two-parameter truncus

Sketch the graph of $f(x) = -\frac{1}{2(x+1)^2}$.

Steps	Working
1 Write the comparison with $y = x^{-2}$.	$f(x) = -0.5(x+1)^{-2}$, so the graph is $y = x^{-2}$ moved 1 to the left, reflected in the x-axis and dilated by a factor of $\frac{1}{2}$ from the x-axis.
2 State the asymptotes.	The asymptotes are at $x = -1$ and $y = 0$.
3 State other reference points.	$\left(-2, -\frac{1}{2}\right)$ and $\left(0, \frac{1}{2}\right)$ are on the curve.
4 Sketch the graph.	

Square root functions

The **square root function**, $f(x) = x^{\frac{1}{2}} = \sqrt{x}$ has the range $[0, \infty]$ as $\sqrt{x}$ is positive (or 0 when $x = 0$).

It is the inverse function of $y = x^2$ with the domain restricted to $[0, \infty]$ to make it one-to-one.

The domain is also $[0, \infty]$ as $\sqrt{-4}$, for example, is not a real number.

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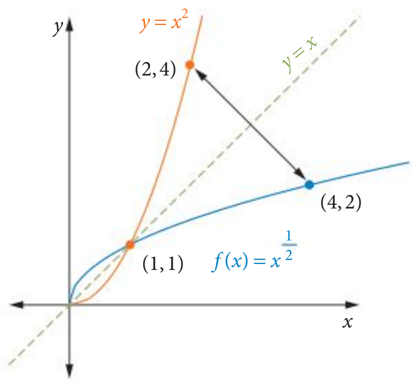
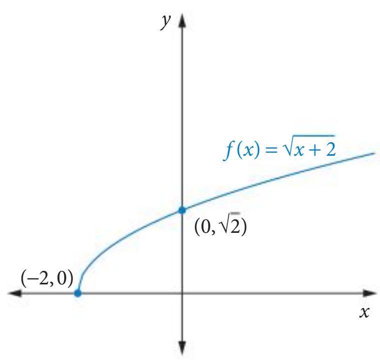
The shape of $f(x)$ is a half-parabola *reflected in* the line $y = x$. The graph starts at (0, 0).



WORKED EXAMPLE 12 Sketching a square root function

Sketch the graph of $f(x) = \sqrt{x+2}$.

Steps	**Working**
1 Compare to $y = \sqrt{x}$.	$f(x)$ is moved 2 units to the left of $y = \sqrt{x}$.
2 State the starting point.	The graph starts at (−2, 0).
3 State the y-intercept.	$f(0) = \sqrt{2}$.
4 Sketch the graph.	

Cube root functions

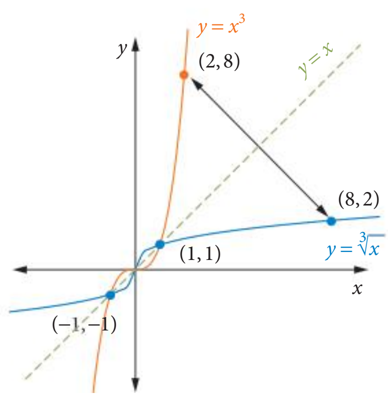
The basic **cube root function**, $f(x) = x^{\frac{1}{3}} = \sqrt[3]{x}$ is the inverse to the cubic function $y = x^3$.

$f(x) = x^{\frac{1}{3}}$ has the same domain and range, R.

The shape is the reflection of $y = x^3$ in the line $y = x$.

There is a point of inflection at the central point (0, 0).

The points (−1, −1) and (1, 1) are on the curve.



Families of cube root functions follow the same rules as other function families.

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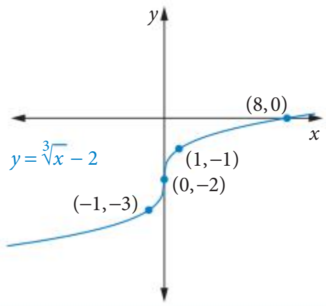
WORKED EXAMPLE 13 Sketching a cube root function

Sketch the graph of $f(x) = \sqrt[3]{x} - 2$.

Steps	Working
1 Compare to $y = \sqrt[3]{x}$.	$f(x)$ is moved 2 units down from $y = \sqrt[3]{x}$.
2 State the central point.	The central point is $(0, -2)$.
3 Find the zero.	$\sqrt[3]{x} - 2 = 0 \Rightarrow x = 8$
4 Sketch the graph.	

EXERCISE 4.1 Power functions

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Mastery

1 Using CAS 1 Use CAS to draw each family of functions and comment on the effect of the parameter.

a $f(x) = (x + b)^4$ **b** $f(x) = x^{-1} + c$ **c** $f(x) = a\sqrt{x}$ **d** $f(x) = \sqrt[3]{x + b}$

2 WORKED EXAMPLES 1–3 Sketch each function.

a $y = 4x^3$ **b** $f(x) = 0.4x^3$ **c** $y = x^3 - 1$ **d** $f(x) = x^3 + 2$

e $f(x) = (x - 2)^3$ **f** $y = (x + 4)^3$ **g** $f(x) = -2x^3 + 1$ **h** $y = (x - 3)^3 + 2$

3 WORKED EXAMPLES 4–6 Sketch each function.

a $f(x) = x^4 + 2$ **b** $y = x^4 - 1$ **c** $f(x) = 0.3x^4$ **d** $y = -2x^4$

e $y = (x + 2)^4$ **f** $f(x) = (x - 1)^4$ **g** $y = 0.5(x + 3)^4$ **h** $f(x) = (x + 2)^4 - 3$

4 WORKED EXAMPLES 7–9 Sketch each function.

a $y = 3x^{-1}$ **b** $f(x) = \frac{1}{2x}$ **c** $y = \frac{1}{x - 2}$ **d** $f(x) = (x + 3)^{-1}$

e $f(x) = \frac{1}{x} - 2$ **f** $y = 3 - x^{-1}$ **g** $f(x) = (x - 1)^{-1} - 2$ **h** $y = \frac{1}{4x} + 2$

5 WORKED EXAMPLES 10, 11 Sketch each function.

a $f(x) = \frac{1}{x^2} - 1$ **b** $y = x^{-2} + 2$ **c** $f(x) = -\frac{1}{3x^2}$ **d** $y = 4x^{-2}$

e $y = (x - 1)^{-2}$ **f** $f(x) = \frac{1}{(x + 3)^2}$ **g** $y = \frac{1}{2(x + 2)^2}$ **h** $f(x) = (x + 2)^{-2} + 1$

6 WORKED EXAMPLE 12 Sketch each function.

a $y = \sqrt{x} + 1$ **b** $f(x) = x^{\frac{1}{2}} - 2$ **c** $y = -2\sqrt{x}$ **d** $f(x) = \frac{\sqrt{x}}{4}$

e $f(x) = \sqrt{x - 3}$ **f** $y = (x + 2)^{\frac{1}{2}}$ **g** $f(x) = 3 - \sqrt{x}$ **h** $y = 2 + \sqrt{x - 1}$

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7 WORKED EXAMPLE 13 Sketch each function.

a $f(x)=\sqrt[3]{x}+2$ **b** $y=\sqrt[3]{x}-3$ **c** $f(x)=(x+3)^{\frac{1}{3}}$ **d** $y=\sqrt[3]{x-1}$

e $y=-2\sqrt[3]{x}$ **f** $f(x)=\frac{1}{3}x^{\frac{1}{3}}$ **g** $y=\sqrt[3]{x+1}-3$ **h** $f(x)=3\sqrt[3]{x}+3$

8 Sketch each function, showing the coordinates of important points.

a $f(x)=(x-2)^{-1}$ **b** $y=(x+3)^4$ **c** $f(x)=-3\sqrt{x}$ **d** $y=\frac{1}{2}\sqrt[3]{x}$

e $y=x^{-2}+3$ **f** $f(x)=(x+1)^3-2$ **g** $y=\frac{3}{(x-2)^2}$ **h** $f(x)=-0.25x^4+3$

Exam practice

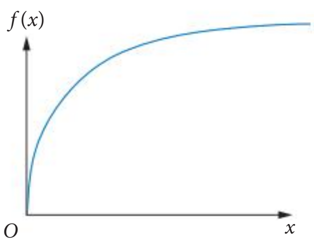
80–100% 60–79% 0–59%

9 TECH-FREE (2 marks) Find the maximal domain and range of

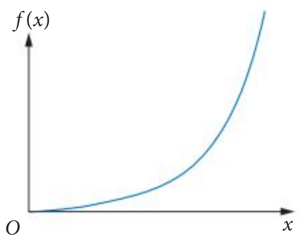
a $f(x)=\frac{1}{(x+2)^2}$ **b** $f(x)=\sqrt{x-4}$

10 © VCAA 2003 1IQ2 97% TECH-FREE The graph of the function $f: R^+ \to R$ with rule $f(x)=x^{\frac{1}{3}}$ could be

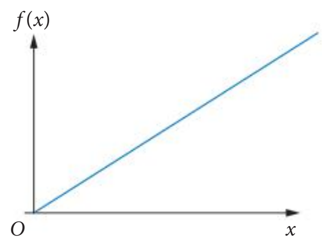
A



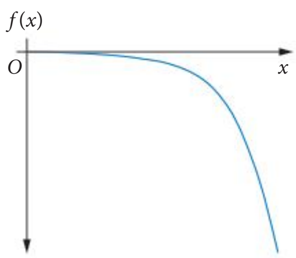
B



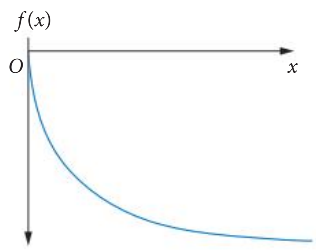
C



D

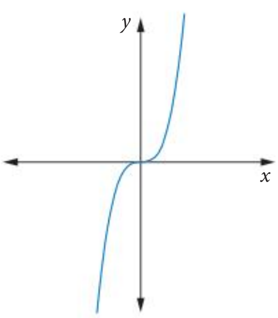


E

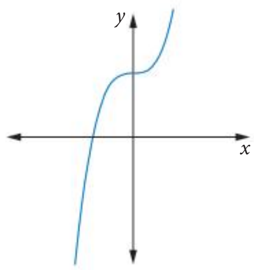


11 Which of the following is most likely to be the graph of $y=(x+2)^3$?

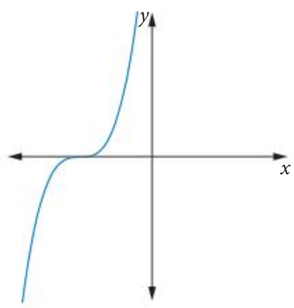
A



B



C



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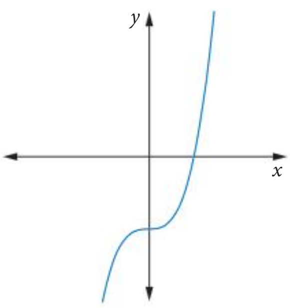
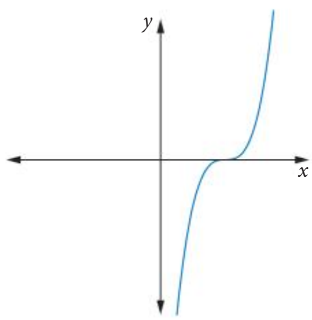
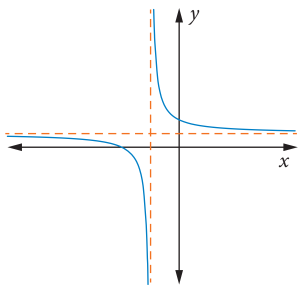


E

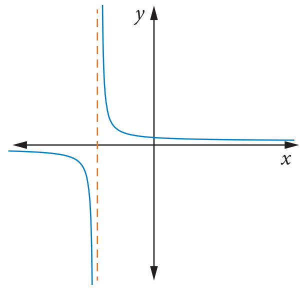


12 Which of the following is most likely to be the graph of $y = \dfrac{1}{x} - 4$?

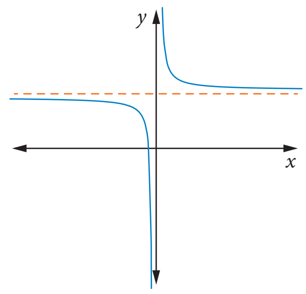
A



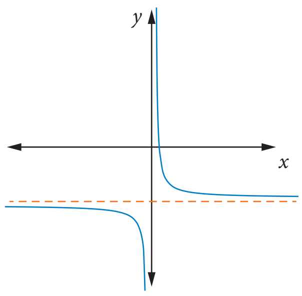
B



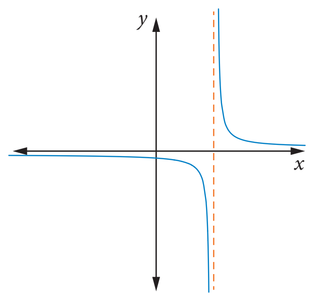
C



D



E



Video playlist Transformations of power functions

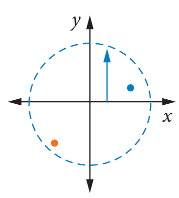
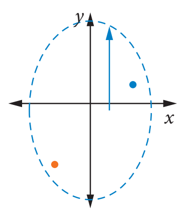
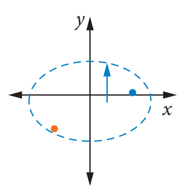
Worksheets Graphing translations of functions

4.2 Transformations of power functions

In the previous section, we sketched families of power functions changed by simple parameters. Changes to the appearance of a graph (or geometric figure) can be regarded as **transformations**.

Dilations from the x- and y-axes

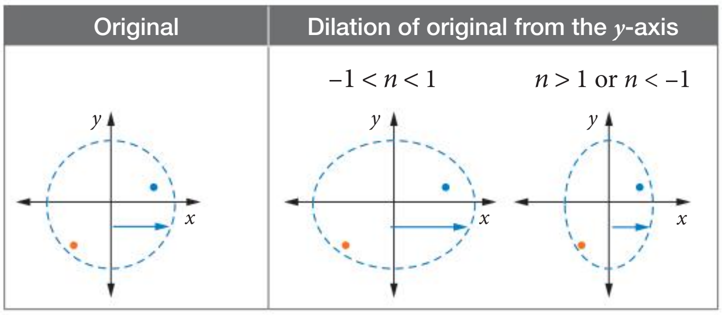
A **dilation** from the x-axis (or parallel to the y-axis) changes the appearance by expansion or compression ('vertically') from the x-axis. Every y-coordinate is multiplied by the same factor a.

Original	Dilation of original from the x-axis	
	$a > 1$ or $a < -1$	$-1 < a < 1$
		

The diagram shows a geometrical shape (circle) and 2 points being dilated from the x-axis (vertically) by the factor a. The dilation is a stretch if $a > 1$ or $a < -1$ and a compression (shrink) if $-1 < a < 1$.

For example, suppose $f(3) = 7$. For a dilation of factor 4 from the x-axis, $(3, 7) \to (3, 28)$. If $f(x) \to g(x)$, then $g(x) = 4f(x)$ so $g(3) = 4f(3) = 4 \times 7 = 28$. In general, under a dilation of factor a from the x-axis, $(x, y) \to (x, ay)$, and $f(x) \to g(x)$ is given by $g(x) = af(x)$.

A dilation from the y-axis (or parallel to the x-axis) changes the appearance by expansion or compression ('horizontally') from the y-axis. Every x-coordinate is multiplied by the same factor n.



The diagram shows a geometric shape (circle) and 2 points being dilated from the y-axis by the factor n, so $x \to nx$. For $n > 1$ or $n < -1$, the x-axis is stretched and for $-1 < n < 1$ it is shrunk.

Suppose $f(3) = 7$. For a dilation of factor 4 from the y-axis, $(3, 7) \to (12, 7)$. If $f(x) \to g(x)$, then $g(12) = f(3) = 7$, so $g(x) = f\left(\frac{x}{4}\right)$. In general, under a dilation of factor n from the y-axis, $(x, y) \to (nx, y)$ and $f(x) \to g(x)$ is given by $g(x) = f\left(\frac{x}{n}\right)$.

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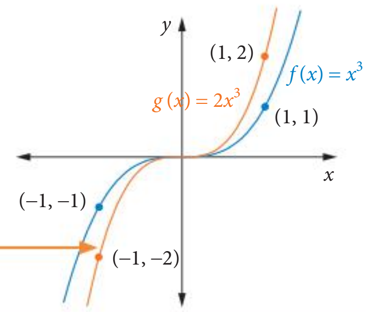
WORKED EXAMPLE 14 Dilations

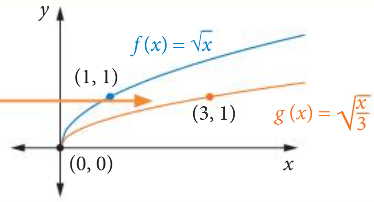
a The function $f(x) = x^3$ is dilated from the x-axis by the factor 2 to give the function $g(x)$. State the rule for $g(x)$ and sketch f and g on the same axes.

b The function $f(x) = \sqrt{x}$ is dilated from the y-axis by the factor 3 to give the function $g(x)$. State the rule for $g(x)$ and sketch f and g on the same axes.

Steps	Working
a **1** Use $g(x) = af(x)$.	$g(x) = 2x^3$
2 State the centres.	Both functions have centre (0, 0).
3 State other points.	f has points $(-1, -1)$ and $(1, 1)$ and g has the corresponding points $(-1, -2)$ and $(1, 2)$.
4 Sketch the functions on the same axes. The graph is stretched parallel to the y-axis (vertically) by factor 2.	
b **1** Use $g(x) = f(nx)$. $\frac{1}{n} = 3$ so $n = \frac{1}{3}$. $g(x) = f\left(\frac{1}{3}x\right)$.	$g(x) = \sqrt{\frac{x}{3}}$
2 State the starting points.	Both functions start at (0, 0).
3 State other points.	f has the point (1, 1) and $g(x)$ has the point (3, 1).

4 Sketch the functions on the same axes.

The graph is stretched parallel to the x-axis (horizontally) by factor 3.



Reflections in the *x*- and *y*-axes

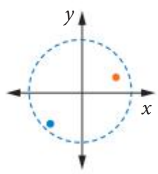
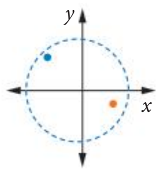
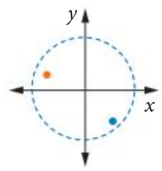
A **reflection** in the x-axis (or parallel to the y-axis) changes the appearance by inverting the shape (flipping it upside-down). The y-coordinate of every point is changed to its negative.

A reflection in the y-axis (or parallel to the x-axis) changes the appearance by swapping the left and right sides (flipping it back-to-front). The x-coordinate of every point is changed to its negative.

The diagram shows a circle and 2 points being reflected in the x-axis (vertically) and y-axis (horizontally).

The negative of the function $f(x)$ given by $g(x) = -f(x)$ is inverted or reflected in the x-axis, compared to $f(x)$.

Similarly, the function given by $g(x) = f(-x)$ is swapped from left-to-right or reflected in the y-axis.

Original	Reflection of original in the x-axis	Reflection of original in the y-axis
		

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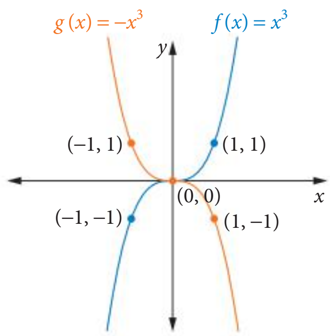
WORKED EXAMPLE 15 Reflections

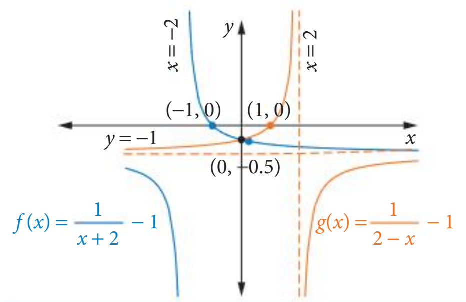
a The function $f(x) = x^3$ is reflected in the x-axis to give the function $g(x)$.

State the rule for $g(x)$ and sketch f and g on the same axes.

b The function $f(x) = \dfrac{1}{x+2} - 1$ is reflected in the y-axis to give the function $g(x)$.

State the rule for $g(x)$ and sketch f and g on the same axes.

Steps	Working
a 1 Use $g(x) = -f(x)$.	$g(x) = -x^3$
2 State the central points.	Both functions have the central points (0, 0).
3 State other points.	f and g both have (0, 0); f has points (−1, −1) and (1, 1); g has points (−1, 1) and (1, −1).
4 Sketch the functions on the same axes.	
b 1 Use $g(x) = f(-x)$	$g(x) = \dfrac{1}{-x+2} - 1 = \dfrac{1}{2-x} - 1$
2 State the asymptotes.	The asymptotes of $f(x)$ are $x = -2$ and $y = 1$; those of $g(x)$ are $x = 2$ and $y = 1$.

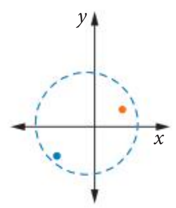
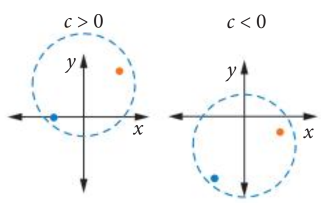
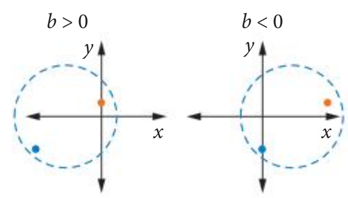
3 State other points.	The intercepts of f are $\left(0, -\frac{1}{2}\right)$ and $(-1, 0)$; for g they are $\left(0, -\frac{1}{2}\right)$ and $(1, 0)$.
4 Sketch the functions on the same axes.	

In part **a** above, we would get the same graph with reflection in the y-axis instead of reflection in the x-axis. For all the power functions in this course except $\sqrt{x}$, x^2 and x^4, reflection in one axis has the same effect on the shape as reflection in the other. Reflection in the y-axis has no effect on the shape of x^{-2}, x^2 or x^4.

Translations

A **translation** in the direction of the y-axis (or from the x-axis) changes the appearance by moving the shape in the y direction. Every point is changed by adding the same amount to the y-coordinate.

A translation in the direction of the x-axis (or from the y-axis) changes the appearance by moving the shape in the x direction. Every point is changed by adding the same amount to the x-coordinate.

Original	Translations in the direction of the y-axis: $f(x) + c$	Translations in the direction of the x-axis: $f(x + b)$
		

Suppose $f(3) = 7$. For a translation of 2 parallel to the y-axis, $(3, 7) \rightarrow (3, 9)$. If $f(x) \rightarrow g(x)$ then $g(3) = f(3) + 2 = 9$, so $g(x) = f(x) + 2$. In general, under a translation of c units in the y direction, $(x, y) \rightarrow (x, y + c)$, and $f(x) \rightarrow g(x)$ is given by $g(x) = f(x) + c$.

For a translation of 2 parallel to the x-axis, $(3, 7) \rightarrow (5, 7)$. If $f(x) \rightarrow g(x)$ then $g(5) = f(3) = 7$ so $g(x) = f(x - 2)$. In general, under a translation of b units in the x direction, $(x, y) \rightarrow (x + b, y)$, and $f(x) \rightarrow g(x)$ is given by $g(x) = f(x - b)$. For $b > 0$, it is to the left (negative direction) and for $b < 0$ it is to the right (positive direction).

WORKED EXAMPLE 16 Translations

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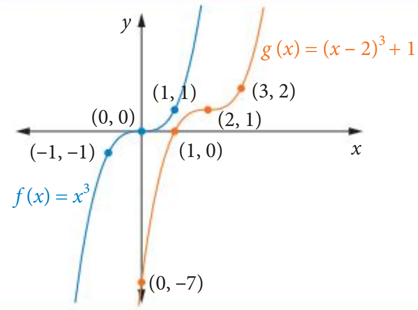
The function $f(x) = x^3$ is translated 2 units to the right and 1 unit up to give the function $g(x)$. State the rule for $g(x)$ and sketch f and g on the same axes.

Steps	Working
1 Use $g(x) = f(x + b) + c$.	$g(x) = (x - 2)^3 + 1$
2 State the central points.	The central points of $f(x)$ and $g(x)$ are $(0, 0)$ and $(2, 1)$, respectively.
3 State other points.	$f(1) = 0$ and $g(3) = 2$. The y-intercepts are 0 and -7 respectively. The zero of $g(x)$ is $(1, 0)$.

4 Sketch the functions on the same axes.



Combined transformations

Combined transformations

Transformations of the form T: $(x, y) \rightarrow (nx + b, ay + c)$ or $f(x) \rightarrow af(n(x + b)) + c$ are performed in the order **dilation**, **reflection**, **translation**. Remember this as DRT or Dr T ('Doctor T').

Transformations in the (horizontal) direction of the x-axis

- The dilation has a factor of $\frac{1}{n}$.
- If $n < 0$, there is a reflection in the y-axis.
- The translation is b units left (if $b < 0$, then the translation is to the right).

Transformations in the (vertical) direction of the y-axis

- The dilation has a factor of a.
- If $a < 0$, there is a reflection in the x-axis.
- The translation is c units up (if $c < 0$, then the translation is downwards).

With the DRT order, it doesn't matter if horizontal or vertical transformations are done first, as long as those for each direction are done in the correct order.

Transformations of the form T: $(x, y) \rightarrow (n(x + b), a(y + c))$ or $f(x) \rightarrow a(f(nx + b) + c)$ follow the reverse order TRD: the translations are first.

The transformation T: $(x, y) \rightarrow (nx + b, ay + c)$ transforms the function $f(x) \rightarrow af\left(\frac{1}{n}(x - b)\right) + c$.

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WORKED EXAMPLE 17 Combined transformations of points

The points $A(3, 7)$, $B(-2, 9)$ and $C(5, -3)$ are on the function $f(x)$.

a Find the new points after a dilation by factor 2 in the x direction and translations 2 units in the positive y direction and 3 units in the negative x direction.

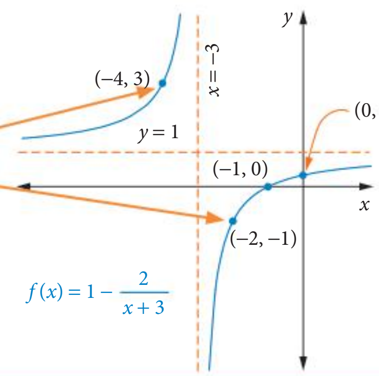
b Find the positions of A, B and C after translations of 4 units down and 2 units to the left and then dilation by factor 0.5 from the x-axis and reflection in the y-axis.

Steps	Working
a **1** Write the transformation.	$(x, y) \rightarrow (2x - 3, y + 2)$
2 Find the points.	$A(3, 7) \rightarrow (2 \times 3 - 3, 7 + 2) = (3, 9)$ $B(-2, 9) \rightarrow (2 \times (-2) - 3, 9 + 2) = (-7, 11)$ $C(5, -3) \rightarrow (2 \times 5 - 3, -3 + 2) = (7, -1)$
3 Write the answer.	The new points are $A(3, 9)$, $B(-7, 11)$ and $C(7, -1)$.

b 1 Write the transformation.	$(x, y) \rightarrow (-(x-2), 0.5(y-4))$
2 Find the points.	$A(3, 7) \rightarrow (-(3-2), 0.5(7-4)) = (-1, 1.5)$ $B(-2, 9) \rightarrow (-(-2-2), 0.5(9-4)) = (4, 2.5)$ $C(5, -3) \rightarrow (-(5-2), 0.5(-3-4)) = (-3, -3.5)$
3 Write the answer.	The new points are $A(-1, 1.5)$, $B(4, 2.5)$ and $C(-3, -3.5)$.

WORKED EXAMPLE 18 Combined transformations of functions

Sketch the graph of **a** $f(x) = 1 - \dfrac{2}{x+3}$ **b** $g(x) = \sqrt{3-2x} - 1$

Steps	**Working**
a 1 State the basic function.	The basic function is $y = x^{-1}$.
2 State the transformations. Transformations parallel to the x-axis first.	Translation 3 to the left (negative direction of the x-axis), reflection in the x-axis, dilation from the x-axis by the factor 2 and translation 1 up (positive direction of the y-axis).
3 State the asymptotes.	The asymptotes are $y = 1$ and $x = -3$.
4 State the y-intercept.	$f(0) = 1 - \dfrac{2}{3} = \dfrac{1}{3}$
5 Find zeros if possible.	$1 - \dfrac{2}{x+3} = 0$ $x + 3 = 2$ $x = -1$, the zero is $(-1, 0)$.
6 Sketch the graph, including coordinates of important points and asymptote equations. You might want to include the coordinates of vertices.	
b 1 State the basic function.	The basic function is $y = \sqrt{x}$.
2 Write the function in the form $af(x+b)+c$.	$g(x) = \sqrt{3-2x} - 1 = \sqrt{-2\left(x - \dfrac{3}{2}\right)} - 1$
3 State the transformations.	Dilation by the factor $\dfrac{1}{2}$ parallel to the x-axis, reflection in the y-axis and translation 1.5 to the right and 1 down.
4 State the starting point.	The starting point is $\left(\dfrac{3}{2}, -1\right)$.
5 State other points.	The y-intercept is $(0, \sqrt{3} - 1)$. The zero is at $(1, 0)$ and $(-3, 2)$ is on the curve.

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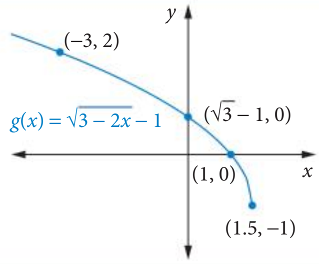
6 Sketch the graph.



We can find the equation of a new function obtained by transformation of a given function.

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WORKED EXAMPLE 19 Finding the equation of a transformed function

Each function $f(x)$ is transformed to $g(x)$ by the operations given. Find each function $g(x)$.

a $f(x) = \sqrt{x}$: dilation parallel to the x-axis by the factor 0.5, reflection in the x-axis and translation 3 up and 1 to the left.

b $f(x) = \dfrac{2}{x-3}$: reflection in the y-axis, translation 5 in the negative direction of the x-axis and 3 in the negative direction of the y-axis, and dilation by the factor $\dfrac{1}{4}$ from the x-axis.

Note that this is not in DRT order.

Steps	Working
a 1 Do the dilation using $\dfrac{1}{0.5} = 2$.	$\sqrt{x} \to \sqrt{2x}$
2 Do the reflection in the x-axis.	$\sqrt{2x} \to -\sqrt{2x}$
3 Do the 2 translations and write $g(x)$.	$-\sqrt{2x} \to -\sqrt{2(x+1)} + 3$ $g(x) = 3 - \sqrt{2(x+1)}$
b 1 Do the reflection in the y-axis.	$\dfrac{2}{x-3} \to \dfrac{2}{-x-3}$
2 Do the translation to the left.	$\dfrac{2}{-x-3} \to \dfrac{2}{(-x-3)+5} = \dfrac{2}{2-x}$
3 Do the translation 3 down.	$\dfrac{2}{2-x} \to \dfrac{2}{2-x} - 3$
4 Do the dilation by a factor of $\dfrac{1}{4}$ from the x-axis.	$\dfrac{2}{2-x} - 3 \to \dfrac{1}{4}\left(\dfrac{2}{2-x} - 3\right) = \dfrac{1}{2(2-x)} - \dfrac{3}{4}$
5 Write $g(x)$.	$g(x) = \dfrac{1}{2(2-x)} - \dfrac{3}{4}$ $\left(\text{or } \dfrac{1}{4-2x} - \dfrac{3}{4}\right)$

When we need to be able to find the transformations that change one function to another, work backwards.

WORKED EXAMPLE 20 Finding the transformations from an equation

What transformations have been performed on $f(x)$ to obtain $g(x)$?

a $f(x) = x^3, g(x) = \frac{(2-x)^3}{4} + 1$ **b** $f(x) = \frac{1}{(2x+3)^2} - 2, g(x) = \frac{3}{(x-2)^2} - 4$

Steps	**Working**
a 1 Express $g(x)$ in the form $af(n(x+b)) + c$.	$g(x) = \frac{(2-x)^3}{4} + 1$ $= \frac{1}{4}(2 + (-x))^3 + 1$ $= \frac{1}{4}(-(x-2))^3 + 1$
2 State any dilations and reflections.	Dilation from the x-axis by the factor $\frac{1}{4}$ and reflection in the y-axis.
3 State any translations.	Translations 2 to the right and 1 up.
4 Put in the correct order. Other answers/orders are possible.	$f(x)$ has been dilated from the x-axis by the factor $\frac{1}{4}$, reflected in the y-axis and translated 2 to the right and 1 up to obtain $g(x)$.
b 1 Express $g(x)$ in terms of f.	$g(x) = \frac{3}{(x-2)^2} - 4$ $= 3\left[\frac{1}{(2 \times 0.5x - 2)^2}\right] - 4$ $= 3\left[\frac{1}{(2(0.5(x-5)) + 3)^2}\right] - 2 - 2$
2 State any dilations and reflections.	Dilation from the y-axis by the factor $\frac{1}{0.5} = 2$ and from the x-axis by the factor 3.
3 State any translations.	Translation 5 in the positive x direction and 2 in the negative y direction.
4 Put in the correct order.	$f(x)$ has been dilated from the y-axis by the factor 2 and from the x-axis by the factor 3, and translated 5 units in the positive x direction and 2 units in the negative y direction to obtain $g(x)$.

4.2

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To find the transformations of a basic function that gives a graph, we also need to work backwards.

WORKED EXAMPLE 21 Finding transformations from a graph

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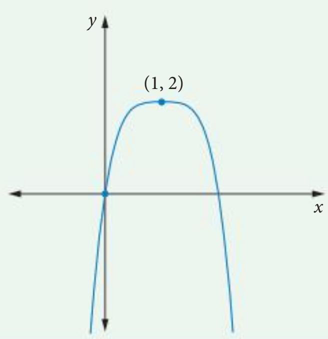
This graph has been obtained by applying transformations to the graph of $y = x^4$.

What transformations have been used and what is the equation of the graph?



Steps	Working
1 Use the turning point to find the translations.	The graph has been flipped upside-down and translated 1 to the right and 2 up.
2 State the form of the equation.	The equation is of the form $f(x) = a(x-1)^4 + 2$. where a is negative.
3 Substitute (0, 0) to find the value of a and the equation.	$0 = a(0-1)^4 + 2$ $0 = a + 2$, so $a = -2$ Equation is $f(x) = -2(x-1)^4 + 2$.
4 State the answer.	The graph has a dilation from the x-axis by a factor of 2, reflection in the x-axis, and translation 1 unit right and 2 units up. Its equation is $y = -2(x-1)^4 + 2$.

With combined transformations, there is usually more than one way of achieving the same result. In the above example, there is another possible solution with a dilation in the y-axis by a factor of $\sqrt[4]{2}$ and translation to the right before the dilation instead of the dilation in the x-axis.

The equation for this is $f(x) = -(\sqrt[4]{2}(x-1))^4 + 2$.

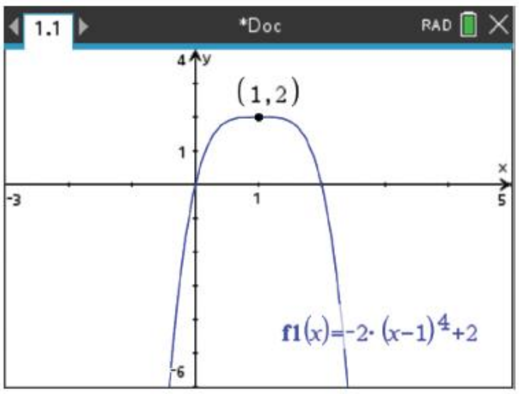
However, it is sensible to use the simplest transformations and corresponding equation.

Use CAS to check the answer when you are finding an equation for a graph.

USING CAS 2 Transformations of power functions

Draw the graph of $y = -2(x-1)^4 + 2$ and find the turning point.

TI-Nspire



1 Add a **Graphs** page.

2 Enter $\boldsymbol{f1(x) = -2(x-1)^4 + 2}$ and press **enter**.

3 Adjust the window settings to suit.

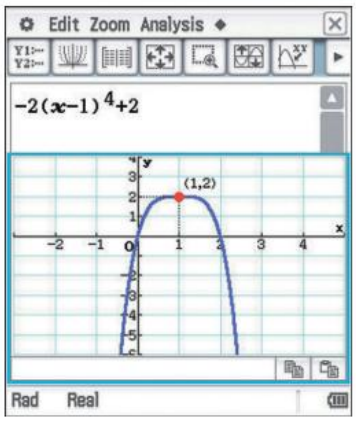
4 Press **menu > Analyze Graph > Maximum**.

5 When prompted for the **lower bound?**, click to the left of the turning point.

6 When prompted for the **upper bound?**, click to the right of the turning point.

7 The coordinates of the turning point will be displayed on the graph.

ClassPad



1 In **Main**, enter and highlight the expression $\boldsymbol{-2(x-1)^4 + 2}$.

2 Tap **Graph** and drag the expression down into the graph window.

3 Adjust the window settings to suit.

4 Tap **Analysis > G-Solve > Max**.

Tap **EXE** to display the label.

5 The coordinates of the turning point will be displayed on the graph.

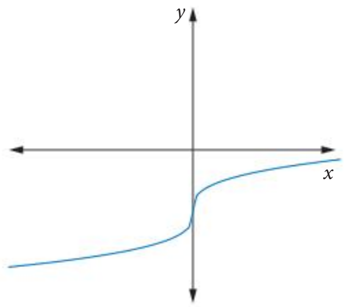
EXERCISE 4.2 Transformations of power functions

ANSWERS p. 509

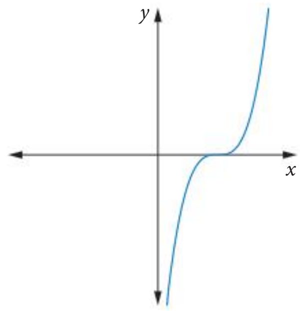
Recap

1 Which of the following is likely to be the graph of $f: R \to R, f(x) = \sqrt[3]{x-2}$?
Note: The graphs are all at the same scale.

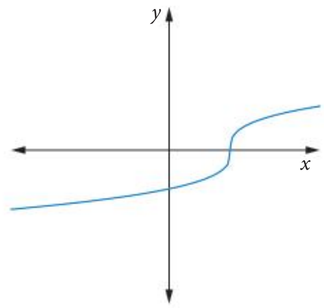
A



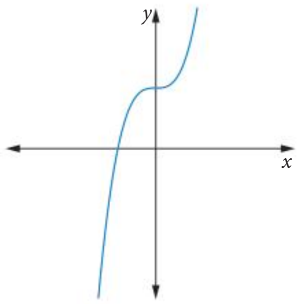
B



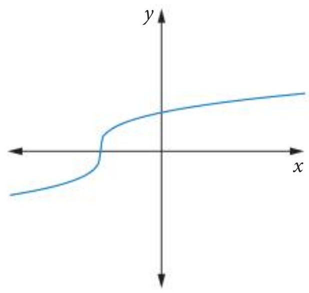
C



D



E



2 The asymptotes of $f(x) = \frac{2}{x}$ over its maximal domain are

A $x = 0$ and $y = 0$ **B** $x = 2$ and $y = 0$ **C** $x = 0$ and $y = 2$

D $x = 0$ and $y = -2$ **E** $x = 2$ and $y = 2$

Mastery

3 WORKED EXAMPLES 14–16 Sketch the graph of each function, showing the coordinates of any intercepts and the equations of any asymptotes.

a $f(x) = 4x^3$ **b** $y = -\frac{1}{x}$ **c** $y = \sqrt{0.5x}$ **d** $f(x) = (x+2)^4 - 8$

e $f(x) = \frac{2}{x^2}$ **f** $f(x) = -\frac{\sqrt[3]{x}}{2}$ **g** $f(x) = \frac{1}{3x}$ **h** $y = 4 + \sqrt{x-1}$

i $y = -x^3$ **j** $y = \frac{1}{(x-2)^2} + 3$ **k** $f(x) = \sqrt{-x}$ **l** $y = \sqrt[3]{1-x}$

4 WORKED EXAMPLE 17 The points $A(3, -2)$, $B(-4, 6)$ and $C(2, 5)$ are on the function $f(x)$. Find their positions after each set of transformations.

a Dilation from the x-axis by the factor 3, from the y-axis by the factor 2, reflection in the y-axis and translation 2 left and 4 up

b Dilation from the y-axis by the factor 4, from the x-axis by the factor 0.5, reflection in both axes and translation 2 to the right and 1 up

c Translation 4 in the positive x direction, reflection in the x-axis and dilation from the y-axis by the factor 3

d Translation 2 down and 4 to the left, then reflection in both axes and dilation by the factor 3 parallel to the y-axis

e Dilation by the factor $\frac{1}{3}$ from the x-axis, reflection in the y-axis and translation 4 to the right and 3 down

5 WORKED EXAMPLE 18 Sketch the graph of each function.

a $f(x) = 2(x+1)^3 - 4$ **b** $y = 3(2-x)^{-1} - 1$ **c** $f(x) = -(2x-1)^4 + 3$

d $y = (1-x)^3 + 2$ **e** $f(x) = 4\sqrt[3]{x+3} - 2$ **f** $y = 5 - 4(0.5x-1)^3$

g $f(x) = (3(x-1))^{-2} - 2$ **h** $y = \dfrac{\sqrt{x+1}}{3} + 2$ **i** $y = \dfrac{2}{1-0.25x} + 3$

j $f(x) = -0.5\sqrt{3x-2} - 1$ **k** $y = 2 - \dfrac{1}{(0.5x+3)^2}$ **l** $f(x) = 1 + \dfrac{\sqrt[3]{2(2-x)}}{3}$

6 WORKED EXAMPLE 19 Find the equation of $g(x)$ from the transformations on $f(x)$.

a $f(x) = x^2$: reflection in the x-axis, dilation by a factor of 3 from the x-axis, translation 1 down and 4 to the right.

b $f(x) = x^4$: translation 1 up, dilation by a factor of 4 parallel to the y-axis, reflection in the x-axis and translation 3 to the left.

c $f(x) = \dfrac{1}{x^2}$: reflection in the x-axis, dilation by a factor of $\dfrac{1}{3}$ from the y-axis, translation 3 in the positive direction of the y-axis and 2 in the positive direction of the x-axis.

d $f(x) = -2(x+4)^2 - 3$: dilation by a factor of $\dfrac{1}{4}$ parallel to the x-axis, translation 2 up and 3 to the right.

e $f(x) = \sqrt[3]{x+3} - 2$: dilation by a factor of 3 from the x-axis, translation 1 in the negative x direction, 3 in the negative y direction and reflection in the y-axis.

f $f(x) = -3(x+4)^{-1} + 4$: translation 3 to the right, reflection in the x-axis, translation 2 down and dilation by a factor of $\dfrac{1}{2}$ from the y-axis.

g $f(x) = 6(x-2)^3 - 3$: translation 6 in the negative x direction, dilation by a factor of $\dfrac{1}{2}$ from the x-axis, reflection in the y-axis and translation 4 in the positive y direction.

h $f(x) = 0.25\sqrt{2-x} + 3$: reflection in the y-axis, dilation by a factor of 4 parallel to the y-axis, translation 3 down and 2 to the right.

7 WORKED EXAMPLE 20 What transformations have been performed on $f(x)$ to obtain $g(x)$?

a $f(x) = x^2, g(x) = 3(5x-2)^2 + 4$ **b** $f(x) = x^{-2}, g(x) = -\left(\dfrac{x}{3} + 4\right)^{-2} + 2$

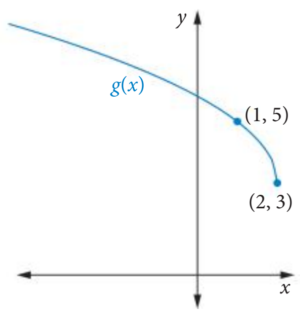
c $f(x) = \sqrt[3]{x}, g(x) = 4\sqrt[3]{3-2x} - 1$ **d** $f(x) = (x+3)^2 - 5, g(x) = 2 - 3x^2$

e $f(x) = \dfrac{1}{2x-3} + 1, g(x) = -\dfrac{1}{x-5} - 3$ **f** $f(x) = -0.5x^4 + 2, g(x) = 3(x-2)^4 + 5$

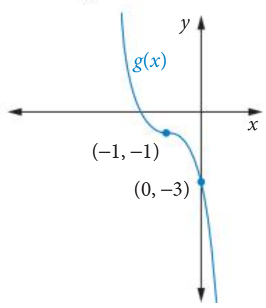
g $f(x) = \dfrac{1}{3x-6} - 4, g(x) = \dfrac{1}{3-x} + 1$ **h** $f(x) = 2\sqrt{x+3} - 1, g(x) = \dfrac{\sqrt{3x-1}}{2} + 3$

8 WORKED EXAMPLE 21 These graphs have been obtained by applying transformations to the given function. What are the transformations and equation of each graph?

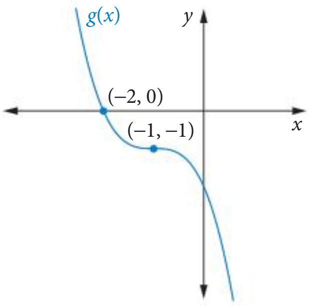
a $f(x) = \sqrt{x}$



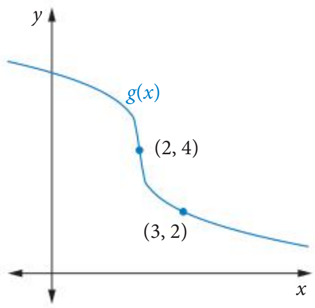
b $f(x) = \dfrac{1}{x}$



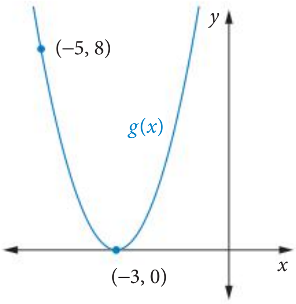
c $f(x) = x^3$



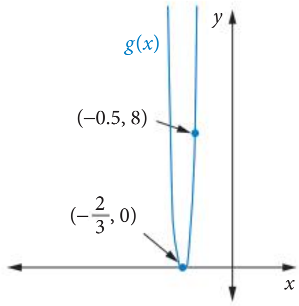
d $f(x) = \sqrt[3]{x}$



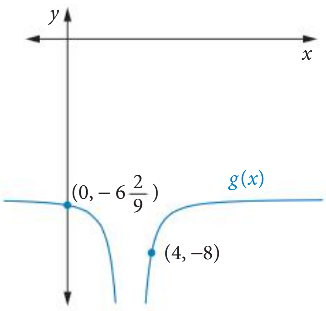
e $f(x) = -4x^2 + 2$



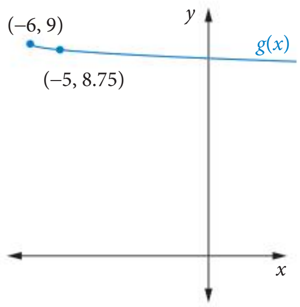
f $f(x) = 2(x + 3)^4$



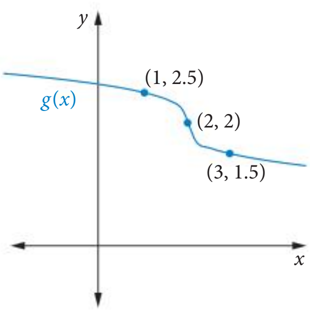
g $f(x) = \dfrac{2}{(x-1)^2} - 3$



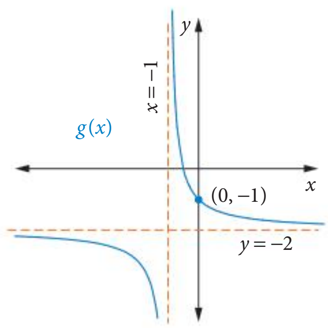
h $f(x) = \sqrt{2 - x} + 4$



i $f(x) = 4\sqrt[3]{2x + 3} - 1$



j $f(x) = 3 - \dfrac{2}{0.25x - 1}$



9 Using CAS 2 Graph each function.

a $f(x) = 0.4(x + 3)^4 - 2$

b $y = 3 - \dfrac{1}{2x - 4}$

c $f(x) = \sqrt{x + 4} - 3$

d $y = \dfrac{2}{(x + 1)^2} + 3$

e $f(x) = 1 - \sqrt[3]{x - 3}$

f $y = 2(x - 1)^3 + 4$

Exam practice

80–100% 60–79% 0–59%

10 © VCAA 2016 1Q3a MODIFIED TECH-FREE (3 marks) Let $f: R\backslash\{3\} \to R$, where $f(x) = 1 + \frac{2}{x-3}$.

Sketch the graph of f. Label the axis intercepts with their coordinates and label any asymptotes with the appropriate equation.

11 © VCAA 2012 2BQ2a 72% TECH-FREE (3 marks) Let $f: R\backslash\{2\} \to R, f(x) = \frac{1}{2x-4} + 3$

Sketch the graph of $y = f(x)$. Label the axes intercepts with their coordinates and label each of the asymptotes with its equation.

12 © VCAA 2014 2AQ1 MODIFIED The point P (4, −3) lies on the graph of a function f. The graph of f is reflected in the x-axis and translated four units vertically up.

The coordinates of the final image of P are

A (−4, 3) **B** (−4, −1) **C** (4, 3) **D** (4, 7) **E** (−4, 1)

13 © VCAA 2016 2AQ12 MODIFIED The graph of a function f is obtained from the graph of the function g with rule $g(x) = 2\sqrt{x-5}$ by a reflection in the x-axis followed by a dilation from the x-axis by a factor of $\frac{1}{2}$. Which one of the following is the rule for the function f?

A $f(x) = \sqrt{5-4x}$

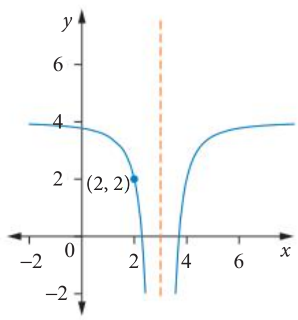
B $f(x) = -\sqrt{x-5}$

C $f(x) = \sqrt{x+5}$

D $f(x) = -\sqrt{4x-5}$

E $f(x) = -\sqrt{4x-10}$

14 Part of the graph of the function with the rule $y = \frac{a}{(x+b)^2} + c$ is shown below.



The values of a, b and c respectively are

A −1, 3, 4 **B** −2, 3, 4 **C** −2, −3, 4 **D** −2, −3, −4 **E** −1, −3, −4

15 The transformation that maps the graph of $y = \sqrt{4x-4}$ onto the graph of $y = \sqrt{x+1}$ is a

A dilation by a factor of 4 from the y-axis and a translation 2 to the right.

B dilation by a factor of $\frac{1}{2}$ from the x-axis and a translation 2 to the right.

C dilation by a factor of 4 from the y-axis and a translation 5 to the left.

D translation of 2 to the left and a dilation by a factor of 4 from the y-axis.

E dilation by a factor of $\frac{1}{2}$ from the x-axis and a translation of 5 to the left.

4.3 Inverse functions

The **inverse of a relation** with ordered pairs (x, y) is the relation formed by reversing all the ordered pairs to (y, x).

If the inverse relation of a function $f(x)$ is also a function, then it is called an **inverse function** and is written as $f^{-1}(x)$. This function links the values of y in $y = f(x)$ back to x. For example, if $f(3) = -5$, then $f^{-1}(-5) = 3$. In general, $f^{-1}(f(x)) = x$ and $f(f^{-1}(x)) = x$.

Consider the function $f(x) = \sqrt{x}$ with ordered pairs (1, 1), (4, 2) and (25, 5). The inverse function to $f(x)$ must have the ordered pairs (1, 1), (2, 4) and (5, 25). The inverse function is $f^{-1}(x) = x^2$.

However, the inverse of the function $y = x^2$ is not a function. Its implied domain is the real numbers and some of its ordered pairs are (−2, 4), (2, 4), (−5, 25) and (5, 25), so the inverse includes (25, 5), (25, −5) etc. The inverse relation has points with 2 different y values for the same x value, so it is not a function.

For an inverse function to exist, the y values of the original function must also be unique. The original function must be a **one-to-one function** because, as well as having only one y value for every x value, there is also only one x value for every y value.

4.3

Video playlist
Inverse functions

WB p. 92

Worksheets
Inverse functions

Inverse functions code puzzle

WORKED EXAMPLE 22 Existence of an inverse function

Determine which of the following functions are one-to-one and have inverse functions.

$f: R \to R, f(x) = x^3$, $g: R \to R, g(x) = x^4$, $h: R \to R, h(x) = 5 - \dfrac{1}{x+2}$

Steps	Working
1 Consider $f(x)$.	Solving $y = x^3$ for x gives $x = \sqrt[3]{y}$, so there is only one possible x value for any y value, so it is one-to-one.
2 Consider $g(x)$.	Solving $y = x^4$ for x gives $x = \pm\sqrt[4]{y}$. When $y = 16$, $x = 2$ or -2, so $g(x)$ is not one-to-one.
3 Consider $h(x)$. $x = -2$ is not in the domain and $y = 5$ is not in the range of $h(x)$, so the restrictions don't matter.	Solving $y = 5 - \dfrac{1}{x+2}$ for x gives $x = \dfrac{9-2y}{y-5}$, so $h(x)$ is one-to-one.
4 Write the answer.	f and h are one-to-one and so have inverse functions, but g is not one-to-one and does not have an inverse function.

We can find the rule for an inverse by solving the function rule to find x, as shown in the example above. Then we can swap the variables to express the inverse function in the normal way.

WB p. 93

WORKED EXAMPLE 23 Finding an inverse function

a Find the rule for the inverse of $f: R \to R, f(x) = 8x^3 - 4$.

b Is the inverse a function?

Steps	Working
a **1** Write with the variable y.	$y = 8x^3 - 4$
2 Solve for x.	$8x^3 = y + 4$ $x^3 = \dfrac{y+4}{8}$ $x = \dfrac{\sqrt[3]{y+4}}{2}$

3 Swap the variables to get the inverse relation.	The inverse is $y = \frac{\sqrt[3]{x+4}}{2}$.
b Determine if it is a function.	$y = \frac{1}{2}\sqrt[3]{x+4}$ is unique $\therefore$ it is a function, so it can be written as $f^{-1}(x) = \frac{1}{2}\sqrt[3]{x+4}$.

We can also swap the variables before rearranging the function rule.

WB

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WORKED EXAMPLE 24 The graph of a function and inverse relation

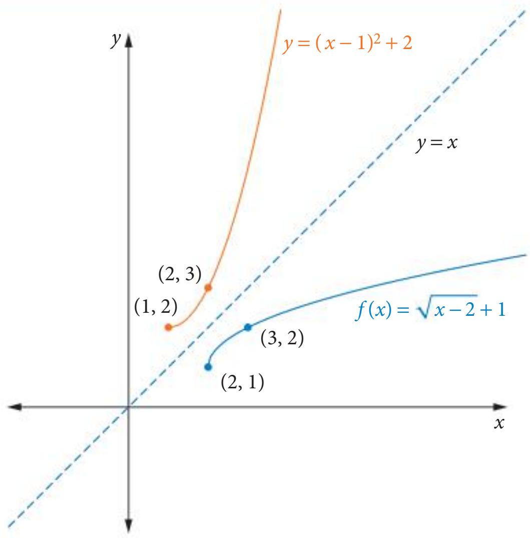
a Find the rule for the inverse of $y = \sqrt{x-2} + 1$.

b Sketch the graph of $y = x$, the function and its inverse on the same axes.

c Comment on the graphs.

d State the domain and range of the function and its inverse.

e Write the inverse formally as a function if possible.

Steps	**Working**
a **1** Write with the variables swapped.	Let $x = \sqrt{y-2} + 1$
2 Solve for y.	$\sqrt{y-2} = x - 1$ $y - 2 = (x-1)^2$ $y = (x-1)^2 + 2$
b Sketch the graphs on the same axes. $f(x)$ is defined only for $x > 2$, so the inverse relation is defined only for $y > 2$. Do not write inverse as $f^{-1}(x)$ before formally stating it is a function.	
c Comment on the graphs.	The graph of $y = (x-1)^2 + 2$ is the reflection of the graph of $f(x)$ in the line $y = x$.
d **1** State the domain and range of $f(x)$. **2** State the domain and range of the inverse.	The domain of f is $[2, \infty)$, and range is $[1, \infty)$. The inverse relation has domain $= [1, \infty)$ and range $= [2, \infty)$.
e The inverse is a function, but does not have its maximal domain.	$f^{-1}: [1, \infty) \rightarrow R, f^{-1}(x) = (x-1)^2 + 2$

Functions and inverses

- We can find the inverse of a function by swapping the x and y variables and rearranging to find y.
- The *domain of the inverse* relation is the *range of the original function.*
- The *range of the inverse* relation is the *domain of the original function.*
- The inverse is a function *only* if the original function is one-to-one.
- The graph of the inverse is the reflection of $y = f(x)$ in the line $y = x$.

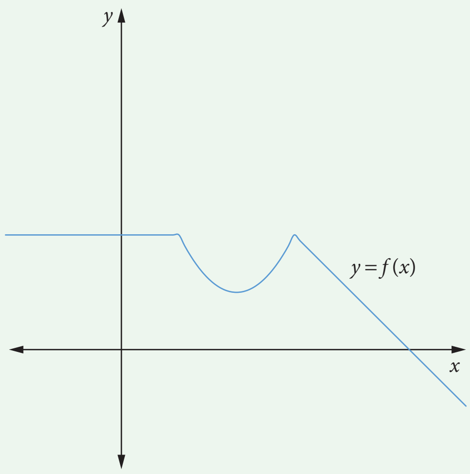
We can sketch the graph of a function and its inverse by reflecting the graph of the function in the line $y = x$. We can determine if the inverse is a function from the graph.

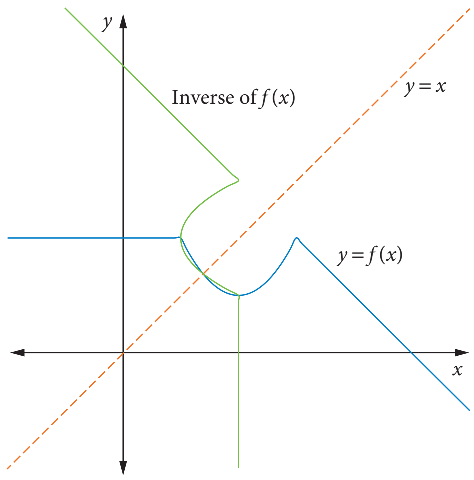
WORKED EXAMPLE 25 Sketching the inverse from the graph

The graph of a function $f(x)$ is shown.

a Sketch the graph of the inverse of $f(x)$.

b Is the inverse a function?



Steps	Working
a Reflect the graph in the line $y = x$.	
b Determine if the inverse (green) is a function.	The inverse is not a function because it has some points directly above others, so the y values are not unique.

4.3

WB

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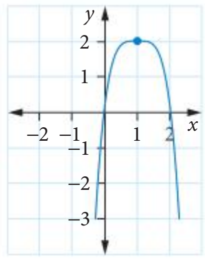
EXERCISE 4.3 Inverse functions

ANSWERS p. 513

Recap

1 Part of the graph of a function is shown.



It is most likely to be

A $f(x) = (x + 1)^4 + 2$

B $f(x) = -(x + 1)^2 + 2$

C $f(x) = -(x - 1)^2 + 2$

D $f(x) = -2(x - 1)^4 + 2$

E $f(x) = -2(x + 1)^4 + 2$

2 The function $f(x) = 4\sqrt{x + 3} - 2$ is reflected in the x-axis, dilated by the factor 2 from the x-axis and translated 2 to the right and 2 up. The new function is

A $8\sqrt{x + 5} - 4$ **B** $-2\sqrt{x + 1} - 4$ **C** $-8\sqrt{x + 1} + 6$

D $-2\sqrt{x + 1} - 4$ **E** $-2\sqrt{x + 1}$

Mastery

3 WORKED EXAMPLE 22 Determine whether each function has an inverse function.

a $f: R \to R, f(x) = 4x + 5$

b $g: R^+ \to R^+, f(x) = 2x^2 + x + 4$

c $h: R \to R, h(x) = \dfrac{1}{(x - 3)^2} + 4$

d $g: R \to R, g(x) = \dfrac{2}{x + 4} - 3$

e $h: R \to R, h(x) = \sqrt{49 - x^2}$

f $f: R^+ \to R, f(x) = 4 - \sqrt[3]{x + 1}$

g $h: [5, \infty) \to R, h(x) = (x - 3)^4 + 5$

h $f: R \to R, f(x) = 2 - 3(x + 8)^3$

4 WORKED EXAMPLE 23 Find the inverse of each function and state if it is a function.

a $f: R \to R, f(x) = 3x + 8$

b $g: R \to R, g(x) = 4x^4 - 6$

c $h: R \to R, h(x) = 5 - 2\sqrt[3]{x + 4}$

d $f: (-4, \infty) \to R, f(x) = \dfrac{2}{(x + 3)^2} - 1$

e $g: [-2, \infty) \to R, g(x) = (x + 2)^2 - 6$

f $h: R^+ \to R, h(x) = 4 - \dfrac{1}{x - 2}$

g $h: [5, \infty) \to R, h(x) = (x - 3)^3 - 7$

h $f: [2, \infty) \to R, f(x) = 4 - 3\sqrt{x^2 - 4}$

5 WORKED EXAMPLE 24 For each function, state the domain and range, find the rule for the inverse and state if it is a function. Sketch the graphs if necessary.

a $f(x) = (x + 2)^3 - 8$

b $g(x) = (x - 5)^2 + 3$

c $h(x) = 3 - \dfrac{2}{x - 1}$

d $f: (2, 5] \to R, f(x) = 4 - 3x$

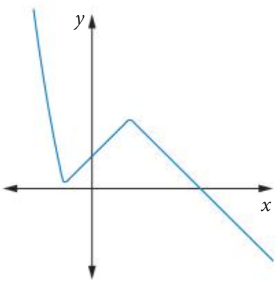
e $g: R^+ \to R, g(x) = \dfrac{3}{x - 4} + 1$

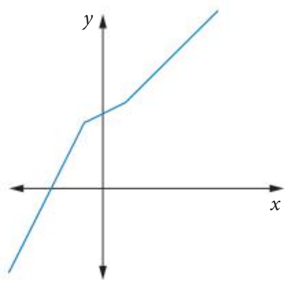
f $h: R^+ \to R, h(x) = 7 - 5x^3$

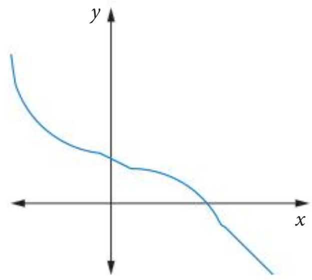
g $h: [5, \infty) \to R, h(x) = 4\sqrt{x - 5} + 1$

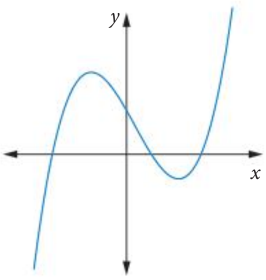
h $f: [-2, \infty) \to R, f(x) = 1 - 4(x - 2)^2$

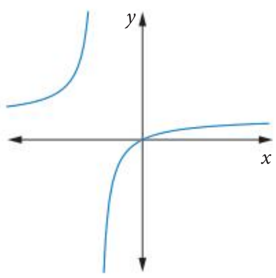
6 WORKED EXAMPLE 25 For each graph, sketch the graph of the inverse and state if the inverse is a function.

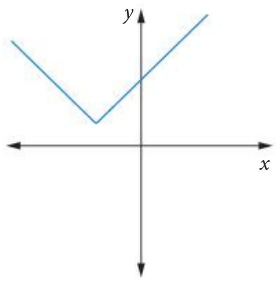
a


b


c


d


e


f


Exam practice

80–100% 60–79% 0–59%

7 © VCAA 2012 1Q3 MODIFIED TECH-FREE (2 marks) The rule for function h is $h(x) = 4x^3 + 3$.
Find the rule for the inverse function h^{-1}.

8 © VCAA 2017N 1Q8 TECH-FREE (8 marks) The rule for a function f is given by $f(x) = \sqrt{2x+3} - 1$, where f is defined on its maximal domain.

a Find the domain and rule of the inverse function f^{-1}. 2 marks

b Solve $f(x) = f^{-1}(x)$. 2 marks

c Let $g: D \to R, g(x) = \sqrt{2x+c} - 1$, where D is the maximal domain of g and c is a real number.

i For what value(s) of c does $g(x) = g^{-1}(x)$ have no real solutions? 2 marks

ii For what value(s) of c does $g(x) = g^{-1}(x)$ have exactly one real solution? 2 marks

9 © VCAA 2014 2AQ9 76% The inverse of the function $f: R^+ \to R, f(x) = \dfrac{1}{\sqrt{x}} + 4$ is

A $f^{-1}: (4, \infty) \to R \qquad f^{-1}(x) = \dfrac{1}{(x-4)^2}$

B $f^{-1}: R^+ \to R \qquad f^{-1}(x) = \dfrac{1}{x^2} + 4$

C $f^{-1}: R^+ \to R \qquad f^{-1}(x) = (x+4)^2$

D $f^{-1}: (-4, \infty) \to R \qquad f^{-1}(x) = \dfrac{1}{(x+4)^2}$

E $f^{-1}: (-\infty, 4) \to R \qquad f^{-1}(x) = \dfrac{1}{(x-4)^2}$

10 © VCAA 2015 2AQ2 MODIFIED The inverse function of $f: (-4, \infty) \to R, f(x) = \frac{1}{\sqrt{x+4}}$ is

A $f^{-1}: R^+ \to R \quad f^{-1}(x) = \frac{1}{x^2} - 4$

B $f^{-1}: R\backslash\{0\} \to R \quad f^{-1}(x) = \frac{1}{x^2} - 4$

C $f^{-1}: R^+ \to R \quad f^{-1}(x) = \frac{1}{x^2} + 4$

D $f^{-1}: (-4, \infty) \to R \quad f^{-1}(x) = x^2 + 4$

E $f^{-1}: (4, \infty) \to R \quad f^{-1}(x) = \frac{1}{x^2 - 4}$

Video playlist
Matrices

Worksheets
Addition and subtraction of matrices

Multiplying matrices

4.4 Matrices

OPTIONAL CONTENT: Since 2023, matrix representation of transformations has been an optional part of the course. According to the VCE Study Design, it 'may be used but is not required'.

A **matrix** is a rectangular array of numbers enclosed by brackets, symbolised by a capital letter, like

$$A = \begin{bmatrix} 2 & -4 & 60 & 1 \\ 5 & 2.8 & -70 & 0 \end{bmatrix}.$$

The **dimensions of the matrix** are the number of rows m and columns n, shown as $m \times n$. The matrix A above is a 2×4 matrix with 8 elements.

A **square matrix** with n rows and n columns has dimension n.

The values in the matrix **elements** are labelled as a_{ij} for the element in row i and column j. In the matrix above, $a_{13} = 60$.

We show a general matrix as $A = (a_{ij})$.

WB
p. 96

WORKED EXAMPLE 26 Matrix dimensions and elements

$$M = \begin{bmatrix} 2 & 4 & 6 & -1 \\ 1 & -2 & 0 & 2 \\ -3 & 5 & 4 & -5 \end{bmatrix}$$

a What are the dimensions of M?

b What are m_{12} and m_{32}?

Steps	Working
a Write the number of rows and columns.	The dimensions are 3×4.
b m_{12} means row 1, column 2.	$m_{12} = 4$
m_{32} means row 3, column 2.	$m_{32} = 5$

Operations with matrices

Scalar multiplication, addition and subtraction

We can **multiply** a constant (number or scalar), x, by a matrix by multiplying the constant by every element in the matrix individually: $xM = (xa_{ij})$.

The **negative** of a matrix M is given by $(-1)M = (-a_{ij})$.

We can **add matrices** of the same size by adding the corresponding elements: $A + B = (a_{ij} + b_{ij})$.

We can **subtract** matrices of the same size by subtracting the corresponding elements:

$A - B = (a_{ij} - b_{ij})$.

Matrices can only be added or subtracted if they have the same dimensions.

WORKED EXAMPLE 27 Simple operations with matrices

$A = \begin{bmatrix} 1 & 2 & -3 \\ -4 & 3 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} -3 & 4 & 5 \\ -1 & 2 & 0 \end{bmatrix}$.

Find $3A$, $A + B$, $-B$, $A - B$ and $2A - 3B$.

Steps	Working
1 Multiply each element in A by 3.	$3A = \begin{bmatrix} 3 & 6 & -9 \\ -12 & 9 & 15 \end{bmatrix}$
2 Add the corresponding elements.	$A + B = \begin{bmatrix} -2 & 6 & 2 \\ -5 & 5 & 5 \end{bmatrix}$
3 Multiply the elements of B by -1.	$-B = \begin{bmatrix} 3 & -4 & -5 \\ 1 & -2 & 0 \end{bmatrix}$
4 Subtract the corresponding elements.	$A - B = \begin{bmatrix} 4 & -2 & -8 \\ -3 & 1 & 5 \end{bmatrix}$
5 Subtract $3B$ from $2A$.	$2A - 3B = \begin{bmatrix} 2 & 4 & -6 \\ -8 & 6 & 10 \end{bmatrix} + \begin{bmatrix} 9 & -12 & -15 \\ 3 & -6 & 0 \end{bmatrix}$ $= \begin{bmatrix} 11 & -8 & -21 \\ -5 & 0 & 10 \end{bmatrix}$

WB p. 97

Multiplication of matrices

We multiply matrices A and B by multiplying the elements of each row of A by the elements of each column of B and adding them together.

So we can only multiply 2 matrices if the number of columns of the first matrix and the number of rows of the second matrix are equal. Matrices that we can multiply are called **conformable matrices**. An $m \times n$ matrix A multiplied by an $n \times p$ matrix B gives an $m \times p$ matrix AB. In general, $AB \neq BA$.

WORKED EXAMPLE 28 Products of matrices

WB p. 98

$P = \begin{bmatrix} -3 & 0 & 2 \\ 1 & -2 & 3 \end{bmatrix}$ and $Q = \begin{bmatrix} 0 & -2 \\ -1 & 3 \\ 4 & 5 \end{bmatrix}$.

Find PQ and QP.

Steps	Working
1 Write the product PQ.	$PQ = \begin{bmatrix} -3 & 0 & 2 \\ 1 & -2 & 3 \end{bmatrix} \begin{bmatrix} 0 & -2 \\ -1 & 3 \\ 4 & 5 \end{bmatrix}$
2 Multiply the numbers in each row of P by the numbers in each column of Q and add the answers.	$= \begin{bmatrix} -3 \times 0 + 0 \times -1 + 2 \times 4 & -3 \times -2 + 0 \times 3 + 2 \times 5 \\ 1 \times 0 + -2 \times -1 + 3 \times 4 & 1 \times -2 + -2 \times 3 + 3 \times 5 \end{bmatrix}$ $= \begin{bmatrix} 8 & 16 \\ 14 & 7 \end{bmatrix}$

3 Write the product QP.	$QP = \begin{bmatrix} 0 & -2 \\ -1 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} -3 & 0 & 2 \\ 1 & -2 & 3 \end{bmatrix}$
4 Multiply the numbers in each row of Q by the numbers in each column of P and add the answers.	$= \begin{bmatrix} 0 \times -3 + -2 \times 1 & 0 \times 0 + -2 \times -2 & 0 \times 2 + -2 \times 3 \\ -1 \times -3 + 3 \times 1 & -1 \times 0 + 3 \times -2 & -1 \times 2 + 3 \times 3 \\ 4 \times -3 + 5 \times 1 & 4 \times 0 + 5 \times -2 & 4 \times 2 + 5 \times 3 \end{bmatrix}$ $= \begin{bmatrix} -2 & 4 & -6 \\ 6 & -6 & 7 \\ -7 & -10 & 23 \end{bmatrix}$

Note that in the above example, PQ and QP are not the same and have different dimensions.

So when multiplying matrices, order is important. In the case of 2×3 and 3×1 matrices, we would just get a 2×1 matrix if multiplying the first matrix by the second, but they are not even conformable the other way round, so we couldn't multiply them at all in the reverse order.

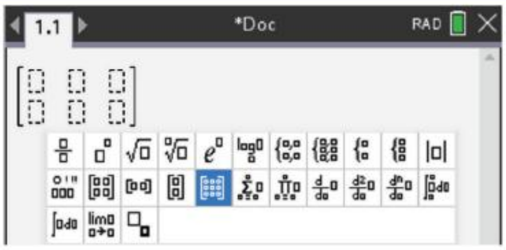
USING CAS 3 Operations with matrices

a Define matrices $A = \begin{bmatrix} 1 & 2 & -3 \\ -4 & 3 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} -3 & 4 & -5 \\ -1 & 2 & 0 \end{bmatrix}$ then find $3A$, $A + B$, $A - B$ and $2A - 3B$.

b Define matrices $P = \begin{bmatrix} -3 & 0 & 2 \\ 1 & -2 & 3 \end{bmatrix}$ and $Q = \begin{bmatrix} 0 & -2 \\ -1 & 3 \\ 4 & 5 \end{bmatrix}$ then determine if $PQ = QP$.

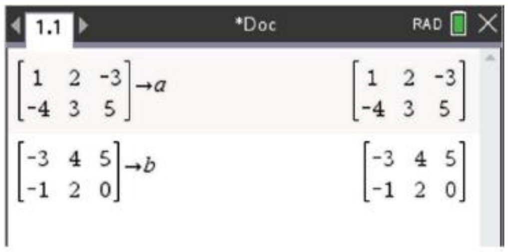
TI-Nspire

a



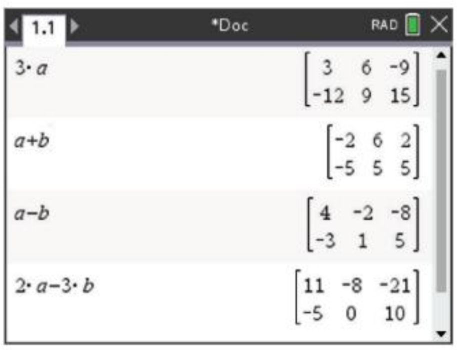
1 Press **template** and select the **3x3** vector template.

2 In the dialogue box, select **2** rows and **3** columns.



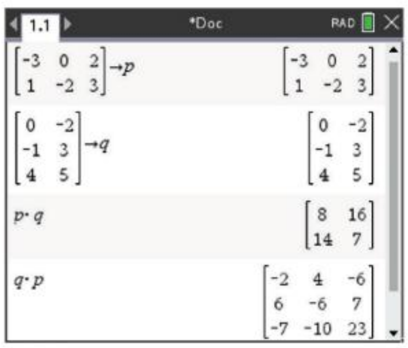
3 Enter the first matrix and press **ctrl + var** to store it into **a**.

4 Repeat to store the second matrix into **b**.



6 Complete the matrix calculations as shown above.

b



1 Create the **2x3** matrix and store it into **p**.

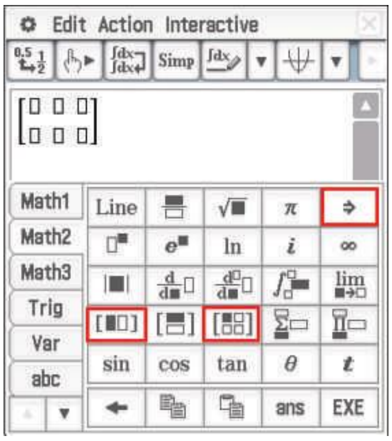
2 Create the **3x2** matrix and store it into **q**.

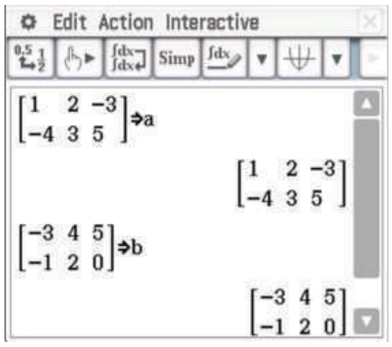
3 Calculate **pxq** and **qxp**.

We can conclude that $PQ \neq QP$.

ClassPad

a





1 Open the **Keyboard > Math2**.

2 Tap the **2×2 matrix** template then the **1×2 matrix** template to create a 2×3 matrix.

Note the location of **store** arrow above, which will be used in the next screen.

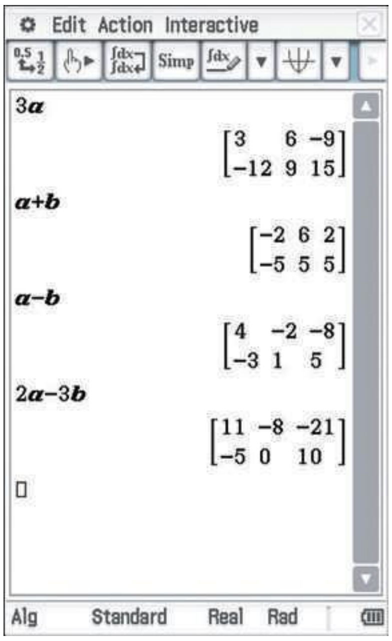
3 Enter the first matrix.

4 Open the **Keyboard > Math1** or **Math2**.

5 Tap **store** followed by ***a***.

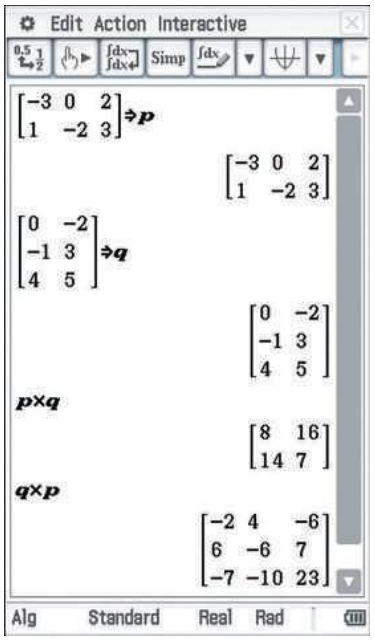
6 Repeat to store the second matrix as **b**.

(Note after using **store** that **a** and **b** remain until the variables are cleared using **Variable Manager**).



7 Complete the matrix calculations as shown above.

b



1 Create the **2×3** matrix and store it into ***p***.

2 Create the **3×2** matrix and store it into ***q***.

3 Calculate ***p*×*q*** and ***q*×*p***.

We can conclude that $PQ \neq QP$.

EXERCISE 4.4 Matrices

ANSWERS p. 514

Recap

1 The inverse function of $f : [7, \infty) \rightarrow R, f(x) = \dfrac{(x-3)^2}{4} + 2$ is

A $f^{-1}: [2, \infty) \rightarrow [3, \infty) \quad f^{-1}(x) = 2\sqrt{x-2} - 3$

B $f^{-1}: [2, \infty) \rightarrow [3, \infty) \quad f^{-1}(x) = 2\sqrt{x+2} + 3$

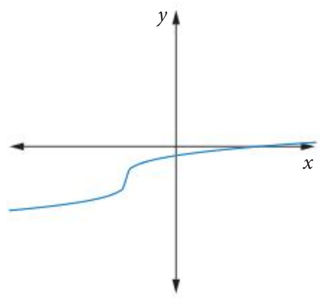
C $f^{-1}: [6, \infty) \rightarrow [7, \infty) \quad f^{-1}(x) = 2\sqrt{x+2} - 3$

D $f^{-1}: [6, \infty) \rightarrow [7, \infty) \quad f^{-1}(x) = 2\sqrt{x-2} + 3$

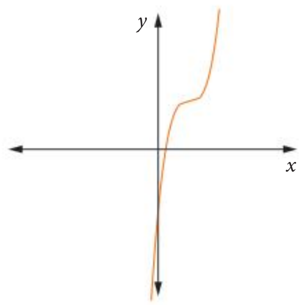
E $f^{-1}: [6, \infty) \rightarrow [7, \infty) \quad f^{-1}(x) = 2\sqrt{x-2} + 3$

2 The graph of $f(x)$ is shown.

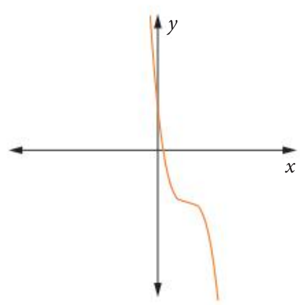
Which of the following best represents the graph of the inverse?



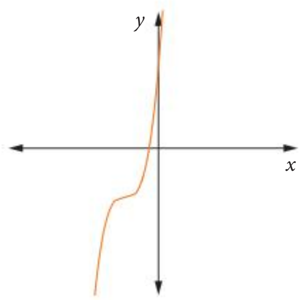
A



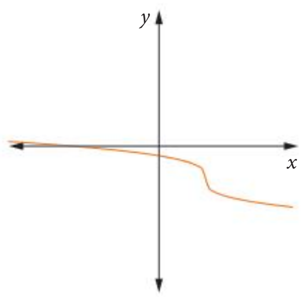
B



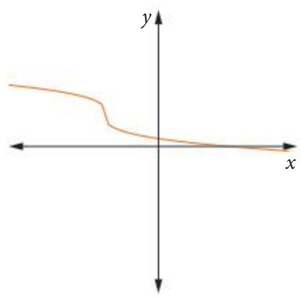
C



D



E



Mastery

3 WORKED EXAMPLE 26 $A = \begin{bmatrix} -2 & 4 & -6 \\ 1 & -3 & -7 \\ -7 & -5 & 9 \end{bmatrix}$, $B = \begin{bmatrix} 0 & -1 \\ 5 & -3 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 2 \\ 6 & -3 \\ -4 & 0 \end{bmatrix}$

a What are the dimensions of A?

b What are a_{13}, a_{23}, a_{31} and a_{32}?

c What are the dimensions of B?

d What are b_{11}, b_{12} and b_{21}?

e What are the dimensions of C?

f What are c_{31}, c_{32}, c_{13} and c_{21}?

4 WORKED EXAMPLE 27 $A = \begin{bmatrix} 2 & 4 \\ -3 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & -2 \\ -2 & 4 \end{bmatrix}$, $C = \begin{bmatrix} -1 & 5 \\ 3 & -4 \end{bmatrix}$, $D = \begin{bmatrix} 1 & 2 \\ 6 & -3 \\ -4 & 0 \end{bmatrix}$ and $E = \begin{bmatrix} 2 & 0 \\ -3 & 1 \\ -1 & 3 \end{bmatrix}$.

Where possible, find each sum or difference.

a $A+B$	**b** $A-C$	**c** $5B$	**d** $3C+2A$	**e** $A+E$
f $D-C$	**g** $2A-3B+4C$	**h** $4D-E$	**i** $A+C$	**j** $C+A$
k $C-A$	**l** $2B-4A$	**m** $D-E$	**n** $3B-A+C$	**o** $2C+4A-3C$

5 WORKED EXAMPLE 28 $A = \begin{bmatrix} -2 & 4 & -6 \\ 1 & -3 & -7 \\ -7 & -5 & 9 \end{bmatrix}$, $B = \begin{bmatrix} 0 & -1 \\ 5 & -3 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 2 \\ 6 & -3 \\ -4 & 0 \end{bmatrix}$, $D = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$ and $E = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$.

Where possible, find each product.

a BD	**b** BC	**c** AB	**d** DE	**e** BE
f BA	**g** CB	**h** AC	**i** DB	**j** EB
k CD	**l** EC	**m** CE	**n** CA	**o** BB

6 Using CAS 3 $A = \begin{bmatrix} 1 & -3 \\ -2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -2 & -5 \\ -1 & 7 \end{bmatrix}$, $C = \begin{bmatrix} 3 & -1 \\ -2 & -3 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 3 \\ 3 & 4 \end{bmatrix}$.

Find

a $A+B$	**b** $A-C$	**c** $3B+2D$	**d** AB	**e** BA
f CD	**g** AD	**h** $5C+BD$	**i** $AC+D$	**j** $CB+2A$

Exam practice

7 TECH-FREE (3 marks) $A = \begin{bmatrix} 2 & 4 \\ 0 & -6 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 5 \\ 4 & -2 \\ -3 & 2 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 0 & 4 \\ -2 & 3 & 5 \end{bmatrix}$.

a Which products are possible? 1 mark

b Find the product(s) with the smallest dimensions. 2 marks

8 $A = \begin{bmatrix} 0 & 3 & -1.3 \\ 6 & -2 & 0.5 \end{bmatrix}$ and $B = \begin{bmatrix} -2.4 & 5 \\ 7 & -3.1 \end{bmatrix}$.

It is possible to calculate

A $A+B$, $A-B$ and AB but not BA.

B AB and BA but not $A+B$ or $A-B$.

C AB but not BA, $A+B$ or $A-B$.

D BA but not AB, $A+B$ or $A-B$.

E BA, $A+B$ or $A-B$ but not AB.

9 $A = \begin{bmatrix} 0 & -4 \\ -2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & -3 \\ 8 & 5 \end{bmatrix}$.

$A-B=$

A $\begin{bmatrix} 1 & -1 \\ -10 & -2 \end{bmatrix}$ **B** $\begin{bmatrix} -1 & 1 \\ 6 & 2 \end{bmatrix}$ **C** $\begin{bmatrix} -1 & -7 \\ 6 & 8 \end{bmatrix}$ **D** $\begin{bmatrix} -32 & -20 \\ 26 & 21 \end{bmatrix}$ **E** $\begin{bmatrix} -1 & 1 \\ 10 & 2 \end{bmatrix}$

10 $A = \begin{bmatrix} 3 & -2 \\ 2 & -3 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 5 \\ -5 & -2 \end{bmatrix}$.

$AB=$

A $\begin{bmatrix} 22 & 19 \\ 23 & 16 \end{bmatrix}$ **B** $\begin{bmatrix} 12 & -10 \\ -10 & 6 \end{bmatrix}$ **C** $\begin{bmatrix} 22 & -23 \\ -19 & 16 \end{bmatrix}$ **D** $\begin{bmatrix} -32 & -20 \\ 26 & 21 \end{bmatrix}$ **E** $\begin{bmatrix} 6 & -5 \\ -10 & -17 \end{bmatrix}$

11 $A = \begin{bmatrix} -2 & 3 \\ 5 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 5 \\ -2 & 4 \end{bmatrix}$.

$AB - BA =$

A $\begin{bmatrix} -6 & -1 \\ 2 & 22 \end{bmatrix}$ **B** $\begin{bmatrix} -25 & 5 \\ -19 & 9 \end{bmatrix}$ **C** $\begin{bmatrix} 23 & -2 \\ 24 & -10 \end{bmatrix}$ **D** $\begin{bmatrix} -22 & 7 \\ -26 & 14 \end{bmatrix}$ **E** $\begin{bmatrix} -31 & 4 \\ -17 & 31 \end{bmatrix}$

Video playlist Matrices and transformations of functions

Worksheet Graphing translations of functions

4.5 Matrices and transformations of functions

OPTIONAL CONTENT

Points in the plane are shown as coordinates like (2, 5). We can use matrices instead, like [2, 5] or $\begin{bmatrix} 2 \\ 5 \end{bmatrix}$.

Matrices can also be used to show transformations in the plane.

Translations

Matrix translations

The point $\begin{bmatrix} x \\ y \end{bmatrix}$ is **translated** a units in the x direction (horizontally) and b units in the y direction (vertically) by adding $\begin{bmatrix} a \\ b \end{bmatrix}$.

$$\begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a + x \\ b + y \end{bmatrix}$$

The **transformation** of all the points in the plane is shown as

$$T: R^2 \rightarrow R^2,\ T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} x \\ y \end{bmatrix}$$

The **image** of a point under a transformation is $\begin{bmatrix} x' \\ y' \end{bmatrix}$, so $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x' \\ y' \end{bmatrix}$.

R^2 means the plane formed by 2 real number axes (at right angles).

We can also use the transformation of a general point to find the result of transforming a function. The transformed function will be given by the relationship between the coordinates of the image point.

WB p. 99

WORKED EXAMPLE 29 Matrix translations

Use matrices to find the image of the point of (3, −5) and the curve $y = 3x^2 + 5$ under the translation 2 to the left and 4 up.

Steps	Working
1 Write the translation as a transformation.	$T: R^2 \rightarrow R^2,\ T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 4 \end{bmatrix} + \begin{bmatrix} x \\ y \end{bmatrix}$
2 Apply the transformation to the point (3, −5).	$\begin{bmatrix} x' \\ y' \end{bmatrix} = T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 4 \end{bmatrix} + \begin{bmatrix} 3 \\ -5 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$
3 Write the image point.	The image of (3, −5) is (1, −1).

4 Apply the transformation to a general point (x, y) of the curve.	$\begin{bmatrix} x' \\ y' \end{bmatrix} = T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 4 \end{bmatrix} + \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x-2 \\ y+4 \end{bmatrix}$
5 Solve the matrix equation for x and y.	$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x-2 \\ y+4 \end{bmatrix}$ so $\begin{matrix} x' = x-2 \text{ and} \\ y' = y+4 \end{matrix}$ so $x = x' + 2$ and $y = y' - 4$.
6 Substitute in the equation of the curve.	$y = 3x^2 + 5$ $y' - 4 = (x' + 2)^2 + 5$
7 Solve for y' to obtain the image of the function.	$y' = (x' + 2)^2 + 9$
8 Write the answer.	The image is $y = (x + 2)^2 + 9$.

4.5

Dilations

Matrix dilations

We can **dilate** the point $\begin{bmatrix} x \\ y \end{bmatrix}$ by the factor a from the y-axis and b from the x-axis by multiplying the point by $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ to give the transformation:

$$T: R^2 \to R^2,\ T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

WORKED EXAMPLE 30 Matrix dilations

WB p. 100

Use matrices to find the image of the curve $y = x^2$ under the dilation by a factor of 3 from the y-axis and 2 from the x-axis.

Steps	Working
1 Write the dilation as a transformation.	$T: R^2 \to R^2,\ T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$
2 Apply the transformation to a general point (x, y) of the curve.	$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3x \\ 2y \end{bmatrix}$
3 Solve the matrix equation for x and y.	$x' = 3x$ and $y' = 2y$ so $x = \frac{x'}{3}$ and $y = \frac{y'}{2}$.
4 Substitute in the equation of the curve.	$\frac{y'}{2} = \left(\frac{x'}{3}\right)^2$
5 Solve for y' to obtain the image of the function.	$y' = \frac{2(x')^2}{9}$
6 Write the answer.	The image is $y = \frac{2x^2}{9}$.

Reflections

Matrix reflections

We can **reflect** the point $\begin{bmatrix} x \\ y \end{bmatrix}$ in the x-axis (vertically) by multiplying the point by $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ to give the transformation:

$$T: R^2 \to R^2,\ T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

We can **reflect** the point $\begin{bmatrix} x \\ y \end{bmatrix}$ in the y-axis (horizontally) by multiplying the point by $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ to give the transformation:

$$T: R^2 \to R^2,\ T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

WB p. 101

WORKED EXAMPLE 31 Matrix reflection

Use matrices to find the curve to which $y = \dfrac{1}{x-3} + 2$ is mapped by reflection in the y-axis.

Steps	Working
1 Write the reflection as a transformation.	$T: R^2 \to R^2,\ T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$
2 Apply the transformation to a general point (x, y) of the curve.	$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix}$
3 Solve the matrix equation for x and y.	$x' = -x$ and $y' = y$ so $x = -x'$.
4 Substitute in the equation of the curve.	$y' = \dfrac{1}{-x'-3} + 2 = 2 - \dfrac{1}{x'+3}$
5 Write the answer.	The curve is mapped to $y = 2 - \dfrac{1}{x+3}$.

Combined transformations

We can combine the matrices for transformations to give a single transformation.

WB p. 102

WORKED EXAMPLE 32 Combined matrix transformations

a What is the transformation for translation 2 in the positive x direction and 4 in the negative y direction, reflection in the x-axis and dilation from the x-axis by a factor of 2?

b Apply the transformation to find what the curve with equation $y = \dfrac{1}{x^2}$ is mapped to under the transformation.

Steps	Working
a 1 The translation is first.	$T: R^2 \to R^2,\ T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2 \\ -4 \end{bmatrix} + \begin{bmatrix} x \\ y \end{bmatrix}$
2 The reflection is second.	$T: R^2 \to R^2,\ T\left(\begin{bmatrix} 2 \\ -4 \end{bmatrix} + \begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\left(\begin{bmatrix} 2 \\ -4 \end{bmatrix} + \begin{bmatrix} x \\ y \end{bmatrix}\right)$

 9780170448260

3 Apply the last transformation. Notice the final order is *right-to-left*.	$T: R^2 \to R^2, T\left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\left(\begin{bmatrix} 2 \\ -4 \end{bmatrix} + \begin{bmatrix} x \\ y \end{bmatrix}\right)\right)$ $= \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\left(\begin{bmatrix} 2 \\ -4 \end{bmatrix} + \begin{bmatrix} x \\ y \end{bmatrix}\right)\right)$
4 Simplify.	$= \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 2+x \\ -4+y \end{bmatrix}\right)$ $= \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2+x \\ 4-y \end{bmatrix}$ $T: R^2 \to R^2, T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2(2+x) \\ 4-y \end{bmatrix}$
b 1 Apply the transformation to a general point (x, y) of the curve.	$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x+2 \\ y-4 \end{bmatrix} = \begin{bmatrix} 2(x+2) \\ 4-y \end{bmatrix}$
2 Express x and y in terms of x' and y'.	$x = 0.5x' - 2$ and $y = 4 - y'$
3 Substitute in the equation of the curve.	$4 - y' = \dfrac{1}{(0.5x' - 2)^2}$
4 Simplify to find y' for the image of the function.	$y' = 4 - \dfrac{1}{(0.5x' - 2)^2} = 4 - \dfrac{4}{(x' - 4)^2}$
5 Write the answer.	The curve is mapped to $y = 4 - \dfrac{4}{(x-4)^2}$.

Notice that a different order of transformations could give a different result. The transformation of dilation from the x-axis by a factor of 2, translation 2 in the positive x direction, reflection in the x-axis and translation 4 in the negative y direction is given by

$T: R^2 \to R^2, = \begin{bmatrix} 0 \\ -4 \end{bmatrix} + \left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\left(\begin{bmatrix} 2 \\ 0 \end{bmatrix} + \left(\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}\right)\right)\right) = \begin{bmatrix} 2x+4 \\ -4-y \end{bmatrix}$ which gives a different function.

The order of operations of transformations follows the usual rules, except that it goes right-to-left when the order is not given by brackets, multiplication etc.

Video
VCE question analysis: Power and inverse functions

VCE QUESTION ANALYSIS

© VCAA 2019 1Q2 2019 Examination 1 Question 2 (8 marks)

a Let $f: R\backslash\left\{\frac{1}{3}\right\} \to R, f(x) = \frac{1}{3x-1}$.

Find the rule of f^{-1}. 2 marks

b State the domain of f^{-1}. 1 mark

c © VCAA MODIFIED Let g be the function obtained by applying the transformation T to the function f, where $T: (x, y) \to (x + c, y + d)$ and $c, d \in R$.

Find the values of c and d given that $g = f^{-1}$. 1 mark

d © VCAA MODIFIED Write the transformation that changes $f(x)$ to its basic function $\frac{1}{x}$ and express in the form $T: (x, y) \to (ax + b, cy + d)$. 2 marks

e © VCAA MODIFIED Write the transformation that changes $g(x)$ to its basic function $\frac{1}{x}$ and express in the form $T: (x, y) \to (ax + b, cy + d)$. 2 marks

Reading the question

- f is a hyperbola. f^{-1} is the inverse function.
- The domain is the x values.
- Transformations are translations, dilations and reflections.
- The expression in part **c** is for a translation.

Thinking about the question

- Swap x and y and solve for the new y to find the inverse.
- The domain of the inverse is the same as the range of the function.
- In part **c**, c is the translation in the x direction and d is in the y direction.
- The transformation in parts **d** and **e** undoes the transformations of $\frac{1}{x}$.

Worked solution (✓ = 1 mark)

a Let $x = \frac{1}{3y-1}$:

$$x(3y-1) = 1$$

$$3y - 1 = \frac{1}{x} \checkmark$$

$$3y = \frac{1}{x} + 1$$

$$y = \frac{1}{3x} + \frac{1}{3}$$

The values of y are unique, so it is a function and $\boldsymbol{f^{-1}(x) = \frac{1}{3x} + \frac{1}{3}}$. ✓

b domain: $R\backslash\{0\}$ ✓

c $\frac{1}{3x-1} \to \frac{1}{3x} + \frac{1}{3} = \frac{1}{3\left(x+\frac{1}{3}\right)-1} + \frac{1}{3} = f\left(x+\frac{1}{3}\right) + \frac{1}{3}$.

So $T: (x, y) \to \left(x - \frac{1}{3}, y + \frac{1}{3}\right)$.

Thus $\boldsymbol{c = -\frac{1}{3}}$ **and** $\boldsymbol{d = \frac{1}{3}}$ ✓

d $\frac{1}{3x-1} \to \frac{1}{x} = 3\left[\frac{1}{3\left(x+\frac{1}{3}\right)-1}\right] = 3f\left(x+\frac{1}{3}\right)$.

The transformation is dilation from the x-axis by the factor 3 and translation left by $\frac{1}{3}$. ✓

The required expression is $\boldsymbol{T}$**: $(x, y) \to \left(x-\frac{1}{3}, 3y\right)$**. ✓

OR $\frac{1}{3x-1} \to \frac{1}{x} = \frac{1}{3\left(\frac{x}{3}+\frac{1}{3}\right)-1} = f\left(\frac{x}{3}+\frac{1}{3}\right) = f\left(\frac{1}{3}(x+1)\right)$.

The transformation is dilation from the y-axis by the factor 3 and translation left by 1. ✓

The required expression is $\boldsymbol{T}$**: $(x, y) \to (3x-1, y)$**. ✓

e $\frac{1}{3x}+\frac{1}{3} \to \frac{1}{x} = 3\left(\frac{1}{3x}+\frac{1}{3}\right)-1 = 3g(x)-1$

The transformation is dilation from the x-axis by the factor 3 and translation down by 1. ✓

The required expression is $\boldsymbol{T}$**: $(x, y) \to (x, 3y-1)$**. ✓

OR $\frac{1}{3x}+\frac{1}{3} \to \frac{1}{x} = \frac{1}{3\left(\frac{x}{3}\right)}+\frac{1}{3}-\frac{1}{3} = g\left(\frac{x}{3}\right)-\frac{1}{3}$.

The transformation is dilation from the y-axis by the factor 3 and translation down by $\frac{1}{3}$. ✓

The required expression is $\boldsymbol{T}$**: $(x, y) \to \left(3x, y-\frac{1}{3}\right)$**. ✓

Student performance

80–100% 60–79% 0–59%

a 76% Well-attempted. Some students had difficulty because of poor notation or made errors with transposing the equation.

b 64% Most students knew domain of f^{-1} = range of f.

c 24% Well-attempted but poorly done. Some students had incorrect signs for c and d while others tried to use dilations instead of translations.

EXERCISE 4.5 Matrices and transformations of functions

ANSWERS p. 514

Recap

1 $A = \begin{bmatrix} 3 & -5 \\ 2 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 6 \\ -3 & 2 \end{bmatrix}$.

$3A - 2B =$

A $\begin{bmatrix} 6 & -23 \\ 11 & -2 \end{bmatrix}$ **B** $\begin{bmatrix} 12 & -33 \\ 15 & 6 \end{bmatrix}$ **C** $\begin{bmatrix} 11 & -27 \\ 12 & 8 \end{bmatrix}$ **D** $\begin{bmatrix} 9 & -28 \\ 11 & 2 \end{bmatrix}$ **E** $\begin{bmatrix} 5 & -17 \\ 8 & 0 \end{bmatrix}$

2 $A = \begin{bmatrix} 1 & 6 \\ 0 & -4 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -3 \\ 4 & -1 \end{bmatrix}$.

$AB =$

A $\begin{bmatrix} 2 & -18 \\ 0 & 4 \end{bmatrix}$ **B** $\begin{bmatrix} -19 & 18 \\ 12 & -8 \end{bmatrix}$ **C** $\begin{bmatrix} 26 & -9 \\ -16 & 4 \end{bmatrix}$ **D** $\begin{bmatrix} 8 & 3 \\ 6 & -14 \end{bmatrix}$ **E** $\begin{bmatrix} 12 & 8 \\ -14 & 20 \end{bmatrix}$

Mastery

3 WORKED EXAMPLE 29

a Express the translation 2 to the right and 3 up as a transformation and apply it to the function $f(x) = x^3$.

b Express the translation 1 in the negative direction of the x-axis and 2 in the negative direction of the y-axis as a transformation and find the image of the function $y = \frac{1}{x}$ under the transformation.

c Express the translation 3 to the right and 4 down as a transformation and find the curve $y = \sqrt{x}$ is mapped to under the transformation.

d Write the translation 3 in the negative x direction and 4 in the positive y direction as a transformation and apply it to the function $f(x) = (x+2)^4 - 1$.

4 WORKED EXAMPLE 30

a Use matrices to find the image of the curve $f(x) = x^4$ under the dilation by a factor of $\frac{1}{2}$ from the y-axis and 3 from the x-axis.

b Express the dilation by a factor of 3 parallel to the y-axis as a transformation and what the function $y = \sqrt[3]{x}$ is mapped to under the transformation.

c Express the dilation by a factor of $\frac{1}{4}$ from the y-axis as a transformation and find the image of the line $y = 4x - 3$ under the transformation.

d Write the dilation by the factor $\frac{1}{2}$ from the x-axis and 3 from the y-axis as a transformation and apply it to the function $f(x) = x^{-2}$.

5 WORKED EXAMPLE 31

a Use matrices to find the image of the function $f(x) = 2 - \frac{1}{(x-3)^2}$ reflected in the x-axis.

b Express reflection in the y-axis as a transformation and find the image of the curve $y = x^3$ under the transformation.

c Express reflections in the x- and y-axes as a single transformation and find the curve that $y = (x+2)^2 - 1$ maps to under the transformation.

d Use matrices to find the image of the curve $f(x) = \sqrt[3]{x+1} - 2$ reflected in the y-axis.

6 WORKED EXAMPLE 32

a Express translation 2 to the left and 1 down, reflection in the y-axis and dilation parallel to the y-axis by a factor of 2 as a single transformation and find the image of the curve with equation $y = \sqrt{x}$ under the transformation.

b Express reflection in the x-axis, dilation from the y-axis by a factor of $\frac{1}{2}$ and translation 3 in the positive x direction and 2 in the positive y direction as a single transformation and find the curve that $y = x^3$ is mapped to under the transformation.

c Use matrices to find the transformation of the curve that $y = \frac{1}{x}$ under translation 2 to the right and 4 down, reflection in both axes and dilation vertically by a factor of $\frac{1}{2}$ and horizontally by a factor of 2.

d Write translation 2 units in the positive direction of the x-axis, reflection in the y-axis, dilation by a factor of 2 from the x-axis and 3 from the y-axis and translation 4 units in the negative direction of the x-axis as a transformation and apply it to the function $f(x) = 4(x+5)^2 + 3$.

7 State the combination of translations, dilations and reflections given these transformations.

a $T: R^2 \to R^2, T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -2 \\ 3 \end{bmatrix}$

b $T: R^2 \to R^2, T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}\left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)$

c $T: R^2 \to R^2, T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -3 \\ -2 \end{bmatrix}$

d $T: R^2 \to R^2, T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \left(\begin{bmatrix} 2 & 0 \\ 0 & -0.5 \end{bmatrix}\left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \end{bmatrix}\right)\right) + \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

Exam practice

80–100% 60–79% 0–59%

8 TECH-FREE (2 marks) The function $y = 6x^3$ is dilated by a factor of 2 from the x-axis, reflected in the y-axis and translated 3 down and 2 to the right by the transformation T.

Write the matrix expression for T and find the transformed function.

9 Consider the transformation T defined as

$$T: R^2 \to R^2, T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 5 \end{bmatrix}$$

The transformation maps the graph of $y = \sqrt[3]{6x+1}$ to

A $y = 3\sqrt[3]{6x+1} + 5$

B $y = 5 - 3\sqrt[3]{6x+1}$

C $y = 3\sqrt[3]{1-6x} + 5$

D $y = 5 - \sqrt[3]{2x+1}$

E $y = \sqrt[3]{6-2x}$

10 © VCAA 2019 2AQ9 57% The point (a, b) is transformed by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -\frac{1}{2} \\ -2 \end{bmatrix}$$

If the image of (a, b) is $(0, 0)$, then (a, b) is

A $(1, 1)$ **B** $(-1, 1)$ **C** $(-1, 0)$ **D** $(0, 1)$ **E** $(1, -1)$

11 © VCAA 2021N 2AQ12 MODIFIED The transformation $T: R^2 \to R^2$ maps the graph of $y = 4x^4 + 6x$ onto the graph of $y = 2(x-1)^4 + 3(x-1) + 2$.

The transformation T could be given by

A $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

B $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

C $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

D $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 4 \end{bmatrix}$

E $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

12 © VCAA 2012 2BQ2e 19% (2 marks) Let $f: R\backslash\{2\} \to R, f(x) = \dfrac{1}{2x-4} + 3$.

A transformation $T: R^2 \to R^2$ that maps the graph of f to the graph of the function

$g: R\backslash\{0\} \to R, g(x) = \dfrac{1}{x}$ has rule $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix}$, where a, c and d are non-zero real numbers.

Find the values of a, c and d.

4 Chapter summary

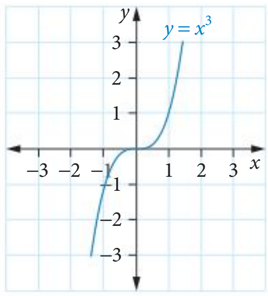
Power functions

Power functions are of the form $y = x^n$.

The basic cubic function is $y = x^3$.

There is a point of inflection at the central point, (0, 0).

A **family** of functions is a set of functions, each of which differs from the others by the value of a particular **parameter**.



The family $y = ax^3$

- $y = ax^3$ is inverted (reflected in the x-axis) for $a < 0$.
- For $a < -1$ or $a > 1$ it is stretched away from the x-axis, steeper than $y = x^3$.
- For $-1 < a < 1$ it is shrunk towards the x-axis, less steep than $y = x^3$.

The family $y = (x + b)^3$

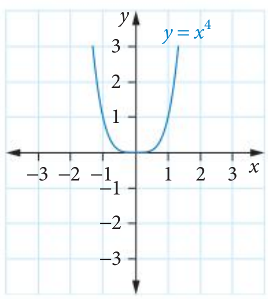
- $y = (x + b)^3$ is moved parallel to the x-axis.
- For $b > 0$ it is moved to the left of $y = x^3$.
- For $b < 0$ it is moved to the right of $y = x^3$.

The family $y = x^3 + c$

- $y = x^3 + c$ is moved parallel to the y-axis.
- For $c > 0$, it is moved upwards, higher than $y = x^3$.
- For $c < 0$, it is moved downwards, lower than $y = x^3$.

The basic **quartic function** is $y = x^4$. It is always positive or 0.

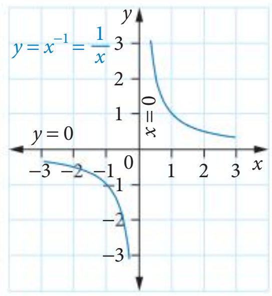
The **turning point** of $y = x^4$ is a **minimum**, while that of $y = -x^4$ is a **maximum**. The families of graphs, $y = ax^4$, $y = (x + b)^4$ and $y = x^4 + c$ are related to $y = x^4$ similarly to the families of $y = x^3$.



The simple **hyperbola** is the graph of the function $f(x) = x^{-1} = \frac{1}{x}$.

There is a **discontinuity**, a break in the graph, at $x = 0$ that separates the graph into 2 **branches**.

The x-axis is the **horizontal asymptote** $y = 0$: as $x \to \pm\infty$, $y \to 0$.



The y-axis is the **vertical asymptote** $x = 0$: as $x \to 0$, $y \to \pm\infty$.

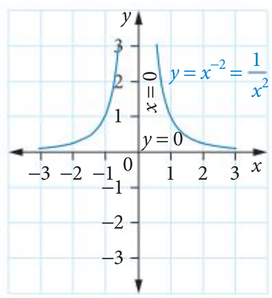
Each of the points $(-1, -1)$ and $(1, 1)$ is called a **vertex**.

The families of hyperbolas given by $f(x) = ax^{-1} = \frac{a}{x}$, $f(x) = (x + b)^{-1} = \frac{1}{x + b}$ and $f(x) = x^{-1} + c = \frac{1}{x} + c$ have the same relationship to the basic hyperbola as the families of other functions.

The basic **truncus** is the graph of the function $f(x) = x^{-2} = \frac{1}{x^2}$.

It is similar to the hyperbola, but both branches are on the positive side of the x-axis.

Its asymptotes are also the axes and its vertexes are also $(-1, -1)$ and $(1, 1)$.

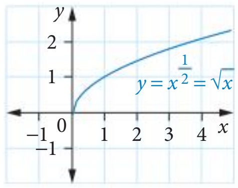


The basic **square root function**, $f(x) = x^{\frac{1}{2}} = \sqrt{x}$ is the inverse of $y = x^2$, whose graph is the graph of $y = x^2$ reflected in the line $y = x$, but only the top half.

It has the domain and range $[0, \infty]$ since the square root of a negative number is not real and $\sqrt{x}$ is positive or 0.

The shape is a half-parabola turned on its side.

It starts at (0, 0) and includes (1, 1).

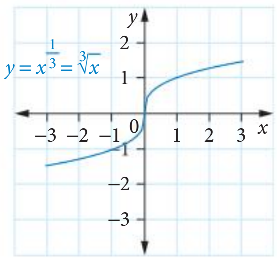


The basic **cube root function**, $f(x) = x^{\frac{1}{3}} = \sqrt[3]{x}$ is the inverse of $y = x^3$.

It has domain and range R.

The shape is that of $y = x^3$ reflected in the line $y = x$, or turned on its side.

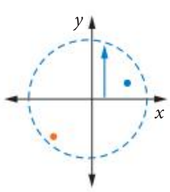
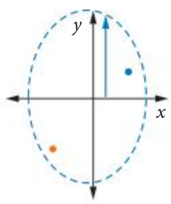
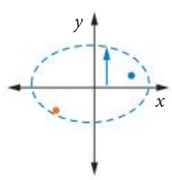
It passes through (−1, −1) and (1, 1).



Transformations of power functions

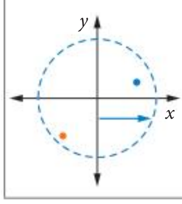
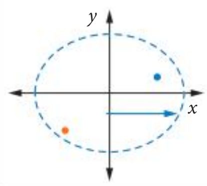
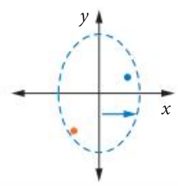
Dilations from the *x*- and *y*-axes

A **dilation from the *x*-axis** (or **parallel to the *y*-axis**) changes the appearance by expansion or compression ('vertically') from the *x*-axis. Every *y* coordinate is multiplied by the same factor *a*: $f(x) \to af(x)$.

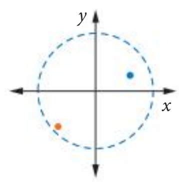
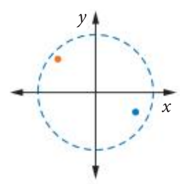
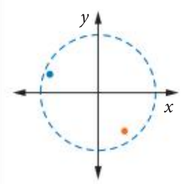
Original	Dilations from the *x*-axis	
	$a > 1$ or $a < -1$	$-1 < a < 1$
		

The family of the function $f(x)$ given by $af(x)$ is dilated from the *x*-axis by the factor *a*. The dilation is a stretch if $a > 1$ or $a < -1$ and a shrink if $-1 < a < 1$.

A **dilation from the *y*-axis** (or **parallel to the *x*-axis**) changes the appearance by expansion or compression ('horizontally') from the *y*-axis.

Original	Dilations from the *y*-axis	
	$n > 1$ or $n < -1$	$-1 < n < 1$
		

The family of the function $f(x)$ given by $f(nx)$ is dilated from the *y*-axis by the factor $\frac{1}{n}$. The dilation is a stretch if $-1 < n < 1$ and a shrink if $n > 1$ or $n < -1$.

Original	Reflection in the *x*-axis	Reflection in the *y*-axis
$f(x)$	$-f(x)$	$f(-x)$
		

Reflections in the *x*- and *y*-axes

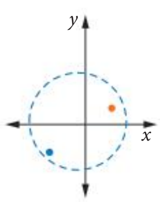
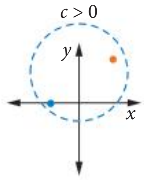
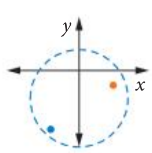
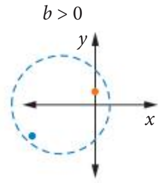
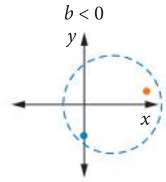
A **reflection in the *x*-axis** (or **parallel to the *y*-axis**) changes the appearance by inverting the shape (flipping it upside-down). The *y* coordinate of every point is changed to its negative: $f(x) \to -f(x)$.

A **reflection in the y-axis** (or **parallel to the *x*-axis**) changes the appearance by swapping the left and right sides (flipping it back-to-front). The *x* coordinate of every point is changed to its negative: $f(x) \to f(-x)$.

Translations

A **translation in the direction of the *y*-axis** (or **from the *x*-axis**) changes the appearance by moving the shape in the *y* direction. Every point is changed by adding the same amount to the *y* coordinate: $f(x) \to f(x) + c$.

A **translation in the direction of the *x*-axis** (or **from the *y*-axis**) changes the appearance by moving the shape in the *x* direction. Every point is changed by adding the same amount to the *x* coordinate: $f(x) \to f(x + b)$.

Original	Translations in the direction of the *y*-axis: $f(x) + c$		Translations in the direction of the *x*-axis: $f(x + b)$	
	$c > 0$	$c < 0$	$b > 0$	$b < 0$
				

Inverse functions

The **inverse of a relation** with ordered pairs (x, y) is the relation formed by reversing all the ordered pairs to (y, x).

If the inverse relation of a function $f(x)$ is also a function, then it is called an **inverse function** and is written as $f^{-1}(x)$.

An inverse function links the values of y in $y = f(x)$ back to x: $f^{-1}(f(x)) = x$ and $f(f^{-1}(x)) = x$.

For an inverse function to exist it must be a **one-to-one function**: *both* the x and y values of the function must be unique.

The domain of the inverse relation is the range of the original function.

The range of the inverse relation is the domain of the original function.

The graph of the inverse is the reflection of $y = f(x)$ in the line $y = x$.

Matrices (optional)

A **matrix** is a rectangular array of numbers enclosed by brackets, symbolised by a capital letter, like $A = \begin{bmatrix} 2 & -4 & 60 & 1 \\ 5 & 2.8 & -70 & 0 \end{bmatrix}$.

The **dimensions** of the matrix are the number of rows m and columns n, shown as $m \times n$. The matrix A has dimensions 2×4.

A **square matrix** with n rows and n columns has **dimension** n.

The values in the **matrix elements** and are labelled as a_{ij} for the element in row i and column j. In the matrix above, $a_{13} = 60$.

We show a general matrix as $A = (a_{ij})$.

We can **multiply** a constant (number or scalar), x, by a matrix by multiplying the constant by every element in the matrix individually: $xM = (xa_{ij})$.

The **negative** of a matrix M is given by $(-1)M = (-a_{ij})$.

We can **add** matrices of the same size by adding the corresponding elements: $A + B = (aij + bij)$.

We can **subtract** matrices of the same size by subtracting the corresponding elements: $A - B = (aij - bij)$.

Matrices can only be added or subtracted if they have the same dimensions.

To **multiply matrices**, multiply the elements of each row of and $m \times n$ matrix A by the elements of each column of an $n \times p$ matrix B and adding them together to give the $m \times p$ matrix AB.

In general, $AB \neq BA$.

Only **conformable matrices** can be multiplied: the number of rows of the first matrix and the number of rows of the second matrix must be the same.

Matrices and transformations of functions (optional)

The point $\begin{bmatrix} x \\ y \end{bmatrix}$ is **translated** a units in the x direction (horizontally) and b units in the y direction (vertically) by adding $\begin{bmatrix} a \\ b \end{bmatrix}$: $\begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a + x \\ b + y \end{bmatrix}$.

The is shown by the transformation T as $T: R^2 \to R^2$, $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} x \\ y \end{bmatrix}$ and $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x' \\ y' \end{bmatrix}$, the **image** point of $\begin{bmatrix} x \\ y \end{bmatrix}$.

The point $\begin{bmatrix} x \\ y \end{bmatrix}$ is **dilated** by the factor a from the y-axis and b from the x-axis by multiplying the point by $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$:

$T: R^2 \to R^2$, $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

The point $\begin{bmatrix} x \\ y \end{bmatrix}$ is **reflected** in the x-axis (vertically) by multiplying the point by $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$:

$T: R^2 \to R^2$, $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

The point $\begin{bmatrix} x \\ y \end{bmatrix}$ is **reflected** in the y-axis (horizontally) by multiplying the point by $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$:

$T: R^2 \to R^2$, $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

9780170448260

Cumulative examination 1

Total number of marks: 10 Reading time: 4 minutes Writing time: 15 minutes

TECH-FREE *Technology is NOT permitted.*

1 © VCAA 2020 2AQ1 MODIFIED (2 marks) Let $p(x) = 5x^3 - 2ax^2 + 7x - 2$, where $a \in R$.
When p is divided by $x + 2$, the remainder is -16. Find the value of a.

2 © VCAA 2020 1Q4 (3 marks) Solve the equation $2\log_2(x + 5) - \log_2(x + 9) = 1$.

3 (3 marks) Sketch the graph of $f: R \to R, f(x) = 2 - \dfrac{1}{x + 3}$, showing all intercepts and the equations of any asymptotes.

4 (2 marks) $f(x) = (x + 5)^3 - 4$ is dilated by a factor of 4 from the x-axis, reflected in the y-axis and translated 3 in the negative direction of the x-axis to give $g(x)$. What is the equation of $g(x)$?

Cumulative examination 2

Total number of marks: 20 Reading time: 4 minutes Writing time: 30 minutes

Approved technology is permitted.

Section A 5 multiple-choice questions

5 marks

1 Which of the following is a factor of $p(x) = 6x^3 + 23x^2 + 9x - 18$?

A $x - 1$ **B** $x - 2$ **C** $x - 3$ **D** $x + 2$ **E** $x + 3$

2 © VCAA 2019N 2AQ3 MODIFIED If $x - a$ is a factor of $8x^3 - 14x^2 - a^2x$, where $a \in R\backslash\{0\}$, then the value of a is

A 6 **B** 3 **C** $\sqrt{6}$ **D** 2 **E** -2

3 Which one of the following is the inverse function of $g: R \to R, g(x) = \sqrt[3]{3x+2}$?

A $g^{-1}: R \to R, g^{-1}(x) = \dfrac{x^3+2}{3}$

B $g^{-1}: R \to R, g^{-1}(x) = \dfrac{x^3-2}{3}$

C $g^{-1}: R \to R, g^{-1}(x) = \dfrac{x^3}{3} + 2$

D $g^{-1}: R \to R, g^{-1}(x) = \dfrac{x^3}{3} - 2$

E $g^{-1}: R \to R, g^{-1}(x) = 3x^3 - 2$

4 © VCAA 2019 2AQ13 The graph of the function f passes through the point $(-2, 7)$.

If $h(x) = f\left(\dfrac{x}{2}\right) + 5$, then the graph of the function h must pass through the point

A $(-1, -12)$ **B** $(-1, 19)$ **C** $(-4, 12)$ **D** $(-4, -14)$ **E** $(3, 3.5)$

5 © VCAA 2021N 2AQ12 MODIFIED The transformation $T: R^2 \to R^2$ maps the graph of $y = x^3 - x$ onto the graph of $y = 2(x-1)^3 - 2(x-1) + 4$.

The transformation T could be given by

A $T: (x, y) \to (x + 1, 2y + 4)$

B $T: (x, y) \to \left(x + 1, \dfrac{1}{2}y + 4\right)$

C $T: (x, y) \to (2x + 1, y + 2)$

D $T: (x, y) \to \left(\dfrac{1}{2}x + 1, y + 2\right)$

E $T: (x, y) \to (2x + 1, y + 2)$

Section B 2 questions

15 marks

1 (8 marks) A cubic polynomial is of the form $p(x) = 2x^3 + ax^2 + bx + c$.

a The polynomial has zeros at $x = 2$ and $x = -3$. Express b and c in terms of a. 4 marks

b When $p(x)$ is divided by $x - 1$, the remainder is -12. What is the remainder when the polynomial is divided by $x + 2$? 3 marks

c Write the polynomial $p(x)$ in full. 1 mark

2 (7 marks) When $f(x)$ is translated 4 to the left and 2 down, reflected in the x-axis and dilated from the x-axis by $\frac{1}{2}$, the new function is given by $g(x) = 1 - 3\sqrt[3]{x+8}$.

a What is the rule for the function f? 3 marks

b Write the transformation applied to f. 2 marks

c Write a transformation that changes $g(x)$ to $\sqrt[3]{1-x} + 3$. 2 marks

CHAPTER

5 PROBABILITY AND COUNTING

Study Design coverage

UNIT 1, AREA OF STUDY 4: DATA ANALYSIS, PROBABILITY AND STATISTICS

- random experiments, sample spaces, outcomes, elementary and compound events, random variables and the distribution of results of experiments
- simulation using simple random generators such as coins, dice, spinners and pseudo-random generators using technology, and the display and interpretation of results, including informal consideration of proportions in samples
- addition and multiplication principles for counting
- combinations including the concept of a selection and computation of ${}^nC_r$ and the application of counting techniques to probability.

STUDY DESIGN COVERAGE

Video playlists (8):
5.1 The language of probability
5.2 Random variables
5.3 Probability simulations
5.4 The addition and multiplication principles
5.5 Permutations
5.6 Combinations
5.7 Applying counting methods to probability
VCE question analysis Probability and counting

Worksheets (10):
5.1 Theoretical probability • Experimental probability • Dartboard probabilities
5.5 Counting techniques • Ordered and unordered selections • Permutation calculations • Permutations • Factorial notation
5.6 Combinations • Combination calculations

Puzzle (1):
5.1 Matching probabilities

Nelson MindTap

To access resources above, visit **cengage.com.au/nelsonmindtap**

5.1 The language of probability

Video playlist The language of probability

Worksheets Theoretical probability

Experimental probability

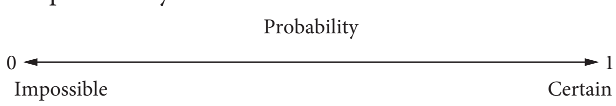
Dartboard probabilities

Puzzle Matching probabilities

Probability is the mathematics of chance. Every day, we are involved with events that may or may not occur with different degrees of certainty.

- Will it be fine today?
- Will I be set homework?
- Will my sports team win?
- Will my teacher be in a cheerful mood?

Probability tells us the likelihood of an event occurring. We can calculate the probability of an event occurring by assigning a number between 0 and 1 to describe the likelihood of the event. An impossible event will have a probability of 0 and a certain event will have a probability of 1. All other events between these two extremes will have a probability between 0 and 1.



Theoretical and experimental probability

The **experimental** or **empirical probability** of an event occurring can be determined by gathering observations from an experiment, with a large number of trials, and counting the number of times our particular event has occurred.

Experimental probability

$$\text{experimental probability of event } (E) \text{ occuring } = \Pr(E) = \frac{\text{number of times event } E \text{ occurs}}{\text{total number of events}}$$

The experimental probability is an estimate of the likelihood of the event occurring.

With **theoretical probability**, we do not have to conduct an experiment. It is calculated based on our knowledge of the situation.

Theoretical probability

$$\text{theoretical probability of event } (E) \text{ occuring} = \Pr(E) = \frac{\text{number of favourable outcomes for event } E}{\text{number of possible outcomes}}$$

An example would be to flip a coin 50 times and record whether it lands on a head or a tail. If a tail is recorded 20 times, the experimental probability would be $\Pr(\text{tail}) = \frac{20}{50} = 0.4$.

In this experiment, there are only 2 possible outcomes {head, tail}, so the theoretical probability is $\Pr(\text{tail}) = \frac{1}{2}$.

Sample space

In probability theory, the **sample space** is a list of all possible outcomes in a probability experiment. It is usually denoted by the set S and is written in set notation using curly brackets { }. For example, when a coin is tossed, the sample space $S = \{H,T\}$. Each outcome is called an **element** of the sample space.

WB

p. 103

WORKED EXAMPLE 1 Finding the experimental and theoretical probabilities

A die is rolled 40 times and the following results are obtained.

6, 6, 1, 4, 3, 5, 1, 3, 6, 2, 5, 6, 2, 3, 1, 6, 1, 1, 4, 6,

6, 2, 2, 1, 1, 5, 1, 3, 2, 6, 1, 6, 4, 2, 6, 5, 2, 2, 6, 6

a Use the results obtained to find the experimental probability of rolling a 6.

b Find the theoretical probability of rolling a 6.

Steps	Working
a **1** Count the number of sixes obtained in the 40 rolls of the die.	There are 12 sixes rolled.
2 Find the probability by substituting into the experimental probability formula.	$\Pr(6) = \frac{\text{number of times a 6 is rolled}}{\text{total number of rolls}}$ $\Pr(6) = \frac{12}{40} = \frac{3}{10}$
b **1** List the sample space when a die is rolled.	$S = \{1, 2, 3, 4, 5, 6\}$
2 Find the probability by substituting into the theoretical probability formula.	$\Pr(6) = \frac{\text{number of outcomes where a 6 occurs}}{\text{number of possible outcomes}}$ $\Pr(6) = \frac{1}{6}$

EXERCISE 5.1 The language of probability ANSWERS p. 515

Mastery

1 WORKED EXAMPLE 1 A die is rolled 30 times and the following results are obtained:

4, 4, 3, 3, 5, 3, 5, 6, 2, 5, 1, 1, 6, 4, 3, 3, 6, 1, 5, 1, 5, 5, 3, 1, 1, 3, 1, 3, 2, 3

a Find the experimental probability of rolling an odd number.

b Find the theoretical probability of rolling an odd number.

2 TECH-FREE A coin is tossed 20 times and the following results are observed.

H, H, H, T, T, H, T, H, T, H, T, T, T, H, H, T, H, T, T, T

a Find the experimental probability of tossing

i a head ii a tail

b Find the theoretical probability of tossing

i a head ii a tail

3 TECH-FREE A 12-sided die, in the shape of a regular dodecahedron, has its faces labelled with the digits 1, 1, 2, 3, 3, 3, 3, 4, 4, 4, 4 and 5. The die is rolled 40 times and the number uppermost is recorded each time.

The results of 40 rolls of the die are

5, 4, 2, 3, 3, 4, 1, 3, 5, 2, 4, 5, 2, 3, 1, 5, 1, 1, 3, 4,

3, 3, 3, 2, 1, 4, 1, 3, 3, 5, 1, 4, 4, 2, 5, 4, 3, 2, 5, 5

a Find the experimental probability of rolling a 3.

b Find the theoretical probability of rolling a 3.

Exam practice

4 TECH-FREE (4 marks) A spinner is divided into 5 sections numbered 1 to 5. The number of times the pointer lands on each section is shown in the table.

Section	1	2	3	4	5
Number	18	22	40	12	8

Find the experimental probability that the pointer lands on

a section 1 — 1 mark

b section 2 — 1 mark

c an even-numbered section — 1 mark

d a section with a number that is a factor of 6. — 1 mark

5 A toy shop packages bags of marbles that contain different numbers of red, green, purple, orange and yellow marbles. A particular bag contains 10 red, 6 green, 5 purple, 9 orange and 5 yellow marbles. The experimental probability of randomly selecting from the bag a marble that is not purple is

A $\frac{1}{7}$ B $\frac{1}{5}$ C $\frac{4}{5}$ D $\frac{5}{6}$ E $\frac{6}{7}$

6 A gardener plants 30 African violet seeds and 9 of them sprout. If 40 African violet seeds are planted, the number of seeds that he would expect to sprout is

A 3 B 9 C 12 D 15 E 18

7 (5 marks) The winner of the most popular photograph in a competition is determined using an online voting app. At the close of voting, contestants 1 to 8 receive the number of votes shown below.

Contestant	1	2	3	4	5	6	7	8
Votes	2000	3200	4600	9000	1000	7200	5000	3000
Experimental probability								

a Copy and complete the table. — 4 marks

b Find the sum of the probabilities. — 1 mark

8 (5 marks) The table below shows the socioeconomic status (SES) of domestic enrolees in Victorian higher education institutions in 2007.

	Number of enrolments			
	Low SES	Middle SES	High SES	Total
University of Ballarat	1013	3048	304	4365
Swinburne University	1214	4729	5020	10 963
Victoria University	3535	7218	3317	14 070
La Trobe University	4156	10 033	5988	20 177
Deakin University	3118	11 820	8317	23 255
RMIT	3266	8980	9606	21 852
Monash University	4609	14 136	15 913	34 658
The University of Melbourne	2546	9495	18 330	30 371
Australian Catholic University*	569	1926	1447	3942
Other providers	445	1502	1378	3325
Total	**24 471**	**72 887**	**69 620**	**166 978**

Source: DEEWR Higher Education Statistics Collection, 2007 and ABS SEIFA 2006

Using the data shown, find the empirical probability, correct to 3 decimal places, that a randomly selected student

a from Victoria University has a low socioeconomic status — 1 mark

b from the University of Melbourne has a low socioeconomic status — 1 mark

c has a low socioeconomic status — 1 mark

d attends Deakin University — 1 mark

e attends Deakin University and has a high socioeconomic status. — 1 mark

9 (5 marks) Australia's population by the top 10 countries of birth in 2020 is shown in the table.

Country of birth	Population in thousands
England	980
India	721
China	651
New Zealand	565
Philippines	310
Vietnam	270
South Africa	200
Italy	178
Malaysia	177
Sri Lanka	147
All overseas-born	**7654**
Australian-born	**18 043**

Source: ABS (Australian Bureau of Statistics) ABS website 2019–2020

Find the empirical probability, correct to 4 decimal places, that a randomly selected Australian resident

a has England as their country of birth — 1 mark

b has Vietnam as their country of birth — 1 mark

c that was born overseas has China as their country of birth — 1 mark

d that was born overseas has New Zealand as their country of birth — 1 mark

e was Australian-born. — 1 mark

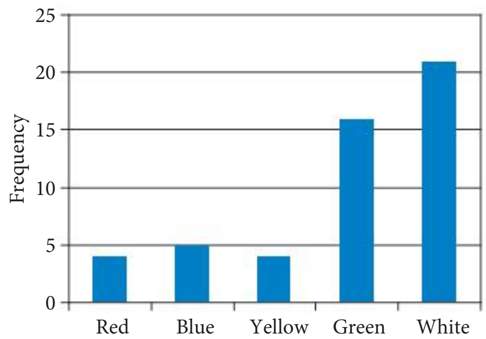
10 (6 marks) Over a 2-minute period at a busy intersection, Brandon observed 30 sedans, 12 station wagons, 14 semi-trailer trucks and 4 motorcycles. Find the empirical probabilities of observing a

a station wagon **b** sedan **c** motorcycle 3 marks

If 180 vehicles of the same types were observed over a 10-minute period, what number would we expect to

d be semi-trailer trucks **e** be sedans **f** have 4 wheels 3 marks

11 (5 marks) A spinner is divided into 12 equal sections. The sections are all coloured in one of the colours red, blue, yellow, green or white. The graph below shows the frequency of the different colours in 50 spins. Use the graph below to determine how many sectors are likely to be coloured red, blue, yellow, green or white.



5.2 Random variables

Video playlist Random variables

A **random variable** is a variable whose possible values are the numerical result of some random process. If we roll a 6-sided die 10 times, the number of 4s that occur will be a random variable. In this situation, the random variable X represents the number of 4s that occur in 10 rolls of a die.

An uppercase letter is used to describe the variable and a lower-case letter is used to represent the values the variable can take.

Simple event probability

A **simple event** is one that has a single outcome, like getting a tail on the flip of one coin or rolling a number greater than 3 on the roll of one die.

WB p. 104

WORKED EXAMPLE 2 Finding simple event probabilities

A number is randomly selected from the set {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15}.

Let the random variable X = the number selected from the set.

Find

a $\Pr(X = 12)$ **b** $\Pr(X < 6)$

Steps	Working
a Use the theoretical probability formula. $\Pr(E) = \dfrac{\text{number of favourable outcomes for event } E}{\text{number of possible outcomes}}$	$\Pr(X = 12) = \dfrac{1}{15}$
b Count the number of digits in the sample space that are less than 6.	$\Pr(X < 6) = \dfrac{5}{15} = \dfrac{1}{3}$

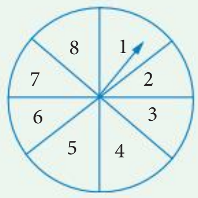
Compound events

A **compound event** is the combination of 2 or more simple events. Examples of compound events would be tossing a coin 3 times or rolling 2 dice. The sample space for compound events can sometimes be found using a **lattice diagram**.

WB
p. 105

WORKED EXAMPLE 3 Finding compound event probabilities

A probability spinner is divided into 8 equal sections numbered 1 to 8 as shown. The pointer is spun twice. The random variable X = the sum of the 2 digits from both spins.



a Illustrate the sample space using a lattice diagram.

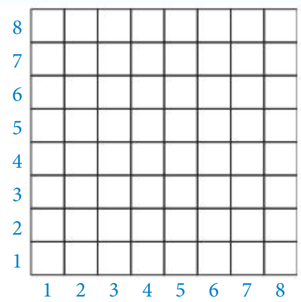
b Find

i $\Pr(X = 8)$

ii $\Pr(X \geq 8)$

Steps | **Working**

a 1 Draw an 8×8 grid. Label the columns 1 to 8 for the first spin and the rows 1 to 8 for the second spin.



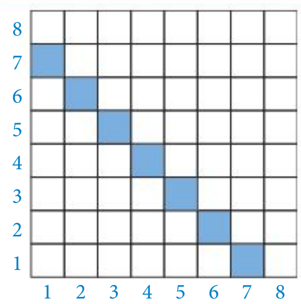
b i Colour every cell where the first and second spins add to 8.

There are 7 combinations that add to 8:

(1,7) (2,6) (3,5) (4,4) (5,3) (6,2) (7,1).

There are $8 \times 8 = 64$ possible combinations.

Find the theoretical probability.

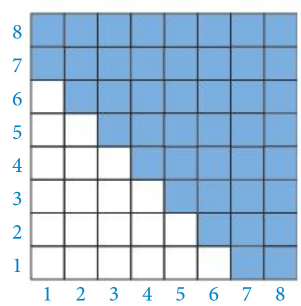


$\Pr(X = 8) = \frac{7}{64}$

ii Colour all the combinations in the lattice that add to 8 or more.

There are 43 combinations that add to 8 or more.

Find the theoretical probability.



$\Pr(X \geq 8) = \frac{43}{64}$

9780170448260

EXERCISE 5.2 Random variables

ANSWERS p. 515

Recap

The following information is used for Questions 1 and 2.

The month of birth for 20 Year 1 students is shown in the table.

Month	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Frequency	3	1	2	4	0	1	0	1	2	0	1	5

1 The experimental probability of a Year 1 student having a birthday in summer is

A $\frac{1}{4}$ **B** $\frac{3}{10}$ **C** $\frac{1}{3}$ **D** $\frac{2}{5}$ **E** $\frac{9}{20}$

2 The theoretical probability of a Year 1 student having a birthday in summer is

A $\frac{1}{4}$ **B** $\frac{3}{10}$ **C** $\frac{1}{3}$ **D** $\frac{2}{5}$ **E** $\frac{9}{20}$

Mastery

3 WORKED EXAMPLE 2 A bag contains 20 numbered discs.

The set {1, 1, 1, 2, 2, 3, 3, 3, 4, 5, 5, 5, 5, 5, 7, 7, 7, 8, 8, 9}, shows the numbers on the discs in the bag. A disc is randomly selected from the bag and the random variable N represents the number chosen.

Find **a** $\Pr(N = 5)$ **b** $\Pr(N > 5)$

4 WORKED EXAMPLE 3 A probability spinner is divided into 6 equal sections numbered 1 to 6. The pointer is spun twice. The random variable X = the sum of the 2 digits from both spins.

a Using a lattice diagram, find $\Pr(X = 6)$.

b Using a lattice diagram, find $\Pr(X \geq 6)$.

5 TECH-FREE A bag contains 5 cubes numbered 1 to 5. A cube is selected, its number is noted and then it is replaced before a second cube is selected. The random variable Y represents the difference when the smaller number is subtracted from the larger number.

Find

a $\Pr(Y = 0)$ **b** $\Pr(Y = 1)$ **c** $\Pr(Y \geq 2)$

6 TECH-FREE A fair 6-sided die is rolled twice. The random variable R represents the remainder when the larger number is divided by the smaller number. If, for example, a 5 and a 6 are rolled then the division $6 \div 5 = 1$ with a remainder of 1.

Find

a $\Pr(R = 0)$ **b** $\Pr(R = 1)$ **c** $\Pr(R = 2)$

7 TECH-FREE A fair coin is tossed 3 times. The random variable X = the number of tails obtained from the 3 tosses. There are 8 different outcomes in the sample space.

a List the sample space.

b Find

i $\Pr(X = 0)$

ii $\Pr(X < 2)$

Exam practice

8 TECH-FREE (4 marks) A tetrahedral die has its faces numbered 1 to 4 and a cube die has its faces numbered 1 to 6. Both dice are rolled and the numbers that are face down are noted.

The random variable N = the sum of the 2 numbers.

Use a lattice diagram to illustrate the sample space and hence find

a $\Pr(N = 6)$ 2 marks

b $\Pr(N < 6)$ 2 marks

9 Two fair coins are tossed and the number of heads that appear is recorded.
If H = the number of heads in the toss of 2 coins then $\Pr(H = 0)$ is equal to

A 0 **B** $\frac{1}{4}$ **C** $\frac{1}{2}$ **D** $\frac{3}{4}$ **E** 1

10 A fair coin is tossed 4 times. The random variable T = the number of tails obtained from the 4 tosses. There are 16 different outcomes in the sample space.

$\Pr(T \geq 1)$ is equal to

A $\frac{1}{16}$ **B** $\frac{1}{4}$ **C** $\frac{7}{16}$ **D** $\frac{13}{16}$ **E** $\frac{15}{16}$

11 (4 marks) Two 6-sided dice are rolled. A fraction is formed by making the number on the first die the numerator and the number on the second die the denominator. Use a lattice diagram to illustrate the sample space and use this to find the probability that the fraction generated is equivalent to

a $\frac{1}{2}$ **b** $\frac{1}{3}$ **c** a number greater than 1 **d** a whole number

12 (3 marks) Two mixed teams of 4 tennis players each select one of their team members to play off in a challenge match. Team A has 3 female players and one male player while team B has an equal number of male and female players. The random variable F is the number of female tennis players in the challenge match.

Find

a $\Pr(F = 0)$ **b** $\Pr(F = 1)$ **c** $\Pr(F = 2)$

Video playlist
Probability simulations

5.3 Probability simulations

Many problems that need to be investigated are too complex to analyse using actual physical experiments and theoretical probability. **Probability simulation** uses methods of generating random numbers to help model real-world situations. Random numbers can be generated using physical means such as rolling a die or tossing a coin. They can also be generated using a spreadsheet or CAS.

How to conduct a simulation

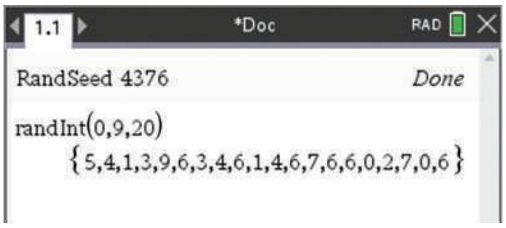
A simulation is useful only if it closely mirrors real-world outcomes. The steps required to produce a useful simulation are presented below.

1 Describe the possible outcomes.
2 Link each outcome to one or more random numbers.
3 Choose a source of random numbers.
4 Generate a set of random numbers.
5 Based on the random number, note the 'simulated' outcome.
6 Repeat steps 4 and 5 multiple times, preferably until the outcomes show a stable pattern.
7 Analyse the simulated outcomes and report results.

USING CAS 1 Random integers

Generate 20 random numbers between 0 and 9.

TI-Nspire



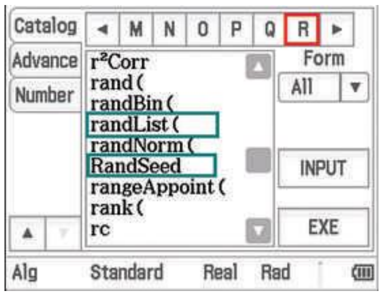
1 Press **menu > Probability > Random > Seed**.

2 Enter any 4-digit number.

3 Press **menu > Probability > Random > Integer**.

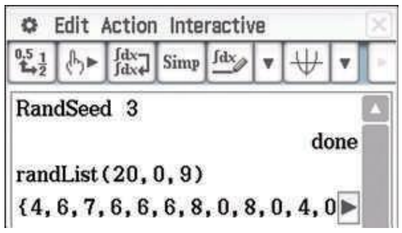
4 Enter **0,9,20**.

ClassPad



1 Open the **Keyboard** and tap the **down arrow** to access the **Catalog**.

2 Tap **R** to access the **RandSeed** and **randList** functions.



3 Tap **RandSeed**.

4 Enter a number from 0 to 9.

5 Open the **Catalog** and tap **RandList**.

6 Enter **20,0,9**.

Tap the arrow or rotate the screen to view all 20 random numbers.

Using simulation to calculate probabilities for equally likely events

Equally likely events have the same theoretical probability or likelihood of occurring. When tossing a coin, heads or tails is equally likely. When rolling a die, the outcomes {1, 2, 3, 4, 5, 6} also have the same chance of occurring. When simulating equally likely events, we allocate one of the outcomes of a random experiment to our item of interest so that the probability of our event and the random experiment are the same.

WORKED EXAMPLE 4 Using simulation to calculate probabilities.

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There are 40 students in Year 6 at a rural primary school, and one Year 6 student is selected at random each day for cleaning duty.

100 random digits between 1 and 40 have been generated below.

29, 36, 15, 5, 19, 40, 28, 22, 26, 38, 27, 32, 23, 14, 9, 25, 36, 5, 19, 19, 12, 33, 40, 20, 17, 10, 26, 25, 33, 31, 19, 38, 25, 14, 19, 33, 35, 11, 29, 19, 20, 9, 4, 15, 26, 6, 27, 1, 24, 12, 1, 37, 34, 11, 30, 10, 5, 30, 9, 24, 33, 38, 20, 4, 17, 26, 15, 33, 30, 1, 16, 15, 10, 8, 10, 32, 19, 14, 29, 38, 30, 6, 16, 23, 24, 14, 9, 12, 36, 8, 21, 14, 29, 39, 37, 26, 28, 1, 5, 32

These random numbers can be grouped into 20 sets of 5 to represent the students selected for 20 weeks.

Find, using this simulation, the probability that in a particular week a student is allocated cleaning duty more than once.

Steps	Working
1 Each student receives a number between 1 and 40. Group the random numbers into 20 groups of 5.	(29, 36, 15, 5, 19) (40, 28, 22, 26, 38) (27, 32, 23, 14, 9) (25, 36, 5, 19, 19) (12, 33, 40, 20, 17) (10, 26, 25, 33, 31) (19, 38, 25, 14, 19) (33, 35, 11, 29, 19) (20, 9, 4, 15, 26) (6, 27, 1, 24, 12) (1, 37, 34, 11, 30) (10, 5, 30, 9, 24) (33, 38, 20, 4, 17) (26, 15, 33, 30, 1) (16, 15, 10, 8, 10) (32, 19, 14, 29, 38) (30, 6, 16, 23, 24) (14, 9, 12, 36, 8) (21, 14, 29, 39, 37) (26, 28, 1, 5, 32)
2 Identify the groups that contain repeated numbers. These groups are weeks where a student has done cleaning duty on more than one day in a week.	The highlighting shows that there are 3 weeks out of the 20 weeks where a student has done cleaning duty twice in the week. $\Pr(\text{cleaning duty more than once}) = \frac{3}{20} = 0.15$

Using simulation to calculate probabilities for non-equally likely events

Suppose a professional basketball player, over her career, has a shooting percentage of 70% on free throws. We can simulate the event of shooting a free throw by generating random numbers between 0 and 9. Three of these digits represent missing the free throw and the other 7 hitting the free throw. If we let the digits {0, 1, 2} represent a miss and {3, 4, 5, 6, 7, 8, 9} represent a hit, then this matches the basketballer's probabilities of 30% miss and 70% hit.

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WORKED EXAMPLE 5 Probabilities for non-equally likely outcomes using simulation

On average, a baseball player hits a home run twice every 10 times at bat, and suppose he gets exactly 2 'at bats' in every game. Using simulation, estimate the likelihood that the player will hit 2 home runs in a single game.

Steps	Working
1 The 2 possible outcomes on each trial are either the player hits a home run or the player does not hit a home run. State which 2 of the 10 random numbers are a home run and which are not a home run.	In the probability simulation: home run = {0, 1} not home run = {2, 3, 4, 5, 6, 7, 8, 9}
2 Generate data for 50 games with 2 innings per game, which is 100 random integers between 0 and 9. Pair off the random numbers so that each pair represents the 2 innings of a game.	(9, 9) (1, 5) (4, 7) (0, 3) (9, 2) (7, 9) (2, 3) (0, 9) (1, 0) (5, 8) (9, 2) (2, 1) (0, 7) (0, 4) (3, 9) (0, 8) (6, 2) (9, 7) (3, 2) (9, 9) (0, 1) (6, 9) (6, 3) (0, 9) (1, 3) (1, 2) (8, 3) (5, 6) (0, 0) (0, 6) (7, 8) (3, 1) (4, 9) (6, 5) (6, 9) (6, 7) (5, 3) (2, 6) (8, 1) (4, 4) (2, 8) (9, 4) (4, 2) (6, 6) (8, 7) (4, 9) (6, 3) (4, 8) (8, 2) (7, 4)
3 Two home runs in a game are represented by a pair of numbers where both digits are either 0 or 1.	The 3 games in which both innings were home runs are highlighted above.
4 Calculated the estimated probability from the simulation.	The estimated probability of scoring 2 home runs in a game is $\frac{3}{50}$. Results in simulations will vary and will be more reliable when a large number of simulations are performed.

EXERCISE 5.3 Probability simulations

ANSWERS p. 516

5.3

Recap

1 A fair coin is tossed 3 times. The random variable H = the number of heads obtained from the 3 tosses. The sample space for this experiment is

S = {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}.

Pr(H = 2) is equal to

A $\frac{1}{8}$ **B** $\frac{3}{8}$ **C** $\frac{1}{2}$ **D** $\frac{3}{4}$ **E** $\frac{7}{8}$

2 A bag contains 5 cubes numbered 2, 4, 6, 8 and 10. A cube is selected, its number is noted and then it is replaced before a second cube is selected. The random variable D represents the difference when the smaller number is subtracted from the larger number.

Pr(D = 4) is equal to

A $\frac{1}{5}$ **B** $\frac{4}{25}$ **C** $\frac{6}{25}$ **D** $\frac{8}{25}$ **E** $\frac{2}{5}$

Mastery

3 Using CAS 1 Generate 20 random numbers between 1 and 6.

4 WORKED EXAMPLE 4 There are 20 students in a Year 7 class, and each day one student is selected at random to act as a classroom monitor. The random numbers below simulate samples for 20 weeks.

Find, using this simulation, the probability that in a particular week a student is allocated classroom monitor duties more than once.

14, 19, 13, 4, 19	10, 3, 9, 3, 9	14, 13, 8, 7, 8	5, 12, 11, 6, 14
9, 10, 12, 6, 8	10, 18, 18, 6, 10	14, 6, 14, 14, 6	8, 8, 18, 3, 13
19, 5, 4, 9, 16	2, 5, 2, 8, 17	10, 17, 16, 8,1	8, 6, 7, 14, 13
12, 14, 4, 17, 7	10, 9, 15, 13, 13	14, 8, 8, 11, 11	11, 17, 11, 14, 18
2, 7, 17, 14, 16	14, 15, 1, 18, 5	17, 8, 1, 8, 4	3, 8, 17, 18, 7

5 WORKED EXAMPLE 5 On average, a basketball player hits a 3-point shot 6 times in every 10 attempts, and suppose she takes four 3-point attempts in every game. Using simulation, estimate the likelihood that the player will hit three 3-pointers in a single game. Simulate the results for 20 games.

Exam practice

6 TECH-FREE (2 marks) An archer shoots an arrow at a target and on average hits the centre of the target once in every 5 attempts. The archer shoots 5 arrows at the target. The random numbers below can be used to complete 10 simulations of this event. If 0 represents a hit and 1–4 a miss, find the probability the archer hits the centre of the target at least twice in a set of 5 shots.

4, 2, 2, 3, 3	3, 1, 2, 4, 1	1, 4, 0, 4, 4	4, 3, 2, 3, 1	3, 4, 2, 0, 2
1, 1, 1, 3, 4	0, 0, 4, 0, 2	4, 2, 1, 3, 3	2, 0, 0, 3, 2	0, 2, 0, 4, 3

7 TECH-FREE (2 marks) Ange attempts the newspaper crossword every day and, on average, gets 3 out of every 5 clues correct. Using the simulation below, find the probability that Ange gets at least 5 of the first 10 clues in the crossword correct.

In the following simulation let 0–2 represent correct and 3 and 4 represent incorrect.

1, 1, 4, 4, 3, 4, 4, 2, 4, 3	2, 4, 0, 4, 2, 0, 1, 3, 0, 1	3, 2, 4, 1, 2, 0, 0, 2, 4, 0	3, 0, 4, 0, 1, 0, 2, 2, 3, 3
4, 1, 2, 2, 2, 3, 1, 0, 3, 3	0, 2, 0, 1, 3, 0, 2, 4, 4, 4	2, 3, 3, 2, 2, 2, 3, 3, 0, 1	3, 4, 1, 3, 3, 0, 2, 3, 0, 4
2, 2, 1, 2, 4, 0, 3, 4, 2, 2	1, 1, 0, 3, 3, 3, 0, 0, 1, 1		

8 The best way to simulate answering 25 multiple-choice questions with 4 options is to

A roll a die 25 times
B spin a spinner with 4 equal sections 25 times
C flip a coin 25 times
D spin a spinner with 25 sections 4 times
E flip a coin 4 times

9 In a particular family with two children, the gender is predicted using simulation.
The first child is equally likely to be a boy or a girl. In the simulation, random digits 0 to 9 are generated.
For the first child: 0–4 = boy and 5–9 = girl.
The gender of the second child is dependent on the gender of the first.
If the first child is a boy: 0–6 = second child is a boy and 7–9 = second child is a girl.
If the first child is a girl: 0–6 = second child is a girl and 7–9 = second child is a boy.
The random numbers 3, 7, 9, 2 show the gender for the 2 children of 2 families.
The genders are

A (B, G) and (B, G) **B** (B, G) and (G, G) **C** (B, G) and (G, B)
D (G, G) and (G, G) **E** (B, B) and (G, G)

10 (6 marks) A condition known as DDD occurs in 30% of female children and 20% of male children.
Simulation code:
First digit: 0–4 represents a boy, second digit: 0 or 1 represents DDD and 2–9 not DDD
First digit: 5–9 represents a girl, second digit: 0–2 represents DDD and 3–9 not DDD
50 random numbers are generated:
26, 65, 34, 74, 14, 34, 56, 35, 31, 35, 33, 26, 69, 57, 20, 45, 61, 53, 16, 17, 33, 01, 87, 51, 73,
33, 95, 69, 54, 52, 95, 23, 76, 34, 04, 38, 39, 80, 53, 96, 09, 02, 32, 88, 65, 87, 66, 32, 07, 95
Copy and complete the table below using the random numbers and the code to determine the probability of a child developing the condition.

	DDD		Not DDD	
	Tally	Frequency	Tally	Frequency
Boy				
Girl				

11 (4 marks) Complete a simulation for the DDD condition described in question 10 if it occurs in 60% of female children and 40% of male children. Complete 100 simulations, showing your simulation code and table of results and hence calculate an estimate of the probability of the condition DDD occurring.

12 (8 marks) Cornie's cereal boxes contain a plastic figurine gift in each box. There are a total of 5 different figurines and the probability of a cereal box containing any one of the figurines is the same. Determine, by completing fifty simulations, the average number of cereal boxes that need to be purchased to collect the full set of 5 figurines.

13 (10 marks) **Investigation: The Infectious Disease Problem**
A person who is unknowingly infected with the Wobbles disease comes to the island of Mambo to visit one of the residents. The Wobbles disease is found to be infectious for a period of one day and after this time the infected individual is immune. The island has 40 residents. On the first day, the resident visited by the person with Wobbles catches the disease, and on the next day this resident visits another resident

to seek help. That resident then becomes infected and on the next day they visit another randomly selected resident. This process continues until a person visits someone who had previously contracted the disease, in which case they would be immune and so the disease dies out. Using probability simulation, find the average number of infected residents.

To start, generate random numbers between 1 and 40.

38, 37, 6, 21, 17, 30, 2, 14, 40, 9, 32, 39, 9

This list shows the residents who have been visited by someone infected with the disease. Continue to generate random numbers until any number repeats.

Because resident 9 has already been visited and is now immune, the disease will not be spread to anyone else. This time 12 residents were infected.

a Collect a total of 50 simulations using the results from your class and hence find the average number of infected residents.

b Repeat this process using populations of 10 and 20 people.

It is also possible to investigate herd immunity for a population of 40 people but assume that there are already some immune people on the island. Use 5, 10 and 20 immune residents for each set of simulations.

5.4

5.4 The addition and multiplication principles

Video playlist The addition and multiplication principles

The addition principle

Maddi is watching a movie tonight and can choose from 3 horror movies, 4 romance movies or 5 science fiction movies.

The total number of possibilities is $3 + 4 + 5 = 12$.

The addition principle

In an event in which we can choose a of one type OR b of another type OR c of another type, the total number of choices can be calculated by adding $a + b + c$.

Exam hack

Generally, adding the number of possibilities is associated with the word 'OR' in a problem.

WORKED EXAMPLE 6 The addition principle

WB p. 108

For breakfast, Elijah can choose one food item from: 3 types of cereals; toasted white or brown bread; or fried, poached or scrambled eggs. How many choices does Elijah have for breakfast?

Steps	Working
1 Elijah can choose only one item and in each case the word OR can be used between each of the choices. The word OR between the choices indicates the addition principle is used.	Elijah can choose between 3 cereals OR 2 toasted breads OR 3 styles of eggs.
2 Calculate the possibilities by adding the different options.	Number of choices $= 3 + 2 + 3$ $= 8$

The multiplication principle

The multiplication principle

When there are 2 or more successive selections to make and we must choose from a of one type AND then from b of another type, the total number of choices can be found by multiplying $a \times b$.

Exam hack

Generally, multiplying the number of possibilities is associated with the word 'AND' in a problem.

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WORKED EXAMPLE 7 The multiplication principle

Freda must do one maths problem and write one English essay for homework. She is told to select the maths problem from 8 different maths problems and the English essay from 4 possible essays. How many different possibilities does she have?

Steps	Working
Freda must choose one maths problem from 8 maths problems AND one English essay from 4 essays. This indicates the multiplication principle is used.	number of possibilities $= 8 \times 4$ $= 32$

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WORKED EXAMPLE 8 The addition and multiplication principles

Dominic visits an Italian restaurant for lunch. He will order either a pizza or pasta. The menu has pizzas with a choice of 3 different bases and 10 different toppings. For pasta, there is a choice of 4 different pastas with 7 different sauces.

a How many different choices of pizza are possible?

b How many different choices of pasta are possible?

c How many different choices does Dominic have altogether?

Shutterstock.com/Timmary

Steps	Working
a For pizza, there are 3 different bases AND 10 different toppings. AND is the multiplication rule.	pizza choices $= 3 \times 10$ $= 30$
b For pasta, there are 4 different pastas AND 7 different sauces. AND is the multiplication rule.	pasta choices $= 4 \times 7$ $= 28$
c Find the total number of choices. Dominic can have pizza OR pasta. OR is the addition rule.	Total number of choices $= 30 + 28$ $= 58$

Applying the addition and multiplication principles

5.4

The probability of an event

$$\Pr(A) = \frac{\text{number of ways } A \text{ can occur}}{\text{total number of possible outcomes}}$$

The number of possibilities can often be determined by applying the multiplication or addition principles.

WORKED EXAMPLE 9 Applying the addition and multiplication principles

William is choosing a meal from a Chinese restaurant and Leah tries to guess which combination he will pick. He has to choose a meat from pork, chicken, duck or beef, a sauce from satay, sweet and sour, vegetable or black bean, and then either rice or noodles. Leah guesses William will choose satay beef with noodles. Find the probability that Leah is correct.

Steps	Working
1 Find the total number of possible combinations. William is choosing from 4 meats AND 4 sauces AND 2 sides.	number of possible combinations $= 4 \times 4 \times 2$ $= 32$
2 Determine the probability. Leah has chosen one combination from 32 possible combinations.	$\Pr(\text{Leah correct}) = \frac{1}{32}$

WB

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EXERCISE 5.4 The addition and multiplication principle

ANSWERS p. 516

Recap

1 The best way to simulate answering 20 questions where the answer is either true or false is to

A roll a die 40 times

B spin a spinner with 3 equal sections 20 times

C flip a coin 20 times

D spin a spinner with 20 sections twice

E flip a coin twice

2 Scientists are studying the environmental impact of temperature changes on python reproduction. If the temperature is hot, the probability of the python egg hatching is 0.7 and if the temperature is cold, the probability of the egg hatching is 0.5.

The probability of a python egg hatching is determined using simulation by generating random 2-digit numbers. The following code is used to interpret the random digits:

First digit	Second digit	
Hot = {0, 1, 2, 3, 4}	Hatch = {0, 1, 2, 3, 4, 5, 6}	Not hatch = {7, 8, 9}
Cold = {5, 6, 7, 8, 9}	Hatch = {0, 1, 2, 3, 4}	Not hatch = {5, 6, 7, 8, 9}

The random digits 26 and 69 would be interpreted as

A temperature hot – first egg hatched and temperature cold – second egg did not hatch

B temperature cold – first egg hatched and temperature cold – second egg did not hatch

C temperature hot – first egg did not hatch and temperature cold – second egg hatched

D temperature cold – first egg hatched and temperature hot – second egg hatched

E temperature hot – first egg hatched and temperature hot – second egg hatched

Mastery

3 WORKED EXAMPLE 6 Andrew is going to read a book during his holidays. He has to choose from 8 different thrillers, 5 science fiction books, 3 biographies and 6 detective novels. How many choices does Andrew have?

4 WORKED EXAMPLE 7 The menu in a restaurant has 5 choices for entrée, 8 for main course and 3 for dessert. How many choices are possible if all three courses are ordered?

5 WORKED EXAMPLE 8 Carol must choose one first-semester unit from 3 units of maths and 2 units of science and one second-semester unit from 2 units of history, 2 units of geography and 3 units of technology. How many different 2 unit courses are possible?

6 WORKED EXAMPLE 9 Samantha is planning her activities on Saturday. In the morning, she will do some exercise and has a choice of going for a run, cycle, swim or walk. In the afternoon, she can go to the football, have a picnic in the park, go shopping or visit the art gallery. In the evening, she can go to the cinema, visit a friend or stay home and read a book. Find the probability Samantha goes for a cycle in the morning, goes shopping in the afternoon and goes to the cinema in the evening.

Exam practice

7 TECH-FREE (1 mark) Gretta is travelling from Melbourne to Mildura. She can take 2 different buses from Melbourne to Mildura, or 2 trains from Melbourne to Bendigo and then 3 connecting buses from Bendigo to Mildura, or she can drive from Melbourne to Mildura. How many ways does Gretta have of making this journey?

8 TECH-FREE (1 mark) Morse code is generated using a sequence of signals called dots and dashes. How many combinations of either 1, 2, 3 or 4 signals are possible?

9 TECH-FREE (1 mark) A restaurant has a \$10 meal deal where customers can choose either a drink and a sandwich or a sandwich and a dessert. The restaurant has a choice of 4 different drinks, 10 varieties of sandwiches and 5 desserts. How many different meal deals are possible?

10 Bunroeun must travel from Melbourne to Darwin. He has a choice of 3 flights from Melbourne to Sydney and 2 different connecting flights from Sydney to Darwin, or four different direct flights from Melbourne to Darwin, or 3 different buses from Melbourne to Darwin. The number of choices Bunroeun has is

A 12 **B** 13 **C** 18 **D** 27 **E** 72

11 Lucy is purchasing a laptop computer. She can purchase a brand of computer that has a choice of 3 different hard drives, three different CPUs and 2 amounts of RAM or she can choose from 8 other computers with set components. The number of choices of computers Lucy has is

A 16 **B** 19 **C** 25 **D** 26 **E** 144

12 Vehicle number plates consist of 3 letters and 3 numbers. The number of different number plates with ‘T’ as the first letter is

A $(26)^2 \times (10)^3$ **B** $(26)^3 \times (10)^3$ **C** $(26)^2 \times (9)^3$

D $2 \times 26 + 3 \times 10$ **E** $3 \times 26 + 3 \times 10$

The following information is used for questions 13 and 14.

Mario's pizza shop sells customised pizzas. Patrons can choose from 3 different crusts, 3 different types of cheese and 5 different combinations of toppings. Mario's also has 10 gourmet pizzas on their menu.

13 The number of possible customised pizzas is

A 14 **B** 18 **C** 30 **D** 40 **E** 45

14 The number of possible pizza choices on Mario's menu is

A 24 **B** 28 **C** 40 **D** 55 **E** 450

15 Emma is completing an English exam that has 3 sections and she must answer one question in each section of the paper. Section A has 5 questions, section B has 2 questions and section C has 4 questions. The probability that Emma answers the first question in section A and the last question in section C correctly is

A $\frac{1}{40}$ **B** $\frac{1}{20}$ **C** $\frac{1}{11}$ **D** $\frac{1}{9}$ **E** $\frac{2}{11}$

5.5

5.5 Permutations

Video playlist Permutations

Worksheets Counting techniques

Ordered and unordered selections

Permutation calculations

Permutations

The **permutations** of a group of objects are the different ways the objects can be arranged.

Consider a race of 6 people where first place and second place are recorded. How many different permutations are possible?

Shutterstock.com/Valery Bareta

1 First and second place can be represented by 2 boxes.

2 Place a 6 in the first box since there are 6 ways to fill first place.

6	

3 Once first place has been filled, there are only 5 ways to fill second place.

Place a 5 in the second box.

6	5

4 Use the multiplication principle.

The total number of possible arrangements is $6 \times 5 = 30$

Permutations

The number of permutations or arrangements of r objects chosen from n objects is ${}^nP_r$.

WB

p. 112

WORKED EXAMPLE 10 Permutations

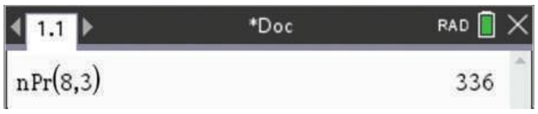
Three family photos are displayed on a wall. In how many ways can this be done if there are 8 photos to choose from?

Steps	Working
1 There are 3 places to fill so draw 3 boxes.	☐ ☐ ☐
2 There are 8 ways to fill the first position.	8 ☐ ☐
3 Once the first place is filled, there are 7 photos left to fill the second position.	8 7 ☐
4 Once the second position has been filled, there are 6 photos left to put in the third position.	8 7 6
5 Use the multiplication principle to calculate the number of possible arrangements.	Total number $= 8 \times 7 \times 6$ $= 336$

USING CAS 2 Permutations

Verify the solution to the above example by calculating ${}^{8}P_{3}$.

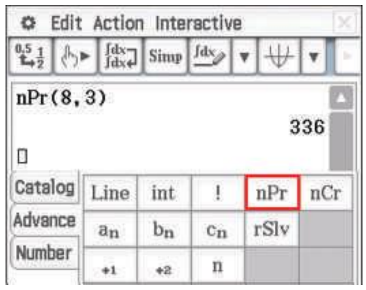
TI-Nspire



1 Press **menu > Probability > Permutations**.

2 Enter **8,3**.

ClassPad



1 Open the **Keyboard > Advance**.

2 Tap **nPr**.

3 Enter **8,3**.

Worksheet
Factorial notation

Factorials

The mathematical notation for n **factorial** is $\boldsymbol{n!}$ where $n! = n \times (n-1) \times (n-2) \times \ldots 3 \times 2 \times 1$.

$n!$ is the number of arrangements of n different objects. $0! = 1$.

WB

p. 113

WORKED EXAMPLE 11 Factorials

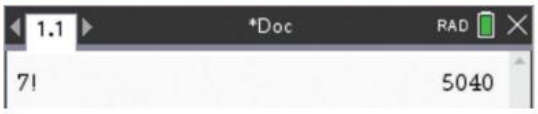
A group of 7 musicians are arranged in a line for a photo. In how many ways can this be done?

Steps	Working
We are required to arrange 7 people in a line. This can be done 7! ways.	The number of arrangements $= 7!$ $= 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$ $= 5040$

USING CAS 3 Factorials

Calculate 7!.

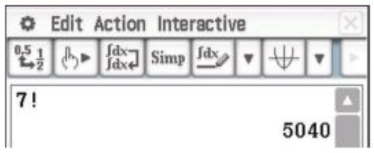
TI-Nspire



1 Enter **7**.

2 Press **menu > Probability > Factorial(!)**.

ClassPad



1 Enter **7**.

2 Open the **Keyboard > Advance**.

3 Tap **!**.

Calculating ^nP_r

We use CAS to calculate ^nP_r, but there is also a formula involving factorials for ^nP_r.

Consider $^8P_3$.

$^8P_3 = 8 \times 7 \times 6 = 336$

This demonstrates a simple method for calculating ^nP_r: find the product of r descending consecutive numbers starting with n. For example, $^{10}P_4 = 10 \times 9 \times 8 \times 7 = 5040$.

This method is useful for calculating permutations without using a calculator.

Permutation formula

$$^nP_r = \frac{n!}{(n-r)!}$$

It is useful to know $^nP_1 = n$ and $^nP_0 = 1$.

WORKED EXAMPLE 12 Calculating permutations

Use the permutation formula to find the value of

a $^6P_2$ **b** $^{10}P_3$

Steps	Working
a Substitute $n = 6$ and $r = 2$ into the formula. $^nP_r = \frac{n!}{(n-r)!}$	$^6P_2 = \frac{6!}{(6-2)!} = \frac{6!}{4!} = \frac{6\times5\times4\times3\times2\times1}{4\times3\times2\times1} = 6\times5 = 30$
b Substitute into the formula and calculate as the product of 3 consecutive descending numbers starting with 10.	$^{10}P_3 = \frac{10!}{(10-3)!} = \frac{10!}{7!} = 10\times9\times8 = 720$

WB p. 114

Arrangements with restrictions

Some permutation problems have a condition or restriction placed on the arrangement.

Arrangements with restrictions

When counting arrangements with restrictions:

1 Determine the number of places to be filled and draw the same number of boxes.

2 Fill all the spaces where the restriction is involved.

3 Fill the rest of the spaces from left to right.

WB
p. 115

WORKED EXAMPLE 13 Arrangements with restrictions

A mum, dad and their 4 children are seated at a football match. How many possible seating arrangements are there if the children must be seated between the parents?

<table>
<tr><th>Steps</th><th>Working</th></tr>
<tr><td>1 There are 6 people to arrange so draw 6 boxes.</td><td><table><tr><td> </td><td> </td><td> </td><td> </td><td> </td><td> </td></tr></table></td></tr>
<tr><td>2 The restriction requires the parents to be seated at either end. The first and last positions are filled first. The first position can be filled in 2 ways as either the mother or father can sit in this position. This leaves one way to fill the last position.</td><td><table><tr><td>2</td><td> </td><td> </td><td> </td><td> </td><td>1</td></tr></table></td></tr>
<tr><td>3 The 4 children can now be seated. There are 4 ways to fill the second position, 3 ways to fill the third position and so on.</td><td><table><tr><td>2</td><td>4</td><td>3</td><td>2</td><td>1</td><td>1</td></tr></table></td></tr>
<tr><td>4 Use the multiplication principle to calculate the number of arrangements.</td><td>The number of seating arrangements
$= 2 \times 4 \times 3 \times 2 \times 1 \times 1$
$= 48$</td></tr>
</table>

Exam hack

Permutations with restrictions *cannot* be calculated using the permutation formula ^nP_r. Each problem must be treated as a special case and solved by filling spaces and applying the multiplication principle.

Arrangements with groups

In some arrangements, we want particular groups of objects kept together.

Arrangements with groups

When counting arrangements with groups:

1. Put all the objects in the group together and count this group as one object.
2. Determine the number of arrangements of the objects.
3. Multiply this by the number of ways of arranging the objects within the group.

WB
p. 116

WORKED EXAMPLE 14 Arrangements with groups

A mum, dad, 3 sisters and a brother are seated at a football match. How many possible seating arrangements are there if the sisters must sit together?

<table>
<tr><th>Steps</th><th>Working</th></tr>
<tr><td>1 Put the 3 sisters in a group and count as one object or ‘seat’.

$S\ S\ S$ $M\ D\ B$

There are 4 objects to arrange.</td><td>The number of ways of arranging the group and the 3 other family members
$= 4!$</td></tr>
<tr><td>2 Find the number of ways of arranging the sisters in the group.</td><td>The number of arrangements of the 3 sisters
$= 3!$</td></tr>
<tr><td>3 Find the total number of possible arrangements by applying the multiplication principle.</td><td>Total number of arrangements
$= 4! \times 3!$
$= 144$</td></tr>
</table>

9780170448260

WORKED EXAMPLE 15 Probability involving arrangements

A 4-digit number is made from the digits {2, 3, 4, 6, 8}, where repeated digits are not allowed. Find the probability the number is an odd number greater than 7000.

Steps	Working
1 Determine the number of possible 4-digit numbers if there is no restriction using the ${}^nP_r$ formula.	Total number of 4-digit numbers $= {}^5P_4 = 120$
2 Determine the number of 4-digit odd numbers greater than 6000. There are 4 spaces to fill and the first and last places must be filled first. There is only one number {8} that can be placed first and one number {3} that can be placed last.	| 1 | | | 1 |
3 Fill the remaining positions with the 3 digits left: {2, 4, 6}. There are 3 numbers that can be placed second, leaving 2 numbers that can be placed third.	| 1 | 3 | 2 | 1 |
4 Use the multiplication principle.	Possible 4-digit odd numbers greater than $7000 = 1 \times 3 \times 2 \times 1 = 6$
5 Calculate the probability.	$\Pr(\text{odd 4-digit} > 7000) = \frac{6}{120} = \frac{1}{20}$

5.5

WB

p. 117

EXERCISE 5.5 Permutations

ANSWERS p. 516

Recap

Questions 1 and 2 relate to the information below.

A restaurant offers a 2-course menu where, for their main course, customers can choose between chicken, fish or steak, served with either chips or mash and either vegetables or salad. They also have a choice of fruit salad, ice cream or cheesecake for dessert.

1 The number of possible choices is

A 10 **B** 15 **C** 21 **D** 24 **E** 36

2 The probability that a customer orders chicken, chips and salad for their main course and cheesecake for dessert is

A $\frac{1}{36}$ **B** $\frac{1}{15}$ **C** $\frac{1}{12}$ **D** $\frac{1}{10}$ **E** $\frac{3}{10}$

Mastery

3 WORKED EXAMPLE 10 Six people compete in a hurdles race. How many different ways can first, second and third placings be filled?

4 WORKED EXAMPLE 11 A 5-digit access code is made using the digits {1, 2, 3, 4, 5}. How many codes are possible if no repetitions are allowed?

5 Using CAS 2 Find the value of **a** ${}^{12}P_3$ **b** ${}^{15}P_1$ **c** ${}^{8}P_5$

6 Using CAS 3 Find the value of **a** 9! **b** 7! **c** 10!

7 WORKED EXAMPLE 12 TECH-FREE Use the permutations formula to find the value of

a ${}^{6}P_{3}$ **b** ${}^{5}P_{2}$ **c** ${}^{12}P_{2}$ **d** ${}^{16}P_{1}$ **e** ${}^{100}P_{0}$

8 WORKED EXAMPLE 13 Three boys and 4 girls sit on a bench. How many different seating arrangements are possible if the girls and boys must alternate?

9 WORKED EXAMPLE 14 The letters of the word HUMBLE are arranged. In how many ways can this be done if the vowels must be together?

10 WORKED EXAMPLE 15 A 3-digit number is made using the digits 1, 2, 5, 7 and 8. If no repetitions are allowed, find the probability that the number made is an odd number greater than 800.

Exam practice

11 TECH-FREE (2 marks) Tess and 4 other class members have been nominated for class captain and vice-captain.

a Find the number of possible combinations of captain and vice-captain. 1 mark

b Find the probability that Tess is elected vice-captain. 1 mark

12 TECH-FREE (3 marks) Six people line up to catch a bus.

a Find the number of different ways this can be done. 1 mark

b Find the number of ways this can be done if 2 of these people must stand next to each other. 1 mark

c Find the number of way the 6 people can line up if 2 people refuse to stand next to each other. 1 mark

13 TECH-FREE (2 marks) A 4-digit number is made from the digits {2, 3, 5, 6, 8}. Find the probability the number is odd and greater than 6000.

14 The number of ways 4 Christmas cards can be arranged on a shelf is

A 3 **B** 4 **C** 12 **D** 15 **E** 24

15 Amy has 6 netball trophies that were won in 2014, 2013, 2010, 2009, 2007 and 2006. If she arranges 4 of these on the mantelpiece with 2006 on the left and 2014 on the right, the number of possible arrangements is

A 6 **B** 12 **C** 4! **D** 6! **E** ${}^{6}P_{4}$

16 The letters of the word PEACH are arranged in a row. The probability that the arrangement starts with a vowel and ends with a vowel is

A $\frac{1}{60}$ **B** $\frac{1}{20}$ **C** $\frac{1}{10}$ **D** $\frac{1}{6}$ **E** $\frac{3}{5}$

17 Julie and Andrea compete in the 50-metre freestyle race with 6 other swimmers. The probability that Julie and Andrea race in adjacent lanes is

A $\frac{1}{8!}$ **B** $\frac{1}{6!}$ **C** $\frac{1}{7}$ **D** $\frac{1}{4}$ **E** $\frac{1}{3}$

18 A 4-digit number is made from the digits {1, 3, 4, 7, 8, 9}. The probability the number is greater than 7000 is

A $\frac{1}{360}$ **B** $\frac{1}{6!}$ **C** $\frac{1}{6}$ **D** $\frac{1}{3}$ **E** $\frac{1}{2}$

19 (3 marks) Three students are to be selected from 4 males and 5 females in the leadership team to fill the roles of school captain, vice-captain and school council representative.

a Find the number of ways this can be done if there are no restrictions. 1 mark

b Find the number of ways this can be done in which a female captain and a male vice-captain are chosen. 1 mark

c Find the number of ways this can be done if the roles are filled with students of the same sex. 1 mark

20 (4 marks) At the senior school sports carnival, the egg and spoon race has 3 Year 11 boys, 2 Year 11 girls, 2 Year 12 boys and one Year 12 girl. How many different finishing positions are there in which

a the girls occupy the first 3 placings? 1 mark

b the Year 11 students occupy the first 5 placings? 1 mark

c Year 12 students occupy first and last place? 1 mark

d a Year 11 girl does not come last? 1 mark

5.6 Combinations

Video playlist Combinations

Worksheets Combinations

Combination calculations

The **combination** of a group of objects is the number of ways the objects can be selected *where the order is not important*.

Consider a team of 8 basketball players, from which a coach must select 5 at any time to play on the court during a match. How many combinations are possible?

1 The 5 places can be represented by 5 boxes.

2 Place an 8 in the first box since there are 8 ways to fill the first place.

8				

3 Fill the other 4 places on the team.

8	7	6	5	4

4 ${}^8P_5 = 8 \times 7 \times 6 \times 5 \times 4$ gives the total number of permutations where the order is important. However, with combinations, the order is *not* important.

The number of possible rearrangements within the team of 5 is $5! = 5 \times 4 \times 3 \times 2 \times 1$.

The number of combinations of 5 players chosen from 8 players is ${}^8P_5 \div 5!$

The number of combinations of 5 from 8 is written as ${}^8C_5$ where ${}^8C_5 = \dfrac{{}^8P_5}{5!} = \dfrac{8 \times 7 \times 6 \times 5 \times 4}{5 \times 4 \times 3 \times 2 \times 1} = 56$.

Calculating ${}^nC_r$

Combinations formula

The number of combinations or selections of r objects chosen from n objects is ${}^nC_r$ where

$${}^nC_r = \frac{{}^nP_r}{r!} = \frac{n!}{(n-r)!\,r!}$$

${}^nC_r$ is sometimes called 'n choose r'.

We can use the shortcut for finding permutations when calculating combinations. To calculate ${}^nC_r$, find the product of r consecutive descending number starting with n and divide by $r!$.

For example, ${}^{10}C_3 = \dfrac{10 \times 9 \times 8}{3 \times 2 \times 1} = 120$. Note that the number of terms in the numerator and denominator is the same. This method is useful for counting combinations without a calculator.

WB p. 118

WORKED EXAMPLE 16 Calculating combinations

Evaluate ${}^7C_3$.

Steps	Working
1 In the combination the numerator is ${}^7P_3$ which is the product of 3 descending numbers starting with 7. The denominator is 3!	${}^7C_3 = \dfrac{{}^7P_3}{3!}$ $= \dfrac{7 \times 6 \times 5}{3 \times 2 \times 1}$
2 Cancel a factor of 6 and evaluate.	$= 7 \times 5$ $= 35$

Pascal's triangle and combinations

Pascal's triangle

				1						Row 0
			1		1					Row 1
		1		2		1				Row 2
	1		3		3		1			Row 3
1		4		6		4		1		Row 4
Term 0		Term 1		Term 2		Term 3		Term 4		

1 + 3 = 4

Add the 2 numbers in the row above to get the numbers in the next row.

Pascal's triangle can also be written using combinations

				${}^0C_0$				
			${}^1C_0$		${}^1C_1$			
		${}^2C_0$		${}^2C_1$		${}^2C_2$		
	${}^3C_0$		${}^3C_1$		${}^3C_2$		${}^3C_3$	
${}^4C_0$		${}^4C_1$		${}^4C_2$		${}^4C_3$		${}^4C_4$

Notice that ${}^4C_3$ is the number from row 4 term 3, so in general ${}^nC_r$ is the number from row n term r.

Properties of ${}^nC_r$

Using symmetry, ${}^nC_r = {}^nC_{n-r}$

This can be useful if r is a large number and $n - r$ is smaller.

$${}^nC_0 = {}^nC_n = 1 \text{ and } {}^nC_1 = {}^nC_{n-1} = n$$

WORKED EXAMPLE 17 Calculating combinations by symmetry

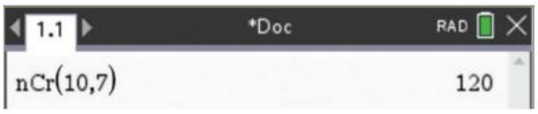
Evaluate ${}^{10}C_7$.

Steps	Working
By symmetry, ${}^{10}C_7 = {}^{10}C_3$. Evaluate ${}^{10}C_3$.	${}^{10}C_3 = \frac{10 \times 9 \times 8}{3 \times 2 \times 1}$ $= \frac{720}{6}$ $= 120$

USING CAS 4 Combinations

Calculate ${}^{10}C_7$.

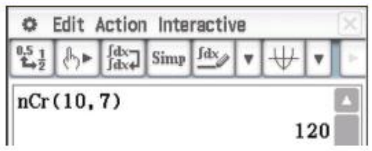
TI-Nspire



1 Press **menu > Probability > Combinations.**

2 Enter **10,7**.

ClassPad



1 Open the **Keyboard > Advance**.

2 Tap **nCr**.

3 Enter **10,7**.

WB p. 119

WORKED EXAMPLE 18 Applying combinations

How many different hands of 5 cards can be dealt from a pack of 52 playing cards?

Steps	Working
The number of different hands of 5 cards is a combination because the order the cards are dealt does not change the hand. We are choosing 5 cards from 52 cards, which can be done ${}^{52}C_5$ ways.	The number of hands $= {}^{52}C_5$ $= 2\ 598\ 960$

WB p. 120

Combinations with restrictions

WORKED EXAMPLE 19 Counting combinations with restrictions

A mixed volleyball team of 9 players is selected from a group of 8 girls and 5 boys. How many different teams are possible if the team contains 5 girls and 4 boys?

Steps	Working
1 The number of different teams is a combination because the order of selection does not impact the team chosen. We are choosing 5 girls from 8 girls AND 4 boys from 5 boys.	The number of ways to choose the girls $= {}^{8}C_5 = 56$ The number of ways to choose the boys $= {}^{5}C_4 = 5$
2 Use the multiplication rule to find the total number of teams.	The number of possible teams $= {}^{8}C_5 \times {}^{5}C_4$ $= 56 \times 5$ $= 280$

WB p. 121

Applications of combinations to probabilities

In the formula $\Pr(A) = \dfrac{\text{number of ways } A \text{ can occur}}{\text{total number of possible outcomes}}$, the number of ways event A can occur is the number of combinations of event A, and the total number of possible outcomes is the number of combinations if there is no restriction.

Exam hack

In a selection without replacement, the order in which the items are selected is not important. Combinations are used to find probabilities in problems of this type.

WB

p. 122

WORKED EXAMPLE 20 Finding probabilities involving combinations

A box contains 15 batteries, of which 5 are defective. A sample of 4 batteries is selected without replacement. Find the probability the sample contains 2 defective batteries.

Steps	Working
1 Count the number of ways of selecting a sample of 4 batteries where 2 are defective. In the sample, we must choose 2 defective batteries from the box that has 5 defective batteries AND choose 2 good batteries from a total of 10 good batteries.	Number of ways of choosing 2 defective batteries in the sample $= {}^5C_2 \times {}^{10}C_2$ $= 10 \times 45$ $= 450$
2 Determine the number of ways of selecting a sample of 4 batteries from a box containing 15 batteries.	Number of ways of selecting a sample of 4 with no restriction $= {}^{15}C_4$ $= 1365$
3 Determine the probability.	$\Pr(2 \text{ defective}) = \dfrac{450}{1365} = \dfrac{30}{91}$

EXERCISE 5.6 Combinations

ANSWERS p. 516

Recap

1 Eight people compete in an egg and spoon race. The number of ways of filling first, second and third places is

A 6 **B** 24 **C** 27 **D** 336 **E** 512

2 The letters of the word 'ORANGE' are arranged in a row into a code word. The probability the code word has all the vowels together is

A $\dfrac{3! \times 3!}{6!}$ **B** $\dfrac{4! \times 3!}{6!}$ **C** $\dfrac{4! \times 3!}{7!}$ **D** $\dfrac{1}{2}$ **E** $\dfrac{1}{6}$

Mastery

3 WORKED EXAMPLE 16 TECH-FREE Evaluate

a ${}^5C_2$ **b** ${}^7C_3$ **c** ${}^4C_1$ **d** ${}^7C_2$ **e** ${}^8C_4$

4 WORKED EXAMPLE 17 TECH-FREE Evaluate

a ${}^{11}C_{10}$ **b** ${}^{12}C_{10}$ **c** ${}^{100}C_{99}$ **d** ${}^{40}C_{38}$ **e** ${}^{15}C_{13}$

5 Using CAS 4 Find the value of

a $^{21}C_4$

b $^{8}P_4 \div {}^{8}C_4$

c $\dfrac{^{6}C_3 \times {}^{4}C_2}{^{10}C_5}$

5.6

6 WORKED EXAMPLE 18 A relay team of 4 runners is selected from a group of 8 athletes. How many different teams are possible?

7 WORKED EXAMPLE 19 A mixed basketball team of 5 players is selected from 7 females and 5 males. How many teams contain 3 female and 2 male players?

8 WORKED EXAMPLE 20 A committee of 4 people is selected from 6 men and 4 women. Find the probability of having equal numbers of men and women on the committee.

Exam practice

80–100% 60–79% 0–59%

9 TECH-FREE (1 mark) A squad of 15 basketball players is reduced to a final team of 12 players. Find the number of possible teams.

10 TECH-FREE (2 marks) A badminton competition has 8 entries. In the draw each player will play everyone else entered in the competition. Find the number of matches required.

11 TECH-FREE (2 marks) Seven people in a room each shake hands with everyone else in the room. How many handshakes will there be?

12 © VCAA 2003 1IQ27 62% A bag contains 12 bread rolls, of which 8 are white and the remainder multigrain. Tony takes 2 bread rolls at random from the bag to eat.

The probability that at least one is a multigrain roll is

A $1-\dfrac{2^{12}}{3^{12}}$

B $1-\dfrac{^{8}C_2}{^{12}C_2}$

C $1-\dfrac{2^{12}}{3^{12}}-12\times\dfrac{1}{3}\times\dfrac{2^{11}}{3^{11}}$

D $1-\dfrac{^{8}C_2}{^{12}C_2}-\dfrac{^{8}C_1 \times {}^{4}C_1}{^{12}C_2}$

E $\dfrac{^{8}C_1 \times {}^{4}C_1}{^{12}C_2}$

13 © VCAA 2005 1IQ4 64% At a party there are six unmarked boxes. Two boxes each have prizes, the other four boxes are empty. When two boxes are selected without replacement, the probability of selecting at least one box with a prize is

A $\dfrac{1}{15}$

B $\dfrac{6}{15}$

C $\dfrac{8}{15}$

D $\dfrac{5}{9}$

E $\dfrac{9}{15}$

14 © VCAA 2014 2AQ11 62% A bag contains five red marbles and four blue marbles. Two marbles are drawn from the bag, without replacement, and the results are recorded.

The probability that the marbles are different colours is

A $\dfrac{20}{81}$

B $\dfrac{5}{18}$

C $\dfrac{4}{9}$

D $\dfrac{40}{81}$

E $\dfrac{5}{9}$

15 © VCAA 2015 2AQ12 60% A box contains five red balls and three blue balls. John selects three balls from the box, without replacing them. The probability that at least one of the balls that John selected is red is

A $\dfrac{5}{7}$

B $\dfrac{5}{14}$

C $\dfrac{7}{28}$

D $\dfrac{15}{56}$

E $\dfrac{55}{56}$

16 © VCAA 2002 1IQ24 58% A bag contains 20 casino chips, which are used to represent cash. Five chips have a value of \$10 and the remainder have a value of \$5.

If 4 casino chips are drawn at random from the bag without replacement, what is the probability that there will be at least 1 chip of value \$10?

A $1-\dfrac{{}^{15}C_4}{{}^{20}C_4}$ **B** $\dfrac{{}^{5}C_1\times{}^{15}C_3}{{}^{20}C_4}$ **C** $\dfrac{1}{4}$

D $1-\left(\dfrac{3}{4}\right)^4$ **E** ${}^{4}C_1\times\left(\dfrac{1}{4}\right)\times\left(\dfrac{3}{4}\right)^3$

17 © VCAA 2003 2Q1d 70% (2 marks) A particular box of 25 rods has 4 rods in it which have **size faults**. A sample of 12 rods is withdrawn without replacement. Find the probability, correct to three decimal places, that the sample contains at least two rods with a size fault.

18 (2 marks) Adam, Luke, Michael, Ben, Anthony and Pat all play for the Melbourne Tigers Basketball Club. Their coach must pick 3 players for their 3-on-3 competitions. How many teams are possible if Michael must be selected?

Video playlist Applying counting methods to probability

5.7 Applying counting methods to probability

Many probability questions can be solved using counting methods. In these problems, we often need to determine if the number of possible outcomes is a permutation or a combination.

Probability problems involving counting methods

- If the question involves arrangements, where order is important, use the permutation methods.
- If the question involves choosing a subgroup, where order is not important, then use combination methods. The problem will often require the choice of a team or committee. The question may also be a selection without replacement.
- Find the number of ways of obtaining the required event.
- Find the number of outcomes where there is no restriction.
- Apply the rule $\Pr(A)=\dfrac{\text{number of ways of event } A \text{ occurring}}{\text{total number of outcomes with no restriction}}$.

WB p. 123

WORKED EXAMPLE 21 Finding the probability using counting methods

A box contains 8 books, of which 5 are fiction and 3 are non-fiction. Ben selects 3 books to read over the school holidays.

a Find the probability he selects at least one fiction book.

b Ben puts the 3 selected books on his shelf. Find the probability that the first and last books are fiction.

Steps	Working
a 1 This is a combination since Ben is CHOOSING 3 books from 8 books. For at least one fiction book, Ben would need to choose 1, 2 or 3 fiction books. It is easier to find the number of combinations where no fiction books are selected and then subtract the probability from 1.	Number of ways of choosing no fiction books and 3 non-fiction books $= {}^{5}C_0\times{}^{3}C_3$ $=1\times 1$ $=1$

2 Determine the number of combinations if there are no restrictions.	Number of ways of selecting a sample of 3 books from 8 books $= {}^8C_3$ $= 56$
3 Determine the probability of at least one fiction book using the rule: Pr(at least 1) = 1 – Pr(none).	$\Pr(\text{at least one fiction}) = 1 - \frac{1}{56} = \frac{55}{56}$
b 1 This is a permutation since Ben is ARRANGING 3 books on a shelf. Count the number of arrangements where the first and last books are fiction. There are 5 fiction books that can go in first place and 4 remaining fiction books that can go in last place. This leaves 6 possible books for the middle position.	Number of arrangements where the first and last books are fiction [5 | 6 | 4] $= 5 \times 6 \times 4$ $= 120$
2 Determine the number of permutations if there is no restriction.	Number of ways of permutations of 3 from a possible 8 books $= {}^8P_3$ $= 336$
3 Determine the probability.	$\Pr(\text{fiction at each end}) = \frac{120}{336} = \frac{5}{14}$

5.7

VCE QUESTION ANALYSIS

A VCE-style question (15 marks)

Video VCE question analysis: Probability and counting

A store blends tea using a mixture of three types of tea: black tea, white tea and oolong tea. These teas are often blended with dried fruits, flowers and spices.

The tea blender has four varieties of black tea, two varieties of white tea and two varieties of oolong tea in the blending room. Each variety is stored in airtight containers on the back wall of the store.

a How many arrangements of the tea containers are possible if

 i there is no restriction? 1 mark

 ii the black teas are always kept together? 1 mark

 iii the varieties of each type of tea are kept together? 1 mark

b Find the probability the varieties of black tea are always kept apart. 2 marks

The store often blends flavoured and scented teas with flowers and spices. The flowers used are jasmine, rose and orchid and the spices used are ginger, cardamom, anise and vanilla.

Customers can create their own individual blends from two options.

Blend 1: Choose a variety of black tea with one type of flower and one type of spice.

Blend 2: Choose any 2 varieties of black tea with 2 types of flower and 2 types of spice.

c Determine the number of different possible blends of

 i Blend 1 1 mark

 ii Blend 2 1 mark

d Find the probability a customer's recipe for blend 1 has the strongest variety of black tea with either jasmine or orchid, plus ginger. 2 marks

e Find the probability a customer's recipe for blend 2 has any two varieties of black tea with the flowers jasmine and rose and with the spices anise and vanilla. 2 marks

The store finds that 40% of customers who create their own blends pick blend 1. Of the customers that pick blend 1, 25% have rose in their mix, while 70% of the customers that pick blend 2 have rose in their mix.

f Find the probability that a customer purchases a blend with rose as an ingredient. 2 marks

g If a customer has a blend with rose as an ingredient, find the probability they ordered blend 2. 2 marks

Reading the question

- Note that the first section of the question is about permutations and the second section is about combinations.
- Highlight number and probability to indicate the type of answer required. Probability answers should be stated as fractions.
- Highlight the conditional probabilities for the tree diagram.
- Highlight the word 'if' in part **g**, which indicates a conditional probability.

Thinking about the question

- You will need to know how to count permutations where some items are kept in groups and also apply permutations to probabilities.
- You will need to know how to count combinations and apply this to finding probabilities.
- Be aware of the conditional probability formula and the use of a tree diagram to illustrate dependent probabilities.

Worked solution (✓ = 1 mark)

a **i** The total number of arrangements of the tea containers = 8! = 40 320 ✓

ii The total number of arrangements if the black teas are kept together = 5! × 4! = 2880 ✓

iii The total number of arrangements if the types of tea are kept together = 3! × 4! × 2! × 2! = 576 ✓

b Arrangements where black (B) tea and not black (NB) tea alternate:

B	NB	B	NB	B	NB	B	NB

or

NB	B	NB	B	NB	B	NB	B

$$\Pr(\text{black teas apart}) = \frac{(4 \times 4 \times 3 \times 3 \times 2 \times 2 \times 1 \times 1) \times 2}{8!} = \mathbf{\frac{1}{35}} \checkmark$$

Correct numerator or denominator ✓

c **i** The number of different combinations of blend 1 $= {}^4C_1 \times {}^3C_1 \times {}^4C_1$

$= \mathbf{48}$ ✓

ii The number of different combinations of blend 2 $= {}^4C_2 \times {}^3C_2 \times {}^4C_2$

$= \mathbf{108}$ ✓

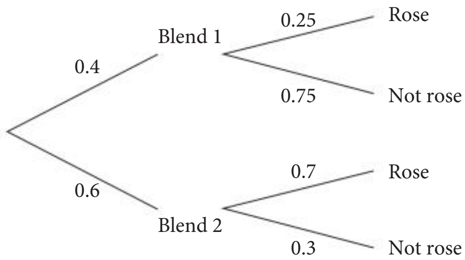
d Pr(strongest, jasmine or orchard, ginger) $= \dfrac{{}^1C_1 \times {}^2C_1 \times {}^1C_1}{48} = \dfrac{\mathbf{1}}{\mathbf{24}}$ ✓

Correct numerator or denominator ✓

e Pr(any tea, jasmine and rose, anise and vanilla) $= \dfrac{{}^4C_2 \times {}^2C_2 \times {}^2C_2}{108} = \dfrac{\mathbf{1}}{\mathbf{18}}$ ✓

Correct numerator or denominator ✓

f Tree diagram for dependent events ✓



Pr(rose) $= 0.4 \times 0.25 + 0.6 \times 0.7 = \mathbf{0.52}$ ✓

g $$\text{Pr(blend 2} \mid \text{rose)} = \frac{\text{Pr(blend 2 and rose)}}{\text{Pr(rose)}}$$

$$= \frac{0.6 \times 0.7}{0.52} \checkmark$$

$$= \frac{\mathbf{21}}{\mathbf{26}}$$

Correct numerator or denominator ✓

EXERCISE 5.7 Applying counting methods to probabilities

ANSWERS p. 516

Recap

1 An ice hockey team of 6 players is selected from 5 boys and 4 girls. The number of different teams that contain an equal number of boys and girls is

A 10 **B** 14 **C** 30 **D** 40 **E** 54

2 © VCAA 2014 2AQ11 MODIFIED A bag contains 6 red marbles and 4 blue marbles. Two marbles are drawn from the bag, without replacement, and the results are recorded.

The probability that the marbles are the same colour is

A $\dfrac{2}{45}$ **B** $\dfrac{1}{10}$ **C** $\dfrac{2}{9}$ **D** $\dfrac{7}{15}$ **E** $\dfrac{5}{9}$

Mastery

3 WORKED EXAMPLE 21 Code words of 5 letters are made from the letters of the word HEXAGON. Find the probability the vowels appear in first, third and fifth place.

4 WORKED EXAMPLE 21 A cricket team of 11 is selected from 7 bowlers, 8 batters and 3 wicketkeepers. Find the probability that at least 1 wicketkeeper is chosen.

5 TECH-FREE Harri is making mixed berry muffins and has a choice of using blueberries, blackberries, strawberries, raspberries, goji berries, boysenberries and acai berries. If she decides to randomly choose 3 of these berry types, find the probability the muffins contain

a 3 types of berries that start with the letter 'b'.

b 2 types of berries that start with the letter 'b'.

6 TECH-FREE Deni has 5 pencils and 3 pens in her pencil case. If she chooses 4 of them at random, find the probability she picks

a 3 pencils

b the same number of pencils as pens

c more pencils than pens

Exam practice

80–100% 60–79% 0–59%

7 TECH-FREE (2 marks) A number is made by selecting one or more digits at random from {1, 3, 5, 6}. Find the probability that the number is

a greater than 500 — 1 mark

b a four-digit even number — 1 mark

8 TECH-FREE (2 marks) Albert receives 10 Christmas cards, one from each of his four children, and the rest from friends. He displays six of these in a row on the shelf. Find the probability that the first four cards displayed are from his children.

9 © VCAA 2002 non-CAS 1IIQ2 61% TECH-FREE (2 marks) In a quality control procedure, a random sample of five motor car alarms is selected at random without replacement from a batch of 100 and tested. The batch is rejected if any of the five alarms is found to be defective.

A particular batch of 100 motor car alarms, which contains 5 defectives, is subjected to this quality control procedure. Find the probability, correct to 3 decimal places, that the batch is accepted.

10 The number of ways of selecting two girls and three boys from a group of four girls and six boys is

A $4! \times 6!$ **B** ${}^4C_2 \times {}^6C_3$ **C** $2! \times 3!$ **D** ${}^4P_2 \times {}^6P_3$ **E** 2×3

11 A 4-digit number is made from the numbers {0, 1, 2, 3, 4}. The total number of possibilities is

A $5!$ **B** ${}^4P_4$ **C** $4!$ **D** $4 \times 4 \times 3 \times 2$ **E** $5 \times 4 \times 3 \times 2$

12 The letters of the word PEACH are arranged in a row. The probability the arrangement starts with a vowel and ends with a vowel is

A $\frac{1}{5!}$ **B** $\frac{2!}{5!}$ **C** $\frac{1}{10}$ **D** $\frac{1}{5}$ **E** $\frac{2}{5}$

13 (2 marks) Four students are randomly chosen by a class each term to act as helpers. If there are 10 boys and 12 girls in the class, what is the probability of choosing at least as many girls as boys?

14 (2 marks) Astrid has 12 sports trophies in a box. Seven of these were awarded in netball and the other 5 were from hockey. She decides to display 4 of the trophies on a shelf in a cabinet. Find the probability that the only netball trophies displayed are at the ends.

15 (2 marks) Kevin, Josie and 10 other students are in a physical education class. The teacher randomly picks a pair of students to demonstrate an activity. Find the probability that the pair selected is Kevin and Josie.

5 Chapter summary

Experimental probability

Experimental probability of event (E) occuring:

$$\Pr(E) = \frac{\text{number of times event } E \text{ occurs}}{\text{total number of events}}$$

Theoretical probability

Theoretical probability of event (E) occuring:

$$\Pr(E) = \frac{\text{number of favourable outcomes for event } E}{\text{number of possible outcomes}}$$

Random variables

A random variable X is a variable whose possible values are the numerical result of some random process. If we roll a 6-sided die 10 times, the number of 4s that occur will be a random variable. In this situation, the random variable X represents the number of 4's that occur in 10 rolls of a die.

Simple event probability

A simple event is one that has a single outcome, like getting a tail on the toss of one coin or rolling a number greater than 3 on the roll of one die.

Compound events

A compound event is the combination of 2 or more simple events. Examples of compound events are tossing a coin 3 times or rolling 2 dice. The sample space for compound events can sometimes be found using a lattice diagram.

Probability simulation

Probability simulation uses methods of generating random numbers to help model real-world situations. Random numbers can be generated using physical means such as rolling a die or tossing a coin. They can also be generated using a spreadsheet or CAS.

The addition principle

- In an event in which we can choose a of one type OR b of another type OR c of another type, the total number of choices can be calculated by adding the number of choices of type a + the number of choices of type b + the number of choices of type c.
- In general, adding the number of possibilities is associated with the word 'OR' in a problem.

The multiplication principle

- When there are 2 or more successive selections to make and we must choose from a of one type AND then from b of another type, the total number of choices is the number of choices of type $a \times$ the number of choices of type b.
- In general, multiplying the number of possibilities is associated with the word 'AND' in a problem.

Permutations

- The number of permutations of a group of objects is the number of different ways the objects can be arranged.
- The number of permutations or arrangements of r objects chosen from n objects is ${}^nP_r$.

Factorials

- The mathematical notation for n factorial is $n!$ where $n! = n \times (n-1) \times (n-2) \times \ldots \times 3 \times 2 \times 1$.
- $n!$ is the number of arrangements of n different objects.

Arrangements with restrictions

The method for finding the number of permutations when there is a restriction is:

1 determine the number of places to be filled

2 treat the restriction first

3 fill all the spaces where the restriction is involved

4 fill the rest of the spaces from left to right.

Arrangements with groups

The method for finding the number of arrangements of objects where a group must be kept together is:

1 Put all the objects in the group together and count this group as one object.

2 Determine the number of arrangements of the objects.

3 Multiply this by the number of ways of arranging the objects within the group.

Combinations

- A combination is a selection of r objects chosen from n objects where the order of selection is NOT important. The mathematical symbol for this selection is ${}^nC_r$.
- ${}^nC_r = \dfrac{n!}{(n-r)!r!}$
- Using symmetry, ${}^nC_r = {}^nC_{n-r}$.

Applications of counting methods to probabilities

Many probability questions can be solved using counting methods. In these problems, we often need to determine if the number of possible outcomes is a permutation or a combination.

When analysing probability problems involving counting methods we need to follow these steps:

1. If the question involves arrangements, where order is important, use the permutation methods.

   If the question involves choosing a subgroup where order is not important then use the combination methods. This will often require the choice of a team or committee. The question may also be a selection without replacement.
2. Find the number of ways of obtaining the required event.
3. Find the number of outcomes where there is no restriction.
4. Apply the rule

$$\Pr(A) = \frac{\text{number of ways of event } A \text{ occurring}}{\text{total number of outcomes with no restriction}}.$$

Cumulative examination 1

Total number of marks: 10 Reading time: 4 minutes Writing time: 15 minutes

TECH-FREE *Technology is NOT permitted.*

1 (3 marks) Two brothers and three of their friends queue for tickets for a soccer game.

a In the queue, how many different arrangements of the five people are possible? 1 mark

b What is the probability that one brother is at the start and the other is at the end of their group? 2 marks

2 (3 marks) Find the coordinates of the points of intersection of the functions $f(x) = x + 1$ and $g(x) = x^2 - 5x + 2$.

3 (2 marks) Find the remainder when the cubic $x^3 - 4x^2 + x$ is divided by $2x - 1$.

4 (2 marks) A committee of 5 students is selected from 4 girls and 5 boys. Find the probability that 3 girls and 2 boys are selected.

Cumulative examination 2

Total number of marks: 20 Reading time: 4 minutes Writing time: 30 minutes

Approved technology is permitted.

Section A 5 multiple-choice questions

5 marks

1 Vehicle number plates consist of a single-digit number and two letters followed by a single-digit number and 2 letters. The number of different number plates with 'Z' as the last letter is

A $26^3 \times 10^2$ **B** $26^4 \times 10^2$ **C** $26^3 \times 9^2$

D $3 \times 26 + 2 \times 10$ **E** $4 \times 26 + 2 \times 10$

2 A three-digit number is made from the digits {1, 3, 4, 7, 8, 9}. The probability that the number is odd and greater than 900 is

A $\frac{1}{6!}$ **B** $\frac{1}{120}$ **C** $\frac{1}{20}$ **D** $\frac{1}{10}$ **E** $\frac{2}{15}$

3 © VCAA 2019N 2AQ11 MODIFIED The function $f: D \to R, f(x) = 5x^2 + 10x + 1$ will have an inverse function for

A $D = R$ **B** $D = (-2, \infty)$ **C** $D = \left(-\infty, \frac{1}{2}\right]$

D $D = (-\infty, -1]$ **E** $D = [-4, \infty)$

4 The transformations that map the function $y = 2f(4x)$ onto the function $y = f(x)$ are

A a dilation, factor 4, from the x-axis then a dilation, factor $\frac{1}{2}$, from the y-axis

B a dilation, factor 2, from the x-axis then a dilation, factor $\frac{1}{4}$, from the y-axis

C a dilation, factor $\frac{1}{2}$, from the x-axis then a dilation, factor 4, from the y-axis

D a dilation, factor $\frac{1}{4}$, from the x-axis then a dilation, factor 2, from the y-axis

E a dilation, factor 2, from the x-axis then a dilation, factor 4, from the y-axis

5 © VCAA 2019N 2AQ4 The graph of the function $f: D \to R, f(x) = \frac{2x-3}{4+x}$, where D is the maximal domain, has asymptotes

A $x = -4, y = 2$ **B** $x = \frac{3}{2}, y = -4$ **C** $x = -4, y = \frac{3}{2}$

D $x = \frac{3}{2}, y = 2$ **E** $x = 2, y = 1$

Section B 2 questions

15 marks

1 (7 marks) In a quality control procedure, to minimise faults in electric motors, a two-stage process is used.

Stage 1: A random sample of 5 motors is selected from a batch of 100 motors. If any of the motors in the sample are defective the batch is rejected.

Stage 2: If the sample of 5 motors from stage 1 contains exactly one defective motor, then a further sample of 5 motors is selected from the remaining motors in the batch. The batch is accepted if there are no defective motors found.

A particular batch of 100 motors contains 4 defective motors.

a Find the number of possible samples selected at stage 1. 1 mark

b Find the possible number of samples selected at stage 2. 1 mark

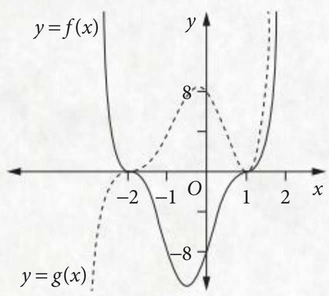
c Find the probability, correct to four decimal places, the batch is accepted at stage 1. 2 marks

d Find the probability, correct to four decimal places, that the stage 1 sample contains exactly one defective motor. 1 mark

e If the stage 1 sample contains exactly one defective motor, then stage 2 of the quality control is implemented. Find, correct to four decimal places, the probability the batch is accepted at stage 2. 1 mark

f Find the probability, correct to four decimal places, the batch of 100 motors passes the quality control test. 1 mark

2 © VCAA 2019N 2BQ1 (8 marks) Parts of the graphs of $f(x) = (x-1)^3(x+2)^3$ and $g(x) = (x-1)^2(x+2)^3$ are shown on the axes below.



The two graphs intersect at three points, $(-2, 0)$, $(1, 0)$ and (c, d). The point (c, d) is not shown in the diagram above.

a Find the values of c and d. 2 marks

b Find the values of x such that $f(x) > g(x)$. 1 mark

c Show that $f(1+m) = f(-2-m)$ for all m. 2 marks

d Find the values of h such that $g(x+h) = 0$ has exactly one negative solution. 2 marks

e Find, correct to three decimal places, the values of k such that $f(x) + k = 0$ has no solutions. 1 mark

CHAPTER

6 RATES OF CHANGE

Study Design coverage

UNIT 1, AREA OF STUDY 3: CALCULUS

- average and instantaneous rates of change in a variety of practical contexts and informal treatment of instantaneous rate of change as a limiting case of the average rate of change
- interpretation of graphs of empirical data with respect to rate of change such as temperature or pollution levels over time, motion graphs and the height of water in containers of different shapes that are being filled at a constant rate, with informal consideration of continuity and smoothness
- use of gradient of a tangent at a point on the graph of a function to describe and measure instantaneous rate of change of the function, including consideration of where the rate of change is positive, negative or zero, and the relationship of the gradient function to features of the graph of the original function.

UNIT 2, AREA OF STUDY 3: CALCULUS

- informal treatment of the gradient of the tangent to a curve at a point as a limit, and the limit definition of the derivative of a function $f'(x) = \lim_{h \to 0} \frac{f(x) - f(x-h)}{h} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$
- the central difference approximation $f'(x) \approx \frac{f(x+h) - f(x-h)}{2h}$ and its graphical interpretation.

Video playlists (5):
- **6.1** Average rate of change
- **6.2** Instantaneous rate of change
- **6.3** The gradient of a tangent
- **6.4** Applying rates of change
- **VCE question analysis** Rates of change

Worksheets (5):
- **6.2** Instantaneous rates of change • Rates of change: gradients of secants
- **6.3** Slopes of curves • Limits
- **6.4** Graphs of change

Nelson MindTap

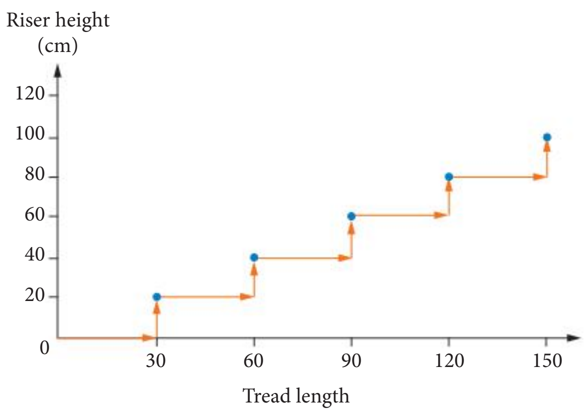
To access resources above, visit **cengage.com.au/nelsonmindtap**

6.1 Average rate of change

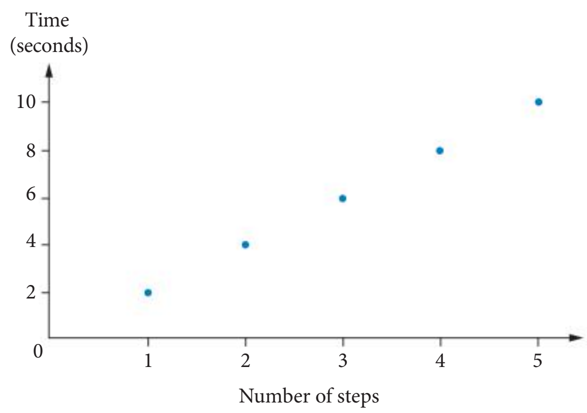
Video playlist Average rate of change

A **rate** indicates how one quantity (variable) changes as a result of a change in another quantity.

For example, suppose there are steps with 30-cm tread lengths (across) and 20-cm risers (up). We might walk up/down one step (one riser per one tread), which as a ratio will be $\frac{20}{30}$. Alternatively, if each step takes 2 seconds to climb, the rate of change will be 2 seconds/step.



Rate: 1 riser (20 cm)/1 tread (30 cm) $= \frac{20\text{ cm}}{30\text{ cm}} = \frac{2}{3}$



Rate: 2 seconds/step

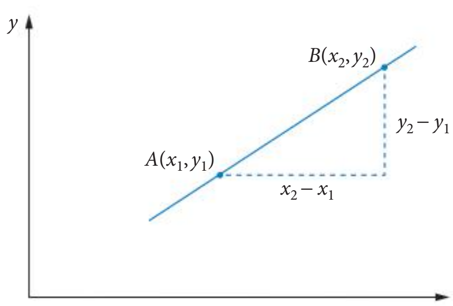
In each case, the rate of change is constant.

For variables x and y, with x as the **independent variable** and y as the **dependent variable**, the rate of change, $\dfrac{\Delta y}{\Delta x}$, is the rate of change in the dependent variable y with respect to the independent variable x, where the symbol Δ represents change in a quantity.

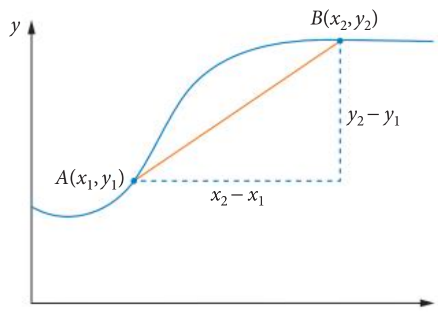
For a **linear function**, the rate of change is constant and represents the **gradient** of the straight line.

For 2 points $A(x_1, y_1)$ and $B(x_2, y_2)$ on the graph of a linear function, the rate of change between them is $\dfrac{\Delta y}{\Delta x} = \dfrac{y_2 - y_1}{x_2 - x_1}$.

For a **non-linear function**, $\dfrac{\Delta y}{\Delta x} = \dfrac{y_2 - y_1}{x_2 - x_1}$ is the **average rate of change** between the 2 points.

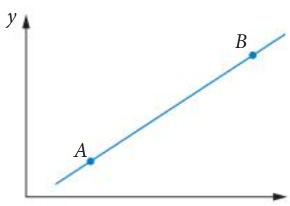


Rate of change
for a linear function



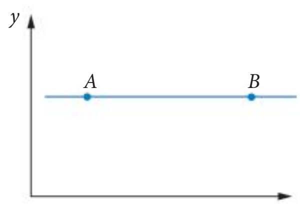
Average rate of change
for a non-linear function

A rate of change can be positive, zero or negative, depending on the gradient of the straight line connecting the 2 points.



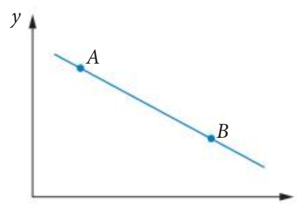
Positive

y = mass of puppy
x = age of puppy



Zero

y = cost of movie ticket
x = height of adult



Negative

y = fuel left
x = distance travelled

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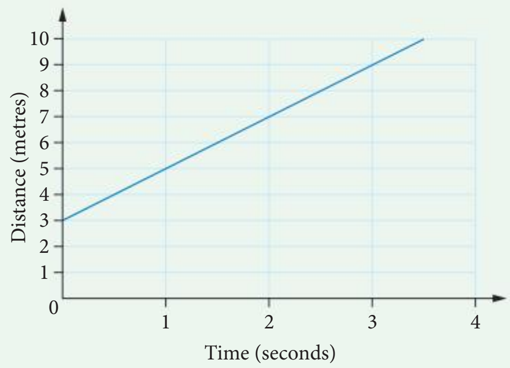
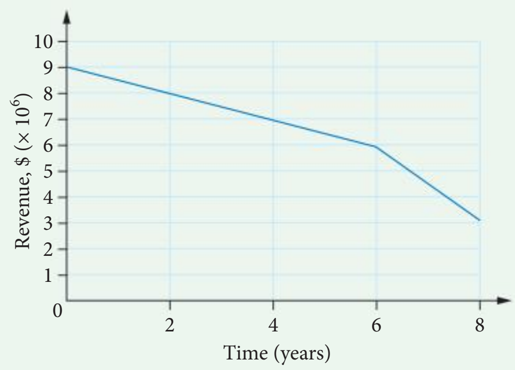
WORKED EXAMPLE 1 Constant rate of change

For each graph, calculate, where possible, the constant rate of change over the entire interval, also stating whether the rate of change is positive, zero or negative. If not possible, give a reason.

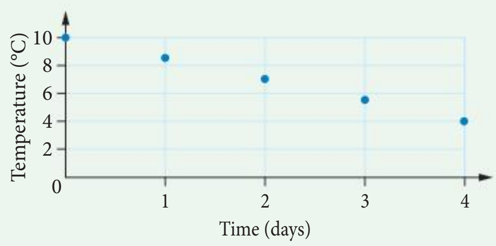
a



b



c



Steps	Working
a **1** Decide if the gradient is constant everywhere on the graph.	The line is increasing uniformly, so the gradient is constant and positive.
2 Calculate the gradient. Choose 2 points on the line, say (0, 3) and (3, 9), and calculate the gradient of the line.	$\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$ $= \frac{9-3}{3-0}$ $= 2$
3 State the rate of change.	The rate of change is 2 metres/second.
b **1** Decide if the gradient is constant everywhere on the graph.	The gradient in the interval from 0 to 6 years is different to the gradient from 6 to 8 years, so the gradient is not constant.
2 State the rate of change.	Since the gradient is not constant, there is no constant rate of change.
c **1** Decide if the gradient is constant everywhere on the graph.	A line connecting the points is decreasing uniformly, so the gradient is constant and negative.
2 Use the points (0, 10) and (4, 4) to calculate the gradient of the line.	$\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$ $= \frac{4-10}{4-0}$ $= -1.5$
3 State the rate of change.	The rate of change is −1.5°C/day.

6.1

WORKED EXAMPLE 2 Rate of change from a linear graph

WB p. 126

The table shows the height, to the nearest centimetre, of a tomato plant over a 6-month period.

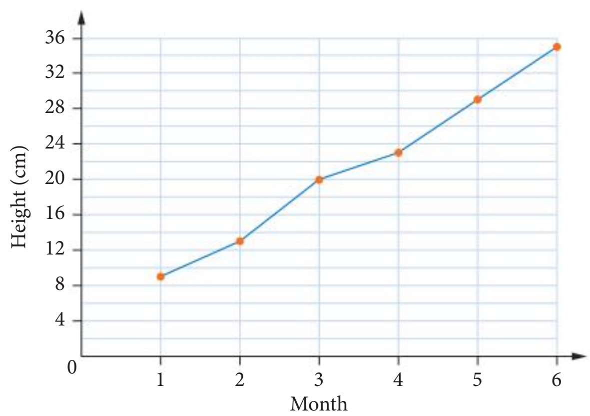
Month	1	2	3	4	5	6
Height (cm)	9	13	20	23	29	35

a Construct a line graph of the data.

b Assuming the graph to be a linear function, describe the type of relationship that exists.

c Calculate the average growth rate of the tomato plant using the first and last points.

d Calculate the growth rate using the middle 2 points and compare it to your answer to part **c**.

Steps	Working
a Plot and connect the points and decide whether the shape generally forms a straight line.	
b Describe the relationship.	The points approximately form a straight line with a positive gradient.

c Calculate the required gradient.	Use the points (1, 9) and (6, 35). $\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{35 - 9}{6 - 1} = \frac{26}{5} = 5.2$ The average growth rate of the tomato plant is 5.2 cm/month.
d **1** Calculate the required gradient.	Use the points (3, 20) and (4, 23). $\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{23 - 20}{4 - 3} = 3$ The average growth rate of the tomato plant is 3 cm/month.
2 Compare answers.	This is significantly less than using the first and last data points because the line joining the middle 2 points is 'flatter' than the line joining the endpoints.

Exam hack

If x and y are independent and dependent variables respectively, it is common to express the rate of change of y with respect to x. Read the question carefully to determine the correct order.

When a graph is not linear, such as a parabola or cubic curve, we can calculate the *average* rate of change.

WB p. 127

WORKED EXAMPLE 3 Average rate of change

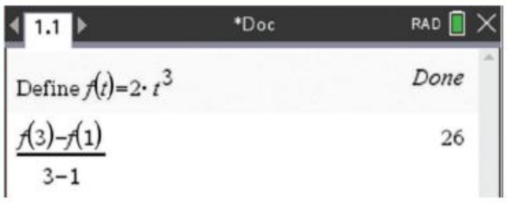
The position of a particle is described by the equation $x = 2t^3$, where x is in metres and t is in minutes. Find the average rate of change of the particle's position between $t = 1$ and $t = 3$.

Steps	Working
1 Work out the required x values.	When $t = 1$, $x = 2 \times 1^3 = 2$. When $t = 3$, $x = 2 \times 3^3 = 54$.
2 Write the coordinates.	$A(t_1, x_1) = (1, 2)$ $B(t_2, x_2) = (3, 54)$
3 Calculate the gradient.	$\frac{\Delta x}{\Delta t} = \frac{x_2 - x_1}{t_2 - t_1} = \frac{54 - 2}{3 - 1} = 26$
4 State the average rate of change.	The average rate of change between $t = 1$ and $t = 3$ is 26 metres/minute.

USING CAS 1 Average rate of change

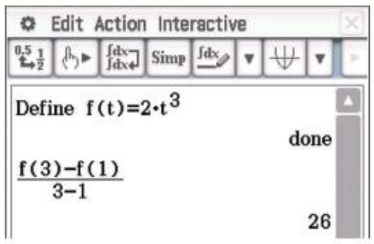
Calculate the average rate of change of $x = 2t^3$, where x is in metres and t is in minutes, between $t = 1$ and $t = 3$.

TI-Nspire



1. Press **menu > Actions > Define** and define the function.
2. Use the formula $\dfrac{f(b)-f(a)}{b-a}$ to calculate the average rate of change.

ClassPad



1. In **Main**, enter and highlight the expression $\mathbf{2t^3}$.
2. Tap **Interactive > Define** and change the **Variable/s** to ***t***.
3. Use the formula $\dfrac{f(b)-f(a)}{b-a}$ to calculate the average rate of change.

Rate of change

The **rate of change** is the gradient of a straight-line segment.

The rate of change can be positive, zero or negative.

The rate of change between the points $A(x_1, y_1)$ and $B(x_2, y_2)$ is $\dfrac{\Delta y}{\Delta x} = \dfrac{y_2 - y_1}{x_2 - x_1}$.

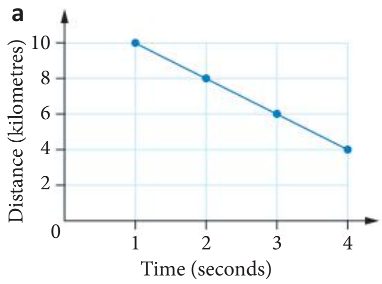
EXERCISE 6.1 Average rate of change

ANSWERS p. 517

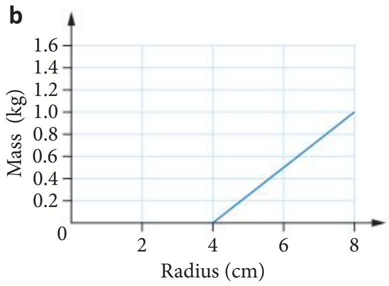
Mastery

1 WORKED EXAMPLE 1 TECH-FREE For each graph, calculate, where possible, the rate of change over the entire interval, also stating whether it is positive, zero or negative. If not possible, give a reason.

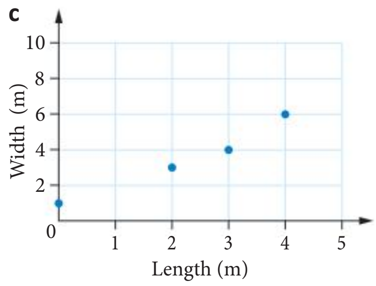
a



b



c



2 WORKED EXAMPLE 2 TECH-FREE The length, in centimetres, of a lighted candle as it burns is shown in the table below.

Time (minutes)	0	6	12	15	18
Length (cm)	25	22	20	18	16

a Construct a line graph of the data.

b Explain why the graph is not linear.

c Assuming the graph to be a linear function, describe the type of relationship that exists.

d Calculate the average decrease in the rate of the length of the candle using the first and last points.

e Calculate the rate of decrease using the last 2 points and compare it to your answer for part **d**.

3 Using CAS 1 Determine the average rate of change of $y = x - \frac{1}{2}x^2$ between $x = 4$ and $x = 8$.

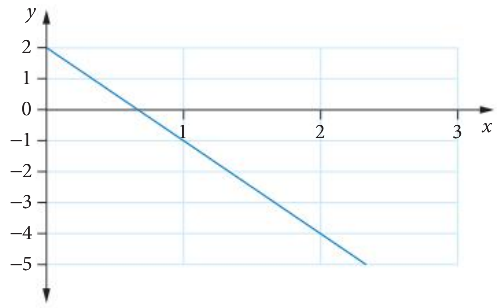
Exam practice

80–100% 60–79% 0–59%

4 WORKED EXAMPLE 3 TECH-FREE (2 marks) The period, T seconds, of a pendulum of length L metres can be approximated by the rule $T = 2\sqrt{L}$.

Find the average rate of change of the pendulum's period between $L = 4$ and $L = 9$.

5 The gradient of this line is



A −3 **B** 0.6 **C** 1 **D** 2 **E** 3

6 After 1.5 minutes, an initially full 4-litre bucket now has 3.4 litres of water remaining. The average rate of change of water volume was

A $-\frac{34}{15}$ litres/minute **B** $-\frac{5}{3}$ litres/minute **C** $\frac{2}{17}$ litres/minute

D $-\frac{2}{5}$ litres/minute **E** $\frac{5}{4}$ litres/minute

7 The gradient of the line connecting the point (–3, 4) to the point (2, –6) is

A –2 **B** $-\frac{2}{5}$ **C** $\frac{2}{5}$ **D** $\frac{1}{2}$ **E** 2

8 Choose the correct statement.

A A negative rate of change means both variables decrease at the same time.

B For the equation $x = 3y$, the average rate of change of y with respect to x is $\frac{\Delta x}{\Delta y}$.

C The gradient of a line of best fit is the average rate of change.

D If M is the dependent variable, its rate of change can be represented as $\frac{\Delta N}{\Delta M}$.

E For a linear graph, the rate of change is always positive.

9 © VCAA 2012 2AQ2 70% For the function with the rule $f(x) = x^3 - 4x$, the average rate of change of $f(x)$ with respect to x on the interval [1, 3] is

A 1 **B** 3 **C** 5 **D** 6 **E** 9

10 © VCAA 2010 2AQ2 69% For $f(x) = x^3 + 2x$, the average rate of change with respect to x for the interval [1, 5] is

A 18 **B** 20.5 **C** 24 **D** 32.5 **E** 33

11 The position x cm of a particle at time t seconds is described by $x = \frac{3}{4}(2t + 1)$. Its average rate of change of distance with respect to time for the first 4 seconds will be

A 1 **B** 1.5 **C** 1.5 cm/s **D** 2.25 cm **E** 2.25 cm/s

12 (2 marks) Calculate the average rate of change between the points $x = \frac{1}{a}$ and $x = \frac{1}{b}$ using $f(x) = \frac{1}{x}$.

13 (3 marks) A particle's position, P metres, at time t seconds can be described by $P = t^2$.

a Find the average rate of change between $t = 1$ and $t = 2$. 1 mark

b Find the average rate of change between $t = 4$ and $t = 5$. 1 mark

c Find the average rate of change of the particle between $t = t_1$ and $t = t_2$. 1 mark

14 (3 marks) From Monday to Friday, Fiona's sales amount is 1.5 times the sales amount of the previous day. On Monday, her sales amount was 400 units.

a Find the average increase in sales between Monday and Wednesday. 1 mark

b Find the average increase in sales between Monday and Friday. 1 mark

c Compare and comment on the ratio of your answers to parts **a** and **b** with 1.5. 1 mark

15 (2 marks) The value, $\$V$, of an investment after t years is given by $V = t^3$.

Show that the average rate of change between 2 consecutive years starting with month t_1 is $3t_1^2 + 3t_1 + 1$.

6.2

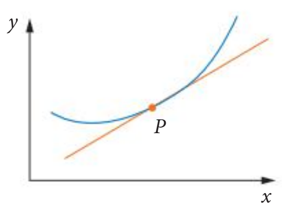
6.2 Instantaneous rate of change

Video playlist Instantaneous rate of change

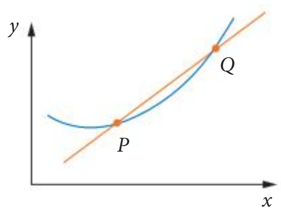
Worksheets Instantaneous rates of change

Rates of change: gradients of secants

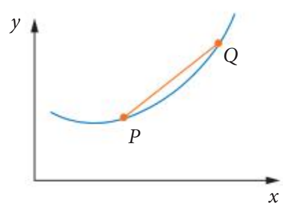
A **tangent** is a straight line that touches a curve at one point. A straight line passing through 2 points on a curve is a **secant** and the interval (line segment) joining 2 points on a curve is a **chord**.



Tangent at point P

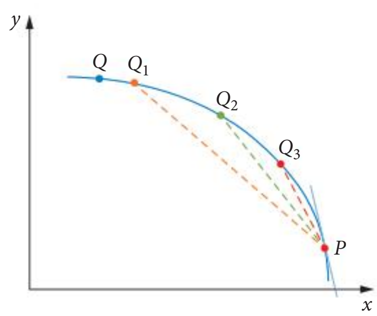


Secant passing through P and Q



Chord from P to Q

In the diagram below, point Q is moving towards point P from the left, and Q_1, Q_2, Q_3 … represent the positions of Q as it approaches P.



The **instantaneous rate of change** at P is the gradient of the tangent line at P.

The gradients of the chords, Q_1P, Q_2P, Q_3P … are approximations for the instantaneous rate of change at P.

WB

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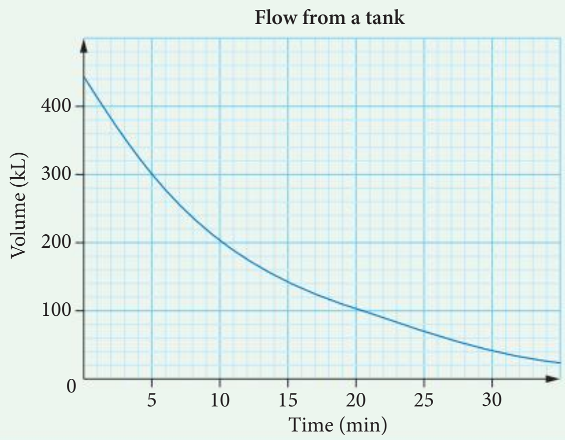
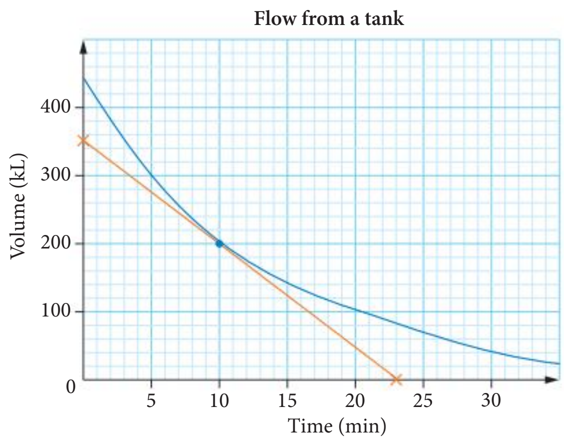
WORKED EXAMPLE 4 Finding the tangent from a graph

This graph shows the amount of water in a tank as it is being drained. Use a tangent to the graph to find the flow rate after 10 minutes.



Steps	**Working**
1 Use a transparent ruler and pencil to draw a tangent at 10 minutes.	
2 Find 2 points on the tangent.	(0, 350) and (23, 0) are on the tangent line.
3 Find the gradient of the tangent.	$m = \frac{y_2 - y_1}{x_2 - x_1}$ $= \frac{0 - 350}{23 - 0}$ ≈ -15 kL/min
4 Write the answer.	The water is flowing out at about 15 kL/min.

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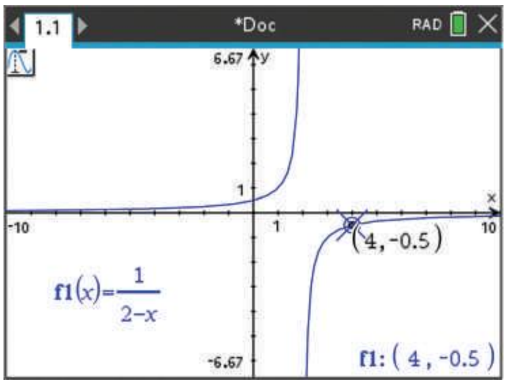
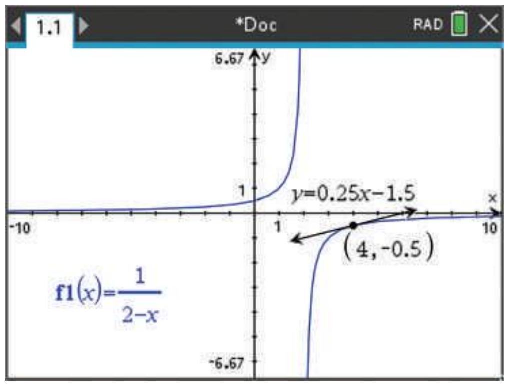
USING CAS 2 Instantaneous rate of change

Graph $f(x) = \dfrac{1}{2 - x}$ and calculate the rate of change at $x = 4$ using the tangent.

TI-Nspire





1 Add a **Graphs** page and graph the function.

2 Press **menu > Trace > Graph Trace**.

3 Enter **4** and press **enter** to jump to the point on the graph where $x = 4$.

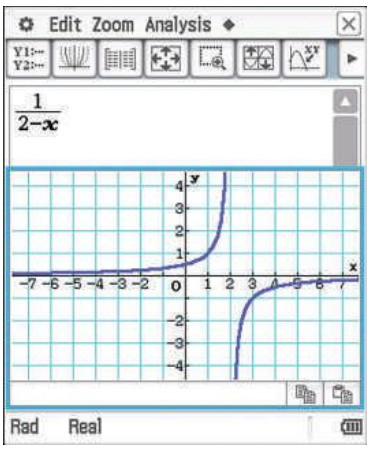
4 Press **enter** to label the coordinates of this point.

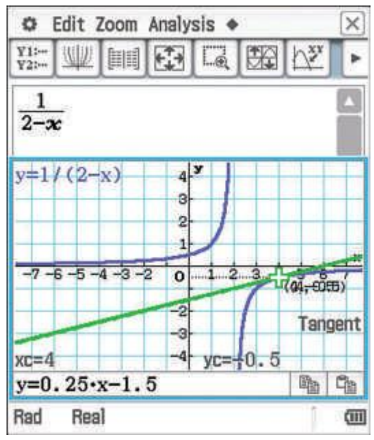
5 Press **menu > Geometry > Points & Lines > Tangent**.

6 Click on the point where $x = 4$ to display the tangent line at this point.

7 Press **enter** to display the equation of the tangent line.

ClassPad





1 In **Main**, enter and highlight the expression.

2 Tap **Graph** and drag the expression down into the Graph window.

3 Tap **Analysis > Sketch > Tangent**.

4 Enter **4** then tap **OK** to display the tangent line when $x = 4$.

5 Press **EXE** to display the equation of the tangent line.

The equation of the tangent line at $x = 4$ is $y = 0.25x - 1.5$.

Therefore, the instantaneous rate of change is 0.25, which is the slope of the tangent line.

WB

p. 129

WORKED EXAMPLE 5 Approximating the instantaneous rate of change

Let $y = 3x^2$.

a Determine approximations for the instantaneous rate of change of y with respect to x between $x = 4$ and $x = 4 + h$, using $h = 0.1, 0.01, 0.001$.

b What value will the approximation approach as the value of h approaches 0?

Steps	Working
a 1 Work out the required y values.	$x = 4, y = 3 \times 4^2 = 48$ $h = 0.1, x = 4.1, y = 3 \times 4.1^2 = 50.43$ $h = 0.01, x = 4.01, y = 3 \times 4.01^2 = 48.2403$ $h = 0.001, x = 4.001, y = 3 \times 4.001^2 = 48.024\,003$
2 Calculate each gradient. Use gradient $= \dfrac{y_2 - y_1}{x_2 - x_1}$.	$h = 0.1$, gradient $= \dfrac{50.43 - 48}{4.1 - 4} = 24.3$ $h = 0.01$, gradient $= \dfrac{48.2403 - 48}{4.01 - 4} = 24.03$ $h = 0.001$, gradient $= \dfrac{48.024\,003 - 48}{4.001 - 4} = 24.003$
b Look at the sequence of approximations.	The values of the approximations are 24.3, 24.03, 24.003. These approach 24 as h approaches 0.

EXERCISE 6.2 Instantaneous rate of change

ANSWERS p. 517

Recap

1 For the function $y = \sqrt{x - 1}$, the average rate of change of y with respect to x from $x = 5$ to $x = 10$ is

A $\frac{1}{5}$ **B** $\frac{2}{5}$ **C** 1 **D** 3 **E** 5

2 For the function $xy = 1$, the average rate of change of y with respect to x from $x = a^2$ to $x = b^2$ is

A $\frac{1}{ab}$ **B** $-ab$ **C** $\frac{ab}{a+b}$

D $-\frac{1}{a^2b^2}$ **E** $-\frac{a+b}{a-b}$

Exam hack

To use the expression that gives the average rate, the function must be in the form $y = f(x)$.

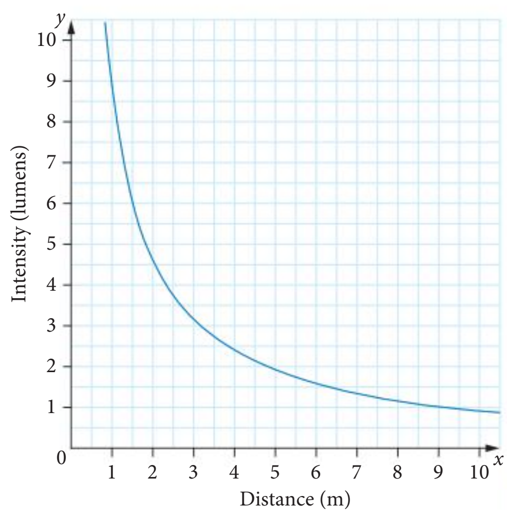
Mastery

3 WORKED EXAMPLE 4 The graph displays the intensity of light (lumens) at a given distance from a candle source.

Calculate the rate of change, correct to one decimal place, of the light intensity at a distance of

a 2 m

b 9 m



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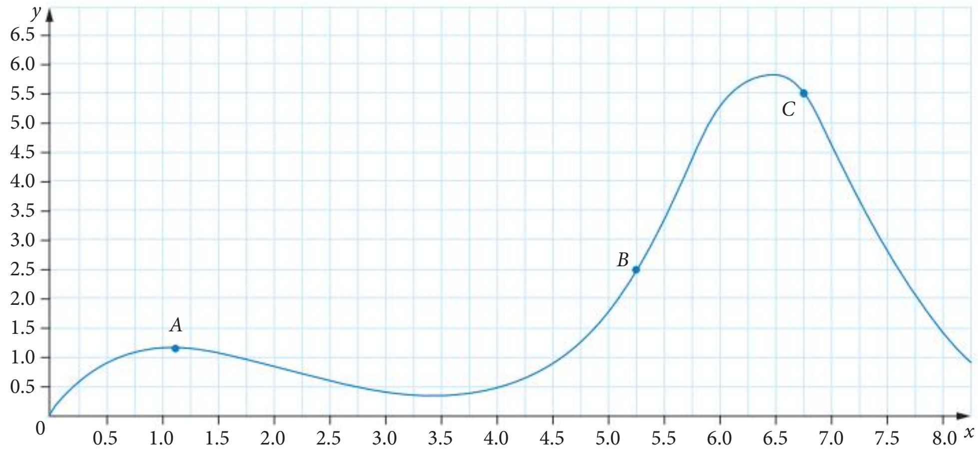
4 By drawing tangent lines, determine the instantaneous rate of change at A, B and C shown in the graph.



6.2

5 Using CAS 2 Use CAS to sketch the graph of $f(x) = x^3 - 3x^2 - 4x + 12$ and calculate the rate of change using a tangent line at $x = 3.5$.

Exam practice

80–100% 60–79% 0–59%

6 TECH-FREE (3 marks)

a State the average rate of change of the function $f(x) = ax^2$ (a is a constant) in the interval $[3, 3 + h]$. 2 marks

b Hence, state the value of a if the instantaneous rate of change is 18. 1 mark

7 TECH-FREE (4 marks)

a Find the average rate of change of the function $f(x) = \frac{1}{x+1}$ from $x = a$ to $x = a + h$, where a and h are numbers. 2 marks

b Hence, by letting the value of h approach zero, show that the instantaneous rate of change is $-\frac{1}{(a+1)^2}$. 2 marks

8 Choose the correct statement.

A The tangent to any point on a graph intersects the graph at 2 points.

B The length of a chord cannot be greater than its corresponding secant.

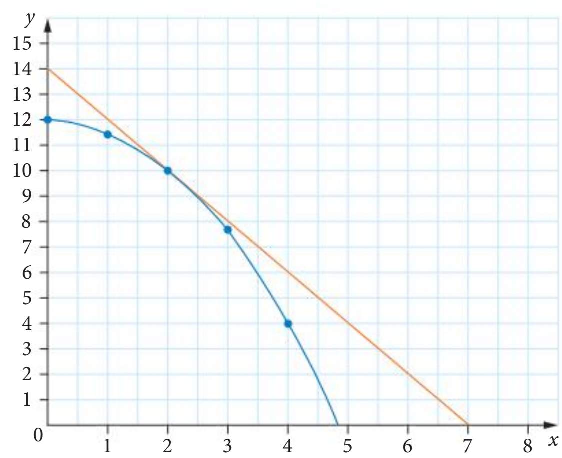
C A chord on a curve cannot be a tangent elsewhere on the same curve.

D Two points on a curve cannot have a tangent that has the same gradient.

E The gradient of a secant cannot be negative.

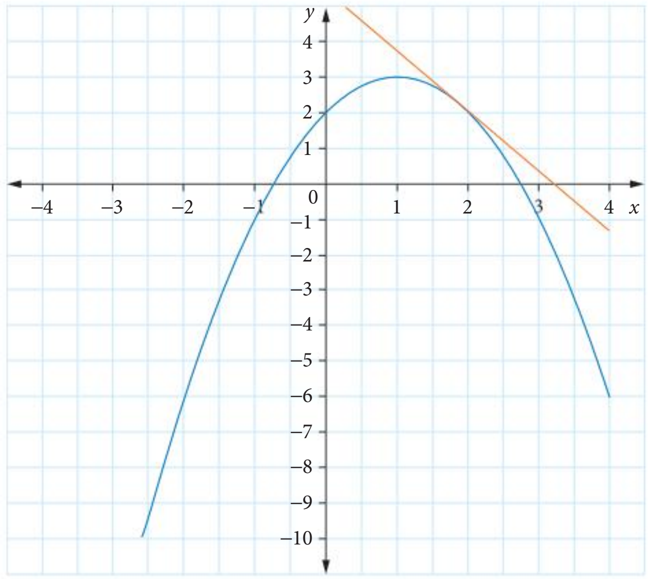
9 The instantaneous rate of change at (2, 10) on the graph is

A –7 B –2 C 5 D 7 E 10



10 The instantaneous rate of change at $x = 2$ on the graph below is closest to

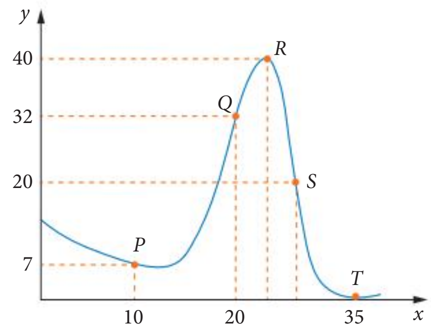
A -2 **B** $-\frac{1}{2}$ **C** $\frac{1}{2}$ **D** 2 **E** 3



11 The gradient of the secant from $x = 10$ to $x = 10.1$ on the curve $y = x^3$ is

A 0.30301 **B** 3.0301 **C** 30.301 **D** 303.01 **E** 3030.1

12 At which point, P, Q, R, S or T on the graph below, is the instantaneous rate of change approximately -0.4?



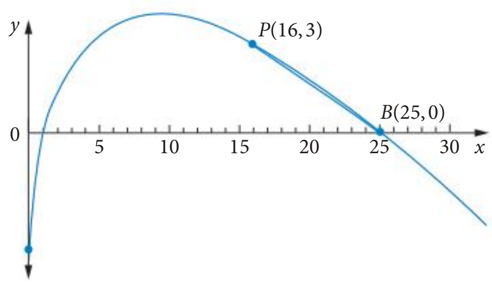
A R **B** S **C** T **D** P **E** Q

13 WORKED EXAMPLE 5 (3 marks) Let $y = x^3$.

a Determine approximations to the instantaneous rate of change of y with respect to x between $x = 5$ and $x = 5 + h$, using $h = 0.1, 0.01, 0.001, 0.0001$. 2 marks

b What value will the approximations approach as the value of h approaches 0? 1 mark

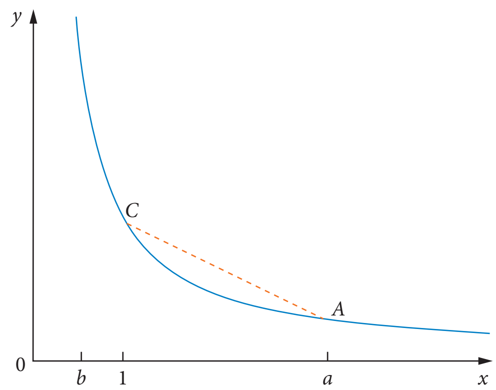
14 © VCAA 2009 2BQ1di 80% (1 mark) The points $P(16, 3)$ and $B(25, 0)$ are labelled on the diagram.



Find m, the gradient of the chord PB. (Exact value to be given.)

15 © VCAA 2008 2BQ2a **50%** (1 mark) The diagram shows part of the graph of the function

$f: R^{+} \to R, f(x) = \dfrac{7}{x}$.



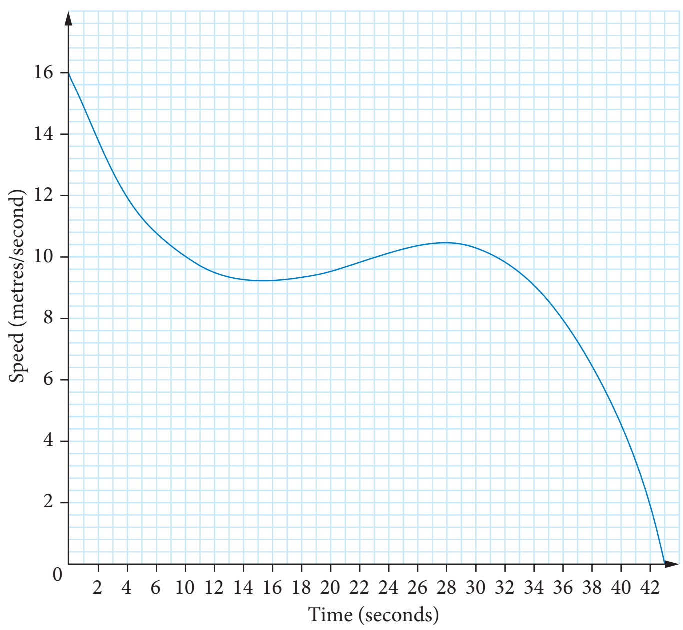
The line segment CA is drawn from the point $C(1, f(1))$ to the point $A(a, f(a))$, where $a > 1$.

Calculate the gradient of CA in terms of a.

Exam hack

Simplify the algebraic fraction obtained as far as possible.

16 (6 marks) A car's speed, measured in m/s and as a function of time, is shown in the graph.



Average acceleration is the change in speed with respect to time.

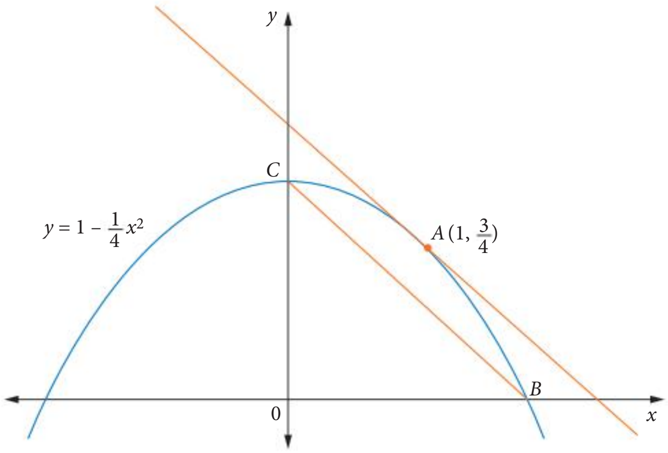
a Determine the car's average acceleration during the first 10 seconds. 2 marks

b What was the average acceleration of the car between 1 second and 4 seconds? 2 marks

c **i** At what time(s) was the acceleration of the car zero? 1 mark

ii What was happening to the car at these times? 1 mark

17 (3 marks) An arch is in the shape of a parabola with equation $y = 1 - \frac{1}{4}x^2$, as shown. Points B and C are the positive x-intercept and y-intercept respectively of the graph of the parabola. There is a tangent line to the parabola at point A, and the tangent is parallel to a string connecting point B to point C.
What is the instantaneous rate of change of y with respect to x at point A?



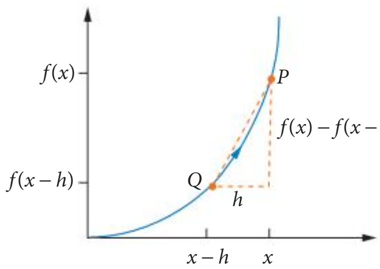
Video playlist The gradient of a tangent

Worksheet Slopes of curves

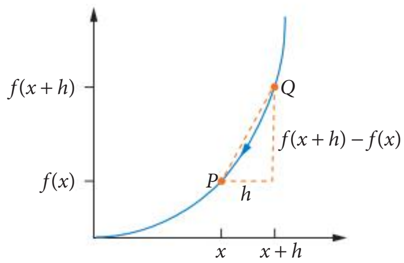
6.3 The gradient of a tangent

Approximating the rate of change

In the previous section, we looked at how the instantaneous rate of change at a point P on a curve is obtained by letting point Q approach P from the left. We can also approach P from the right.



From the left



From the right

Approximate rate of change

For a small increase or decrease h to the value of x, the approximate rate of change at P is:

$$\frac{\Delta y}{\Delta x} = \frac{f(x) - f(x-h)}{h} \text{ (from the left)}$$

$$\frac{\Delta y}{\Delta x} = \frac{f(x+h) - f(x)}{h} \text{ (from the right).}$$

WORKED EXAMPLE 6 Approximating the rate of change from the right

Find the approximate rate of change of $f(x) = x^3 - x^2 + 2x + 3$ between the points with x-coordinates $x = 2$ and $x = 2.1$, approaching $x = 2$ from the right.

Steps	Working
1 Work out the value of h.	Find the difference between the two x values. $h = 2 - 2.1 = 0.1$
2 Evaluate Δy.	Find the difference between the two y values. $\Delta y = f(x+h) - f(x)$ $= f(2.1) - f(2)$ $= 12.051 - 11$ $= 1.051$
3 Calculate $\frac{\Delta y}{\Delta x}$.	$\frac{\Delta y}{\Delta x} = \frac{f(x+h) - f(x)}{h} = \frac{1.051}{0.1} = 10.51$
4 State the answer.	The approximate rate of change is 10.51.

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WORKED EXAMPLE 7 Approximating the rate of change from the left

Use $h = 0.2$ to find the approximate rate of change of $f(x) = 5x^2 - x + 1$ at $x = 3$ by approaching $x = 3$ from the left.

Steps	Working
1 Find $x - h$.	$x - h = 3 - 0.2 = 2.8$
2 Calculate $f(x - h)$ and $f(x)$.	$f(x-h) = f(2.8) = 5(2.8)^2 - 2.8 + 1 = 37.4$ $f(x) = f(3) = 5(3)^2 - 3 + 1 = 43$
3 Calculate $f(x) - f(x - h)$.	$f(x) - f(x-h) = 43 - 37.4$ $= 5.6$
4 Calculate $\frac{\Delta y}{\Delta x}$.	$\frac{\Delta y}{\Delta x} = \frac{f(x) - f(x-h)}{h} = \frac{5.6}{0.2} = 28$
5 State the answer.	The approximate rate of change is 28.

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By letting the value of h approach zero, the rate of change approximations $\frac{\Delta y}{\Delta x} = \frac{f(x) - f(x-h)}{h}$ and $\frac{\Delta y}{\Delta x} = \frac{f(x+h) - f(x)}{h}$ approach the instantaneous rate of change or the **derivative**, $f'(x)$, at point x.

Thus, we have

Worksheet
Limits

The derivative of a function $f(x)$

$$f'(x) = \lim_{h \to 0} \frac{f(x) - f(x-h)}{h} \quad \text{(from the left)}$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \quad \text{(from the right)}$$

The value of the derivative can be determined, as shown in Using CAS 2: Instantaneous rate of change, or it can be deduced by approaching the given value of x using a number of small h values.

WB

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WORKED EXAMPLE 8 The derivative at a point

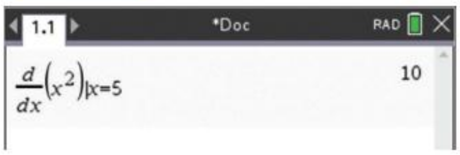
a Use $h = 0.1, 0.01, 0.001$ to find approximations for the instantaneous rate of change of $f(x) = x^2$ at $x = 5$ by approaching $x = 5$ from the right.

b Hence, state the derivative of $f(x)$ at $x = 5$.

c Verify your answer using CAS.

Steps	Working
a Find each approximation using $\frac{\Delta y}{\Delta x} = \frac{f(x+h) - f(x)}{h}$.	$h = 0.1$ $\frac{f(x+h) - f(x)}{h} = \frac{(5+0.1)^2 - 5^2}{0.1} = 10.1$ $h = 0.01$ $\frac{f(x+h) - f(x)}{h} = \frac{(5+0.01)^2 - 5^2}{0.01} = 10.01$ $h = 0.001$ $\frac{f(x+h) - f(x)}{h} = \frac{(5+0.001)^2 - 5^2}{0.001} = 10.001$
b Deduce the derivative at the required value.	As h becomes smaller, the rates of change approach 10, so the instantaneous rate of change is 10. That is, $\lim\limits_{h \to 0} \frac{f(5+h) - f(x)}{h} \to 10$, so the derivative at $x = 5$ is given as $f'(5) = 10$.

c **TI-Nspire**

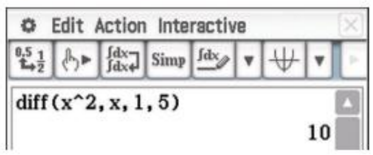


1 Press **menu > Calculus > Derivative at a Point**.

2 In the dialogue box **Value:** field, enter **5**.

3 In the template, enter $\boldsymbol{x^2}$.

ClassPad



1 Enter and highlight $\boldsymbol{x^2}$.

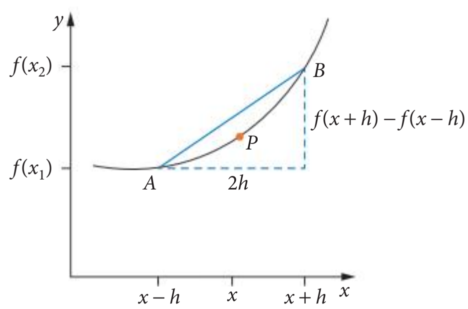
2 Tap **Interactive > Calculation > diff**.

3 In the dialogue box, tap to select **Derivative at value**.

4 In the **Value:** field, enter **5**.

Central difference approximation

Another way to estimate the rate of change at a point is to use **central difference approximation**. Two equidistant points from point x ($x_1 = x - h$ and $x_2 = x + h$) represent the endpoints of a line segment whose gradient can be determined. Here, h is a given value whose magnitude may affect the accuracy of the estimate.



The gradient of the line segment AB is an estimate of the gradient at point x. The gradient of AB is $\frac{\Delta y}{\Delta x} = \frac{f(x+h) - f(x-h)}{2h}$.

The central difference approximation

$$f'(x) \approx \frac{f(x+h) - f(x-h)}{2h}$$

6.3

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WORKED EXAMPLE 9 Central difference approximation

a Use $h = 0.5$ to find the central difference approximation of the gradient of $f(x) = x^2 + x$ at $x = 3$.

b Evaluate the gradient by approaching the function from the left and from the right.

c Compare the 3 methods, given that the exact value of the gradient is 6.

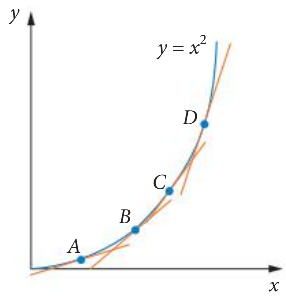
Steps	Working
a 1 Use the given value of h to evaluate $f(x+h)$ and $f(x-h)$.	$x = 3, f(x) = x^2 + x$ $h = 0.5$ $f(x+h) = (3.5)^2 + 3.5 = 15.75$ $f(x-h) = (2.5)^2 + 2.5 = 8.75$
2 Apply the formula.	$f'(x) \approx \frac{f(x+h) - f(x-h)}{2h}$ $= \frac{15.75 - 8.75}{2 \times 0.5}$ $= 7$
b 1 To approach from the left, find $f(x)$, $f(x-h)$ and apply the formula.	$f(x) = f(3) = 12$ $f(x-h) = f(2.5) = 8.75$ $f'(x) \approx \frac{f(x) - f(x-h)}{h}$ $= \frac{12 - 8.75}{0.5}$ $= 6.5$
2 To approach from the right, find $f(x)$, $f(x+h)$ and apply the formula.	$f(x) = 12$ $f(x+h) = f(3.5) = 15.75$ $f'(x) \approx \frac{f(x+h) - f(x)}{h}$ $= \frac{15.75 - 12}{0.5}$ $= 7.5$
c Compare each approximation to the exact answer.	The central difference method gave the exact answer. The approach from the left was less than the exact gradient, and the approach from the right gave a greater value than the gradient.

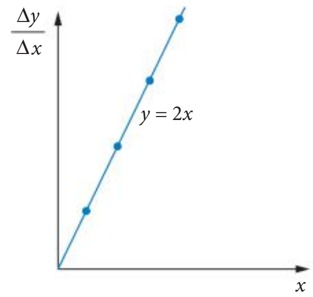
The gradient function

The **gradient function** gives the rate of change, or the gradient of the tangent, for any value of x.

In the first graph below, the tangents at points A, B, C and D represent the instantaneous rate of change on the curve $y = x^2$ at each point. Notice that the gradient becomes steeper as we move from A to D.

This means that the rate of change of the gradient is increasing. Plotting each gradient, $\frac{\Delta y}{\Delta x}$, with respect to its x-coordinate produces a linear graph of gradient 2 (the second graph below).





To confirm this, suppose we approach each point from the right and calculate the rate of change of the function $y = f(x) = x^2$ using the x values 0, 1, 2, 3, 4 and $h = 0.02$.

The results for $\frac{\Delta y}{\Delta x} = \frac{f(x+h) - f(x)}{h}$ are summarised below.

$x = 0, \quad \frac{\Delta y}{\Delta x} = \frac{(0 + 0.02)^2 - 0}{0.02} \approx 0$

$x = 1, \quad \frac{\Delta y}{\Delta x} = \frac{(1 + 0.02)^2 - 1}{0.02} \approx 2$

$x = 2, \quad \frac{\Delta y}{\Delta x} = \frac{(2 + 0.02)^2 - 2}{0.02} \approx 4$

$x = 3, \quad \frac{\Delta y}{\Delta x} = \frac{(3 + 0.02)^2 - 3}{0.02} \approx 6$

$x = 4, \quad \frac{\Delta y}{\Delta x} = \frac{(4 + 0.02)^2 - 4}{0.02} \approx 8$

The results show that the rate of change is twice the value of the x-coordinate.

If we let $R(x)$ be the gradient function, then $R(x) = 2x$.

Exam hack

The relationship between the x value and its rate of change may not be immediately obvious. Depending on h and the x values used, the rate of change may be a decimal requiring rounding before a relationship becomes apparent. Look for a general pattern in the values.

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WORKED EXAMPLE 10 Finding the gradient function using approximations

Use $h = 0.1$ to find the gradient function $R(x)$ of $f(x) = x^2 + x$ for the x values 1, 2 and 3, approaching each x value from the right.

Steps	Working
1 Obtain $\frac{\Delta y}{\Delta x}$ for each x value given.	Substitute each x value into $\frac{\Delta y}{\Delta x} = \frac{f(x+h) - f(x)}{h}$. $x = 1, \quad \frac{\Delta y}{\Delta x} = \frac{[(1.1)^2 + 1.1] - [1^2 + 1]}{0.1} \approx 3$ $x = 2, \quad \frac{\Delta y}{\Delta x} = \frac{[(2.1)^2 + 2.1] - [2^2 + 2]}{0.1} \approx 5$ $x = 3, \quad \frac{\Delta y}{\Delta x} = \frac{[(3.1)^2 + 3.1] - [3^2 + 3]}{0.1} \approx 7$

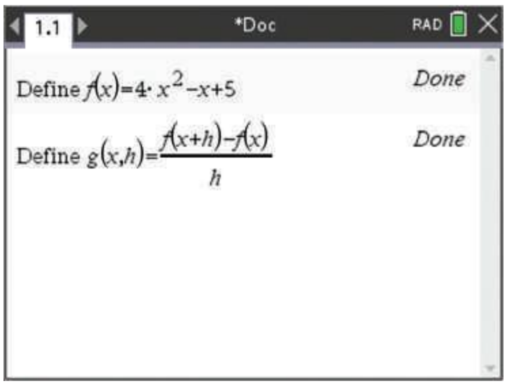
2 Identify a relationship between the x value used and the gradient.	It appears that the gradient is double the x value plus 1.
3 State the gradient function, $R(x)$.	$R(x) = 2x + 1$

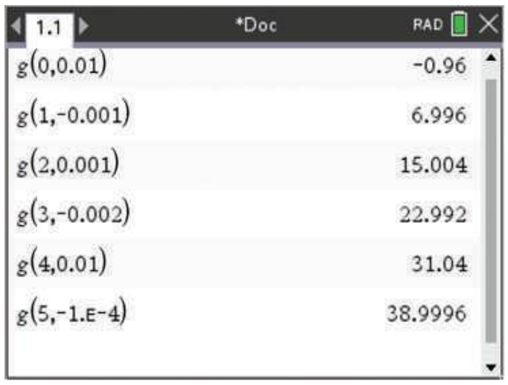
Using CAS makes the computations in the previous worked example much easier.

USING CAS 3 The gradient function

Find approximate rates of change for the function $f(x) = 4x^2 - x + 5$ at various values of x to suggest a gradient function $g(x)$ for the rate of change of $f(x)$.

TI-Nspire

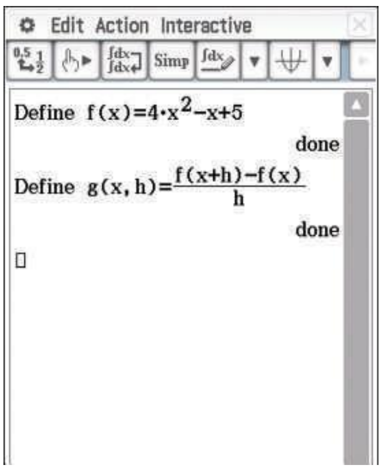


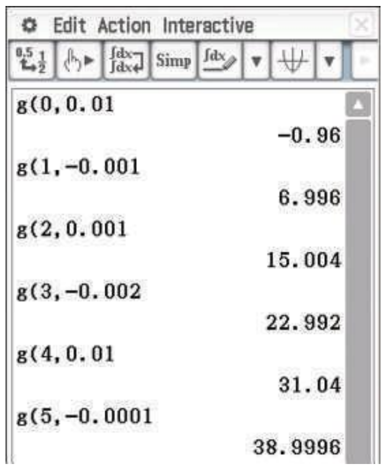


1 Define the functions ***f*(x)** and ***g*(x,h)** as shown above.

2 Find values of $g(x,h)$ for x from 0 to 5, using $h = 0.01$ or $h = -0.01$ or less.

ClassPad





1 Define the functions ***f*(x)** and ***g*(x,h)** as shown above.

2 Find values of $g(x,h)$ for x from 0 to 5, using $h = 0.01$ or $h = -0.01$ or less.

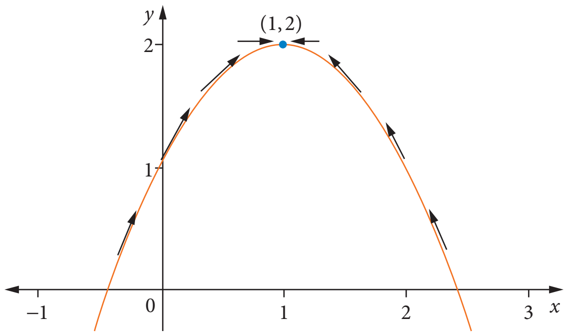
Make a table of values. Approximate values of $g(x)$.

x	0	1	2	3	4	5
$g(x)$	–1	7	15	23	31	39

The function goes up in steps of 8. The function seems like $g(x) = 8x - 1$.

The gradient of the tangent line at a **stationary point** (or **turning point**) is 0.

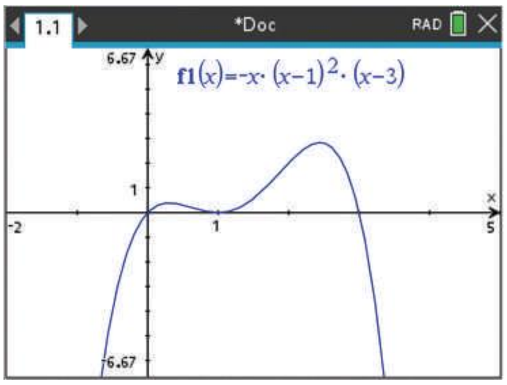
The function shown below has equation $y = 1 + 2x - x^2$. As x approaches 1 from the left side and from the right side, the gradient of the tangent line approaches 0.

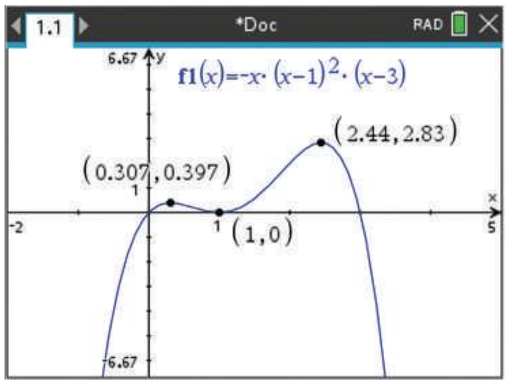


USING CAS 4 Stationary points

Find the coordinates in the interval [0, 3] where the gradient of the tangent line to the function $f(x) = -x(x - 1)^2(x - 3)$ is 0. Give answers correct to 2 decimal places where necessary.

TI-Nspire





1 Graph the function.

2 Adjust the window settings to suit.

3 Press **menu > Analyze Graph > Maximum**.

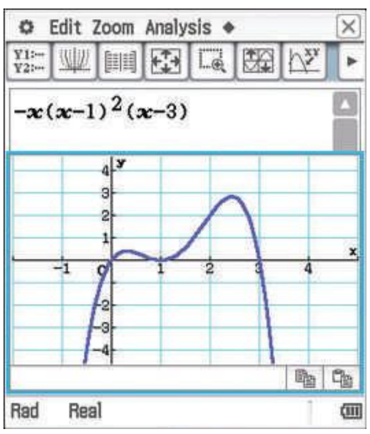
4 When prompted for the **lower bound?**, click to the left of the first local maximum.

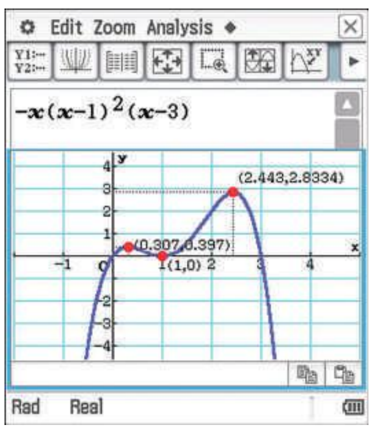
5 When prompted for the **upper bound?**, click to the right of the point.

6 Repeat to find the coordinates of the second local maximum.

7 Press **menu > Analyze Graph > Minimum** to find the coordinates of the local minimum.

ClassPad





1 In **Main**, enter and highlight the expression **$-x(x - 1)^2(x - 3)$**.

2 Tap to open the **Graph** window and drag the expression into it.

3 Adjust the window settings to suit.

4 Tap **Analysis > G-Solve > Max**.

5 The cursor will jump to the first local maximum.

6 Press **EXE** to display the coordinates of this point.

7 Press the **right arrow** to jump to the second local maximum and press **EXE**.

8 Tap **Analysis > G-Solve > Min** to find the coordinates of the local minimum.

(Highlight and drag the rule into the graph to get rid of doubling up of labelled points.)

At local maximums and minimums, the gradient of the tangent line is equal to 0.

The coordinates of the local maximums are approximately (0.31, 0.40) and (2.44, 2.83).

The coordinates of the local minimum are (1, 0).

6.3

EXERCISE 6.3 The gradient of a tangent

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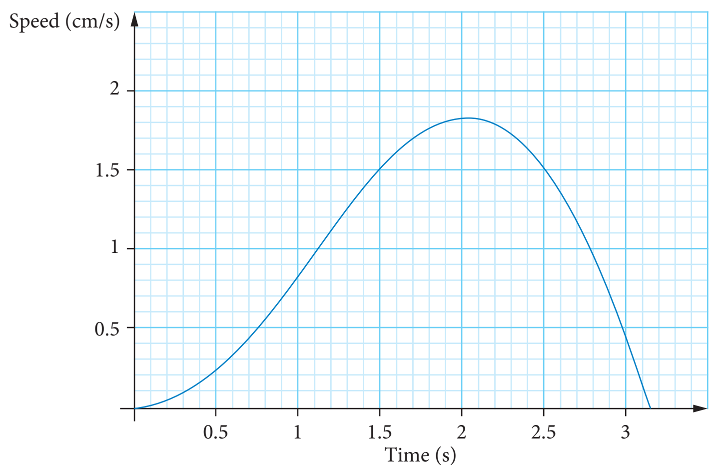
Recap

1 The graph shows the speed of a particle as a function of time.



The average acceleration of the particle between 1.5 seconds and 2.5 seconds is closest to

A -3 cm/s^2 **B** -1 cm/s^2 **C** 0 cm/s^2

D 1 cm/s^2 **E** 3 cm/s^2

2 The approximation for the instantaneous rate of change of y with respect to x using the function $f(x) = x^2 - 2$ from x to $x + 0.1$ is

A $2x + 0.1$ **B** $x - 0.1$ **C** $x^2 + 0.1$

D $2x - 0.1$ **E** $x^2 - 0.1$

Mastery

3 WORKED EXAMPLE 6 Find the approximate rate of change of $f(x) = x^3 + x^2 - 3x - 1$ between the points $x = 4$ and $x = 4.2$ using $f'(x) \approx \dfrac{f(x+h) - f(x)}{h}$.

4 WORKED EXAMPLE 7 Calculate the approximate rate of change of $f(x) = 3x^2 + 2x + 2$ using $h = 0.1$ and by approaching $x = 5$ from the left.

5 WORKED EXAMPLE 8 Using $h = 0.1, 0.01, 0.001, 0.0001$, state the approximations for the instantaneous rate of change of $f(x) = 2x^3 - 1$ at $x = -2$ by approaching the point from the left. Hence, state the derivative of $f(x)$ at $x = -2$.

6 WORKED EXAMPLE 9 Use central difference approximation with $h = 0.4$ to find the gradient of $f(x) = x^2 - x + 2$ at $x = 4$.

7 Using CAS 3 By trying various values of x, find approximate rates of change for $f(x) = 4x - 3x^2$ and use your results to determine the gradient function. Use the method of approaching each value of x from the right.

8 Using CAS 4 Find the coordinates where the gradient of the tangent line to the function $f(x) = x^3 - 6x^2 + 11x - 6$ is zero. Give answers correct to 2 decimal places.

Exam practice

9 WORKED EXAMPLE 10 TECH-FREE (2 marks) Using $h = 0.2$ and by approaching each x value from the right, find the gradient function $R(x)$ of $f(x) = 2x^2 - 1$ for the x values 0, 1, 2 and 3.

10 TECH-FREE (2 marks) Approach $x = a$ from the right to show that the gradient function of $f(x) = x^3$ at $x = a$ is $3a^2$.

11 Which formula shows the approximate rate of change of $f(x)$ at $x = 1$ from the right with $h = 0.5$?

A $f'(x) \approx \dfrac{f(1.1) - f(0.5)}{0.5}$ **B** $f'(x) \approx \dfrac{f(1) - f(1.5)}{0.5}$ **C** $f'(x) \approx \dfrac{f(1.5) - f(1.1)}{0.5}$

D $f'(x) \approx \dfrac{f(1.5) - f(0.5)}{0.5}$ **E** $f'(x) \approx \dfrac{f(1.5) - f(1)}{0.5}$

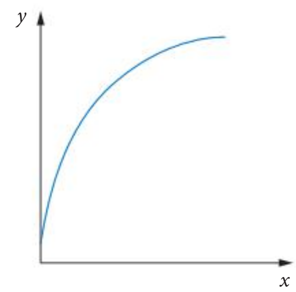
12 For $y = 3 - 2x^3$, using $h = 1$ and by approaching $x = 10$ from the right, $\dfrac{\Delta y}{\Delta x}$ is

A −994 **B** −662 **C** −559 **D** 1328 **E** 1331

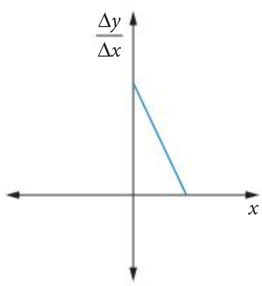
13 For $y = -x^2$, using $h = 0.2$ and by approaching $x = 2$ from the left, $\dfrac{\Delta y}{\Delta x}$ is

A −3.8 **B** −0.76 **C** 0.38 **D** 0.76 **E** 3.8

14 The gradient function for this curve is most likely to be



A



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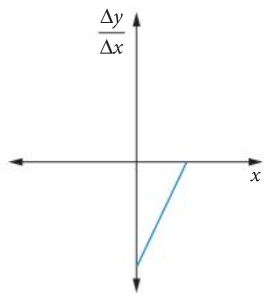

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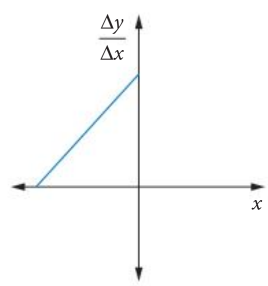

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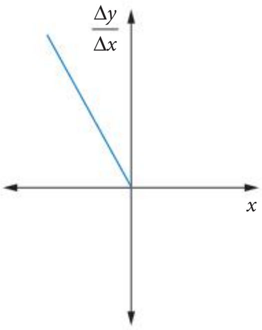
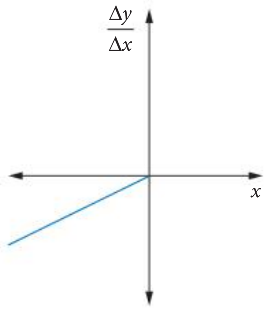


E



15 Using $h = 0.1$ and by approaching each x value from the right, the approximate rates of change of $y = x^2 - x$ using $x = 1, 2, 3$ are, respectively

A 2, 3, 6 **B** 2, 4, 5 **C** 1, 3, 6 **D** 1, 3, 5 **E** 4, 5, 6

16 Some approximations to the rate of change of a function are shown below.

$x = 0, \frac{\Delta y}{\Delta x} = 0.82$ $\quad x = 1, \frac{\Delta y}{\Delta x} = 6.89$ $\quad x = 2, \frac{\Delta y}{\Delta x} = 12.77$ $\quad x = 3, \frac{\Delta y}{\Delta x} = 18.49$

The gradient function is closest to

A $y = 6x - 1$ **B** $y = 6x + 1$ **C** $y = 2x + 1$ **D** $y = 4x - 3$ **E** $y = 7x + 1$

17 (4 marks) The height, $H(t)$ metres, of a toy rocket t seconds after launch is described by the function $H(t) = -t(t - 10)$.

a Speed is the change in distance divided by the change in time. Estimate the speed of the rocket between 2 seconds and 4 seconds. 2 marks

b What is the rocket's rate of change of height between 3 and 3.01 seconds? 2 marks

18 (4 marks)

a Copy and complete the table using the function $f(x) = x^2 + 3x + 1$ and $h = 0.03$ to calculate the approximate rate of change by approaching x from the left. 3 marks

x	$f(x)$	$x - h$	$f(x - h)$	$\frac{\Delta y}{\Delta x}$ (rounded to nearest whole number)
0				
2				
4				

b Use the results in the last column to suggest the gradient function, $R(x)$. 1 mark

19 (4 marks)

a Show that $\frac{\Delta y}{\Delta x} = \frac{f(x+h) - f(x)}{h} = 2ax + b + h$ for $f(x) = ax^2 + bx + c$, where a, b, c are given constants. 3 marks

b If h approaches 0, what is the suggested gradient function, $R(x)$? 1 mark

20 (5 marks)

a Use $\frac{\Delta y}{\Delta x} = \frac{f(x) - f(x-h)}{h}$ with $f(x) = x^3$ to obtain an expression for $\frac{\Delta y}{\Delta x}$. 2 marks

b Use $\frac{\Delta y}{\Delta x} = \frac{f(x+h) - f(x)}{h}$ with $f(x) = x^3$ to obtain an expression for $\frac{\Delta y}{\Delta x}$. 2 marks

c If h approaches 0, what is the suggested gradient function, $R(x)$, for each expression in parts **a** and **b**? 1 mark

21 (3 marks) Use the function $f(x) = x^3 - x$ to show that $\lim_{h \to 0} \frac{\Delta y}{\Delta x} = 3x^2 - 1$ for each of

$$\frac{\Delta y}{\Delta x} = \frac{f(x) - f(x-h)}{h}, \frac{\Delta y}{\Delta x} = \frac{f(x+h) - f(x)}{h} \text{ and } \frac{\Delta y}{\Delta x} = \frac{f(x+h) - f(x-h)}{2h}.$$

Video playlist Applying rates of change

Worksheet Graphs of change

6.4 Applying rates of change

There are many real-life examples of rates of change. Many rates of change have time as the independent variable. For example:

- the rate at which water passes through a pipe of a given diameter
- the rate of change of heartbeat rate after strenuous exercising
- the rate at which bacteria multiply.

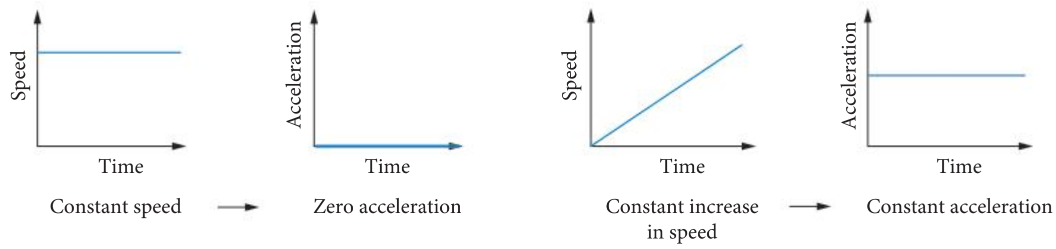
Speed is the rate of change of distance over time. Signed speed (positive or negative) is called **velocity**, while signed distance is called **displacement**.

If s is speed, d is distance travelled and t is the time taken, then $s = \frac{\Delta d}{\Delta t}$.

Acceleration is the rate of change of speed with respect to time. If a is acceleration, then $a = \frac{\Delta s}{\Delta t}$.

Constant speed means zero acceleration.

A constant increase or decrease in the rate of change of the speed means constant acceleration.



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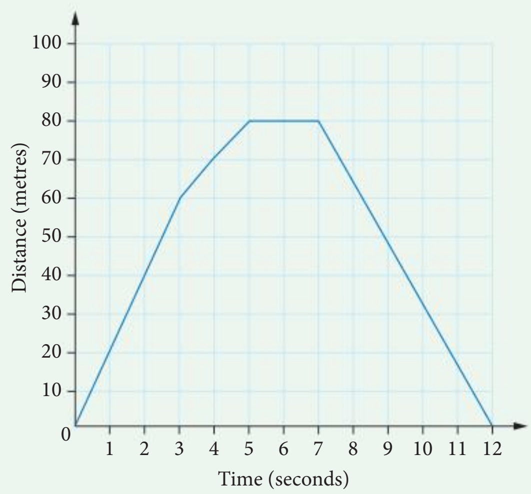
WORKED EXAMPLE 11 Interpreting a distance–time graph

Part of Zhao's car journey is represented by the graph.



a How fast was Zhao driving during the time interval 3 seconds to 5 seconds?

b For how long was Zhao stationary?

c Find the speed during the return journey.

d Determine Zhao's maximum speed.

Steps			Working
a	**1**	Calculate the speed in the required interval.	For the interval 3 seconds to 5 seconds, $\Delta d = 80 - 60 = 20$ and $\Delta t = 5 - 3 = 2$. $s = \frac{\Delta d}{\Delta t} = \frac{20}{2} = 10$
	2	State the answer, including units.	The speed was 10 metres/second.
b	**1**	Zero speed means the gradient is zero.	The horizontal line from $t = 5$ to $t = 7$ represents zero speed for $7 - 5 = 2$ seconds.
	2	State the answer.	Zhao was stationary for 2 seconds.
c	**1**	Calculate the speed in the required interval. Negative speed because the car is travelling towards 0.	The return journey was from $t = 7$ to $t = 12$. $\Delta d = 0 - 80 = -80$ and $\Delta t = 12 - 7 = 5$. $s = \frac{\Delta d}{\Delta t} = \frac{-80}{5} = -16$
	2	State the answer, including units.	The speed was 16 metres/second.
d	**1**	Maximum speed is the section with the steepest gradient.	The steepest section is during the first 3 seconds.
	2	Calculate the speed in the required interval.	$\Delta d = 60 - 0 = 60$ and $\Delta t = 3 - 0 = 3$ $s = \frac{\Delta d}{\Delta t} = \frac{60}{3} = 20$
	3	State the answer, including units.	The maximum speed was 20 metres/second.

WORKED EXAMPLE 12 Interpreting a speed–time graph

WB p. 136

The speed–time graph of a car is shown.

a State the acceleration of the car between

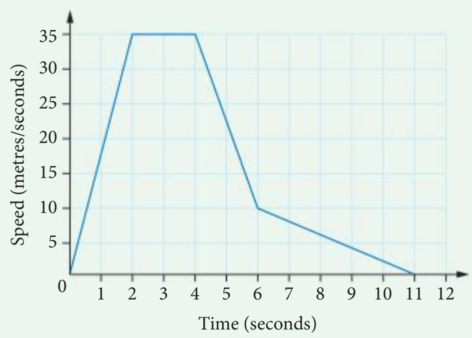
i 0 and 2 seconds

ii 2 and 4 seconds

iii 6 and 11 seconds

b In which time interval was the car's acceleration greatest?

c Calculate the acceleration of the car, in km/h^2, between 4 and 6 seconds.



Steps				Working
a	**i**	**1**	Use $a = \frac{\Delta s}{\Delta t}$ between $t = 0$ and $t = 2$.	$\Delta s = 35 - 0 = 35$ $\quad \Delta t = 2 - 0 = 2$ $a = \frac{\Delta s}{\Delta t} = \frac{35}{2} = 17.5$
		2	State the answer, including units.	The acceleration is 17.5 m/s^2.
	ii	**1**	Use $a = \frac{\Delta s}{\Delta t}$ between $t = 2$ and $t = 4$.	The speed is constant in the time interval 2 to 4, so the acceleration is zero.
		2	State the answer, including units.	The acceleration is 0 m/s^2.
	iii	**1**	Use $a = \frac{\Delta s}{\Delta t}$ between $t = 6$ and $t = 11$.	$\Delta s = 0 - 10 = -10$ $\quad \Delta t = 11 - 6 = 5$ $a = \frac{\Delta s}{\Delta t} = \frac{-10}{5} = -2$
		2	State the answer, including units.	The acceleration is -2 m/s^2.

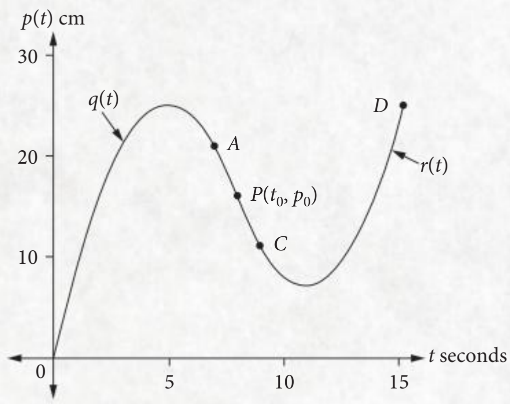
b Consider the slope of the graph at each section.	The steeper the slope, the greater the acceleration. The rate of change of speed is greatest from $t = 0$ to $t = 2$.
c **1** Find Δs and Δt between $t = 4$ and $t = 6$.	$\Delta s = 10 - 35 = -25$ m $\quad \Delta t = 6 - 4 = 2$ seconds
2 Convert to km and hours.	-25 m $= -25 \div 1000 = -0.025$ km 2 seconds $= \dfrac{2}{3600} = \dfrac{1}{1800}$ hours
3 Use $a = \dfrac{\Delta s}{\Delta t}$.	$a = \dfrac{-0.025}{\frac{1}{1800}} = -0.025 \times 1800 = -45$
4 State the answer, including units.	The acceleration is -45 km/h^2.

Video
VCE question analysis: Rates of change

VCE QUESTION ANALYSIS

A VCE-style question (13 marks)

The position, $p(t)$ centimetres, of a particle from a fixed point after t seconds is shown in the graph below. The graph of $p(t)$ is such that $p(t) = q(t) + r(t)$, where the curve OP is a parabola given by $q(t) = -t(at - b)$ and the curve PD is described by the parabola $r(t) = t^2 - 22at + 12b + 8$, with a and b being constant values.



a Given that $q(1) = 9$ and $r(1) = 107$, show that $a = 1$ and $b = 10$. 2 marks

b Find the coordinates of point P. 2 marks

c The particle takes $t_0 - 1$ seconds to reach point A from point O and $t_0 + 1$ seconds to reach point C from point O.

Determine the average rate of change between A and C. 2 marks

d $\dfrac{f(t+h) - f(t)}{h}$ gives an approximation for the rate of change of the function $f(t)$ from t to $t + h$.

i Use $h = 0.001$ to find approximate rates of change for the function $q(t)$ using $t = 6$, $t = 7$ and $t = 8$. Give answers correct to the nearest integer. 2 marks

ii Use your results from part **i** to find the gradient function, $a(t)$, for the rate of change of $q(t)$. 1 mark

e **i** Use $h = 0.001$ to find approximate rates of change for the function $r(t)$ using $t = 8$, $t = 9$ and $t = 10$. Give answers correct to the nearest integer. 2 marks

ii Use your results from part **i** to find the gradient function, $b(t)$, for the rate of change of $r(t)$. 1 mark

f Show that the instantaneous rate of change of the particle at point P is the same using either of $a(t)$ or $b(t)$. 1 mark

Reading the question

- When reading the questions, try to understand how they are related.
- The graph consists of 2 different parabolas, whose equations must be found first.
- Understand key words and terms, because they often indicate the approach or formula needed to answer the question. For instance, average rate of change and instantaneous rate of change use different formulas.
- Ensure that answers are written to the required levels of accuracy.

Thinking about the question

- Working must be shown for a question worth more than 1 mark.
- Decide when to use algebra or CAS, such as in parts **d** and **e**.
- Avoid unnecessary calculations. For example, 'Show that' in part **a** means it is sufficient only to verify using substitution that the given values for a and b are correct. It is not necessary to solve the simultaneous equations.
- For points $A(t, f(t))$ and $B(t+h, f(t+h))$, the average rate of change is found using $\dfrac{f(t+h)-f(t)}{h}$.

Worked solution (✓ = 1 mark)

a

$q(t) = -t(at-b)$

$q(1) = -(a-b) = 9$

Test $a = 1, b = 10$:

LHS $= -(1-10) = 9 =$ RHS ✓

$r(t) = t^2 - 22at + 12b + 8$

$r(1) = 1 - 22a + 12b + 8 = 107$

$-22a + 12b = 98$

Test $a = 1$, $b = 10$:

LHS $= -22 \times 1 + 12 \times 10 = 98 =$ RHS ✓

b

$q(t) = -t(t-10), r(t) = t^2 - 22t + 128$

$q(t) = r(t)$

$-t(t-10) = t^2 - 22t + 128$

$2t^2 - 32t + 128 = 0$ ✓

$t^2 - 16t + 64 = 0$

$(t-8)^2 = 0$

$t = 8$

$q(8) = 16$

The coordinates of P are **(8, 16)**. ✓

c

$q(7) = -7(7-10) = 21, r(9) = 9^2 - 22 \times 9 + 128 = 11$

The coordinates of A are (7, 21). The coordinates of C are **(9, 11)**. ✓

Average rate of change between A and C: $\dfrac{11-21}{9-7} = \mathbf{-5\ cm/s}$ ✓

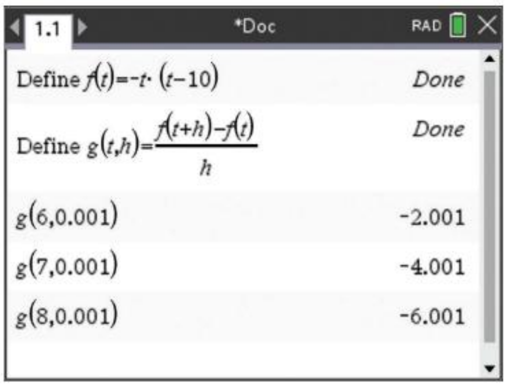
d i

$t = 6, \dfrac{q(6+0.001)-q(6)}{0.001} \approx -2$

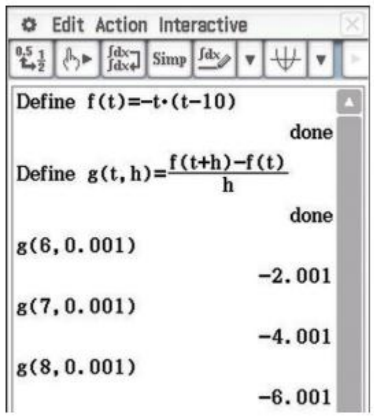
$t = 7, \dfrac{q(7+0.001)-q(7)}{0.001} \approx -4$

$\boldsymbol{t = 8, \dfrac{q(8+0.001)-q(8)}{0.001} \approx -6}$ ✓✓ (for all 3 answers)

TI-Nspire



ClassPad

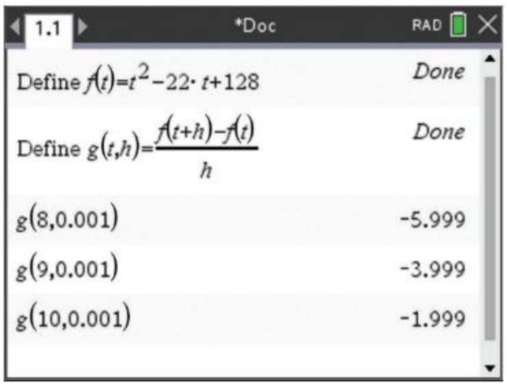


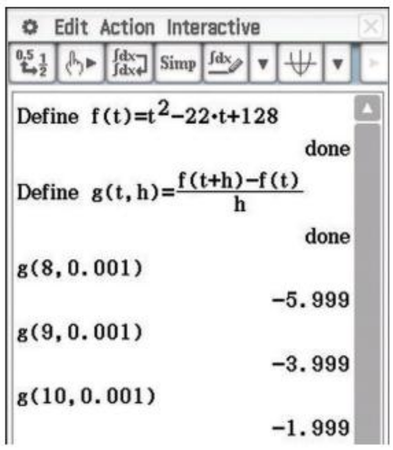
ii $a(t) = -2t + 10$ ✓

e i $t = 8, \dfrac{r(8 + 0.001) - r(8)}{0.001} \approx -6$

$t = 9, \dfrac{r(9 + 0.001) - r(9)}{0.001} \approx -4$

$t = 10, \dfrac{r(10 + 0.001) - r(10)}{0.001} \approx -2$ ✓✓ (for all 3 answers)





ii $b(t) = 2t - 22$ ✓

f $a(8) = -2 \times 8 + 10 = -6$ cm/s

$b(8) = 2 \times 8 - 22 = -6$ cm/s ✓ (for both answers)

EXERCISE 6.4 Applying rates of change

ANSWERS p. 518

Recap

1 The approximate rate of change when approaching $x = 1$ from the left of the function $f(x) = \dfrac{1}{1 + x^2}$ using $h = 0.1$ is closest to

A -0.52 **B** -0.48 **C** 0.52 **D** 5.2 **E** 10.52

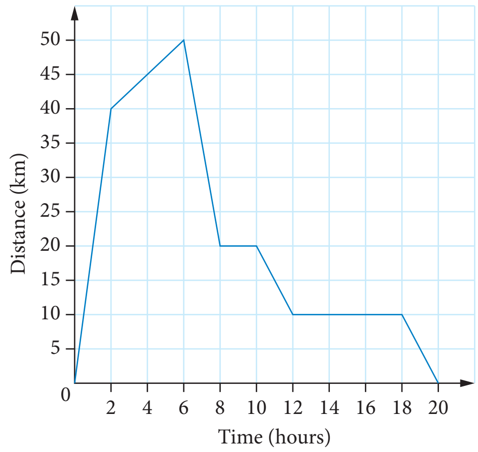
2 The rate of change when approaching $x = a$ from the right of the function $f(x) = x(x + 1)$ using h as the x increment is

A $2a + h$ **B** $a + h - 1$ **C** $2a - h + 1$

D $2a + h + 1$ **E** $a + h$

Mastery

3 WORKED EXAMPLE 11 This graph shows Justine's overnight bike ride in the country. She began her trip from base camp at 7 am Monday.



a When did Justine return to base camp?

b Calculate Justine's average speed from base camp to the furthest point she reached.

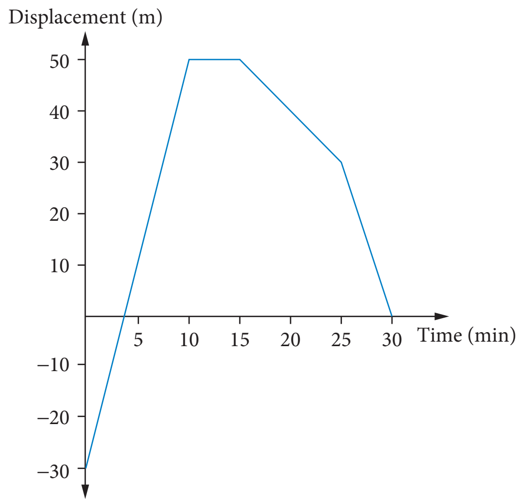
c What was her greatest speed during the return journey?

d When did Justine have an extended rest?

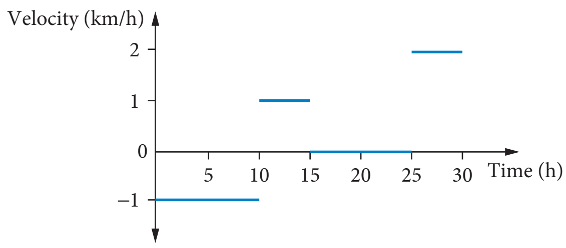
e What distance did Justine travel from the start of her return trip to her first rest stop?

f Not including rest stops, calculate the average overall speed for the bike ride.

4 TECH-FREE Use the displacement–time graph shown below to draw the corresponding velocity–time graph.



5 TECH-FREE Construct a displacement–time graph for the velocity–time graph given below. Assume that the initial position is at zero.



6 Which statement is correct?

A distance $= \dfrac{\text{speed}}{\text{time}}$

B Acceleration is the rate of change of distance over time.

C Velocity is represented by the gradient of a distance–time graph.

D Speed is the rate of change of acceleration over time.

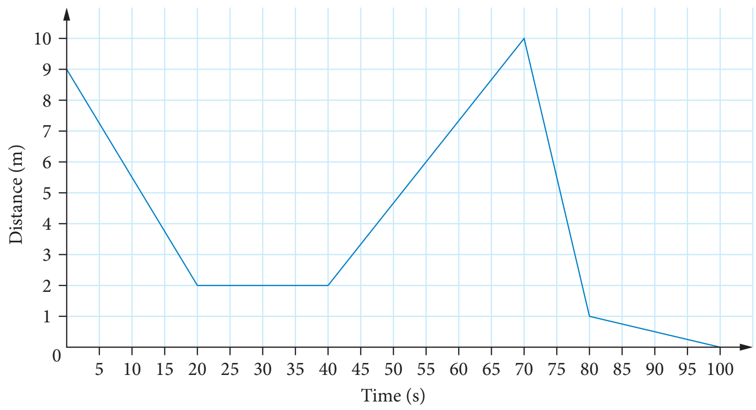
E If distance is in millimetres and time is in minutes, then speed will be in minutes/millimetre.

7 Nami drives 50 km north-east then 100 km south-west. Her position from her starting point is now

A 100 km south-west **B** 50 km south-east **C** 150 km south-west

D 50 km south-west **E** 100 km south-east

8 Milos left home to go for a walk. According to the graph below, the total distance he travelled was



A 20 m **B** 23 m **C** 25 m **D** 26 m **E** 29 m

9 According to the graph in Question 8, Milos' fastest walking speed was

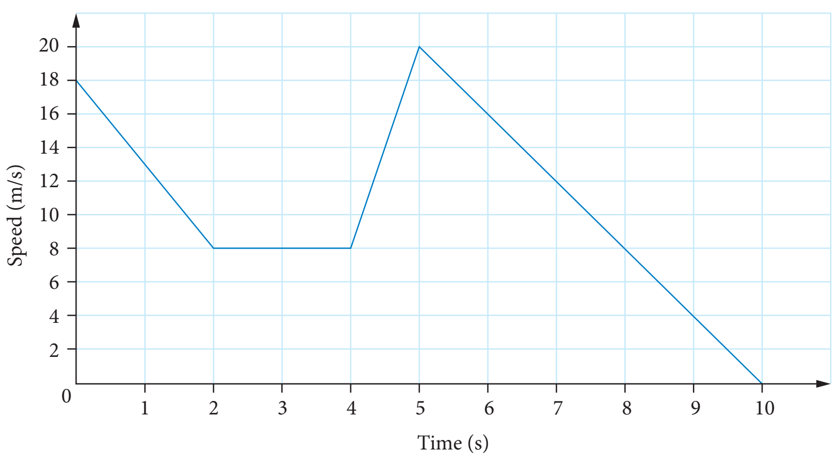
A 0.22 m/s **B** 0.34 m/s **C** 0.52 m/s **D** 0.65 m/s **E** 0.9 m/s

10 The speed of a meteor changed from 4000 m/s to 3100 m/s in 6 seconds as it entered the atmosphere. What was the average rate of decrease of its speed?

A 54 m/s^2 **B** 150 m/s^2 **C** 300 m/s^2 **D** 600 m/s^2 **E** 900 m/s^2

11 (3 marks) A runner in training does four 500-metre laps of an oval, with lap times of 95 seconds, 102 seconds, 105 seconds and 103 seconds. Find, correct to 2 decimal places, the average speed for each lap and the average speed for the whole 2 km.

12 WORKED EXAMPLE 12 (4 marks) The speed–time graph of a go-kart is shown.



a When was the go-kart moving at a constant speed? 1 mark

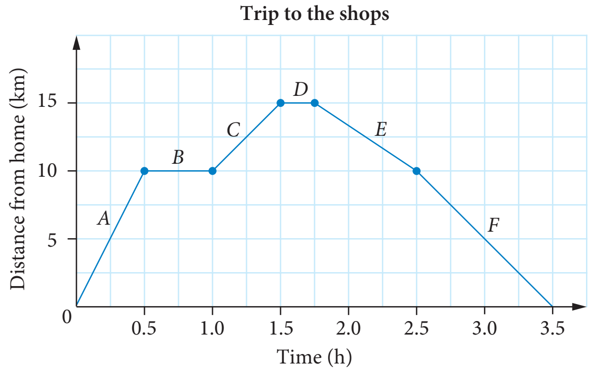
b Calculate the average speed for the first 10 seconds. 1 mark

c Calculate the rate of change of speed from 5 seconds to 10 seconds. 1 mark

d Find the magnitude of the maximum acceleration in m/s^2. 1 mark

6.4

13 (5 marks) The graph shows the distance of a cyclist from her home at various times on a trip to a shop 15 km away and back again.



a When was she moving at the greatest speed? 1 mark

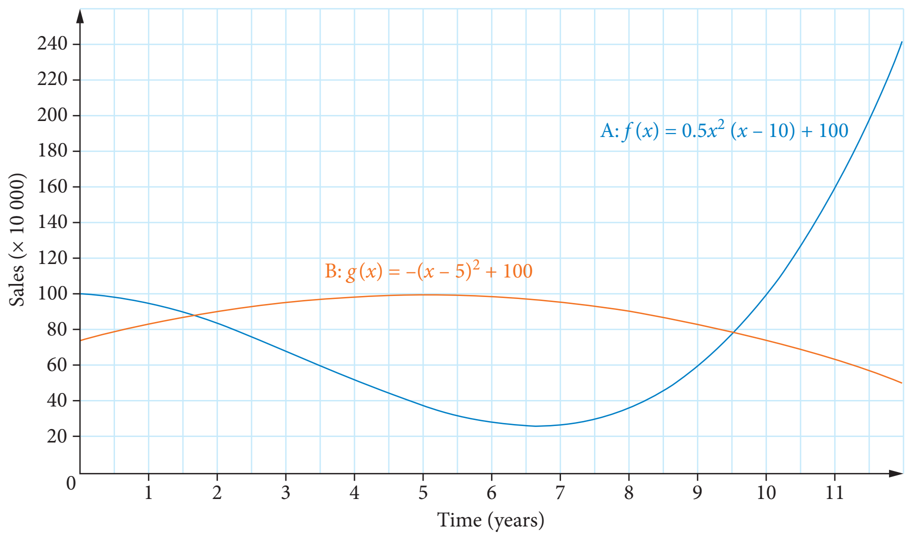
b What was her average speed to the shop? 1 mark

c How long did she stop at the shop? 1 mark

d What was her average speed in section *E*? 1 mark

e What was her average speed for the whole trip? Answer correct to one decimal place. 1 mark

14 (7 marks) The number of sales by two companies, A and B, over a number of years is shown below. The sales $f(x)$, ($\times$ 10 000) for company A followed the rule $f(x) = 0.5x^2(x - 10) + 100$, where x is the number of years from the start of year 2010. The sales $g(x)$, ($\times$ 10 000) for company B followed $g(x) = -(x - 5)^2 + 100$.



a Find the rate of change of sales of each company when they had both achieved the same number of sales for the first time. 2 marks

b In what year and month did company A's rate of sales begin to increase? 1 mark

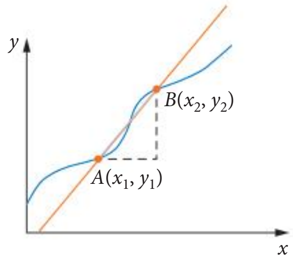
c What was company A's rate of sales that produced sales of 200 000 units? 2 marks

d Determine company B's sales rate in June 2013. 2 marks

6 Chapter summary

Average rate of change

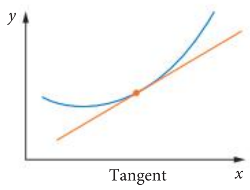
- A rate of change occurs when a change in one variable produces a change in another variable.

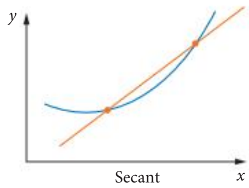


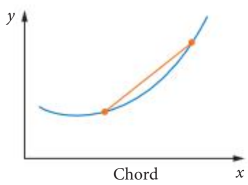
- The average rate of change is the gradient of the straight line.
- The average rate of change $\frac{\Delta y}{\Delta x}$ between $A(x_1, y_1)$ and $B(x_2, y_2)$ is $\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$.

Instantaneous rate of change

- A straight line that touches a curve at one point is a **tangent**.
- If a straight line passes through 2 points on a curve, it is a **secant**.
- The interval (line segment) joining 2 points on a curve is a **chord**.





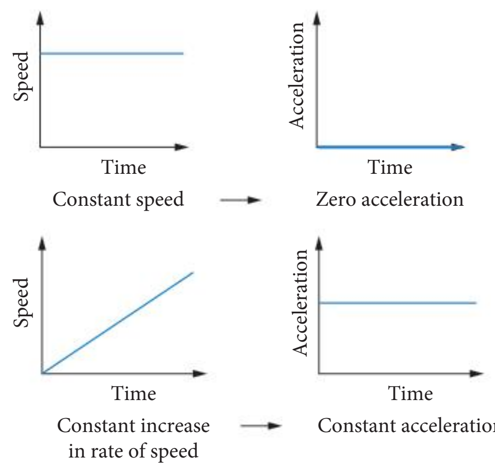


The gradient of a tangent

- The gradient of the tangent at a point on the curve is the instantaneous rate of change.
- An approximation $\frac{\Delta y}{\Delta x}$ for the instantaneous rate of change at point x is found using one of these formulas:
- $\frac{\Delta y}{\Delta x} = \frac{f(x) - f(x-h)}{h}$ (from the left)
- $\frac{\Delta y}{\Delta x} = \frac{f(x+h) - f(x)}{h}$ (from the right)
- $\frac{\Delta y}{\Delta x} = \frac{f(x+h) - f(x-h)}{2h}$ is the central difference approximation.
- The **gradient function** gives the instantaneous rate of change (or gradient of the tangent) at any value of x.
- A **stationary point** (or **turning point**) on a curve has a zero gradient.
- The **derivative**, $f'(x)$, of a function $f(x)$ can be found using $f'(x) = \lim\limits_{h \to 0} \frac{f(x) - f(x-h)}{h}$ or $f'(x) = \lim\limits_{h \to 0} \frac{f(x+h) - f(x)}{h}$.

Applying rates of change

- The speed (s) of an object is its rate of change of distance (d) with respect to time (t) and is given by $s = \frac{\Delta d}{\Delta t}$.
- The acceleration is the rate of change of speed with respect to time and is found using $a = \frac{\Delta s}{\Delta t}$.
- Constant speed means zero acceleration.
- A constant increase or decrease in the rate of change in the speed means constant acceleration.



Cumulative examination 1

Total number of marks: 10 Reading time: 4 minutes Writing time: 15 minutes

TECH-FREE *Technology is NOT permitted.*

1 (2 marks) The temperature, T°C, inside a greenhouse at time t hours after midday is given by the function $T(t) = \frac{1}{10}(x-1)(x-2)^2 + 5, 0 \le t \le 5$.

What is the average rate of change of the temperature in the greenhouse between 2 pm and 5 pm?

2 © VCAA 2017N 2BQ2 (1 mark) Let $f: R \to R$, where $f(x) = (x-2)^2(x-5)$.

Find the gradient of the line segment joining the points on the graph of $y = f(x)$ where $x = 1$ and $x = 5$.

3 (2 marks) The nth term, $t(n)$, of a sequence is given by $t(n) = r^{n-1}$, where r is a non-zero constant.

Show that the average rate of change from the ath term to the bth term is $\frac{r^b - r^a}{r(b-a)}$.

4 (1 mark) Given the polynomial $p(x) = 4x^3 + ax + 1$, what is the value of a if $p\left(\frac{1}{2}\right) = \frac{5}{4}$?

5 (2 marks) A bag contains a red balls, b blue balls and c green balls. Two balls are randomly drawn one at a time without replacement. Determine the probability that the two balls drawn do not have the same colour.

6 (2 marks) Find the values of k so that the parabola with equation $y = kx^2 - x - 4$ intersects the parabola with equation $y = -2x^2 + (k-2)x - 1$ at two distinct points.

Cumulative examination 2

Total number of marks: 20 Reading time: 4 minutes Writing time: 30 minutes

Approved technology is permitted.

Section A 5 multiple-choice questions

5 marks

1 The average rate of change of the function f with rule $f(x) = 1 + \frac{5}{x}$ between $x = 1$ and $x = 5$ is

A -1 **B** $-\frac{1}{3}$ **C** $\frac{2}{3}$ **D** $\frac{5}{6}$ **E** 1

2 A cylindrical tank completely filled with 80 litres of water is being emptied at a constant rate. After two minutes, 75% of the water remains in the tank. The rate of water flow in litres/minute is

A 10 **B** 15 **C** 20 **D** 25 **E** 40

3 Function $f(x)$ has a minimum at $x = -2$ and a maximum at $x = 3$. Choose the correct statement.

A The instantaneous rate of change at $x = 3$ is $f(3)$.

B The gradient on the curve where $x = -2$ is negative.

C The minimum and the maximum points on the curve are not stationary points.

D The average rate of change from the minimum point to the maximum point is negative.

E When $x = a$ approaches the value $x = 3$, the gradient of the line connecting $f(a)$ to $f(3)$ approaches zero.

4 The magnitude of v in $E = \frac{1}{2}mv^2$ when $m = 3$ and $E = 5$ is closest to

A 1.095 **B** 1.826 **C** 2.236 **D** 2.752 **E** 3.333

5 The domain of the function $f(x) = 2\sqrt{x+3} + 1$ is

A $x \geq 3$ **B** $y \geq 1$ **C** $x < 3$ **D** $y < 2$ **E** $x \geq -3$

Section B 6 questions

15 marks

1 (3 marks) The average rate of change of the function f with rule $f(x) = x^2 + x$ between $x = 2$ and $x = k$ is 7, with $k > 2$.

a Show that $k^2 - 6k + 8 = 0$. 2 marks

b Hence, find the value of k. 1 mark

2 © VCAA 2014 2BQ3aci (3 marks) In a controlled experiment, Juan took some medicine at 8 pm. The concentration of medicine in his blood was then measured at regular intervals.

The concentration of medicine in Juan's blood is modelled by the function $c(t) = \frac{5}{2}te^{-\frac{3t}{2}}, t \geq 0$, where c is the concentration of medicine in his blood, in milligrams per litre, t hours after 8 pm.

> e is a constant that is approximately 2.718 (discussed in Chapter 11, *Trigonometric and exponential equations*). To calculate and graph $c(t)$ to answer the questions, use $e \approx 2.718$ or the e function found in the menu of CAS.

Part of the graph of the function c is shown below.

a What was the maximum value of the concentration of medicine in Juan's blood, in milligrams per litre, correct to two decimal places? 1 mark

b What was the value of the average rate of change of the concentration of medicine in Juan's blood over the interval $\left[\frac{2}{3}, 3\right]$? Express the answer in milligrams per litre per hour, correct to two decimal places. 2 marks

3 (3 marks)

a Find the coordinates of the two turning points on the curve given by the equation $y = 10(x - 1)x^2$. State the answers to two decimal places where appropriate. 2 marks

b Calculate, correct to two decimal places, the average rate of change of y with respect to x between the x-coordinates corresponding to the maximum and the minimum points. 1 mark

4 (2 marks) Let $P(x) = (ax + 1)^2$ and $Q(x) = (x + b)^2$, where a and b are positive numbers.

Determine the values of a and b in exact form given that $P(x) + Q(x) = 3x^2 + 2(\sqrt{2} - \sqrt{3})x + 4$.

5 (2 marks) Given that $P(x) = 4x^2 - 2x + 4$ and $\frac{P(x)}{x + k} = 6$, show that the sum of the solutions for k is $-\frac{1}{2}$.

6 (2 marks) The line with equation $f(x) = mx + c$ makes an angle of 25° with the positive x-axis and passes through the point (0, −1).

Find the equation of the perpendicular line that passes through the point where $x = 10$. Write the equation in the form $y = mx + c$, with m and c rounded to two decimal places.

CUMULATIVE EXAMINATION 2

CHAPTER

7 THE EXPONENTIAL AND LOGARITHMIC FUNCTIONS

Study Design coverage

UNIT 2, AREA OF STUDY 1: FUNCTIONS, RELATIONS AND GRAPHS

- exponential functions of the form $f: R \to R, f(x) = Aa^{kx} + c$ and their graphs, where $a \in R^+$, $A, k, c \in R, A \neq 0$
- logarithmic functions of the form $f: R^+ \to R, f(x) = \log_a(x)$, where $a > 1$, and their graphs, and as the inverse function of $y = a^x$, including the relationships $a^{\log_a(x)} = x$ and $\log_a\left(a^x\right) = x$
- simple applications of exponential functions of the above form, with examples from various modelling contexts, and the interpretation of initial value, rate of growth or decay, half-life, doubling time and long run value in these contexts and their relationship to the parameters A, k and c.

UNIT 2, AREA OF STUDY 2: ALGEBRA, NUMBER AND STRUCTURE

- exponent laws and logarithm laws, including their application to the solution of simple exponential equations.

STUDY DESIGN COVERAGE

Video playlists (7):
- 7.1 The exponential function $y = a^x$
- 7.2 Transformations of the exponential function
- 7.3 Exponential growth and decay
- 7.4 Logarithms
- 7.5 Logarithm laws
- 7.6 The logarithm function $y = \log_a(x)$
- **VCE question analysis** The exponential and logarithmic functions

Worksheets (7)
- 7.1 Exponential functions • Graphing exponentials
- 7.2 Translating exponential functions
- 7.3 Using exponential models
- 7.5 Logarithm laws • Logarithms review
- 7.6 Plotting log functions

Puzzles (3)
- 7.4 Logarithms
- 7.6 Logarithmic graphs match-up • Exponential and log graphs match-up

Nelson MindTap

To access resources above, visit **cengage.com.au/nelsonmindtap**

7.1 The exponential function $y = a^x$

Video playlist The exponential function $y = a^x$

Worksheets Exponential functions

Graphing exponentials

An epidemic occurs when new cases of a disease appear at a much greater rate than normally expected. Famous examples of epidemics include the bubonic plague (Black Death) in 14th century Europe, the Spanish influenza (flu) in 1918–19 and sudden acute respiratory syndrome (SARS) in Asia in 2003.

One of the reasons influenza spreads so quickly is that the flu virus multiplies rapidly. Suppose a culture containing 20 mg of flu virus increases by 30% every 10 minutes. Growth at this rate for an hour is shown in the following table.

10-minute periods	Mass of virus (mg)
0	20
1	20×1.3
2	$20 \times 1.3 \times 1.3 = 20 \times (1.3)^2$
3	$20 \times (1.3)^2 \times 1.3 = 20 \times (1.3)^3$
4	$20 \times (1.3)^3 \times 1.3 = 20 \times (1.3)^4$
5	$20 \times (1.3)^4 \times 1.3 = 20 \times (1.3)^5$
6	$20 \times (1.3)^5 \times 1.3 = 20 \times (1.3)^6$

Adding an increase of 30% is the same as multiplying by 130% or 1.3.

After an hour, the virus culture has grown from 20 mg to $20 \times (1.3)^6 \approx 97$ mg, almost 5 times its original size.

From the table, it can be seen that the mass in mg of the virus can be modelled as

$$M = 20 \times (1.3)^t$$

where t is the number of 10-minute periods.

$M = 20 \times (1.3)^t$ is an example of an **exponential function**, where the variable is a power. The **exponent** is another name for power or index.

The general form of the exponential function

The general form of a simple exponential function is

$$y = b(a^x)$$

where x is the independent variable

a and b are constants

a is the base, where $a > 0$ and $a \neq 1$.

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WORKED EXAMPLE 1 Substituting and evaluating exponential functions

An exponential function is given by $f(x) = 12 \times 3^x$. Calculate the value of

a $f(0)$ **b** $f(2)$ **c** $f(-2)$

Steps	Working
a Substitute $x = 0$ into the function and evaluate.	$f(0) = 12 \times 3^0 = 12$
b Substitute $x = 2$.	$f(2) = 12 \times 3^2 = 108$
c Substitute $x = -2$.	$f(-2) = 12 \times 3^{-2} = 12 \times \frac{1}{3^2} = \frac{12}{9} = \frac{4}{3}$

The index laws

We use the index laws when working with exponential functions.

$a^m \times a^n = a^{m+n}$ $a^m \div a^n = a^{m-n}$ $(a^m)^n = a^{mn}$

$(ab)^n = a^n b^n$ $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$ $a^0 = 1$

$a^{-n} = \frac{1}{a^n}$ $a^{\frac{1}{q}} = \sqrt[q]{a}$ $a^{\frac{p}{q}} = \sqrt[q]{(a)^p} = \left(\sqrt[q]{a}\right)^p$

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WORKED EXAMPLE 2 Simplifying exponential expressions

Simplify $3^{2x-5} \times 3^{6-3x}$.

Steps	Working
Add powers and simplify.	$3^{2x-5} \times 3^{6-3x} = 3^{2x-5+6-3x} = 3^{1-x}$

Exam hack

When simplifying the exponentials, index laws can only be applied when the bases are the same.

EXERCISE 7.1 The exponential function $y = a^x$

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7.1

Mastery

1 Which one of the following is not an exponential function?

A $y = 2.7^h + 5$ **B** $h(x) = 5 \times 2^{3x}$ **C** $y = \left(\frac{1}{2}\right)^{x+1} + 3$

D $f(x) = 5x - 2x^7$ **E** $y = 2^{-x^2}$

2 Evaluate each expression.

a $16^{\frac{3}{4}}$ **b** 9^0 **c** $8^{\frac{1}{3}}$ **d** 36^0 **e** $36^{\frac{1}{2}}$

f $27^{\frac{2}{3}}$ **g** 17^0 **h** $81^{\frac{1}{2}}$ **i** $49^{\frac{3}{2}}$ **j** $36^{\frac{6}{4}}$

k $64^{\frac{2}{3}}$ **l** $32^{\frac{6}{5}}$ **m** $8^{-\frac{2}{3}}$ **n** $\left(\frac{3}{4}\right)^{-2}$ **o** $\left(\frac{1}{16}\right)^{-4}$

p 18^0 **q** $\left(\frac{9}{16}\right)^{-\frac{3}{2}}$ **r** $216^{-\frac{2}{3}}$ **s** $25^{1\frac{1}{2}}$ **t** $\left(6\frac{1}{4}\right)^{-\frac{1}{2}}$

u $\left(2\frac{1}{4}\right)^{\frac{3}{2}}$ **v** $4^{\frac{1}{3}} \times \sqrt[3]{16}$ **w** $(0.01)^{-\frac{3}{2}}$ **x** $1000^{-\frac{4}{3}}$

3 WORKED EXAMPLE 1 If $f(x) = 3 \times 4^x$, find, correct to 4 decimal places, the value of

a $f(3)$ **b** $f(8)$ **c** $f(1.5)$ **d** $f(4)$ **e** $f(-2)$

4 If $h(t) = 1.85 \times (2.3)^t$, find, correct to 4 decimal places, the value of

a $h(4)$ **b** $h(2)$ **c** $h(10)$ **d** $h(4.7)$ **e** $h(-1)$

5 If $m(x) = 12.9 \times (0.85)^x$, find, correct to 4 decimal places, the value of

a $m(3)$ **b** $m(7)$ **c** $m(2.5)$ **d** $m(10)$ **e** $m(-3)$

6 WORKED EXAMPLE 2 Simplify each expression in index form.

a $2^{2x+5} \times 2^{3x-4}$ **b** $5^{4n-7} \times 5^{5n+3}$ **c** $7^{6m-8} \times 7^{m-2}$

d $3^{7y-2} \div 3^{2y+5}$ **e** $11^{3p+4} \times 11^{2p+8}$ **f** $13^{6q-3} \div 13^{4q-7}$

g $2^{4v-3} \times 4^{5v+2}$ **h** $3^{7m+6} \times 9^{4m+5}$ **i** $3^{3x+10} \div 15^{4x-6}$

j $8^{4n-5} \times 16^{6n-7}$ **k** $12^{8r+3} \div 24^{7r-2}$ **l** $21^{4z-8} \times 35^{2z+11}$

Exam practice

7 (12 marks) Simplify each expression.

a $\dfrac{2^{n+2} \times 2^{n-3}}{2^{2n-5}}$ **b** $\dfrac{5^{2k} \times 5^{k-1}}{5^{2k-2}}$ **c** $\dfrac{3^{j+1} \times 2^{j+3}}{6^j}$

d $\dfrac{6^{2n-m} \times 9^{n+m}}{36^{n-m}}$ **e** $\dfrac{15^{4i-5} \times 21^{2i+1}}{35^{2i-2} \times 45^{2i-3}}$ **f** $\dfrac{8^{2f-3} \times 16^{4-f}}{4^{f+2}}$

Video playlist Transformations of the exponential function

Worksheet Translating exponential functions

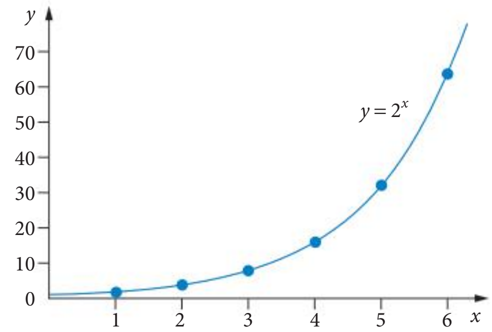
7.2 Transformations of the exponential function

Most plants and animals begin as a single cell. This cell divides into 2, the subsequent cells divide, and so on. The process continues in this way until (in some organisms) the cells begin to specialise, when some will divide faster or slower than others. The division of cells and the number of cells is shown in this table.

Reproductions, x	0	1	2	3	4	5	6
Cells, y	1	2	4	8	16	32	64

The rule for the number of cells is $y = 2^x$, an exponential function.

Its graph is shown here, with the y values doubling as x increases by 1.



Sketching an exponential function

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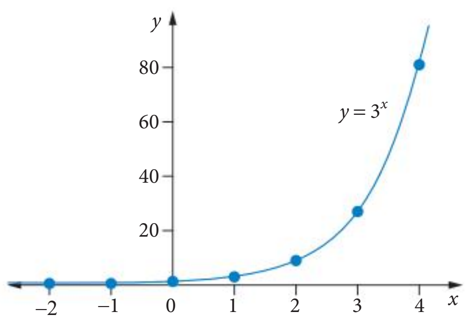
WORKED EXAMPLE 3 Sketching an exponential function $y = a^x$

Graph $y = 3^x$ for $[-2, 4]$ and comment on the graph's features.

Steps	Working
a **1** Draw up a table of values using the domain values given.	see table below
2 Plot the points and join with a smooth curve.	see graph below
3 Examine the features of the graph.	The graph has a y-intercept of 1. The graph is always positive and is increasing as x increases. The gradient of the graph is also increasing as x increases.

x	-2	-1	0	1	2	3	4
y	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9	27	81



Properties of the exponential function $y = a^x$ where $a > 1$

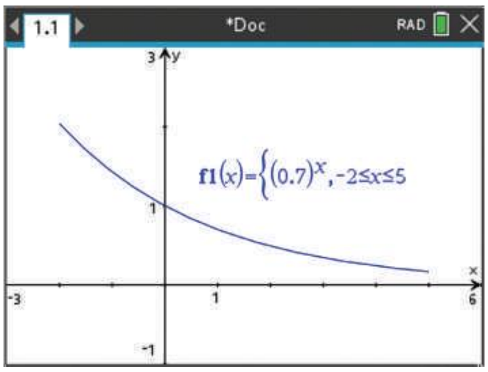
- An increasing function: increasing slowly at first, then more quickly.
- The gradient of the graph is also always increasing.
- y-intercept of 1 (because $a^0 = 1$).
- Domain: $x \in R$
- Range: $y \in R^+$: y is always positive.
- The x-axis is an asymptote.
- As x increases, y approaches ∞.
- As x decreases, y approaches 0.

Now let's graph $y = a^x$ where a is a fraction: $0 < a < 1$.

USING CAS 1 Graphing exponential functions

Graph $y = 0.7^x$ for $-2 \le x \le 5$ and comment on the features of the graph.

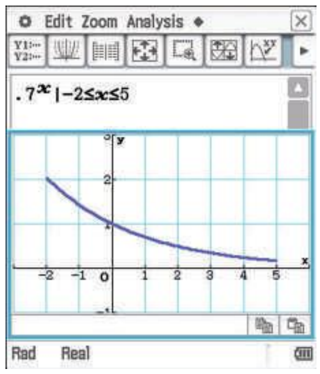
TI-Nspire



1 Add a **Graphs** page and graph the function over the domain [–2, 5].

2 Adjust the window settings to suit.

ClassPad



1 In **Main**, enter the expression using the **Math3** symbols | and ≤ for the domain [–2, 5].

2 Highlight and drag the expression into the Graph window.

3 Adjust the window settings to suit.

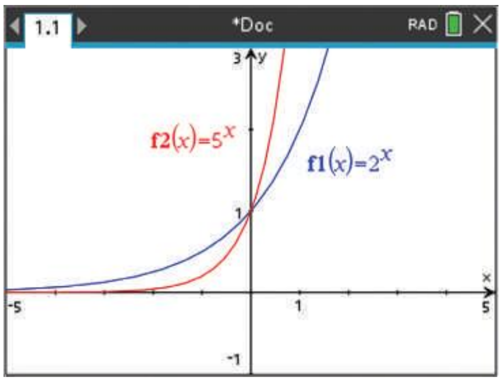
The graph has a y-intercept of 1. The graph is always positive but it is decreasing as x increases, getting closer to 0. The gradient of the graph is also increasing (becoming less negative) as x increases.

Properties of the exponential function $y = a^x$ where $0 < a < 1$

- A decreasing function: decreasing quickly at first, then more slowly.
- The gradient of the graph is always increasing (becoming less negative).
- y-intercept of 1 (because $a^0 = 1$).
- Domain: $x \in R$
- Range: $y \in R^+$: y is always positive.
- The x-axis is an asymptote.
- As x increases, y approaches 0.
- As x decreases, y approaches ∞.

These CAS screens compare the graphs of $y = 2^x$ and $y = 5^x$.

TI-Nspire



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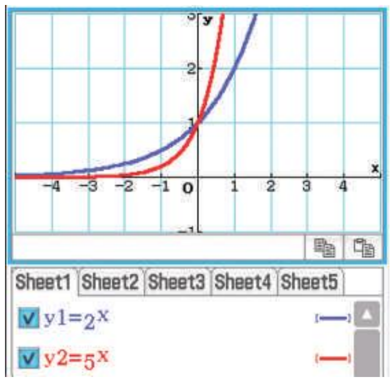
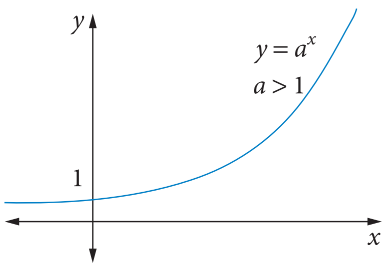


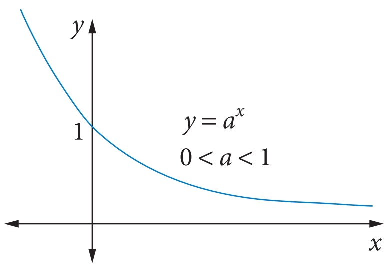
Comment on the similarities and differences between the above graphs. Describe how the graph of $y = a^x$ changes as the value of a changes.

Exponential functions of the form $y = a^x$ have two basic shapes.

Increasing exponential function



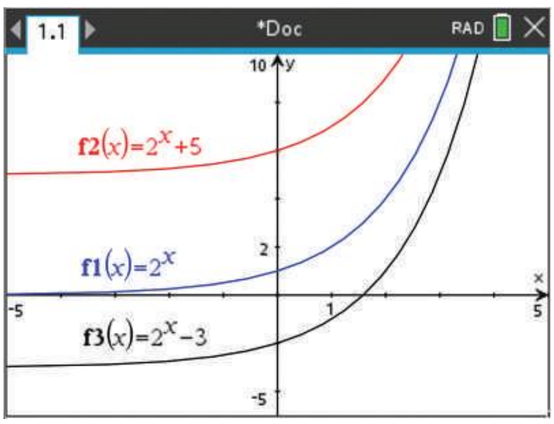
Decreasing exponential function



- y is always positive (the graph is above the x-axis).
- The x-axis is an asymptote.
- The y-intercept is 1.

These CAS screens compare the graphs of $y = 2^x$, $y = 2^x + 5$ and $y = 2^x - 3$.

TI-Nspire



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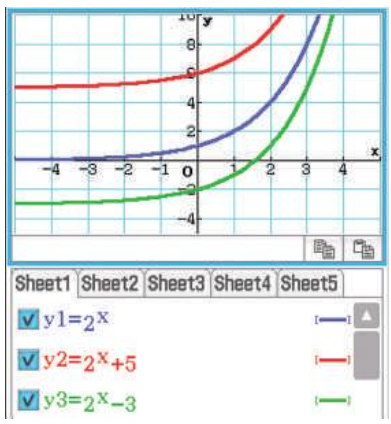

As an investigation, graph other exponential functions of the form $y = a^x + c$, and explain the significance of the value of k when comparing the graphs of $y = a^x$ and $y = a^x + c$.

A more general form of the exponential function is

$$y = Aa^{n(x+b)} + c$$

where the constants A, b, n and c refer to transformations to the basic graph of $y = a^x$.

- The effect of A is a (vertical) **dilation from the x-axis** (or **parallel to the y-axis**). For $A > 1$, the graph is stretched, dilated by a factor of A. For $0 < A < 1$, the graph is shrunk. If $A < 0$, then the graph is also reflected (vertically) in the x-axis.
- The effect of n is a (horizontal) **dilation from the y-axis** (or **parallel to the x-axis**). For $n > 1$, the graph is shrunk, dilated by a factor of $\frac{1}{n}$. If $0 < n < 1$, the graph is stretched.
- The effect of c is a (vertical) **translation in the direction of the y-axis**. For $c > 0$, the graph is moved up c units (in the positive y direction), and for $c < 0$, the graph is moved down k units (in the negative y direction). The new horizontal asymptote is $y = c$.
- The effect of b is a (horizontal) **translation in the direction of the x-axis**. For $b > 0$, the graph is moved left h units (in the negative x direction), and for $b < 0$, the graph is moved right b units (in the positive x direction).

7.2

Exam hack

When graphing exponential functions of the form $y = Aa^{n(x+b)} + c$, remember that the function is increasing if the base $a > 1$ and decreasing if $0 < a < 1$.

WORKED EXAMPLE 4 Sketching graphs of exponential functions

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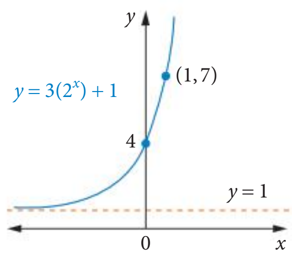
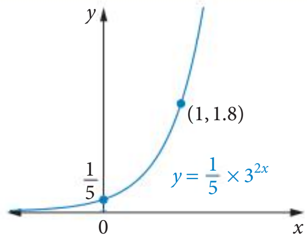
Sketch the graph of

a $y = 3(2^x) + 1$ **b** $y = \frac{1}{5} \times 3^{2x}$

Steps	Working
a **1** Compare $y = 3(2^x) + 1$ to $y = Aa^{nx} + c$ and identify each transformation.	$A = 3$: Dilation from the x-axis (vertical) by a factor of 3. $c = 1$: Translation 1 unit up (vertical).
2 Consider the new y-intercept and asymptote.	When $x = 0$, $y = 3(2^0) + 1 = 4$. asymptote: $y = 1$
3 Find an important point for scaling.	When $x = 1$, $y = 3(2^1) + 1 = 7$. (1, 7)
4 Sketch the graph.	
b **1** Compare $y = \frac{1}{5} \times 3^{2x}$ to $y = Aa^{nx} + c$ and identify each transformation.	$A = \frac{1}{5}$: Dilation from the x-axis by a factor of $\frac{1}{5}$. $n = 2$: Dilation from the y-axis by a factor of $\frac{1}{2}$.
2 Consider the new y-intercept and asymptote.	When $x = 0$, $y = \frac{1}{5}(3^{2\times 0}) = \frac{1}{5}$. asymptote: $y = 0$ (no change)
3 Find an important point for scaling.	When $x = 1$, $y = \frac{1}{5}(3^{2\times 1}) = 1.8$. (1, 1.8)

4 Sketch the graph.



Finding equations of exponential functions

Finding the equation of exponential functions is often a multi-step process, and every problem is different based upon the information and type of graph we are given. Given the graph of exponential functions, we need to be able to take some information from the graph itself, and then solve for what we are unable to take directly from the graph.

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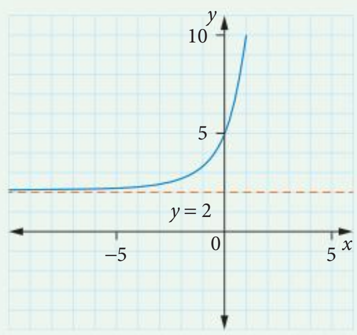
WORKED EXAMPLE 5 Finding the rule of an exponential function

The rule for the function with the graph shown is of the form

$y = A(4^x) + c$

a Find the values of A and c.

b Hence, state the equation of the function.



Steps	Working
a 1 Using the horizontal asymptote find the value of c.	$y = 2$ is a vertical translation of the asymptote 3 units in the positive direction of the y-axis. Therefore $c = 2$.
2 Substitute $(0, 5)$ into the equation to find A.	$y = A(4^x) + 2$ $5 = A(4^0) + 2$ $5 = A(1) + 2$ $A = 3$
b Write the equation of the function.	$y = 3(4^x) + 2$

EXERCISE 7.2 Transformations of the exponential function

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Recap

1 Simplify $5^{4n-7} \times 5^{5n+3}$.

2 Evaluate $(49)^{\frac{3}{2}}$.

Mastery

3 WORKED EXAMPLE 3 TECH-FREE Sketch each function for the stated domain. Comment on the features of the graph.

a $y = 2^x$ for $-3 \le x \le 4$

b $y = 5 \times 2^x$ for $-4 \le x \le 3$

c $y = 4 \times 0.6^x$ for $-2 \le x \le 4$

d $y = 10 \times 1.8^x$ for $-3 \le x \le 3$

e $y = 40 \times 0.2^x$ for $-2 \le x \le 4$

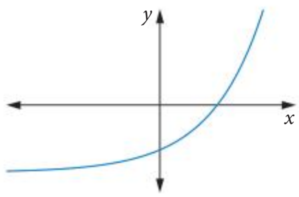
f $y = 10^x$ for $-2 \le x \le 3$

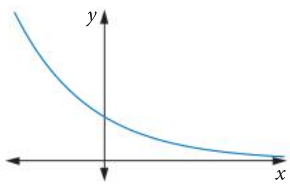
4 Using CAS 1 Use CAS to graph each function. Comment on the features of the graph.

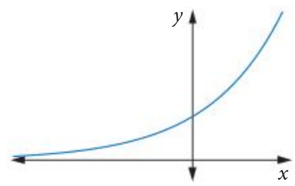
a $y = 8.5 \times 2.5^x$ for $-3 \le x \le 3$
b $y = 20 \times 0.9^x$ for $-5 \le x \le 5$
c $y = 16 \times 1.2^x$ for $-5 \le x \le 5$
d $y = 16 \times 2^x$ for $-5 \le x \le 5$
e $y = 16 \times 0.1^x$ for $-3 \le x \le 3$
f $y = 8 \times 10^x$ for $-3 \le x \le 3$

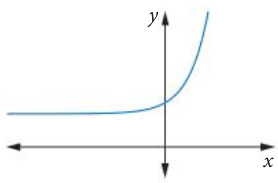
5 Match each function to its graph below.

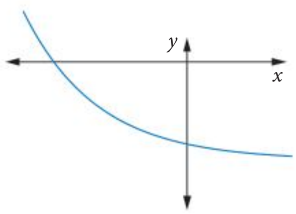
a $y = 2^x + 3$
b $y = 2^x$
c $y = 2^x - 3$
d $y = \left(\frac{1}{2}\right)^x$

A


B


C


D


E


6 WORKED EXAMPLE 4 TECH-FREE For each function, find its y-intercept and asymptote, then graph the function.

a $y = 4^x - 3$
b $y = 2^{3x} - 1$
c $y = 3(2^x) + 2$
d $y = \left(\frac{1}{2}\right)^x - 1$
e $y = 5^{2x} - 4$
f $y = 2^{-x} + 3$
g $y = 5^{x-2} + 1$
h $y = 3^{-2x-1} - 1$

7 Which of the following is an increasing exponential function?

A $y = 3 \times 0.9^x$
B $y = 7 \times 0.8^x$
C $y = 2 \times 3.4^x$
D $y = 5 \times 0.1^x$
E $y = 1.1 \times 0.7^x$

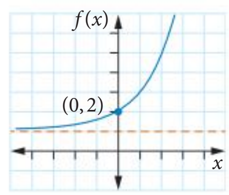
8 Which of the following is true for the graph of $y = 2.5 \times 1.8^x$?

A The graph has a y-intercept at 1.8.
B As x increases, y gets closer to 0.
C The graph is decreasing.
D As x decreases, y gets closer to 0.
E The graph has an x-intercept at 2.5.

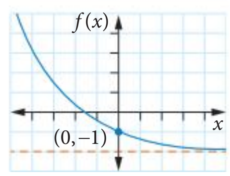
9 Which of the following graphs of exponential functions rises the most steeply?

A $y = 4 \times 2.1^x$
B $y = 4 \times 1.8^x$
C $y = 4 \times 4.5^x$
D $y = 4 \times 3.2^x$
E $y = 4 \times 6.3^x$

10 WORKED EXAMPLE 5 The rule for the function with the graph shown is of the form $y = A(3)^x + c$. Find the values of A and c. Hence, state the equation of the function.



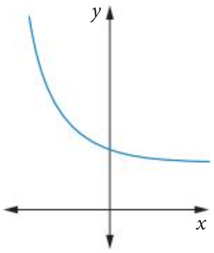
11 The rule for the function with the graph shown is of the form $y = A(2)^{-x} + c$. Find the equation of the function.



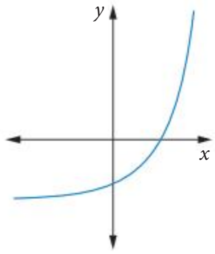
80–100% 60–79% 0–59%

12 Which graph best represents the function $y = a^x - b$, where $a > 1$ and $b > 0$?

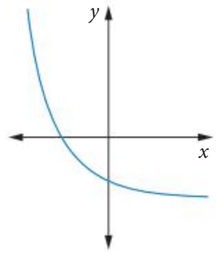
A



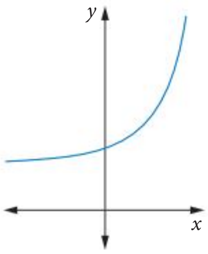
B



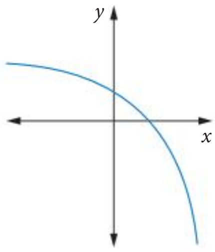
C



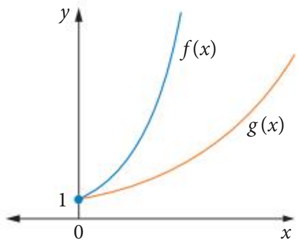
D



E



13 Which pair of equations best describes the graphed functions for $x \geq 0$?



A $f(x) = \left(\frac{1}{2}\right)^x$ and $g(x) = \left(\frac{1}{5}\right)^x$

B $f(x) = a^x$ and $g(x) = b^x$, where $0 < a < b$

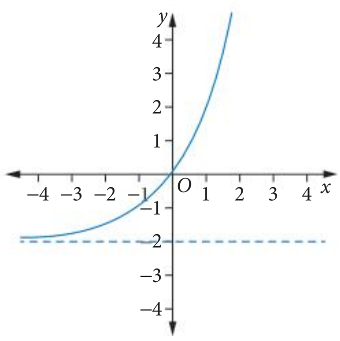
C $f(x) = a^x$ and $g(x) = b^x$, where $a = b$

D $f(x) = a^x$ and $g(x) = b^x$, where $0 < a < b < 1$

E $f(x) = a^x$ and $g(x) = b^x$, where $a > b > 1$

14 © VCAA 2017 2BQ4a 62% (2 marks) Let $f: R \to R: f(x) = 2^{x+1} - 2$. Part of the graph of f is shown below.

The transformation $T: R^2 \to R^2$, $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix}$ maps the graph of $y = 2^x$ onto the graph of f. State the values of c and d.



7.3 Exponential growth and decay

Any quantity that **increases** according to the exponential function $y = A(a^x)$, where $a > 1$, is showing **exponential growth**. Something that grows exponentially increases slowly at first, then more quickly. Examples of exponential growth are population size (people, animals, bacteria), an investment attracting compound interest, the size of a bushfire and a flu or computer virus.

Any quantity that **decreases** according to the exponential function $y = A(a^x)$, where a is between 0 and 1, is showing **exponential decay**. Something that decays exponentially decreases quickly at first, then more slowly. Examples of exponential decay are radioactive decay, the cooling of substances, the intensity of light in water and the dampening of vibrations.

Exponential growth and decay

The general form of an exponential growth or decay function is

$$y = A(a^x)$$

where the y-intercept A is the initial value or starting point and a is the growth factor.

For a growth function, $a > 1$.

For a decay function, $0 < a < 1$.

Video playlist Exponential growth and decay

Worksheet Using exponential models

WB p. 142

WORKED EXAMPLE 6 Exponential growth modelling

A bacterial colony has about 2000 bacteria. The number of bacteria triples every day. Assuming that growth continues in this way, find how many bacteria will be in the colony

a after 5 days **b** after 60 hours

Steps	Working
This is an example of exponential growth.	growth factor $a = 3$
The number of bacteria triples every day, or the new number of bacteria is 300% of the previous number.	initial number $A = 2000$
The initial number is 2000.	
Write the growth function for the number of bacteria after d days.	$N(d) = 2000 \times 3^d$
a For the number of bacteria after 5 days, substitute $d = 5$.	$N(5) = 2000(3^5) = 486\,000$ There will be 486 000 bacteria after 5 days.
b For the number of bacteria after 60 hours, convert to days first. 60 hours $= \frac{60}{24} = 2\frac{1}{2}$ days, so substitute $d = 2.5$.	$N(2.5) = 2000(3^{2.5}) = 31\,176.914\ldots$ There will be about 31 177 bacteria after 60 hours.

WB p. 143

WORKED EXAMPLE 7 Exponential decay modelling

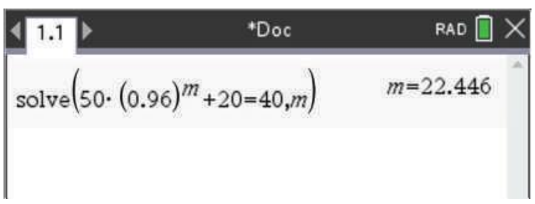
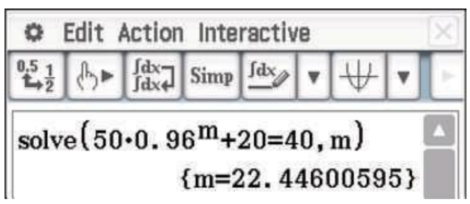
The amount of air left (and thus pressure) in a leaking spaceship is decreasing by 30% every minute. When the leak first occurred, the air pressure was 105 kPa (kilopascals). What will be the pressure (correct to one decimal place) after 6 minutes?

Steps	Working
1 This is an example of exponential decay. The pressure decreases by 30% every minute, or the new pressure is 70% of the previous value. The initial value, A, is 105 kPa.	growth factor $a = 0.7$ $A = 105$
2 Write the decay function for pressure after m minutes.	$P(m) = 105 \times (0.7)^m$
3 For the air pressure after 6 minutes, substitute $m = 6$.	$P(6) = 105(0.7)^6 \approx 12.4$ After 6 minutes, the pressure will be about 12.4 kPa.

WB p. 144

WORKED EXAMPLE 8 Further exponential decay modelling

The difference between the temperature of a cup of coffee and its surroundings decreases by 4% every minute. A fresh cup has a temperature of 70°C, and the room temperature is 20°C. A cup of coffee is considered too cold to drink if it is below 40°C. Model the temperature of the cup of coffee and find how long (to the nearest minute) it takes to become undrinkable.

Steps	Working
1 This is an example of exponential decay. The *difference* in temperature between the coffee and the room decreases by 4% each minute. Write the relationship between the functions. Room temperature = 20°C	Let $C(m)$ be the coffee temperature after m minutes. Let $D(m)$ be the difference between the coffee and room temperatures. $D(m) = C(m) - 20$
2 Write $D(m)$ as an exponential function. $D(m)$ decreases by 4% each minute so $a = 96\% = 0.96$. The initial difference was 70°C – 20°C.	growth factor $a = 0.96$ $A = 70 - 20 = 50$ $D(m) = 50(0.96)^m$
3 Make $C(m)$ the subject.	$D(m) = C(m) - 20 = 50(0.96)^m$ $C(m) = 50(0.96)^m + 20$
4 To find the time taken for the coffee to become undrinkable, we want $C(m) = 40$.	$C(5) = 50(0.96)^5 + 20 \approx 60.8$ $C(10) \approx 53.2$ $C(20) \approx 42.1$
5 Try some different values of m until $C(m)$ is close to 40. or use CAS to solve.	$C(25) \approx 38.0$ $C(22) \approx 40.4$ $C(23) \approx 39.6$ $C(22.5) \approx 39.96$
TI-Nspire	**ClassPad**
	
6 Answer the question.	It takes about 22 minutes for the coffee to become undrinkable.

An exponential growth or decay function can sometimes be modelled from a table of values. Look at the values below.

x	0	1	2	3	4
y	5	10	20	40	80

We can see that as x increases by 1, y doubles. Clearly, $y = 5 \times 2^x$.
We should check that this is true for the whole table.

EXERCISE 7.3 Exponential growth and decay

ANSWERS p. 521

Recap

1 Which of the following exponential functions has a graph that rises the most steeply?

A $y = 5 \times 3.1^x$ **B** $y = 5 \times 1.9^x$ **C** $y = 5 \times 4.8^x$

D $y = 5 \times 4.2^x$ **E** $y = 5 \times 6.3^x$

2 Which of the following is an increasing exponential function?

A $y = 5 \times 0.9^x$ **B** $y = 7 \times 0.8^x$ **C** $y = 2 \times 4.4^x$

D $y = 5 \times 0.2^x$ **E** $y = 1.3 \times 0.7^x$

Mastery

3 For each exponential function

i find the initial amount

ii find the growth factor.

a $f(x) = 45 \times (1.15)^x$

b $f(t) = 1100 \times (0.82)^t$

4 WORKED EXAMPLE 6 An algal bloom in a polluted lake covers an area of $3\,\text{m}^2$. The bloom grows in size by 5% every day.

a Set up an exponential model for this situation.

b Find the area covered by the algal bloom after

i 3 days **ii** 10 days **iii** 20 days

c Find when the algal bloom will cover $500\,\text{m}^2$.

5 An investment of \$3000 at compound interest grows by 8% every year.

a Set up an exponential model for the value of the investment.

b Find the value of the investment after

i 3 years **ii** 10 years **iii** 25 years

c Find the time taken for the investment to double in value.

6 WORKED EXAMPLE 7 The value of a car depreciates by 15% each year. The car was originally worth \$24 900.

a Set up an exponential model for the value of the car.

b Find the value of the car after

i 3 years **ii** 5 years **iii** 10 years

c Find the time taken for its value to reduce to \$1000.

7 WORKED EXAMPLE 8 An oven has a maximum temperature of 500°C. The temperature of the room is 20°C. When the oven is switched on, its temperature T increases each minute by 10% of the difference between its temperature and the theoretical maximum; that is, $(500 - T)$ decreases by 10% each minute.

a Set up a model of the oven temperature m minutes after it is switched on.

b Find the temperature of the oven after

i 5 minutes **ii** 10 minutes **iii** 25 minutes

c Find the time taken for the oven to reach a temperature of 400°C.

8

m	0	1	2	3	4
p	3	6	12	24	48

For this table of values,

a find the rule for p as a function of m

b find the value of p when $m = 8$.

9

x	0	1	2	3	4
q	24	43.2	77.76	139.968	251.9424

For this table of values,

a find the rule for q as a function of x

b find the value of q when $x = 2.7$.

Exam practice

10 (3 marks) The population of Doreen can be modelled by $P = 6191(1.04)^t$, where t is the number of years since 1990.

a What was the population in 1990? 1 mark

b By what percent did the population increase by each year? 2 marks

11 (2 marks) Find a bank account balance if the account starts with \$100, has an annual rate of 4% and the money is left in the account for 12 years.

12 (2 marks) Bacteria can multiply at an alarming rate when each bacteria splits into 2 new cells, thus doubling. If we start with only one bacteria, which can double every hour, how many bacteria will we have by the end of one day?

13 (2 marks) In 1985, there were 285 mobile phone subscribers in the small town of Centreville. The number of subscribers increased by 75% per year after 1985. How many mobile phone subscribers were in Centreville in 1994?

Video playlist Logarithms

Puzzle Logarithms

7.4 Logarithms

$3^{16} = 43\,046\,721$

This means that when we multiply 3 by itself 16 times, we get 43 046 721.

Another way of writing this is by using a **logarithm**, which is abbreviated as **log**.

$\log_3 (43\,046\,721) = 16$, which is read 'the logarithm of 43 046 721, to base 3, is 16'.

The logarithm of a number to base a is the **power** to which a must be raised to give that number.

Now consider $\log_5(625)$, which means the power of 5 needed to get 625.

Since $5^4 = 625$, $\log_5(625) = 4$.

If $a^x = b$, then $\log_a(b) = x$.

The logarithm of b to the base a is the power x such that $a^x = b$, where

$a > 0$ and $a \neq 1$, and

$b > 0$: we cannot find the logarithm of a negative number or 0.

Logarithms

If $a^x = b$, then $\log_a(b) = x$

7.4

WB p. 145

WORKED EXAMPLE 9 Converting to logarithmic form

Write each statement in logarithmic form.

a $5^2 = 25$ **b** $2^{-3} = \frac{1}{8}$

Steps	Working
a Write as $\log_a(b) = x$, where a is the base and x is the power. base = 5, power = 2	$5^2 = 25$ $\log_5(25) = 2$
b base = 2, power = −3	$2^{-3} = \frac{1}{8}$ $\log_2\left(\frac{1}{8}\right) = -3$

WB p. 146

WORKED EXAMPLE 10 Converting to exponential form

Write each statement in index form.

a $\log_2(64) = 6$ **b** $\log_7\left(\frac{1}{7}\right) = -1$

Steps	Working
a Write as $a^x = b$, where a is the base and x is the power. base = 2, power = 6	$\log_2(64) = 6$ $2^6 = 64$
b base = 7, power = −1	$\log_7\left(\frac{1}{7}\right) = -1$ $7^{-1} = \frac{1}{7}$

Exam hack

When you have to find the logarithm of a number to a particular base, you need to think 'What power of this will give the number?' or 'How many times do I need to multiply the base to get the number?'

WB p. 147

WORKED EXAMPLE 11 Evaluating logarithms

Evaluate each logarithm.

a $\log_4(64)$ **b** $\log_3(9)$ **c** $\log_{\frac{1}{2}}\left(\frac{1}{32}\right)$

d $\log_6\left(\frac{1}{216}\right)$ **e** $\log_{\frac{1}{5}}(25)$ **f** $\log_9(1)$

Steps	Working
a **1** Think $4^? = 64$.	$4^3 = 64$
2 Evaluate the logarithm.	So $\log_4(64) = 3$.
b **1** Think $3^? = 9$.	$3^2 = 9$
2 Evaluate the logarithm.	So $\log_3(9) = 2$.
c **1** Think $\left(\frac{1}{2}\right)^? = \frac{1}{32}$.	$\left(\frac{1}{2}\right)^5 = \frac{1}{32}$
2 Evaluate the logarithm.	So $\log_{\frac{1}{2}}\left(\frac{1}{32}\right) = 5$.
d **1** Think $6^? = \frac{1}{216}$. The power must be negative.	$6^{-3} = \frac{1}{216}$
2 Evaluate the logarithm.	So $\log_6\left(\frac{1}{216}\right) = -3$.
e **1** Think $\left(\frac{1}{5}\right)^? = 25$. The power must be negative.	$\left(\frac{1}{5}\right)^{-2} = 25$
2 Evaluate the logarithm.	So $\log_{\frac{1}{5}}(25) = -2$.
f **1** Think $9^? = 1$.	$9^0 = 1$ $\log_a(1) = 0$ always, because $a^0 = 1$.
2 Evaluate the logarithm.	So $\log_9(1) = 0$.

Exam hack

Your calculator has a log function for calculating $\log_{10}(x)$; that is, logarithms to base 10. Use it to check the answer to $\log_{10}(100\,000)$.

EXERCISE 7.4 Logarithms

ANSWERS p. 521

Recap

1 An investment of \$2000 at compound interest grows by 9% every year. Write an exponential model for the value of the investment.

2 A purchase of \$2000 at compound interest depreciates by 9% every year. Write an exponential model for the value of the investment.

Mastery

3 WORKED EXAMPLE 9 Write each statement in logarithmic form.

a $7^2 = 49$ **b** $3^3 = 27$ **c** $2^4 = 16$ **d** $5^3 = 125$

e $11^0 = 1$ **f** $(2)^0 = 1$ **g** $5^{-2} = \frac{1}{25}$ **h** $4^{-2} = \frac{1}{16}$

i $10^{-3} = \frac{1}{1000}$ **j** $\left(\frac{1}{3}\right)^4 = \frac{1}{81}$ **k** $\left(\frac{1}{4}\right)^3 = \frac{1}{64}$ **l** $\left(\frac{1}{2}\right)^3 = \frac{1}{8}$

m $6^{\frac{2}{3}} = \sqrt[3]{36}$ **n** $7^{\frac{3}{5}} = \sqrt[5]{343}$ **o** $a^k = r$ **p** $b^3 = d$

4 WORKED EXAMPLE 10 Write each statement in index form.

a $\log_5(25) = 2$ **b** $\log_4(16) = 2$ **c** $\log_5(125) = 3$ **d** $\log_2(16) = 4$

e $\log_3(3) = 1$ **f** $\log_7(49) = 2$ **g** $\log_2(128) = 7$ **h** $\log_5(1) = 0$

i $\log_8(2) = \frac{1}{3}$ **j** $\log_4\left(\frac{1}{2}\right) = -\frac{1}{2}$ **k** $\log_4\left(\sqrt{2}\right) = \frac{1}{4}$ **l** $\log_3\left(\frac{1}{\sqrt[3]{3}}\right) = -\frac{1}{3}$

m $\log_2\left(\frac{\sqrt{2}}{4}\right) = -\frac{3}{2}$ **n** $\log_a(b) = c$ **o** $\log_c\left(\sqrt{a}\right) = 3m$

5 WORKED EXAMPLE 11 TECH-FREE Evaluate each logarithm.

a $\log_2(64)$ **b** $\log_9(81)$ **c** $\log_3(81)$ **d** $\log_7(343)$

e $\log_6(216)$ **f** $\log_5(1)$ **g** $\log_3(3)$ **h** $\log_{10}(100\,000)$

i $\log_3(243)$ **j** $\log_4(1024)$ **k** $\log_{\frac{1}{2}}\left(\frac{1}{16}\right)$ **l** $\log_5\left(\frac{1}{125}\right)$

m $\log_{\frac{1}{4}}(16)$ **n** $\log_{\frac{1}{4}}\left(\frac{1}{256}\right)$ **o** $\log_2\left(\frac{1}{128}\right)$ **p** $\log_{\frac{1}{2}}(512)$

q $\log_{\frac{1}{3}}\left(\frac{1}{81}\right)$ **r** $\log_7\left(\frac{1}{7}\right)$ **s** $\log_{\frac{1}{3}}(27)$ **t** $\log_{\frac{1}{10}}(0.001)$

Exam practice

6 TECH-FREE (10 marks) Evaluate each logarithm.

a $\log_2\left(\sqrt{2}\right)$ **b** $\log_9\left(9\sqrt{9}\right)$ **c** $\log_4\left(\sqrt{64}\right)$ **d** $\log_7\left(\sqrt{343}\right)$

e $\log_6\left(\sqrt[3]{36}\right)$ **f** $\log_4(8)$ **g** $\log_9(27)$ **h** $\log_8(16)$

i $\log_{64}(32)$ **j** $\log_{27}(243)$

7 (2 marks) If $x = \sqrt[m]{a^p}$, show that $\log_a(x^m) = p$.

7.5 Logarithm laws

Video playlist Logarithm laws

Worksheets Logarithm laws

Logarithms review

Logarithms

$\log_a(1) = 0$ because $a^0 = 1$.

$\log_a(a) = 1$ because $a^1 = a$.

$\log_a\left(\frac{1}{a}\right) = -1$ because $a^{-1} = \frac{1}{a}$.

$\log_a(a^x) = x$

WB

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WORKED EXAMPLE 12 Evaluating logarithms using log laws

Evaluate each logarithm.

a $\log_3(1)$	**b** $\log_7(7)$	**c** $\log_5(0)$
d $\log_2(-3)$	**e** $\log_1(5)$	**f** $\log_{10}(0.1)$

Steps	Working
a The log of 1 to any base is 0.	$\log_3(1) = 0$
b The log of any number to its own base is 1.	$\log_7(7) = 1$
c We can only have logarithms of *positive* numbers. We cannot find the logarithm of a negative number or 0. The base of a logarithm cannot be negative, 0 or 1.	$\log_5(0)$ is undefined.
d We can only have logarithms of *positive* numbers.	$\log_2(-3)$ is undefined.
e The base of a log cannot be 1.	$\log_1(5)$ is undefined.
f The logarithm of the reciprocal of any number to its own base is −1, and $0.1 = \frac{1}{10}$.	$\log_{10}(0.1) = -1$

Since $3^2 = 9$, $3^4 = 81$ and $3^6 = 729$, then

$\log_3(9) = 2$, $\log_3(81) = 4$ and $\log_3(729) = 6$.

Notice that $9 \times 81 = 729$ and $2 + 4 = 6$,
so $\log_3(9 \times 81) = \log_3(9) + \log_3(81)$.

Logarithm of a product

$$\log_a(xy) = \log_a(x) + \log_a(y)$$

This can be proved using the index law $a^m \times a^n = a^{m+n}$.

Let $x = a^m$ and $y = a^n$.

Then $m = \log_a(x)$ and $n = \log_a(y)$.

Now $\log_a(xy) = \log_a(a^m \times a^n) = \log_a(a^{m+n}) = m + n$.

But $m = \log_a(x)$ and $n = \log_a(y)$,
so $\log_a(xy) = \log_a(x) + \log_a(y)$.

Logarithm of a quotient

$$\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)$$

This can be proved using the index law $\frac{a^m}{a^n} = a^{m-n}$.

Let $x = a^m$ and $y = a^n$.

Then $m = \log_a(x)$ and $n = \log_a(y)$.

$$\text{Now } \log_a\left(\frac{x}{y}\right) = \log_a\left(\frac{a^m}{a^n}\right)$$

$$= \log_a(a^{m-n}) = m - n = \log_a(x) - \log_a(y).$$

Logarithm of a power

$$\log_a(x^n) = n\log_a(x)$$

This can be proved using the index law $(a^m)^n = a^{mn}$.

Let $x = a^m$ and $y = a^n$.

Then $m = \log_a(x)$ and $n = \log_a(y)$.

Now $\log_a(x^n) = \log_a[(a^m)^n] = \log_a(a^{mn}) = mn = nm$.

But $m = \log_a(x)$,

so $\log_a(x^n) = n\log_a(x)$.

Logarithm laws

$$\log_a(xy) = \log_a(x) + \log_a(y)$$

$$\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)$$

$$\log_a(x^n) = n\log_a(x)$$

7.5

WORKED EXAMPLE 13 Simplifying and evaluating logarithms using log laws

WB
p. 149

Simplify each expression.

a $\log_4(2)$ **b** $\log_6(18) + \log_6(12)$ **c** $\dfrac{\log_2(27)}{\log_2(81)}$

Steps	Working
a Write 2 as a power of 4 and simplify.	$\log_4(2) = \log_4\left(\sqrt{4}\right) = \log_4\left(4^{\frac{1}{2}}\right) = \frac{1}{2}$
b 1 Use $\log_a(x) + \log_a(y) = \log_a(xy)$.	$\log_6(18) + \log_6(12) = \log_6(18 \times 12)$
2 Write 216 as a power of 6 and simplify.	$= \log_6(216)$ $= \log_6(6^3) = 3$
c 1 Write in terms of common powers of 3.	$\dfrac{\log_2(27)}{\log_2(81)} = \dfrac{\log_2(3^3)}{\log_2(3^4)}$
2 Use $\log_a(x^n) = n\log_a(x)$ and simplify.	$= \dfrac{3\log_2(3)}{4\log_2(3)}$ $= \dfrac{3}{4}$

WORKED EXAMPLE 14 Simplifying logarithms into a single expression

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p. 150

Simplify each expression to a single logarithm.

a $3\log_2(x) - 4\log_2(x+3) + \log_2(y)$ **b** $2\log_5\left(\sqrt{x}\right) - \log_5(5x)$

Steps	Working
a 1 Use $n\log_a(x) = \log_a(x^n)$.	$3\log_2(x) - 4\log_2(x+3) + \log_2(y)$ $= \log_2(x^3) - \log_2(x+3)^4 + \log_2(y)$
2 Use $\log_a(x) + \log_a(y) = \log_a(xy)$.	$= \log_2(x^3y) - \log_2(x+3)^4$
3 Use $\log_a(x) - \log_a(y) = \log_a\left(\frac{x}{y}\right)$.	$= \log_2\left[\dfrac{x^3y}{(x+3)^4}\right]$

b **1** Use $n \log_a(x) = \log_a(x^n)$.

$$2\log_5\left(\sqrt{x}\right) - \log_5(5x)$$
$$= \log_5\left[\left(\sqrt{x}\right)^2\right] - \log_5(5x)$$
$$= \log_5(x) - \log_5(5x)$$

2 Use $\log_a(x) - \log_a(y) = \log_a\left(\frac{x}{y}\right)$ and simplify.

$$= \log_5\left(\frac{x}{5x}\right)$$
$$= \log_5\left(\frac{1}{5}\right)$$
$$= -1$$

The change of base law

Logarithms were invented in 1614 by the Scottish mathematician John Napier for calculation purposes. Before the introduction of calculators in the 1970s, the use of logarithm tables was taught in schools, particularly for multiplying and dividing large numbers.

Calculators can evaluate a logarithm to any base, but it has two main logarithm functions, **common logarithms** (base 10) shown by **log** and **natural logarithms** (base e) shown by **ln**, which you will learn more about in Year 12.

Let $x = b^m$, then $m = \log_b(x)$.

Now $\log_a(x) = \log_a(b^m) = m\log_a(b)$.

Making m the subject:

$$m = \frac{\log_a(x)}{\log_a(b)}$$

But $m = \log_b(x)$,

so $\log_b(x) = \dfrac{\log_a(x)}{\log_a(b)}$.

Change of base law

To convert a logarithm from base a to base b, use this formula:

$$\log_b(x) = \frac{\log_a(x)}{\log_a(b)}$$

This formula is useful for calculating logarithms to bases other than 10 and e.

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WORKED EXAMPLE 15 **Evaluating logarithms using the change of base law**

Use the change of base theorem to find the value of $\log_5(27)$ correct to 3 decimal places.

Steps	Working
1 Use the change of base theorem.	$\log_5(27) = \dfrac{\log_{10}(27)}{\log_{10}(5)}$
2 Solve using CAS.	

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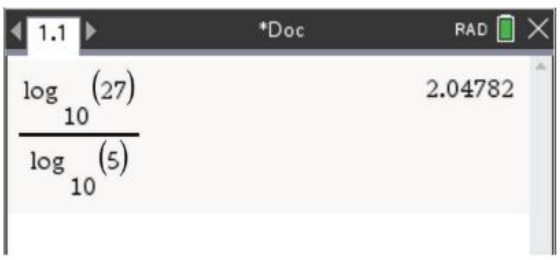
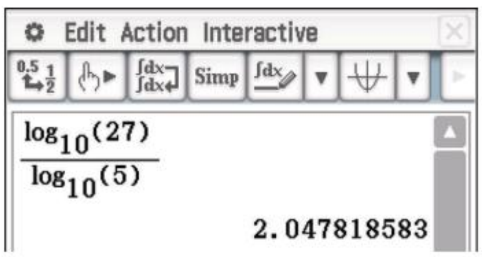


ClassPad



3 Write the answer.	$\log_5(27) \approx 2.048$

EXERCISE 7.5 Logarithm laws

ANSWERS p. 522

7.5

Recap

1 Evaluate $\log_5(1)$.

2 Write $\log_5(25) = 2$ in index form.

Mastery

3 WORKED EXAMPLE 12 TECH-FREE Evaluate each logarithm.

a $\log_3(1)$ b $\log_3(3)$ c $5\log_a(a)$ d $\log_x(1)$

e $\log_x(x)$ f $\log_7(49)$ g $\log_4\left(\frac{1}{4}\right)$ h $-3\log_5(5)$

i $\log_3(-3)$ j $\log_5(0) + 10$ k $[\log_3(1)]^2$ l $\log_2\left(\frac{1}{4}\right)$

m $\log_r(r^2)$ n $\log_5(1) - 4$ o $\log_e(e)$ p $\log_7(7^a)$

q $10\log_0(10)$ r $2 - \log_2(1)$ s $7 + \log_2(0.5)$ t $4\log_{10}(0.01)$

4 WORKED EXAMPLE 13 TECH-FREE Evaluate each logarithm.

a $\log_2(64)$ b $\log_4(64)$ c $\log_3\left(\sqrt{3}\right)$ d $\log_5\left(5\sqrt{5}\right)$

e $\log_7\left(\frac{1}{\sqrt{7}}\right)$ f $\log_6\left(\frac{1}{216}\right)$ g $\log_2\left(\frac{1}{16}\right)$ h $2\log_5(125)$

5 WORKED EXAMPLE 14 TECH-FREE Simplify and evaluate each expression.

a $\log_4(10) + \log_4(2) - \log_4(5)$

b $\log_5(25) + \log_5(125) - \log_5(625)$

c $\log_8\left(\frac{1}{8}\right) + \log_8(4)$

d $\log_2(16) + \log_2(4) + \log_2(8)$

e $\log_{10}(400) + \log_{10}(10) - \log_{10}(4)$

f $\log_5(8) - \log_5(4) - \log_5(2)$

g $\log_8(2) - \log_8\left(\frac{1}{4}\right)$

h $\log_4(256) - \log_4(32) + \log_4(2)$

6 WORKED EXAMPLE 15 Evaluate each logarithm using the change of base formula, correct to four significant figures.

a $\log_3(6)$ b $\log_{12}(2)$ c $\log_5(15)$

d $\log_{25}(4)$ e $\log_8(1.3)$ f $\log_2(100)$

Exam practice

7 (12 marks) Simplify each expression as a single logarithm.

a $5\log_4(x) + \log_4(x^2) - \log_4(x^3)$

b $3\log_7(x) - 5\log_7(x) + 4\log_7(x)$

c $4\log_6(x) - \log_6(x^2) - \log_6(x^3)$

d $\log_2(x+2) - \log_2(x+2)^2$

e $\log_4[(x-1)^3] - \log_4[(x-1)^2]$

f $\log_3(x-3) + \log_3(x+3) - \log_3(x^2-9)$

8 (8 marks) Expand each expression using the logarithm laws.

a $\log_{10}\left(\frac{12a}{10}\right)$ b $\log_6\left[\left(\frac{a}{b}\right)^5\right]$ c $\log_3\left(\sqrt[5]{10x^3}\right)$ d $\log_4\left(\frac{\sqrt[3]{x^2a}}{y^2}\right)$

Video playlist
The logarithmic function $y = \log_a(x)$

Worksheet
Plotting log functions

Puzzles
Logarithmic graphs match-up

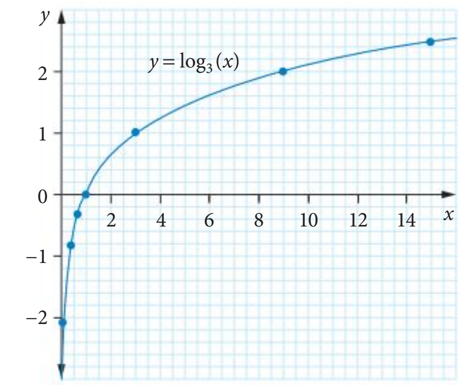
Exponential and log graphs match-up

7.6 The logarithmic function $y = \log_a(x)$

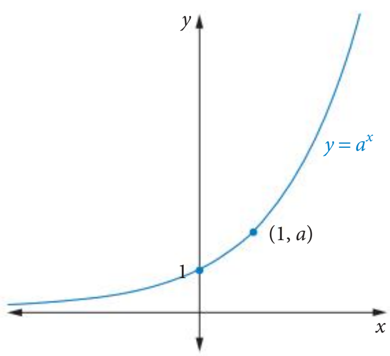
A **logarithmic function** takes the form $y = \log_a(x)$.

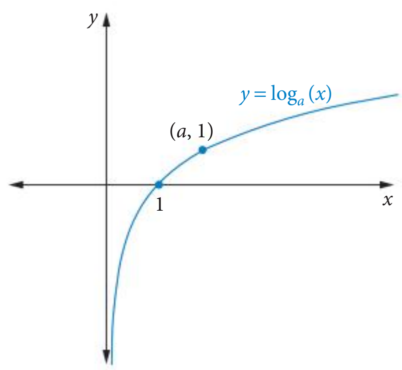
A table of values and the graph of $y = \log_3(x)$ are shown below:

x	$\frac{1}{9}$	$\frac{1}{3}$	0.7	1	3	9	15
y	-2	-1	-0.32	0	1	2	2.46

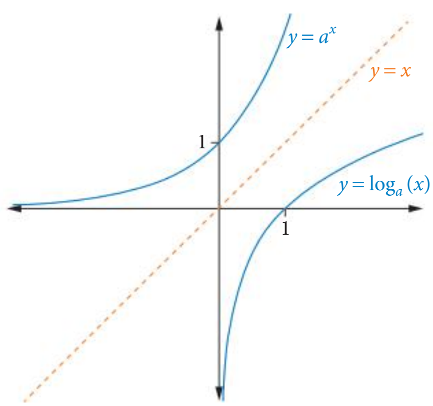


$y = \log_a(x)$ is the **inverse function** of $y = a^x$. The inverse function exists because $y = a^x$ is a **one-to-one function**; that is, each value in the range has only one ordered pair (x, y). In other words, as well as having only one y value for every x value, it also has only one x value for every y value.





The domain of $y = \log_a(x)$ is the range of $y = a^x$ and the range of $y = \log_a(x)$ is the domain of $y = a^x$.
The domain of $y = \log_a(x)$ is $x \in R^+$ and the range is $y \in R$. Because the domain and range swap, the graph of $y = \log_a(x)$ is the reflection of the graph of $y = a^x$ in the line $y = x$.



Furthermore, because the exponential and logarithmic functions are inverse functions,

$a^{\log_a(x)} = x$ and $\log_a(a^x) = x$

7.6

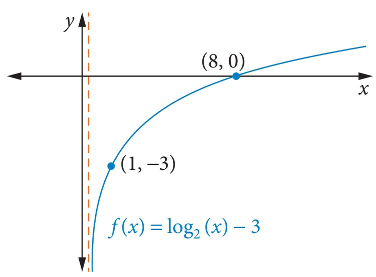
Properties of the logarithmic function $y = \log_a(x)$

- An increasing function: increasing quickly at first, then more slowly.
- The gradient of the graph is always decreasing.
- A one-to-one function.
- x-intercept of 1 (because $\log_a(1) = 0$).
- Domain: $x \in R^+$, x is always positive.
- Range: $y \in R$
- The y-axis is an asymptote.
- As x increases, y approaches ∞.
- As x approaches 0, y approaches $-\infty$.

WORKED EXAMPLE 16 Graphing logarithmic functions

WB p. 152

Sketch the graph of $f(x) = \log_2(x) - 3$, labelling important features.

Steps	Working
1 State the effect of the '−3'.	$\log_2(x)$ is translated 3 units in the negative y direction.
2 Find the x-intercept by solving $f(x) = 0$.	$f(x) = \log_2(x) - 3 = 0$ $\log_2(x) = 3$ $x = 2^3$ $= 8$ The x-intercept is 8.
3 State the asymptote.	The asymptote is at $x = 0$.
4 Since there is no y-intercept, state another point, or notice that the x-intercept (1, 0) is translated down 3 to (1, −3).	When $x = 1$, $y = \log_2(1) - 3$ $= 0 - 3$ $= -3$ The point (1, −3) is on the graph.
5 Sketch the graph.	

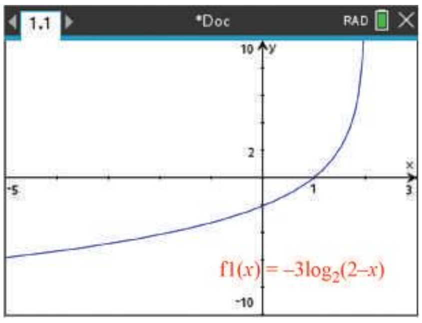
Exam hack

If a y-intercept does not exist on the graph, an extra important point known as a **guiding point** should be identified and plotted for scaling.

USING CAS 2 Graphing logarithmic functions

Graph $y = -3\log_2(2-x)$.

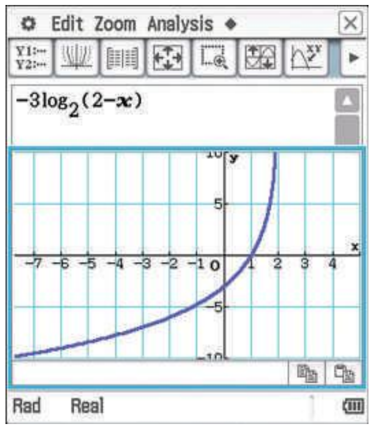
TI-Nspire



1 Add a **Graphs** page and enter the function as shown above.

2 Adjust the window settings to suit.

ClassPad



1 Using **Main**. enter the required expression using the **log** template from **Math1**.

2 Highlight and drag the expression down into the Graph window.

3 Adjust the window settings to suit.

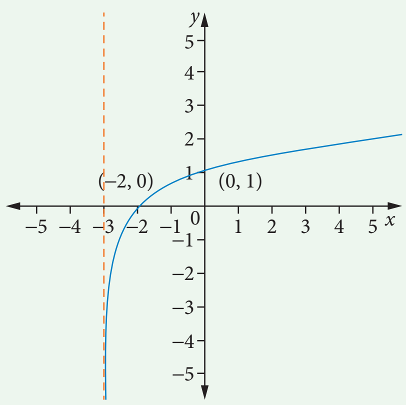
Finding equations of logarithmic functions

Finding the equation of logarithmic functions is often a multi-step process, and every problem is different based upon the information and type of graph we are given. Given the graph of a logarithmic function, we need to be able to take some information from the graph itself, and then solve for what we are unable to take directly from the graph.

WB

p. 153

WORKED EXAMPLE 17 Finding the rule of a logarithmic function



This is the graph of $y = a\log_3(x-b)$. Find the values of a and b, and hence state the equation of the function.

Steps	Working
1 Substitute (0, 1) into the equation.	$1 = a\log_3(0-b)$ $1 = a\log_3(-b)$
2 Substitute (−2, 0) into the equation.	$0 = a\log_3(-2-b)$

3 Solve for b.	$0 = \log_3(-2 - b)\ (a \neq 0)$ $3^0 = -2 - b$ $1 = -2 - b$ $b = -2 - 1$ $= -3$
4 Substitute $b = -3$ into the equation in step 1 to find a.	$1 = a\log_3(-(-3))$ $1 = a\log_3(3)$ $1 = a(1)$ $a = 1$
5 Write the equation.	$y = 1\log_3(x - (-3))$ $y = \log_3(x + 3)$

Applying logarithmic functions

Logarithmic functions are used in physical applications such as describing the magnitude of earthquakes, the intensity of sound and the acidity of a solution. A logarithmic model initially changes rapidly but the growth slows as it continues.

The Richter scale is a logarithmic scale that measures the size of an earthquake. One of the Richter formulas used to measure the magnitude of an earthquake is $M = \log_{10}\left(\frac{x}{a}\right)$, where x is the seismographic reading of the earthquake 100 km from its epicentre and $a = 0.001$ is the magnitude of an earthquake with a seismographic reading of 0.001 mm 100 km from its epicentre.

WORKED EXAMPLE 18 Applying logarithmic functions

WB p. 154

The San Francisco earthquake of 1906 had a seismographic reading of 7.943 metres 100 kilometres from its epicentre. What was its magnitude, correct to one decimal place?

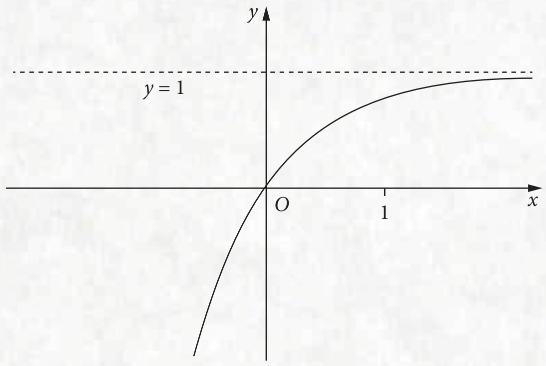
Steps	Working
1 Substitute $x = 7943$ into the formula. (7.943 m = 7943 mm)	$M = \log_{10}\left(\frac{7943}{0.001}\right)$ $= \log_{10}(7\,943\,000)$ $= 6.8999\ldots$ ≈ 6.9
2 Write the answer.	The magnitude was 6.9.

Video
VCE question analysis: The exponential and logarithmic functions

VCE QUESTION ANALYSIS

 2007 2BQ4 MODIFIED 2007 Examination 2 Section B Question 4 MODIFIED (10 marks)

Part of the graph of the function $f: R \to R, f(x) = a + b(10^{-x})$ is shown below. The line with equation $y = 1$ is an asymptote and the graph passes through the origin.



a Explain why $a = 1$ and $b = -1$. 2 marks

Let h be the function h: $[0, 2] \to R, h(x) = f(x)$.

b State the range of h using exact values. 2 marks

c **i** Find the inverse function h^{-1}. 1 mark

ii Copy the graph above and on it sketch and label the graph of the inverse function h^{-1}. 2 marks

d For $f(x) = 1 - 10^{-x}$

i show that $f(u)f(v) = f(u) + f(v) - f(u + v)$ where u and v are real numbers. 2 marks

ii show that $f(u)f(-u) = f(u) + f(-u)$ where u is real number. 1 mark

Reading the question

- The curve is an exponential function, so be prepared to apply knowledge of these function properties.
- The inverse function is a logarithmic function, so be prepared to use knowledge of this.
- Take note of formulas required, which questions will require working to be shown, and the specified accuracy of answers.
- Appreciate, in the context of the question, the significance of keywords and phrases such as **function**, **show**, **real**, **inverse** and **exact**.

Thinking about the question

- This question increases in difficulty as the question progresses, as do most extended-response questions.
- Many of the questions are 2-mark questions. Even though technology may be used, these questions require some working out.
- If you cannot answer some parts of the question, look for other sections of the question which are not directly linked and see if you are able to do these questions.
- Define any given function on CAS at the start of the question. You can use this defined function to calculate values, solve equations and sketch.

Worked solution (✓ = 1 mark)

a For an exponential function of the form $f(x) = a + b(10)^{-x}$, the asymptote is $y = a$. Therefore $a = 1$. ✓

Using (0, 0) $\quad 0 = 1 + b(10)^{-0}$

$\quad \boldsymbol{b = -1}$ ✓

b $\boldsymbol{h(0) = 1 - 10^0 = 0}$ (or given in question/graph)

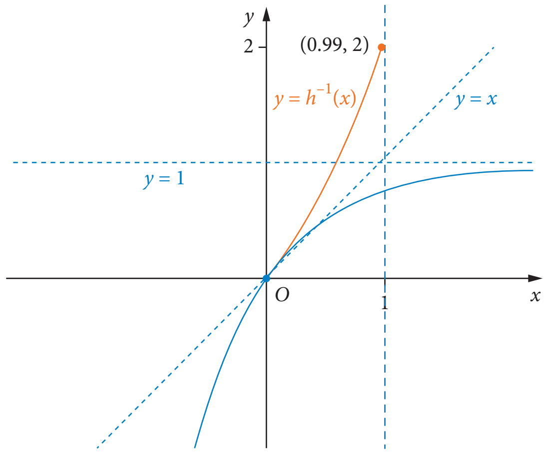
$h(0) = 1 - 10^{-2} = 1 - \frac{1}{100} = \frac{99}{100} = 0.99$ ✓

range $=$ **[0, 0.99]** ✓

c **i**
$$x = 1 - 10^{-y}$$
$$x - 1 = -10^{-y}$$
$$-x + 1 = 10^{-y}$$
$$\log_{10}(-x+1) = -y$$
$$y = -\log_{10}(-x+1)$$
$$h^{-1}(x) = \boldsymbol{-\log_{10}(1-x)}$$

where h^{-1}: $[0, 0.99] \to \boldsymbol{R}$ ✓

ii



✓✓

d **i** LHS $= f(u)f(v)$

$= (1 - 10^{-u})(1 - 10^{-v})$

$= 1 - 10^{-v} - 10^{-u} + 10^{-u}10^{-v}$

$\boldsymbol{= 1 - 10^{-v} - 10^{-u} + 10^{-u-v}}$ ✓

RHS $= f(u) + f(v) - f(u+v)$

$= 1 - 10^{-u} + 1 - 10^{-v} - (1 - 10^{-(u+v)})$

$= 1 - 10^{-u} - 10^{-v} + 10^{-u-v}$

$\boldsymbol{= 1 - 10^{-v} - 10^{-u} + 10^{-u-v}}$

$=$ **LHS** ✓

ii LHS $= f(u)f(-u)$

$= (1 - 10^{-u})(1 - 10^{u})$

$= 1 - 10^{u} - 10^{-u} + 10^{0}$

$\boldsymbol{= 2 - 10^{u} - 10^{-u}}$

RHS $= f(u) + f(-u)$

$= 1 - 10^{-u} + 1 - 10^{u}$

$= 2 - 10^{u} - 10^{-u}$

$=$ **LHS** ✓

Student performance 80–100% 60–79% 0–59%

a 45% Many students correctly explained why $a = 1$ but were not able to explain why $b = -1$.

b 66% Most students correctly answered this question. Common errors were using round brackets instead of square and not expressing the answer in an exact form.

c **i** 65% Always include the domain if an inverse function is required, unless only the rule is asked for in the question. Some students did not score full marks for leaving the domain out.

ii 40% Most students did not answer this question well. Most failed to restrict the domain.

d **i** 37% Students struggled with this unfamiliar ‘show that’ question. They need more practice with setting out the proof and not assume that LHS = RHS from the start.

ii 29% Many students were able to substitute correctly but unable to complete the question.

EXERCISE 7.6 The logarithmic function $y = \log_a(x)$

ANSWERS p. 522

Recap

1 Evaluate $\log_3(8) - \log_3(4) - \log_3(2)$.

2 Simplify $3\log_4(x) - 5\log_4(x) + 4\log_4(x)$.

Mastery

3 WORKED EXAMPLE 16 Graph each logarithmic function.

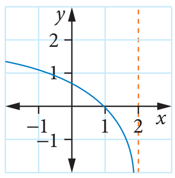
a $y = \log_2(x)$ **b** $y = \log_5(x)$ **c** $y = \log_4(x) + 1$ **d** $y = \log_3(x) - 2$

4 Using CAS 2 Graph each logarithmic function.

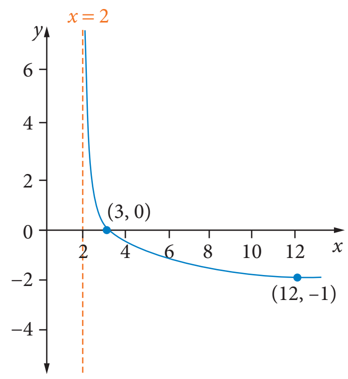
a $y = 2\log_2(x+2)$ **b** $y = -\log_2(x) - 4$

c $y = -3\log_2(1-x)$ **d** $y = 2\log_2(4-5x) + 3$

5 WORKED EXAMPLE 17 The rule for the function with the graph shown is of the form $y = \log_2[a(x-b)]$. Find the values of a and b, and state the equation of the function.



6 The rule for the function with the graph shown is of the form $y = a\log_{10}(x-b)$. Find the equation of the function.



7 WORKED EXAMPLE 18 The weakest soundwave that a human ear can hear has an intensity of 1×10^{-12} W/m^2 and is called the **threshold of human hearing, I_0**. To compare relative sound intensities, S, we use a sound level (loudness) scale called **decibels**, dB, which is calculated by the formula $S = 10\log_{10}\left(\frac{I}{I_0}\right)$. Use the formula to find the sound level reading of a fire alarm if its sound wave intensity is 3.16 W/m^2. Answer to the nearest dB.

8 Biologists use the logarithmic model $n = 1037\log_{10}(A)$ to estimate the number of species (n) that live in an area (A km^2).

a How many species live in a rainforest of 500 km^2?

b How many species will be left when half of this rainforest is destroyed by logging?

9 One measure of a star's brightness is its magnitude – the dimmer the star, the larger its magnitude. A formula for the limiting magnitude L of a telescope, that is, the magnitude of the dimmest star that it can be used to view, is $L = 9 + 5.1 \log_{10}(d)$, where d is the diameter (in cm) of the lens. What (correct to 4 decimal places) is the limiting magnitude of a telescope 6 cm in diameter?

10 The Richter scale can be given by $R = 0.67 \log_{10}(0.37E) + 1.46$, where E is the energy (in kilowatt-hours) released by an earthquake. An earthquake releases 15 500 000 000 kilowatt-hours of energy. What is the earthquake's magnitude?

11 Desalination is the process of producing fresh water from salt water. The formula for desalination is $W = 18 + 7\log_2(t)$, where W is the amount of fresh water produced (in litres) in time t (hours). How much fresh water (correct to 2 decimal places) can be produced after 10 hours from a desalination process?

12 In chemistry, pH scale measures the acidity of a solution using the concentration of hydrogen ions. Its formula is $\text{pH} = -\log_{10}(a_{H^+})$, where a_{H^+} is the concentration of hydrogen ions in moles/litre.

Solutions with a pH less than 7 are acidic. In pure water, the concentration is 10^{-7} moles/L.

What is the pH of pure water?

Exam practice

80–100% 60–79% 0–59%

13 © VCAA 2009 2AQ3 86% The maximal domain D of the function $f: D \to R$ with rule $f(x) = \log_{10}(2x+1)$ is

A $R\backslash\left\{\frac{1}{2}\right\}$ B $\left(-\frac{1}{2}, \infty\right)$ C R D $(0, \infty)$ E $\left(-\infty, -\frac{1}{2}\right)$

14 © VCAA 2012 2AQ5 66% Let the rule for a function g be $g(x) = \log_e[(x-2)^2]$.

For the function g, the

A maximal domain $= R^+$ and range $= R$

B maximal domain $= R\backslash\{2\}$ and range $= R$

C maximal domain $= R\backslash\{2\}$ and range $= (-2, \infty)$

D maximal domain $= [2, \infty)$ and range $= (0, \infty)$

E maximal domain $= [2, \infty)$ and range $= [0, \infty)$

7 Chapter summary

The index laws

$a^m \times a^n = a^{m+n}$ $a^m \div a^n = a^{m-n}$ $(a^m)^n = a^{mn}$

$(ab)^n = a^n b^n$ $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$

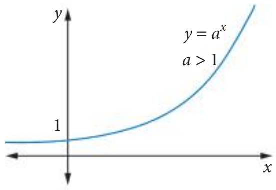
$a^0 = 1$ $a^{-n} = \frac{1}{a^n}$

$a^{\frac{1}{q}} = \sqrt[q]{a}$ $a^{\frac{p}{q}} = \sqrt[q]{(a)^p} = \left(\sqrt[q]{a}\right)^p$

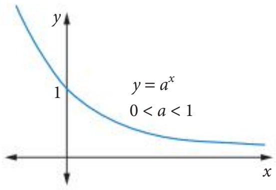
The exponential function $y = a^x$

- a is the base, where $a > 0$ and $a \neq 1$.
- An increasing function if $a > 1$, increasing slowly at first, then more quickly.
- A decreasing function if $0 < a < 1$, decreasing quickly at first, then more slowly.
- The gradient of the graph is also always increasing.
- A one-to-one function.

- y-intercept of 1 (because $a^0 = 1$).
- Domain: $x \in R$
- Range: $y \in R^+$: y is always positive (the graph is above the x-axis).
- The x-axis is an asymptote.



Increasing exponential function



Decreasing exponential function

Transformations of the exponential function

A general form of the exponential function is

$$y = Aa^{n(x+b)} + c$$

where the constants A, b, n and c refer to transformations to the basic graph of $y = a^x$.

- The effect of A is a (vertical) **dilation from the x-axis** (or **parallel to the y-axis**). For $A > 1$, the graph is stretched, dilated by a factor of A. For $0 < A < 1$, the graph is shrunk. If $A < 0$, then the graph is also reflected (vertically) in the x-axis.
- The effect of n is a (horizontal) **dilation from the y-axis** (or **parallel to the x-axis**). For $n > 1$, the graph is shrunk, dilated by a factor of $\frac{1}{n}$. If $0 < n < 1$, the graph is stretched.
- The effect of c is a (vertical) **translation in the direction of the y-axis**. For $c > 0$, the graph is moved up c units (in the positive y direction), and for $c < 0$, the graph is moved down k units (in the negative y direction). The new horizontal asymptote is $y = c$.
- The effect of b is a (horizontal) **translation in the direction of the x-axis**. For $b > 0$, the graph is moved left h units (in the negative x direction), and for $b < 0$, the graph is moved right b units (in the positive x direction).

Exponential growth and decay

$y = A(a^x)$, where the y-intercept A is the **initial value** and a is the **growth factor**.

For a **growth function**, $a > 1$.

For a **decay function**, $0 < a < 1$.

Logarithms

If $a^x = b$, then $\log_a(b) = x$.

For example, $5^4 = 625$, so $\log_5(625) = 4$.

The logarithm of b to the base a is the power x such that $a^x = b$.

$a > 0$ and $a \neq 1$.

$b > 0$: we cannot find the logarithm of a negative number or 0.

Calculators can calculate logarithms to any base.

Logarithm laws

$\log_a(1) = 0$

$\log_a(a) = 1$

$\log_a\left(\frac{1}{a}\right) = -1$

$\log_a(a^x) = x$

$\log_a(xy) = \log_a(x) + \log_a(y)$

$\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)$

$\log_a(x^n) = n\log_a(x)$

Change of base law

$$\log_b(x) = \frac{\log_a(x)}{\log_a(b)}$$

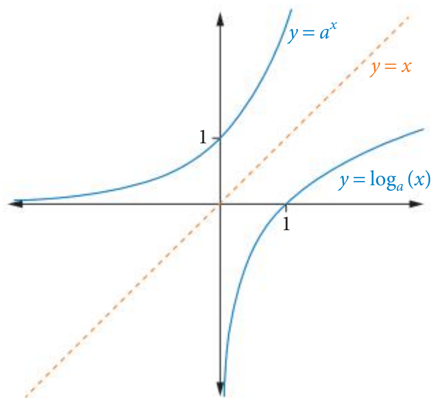
This formula is useful for evaluating logarithms on a calculator for a range of different bases.

The logarithmic function $y = \log_a(x)$, where $a > 1$.

The logarithmic function

A **logarithmic function** takes the form $y = \log_a(x)$.

- $y = \log_a(x)$ is the **inverse function** of $y = a^x$.



- An increasing function, increasing quickly at first, then more slowly.
- The gradient of the graph is always decreasing.
- A one-to-one function.
- x-intercept of 1 (because $\log_a(1) = 0$).
- Domain: $x \in R^+$: x is always positive.
- Range: $y \in R$
- The y-axis is an asymptote.

Cumulative examination 1

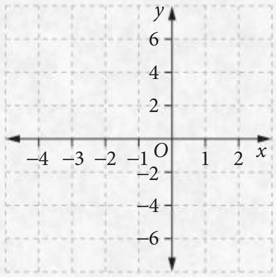
Total number of marks: 10 Reading time: 4 minutes Writing time: 15 minutes

TECH-FREE *Technology is NOT permitted.*

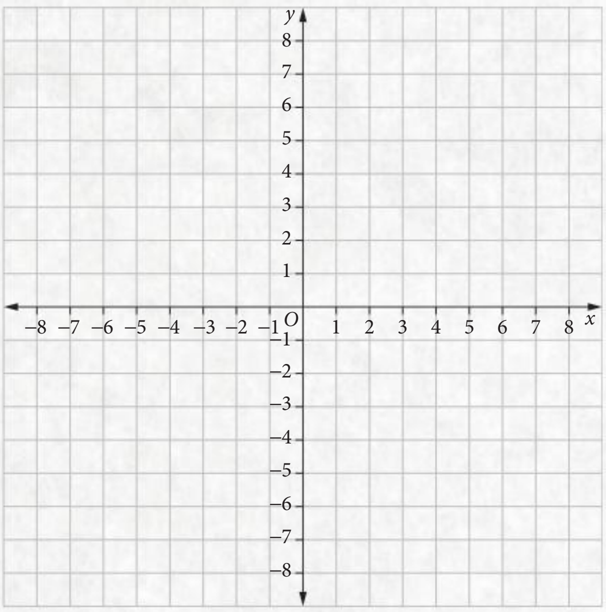
1 © VCAA 2017 1Q3 (4 marks) Let $f: [-3, 0] \to R, f(x) = (x+2)^2(x-1)$.

a Show that $(x+2)^2(x-1) = x^3 + 3x^2 - 4$. 1 mark

b Copy the axes below and on them sketch the graph of f. Label the axis intercepts and any stationary points with their coordinates. 3 marks



2 © VCAA 2015 1Q4b MODIFIED (2 marks) Copy the axes below and on them sketch the graph of $f(x) = \frac{1}{2}(x-1)(x+2)^2$, clearly indicating axis intercepts and turning points. Label the endpoints with their coordinates.



3 © VCAA 2007 1Q2a MODIFIED (1 mark) Simplify $\log_{10}(3x+5) + \log_{10}(2)$.

4 © VCAA 2006 1Q2 MODIFIED (3 marks) For the function $f: R \to R, f(x) = 3(4)^{2x} - 1$,

a find the rule for the inverse function f^{-1} 2 marks

b find the domain of the inverse function f^{-1}. 1 mark

Cumulative examination 2

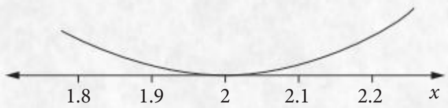
Total number of marks: 20 Reading time: 4 minutes Writing time: 30 minutes

Approved technology is permitted.

Section A 6 multiple-choice questions

6 marks

1 © VCAA 2003 1Q1 A polynomial function p has degree three. A part of its graph, near the point on the graph with coordinates (2, 0), is shown.



Which one of the following could be the rule for the third degree polynomial p?

A $p(x) = x(x + 2)^2$

B $p(x) = (x - 2)^3$

C $p(x) = x^2(x - 2)$

D $p(x) = (x - 1)(x - 2)^2$

E $p(x) = -x(x - 2)^2$

2 © VCAA 2020 2AQ2 Let $p(x) = x^3 - 2ax^2 + x - 1$, where $a \in R$. When p is divided by $x + 2$, the remainder is 5. The value of a is

A 2 **B** $-\frac{7}{4}$ **C** $\frac{1}{2}$ **D** $-\frac{3}{2}$ **E** -2

3 © VCAA 2015 2AQ20 If $f(x - 1) = x^2 - 2x + 3$, then $f(x)$ is equal to

A $x^2 - 2$ **B** $x^2 + 2$ **C** $x^2 - 2x + 2$ **D** $x^2 - 2x + 4$ **E** $x^2 - 4x + 6$

4 © VCAA 2011 2AQ2 The gradient of a line **perpendicular** to the line which passes through $(-2, 0)$ and $(0, -4)$ is

A $\frac{1}{2}$ **B** -2 **C** $-\frac{1}{2}$ **D** 4 **E** 2

5 © VCAA 2008 2AQ12 MODIFIED Let $f: R \to R, f(x) = 3^x + 3^{-x}$. For all $u \in R$, $f(2u)$ is equal to

A $f(u) + f(-u)$

B $2f(u)$

C $(f(u))^2 - 2$

D $(f(u))^2$

E $(f(u))^2 + 2$

6 © VCAA 2020 2AQ10 MODIFIED Given that $\log_2(n + 1) = x$, the values of n for which x is a positive integer are

A $n = 2^k, k \in Z^+$

B $n = 2^k - 1, k \in Z^+$

C $n = 2^{k-1}, k \in Z^+$

D $n = 2k - 1, k \in Z^+$

E $n = 2k, k \in Z^+$

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Section B 2 questions

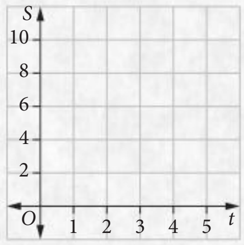
14 marks

1 © VCAA 2015 2BQ5 MODIFIED (5 marks) Let $S(t) = 2(10)^{\frac{t}{3}} + 8(10)^{-\frac{2t}{3}}$ where $0 \le t \le 5$.

a Find $S(0)$ and $S(5)$. 1 mark

b The minimum value of S occurs when $t = \log_{10}(c)$. State the value of c and the minimum value of S. 2 marks

c Copy the axes below and on them sketch the graph of S against t for $0 \le t \le 5$. Label the endpoints and the minimum point with their coordinates for $0 \le t \le 5$. 2 marks

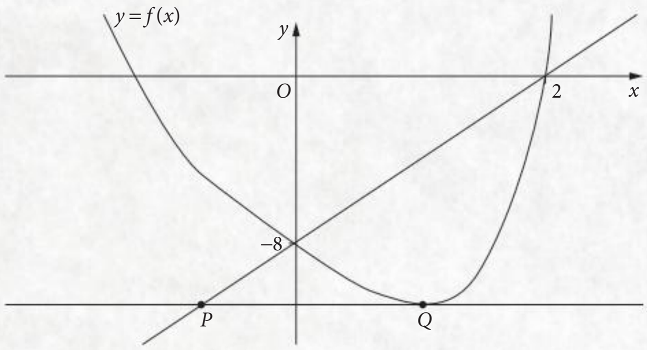


2 © VCAA 2018N 2BQ1 (9 marks) Let $f : R \rightarrow R, f(x) = x^4 - 4x - 8$.

a Given $f(x) = (x - 2)(x^3 + ax^2 + bx + c)$, find a, b and c. 1 mark

b Find two consecutive integers m and n such that a solution to $f(x) = 0$ is in the interval (m, n), where $m < n < 0$. 2 marks

The diagram below shows part of the graph of f and a straight line drawn through the points $(0, -8)$ and $(2, 0)$. A second straight line is drawn parallel to the horizontal axis and it touches the graph of f at the point Q. The two straight lines intersect at the point P.



c **i** Find the equation of the line through $(0, -8)$ and $(2, 0)$. 1 mark

ii State the equation of the line through the points P and Q. 1 mark

iii State the coordinates of the points P and Q. 2 marks

d © VCAA MODIFIED A transformation $T: (x, y) \rightarrow (x + d, y)$ is applied to the graph of f.

i Find the value of d for which P is the image of Q. 1 mark

ii Let $(m', 0)$ and $(n', 0)$ be the images of $(m, 0)$ and $(n, 0)$ respectively, under the transformation T, where m and n are defined in part **b**.

Find the values of m' and n'. 1 mark

CHAPTER

8 PROBABILITY THEORY

Study Design coverage

UNIT 2, AREA OF STUDY 4: DATA ANALYSIS, PROBABILITY AND STATISTICS

- probability of elementary and compound events and their representation as lists, grids, Venn diagrams, tables and tree diagrams
- the addition rule for probabilities, $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$, and the relation that for mutually exclusive events $\Pr(A \cap B) = 0$, hence $\Pr(A \cup B) = \Pr(A) + \Pr(B)$
- conditional probability in terms of reduced sample space, the relations $\Pr(A|B) = \dfrac{\Pr(A \cap B)}{\Pr(B)}$ and $\Pr(A \cap B) = \Pr(A|B) \times \Pr(B)$
- the law of total probability for two events $\Pr(A) = \Pr(A|B)\Pr(B) + \Pr(A|B')\Pr(B')$
- the relations for pairwise independent events A and B, $\Pr(A|B) = \Pr(A)$, $\Pr(B|A) = \Pr(B)$ and $\Pr(A \cap B) = \Pr(A) \times \Pr(B)$
- simulation to estimate probabilities involving selection with and without replacement.

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STUDY DESIGN COVERAGE

Video playlists (5):
8.1 Venn diagrams and two-way tables
8.2 Tree diagrams and compound events
8.3 Conditional probability
8.4 The addition rule for probabilities
VCE question analysis Probability theory

Worksheets (9)
8.1 Venn diagrams 1 • Set operations • Venn diagrams 2 • Venn diagrams matching activity • Two-way tables
8.2 Tree diagrams 1 • Tree diagrams 2 • And/or problems
8.3 Conditional probability

Puzzles (2)
8.1 Combined events: Two-way tables
8.3 Conditional probability: Two-way tables

Nelson MindTap

To access resources above, visit **cengage.com.au/nelsonmindtap**

8.1 Venn diagrams and two-way tables

Video playlist Venn diagrams and two-way tables

The probability associated with a simple event was covered in detail in Chapter 5, *Probability and counting*. In this chapter, we will examine other techniques of organising the sample space for other types of simple events.

Probability of a simple event

$$\Pr(A) = \frac{\text{number of ways } A \text{ can occur}}{\text{total number of possible outcomes}}$$

WORKED EXAMPLE 1 Simple event probability

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Ashleigh has a spinner with sections numbered 1 to 5. The probability of the spinner landing on one of these numbers on any of these spins is shown in the table below. Find the probability that Ashleigh spins a number greater than 2.

Number, x	1	2	3	4	5
Probability, $\Pr(X = x)$	0.25	0.22	0.3	0.18	0.05

Steps	Working
1 The possible numbers that are greater than 2 are 3, 4 and 5.	$\Pr(X > 2) = \Pr(X = 3) + \Pr(X = 4) + \Pr(X = 5)$
2 Use the values from the table.	$\Pr(X > 2) = 0.3 + 0.18 + 0.05$ $= 0.53$

Set theory and Venn diagrams

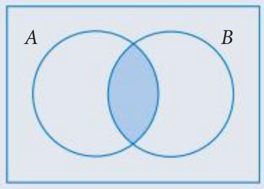
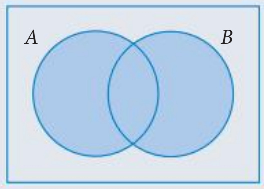
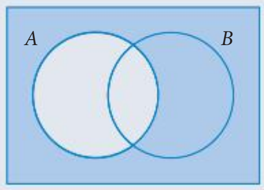
Worksheets
Venn diagrams 1
Set operations
Venn diagrams 2
Venn diagrams matching activity

The **sample space** is a set that contains all the possible outcomes in a probability experiment. In some cases, the set of elements in the sample space can be divided into a number of subsets that will usually have some elements in common. For example, a bag of fruits may contain red apples, green apples and tomatoes. If A is the set of apples and R is the set of red fruit, then these sets will be subsets of the bag of fruits that is our sample space. The elements in the sample space can be illustrated using a diagram.

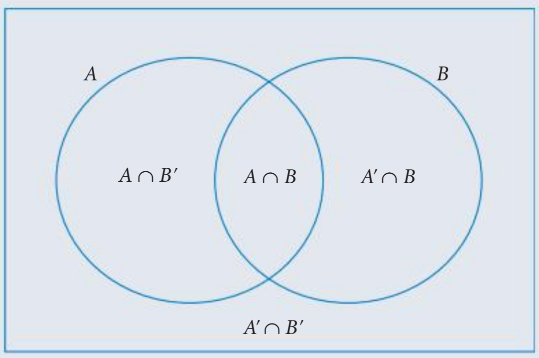
Shutterstock.com/Tiplyashina Evgeniya

A **Venn diagram** illustrates the sample space by showing each subset as a circle or oval inside a rectangle that represents the universal set (all outcomes). It is a very useful tool for analysing simple event probability problems where the sample space contains different groups with elements in common.

Set theory notation

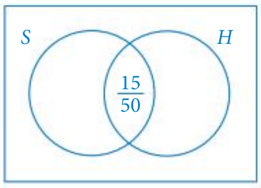
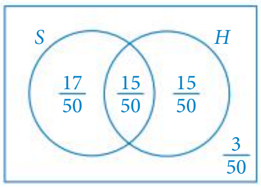
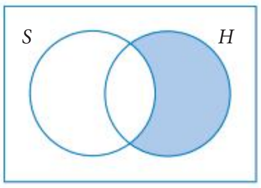
Intersection $A \cap B$	The **intersection of sets** A and B is a set that contains the elements that are in both sets.	
Union $A \cup B$	The **union of sets** A and B is obtained by combining or uniting both sets. The elements in the union are in set A or set B or both.	
Complement A'	The **complement of set** A is the set that contains all the elements that are not in set A.	

Sections of the Venn diagram



WORKED EXAMPLE 2 Finding probabilities using a Venn diagram 1

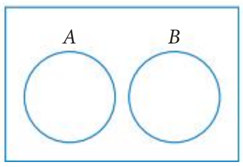
In a group of 50 students, 32 students are wearing sunglasses, 30 students are wearing hats and 15 students are wearing both sunglasses and hats. Use a Venn diagram to find the probability of selecting a student in the group who is wearing a hat but not sunglasses.

Steps	Working
1 Let S represent the set of students wearing sunglasses and H represent the set of students wearing hats. Summarise the information given. $n(U)$ is the number of elements in the universal (total) set.	$n(U) = 50$ $n(S) = 32$ $n(H) = 30$ $n(S \cap H) = 15$
2 Enter the probabilities in a Venn diagram. Fill the centre (intersection) first.	
3 In set S, there are 32 students. However, 15 have already been included in the intersection. The remaining part of set S contains $32 - 15 = 17$ students. Similarly, the remaining part of set H contains $30 - 15 = 15$ students. The total must be 50, leaving $50 - 17 - 15 = 3$ students outside sets S and H.	
4 The probability of selecting a person wearing a hat but not sunglasses is in the shaded section. 	$\Pr(H \cap S') = \frac{15}{50}$ $= \frac{3}{10}$

WB p. 156

Mutually exclusive events

Two events A and B are mutually exclusive if $\Pr(A \cap B) = 0$; that is, sets A and B have no elements in common. It is impossible for **mutually exclusive events** to occur at the same time. An example of mutually exclusive events would be with the roll of a die if A represents rolling an odd number and B represents rolling a 6.



WB
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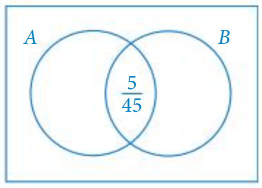
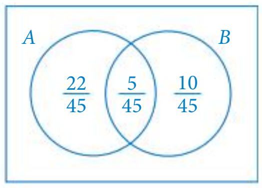
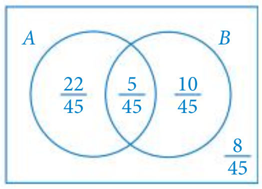
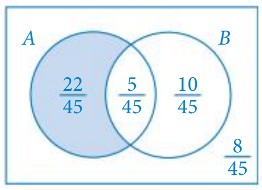
WORKED EXAMPLE 3 Finding probabilities a using Venn diagram 2

If $\Pr(A) = \frac{3}{5}$, $\Pr(B) = \frac{1}{3}$ and $\Pr(A \cap B) = \frac{1}{9}$, draw a Venn diagram and hence find $\Pr(A \cap B')$

Steps	Working
1 Express all fractions with a common denominator.	$\Pr(A) = \frac{3}{5} = \frac{27}{45}$ $\Pr(B) = \frac{1}{3} = \frac{15}{45}$ $\Pr(A \cap B) = \frac{1}{9} = \frac{5}{45}$
2 Illustrate the probabilities with a Venn diagram. Fill the centre (intersection) first.	
3 Complete the remaining part of set A and set B. $\Pr(A \cap B') = \frac{27}{45} - \frac{5}{45} = \frac{22}{45}$ $\Pr(A' \cap B) = \frac{15}{45} - \frac{5}{45} = \frac{10}{45}$	
4 All the probabilities must sum to 1 or $\frac{45}{45}$. Complete the remaining probability in the universal set. $\Pr(A' \cap B') = \frac{45}{45} - \frac{22}{45} - \frac{5}{45} - \frac{10}{45} = \frac{8}{45}$	
5 Find $\Pr(A \cap B')$, which is the part of set A that is not in set B.	 $\Pr(A \cap B') = \frac{22}{45}$

WB
p. 158

WORKED EXAMPLE 4 Probabilities for mutually exclusive events

Events A and B are mutually exclusive with $\Pr(A) = \frac{1}{2}$ and $\Pr(B) = \frac{5}{24}$.
Draw a Venn diagram and hence find $\Pr(A' \cap B')$.

Steps	Working
1 Express all fractions with a common denominator.	$\Pr(A) = \frac{1}{2} = \frac{12}{24}$ $\Pr(B) = \frac{5}{24}$

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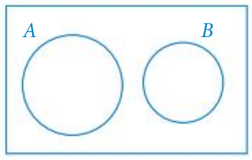
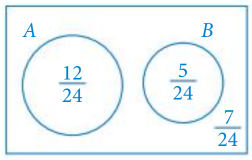
2 Illustrate the probabilities with a Venn diagram.

Since A and B are mutually exclusive, the circles will not intersect.



3 Enter the probabilities for A and B and calculate the remaining part of the universal set.

$\Pr(A' \cap B') = \frac{24}{24} - \frac{12}{24} - \frac{5}{24} = \frac{7}{24}$



4 Find the probability noting that $A' \cap B'$ is the region of the Venn diagram outside sets A and B.

$\Pr(A' \cap B') = 1 - \frac{12}{24} - \frac{5}{24} = \frac{7}{24}$

8.1

Two-way tables

Two-way tables or **Karnaugh maps** compare two intersection sets. When these tables include probabilities they are also referred to as probability tables.

A two-way probability table can be used for two intersecting sets A and B.

Set A is divided into the part that is in B $(A \cap B)$ and the part that is not in B $(A \cap B')$.

Set B is divided into the part that is in A $(A \cap B)$ and the part that is not in A $(A' \cap B)$.

The elements that are not in set A nor set B belong to the set $A' \cap B'$.

The 4 intersections (coloured yellow in the table) are also the 4 sections in the Venn diagram.

	B	B'	Totals
A	$A \cap B$	$A \cap B'$	Total of set A
A'	$A' \cap B$	$A' \cap B'$	Total of set A'
Totals	Total of set B	Total of set B'	1.0

Worksheet Two-way tables

Puzzle Combined events: Two-way tables

WORKED EXAMPLE 5 Finding probabilities using two-way tables 1

For two events A and B, $\Pr(A \cap B) = 0.15$, $\Pr(B) = 0.35$ and $\Pr(A' \cap B') = 0.2$. Find $\Pr(A \cap B')$.

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Steps / **Working**

1 Enter the given probabilities into a probability table.

	B	B'	**Totals**
A	0.15		
A'		0.2	
Totals	0.35		1.0

2 $\Pr(B')$ in the bottom row can be found first.

$0.35 + \Pr(B') = 1.0$

$\Pr(B') = 0.65$

	B	B'	**Totals**
A	0.15		
A'		0.2	
Totals	0.35	0.65	1.0

3 Complete the remainder of the table using the row totals and column totals.

	B	B'	**Totals**
A	0.15	0.45	0.6
A'	0.2	0.2	0.4
Totals	0.35	0.65	1.0

4 Find the value of $\Pr(A \cap B')$ in the table.

$\Pr(A \cap B') = 0.45$

WB

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WORKED EXAMPLE 6 Finding probabilities using two-way tables 2

For two events A and B, $\Pr(A \cap B) = p$, $\Pr(B') = 0.4$, $\Pr(A) = 1.5p$ and $\Pr(A' \cap B) = 2p$. Find the value of p and hence find $\Pr(A' \cap B')$.

Steps | **Working**

1 Enter the given probabilities into a probability table.

	B	B'	**Totals**
A	p		$1.5p$
A'	$2p$		
Totals		0.4	1.0

2 Pr (B) can be found two ways.

Adding the entries in the B column $p + 2p = 3p$ and subtracting 0.4 from 1.0

$1.0 - 0.4 = 0.6$.

	B	B'	**Totals**
A	p		$1.5p$
A'	$2p$		
Totals	$3p = 0.6$	0.4	1.0

$3p = 0.6$

$p = 0.2$

3 Substitute $p = 0.2$ and calculate the other values.

	B	B'	**Totals**
A	0.2	0.1	0.3
A'	0.4	0.3	0.7
Totals	0.6	0.4	1.0

4 Find the value of $\Pr(A' \cap B')$ in the table.

$\Pr(A' \cap B') = 0.3$

When a group can be categorised into two subgroups or sets and a single selection is made, it is possible to analyse the problem with a Venn diagram or a two-way probability table.

Independent events

Two events A and B are independent if one event does not influence the outcome of the other event. An example of **independent events** would be the Hawks winning a football match and the Tigers winning a different match across town.

Independent events

For two independent events A and B,

$$\Pr(A \cap B) = \Pr(A) \times \Pr(B)$$

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WORKED EXAMPLE 7 Finding probabilities using two-way tables 3

For two events A and B, Pr (A) = 0.35 and $\Pr(B') = 0.2$. If A and B are independent, find $\Pr(A \cap B')$.

Steps | **Working**

1 Enter the given probabilities into a probability table and complete the other totals.

	B	B'	**Totals**
A			0.35
A'			0.65
Totals	0.8	0.2	1.0

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2 Use the rule for independent events to find $\Pr(A \cap B)$

$$\Pr(A \cap B) = \Pr(A) \times \Pr(B)$$
$$= 0.8 \times 0.35$$
$$= 0.28$$

	B	B'	Totals
A	0.28		0.35
A'			0.65
Totals	0.8	0.2	1.0

3 Calculate the remaining probabilities in the table by subtracting from the totals.

> If A and B are independent, then A and B' are also independent.
>
> $\Pr(A \cap B')$ can also be found using the formula
>
> $\Pr(A \cap B') = \Pr(A) \times \Pr(B')$

	B	B'	Totals
A	0.28	0.07	0.35
A'	0.52	0.13	0.65
Totals	0.8	0.2	1.0

4 Find the value of $\Pr(A \cap B')$ in the table. $\Pr(A \cap B') = 0.07$

EXERCISE 8.1 Venn diagrams and two-way tables

ANSWERS p. 523

Mastery

1 WORKED EXAMPLE 1 A 12-sided die has the numbers from 1 to 9, an extra 2 and two extra 5s. The probability of it landing on a 5 is

A $\frac{1}{12}$ **B** $\frac{1}{10}$ **C** $\frac{1}{9}$ **D** $\frac{1}{6}$ **E** $\frac{1}{4}$

2 WORKED EXAMPLE 2 The probability of a student studying Art or Music is $\frac{3}{10}$. If $\frac{1}{8}$ of the students study Music and $\frac{1}{4}$ study Art, find the probability of a student studying both Art and Music.

A $\frac{3}{40}$ **B** $\frac{1}{8}$ **C** $\frac{7}{40}$ **D** $\frac{17}{40}$ **E** $\frac{9}{20}$

3 WORKED EXAMPLE 3 For two events A and B, $\Pr(A) = \frac{3}{4}$, $\Pr(B) = \frac{1}{4}$ and $\Pr(A \cap B) = \frac{1}{5}$. Illustrate this with a Venn diagram and hence find $\Pr(A' \cap B')$.

4 WORKED EXAMPLE 4 Events A and B are mutually exclusive with $\Pr(A) = \frac{3}{4}$ and $\Pr(A' \cap B') = \frac{1}{5}$. Draw a Venn diagram and hence find $\Pr(A \cup B')$.

5 WORKED EXAMPLE 5 For two events A and B, $\Pr(A \cap B) = 0.32$, $\Pr(B') = 0.46$ and $\Pr(A' \cap B') = 0.11$. Find $\Pr(A')$.

6 WORKED EXAMPLE 6 For two events A and B, $\Pr(A \cap B) = p + 0.25$, $\Pr(B') = p$, $\Pr(A') = p - 0.15$ and $\Pr(A \cap B') = 0.2$. Find the value of p and hence find $\Pr(A' \cap B)$.

7 WORKED EXAMPLE 7 For two independent events A and B, $\Pr(A) = 0.25$ and $\Pr(A \cap B) = 0.15$. Find $\Pr(A \cap B')$.

Exam practice

80–100% 60–79% 0–59%

8 © VCAA 2002 1IIQ2a 65% TECH-FREE (1 mark) A roulette wheel has 37 slots, numbered 0 to 36. Slot 0 is green. Eighteen of the remaining slots are black, and the other 18 are red. In a single game, a person spins the wheel and at the same time rolls a ball around the wheel in the opposite direction. As the wheel slows, the ball falls into one of the slots. The wheel is carefully balanced so that the ball is equally likely to fall into any of the slots.

In a single game, what is the probability, correct to three decimal places, that the ball falls into a red slot?

9 © VCAA 2007 1Q6 63% TECH-FREE (2 marks) Two events, A and B, from a given event space, are such that $\Pr(A) = \frac{1}{5}$ and $\Pr(B) = \frac{1}{3}$.

a Calculate $\Pr(A' \cap B)$ when $\Pr(A \cap B) = \frac{1}{8}$. 1 mark

b Calculate $\Pr(A' \cap B)$ when A and B are mutually exclusive events. 1 mark

10 TECH-FREE (2 marks) A drug company develops a test for a particular disease. It is found that

- The probability of the test being positive is 0.3.
- The probability of a person having the disease is 0.29.
- The probability of the drug testing positive and the patient not having the disease is 0.02.

Find the probability that the test result is negative, and the patient has the disease.

11 TECH-FREE (3 marks) At a recent swimming carnival there were 40 competitors. The list of entries showed that

- 14 entered backstroke.
- 22 entered freestyle.
- 17 entered butterfly.
- 5 entered all three events.
- 7 entered backstroke and freestyle.
- 9 entered freestyle and butterfly.
- 6 entered backstroke and butterfly.

Find the probability that a swimmer entered

a only butterfly 1 mark

b butterfly and backstroke but not freestyle 1 mark

c exactly one event 1 mark

12 TECH-FREE (2 marks) Two independent events, A and B, from a given event space, are such that $\Pr(A') = 0.6$ and $\Pr(A' \cap B) = 0.18$. Find $\Pr(A \cap B)$.

13 © VCAA 2005 1Q1 90% Over a 30-day period, Timothy recorded the number of hours each day that he spent cycling on his bicycle. The results are shown in the table below.

Number of hours each day spent cycling (x)	0	1	2	3	4
Proportion of days on which x hours were spent cycling	$\frac{3}{30}$	$\frac{6}{30}$	$\frac{7}{30}$	$\frac{10}{30}$	$\frac{4}{30}$

During this 30-day period, the proportion of days on which Timothy cycled for at least two hours is

A $\frac{7}{30}$ **B** $\frac{14}{30}$ **C** $\frac{17}{30}$ **D** $\frac{21}{30}$ **E** $\frac{27}{30}$

14 © VCAA 2004 1Q1 84% Mollie has constructed a spinner that will randomly display an integer between 0 and 4, inclusive, with the following probabilities.

Number	x	0	1	2	3	4
Probability	$\Pr(X = x)$	0.2	0.3	0.15	0.25	0.1

If Mollie spins the spinner once, the probability of obtaining an odd number is

A 0.075 **B** 0.25 **C** 0.55 **D** 0.6 **E** 0.75

15 © VCAA 2013 2AQ10 51% For events A and B, $\Pr(A \cap B) = p$, $\Pr(A' \cap B) = p - \frac{1}{8}$ and $\Pr(A \cap B') = \frac{3p}{5}$.

If A and B are independent, then the value of p is

A 0 **B** $\frac{1}{4}$ **C** $\frac{3}{8}$ **D** $\frac{1}{2}$ **E** $\frac{3}{5}$

16 © VCAA 2010 2AQ21 43% Events A and B are mutually exclusive events of a sample space with $\Pr(A) = p$ and $\Pr(B) = q$ where $0 < p < 1$ and $0 < q < 1$.

$\Pr(A' \cap B')$ is equal to

A $(1 - p)(1 - q)$ **B** $1 - pq$ **C** $1 - (p + q)$

D $2 - p - q$ **E** $1 - (p + q - pq)$

8.2

8.2 Tree diagrams and compound events

Video playlist Tree diagrams and compound events

Worksheets Tree diagrams 1

Tree diagrams 2

And/or problems

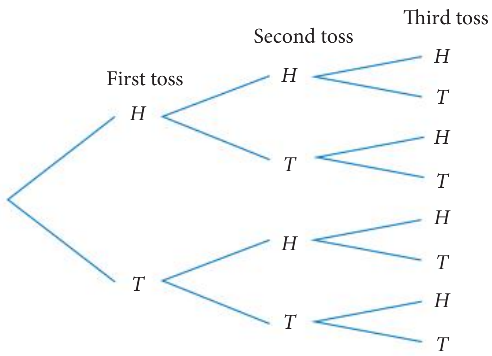
A **compound event** consists of 2 or more simple events that occur in succession. Tossing a coin 3 times is an example of a compound event. These problems are often solved using a tree diagram as an organiser.

Tree diagrams for independent events

A **tree diagram** is a useful way of representing a series of simple events. Each simple event is represented by another stage of branches with the number of branches in each set equal to the number of possible outcomes for the event.

A coin tossed 3 times is 3 simple independent events. Each toss of the coin has 2 different outcomes. This would be represented by a tree diagram with 3 stages and 2 branches produced from every outcome in the previous stage as shown.

Shutterstock.com/Khakimullin Aleksandr



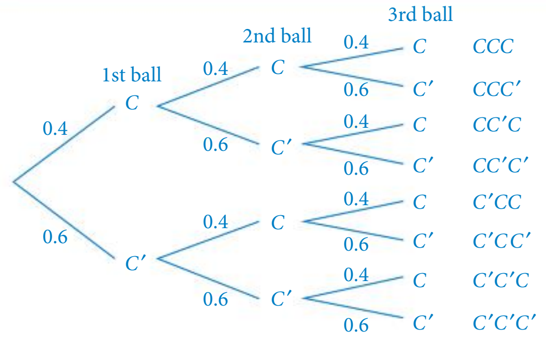
WORKED EXAMPLE 8 Using a tree diagram to find compound event probabilities

WB p. 162

A barrel contains 20 balls, of which 8 are multi-coloured. Three balls are randomly selected from the barrel, **with replacement**. This means that a ball is selected, its colour noted, and the ball is replaced **before** the next ball is selected. Find the probability that 2 of the 3 balls are multi-coloured.

Steps	Working
1 On each selections the ball selected can be multi-coloured (C).	$\Pr(C) = \frac{8}{20} = \frac{2}{5} = 0.4$
Or the ball can be not multi-coloured (C').	$\Pr(C') = 1 - 0.4 = 0.6$

2 Represent with a tree diagram.



3 Identify the branches where there are 2 multi-coloured balls and one ball that is not multi-coloured.

Pr(2 multi-coloured) balls
$= \Pr(CCC') + \Pr(CC'C) + \Pr(C'CC)$

4 Multiply the probabilities along the branches and add the products.

Pr(2 multi-coloured balls)
$= 0.4 \times 0.4 \times 0.6 + 0.4 \times 0.6 \times 0.4 + 0.6 \times 0.4 \times 0.4$
$= 0.288$

Independent compound events

In cases when there are a large number of independent events. it is impractical to draw a tree diagram. However, we can multiply probabilities to determine the final probability.

WB
p. 163

WORKED EXAMPLE 9 Finding probabilities for independent events

Mark is competing in an archery competition. Each of his shots is independent and his probability of hitting a bullseye is 0.35. Find the probability (correct to 3 decimal places) that Mark's fifth shot is the first bullseye he hits.

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Steps	Working
1 Write the sequence of events where the fifth shot is the first bullseye.	B represents hitting a bullseye. B' represents not hitting a bullseye. Pr (5$^{\text{th}}$ shot bullseye) = Pr $(B'B'B'B'B)$
2 Since each event is independent, we can multiply the probabilities.	$\Pr(B'B'B'B'B)$ $= \Pr(B') \times \Pr(B') \times \Pr(B') \times \Pr(B') \times \Pr(B)$
3 $\Pr(B) = 0.35$ and $\Pr(B') = 1 - 0.35 = 0.65$.	$= 0.65 \times 0.65 \times 0.65 \times 0.65 \times 0.35$ $= 0.062$

9780170448260

Selections without replacement

Selection without replacement occurs when an item is selected from a sample and is not returned to the sample, then another item is selected. When this occurs, the sample space reduces after each selection. A probability in the tree diagram will change depending on which items remain in the sample space, so the events are not independent. They are **dependent events**.

8.2

WB p. 164

WORKED EXAMPLE 10 Finding probabilities for selections without replacement

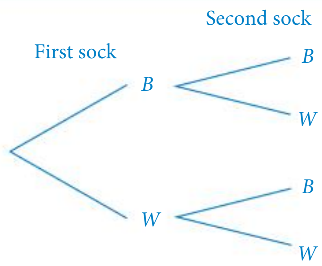
Zoe has 4 pairs of black socks and 2 pairs of white socks. Her socks are randomly mixed in her drawer. Zoe takes 2 individual socks at random from the drawer in the dark.

Find the probability she selects a matching pair.

Steps | **Working**

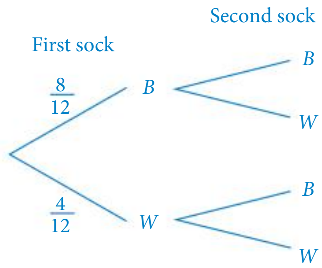
1 There are 2 socks selected without replacement from 8 black socks and 4 white socks.

Draw a tree diagram with 2 stages and 2 branches in each set.



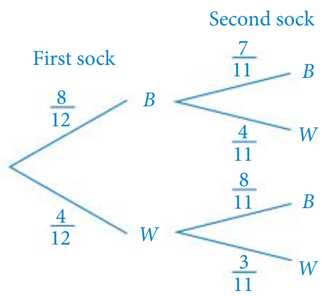
2 On the first selection

$\Pr(B) = \dfrac{8}{12}$, $\Pr(W) = \dfrac{4}{12}$



3 If a black sock is selected, there will be 7 black and 4 white socks.

If a white sock is selected, there will be 8 black and 3 white socks.



4 There will be 2 matching socks if BB or WW is selected.

Multiply probabilities along the branches and add the products.

$$\begin{aligned}&\Pr(\text{matching pair})\\&= \Pr(BB) + \Pr(WW)\\&= \frac{8}{12} \times \frac{7}{11} + \frac{4}{12} \times \frac{3}{11}\\&= \frac{17}{33}\end{aligned}$$

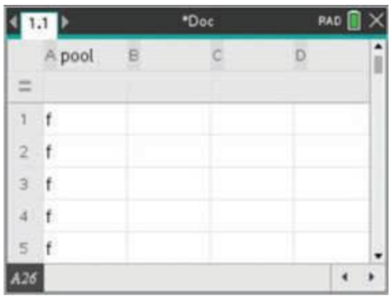
Exam hack

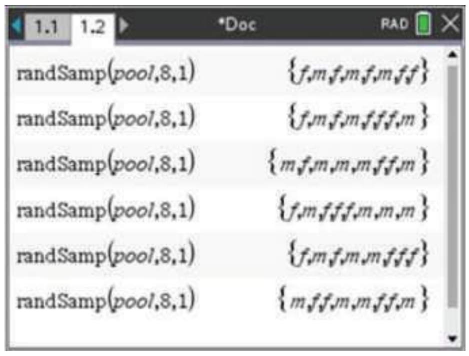
In probability questions, if you read 'and', multiply the probabilities, and if you read 'or', add the probabilities.

USING CAS 1 Simulating selections without replacement

Two debating teams of 4 students are randomly chosen from a class of 15 female and 10 male students. Using probability simulation, randomly generate 25 groups of 8 students and hence determine the approximate probability that there are 6 female students chosen in the group of 8 debaters.

TI-Nspire





1 Add **Lists & Spreadsheet**.

2 Label column **A 'pool'** and enter **'f'** in cells **A1** to **A15** and **'m'** in cells **A16** to **A25**.

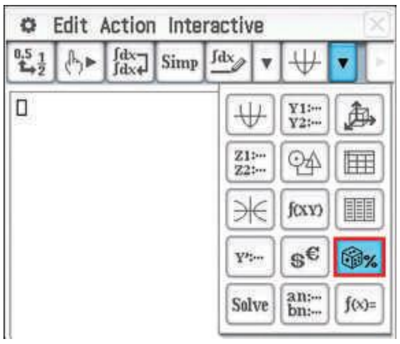
3 Add a **Calculator** page.

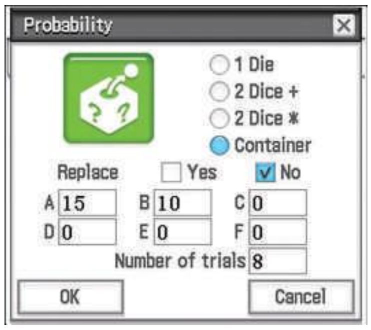
4 Press **menu > Probability > Random > Sample**.

5 Enter **pool,8,1)**.

6 Continue to press **enter** until 25 samples have been generated.

ClassPad





1 From **Main**, tap the **down arrow** to access the tool options and tap the **probability** icon.

2 Tap **Container > No**.

3 Enter **15** for category **A**, **10** for category **B** and **8** for the **Number of trials**.

4 Tap **OK**.

5 Repeat this process until 25 samples have been completed.

The results of 25 simulations are shown in the table.

Females	Frequency
3	2
4	12
5	4
6	5
7	1
8	1

The table shows the number of females in the debating teams of 8 students for 25 simulations.

The approximate probability that a team of 8 students contains 6 females is $\frac{5}{25} = 0.2$.

EXERCISE 8.2 Tree diagrams and compound events

ANSWERS p. 523

Recap

1 A basket of 50 toys contains 20 toys that have wheels, 18 toys that are red and 12 red toys that have wheels. If a toy is randomly selected from the basket, the probability the toy is not red and does not have wheels is

A 0.16 **B** 0.24 **C** 0.36 **D** 0.48 **E** 0.76

2 For two events A and B, $\Pr(A \cap B') = p$, $\Pr(A) = 3p$ and $\Pr(A \cap B) = 0.24$.

The value of p is

A 0.12 **B** 0.36 **C** 0.4 **D** 0.6 **E** 0.76

Mastery

3 WORKED EXAMPLE 8 A bag contains 3 red marbles and 2 green marbles. A marble is selected, its colour is noted, and then it is replaced. A second marble is then selected.

Use a tree diagram to find the probability of selecting 2 marbles the same colour.

4 April has 3 balls that she attempts to throw into a basket. The probability of any of her throws successfully landing in the basket is 0.2 and the success of any throw is independent of the outcome on the previous throw. Find the probability of April landing exactly 2 balls in the basket.

5 WORKED EXAMPLE 9 On a biased coin, the probability of obtaining a head is $\frac{1}{4}$. If the coin is tossed 4 times, find the probability of obtaining a head for the first time on the last toss.

6 WORKED EXAMPLE 10 A committee of 5 Year 12 students and 6 Year 11 students elect a captain and a vice captain from amongst itself. Illustrate this with a tree diagram and hence find the probability of electing a captain and a vice captain that are both in the same year level.

7 Using CAS 1 Six students are randomly chosen from a class of 12 female and 8 male students. Using probability simulation, randomly generate 25 groups of 6 students and hence determine the approximate probability that the group chosen contains 2 female students.

8 A mixed table tennis team of 4 players is randomly chosen from a physical education class of 8 female and 11 male students. Use probability simulation to randomly generate 40 groups of 4 students and hence determine the approximate probability that the table tennis team chosen contains more female students than male students.

9 A random sample of 4 cubes is selected, without replacement, from a bag containing 7 green, 5 white and 8 red cubes. Use probability simulation to randomly generate 40 samples of 4 cubes and hence determine the approximate probability there are more white cubes in the sample than red cubes.

Exam practice

80–100% 60–79% 0–59%

10 © VCAA 2009 1Q5 TECH-FREE (2 marks) Four identical balls are numbered 1, 2, 3 and 4 and put into a box. A ball is randomly drawn from the box, and not returned to the box. A second ball is then randomly drawn from the box.

a 70% What is the probability that the first ball drawn is numbered 4 and the second ball drawn is numbered 1? 1 mark

b 58% What is the probability that the sum of the numbers on the two balls is 5? 1 mark

11 © VCAA 2008 1Q7 55% TECH-FREE (2 marks) Jane drives to work each morning and passes through three intersections with traffic lights. The number X of traffic lights that are red when Jane is driving to work is a random variable with probability distribution given by

x	0	1	2	3
$\Pr(X = x)$	0.1	0.2	0.3	0.4

Jane drives to work on two consecutive days. What is the probability that the number of traffic lights that are red is the same on both days?

12 © VCAA 2011 1Q7 TECH-FREE (4 marks) A biased coin is tossed three times. The probability of a head from a toss of this coin is p.

a Find, in terms of p, the probability of obtaining

i 71% three heads from the three tosses 1 mark

ii 41% two heads and a tail from the three tosses. 1 mark

b 47% If the probability of obtaining three heads equals the probability of obtaining two heads and a tail, find p. 2 marks

13 © VCAA 2010 2AQ14 81% A bag contains four white balls and six black balls. Three balls are drawn from the bag without replacement. The probability that they are all black is

A $\frac{1}{6}$ **B** $\frac{27}{125}$ **C** $\frac{24}{29}$ **D** $\frac{3}{500}$ **E** $\frac{8}{125}$

14 © VCAA 2006 2AQ5 68% A bag contains three white balls and seven yellow balls. Three balls are drawn one at a time from the bag without replacement.

The probability that they are all yellow is

A $\frac{3}{500}$ **B** $\frac{27}{1000}$ **C** $\frac{21}{100}$ **D** $\frac{7}{24}$ **E** $\frac{243}{1000}$

15 © VCAA 2005 1IQ4 65% At a party there are six unmarked boxes. Two boxes each have prizes, the other four boxes are empty. When two boxes are selected without replacement, the probability of selecting at least one box with a prize is

A $\frac{1}{15}$ **B** $\frac{6}{15}$ **C** $\frac{8}{15}$ **D** $\frac{5}{9}$ **E** $\frac{9}{15}$

16 © VCAA 2014 2AQ11 62% A bag contains five red marbles and four blue marbles. Two marbles are drawn from the bag, without replacement, and the results are recorded.

The probability that the marbles are different colours is

A $\frac{20}{81}$ **B** $\frac{5}{18}$ **C** $\frac{4}{9}$ **D** $\frac{40}{81}$ **E** $\frac{5}{9}$

8.2

17 © VCAA 2004 1IQ3 47% Christopher has five pairs of identical purple socks and three pairs of identical green socks. His socks are randomly mixed in his drawer. He takes two individual socks at random from the drawer in the dark.

The probability that he obtains a matching pair is

A $\frac{1}{8}$ **B** $\frac{5}{14}$ **C** $\frac{3}{8}$ **D** $\frac{13}{28}$ **E** $\frac{1}{2}$

18 © VCAA 2017N 2AQ11 A bag contains five blue marbles and four red marbles. A sample of four marbles is taken from the bag, without replacement. The probability that the proportion of blue marbles in the sample is greater than $\frac{1}{2}$ is

A $\frac{1}{2}$ **B** $\frac{2}{9}$ **C** $\frac{5}{14}$ **D** $\frac{5}{9}$ **E** $\frac{25}{63}$

19 © VCAA 2014 2AQ22 37% John and Rebecca are playing darts. The result of each of their throws is independent of the result of any other throw. The probability that John hits the bullseye with a single throw is $\frac{1}{4}$. The probability that Rebecca hits the bullseye with a single throw is $\frac{1}{2}$. John has four throws and Rebecca has two throws.

The ratio of the probability of Rebecca hitting the bullseye at least once to the probability of John hitting the bullseye at least once is

A $1:1$ **B** $32:27$ **C** $64:85$ **D** $2:1$ **E** $192:175$

20 © VCAA 2012 2BQ3ai 70% (1 mark) Steve, Katerina and Jess are three students who have agreed to take part in a psychology experiment. Each student is to answer several sets of multiple-choice questions. Each set has the same number of questions, n, where n is a number greater than 20. For each question there are four possible options (A, B, C or D), of which only one is correct.

Steve decides to guess the answer to every question, so that for each question he chooses A, B, C or D at random.

Let the random variable X be the number of questions that Steve answers correctly in a particular set.

What is the probability that Steve will answer the first three questions of this set correctly?

21 © VCAA 2008 2BQ1ai 70% (1 mark) Sharelle is the goal shooter for her netball team. During her matches, she has many attempts at scoring a goal.

Assume that each attempt at scoring a goal is independent of any other attempt. In the long term, her scoring rate has been shown to be 80% (that is, 8 out of 10 attempts to score a goal are successful).

What is the probability, correct to four decimal places, that her first 8 attempts at scoring a goal in a match are successful?

22 © VCAA 2004 1IIQ6a 40% (1 mark) A barrel contains 100 balls, some of which are rainbow-coloured. Four balls are randomly selected from the barrel, **with replacement**. This means that a ball is selected, its colour noted, and the ball replaced **before** the next ball is selected. (that is, a ball is selected, its colour noted, and the ball replaced **before** the next ball is selected).

Let p be the proportion of rainbow-coloured balls in the barrel.

Write down an expression for the probability that exactly one of the four balls selected is rainbow-coloured.

23 © VCAA 2002 1IIQ2 (4 marks) A roulette wheel has 37 slots, numbered 0 to 36. Slot 0 is green. Eighteen of the remaining slots are black, and the other 18 are red. In a single game, a person spins the wheel and at the same time rolls a ball around the wheel in the opposite direction. As the wheel slows, the ball falls into one of the slots. The wheel is carefully balanced so that the ball is equally likely to fall into any of the slots.

a 65% In a single game, what is the probability, correct to three decimal places, that the ball falls into a red slot? 1 mark

Several games are played one after the other. Assume that the result of each game is independent of the result of any other game.

b 32% What is the probability, correct to three decimal places, that the first time that the ball falls into a red slot is in the fifth game? 1 mark

c 11% What is the probability, correct to three decimal places, that the ball falls into a red slot before the fifth game? 2 marks

Video playlist
Conditional probability

Worksheet
Conditional probability

Puzzle
Conditional probability: Two-way tables

8.3 Conditional probability

A **conditional probability** is the probability of an event occurring if another event has also occurred. In these situations, the sample space is reduced because of the condition placed on the probability.

The probability of event A occurring given event B occurs is written as $\Pr(A|B)$

Conditional probability formula

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$

where $\Pr(A|B)$ means 'the probability of A occurring given B occurs'.

Conditional probabilities from simple events

When calculating conditional probabilities from simple events, it is easier to use a reduced sample space approach. The probability of event A occurring given event B occurs can be determined by reducing the sample space to event B.

Conditional probability

$$\Pr(A|B) = \frac{n(A \cap B)}{n(B)} \text{ or } \Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$

where $n(A \cap B)$ is the number of outcomes in events A and B and $n(B)$ is the number of outcomes in event B.

Exam hack

Look for the key words 'if' or 'given that' to identify conditional probability problems. In the conditional probability formula, the condition always goes second and the required probability goes first.

$\Pr(\text{required probability} \mid \text{condition})$

WORKED EXAMPLE 11 Using the conditional probability formula

For events A and B from a sample space, $\Pr(A \cap B') = 0.04$, $\Pr(B) = 0.6$ and $\Pr(A') = 0.48$.
Find $\Pr(A'|B')$.

Steps | **Working**

1 Enter the given probabilities into a probability table and complete the other totals.

	B	B'	Totals
A		0.04	
A'			0.48
Totals	0.6		1.0

2 Complete the probability table.

	B	B'	Totals
A	0.48	0.04	0.52
A'	0.12	0.36	0.48
Totals	0.6	0.4	1.0

3 Substitute into the conditional probability formula.

$$\Pr(A'|B') = \frac{\Pr(A' \cap B')}{\Pr(B')}$$
$$= \frac{0.36}{0.4}$$
$$= 0.9$$

WB p. 165

WORKED EXAMPLE 12 Conditional probability using a Venn diagram

A group of 50 performers contains 18 tap dancers and 6 performers that sing and tap dance. If all the performers either tap dance or sing, find the probability of randomly selecting a tap dancer given that a singer is selected.

Steps | **Working**

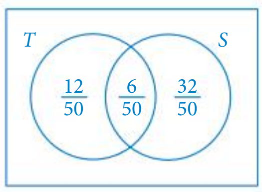
1 Express the information given using mathematical notation.

T represents tap dancers and S represents singers.

$$\Pr(T) = \frac{18}{50}$$
$$\Pr(T \cap S) = \frac{6}{50}$$

2 Enter and complete a Venn diagram.

$$\Pr(T \cap S') = \frac{18}{50} - \frac{6}{50} = \frac{12}{50}$$
$$\Pr(T' \cap S) = 1 - \frac{6}{50} - \frac{12}{50} = \frac{32}{50}$$



3 Write the conditional probability formula for this problem.

Probability of selecting a tap dancer given that a singer is selected.

$$\Pr(T|S) = \frac{n(T \cap S)}{n(S)}$$

4 Calculate the probability.
We are only selecting from the 38 singers and 6 of those tap dance.
It is also possible to substitute into the formula to obtain the same answer.

$$\Pr(T|S) = \Pr(T \cap S) \div \Pr(S) = \frac{6}{50} \div \frac{38}{50}$$
$$\Pr(T|S) = \frac{6}{50} \times \frac{50}{38}$$
$$\Pr(T|S) = \frac{6}{38} = \frac{3}{19}$$

$$= \frac{6}{38}$$
$$= \frac{3}{19}$$

WB p. 166

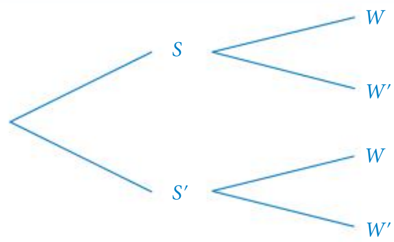
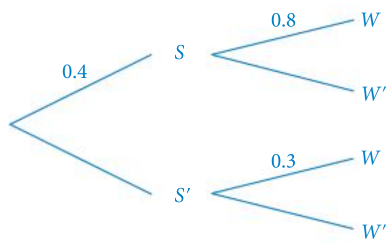
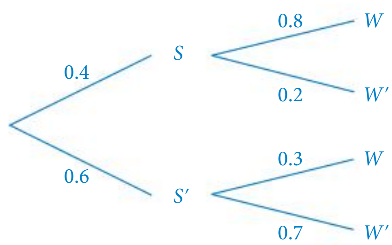
Conditional probability tree diagrams

Two events are dependent if the outcome of the first event influences the outcome of the second event. When drawing the tree diagram, the outcomes for the dependent event go second and the outcomes for the event that influences it go first.

WB
p. 167

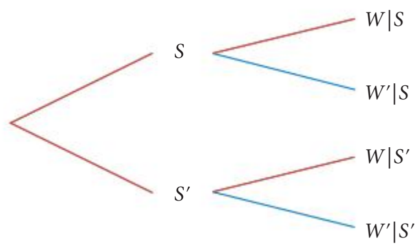
WORKED EXAMPLE 13 Conditional probability from a tree diagram

A football team finds that they win 80% of their games when their star player plays and 30% of their games when he does not play. Their star player was injured at training during the week and has a 40% chance of playing on the weekend. Find the probability that in their next game the star player plays given that they will win.

Steps	Working
1 Set up the tree diagram. S represents the star plays. S' represents the star player does not play. W represents the team wins. W' represents the team does not win. The team winning depends on the star player playing, so S and S' go first in the tree diagram.	
2 Enter the given probabilities on the tree diagram.	
3 Enter the remaining probabilities on the tree diagram. The probabilities on each set of branches must add to 1.0.	
4 Write the required probability using probability notation and express as a formula.	Find the probability that in their next game the star player plays given that they win. $\Pr(S\|W) = \dfrac{\Pr(S \cap W)}{\Pr(W)}$
5 Find the probability of $\Pr(S \cap W)$ and $\Pr(W)$ from the tree diagram. There are 2 possibilities where the teams wins: SW and $S'W$.	$\Pr(S \cap W) = 0.4 \times 0.8 = 0.32$ $\Pr(W) = 0.4 \times 0.8 + 0.6 \times 0.3$ $\Pr(W) = 0.5$
6 Substitute into the conditional probability formula.	$\Pr(S\|W) = \dfrac{\Pr(S \cap W)}{\Pr(W)}$ $\Pr(S\|W) = \dfrac{0.32}{0.5}$ $\Pr(S\|W) = 0.64$

The law of total probabilities

The tree diagram for Worked example 13 can also be drawn as



The branches on the tree diagram that represent a win are shown in red.

The top branch is a win when the star player plays = $\Pr(W \mid S) \times \Pr(S)$.

The bottom branch is a win when the star player does not play = $\Pr(W \mid S') \times \Pr(S')$.

$\Pr(W) = \Pr(W|S) \times \Pr(S) + \Pr(W|S') \times \Pr(S')$

The law of total probability breaks up probability calculations into distinct parts. It is used to find the probability of an event, A, when we don't know enough about A's probabilities to calculate it directly. Instead, we take a related event, B, and use that to calculate the probability for A.

The law of total probability

$\Pr(A) = \Pr(A \mid B)\Pr(B) + \Pr(A \mid B')\Pr(B')$

WORKED EXAMPLE 14 Conditional probability using the law of total probability

WB p. 168

A soccer team wins 60% of its games when it scores the first goal, and 10% of its games when the opposing team scores first. If the team scores the first goal about 30% of the time, what fraction of games does it win?

Steps	Working
1 Define variables.	F = the team scores the first goal. F' = the team does not score the first goal. $W \mid F$ = the team wins given they score the first goal. $W \mid F'$ = the team wins given they did not score the first goal.
2 Write the probabilities for each of the defined events.	$\Pr(F) = 0.3$ $\Pr(F') = 0.7$ $\Pr(W \mid F) = 0.6$ $\Pr(W \mid F') = 0.1$
3 Substitute into the law of total probabilities. $\Pr(A) = \Pr(A \mid B)\Pr(B) + \Pr(A \mid B')\Pr(B')$	$\Pr(W) = \Pr(W \mid F) \times \Pr(F) + \Pr(W \mid F') \times \Pr(F')$ $\Pr(W) = 0.3 \times 0.6 + 0.7 \times 0.1$ $\Pr(W) = 0.25$ The team wins one quarter of its games.

Repeated conditional events

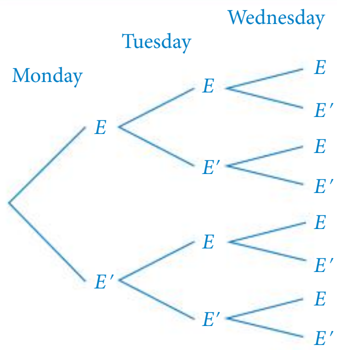
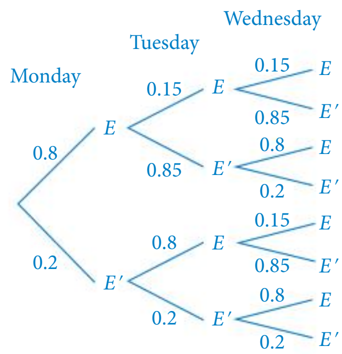
A sequence of probabilities can be generated where each trial has 2 possibilities and the probabilities are dependent on the previous outcome. These problems can be solved using tree diagrams.

WB

p. 169

WORKED EXAMPLE 15 Calculating probabilities for repeated events

Beryl is the oldest hen at the farm and is a very fussy layer. If she lays an egg on one day, there is a probability of 0.15 she will lay an egg the next day. If she does not lay on one day, the probability of laying an egg the next day is 0.8. On Sunday, Beryl is feeling a bit indifferent and does not lay an egg. Find the probability that she lays 2 eggs in the next 3 days.

Steps	Working
1 Set up the tree diagram. E represents Beryl lays an egg. E' represents Beryl does not lay an egg.	
2 Enter the given probabilities on the tree diagram. Since Beryl does not lay an egg on Sunday, the probabilities for Monday are $\Pr(E) = 0.8$ and $\Pr(E') = 0.2$.	
3 Identify the branches where there are exactly 2 eggs.	$\Pr(2 \text{ eggs}) = \Pr(EEE') + \Pr(EE'E) + \Pr(E'EE)$
4 Calculate the probabilities.	$= 0.8 \times 0.15 \times 0.85 + 0.8 \times 0.85 \times 0.8 + 0.2 \times 0.8 \times 0.15$ $= 0.67$

EXERCISE 8.3 Conditional probability

ANSWERS p. 524

Recap

1 The weather bureau predicts a 30% chance of frost on a Monday and a 40% chance of frost on Tuesday. If the occurrence of a frost occurring on Tuesday is independent of a frost occurring on Monday, then the probability of frost on neither Monday nor Tuesday is

A 0.12 **B** 0.24 **C** 0.3 **D** 0.42 **E** 0.7

2 A bag contains 3 white balls and 5 black balls. Three balls are drawn from the bag without replacement. The probability that they are all black is

A $\frac{1}{28}$ **B** $\frac{27}{512}$ **C** $\frac{5}{28}$ **D** $\frac{125}{512}$ **E** $\frac{5}{8}$

Mastery

3 WORKED EXAMPLE 11 For events A and B from a sample space, $\Pr(A' \cap B) = 0.05$, $\Pr(B') = 0.45$ and $\Pr(A) = 0.7$. Find $\Pr(A'|B')$.

4 For events A and B from a sample space, $\Pr(A \cap B) = \frac{7}{24}$, $\Pr(A \cap B') = \frac{1}{8}$ and $\Pr(A' \cap B) = \frac{5}{12}$. Find $\Pr(A|B')$.

5 WORKED EXAMPLE 12 A survey of 70 people shows that 55 own a smartphone, 33 own a tablet and 22 own both items. Find the probability that a randomly selected person owns a smartphone if it is known that the person selected owns a tablet.

6 Carmen has 40 different ribbons in her drawer. 15 of these ribbons are silk, 25 are green and 10 are silk but are not green. If Carmen randomly selects a ribbon that is green, find the probability that it is not silk.

7 WORKED EXAMPLE 13 Sleepy Sam arrives at school on time 90% of occasions when his alarm goes off in the morning but only arrives on time 35% of occasions when his alarm does not ring. His alarm is faulty and will only ring 70% of school day mornings. Find the probability that, if Sleepy Sam arrives at school on time, his alarm rang that morning.

8 A company manufactures tyres at two different factories. Factory A produces 400 tyres each week with a defective rate of 5% while factory B produces 600 tyres each week with a defective rate of 8%. One week a tyre is randomly selected from one of the factories and found to be defective. Find the probability the tyre was manufactured at factory B.

9 WORKED EXAMPLE 14 At Nelson Secondary College, 46% of Year 10 students are male and 54% are female. If 18% of the male Year 10 students study Japanese and 25% of the female Year 10 students study Japanese, find the probability that a randomly selected Year 10 student from Nelson Secondary College studies Japanese.

10 Marnie arrives at school on time 90% of occasions when she remembers to set her alarm, but only arrives on time 35% of occasions when she forgets to set her alarm. Marnie remembers to set her alarm on 70% of school days. Find the probability that Marnie arrives at school on time.

11 Alex and Roxanne work on quality control, checking all the balloons produced by a party balloons manufacturer. On a particular day, Alex checks 400 balloons, of which 5% are faulty, and Roxanne checks 600 balloons and finds that 2% of her batch are faulty. Find the probability that a randomly selected balloon manufactured that day is faulty.

12 WORKED EXAMPLE 15 Eleanor's mum always bakes a cake on Eleanor's birthday. Her two favourite cakes are chocolate and orange. If Eleanor has a chocolate cake one year, there is a 75% chance she will get an orange cake the next year. If she has an orange cake one year, there is a 60% chance she will get a chocolate cake the next year. Eleanor has a chocolate cake on her 10^{th} birthday.

Find, correct to 3 decimal places, the probability Eleanor has exactly 2 chocolate birthday cakes before her 14^{th} birthday.

13 Each afternoon, Aldo buys a piece of fruit on his way home from school. If he buys an apple one day, there is a 40% chance he will buy an apple the next day. If he buys a peach one day, there is a 70% chance he will buy a peach the next day. On Monday, Aldo buys a peach. Find the probability that Aldo will buy a total of exactly 2 peaches on Tuesday, Wednesday and Thursday.

Exam practice

80–100% 60–79% 0–59%

14 © VCAA 2015 1Q8 60% TECH-FREE (3 marks) For events A and B from a sample space, $\Pr(A|B) = \frac{3}{4}$ and $\Pr(B) = \frac{1}{3}$.

a Calculate $\Pr(A \cap B)$. 1 mark

b Calculate $\Pr(A' \cap B)$, where A' denotes the complement of A. 1 mark

c If A and B are independent, calculate $\Pr(A \cup B)$. 1 mark

15 © VCAA 2008 1Q8 60% TECH-FREE (3 marks) Every Friday Jean-Paul goes to see a movie. He always goes to one of two local cinemas – the Dandy or the Cino.

If he goes to the Dandy one Friday, the probability that he goes to the Cino the next Friday is 0.5. If he goes to the Cino one Friday, then the probability that he goes to the Dandy the next Friday is 0.6.

On any given Friday the cinema he goes to depends only on the cinema he went to on the previous Friday.

If he goes to the Cino one Friday, what is the probability that he goes to the Cino on **exactly two** of the next three Fridays?

16 © VCAA 2014 1Q9 50% TECH-FREE (6 marks) Sally aims to walk her dog, Mack, most mornings.

If the weather is pleasant, the probability that she will walk Mack is $\frac{3}{4}$, and if the weather is unpleasant, the probability that she will walk Mack is $\frac{1}{3}$.

Assume that pleasant weather on any morning is independent of pleasant weather on any other morning.

a In a particular week, the weather was pleasant on Monday morning and unpleasant on Tuesday morning.

Find the probability that Sally walked Mack on at least one of these two morning. 2 marks

b In the month of April, the probability of pleasant weather in the morning was $\frac{5}{8}$.

i Find the probability that on a particular morning in April, Sally walked Mack. 2 marks

ii Using your answer from part **b i**, or otherwise, find the probability that on a particular morning in April, the weather was pleasant, given that Sally walked Mack that morning. 2 marks

17 © VCAA 2016 1Q7 47% TECH-FREE (3 marks) A company produces motors for refrigerators. There are two assembly lines, Line A and Line B. 5% of the motors assembled on Line A are faulty and 8% of the motors assembled on Line B are faulty. In 1 hour, 40 motors are produced from Line A and 50 motors are produced from Line B. At the end of an hour, one motor is selected at random from all the motors that have been produced during that hour.

a What is the probability that the selected motor is faulty? Express your answer in the form $\frac{1}{b}$, where b is a positive integer. 2 marks

b The selected motor is found to be faulty. What is the probability that it was assembled on Line A? Express your answer in the form $\frac{1}{c}$, where c is a positive integer. 1 mark

18 © VCAA 2012 1Q4 43% TECH-FREE (4 marks) On any given day, the number X of telephone calls that Daniel receives is a random variable with probability distribution given by

x	0	1	2	3
$\Pr(X = x)$	0.2	0.2	0.5	0.1

a What is the probability that Daniel receives only one telephone call on each of three consecutive days? 1 mark

b Daniel receives telephone calls on both Monday and Tuesday.

What is the probability that Daniel receives a total of four calls over these two days? 3 marks

19 © VCAA 2015 1Q9 35% TECH-FREE (4 marks) An egg marketing company buys its eggs from farm A and farm B. Let p be the proportion of eggs that the company buys from farm A. The rest of the company's eggs come from farm B. Each day, the eggs from both farms are taken to the company's warehouse. Assume that $\frac{3}{5}$ of all eggs from farm A have white eggshells and $\frac{1}{5}$ of all eggs from farm B have white eggshells.

a An egg is selected at random from the set of all eggs at the warehouse. Find, in terms of p, the probability that the egg has a white eggshell. 1 mark

b Another egg is selected at random from the set of all eggs at the warehouse.

i Given that the egg has a white eggshell, find, in terms of p, the probability that it came from farm B. 2 marks

ii If the probability that this egg came from farm B is 0.3, find the value of p. 1 mark

20 © VCAA 2019 1Q3 MODIFIED TECH-FREE (3 marks) The only possible outcomes when a coin is tossed are a head or a tail. When an unbiased coin is tossed, the probability of tossing a head is the same as the probability of tossing a tail. Jo has four coins in her pocket; three are unbiased and one is biased. When the biased coin is tossed, the probability of tossing a head is $\frac{1}{3}$. Jo randomly selects a coin from her pocket and tosses it.

a Find the probability that she tosses a tail. 2 marks

b Find the probability that she selected an unbiased coin, given that she tossed a tail. 1 mark

21 © VCAA 2009 1Q5c 20% TECH-FREE (2 marks) Four identical balls are numbered 1, 2, 3 and 4 and put into a box. A ball is randomly drawn from the box, and not returned to the box. A second ball is then randomly drawn from the box.

Given that the sum of the numbers on the two balls is 5, what is the probability that the second ball drawn is numbered 1?

22 © VCAA 2012 2AQ12 72% Demelza is a badminton player. If she wins a game, the probability that she will win the next game is 0.7. If she loses a game, the probability that she will lose the next game is 0.6. Demelza has just won a game.

The probability that she will win exactly one of her next two games is

A 0.33 **B** 0.35 **C** 0.42 **D** 0.49 **E** 0.82

23 © VCAA 2012 2AQ13 52% A and B are events of a sample space S.

$\Pr(A \cap B) = \frac{2}{5}$ and $\Pr(A \cap B') = \frac{3}{7}$.

$\Pr(B' | A)$ is equal to

A $\frac{6}{35}$ **B** $\frac{15}{29}$ **C** $\frac{14}{35}$ **D** $\frac{29}{35}$ **E** $\frac{2}{3}$

8.3

24 © VCAA 2014 2AQ14 45% If X is a random variable such that $\Pr(X > 5) = a$ and $\Pr(X > 8) = b$, then $\Pr(X < 5 \mid X < 8)$ is

A $\frac{a}{b}$ **B** $\frac{a-b}{1-b}$ **C** $\frac{1-b}{1-a}$ **D** $\frac{ab}{1-b}$ **E** $\frac{a-1}{b-1}$

25 © VCAA 2008 2BQ1 60% (4 marks) Sharelle is the goal shooter for her netball team. During her matches, she has many attempts at scoring a goal.

Assume instead that the success of an attempt to score a goal depends only on the success or otherwise of her previous attempt at scoring a goal.

If an attempt at scoring a goal in a match is successful, then the probability that her next attempt at scoring a goal in the match is successful is 0.84. However, if an attempt at scoring a goal in a match is unsuccessful, then the probability that her next attempt at scoring a goal in the match is successful is 0.64.

Her first attempt at scoring a goal in a match is successful.

a What is the probability, correct to four decimal places, that her next 7 attempts at scoring a goal in the match will be successful? 1 mark

b What is the probability, correct to four decimal places, that exactly 2 of her next 3 attempts at scoring a goal in the match will be successful? 3 marks

26 © VCAA 2010 2BQ2 57% (6 marks) Victoria Jones runs a small business making and selling statues of her cousin the adventurer Tasmania Jones.

The statues are made in a mould, then finished (smoothed and then hand-painted using a special gold paint) by Victoria herself. Victoria sends the statues **in order of completion** to an inspector, who classifies them as either 'Superior' or 'Regular', depending on the quality of their finish.

If a statue is Superior then the probability that the next statue completed is Superior is p.

If a statue is Regular then the probability that the next statue completed is Superior is $p - 0.2$.
On a particular day, Victoria knows that $p = 0.9$.

On that day

a if the **first statue inspected is Superior**, find the probability that the third statue is Regular 2 marks

b if the **first statue inspected is Superior**, find the probability that the next three statues are Superior. 1 mark

On another day, Victoria finds that if the **first statue inspected is Superior** then the probability that the third statue is Superior is 0.7.

c Show that the value of p on this day is 0.75. 3 marks

27 © VCAA 2006 2BQ2a 50% (3 marks) Each night Kim goes to the gym or the pool. If she goes to the gym one night, the probability she goes to the pool the next night is 0.4, and if she goes to the pool one night, the probability she goes to the gym the next night is 0.7.

Suppose she goes to the gym one Monday night.

a What is the probability that she goes to the pool on each of the next three nights? (The exact value must be given.) 1 mark

b What is the probability that she goes to the pool on exactly two of the next three nights? (The exact value must be given.) 2 marks

28 © VCAA 2012 2BQ3bi 43% (3 marks) If Katerina answers a question correctly, the probability that she will answer the next question correctly is $\frac{3}{4}$. If she answers a question incorrectly, the probability that she will answer the next question incorrectly is $\frac{2}{3}$.

In a particular set, Katerina answers Question 1 incorrectly. Calculate the probability that Katerina will answer Questions 3, 4 and 5 correctly.

29 © VCAA 2003 1BQ6 MODIFIED (4 marks) Kim goes to the sports centre each evening and either works out in the gym or has a swim. She never has a swim two evenings in a row. If she has a work-out in the gym one evening, then the next evening she is twice as likely to have a swim as she is to have a work-out in the gym. On a particular Monday evening, she works out in the gym.

a What is the probability that she works out in the gym on both the Tuesday and Wednesday evenings of that week? 1 mark

b What is the probability that she has a swim on Wednesday of that week? 3 marks

Exam hack

On a tree diagram, the probabilities on any set of branches MUST sum to 1.

8.4

8.4 The addition rule for probabilities

Video playlist
The addition rule for probabilities

The mathematical notation for the probability of events A OR B (or both) occurring is $\Pr(A \cup B)$.

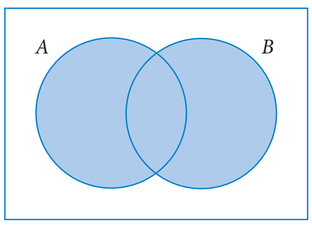
The mathematical notation for the probability of events A AND B occurring is $\Pr(A \cap B)$.

The Venn diagram and probability table below show that when we add $\Pr(A)$ and $\Pr(B)$, the intersection $\Pr(A \cap B)$ is counted twice, so we must subtract one $\Pr(A \cap B)$. This is called the **addition rule for probabilities**.

The addition rule for probabilities

$$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$$

	B	B'	
A			
A'			



WORKED EXAMPLE 16 Finding the probability of the union of two events from a two-way table

WB p. 170

If $\Pr(A) = 0.25$, $\Pr(B) = 0.3$ and $\Pr(A \cap B) = 0.12$, find $\Pr(A' \cup B)$.

Steps | **Working**

1 Enter the probabilities into a probability table.

	B	B'	Totals
A	0.12		0.25
A'			
Totals	0.3		1.0

2 Complete the table.

	B	B'	Totals
A	0.12	0.13	0.25
A'	0.18	0.57	0.75
Totals	0.3	0.7	1.0

3 Calculate $\Pr(A' \cup B)$ using the addition rule.

This probability can also be found by adding the shaded values in the table or by calculating 1 – 0.13.

$$\Pr(A' \cup B) = \Pr(A') + \Pr(B) - \Pr(A' \cap B)$$
$$= 0.75 + 0.3 - 0.18$$
$$= 0.87$$

WB
p. 171

WORKED EXAMPLE 17 Finding probabilities using the addition rule

If $\Pr(A \cup B) = \frac{2}{5}$, $\Pr(A \cap B) = \frac{1}{5}$ and $\Pr(A) = 2 \times \Pr(B,)$ find $\Pr(A)$.

Steps	Working
1 Let $\Pr(B) = b$. Write $\Pr(A)$ in terms of b.	Let $\Pr(B) = b$. $\Pr(A) = 2 \times \Pr(B)$ $\Pr(A) = 2b$
2 Substitute the given values into the addition rule.	$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$ $\frac{2}{5} = 2b + b - \frac{1}{5}$ $\frac{3}{5} = 3b$ $b = \frac{1}{5}$
3 Substitute into $\Pr(A) = 2b$.	$\Pr(A) = 2 \times \frac{1}{5} = \frac{2}{5}$

WB
p. 172

WORKED EXAMPLE 18 Finding $\Pr(A \cup B)$

A card is selected at random from a pack of 52 playing cards. Find the probability of selecting a king or a heart.

Steps	Working
1 $\Pr(\text{king or heart}) = \Pr(K \cup H)$ The addition rule for probabilities can be used to solve this problem.	Let K represent set of kings and H represent the set of hearts. $\Pr(K \cup H) = \Pr(K) + \Pr(H) - \Pr(K \cap H)$
2 Find the probabilities required in the formula. $\Pr(K \cap H) = \Pr(\text{a king and a heart})$ $= \Pr(\text{the king of hearts})$	$\Pr(K) = \frac{4}{52}$ $\Pr(H) = \frac{13}{52}$ $\Pr(K \cap H) = \frac{1}{52}$
3 Substitute into the addition rule.	$\Pr(K \cup H) = \frac{4}{52} + \frac{13}{52} - \frac{1}{52}$ $\Pr(K \cup H) = \frac{16}{52}$ $\Pr(K \cup H) = \frac{4}{13}$

Exam hack

The probability of A or B occurring can be calculated using

- a Venn diagram
- a two-way probability table
- a tree diagram
- the addition rule.

You will need to determine the most efficient method for each individual question.

Exam hack

It is useful to know that $(A' \cup B') = (A \cap B)'$ and $(A' \cap B') = (A \cup B)'$.

8.4

Video
VCE question analysis: Probability theory

VCE QUESTION ANALYSIS

 2005 2Q2 MODIFIED 2005 Examination 2 Question 2 MODIFIED (10 marks)

Oz-Online is an internet provider company, with more than one million customers, and has two internet access plans.

- The Dial-up Plan
- The Cable-modem Plan

This question from 2005 predates broadband internet access

80% of customers have the Dial-up Plan and 20% of customers the Cable-modem Plan.

a If three customers of Oz-Online are selected at random, find the probability correct to three decimal places that

- **i** all three customers are on the Dial-up Plan. 1 mark
- **ii** two customers are on the Dial-up Plan. 1 mark
- **iii** at most two customers are on the Dial-up Plan. 1 mark
- **iv** exactly two customers are on the Dial-up Plan if it is known that at least one customer is on the Dial-up Plan. 2 marks

b To try to increase the percentage of customers using a cable modem, Oz-Online offers all its **present** customers the Cable-modem Plan with unlimited hours, and at a reduced cost.

The marketing manager estimates that each month 20% of the customers on the Dial-up Plan will change to the Cable-modem Plan, and 5% of the customers on the Cable-modem Plan will switch to the Dial-up Plan. No **present** customers stop using Oz-Online.

- **i** What proportion of present customers are on Dial-up after one month? 2 mark
- **ii** After two months what proportion of the **present** customers, correct to two decimal places, will now be on the Dial-up Plan? 3 marks

Reading the question

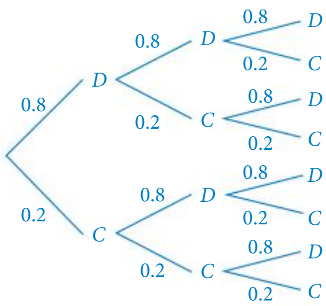
- Highlight the information that identifies the options for each customer and the respective probabilities.
- Highlight the number of customers. This information will indicate the dimensions of the tree diagram.
- Highlight the number of decimal places in the 4 answers in part **a**.
- In part **a iv**, 'if' is highlighted since this indicates **conditional probability**.
- The information required for the new conditional tree diagram in part **b** is also highlighted.

Thinking about the question

- The probabilities in successive trials can be either independent or dependent. You will need to know how to use this information to draw a tree diagram.
- You will need to be able to use a tree diagram to calculate probabilities.
- You will also need to recognise conditional probabilities and be able to use the conditional probability formula.

Worked solution (✓ = 1 mark)

a **i** D represents customers on the Dial-up Plan and C represents the customers on the Cable-modem Plan.



$\Pr(DDD) = 0.8^3 = \mathbf{0.512}$ ✓

ii $\Pr(D = 2)$

$= \Pr(DDC) + \Pr(DCD) + \Pr(CDD)$

$= 0.8 \times 0.8 \times 0.2 + 0.8 \times 0.2 \times 0.8 + 0.2 \times 0.8 \times 0.8$

$= 0.8^2 \times 0.2 \times 3$

$= \mathbf{0.384}$ ✓

iii $\Pr(D \le 2)$

$= 1 - \Pr(DDD)$

$= 1 - 0.512$

$= \mathbf{0.488}$ ✓

iv $\Pr(D = 2 \mid D \ge 1) = \dfrac{\Pr(D = 2)}{\Pr(D \ge 1)}$

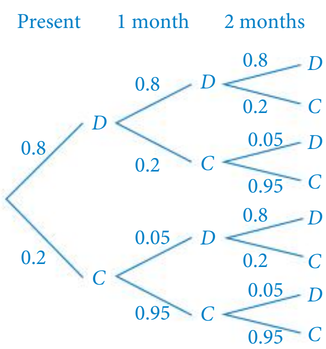
$= \dfrac{\Pr(D = 2)}{1 - \Pr(D = 0)}$

$= \dfrac{\mathbf{0.384}}{\mathbf{1 - 0.2^2}}$

$\approx \mathbf{0.4}$ ✓

Correct numerator or denominator ✓

b **i**



Proportion of customers on the Dial-up Plan after 1 month

$= \mathbf{\Pr(DD) + \Pr(CD)}$ ✓

$= 0.8 \times 0.8 + 0.2 \times 0.05$

$= \mathbf{0.65}$ ✓

ii Proportion of customers on the Dial-up Plan after 2 months

$= \mathbf{\Pr(DDD) + \Pr(DCD) + \Pr(CDD) + \Pr(CCD)}$ ✓

$= 0.8 \times 0.8 \times 0.8 + 0.8 \times 0.2 \times 0.05 + 0.2 \times 0.05 \times 0.8 + 0.2 \times 0.95 \times 0.05$ ✓

$= 0.5375$

$\approx \mathbf{0.54}$ ✓

EXERCISE 8.4 The addition rule for probabilities

ANSWERS p. 524

Recap

1 Danni is a tennis player. If she wins a game, the probability that she will win the next game is 0.8. If she loses a game, the probability that she will lose the next game is 0.9. Danni has just lost a game.

The probability that she will win exactly one of her next 2 games is

A 0.1 **B** 0.11 **C** 0.17 **D** 0.19 **E** 0.81

2 For events A and B from a sample space, $\Pr(A|B') = \frac{3}{4}$, $\Pr(A) = \frac{2}{3}$ and $\Pr(B) = \frac{1}{3}$.

The value of $\Pr(A \cap B)$ is

A $\frac{5}{36}$ **B** $\frac{1}{6}$ **C** $\frac{1}{4}$ **D** $\frac{1}{3}$ **E** $\frac{2}{3}$

Mastery

3 WORKED EXAMPLE 16 For two events A and B from a sample space, $\Pr(A') = 0.44$, $\Pr(A \cap B') = 0.25$ and $\Pr(A' \cap B) = 0.17$. Find $\Pr(A' \cup B')$.

4 For two events X and Y from a sample space, $\Pr(X' \cap Y) = 0.38$, $\Pr(X \cup Y) = 0.78$ and $\Pr(Y) = 0.61$. Find $\Pr(X' \cup Y')$.

5 WORKED EXAMPLE 17 For two independent events A and B from a sample space, $\Pr(A') = 0.4$ and $\Pr(B) = 0.15$. Find $\Pr(A \cup B')$.

6 For two independent events X and Y from a sample space, $\Pr(Y) = 0.8$, $\Pr(X \cap Y) = 0.24$ and $\Pr(X \cup Y) = 0.86$. Find $\Pr(X)$.

7 WORKED EXAMPLE 18 A card is selected at random from a pack of 52 playing cards. Find the probability of selecting a red card or a picture card (king, queen, jack).

8 A survey of 80 shoppers at the Bridge Street Mall showed that 50 people were shopping for a bargain, 45 people were shopping for a present and 12 were not shopping for either. Find the probability that a randomly selected shopper is looking for a bargain or a present.

Exam practice

9 On Michael's bookshelf there are 6 reference books, 14 fiction books and 5 non-fiction books. If a book is selected at random the probability that a reference book or a fiction book is selected is

A $\frac{1}{5}$ **B** $\frac{6}{25}$ **C** $\frac{14}{25}$ **D** $\frac{4}{5}$ **E** 20

10 At activities week 80 students can select a range of outdoor activities. It is found that 60 students chose cycling, 35 chose bushwalking and 10 chose neither of these two activities.

The probability that a randomly selected student chose cycling or bushwalking is

A $\frac{1}{8}$ **B** $\frac{5}{16}$ **C** $\frac{7}{16}$ **D** $\frac{3}{4}$ **E** $\frac{7}{8}$

11 For two independent events A and B, $\Pr(A') = 0.6$ and $\Pr(B') = 0.7$. $\Pr(A \cup B)$ is equal to

A 0.12 **B** 0.21 **C** 0.42 **D** 0.58 **E** 0.7

12 Two dice are rolled and the sum of the numbers uppermost is recorded. The probability of obtaining a sum greater than 8, or a 6, on one of the dice is

A $\frac{10}{36}$ **B** $\frac{11}{36}$ **C** $\frac{14}{36}$ **D** $\frac{18}{36}$ **E** $\frac{22}{36}$

13 (6 marks) The library conducted a survey to determine the most popular style of books. Of the 100 students surveyed

- 60 liked fantasy
- 18 liked only romance
- 11 liked all three styles
- 13 liked adventure and romance
- 16 liked fantasy and romance
- 26 liked exactly two of these styles
- 36 liked adventure but not romance

Find the probability that a student chosen at random likes

a only fantasy 1 mark

b none of these three styles 1 mark

c fantasy or adventure 2 marks

d fantasy or adventure or romance 2 marks

14 (3 marks) Lizzie travels to work by car, bus and bicycle. The proportion of time she uses each form of transport is 0.6, 0.25 and 0.15. The probability of her being late is 0.1 when she travels by car, 0.2 when she travels by bus and 0.45 when she cycles. Find the probability that Lizzie

a is late for work 1 mark

b travels by car and is on time for work 1 mark

c travels by car or is on time for work 1 mark

15 (4 marks)

	B	B'	**Totals**
A	0.35	0.09	
A'	0.25	0.31	
Totals			

Use the probability table to find

a $\Pr(A \cap B')$ 1 mark

b $\Pr(A')$ 1 mark

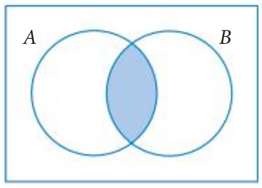
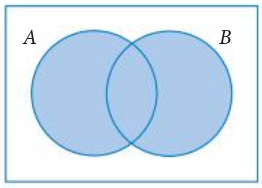
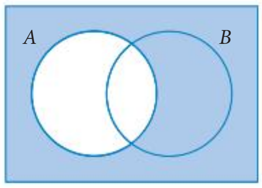
c $\Pr(A \cup B)$ 1 mark

d $\Pr(B|A')$ 1 mark

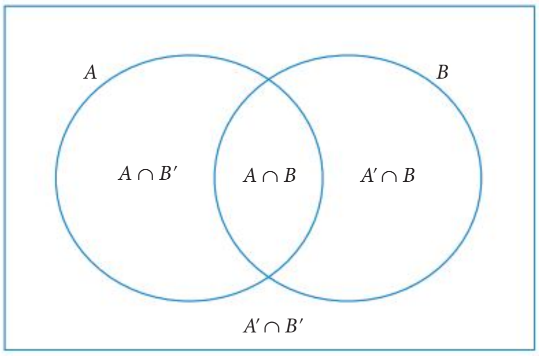
8 Chapter summary

The probability of an event occurring must have a value between 0 and 1 where 0 represents an impossible event and 1 represents a certain event.

$$\Pr(A) = \frac{\text{number of ways } A \text{ can occur}}{\text{total number of possible outcomes}}$$

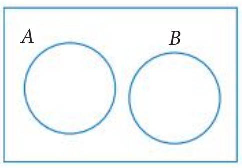
Intersection $A \cap B$	The intersection of sets A and B is a set that contains the elements that are in both sets.	
Union $A \cup B$	The union of sets A and B is obtained by combining or uniting both sets. The elements in the union are in set A or set B or both.	
Complement A'	The complement of set A is a set that contains all the elements that are not in set A.	

Sections of the Venn diagram



Mutually exclusive events

Two events A and B are mutually exclusive if $\Pr(A \cap B) = 0$. It is impossible for mutually exclusive events to occur simultaneously.



Independent events

For two independent events A and B,

$\Pr(A \cap B) = \Pr(A) \times \Pr(B)$

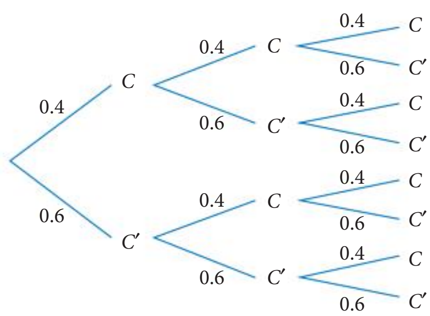
Two-way probability tables

A two-way probability table compares two intersecting sets.

	B	B'	**Totals**
A	$A \cap B$	$A \cap B'$	Total of set A
A'	$A' \cap B$	$A' \cap B'$	Total of set A'
Totals	Total of set B	Total of set B'	1.0

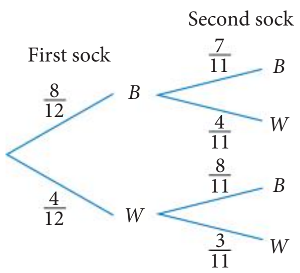
Tree diagrams

A tree diagram is a useful way of representing a series of simple events. Each simple event is represented by another stage of branches with the number of branches in each set equal to the number of possible outcomes for the event.



Selections without replacement

Selection without replacement occurs when an item is selected from a sample and is not replaced, then another item is selected. The events are dependent.



Conditional probability

The mathematical notation for 'find the probability of A occurring given B occurs' or 'if event B occurs find the probability of A occurring' is $\Pr(A|B)$ where $\Pr(A|B) = \dfrac{\Pr(A \cap B)}{\Pr(B)}$ or $\Pr(A|B) = \dfrac{n(A \cap B)}{n(B)}$.

The law of total probability

$\Pr(A) = \Pr(A \mid B)\Pr(B) + \Pr(A \mid B')\Pr(B')$

The addition rule for probabilities

$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$

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Cumulative examination 1

Total number of marks: 10 Reading time: 4 minutes Writing time: 15 minutes

TECH-FREE *Technology is NOT permitted.*

1 © VCAA 2018 1Q5 MODIFIED (2 marks) Let $f: (-2, \infty) \to R, f(x) = \dfrac{1}{(x+2)^2} + 1.$

Find the rule and domain of f^{-1}.

2 © VCAA 2018 1Q6 (4 marks) Two boxes each contain four stones that differ only in colour. Box 1 contains four black stones. Box 2 contains two black stones and two white stones. A box is chosen randomly and one stone is drawn randomly from it. Each box is equally likely to be chosen, as is each stone.

a What is the probability that the randomly drawn stone is black? 2 marks

b It is not known from which box the stone has been drawn. Given that the stone that is drawn is black, what is the probability that it was drawn from Box 1? 2 marks

3 (4 marks) Let $f: [0, \infty) \to R, f(x) = \sqrt{x+4}.$

a Find the value of a if the range of f is $[a, \infty)$. 1 mark

b Let $g: [c, \infty) \to R, g(x) = x^2 + 6x - 5.$

i Write g in the form $g(x) = (x - h)^2 + k$. 1 mark

ii Find the value of c if the range of g is $[a, \infty)$. 2 marks

Cumulative examination 2

Total number of marks: 20 Reading time: 4 minutes Writing time: 30 minutes

Approved technology is permitted.

Section A 5 multiple-choice questions

5 marks

1 © VCAA 2019N 2AQ3 If $x + a$ is a factor of $8x^3 - 14x^2 - a^2x$, where $a \in R\backslash\{0\}$, then the value of a is

A 7 **B** 4 **C** 1 **D** -2 **E** -1

2 © VCAA 2017 2AQ8 If $y = a^{b-4x} + 2$, where $a > 0$, then x is equal to

A $\frac{1}{4}\left(b - \log_a(y-2)\right)$

B $\frac{1}{4}\left(b - \log_a(y+2)\right)$

C $\left(b - \log_a\left(\frac{1}{4}(y+2)\right)\right)$

D $\frac{b}{4} - \log_a(y-2)$

E $\frac{1}{4}\left(b + 2 - \log_a(y)\right)$

3 © VCAA 2019N 2AQ9 At the start of a particular week, Kim has three red apples and two green apples. She eats one apple every day. On Monday, Tuesday and Wednesday of that week, she randomly selects an apple to eat.

In this three-day period, the probability that Kim does not eat an apple of the same colour on any two consecutive days is

A $\frac{1}{10}$ **B** $\frac{1}{5}$ **C** $\frac{3}{10}$ **D** $\frac{2}{5}$ **E** $\frac{6}{25}$

4 © VCAA 2019N 2AQ11 MODIFIED The function $f: D \to R, f(x) = 2x^2 + 4x + 1$ will have an inverse function for

A $D = R$

B $D = (-2, \infty)$

C $D = \left(-\infty, \frac{1}{2}\right]$

D $D = (-\infty, -1]$

E $D = [-4, \infty)$

5 © VCAA 2017 2AQ3 MODIFIED A box contains four red marbles and two yellow marbles. Two marbles are drawn at random from the box without replacement.

The probability that the marbles are of **different** colours is

A $\frac{5}{9}$ **B** $\frac{7}{15}$ **C** $\frac{7}{18}$ **D** $\frac{8}{15}$ **E** $\frac{2}{9}$

Section B 2 questions

15 marks

1 (8 marks) A laboratory is studying the growth of a particular strain of bacteria by establishing a colony in a controlled environment and measuring the growth over a number of days.

The number of bacteria in the colony is modelled by the equation $N = N_0 \times 3^{kt}$ where N is the number of bacteria and t is the number of days after the colony was established.

They found there were 6000 bacteria after 2 days and 18 000 bacteria after 4 days.

a Write two equations in terms of N_0 and k. 2 marks

b Find the values of N_0 and k. 2 marks

c Find the number of bacteria after 6 days. 1 mark

d If the bacteria is only counted once a day, find the number of days required for the bacteria population to first exceed 1 000 000. 1 mark

e Another population of bacteria doubles every three days. If the initial population was 5000 bacteria, write the equation to model this situation in the form $M = M_0 \times 2^{\frac{t}{k}}$ and hence determine the number of days required for the population to increase by 35 000. 2 marks

2 © VCAA 2013 2BQ2 MODIFIED (7 marks) FullyFit is an international company that owns and operates many fitness centres (gyms) in several countries. At every one of FullyFit's gyms, each member agrees to have his or her fitness assessed every month by undertaking a set of exercises called **S**. There is a five-minute time limit on any attempt to complete **S** and if someone completes **S** in less than three minutes, they are considered fit.

a At FullyFit's Melbourne gym, it has been found that the probability that any member will complete **S** in less than three minutes is $\frac{5}{8}$. This is independent of any other member.

In a particular week, three members of this gym attempt **S**.

i Find the probability, correct to four decimal places, that two of these three members will complete **S** in less than three minutes. 2 marks

ii Given that at least two of these three members complete **S** in less than three minutes, what is the probability, correct to three decimal places, that exactly two members complete **S** in less than three minutes? 2 marks

Paula is a member of FullyFit's gym in San Francisco. She completes **S** every month as required, but otherwise does not attend regularly and so her fitness level varies over many months. Paula finds that if she is fit one month, the probability that she is fit the next month is $\frac{3}{4}$, and if she is not fit one month, the probability that she is not fit the next month is $\frac{1}{2}$.

b If Paula is not fit in one particular month

i What is the probability that she is fit the next two months? 1 mark

ii What is the probability that she is fit in exactly two of the next three months? 2 marks

CHAPTER

9 THE CIRCULAR FUNCTIONS

STUDY DESIGN COVERAGE

Study Design coverage

UNIT 2, AREA OF STUDY 1: FUNCTIONS, RELATIONS AND GRAPHS

- the unit circle, radians, arc length and sine, cosine and tangent as functions of a real variable
- the relationships $\sin(x) \approx x$ for small values of x, $\sin^2(x) + \cos^2(x) = 1$ and $\tan(x) = \frac{\sin(x)}{\cos(x)}$
- exact values for sine, cosine and tangent of $\frac{n\pi}{6}$ and $\frac{n\pi}{4}$, $n \in Z$
- symmetry properties, complementary relations and periodicity properties for sine, cosine and tangent functions
- circular functions of the form $y = Af(nx) + c$ and their graphs, where f is the sine, cosine or tangent function, and $A, n, c \in R$ with $A, n \neq 0$
- simple applications of sine and cosine functions of the above form, with examples from various modelling contexts, the interpretation of period, amplitude and mean value in these contexts and their relationship to the parameters A, n and c.

VCE Mathematics Study Design 2023–2027 p. 51, © VCAA 2022

Video playlists (7):

9.1 Radian measure and arc length
9.2 The circular functions
9.3 The exact values
9.4 Graphs of circular functions
9.5 Transformations of the circular functions
9.6 Applying circular functions
VCE question analysis The circular functions

Worksheets (8):

9.1 Converting degrees and radians
9.2 Angles of any magnitude • Radians of any magnitude
9.5 Sketching periodic functions – amplitude and period • Sketching periodic functions – phase and vertical shift • Sine and cosine curves • Trigonometric graphs
9.6 Applying trigonometric functions

Puzzle (1):

9.5 Trigonometric graphs match-up

Nelson MindTap

To access resources above, visit **cengage.com.au/nelsonmindtap**

9.1 Radian measure and arc length

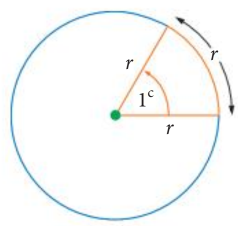
Radian measure

Video playlist Radian measure and arc length

Worksheet Converting degrees and radians

We use degrees to measure the size of an angle, but we can also use **radians**.

One radian is the angle at the centre of a circle of radius r that is opposite an arc of length r. The length of the arc opposite an angle of 1 radian is equal to the radius of the circle.



1 radian is about 57°.

One radian can be written as 1^c or 1 rad, but it is usually written simply as 1, without units. Radian measure is also called **circular measure**, because it is based on the circle.

1^c corresponds to an arc length of r.

Multiplying both values by 2π gives the circumference of a circle:

$2\pi^c$ corresponds to an arc length of $2\pi r$.

This means that $2\pi^c = 360°$ (a **revolution**).

So $\pi^c = 180°$.

Degrees and radians

The relationship between degrees and radians is

$$180° = \pi$$

We can use this to convert between degrees and radians.

$90° = \dfrac{\pi}{2}$ $\quad 45° = \dfrac{\pi}{4}$ $\quad 60° = \dfrac{\pi}{3}$ $\quad 30° = \dfrac{\pi}{6}$ $\quad 270° = \dfrac{3\pi}{2}$ $\quad 360° = 2\pi$

WB

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WORKED EXAMPLE 1 Converting between degrees and radians

Convert

a $\dfrac{3\pi}{4}$ to degrees **b** $225°$ to radians **c** $60°$ to radians

Steps	Working
a Substitute $\pi = 180°$ and simplify.	$\dfrac{3\pi}{4} = \dfrac{3}{4} \times 180° = 135°$
b **1** Start with $180° = \pi$ and divide to find $1°$.	$180° = \pi$ $1° = \dfrac{\pi}{180}$
2 Multiply to find $225°$ and simplify.	$225° = 225° \times \dfrac{\pi}{180°} = \dfrac{225\pi}{180} = \dfrac{5\pi}{4}$
c Use same method as above: multiply by $\dfrac{\pi}{180°}$.	$60° = 60° \times \dfrac{\pi}{180°} = \dfrac{\pi}{3}$

Degrees to radians

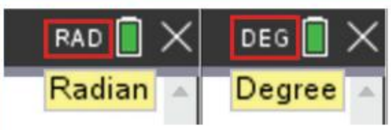
To convert degrees to radians, multiply by $\dfrac{\pi}{180°}$.

Exam hack

Set your calculator to either degree mode or radian mode to satisfy the requirements of the question.

USING CAS 1 Setting degree and radian mode

TI-Nspire



1 In the top right-hand corner of the screen, click to toggle between **RAD** and **DEG** mode.

2 Press π to access the $^{\circ}$ and ^{r} symbols.

ClassPad

1 In the bottom right-hand corner of the screen, tap to toggle between **Deg** and **Rad** mode (ignore **Gra** mode, which stands for gradians).

2 Tap **Math1** or **Trig** to access the $^{\circ}$ and ^{r} symbols.

Trigonometric ratios

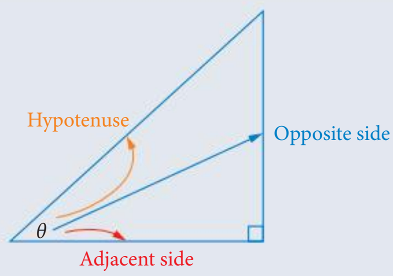
For a right-angled triangle, the trigonometric ratios **sine**, **cosine** and **tangent** (abbreviated as sin, cos and tan respectively) are defined in terms of one of the acute angles and two sides related to the angle. The angle, θ, is measured in degrees or radians.

The trigonometric ratios

$$\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\tan(\theta) = \frac{\text{opposite}}{\text{adjacent}}$$



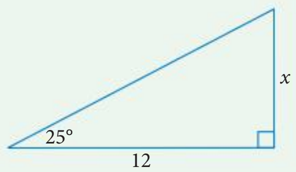
Trigonometric ratios can be used to find the size of an unknown side or angle in a triangle.

WB p. 174

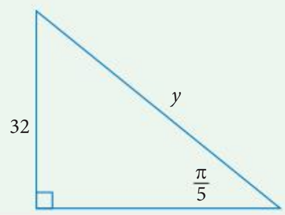
WORKED EXAMPLE 2 Finding unknown sides in right-angled triangles

Find, correct to three decimal places, the value of the pronumeral in each triangle.

a



b



Steps	Working
a **1** Set the calculator to degree mode.	
2 Decide which trigonometric ratio is required. With 25°, x is the opposite side and 12 is the adjacent side, so use tan.	$\tan(\theta) = \frac{\text{opposite}}{\text{adjacent}}$ $\tan(25°) = \frac{x}{12}$
3 Multiply both sides by 12 to solve the equation.	$12\tan(25°) = x$
4 Evaluate and round the answer.	$x = 5.59569\ldots$ ≈ 5.596

From the diagram, an answer of $x \approx 5.596$ looks reasonable.

b **1** Set the calculator to radian mode.

2 Decide which trigonometric ratio is required. With $\frac{\pi}{5}$, y is the hypotenuse and 32 is the opposite side, so use sin.

$$\sin(\theta) = \frac{\text{opposite side}}{\text{hypotenuse}}$$

$$\sin\left(\frac{\pi}{5}\right) = \frac{32}{y}$$

3 Multiply both sides by y.

$$y\sin\left(\frac{\pi}{5}\right) = 32$$

4 Divide both sides by $\sin\left(\frac{\pi}{5}\right)$ to solve the equation.

From the diagram, an answer of $y \approx 46.9$ looks reasonable.

$$y = \frac{32}{\sin\left(\frac{\pi}{5}\right)}$$

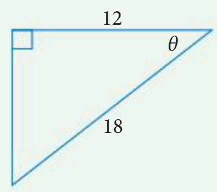
$$= 54.441\,65\ldots$$

$$\approx 54.442$$

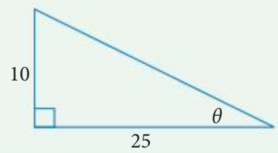
WB p. 175

WORKED EXAMPLE 3 Finding an unknown angle in a right-angled triangle

a Find, to the nearest degree, the size of angle θ.



b Find, correct to one decimal place, angle θ in radians.



Steps	Working
a **1** Set the calculator to degree mode.	
2 Decide which trigonometric ratio is required. With θ, 12 is the adjacent side and 18 is the hypotenuse, so use cos.	$\cos(\theta) = \frac{12}{18}$
3 The angle is the inverse of the trigonometric function.	Use $\cos^{-1}$ $\theta = \cos^{-1}\left(\frac{12}{18}\right)$
4 Evaluate and round the answer.	$= 48.189\ldots°$ $\approx 48°$ From the diagram, an answer of $x \approx 48°$ looks reasonable.
b **1** Set the calculator to radian mode.	
2 Decide which trigonometric ratio is required. With θ, 10 is the opposite side and 25 is the adjacent side, so use tan.	$\tan(\theta) = \frac{10}{25}$

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3 The angle is the inverse of the trigonometric function.	Use $\tan^{-1}$. $\theta = \tan^{-1}\left(\frac{10}{25}\right)$
4 Evaluate and round the answer.	$= 0.3805\ldots^c$ $\approx 0.4^c$

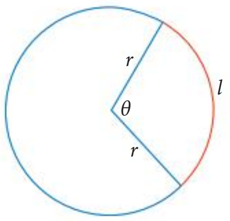
9.1

Arc length

When radians are used, finding the length (l) of an arc in a circle is simple.

l is a fraction of the circumference of the circle $C = 2\pi r$.

θ is a fraction of a revolution 2π.



So $\frac{l}{2\pi r} = \frac{\theta}{2\pi}$.

$$l = \frac{\theta}{2\pi} \times 2\pi r = r\theta$$

Arc length

The arc length l that subtends an angle of θ radians at the centre of the circle of radius r is

$$l = r\theta$$

WORKED EXAMPLE 4 Arc length 1

WB
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a Find the exact length of the arc formed by an angle of $\frac{\pi}{3}$ in a circle of radius 6 cm.

b Find, correct to two decimal places, the angle in radians subtended at the centre of a circle of radius 5.4 cm by an arc 3.6 cm long.

Subtended means 'sits opposite'.

c Find the exact length of the arc formed by an angle of 45° in a circle of radius 2 cm.

Steps	Working
a Use $l = r\theta$ with $r = 6$ cm and $\theta = \frac{\pi}{3}$.	$l = r\theta$ $= 6 \times \frac{\pi}{3}$ $= 2\pi$ cm

'Exact length' means leave the answer in terms of π.

b	**1** Use $l = r\theta$ with $l = 3.6$ cm, $r = 5.4$ cm.	$3.6 = 5.4\theta$
	2 Divide both sides by 5.4.	$\theta = \frac{3.6}{5.4}$ $= 0.6666\ldots$ ≈ 0.67
c	**1** Convert the angle into radians first.	$45° = 45° \times \frac{\pi}{180°} = \frac{\pi}{4}$
	2 Use $l = r\theta$ with $r = 2$ cm and $\theta = \frac{\pi}{4}$.	$l = 2 \times \frac{\pi}{4}$ $= \frac{\pi}{2}$ cm

WB

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WORKED EXAMPLE 5 Arc length 2

The minute hand of a clock is 15 cm long. If the end of the hand travelled an arc of 10.5 cm, through what angle, to the nearest degree, did the hand rotate?

Steps	Working
1 Use $l = r\theta$ with $l = 10.5$ cm, $r = 15$ cm.	$10.5 = 15\theta$
2 Divide both sides by 15.	$\theta = \frac{10.5}{15}$ $= 0.7$
3 Convert this angle from radians to degrees. For the formula $l = r\theta$, θ must be in radians.	$180° = \pi$ $\frac{180°}{\pi} = 1$ rad $0.7 \text{ rad} = 0.7 \times \frac{180°}{\pi}$ $= 40.1070\ldots$ $\approx 40°$

EXERCISE 9.1 Radian measure and arc length

ANSWERS p. 524

Mastery

1 WORKED EXAMPLE 1 Convert each angle to degrees, correct to one decimal place for parts **k** and **l**.

a $\frac{\pi}{5}$ **b** $\frac{2\pi}{3}$ **c** $\frac{5\pi}{4}$ **d** $\frac{7\pi}{6}$

e 3π **f** $\frac{7\pi}{9}$ **g** $\frac{4\pi}{3}$ **h** $\frac{7\pi}{3}$

i $\frac{\pi}{9}$ **j** $\frac{5\pi}{18}$ **k** 1.09 **l** 0.768

Exam hack

A fast way is to replace π by 180 and simplify.

2 WORKED EXAMPLE 1 Convert each angle to radians, correct to 2 decimal places.

a 56° **b** 68° **c** 127° **d** 289° **e** 312°

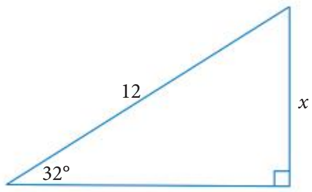
3 Convert each angle to radians in terms of π.

a $135°$ **b** $30°$ **c** $150°$ **d** $240°$ **e** $300°$

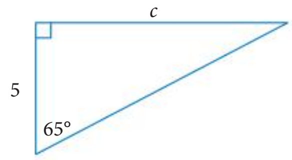
f $63°$ **g** $15°$ **h** $450°$ **i** $225°$ **j** $120°$

4 WORKED EXAMPLE 2 Find the value of each variable correct to 3 decimal places.

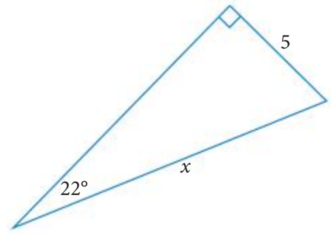
a



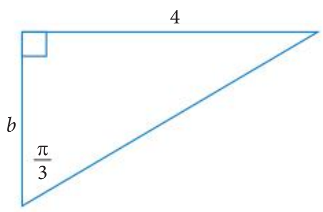
b



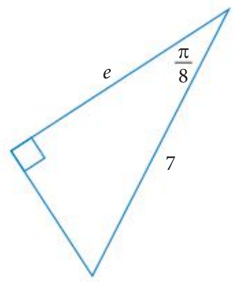
c



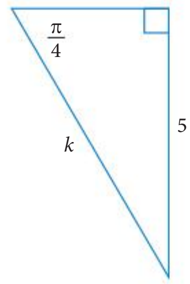
d



e



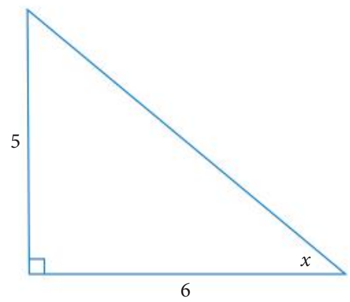
f



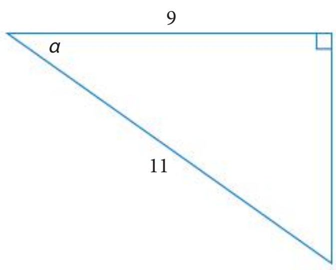
5 WORKED EXAMPLE 3 Find the size of each angle

i correct to the nearest degree **ii** in radians, correct to one decimal place

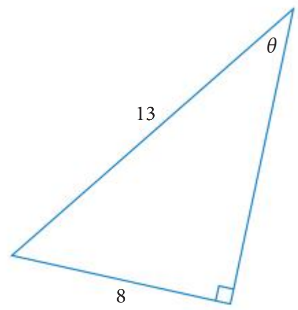
a



b



c



6 WORKED EXAMPLE 4 Find the arc length of the circle for each given radius and angle subtended at the centre. For parts **a** to **e**, write the answer in terms of π. For parts **f** to **h**, write the answer correct to 2 decimal places.

a 4 cm, π **b** 3 m, $\frac{\pi}{3}$ **c** 10 cm, $\frac{5\pi}{6}$ **d** 3 cm, $30°$

e 7 mm, $45°$ **f** 1.5 m, 0.43 **g** 3.21 cm, 1.22 **h** 7.2 mm, $55°$

7 Find the angle (in radians) subtended at the centre of a circle (correct to 2 decimal places where necessary) by an arc with

a radius 16 m, length 28 m **b** radius 3 cm, length $\frac{2\pi}{7}$ cm

c radius $\frac{\pi}{3}$ m, length $\frac{\pi}{2}$ m **d** radius 18 cm, length 28 cm

e radius 194 mm, length 218 mm **f** radius 9 m, length 6 m

g radius 19.2 cm, length 12 cm **h** radius π cm, length 2π cm

8 WORKED EXAMPLE 5 The string of a pendulum is 50 cm in length and the bob at its end rotates through an angle of $\frac{1}{4}^c$.

a What arc length does the bob sweep out as it swings?

b If the angle of swing is now 20°, how much further will the bob travel? State the answer correct to one decimal place.

Exam practice

9 TECH-FREE (2 marks) Tim is on level ground and measures the angle of elevation of the top of a building of height 25 metres to be $\frac{2\pi}{7}^c$.

How far in front of the building is he standing? Give your answer correct to 2 decimal places.

10 TECH-FREE (2 marks) State the exact answer to $\dfrac{\frac{2\pi^c}{3}+60^\circ}{\frac{5\pi^c}{6}-\frac{\pi^c}{4}-15^\circ}$.

11 In a right-angled triangle ABC, $\angle A = 90°$, $\angle B = \theta$, $AC = x$, $AB = y$ and $BC = z$. Which of the following statements is correct?

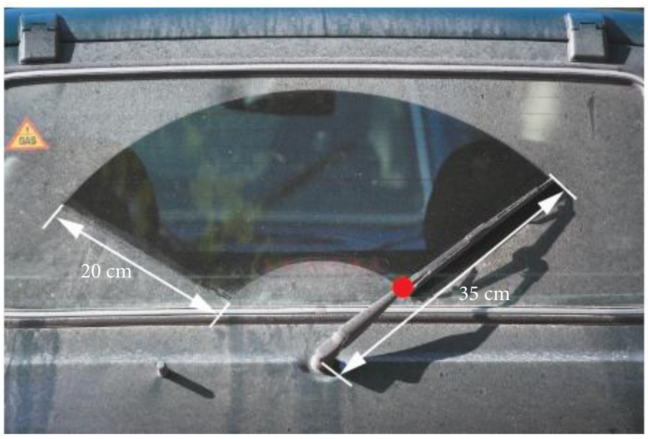
A $\sin(\theta) = \frac{z}{x}$ **B** $\cos(\theta) = \frac{x}{y}$ **C** $\tan(\theta) = \frac{y}{x}$

D $\cos(90° - \theta) = \frac{z}{y}$ **E** $\sin(90° - \theta) = \frac{y}{z}$

12 TECH-FREE A circle of radius $\frac{9}{\pi}$ cm has arc length 8 cm. The angle the arc subtends at the centre of the circle is

A 120° **B** 130° **C** 140° **D** 150° **E** 160°

13 (2 marks) This car's windscreen wiper sweeps through an angle of 110°. Calculate, correct to the nearest centimetre, the distance the red marker shown on the wiper covers in one sweep.



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9.2 The circular functions

9.2

Video playlist
The circular functions

Worksheets
Angles of any magnitude

Radians of any magnitude

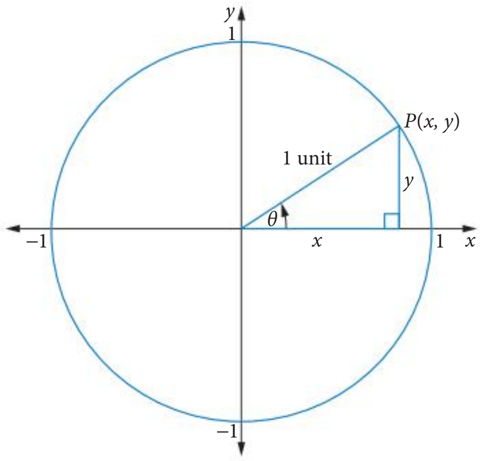
The sine and cosine functions

Trigonometric functions are also called **circular functions** because they are based on the **unit circle**.

Right-angled triangles can be constructed within a **quadrant** in a unit circle to determine the trigonometric values of angles of any size.

Consider the unit circle below with equation $x^2 + y^2 = 1$, centre (0, 0) and radius 1. We can use it to define sine, cosine and tangent of any angle size θ.

Let $P(x, y)$ be a point on the unit circle that makes an angle of θ with the positive direction of the x-axis.



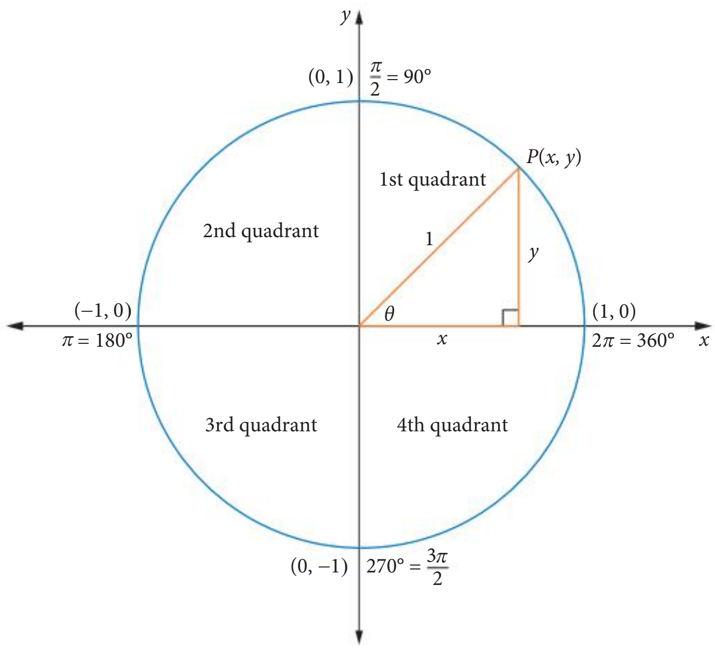
Then $\sin(\theta) = \dfrac{\text{opposite}}{\text{hypotenuse}} = \dfrac{y}{1} = y$

and $\cos(\theta) = \dfrac{\text{adjacent}}{\text{hypotenuse}} = \dfrac{x}{1} = x$.

So for any angle size θ corresponding to a point $P(x, y)$ on the unit circle,

$\sin(\theta) = y$-coordinate, $\cos(\theta) = x$-coordinate.

We can extend this definition of the sine and cosine functions to angles in the 2nd, 3rd and 4th quadrants, and therefore angles of any size.



By examining the signs of x and y in each quadrant, it follows that:

- in the 1st quadrant, $\cos(\theta)$ is positive and $\sin(\theta)$ is positive.
- in the 2nd quadrant, $\cos(\theta)$ is negative and $\sin(\theta)$ is positive.
- in the 3rd quadrant, $\cos(\theta)$ is negative and $\sin(\theta)$ is negative.
- in the 4th quadrant, $\cos(\theta)$ is positive and $\sin(\theta)$ is negative.

From the right-angled triangle in the diagram, using **Pythagoras' theorem** we obtain $x^2 + y^2 = 1$, the equation of the circle.

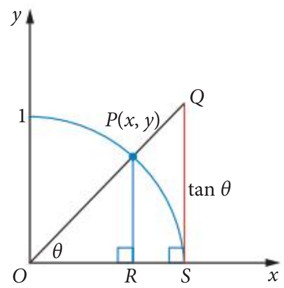
Therefore, $[\cos(\theta)]^2 + [\sin(\theta)]^2 = 1$, which is usually written with $\sin(\theta)$ first as $\sin^2(\theta) + \cos^2(\theta) = 1$ and known as the **Pythagorean identity**. This is an example of an **identity** because it is true for all angles θ.

The Pythagorean identity

$\sin^2(\theta) + \cos^2(\theta) = 1$ for all angles θ.

The tangent function

In the diagram, OQ passes through P, QS is a vertical tangent to the circle at S and ΔQSO is right-angled.



Then $\tan(\theta) = \dfrac{\text{opposite}}{\text{adjacent}} = \dfrac{QS}{OS} = \dfrac{y\text{-coordinate of } Q}{1} = y\text{-coordinate of } Q.$

But also, $\tan(\theta) = \dfrac{\text{opposite}}{\text{adjacent}} = \dfrac{PR}{OR} = \dfrac{y\text{-coordinate of } P}{x\text{-coordinate of } P} = \dfrac{\sin(\theta)}{\cos(\theta)}.$

The tangent identity

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$$

By examining the signs of $\sin(\theta)$ and $\cos(\theta)$ in each quadrant, it follows that:

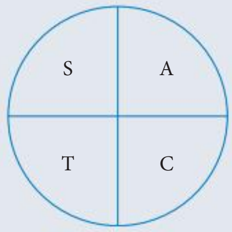
- in the 1st quadrant, $\tan(\theta)$ is positive.
- in the 2nd quadrant, $\tan(\theta)$ is negative.
- in the 3rd quadrant, $\tan(\theta)$ is positive.
- in the 4th quadrant, $\tan(\theta)$ is negative.

1st quadrant	2nd quadrant	3rd quadrant	4th quadrant
$\sin(\theta)$ is positive $\cos(\theta)$ is positive $\tan(\theta)$ is positive	$\sin(\theta)$ is positive $\cos(\theta)$ is negative $\tan(\theta)$ is negative	$\sin(\theta)$ is negative $\cos(\theta)$ is negative $\tan(\theta)$ is positive	$\sin(\theta)$ is negative $\cos(\theta)$ is positive $\tan(\theta)$ is negative

Note that in the 1st quadrant, all the circular functions are positive, but in the 2nd, 3rd and 4th quadrants, only one of the circular functions is positive. This can be summarised by the **ASTC rule**.

The ASTC rule

The ASTC rule gives the signs of the circular functions in each quadrant. We can remember it using a mnemonic such as 'All Stations To Cragieburn'.



Alternatively, we could remember the word CAST.

- **All** the circular functions are positive in the 1st quadrant.
- **Sine** is positive and the others negative in the 2nd quadrant.
- **Tangent** is positive and the others negative in the 3rd quadrant.
- **Cosine** is positive and the others negative in the 4th quadrant.

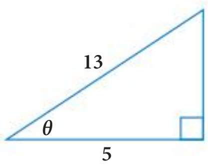
9.2

WORKED EXAMPLE 6 Using the Pythagorean identity

WB p. 178

Given $\cos(\theta) = -\frac{5}{13}$ and $\pi < \theta < \frac{3\pi}{2}$, find the exact value of

a $\sin(\theta)$ **b** $\tan(\theta)$

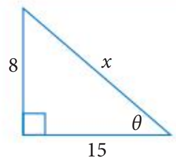
Steps	Working
a 1 Use the identity $\sin^2(\theta) + \cos^2(\theta) = 1$ and solve to find $\sin(\theta)$. This question can also be answered by first drawing a right-angled triangle with adjacent side 5 and hypotenuse 13 and using Pythagoras' theorem to find the opposite side. 	$\sin^2(\theta) + \left(-\frac{5}{13}\right)^2 = 1$ $\sin^2(\theta) + \frac{25}{169} = 1$ $\sin^2(\theta) = 1 - \frac{25}{169} = \frac{144}{169}$ $\sin(\theta) = \pm\sqrt{\frac{144}{169}} = \pm\frac{12}{13}$
2 Use the quadrant to determine if $\sin(\theta)$ is positive or negative. $\pi < \theta < \frac{3\pi}{2}$ means 3rd quadrant.	θ is in the 3rd quadrant, so $\sin(\theta)$ is negative. $\sin(\theta) = -\frac{12}{13}$
b Use $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$.	$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} = \frac{-\frac{12}{13}}{-\frac{5}{13}} = \frac{12}{5}$ $\cos(\theta)$ is negative in the 3rd quadrant, and $\tan(\theta)$ is positive.

WB

p. 179

WORKED EXAMPLE 7 Applying Pythagoras' theorem to find a trigonometric ratio

Given $\tan(\theta) = -\frac{8}{15}$ and $90° < \theta < 180°$, find the exact value of $\cos(\theta)$.

Steps	Working
1 Draw a right-angled triangle with opposite side 8 and adjacent side 15, and use Pythagoras' theorem to find the hypotenuse.	 $x^2 = 8^2 + 15^2 = 289$ $x = \sqrt{289} = 17$
2 Find $\cos(\theta)$.	$\cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{15}{17}$
3 Use the quadrant to determine if $\cos(\theta)$ is positive or negative. $90° < \theta < 180°$ means 2nd quadrant.	θ is in the 2nd quadrant, so $\cos(\theta)$ is negative. $\cos(\theta) = -\frac{15}{17}$

Angles in the 4 quadrants

Trigonometric ratios involving the 2nd, 3rd and 4th quadrants can be expressed in terms of trigonometric ratios in the 1st quadrant.

Let the angle θ be formed using the point $P(a, b)$ on the unit circle, where θ is the angle that point P makes with the positive direction of the x-axis.

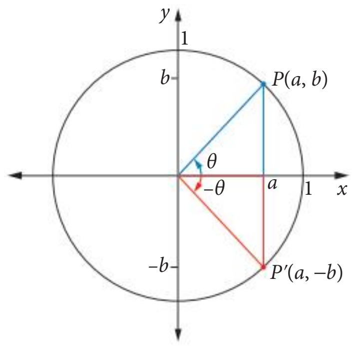
Consider a **negative angle**, $-\theta$, formed using the point $P'(a, -b)$.

P and P' have the same x-coordinate, but the y-coordinate of P' is the negative of the y-coordinate of P, so

$$\cos(-\theta) = \cos(\theta)$$

and $$\sin(-\theta) = -\sin(\theta).$$

Also, $$\tan(-\theta) = \frac{\sin(-\theta)}{\cos(-\theta)} = \frac{-\sin(\theta)}{\cos(\theta)} = -\tan(\theta).$$



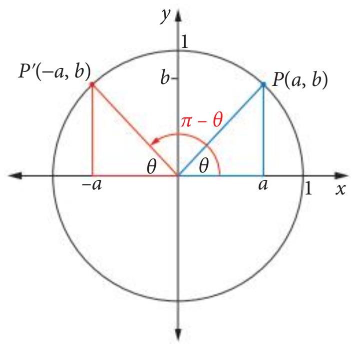
Consider an **angle in the 2nd quadrant**, $\pi - \theta$, formed using the point $P'(-a, b)$.

P and P' have the same y-coordinate, but the x-coordinate of P' is the negative of the x-coordinate of P, so

$$\cos(\pi - \theta) = -\cos(\theta)$$

and $$\sin(\pi - \theta) = \sin(\theta).$$

Also, $$\tan(\pi - \theta) = \frac{\sin(\pi - \theta)}{\cos(\pi - \theta)} = \frac{\sin(\theta)}{-\cos(\theta)} = -\tan(\theta).$$



In the 2nd quadrant, sin is positive.

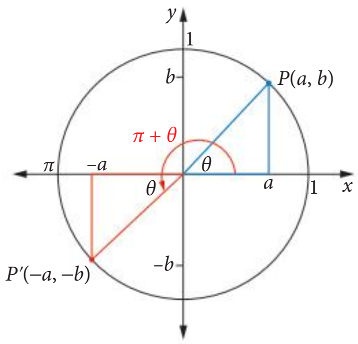
Consider an **angle in the 3rd quadrant**, $\pi + \theta$, formed using the point $P'(-a, -b)$.

The x- and y-coordinates of P' are the negative of the x- and y-coordinates of P respectively, so

$$\cos(\pi + \theta) = -\cos(\theta)$$

and $$\sin(\pi + \theta) = -\sin(\theta).$$

Also, $\tan(\pi + \theta) = \dfrac{\sin(\pi + \theta)}{\cos(\pi + \theta)} = \dfrac{-\sin(\theta)}{-\cos(\theta)} = \tan(\theta)$.



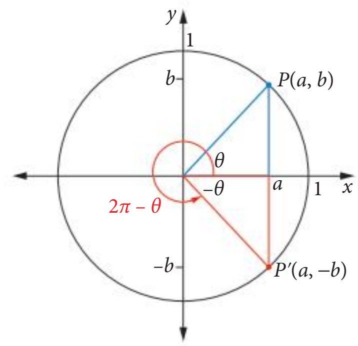
In the 3rd quadrant, tan is positive.

Consider an **angle in the 4th quadrant**, $2\pi - \theta$, formed using the point $P'(a, -b)$. This is similar to the negative angles described earlier.

$$\cos(2\pi - \theta) = \cos(\theta)$$

and $$\sin(2\pi - \theta) = -\sin(\theta).$$

Also, $\tan(2\pi - \theta) = \dfrac{\sin(2\pi - \theta)}{\cos(2\pi - \theta)} = \dfrac{-\sin(\theta)}{\cos(\theta)} = -\tan(\theta)$.



In the 4th quadrant, cos is positive.

2nd quadrant	3rd quadrant	4th quadrant	Negative angles
$\sin(\pi - \theta) = \sin(\theta)$	$\sin(\pi + \theta) = -\sin(\theta)$	$\sin(2\pi - \theta) = -\sin(\theta)$	$\sin(-\theta) = -\sin(\theta)$
$\cos(\pi - \theta) = -\cos(\theta)$	$\cos(\pi + \theta) = -\cos(\theta)$	$\cos(2\pi - \theta) = \cos(\theta)$	$\cos(-\theta) = \cos(\theta)$
$\tan(\pi - \theta) = -\tan(\theta)$	$\tan(\pi + \theta) = \tan(\theta)$	$\tan(2\pi - \theta) = -\tan(\theta)$	$\tan(-\theta) = -\tan(\theta)$

Note that $2\pi + \theta$ is in the 1st quadrant.

Hence, $\sin(2\pi + \theta) = \sin(\theta)$, $\cos(2\pi + \theta) = \cos(\theta)$ and $\tan(2\pi + \theta) = \tan(\theta)$.

WORKED EXAMPLE 8 Equivalent ratios

WB p. 180

State the equivalent trigonometric function in quadrant 1 for each of the following.

a $\sin(360° - 30°)$ **b** $\tan(180° + 26°)$ **c** $-\cos\left(\dfrac{\pi}{8} - \pi\right)$

Steps	Working
a Use $\sin(360° - \theta°) = -\sin(\theta°)$.	$\sin(360° - 30°) = -\sin(30°)$
b Use $\tan(180° + \theta°) = \tan(\theta°)$.	$\tan(180° + 26°) = \tan(26°)$
c **1** Write in the form $\cos(\pi - \theta)$.	$-\cos\left(\dfrac{\pi}{8} - \pi\right) = -\cos\left(-\left(\pi + \dfrac{\pi}{8}\right)\right)$
2 Use $\cos(-\theta) = -\cos(-\theta)$. Apply $\cos(\pi + \theta) = -\cos(\theta)$.	$-\cos\left(-\left(\pi + \dfrac{\pi}{8}\right)\right) = \cos\left(\pi + \dfrac{\pi}{8}\right)$ $\cos\left(\pi + \dfrac{\pi}{8}\right) = -\cos\left(\dfrac{\pi}{8}\right)$

WB

p. 181

WORKED EXAMPLE 9	Rewrite the angle to find equivalent ratios
State the equivalent trigonometric function in quadrant 1 for each of the following.	
a $\cos\left(\frac{3\pi}{4}\right)$ **b** $\sin(240°)$ **c** $\tan\left(\frac{7\pi}{3}\right)$	
Steps	**Working**
a 1 Write as $\cos(\pi - \theta)$.	$\cos\left(\frac{3\pi}{4}\right) = \cos\left(\pi - \frac{\pi}{4}\right)$
2 Use $\cos(\pi - \theta) = -\cos(\theta)$.	$\cos\left(\pi - \frac{\pi}{4}\right) = -\cos\left(\frac{\pi}{4}\right)$
b 1 Write as $\sin(\pi + \theta)$.	$\sin(240°) = \sin(180° + 60°)$
2 Use $\sin(\pi + \theta) = -\sin(\theta)$.	$\sin(180° + 60°) = -\sin(60°)$
c 1 Write in the form $\tan(2\pi + \theta)$.	$\tan\left(\frac{7\pi}{3}\right) = \tan\left(2\pi + \frac{\pi}{3}\right)$
2 Use $\tan(2\pi + \theta) = \tan(\theta)$.	$= \tan\left(\frac{\pi}{3}\right)$

EXERCISE 9.2 The circular functions

ANSWERS p. 524

Recap

1 In a right-angled triangle, the hypotenuse has length 5 cm and the adjacent side has length 2 cm. The angle between the adjacent side and the hypotenuse is closest to

A 21.80° **B** 23.30° **C** 23.58° **D** 66.42° **E** 66.70°

2 An arc of length 2 cm on a circle subtends an angle of 10° at the circle's centre. The radius of the circle is

A $\frac{\pi}{36}$ cm **B** 0.2π cm **C** 2π cm **D** $\frac{36}{\pi}$ cm **E** 5π cm

Mastery

3 WORKED EXAMPLE 6 TECH-FREE Given $\sin(\theta) = -\frac{4}{5}$ and $270° < \theta < 360°$, find the exact value of

a $\cos(\theta)$

b $\tan(\theta)$

4 WORKED EXAMPLE 7 Given $\sin(\theta) = -\frac{24}{25}$ and θ is not in the 3rd quadrant, find the exact value of $\tan(\theta)$.

5 WORKED EXAMPLE 8 State the equivalent trigonometric ratio in the 1st quadrant for each expression.

a $\cos(180° + 80°)$ **b** $\tan\left(\pi - \frac{\pi}{7}\right)$ **c** $\sin(360° + 3°)$

6 WORKED EXAMPLE 9 State the equivalent trigonometric ratio in the 1st quadrant for each expression.

a $\sin(100°)$ **b** $\cos\left(\frac{9\pi}{4}\right)$ **c** $\tan\left(\frac{11\pi}{6}\right)$

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Exam practice

80–100% 60–79% 0–59%

9.2

7 TECH-FREE (3 marks) Show that $\dfrac{\sin(2\pi - x) + \cos(\pi + x)}{\sin(x) + \cos(x)} = -1$.

8 TECH-FREE (4 marks) Given $\tan(\theta) = \dfrac{12}{5}$ and $\pi < \theta < 2\pi$, find the exact value of

a $\sin(\theta)$ — 2 marks

b $\cos(\theta)$ — 2 marks

9 Choose the correct statement.

A An angle of $\dfrac{3\pi}{2}$ is in the 2nd quadrant.

B An angle of $-\dfrac{5\pi}{6}$ is in the 4th quadrant.

C -3.25 revolutions of a circle produces an angle of 585°.

D $\dfrac{2\pi}{9}$ and $-320°$ are equivalent angles.

E $\pi° = \left(\dfrac{180}{\pi^2}\right)^c$

10 The equivalent 1st quadrant angle for 209° is

A $-209°$ **B** $-29°$ **C** $29°$ **D** $61°$ **E** $331°$

11 $\dfrac{\cos(2\pi - \theta)}{\sin(\pi + \theta)}$ is equivalent to

A $\tan(\theta)$ **B** $\cos(\theta)$ **C** $-\dfrac{1}{\tan(\theta)}$ **D** $\sin(\theta)$ **E** $\dfrac{1}{\cos(\theta)}$

12 $\cos\left(\dfrac{2\pi}{3}\right) + \sin\left(\dfrac{7\pi}{6}\right) - \tan\left(\dfrac{3\pi}{4}\right)$ is

A -1 **B** 0 **C** 1 **D** $\sqrt{2}$ **E** 2

13 © VCAA 2012 2AQ19 45% A function f has the following two properties for all real values of θ.

$f(\pi - \theta) = -f(\theta)$ and $f(\pi - \theta) = -f(-\theta)$

A possible rule for f is

A $f(x) = \sin(x)$ **B** $f(x) = \cos(x)$ **C** $f(x) = \tan(x)$

D $f(x) = \sin\left(\dfrac{x}{2}\right)$ **E** $f(x) = \tan(2x)$

14 (2 marks) If $\sin(\theta) = -\dfrac{\sqrt{3}}{2}$ and $0 < \theta < 2\pi$, find the two possible values of θ.

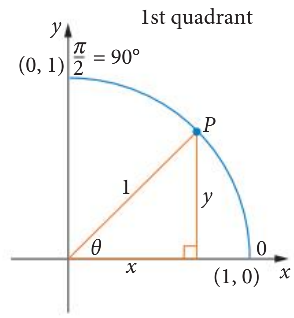
15 (3 marks) If $\cos(\theta) = \dfrac{\sqrt{3}}{2}$, show that the exact value of $\cos(2\pi + \theta) + \sin^2(\theta)$ is $\dfrac{2\sqrt{3} + 1}{4}$.

Video playlist
The exact values

9.3 The exact values

We can find **exact values** of circular functions for the angles 0, $\frac{\pi}{6}$ (30°), $\frac{\pi}{4}$ (45°), $\frac{\pi}{3}$ (60°) and $\frac{\pi}{2}$ (90°).

Points on the axes



The values for 0 and $\frac{\pi}{2}$ follow immediately from the unit circle definitions.

For the angle 0, P is at (1, 0) on the x-axis, so:

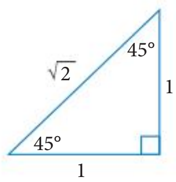
$\sin(0) = y = 0$, $\cos(0) = x = 1$, $\tan(0) = \frac{y}{x} = \frac{0}{1} = 0$.

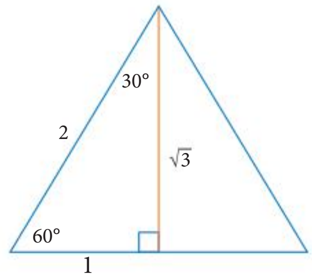
For the angle $\frac{\pi}{2}$ (90°), P is at (0, 1) on the y-axis, so:

$\sin\left(\frac{\pi}{2}\right) = y = 1$, $\cos\left(\frac{\pi}{2}\right) = x = 0$, $\tan\left(\frac{\pi}{2}\right) = \frac{y}{x} = \frac{1}{0}$, which is not defined.

The reference triangles

The exact values for $\frac{\pi}{6}$, $\frac{\pi}{4}$ and $\frac{\pi}{3}$ can be found using special right-angled triangles that have angles of size 30°, 45° and 60°. These are called the **reference triangles**.





$\sin\left(\frac{\pi}{4}\right) = \sin(45°) = \frac{1}{\sqrt{2}}$ $\quad$ $\cos\left(\frac{\pi}{4}\right) = \cos(45°) = \frac{1}{\sqrt{2}}$ $\quad$ $\tan\left(\frac{\pi}{4}\right) = \tan(45°) = \frac{1}{1} = 1$

$\sin\left(\frac{\pi}{6}\right) = \sin(30°) = \frac{1}{2}$ $\quad$ $\cos\left(\frac{\pi}{6}\right) = \cos(30°) = \frac{\sqrt{3}}{2}$ $\quad$ $\tan\left(\frac{\pi}{6}\right) = \tan(30°) = \frac{1}{\sqrt{3}}$

$\sin\left(\frac{\pi}{3}\right) = \sin(60°) = \frac{\sqrt{3}}{2}$ $\quad$ $\cos\left(\frac{\pi}{3}\right) = \cos(60°) = \frac{1}{2}$ $\quad$ $\tan\left(\frac{\pi}{3}\right) = \tan(60°) = \sqrt{3}$

	0	$\frac{\pi}{6}$(30°)	$\frac{\pi}{4}$(45°)	$\frac{\pi}{3}$(60°)	$\frac{\pi}{2}$(90°)
$\sin(\theta)$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos(\theta)$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan(\theta)$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	not defined

You can check these answers on your calculator, but ensure that your calculator is in radian mode for angles.

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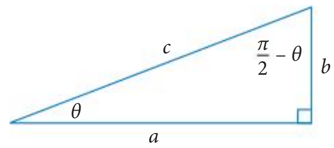
The values of $\sin(\theta)$ across the table are easily remembered as $\frac{\sqrt{0}}{2}, \frac{\sqrt{1}}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{2}, \frac{\sqrt{4}}{2}$.

The values of $\cos(\theta)$ are in the reverse order, and the values of $\tan(\theta)$ are given by $\frac{\sin(\theta)}{\cos(\theta)}$.
As sine and cosine are complementary functions,

$\sin\left(\frac{\pi}{2} - \theta\right) = \sin(90° - \theta) = \cos(\theta)$



$\cos\left(\frac{\pi}{2} - \theta\right) = \cos(90° - \theta) = \sin(\theta)$

so, for example, $\sin\left(\frac{\pi}{6}\right) = \cos\left(\frac{\pi}{3}\right)$ and $\sin(45°) = \cos(45°)$.

WORKED EXAMPLE 10 Exact values

Find the exact value of each trigonometric ratio.

a $\sin\left(\frac{3\pi}{4}\right)$ **b** $\cos(-240°)$ **c** $\tan\left(\frac{11\pi}{6}\right)$

WB p. 182

Steps	Working
a **1** Identify the quadrant of the angle and use ASTC to establish the sign.	$\frac{3\pi}{4}$ is in the 2nd quadrant, so $\sin\left(\frac{3\pi}{4}\right)$ is positive.
2 For the 2nd quadrant, write the angle in the form $\pi - \theta$ using an acute angle θ.	$\sin\left(\frac{3\pi}{4}\right) = \sin\left(\pi - \frac{\pi}{4}\right)$ $= \sin\left(\frac{\pi}{4}\right)$ $= \frac{1}{\sqrt{2}}$
b **1** Use $\cos(-\theta) = \cos(\theta)$.	$\cos(-240°) = \cos(240°)$ $\cos(-240°)$ on the unit circle is also equivalent to $\cos(360° - 240°) = \cos(120°)$.
2 Identify the quadrant of the angle and use ASTC to establish the sign.	240° is in the 3rd quadrant, so $\cos(240°)$ is negative.
3 For the 3rd quadrant, write the angle in the form $180° + \theta$, using an acute angle for θ.	$\cos(240°) = \cos(180° + 60°)$ $= -\cos(60°)$ $= -\frac{1}{2}$

c **1** Identify the quadrant of the angle and use ASTC to establish the sign.	$\frac{11\pi}{6}$ is in the 4th quadrant, so $\tan\left(\frac{11\pi}{6}\right)$ is negative.
2 For the 4th quadrant, write the angle in the form $2\pi - \theta$ using an acute angle for θ.	$\tan\left(\frac{11\pi}{6}\right) = \tan\left(2\pi - \frac{\pi}{6}\right)$ $= -\tan\left(\frac{\pi}{6}\right)$ $= -\frac{1}{\sqrt{3}}$

EXERCISE 9.3 The exact values

ANSWERS p. 525

Recap

1 For angle θ, if $\tan(\theta) = -5$ and $0 < \theta < \pi$, then $\cos(\theta)$ is

A $\frac{\sqrt{24}}{5}$ **B** $\frac{1}{\sqrt{24}}$ **C** $\frac{1}{\sqrt{26}}$ **D** $-\frac{5}{\sqrt{26}}$ **E** $-\frac{1}{\sqrt{26}}$

2 If $\cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$, then $\cos\left(\frac{9\pi}{4}\right) - \cos\left(\pi + \frac{\pi}{4}\right) + \cos\left(2\pi - \frac{\pi}{4}\right)$ will simplify to

A $\sqrt{2}$ **B** $-\frac{\sqrt{2}}{2}$ **C** $2\sqrt{2}$ **D** $\frac{3\sqrt{2}}{2}$ **E** $-3\sqrt{2}$

Mastery

3 WORKED EXAMPLE 10 TECH-FREE Find the exact value of

a $\sin\left(\frac{11\pi}{6}\right)$ **b** $\cos(-150°)$ **c** $\tan\left(\frac{5\pi}{6}\right)$

4 TECH-FREE Find the value of x in each case if $x \in \left[0, \frac{\pi}{2}\right]$.

a $\sin(2\pi - x) = -\frac{1}{2}$ **b** $\tan(\pi - x) = -1$ **c** $\cos(x - 2\pi) = \frac{1}{2}$

5 TECH-FREE

a For $0 < \theta < \frac{\pi}{2}$, simplify $\cos^2(\pi - \theta) - \sin^2(\pi + \theta)$.

b Simplify $\cos(2\pi + 2\theta)$.

c Evaluate your answers to parts **a** and **b** using $\theta = \frac{\pi}{6}$ and comment on your results.

6 TECH-FREE

a Find values of a so that $\frac{a}{3}+\frac{b}{4}=\frac{7\pi}{12}$.

b Given the identity $\cos(\theta+\phi)=\cos(\theta)\cos(\phi)-\sin(\theta)\sin(\phi)$, show that $\cos\left(\frac{7\pi}{12}\right)=\frac{1-\sqrt{3}}{2\sqrt{2}}$.

7 TECH-FREE

a Simplify $\frac{\sin(90n-\theta)^\circ}{\cos(90n-\theta)^\circ}$, where n is a multiple of 4 and $0<\theta<90$.

b Find the value of the expression in part **a** if $n=4$ and $\theta=30°$.

Exam practice

8 TECH-FREE (3 marks) Simplify $\frac{\sin\left(\frac{5\pi}{2}\right)\times\cos\left(\frac{3\pi}{4}\right)\times\sin\left(\frac{\pi}{4}\right)}{\sqrt{3}\times\cos\left(\frac{5\pi}{6}\right)}$.

9 TECH-FREE (3 marks) Solve the equation $2\sin(x)-1=0$, $0°<x<180°$.

10 If $\theta=\frac{7\pi}{2}$, then

A $\tan(\theta)=0$ B $\sin(\theta)=-1$ C $\cos(\theta)$ is undefined

D $\sin(\theta)=0$ E $\cos(\theta)=-1$

11 Angle ϕ in the second quadrant is reflected in the x-axis followed by a reflection in the y-axis. The new angle is θ. Which statement is true?

A $\tan(\theta)<0, \cos(\theta)<0$ B $\cos(\theta)<0, \sin(\theta)<0$ C $\sin(\theta)>0, \tan(\theta)>0$

D $\cos(\theta)>0, \sin(\theta)<0$ E $\tan(\theta)>0, \cos(\theta)>0$

12 Evaluate $\frac{2\sin\left(\frac{\pi}{3}\right)}{\cos\left(\frac{\pi}{4}\right)\tan\left(-\frac{3\pi}{4}\right)}$.

A $2\sqrt{3}$ B $-\sqrt{3}$ C $\sqrt{6}$ D $3\sqrt{2}$ E $-\sqrt{6}$

13 (4 marks)

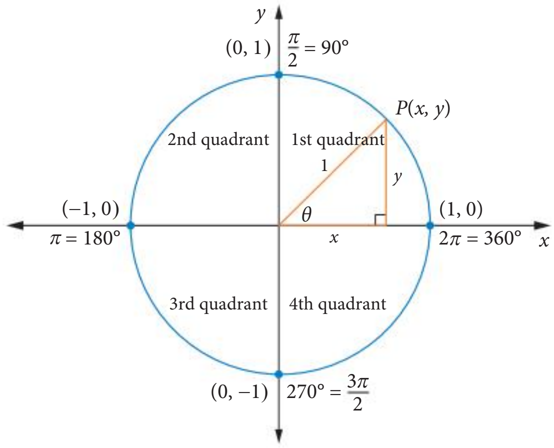
a Show that $\cos^2(\theta)+\sin^2(\phi)=1$ becomes the Pythagorean identity $\cos^2(\theta)+\sin^2(\theta)=1$ when θ and ϕ are supplementary angles. 2 marks

b If θ and ϕ are complementary angles, show that $\cos^2(\theta)+\sin^2(\phi)=1$ is only true if $\theta=\frac{\pi}{4}$ and $\phi=\frac{\pi}{4}$. 2 marks

Video playlist
Graphs of circular functions

9.4 Graphs of circular functions

We defined the circular functions according to the coordinates of a point P on the unit circle: $\sin(\theta) = y$, $\cos(\theta) = x$, $\tan(\theta) = \frac{y}{x}$. By examining the behaviour of the coordinates of $P(x, y)$ in different positions, the shape of the graphs of the circular functions can be determined.



The sine curve

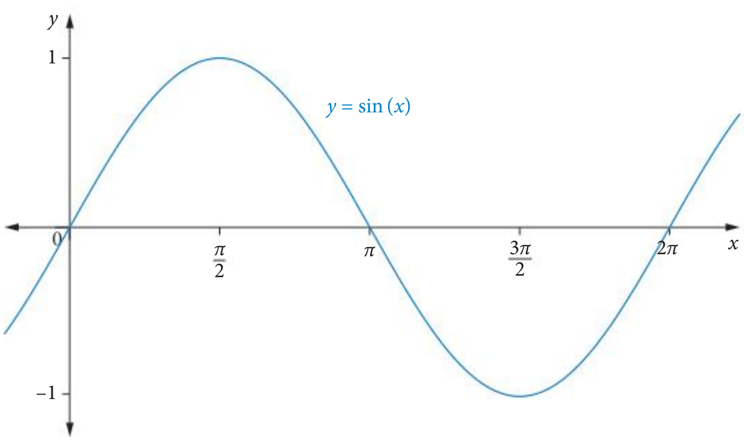
$\sin(\theta) = y$, which means that $\sin(\theta)$ is the 'signed' **height** of the right-angled triangle as P moves from (1, 0) in an anti-clockwise direction.

	0	$\frac{\pi}{2}$ (90°)	π (180°)	$\frac{3\pi}{2}$ (270°)	2π (360°)
$\sin(\theta)$	0	1	0	−1	0

- At $\theta = 0$, the height of the triangle, y, is 0, but as P moves through the 1st quadrant, $\sin(\theta)$ increases from 0 to 1.
- At $\theta = \frac{\pi}{2}$, $y = 1$, but as P moves through the 2nd quadrant, $\sin(\theta)$ decreases from 1 to 0.
- At $\theta = \pi$, $y = 0$, but as P moves through the 3rd quadrant, $\sin(\theta)$ decreases from 0 to −1.
- At $\theta = \frac{3\pi}{2}$, $y = -1$, but as P moves through the 4th quadrant, $\sin(\theta)$ increases from −1 back to 0.
- At $\theta = 2\pi$, $y = 0$ again, and the cycle repeats.

Search the internet for animations of these circular function graphs created from the unit circle.

The graph of $y = \sin(x)$ looks like this:



Note the use of x in place of variable θ. This is because we commonly use y as a function of x.

- Note that sin is positive (above the x-axis) in the 1st and 2nd quadrants, which is consistent with the ASTC rule.
- The domain of $y = \sin(x)$ is R, and the range is $[-1, 1]$.
- The graph repeats itself after 2π, so we say that it has a **period** of 2π.
- The **amplitude** of a circular function is the vertical distance from its centre to its turning points, so the amplitude of $y = \sin(x)$ is 1.

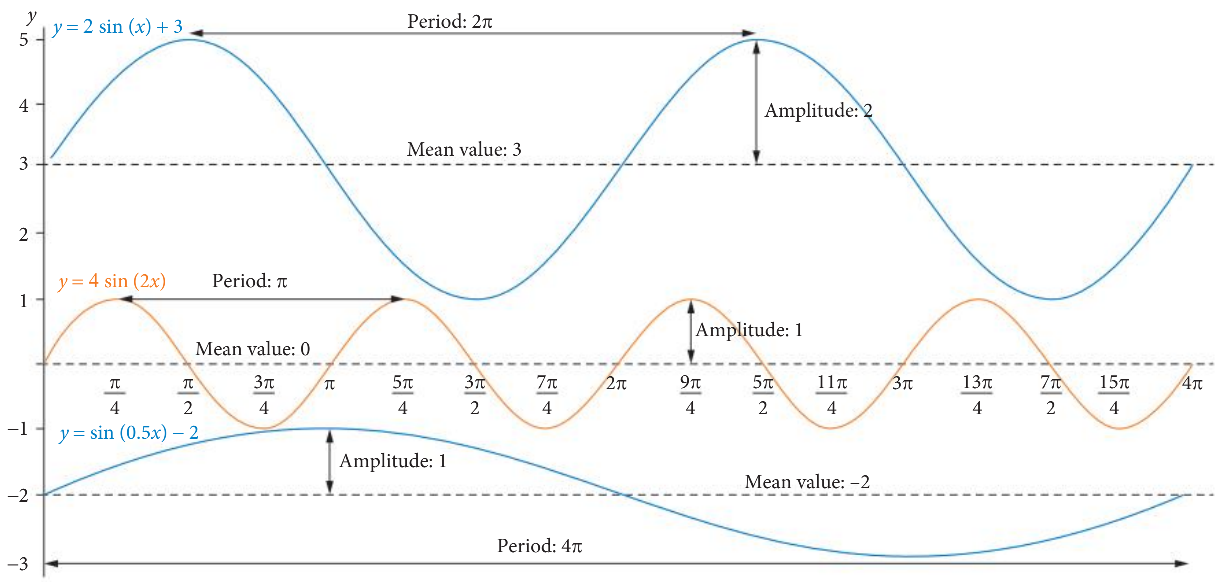
> Cycle and period are not the same. Cycle refers to the repeating pattern and period refers to the horizontal length of the cycle.

- The y value of the sine function moves between −1 and 1, so we say it has a **mean (central) value** of 0.

Amplitude, period and mean value of a sine function

For $y = a\sin(nx) + c$, where $a > 0$ and $n > 0$,

the amplitude is a, the period is $\dfrac{2\pi}{n}$ and the mean value is c.



When technology is not used, the graph can be drawn by determining its general shape and properties, including amplitude, period and the mean value.

WORKED EXAMPLE 11 Amplitude, period and mean

a State the amplitude, period and mean value of $y = 2\sin(3x) + 1$.

b Sketch the graph of $y = 2\sin(3x) + 1$ in $[0, \pi]$.

Steps	Working
a $y = a\sin(nx) + c$ has amplitude a, period is $\dfrac{2\pi}{n}$ and mean value c.	$y = 2\sin(3x) + 1$ has amplitude 2, period $\dfrac{2\pi}{3}$ and mean value 1.

WB p. 183

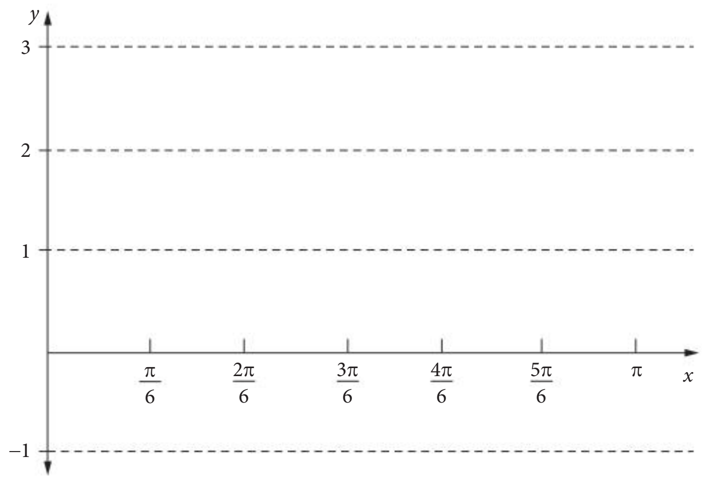
b **1** Decide on a scale for the x-axis.

The maximum occurs when $\sin(3x) = 1$. This is when $x = \frac{\pi}{6}$, so choose increments of $\frac{\pi}{6}$ from 0 to π.

2 Decide on a range for the y-axis.

The amplitude is 2 and the mean value is 1, so the minimum is $1 - 2 = -1$ and the maximum is $1 + 2 = 3$.

3 Draw the grid and indicate the mean value, the maximum and the minimum.



4 Find some critical points, including the minimum.

Use the period to find other values.

$x = 0$, $y = 2\sin(0) + 1 = 1$. That is, $(0, 1)$.

The period is $\frac{2\pi}{3}$, so another point is $\left(0 + \frac{2\pi}{3}, 1\right)$.

$x = \frac{\pi}{6}$, $y = 2\sin\left(3 \times \frac{\pi}{6}\right) + 1 = 3$. That is, $\left(\frac{\pi}{6}, 3\right)$.

Another point is $\left(\frac{\pi}{6} + \frac{2\pi}{3}, 3\right)$, or $\left(\frac{5\pi}{6}, 3\right)$.

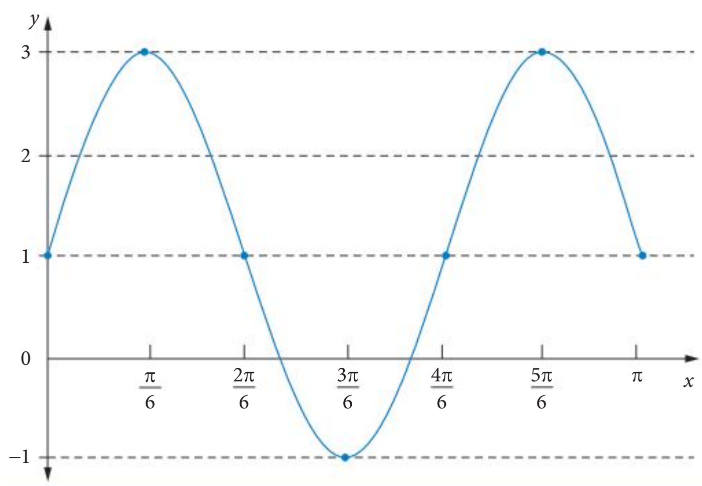
$x = \frac{\pi}{3}$, $y = 2\sin\left(3 \times \frac{\pi}{3}\right) + 1 = 1$. That is, $\left(\frac{\pi}{3}, 1\right)$.

Another point is $\left(\frac{\pi}{3} + \frac{2\pi}{3}, 1\right)$ or $(\pi, 1)$.

The minimum occurs when $\sin(3x) = -1$.

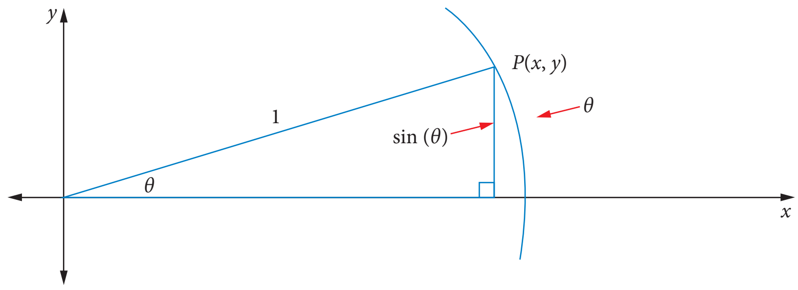
This is when $x = \frac{3\pi}{6}$.

5 Plot and connect the points smoothly, keeping in mind the general shape of a sine graph.

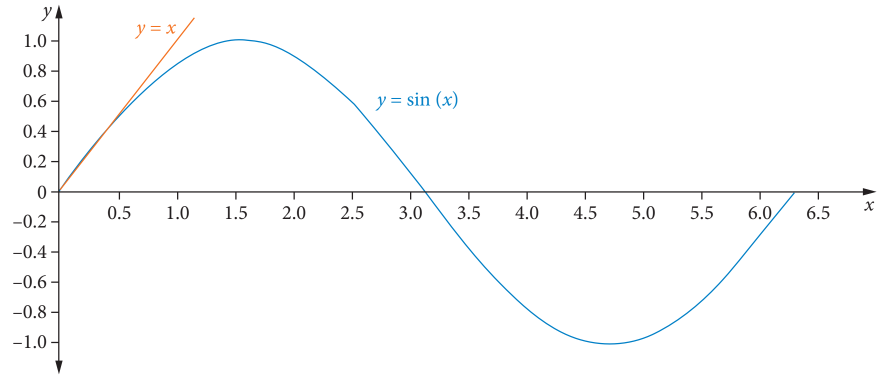


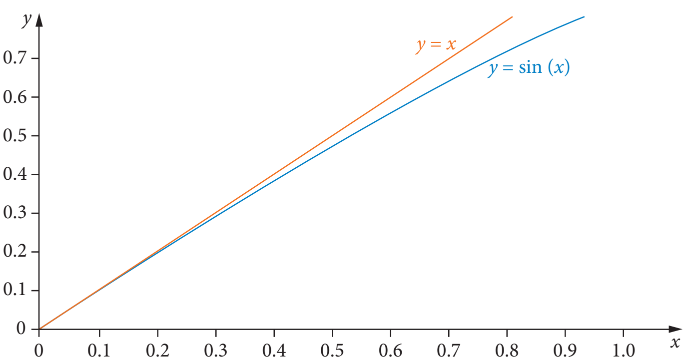
Approximation to sin (*x*) for small *x*

As P and θ move around the unit circle, $\sin(\theta)$ is the **height** of the right-angled triangle and θ is the arc length. For small values of θ, $\sin(\theta) \approx \theta$.



The graphs below compare the functions $y = x$ and $y = \sin(x)$. Notice from the second graph that for $x = 0$ to $x = 0.3$, the graph of $y = x$ appears to be identical to the graph for $y = \sin(x)$. For small values of x, $\sin(x) \approx x$. Note that x is measured in radians.





WORKED EXAMPLE 12 Approximating sin (*x*)

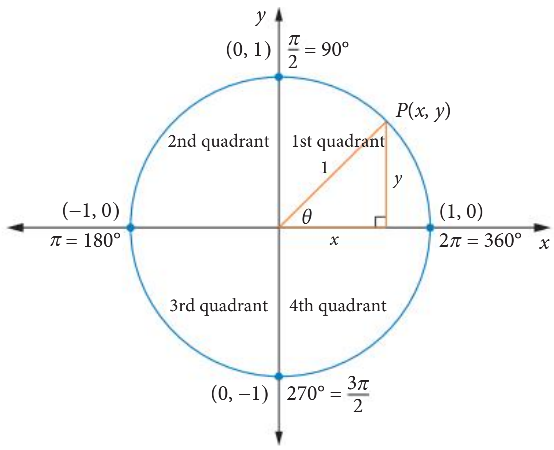
WB p. 185

Calculate the percentage error, correct to 3 decimal places, in the approximation to $\sin(x)$ using $x = 0.1$.

Steps	Working
1 Use $\sin(x) \approx x$ to obtain the approximation at the given value.	For $x = 0.1$, $\sin(x) \approx 0.1$.
2 Calculate the value with a calculator.	$\sin(0.1) = 0.0998334\ldots$
3 Calculate the error, the difference between the exact value and the approximation.	$\text{error} = 0.1 - 0.0998334\ldots$ $= 0.0001665\ldots$
4 Calculate the percentage error, the error as a percentage of the exact value.	$\text{percentage error} = \frac{0.0001665\ldots}{0.0998334\ldots} \times 100\%$ $= 0.16686\ldots\%$ $\approx 0.167\%$

The cosine curve

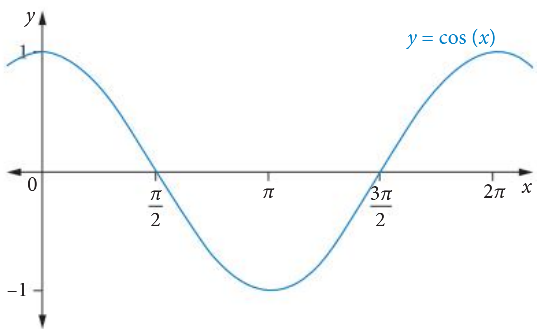
On the unit circle, $\cos(\theta) = x$, the 'signed' base length of the right-angled triangle as point P moves from $(1, 0)$ around the unit circle.



	0	$\frac{\pi}{2}$ (90°)	π (180°)	$\frac{3\pi}{2}$ (270°)	2π (360°)
$\cos(\theta)$	1	0	−1	0	1

- At $\theta = 0$, the base length of the triangle, x, is 1, but as P moves through the 1st quadrant, $\cos(\theta)$ decreases from 1 to 0.
- At $\theta = \frac{\pi}{2}$, $x = 0$, but as P moves through the 2nd quadrant, $\cos(\theta)$ decreases from 0 to −1.
- At $\theta = \pi$, $x = -1$, but as P moves through the 3rd quadrant, $\cos(\theta)$ increases from −1 to 0.
- At $\theta = \frac{3\pi}{2}$, $x = 0$, but as P moves through the 4th quadrant, $\cos(\theta)$ increases from 0 back to 1.
- At $\theta = 2\pi$, $x = 1$ again, and the cycle repeats.

The graph of $y = \cos(x)$ looks like this:



- Note that cos is positive in the 1st and 4th quadrants, consistent with the ASTC rule.
- The graph has a similar shape to the sine graph and also repeats itself after 2π, so it has a **period** of 2π.
- The domain of $y = \cos(x)$ is R, the range is $[-1, 1]$.
- It has a mean (central) value of 0 and an amplitude of 1.

9780170448260

The amplitude, period and mean value of a cosine function are the same as those of a sine function.

Amplitude, period and mean value of a cosine function

For $y = a\cos(nx) + c$, where $a > 0$ and $n > 0$, the amplitude is a, the period is $\frac{2\pi}{n}$ and the mean value is c.

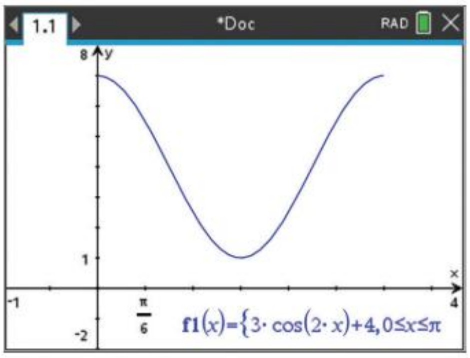
USING CAS 2 Graphing cosine

a Draw the graph of $3\cos(2x) + 4$ in $[0, \pi]$.

b Find the coordinates of the maximum and the minimum values.

TI-Nspire

a

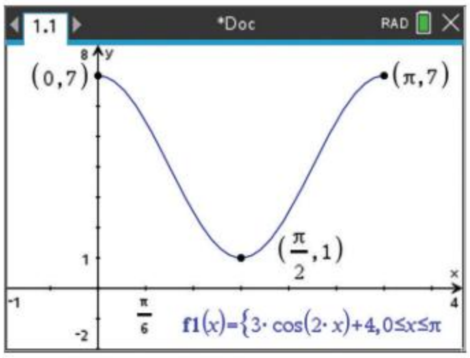


1 Set the mode to **radian**.

2 Graph the cosine function over the domain $[0, \pi]$.

3 Adjust the window settings to suit.

b



4 To find the exact coordinates, press **menu > Geometry > Points & Lines > Point On**. (Shortcut: press **p**)

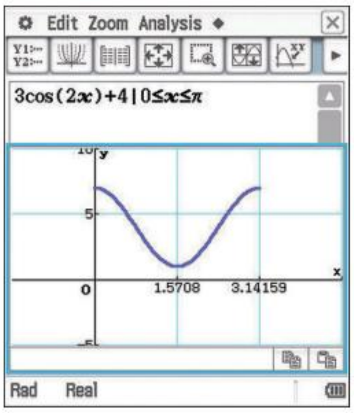
5 Click to place three points on the curve.

6 Press **esc** to remove the **Point On** tool.

7 Click on the **x-coordinate** of each point and enter the values of **0**, $\frac{\pi}{2}$ and π as shown above.

ClassPad

a



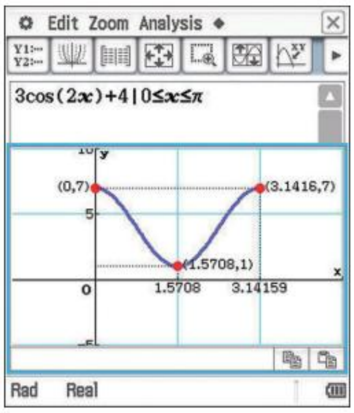
1 Using graph in **Main**, set the mode to **Rad**.

2 Enter the cosine function over the domain $[0, \pi]$ using **Math3** for | and ≤.

3 Drag down into the Graph screen from the left-hand side.

4 Adjust the window settings to suit. It is useful to alter the x scale with fractions of radians.

b



5 Tap **Analysis > G-Solve > fMax** to go to the first maximum.

6 Press **EXE** to display the coordinates.

7 Press the right arrow to jump to the next maximum.

8 Press **EXE** to display the coordinates.

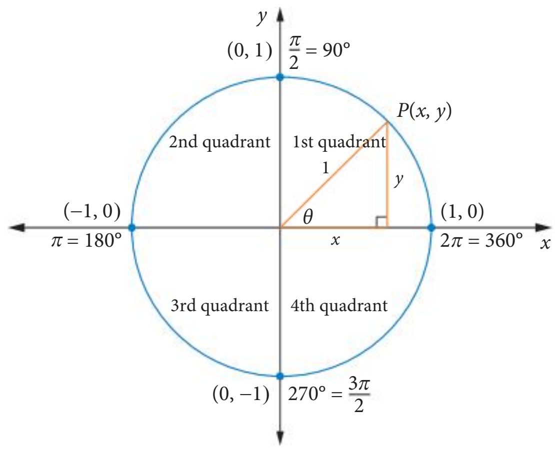
9 Tap **Analysis > G-Solve > fMin** to find to the minimum.

The decimal values of **0**, $\frac{\pi}{2}$ and π are shown above

Use **Main** to find the exact coordinates.

The tangent curve

$\tan(\theta) = \frac{y}{x} = \frac{\sin(\theta)}{\cos(\theta)}$, which means that $\tan(\theta)$ is the **gradient** $\left(\frac{\textbf{rise}}{\textbf{run}}\right)$ of the hypotenuse of the right-angled triangle as P moves around the unit circle.

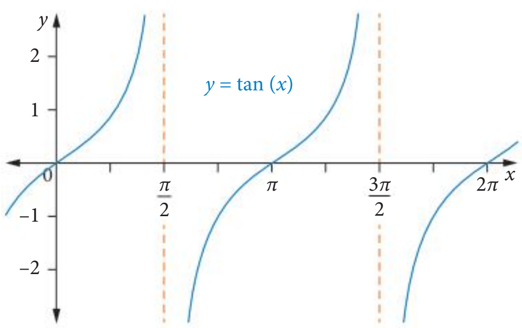


$\tan(\theta)$ is also the height of QS, the vertical tangent to the circle shown on page 336.

θ	0	$\frac{\pi}{2}$ (90°)	π (180°)	$\frac{3\pi}{2}$ (270°)	2π (360°)
$\sin(\theta)$	0	1	0	−1	0
$\cos(\theta)$	1	0	−1	0	1
$\tan(\theta)$	0	undefined	0	undefined	0

- At $\theta = 0$, the gradient of the hypotenuse, or $\frac{y}{x}$, is 0, but as P moves through the 1st quadrant, $\tan(\theta)$ increases from 0 towards infinity (∞).
- At $\theta = \frac{\pi}{2}$, the gradient $\frac{y}{x}$ is undefined, but as P moves through the 2nd quadrant, $\tan(\theta)$ increases from $-\infty$ (negative infinity) back to 0.
- At $\theta = \pi$, the gradient $\frac{y}{x} = 0$, but as P moves through the 3rd quadrant, the cycle starts again and $\tan(\theta)$ increases from 0 towards ∞ again.
- At $\theta = \frac{3\pi}{2}$, the gradient $\frac{y}{x}$ is undefined, but as P moves through the 4th quadrant, $\tan(\theta)$ increases from $-\infty$ back to 0.
- At $\theta = 2\pi$, the gradient $\frac{y}{x} = 0$ again, and the cycle repeats.

The graph of $y = \tan(x)$ looks like this:



Asymptotes at $x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2} \ldots$

- Note that tan is positive in the 1st and 3rd quadrants, consistent with the ASTC rule.
- The graph repeats itself after π, so it has a period of π.
- The domain of $y = \tan(x)$ is $R \backslash \left\{\frac{k\pi}{2}\right\}$ for odd integers k and the range is R.
- It has a mean (central) value of 0 and no amplitude.

Because k is odd, sometimes it is written as $2n + 1$ (where n is an integer).

Period and mean value of a tangent function

The period of $y = \tan(nx) + c$ is $\frac{\pi}{n}$, where $n > 0$,

there is no amplitude and the mean value is c.

9.4

EXERCISE 9.4 Graphs of circular functions

ANSWERS p. 525

Recap

1 The exact value of $\sin\left(\frac{13\pi}{6}\right) + \cos\left(\frac{8\pi}{3}\right)$ is

A -1 **B** $-\frac{1}{2}$ **C** 0 **D** $\frac{1}{\sqrt{2}}$ **E** $\frac{2}{\sqrt{3}}$

2 If $\tan(\pi + x) = 1$ and $\cos(x) > 0$, then one possible value of x is

A $-\pi$ **B** $-\frac{\pi}{4}$ **C** $\frac{\pi}{6}$ **D** $\frac{\pi}{3}$ **E** $\frac{7\pi}{4}$

Mastery

3 WORKED EXAMPLE 11 TECH-FREE

a State the amplitude, period and mean value of $y = 4\sin(x) - 3$.

b Sketch the graph of $y = 4\sin(x) - 3$ in $[-\pi, \pi]$.

4 Given that $\sin(2x) = 2\sin(x)\cos(x)$, state the amplitude, period and mean of $y = 15 - 20\sin(4x)\cos(4x)$.

5 WORKED EXAMPLE 12 Calculate the percentage error, correct to three decimal places, in the approximation to $\sin(x)$ using $x = 0.3$.

6 Using CAS 2

a Draw the graph of $y = 2\cos(1.5x) - 1$ in $[0, \pi]$.

b Find the coordinates of the maximum and the minimum values, correct to two decimal places where appropriate.

Exam practice

80–100% 60–79% 0–59%

7 © VCAA 2011 1Q3a 66% TECH-FREE (2 marks) State the range and period of the function

$h: R \to R, h(x) = 4 + 3\cos\left(\frac{\pi x}{2}\right)$.

8 TECH-FREE (2 marks) Determine the value of p given that the period of $y = \sin\left(\frac{5\pi}{2p+7}\right)x$ is 6.

9 TECH-FREE (2 marks) Determine the value(s) of p given that $y = \cos(p\pi x)$ has the same period as $y = \tan\left(\frac{\pi x}{p-1}\right)$ and $p > 0$.

10 © VCAA 2017 2AQ1 **92%** Let $f: R \to R, f(x) = 5\sin(2x) - 1$.

The period and range of this function are respectively

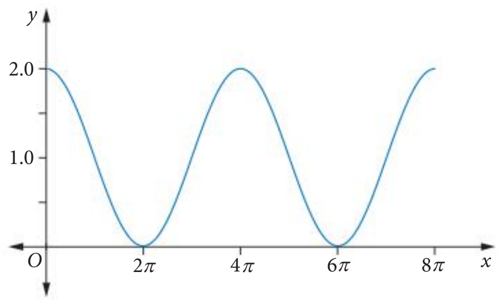
A π and $[-1, 4]$ **B** 2π and $[-1, 5]$ **C** π and $[-6, 4]$

D 2π and $[-6, 4]$ **E** 4π and $[-6, 4]$

11 © VCAA 2012 2AQ1 **91%** The function with rule $f(x) = -3\sin\left(\frac{\pi x}{5}\right)$ has period

A 3 **B** 5 **C** 10 **D** $\frac{\pi}{5}$ **E** $\frac{\pi}{10}$

12 © VCAA 2002 1IQ1 **88%** The diagram shows part of the graph of a circular function.

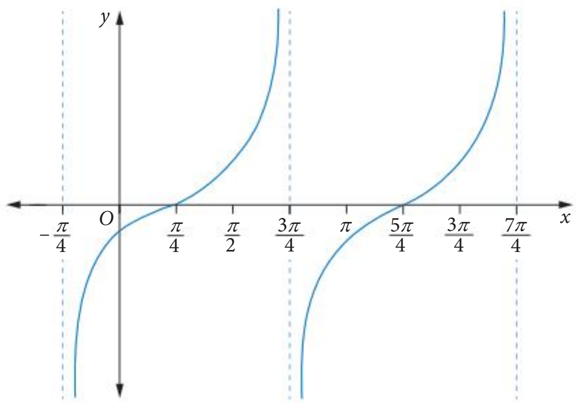


A possible equation for the function whose graph is shown is

A $y = 1 + \sin(x)$ **B** $y = 1 + \sin\left(\frac{x}{2}\right)$ **C** $y = 1 + \cos(x)$

D $y = 1 - \cos\left(\frac{x}{2}\right)$ **E** $y = 1 + \cos\left(\frac{x}{2}\right)$

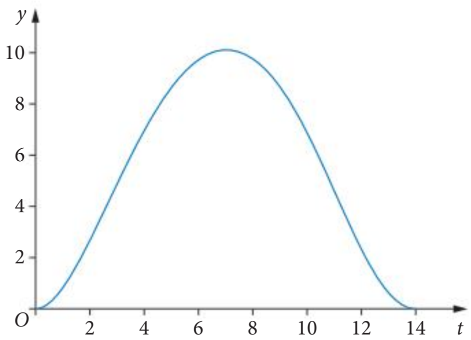
13 © VCAA 2003 1IQ7 **87%** The diagram shows two cycles of the graph of a circular function.



The period of the circular function is

A $\frac{\pi}{2}$ **B** $\frac{3\pi}{4}$ **C** π **D** $\frac{7\pi}{4}$ **E** 2π

14 © VCAA 2016 2AQ8 77% The UV index, y, for a summer day in Melbourne is illustrated in the graph below, where t is the number of hours after 6 am. The graph is most likely to be the graph of



A $y = 5 + 5\cos\left(\dfrac{\pi t}{7}\right)$ **B** $y = 5 - 5\cos\left(\dfrac{\pi t}{7}\right)$ **C** $y = 5 + 5\cos\left(\dfrac{\pi t}{14}\right)$

D $y = 5 - 5\cos\left(\dfrac{\pi t}{14}\right)$ **E** $y = 5 + 5\sin\left(\dfrac{\pi t}{14}\right)$

15 © VCAA 2004 1IQ6 76% A sine function f has an amplitude of 3 and a period of $\dfrac{1}{10}$. The rule for f could be

A $f(t) = 3\sin\left(\dfrac{\pi t}{5}\right)$ **B** $f(t) = 3\sin(20\pi t)$ **C** $f(t) = 3\sin(10t)$

D $f(t) = 1.5\sin\left(\dfrac{\pi t}{5}\right)$ **E** $f(t) = 1.5\sin(20\pi t)$

16 © VCAA 2010 2AQ1 60% The function with rule $f(x) = 4\tan\left(\dfrac{x}{3}\right)$ has period

A $\dfrac{\pi}{3}$ **B** 6π **C** 3 **D** 3π **E** $\dfrac{2\pi}{3}$

17 © VCAA 2005 non-CAS 1IIQ3 73% (3 marks) Sketch the graph of the function

$f: (-\pi, \pi) \to R,\ f(x) = \tan\left(\dfrac{x}{2}\right) + 1.$

Clearly label any axes intercepts, and any asymptotes with their equations.

18 (4 marks)

a State the amplitude and period of the function $y = -3\sin(2x) + 5$. 2 marks

b State the transformations required to obtain $y = -3\sin(2x) + 5$ from $y = \sin(x)$. 2 marks

19 (2 marks) Sketch one cycle of the graph $y = -2\cos(4x) + 2$ that starts at $x = 0$.

Video playlist Transformations of the circular functions

Worksheets Sketching periodic functions – amplitude and period

Sketching periodic functions – phase and vertical shift

Sine and cosine curves

Trigonometric graphs

Puzzle Trigonometric graphs match-up

9.5 Transformations of the circular functions

The **transformations** needed for one graph to look like another graph can be determined from the function. The types of transformations are connected to the amplitude, period and mean value.

The main types of transformations are **translation**, **dilation** and **reflection**.

Transformations of the sine and cosine functions

The general forms of the sine and cosine functions are respectively:

$$y = a\sin(nx) + c \quad \text{and} \quad y = a\cos(nx) + c$$

where the constants a, n and c refer to transformations to the basic graphs of $y = \sin(x)$ and $y = \cos(x)$.

The effect of a is a (vertical) **dilation from the x-axis** or **parallel to the y-axis** since it multiplies all the y values by a, affecting the amplitude.

- For $a > 1$ or $a < -1$, the graph is stretched vertically by a factor of a.
- For $-1 < a < 1$, the graph is shrunk vertically. The amplitude of the new graph is a.
- If $a < 0$, then the graph is also reflected in the x-axis (vertically).

The effect of n is a (horizontal) **dilation from the y-axis** or **parallel to the x-axis** since it rescales the x values, affecting the period.

- For $n > 1$ or $n < -1$, the graph is shrunk horizontally by a factor of $\frac{1}{n}$.
- If $-1 < n < 1$, the graph is stretched horizontally. The period of the new graph is $\frac{2\pi}{n}$.
- If $n < 0$, then the graph is also reflected in the y-axis (horizontally).

Replacing x by $-x$ produces a **reflection in the y-axis**.

Replacing y by $-y$ produces a **reflection in the x-axis**.

The effect of c is a (vertical) **translation** up or down, **in the direction of the y-axis,** affecting the mean value.

- For $c > 0$, the graph is moved up c units, in the positive y direction.
- For $c < 0$, the graph is moved down c units, in the negative y direction.

The mean value of the new graph is c.

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WORKED EXAMPLE 13 Transformations applied to $\cos(x)$

a State the transformations required for $y = \cos(x)$ to become $y = 3\cos(2x) + 1$.

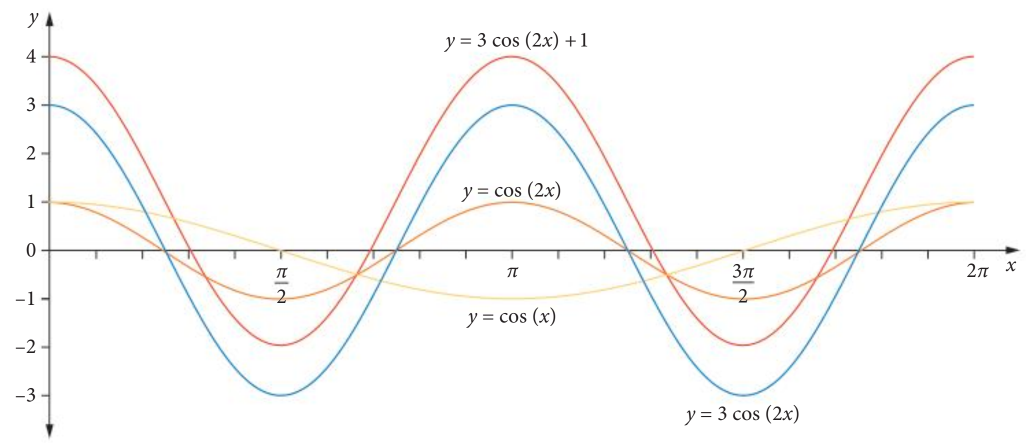
b Sketch each transformation in the interval $[0, 2\pi]$.

c State the domain and range of $y = 3\cos(2x) + 1$ in the interval $[0, 2\pi]$.

Steps	Working
a Compare $y = 3\cos(2x) + 1$ to $y = a\cos(nx) + c$ and identify each transformation. **Exam hack** Perform transformations in the order: Dilation, Reflection, Translation. Remember DRT ('Doctor T').	$a = 3$: Dilation from the x-axis (vertically) by a factor of 3. $n = 2$: Dilation from the y-axis (horizontally) by a factor of $\frac{1}{2}\left(\text{period} = \frac{2\pi}{2} = \pi\right)$. $c = 1$: Translation 1 unit up (vertically, in the positive direction of the y-axis).

9780170448260

b Draw the graph of $y = \cos(x)$ with the graph of each transformation.	$y = \cos(x)$ in yellow $y = \cos(2x)$ in brown $y = 3\cos(2x)$ in blue $y = 3\cos(2x) + 1$ in red



c State the domain and range.	Domain is $[0, 2\pi]$. Range is $[-3 + 1, 3 + 1] = [-2, 4]$.

9.5

Transformations of the tangent function

The general form of the tangent function is $y = a\tan(nx) + c$, where the constants a, n and c refer to transformations to the graph of $y = \tan(x)$.

The effect of a is a (vertical) dilation from the x-axis.

- For $a > 1$ or $a < -1$, the graph is stretched vertically by a factor of a.
- For $-1 < a < 1$, the graph is shrunk vertically.
- If $a < 0$, then the graph is also reflected in the x-axis (vertically).

The effect of n is a (horizontal) dilation from the y-axis.

- For $n > 1$ or $n < -1$, the graph is shrunk horizontally by a factor of $\frac{1}{n}$.
- If $-1 < n < 1$, the graph is stretched horizontally.
- If $n < 0$, then the graph is also reflected in the y-axis (horizontally).

The period of the new graph is $\frac{\pi}{n}$.

The effect of c is a (vertical) translation in the direction of the y-axis.

- For $c > 0$, the graph is moved up c units.
- For $c < 0$, the graph is moved down c units.

The mean value of the new graph is c.

Function	Amplitude	Period	Average (central) value	Domain	Range
$y = \sin(x)$	1	2π	0	R	$[-1, 1]$
$y = \cos(x)$	1	2π	0	R	$[-1, 1]$
$y = \tan(x)$	No amplitude	π	0	$R\backslash\left\{\frac{k\pi}{2}\right\}$ for odd k	R
$y = a\sin(nx) + c$	a	$\frac{2\pi}{n}$	c	R	$[-a + c, a + c]$
$y = a\cos(nx) + c$	a	$\frac{2\pi}{n}$	c	R	$[-a + c, a + c]$
$y = a\tan(nx) + c$	No amplitude	$\frac{\pi}{n}$	c	$R\backslash\left\{\frac{k\pi}{2n}\right\}$ for odd k	R

WB

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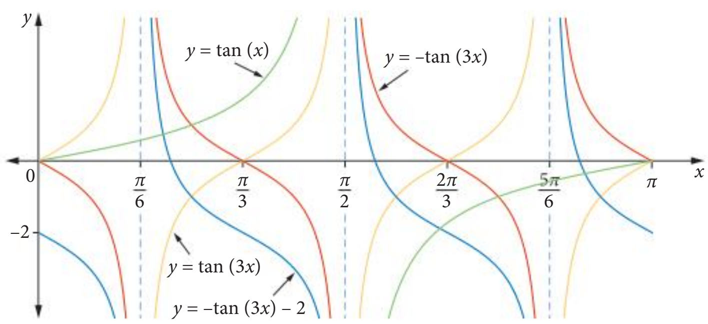
WORKED EXAMPLE 14 Transformations applied to tan (x)

a State the transformations required to transform the graph of $y = \tan(x)$ to the graph given by $y = -\tan(3x) - 2$ in the interval $[0, \pi]$.

b Find the equations of all asymptotes in the given interval.

c Sketch $y = -\tan(3x) - 2$, including asymptotes.

d State the domain and range of $y = -\tan(3x) - 2$ in the given interval.

Steps	Working
a Compare $y = -\tan(3x) - 2$ to $y = a\tan(nx) + c$ and identify each transformation.	$a = -1$: Reflection about the x-axis. $n = 3$: Horizontal dilation by a factor of $\frac{1}{3}$ $\left(\text{period} = \frac{\pi}{3}\right)$. $c = -2$: Vertical translation 2 units down.
b Asymptotes are $x = \frac{k\pi}{2n}$ for odd k.	Use $n = 3$ to get $x = \frac{\pi}{6}, \frac{3\pi}{6}, \frac{5\pi}{6}$ for $[0, \pi]$. So $x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$ are the asymptotes.
c Draw the graph of $y = \tan(x)$ with the graph of each transformation. $y = \tan(x)$ in green, $y = \tan(3x)$ in yellow, $y = -\tan(3x)$ in red, $y = -\tan(3x) - 2$ in blue.	
d State the domain and range.	Domain is $[0, \pi] \backslash \left\{\frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}\right\}$. Range is R.

EXERCISE 9.5 Transformations of the circular functions

ANSWERS p. 526

9.5

Recap

1 The period of a trigonometric function is 3 and its mean value is 1. A possible function is

A $4 \sin(3x) + 1$ **B** $5(\cos(\pi x) + 1)$ **C** $2 \tan(3x) + 1$

D $4\left(0.25 - 0.5 \tan\left(\frac{\pi x}{3}\right)\right)$ **E** $1 - \sin\left(\frac{x}{3}\right)$

2 A cosine function has the form $f(x) = a \cos(nx) + b$, with a, n, b positive numbers.

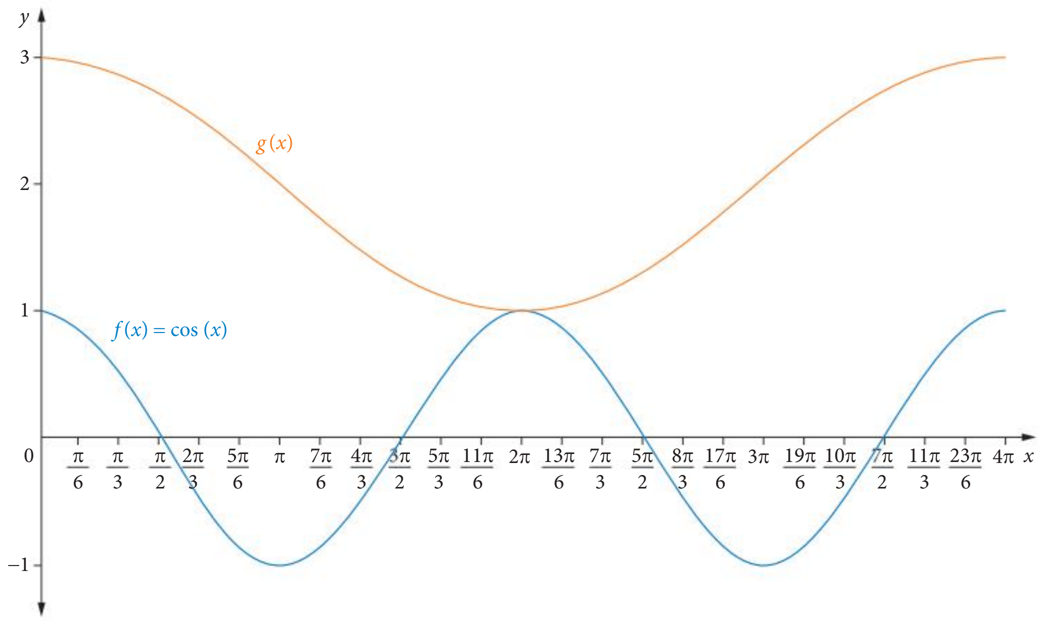
If its minimum and maximum values are 1 and 7 respectively and its period is 2, a possible function for $f(x)$ is

A $4 \cos(x) - 3$ **B** $-3 \cos(\pi x) + 4$ **C** $2 \cos(\pi x) + 5$

D $3 \cos(\pi x) - 4$ **E** $-4 \cos(x) - 3$

Mastery

3 List the transformations for the graph of $f(x)$ to become the graph of $g(x)$.



4 WORKED EXAMPLE 13 TECH-FREE

a State the transformations required for $y = \sin(x)$ to become $y = 2 \sin(2x) - 3$.

b Sketch each transformation in the interval $[0, \pi]$.

c State the domain and range of $y = 2 \sin(2x) - 3$ in the interval $[0, \pi]$.

5 WORKED EXAMPLE 14 TECH-FREE

a State the transformations required for the graph $y = \tan(x)$ to become the graph $y = -2 \tan(x) + 1$ in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

b Find the equations of all asymptotes in the given interval.

c Sketch $y = -2 \tan(x) + 1$, including asymptotes.

d State the domain and range of $y = -2 \tan(x) + 1$ in the given interval.

6 a In terms of transformations, explain why the graph of the function $y = \sin(x)$ is the same as the graph of the function $y = -\sin(-x)$.

b Explain whether the function $y = \cos(x)$ is the same as the graph of the function $y = -\cos(-x)$.

Exam practice

7 TECH-FREE (2 marks) The graph of the function $y = 1.5\sin\left(\frac{\pi}{3}x\right) + 6$ is dilated from the x-axis by a factor of 4 and a translation in the y-direction of -1 is applied. Finally, the graph is dilated from the y-axis by a factor of $\frac{1}{9}$. Find the equation of the transformed function.

8 TECH-FREE (2 marks) The point $P(0, 1)$ lies on the graph of the function $y = \tan(x) + 1$. The function is transformed by a dilation from the y-axis by a factor of 2, then a dilation from the x-axis by a factor of -2, and a translation vertically by -2 units. Find the new coordinates of P.

9 The graph of the function $y = 6\sin(x)$ is dilated from the x-axis with scale factor 0.5, reflected in the x-axis and then dilated from the y-axis by a factor of 4.

The transformed function is

A $y = -3\sin(0.5x)$
B $y = -\sin(4x)$
C $y = 6\sin(0.5x)$
D $y = 3\sin(0.5x) + 4$
E $y = -3\sin(4x)$

10 Choose the correct statement.

A A dilation from the y-axis using a scale factor of 2 doubles the period of the function.

B The mean value of a function cannot be the same as the amplitude.

C A translation in the y-direction cannot be applied to the function $y = \frac{1}{\tan(x)}$.

D After a dilation from the y-axis by a factor of $\frac{\pi}{3}$, the period of $y = \tan(\pi x)$ triples.

E A translation of 1 unit upwards has the same effect as a dilation from the y-axis by a factor of 1.

11 Which transformations have been applied to the function $f(x)$ to obtain the function $g(x)$?

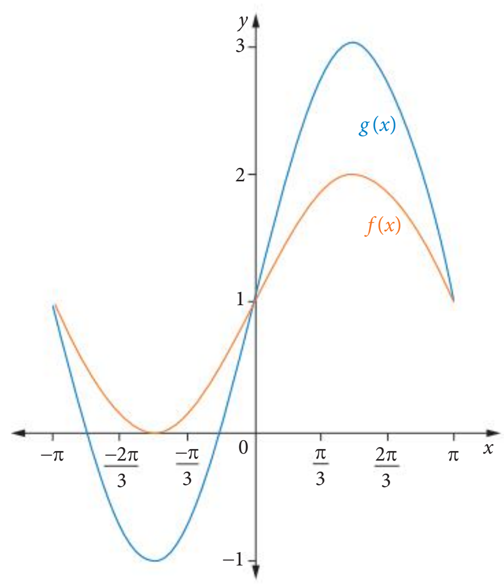
A Dilation from the y-axis of factor 2, vertical translation 1 unit up.

B Dilation from the y-axis of factor $\frac{1}{2}$, vertical translation 2 units up.

C Dilation from the x-axis of factor $\frac{1}{2}$, vertical translation 1 unit up.

D Reflection in the x-axis, vertical translation 2 units up.

E Dilation from the x-axis of factor 2, vertical translation 1 unit down.



12 The value of the function $y = a \tan(\pi bx)$, where a and b are constants, is 1 at $x = 0.25$ and it has a period of 1.

After a dilation by a factor of a from the x-axis and a vertical translation of $-b^2$, the function becomes

A $y = \tan(2\pi x) - 1$ **B** $y = 3\tan(\pi x) - 2$ **C** $y = \tan(\pi x) - 1$

D $y = -\tan(3\pi x) + 2$ **E** $y = -2\tan(2\pi x) - 1$

13 (3 marks) The period of the function $y = 2\sin(x) - 1$ is doubled and its graph is reflected in the y-axis.

a State the equation of the transformed function. 1 mark

b Sketch the graph of the transformed function in $[0, 2\pi]$. 2 marks

14 (4 marks) The minimum and the maximum value of the function $y = a\cos(nx) + c$ is -2 and 8 respectively, where a, n and c are constants and a > 0. The mean value is 3 and after a dilation of factor $\frac{1}{2}$ from the y-axis, the period is $\frac{\pi}{2}$.

a State the equation of the original function. 2 marks

b Sketch one cycle of the transformed function starting at $x = 0$. 2 marks

9.6

9.6 Applying circular functions

Video playlist Applying circular functions

Circular functions, particularly sine and cosine, can be used to model real-life phenomena where cycles and periodicity are observed, such as in the rise and fall of tides and in regular rotation of objects and motors.

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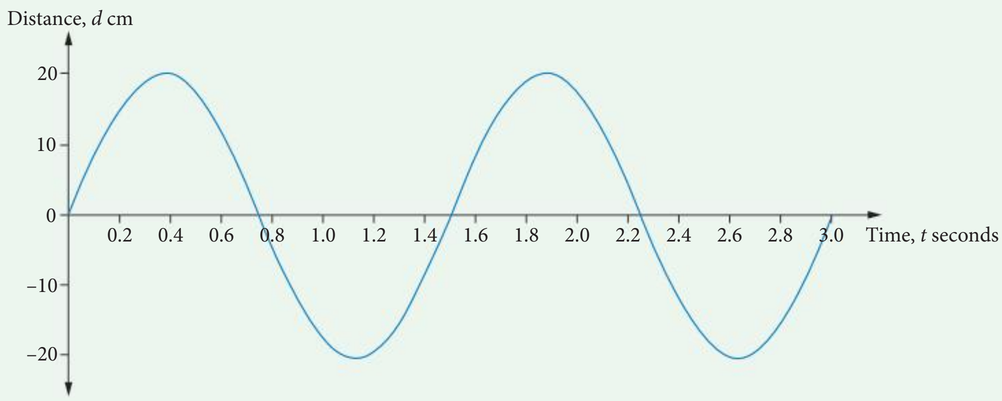
Worksheet Applying trigonometric functions

WORKED EXAMPLE 15 Applying circular functions

The graph shows the vertical distance between a fixed point on the rim of a rotating bicycle wheel and the axle (centre) of the wheel, over time.



a What is the radius of the wheel?

b How long does it take the wheel to make one revolution?

c What is the minimum vertical distance reached by the point? Where on the wheel is this?

d What do the t-intercepts on the graph represent?

Steps	Working
a Find the amplitude.	The radius is the amplitude of the graph. From the graph, this is 20 cm.
b One revolution of the wheel corresponds to 1 period on the graph.	Two periods correspond to 3 s, so 1 period is 1.5 s. It takes the wheel 1.5 s to make a revolution.
c Find the lowest point on the graph.	Minimum distance is –20 cm, which represents when the point is at ground level, 20 cm below the centre of the wheel.
d The t-intercepts represent when $d = 0$, when the point is 0 cm vertically from the centre of the wheel.	The t-intercepts represent when the point is level with the centre of the wheel.

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WORKED EXAMPLE 16 Water level described by a trigonometric function

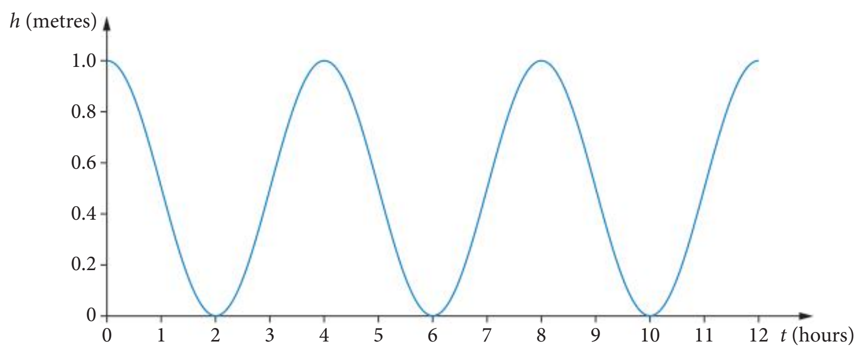
The water level (above sea level) of a lagoon changes periodically and is described by the function $h = 0.5\cos\left(\frac{\pi t}{2}\right) + 0.5$, where h is in metres and t is the time in hours after midnight.

a Sketch the graph of the function in the interval [0, 12].

b At what times after midnight is the water level at its maximum?

c What is the water level at 3.45 am? Give your answer correct to 3 decimal places.

Steps	Working
a Identify the amplitude, period and transformations, and use them to sketch the graph.	Amplitude is 0.5. Period is $2\pi \div \frac{\pi}{2} = 4$. Vertical translation: 0.5 units up.



Steps	Working
b Find the t values after $t = 0$ corresponding to the maximum points on the graph.	The maximum water level occurs at $t = 4, 8, 12$. This corresponds to 4 am, 8 am and 12 pm.
c Determine the t value corresponding to 3.45 am and substitute to find h.	3.45 am is 3.75 hours after midnight, so $t = 3.75$. When $t = 3.75$, $h = 0.5\cos\left(\frac{\pi \times 3.75}{2}\right) + 0.5$ $= 0.9619\ldots$ ≈ 0.962 m

Exam hack

For circular functions, make sure that your calculator is in radian mode, not degree mode.

USING CAS 3 Applying circular functions

The vertical displacement d cm at time t seconds of an old engine's piston is modelled by the equation $d = 5\sin(9.5\pi t)$.

a Sketch the graph of the function in the interval $[0, 1]$.

b How many times does the piston move away from its centre position in 1 second?

c Calculate the distance the piston travels each $\frac{1}{5}$ of a second.

d During the first 0.1 second, when was the piston at a distance of more than 2 cm away from its centre position? Give your answer correct to 2 decimal places.

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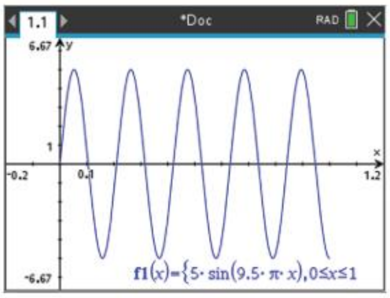
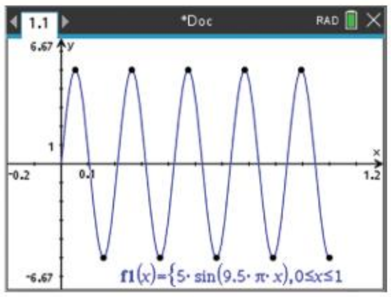
a



1 Graph the function over the domain [0, 1].

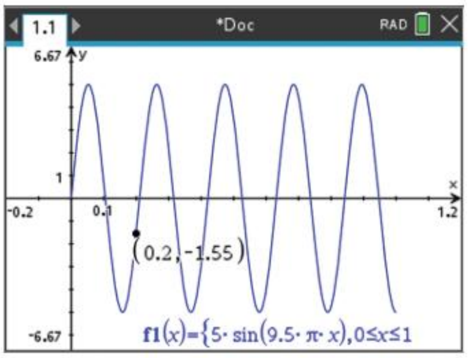
2 Adjust the window settings to suit.

b



1 Count the number of maximums and minimums to get a total of 10.

c

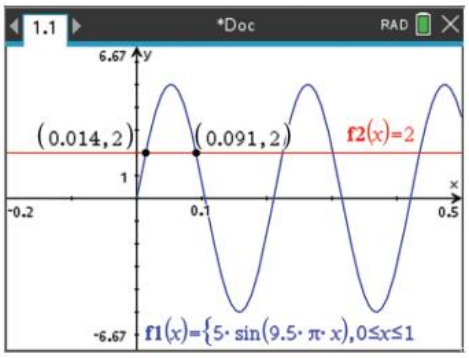


1 Press **menu > Trace > Graph Trace**.

2 Enter $\frac{1}{5}$ and the graph will jump to the point **(0.2, −1.55)**.

3 The distance travelled is 4 amplitudes − 1.55 $= 4 \times 5 - 1.55 = 18.45$ cm.

d



1 Graph **$f(x) = 2$**.

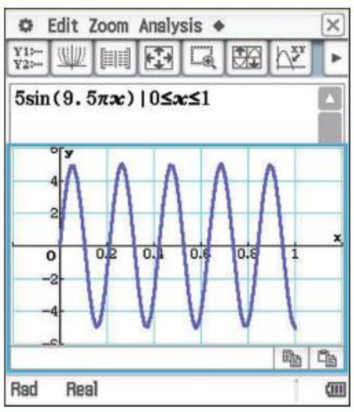
2 Press **menu > Analyze Graph > Intersection**.

3 Determine the coordinates of the first two points of intersection.

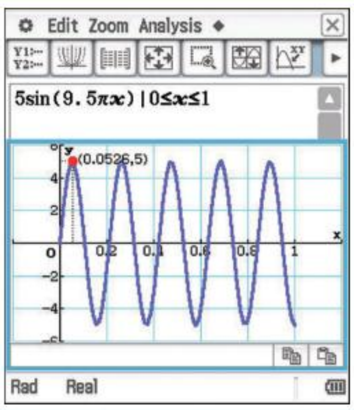
4 The piston was over 2 cm away for $0.091 - 0.014 \approx 0.08$ s.

ClassPad

a



b



1 In **Main**, set the mode to **Rad.**

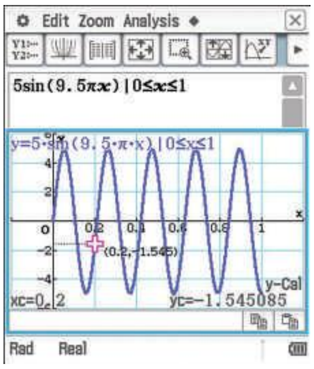
2 Enter the sine function over the domain [0, 1] using **Math3** for | and ≤.

3 Drag down into the Graph screen from the **left hand** side.

4 Adjust the window settings to suit.

Count the number of maximums and minimums to get a total of 10.

c

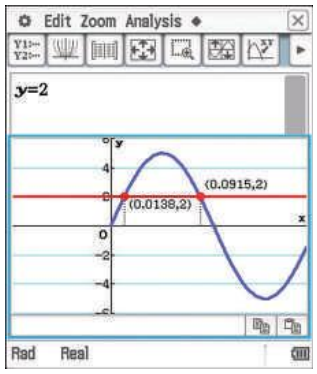


1 Tap **Analysis > G-Solve > x-Cal/y-Cal > y-Cal** and in the dialogue box enter **1/5**.

2 Tap **OK** and the graph will jump to the point **(0.2, −1.55)**.

3 The distance travelled is 4 amplitudes − 1.55 = 4 × 5 − 1.55 = 18.45 cm.

d



1 In the upper window, enter **y = 2**.

2 Drag down both graphs into the Graph window.

3 Tap **Analysis > G-Solve > Intersection**.

4 Press **EXE** to label the first point of intersection.

5 Press the **right arrow**, then **EXE** to label the second point of intersection.

6 The piston was over 2 cm away for 0.0915 − 0.0138 ≈ 0.077 ≈ 0.08 s.

Video
VCE question analysis: The circular functions

VCE QUESTION ANALYSIS

© VCAA 2013 2BQ1 2013 Examination 2 Section B Question 1 (7 marks)

Trigg the gardener is working in a temperature-controlled greenhouse. During a particular 24-hour time interval, the temperature (T°C) is given by $T(t) = 25 + 2\cos\left(\frac{\pi t}{8}\right), 0 \le t \le 24$, where t is the time in hours from the beginning of the 24-hour time interval.

a State the maximum temperature in the greenhouse and the values of t when this occurs. 2 marks

b State the period of the function T. 1 mark

c Find the smallest value of t for which $T = 26$. 2 marks

d For how many hours during the 24-hour time interval is $T \ge 26$? 2 marks

Reading the question

- Read all parts of each question first to see where CAS will be used.
- Notice the key words and phrases. For example, 'the beginning of the 24-hour time interval' means $t = 0$, and 'how many hours' means the total number of hours in the 24-hour period.
- A cosine function means that the temperature will be cyclic.
- Give answers to the accuracy required. If the question does not specify the accuracy, answers must be exact values.

Thinking about the question

- This is the first question of Section B of Exam 2, so don't expect involved calculations or information that requires a high level of interpretation.
- For questions worth more than 1 mark, sufficient working needs to be shown, or if CAS is used, clear explanations of how the solution was obtained.
- Decide when it will be more efficient to use algebra, particularly when exact answers are required.
- Since calculations using $T(t) = 25 + 2\cos\left(\frac{\pi t}{8}\right)$ involve radians, ensure that calculator settings are also in radian mode.

Worked solution (✓ = 1 mark)

a Maximum temperature when $\cos\left(\frac{\pi t}{8}\right) = 1$.

Maximum temperature is $25 + 2 \times 1 = 27\,°\text{C}$. ✓

Maximum temperature when

$$\cos\left(\frac{\pi t}{8}\right) = 1$$

Then $\frac{\pi t}{8} = 0, 2\pi, 4\pi \ldots$

$t = 0, 16, 32 \ldots$

$t = 0$ (midnight) and $t = 16$ (4 pm) are in the interval **[0, 24]**. ✓

Alternatively, use CAS to find the maximum value.

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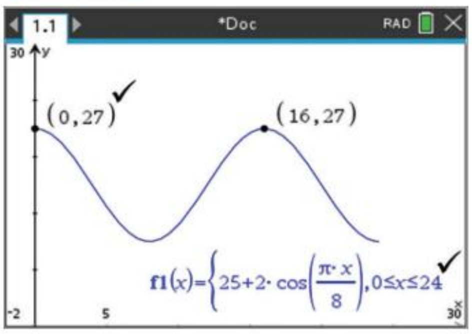

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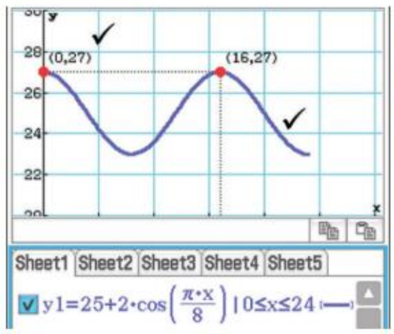


b The period of $\cos(nt)$ is $\frac{2\pi}{n}$, so the period of $\cos\left(\frac{\pi t}{8}\right)$ is $2\pi \div \frac{\pi}{8} = 16$.

Hence, the period of $T(t)$ is **16 hours**. ✓

c $T = 26$, so $26 = \mathbf{25 + 2\cos\left(\frac{\pi t}{8}\right)}$ ✓

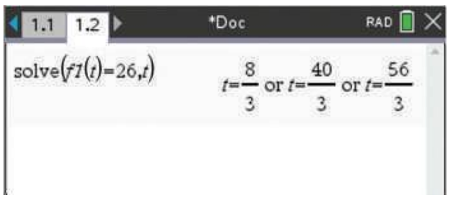
$$\cos\left(\frac{\pi t}{8}\right) = \frac{1}{2}$$

Solve for t.

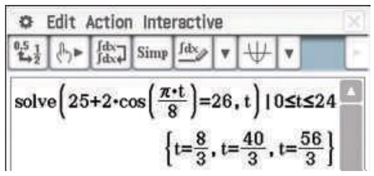
$\frac{\pi t}{8} = \frac{\pi}{3} \Rightarrow t = \mathbf{\frac{8}{3}}$ **hours**. ✓

Confirm using CAS (see next page).

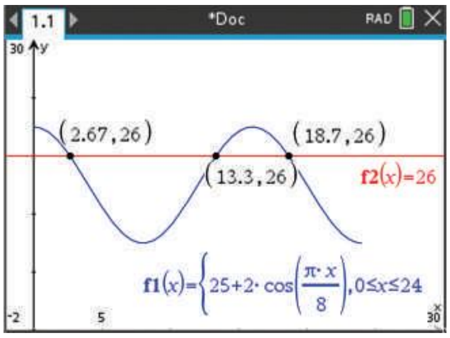
TI-Nspire

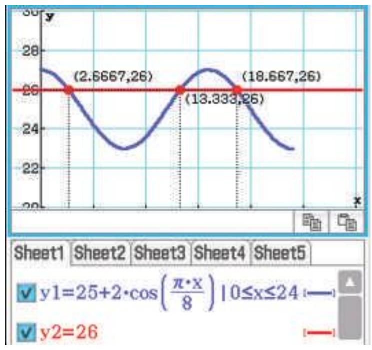


ClassPad



d Using CAS, find the times when the temperature is 26°C.





Determine the range of hours where the temperature is above 26°C.

The temperature is 26°C when $t = \frac{8}{3}, \frac{40}{3}, \frac{56}{3}$. ✓

From inspection of the graph, the total time when $T > 26$ is $\mathbf{\frac{8}{3} + \left(\frac{56}{3} - \frac{40}{3}\right) = 8}$ **hours**. ✓

Student performance

80–100% 60–79% 0–59%

a 71% Questions should be carefully read. Some students did not answer both parts of the question. The question asks for both the maximum temperature and the times when this occurs, but students gave the required temperature only.

b 88% Most students answered this question well. Students who did not provide the correct answer may have used wrong values in the expression for the period or incorrectly simplified the expression. Verifying the computation using a calculator may avoid this problem.

c 76% 64% of students achieved full marks in this question. Other students achieved fewer marks possibly because they rounded their answers rather than providing the exact value. Remember that unless the question asks for an answer to be given to a specified number of decimal places, you should leave it in exact form, even if it involves fractions, surds and so on.

d 54% The question contains the words 'how many hours', thus suggesting an interval of time. Some students did not appreciate this requirement and just gave the values of t without also finding the difference between them.

EXERCISE 9.6 Applying circular functions

ANSWERS p. 526

Recap

1 To transform the graph of $y = 0.5\cos(x) - 1$ to the graph $y = \cos(x)$ involves

A a translation of 1 unit down and a dilation from the y-axis by a factor of 2.

B dilating the graph from the x-axis by a factor of 2 and a vertical translation 1 unit down.

C translating the graph 1 unit left followed by a dilation from the x-axis by a factor of 2.

D translating the graph 1 unit up followed by a dilation from the x-axis with factor 2.

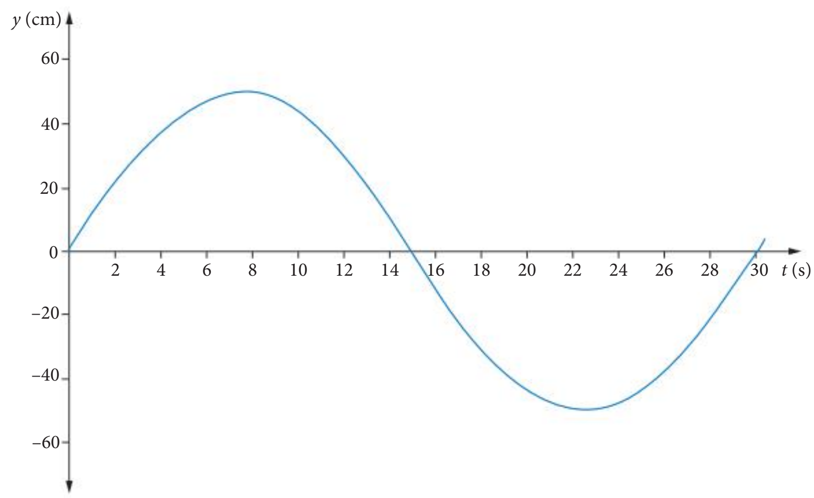
E dilating the graph from the x-axis by a factor of 0.5 and a translation 1 unit right.

2 The dilation factor parallel to the x-axis needed to transform $f(x) = 4 - 3\sin\left(\frac{\pi x}{5}\right)$ to $g(x) = 2 - \frac{1}{5}\sin\left(\frac{5x}{4\pi}\right)$ is

A $\frac{25}{4\pi^2}$ **B** $\frac{4\pi}{25}$ **C** $\frac{4\pi^2}{25}$ **D** $\frac{25}{4\pi}$ **E** $\frac{4}{25}$

Mastery

3 WORKED EXAMPLE 15 The function $y = 50\sin\left(\frac{\pi t}{15}\right)$ shows the height of a horse above its rest position on a merry-go-round over 30 seconds.



The merry-go-round itself takes 40 seconds to make one revolution.

a How many times does the horse go up and down after 10 revolutions of the merry-go-round?

b Find, correct to the nearest centimetre, the height of the horse after 1.25 revolutions of the merry-go-round.

c Calculate, correct to two decimal places, the amount of time during the first revolution of the merry-go-round, the horse is moving upward at more than 40 cm away from its rest position.

4 WORKED EXAMPLE 16 The water level in a dam is controlled by adding and releasing water according to the function $H = 3.5\sin\left(\frac{\pi t}{3}\right) + 6.5$. *The* depth of the water is H metres at time t hours after midday.

a Sketch the graph of the function in the interval $[0, 10]$.

b What is the minimum and the maximum level of water in the dam at any time?

c At what time of day is the water level first at half its maximum capacity?

d By how much does the water level change between 2 pm and 5 pm? Give your answer correct to two decimal places.

5 Using CAS 3 When a mass is suspended on the end of a spring, pulled down and released, it executes simple harmonic motion (moves up and down). The amount of extension and contraction of the spring is described by the equation $d = -4.5\cos(6t)$, where d is the position of the mass in centimetres and t is the time in seconds. A positive value for d means contraction of the spring (moving up).

a Sketch the graph of the function in the interval [0, 1.5].

b Describe the initial position of the mass.

c For how many seconds during each oscillation (that is, for one up-and-down cycle) is the spring stretched? Give the answer correct to two decimal places.

d Calculate correct to one decimal place the position of the mass after 1 s.

Exam practice

6 TECH-FREE (2 marks) The temperature, T °C, in a greenhouse t hours after 6 am is described by the equation $T = 10\cos\left(\frac{\pi}{4}t\right) + 25$. State the maximum and the minimum temperature, and the times of the day from 6 am to 6 pm when they occur.

7 TECH-FREE (2 marks) The motion of a girl on a swing can be approximated by the equation $d = 1.5\sin\left(\frac{\pi}{2}t\right) + 2$, where d metres is her vertical height above the ground at time t seconds. For how long during the first 30 seconds is the girl at least 2 metres above the ground?

8 A cork is bobbing periodically on the surface of the sea according to the equation $h = 8\sin(2\pi t)$, where h cm is the cork's height at time t seconds. A calm sea has zero height.

The number of cycles the cork makes in 2 seconds is

A 1 **B** 2 **C** 3 **D** 4 **E** 8

9 The water depth, D metres, of a river is described by $D = 1.2\sin\left(\frac{\pi}{3}t\right) + 1.8$, where t is the number of hours after midnight. The water level at 2.30 am will be

A 0.6 m **B** 1.2 m **C** 1.8 m **D** 2.4 m **E** 3 m

10 The function $h = 0.1\cos\left(\frac{1}{2}\pi t\right) + 0.5$ gives the vertical height h m of a bicycle pedal at time t seconds.

The height of the pedal when it is at its lowest value is

A 0.1 m **B** 0.15 m **C** 0.4 m **D** 0.5 m **E** 0.6 m

11 The vertical position of a toy train above the centre of a circular track of radius 46 cm is given by the equation $P = 46\sin(at)$, where P cm is the position and t is time in seconds. The train takes 10 seconds to complete one half of the track. The value of a is

A 2π **B** $\frac{\pi}{2}$ **C** $\frac{\pi}{46}$ **D** 46π **E** $\frac{\pi}{10}$

12 The path of a rocket can be described by $h = 36\tan\left(\frac{\pi}{4}x\right)$, where x km is its horizontal distance from the take-off site and h is its height. What horizontal distance does the rocket approach as its height increases?

A $\frac{\pi}{4}$ **B** 2 **C** 4 **D** 9 **E** 9π

13 (5 marks) A pendulum's horizontal displacement (y cm) from its rest position is modelled by $y = 6.5\cos(3.5t)$, where t is the time in seconds.

Initially, the pendulum is released from its maximum possible height.

a How many complete swings does the pendulum make per minute? 1 mark

Note: One swing means the pendulum moves from one side to the other side and back.

b Find the horizontal displacement after the first 0.2 seconds, correct to 2 decimal places. 2 marks

c Determine for how long, during one swing, the pendulum's horizontal displacement is less than 3 cm. State your answer correct to two decimal places. 2 marks

14 (5 marks) The vertical displacement, x cm, of an engine piston from its casing is described by $x = 4\sin\left(\frac{3\pi}{4}t\right)$, where t milliseconds is the time after the engine is switched on.

a What is the maximum distance of the piston from the casing? 1 mark

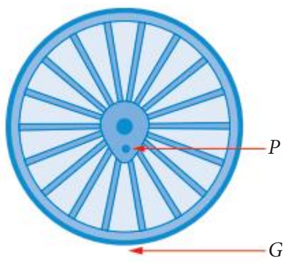
b State the number of cycles (periods) the piston makes in 24 milliseconds. 1 mark

c How far does the piston travel in 4 seconds? 1 mark

d Determine the first time after the engine starts that the piston's position is $2\sqrt{2}$ cm from its casing. 2 marks

15 (9 marks) P is a point on the wheel of a train that rotates as the train moves.

The vertical distance y cm of P in relation to point G (at ground level) is given by $y = 4\cos(at) + r$, where t is measured in seconds and r is the radius of the wheel in centimetres.



a The circumference of the wheel is 40π cm and it rotates with a period of $\frac{\pi}{40}$ seconds. Find the value of r and a. 2 marks

b Sketch the graph of $y = 4\cos(at) + r$ in the interval [0, 2]. 3 marks

c How many full rotations does the wheel make in the first 2 seconds? 2 marks

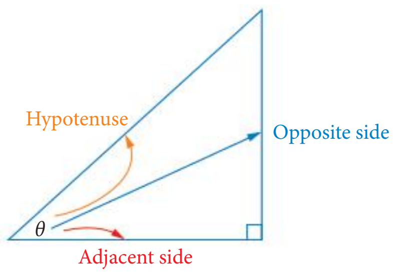
d In relation to point G, describe the position of P after 1.2 seconds. State your answer correct to one decimal place. 2 marks

9 Chapter summary

Trigonometry of right-angled triangles

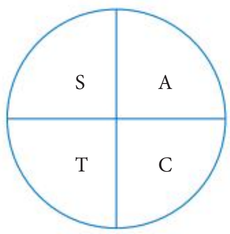
- $\sin(\theta) = \dfrac{\text{opposite}}{\text{hypotenuse}}$ $\quad\cos(\theta) = \dfrac{\text{adjacent}}{\text{hypotenuse}}$

  $\tan(\theta) = \dfrac{\text{opposite}}{\text{adjacent}}$

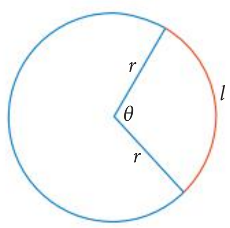


- $180° = \pi$
- To convert degrees to radians, multiply by $\dfrac{\pi}{180}$.
- To convert radians to degrees, multiply by $\dfrac{180}{\pi}$.

The ASTC rule

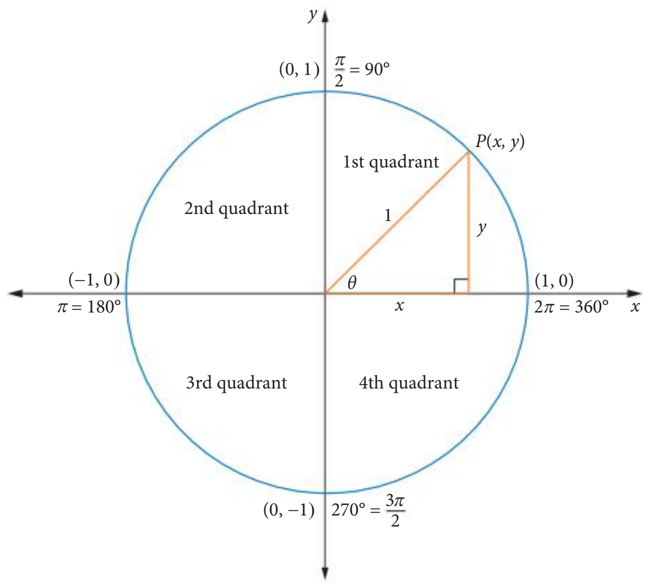


- **All** the circular functions are positive in the 1st quadrant.
- **Sine** is positive and the others are negative in the 2nd quadrant.
- The arc length l that subtends an angle of θ radians at the centre of the circle of radius r is $l = r\theta$.



The circular functions

- The circular functions **sine**, **cosine** and **tangent** are defined for any angle using the point $P(x, y)$ on the **unit circle**.
- $\sin(\theta) = y$
- $\cos(\theta) = x$
- $\tan(\theta) = \dfrac{y}{x}$



- **Tangent** is positive and the others are negative in the 3rd quadrant.
- **Cosine** is positive and the others are negative in the 4th quadrant.

Trigonometric identities and properties

- $\tan(\theta) = \dfrac{\sin(\theta)}{\cos(\theta)}$
- $\sin^2(\theta) + \cos^2(\theta) = 1$
- $\sin\left(\dfrac{\pi}{2} - \theta\right) = \cos(\theta)$
- $\cos\left(\dfrac{\pi}{2} - \theta\right) = \sin(\theta)$

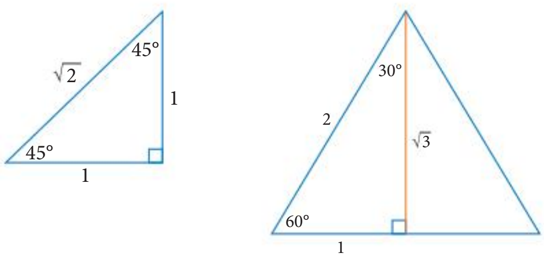
2nd quadrant	3rd quadrant	4th quadrant	Negative angles
$\sin(\pi - \theta) = \sin(\theta)$	$\sin(\pi + \theta) = -\sin(\theta)$	$\sin(2\pi - \theta) = -\sin(\theta)$	$\sin(-\theta) = -\sin(\theta)$
$\cos(\pi - \theta) = -\cos(\theta)$	$\cos(\pi + \theta) = -\cos(\theta)$	$\cos(2\pi - \theta) = \cos(\theta)$	$\cos(-\theta) = \cos(\theta)$
$\tan(\pi - \theta) = -\tan(\theta)$	$\tan(\pi + \theta) = \tan(\theta)$	$\tan(2\pi - \theta) = -\tan(\theta)$	$\tan(-\theta) = -\tan(\theta)$

The exact values

Points on the axes

	$\sin(\theta)$	$\cos(\theta)$	$\tan(\theta)$
$0°\ (0)$	0	1	0
$90°\ \left(\frac{\pi}{2}\right)$	1	0	undefined
$180°\ (\pi)$	0	-1	0
$270°\ \left(\frac{3\pi}{2}\right)$	-1	0	undefined
$360°\ (2\pi)$	0	1	0

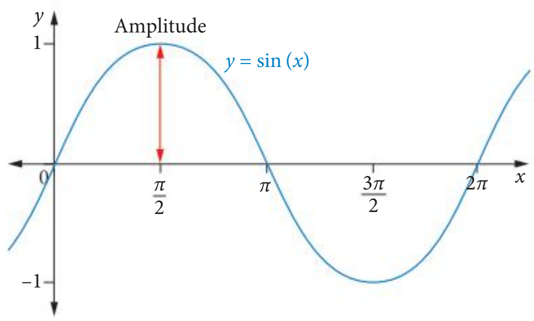
The reference triangles



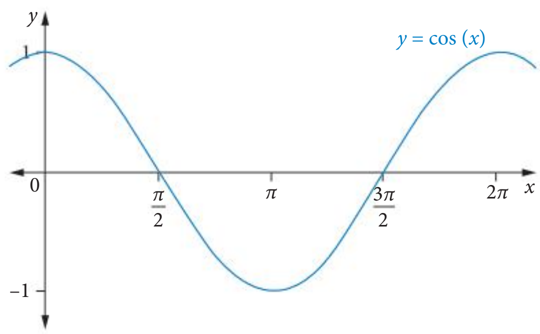
	0	$\frac{\pi}{6}$ (30°)	$\frac{\pi}{4}$ (45°)	$\frac{\pi}{3}$ (60°)	$\frac{\pi}{2}$ (90°)
$\sin(\theta)$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos(\theta)$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan(\theta)$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	not defined

Graphs of circular functions

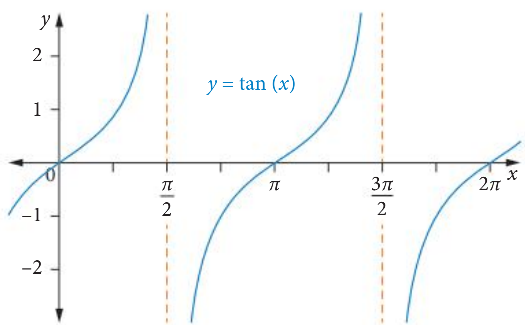
- $y = \sin(x)$ has period 2π and amplitude 1.



- $y = \cos(x)$ has period 2π and amplitude 1.



- $y = \tan(x)$ has period π and no amplitude.



Function	Amplitude	Period	Average (central) value	Domain	Range
$y = \sin(x)$	1	2π	0	R	$[-1, 1]$
$y = \cos(x)$	1	2π	0	R	$[-1, 1]$
$y = \tan(x)$	No amplitude	π	0	$R\backslash\left\{\frac{k\pi}{2}\right\}$ for odd k	R
$y = a\sin(nx) + c$	a	$\frac{2\pi}{n}$	c	R	$[-a + c, a + c]$
$y = a\cos(nx) + c$	a	$\frac{2\pi}{n}$	c	R	$[-a + c, a + c]$
$y = a\tan(nx) + c$	No amplitude	$\frac{\pi}{n}$	c	$R\backslash\left\{\frac{k\pi}{2n}\right\}$ for odd k	R

Transformations of the circular functions

For the functions $y = a\sin(nx) + c$ and $y = a\cos(nx) + c$, the constants a, n and c refer to the following transformations.

- a is a (vertical) dilation from the x-axis.

  If $a > 1$ or $a < -1$, the graph is stretched vertically.

  If $-1 < a < 1$, the graph is shrunk vertically.

  If $a < 0$, the graph is reflected (vertically) in the x-axis.
- n is a (horizontal) dilation from the y-axis.

  If $n > 1$ or $n < -1$, the graph is shrunk horizontally.

  If $-1 < n < 1$, the graph is stretched horizontally.

  If $n < 0$, the graph is reflected (horizontally) in the y-axis.
- c is a (vertical) translation parallel to the y-axis.

  If $c > 0$, the graph moves up c units.

  For $c < 0$, the graph moves down c units.

Applying circular functions

Real-life situations involving periodicity are modelled using circular functions. This includes tides, weather, motors, light and soundwaves.

For example, the function $h(t) = 2\sin\left(\frac{\pi t}{6}\right) + 0.5$, where t represents the number of hours after midnight, models the water level, in relation to a fixed level, of daily tides. The period of each tide is 12 hours, the high tide level is 2.5 m at 3 am and 3 pm, and the low tide height is 1.5 m occurring at 9 am and at 9 pm.

Cumulative examination 1

Total number of marks: 10 Reading time: 4 minutes Writing time: 15 minutes

TECH-FREE *Technology is NOT permitted.*

1 (2 marks) Simplify $\frac{\sin\theta[\sin(2\pi+\theta)-\sin(\pi-\theta)+1]}{\cos(-\theta)}$.

2 © VCAA 2015 1Q5 (3 marks) On any given day, the depth of water in a river is modelled by the function

$$h(t) = 14 + 8\sin\left(\frac{\pi t}{12}\right), 0 \le t \le 24$$

where h is the depth of water, in metres, and t is the time, in hours, after 6 am.

a Find the minimum depth of the water in the river. 1 mark

b © VCAA MODIFIED Calculate as an exact value the depth of water at 10 am. 2 marks

3 (3 marks)

a Find the approximate rate of change in the function $f(x) = 4-(x-2)^2$ when approaching $x = 2$ from the right using $h = 0.1$, $h = 0.01$ and $h = 0.001$. 1 mark

b Find the approximate rate of change when approaching $x = 2$ from the left of $f(x)$ using $h = 0.1$, $h = 0.01$ and $h = 0.001$. 1 mark

c What can be concluded about the instantaneous rate of change of $f(x)$ at $x = 2$? 1 mark

4 (2 marks) $P(x) = x^3-(a+1)x^2+(a+b)x-b$ and $P(1) = 0$, where $a > 0$.

Show that $P(x) = 0$ will have three solutions for x when $b < \frac{a}{4}$.

Cumulative examination 2

Total number of marks: 20 Reading time: 4 minutes Writing time: 30 minutes

Approved technology is permitted.

Section A 5 multiple-choice questions

5 marks

1 Choose the correct statement.

A The amplitude of $2\tan(x)$ is 2.

B The maximum value of $3 - 4\sin(5x)$ is -1.

C $\dfrac{7\pi}{6}$ radians is 210°.

D The function $y = 3\cos(x)$ is a horizontal dilation from the y-axis of the function $y = \cos(x)$.

E The length of the arc formed by an angle of 30° in a circle of radius 12 cm is π.

2 © VCAA 2016 2AQ2 Let $f: R \to R, f(x) = 1 - 2\cos\left(\dfrac{\pi x}{2}\right)$.

The period and range of this function are respectively

A 4 and $[-2, 2]$

B 4 and $[-1, 3]$

C 1 and $[-1, 3]$

D 4π and $[-1, 3]$

E 4π and $[-2, 2]$

3 © VCAA 2013 2AQ1 The function with rule $f(x) = -3\tan(2\pi x)$ has period

A $\dfrac{2}{\pi}$ **B** 2 **C** $\dfrac{1}{2}$ **D** $\dfrac{1}{4}$ **E** 2π

4 Which of the following represents a one-to-one function?

A $f: [0,5] \to R,\ f(x) = 2$

B $f: [-4,4] \to R,\ f(x) = x^2$

C $x^2 + y^2 = 1,\ \{x: 0 \le x \le 1\}$

D $f: [-3,0] \to R,\ f(x) = 1 - x^2$

E $f: [-1,1] \to R,\ f(x) = 3x^4$

5 The graph of $f(x) = \dfrac{3x-1}{x-2}$ has asymptotes at

A $x = 2$ and $y = 3$

B $x = \dfrac{1}{3}$ and $x = 2$

C $x = 2$ and $y = 5$

D $x = 2$ and $y = -1$

E $x = -2$ and $y = -\dfrac{3}{2}$

Section B 4 questions

15 marks

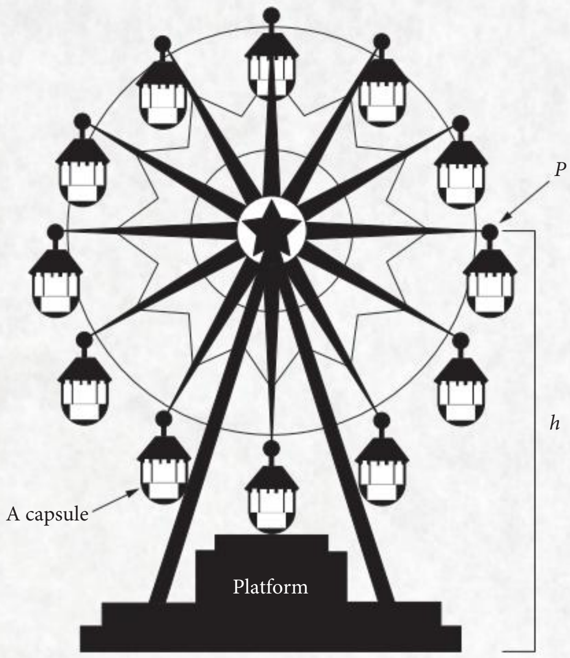
1 © VCAA 2014 2BQ1 (5 marks) The population of wombats in a particular location varies according to the rule $n(t) = 1200 + 400\cos\left(\frac{\pi t}{3}\right)$, where n is the number of wombats and t is the number of months after 1 March 2013.

a Find the period and amplitude of the function n. 2 marks

b Find the maximum and minimum populations of wombats in this location. 2 marks

c Find $n(10)$. 1 mark

2 © VCAA 2017 2BQ2 (4 marks) Sammy visits a giant Ferris wheel. Sammy enters a capsule on the Ferris wheel from a platform above the ground. The Ferris wheel is rotating anticlockwise.
The capsule is attached to the Ferris wheel at point P. The height of P above the ground, h, is modelled by $h(t) = 65 - 55\cos\left(\frac{\pi t}{15}\right)$, where t is the time in minutes after Sammy enters the capsule and h is measured in metres. Sammy exits the capsule after one complete rotation of the Ferris wheel.



a State the minimum and maximum heights of P above the ground. 1 mark

b For how much time is Sammy in the capsule? 1 mark

c © VCAA MODIFIED If the radius of the Ferris wheel is 5 metres, what is the arc length between any two consecutive capsules? Give the answer to the nearest centimetre. 2 marks

3 (5 marks) A container is filled with water so that the time, t seconds, taken to fill the container to a height of h mm is given by the equation $t = \sqrt{h^3}$.

a Determine the average rate of change of height with respect to time between the 1st and the 27th second. Give the answer to two decimal places. 2 marks

b Find an approximation from the left for the instantaneous rate of change of height with respect to time midway between the 1st and the 27th second. Use a time step size of 0.1 and give the answer to two decimal places. 2 marks

c Comment on your answers to parts **a** and **b**. Explain with reference to the shape of the graph. 1 mark

4 (1 mark) A coin is tossed so that $3 \times \Pr(\text{head}) = 2 \times \Pr(\text{tail})$. If the coin is tossed twice, find the probability of getting a head and a tail in no particular order.

CHAPTER

10 DIFFERENTIATION

STUDY DESIGN COVERAGE

Study Design coverage

UNIT 2, AREA OF STUDY 3: CALCULUS

- informal treatment of the gradient of the tangent to a curve at a point as a limit, and the limit definition of the derivative of a function $f'(x) = \lim_{h \to 0} \frac{f(x) - f(x-h)}{h} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$
- the central difference approximation $f'(x) \approx \frac{f(x+h) - f(x-h)}{2h}$ and its graphical interpretation
- the derivative as the gradient of the graph of a function at a point and its representation by a gradient function, and as a rate of change
- differentiation of polynomial functions by rule.

Video playlists (6):

10.1 Continuity and limits
10.2 Differentiation by first principles
10.3 Differentiating polynomial functions
10.4 The central difference
10.5 Equation of a tangent
VCE question analysis Differentiation

Worksheets (9):

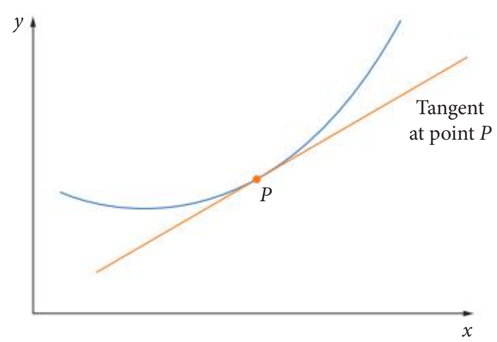
10.1 Limits
10.2 Finding derivatives from first principles • Differentiation from first principles
10.3 Derivatives of linear products • Derivative of a sum of terms • Derivatives of polynomials • Basic differentiation
10.5 Slopes of curves • Tangents to a curve

Nelson MindTap

To access resources above, visit **cengage.com.au/nelsonmindtap**

10.1 Continuity and limits

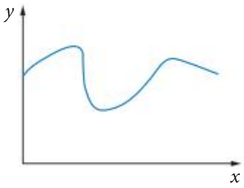
Video playlist Continuity and limits



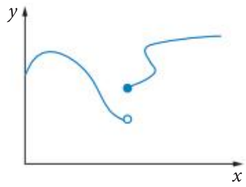
The **rate of change** describes how one quantity varies as a result of a change in another quantity. The growth of a plant and blood pressure can provide a rate of change when each is monitored over time.

In Chapter 6, *Rates of change*, we saw that the **instantaneous rate of change** of a function at a point $P(x, y)$ is the **gradient** of the **tangent** at that point.

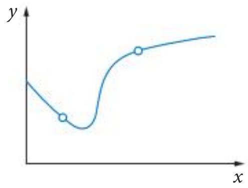
A function is **continuous** if its graph has no gaps and it can be drawn without taking the pen off the paper. Vertical asymptotes can also provide 'gaps'.



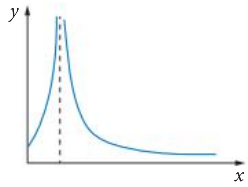
Continuous



Not continuous
('jumps')



Not continuous
('gaps')



Not continuous
('gap at a vertical asymptote')

For a rate of change to exist for a particular value of x, the function must be **continuous** at that value of x.

Worksheet
Limits

Limits

What happens to the function $f(x) = x + 1$ as $x \to 3$?

As we approach $x = 3$ **from the left**, we look at x values less than 3, such as 0, 1, 2.9 and 2.999, and find the corresponding values for $f(x)$: 1, 2, 3.9 and 3.999.

Thus, as x approaches 3 from the left, $f(x)$ approaches $3 + 1 = 4$. We write this as $\lim\limits_{x \to 3^-} f(x) = 4$.

Similarly, if we approach $x = 3$ **from the right**, we use any x value greater than 3, such as 3.1, 3.01 and 3.001, and find the corresponding values for $f(x)$: 4.1, 4.01 and 4.001.

As x approaches 3 from the right, $f(x)$ approaches $3 + 1 = 4$. This is written as $\lim\limits_{x \to 3^+} f(x) = 4$.

In general, if $f(x) \to L$ as $x \to a^-$ and $f(x) \to L$ as $x \to a^+$, we write the **limit**:

$$\lim_{x \to a} f(x) = L$$

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WORKED EXAMPLE 1 Limit of a function 1

Find the value of $\lim\limits_{x \to 4} f(x)$, $f(x) = \dfrac{x^2 - 16}{x - 4}$.

Steps	Working
1 Simplify to obtain a linear function.	$f(x) = \dfrac{x^2 - 16}{x - 4} = \dfrac{(x-4)(x+4)}{x - 4} = x + 4, \quad x \neq 4$
2 Decide if the limit is the same from the left and right sides.	$\lim\limits_{x \to 4^-} f(x) = 4 + 4 = 8$ $\lim\limits_{x \to 4^+} f(x) = 4 + 4 = 8$
3 Answer the question.	$\lim\limits_{x \to 4} \dfrac{x^2 - 16}{x - 4} = 8$

To say that $\lim\limits_{x \to a} f(x) = f(a)$ is not always true. It is true, for example, when $f(x) = x + 1$, but false if

$f(x) = \dfrac{x^2 - 16}{x - 4}$, because in this case $f(4) = \dfrac{4^2 - 16}{4 - 4} = \dfrac{0}{0}$, which is undefined.

It is better to simplify the function wherever possible and investigate the limit from the left and from the right.

WORKED EXAMPLE 2 Limit of a function 2

Evaluate $\lim\limits_{x \to 2} \dfrac{1}{x-2}$.

Steps	Working
1 Look at the limit from the left.	As x approaches 2 from the negative side, $x - 2 < 0$, so $\dfrac{1}{x-2}$ becomes larger and more negative. It approaches $-\infty$.
2 Look at the limit from the right.	As x approaches 2 from the positive side, $x - 2 > 0$, so the value of $\dfrac{1}{x-2}$ becomes larger and more positive. It approaches ∞.
3 Decide if the limit exists.	There are two different limits, so the limit does not exist as x approaches 2.

USING CAS 1 Limits

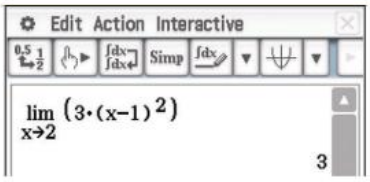
Evaluate $\lim\limits_{x \to 2} 3(x-1)^2$.

TI-Nspire

1 Press **menu > Calculus > Limit**.

2 Enter the expression **3(x–1)²** and the limit of **2** as shown above.

ClassPad



1 Enter and highlight the expression.

2 Tap **Interactive > Calculation > lim**.

3 In the **Point:** field, enter **2** then tap **OK**.

Answer: 3

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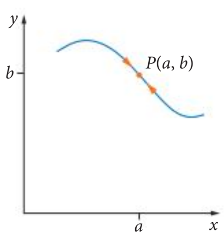
10.1

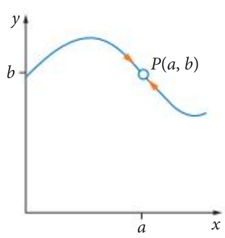
Continuity

For a function to be continuous at $x = a$:

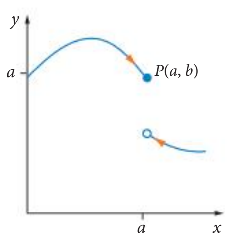
- the function must exist at $x = a$; that is, $f(a) = b$ for a real value of b.
- $\lim\limits_{x \to a} f(x) = f(a)$ when the point is approached from the left and from the right.



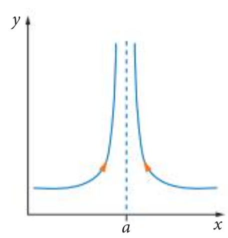
$f(a)$ exists
$\lim\limits_{x \to a} f(x) = f(a)$
continuous at $x = a$



$f(a)$ does not exist
discontinuous (not continuous) at $x = a$



$f(a)$ exists
$\lim\limits_{x \to a^+} f(x) \neq f(a)$
discontinuous at $x = a$



$f(a)$ does not exist
discontinuous at $x = a$

Exam hack

Look for x values for which the denominator of the function becomes zero. These x values will be points of discontinuity or asymptotes.

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WORKED EXAMPLE 3 Testing for a continuous function

Determine whether each function is continuous at the given value.

a $f(x) = \dfrac{x^2 - 25}{x - 5}$ at $x = 5$ **b** $f(x) = \dfrac{1}{x}$ at $x = 2$

Steps	Working
a **1** Check whether $f(5)$ exists.	$f(5) = \dfrac{5^2 - 25}{5 - 5} = \dfrac{0}{0}$, which is undefined because of division by 0.
2 Answer the question.	The function is not continuous at $x = 5$.
b **1** Check whether $f(2)$ exists.	$f(2) = \dfrac{1}{2}$
2 Evaluate $\lim\limits_{x \to 2} f(x)$. When x approaches 2 from the negative side $(x \to 2^-)$, $\dfrac{1}{x}$ becomes closer to $\dfrac{1}{2}$. When x approaches 2 from the positive side $(x \to 2^+)$, $\dfrac{1}{x}$ becomes closer to $\dfrac{1}{2}$.	$\lim\limits_{x \to 2} f(x) = \dfrac{1}{2}$
3 Check whether $\lim\limits_{x \to 2} f(x) = f(2)$.	So $\lim\limits_{x \to 2} f(x) = f(2) = \dfrac{1}{2}$.
4 Answer the question.	The function is continuous at $x = 2$.

EXERCISE 10.1 Continuity and limits

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Mastery

1 WORKED EXAMPLE 1 Find the value of

a $\lim\limits_{x \to 3} f(x), f(x) = x^3 - 4$ **b** $\lim\limits_{x \to -2} \dfrac{4 - x^2}{2 + x}$

2 WORKED EXAMPLE 2 Evaluate

a $\lim\limits_{x \to 1} \sqrt{x - 4}$ **b** $\lim\limits_{x \to -3} \dfrac{5}{2x + 6}$

3 WORKED EXAMPLE 3 Decide whether each function is continuous at the given value.

a $f(x) = \dfrac{-2}{5 + x}$ at $x = -5$.

b $f(x) = \dfrac{3x^2 - 3}{1 + x}$ at $x = 1$.

c $f(x) = -2x$ at $x = 0$.

Exam hack

If there is a value of x that makes the denominator zero, then $f(x)$ is not continuous.

9780170448260

Exam practice

80–100% 60–79% 0–59%

4 TECH-FREE (2 marks) By factorising $x^2 - 7x + 12$, show that $\lim\limits_{x \to 3} \dfrac{x^2 - 7x + 12}{(x-3)(x+4)} = -\dfrac{1}{7}$.

5 TECH-FREE (2 marks) If $f(x) = x^2$ and $g(x) = x + 1$, show that $\lim\limits_{x \to k} \left(f(x) + g(x)\right) = \lim\limits_{x \to k} f(x) + \lim\limits_{x \to k} g(x)$.

6 A point of discontinuity on the curve of the function $f(x) = \dfrac{x+2}{5(3x+12)}$ is

A $x = -12$ **B** $x = -4$ **C** $x = -2$ **D** $x = 0$ **E** $x = 5$

7 For the function $f(x) = \dfrac{1}{x^2}$

A $f(x)$ is continuous at $x = 0$

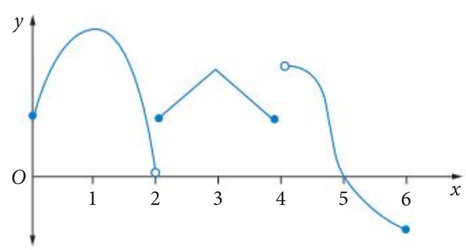
B $f(x)$ is not continuous at $x = 1$

C $\lim\limits_{x \to 0} f(x) = 1$

D $\lim\limits_{x \to 0} f(x) = 0$

E $f(x)$ is discontinuous at $x = 0$

8 © VCAA 2008 2BQ22 53% The graph of the function f with domain [0, 6] is shown below.



Which one of the following is **not** true?

A The function is not continuous at $x = 2$ and $x = 4$.

B The function exists for all values of x between 0 and 6.

C $f(x) = 0$ for $x = 2$ and $x = 5$.

D The function is positive for $x \in [0, 5)$.

E The gradient of the function is not defined at $x = 4$.

9 $\lim\limits_{x \to -3} (x+2) + \lim\limits_{x \to 2} \left(\dfrac{x(x-2)}{x-2}\right)$ is

A -3 **B** -1 **C** 1 **D** 2 **E** 4

10 Using CAS 1 (3 marks) An approximation to Euler's number $e = 2.718\,28\ldots$ can be obtained using $e = \lim\limits_{n \to \infty} \left(1 + \dfrac{1}{n}\right)^n$, where n is a positive whole number.

Show that the percentage error using $n = 20$ is approximately 2.39%.

11 (2 marks) Explain whether $f(x) = 2^x$ is continuous in the interval [0, 3].

12 (2 marks) Determine if the function $f(x) = -\dfrac{1}{4(2x-1)}$ is continuous in the interval (0, 1).

13 (2 marks)

a State the point of discontinuity of the function $f(x) = \dfrac{1}{x}$. 1 mark

b Explain why the limit of the function does not exist at the point of discontinuity. 1 mark

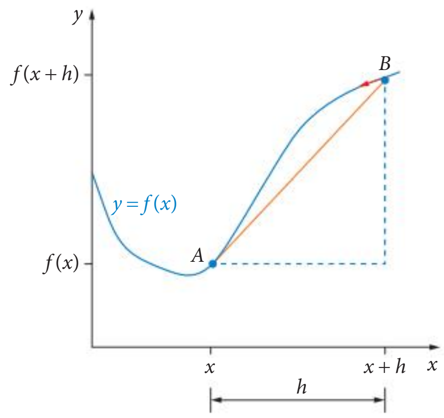
Video playlist
Differentiation by first principles

Worksheets
Finding derivatives from first principles

Differentiation from first principles

10.2 Differentiation by first principles

Suppose we want to find the instantaneous rate of change at point A shown on the graph. We can use limits to see what happens to the gradient of the chord AB as B moves closer to A.



Points A and B lie on the curve $y = f(x)$ and have coordinates $A(x, f(x))$ and $B(x + h, f(x + h))$ respectively.

The gradient of the line connecting A to B is $\dfrac{f(x+h) - f(x)}{h}$.

As B approaches A, h becomes smaller. In the limiting case, we let h approach 0 and AB becomes the tangent to the curve at A.

We write $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$, where $f'(x)$ is the **gradient function** or the **derivative** of $f(x)$. This is a function that gives the gradient of the tangent (or instantaneous rate of change) at the point $(x, f(x))$. Finding the gradient function in this way is called **differentiation by first principles**.

$f'(x)$ is also written as $\dfrac{dy}{dx}$, meaning the derivative of y with respect to x. The 'd' stands for 'difference', so the derivative can be interpreted as 'difference in y' over 'difference in x'.

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WORKED EXAMPLE 4 Differentiation by first principles

Use differentiation by first principles to find the gradient function for $f(x) = 4x^2$.

Steps	Working
1 Write the function.	$f(x) = 4x^2$
2 Find an expression for $f(x + h)$. Expand and simplify.	$f(x+h) = 4(x+h)^2$ $= 4x^2 + 8xh + 4h^2$
3 Find an expression for $\dfrac{f(x+h) - f(x)}{h}$ and simplify.	$\dfrac{f(x+h) - f(x)}{h} = \dfrac{(4x^2 + 8xh + 4h^2) - 4x^2}{h}$ $= \dfrac{8xh + 4h^2}{h}$ $= 8x + 4h$
4 Find $\lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$.	$\lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h} = \lim\limits_{h \to 0}(8x + 4h)$ $= 8x + 4 \times 0$ $= 8x$
5 State the gradient function.	The derivative of $f(x) = 4x^2$ is $8x$.

WORKED EXAMPLE 5 Derivative at a point by first principles

Use differentiation by first principles to find $f'(x)$ for $f(x) = -x^3$, then evaluate $f'(-3)$.

Steps	Working
1 Write the function.	$f(x) = -x^3$
2 Find an expression for $f(x+h)$. Expand and simplify.	$f(x+h) = -(x+h)^3$ $= -x^3 - 3x^2h - 3xh^2 - h^3$
3 Find an expression for $\frac{f(x+h)-f(x)}{h}$ and simplify.	$\frac{f(x+h)-f(x)}{h} = \frac{(-x^3 - 3x^2h - 3xh^2 - h^3) - (-x^3)}{h}$ $= \frac{-3x^2h - 3xh^2 - h^3}{h}$ $= -3x^2 - 3xh - h^2$
4 Find $f'(x)$.	$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ $= \lim_{h\to 0}(-3x^2 - 3xh - h^2)$ $= -3x^2 - 3x \times 0 - 0^2$ $= -3x^2$
5 Evaluate $f'(x)$ using the given value.	$f'(-3) = -3 \times (-3)^2 = -27$

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10.2

Exam hack

When simplifying $\frac{f(x+h)-f(x)}{h}$, all terms in $f(x)$ always cancel with terms from $f(x+h)$.

If this does not happen, check your working.

EXERCISE 10.2 Differentiation by first principles

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Recap

1 The function $f(x) = \frac{3(x-2)}{4(x^2-9)}$ is not continuous at

A $x = 2$ **B** $x = 3, x = -3$ **C** $x = 4$ **D** $x = 6$ **E** $x = 9$

2 $\lim_{x\to 3}\left(\frac{3-x}{x}\right) + \lim_{x\to -3}\left(x^2 - 9\right)$ is

A -18 **B** -6 **C** 0 **D** 6 **E** 12

Mastery

3 WORKED EXAMPLE 4 Use differentiation by first principles to find the gradient function for $f(x) = \frac{1}{3}x^3$.

4 WORKED EXAMPLE 5 Use differentiation by first principles to evaluate $f'\left(\frac{2}{3}\right)$ for $f(x) = 6x$.

Exam practice

5 TECH-FREE (2 marks) Use differentiation by first principles to show that $f'(-1) = 2$ when $f(x) = -2 - x^2$.

6 TECH-FREE (2 marks) Given $f(x) = \frac{x^2 - a^2}{x+a}, a \in R$, use differentiation by first principles to show that $f'(x) = 1$ for all values of $x \neq -a$.

7 TECH-FREE The gradient function of $f(x) = -7x^2$ is

A $-14x^2$ **B** $-14x$ **C** $-7x$ **D** $-2x$ **E** $7x$

8 TECH-FREE For the function $f(x) = 2 - 3x$, $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ simplifies to

A $\lim_{h \to 0} -3$

B $\lim_{h \to 0} -3h$

C $\lim_{h \to 0} \frac{-3x - 3h}{h}$

D $\lim_{h \to 0} \frac{2 + h - 3x}{h}$

E $\lim_{h \to 0} \frac{-3h + 2}{h}$

9 To differentiate $f(x) = (2x + 3)^2$ by first principles, we start with

A $\lim_{h \to 0} \frac{(2x + 2h + 3)^2 - (2x + 3)^2}{h}$

B $\lim_{h \to 0} \frac{(2x + 2h + 3h)^2 - (2x + 3)}{h}$

C $\lim_{h \to 0} \frac{(2x + h + 3)^2 - (2x + 3)^2}{h}$

D $\lim_{h \to 0} \frac{(2x + 3h)^2 - (2x + 3)^2}{h}$

E $\lim_{h \to 0} \frac{(2x + 3h)^2 - (2x + h + 3)^2}{h}$

10 (2 marks) Use differentiation by first principles to find the gradient function of the linear function $f(x) = mx + c$, where m and c are constants.

11 (2 marks) Use differentiation by first principles to find the derivative of the quadratic function $f(x) = ax^2 + bx + c$, where a, b and c are constants.

12 (3 marks) Use differentiation by first principles to find the gradient function of $f(x) = x^4$.

13 (4 marks) Use differentiation by first principles to show that for $f(x) = x^n$, $n = 1, 2, 3 \ldots$ $f'(x) = nx^{n-1}$.

Hint: $(x + h)^n = x^n + nx^{n-1}h + k_2x^{n-2}h^2 + k_3x^{n-3}h^3 + \ldots + h^n$, where n, k_2, k_3 … are constants.

10.3 Differentiating polynomial functions

Conditions for differentiability

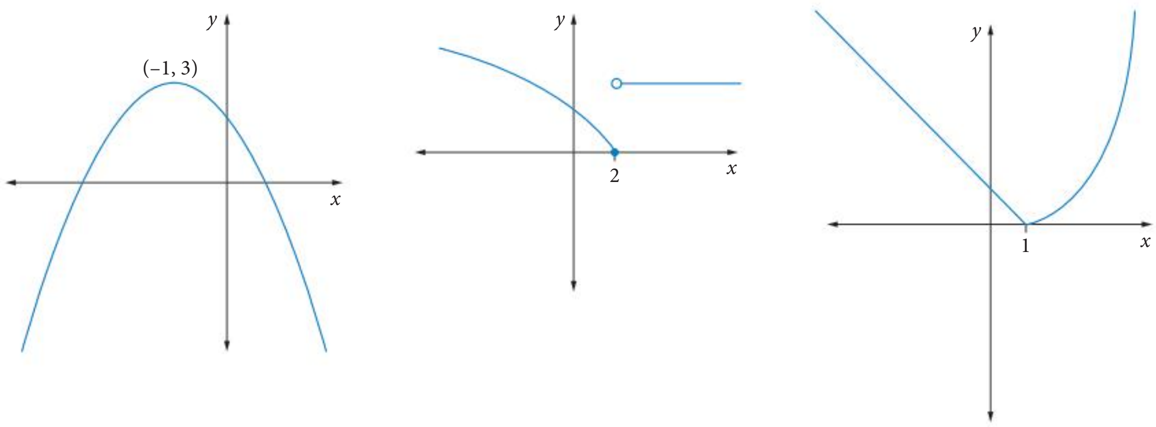
Not all functions have a derivative at all points in their domain. The functions $y = \sqrt{x}$ and $y = \frac{1}{x}$ are not differentiable at $x = 0$.

Conditions for differentiability

For a function to be differentiable at the point where $x = a$, two conditions must be met:

- $f(x)$ must be continuous at $x = a$.
- $f(x)$ must have a unique tangent at $x = a$; that is, the tangent at $x = a$ is the same when $x = a$ is approached from either side.

Because $f'(x)$ measures the gradient of the tangent at a point on $y = f(x)$, the derivative won't exist at points where the function is discontinuous, where the tangent cannot be drawn or where the tangent is vertical (so its gradient is undefined).



Continuous at $x = -1$
Differentiable at $x = -1$

Discontinuous at $x = 2$
No unique tangent at $x = 2$
Not differentiable at $x = 2$

Continuous but with a sharp point at $x = 1$
No unique tangent at $x = 1$
Not differentiable at $x = 1$

Exam hack

One way of determining whether a function is differentiable at a point is to imagine its graph as a railway track, and whether a train could travel along it safely. If it is continuous and smooth, then the train can run on it and the function is differentiable. If there are gaps, sharp corners or vertical sections, then the train can't run on it and the function is not differentiable.

Video playlist Differentiating polynomial functions

Worksheets Derivatives of linear products

Derivative of a sum of terms

Derivatives of polynomials

Basic differentiation

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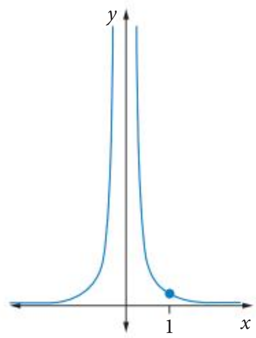
WORKED EXAMPLE 6 Checking for differentiability

Decide if each function is differentiable at the given point.

a $f(x) = \dfrac{(5-x)(5+x)}{5-x}$ at $x = 5$

b $f(x) = \dfrac{1}{x^2}$ at $x = 1$

Steps	Working
a 1 Simplify the function.	$f(x) = \dfrac{(5-x)(5+x)}{5-x} = 5 + x$, but the denominator cannot be zero, so $x \neq 5$.
2 Decide if the function is continuous.	The function is not continuous at $x = 5$.
3 Decide if the function is differentiable.	Since $f(x)$ is not continuous at $x = 5$, it is not differentiable at $x = 5$.
b 1 Decide if the function is continuous at $x = 1$.	 $y = \dfrac{1}{x^2}$ $f(x)$ is continuous at $x = 1$.
2 Decide if the function has a unique tangent at $x = 1$.	There is only one tangent line that can be drawn at $x = 1$.
3 Decide if the function is differentiable.	$f(x)$ is differentiable.

Differentiating powers of *x*

When we differentiate powers of x by first principles, we obtain the derivatives shown in the table.

$f(x)$	$f'(x)$
x	1
x^2	$2x$
x^3	$3x^2$
x^4	$4x^3$

Can you see a pattern in finding the derivative of $f(x) = x^n$, where $n = 1, 2, 3 \ldots$?

To differentiate any power of x, make the power become the coefficient of x and subtract 1 from the power.

The derivative of x^n

If $f(x) = x^n$, then $f'(x) = nx^{n-1}$.

For example, for $f(x) = x^{99}$, $f'(x) = 99x^{98}$.

What if there is a coefficient of x other than 1 involved, such as in $f(x) = 4x^2$?

From Worked example 4 on page 382, we found that $f'(x) = 8x$.

Notice that $f'(x) = 4 \times 2x^1 = 8x$

The derivative of ax^n

If $f(x) = ax^n$, then $f'(x) = anx^{n-1}$.

To differentiate $y = ax^n$, multiply the coefficient a by the power and subtract 1 from the power.

For example, the derivative of $f(x) = 3x^4$ is $f'(x) = 3 \times 4x^{4-1} = 12x^3$.

Furthermore, the derivative of a constant (number), c, is 0, because $c = c \times 1 = cx^0$, so its derivative is $c \times 0x^{-1} = 0$.

The derivative of a constant

If $f(x) = c$, then $f'(x) = 0$.

For example, the derivative of $f(x) = 7$ is $f'(x) = 0$.

This makes sense because if a function is a constant (number), such as $y = 7$, then its graph is a horizontal (flat) line, so its gradient is 0 at every point.

Differentiating polynomial functions

We can differentiate polynomial functions by differentiating each term separately. This is called **term-by-term differentiation**.

For example, if $y = 4x^3 - x^2 + 5$:

$\frac{d}{dx}$ is another way of writing the derivative.

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(4x^3) + \frac{d}{dx}(-x^2) + \frac{d}{dx}(5)\\ &= 4 \times 3x^{3-1} + (-2 \times x^{2-1}) + 0\\ &= 12x^2 - 2x\end{aligned}$$

The derivative of a constant is always 0.

WORKED EXAMPLE 7 Term-by-term differentiation

Differentiate each function.

a $f(x) = -x^8$ **b** $f(x) = 5x^4 - \frac{3}{2}x^2 + 1$

Steps	Working
a Differentiate using $f'(x) = a \times nx^{n-1}$.	$f(x) = -x^8$ $f'(x) = -1 \times 8x^{8-1} = -8x^7$
b Differentiate each term separately using $f'(x) = anx^{n-1}$.	$f(x) = 5x^4 - \frac{3}{2}x^2 + 1$ $f'(x) = 5 \times 4x^{4-1} - \frac{3}{2} \times 2x^{2-1} + 0$ $= 20x^3 - 3x$

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The rule $f'(x) = anx^{n-1}$ applies to all types of **powers of** x (integers, fractions, surds and irrationals).
To differentiate functions involving square roots and other types of roots, write them as fractional powers of x first. For example, write $f(x) = 2\sqrt{x^3}$ as $f(x) = 2x^{\frac{3}{2}}$ and differentiate to get $f'(x) = 3x^{\frac{1}{2}}$, which can be written as $f'(x) = 3\sqrt{x}$.
We can also differentiate functions that use variables other than x and y.

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WORKED EXAMPLE 8 Differentiation using different variables

Differentiate each function.

a $f(t) = -5t^{-6}$ **b** $y = \dfrac{15}{2u^{\frac{2}{3}}}$ **c** $v = r\sqrt{2r^{\frac{3}{2}}}$

Steps	Working
a Differentiate with respect to t.	$f(t) = -5t^{-6}$ $f'(t) = -5 \times \left(-6t^{-6-1}\right)$ $= 30t^{-7}$ $\left(\text{or } \dfrac{30}{t^7}\right)$
b **1** Write the function in the form $y = au^n$.	$y = \dfrac{15}{2u^{\frac{2}{3}}} = \dfrac{15}{2}u^{-\frac{2}{3}}$
2 Differentiate with respect to u.	$\dfrac{dy}{du} = \dfrac{15}{2} \times \left(-\dfrac{2}{3}u^{-\frac{2}{3}-1}\right)$ $= -5u^{-\frac{5}{3}}$
c **1** Write the function in the form $v = ar^n$.	$v = r\sqrt{2r^{\frac{3}{2}}} = r\left(2r^{\frac{3}{2}}\right)^{\frac{1}{2}} = r \times 2^{\frac{1}{2}}r^{\frac{3}{4}}$ $= \sqrt{2}r^{\frac{7}{4}}$
2 Differentiate using $v'(r) = anr^{n-1}$.	$v'(r) = \sqrt{2} \times \dfrac{7}{4}r^{\frac{7}{4}-1} = \dfrac{7\sqrt{2}r^{\frac{3}{4}}}{4}$

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WORKED EXAMPLE 9 The derivative at a given point

Calculate $f'(x)$ for the given value of x.

a $f(x) = 4x^2 - 2x + 1$, $f'\left(\dfrac{1}{2}\right)$ **b** $f(x) = \dfrac{x^{\frac{7}{6}}}{x^{\frac{2}{3}}}$, $f'\left(\dfrac{9}{16}\right)$

Steps	Working
a **1** Differentiate each term.	$f(x) = 4x^2 - 2x + 1$ $f'(x) = 8x - 2$
2 Evaluate $f'\left(\dfrac{1}{2}\right)$.	$f'\left(\dfrac{1}{2}\right) = 8 \times \dfrac{1}{2} - 2 = 2$

b **1** Simplify $f(x)$ using index laws.

$$f(x) = \frac{x^{\frac{7}{6}}}{x^{\frac{2}{3}}} = x^{\frac{7}{6}-\frac{2}{3}} = x^{\frac{1}{2}}$$

2 Differentiate using $f'(x) = anx^{n-1}$.

$$f'(x) = \frac{1}{2}x^{\frac{1}{2}-1}$$
$$= \frac{1}{2}x^{-\frac{1}{2}} \quad \left(\text{or } \frac{1}{2\sqrt{x}}\right)$$

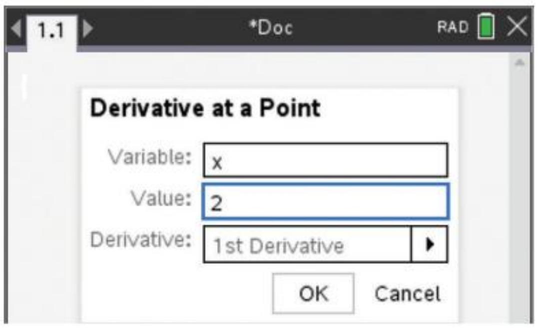
3 Evaluate $f'\left(\frac{9}{16}\right)$.

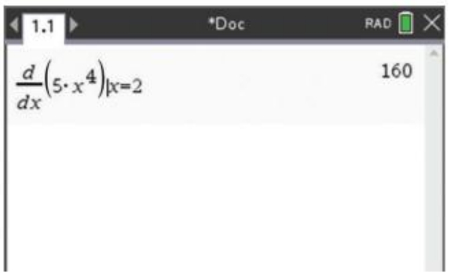
$$f'\left(\frac{9}{16}\right) = \frac{1}{2}\left(\frac{9}{16}\right)^{-\frac{1}{2}}$$
$$= \frac{1}{2\sqrt{\frac{9}{16}}}$$
$$= \frac{1}{2 \times \frac{3}{4}}$$
$$= \frac{2}{3}$$

USING CAS 2 Differentiation

Find $f'(2)$ for $f(x) = 5x^4$.

TI-Nspire



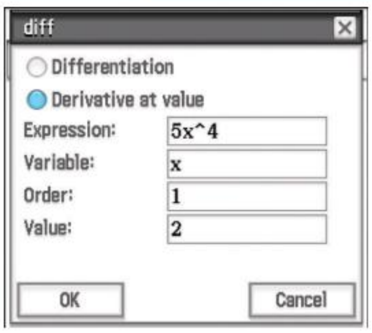


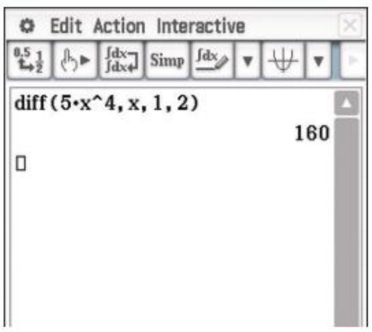
1 Press **menu > Calculus > Derivative at a Point**.

2 In the dialogue box, enter the value **2**.

3 In the derivative template, enter the expression **5x⁴**.
The value of $f'(2)$ will be displayed.

ClassPad





1 In **Main**, enter and highlight the expression **$5x^4$**.

2 Tap **Interactive > Calculation > diff > Derivative at value**.

3 In the **Value:** field, enter **2**.

4 Tap **OK**. The value of $f'(2)$ will be displayed.

$f'(2) = 160$

WB
p. 199

WORKED EXAMPLE 10 Finding a constant given the derivative at a given point

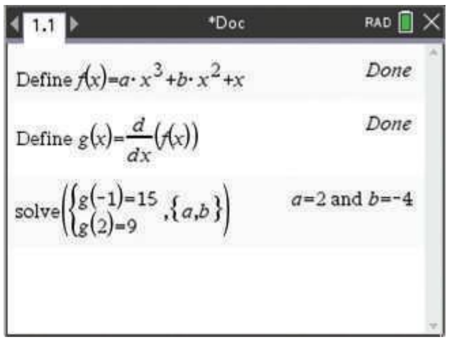
For the function $f(x) = x^4 + ax^2 + 5$, $f'(-2) = -44$. Find the value of the constant, a.

Steps	**Working**
1 Find the derivative of $f(x)$.	Differentiate $f(x)$ term by term. $f'(x) = 4x^3 + 2ax$
2 Form an equation.	$f'(-2) = 4(-2)^3 + 2a \times (-2)$ $= -32 - 4a$ $f'(-2) = -44 \Rightarrow -32 - 4a = -44$
3 Solve for the unknown.	Solve the equation to find a. $-32 - 4a = -44$ $-4a = -12$ $a = 3$

USING CAS 3 Finding the function from the derivative

For the function $f(x) = ax^3 + bx^2 + x$, $f'(-1) = 15$ and $f'(2) = 9$. Find the value of the constants a and b.

TI-Nspire



1 Define the function $\boldsymbol{f(x) = a \cdot x^3 + b \cdot x^2 + x}$. Be sure to insert multiplication signs after the a and the b.

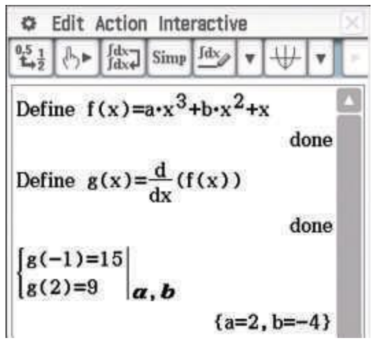
2 Define **g(x)** as the derivative of **f(x)**.

3 Press **menu > Algebra > Solve System of Equations > Solve System of Equations**.

4 In the dialogue box, change the **Variables:** to **a,b**.

5 Enter the two derivative equations into the template as shown above.

ClassPad



1 Define the function $\boldsymbol{f(x) = ax^3 + bx^2 + x}$ using **Var** for variables. There is no need to insert multiplication signs when using **Var**.

2 Define **g(x)** as the derivative of **f(x)**.

3 Open the **Keyboard > Math1** and insert the **simultaneous equations** template.

4 Enter the two derivative equations and the variables **a,b** using **Var** into the template as shown above.

The function is $f(x) = 2x^3 - 4x^2 + x$.

EXERCISE 10.3 Differentiating polynomial functions

ANSWERS p. 528

Recap

1 If $f(x) = 3x^2 + x$, then $\dfrac{f(x+h) - f(x)}{h}$ is

A $3x + h + 1$

B $6x + 1$

C $3x + 3h$

D $6x + 3h + 1$

E $6x + h + 1$

2 A possible step in finding the gradient function of $f(x) = 1 - x^2$ is

A $\lim\limits_{h \to 0} \dfrac{(x+h)^2 + 1 - x^2}{h}$

B $\lim\limits_{h \to 0} -\dfrac{(x+h)^2 + x^2}{h}$

C $\lim\limits_{h \to 0} \dfrac{(x-h)^2 - x^2}{h}$

D $\lim\limits_{h \to 0} \dfrac{(x+h)^2 + 1 + x^2}{h}$

E $\lim\limits_{h \to 0} \dfrac{(x-h)^2 + x^2}{h}$

Mastery

3 WORKED EXAMPLE 6 TECH-FREE Decide if each function is differentiable at the given point.

a $f(x) = \dfrac{x(x+2)}{x+2}$ at $x = 2$

b $f(x) = \dfrac{-3}{x^2 - 5}$ at $x = \sqrt{5}$

c $f(x) = \tan(x)$ at $x = \pi$

4 WORKED EXAMPLE 7 TECH-FREE Differentiate each function.

a $y = x^3 + x^2 + x$

b $y = 6x^7 + 10x^9 - 5x^{12} + 8x^6$

c $y = 14 - 2x^6 + 9x^8 + x^{11}$

d $y = 3x^8 - 5x^{12} + 31$

e $y = 6x^{12} + 11x^{13} - 24x^5$

f $y = 12x^3 + 7x^4 - 11x^6 + 38$

g $y = 4x^8 - 15x + 28x^2 + 14x^3$

h $y = 10 - 6x + 7x^2 + 14x^3 - 9x^4$

i $y = 15x^8 - 7x^5 + 4x^7 + 12x - 21$

j $y = 16x^2 - 9x^4 + 7x^3 + 18x - 32$

5 WORKED EXAMPLE 8 TECH-FREE Differentiate each function.

a $f(x) = 4x^{-3}$

b $f(x) = -\dfrac{9x^{\frac{4}{9}}}{4}$

c $f(x) = \dfrac{4x}{\sqrt{x}}$

6 WORKED EXAMPLE 9 TECH-FREE Calculate $f'(x)$ for the given value of x.

a $f(x) = -2x^2 + \dfrac{2}{3}x - 1, f'\left(\dfrac{1}{8}\right)$

b $f(x) = \dfrac{\sqrt{x^3} + x}{2x}, f'(25)$

7 Using CAS 2 Use CAS to find the derivative of $y = 2x^2 - \dfrac{1}{2x^2}$ and evaluate $\dfrac{dy}{dx}$ at $x = 6$.

8 WORKED EXAMPLE 10 TECH-FREE For the function $f(x) = 3 + ax^2 - x^4$, $f'(-1) = -2$. Find the value of the constant, a.

9 Find the values of the constants a and b given that for the function $f(x) = (ax + b)^2$, $a + b > 0$, $f(1) = 16$ and $f'(1) = 40$.

10 Find each derivative.

a $\frac{d}{dx}(3x^4 - 6x^2 + 18)$

b $\frac{d}{du}(3u^4 - 6u^2 + 18)$

c $\frac{d}{dm}(5m^6 + 8m^5 - 24m + 8)$

d $\frac{d}{dy}(9y + 6y^5 - 4y^3 + 2y^7)$

e $\frac{d}{dz}(3z^2 - 9z^3 + 38z - 5)$

f $\frac{d}{dt}(5t^7 + 9t^4 - 16t)$

g $\frac{d}{dr}(21 - 13r - 6r^2 + 9r^3)$

h $\frac{d}{da}(-6a^4 + 2a^2 - 10)$

i $\frac{d}{dp}(8p - 3p^4 - 6p^7 - 5p^9)$

j $\frac{d}{dt}(8 + 4t^2 - 2t^3 + 11t^4)$

Exam practice

80–100% 60–79% 0–59%

11 TECH-FREE (2 marks) Find the values of a and b in the function $f(x) = ax^2 + bx + 1$ given that $f(1) = 0$ and $f'(-1) = -10$.

12 TECH-FREE (3 marks) Determine the derivative of the function $f(x) = \frac{5(x-2)(x^2+2x+4)(x+3)}{x^2+x-6}$ by first simplifying $f(x)$.

13 The function $\frac{x-2}{x^2-2x}$ is not differentiable at the points

A $x = 0, x = 2$
B $x = 0, x = -2$
C $x = 0$ only
D $x = 2$ only
E in the interval $[0, 2]$

14 The derivative of $0.25x^{12}$ is

A $3x^{11}$
B $0.25x^{11}$
C $2.75x^{11}$
D $3x^{13}$
E $2.75x^{12}$

15 For $f(x) = 1 + 2x - \frac{1}{4}x^8 - \frac{5}{18}x^9$, $f'(x)$ is

A $3 - 2x^8 - \frac{5}{2}x^8$
B $2 - \frac{1}{32}x^7 - 2.5x^8$
C $1 - 2x^7 - 2.5x^8$
D $2 - \frac{1}{4}x^7 - 2.5x^8$
E $2 - 2x^7 - 2.5x^8$

16 If $f(x) = \sqrt{x} - \frac{x^2}{10}$, then $f'(4)$ is

A -4.65
B -3.05
C -0.55
D 1.45
E 2.75

17 For $y = \sqrt{x}\left(\frac{2}{x} + \sqrt{x^3}\right)$, $\frac{dy}{dx}$ at $x = 4$ is

A $2\frac{3}{5}$
B $5\frac{3}{4}$
C $6\frac{1}{2}$
D $7\frac{7}{8}$
E $8\frac{1}{3}$

18 © VCAA 2004 non-CAS 1IQ19 31% If $y = x^2 + 2x + 4$, the rate of change of y with respect to x at $x = k$ is

A $2k + 2$
B $2k + 6$
C $k^2 + 2k$
D $k^2 + 2k + 4$
E $\frac{k^3}{3} + k^2 + 4k$

19 If $y = 3x^2 + 7$ and $z = 9 - 4x$, which of the following is the derivative of $y + z$?

A $6x - 4$ **B** $2x$ **C** $4 + 6x$ **D** $2x + 16$ **E** $-24x$

20 If $f(x) = 3x - 5$ and $g(x) = 2x + 4$, which of the following is the derivative of $f(x) \times g(x)$?

A $6x + 1$ **B** -6 **C** $12x - 2$ **D** 6 **E** $2(6x + 1)$

21 (2 marks) Find the derivative of $g(a) = (a - 6)^2$.

22 © VCAA 2011 2BQ3 (2 marks) Consider the function $f: R \rightarrow R, f(x) = 4x^3 + 5x - 9$.

a 95% Find $f'(x)$. 1 mark

b 47% Explain why $f'(x) \geq 5$ for all x. 1 mark

Exam hack

For part **b**, consider the minimum value of the derivative function.

10.4

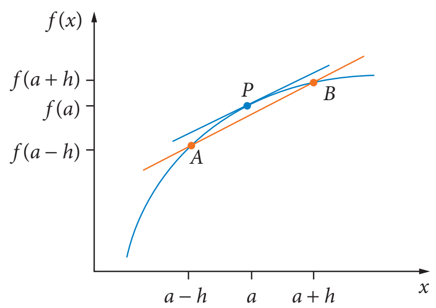
10.4 The central difference

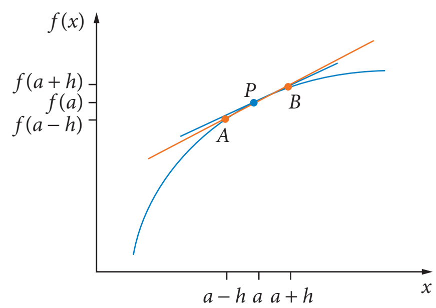
Video playlist
The central difference

Approximating the gradient

In the first diagram below, A, P and B are points on the curve $f(x)$ with coordinates $(a - h, f(a - h))$, $(a, f(a))$ and $(a + h, f(a + h))$ respectively. Here, the **step size** h is some arbitrary small value. The tangent to P is shown, and the line passing through A and B is an approximation to the tangent.

In the second diagram below, by letting the value of h become smaller, A and B move closer to P on the curve and the line passing through A and B becomes a better estimate for the gradient at P.





The gradient of the line connecting A to B is $\dfrac{\text{rise}}{\text{run}} = \dfrac{f(a + h) - f(a - h)}{2h}$. This expression approximates the gradient at a given point, and is called the **central difference formula**.

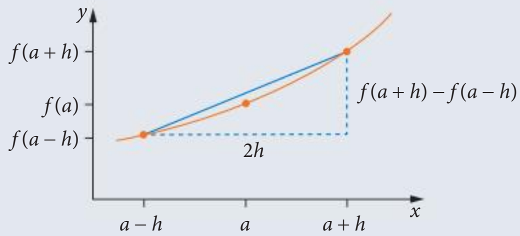
The central difference formula

The central difference formula

An approximation to the gradient, m, of a point $(a, f(a))$ on the curve is found using the **central difference formula**

$$m = \frac{f(a+h) - f(a-h)}{2h}$$

for small values of h.



Note that the limiting case is when h approaches the value 0. The estimate then becomes the **derivative, $f'(x)$**, and provides the exact value of the gradient.

Applying this formula to a point on a displacement–time graph gives an estimate for the instantaneous velocity, and on a velocity–time graph it estimates the instantaneous acceleration.

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WORKED EXAMPLE 11 Applying the central difference formula to find gradient

Use the central difference formula with $h = 0.1$ to find an estimate for the gradient at the point on the curve $f(x) = x^3$ when $x = 2$.

Steps	Working
1 Evaluate $f(a-h)$ and $f(a+h)$ using the known information.	$a = 2, h = 0.1, f(x) = x^3$ $f(a-h) = f(1.9) = 1.9^3 = 6.859$ $f(a+h) = f(2.1) = 2.1^3 = 9.261$
2 Substitute the values into the formula.	$m = \frac{f(a+h) - f(a-h)}{2h}$ $= \frac{9.261 - 6.859}{2 \times 0.1}$ $= 12.01$

Note that this estimate is quite close to the exact value of $f'(2) = 12$. For a smaller value of h, say, 0.01, the estimate for the gradient will be more accurate, about 12.005.

WORKED EXAMPLE 12 Using the central difference formula to find velocity and acceleration

The position, $f(t)$ metres, of a particle at time t seconds is described by the equation $f(t) = t^2 - 2t + 1$.

a Use a value of $h = 0.01$ to estimate the particle's instantaneous velocity after

i 10 seconds **ii** 20 seconds

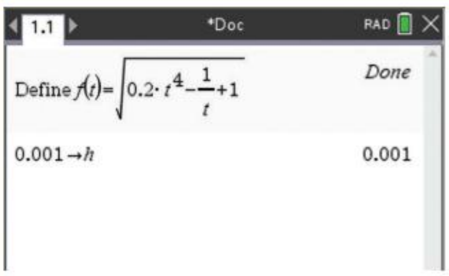
b Hence, estimate the particle's average acceleration from 10 seconds to 20 seconds.

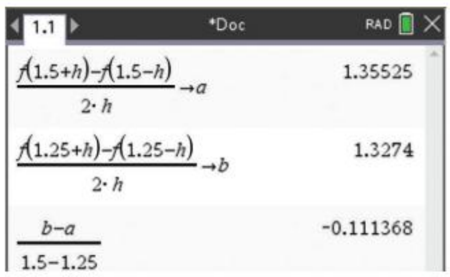
Steps	Working
a **i** Use the central difference formula to estimate the gradient at $t = 10$ s.	$a = 10,\ h = 0.01,\ f(t) = t^2 - 2t + 1.$ $f(a-h) = f(9.99) = 80.8201$ $f(a+h) = f(10.01) = 81.1801$ $m = \dfrac{81.1801 - 80.8201}{2 \times 0.01} = 18$
ii Use the central difference formula to estimate the gradient at $t = 20$ s.	$a = 20$ $f(a-h) = f(19.99) = 360.6201$ $f(a+h) = f(20.01) = 361.3801$ $m = \dfrac{361.3801 - 360.6201}{2 \times 0.01} = 38$
b Use the estimated values for the instantaneous velocity to obtain the average acceleration.	$\dfrac{\Delta v}{\Delta t} = \dfrac{\text{change in velocity}}{\text{change in time}}$ $= \dfrac{38 - 18}{20 - 10}$ $= 2 \text{ m/s}^2$ The average acceleration is 2 m/s^2.

USING CAS 4 Finding acceleration using the central difference

The displacement $f(t)$ cm of a moving object at time t hours is $f(t) = \sqrt{0.2t^4 - \dfrac{1}{t} + 1}$. Using a value of $h = 0.001$, obtain an approximation for the average acceleration of the object between $t = 1.25$ hours and $t = 1.5$ hours. Give your answer correct to 2 decimal places.

TI-Nspire





1 Press **menu > Algebra > Define** and define ***f*(*t*)** as shown above.

2 Enter **0.001** and **store** into ***h***.

3 Apply the central difference formula for ***t* = 1.5** and **store** into ***a***.

4 Repeat for ***t* = 1.25** and **store** into ***b***.

5 Use the formula $m = \dfrac{y_2 - y_1}{x_2 - x_1}$ to determine the approximate gradient.

The average acceleration of the object between $t = 1.25$ hours and $t = 1.5$ hours is 0.11 cm/h^2.

10.4

WB

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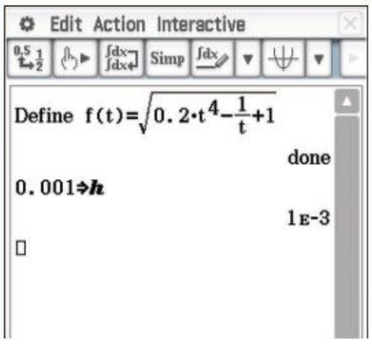
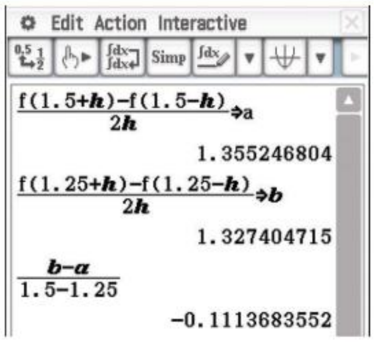
ClassPad





1 Enter and highlight the expression in ***t***.

2 Tap **Interactive > Define** to define ***f*(*t*)**, changing your variable to ***t***.

3 Enter **0.001** and **store** as ***h***.

4 Apply the central difference formula for ***t* = 1.5** and store it as ***a***.

5 Repeat for ***t* = 1.25** and **store** it as ***b***.

6 Use the formula $m = \dfrac{y_2 - y_1}{x_2 - x_1}$ to determine the approximate gradient.

(Note when using this storing method, ***h***, ***a*** and ***b*** will remain as these values until you clear variables under **Variable Manager**).

EXERCISE 10.4 The central difference

ANSWERS p. 528

Recap

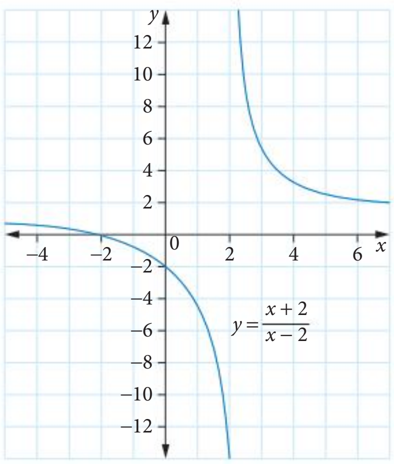
1 If $f(x) = \dfrac{1}{3}a^2x^3 + \dfrac{5}{2}ax^2 + 6x$ and $f'(-1) = 2$, the possible values for the constant a are

A $a = -1, a = 4$ **B** $a = 2, a = 3$ **C** $a = 2, a = -\dfrac{1}{3}$

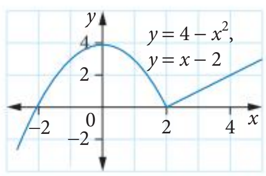
D $a = -2, a = 4$ **E** $a = 2, a = 5$

2 Which graph represents a function that is differentiable at $x = 2$?

A



B



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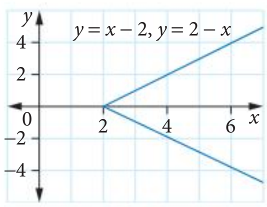

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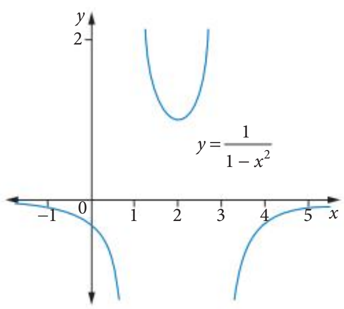
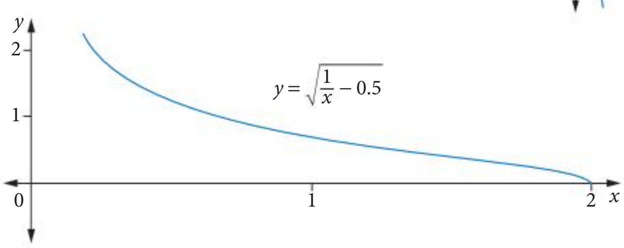


E



Mastery

10.4

3 WORKED EXAMPLE 11 TECH-FREE Estimate the gradient using the central difference formula for the curve $f(x) = 5x^2$ at the point where $x = 4$ using $h = 0.2$.

4 WORKED EXAMPLE 12 TECH-FREE The position, $g(t)$ cm, of a particle at time t seconds is given by $g(t) = 5 - 4t^2$.

a Use a value of $h = 0.04$ to calculate the particle's instantaneous velocity

i after 5 seconds **ii** after 10 seconds

b Hence, estimate the particle's average acceleration from 5 seconds to 10 seconds.

5 Using CAS 4 A particle's position $f(t)$ m at time t seconds is specified by $f(t) = \sqrt{\dfrac{5}{1+t^2}}$. Using a value of $h = 0.005$, obtain an approximation for the average acceleration of the particle between $t = 0.5$ seconds and $t = 2.5$ seconds. Give your answer correct to 2 decimal places.

Exam practice

6 TECH-FREE (2 marks) Find the simplified expression to $\dfrac{f(a+h)-f(a-h)}{2h}$ using $f(x) = x^2 - x + 1$.

7 TECH-FREE (2 marks) Show that if $f(x) = e^x$, then $\dfrac{f(h)-f(-h)}{2h} = \dfrac{e^{2h}-1}{2he^h}$.

8 Using the central difference formula, an estimate for the gradient on the curve $y = \sqrt{x}$ at $x = 10$ using $h = 0.05$ is closest to

A -0.375 **B** -0.144 **C** 0.158 **D** 0.205 **E** 0.212

9 An object's position in metres at time t seconds is described by $g(t) = \sqrt{1+t^2}$. The central difference formula with $h = 0.1$ will give an average velocity (in m/s) at 1 second that is closest to

A 0.39 **B** 0.54 **C** 0.63 **D** 0.71 **E** 0.87

10 For $f(x) = ax + b$, with a and b given constants, the value of $\dfrac{f(a+h)-f(a-h)}{2h}$ is

A $3a$ **B** $-2a$ **C** a **D** $-4a$ **E** $\dfrac{1}{2}a$

11 (2 marks) The estimate for the gradient of the tangent at a point P on the curve $f(x) = x^2$ is 10. For a value of $h = 0.1$, determine the coordinates of P.

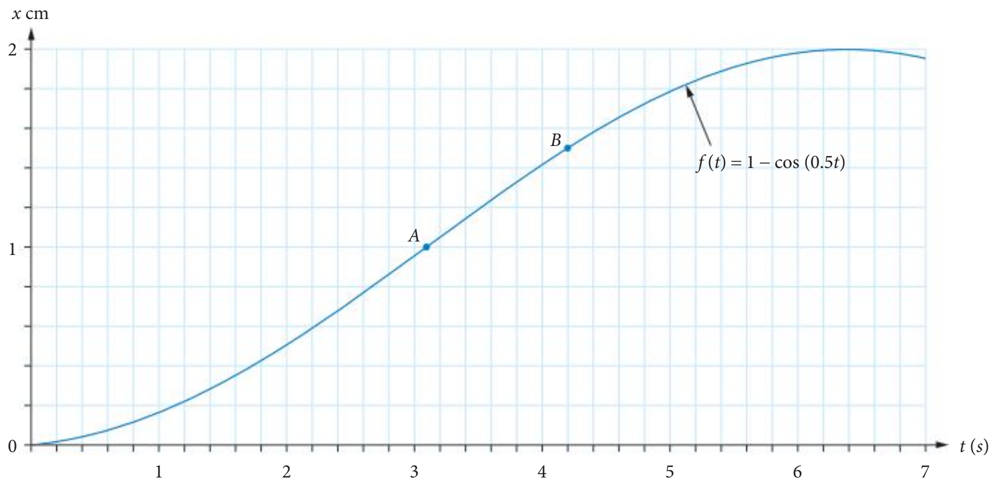
12 (3 marks) A stunt pilot dramatically reduces her air speed, $v(t)$ m/s, according to the velocity–time equation $v(t) = 80e^{-t}$. Correct to 2 decimal places, give an estimate for the instantaneous acceleration of the plane after 5 seconds if a value of $h = 0.5$ is used with the central difference formula.

Exam hack

Determine the sign and estimate the size of the gradient by looking at the graph of the function.

13 (3 marks) Show that for $f(t) = \sqrt{t}$ and for given h, an estimate for the instantaneous velocity at the point $t = h$ is $\dfrac{1}{\sqrt{2h}}$.

14 (5 marks) An object's displacement with respect to time is shown below.



a Find, correct to 3 decimal places, the t values corresponding to points A and B whose displacements are 1 cm and 1.5 cm respectively. 2 marks

b Estimate, using $h = 0.1$, the average acceleration as the particle moves in the time interval $[A, B]$. Give your answer correct to 2 decimal places. 3 marks

15 (8 marks)

a Show that for $f(x) = x$, $\lim_{h \to 0} \dfrac{f(a+h) - f(a-h)}{2h} = 1$. 2 marks

b Show that for $f(x) = x^2$, $\lim_{h \to 0} \dfrac{f(a+h) - f(a-h)}{2h} = 2x$. 2 marks

c Show that for $f(x) = x^3$, $\lim_{h \to 0} \dfrac{f(a+h) - f(a-h)}{2h} = 3x^2$. 2 marks

d If $f(x) = x^n$, where n is a positive whole number, what would be the value of $\lim_{h \to 0} \dfrac{f(a+h) - f(a-h)}{2h}$? 2 marks

16 (3 marks) The position of a particle at time t is given by $f(t) = \dfrac{1}{t}$. Show that an estimate for the instantaneous velocity using the central difference formula is $\dfrac{100}{1 - 100t^2}$ when $h = \dfrac{1}{10}$.

17 (3 marks) Show that the estimate for the rate of change of y with respect to x at $x = 0.5$ on the upper half of the circle $x^2 + y^2 = 1$ using $h = 0.1$ is $4 - \sqrt{21}$.

18 (4 marks) Follow the steps described below to show that central difference can be used to verify

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x-h)}{2h}.$$

Taylor expansion $f(x+h) = f(x) + hf^1(x) + \dfrac{h^2}{2!} f^2(x) + \dfrac{h^3}{3!} f^3(x) + \dfrac{h^4}{4!} f^4(x) + \ldots,$

where $f^n(x)$ means the n^{th} derivative of $f(x)$ and $n! = n(n-1)(n-2)(n-3)\ldots 3 \times 2 \times 1$.

Step 1: Replace h with $-h$ in the Taylor expansion to find the expansion for $f(x-h)$.

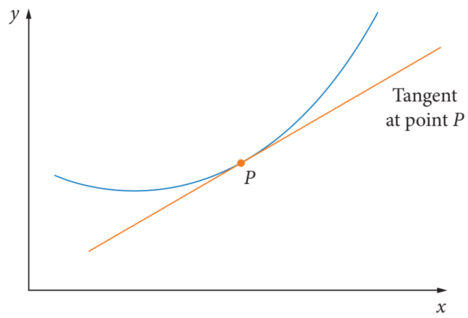
Step 2: Simplify the expression $\dfrac{f(x+h) - f(x-h)}{2h}$.

Step 3: Evaluate $\lim_{h \to 0} \dfrac{f(x+h) - f(x-h)}{2h}$.

10.5 Equation of a tangent

10.5

The derivative is a gradient function that gives the gradient of the tangent (or instantaneous rate of change) to the graph of $y = f(x)$ at the point $(x, f(x))$. It can be used to find the equation of the tangent to the graph at any point on the graph.



In Chapter 2, *Functions and graphs,* we studied the equations of a straight line.

- The **gradient–intercept form** of a straight line with gradient m and y-intercept c is $y = mx + c$.
- The **point–gradient form** of a straight line with gradient m that passes through the point (x_1, y_1) is $y - y_1 = m(x - x_1)$.

Video playlist
Equation of a tangent

Worksheets
Slopes of curves

Tangents to a curve

Equation of a tangent at a point

To find the **equation of the tangent** to the graph of $y = f(x)$ at the point (x_1, y_1):

1. Find $f'(x)$ by differentiation.
2. Find the gradient of the tangent by evaluating $f'(x_1)$.
3. Use the point to find the equation of the line, using either $y - y_1 = m(x - x_1)$ or $y = mx + c$.

WORKED EXAMPLE 13 Equation of the tangent at a given point

WB

p. 202

Find the equation of the tangent to the curve $f(x) = x^2 + x + 1$ at the point (1, 3).

Steps	Working
1 Find the gradient function by differentiation.	$f'(x) = 2x + 1$
2 Find the gradient of the tangent by substituting $x = 1$ in $f'(x)$.	$f'(1) = 2 \times 1 + 1 = 3$ The gradient is $m = 3$.
3 Find the equation of the tangent using $y - y_1 = m(x - x_1)$.	Substitute $m = 3$, $x_1 = 1$ and $y_1 = 3$: $y - 3 = 3(x - 1) = 3x - 3$ $y = 3x$
or	
Find the equation of the tangent using $y = mx + c$.	Substitute $m = 3$: $y = 3x + c$ To find the value of c, substitute $x = 1$ and $y = 3$. $3 = 3 \times 1 + c$ $c = 0$ $y = 3x + 0$ $y = 3x$

WB

p. 203

WORKED EXAMPLE 14 Finding points on a curve that have a given gradient

Find the point(s) on the graph of $f(x) = \frac{1}{3}x^3 - x^2 - 4x + 1$ where the tangent

a has a gradient of -1.

b is parallel to the line $y = 4x + 1$.

Steps	Working
a **1** Find the gradient function by differentiation.	$f'(x) = x^2 - 2x - 4$
2 Solve $f'(x) = -1$.	$x^2 - 2x - 4 = -1$ $x^2 - 2x - 3 = 0$ $(x - 3)(x + 1) = 0$ $x = 3$ or $x = -1$
3 Find the corresponding y values.	$f(3) = \frac{1}{3}(3)^3 - 3^2 - 4(3) + 1 = -11$ $f(-1) = \frac{1}{3}(-1)^3 - (-1)^2 - 4(-1) + 1 = 3\frac{2}{3}$
4 State the coordinates required.	The gradient of the tangent is -1 at $(3, -11)$ and $\left(-1, 3\frac{2}{3}\right)$.
b **1** Determine the gradient value needed.	Parallel lines have the same gradient. The gradient of $y = 4x + 1$ is 4.
2 Solve $f'(x) = 4$.	$x^2 - 2x - 4 = 4$ $x^2 - 2x - 8 = 0$ $(x - 4)(x + 2) = 0$ $x = 4$ or $x = -2$
3 Find the corresponding y values.	$f(4) = \frac{1}{3}(4)^3 - 4^2 - 4(4) + 1 = -9\frac{2}{3}$ $f(-2) = \frac{1}{3}(-2)^3 - (-2)^2 - 4(-2) + 1 = 2\frac{1}{3}$
4 State the coordinates required.	The tangent is parallel to $y = 4x + 1$ at $\left(4, -9\frac{2}{3}\right)$ and $\left(-2, 2\frac{1}{3}\right)$.

VCE QUESTION ANALYSIS

10.5

VCE-style exam question (11 marks)

A quadratic function is given by $f(x) = (ax + 3)(x + b)$, where a and b are constants.

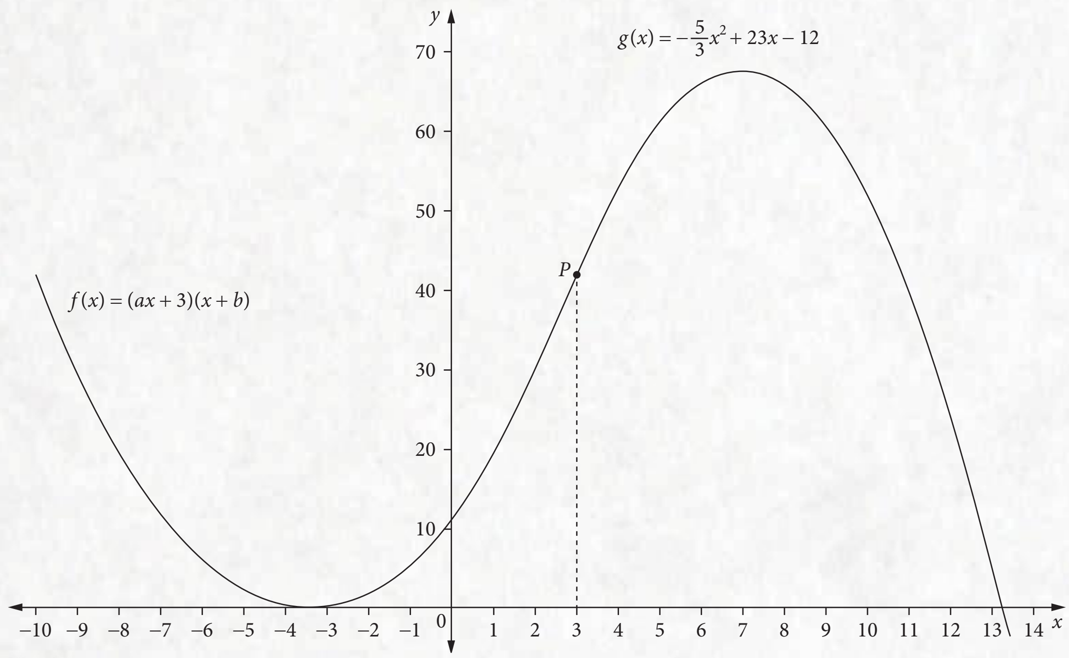
a Use differentiation by first principles to show that the derivative of $f(x)$ is $f'(x) = 2ax + ab + 3$. — 2 marks

b Find the values of a and b given that $f'(-2) = 3$ and $f'(4) = 15$ — 2 marks

c Calculate $f'(-3.5)$.
What can be concluded about the graph of $y = f(x)$ at $x = -3.5$? — 2 marks

d Determine the equation of the tangent at $x = 1$. — 2 marks

e The two parabolas, $f(x)$ and $g(x)$, have point P in common, as shown in the graph below. The graph can be defined by the function $h(x)$, with $h(x) = f(x)$ for $x \le 3$ and $h(x) = g(x)$ for $x > 3$. Show that $h(x)$ is differentiable at P. — 3 marks



Video
VCE question analysis: Differentiation

Reading the question

- Read each question carefully to understand what method you should use to find the solution.
- Identify key words that suggest what approach is needed. For example, for part **a**, do not use the fast method of differentiation when differentiation by first principles is required.
- In part **b**, it is the values of the derivative functions at the given points and not the value of the functions.
- In part **d**, the equation for the tangent requires the gradient at that point.

Thinking about the question

- Each question is worth 2 marks, so show working, even if technology is used.
- In part **c**, consider the significance of the question in terms of the gradient of the tangent line at that point.
- Decide if it will be more effective to use technology. For instance, in part **b**, the values for a and b can be found either algebraically or using the calculator.
- In part **e**, in addition to point P satisfying $f(x)$ and $g(x)$, what must also be true if the function $h(x) = f(x) + g(x)$ is to be differentiable?

Worked solution (✓ = 1 mark)

a $f(x) = ax^2 + abx + 3x + 3b$

$\quad = ax^2 + (ab+3)x + 3b$ ✓

$$\frac{f(x+h) - f(x)}{h}$$

$$= \frac{[a(x+h)^2 + (ab+3)(x+h) + 3b] - [ax^2 + (ab+3)x + 3b]}{h}$$

$$= \frac{[ax^2 + 2axh + ah^2 + (ab+3)x + (ab+3)h + 3b] - [ax^2 + (ab+3)x + 3b]}{h}$$

$$= \frac{[2axh + ah^2 + (ab+3)h]}{h}$$

$$= 2ax + ah + ab + 3$$

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} [2ax + ah + (ab+3)] \checkmark$$

$$= 2ax + ab + 3$$

b $f'(x) = 2ax + ab + 3$

$f'(-2) = 2a \times (-2) + ab + 3 = 3$

$f'(4) = 2a \times (4) + ab + 3 = 15$

Simplify to get

$-4a + ab = 0$

$8a + ab = 12$ ✓

The solution can be found algebraically or by using CAS.

Subtract the first equation from the second equation.

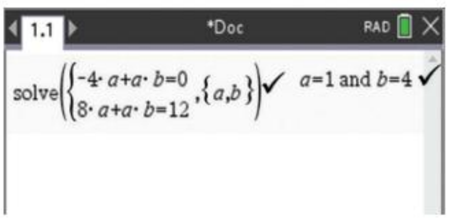
$8a - (-4a) + ab - ab = 12 - 0$

$12a = 12 \Rightarrow a = 1$

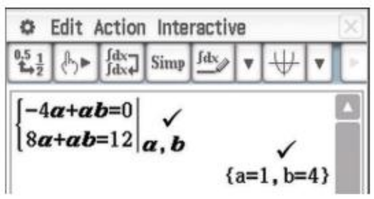
Substitute for a in $-4a + ab = 0$.

$-4 \times 1 + 1 \times b = 0 \Rightarrow \boldsymbol{b = 4}$ ✓

TI-Nspire



ClassPad



c $f'(x) = 2ax + ab + 3$

$\quad = 2x + 7$, using $a = 1$, $b = 4$

$f'(-3.5) = 2 \times (-3.5) + 7$

$\quad = 0$ ✓

The derivative is zero, which means the gradient of the tangent at $x = -3.5$ is zero.

At $x = -3.5$, the graph is flat or has a stationary point (The graph shows that it is a minimum point). ✓

d The equation of a straight line of gradient m that passes through the point (x_1, y_1) is

$y - y_1 = m(x - x_1)$. ✓

The gradient at $x = 1$ is $f'(1) = 2 \times 1 + 7 = 9$ and $f(1) = (1 + 3)(1 + 4) = 20$.

$m = 9, x_1 = 9, y_1 = 20$

$y - 20 = 9(x - 1)$

$\mathbf{y = 9x + 11}$ ✓

e There are no gaps anywhere on the curve, so $h(x)$ is continuous. ✓

For $f(x)$, the gradient at $x = 3$ is $f'(3) = 2 \times 3 + 7 = 13$

For $g(x)$, $g'(x) = -\frac{10}{3}x + 23$ and $g'(3) = 13$. ✓

The gradient at $x = 3$ is 13 for both $f(x)$ and $g(x)$, so $h(x)$ is differentiable at P. ✓

EXERCISE 10.5 Equation of a tangent

ANSWERS p. 528

Recap

1 Using step size h, applying the central difference formula to $f(x) = 3x^2 - 1$ at $x = a$ gives

A $2a$ **B** $3a$ **C** $4a$ **D** $5a$ **E** $6a$

2 The position, $x(t)$ metres, of a particle at time t seconds is given by the equation $x(t) = t^3 + 2$.

Using the central difference formula with $h = 0.01$, an estimate for the particle's instantaneous velocity after 2.5 seconds is

A 17.62 **B** 18.75 **C** 35.15 **D** 37.50 **E** 56.68

Mastery

3 WORKED EXAMPLE 13 Find the equation of the tangent to the curve $f(x) = 6x^2 - x - 4$ at the point $(-1, 3)$.

4 Find the equation of the tangent to the graph of each function at the given point.

a $y = 5x^2 - 2x - 4$ at $x = 1$

b $y = x - x^3 - 5$ at $x = 2$

c $y = x^2 - x^3 + 3x + 2$ at $x = -1$

d $y = x^2 - 2x - 3$ at $x = 4$

5 WORKED EXAMPLE 14 Find the point(s) on the graph of $f(x) = x^3 - 3x^2 + 3x + 4$ where the tangent

a has a gradient of 12

b is parallel to the line $y = 3x$

c is perpendicular to $y = -\frac{1}{27}x + 4$

d is parallel to the x-axis

6 Show that the tangent to the graph of $f(x) = -2x^4 + 2x^3 + 5x^2 - 4x - 4$ at $x = -1$ is parallel to the x-axis.

Parallel to the x-axis means the gradient is zero.

Exam practice

7 TECH-FREE (2 marks) Find the coordinates of the point where the gradient of the tangent to the parabola $y = 5x^2 - x$ is 2.

8 TECH-FREE (2 marks) Show that $\frac{-1 \pm \sqrt{10}}{3}$ are the values of x where the tangent to the curve with equation $y = x^2(x + 1)$ is parallel to the line $y = 3x - 1$.

9 The gradient of the tangent to the graph of $f(x) = x - x^{\frac{1}{2}}$ at $x = 4$ is

A $-\frac{2}{3}$ **B** $-\frac{1}{4}$ **C** $\frac{1}{3}$ **D** $\frac{1}{2}$ **E** $\frac{3}{4}$

10 The equation of the tangent to the curve of the function $f(x) = x - x^3$ at the point $(-2, 6)$ is

A $y = 11x - 16$ **B** $y = 16x - 11$ **C** $y = -16x - 11$

D $y = -11x + 16$ **E** $y = -11x - 16$

11 (2 marks) Find the equation of the tangent to the graph of $f(x) = (x - 4)(x + 6)$ at its positive x-intercept.

12 (2 marks) Show that the tangent to the graph of $f(x) = x^2 - 8x + 12$ at the point $(5, 11)$ is parallel to the line $y = 2x + 1$.

13 (3 marks) Show that the tangent to the graph of the general quadratic function $f(x) = ax^2 + bx + c$, where a, b and c are constants, is parallel to the x-axis when $x = -\frac{b}{2a}$.

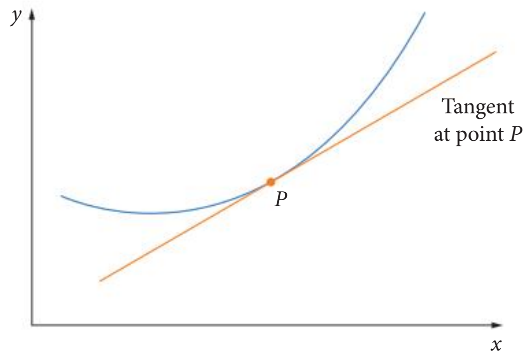
14 (3 marks) Find the coordinates of the point on the graph of $f(x) = 2x^{\frac{1}{4}} - \sqrt{x}$ where the tangent is parallel to the x-axis.

15 (3 marks) The function $f(x) = x^3 + 3x^2 - 2$ has a tangent parallel to the x-axis at 2 points. Find the coordinates of these points.

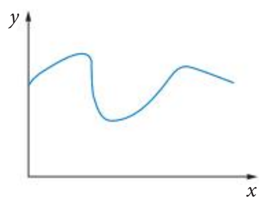
10 Chapter summary

Continuity and limits

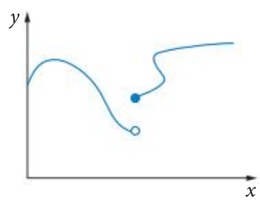
- The **rate of change** describes how one variable changes in response to another variable.
- The **instantaneous rate of change** of a function for a particular value of x is measured by the **gradient of the tangent** to the graph of the function at the point $P(x, y)$.



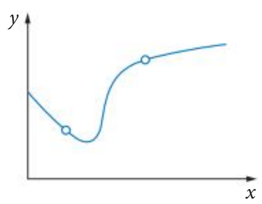
- For a rate of change to exist for a particular value of x, the function must be **continuous** at that value of x.



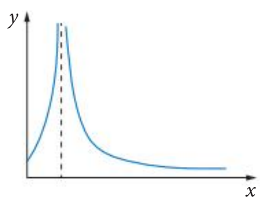
Continuous



Not continuous ('jumps')

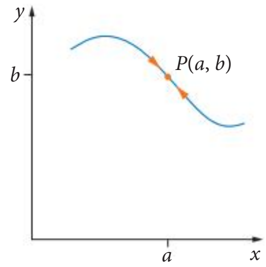


Not continuous ('gaps')



Not continuous ('gap at a vertical asymptote')

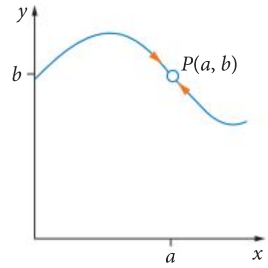
- If $f(x) \to L$ as $x \to a^-$ and $x \to a^+$, we write the **limit**: $\lim\limits_{x \to a} f(x) = L$.
- For a function to be **continuous** at $x = a$:
  - the function must exist at $x = a$, that is, $f(a)$ exists.
  - $\lim\limits_{x \to a} f(x) = f(a)$ when the point is approached from the left and from the right.



$f(a)$ exists

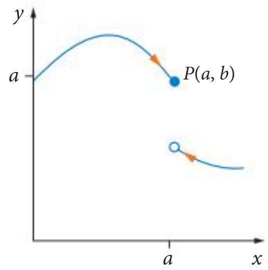
$\lim\limits_{x \to a} f(x) = f(a)$

Continuous at $x = a$



$f(a)$ does not exist

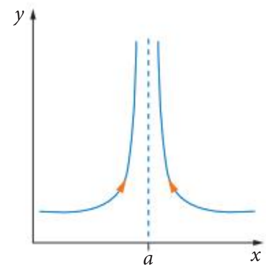
Discontinuous at $x = a$



$f(a)$ exists

$\lim\limits_{x \to a^+} f(x) \neq f(a)$

Discontinuous at $x = a$

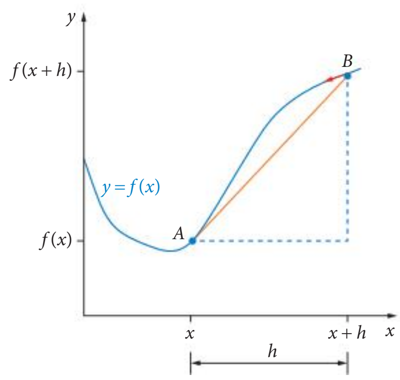


$f(a)$ does not exist

Discontinuous at $x = a$

Differentiation by first principles

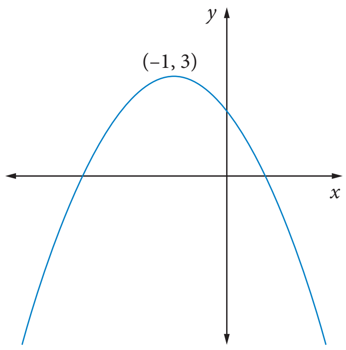
- Finding the gradient function using the formula $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$ is called **differentiation by first principles**, where $f'(x)$ or $\dfrac{dy}{dx}$ is the **derivative** of $f(x)$. This is a function that gives the gradient of the tangent (or **instantaneous rate of change**) at the point $(x, f(x))$.



- For a function to be differentiable at the point where $x = a$:

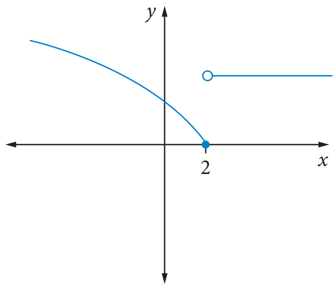
  − $f(x)$ must be continuous at $x = a$.

  − $f(x)$ must have a unique tangent at $x = a$; that is, the tangent at $x = a$ is the same when $x = a$ is approached from either side.



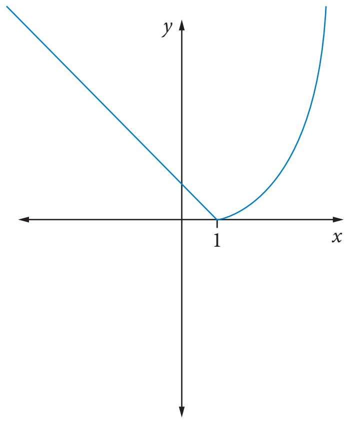
Continuous at $x = -1$

Differentiable at $x = -1$



Discontinuous at $x = 2$

No unique tangent at $x = 2$
Not differentiable at $x = 2$



Continuous but with a sharp point at $x = 1$

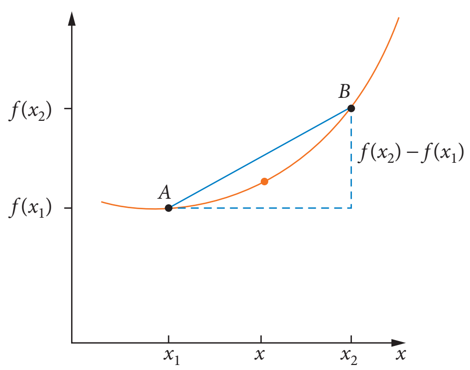
No unique tangent at $x = 1$
Not differentiable at $x = 1$

Differentiating polynomial functions

- If $f(x) = x^n$, then $f'(x) = nx^{n-1}$.
- If $f(x) = ax^n$, then $f'(x) = a \times nx^{n-1}$.
- If $f(x) = c$, then $f'(x) = 0$.
- We can differentiate polynomial functions by differentiating each term separately. This is called **term-by-term differentiation**.

The central difference

The central difference formula $\dfrac{f(a+h) - f(a-h)}{2h}$ is the gradient of the line joining A to B and approximates the gradient at the point $(a, f(a))$.



In the limiting case, letting h approach 0 gives the **derivative, $f'(x)$** of $f(x)$ and is the exact value of the gradient.

On a displacement–time graph, the central difference estimates the instantaneous velocity, and when the central difference is used on a velocity–time graph, an approximation to the instantaneous acceleration is obtained.

Equation of a tangent

- To find the **equation of the tangent** to the graph of $y = f(x)$ at the point (x_1, y_1):
  1. Differentiate $f(x)$ to get $f'(x_1)$.
  2. Find the gradient of the tangent by evaluating $f'(x_1)$.
  3. Use the point to find the equation of the line.
- The equation of the tangent line of gradient m is found using:
  - the **gradient–intercept form** $y = mx + c$, where c is the y-intercept.
  - the **point–gradient form** $y - y_1 = m(x - x_1)$, where (x_1, y_1) is a point on the line.

Cumulative examination 1

Total number of marks: 10 Reading time: 4 minutes Writing time: 15 minutes

TECH-FREE *Technology is NOT permitted.*

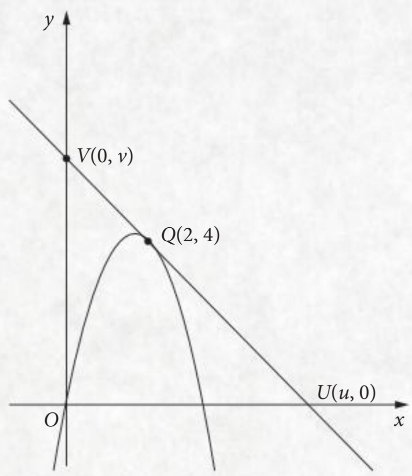
1 (2 marks) The function $f(x)$ is defined as $f(x) = \frac{16 - x^2}{x + 4}$.

a When is the function discontinuous? 1 mark

b Evaluate $\lim\limits_{x \to -4} \frac{16 - x^2}{x + 4}$. 1 mark

2 © VCAA 2014 1Q10 (3 marks) A line intersects the coordinate axes at the points U and V with coordinates $(u, 0)$ and $(0, v)$, respectively, where u and v are positive real numbers and $\frac{5}{2} \le u \le 6$.

When $u = 6$, the line is a tangent to the graph of $y = ax^2 + bx$ at the point Q with coordinates $(2, 4)$, as shown.



If a and b are non-zero real numbers, find the values of a and b.

3 (1 mark) Solve the equation $(x - 3)(x^2 - 8x + 16) = 0$.

4 (2 marks) Using any of the digits 1, 2, 5, 7, 9, how many even numbers can be formed if repetition of a digit in each combination is not allowed?

5 (2 marks) Find non-zero integer values for a and b if $(x - 3a)(x - 2b) = x^2 - (7 - 6a)x + 6b$.

Cumulative examination 2

Total number of marks: 20 Reading time: 4 minutes Writing time: 30 minutes

Approved technology is permitted.

Section A 5 multiple-choice questions

5 marks

1 The function with rule $f(x) = \frac{9 - x^2}{3 + x} + \frac{4 - x}{x}$ is discontinuous at

A $x = 0$ only **B** $x = 3$ only **C** $x = 3, x = -3$
D $x = 4$ only **E** $x = 0$ and $x = -3$

2 Evaluate $\lim_{x \to 6} \frac{12 - 2x}{x^2 - 36}$.

A -6 **B** $-\frac{1}{6}$ **C** 0 **D** $\frac{1}{6}$ **E** 6

3 For $f(x) = \frac{1}{2}x^2 - x$, $\frac{f(x+h) - f(x)}{h}$ is

A $x - \frac{1}{2}h + 1$ **B** $\frac{1}{2}x + h - 1$ **C** $x + \frac{1}{2}h - 1$
D $x - \frac{1}{2}h - 1$ **E** $x - h - \frac{1}{2}$

4 If $P(x) = (x + 3)(x - 1)$, then $P(-a)$ is

A $a^2 + 2a - 3$ **B** $a^2 - 2a - 3$ **C** $a^2 - 2a - 1$
D $a^2 - a + 1$ **E** $a^2 - 3a - 1$

5 The solution to the equation $2x^2 + x + 5 = 3x^2 - x + 6$ is $x =$

A 1 **B** 2 **C** 3 **D** 4 **E** 5

Section B 5 questions

15 marks

1 (6 marks)

a Find $\lim_{h\to 0}\frac{f(2+h)-f(2)}{h}$ using the function $f(x)=x^2+2$. 2 marks

b Find $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$. 2 marks

c Equate and solve for x your answers to parts **a** and **b**. Explain the significance of the answer. 2 marks

2 (2 marks) The line with equation $f(x)=mx+c$ is a tangent to the parabola $g(x)=x^2+4$ at $x=5$ and $\lim_{x\to 2} f(x)=-1$.

Determine the values of m and c.

3 (1 mark) The function $f(x)=x^2-7x+12$ has an x-intercept in the interval [2.6, 3.2].

Using the bisection method, find the next interval to use.

4 (4 marks)

a By letting $y=2^x$, express $3\times 2^{2x}-18\times 2^x+24$ as a quadratic expression involving y. 1 mark

b Factorise the quadratic expression in part **a** in the form $(ay+b)(cy+d)$, where a, b, c and d are constants. 1 mark

c Solve the equation $(ay+b)(cy+d)=0$ for y. 1 mark

d Hence, solve $3\times 2^{2x}-18\times 2^x+24=0$. 1 mark

5 (2 marks) Solve the equation $\frac{18^{4x-1}}{16^{\frac{x}{2}}\times 8^{2x}}=\frac{3^2}{2^3}$.

CHAPTER

11 TRIGONOMETRIC AND EXPONENTIAL EQUATIONS

Study Design coverage

UNIT 2, AREA OF STUDY 2: ALGEBRA, NUMBER AND STRUCTURE

- use of inverse functions and transformations to solve equations of the form $Af(nx) + c = k$, where $A, n, c, k \in R$ for $A, n \neq 0$, and f is sine, cosine, tangent or a^x, using exact or approximate values on a given domain
- exponent laws and logarithm laws, including their application to the solution of simple exponential equations
- numerical approximation of roots of cubic polynomial functions using the Newton's method algorithm.

STUDY DESIGN COVERAGE

Video playlists (5):

11.1 Trigonometric equations
11.2 Exponential equations
11.3 Logarithmic equations
11.4 Newton's method
VCE question analysis Trigonometric and exponential equations

Worksheets (5):

11.1 Trigonometric equations
11.2 Solving exponential equations
11.3 Logarithms - Solving equations 1
- Logarithms - Solving equations 2
- Logarithmic and exponential equations

Nelson MindTap

To access resources above, visit **cengage.com.au/nelsonmindtap**

11.1 Trigonometric equations

Video playlist Trigonometric equations

Worksheet Trigonometric equations

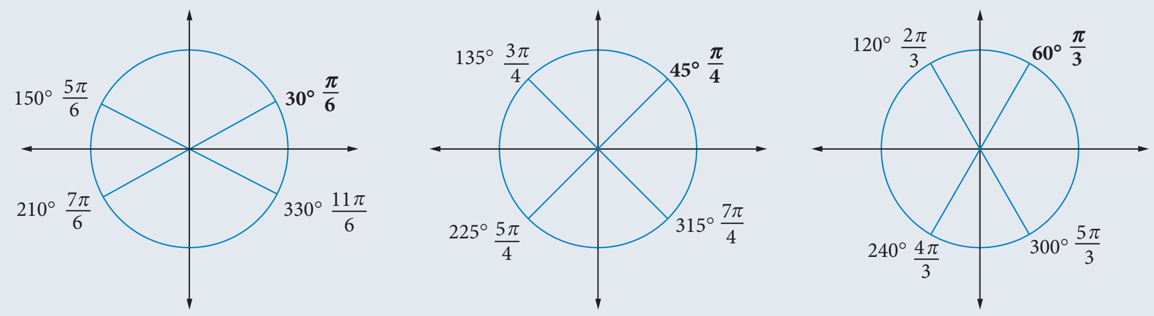
Exact values

	$0°$	$30°\left(\frac{\pi}{6}\right)$	$45°\left(\frac{\pi}{4}\right)$	$60°\left(\frac{\pi}{3}\right)$	$90°\left(\frac{\pi}{2}\right)$
sin θ	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cos θ	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tan θ	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	Not defined

Exact value angles in the 4 quadrants

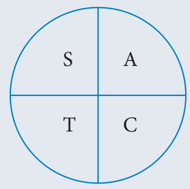
The exact value angles $\left\{\frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}\right\}$ in quadrant 1 can also occur in quadrants 2, 3 and 4, but their sign will be determined by their quadrant and the **ASTC rule**.

The exact values in the 4 quadrants



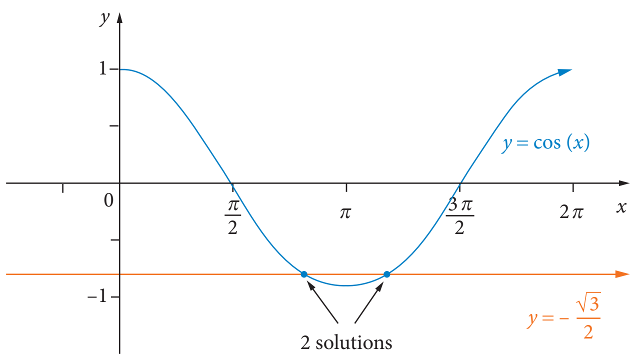
The ASTC rule gives the signs of the circular functions in each quadrant.

- All the circular functions are positive in the 1st quadrant.
- Sine is positive and the others negative in the 2nd quadrant.
- Tangent is positive and the others negative in the 3rd quadrant.
- Cosine is positive and the others negative in the 4th quadrant.



Solving trigonometric equations

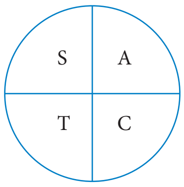
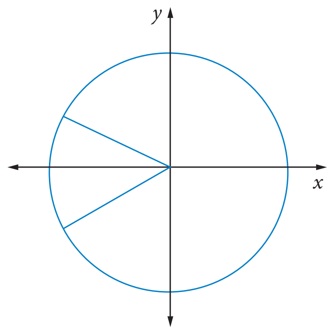
To solve the equation $\cos x = -\frac{\sqrt{3}}{2}$ over the domain $[0, 2\pi]$, we can examine the points of intersection of the graphs of $y = \cos x$ and $y = -\frac{\sqrt{3}}{2}$.



We can see from these graphs that there will be 2 solutions over the given domain.

WORKED EXAMPLE 1 Solving trigonometric equations.

Solve $\cos(x) = -\frac{\sqrt{3}}{2}$, $x \in [0, 2\pi]$ for x.

Steps	Working
1 Ignoring the sign of $-\frac{\sqrt{3}}{2}$, find the reference angle (the acute angle in the 1st quadrant). Refer to the exact value table.	reference angle $= \frac{\pi}{6}$
2 Identify the quadrants where there are solutions. Find the quadrants where $\cos(x)$ is negative.  	 Solutions in 2nd and 3rd quadrants.
3 Find the solutions. Find the equivalent angle to $\frac{\pi}{6}$ in quadrants 2 and 3.	$x = \left\{\pi - \frac{\pi}{6}, \pi + \frac{\pi}{6}\right\}$ $x = \left\{\frac{5\pi}{6}, \frac{7\pi}{6}\right\}$

Exam hack

For trigonometric equations involving radians, make sure the angle settings of your calculator are in radians not degrees.

Use the unit circle, the ASTC rule and the following steps to solve trigonometric equations such as $\cos(nx) = c$.

Method for solving trigonometric equations

Step 1 Find the **reference angle** in the 1st quadrant, the solution to the equation if the sign of c is ignored. This can be an exact value angle.

Step 2 Use the sign of c to identify the 2 quadrants where there are solutions.

Step 3 Identify the domain for nx.

For example, if $x \in [0, 2\pi]$, then $3x \in [0, 6\pi]$.

Step 4 Solve for nx that lie in the domain: this will involve adding or subtracting the reference angle from π or 2π, and may require more than one revolution of the unit circle.

Step 5 Solve for x.

11.1

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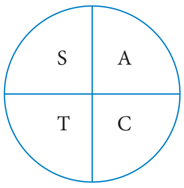
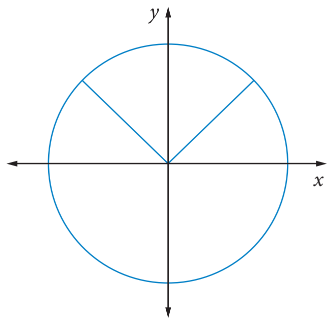
WORKED EXAMPLE 2 Solving trigonometric equations in the form sin(nx) = c

Solve $\sin(2x) = \frac{1}{\sqrt{2}}$, $x \in [0, 2\pi]$ for x.

Steps	Working
1 Find the reference angle. Refer to the exact value table.	reference angle $= \frac{\pi}{4}$
2 Identify the quadrants where there are solutions. Find the quadrants where sin (2x) is positive. 	 Solutions in 1st and 2nd quadrants.
3 Determine the domain for 2x. If $x \in [0, 2\pi]$, then $2x \in [0, 4\pi]$. A domain of $[0, 4\pi]$ will produce 4 solutions.	$2x \in [0, 4\pi]$
4 Solve the equation for 2x. $\frac{\pi}{4}$ and $\frac{3\pi}{4}$ are in the first revolution. To obtain the next 2 solutions add 2π or $\frac{8\pi}{4}$ to these values.	$2x = \left\{\frac{\pi}{4}, \pi - \frac{\pi}{4}\right\}$ $2x = \left\{\frac{\pi}{4}, \frac{3\pi}{4}\right\}$ $2x = \left\{\frac{\pi}{4}, \frac{3\pi}{4}, \frac{9\pi}{4}, \frac{11\pi}{4}\right\}$
5 Divide by 2 to solve for x.	$x = \left\{\frac{\pi}{8}, \frac{3\pi}{8}, \frac{9\pi}{8}, \frac{11\pi}{8}\right\}$
6 Confirm by CAS. Be sure to include the domain.	

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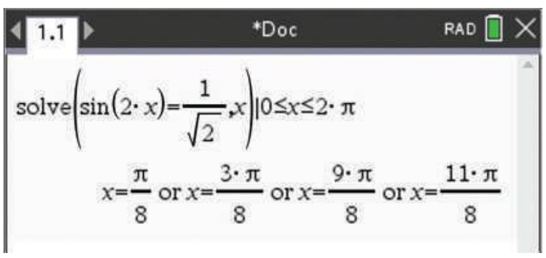
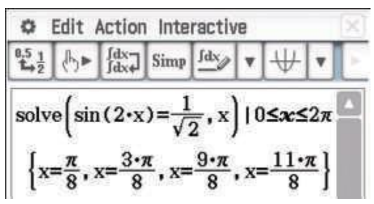


ClassPad



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WORKED EXAMPLE 3 Transposing and solving trigonometric equations

Solve $3\tan(2x) + 3 = 0$, $x \in [-\pi, 0]$ for x.

Steps	Working
1 Make tan (2x) the subject of the equation.	$3\tan(2x) + 3 = 0$ $3\tan(2x) = -3$ $\tan(2x) = -1$
2 Find the reference angle.	reference angle $= \frac{\pi}{4}$

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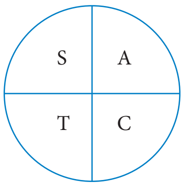
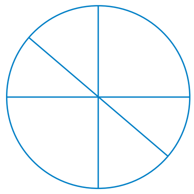
3 Identify the quadrants where tan $(2x)$ is negative. 	 Solutions in quadrants 2 and 4.
4 Determine the domain for $2x$. If $x \in [-\pi, 0]$, then $2x \in [-2\pi, 0]$.	$2x \in [-2\pi, 0]$
5 Solve the equation for nx.	$2x = \left\{-\frac{\pi}{4}, -\frac{\pi}{4} - \pi\right\}$
For negative angles, start at 0 on the x-axis and rotate clockwise.	$2x = \left\{-\frac{\pi}{4}, -\frac{5\pi}{4}\right\}$
6 Divide by 2 to solve for x.	$x = \left\{-\frac{\pi}{8}, -\frac{5\pi}{8}\right\}$

Solving equations of the form sin(nx) = a cos(nx)

$\tan(nx) = \dfrac{\sin(nx)}{\cos(nx)}$ is used to solve equations of the form $\sin(nx) = a\cos(nx)$.

WORKED EXAMPLE 4 Solving equations of the form sin (nx) = a cos(nx)

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Solve $\sqrt{3}\sin(3x) = -\cos(3x), 0 \le x \le 2\pi$ for x.

Steps	**Working**
1 Divide both sides by cos $(3x)$.	$\frac{\sqrt{3}\sin(3x)}{\cos(3x)} = -\frac{\cos(3x)}{\cos(3x)}$ $\sqrt{3}\tan(3x) = -1$ $\tan(3x) = -\frac{1}{\sqrt{3}}$
2 Determine the reference angle.	reference angle $= \frac{\pi}{6}$
3 Identify the quadrants where tan $(3x)$ is negative.	Solutions in quadrants 2 and 4.
4 Determine the domain for $3x$. If $x \in [0, 2\pi]$ then $3x \in [0, 6\pi]$. There will be 6 solutions.	$3x \in [0, 6\pi]$.
5 Solve the equation for $3x$.	$3x = \left\{\pi - \frac{\pi}{6}, 2\pi - \frac{\pi}{6}\right\}$
$\frac{5\pi}{6}$ and $\frac{11\pi}{6}$ are in the 1st revolution.	$3x = \left\{\frac{5\pi}{6}, \frac{11\pi}{6}\right\}$
For the 2nd revolution, add 2π or $\frac{12\pi}{6}$ to these values. For the 3rd revolution, add another 2π or $\frac{12\pi}{6}$.	$3x = \left\{\frac{5\pi}{6}, \frac{11\pi}{6}, \frac{17\pi}{6}, \frac{23\pi}{6}, \frac{29\pi}{6}, \frac{35\pi}{6}\right\}$
6 Divide by 3 to solve for x.	$x = \left\{\frac{5\pi}{18}, \frac{11\pi}{18}, \frac{17\pi}{18}, \frac{23\pi}{18}, \frac{29\pi}{18}, \frac{35\pi}{18}\right\}$

EXERCISE 11.1 Trigonometric equations

ANSWERS p. 529

Mastery

1 WORKED EXAMPLE 1 TECH-FREE Solve each equation for x.

a $\cos(x) = -\frac{\sqrt{3}}{2}, x \in [0, 2\pi]$

b $\sin(x) = \frac{\sqrt{3}}{2}, x \in [0, 2\pi]$

c $\sin(x) = -\frac{1}{\sqrt{2}}, x \in [0, 2\pi]$

d $\tan(x) = -1, x \in [0, 2\pi]$

2 WORKED EXAMPLE 2 TECH-FREE Solve each equation for x.

a $\sin(3x) = \frac{1}{\sqrt{2}}, x \in [0, 2\pi]$

b $\tan(2x) = \sqrt{3}, x \in [0, 2\pi]$

c $\cos(2x) = \frac{1}{2}, x \in [0, 2\pi]$

d $\cos(3x) = -\frac{\sqrt{3}}{2}, x \in [0, 2\pi]$

3 WORKED EXAMPLE 2 Solve each equation, correct to three decimal places.

a $\sin(x) = 0.4, 0 \le x \le 2\pi$

b $\cos(2x) = -0.65, 0 \le x \le \pi$

4 WORKED EXAMPLE 3 TECH-FREE Solve each equation.

a $\sqrt{3}\tan(2x) = -1, x \in [-\pi, \pi]$

b $\sqrt{2}\sin(3x) = -1, x \in [0, \pi]$

c $2\cos(2x) - 1 = 0, x \in [-2\pi, 0]$

d $4\tan(4x) + 4 = 0, x \in [0, \pi]$

5 WORKED EXAMPLE 4 TECH-FREE Solve each equation

a $\cos(x) = \sin(x), x \in [0, 2\pi]$

b $-\sqrt{3}\cos(x) = \sin(x), x \in [-\pi, \pi]$

c $\cos(2x) - \sqrt{3}\sin(2x) = 0, x \in [0, \pi]$

d $\cos(3x) + \sin(3x) = 0, x \in [0, \pi]$

Exam practice

80–100% 60–79% 0–59%

6 © VCAA 2012 1Q6 75% TECH-FREE (3 marks)

The graphs of $y = \cos(x)$ and $y = a\sin(x)$, where a is a real constant, have a point of intersection at $x = \frac{\pi}{3}$.

a Find the value of a. 2 marks

b If $x \in [0, 2\pi]$, find the x-coordinate of the other point of intersection of the two graphs. 1 mark

7 © VCAA 2014 1Q3 70% TECH-FREE (2 marks) Solve $2\cos(2x) = -\sqrt{3}$ for x where $0 \le x \le \pi$.

8 TECH-FREE (2 marks) Solve the equation $\cos(3x) = -\frac{1}{\sqrt{2}}, x \in [0, 2\pi]$.

9 TECH-FREE (2 marks) Solve $2\sin(2x) + \sqrt{3} = 0, x \in [0, 2\pi]$.

10 © VCAA 2013 1Q4 65% TECH-FREE (2 marks) Solve the equation $\sin\left(\frac{x}{2}\right) = -\frac{1}{2}$ for $x \in [2\pi, 4\pi]$.

11 © VCAA 2019 1Q4 63% TECH-FREE (2 marks) Solve $1 - \cos\left(\frac{x}{2}\right) = \cos\left(\frac{x}{2}\right)$, for $x \in [-2\pi, \pi]$.

12 © VCAA 2009 1Q4 63% TECH-FREE (3 marks)

Solve the equation $\tan(2x) = \sqrt{3}$

for $x \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right) \cup \left(\frac{\pi}{4}, \frac{3\pi}{4}\right)$.

13 © VCAA 2003 1Q3 62% TECH-FREE (2 marks)

Find the exact solutions of the equation $\sin(2\pi x) = -\sqrt{3}\cos(2\pi x),\ 0 \le x \le 1$.

Exam hack

The function $y = \tan(2x)$ has asymptotes at $x = -\frac{\pi}{4}$, $x = \frac{\pi}{4}$ and $x = \frac{3\pi}{4}$. The domain $x \in \left[-\frac{\pi}{4}, \frac{3\pi}{4}\right]$ is equivalent to the one shown in Question 12 once these values are removed.

11.1

14 © VCAA 2002 1Q6 MODIFIED 61% TECH-FREE (2 marks) Given $f: \left[0, \frac{\pi}{2}\right] \to R,\ f(x) = 2\sin(3x) - \frac{3}{2}$, find the exact values of x for which $f(x) = -\frac{1}{2}$

15 © VCAA 2008 1Q3 60% TECH-FREE (2 marks) Solve the equation $\cos\left(\frac{3x}{2}\right) = \frac{1}{2}$ for $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

16 © VCAA 2007 1Q8 55% TECH-FREE (2 marks) Let $f: R \to R,\ f(x) = \sin\left(\frac{2\pi x}{3}\right)$.

Solve the equation $\sin\left(\frac{2\pi x}{3}\right) = -\frac{\sqrt{3}}{2}$ for $x \in [0, 3]$.

17 The solutions for the equation $\sin(x) = \frac{1}{\sqrt{2}},\ x \in [0, 2\pi]$ are

A $x = \frac{5\pi}{4}, x = \frac{7\pi}{4}$ **B** $x = \frac{\pi}{4}, x = \frac{5\pi}{4}$ **C** $x = \frac{\pi}{3}, x = \frac{2\pi}{3}$

D $x = \frac{\pi}{4}, x = \frac{3\pi}{4}$ **E** $x = \frac{\pi}{6}, x = \frac{5\pi}{6}$

18 The solutions for the equation $\cos(x) = \frac{1}{2},\ x \in [0, 2\pi]$ are

A $x = \frac{\pi}{3}, x = \frac{5\pi}{3}$ **B** $x = \frac{\pi}{4}, x = \frac{7\pi}{4}$ **C** $x = \frac{\pi}{3}, x = \frac{2\pi}{3}$

D $x = \frac{2\pi}{3}, x = \frac{4\pi}{3}$ **E** $x = \frac{\pi}{6}, x = \frac{11\pi}{6}$

19 The solutions for the equation $\tan(x) = -1,\ x \in [-\pi, \pi]$ are

A $x = -\frac{\pi}{4}, x = \frac{3\pi}{4}$ **B** $x = \frac{3\pi}{4}, x = \frac{7\pi}{4}$ **C** $x = -\frac{\pi}{4}, x = -\frac{3\pi}{4}$

D $x = -\frac{\pi}{3}, x = \frac{2\pi}{3}$ **E** $x = -\frac{\pi}{6}, x = \frac{5\pi}{6}$

20 The solutions for the equation $\tan(2x) = \frac{1}{\sqrt{3}},\ x \in [0, 2\pi]$ are

A $x = \left\{\frac{\pi}{6}, \frac{2\pi}{3}, \frac{7\pi}{6}, \frac{5\pi}{3}\right\}$ **B** $x = \left\{\frac{\pi}{6}, \frac{7\pi}{6}, \frac{13\pi}{6}, \frac{19\pi}{6}\right\}$ **C** $x = \left\{\frac{\pi}{12}, \frac{7\pi}{12}, \frac{13\pi}{12}, \frac{19\pi}{12}\right\}$

D $x = \left\{\frac{5\pi}{12}, \frac{11\pi}{12}, \frac{17\pi}{12}, \frac{23\pi}{12}\right\}$ **E** $x = \left\{\frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}, \frac{13\pi}{8}\right\}$

21 The solutions for the equation $2\sin(2x) = -1, x \in [0, 2\pi]$ are

A $x = \left\{\frac{7\pi}{6}, \frac{11\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}\right\}$ **B** $x = \left\{\frac{7\pi}{3}, \frac{11\pi}{3}, \frac{19\pi}{3}, \frac{23\pi}{3}\right\}$ **C** $x = \left\{\frac{\pi}{12}, \frac{7\pi}{12}, \frac{13\pi}{12}, \frac{19\pi}{12}\right\}$

D $x = \left\{\frac{5\pi}{12}, \frac{11\pi}{12}, \frac{17\pi}{12}, \frac{23\pi}{12}\right\}$ **E** $x = \left\{\frac{7\pi}{12}, \frac{11\pi}{12}, \frac{19\pi}{12}, \frac{23\pi}{12}\right\}$

22 The solutions for the equation $2\cos(2x) + \sqrt{3} = 0,\ x \in [-\pi, 0]$ are

A $x = \left\{-\frac{7\pi}{3}, -\frac{5\pi}{3}\right\}$ **B** $x = \left\{-\frac{7\pi}{3}, -\frac{11\pi}{3}\right\}$ **C** $x = \left\{\frac{5\pi}{12}, \frac{7\pi}{12}\right\}$

D $x = \left\{-\frac{3\pi}{8}, -\frac{5\pi}{8}\right\}$ **E** $x = \left\{-\frac{7\pi}{12}, -\frac{5\pi}{12}\right\}$

23 © VCAA 2006 2AQ2 **84%** The smallest positive value of x for which $\tan(2x) = 1$ is

A 0 **B** $\frac{\pi}{8}$ **C** $\frac{\pi}{4}$ **D** $\frac{\pi}{2}$ **E** π

24 © VCAA 2003 1Q6 **78%** The number of solutions of the equation $0.5\cos(2x) = 1$, for $x \in [-\pi, \pi]$, is

A 0 **B** 1 **C** 2 **D** 3 **E** 4

25 © VCAA 2004 1Q8 **73%** The **sum** of the solutions of $\cos\left(\frac{x}{2}\right) = \frac{\sqrt{3}}{2}$, for $0 \le x \le 4\pi$, is

A $\frac{\pi}{6}$ **B** $\frac{\pi}{3}$ **C** $\frac{11\pi}{3}$ **D** 4π **E** 8π

26 © VCAA 2004 1AQ7 **70%** The depth of water near the Lorne Pier changes with the tides according to the rule

$$h(t) = 7 - 2\sin\left(\frac{4\pi t}{25} + \frac{\pi}{2}\right)$$

where t is the time in hours after low tide and h is the depth in metres. A low tide occurred at midnight. The time of the **next high** tide is

A 6.00 am **B** 6.15 am **C** 12.30 pm

D 6.30 pm **E** 1.00 am on the next day

27 © VCAA 2002 2AQ2 **65%** For the equation $\sin(2x) = 1$, the sum of the solutions in the interval $[0, 4\pi]$ is

A $\frac{\pi}{4}$ **B** 1.099 75 **C** $\frac{(4n+1)\pi}{4}$ **D** $\frac{3\pi}{2}$ **E** 7π

28 © VCAA 2019 2AQ19 **25%** Given that $\tan(\alpha) = d$, where $d > 0$ and $0 < \alpha < \frac{\pi}{2}$, the sum of the solutions to $\tan(2x) = d$, where $0 < x < \frac{5\pi}{4}$, in terms of α, is

A 0 **B** 2α **C** $\pi + 2\alpha$ **D** $\frac{\pi}{2} + \alpha$ **E** $\frac{3(\pi + \alpha)}{2}$

11.2 Exponential equations

An **exponential equation** is an equation involving a^x, where the variable (x) is in the **exponent** (power). For example $8^x = 128$.

We can solve exponential equations by expressing all terms with a common base, usually a prime number, such as 2, 3 or 5. Some knowledge of the powers of prime numbers will help when solving exponential equations.

Powers of prime numbers

Powers of 2	
2^0	1
2^1	2
2^2	4
2^3	8
2^4	16
2^5	32
2^6	64

Powers of 3	
3^0	1
3^1	3
3^2	9
3^3	27
3^4	81
3^5	243
3^6	729

Powers of 5	
5^0	1
5^1	5
5^2	25
5^3	125
5^4	625
5^5	3125

11.2

Video playlist Exponential equations

WORKED EXAMPLE 5 Solving simple exponential equations

Solve $8^x = 128$ for x.

Steps	Working
1 Write 8 and 128 as powers of 2.	$8^x = 128$ $\left(2^3\right)^x = 2^7$ $2^{3x} = 2^7$
2 Equate the powers.	$3x = 7$
3 Solve the linear equation.	$x = \frac{7}{3}$

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Worksheet Solving exponential equations

WORKED EXAMPLE 6 Solving exponential equations

Solve $4^{x-2} = 32^{1-x}$ for x.

Steps	Working
1 Write 4 and 32 as powers of 2.	$4^{x-2} = 32^{1-x}$ $\left(2^2\right)^{x-2} = \left(2^5\right)^{1-x}$
2 Expand the brackets and multiply the powers.	$2^{2x-4} = 2^{5-5x}$
3 Equate the powers.	$2x - 4 = 5 - 5x$
4 Solve the linear equation.	$7x - 4 = 5$ $7x = 9$ $x = \frac{9}{7}$

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WORKED EXAMPLE 7 Solving exponential equations using index laws

Solve $9^x \times 27^{x+1} = 81^{1-x}$ for x.

Steps	Working
1 Write 9, 27 and 81 as powers of 3.	$9^x \times 27^{x+1} = 81^{1-x}$ $\left(3^2\right)^x \times \left(3^3\right)^{x+1} = \left(3^4\right)^{1-x}$
2 Expand the brackets, multiply the powers then simplify.	$3^{2x} \times 3^{3x+3} = 3^{4-4x}$ $3^{2x+3x+3} = 3^{4-4x}$ $3^{5x+3} = 3^{4-4x}$
3 Equate the powers and solve the linear equation.	$5x + 3 = 4 - 4x$ $9x + 3 = 4$ $9x = 1$ $x = \frac{1}{9}$
4 Confirm by CAS.	

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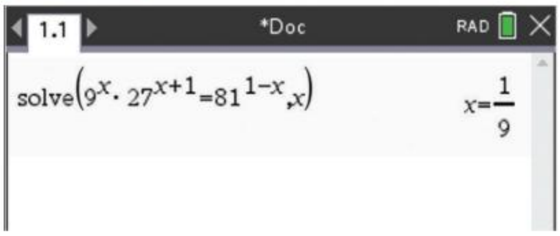

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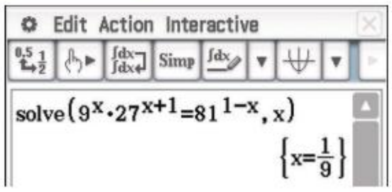

Exponential equations that factorise as quadratic equations

An exponential equation of the form $a(m^{2x}) + b(m^x) + c = 0$ can be factorised and solved as a quadratic equation.

WB p. 211

WORKED EXAMPLE 8 Solving exponential quadratic equations 1

Solve $3^{2x} - 12(3^x) + 27 = 0$ for x.

Steps	Working
1 Let a represent the exponential term.	Let $a = 3^x$ $a^2 = (3^x)^2 = 3^{2x}$
2 Express the equation in terms of a.	$a^2 - 12a + 27 = 0$
3 Factorise and solve for a.	$(a-9)(a-3) = 0$ $a = 9, a = 3$
4 Rewrite the solutions in terms of x.	$3^x = 9, 3^x = 3$
5 Solve by writing all terms as a power of 3 and equating the powers.	$3^x = 3^2, 3^x = 3^1$ $x = 1, 2$

WORKED EXAMPLE 9 Solving exponential quadratic equations 2

Solve $4(2^{2x}) + 31(2^x) - 8 = 0$ for x.

Steps	Working
1 Let k represent the exponential term.	Let $k = 2^x$ $k^2 = 2^{2x}$
2 Express the equation as a quadratic in terms of k.	$4k^2 + 31k - 8 = 0$
3 Factorise and solve for k.	$(4k - 1)(k + 8) = 0$ $k = \frac{1}{4}, k = -8$
4 Rewrite the solutions in terms of x.	$2^x = \frac{1}{4}, 2^x = -8$
5 Solve by writing all terms as a power of 2 and equating the powers.	$2^x = 2^{-2}$, no solution $x = -2$

11.2

WB

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EXERCISE 11.2 Exponential equations

ANSWERS p. 529

Recap

1 The exact solutions of the equation $2\cos(2x) = 1$ for x, where $0 \le x \le \pi$, are

A $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$ **B** $\frac{\pi}{6}, \frac{5\pi}{6}$ **C** $\frac{\pi}{3}, \frac{5\pi}{3}$

D $\frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}$ **E** $\frac{\pi}{12}, \frac{5\pi}{12}$

2 © VCAA 2003 1IIQ2 MODIFIED The exact solutions of the equation $\sin(3\pi x) = -\cos(3\pi x)$ for $0 \le x \le 1$, are

A $\frac{1}{4}, \frac{7}{12}, \frac{11}{12}$ **B** $\frac{\pi}{4}, \frac{7\pi}{12}, \frac{11\pi}{12}$ **C** $\frac{1}{4}, \frac{7}{12}, \frac{11}{12}$

D $\frac{1}{12}, \frac{5}{12}, \frac{7}{12}$ **E** $\frac{1}{12}, \frac{5}{12}$

Mastery

3 WORKED EXAMPLE 5 Solve each equation.

a $81^x = 243$ **b** $16^x = 512$ **c** $125^x = 25$ **d** $8^x = \frac{1}{4}$

4 WORKED EXAMPLE 6 Solve each equation.

a $27^{x-1} = 81^{3-2x}$ **b** $64^{x+1} = 32^{x+2}$ **c** $4^{5x+3} = 8^{1-3x}$ **d** $125^{x-2} - 25^x = 0$

5 WORKED EXAMPLE 7 Solve each equation.

a $9 \times 27^{x-1} = 81$ **b** $64^{x+1} \times 32^{x+2} = 4 \times 16^x$

c $16 = 64^{x+1} \times 8^{x-1}$ **d** $125^{x-2} \times 25^x = 625$

6 WORKED EXAMPLE 8 Solve each equation.

a $2^{2x} - 12\left(2^x\right) + 32 = 0$ **b** $3^{2x} - 82(3^x) + 81 = 0$

c $5^{2x} + 120(5^x) - 625 = 0$ **d** $2^{2x} - 2(2^x) - 8 = 0$

7 Solve each equation for x, correct to two decimal places.

a $8^x = 5$ b $3^{x+1} = 21$ c $14^{x-2} = 7^{x+2}$

8 WORKED EXAMPLE 9 Solve each equation for x.

a $9\left(3^{2x}\right) - 28\left(3^x\right) + 3 = 0$ b $4(2^{2x}) - 5(2^x) + 1 = 0$

c $2(2^{2x}) - 9(2^x) + 4 = 0$ d $8(2^{2x}) + 31(2^x) - 4 = 0$

Exam practice

80–100% 60–79% 0–59%

9 © VCAA 2014 1Q4 85% TECH-FREE (2 marks) Solve the equation $2^{3x-3} = 8^{2-x}$ for x.

10 © VCAA 2013 1Q5b 75% TECH-FREE (2 marks) Solve the equation $3^{-4x} = 9^{6-x}$ for x.

11 TECH-FREE (8 marks) Solve each equation.

a $8^{x+3} \times 32^x = 64$ b $\dfrac{8^{3x+2}}{4^{3-x}} = 256$

c $4\left(2^{2x}\right) - 5\left(2^x\right) + 1 = 0$ d $3^{2x} - 6\left(3^x\right) - 27 = 0$

12 (2 marks) Solve each equation correct to 3 decimal places.

a $5^x = 7$ b $10^{x-4} = 9^{x+5}$

13 The solution of the equation $128^x = 8$ is

A $x = \dfrac{1}{16}$ B $x = \dfrac{3}{7}$ C $x = \dfrac{1}{2}$ D $x = 2$ E $x = 16$

14 The solution of the equation $25^{2x} = 125^{x+2}$ is

A $x = -2$ B $x = 0$ C $x = 2$ D $x = 5$ E $x = 6$

15 The solution of the equation $16 \times 2^{2x} = 128^{2-x}$ is

A $x = \dfrac{14}{17}$ B $x = \dfrac{14}{15}$ C $x = \dfrac{15}{9}$ D $x = \dfrac{10}{9}$ E $x = \dfrac{10}{3}$

16 The solution of the equation $\dfrac{81^{2x+3}}{9} = 27^x$ is

A $x = -6$ B $x = -2$ C $x = 1$ D $x = 3$ E $x = 6$

17 The solutions to the equation $5^{2x} - 26(5^x) + 25 = 0$ are

A $x = 25,\ x = 1$ B $x = 2,\ x = 1$ C $x = 2,\ x = -2$

D $x = 2,\ x = 0$ E $x = 5,\ x = 0$

18 The solutions to the equation $9^x - 30(3^x) + 81 = 0$ are

A $x = 9,\ x = 1$ B $x = 10,\ x = 3$ C $x = 3,\ x = -1$

D $x = 27,\ x = 3$ E $x = 1,\ x = 3$

11.3 Logarithmic equations

Logarithms and logarithm laws

If $a^x = b$ then $\log_a(b) = x$.

The logarithm of b to base a is the power x, such that $a^x = b$ where $a > 0$ and $a \neq 1$.

$\log_a(1) = 0$

$\log_a(a) = 1$

$\log_a(a^x) = x$

$\log_a(x) + \log_a(y) = \log_a(xy)$

$\log_a(x) - \log_a(y) = \log_a\left(\dfrac{x}{y}\right)$

$n\log_a(x) = \log_a(x^n)$

Video playlist
Logarithmic equations

To solve a **logarithmic equation**, use logarithm laws to simplify the expressions and change the logarithmic form to index form.

WORKED EXAMPLE 10 Solving logarithmic equations

Solve $\log_3(x-1) = 2$ for x.

Steps	Working
1 Change the equation from logarithmic form to index form.	$\log_3(x-1) = 2$ $x - 1 = 3^2$
2 Simplify and solve the equation.	$x - 1 = 9$ $x = 10$

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Worksheets
Logarithms - Solving equations 1
Logarithms - Solving equations 2
Logarithmic and exponential equations

WORKED EXAMPLE 11 Solving logarithmic equations using logarithm laws

Solve $\log_2(x-1) + 2\log_2(5) = 2$ for x.

Steps	Working
1 Use logarithm laws to express the left-hand side of the equation as a single logarithm.	$\log_2(x-1) + 2\log_2(5) = 2$ $\log_2(x-1) + \log_2(5)^2 = 2$ $\log_2(x-1) + \log_2(25) = 2$ $\log_2 25(x-1) = 2$
2 Change the equation from logarithm form to index form.	$25(x-1) = 2^2$
3 Simplify and solve the equation.	$25x - 25 = 4$ $25x = 29$ $x = \dfrac{29}{25}$

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For equations where every term is a logarithm, first simplify into the form $\log_n(a) = \log_n(b)$.

Then solve the equation by 'crossing out' the logs and solving $a = b$.

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WORKED EXAMPLE 12 Solving equations where every term is a logarithm

Solve $\log_7(x) = \log_7(3) + \log_7(6)$ for x.

Steps	Working
1 Use logarithm laws to express the right-hand side of the equation as a single logarithm.	$\log_7(x) = \log_7(3) + \log_7(6)$ $\log_7(x) = \log_7(3 \times 6)$ $\log_7(x) = \log_7(18)$
2 Equate the brackets.	$x = 18$

Euler's number, e

Euler's number (pronounced 'oiler') is an irrational constant (number) like π and is used in exponential and logarithmic functions. It is represented by the letter ***e***, where $e = 2.718\,28\ldots$

Logarithms to base e, called **natural logarithms**, are written as $\log_e(x)$ and on a calculator as $\ln(x)$.

Exam hack

When solving logarithmic equations to base e, express your answer in exact form in terms of e, unless a decimal approximation is required.

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WORKED EXAMPLE 13 Solving natural logarithmic equations 1

Solve $\log_e(x-1) + \log_e(x) = \log_e(6)$ for x.

Steps	Working
1 Use logarithm laws to express the left-hand side of the equation as a single logarithm.	$\log_e(x-1) + \log_e(x) = \log_e(6)$ $\log_e(x(x-1)) = \log_e(6)$ $\log_e(x^2 - x) = \log_e(6)$
2 Equate the brackets and solve the quadratic.	$x^2 - x = 6$ $x^2 - x - 6 = 0$ $(x-3)(x+2) = 0$ $x = 3, x = -2$
3 Check to see if both solutions are valid in the original equation.	For $x = 3$, $\text{LHS} = \log_e(2) + \log_e(3)$ For $x = -2$, $\text{LHS} = \log_e(-3) + \log_e(-3)$ Logarithms only apply to positive numbers so the solution $x = -2$ is not valid.
4 Write the solution.	The solution is $x = 3$.

Exam hack

When solving logarithmic equations, always check the validity of your solutions in the original equation.

WORKED EXAMPLE 14 Solving natural logarithmic equations 2

Solve $2\log_e(3) + \log_e(x) = 10$ for x.

Steps	Working
1 Use log laws to express the left-hand side of the equation as a single logarithm.	$2\log_e(3)+\log_e(x)=10$ $\log_e(3^2)+\log_e(x)=10$ $\log_e(9)+\log_e(x)=10$ $\log_e(9x)=10$
2 Change the equation from log form to index form.	$e^{10} = 9x$
3 Solve the equation, expressing the answer in exact form.	$x = \dfrac{e^{10}}{9}$

WORKED EXAMPLE 15 Solving quadratic exponential equations

Solve $e^{2x} - 5e^x + 6 = 0$ for x.

Steps	Working
1 The equation can be written as a quadratic by substituting $a = e^x$	$e^{2x} - 5e^x + 6 = 0$ Let $a = e^x$, $a^2 - 5a + 6 = 0$
2 Solve the equation for a.	$(a-2)(a-3) = 0$ $a = 2, a = 3$
3 Rewrite the solution in terms of x.	$e^x = 2, e^x = 3$
4 Solve the equation, expressing the answers in exact form.	$x = \log_e(2), \log_e(3)$
5 Confirm using CAS. Use e^x, not the letter e.	

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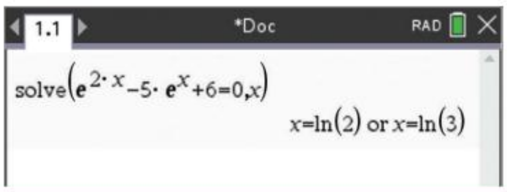

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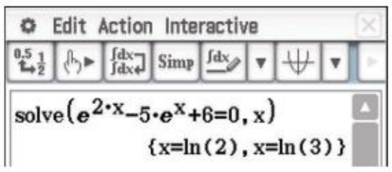

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EXERCISE 11.3 Logarithmic equations

ANSWERS p. 529

Recap

1 The solution of the equation $32^{3x-2} = 256$ is

A $x = \dfrac{8}{15}$ **B** $x = \dfrac{2}{3}$ **C** $x = \dfrac{6}{5}$ **D** $x = 2$ **E** $x = \dfrac{10}{3}$

2 The solution of the equation $3e^{2x} = 12$ is

A $\dfrac{e^4}{2}$ **B** $\log_e(36)$ **C** $\dfrac{1}{2}\log_e(4)$

D $\dfrac{1}{3}\log_e(12)$ **E** $\log_e(4)$

Mastery

3 WORKED EXAMPLE 10 Solve each logarithmic equation.

a $\log_3(4x-3)=2$ **b** $\log_2(2x-6)=5$ **c** $\frac{1}{3}\log_5(x+4)=1$

4 WORKED EXAMPLE 11 Solve each logarithmic equation.

a $\log_3(x+7)+\log_3(2)=3$

b $2\log_2(3)+\log_2(x+1)=4$

c $\log_2(3x-2)-2\log_2(4)=3$

5 WORKED EXAMPLE 12 Solve each logarithmic equation.

a $\log_5(x)+\log_5(3)-\log_5(2)=\log_5(6)$

b $\log_2(x)+\log_2(6)=\log_2(3)+\log_2(x+7)$

c $\log_2(x)-3\log_2(2)=\log_2(x+1)-2\log_2(5)$

6 WORKED EXAMPLE 13 Solve each logarithmic equation.

a $\log_e(2x+3)+\log_e(x)=\log_e(5)$

b $\log_e(x)+\log_e(x-2)=\log_e(8)$

c $\log_2(x+1)+\log_2(x+3)=3$

7 WORKED EXAMPLE 14 Solve each logarithmic equation.

a $\log_e(x)-3\log_e(2)=5$

b $\log_e(x+3)+\log_e(4)=3$

c $\log_e(x)+\log_e(12)-\log_e(3)=4$

8 WORKED EXAMPLE 15 Solve each exponential equation.

a $e^{2x}+4e^x-21=0$ **b** $e^{2x}-7e^x+12=0$ **c** $3e^{2x}-10e^x-8=0$

9 Solve each exponential equation, correct to 3 decimal places.

a $e^{2x}+7e^x-2=0$ **b** $e^{2x}-10e^x+3=0$ **c** $2e^{2x}+3e^x-1=0$

Exam practice

80–100% 60–79% 0–59%

10 TECH-FREE (6 marks) Solve each logarithmic equation for x.

a $\log_4(x)-\frac{1}{2}\log_4(9)=1$ 2 marks

b $\log_2(x)=\frac{1}{3}\log_2(125)+\frac{1}{2}\log_2(16)$ 2 marks

c $\log_2(x)-\frac{1}{2}\log_2(64)=4$ 2 marks

11 © VCAA 2013 1Q5a 70% TECH-FREE (2 marks) Solve the equation $2\log_3(5)-\log_3(2)+\log_3(x)=2$ for x.

12 © VCAA 2007 1Q2a 70% TECH-FREE (2 marks) Solve the equation $\log_e(3x+5)+\log_e(2)=2$, for x.

13 © VCAA 2012 1Q7 67% TECH-FREE (3 marks) Solve the equation $2\log_e(x+2)-\log_e(x)=\log_e(2x+1)$, where $x>0$, for x.

14 © VCAA 2020 1Q4 61% TECH-FREE (3 marks) Solve the equation $2\log_2(x+5)-\log_2(x+9)=1$.

15 © VCAA 2014 1Q6 60% TECH-FREE (2 marks) Solve $\log_e(x)-3=\log_e(\sqrt{x})$ for x, where $x>0$.

16 © VCAA 2009 1Q9 55% TECH-FREE (4 marks)

Solve the equation $2\log_e(x)-\log_e(x+3)=\log_e\left(\frac{1}{2}\right)$ for x.

17 The solution to the equation $\log_7(5x-1)=2$ is

A $x=0.6$ B $x=9$ C $x=10$ D $x=25.8$ E $x=49$

18 The solution to the equation $\log_3(x)+3\log_3(4)=0$ is

A $x=\frac{1}{64}$ B $x=\frac{3}{64}$ C $x=\frac{1}{12}$ D $x=\frac{1}{4}$ E $x=1$

19 The solution to the equation $\log_x(243)=5$ is

A $x=2$ B $x=3$ C $x=5$ D $x=6$ E $x=9$

20 The solution to the equation $\log_2(2x)+\log_2(5)-2\log_2(3)=\log_2(90)$ is

A $x=1$ B $x=18$ C $x=81$ D $x=90$ E $x=100$

21 The solution to the equation $e^{2x}+3e^x-54=0$ is

A $x=\log_e(9)$ B $x=\log_e(54)$ C $x=\log_e(6)$

D $x=6$ E $x=9$

22 © VCAA 2005 1Q8 89% The solution of the equation $3e^{2x}=4$ is closest to

A -0.406 B -0.405 C 0.143 D 0.144 E 0.575

23 © VCAA 2008 1Q17 89% The graph of the function $f(x)=e^{2x}-2$ intersects the graph of $g(x)=e^x$ where

A $x=-1$ B $x=\log_e(2)$ C $x=2$

D $x=\frac{1\pm\sqrt{7}}{2}$ E $x=\log_e\left(\frac{1+\sqrt{7}}{2}\right)$

24 © VCAA 2007 2AQ11 82% The solution set of the equation $e^{4x}-5e^{2x}+4=0$ over R is

A $\{1, 4\}$ B $\{-4, -1\}$ C $\{-2, -1, 1, 2\}$

D $\{-\log_e(2), 0, \log_e(2)\}$ E $\{0, \log_e(2)\}$

25 © VCAA 2003 1AQ22 80% If $2\log_e(x)-\log_e(x+2)=1+\log_e(y)$, then y is equal to

A $\frac{x^2}{10(x+2)}$ B $\frac{2x}{x+2}-1$ C $\frac{x^2}{x+2}$

D $\frac{2x}{x+2}$ E $\frac{x^2}{e(x+2)}$

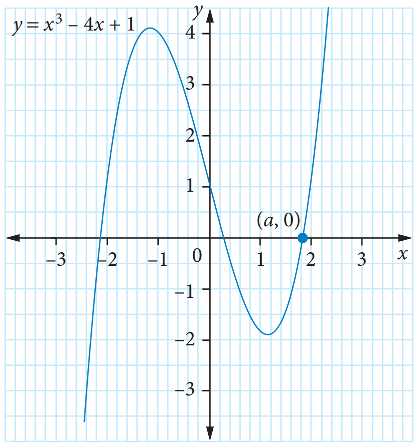
11.3

Video playlist
Newton's method

11.4 Newton's method

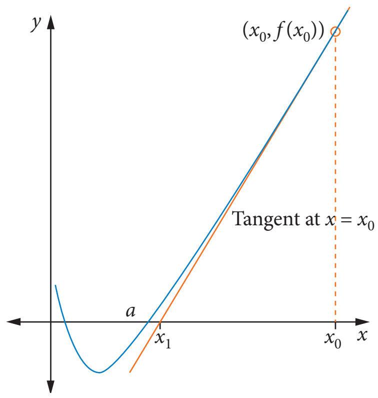
In Chapter 3, *Polynomials,* we learned about the bisection method of solving polynomial equations of degree 3 or higher. A numerical approximation method starts with a good guess of a solution and then repeats the procedure to improve on the guesses. It is also called an iterative method, where iterative means repeated.

Newton's method is another powerful numerical approximation technique for solving polynomial equations. Like the bisection method, it involves a number of **iterations** (repeated steps) that continually improve the approximation, but it locates the solution(s) of the equation faster than the bisection method.



Consider the function $y = x^3 - 4x + 1$, which has a zero at $x = a$.

Newton's method can be used to approximate the value of a.



Newton's method uses the tangent to the graph of $y = f(x)$ at the point $\left(x_0, f(x_0)\right)$, close to the zero of the function. Note from the diagram that the x-intercept, x_1, of the tangent is much closer to the zero than the initial estimate x_0.

The tangent at $\left(x_0, f(x_0)\right)$ has a gradient of $m = f'(x_0)$.

This tangent line also passes through the points $\left(x_0, f(x_0)\right)$ and $(x_1, 0)$.

The equation of the tangent is

$$y - y_1 = m(x - x_1)$$
$$y - f(x_0) = f'(x_0)(x - x_0)$$

The x-intercept is $\left(x_1, 0\right)$.

$$0 - f(x_0) = f'(x_0)(x_1 - x_0)$$
$$-\frac{f(x_0)}{f'(x_0)} = x_1 - x_0$$
$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

We can use the tangent at $(x_1, f(x_1))$ to find a point even closer to the zero at x_2, and so on.

When using Newton's method, start with a guess or initial estimate (x_0) for the value of the zero. Using this guess, Newton's method produces an improved estimate (x_1). We then use the formula again on x_1 to produce an improved estimate x_2. As we continue, the estimates improve and will approximate the value of the zero at a.

11.4

Newton's method

The formula for Newton's method is $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$.

Using Newton's method, where x_0 = the initial estimate:

- the first approximation x_1 is given by $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$
- the second approximation x_2 is given by $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$.

WORKED EXAMPLE 16 The first iteration of Newton's method

WB
p. 219

Use Newton's method to approximate a zero for the cubic $y = x^3 - 4x + 1$. If $x_0 = 2$, find the value of x_1.

Steps	Working
1 Differentiate $y = x^3 - 4x + 1$.	$f'(x) = 3x^2 - 4$
2 Calculate $f(x_0)$ and $f'(x_0)$ where $x_0 = 2$.	$f(2) = 2^3 - 4(2) + 1 = 1$ $f'(2) = 3(2)^2 - 4 = 8$
3 Use Newton's method to find x_1. $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$	$x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$ $= 2 - \dfrac{f(2)}{f'(2)}$ $= 2 - \dfrac{1}{8}$ $= \dfrac{15}{8}$

USING CAS 1 Newton's method

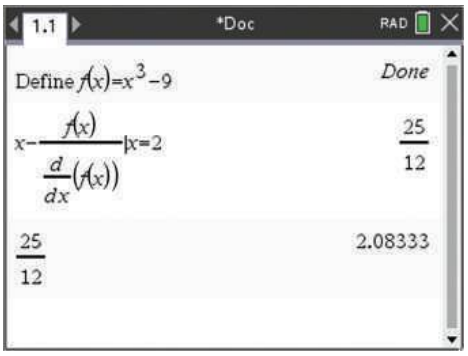
Use Newton's method to solve $x^3 - 9 = 0$.

If $x_0 = 2$, find, to 4 decimal places

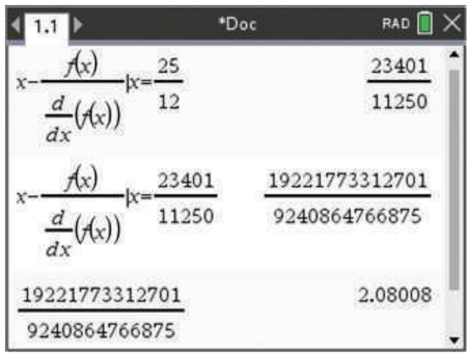
a x_1 **b** x_3

TI-Nspire

a



b



1 Define the function $f(x) = x^3 - 9$.

2 Apply Newton's method for the initial value of $x_0 = 2$, as shown above.

3 Press **ctrl + =** for the approximate answer.

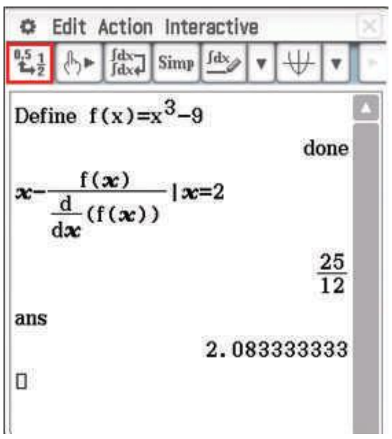
4 Copy and paste the expression and apply Newton's method using the exact value of x_1.

5 Repeat to apply Newton's method using the exact value of x_2.

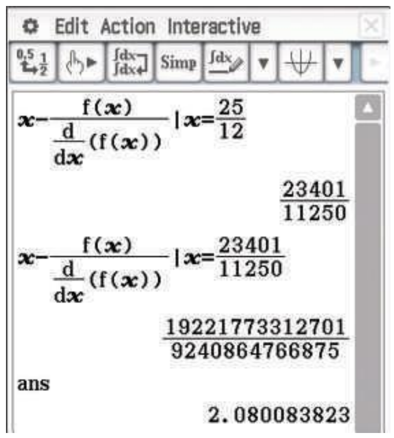
6 Press **ctrl + =** for the approximate answer.

ClassPad

a



b



1 Define the function $f(x) = x^3 - 9$.

2 Apply Newton's method for the initial value of $x_0 = 2$, as shown above.

3 Tap **EXE**, then **Convert** for the decimal answer.

4 Copy and paste to expression and apply Newton's method using the exact value of x_1.

5 Repeat to apply Newton's method using the exact value of x_2.

6 Tap to convert for the decimal answer.

a $x_1 \approx 2.0833$

b $x_3 \approx 2.0801$

WORKED EXAMPLE 17 Two iterations of Newton's method

Complete two iterations of Newton's method to solve $x^3 - 2 = 0$ where $x_0 = 1$.

Steps	Working
1 Differentiate $y = x^3 - 2$.	$f'(x) = 3x^2$
2 Calculate $f(x_0)$ and $f'(x_0)$ where $x_0 = 1$.	$f(1) = 1^3 - 2 = -1$ $f'(1) = 3(1)^2 = 3$
3 Use Newton's method to find x_1: $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$	$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$ $= 1 - \frac{-1}{3}$ $= \frac{4}{3}$
4 Use Newton's method to find x_2: $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$ where $x_1 = \frac{4}{3}$	$x_2 = \frac{4}{3} - \frac{f\left(\frac{4}{3}\right)}{f'\left(\frac{4}{3}\right)}$
5 Calculate $f\left(\frac{4}{3}\right)$ and $f'\left(\frac{4}{3}\right)$.	$f\left(\frac{4}{3}\right) = \left(\frac{4}{3}\right)^3 - 2 = \frac{10}{27}$ $f'\left(\frac{4}{3}\right) = 3\left(\frac{4}{3}\right)^2 = \frac{16}{3}$
6 Substitute in the x_2 formula.	$x_2 = \frac{4}{3} - \frac{\frac{10}{27}}{\frac{16}{3}} = \frac{91}{72}$

11.4

WB

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VCE QUESTION ANALYSIS

A VCE-style question (10 marks)

Video
VCE question analysis: Trigonometric and exponential equations

The Acme Seed Company is about to bring out a new product called 'The Temperature Dwarf Bean'. Each packet contains 100 bean seeds. Fifty of the seeds are of type A and grow well at temperatures between 0°C and 40°C, while the other 50 are of type B and grow well at temperatures above 30°C.

Vic, a research assistant at Acme Seed, conducted tests on the two seeds. In each test, he attempted to grow 50 seeds at a constant temperature and observed the number of plants that grew. The number of plants that grew are approximated by the models

Type A: $A(t) = 30 + 15\sin\left(\frac{\pi t}{30}\right), 0 \le t \le 60$

Type B: $B(t) = a\log_e(3t + 16), 0 \le t \le 60$

where $A(t)$ and $B(t)$ represent the number of seeds of each type that produce plants and t represents the temperature in degrees Celsius.

a State the period and amplitude of the type A function. 2 marks

b Vic observed that all 50 of the type B beans grew at 60°C.

Show that $a = \frac{25}{\log_e 14}$. 2 marks

c Determine the values of t, correct to two decimal places, for which at least 35 of type A seeds produced plants. 1 mark

d Determine the exact value of t, when 30 type B seeds produce plants. 1 mark

e Determine (to the nearest degree) the temperature when both plants produce the same number of seeds. 2 marks

f Sketch the graphs of $A(t)$ and $B(t)$ on the same axes. Hence determine the temperatures required, correct to two decimal places, for which the type B seeds produce more plants than the type A seeds. 2 marks

Reading the question

- Highlight 'show' since this indicates mathematical working must be shown. It is not sufficient to obtain this value of 'a' using CAS.
- Highlight the form of answer required. This may be an approximation, an exact value or a graph.
- Highlight information in the question that indicates if you are solving an equation or an inequality.

Thinking about the question

- Define any functions in the analysis question. This will make substituting, solving and graphing easier.
- Many parts of this question require the solution of a trigonometric equation or a logarithmic equation. CAS may be used to obtain the solutions unless the question indicates you must 'show' a particular solution exists.
- Both functions are defined over the same domain so be sure to take this into account when graphing and solving. This is especially important for trigonometric functions.
- Some parts of this question involve solving an inequality. The graph of the function(s) will help you to determine the correct interval for your answer.

Worked solution (✓ = 1 mark)

a $A(t) = 30 + 15\sin\left(\frac{\pi t}{30}\right), 0 \le t \le 60$

$\text{period} = 2\pi \div \frac{\pi}{30} = 2\pi \times \frac{30}{\pi} = \mathbf{60}$ ✓

$\text{amplitude} = \mathbf{15}$ ✓

b $B(60) = \mathbf{50}$ ✓

$50 = a\log_e(3 \times 60 + 16)$

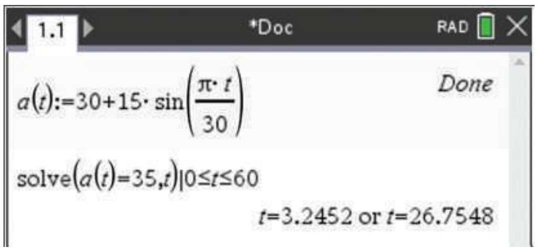
$50 = a\log_e(196)$

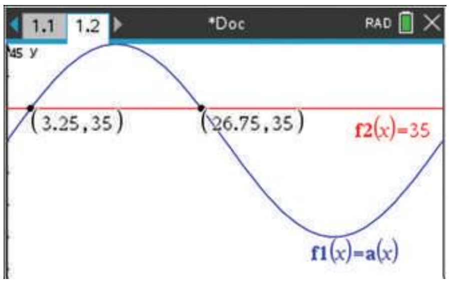
$a = \frac{50}{\log_e(196)}$

$a = \frac{50}{\log_e(14^2)} = \frac{50}{2\log_e(14)} = \frac{\mathbf{25}}{\mathbf{\log_e(14)}}$ ✓

c $30 + 15\sin\left(\frac{\pi t}{30}\right) \geq 35$

TI-Nspire





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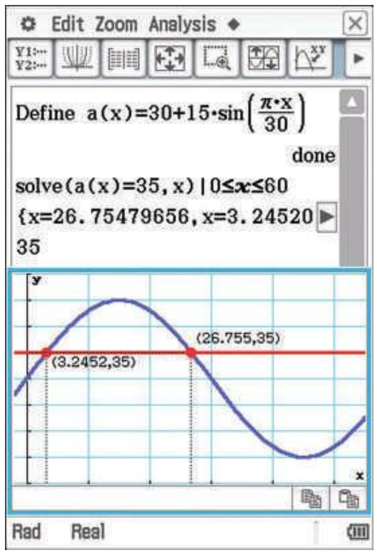


At least 35 type A seeds produced plants for $\boldsymbol{t \in [3.25, 26.75]}$. ✓

d

$$30 = \frac{25\log_e(3t + 16)}{\log_e(14)}$$

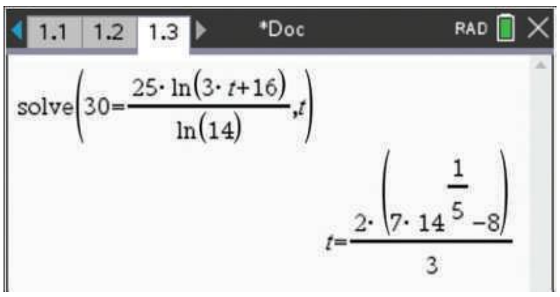
$$\log_e(3t + 16) = \frac{30\log_e(14)}{25}$$

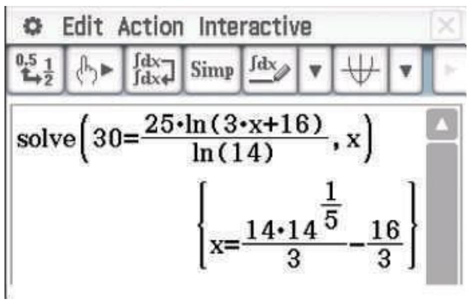
$$\log_e(3t + 16) = \frac{6\log_e(14)}{5}$$

$$3t + 16 = e^{1.2\log_e(14)}$$

$$\boldsymbol{t = \frac{1}{3}\left(e^{1.2\log_e(14)} - 16\right)} \checkmark$$

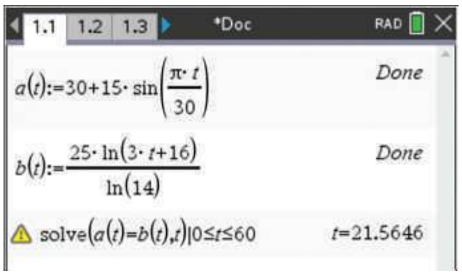
There are many alternative equivalent forms of the value of t, as shown by the CAS below.





e Solve $A(t) = B(t)$ over the domain $0 \le t \le 60$. ✓

TI-Nspire



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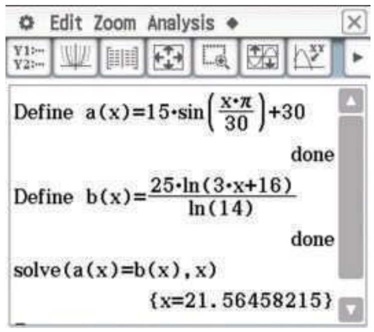
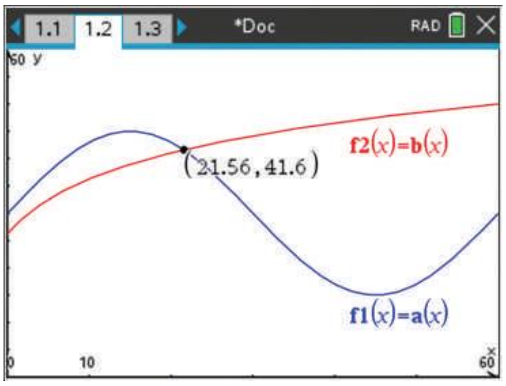
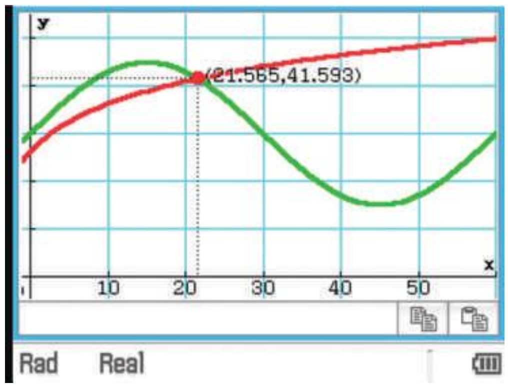


The temperature is **22°C** when the number of type A and type B seeds produce the same number of plants. ✓

f



✓



The type B seeds produce more plants than the type A seeds for $t \in$ **(21.56, 60]**. ✓

EXERCISE 11.4 Newton's method

ANSWERS p. 530

Recap

1 The solution to $\log_4(x) - \frac{1}{2}\log_4(25) = 2$ is

A 5 **B** 10 **C** 16 **D** 64 **E** 80

2 © VCAA 2007 2AQ11 MODIFIED The solution set of the equation $e^{2x} - 4e^{x} + 3 = 0$ is

A $\{\log_e(3), 1\}$ **B** $\{\log_e(3), 0\}$ **C** $\{3, 1\}$

D $\{\log_e(4)\}$ **E** $\{\log_e(3), \log_e(4)\}$

Mastery

3 WORKED EXAMPLE 16 Use Newton's method to approximate a zero for the equation $f(x) = x^2 - x - 1$. If the initial estimate is $x_0 = 1$, find the value of

a x_1 **b** x_2 **c** x_3

4 Using CAS 1 Use Newton's method to approximate the value of x for the equation $x^3 - 5 = 0$. If $x_0 = 1$, find, correct to 4 decimal places, the value of

a x_1 **b** x_2 **c** x_3

5 WORKED EXAMPLE 17 Using Newton's method, complete 2 iterations to find approximate solutions for the equation $x^3 + 2x + 10 = 0$ where $x_0 = -2$. Round your answers to 4 decimal places.

11.4

Exam practice

6 Using Newton's method, 2 iterations are completed to find a zero for $f(x) = x^2 - 2$.
If the initial estimate $x_0 = 1$, then the value of x_2 is given by

A $x_2 = 1 - \dfrac{f(1)}{f'(1)}$

B $x_2 = \dfrac{3}{2} - \dfrac{f\left(\frac{3}{2}\right)}{f'\left(\frac{3}{2}\right)}$

C $x_2 = \dfrac{1}{2} - \dfrac{f\left(\frac{1}{2}\right)}{f'\left(\frac{1}{2}\right)}$

D $x_2 = 2 - \dfrac{f(2)}{f'(2)}$

E $x_2 = \dfrac{3}{2} - \dfrac{f(1)}{f'(1)}$

7 A zero for the equation $x^3 = -x + 8$ is found using Newton's method with an initial estimate of $x_0 = 2$. The value of x_3 correct to 5 decimal places is

A 1.833 70 **B** 1.833 71 **C** 1.833 75

D 1.833 82 **E** 1.846 15

8 A zero for the equation $x^3 - 25 = 0$ is found using Newton's method with an initial estimate of $x_0 = 3$. The number of iterations required to obtain a solution of 2.924 when rounded to 3 decimal places is

A 0 **B** 1 **C** 2 **D** 3 **E** 4

9 The converging solution to the equation $x^3 - 5x - 3 = 0$, to 4 decimal places, is found using Newton's method. If the initial estimate is $x_0 = 2$, the solution is

A 2.4908 **B** 2.4909 **C** 2.4911 **D** 2.4912 **E** 2.5140

10 Newton's method is used to find a zero for the function $f(x) = x^3 - 3x - 3$, using an initial estimate of $x_0 = a$, where a is an integer. If the first estimate is $x_1 = \dfrac{19}{9}$, the value of a is

A 1 **B** 2 **C** 3 **D** 4 **E** 5

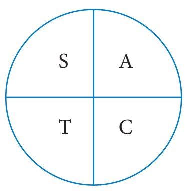
11 Chapter summary

Exact values

	0°	30° $\left(\frac{\pi}{6}\right)$	45° $\left(\frac{\pi}{4}\right)$	60° $\left(\frac{\pi}{3}\right)$	90° $\left(\frac{\pi}{2}\right)$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	Not defined

The ASTC rule

- All the circular functions are positive in the 1st quadrant.
- Sine is positive and the others negative in the 2nd quadrant.
- Tangent is positive and the others negative in the 3rd quadrant.
- Cosine is positive and the others negative in the 4th quadrant.



Method for solving $\sin(nx) = c$, $\cos(nx) = c$ or $\tan(nx) = c$

Step 1 Find the reference angle in the 1st quadrant, the solution to the equation if the sign of c is ignored. This can be an exact value angle.

Step 2 Use the sign of c to identify the 2 quadrants where there are solutions using the ASTC rule.

Step 3 Identify the domain for nx.

For example, if $x \in [0, 2\pi]$, then $3x \in [0, 6\pi]$.

Step 4 Solve for nx that lie in the domain: this will involve adding or subtracting the reference angle from π or 2π, and may require more than one revolution of the unit circle.

Step 5 Solve for x.

$\tan(nx) = \dfrac{\sin(nx)}{\cos(nx)}$ is used to solve equations of the form $\sin(nx) = a\cos(nx)$.

Exponential equations with a common base

- Exponential equations can be solved by expressing all terms with a common base, usually a prime number like 2, 3 or 5.
- An exponential equations of the form $m^{2x} + b(m^x) + c = 0$ can be factorised and solved as a quadratic.

Solving logarithmic equations

- If $a^x = b$ then $\log_a(b) = x$.
- The logarithm of b to base a is the power x, such that $a^x = b$ where $a > 0$ and $a \neq 1$.
- $\log_a(1) = 0$
- $\log_a(a) = 1$
- $\log_a(a^x) = x$
- $\log_a(x) + \log_a(y) = \log_n(xy)$
- $\log_a(x) - \log_a(y) = \log_a\left(\frac{x}{y}\right)$
- $n\log_a(x) = \log_a(x^n)$

Euler's number, e

Euler's number is $e = 2.718\,28\ldots$

Logarithms to base e, called natural logarithms, are written as $\log_e(x)$ and on a calculator as $\ln(x)$.

Newton's method

- Like the bisection method, Newton's method is a numerical approximation method for finding the root (solution) of a polynomial equation.

  It involves a number of iterations or repeated steps that continually improve the approximation.
- The formula for Newton's method is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

The initial estimate is x_0.

Cumulative examination 1

Total number of marks: 10 Reading time: 4 minutes Writing time: 15 minutes

TECH-FREE *Technology is NOT permitted.*

1 © VCAA 2013 1Q4 MODIFIED (2 marks) Solve the equation $\cos\left(\frac{x}{2}\right) = -\frac{1}{2}$ for $x \in [2\pi, 6\pi]$.

2 © VCAA 2014 1Q4 MODIFIED (2 marks) Solve the equation $3^{3x-3} = 27^{2-x}$ for x.

3 © VCAA 2007 1Q2 MODIFIED (2 marks) Solve the equation $\log_e(2x-7) + \log_e(2) = 3$, for x.

4 (2 marks) Find the equation of the tangent to $y = x^2 + 4x - 3$ at the point where $x = 2$.

5 (2 marks) If $f(x) = \frac{1}{x^2}$, find $f'(-3)$.

Cumulative examination 2

Total number of marks: 20   Reading time: 4 minutes   Writing time: 30 minutes

Approved technology is permitted.

Section A 5 multiple-choice questions

5 marks

1 The solutions for the equation $\sin(x) = -\frac{1}{\sqrt{2}}, x \in [0, 2\pi]$ are

A $x = \frac{5\pi}{4}, x = \frac{7\pi}{4}$

B $x = \frac{\pi}{4}, x = \frac{5\pi}{4}$

C $x = \frac{\pi}{3}, x = \frac{2\pi}{3}$

D $x = \frac{\pi}{4}, x = \frac{3\pi}{4}$

E $x = \frac{\pi}{6}, x = \frac{5\pi}{6}$

2 © VCAA 2006 2AQ2 MODIFIED The smallest positive value of x for which $\tan(2x) = -1$ is

A 0 **B** $\frac{\pi}{8}$ **C** $\frac{3\pi}{8}$ **D** $\frac{3\pi}{4}$ **E** π

3 © VCAA 2003 1IQ22 MODIFIED If $\log_e(x) - 2\log_e(x+2) = 1 - \log_e(y)$, then y is equal to

A $\frac{(x+2)^2}{x}$

B $\frac{x}{e(x+2)^2}$

C $x - 4$

D $\frac{x}{(x+2)^2}$

E $\frac{e(x+2)^2}{x}$

4 The vertex of the parabola $y = 2x^2 - 6x + 1$ is

A $\left(\frac{3}{2}, -\frac{7}{2}\right)$ **B** $\left(-\frac{3}{2}, -\frac{7}{2}\right)$ **C** $\left(-\frac{3}{2}, \frac{29}{2}\right)$ **D** $(0, 1)$ **E** $(3, 1)$

5 The polynomial $p(x) = ax^3 + bx^2 - cx + 12$ has a factor of $x - 3$. The remainder when the cubic $ax^3 + bx^2 - cx$ is divided by $x - 3$ is

A -12 **B** -3 **C** 0 **D** 3 **E** 12

9780170448260

Section B 2 questions

15 marks

1 © VCAA 2018N 2BQ1 MODIFIED (11 marks) Let $f: R \rightarrow R, f(x) = x^3 - 4x^2 - 9x + 36$.

a Given $f(x) = (x - 3)(ax^2 + bx + c)$, find a, b and c. 2 marks

b Find the solution to the equation $f(x) = 0$. 1 mark

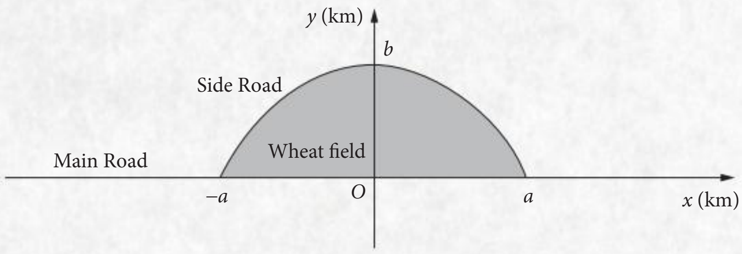
c Find two consecutive integers m and n such that a solution to $f(x) = 0$ is in the interval (m, n), where $m < n < 0$. 2 marks

A straight line is drawn between the points A and B on the function $f(x)$ that have x-coordinates of -2 and 4 respectively.

d Show the equation of the line between A and B is $y = 20 - 5x$. 4 marks

e Find the coordinates of the third point of intersection between $f(x)$ and $y = 20 - 5x$. 2 marks

2 © VCAA 2006 2BQ3 MODIFIED (4 marks)



Tasmania Jones' wheat field lies between two roads as shown in the diagram above.

Main Road lies along the x-axis and Side Road lies along the curve with equation $y = 3 - e^x - e^{-x}$.

a Show, using a suitable substitution, that the equation $3 - e^x - e^{-x} = 0$ can be written as $k^2 - 3k + 1 = 0$. 1 mark

b Solve $k^2 - 3k + 1 = 0$ and hence find the exact value of a. 3 marks

CHAPTER

12 APPLICATIONS OF DIFFERENTIATION

Study Design coverage

UNIT 2, AREA OF STUDY 3: CALCULUS

- applications of differentiation, including finding instantaneous rates of change, stationary values of functions, local maxima or minima, points of inflection, analysing graphs of functions including motion graphs, and solving maximum and minimum problems with consideration of modelling domain and local and global maxima and minima
- anti-differentiation as the inverse process of differentiation and identification of families of curves with the same gradient function, and the use of a boundary condition to determine a specific anti-derivative of a given function.

STUDY DESIGN COVERAGE

Video playlists (9):

12.1 Straight-line motion
12.2 Increasing and decreasing functions
12.3 Stationary points
12.4 Maximum and minimum points
12.5 Curve sketching
12.6 Maximum and minimum problems
12.7 The anti-derivative
12.8 Applying the anti-derivative
VCE question analysis Applications of differentiation

Worksheets (18):

12.1 Straight-line motion 1 • Rates of change • Graphs of rates of change • Instantaneous rates of change
12.2 The sign of the derivative
12.3 Stationary points
12.5 Gradient functions • Sketching curves
12.6 Greatest and least values • Maximum and minimum problems • Optimisation problems • Further optimisation problems • Starting maxima and minima problems • Applications of optimisation
12.7 Anti-derivatives • Antiderivatives • Antidifferentiation
12.8 Straight-line motion 2

Nelson MindTap

To access resources above, visit **cengage.com.au/nelsonmindtap**

12.1 Straight-line motion

Kinematics is a study of straight-line motion using

- **displacement** $= x(t) =$ position of a particle at time t from a chosen origin
- **velocity** $= v(t) =$ velocity at time t, the rate of change of displacement
- **acceleration** $= a(t)$, acceleration at time t, the rate of change of velocity.

Displacement is a 'signed distance' that can be positive or negative.
Velocity is a 'signed speed' that can be positive or negative.

Hence, velocity is the derivative of displacement, and acceleration is the derivative of velocity (and the second derivative of displacement).

Instantaneous velocity and acceleration

$$\text{instantaneous velocity} = \frac{dx}{dt} = v$$

$$\text{instantaneous acceleration} = \frac{d^2x}{dt^2} = \frac{dv}{dt} = a$$

Video playlist Straight-line motion

Worksheets Straight-line motion 1

Rates of change

Graphs of rates of change

Instantaneous rates of change

From the above equations, usually

- x is measured in metres
- t is measured in seconds
- velocity v is measured in m/s (metres per second)
- acceleration a is measured in m/s/s = m/s^2 (metres per second squared).

For example, consider an ant that is travelling over the ground. The ant takes 20 seconds to travel 80 cm.

This means the ant is travelling, on average, $\frac{80}{20} = 4$ cm/s. This is considered the **average rate of change** of the ant's displacement with respect to time.

Average velocity, acceleration and speed

$$\text{average velocity} = \frac{x_2 - x_1}{t_2 - t_1}$$

$$\text{average acceleration} = \frac{v_2 - v_1}{t_2 - t_1}$$

$$\text{average speed} = \frac{\text{distance travelled}}{\text{time taken}}$$

If the ant is not travelling steadily, it is more likely to speed up and slow down when it meets food, so here the ant at a particular point may be travelling using **instantaneous rate of change** at a particular time.

WB

p. 221

WORKED EXAMPLE 1 Differentiating for kinematics

The displacement of a particle travelling in a straight line is given by $x(t) = 2t^3 + t$, with x in metres and t in seconds.

a Find an expression for $v(t)$.

b Find the acceleration of the particle at $t = 3$.

Steps	Working
a Write the function for displacement, then use $\frac{dx}{dt}$ to find the velocity.	$x(t) = 2t^3 + t$ $\therefore\ v(t) = \frac{dx}{dt} = 6t^2 + 1$
b **1** Use $\frac{dv}{dt}$ to find a.	$v(t) = 6t^2 + 1$ $\therefore\ a(t) = \frac{dv}{dt} = 12t$
2 Find acceleration at $t = 3$.	$a(3) = 36$ m/s^2

EXERCISE 12.1 Straight-line motion

ANSWERS p. 530

Mastery

1 WORKED EXAMPLE 1 TECH-FREE The displacement of a particle travelling in a straight line is given by $x(t) = 2t^2 + 1$, where x is in metres and t is in seconds.

a Find an expression for $v(t)$.

b Find the velocity of the particle at $t = 3$.

2 TECH-FREE The displacement of a particle travelling in a straight line is given by $x(t) = 3t^2 + 2t$, where x is in metres and t is in seconds.

- **a** Find an expression for $v(t)$.
- **b** Find the acceleration of the particle at $t = 6$.

3 The displacement of a particle travelling in a straight line is given by $x(t) = -3t^3 + 2t$, where x is in metres and t is in seconds. Its velocity at $t = 1$ is

A $-18t$ **B** -18 **C** $-9t^2 + 2$ **D** -7 **E** 5

4 The above particle's acceleration at $t = 1$ is

A $-18t$ **B** -18 **C** $-9t^2 + 2$ **D** -7 **E** 5

Exam practice

80–100% 60–79% 0–59%

5 © VCAA 2017 1Q1 TECH-FREE (4 marks)

- **a** 79% Let $f: (-2, \infty) \to R, f(x) = \dfrac{x}{x+2}$. Differentiate f with respect to x. 2 marks
- **b** 77% Let $g(x) = (2 - x^3)^3$. Evaluate $g'(1)$. 2 marks

6 © VCAA 2018 1Q1 TECH-FREE (3 marks)

- **a** 58% If $y = (-3x^3 + x^2 - 64)^3$, find $\dfrac{dy}{dx}$. 1 mark
- **b** 71% Let $f(x) = \dfrac{e^x}{\cos(x)}$. Evaluate $f'(\pi)$. 2 marks

7 TECH-FREE (3 marks) A marble dropped into a barrel of water falls such that its height, m, after t seconds is given by $h(t) = 1.5 - 0.1t^2$. Find the height, velocity and acceleration of the marble at 3 seconds.

8 TECH-FREE (5 marks) The displacement in centimetres after time t seconds of a particle moving in a straight line is given by $x = 2 - t - t^2$, $t \geq 0$.

- **a** Find the initial displacement. 1 mark
- **b** Find when the particle will be at the origin. 1 mark
- **c** Find the displacement at 2 seconds. 1 mark
- **d** How far will the particle move in the first 2 seconds? 1 mark
- **e** Find its velocity at 3 seconds. 1 mark

9 TECH-FREE (3 marks) The displacement of a particle is given by $s = t^3 - 4t^2 + 3t$, where s is in metres and t is in seconds.

- **a** Find the initial velocity. 1 mark
- **b** Find the times when the particle will be at the origin. 1 mark
- **c** Find the acceleration at 3 seconds. 1 mark

10 TECH-FREE (4 marks) An object is travelling along a straight line over time t seconds, with displacement (in metres) according to the equation $x = t^3 + 6t^2 - 2t + 1$.

- **a** Find the equations for its velocity and acceleration. 1 mark
- **b** What will its displacement be at 5 seconds? 1 mark
- **c** What will its velocity be at 5 seconds? 1 mark
- **d** Find its acceleration at 5 seconds. 1 mark

11 TECH-FREE (6 marks) The displacement of an object from a particular point is given by $x(t) = t^3 - 5t^2 + 6t + 10$, where $x(t)$ is in metres and t is in seconds.

a Find the object's average velocity from 2 seconds to 4 seconds. 1 mark

b Find the velocity at 2 seconds. 1 mark

c Find the velocity at 4 seconds. 1 mark

d Find the average of the velocities at 2 seconds and 4 seconds and compare it to the answer to part **a**. 1 mark

e Find the acceleration at 2 seconds. 1 mark

f Find the acceleration at 5 seconds. 1 mark

12 TECH-FREE (4 marks) A projectile is fired into the air and its height in metres is given by $h = 40t - 5t^2 + 4$, where t is in seconds.

a Find its initial height. 1 mark

b Find the initial velocity. 1 mark

c Find the height after 1 seconds. 1 mark

d What is the maximum height of the projectile? 1 mark

13 TECH-FREE (6 marks) A particle is moving such that its displacement is given by $s = 2t^2 - 8t + 3$, where s is in metres and t is in seconds.

a Find its initial velocity. 1 mark

b Show that its acceleration is constant and find its value. 1 mark

c Find the displacement after 5 seconds. 1 mark

d Find when the particle will be at rest. 1 mark

e What will the particle's displacement be at that time? 1 mark

f Sketch the graph of the displacement against time. 1 mark

14 TECH-FREE (4 marks) The displacement of a particle is given by $x = t^3 - 9t$ cm, where t is in seconds.

a Find the velocity of the particle after 3 seconds. 2 marks

b Find the acceleration after 2 seconds. 2 marks

15 TECH-FREE (4 marks) A car is travelling on a day's outing with displacement expressed by $x(t) = 10(t + 1)^2$, where x is the displacement from the driver's home in kilometres and t is the time taken in hours.

a Find its velocity exactly 5 hours from home. 2 marks

b Find its acceleration exactly 5 hours from home. 2 marks

16 TECH-FREE (2 marks) A toy car is travelling with a velocity expressed by $v(t) = 2t^2 + 1$ m/s. Find its acceleration after travelling for 5 seconds.

17 The displacement of a particle travelling in a straight line is given by $x(t) = t^3 + 3t^2$, where x is in metres and t is in seconds. Its velocity, in m/s, is

A 4 **B** $3t^2 + 6t$ **C** $6t + 6$ **D** 9 **E** 12

18 The above particle's velocity, in m/s, at $t = 1$ is

A 4 **B** $3t^2 + 6t$ **C** $6t + 6$ **D** 9 **E** 12

19 The velocity of a particle travelling in a straight line is given by $v = (2t - 1)^2$, where v is in m/s and t is in seconds. Its acceleration, in m/s^2, at $t = 1$ is

A 0 **B** $2(2t - 1)$ **C** 2 **D** 3 **E** 4

12.2 Increasing and decreasing functions

The **gradient** of a function describes the direction of its graph. When a function is increasing, it is moving upwards to the right. When a function is decreasing, it is moving downwards to the right. We need to identify the intervals over which a function is **constant**, **stationary**, **strictly increasing** or **strictly decreasing**.

12.2

Video playlist Increasing and decreasing functions

Worksheet The sign of the derivative

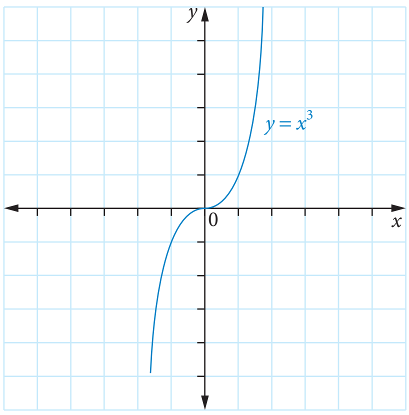
Positive and negative gradients

A positive gradient points up:

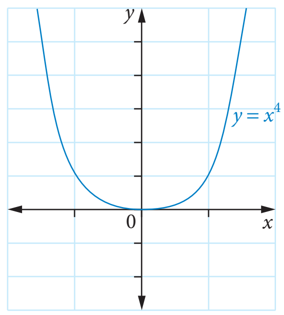
A negative gradient points down:

The gradient of the graph of $y = f(x)$ is the derivative function $f'(x)$ or $\dfrac{dy}{dx}$.

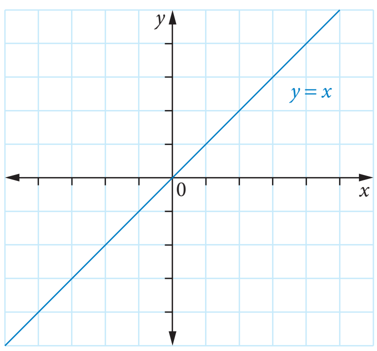
1 Consider the graph of $y = x^3$. All of this graph except at $x = 0$, has a positive gradient, pointing up to the right. At $x = 0$ the gradient is 0. So the graph has a positive gradient for $x \in (-\infty, 0) \cup (0, \infty)$.



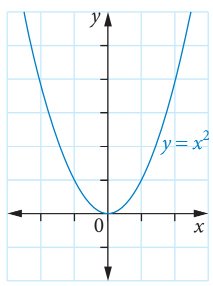
2 Consider the graph of $y = x^4$. Only the right half of the graph has a positive gradient, pointing up to the right. The left half of the graph has a negative gradient, and at $x = 0$, the gradient is 0. So the graph has a positive gradient for $x \in (0, \infty)$.



3 Consider the graph of $y = x$. This entire graph (line) has a positive gradient. So the graph has a positive gradient for $x \in (-\infty, \infty)$.



4 Consider the graph of $y = x^2$. Only the right half of the graph has a positive gradient. The left half of the graph has a negative gradient, and at $x = 0$, the gradient is 0. So the graph has a positive gradient for $x \in (0, \infty)$.



WB

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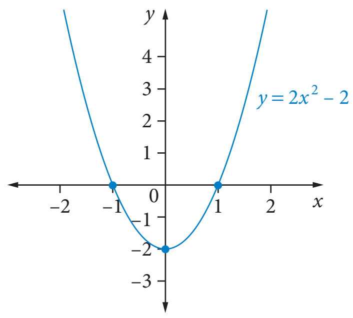
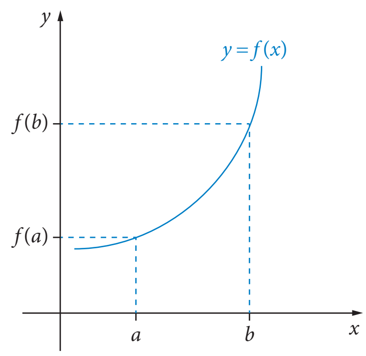
WORKED EXAMPLE 2 Positive and negative gradients

State the intervals over which the graph with the rule $y = 2x^2 - 2$ has a positive and negative gradient.

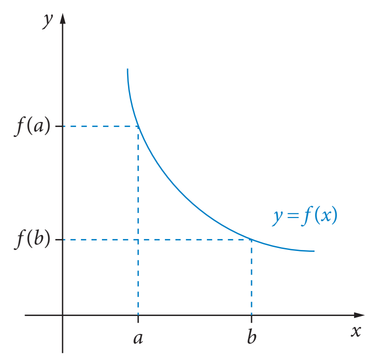
Steps	Working
1 Sketch the graph of $y = 2x^2 - 2$.	
2 Find the x value of the turning point from the graph.	Turning point at $x = 0$.
3 Identify ↗ or ↘ gradients. Positive for $x > 0$, negative for $x < 0$.	Gradient is positive for $x \in (0, \infty)$. Gradient is negative for $x \in (-\infty, 0)$.

Strictly increasing and decreasing

A function is **strictly increasing** where the values of y are getting larger: if $a < b$, then $f(a) < f(b)$. Similarly, a function is **strictly decreasing** where the values of y are getting smaller: if $a < b$, then $f(a) > f(b)$.

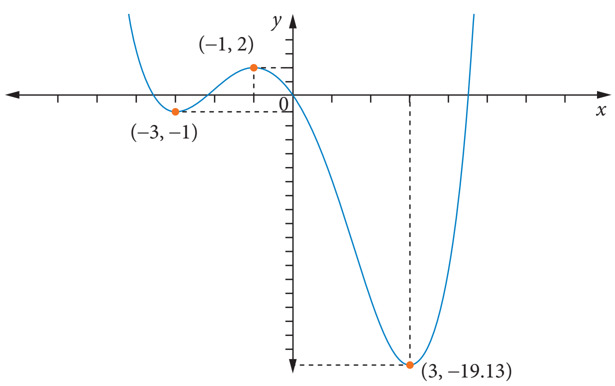


When $f(x)$ is strictly increasing
$f(a) < f(b)$



When $f(x)$ is strictly decreasing
$f(a) > f(b)$

Consider the graph of a quartic function as shown below



The graph is strictly increasing in two segments: $x \in [-3, -1] \cup [3, \infty)$.

The graph is strictly decreasing in two segments: $x \in (-\infty, -3] \cup [-1, 3]$.

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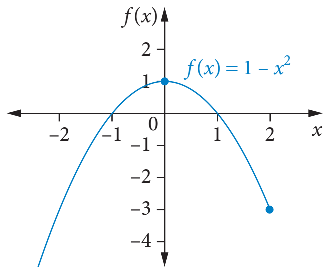
For strictly increasing and strictly decreasing, the endpoints of the intervals are *included*, so use [ ], except for ∞ or –∞ when we use ( ).

WORKED EXAMPLE 3 Strictly increasing and decreasing

State the intervals over which the function with the rule $f(x) = 1 - x^2$ for $x \leq 2$ is strictly increasing and strictly decreasing.

Steps	Working
1 Sketch the graph of $f(x) = 1 - x^2$ with the stated domain. **Exam hack** Always sketch a graph.	
2 Find the x value of the turning point from the graph.	Turning point at $x = 0$.
3 From the graph, $f(x)$ is strictly increasing for $x \leq 0$, and strictly decreasing for $0 \leq x \leq 2$.	Strictly increasing for $x \in (-\infty, 0]$. Strictly decreasing for $x \in [0, 2]$.

EXERCISE 12.2 Increasing and decreasing functions

ANSWERS p. 530

Recap

1 The displacement of a particle travelling in a straight line is given by $x(t) = 2t^2 + 1$, where x is in metres and t is in seconds.

a Find an expression for $v(t)$.

b Find $a(t)$.

2 The average rate of change of $y = 2\log_e(x + 1)$ for $x \in [0, 2]$ is

A $2\log_e(2)$ B $\log_e(2)$ C $\log_e(3)$ D 0 E 1

Mastery

3 WORKED EXAMPLE 2 TECH-FREE State the interval over which the graph with the rule $y = x^2 - 2$ has a positive gradient.

4 TECH-FREE State the interval over which the graph with the rule $y = x^2 - 2$ has a negative gradient.

5 WORKED EXAMPLE 3 TECH-FREE State the intervals over which the graph with the rule $y = \frac{x^4}{4} - 2x^2$ is strictly increasing.

6 The interval over which the graph with the rule $y = 2x^4 - 32$ has a negative gradient is

A $(-\infty, 0) \cup (0, \infty)$ B $(-\infty, 0)$ C $(0, \infty)$

D $(-\infty, -2) \cup (2, \infty)$ E $(-\infty, 0) \cup (2, \infty)$

7 The interval over which the graph with the rule $y = 2x^4 - 32$ is strictly increasing is

A $(-\infty, 0) \cup (0, \infty)$ B $(-\infty, 0]$ C $[0, \infty)$

D $(-\infty, -2] \cup [2, \infty)$ E $(-\infty, 0] \cup [2, \infty)$

12.2

WB

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Exam practice

80–100% 60–79% 0–59%

8 TECH-FREE (3 marks) State the interval where the function $f(x) = 4x^3 - 12x^2 + 12x$

a has a positive gradient 2 marks

b is strictly increasing. 1 mark

9 TECH-FREE (3 marks) Consider the function $f(t) = t^3 - 3t^2 + t - 5$. Show that the function is

a increasing when $t = -1$ **b** decreasing when $t = 1$ **c** increasing at $t = 3$

10 © VCAA 2010S 1Q4 TECH-FREE (3 marks) State the subset of R for which the graph of the function $f(x) = x^4 - x^2$ is strictly decreasing.

11 TECH-FREE (3 marks) For $w = m^3 - 12m + 9$, state whether the function is strictly increasing or strictly decreasing at

a $m = -4$ **b** $m = -1$ **c** $m = 0$

12 TECH-FREE (5 marks) For $y = x^2 - 4x$, state whether y is strictly increasing or strictly decreasing at

a $x = -4$ **b** $x = -3$ **c** $x = -2$ **d** $x = 1$ **e** $x = 3$

13 TECH-FREE (3 marks) Find values of x for which the curve $y = x^2 - 3x - 4$

a has a negative gradient **b** has a positive gradient **c** has zero gradient.

14 TECH-FREE (3 marks) Find the domain over which the function $f(x) = 4 - x^2$ is strictly increasing.

15 TECH-FREE (3 marks) Find all values of x for which the curve $y = 2x^2 - x$ is strictly decreasing.

16 © VCAA 2009 2AQ8 79% For the function $f: R \to R, f(x) = (x + 5)^2(x - 1)$, the subset of R for which the gradient of f is negative is

A $(-\infty, 1)$ **B** $(-5, 1)$ **C** $(-5, -1)$ **D** $(-\infty, -5)$ **E** $(-5, 0)$

17 © VCAA 2010 2AQ16 78% The gradient of the function $f: R \to R, f(x) = \dfrac{5x}{x^2 + 3}$ is negative for

A $-\sqrt{3} < x < \sqrt{3}$ **B** $x > 3$ **C** $x \in R$

D $x < -\sqrt{3}$ and $x > \sqrt{3}$ **E** $x < 0$

18 © VCAA 2007 2AQ13 68% For the graph of $y = 4x^3 + 27x^2 - 30x + 10$, the subset of R for which the gradient is negative is given by the interval

A $(0.5, 5.0)$ **B** $(-4.99, 0.51)$ **C** $\left(-\infty, \dfrac{1}{2}\right)$ **D** $\left(-5, \dfrac{1}{2}\right)$ **E** $(2.25, \infty)$

19 The interval over which the function with the rule $f(x) = (x - 1)^3(x + 2)$ is strictly increasing is

A $[-1, 1.25]$ **B** $[-1.25, 1]$ **C** $[-1.25, \infty)$

D $[-1.25, -1] \cup [1, \infty)$ **E** $(-\infty, -1] \cup [1, \infty)$

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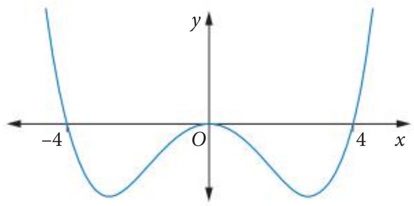
20 The interval over which the graph with the rule $y = -3(x-1)^3$ has a negative gradient is

A $[-1, 1)$ **B** $[-1, 1]$ **C** $R \setminus \{1\}$ **D** $R \setminus \{0\}$ **E** $(-\infty, -1)$

21 © VCAA 2021N 2AQ8 Part of the graph of a polynomial function f is shown below. This graph has turning points at $\left(-2\sqrt{2}, -1\right)$ and $\left(2\sqrt{2}, -1\right)$.



$f(x)$ is strictly decreasing for

A $x \in (-\infty, -4] \cup [4, \infty)$

B $x \in [-4, 4]$

C $x \in \left[-2\sqrt{2}, 2\sqrt{2}\right]$

D $x \in \left(-\infty, -2\sqrt{2}\right] \cup \left[0, 2\sqrt{2}\right]$

E $x \in \left[-2\sqrt{2}, 0\right] \cup \left[2\sqrt{2}, \infty\right)$

12.3

12.3 Stationary points

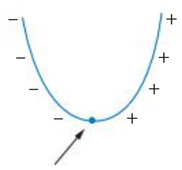
Video playlist
Stationary points

Worksheet
Stationary points

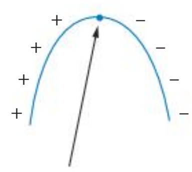
A **stationary point** on the graph of a function is where there is zero gradient, $f'(x) = 0$, such that the graph is flat, neither increasing nor decreasing. There are 3 types of stationary points: a **local minimum,** a **local maximum** (both called **turning points**) or a **stationary point of inflection**.

Turning points

A turning point is found where $f'(x) = 0$ and the sign of the gradient changes on either side of the stationary point.



Local minimum point



Local maximum point

Consider the graph of $f(x) = 2x^3 - 6x$.

The turning points are found where $f'(x) = 0$.

$f'(x) = 6x^2 - 6 = 0$

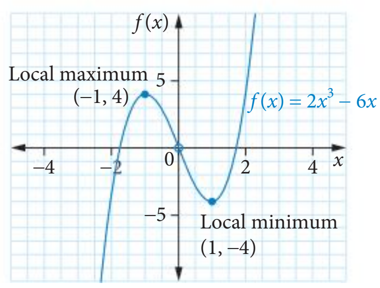
$6(x-1)(x+1) = 0.$

$\therefore\ x = 1$ or $x = -1$

Substitute $x = 1$ and $x = -1$ into $f(x)$ to get the coordinates of the stationary points as $(1, -4)$ and $(-1, 4)$.

On either side of the point $(-1, 4)$, the gradient changes from positive to negative, giving a local maximum turning point.

On either side of the point $(1, -4)$, the gradient changes from negative to positive, giving a local minimum turning point.



WB

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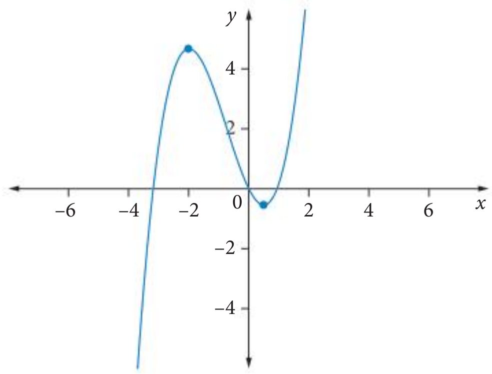
WORKED EXAMPLE 4 Turning points

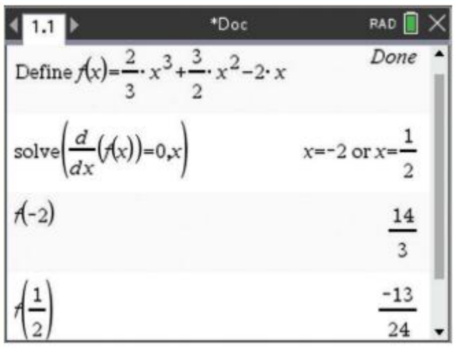
Find the coordinates of the turning points in the graph $f(x) = \frac{2}{3}x^3 + \frac{3}{2}x^2 - 2x$.

Steps	Working
1 Sketch the graph of $f(x) = \frac{2}{3}x^3 + \frac{3}{2}x^2 - 2x$.	
2 Find $f'(x)$ and solve for zero.	$f'(x) = 2x^2 + 3x - 2$ $f'(x) = 0$ for $x = -2$ and $x = \frac{1}{2}$.
3 Identify if these are turning points by checking whether the sign of the gradient changes on either side.	At $x = -2$, gradient changes from positive to negative. At $x = \frac{1}{2}$, gradient changes from negative to positive. Both are turning points.
4 Substitute x values into $f(x)$ for coordinates.	$f(-2) = \frac{14}{3}$ $\therefore \left(-2, \frac{14}{3}\right)$ is a local maximum turning point. $f\left(\frac{1}{2}\right) = -\frac{13}{24}$ $\therefore \left(\frac{1}{2}, -\frac{13}{24}\right)$ is a local minimum turning point.

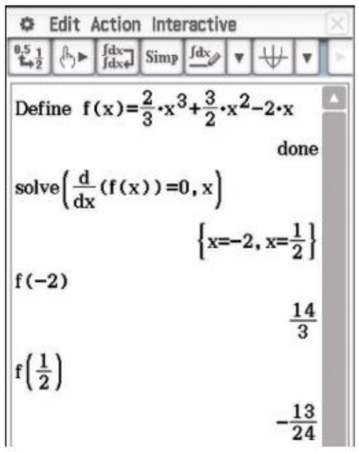
USING CAS 1 Turning points

Confirm the coordinates of the turning points for the graph $f(x) = \frac{2}{3}x^3 + \frac{3}{2}x^2 - 2x$ from the previous example.

TI-Nspire



ClassPad



1 Define the function **$f(x)$**.

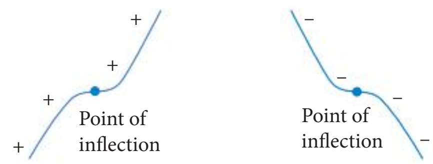
2 Set the **derivative of $f(x)$** equal to **0** and **solve** to find the x-coordinates of the turning points.

3 Substitute the solutions into **$f(x)$** to determine the corresponding y-coordinates.

The coordinates of the turning points are $\left(-2, \frac{14}{3}\right)$ and $\left(\frac{1}{2}, -\frac{13}{24}\right)$.

Stationary points of inflection

A **stationary point of inflection** is a flat 'bend' in the graph where $f'(x) = 0$ and the concavity of the graph changes: from concave down to concave up, or concave up to concave down. Also, the sign of the gradient stays *the same* on both sides of the stationary point. A stationary point is a point where the graph is instantaneously zero.



Concave down to concave up

Concave up to concave down

Consider the graph of $f(x) = (x - 2)^3 - 27$.

$f'(x) = 3(x - 2)^2 = 0$ for stationary points.

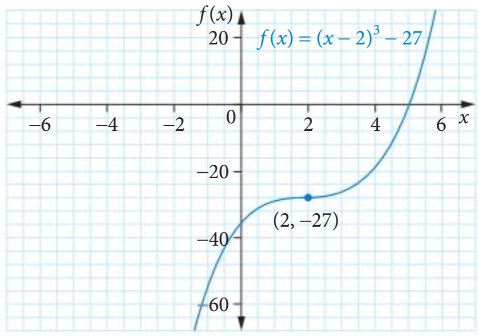
Solving, we get $x = 2$.

The points of inflection are found where $f''(x) = 6(x - 2) = 0$ and the concavity of the graph changes.

Again, solving we get $x = 2$.

Substitute $x = 2$ into $f(x)$ to get the y-coordinate of the stationary point of inflection as $f(2) = -27$.

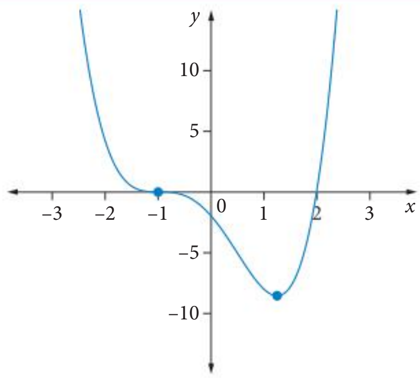
In the graph below, the graph changes from concave down to concave up at (2, –27) and the sign of the gradient stays positive on both sides.

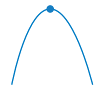
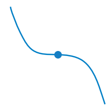


WB
p. 225

WORKED EXAMPLE 5 Stationary points

Find the coordinates of any stationary points of inflection in the graph of $f(x) = x^4 + x^3 - 3x^2 - 5x - 2$.

Steps	Working
1 Sketch the graph of $f(x) = x^4 + x^3 - 3x^2 - 5x - 2$.	
2 Find $f'(x)$ and solve for zero.	$f'(x) = 4x^3 + 3x^2 - 6x - 5 = 0$ $f'(x) = 0$ for $x = -1$ and $x = \frac{5}{4}$.
3 Check whether the sign of the gradient stays the same on either side to identify the stationary point of inflection.	At $x = -1$, gradient is negative on either side, so it is a stationary point of inflection. At $x = \frac{5}{4}$, gradient changes from negative to positive so it is a turning point.
4 Alternatively, find $f''(x)$ and solve for zero.	$f''(x) = 12x^2 + 6x - 6 = 0$ $f''(x) = 0$ for $x = -1$ and $x = \frac{1}{2}$. Identify $f'(x) = 0$ and $f''(x) = 0$ for $x = -1$.
5 Find the coordinates of the stationary point of inflection.	$f(-1) = 0$ Stationary point of inflection at $(-1, 0)$.

$f'(x)$ before the stationary point	$f'(x)$ at the stationary point	$f'(x)$ after the stationary point	Type of point
+	0	−	Local maximum point 
−	0	+	Local minimum point 
+	0	+	Stationary point of inflection 
−	0	−	Stationary point of inflection 

EXERCISE 12.3 Stationary points

ANSWERS p. 531

Recap

1 TECH-FREE

a Solve, for x, the equation $(2x+3)^2 = 0$.

b Hence, state the interval where the graph of $y = (2x+3)^2$ is strictly increasing.

c State the interval where the graph of $y = (2x+3)^2$ has a positive gradient.

2 TECH-FREE Differentiate the expression $4m^2 - 16m + 1$.

Mastery

3 WORKED EXAMPLE 4 TECH-FREE Find the coordinates of the turning points in the graph of $f(x) = x^3 + x^2 - 2$.

4 WORKED EXAMPLE 5 TECH-FREE Find the coordinates of any stationary point of inflection in the graph of $f(x) = (x+2)^3 - 2$.

5 Using CAS 1 The turning point of the graph of $f(x) = 4x^2 - 16x + 2$ is

A $(2, 0)$ **B** $(2, 14)$ **C** $(-2, 50)$ **D** $(-2, 0)$ **E** $(2, -14)$

6 The stationary point of the graph of $y = 4m^2 - 16m + 1$ is at

A $m = -4$ **B** $m = -2$ **C** $m = 2$ **D** $m = \pm 2$ **E** $m = 4$

7 TECH-FREE Find any stationary points of each function.

a $f(x) = 2x^2 - 8x + 3$ **b** $f(x) = x^3 - x^2 - 8x + 10$

Exam practice

80–100% 60–79% 0–59%

8 TECH-FREE (2 marks) Find any stationary points on the curve $y = (x-2)^4$.

9 TECH-FREE (3 marks) The function $f(x) = 2x^2 + px + 7$ has a stationary point at $x = 3$. Find p.

10 The turning point of the graph with the rule $y = x^2 - 3x + 2$ is at

A $x = \frac{2}{3}$ **B** $x = 1$ **C** $x = \frac{3}{2}$ **D** $x = 2$ **E** $x = 3$

11 The stationary point of inflection of the graph with the rule $y = 2(x + 1)^3 + 7$ is at

A $x = -1$ **B** $x = 1$ **C** $x = 2$ **D** $x = 7$ **E** $(3x - 5)(3x - 5)$

12 The stationary point of inflection of the graph with the rule $y = -2(x + 1)^3 + 7$ has coordinates

A $(1, 0)$ **B** $(1, 7)$ **C** $(-1, 0)$ **D** $(-1, 7)$ **E** $(-2, 7)$

13 The turning point of the graph with the rule $y = -2(x + 1)^2 + 7$ has coordinates

A $(1, 0)$ **B** $(1, 7)$ **C** $(-1, 0)$ **D** $(-1, 7)$ **E** $(-2, 7)$

14 The stationary point(s) of the graph with the rule $y = \frac{1}{3}x^3 + x^2 - 3x + 1$ is/are

A $(-3, 1)$ and $(1, -3)$ **B** $(-3, 10)$ and $\left(1, -\frac{2}{3}\right)$ **C** $(10, -3)$ and $\left(-\frac{2}{3}, 1\right)$

D $\left(-1, \frac{14}{3}\right)$ **E** $(-3, 10)$, $\left(1, -\frac{2}{3}\right)$ and $\left(-1, \frac{14}{3}\right)$

15 © VCAA 2009 2AQ2 96% At the point $(1, 1)$ on the graph of the function with rule $y = (x - 1)^3 + 1$

A there is a local maximum.
B there is a local minimum.
C there is a stationary point of inflection.
D the gradient is not defined.
E there is a point of discontinuity.

16 © VCAA 2007 2AQ12 69% Let $f: R \to R$ be a differentiable function such that

- $f'(3) = 0$
- $f'(x) < 0$ when $x < 3$ and when $x > 3$

When $x = 3$, the graph of f has a

A local minimum.
B local maximum.
C stationary point of inflection.
D point of discontinuity.
E gradient of 3.

17 © VCAA 2018 2AQ5 67% Consider $f(x) = x^2 + \frac{p}{x}, x \neq 0, p \in R$. There is a stationary point on the graph of f when $x = -2$. The value of p is

A -16 **B** -8 **C** 2 **D** 8 **E** 16

18 © VCAA 2018 2AQ9 57% A tangent to the graph of $y = \log_e(2x)$ has a gradient of 2. This tangent will cross the y-axis at

A 0 **B** -0.5 **C** -1 **D** $-1 - \log_e(2)$ **E** $-2\log_e(2)$

19 © VCAA 2009 2AQ21 43% A cubic function has the rule $y = f(x)$. The graph of the derivative function f' crosses the x-axis at $(2, 0)$ and $(-3, 0)$. The maximum value of the derivative function is 10. The value of x for which the graph of $y = f(x)$ has a local maximum is

A -2 **B** 2 **C** -3 **D** 3 **E** $-\frac{1}{2}$

20 © VCAA 2013 2AQ21 29% The cubic function $f: R \to R, f(x) = ax^3 - bx^2 + cx$, where a, b and c are positive constants, has no stationary points when

A $c > \frac{b^2}{4a}$ **B** $c < \frac{b^2}{4a}$ **C** $c < 4b^2a$ **D** $c > \frac{b^2}{3a}$ **E** $c < \frac{b^2}{3a}$

12.3

21 The function $f(x)$ has a derivative that can be factorised to $f'(x) = (x + 2)(1 - x)$.
$f(x)$ has

A a minimum at $x = 2$ and a maximum at $x = -1$

B a maximum at $x = 2$ and a minimum at $x = -1$

C a minimum at $x = -2$ and a maximum at $x = 1$

D a maximum at $x = -2$ and a minimum at $x = -1$

E points of inflection at $x = -2$ and $x = -1$.

22 The function $f(x) = 2x^3 + 3x^2 - 36x + 7$ has a

A minimum at $(-3, 88)$ **B** maximum at $(-3, 88)$

C minimum at $(3, -88)$ **D** maximum at $(3, 88)$

E maximum at $(2, 37)$.

23 A cubic function has

A at most 1 stationary point **B** at most 2 stationary points

C any number of stationary points **D** at least 2 stationary points

E 3 stationary points.

24 © VCAA 2021N 2AQ4 The graph of $f: R \to R, f(x) = x^3 + ax^2 + bx + c$ has a turning point at $x = 3$ and a y-intercept at $y = 9$. The values of a, b and c could be, respectively

A -5, 3 and 9 **B** 7, -15 and -9 **C** -2, $-\frac{3}{2}$ and 9

D 5, -3 and -9 **E** -1, -3 and 9

25 a © VCAA 2006S 2BQ1 (6 marks) Consider the function $f: [0, 2] \to R, f(x) = ax^3 + bx^2 + cx + 2$.
The graph of $y = f(x)$ has a turning point at the point with coordinates $(1, 1)$.

i Find the values of a and b in terms of c. 2 marks

ii If also $f(2) = 0$, find the value of c. 1 mark

b Consider the function $f: [0, 2] \to R, f(x) = (x - 1)^2(x - 2) + 1$.

i Find $f'(x)$. 1 mark

ii The coordinates of the turning point of the graphs of $y = f(x)$ occur at $(m, 1)$ and $\left(n, \frac{23}{27}\right)$. Find the values of m and n. 2 marks

26 © VCAA 2011 2BQ3a bi di (6 marks)

a Consider the function $f: R \to R, f(x) = 4x^3 + 5x - 9$.

i 95% Find $f'(x)$. 1 mark

ii 47% Explain why $f'(x) \geq 5$ for all x. 1 mark

b 9% The cubic function p is defined by $p: R \to R, p(x) = ax^3 + bx^2 + cx + k$, where a, b, c and k are real numbers.

If p has m stationary points, what possible values can m have? 1 mark

c 44% The cubic function g is defined by $g: R \to R, g(x) = x^3 + 2x^2 + cx + k$, where c and k are real numbers. If g has exactly one stationary point, find the value of c. 3 marks

27 © VCAA 2010 2BQ4 (14 marks) Consider the function $f: R \to R, f(x) = \frac{1}{27}(2x-1)^3(6-3x)+1$.

a 70% Find the x-coordinate of each of the stationary points of f and state the nature of each of these stationary points. 4 marks

In the following, f is the function $f: R \to R, f(x) = \frac{1}{27}(ax-1)^3(b-3x)+1$ where a and b are real constants.

b 46% Write down, in terms of a and b, the possible values of x for which $(x, f(x))$ is a stationary point of f. 3 marks

c 47% For what value of a does f have no stationary points? 1 mark

d 16% Find a in terms of b if f has one stationary point. 2 marks

e 37% What is the maximum number of stationary points that f can have? 1 mark

f 9% Assume that there is a stationary point at $(1, 1)$ and another stationary point (p, p), where $p \neq 1$. Find the value of p. 3 marks

Video playlist Maximum and minimum points

12.4 Maximum and minimum points

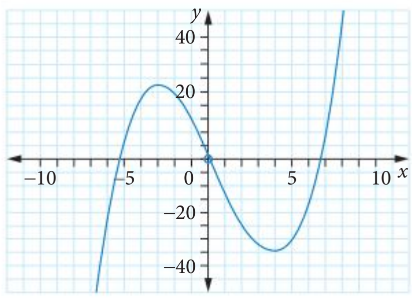
A stationary point on the graph of a function is where there is zero gradient, $f'(x) = 0$, such that the graph is flat, neither increasing nor decreasing. There are 3 types of stationary points: a local minimum, local maximum called turning points or a stationary point of inflection.

A turning point is found where $f'(x) = 0$ and the sign of the gradient changes on either side of the stationary point.

Turning points are local maximum or local minimum points.

Consider a simple cubic graph with the rule $y = \frac{1}{3}x^3 - \frac{1}{2}x^2 - 12x$.

The graph will have the shape of the positive cubic graph where we will expect a local maximum turning point then a local minimum turning point.



If $y = \frac{1}{3}x^3 - \frac{1}{2}x^2 - 12x$, then $\frac{dy}{dx} = x^2 - x - 12$.

Factorising this we get $\frac{dy}{dx} = (x+3)(x-4)$.

Solving for zero, we get stationary points at $x = -3$ and $x = 4$.

Substituting into the rule, we can see from the shape of the graph, we get

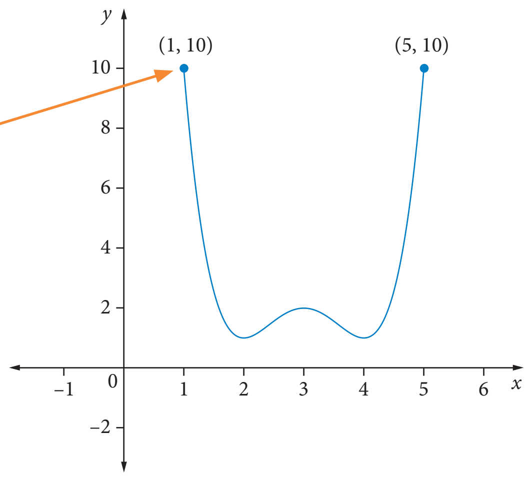
- local maximum turning point at $\left(-3, \frac{45}{2}\right)$
- local minimum turning point at $\left(4, -\frac{104}{3}\right)$.

Exam hack

Remember to always sketch a graph since it gives information about where the gradient changes either side of the turning point.

WORKED EXAMPLE 6 Maximum and minimum points

Find the local maximum and minimum points for the function $h(x) = (x-2)^2(x-4)^2 + 1$ for the interval $x \in [1, 5]$.

Steps	Working
1 Sketch the graph of $h(x) = (x-2)^2(x-4)^2 + 1$ for $x \in [1, 5]$. Notice that the highest value of this restricted graph is $y = 10$. This is called the **global** maximum point, not the **local** maximum point.	
2 Expand $h(x)$ to find $h'(x)$ and solve for zero.	$h(x) = (x-2)^2(x-4)^2 + 1$ $= x^4 - 12x^3 + 52x^2 - 96x + 65$ $h'(x) = 4x^3 - 36x^2 + 104x - 96$ $= 4(x^3 - 9x^2 + 26x - 24)$ $= 4(x-2)(x-3)(x-4)$ $h'(x) = 0$ for $x \in [1, 5]$ gives $x = 2, 3, 4$.
3 Identify the local maximum points.	From the graph, it can be seen that $x = 3$ is a local maximum point. $h(3) = 2$
4 Identify the local minimum points.	From the graph, it can be seen that $x = 2$ and $x = 4$ are local minimum points. $h(2) = 1$, $h(4) = 1$
5 Identify the turning points.	The local maximum point is $(3, 2)$. The local minimum points are $(2, 1)$ and $(4, 1)$.

12.4

WB

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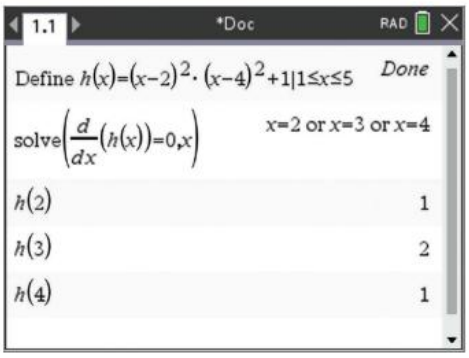
USING CAS 2 Maximum and minimum points

a Confirm the local maximum and minimum points for the function $h(x) = (x - 2)^2(x - 4)^2 + 1$ for the interval $x \in [1, 5]$ from the previous example.

b Find the stationary points of the graph of $y = \frac{1}{3}x^3 - \frac{1}{2}x^2 - 12x$.

TI-Nspire

a

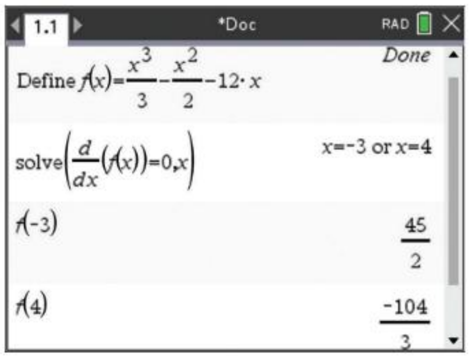


1 Define the function ***h*(*x*)**.

2 Set the **derivative of *h*(*x*)** equal to **0** and **solve** to find the x-coordinates of the stationary points.

3 Substitute the solutions into ***h*(*x*)** to determine the corresponding y-coordinates.

b



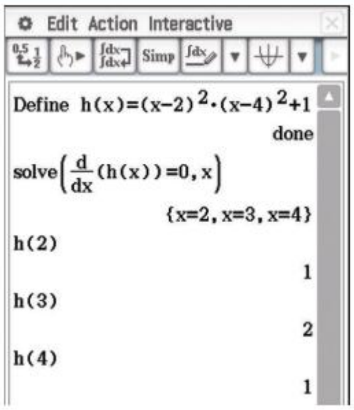
1 Define the function ***h*(*x*)**.

2 Set the **derivative of *h*(*x*)** equal to **0** and **solve** to find the x-coordinates of the stationary points.

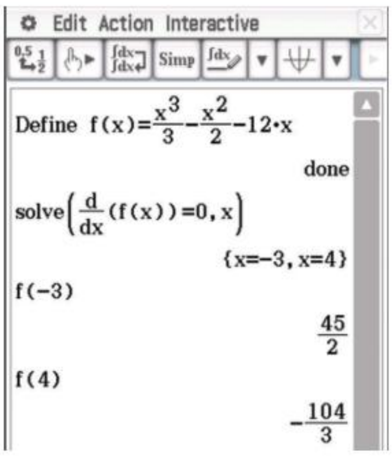
3 Substitute the solutions into ***h*(*x*)** to determine the corresponding y-coordinates.

ClassPad

a



b



Follow the same steps as in part **a** but use ***f*(*x*)** instead of $h(x)$ for the function y.

Follow the same steps as in part **a** but use ***f*(*x*)** instead of $h(x)$ for the function y.

a The coordinates of the stationary points are (2, 1), (3, 2) and (4, 1).

b The coordinates of the stationary points are $\left(-3, \frac{45}{2}\right)$ and $\left(4, -\frac{104}{3}\right)$.

EXERCISE 12.4 Maximum and minimum points

ANSWERS p. 531

12.4

Recap

1 The equation $e^x(e^x - 9) = 0$ gives the solution(s)

A $x = 0, x = 9$ **B** $x = 0, x = -9$ **C** $x = 0, x = \log_e(9)$
D $x = 2\log_e(3)$ **E** $x = 0, x = \pm 3$

2 The stationary point on the graph of $y = 2x^2 - 1$ is at

A $x = -1$ **B** $x = 0$ **C** $x = \frac{1}{2}$ **D** $x = 1$ **E** $x = 2$

Mastery

3 TECH-FREE Find the coordinates of the turning point of $y = (x - 5)^2$.

4 TECH-FREE Find the x value of the turning points of a graph given that $f'(x) = (x - 5)(x + 5)$.

5 TECH-FREE Find the x values of the turning points of a graph given that $f'(x) = (4x - 5)(4x + 5)$.

6 TECH-FREE Find the coordinates of the stationary point of the graph given $f(x) = x^2 + 4x - 5$.

7 WORKED EXAMPLE 6 TECH-FREE Find the local minimum point for the function $h(x) = (x - 2)(x - 4)^2$ for the interval $x \in [1, 5]$.

8 TECH-FREE Find the local maximum point for the function $h(x) = (x - 2)(x - 4)^2$ for the interval $x \in [1, 5]$.

9 The turning point in the graph of $y = -2(x + 2)^2 + 1$ is

A (2, 0) **B** (–2, 1) **C** (–2, 0) **D** (2, –1) **E** (1, –2)

10 The local minimum turning point in the graph of $y = 3x^2 - 9x$ is

A (3, 0) **B** (1.5, 0) **C** (1.5, –6.75)
D (–6.75, 1.5) **E** (3, 0) and (0, 0)

11 Using CAS 2 The turning points of the graph of $y = x^3 - x^2 - x$ are

A (1, 0) and $\left(-\frac{1}{3}, 0\right)$ **B** (1, –1) only **C** $\left(-\frac{1}{3}, \frac{5}{27}\right)$ only
D $\left(-\frac{1}{3}, 0\right)$ only **E** $\left(-\frac{1}{3}, \frac{5}{27}\right)$ and (1, –1)

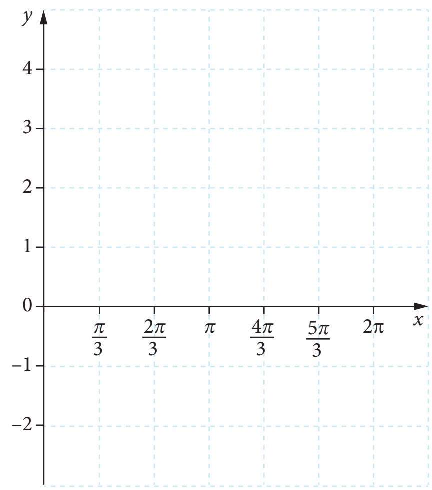
Exam practice

80–100% 60–79% 0–59%

12 © VCAA 2018 1Q3 TECH-FREE (5 marks) Let $f: [0, 2\pi] \to R, f(x) = 2\cos(x) + 1$.

a 81% Solve the equation $2\cos(x) + 1 = 0$ for $0 \le x \le 2\pi$. 2 marks

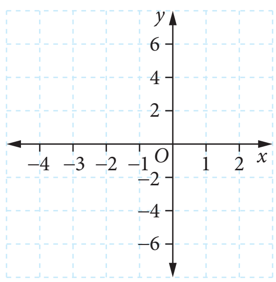
b 78% Copy the axes below and on them sketch the graph of the function f. Label the endpoints and local minimum point with their coordinates. 3 marks



13 © VCAA 2017 1Q3 TECH-FREE (4 marks) Let $f: [-3, 0] \to R, f(x) = (x+2)^2(x-1)$.

a 79% Show that $(x+2)^2(x-1) = x^3 + 3x^2 - 4$. 1 mark

b 68% Copy the axes below and on them sketch the graph of f. Label the axis intercepts and any stationary points with their coordinates. 3 marks



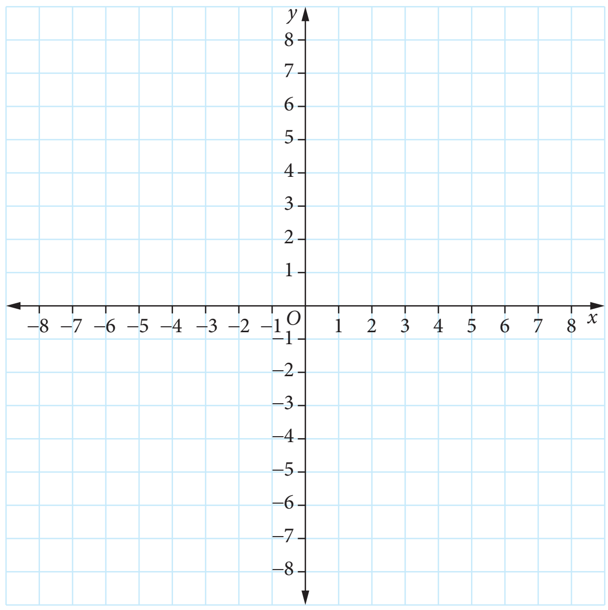
14 © VCAA 2015 1Q4 TECH-FREE (4 marks) Consider the function

$$f: [-3, 2] \to R, f(x) = \frac{1}{2}(x^3 + 3x^2 - 4)$$

a 74% Find the coordinates of the stationary points of the function. 2 marks

The rule for f can also be expressed as $f(x) = \frac{1}{2}(x-1)(x+2)^2$.

b 69% On the axes below, sketch the graph of f, clearly indicating axis intercepts and turning points. Label the endpoints with their coordinates. 2 marks

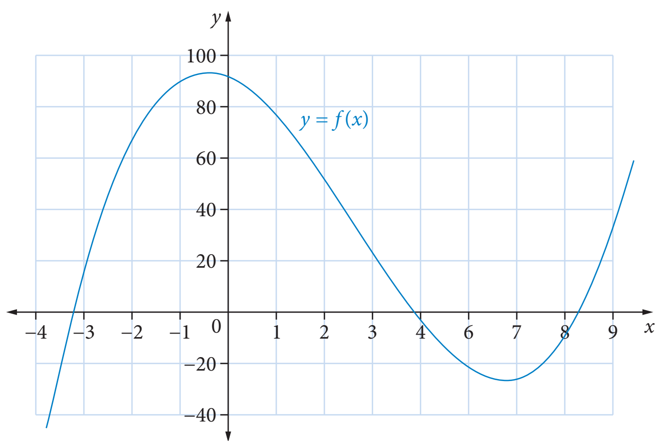


15 TECH-FREE (3 marks) The graph of $f(x)$ is shown. Use the graph to estimate

a the coordinates of stationary points. 1 mark

b the y value at $x = 1$. 1 mark

c the rate of change at $x = 2$. 1 mark



16 TECH-FREE (4 marks)

a Draw the graph of $y = 5x - 3x^2 - 2x^3$ from $x = -10$ to $x = 10$ and hence estimate the coordinates of the stationary points. 3 marks

b Estimate where the rate of change is negative. 1 mark

17 The height of a roller coaster is defined by the function $H(x) = -0.2x(x-2)(x-4) + 10$ for the domain $x \in [0, 10]$, where $H(x)$ and x are both measured in metres. The maximum height of the roller coaster is closest to

A 1 m **B** 3 m **C** 9 m **D** 10 m **E** 11 m

18 The local maximum point of the graph with the rule $y = (x - 2)^2(x - 4)^2$ is found at

A $x = 0$ **B** $x = 1$ **C** $x = 2$ **D** $x = 3$ **E** $x = 4$

19 The local maximum value of the graph with the rule $y = (x - 2)^2(x - 4)^2$ is

A $y = 0$ **B** $y = 1$ **C** $y = 2$ **D** $y = 3$ **E** $y = 4$

20 The local minimum points of the graph with the rule $y = x^4 - 3x^3 - 3x^2 + 7x + 6$ are found at

A $x = -1$ **B** $x = -1, x = \frac{13 + \sqrt{57}}{8}$ **C** $x = -1, x = \frac{13 + \sqrt{57}}{8}$

D $x = \frac{13 + \sqrt{57}}{8}$ **E** $x = \frac{13 \pm \sqrt{17}}{4}$

21 © VCAA 2014 2AQ4 65% Let f be a function with domain R such that $f'(5) = 0$ and $f'(x) < 0$ when $x \neq 5$. At $x = 5$, the graph of f has a

A local minimum. **B** local maximum. **C** gradient of 5.

D gradient of -5. **E** stationary point of inflection.

22 © VCAA 2010 2AQ17 60% The function f is differentiable for all $x \in R$ and satisfies the following conditions.

- $f'(x) < 0$ where $x < 2$
- $f'(x) = 0$ where $x = 2$
- $f'(x) = 0$ where $x = 4$
- $f'(x) > 0$ where $2 < x < 4$
- $f'(x) > 0$ where $x > 4$

Which one of the following is true?

A The graph of f has a local maximum point where $x = 4$.

B The graph of f has a stationary point of inflection where $x = 4$.

C The graph of f has a local maximum point where $x = 2$.

D The graph of f has a local minimum point where $x = 4$.

E The graph of f has a stationary point of inflection where $x = 2$.

23 © VCAA 2012 2AQ16 34% The graph of a cubic function f has a local maximum at $(a, -3)$ and a local minimum at $(b, -8)$. The values of c, such that the equation $f(x) + c = 0$ has exactly one solution, are

A $3 < c < 8$ **B** $c > -3$ or $c < -8$ **C** $-8 < c < -3$

D $c < 3$ or $c > 8$ **E** $c < -8$

24 © VCAA 2006S 2AQ15 A function $f: R \to R$ is such that

$f'(x) = 0$ at $x = 0$ and $x = 2$

$f'(x) < 0$ for $0 < x < 2$ and $x > 2$

$f'(x) > 0$ for $x < 0$

Which one of the following is true?

A The graph of f has a stationary point of inflection at $x = 0$.

B The graph of f has a local maximum point at $x = 2$.

C The graph of f has a stationary point of inflection at $x = 2$.

D The graph of f has a local minimum point at $x = 0$.

E The graph of f has a local minimum point at $x = 2$.

25 © VCAA 2019N 2AQ13 The graph of $f(x) = x^3 - 6(b-2)x^2 + 18x + 6$ has exactly two stationary points for

A $1 < b < 2$

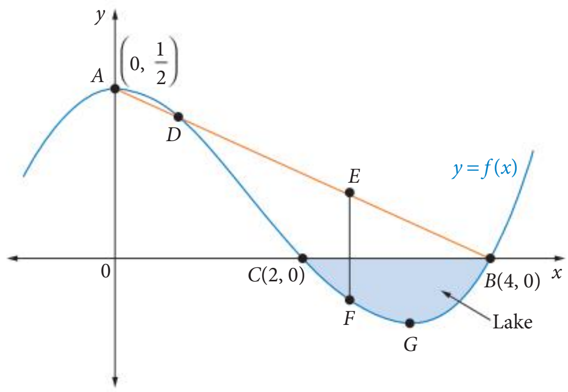
B $b = 1$

C $b = \dfrac{4 \pm \sqrt{6}}{2}$

D $\dfrac{4-\sqrt{6}}{2} \le b \le \dfrac{4+\sqrt{6}}{2}$

E $b < \dfrac{4-\sqrt{6}}{2}$ or $b > \dfrac{4+\sqrt{6}}{2}$

26 © VCAA 2013 2BQ3aefgh (10 marks) Tasmania Jones is in Switzerland. He is working as a construction engineer and he is developing a thrilling train ride in the mountains. He chooses a region of a mountain landscape, the cross-section of which is shown in the diagram below.



The cross-section of the mountain and the valley shown in the diagram (including a lake bed) is modelled by the function with rule $f(x) = \dfrac{3x^3}{64} - \dfrac{7x^2}{32} + \dfrac{1}{2}$.

Tasmania knows that A $\left(0, \dfrac{1}{2}\right)$ is the highest point on the mountain and that $C(2, 0)$ and $B(4, 0)$ are the points at the edge of the lake, situated in the valley. All distances are measured in kilometres.

a 79% Find the coordinates of G, the deepest point in the lake. 3 marks

Tasmania's train travels down the railway line from A to B. The speed, in km/h, of the train as it moves down the railway line is described by the function

$$V: [0, 4] \to R,\ V(x) = k\sqrt{x} - mx^2$$

where x is the x-coordinate of a point on the front of the train as it moves down the railway line, and k and m are positive real constants.

The train begins its journey at $A\left(0, \dfrac{1}{2}\right)$. It increases its speed as it travels down the railway line.

The train then slows to a stop at $B(4, 0)$, that is $V(4) = 0$.

b 64% Find k in terms of m. 1 mark

c 53% Find the value of x for which the speed, V, is a maximum. 2 marks

Tasmania is able to change the value of m on any particular day. As m changes, the relationship between k and m remains the same.

d 43% If, on one particular day, $m = 10$, find the maximum speed of the train, correct to one decimal place. 2 marks

e 38% If, on another day, the maximum value of V is 120, find the value of m. 2 marks

Video playlist
Curve sketching

Worksheets
Gradient functions

Sketching curves

12.5 Curve sketching

By the end of Year 11, recognising and sketching a range of graphs is considered assumed knowledge for students. It is important to thoroughly understand how to deal with graphs of all types.

Key features of a graph

When sketching a graph, it is important to identify the **key features**:

- recognised general shape
- domain and range
- y-intercept
- x-intercepts
- stationary points
- endpoint values.

> Don't just rely on CAS for graphing. You should have a general idea of the shape of the graph.

WB
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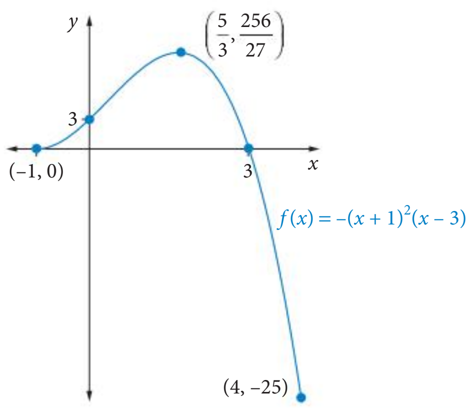
WORKED EXAMPLE 7 Curve sketching

a Sketch the graph of $f: [-1, 4] \to R, f(x) = -(x + 1)^2(x - 3)$, labelling key features.

b State the range of f.

Steps	Working
a **1** Explore the general features of the graph.	Note from its equation that this is a cubic function with a leading coefficient of -1 and x-intercepts at -1 and 3.
2 Find the y-intercept.	$f(0) = -(0 + 1)^2(0 - 3) = 3$
3 Find the x-intercepts.	$-(x + 1)^2(x - 3) = 0$ $x = -1, x = 3$
4 Find $f'(x)$ and solve for zero for stationary points.	$f'(x) = -3x^2 + 2x + 5 = 0$ $x = -1, x = \frac{5}{3}$ $f(-1) = 0, f\left(\frac{5}{3}\right) = \frac{256}{27}$
5 Find the endpoint values of the domain $[-1, 4]$.	$f(-1) = 0$ $f(4) = -25$
6 Sketch the graph of $f: [-1, 4] \to R, f(x) = -(x + 1)^2(x - 3)$. **Exam hack** The range is not always the y values of the endpoints. You must also compare them with the y values of the local maximum and minimum points.	
b State the range.	$\left[-25, \frac{256}{27}\right]$

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12.5

Sometimes, the turning points found on a function may not actually be the maximum or minimum point for the whole function being considered (called the **global** or **absolute** maximum or minimum). We need to compare the turning points to the **interval endpoint values** to find the **global maximum** or **minimum point** for the given interval.

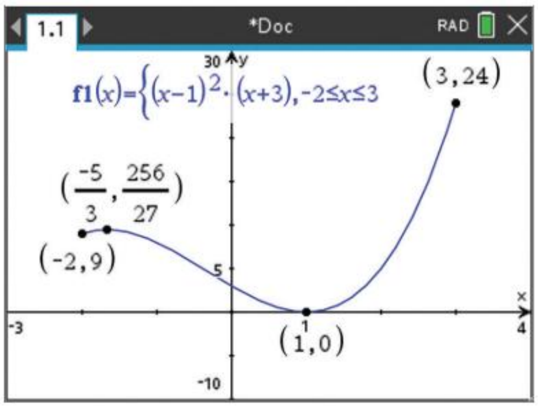
WB
p. 228

WORKED EXAMPLE 8 Minimum value

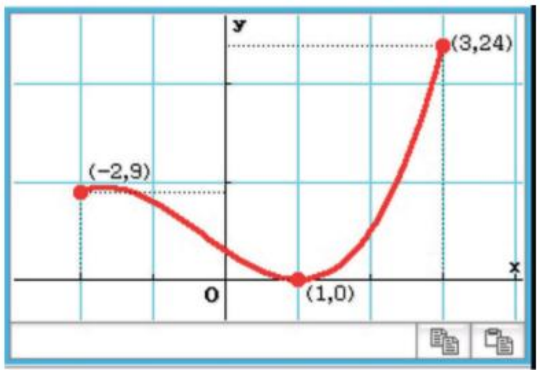
Find the minimum value of the graph of f: $[-2, 3] \rightarrow R, f(x) = (x-1)^2(x+3)$.

Steps	Working
1 Find endpoint values.	$f(-2) = 9, f(3) = 24$
2 Find the y-intercept.	$f(0) = (0-1)^2(0+3) = 3$
3 Find $f'(x)$ and solve for zero for $[-2, 3]$.	$f'(x) = 3x^2 + 2x - 5 = (3x+5)(x-1) = 0$
Note from its equation that this is a cubic function with a leading coefficient of 1 and x-intercepts at 1 and -3.	$\therefore x = 1, x = -\frac{5}{3}$ $f(1) = 0, f\left(-\frac{5}{3}\right) = \frac{256}{27}$ Minimum point is (1, 0).
4 Compare endpoint minimum with local minimum.	Endpoint minimum is 9, so global minimum at $x = 1$.
5 Use CAS to sketch the graph to confirm.	

TI-Nspire



ClassPad



6 State the minimum value (at 1, 0).	Minimum value = 0

EXERCISE 12.5 Curve sketching

ANSWERS p. 531

Recap

1 The local minimum point of the graph with rule $y = x^2 - 2x - 8$ is

A (4, –8) **B** (2, –8) **C** (1, 0) **D** (4, 0) **E** (1, –9)

2 For f: $[-2, 2] \rightarrow R, f(x) = x^2 - 2x - 8$, the global maximum point is

A (2, –8) **B** (–2, 0) **C** (1, 0) **D** (4, 0) **E** (1, –9)

Mastery

3 WORKED EXAMPLE 7 TECH-FREE

a Sketch the graph of g: $[-4, 0] \rightarrow R, g(x) = -\frac{1}{2}(x+2)^2(x+3)$, labelling key features.

b State the range of g.

4 TECH-FREE Find the roots of the graph of $g(x) = -\frac{1}{2}(x+2)^2(x+3)$.

5 WORKED EXAMPLE 8 TECH-FREE Find the minimum value of the graph of

$$g: [-4, 0] \to R,\ g(x) = -\frac{1}{2}(x+2)^2(x+3)$$

6 The range of the graph of $h: [-2, 2] \to R,\ h(x) = -x^3 + 8$ is

A 2 **B** -2 **C** $[-2, 2]$ **D** $[0, 16]$ **E** $[16, 0]$

7 The graph of $g(x) = -\frac{1}{2}(x+2)^2(x+3)$ for $-3 \le x \le 0$ has the maximum value of

A -6 **B** -3 **C** -2 **D** 0 **E** 6

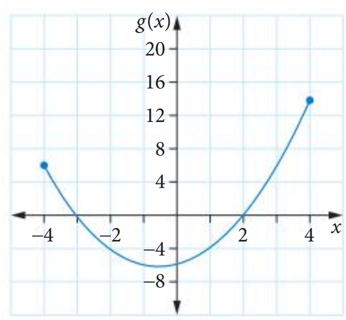
Exam practice

80–100% 60–79% 0–59%

8 TECH-FREE (3 marks) Sketch the graph of $f: [0, 5] \to R,\ f(x) = 2(x-2)^2$, labelling key features.

9 TECH-FREE (3 marks) Sketch the graph of $f: [-1, 3] \to R,\ f(x) = -\frac{3}{2}(x-2)^2(x+1)^2$, labelling key features.

10 TECH-FREE (7 marks) The graph of $g(x)$ is shown.



Use the graph to estimate

a the coordinates of the stationary points — 1 mark

b the y value at $x = -4$ — 1 mark

c the y value at $x = 1$ — 1 mark

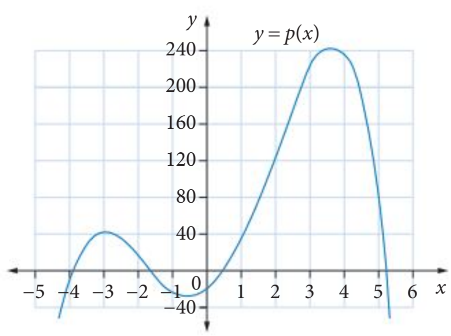
d the intervals where the function is strictly decreasing — 1 mark

e the intervals where the function is strictly increasing — 1 mark

f the rate of change at $x = 1$ — 1 mark

g the range — 1 mark

11 TECH-FREE (5 marks) The graph of $p(x)$ is shown.



Use the graph to estimate

a the coordinates of the stationary points — 1 mark

b the y value at $x = 1$ — 1 mark

c the y value at $x = 2$ — 1 mark

d the intervals where the rate of change of the function is positive — 1 mark

e the intervals where the rate of change of the function is negative — 1 mark

12 © VCAA 2014 1Q5 TECH-FREE (4 marks) Consider the function $f: [-1, 3] \to R,\ f(x) = 3x^2 - x^3$.

a 76% Find the coordinates of the stationary points of the function. — 2 marks

b 71% Sketch the graph of f. Label any endpoints with their coordinates. — 2 marks

13 TECH-FREE (4 marks) Find any stationary points on the curve $y = 2x^3 - 9x^2 - 24x + 30$ and sketch its graph.

14 TECH-FREE (4 marks) Determine any stationary points on the curve $y = x^3 + 6x^2 - 7$ and sketch the curve.

15 TECH-FREE (4 marks) Find the stationary points on the curve $y = (x - 4)(x + 2)^2$ and hence sketch the curve.

16 The function $f(x) = -\frac{1}{2}(x + 1)^2$ has

A a range of R **B** a local maximum at $x = 1$ **C** a local minimum at $x = -1$

D a local maximum at $x = -1$ **E** a maximal domain of $[-1, 1]$.

17 The maximum point of the graph of $f(x) = -\frac{1}{3}(x + 1)^3(x - 3)$ is

A $(2, 9)$ **B** $(9, 2)$ **C** $(-2, 9)$ **D** $(-1, 0)$ **E** $\left(1, \frac{16}{3}\right)$

18 The function $f(x) = -\frac{1}{2}(x + 1)^2(x - 2)$ has

A a range of $[-1, 2]$ **B** a local maximum at $x = 2$ **C** a local minimum at $x = -1$

D a local maximum at $x = -1$ **E** an x-intercept at $x = 1$.

19 The local maximum point of the graph of $f(x) = -\frac{1}{2}(x + 1)^2(x - 2)$ is

A $(1, 2)$ **B** $(2, 1)$ **C** $(1, 0)$ **D** $(-1, 0)$ **E** $\left(1, \frac{1}{2}\right)$

Video playlist
Maximum and minimum problems

Worksheets
Greatest and least values

Maximum and minimum problems

Optimisation problems

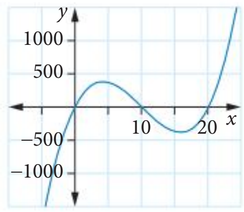
Further optimisation problems

Starting maxima and minima problems

Applications of optimisation

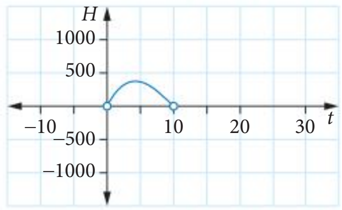
12.6 Maximum and minimum problems

Differentiation techniques are used to solve problems that ask for maximum or minimum points of a graph, or the maximum or minimum values of a function. Such problems are quite common in VCE exams, often appearing more than once and in a range of contexts, and question types and length.



Consider the graph of $f(x) = x(20 - x)(10 - x)$.

If we differentiate and solve for zero, we find the stationary points of the graph. This question could apply to a common real-life problem with the height of liquid in a container, written as $H(t) = t(20 - t)(10 - t)$, where $H(t)$ is the height of liquid in cm, at $t > 0$.



In reality, the function $H(t)$ will only exist for $H, t > 0$

The maximum height can be found by finding the local maximum of $H(t)$.

$H'(t) = 0$, giving the local maximum turning point (4.2, 384.9) correct to one decimal place.

The maximum height equals 384.9 cm.

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WORKED EXAMPLE 9 Maximum and minimum points

Find the coordinates of the local maximum and minimum points in the graph of

$$f: [-5, 2] \to R, f(x) = \frac{1}{5}(x-1)^2(x+4)$$

Steps

Working

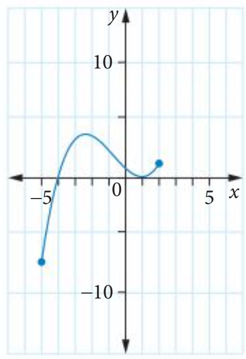
1 Find $f'(x)$ and solve for zero.

$f'(x) = \dfrac{3x^2 + 4x - 7}{5} = \dfrac{1}{5}(3x+7)(x-1) = 0$

$\therefore x = -\dfrac{7}{3}, x = 1$

2 Consider the graph shape to decide which is maximum or minimum.

> Note from its equation that this is a cubic function with a leading coefficient of $\frac{1}{5}$ and x-intercepts at 1 and −4.



Local maximum at $x = -\dfrac{7}{3}$.

Local minimum at $x = 1$.

3 Find the y values of the turning points.

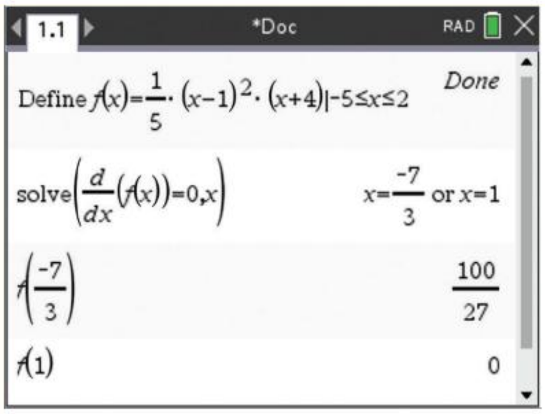
$f\left(-\dfrac{7}{3}\right) = \dfrac{100}{27}, f(1) = 0$

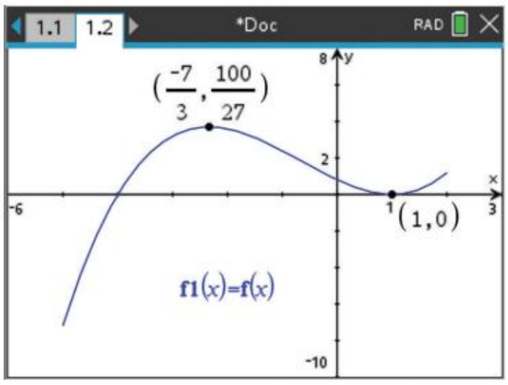
Local maximum $\left(-\dfrac{7}{3}, \dfrac{100}{27}\right)$ and local minimum (1, 0).

USING CAS 3 Maximum and minimum points

Confirm the local maximum and minimum points in the graph of f: $[-5, 2] \to R$, $f(x) = \frac{1}{5}(x-1)^2(x+4)$ from the previous example.

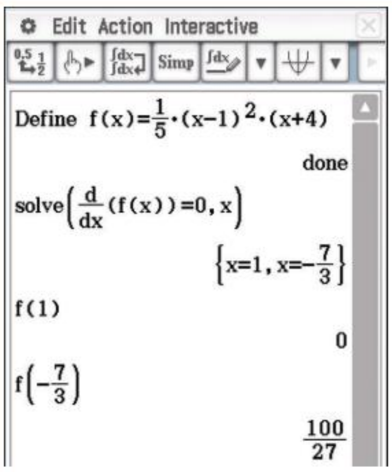
TI-Nspire

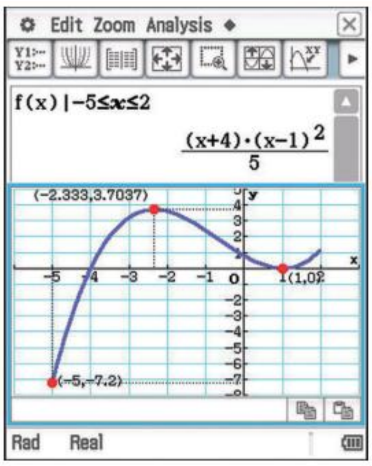




1. Define the function ***f*(*x*)**.
2. Set the **derivative of *f*(*x*)** equal to **0** and **solve** to find the *x*-coordinates of the turning points.
3. Substitute the solutions into ***f*(*x*)** to determine the corresponding *y*-coordinates.
4. To visualize these solutions, add a **Graphs** page and graph ***f*(*x*)**.
5. Adjust the window settings to suit.
6. To find the exact coordinates of the turning points, press **menu > Geometry > Points & Lines > Point On**.
7. Click to add 2 points onto the curve then press **esc** to remove the **Point On** tool.
8. Click on the *x*-coordinates of each point and enter the values as shown above.

ClassPad





1. Define the function ***f*(*x*)**.
2. Set the **derivative of *f*(*x*)** equal to **0** and **solve** to find the *x*-coordinates of the turning points.
3. Substitute the solutions into ***f*(*x*)** to determine the corresponding *y*-coordinates.
4. To visualise these solutions, graph ***f*(*x*)** over the domain **[–5, 2]**.
5. Adjust the window settings to suit.
6. Tap **Interactive > G-Solve > Max** to find the local maximum.
7. Tap **Interactive > G-Solve > Min** to find the local minimum.

The local maximum is $\left(-\frac{7}{3}, \frac{100}{27}\right)$. The local minimum is (1, 0).

Applying maximum and minimum points

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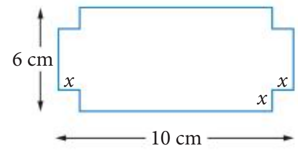
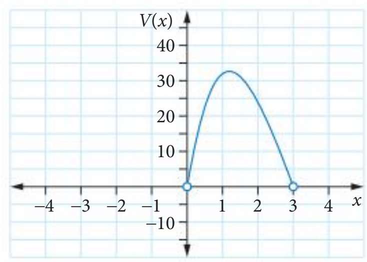
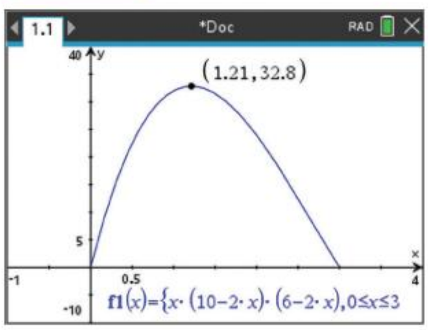
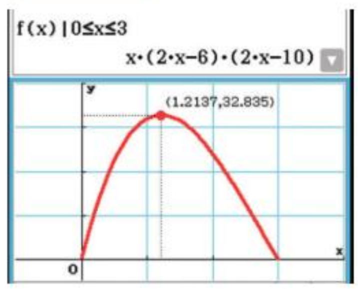
WORKED EXAMPLE 10 Maximum and minimum problems

A rectangular piece of cardboard, measuring 10 cm by 6 cm, has square corners of length x units cut out at each corner. These corners are folded up to make a small packing box for jewellery. Find the maximum possible volume, in cm^3 correct to one decimal place, for these jewellery boxes.

Steps	Working
1 Sketch a diagram.	
2 Set up an equation to describe the volume of the box.	Length $= 10 - 2x$, width $= 6 - 2x$, height $= x$. $V(x) = x(10 - 2x)(6 - 2x)$
3 Identify the domain for this problem.	x is the length of a square corner so it must be positive and less than half of the width 6 cm. The domain is $(0, 3)$.
4 Find $V'(x)$ and solve for zero.	$V(x) = x(10 - 2x)(6 - 2x) = 4x^3 - 32x^2 + 60x$ $V'(x) = 12x^2 - 64x + 60 = 0$ $\therefore\ x = \dfrac{8 \pm \sqrt{19}}{3}$ $\dfrac{8 - \sqrt{19}}{3} \approx 1.2137$ for domain $(0, 3)$
5 Test whether this is a maximum point by using CAS to sketch a graph or by checking whether the sign of $V'(x)$ changes from positive to negative.	The point is a maximum. 
TI-Nspire 	**ClassPad** 
6 Substitute this value of x into $V(x)$ to find the maximum volume of the jewellery box.	$V(1.2137) = 32.835$ Maximum possible volume is about $32.8\ cm^3$.

Exam hack

For maximum and minimum problems, re-read the question to make sure that you have answered it. One common mistake students make in exams is to forget to answer the actual question, even though all the working is correct.

EXERCISE 12.6 Maximum and minimum problems

ANSWERS p. 532

12.6

Recap

1 TECH-FREE Find the maximum value of the function $y = -4x(x-1)$.

2 TECH-FREE Solve each equation.

a $x^2 + 5x - 50 = 0$ b $\log_e(x+1) = 4$ c $e^{2x} - 5e^x + 4 = 0$

Mastery

80–100% 60–79% 0–59%

3 WORKED EXAMPLE 9 TECH-FREE Find the coordinates of the local maximum and minimum points in the graph of $f: R \to R, f(x) = (x-1)(x+3)^2$.

4 WORKED EXAMPLE 10 TECH-FREE The height of water in a dam is modelled by the equation $h(t) = 20\cos(2\pi t) + 60$, with h in metres and t in days, $t \geq 0$. Find the maximum and minimum height of water in the dam for $t \in [0, 10]$.

5 © VCAA 2007 1Q12 26% TECH-FREE (4 marks) P is the point on the line $2x + y - 10 = 0$ such that the length of OP, the line segment from the origin O to P, is a minimum. Find the coordinates of P **and** this minimum length.

6 TECH-FREE A rectangular field shares one side with an existing paddock and so this requires no fence. There is only 1000 m of fencing material to fence the remaining sides. Find the maximum possible area of the field.

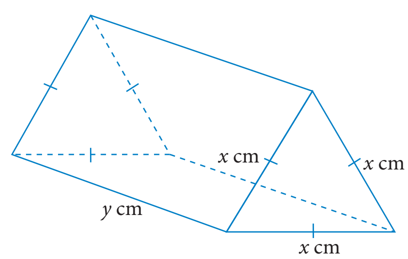
7 TECH-FREE The perimeter of a rectangle is 60 m and its length is x m. Show that the area of the rectangle is given by the equation $A = 30x - x^2$. Hence, find the greatest area of the rectangle.

8 TECH-FREE Find two numbers whose sum is 28 and whose product is a maximum.

Exam practice

80–100% 60–79% 0–59%

9 © VCAA 2008 1Q9 TECH-FREE (7 marks) A plastic brick is made in the shape of a right triangular prism. The triangular end is an equilateral triangle with side length x cm and the length of the brick is y cm.



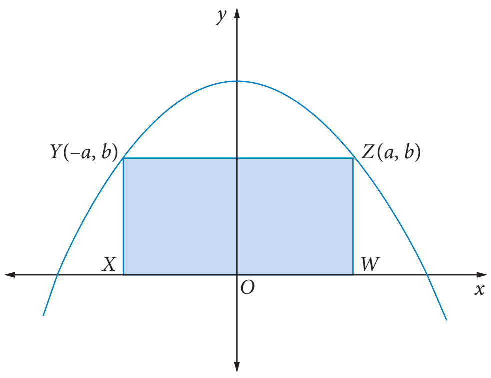
The volume of the brick is 1000 cm^3.

a 56% Find an expression for y in terms of x. 2 marks

b 45% Show that the total surface area, A cm^2, of the brick is given by $A = \dfrac{4000\sqrt{3}}{x} + \dfrac{\sqrt{3}x^2}{2}$. 2 marks

c 42% Find the value of x for which the brick has minimum total surface area. (You do not have to find this minimum.) 3 marks

10 © VCAA 2006 1Q9 TECH-FREE (4 marks) A rectangle $XYZW$ has two vertices, X and W, on the x-axis and the other two vertices, Y and Z, on the graph of $y = 9 - 3x^2$, as shown in the diagram below. The coordinates of Z are (a, b), where a and b are positive real numbers.



a 37% Find the area, A, of rectangle $XYZW$ in terms of a. 1 mark

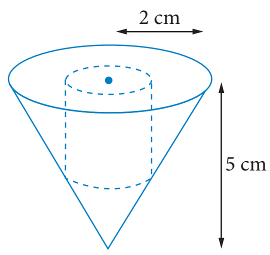
b 29% Find the maximum value of A and the value of a for which this occurs. 3 marks

11 TECH-FREE (3 marks) Find the greatest volume of a rectangular box with a square base if the sum of the height and the side of the base must not exceed 24 cm.

12 TECH-FREE (3 marks) The cost per hour of a bike ride is given by the formula $C = x^2 - 15x + 70$, where x is the distance travelled in kilometres. Find the distance that gives the lowest cost.

13 TECH-FREE (3 marks) The height, in metres, of a ball is given by the equation $h = 16t - 4t^2$, where t is the time in seconds. Find when the ball will reach its greatest height, and what the greatest height will be.

14 © VCAA 2010 1Q11 TECH-FREE (5 marks) A cylinder fits exactly in a right circular cone so that the base of the cone and one end of the cylinder are in the same plane as shown in the diagram below. The height of the cone is 5 cm and the radius of the cone is 2 cm.



The radius of the cylinder is r cm and the height of the cylinder is h cm. For the cylinder inscribed in the cone as shown

a 22% find h in terms of r 2 marks

The total surface area, S cm^2, of a cylinder of height h cm and radius r cm is given by the formula $S = 2\pi rh + 2\pi r^2$.

b 47% Find S in terms of r. 1 mark

c 46% Find the value of r for which S is a maximum. 2 marks

12.6

15 TECH-FREE (3 marks) In an experiment, it is found that the temperature (°C) of an object is approximated by the function $f(t) = 40 - 24t + 9t^2 - t^3$, where t is the time in minutes after the start of the experiment, given that $\frac{3}{2} \le t \le 6$. Find the highest and lowest temperatures of the object during the given time interval.

16 The local maximum point on the graph of $f: R \to R$, $f(x) = -(x-1)(x+3)^2$ is

A $\left(-\frac{1}{3}, -\frac{256}{27}\right)$ **B** $(-3, 0)$ **C** $\left(-\frac{1}{3}, \frac{256}{27}\right)$ **D** $(3, 0)$ **E** $\left(\frac{1}{3}, \frac{200}{27}\right)$

17 Using CAS 3 The local minimum point on the graph of $f: [-3, 3] \to R$, $f(x) = 2(x+1)^2(x+2)$ is at

A $x = -3$ **B** $x = -1$ **C** $x = 0$ **D** $x = 1$ **E** $x = 3$

18 The global minimum point on the graph of $f: [-3, 3] \to R$, $f(x) = 2(x+1)^2(x+2)$ is at

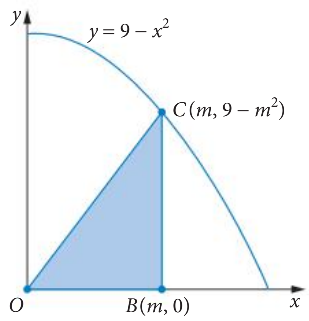
A $x = -3$ **B** $x = -1$ **C** $x = 0$ **D** $x = 1$ **E** $x = 3$

19 The equation $2(x-1)^2(x+3) = 0$ gives the solution(s)

A $x = 1, x = -3$ **B** $x = 1, x = \pm 3$ **C** $x = 0, x = 3$
D $x = -3$ only **E** $x = 0, x = \pm 3$

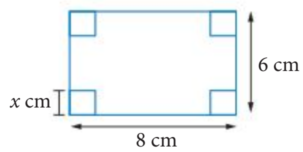
20 © VCAA 2020 2AQ16 53% A right-angled triangle, OBC, is formed using the horizontal axis and the point $C(m, 9 - m^2)$, where $m \in (0, 3)$, on the parabola $y = 9 - x^2$, as shown below.

The maximum area of the triangle OBC is



A $\frac{\sqrt{3}}{3}$ **B** $\frac{2\sqrt{3}}{3}$ **C** $\sqrt{3}$ **D** $3\sqrt{3}$ **E** $9\sqrt{3}$

21 © VCAA 2014 2AQ15 44% Zoe has a rectangular piece of cardboard that is 8 cm long and 6 cm wide. Zoe cuts squares of side length x centimetres from each of the corners of the cardboard, as shown in the diagram below.

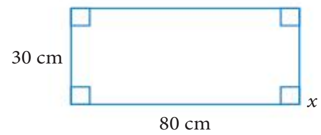


Zoe turns up the sides to form an open box.

The value of x for which the volume of the box is a maximum is closest to

A 0.8 **B** 1.1 **C** 1.6 **D** 2.0 **E** 3.6

22 (6 marks) A box is made from an 80 cm by 30 cm rectangle of cardboard by cutting out 4 equal squares of size x cm from each corner. The edges are turned up to make an open box.

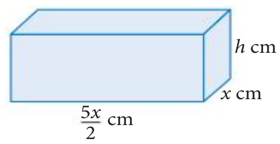


a Show that the volume of the box is given by the equation $V = 4x^3 - 220x^2 + 2400x$. 2 marks

b Find the value of x that gives the box its greatest volume. 2 marks

c Find the maximum volume of the box, correct to one decimal place. 2 marks

23 © VCAA 2012 2BQ1 (9 marks) A solid block in the shape of a rectangular prism has a base of width x cm. The length of the base is two-and-a-half times the width of the base.



The block has a total surface area of 6480 sq. cm.

a 66% Show that if the height of the block is h cm, $h = \dfrac{6480 - 5x^2}{7x}$. 2 marks

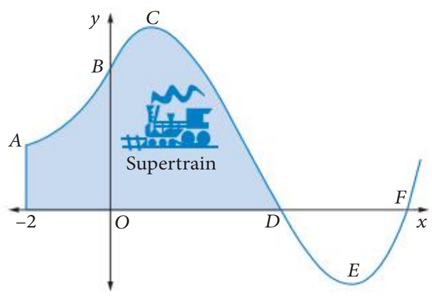
b 52% The volume, V cm^3, of the block is given by $V(x) = \dfrac{5x\left(6480 - 5x^2\right)}{14}$.

Given that $V(x) > 0$ and $x > 0$, find the possible values of x. 2 marks

c 73% Find $\dfrac{dV}{dx}$, expressing your answer in the form $\dfrac{dV}{dx} = ax^2 + b$, where a and b are real numbers. 3 marks

d 57% Find the exact values of x and h if the block is to have maximum volume. 2 marks

24 © VCAA 2006 2BQ4 (11 marks) A part of the track for Tim's model train follows the curve passing through A, B, C, D, E and F shown. Tim has designed it by putting axes on the drawing as shown. The track is made up of two curves, one to the left of the y-axis and the other to the right.



B is the point (0, 7). The curve from B to F is part of the graph of $f(x) = px^3 + qx^2 + rx + s$, where p, q, r and s are constants and $f'(0) = 4.25$.

a **i** 86% Show that $s = 7$. 1 mark

ii 78% Show that $r = 4.25$. 1 mark

The furthest point reached by the track in the positive y direction occurs when $x = 1$. Assume that $p > 0$.

b **i** 46% Use this information to find q in terms of p. 2 marks

ii 34% Find $f(1)$ in terms of p. 1 mark

iii 27% Find the value of a in terms of p for which $f'(a) = 0$, where $a > 1$. 1 mark

iv 33% If $a = \frac{17}{3}$, show that $p = 0.25$ and $q = -2.5$. 2 marks

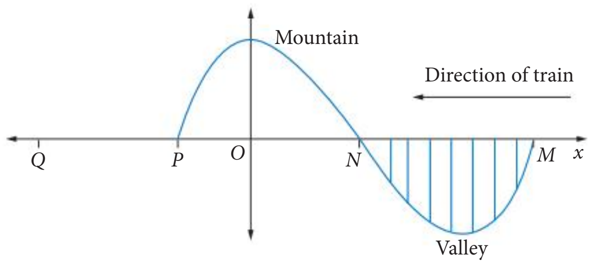
For the following, assume $f(x) = 0.25x^3 - 2.5x^2 + 4.25x + 7$.

c 64% Find the exact coordinates of D and F. 2 marks

d 55% Find the greatest distance that the track is from the x-axis when it is below the x-axis, correct to two decimal places. 1 mark

12.6

25 © VCAA 2009 2BQ2 (9 marks)



A train is travelling at a constant speed of w km/h along a straight level track from M towards Q. The train will travel along a section of track $MNPQ$.

Section MN passes along a bridge over a valley. Section NP passes through a tunnel in a mountain. Section PQ is 6.2 km long.

From M to P, the curve of the valley and the mountain, directly below and above the train track, is modelled by the graph of $y = \frac{1}{200}(ax^3 + bx^2 + c)$ where a, b and c are real numbers. All measurements are in kilometres.

a The curve defined from M to P passes through $N(2, 0)$. The gradient of the curve at N is -0.06 and the curve has a turning point at $x = 4$.

i 45% From this information write down three simultaneous equations in a, b and c. 3 marks

ii 41% Hence show that $a = 1$, $b = -6$ and $c = 16$. 2 marks

b Find, giving exact values

i 59% the coordinates of M and P 2 marks

ii 41% the length of the tunnel 1 mark

iii 50% the maximum depth of the valley below the train track. 1 mark

Video playlist
The anti-derivative

Worksheets
Anti-derivatives

Antiderivatives

Antidifferentiation

12.7 The anti-derivative

The **anti-derivative** of a function $f(x)$ is the function $F(x)$ whose derivative is $f(x)$. In other words, $F'(x) = f(x)$. The process of finding a function from its derivative is called **anti-differentiation**, the reverse process of differentiation.

When we differentiate a **constant** term (number), we get 0, so when anti-differentiating, we must include '$+ c$' in the answer, where c stands for any real number.

The general anti-derivative of $2x$ is $x^2 + c$, where c is a constant.

In algebraic terms, this is written as $\int (2x)\, dx = x^2 + c$.

This is called the **anti-derivative** of $2x$, the **integral** of $2x$, or the **primitive** of $2x$.

- The symbol $\int$ is read as 'the integral of'.
- 'dx' means 'with respect to x'.
- Another name for the process of anti-differentiation is **integration**.

The anti-derivative (or integral) of ax^n

If $\frac{dy}{dx} = ax^n$, then $y = \int (ax^n)\, dx = a\int (x^n)\, dx = \frac{ax^{n+1}}{n+1} + c$

where ax^n is called the **integrand**.

Consider the function $f(x) = 5 + 4x + 9x^2$.

To anti-differentiate, we 'add 1 to the power, divide by the new power'.

The term 5 becomes $5x$.

The term $4x$ becomes $2x^2$.

The term $9x^2$ becomes $3x^3$.

$$\int \left(5 + 4x + 9x^2\right) dx = \frac{5x^1}{1} + \frac{4x^2}{2} + \frac{9x^3}{3} + c$$

So the anti-derivative of the expression $5 + 4x + 9x^2$ becomes $5x + 2x^2 + 3x^3 + c$.

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WORKED EXAMPLE 11 The anti-derivative

Find an anti-derivative of the function $f(x) = 3x^2 + 2x - 4$.

Exam hack

You can always check your answer by differentiating it to see whether you get the original function.

Steps	Working
1 Write the function as a derivative.	$\frac{dy}{dx} = 3x^2 + 2x - 4$
2 Integrate each term using the formula $\int ax^n\, dx = \frac{ax^{n+1}}{n+1} + c$ to find y.	$y = \frac{3x^3}{3} + \frac{2x^2}{2} - 4x$ $\therefore y = x^3 + x^2 - 4x$ In this case, the $+ c$ is not required since we are asked for *an* anti-derivative.

Exam hack

If the question asks for *an* anti-derivative instead of *the* anti-derivative, you should not add the $+ c$.

If we leave in the $+ c$, it is called an **indefinite integral.**

To find the + c, we are given an extra piece of information.

For example, if $\dfrac{dy}{dx} = 3x^3 + 2x^2 + 4x + 1$,

we anti-differentiate to find $y = \dfrac{3}{4}x^4 + \dfrac{2}{3}x^3 + 2x^2 + x + c$.

Given an extra piece of information, when $x = 1$, $y = 2$, substitute to get

$$2 = \frac{3}{4}(1)^4 + \frac{2}{3}(1)^3 + 2(1)^2 + (1) + c$$

$$\therefore\ 2 = \frac{3}{4} + \frac{2}{3} + 2 + 1 + c.$$

Giving $c = -\dfrac{29}{12}$.

So in this case, $y = \dfrac{3}{4}x^4 + \dfrac{2}{3}x^3 + 2x^2 + x - \dfrac{29}{12}$.

When more functions are learned, we have a series of rules to find the anti-derivative.

For example, $\int (\cos(2x))\, dx = \dfrac{1}{2}\sin(2x) + c$

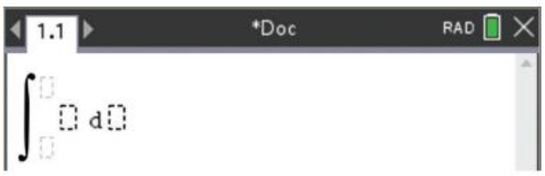
or $\int \left(e^{2x} + x\right) dx = \dfrac{1}{2}e^{2x} + \dfrac{x^2}{2} + c$.

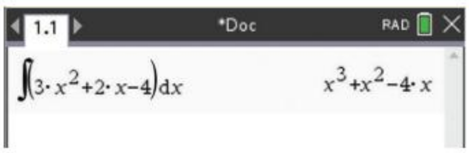
Using CAS, we can confirm these results.

USING CAS 4 The anti-derivative

Confirm the anti-derivative of $3x^2 + 2x - 4$ from the previous example.

TI-Nspire





1 Press **menu > Calculus > Integral**.

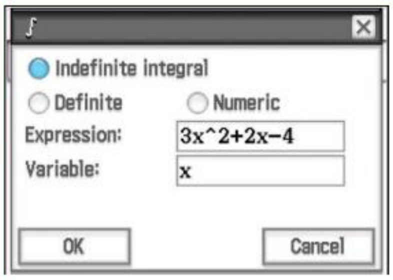
2 The **Integral** template will be displayed.

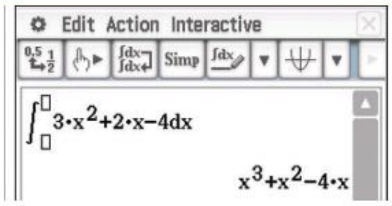
Alternatively, press **template** to access the Integrate template.

3 Enter the expression, including the ***x*** at the end as shown above.

4 Tap **enter** to display the anti-derivative.

ClassPad





1 Enter and highlight the expression.

2 Tap **Interactive > Calculation > ∫**.

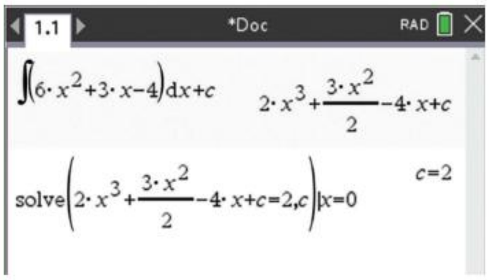
3 In the dialogue box, tap **OK**.

4 The anti-derivative will be displayed.

USING CAS 5 The anti-derivative with a value

Find the anti-derivative of $f'(x) = 6x^2 + 3x - 4$, given that $f(0) = 2$.

TI-Nspire



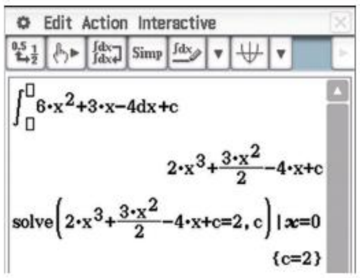
1 Press **menu > Calculus > Integrate**.

2 Enter the expression, including the ***x***.

3 Enter **+*c*** at the end of the integral as shown above.

4 Set the anti-derivative expression equal to **2** and solve for ***c***. Include the condition that ***x* = 0**.

ClassPad



1 Enter and highlight the expression.

2 Tap **Interactive > Calculation > ∫**.

3 Enter **+*c*** at the end of the integral as shown above.

4 Set the anti-derivative expression equal to 2 and solve for ***c***. Include the condition that ***x* = 0**.

Note that the CAS does not give you the + *c*.

EXERCISE 12.7 The anti-derivative

ANSWERS p. 533

Recap

1 TECH-FREE Find any stationary points in the graph of

a $y = \frac{1}{2}x^4 + 3x^3 - 3x^2 - 5x$

b $f(x) = (x - 5)(x + 5)$

2 The derivative of $y = 3x^3 + 2x^2$ equals

A $9x^2 + 4x$ **B** $9x^3 + 4x^2$ **C** $9x^2 - 4x$ **D** $3x^4 + 2x^3$ **E** $\frac{3}{4}x^4 + \frac{2}{3}x^3$

Mastery

3 WORKED EXAMPLE 11 TECH-FREE Find an anti-derivative of the function $f(x) = 3x^3 + 2x^2$.

4 Find the anti-derivative of $f'(x) = x^2 + 3x + 7$, given that $f(1) = 2$.

5 Using CAS 4 An anti-derivative of the function $y = -2(3x+2)^2$ is

A $6(3x+2)$ **B** $6(3x+2)^3$ **C** $-2(3x^3 + 6x^2 - 4x)$

D $6x^3 + 12x^2 + 8x$ **E** $-6x^3 - 12x^2 - 8x$

6 Using CAS 5 The function $\frac{dy}{dx} = x^2 - x$, where $x = 1$ when $y = 0$, has an anti-derivative of

A $\frac{1}{3}x^3 - \frac{1}{2}x^2 + c$ **B** $\frac{1}{3}x^3 - \frac{1}{2}x^2$ **C** $\frac{1}{3}x^3 - \frac{1}{2}x^2 + \frac{1}{6}$

D $\frac{1}{3}x^3 - \frac{1}{2}x^2 - \frac{1}{6}$ **E** $2x - 1$

Exam practice

80–100% 60–79% 0–59%

7 © VCAA 2019N 1Q2 TECH-FREE (2 marks) Find $f(x)$ given that $f(1) = -\frac{7}{4}$ and $f'(x) = 2x^2 - \frac{1}{4}x^{-\frac{2}{3}}$.

8 © VCAA 2018 1Q2 59% TECH-FREE (3 marks) The derivative with respect to x of the function $f: (1, \infty) \rightarrow R$ has the rule $f'(x) = \frac{1}{2} - \frac{1}{2x-2}$. Given that $f(2) = 0$, find $f(x)$ in terms of x.

9 TECH-FREE (2 marks) Find an anti-derivative of $\frac{1}{(x-1)^3}$ with respect to x.

10 TECH-FREE (3 marks) The function with the rule $g(x)$ has derivative of $g'(x) = 3x^2 + 1$. Given that $g(1) = 2$, find $g(x)$.

11 TECH-FREE (4 marks) Find an anti-derivative of

a x^2 **b** $x^3 + x^2$ **c** $4x^3 + 2x - 4$ **d** $3x^2 + 5x + 7$

12 An anti-derivative of the function $y = 8x^3 + 2x^2 - 7$ is

A $y = 8x^3 + 2x^2 - 7$ **B** $y = 8x^4 + 2x^3 - 7x$ **C** $y = 2x^4 + \frac{2}{3}x^3 - 7x$

D $y = 24x^2 + 4x$ **E** $y = 2x^4 + \frac{1}{3}x^3 - 7x$

13 © VCAA 2020 2AQ3 86% Let $f'(x) = \frac{2}{\sqrt{2x-3}}$.

If $f(6) = 4$, then

A $f(x) = 2\sqrt{2x-3}$ **B** $f(x) = \sqrt{2x-3} - 2$ **C** $f(x) = 2\sqrt{2x-3} - 2$

D $f(x) = \sqrt{2x-3} + 2$ **E** $f(x) = \sqrt{2x-3}$

Video playlist
Applying the anti-derivative

WB
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12.8 Applying the anti-derivative

When using the anti-derivative in a real-life context, it is usually necessary to find the $+c$.

WORKED EXAMPLE 12 Applying the anti-derivative

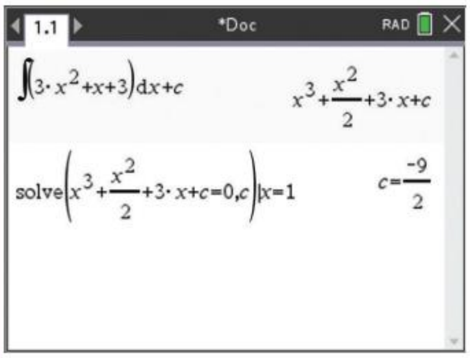
Find y if $\frac{dy}{dx} = 3x^2 + x + 3$ and $x = 1$ when $y = 0$.

Steps	Working
1 Write the expression as a derivative.	$\frac{dy}{dx} = 3x^2 + x + 3$
2 Integrate each term and simplify.	$y = \frac{3x^3}{3} + \frac{x^2}{2} + 3x + c$ $\therefore\ y = x^3 + \frac{x^2}{2} + 3x + c$
3 Find the value of c by substituting $x = 1$ when $y = 0$.	$0 = 1^3 + \frac{1^2}{2} + 3(1) + c$ $0 = \frac{9}{2} + c$ $\therefore\ c = -\frac{9}{2}$
4 State the answer including the value of c.	$y = x^3 + \frac{x^2}{2} + 3x - \frac{9}{2}$

USING CAS 6 Applying the anti-derivative

Confirm the expression for y if $\frac{dy}{dx} = 3x^2 + x + 3$ and $x = 1$ when $y = 0$ from the previous example.

TI-Nspire



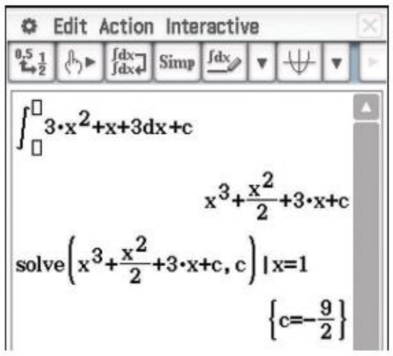
1 Press **menu > Calculus > Integrate**.

2 Enter the expression, including the ***x***.

3 Enter **+*c*** at the end of the integral as shown above.

4 Set the anti-derivative expression equal to **0** and **solve** for ***c***. Include the condition that ***x* = 1**.

ClassPad



1 Enter and highlight the expression.

2 Tap **Interactive > Calculation > ∫**.

3 Enter **+*c*** at the end of the integral, as shown above.

4 Set the anti-derivative expression equal to **0** and **solve** for ***c***. Include the condition that ***x* = 1**.

Straight-line motion

12.8

Kinematics is a study of motion using displacement (x), velocity (v) and acceleration (a), all in terms of time.

We know that $v = \frac{dx}{dt}$ and $a = \frac{d^2x}{dt^2} = \frac{dv}{dt}$.

Conversely,

velocity $= v = \int a(t)\,dt$

and displacement $= x = \int v(t)\,dt$.

WORKED EXAMPLE 13 Anti-differentiating for kinematics

The acceleration of a particle travelling in a straight line is given by $a(t) = 2t$.

a Find an expression for the velocity $v(t)$ if $t = 3$ when $v = 0$.

b Find an expression for the displacement $x(t)$ if the particle started at the origin.

Steps	Working
1 State the rule for acceleration.	$a(t) = 2t.$
2 Use velocity $= v = \int a(t)\,dt$.	$v = \int (2t)\,dt$ $\therefore\ v = t^2 + c$
3 Find c using $t = 3$ when $v = 0$.	Solve $0 = 3^2 + c$ $\therefore\ c = -9$
4 State the velocity function.	$v = t^2 - 9$
5 Use $x = \int v(t)\,dt$. We are using d for the constant because we have already used c above.	$x = \int (t^2 - 9)\,dt$ $= \frac{t^3}{3} - 9t + d$
6 Find d using $t = 0$ when $x = 0$ (starting at the origin).	$0 = \frac{0^3}{3} - 9(0) + d$ $\therefore\ d = 0$
7 State the displacement function.	$x = \frac{t^3}{3} - 9t$

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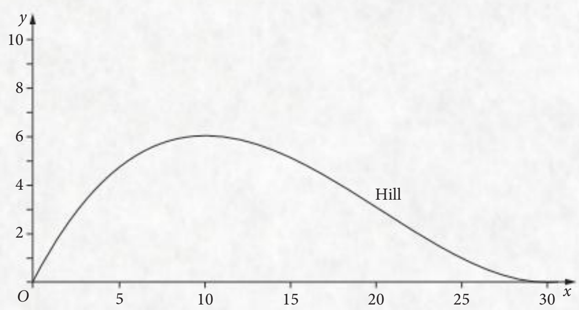
Worksheet Straight-line motion 2

VCE QUESTION ANALYSIS

© VCAA 2019 2BQ2 2019 Examination 2 Section B Question 2 (11 marks)

An amusement park is planning to build a zip-line above a hill on its property.

The hill is modelled by $y = \frac{3x(x-30)^2}{2000}$, $x \in [0, 30]$, where x is the horizontal distance, in metres, from an origin and y is the height, in metres, above this origin as shown in the graph below.

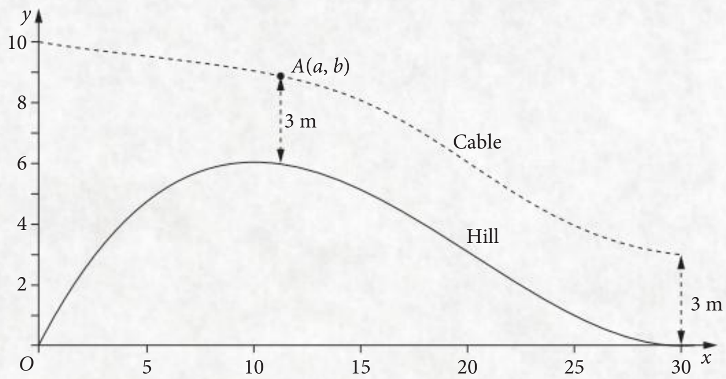


Video VCE question analysis: Applications of differentiation

a Find $\frac{dy}{dx}$. 1 mark

b State the set of values for which the gradient of the hill is strictly decreasing. 1 mark

The cable for the zip-line is connected to a pole at the origin at a height of 10 m and is straight for $0 \le x \le a$, where $10 \le a \le 20$. The straight section joins the curved section at $A(a, b)$. The cable is then exactly 3 m vertically above the hill from $a \le x \le 30$, as shown in the graph below.



c State the rule, in terms of x, for the height of the cable above the horizontal axis for $x \in [a, 30]$. 1 mark

d Find the values of x for which the gradient of the cable is equal to the average gradient of the hill for $x \in [10, 30]$. 3 marks

The gradients of the straight and curved sections of the cable approach the same value at $x = a$, so there is a continuous and smooth join at A.

e **i** State the gradient of the cable at A, in terms of a. 1 mark

ii Find the coordinates of A, with each value correct to two decimal places. 3 marks

iii Find the value of the gradient at A, correct to one decimal place. 1 mark

Reading the question

- The question is about a hill modelled by a cubic graph.
- The question asks where the average rate of change is equal to the instantaneous rate of change.
- The first section asks for the gradient of the graph and where the graph is strictly decreasing.
- The next section asks where gradients are equal.

Thinking about the question

- As this question is technology active the questions are much easier to solve than it looks on initial reading. Make sure your CAS is on 'standard' to get exact values, even if they are surds.
- Part **b** is very hard since it asks where the gradient is strictly decreasing and not the graph itself being strictly decreasing.
- You should define $f(x)$ on your calculator then, for example, we want to find $\frac{dy}{dx}$ of the graph f, the answer will be given in one step.
- In part **e**, remember how to write A, in terms of a.

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Worked solution (✓ = 1 mark)

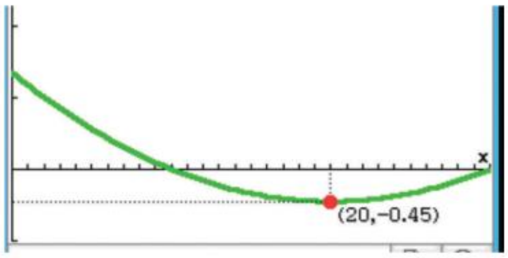
a $y = \dfrac{3x(x-30)^2}{2000}$, $x \in [0, 30]$

$\dfrac{dy}{dx} = \dfrac{9(x-10)(x-30)}{2000}$ ✓

b We need to find x values where the gradient of the hill is strictly decreasing.

Solve $\dfrac{dy}{dx} = \dfrac{9(x-10)(x-30)}{2000} = 0$ gives $x = 10, x = 30$. ✓

The gradient graph shows the gradient is strictly decreasing for $x = 0$ to $x = 20$.



Within the domain [0, 30], strictly decreasing for $x \in (0, 20]$. ✓

c Height of cable for $x \in [a, 30]$ is $h(x) = \dfrac{3x(x-30)^2}{2000} + 3$

$= \dfrac{3x^3}{2000} - \dfrac{9x^2}{100} + \dfrac{27x}{20} + 3$. ✓

d Average gradient $= \dfrac{f(30) - f(10)}{30 - 10}$ ✓

$= \dfrac{0-6}{20}$

$= -\dfrac{3}{10}$ ✓

Gradient of cable $= f'(x) = \dfrac{9(x-10)(x-30)}{2000} = -\dfrac{3}{10}$ Gradient of cable = gradient of hill

$x = \dfrac{60 \pm 10\sqrt{3}}{3}$. ✓ Both values of x are between 10 and 30.

e **i** Two possible correct answers.

Gradient of cable at $A = f'(a)$ Gradient of cable = gradient of hill

$= \dfrac{9(a-10)(a-30)}{2000}$ ✓

OR Gradient of cable at A = gradient of line from (0, 10) to (a, b)

$= \dfrac{b-10}{a-0}$

$= \dfrac{h(a)-10}{a}$

$= \dfrac{3a^2}{2000} - \dfrac{9a}{100} - \dfrac{7}{a} + \dfrac{27}{20}$ ✓

ii Solve $\dfrac{3a^2}{2000}-\dfrac{9a}{100}-\dfrac{7}{a}+\dfrac{27}{20}$ ✓ $=\dfrac{9(a-10)(a-30)}{2000}$ ✓

$a = 11.1157\ \ldots$ or $26.7356\ \ldots$, but a is between 10 and 20. $a \approx 11.12$

$h(11.1157\ldots) \approx 8.95$

(11.12, 8.95) to two decimal places ✓

iii $f'(11.1157\ldots) = \dfrac{9(11.1157\ldots-10)(11.1157\ldots-30)}{2000}$

$\approx$ **−0.1** to one decimal place ✓

Student performance 80–100% 60–79% 0–59%

a 93% This question was answered well.

b 3% This question was very hard since it asked where the gradient was strictly decreasing and not the graph itself being strictly decreasing. Students performed poorly in this question.

c 62% This question was generally well done. Some students did not give an equation. Others added 10, instead of 3, to f.

d 35% Students performed poorly in this question. A common incorrect answer for the average gradient was $\dfrac{3}{10}$ rather than $-\dfrac{3}{10}$. Some gave approximate answers for the x values, other students did not use brackets correctly.

e **i** 52% Some students got the correct answer but did not write the gradient in terms of a as required by the question.

ii 22% A challenging question that required the correct answer to part **i**. Some students found the value of a but not b. Other students rounded their answers incorrectly.

iii 20% If students obtained the correct value for a in part **ii**, they were generally successful in part **iii**.

EXERCISE 12.8 Applying the anti-derivative

ANSWERS p. 533

Recap

1 TECH-FREE Find the anti-derivative of $10x^4 + 4x^3 - 3x^2 - 2x$.

2 $2x^2 + 3x - 2$ has the anti-derivative

A $6x + 3$ **B** $6x + 3x - 2$ **C** $\dfrac{2x^3}{3}+\dfrac{3x^2}{2}-2x+c$

D $9x$ **E** $2x^2 + 3x$

Mastery

3 WORKED EXAMPLE 12 TECH-FREE Find an anti-derivative of $f'(x) = 3x^3 + 2x^2$ if $x = 1$ when $y = 0$.

4 WORKED EXAMPLE 13 TECH-FREE The velocity, m/s, of a particle is expressed by $v(t) = t^2 + 3t$. Find the displacement, x metres, given that $x(0) = 2$.

5 For the function $\dfrac{dy}{dx} = -2(x + 2)$, y could be expressed as

A $6(x + 2)$ **B** $6(x + 2)^2$ **C** -2

D $x^2 + 4x$ **E** $-x^2 - 4x$

6 Using CAS 6 The acceleration of a toy car, starting at the origin, is 20 m/s^2 and at $t = 1$ its velocity is 10 m/s. Its displacement at $t = 30$ is

A 20 m **B** $20t + c$ **C** $10t^2 - 10t + c$ **D** 8700 m **E** 9000 m

7 An anti-derivative of $y = 3x^3 + 2x^2$ equals

A $9x^2 + 4x$ **B** $9x^3 + 4x^2$ **C** $9x^2 - 4x$ **D** $3x^4 + 2x^3$ **E** $\frac{3}{4}x^4 + \frac{2}{3}x^3$

12.8

8 TECH-FREE The acceleration of an object is given by $a = 6t + 4$ cm/s^2. The particle is initially at rest at the origin. Find

a its velocity after 5 seconds. **b** its displacement after 5 seconds.

9 TECH-FREE The velocity of a particle is given by $v = 8t^3 - 3t^2$ cm/s. If the particle is initially at the origin, find

a its acceleration after 5 seconds. **b** its displacement after 3 seconds.

c when it will be at the origin again.

Exam practice

10 TECH-FREE (4 marks) The velocity (in cm/s) of an object is given by $v = t^2 (t^3 + 1)^2$ and the object is initially 2 centimetres to the right of the origin.

a Find its acceleration after 1 second. 2 marks

b Find the displacement of the object after 2 second. 2 marks

11 TECH-FREE (4 marks) A particle has a constant acceleration of 12 m/s. If the particle has a velocity of 2 m/s and is 3 metres from the origin after 5 seconds, find its displacement after 10 seconds.

12 TECH-FREE (4 marks) A particle is accelerating according to the equation $a = (3t + 1)^2$ m/s^2. If the particle is initially at rest 2 metres to the left of the origin, find its displacement after 4 seconds.

13 TECH-FREE (4 marks) A particle has a velocity given by $\frac{dx}{dt} = 2t - 3$ m/s. After 3 seconds, the particle is at the origin. Find its displacement after 7 seconds.

14 TECH-FREE (4 marks) The velocity of a particle is given by $v = 4 - 3t$ cm/s. If the particle is 5 centimetres from the origin after 2 seconds, find the displacement after 7 seconds.

15 TECH-FREE (4 marks) The velocity of a particle is given by $v = 3t^2 - 5$ cm/s. If the particle is initially at the origin, find its displacement after 3 seconds.

16 If $\frac{dy}{dx} = 2x^3 + 1$, and $y = 1$ when $x = 2$, then y equals

A $\frac{x^4}{2} + x$ **B** $\frac{x^4}{2} + x + 2$ **C** $2x^3 + x + 17$ **D** $\frac{x^4}{2} + x - 9$ **E** $\frac{x^4}{4} + x - 5$

17 A toy car is travelling with a velocity expressed by $v(t) = 2t + 1$ m/s. If the toy car started at the origin, its position after travelling for 5 seconds is

A 30 m **B** 11 m **C** $\frac{325}{6}$ m **D** $\frac{t^2}{2} + t + 5$ **E** $2t + 1 + c$

18 If $\frac{dy}{dx} = 2x + 1$, and $y = 1$ when $x = 0$, then y equals

A $x^2 + x + c$ **B** $x^2 + x + 1$ **C** $x^2 + x$ **D** $x^2 + x + 2$ **E** $2x + 1 + c$

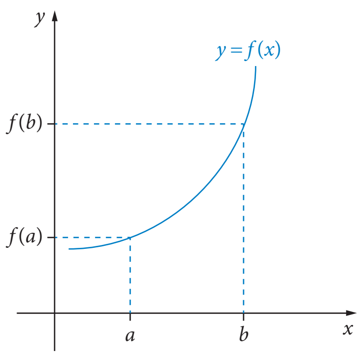
19 A car is travelling with acceleration expressed by $a(t) = 2t + 1$ m/s^2. If the car's velocity at $t = 1$ is 40 m/s, its velocity can be expressed as

A $2t + 1 + c$ **B** $t^2 + t$ **C** $t^2 + t + c$ **D** $t^2 + t + 38$ **E** $t^2 + t - 1639$

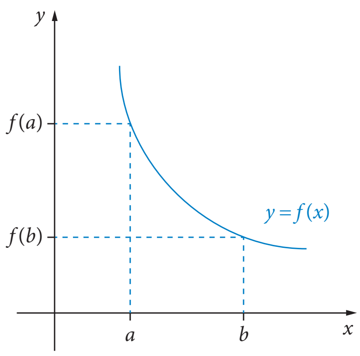
12 Chapter summary

Increasing and decreasing functions

- A positive gradient points up: $\nearrow$
- A negative gradient points down: $\searrow$
- The gradient of the graph $f(x)$ is the derivative function $f'(x)$ or $\dfrac{dy}{dx}$.
- A function is **strictly increasing** where the values of y are getting larger: as x increases, $f(a) < f(b)$.
- A function is **strictly decreasing** where the values of y are getting smaller: as x increases, $f(a) > f(b)$.



When $f(x)$ is strictly increasing
$f(a) < f(b)$

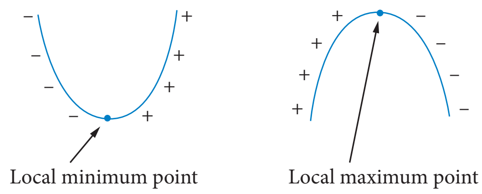


When $f(x)$ is strictly decreasing
$f(a) > f(b)$

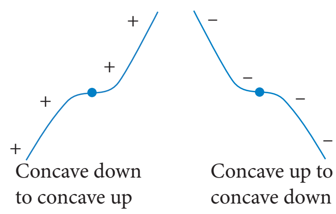
- For strictly increasing and strictly decreasing, the endpoints of the intervals are *included*, so use [ ], except for ∞ or $-\infty$ when we use ( ).

Stationary points

- A turning point is found where $f'(x) = 0$ and the sign of the gradient changes on either side of the stationary point.



- A stationary point of inflection is a flat 'bend' in the graph where the concavity changes and the sign of the gradient stays *the same* on both sides of the stationary point.



Maximum and minimum points

- A turning point is also called a **local maximum** or **minimum point**.
- To find the **global maximum** or **minimum point** of the graph of a function over an interval, we need to compare the **interval endpoint values** to the local maximum and minimum points.

Curve sketching

Identify **key features** of a graph when sketching:

- recognised general shape
- domain and range
- y-intercept
- x-intercepts
- stationary points
- endpoint values.

Maximum and minimum problems

Techniques of finding maximum and minimum points of graphs will help solve maximum and minimum problems.

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The anti-derivative

- The **anti-derivative** of a function $f(x)$ is the function $F(x)$ whose derivative is $f(x)$. In other words, $F'(x) = f(x)$. The process of finding a function from its derivative is called **anti-differentiation** or **integration**.
- If $\frac{dy}{dx} = ax^n$, then $y = \int ax^n dx = \frac{ax^{n+1}}{n+1} + c$, where ax^n is the **integrand** and dx means with respect to x.
- Notations for an **anti-derivative**, **primitive or integral** of a function are $F(x)$ and $\int f(x)dx$.
- If we are given extra information about the anti-derivative function, then we can find the value of c.

Straight-line motion

- Kinematics is the study of motion using displacement (x), velocity (v) and acceleration (a), all in terms of time.
- $\frac{dx}{dt} = v$ and $\frac{d^2x}{dt^2} = \frac{dv}{dt} = a$
- velocity $= v = \int a(t)\, dt$

  displacement $= x = \int v(t) dt$

Cumulative examination 1

Total number of marks: 10 Reading time: 4 minutes Writing time: 15 minutes

TECH-FREE *Technology is NOT permitted.*

1 (3 marks) Let $f: [-2, 2] \rightarrow R$, where $f(x) = 2x^3 - 1$.

a Calculate the average rate of change of f between $x = -1$ and $x = 2$. 1 mark

b Calculate the instantaneous rate of change of f at $x = -1$. 2 marks

2 (2 marks) Solve the equation $\log_2(2x^2) - \log_2(x) = 8$ for x.

3 (3 marks)

a Differentiate $y = 2x^{-3} - x$, with respect to x. 1 mark

b Evaluate $f'(4)$, where $f(x) = x - 3\sqrt{x}$. 2 marks

4 © VCAA 2017N 1Q2c (2 marks) Find $f(x)$ given that $f(4) = 25$ and $f'(x) = \frac{3}{8}x^2 - 10x^{-\frac{1}{2}} + 1$, $x > 0$.

Cumulative examination 2

Total number of marks: 20 Reading time: 4 minutes Writing time: 30 minutes

Approved technology is permitted.

Section A 6 multiple-choice questions

6 marks

1 When simplified, the expression $p^{-2}q^4p^6q^{-2}$ equals

A p^4q^2 **B** p^4q **C** p^4q^6 **D** $(pq)^4$ **E** p^2q^2

2 © VCAA 2019 2AQ5 Let $f'(x) = 3x^2 - 2x$ such that $f(4) = 0$. The rule of f is

A $f(x) = x^3 - x^2$ **B** $f(x) = x^3 - x^2 + 48$ **C** $f(x) = x^3 - x^2 - 48$

D $f(x) = 6x - 2$ **E** $f(x) = 6x - 24$

3 © VCAA 2018 2AQ5 Consider $f(x) = x^2 + \frac{p}{x}$, $x \neq 0$, $p \in R$. There is a stationary point on the graph of f when $x = -2$. The value of p is

A -16 **B** -8 **C** 2 **D** 8 **E** 16

4 When simplified, the expression $\log_2(x+2) - \log_2(x) - \log_2(2)$ equals

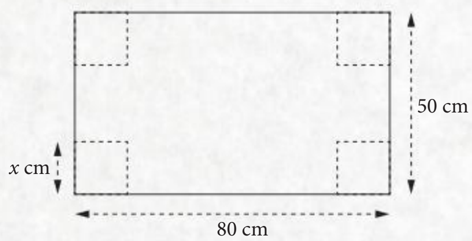
A $\log_2\left(\frac{2(x+2)}{x}\right)$ **B** $\log_2\left(\frac{2x+2}{x+2}\right)$ **C** $\log_2\left(\frac{2x}{x+2}\right)$

D $\log_2\left(\frac{x+2}{2x}\right)$ **E** $\log_2(1)$

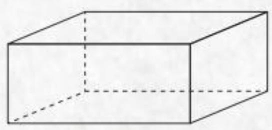
5 The solution(s) to the equation $2x^2 + 6x = 0$ are

A $\{-3, 0\}$ **B** $\{3, -3\}$ **C** $\{-3\}$ **D** $\{3\}$ **E** $\{0, 3\}$

6 © VCAA 2019 2AQ6 A rectangular sheet of cardboard has a length of 80 cm and a width of 50 cm. Squares, of side length x centimetres, are cut from each of the corners, as shown in the diagram below.



A rectangular box with an open top is then constructed, as shown in the diagram below.



The volume of the box is a maximum when x is equal to

A 10 **B** 20 **C** 25 **D** $\frac{100}{3}$ **E** $\frac{200}{3}$

Section B 2 questions

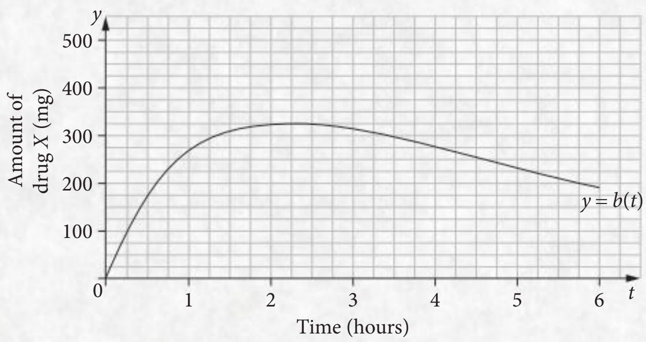
14 marks

1 © VCAA 2018 2BQ2 (8 marks) A drug, X, comes in 500 milligram (mg) tablets. The amount, b, of drug X in the bloodstream, in milligrams, t hours after one tablet is consumed is given by the function

$$b(t) = \frac{4500}{7}\left(e^{\left(-\frac{t}{5}\right)} - e^{\left(-\frac{9t}{10}\right)}\right)$$

a Find the time, in hours, it takes for drug X to reach a maximum amount in the bloodstream after one tablet is consumed. Express your answer in the form $a \log_e(c)$, where $a, c \in R$. 2 marks

The graph of $y = b(t)$ is shown below for $0 \le t \le 6$.



b Find the average rate of change of the amount of drug X in the bloodstream, in milligrams per hour, over the interval [2, 6]. Give your answer correct to one decimal place. 2 marks

c © VCAA MODIFIED Find the amount of drug X in the bloodstream, in milligrams, 6 h after one tablet is consumed. Give your answer correct to the nearest milligram. 2 marks

d © VCAA MODIFIED Find the rate at which the amount of drug X in the bloodstream is increasing exactly 2 hours after one tablet is consumed. Give your answer correct to one decimal place. 2 marks

2 © VCAA 2021 2BQ3a-d (6 marks) Let $q(x) = \log_e(x^2 - 1) - \log_e(1 - x)$.

a State the maximal domain and the range of q. 2 marks

b **i** Find the equation of the tangent to the graph of q when $x = -2$. 1 mark

ii Find the equation of the line that is perpendicular to the graph of q when $x = -2$ and passes through the point $(-2, 0)$. 1 mark

Let $p(x) = e^{-2x} - 2e^{-x} + 1$.

c Explain why p is not a one-to-one function. 1 mark

d Find the gradient of the tangent to the graph of p at $x = a$. 1 mark

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Answers

CHAPTER 1

EXERCISE 1.1

1 a^3b^8
2 12
3 a 2^{15x} b 2^{16x} c 2^{4x}
d $32a^7$ e b^{5x} f a^{3x}
4 a $\frac{1}{2a}$ b b^x c 1
d 3^{-7} e 2^{3x} f 2^{-4x}
5 A 6 B 7 C 8 E
9 $x = \frac{3}{2}$
10 $x = -6$
11 E 12 D 13 E 14 E
15 B 16 A 17 C 18 C

EXERCISE 1.2

1 $x = \frac{1}{4}$
2 C
3 a $4x + 6$ b $8x - 12$
c $x^2 - x - 6$ d $6x^2 + 13x + 6$
4 $x^2 + 6x + 5$
5 $4x^2 + 12x + 5$
6 a $x^2 - 1$ b $4x^2 - 4$
c $x^2 - 9$ d $6x + 3$
e $-4x + 4$ f $-16x^2 + 48x - 36$
7 a $6x^2 - 39x + 63$
b $6x^2 + 7x - 24$
c $m^2 - 10mn + 25n^2$
d $49p^2 - 16$
e $2x^2 - 3xy + 11x - 6y + 14$
8 C 9 D 10 C
11 D 12 B

EXERCISE 1.3

1 a $2x^2 + 5x + 3$ b $8x^2 - 4x - 12$
c $x^2 + 5x + 6$ d $6x^2 - 5x - 6$
2 B
3 $x^2 + 6x + 9$
4 $(x + 5)^2$
5 B
6 a $3x(x + 1)$ b $4x(x - 1)$
c $-3x(x + 3)$ d $-3(x^2 + 3)$
e $3(x + 1)^2$ f $4(x - 2)^2$
g $(x + 3)(x - 3)$ h $(x + 4)(x - 4)$
i $(x - y)(x + y)$ j $3(x + 2)(x - 2)$
k $4x^2(x - 4)$ l $2(x - 3a)(x + 3a)$
m $(x - 1)^2$ n $(x + 4)^2$
o $2(x + 1)^2$
7 a $-3(x + 3)(x - 1)$ b $3(x^2 + 1)(x + 2)$
c $4(x^2 + 1)(x - 4)$ d $a(ax - 2)(x + 3)(x - 3)$
e $3(x - 2)^2$ f $(x + 1)(x - 3)$
g $(x + 8)(x - 4)$ h $(2x + 3)(x - 5)$
i $(3x + 2)(x - 5)$
8 a $(2x + 7)^2$ b $(2x - 3)^2$ c $(m + 3n)^2$
9 B 10 E 11 A 12 E

EXERCISE 1.4

1 $a(x + 2)(x - 2)(ax + 1)$
2 $(x + 3)(x - 3)(ax + 2)$
3 $x^3 + 6x^2 + 12x + 8$
4 $(x - 1)^3$
5 $x = -12$
6 $x = 0, x = -9$
7 A 8 C 9 D
10 $x = -1, x = 0, x = -k$
11 D 12 C 13 A 14 C 15 E

EXERCISE 1.5

1 a $x = -\frac{1}{3}$ b $x = -2, x = \frac{5}{2}$
2 D
3 a $k = -5, 1$ b $v = -7, 5$ c $x = 1, 8$
4 $x = \frac{1}{2}, x = -3$
5 $x = -3 \pm 2\sqrt{2}$
6 $x = \frac{3}{2}, x = -1$
7 a $\Delta = 25$
b $\Delta =$ perfect square $\therefore$ 2 rational roots
8 $\Delta = -55 < 0$
9 $k = 0, k = 6$ 10 D
11 B 12 D 13 E 14 B
15 a $x = -3, x = \frac{4}{3}$ b $x = 1, x = 11$
c $x = -4, x = -1$
16 a $x = \frac{-1 \pm \sqrt{17}}{2}$ b $x = \frac{5 \pm \sqrt{13}}{6}$
c $q = 2 \pm \sqrt{7}$
17 C 18 B 19 E 20 B
21 A 22 B 23 B 24 A

EXERCISE 1.6

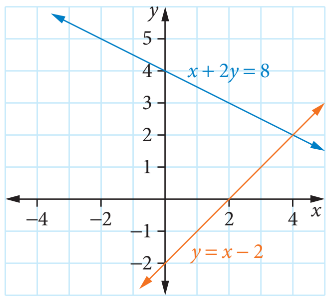
1 a $\Delta = 0$ b one real solution
2 E
3 $r = \frac{S - 2}{4\pi h^2}$
4 a $A = \frac{3V}{h}$ b $a = \frac{2s + nd - n^2d}{2n}$
c $s = \frac{v^2 - u^2}{2a}$ d $b = \sqrt{c^2 - a^2}$
5 D 6 B 7 D

EXERCISE 1.7

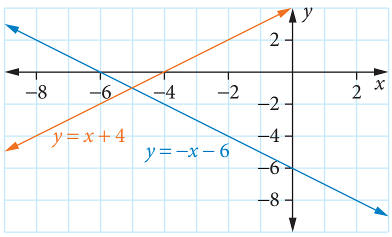
1 $a = \frac{3}{m} - 4$

2 $p = \frac{v}{2v - 1}$

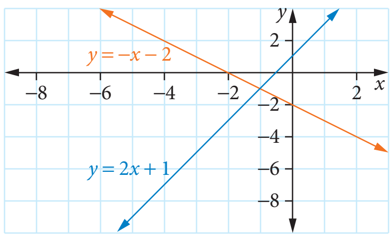
3 (4, 2)



4 a (−5, −1)



b (−1, −1)



5 a (2, 3) b (4, 1)

6 (−1, 0) and (2, 3)

7 a $x = \frac{25}{4}, y = \frac{15}{4}$ b $x = 3, y = -1$

c $x = -\frac{17}{5}, y = \frac{39}{5}$ d $x = 4, y = -3$

e $x = 0, y = -5$ f $x = 1, y = 0$

8 $\{x = -1, y = 0, a = -\frac{1}{3}, b = -\frac{1}{3}\}$

or $\{x = \frac{3}{2}, y = \frac{5}{4}, a = \frac{1}{2}, b = \frac{1}{2}\}$

9 C 10 C 11 D

12 a $(x - 2)(x^3 + 2x^2 + 4x + 4)$

b (0,−8) and (2, 0)

EXERCISE 1.8

1 $x = 4, y = -3$

2 $x = \sqrt{6} - 1, y = -2\sqrt{6} + 8$ or $x = -\sqrt{6} - 1, y = 2\sqrt{6} + 8$

3 $x \geq \frac{1}{2}$ 4 $x < 0$ 5 B 6 D

7 C 8 C 9 C 10 C

CUMULATIVE EXAMINATION 1

1 a $y = 3$ b $x = 0, x = 2$

c $\frac{3}{2}$ d $a = 2, b = 3$

2 $a = -\frac{39}{64}$ 3 $x = -1, x = 3$

CUMULATIVE EXAMINATION 2

Section A

1 D 2 D 3 B 32%

4 E 5 D

Section B

1 a $x = 0, x = 5$ b (2.5, −6.25)

c i $y = -5x + 1$ ii $2\sqrt{26}$

d i $2\sqrt{k^2 + 1}$ ii $k = \frac{3}{4}$

e (0, 0) and (6, 6) f (0, 0) and $(k + 1, k + 1)$

CHAPTER 2

EXERCISE 2.1

1 a {(−1, 5), (4, 3), (6, 3), (7, 5)}, domain: {−1, 4, 6, 7}, range: {3, 5}, function, many-to-one

b {(2, 7), (2, 3), (5, 3), (5, −2), (8, −2), (8, 4)}, domain: {2, 5, 8}, range: { −2, 3, 4, 7}, not a function

c {(4, 1), (−6, −4), (6, 2), (3, −1)}, domain: {−6, 3, 4, 6}, range: {−4, −1, 1, 2}, function, one-to-one

d {(7, 5), (−4, 3), (3, 3), (3, 6)}, domain: {−4, 3, 7}, range: {1, 3, 5, 6}, not a function

2 a function, many-to-one b not a function

c not a function d function, one-to-one

3 a function b not a function

c not a function d function

4 a not a function b function

c function d not a function

e not a function f function

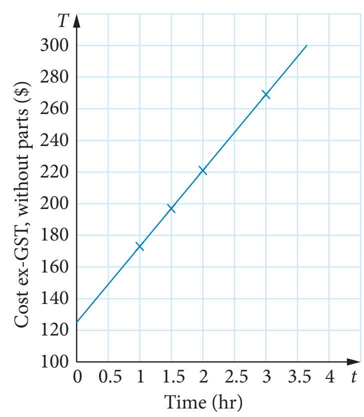
g function h function

5 E 6 A 7 D

EXERCISE 2.2

1 D 2 B

3 a



b The equation is $T = 48t + 125$.

c The hourly rate is \$48/hr.

d The callout fee is \$125 + \$24 = \$149.

e The cost is \$398, ex GST.

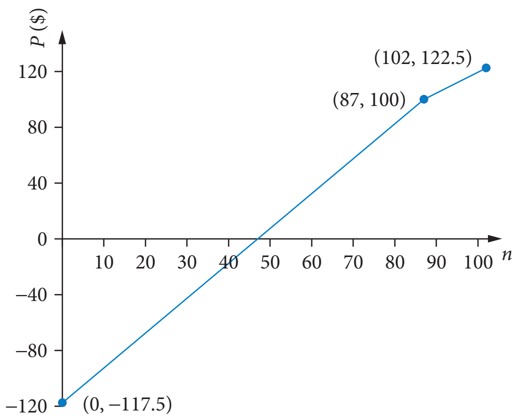
f The cost with GST is \$437.80.

4 a 2.5

b $P = 2.5n - 2.5 \times 47 = 2.5n - 117.5$

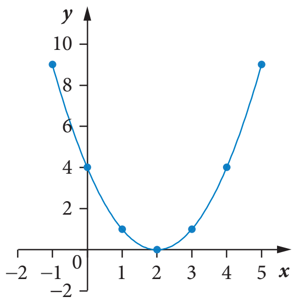
c \$100

d

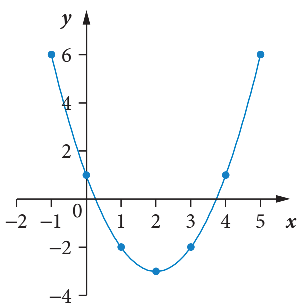


e 4%

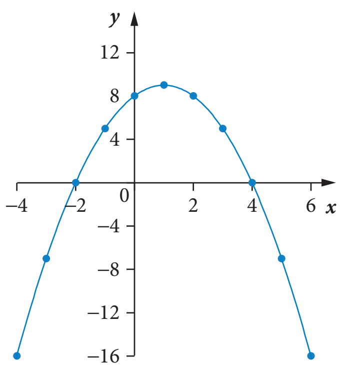
5 a



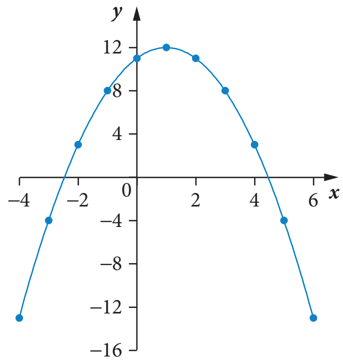
b



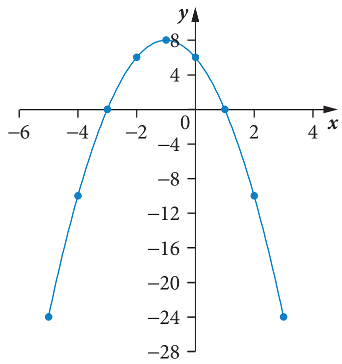
c



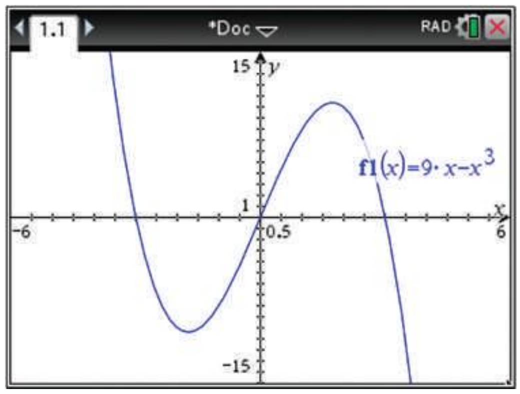
d



e

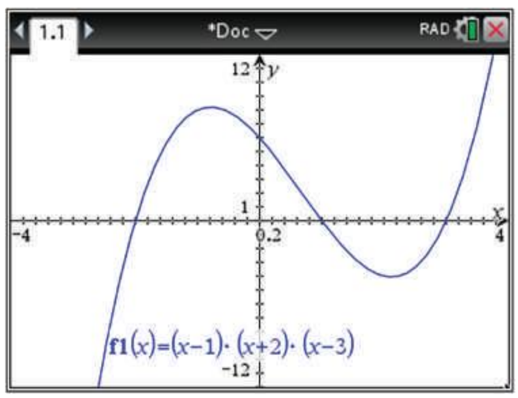


6 a

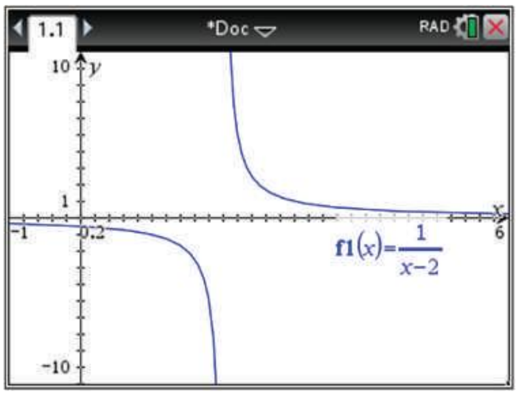


minimum (−1.7, −10.4), point of inflection (0, 0), maximum (1.7, 10.4), y-intercept 0, zeros −3, 0, 3

b

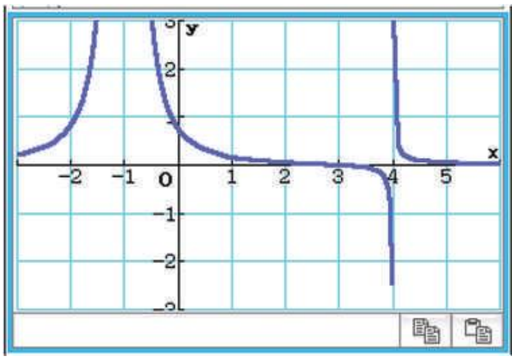


maximum (−0.8, 8.2), point of inflection $\left(\frac{2}{3}, \frac{56}{27}\right)$, minimum (2.1, −4.1), y-intercept 6, zeros −2, 1, 3 c



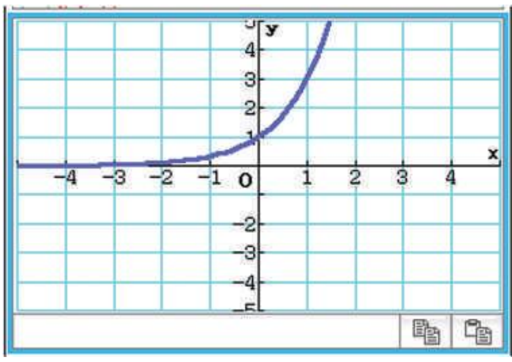
horizontal asymptote $y = 0$, vertical asymptote $x = 2$, y-intercept -0.5

d



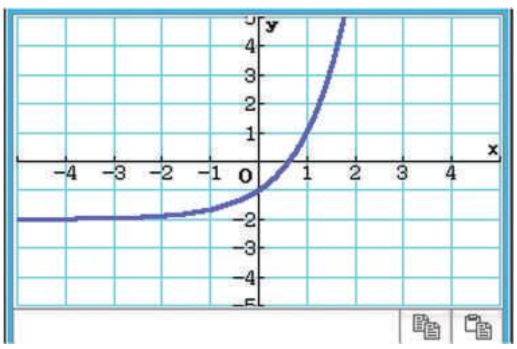
horizontal asymptote $y = 0$, vertical asymptotes $x = -1$, $x = 4$, point of inflection, $(2.6, 0)$, y-intercept 0.75, zero 3

e



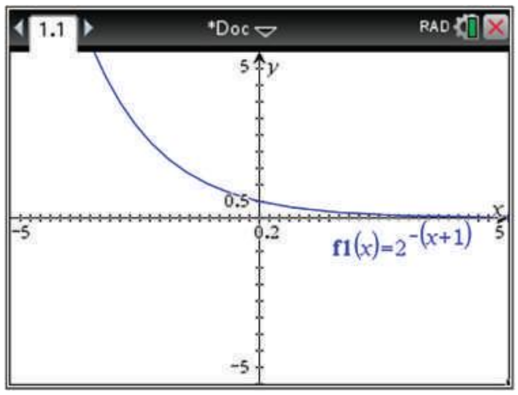
horizontal asymptote $y = 0$, y-intercept 1

f



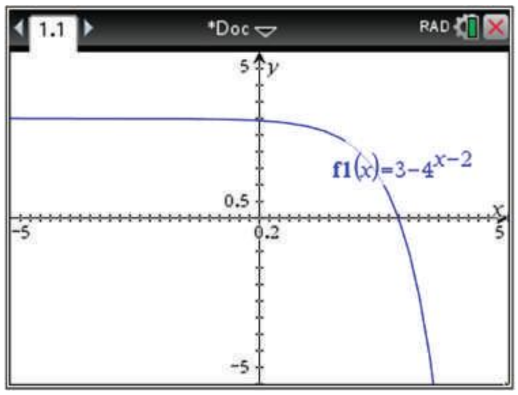
horizontal asymptote $y = -2$, y-intercept -1, zero 0.6

g



horizontal asymptote $y = 0$, y-intercept 0.5

h



horizontal asymptote $y = 3$, y-intercept $2\frac{15}{16}$, zero 2.8

7 **a** horizontal asymptote $y = -1$, y-intercept -0.9, zero 2

b stationary point of inflection $(-1, -2)$, y-intercept -4, zero -2

c horizontal asymptote $y = 3$, vertical asymptote $x = 2$, y-intercept 2.5, zeros 1, 3

d minimum at $(-1, -16)$, maximum at $(2, 7.2)$, point of inflection $(0.4, -6)$, y-intercept -11, zeros -2.1, 0.9, 2.8

e vertical asymptote $x = 0.75$, zero 3.4

f horizontal asymptote $y = 2$, vertical asymptotes $x = -1$, $x = 1$, minimum $(-0.3, 2.8)$, y-intercept 3, zero -1.2

8 There is a maximum in March and a minimum in August. There is a point of inflection in about May. The population increases during spring and summer and decreases during autumn and winter.

9 **a** They have maxima at the end of the exercise period.

b They started exercising after 5 minutes.

c The recovery time of the fit person is much less than the recovery time of the unfit person.

10 **a** The inverse is a function.

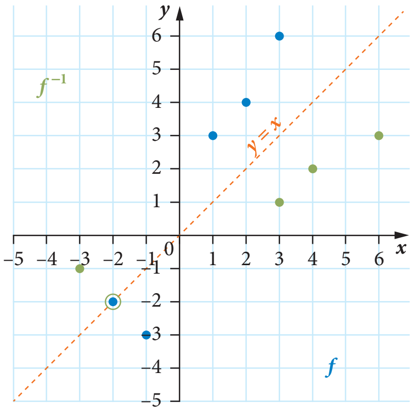
b The inverse is not a function. Restrict the domain where there are no duplicate values; for example. $\{-4, -2, 1, 5, 8\}$.

c The inverse is a function.

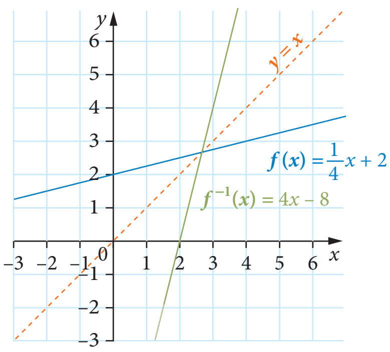
d The inverse is not a function. Restrict the domain to $x \le 2$ or $x \ge 2$.

e The inverse is a function.

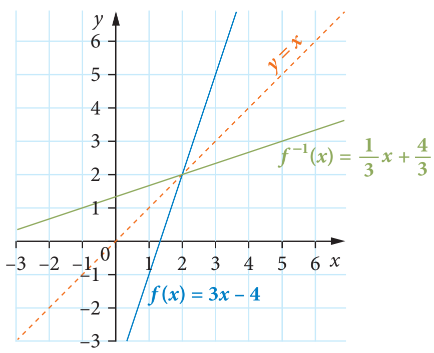
11 a f^{-1}: (−3, −1), (−2, −2), (3, 1), (4, 2), (6, 3), (−4, 5)



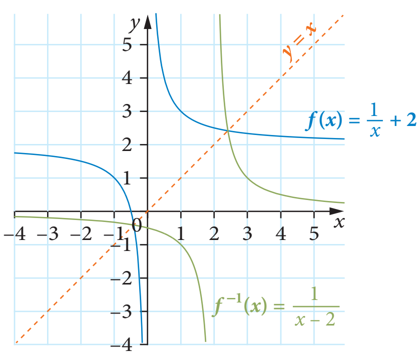
b $f^{-1}(x) = 4x - 8$



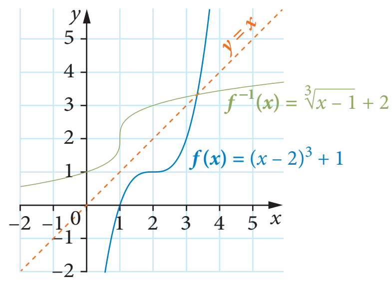
c $f^{-1}(x) = \frac{1}{3}x + \frac{4}{3}$



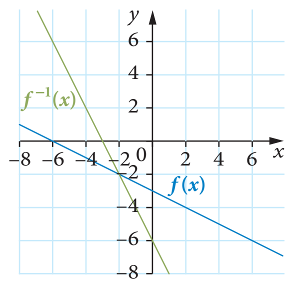
d $f^{-1}(x) = \dfrac{1}{x-2}$



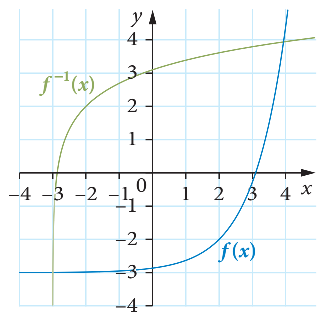
e $f^{-1}(x) = \sqrt[3]{x-1} + 2$



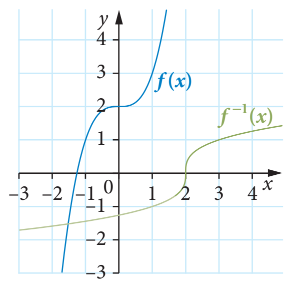
12 a



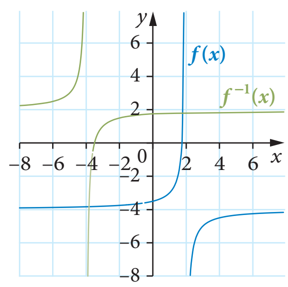
b



c



d



13 a $b = 10, c = -25$

b $b = c = 4.5$

c $2f^3 - 2f^2 - gf + g - 5 = 0$ or $g = \dfrac{2f^3 - 2f^2 - 5}{1 - f}$

14 $f^{-1}: R \to R^+, f^{-1}(x) = (6 - 2x)^2$

15 C

16 A

EXERCISE 2.3

1 E **2** C

3 **a** $R\backslash\{-5\}$ **b** R

c $R\backslash\{-2, 2\}$ **d** R

e $(-\infty, 4] \cup [4, \infty)$ **f** $[0, \infty)$

g $(-\infty, -2] \cup [3, \infty)$ **h** $[-3, 5]$

i $(-\infty, 7)$

4 **a** $(-\infty, 25]$ **b** $[-1, \infty)$ **c** $(-\infty, 9]$

d $R\backslash\{0\}$ **e** R^+ **f** $[-4. 4]$

g $(-17, 23)$ **h** $(-21, 18]$ **i** $[4, \infty)$

j $[-16, 20]$

5 **a** $[-14, 6.25]$ **b** $[-20.25, 22)$

c $[5.25, 21]$ **d** $\left[-\frac{4}{11}, -\frac{1}{9}\right]$

e $(\log_{10}(2), \log_{10}(8)]$ **f** $[\approx 13.1285, 168)$

6 **a** $(-2, 2]$ **b** $(1, 6]$

c $\left[-\frac{4}{3}, \frac{1}{3}\right)$ **d** $[0.5, 2.75)$

e $[0.1, 5)$ **f** $[-2, -1) \cup (2.25, 4]$

7 $[-3, \infty)$ **8** $(-\infty, 3)$

9 domain: $(-\infty, -2] \cup [2, \infty)$, range: $[0, \infty)$

10 A **11** D **12** D **13** C **14** A

EXERCISE 2.4

1 D **2** B

3 **a** $2x - y + 1 = 0$ **b** $3x + 4y - 14 = 0$

c $x + 5y - 33 = 0$ **d** $2x - y - 4 = 0$

e $y = 5$ **f** $x + y + 1 = 0$

g $x + 3y + 3 = 0$ **h** $x = -5$

i $3x + 4y + 12 = 0$ **j** $4x + 5y + 6 = 0$

k $8x - 12y - 3 = 0$ **l** $x - 3y - 5 = 0$

m $x + 3y - 6 = 0$ **n** $x - 3y + 6 = 0$

o $2x - y + 1 = 0$ **p** $x + y - 3 = 0$

q $x = 1$ **r** $5x - 2y - 19 = 0$

s $3x + y - 2 = 0$ **t** $y = -6$

4 **a** $2x + y - 1 = 0$ **b** $3x - 2y + 6 = 0$

c $x + 2y - 6 = 0$ **d** $2x - 3y = 0$

5 **a** 45° **b** 117° **c** 30° **d** 18°

e 135° **f** 139°

6 **a** $(5, 2)$ **b** $(-0.5, -3.5)$ **c** $(1.5, 3)$

d $(-5, 4)$ **e** $(3.5, 6.5)$ **f** $(0, -6)$

7 $(3, -3)$

8 **a** $\sqrt{10}$ **b** $2\sqrt{5}$ **c** 5

d $10\sqrt{2}$ **e** 17 **f** $2\sqrt{13}$

9 $a = -1$ or $a = 11$

10 $(a - 7)^2 + (b - 5)^2 = 100$; a and b are the coordinates (a, b) of a circle of radius 10 with centre at $(7, 5)$.

11 **a** $5x - 3y - 17 = 0$ **b** $x + 2y + 6 = 0$

c $3x - 4y + 25 = 0$ **d** $3x + y - 11 = 0$

12 **a** $x + 2y - 12 = 0$ **b** $x - 3y - 6 = 0$

c $3x - 2y - 6 = 0$ **d** $5x + 3y + 19 = 0$

13 **a** L_2 and L_4 **b** L_3 and L_6

14 Proof: See worked solutions

15 $(4, 2)$ **16** A **17** A

18 B **19** A

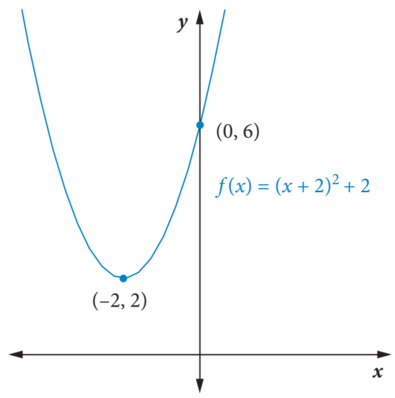
EXERCISE 2.5

1 E **2** A

3 **a** $(-2, 2)$ **b** $(3, -3)$ **c** $(1, -5)$

d $(-2, 6)$ **e** $(-3, -22)$ **f** $(3, 21)$

4 **a**



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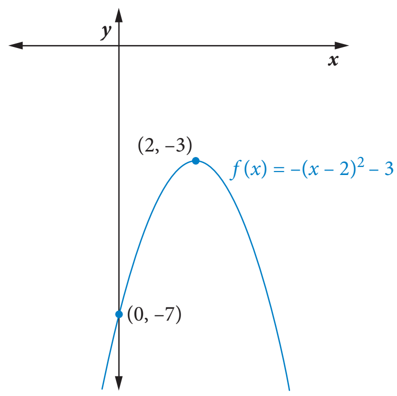

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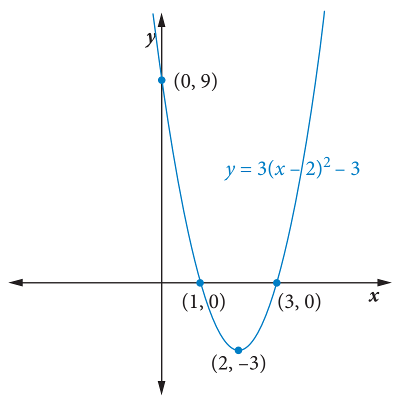

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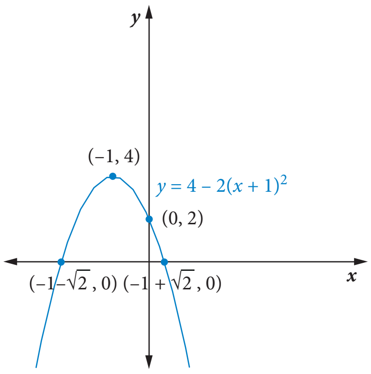

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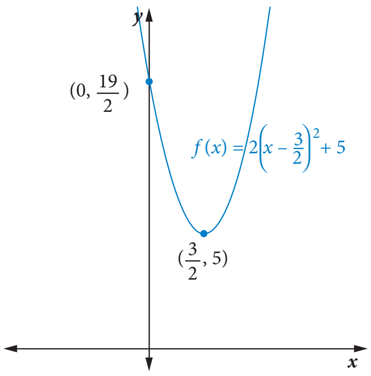

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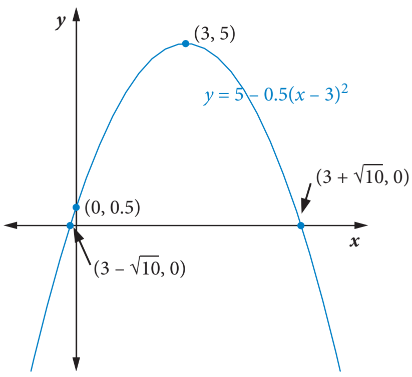

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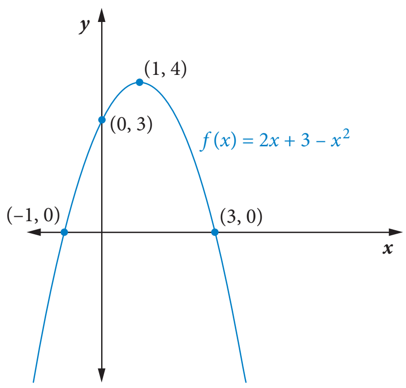

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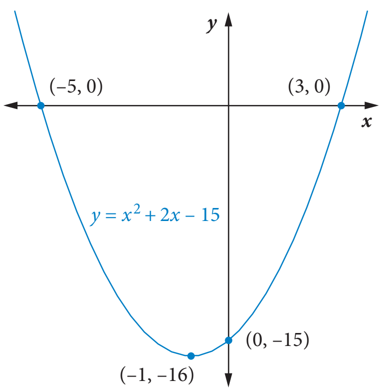

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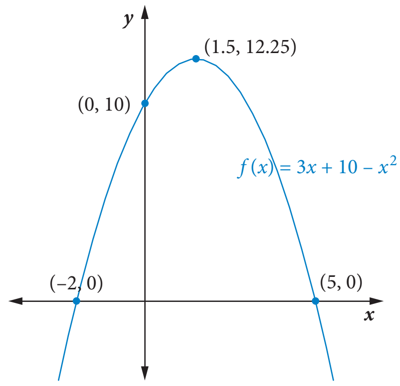

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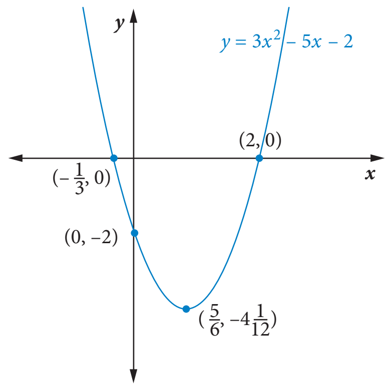

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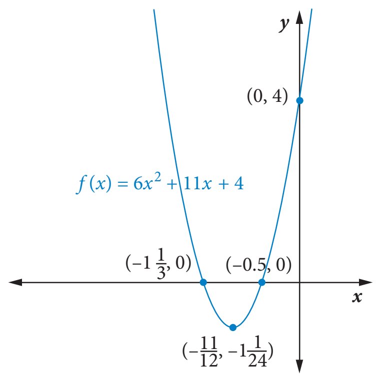

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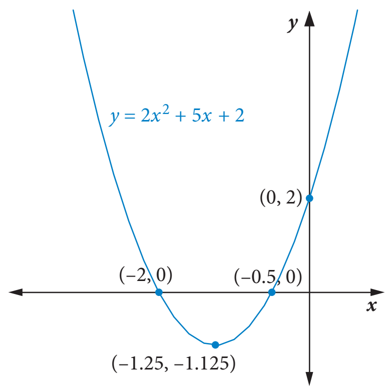

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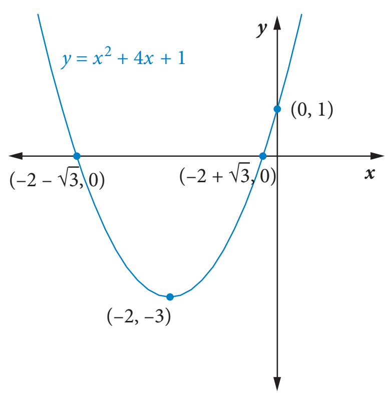
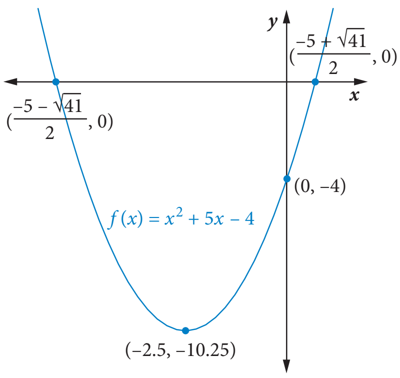


ANSWERS

b



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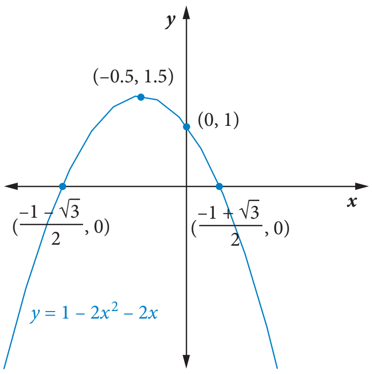

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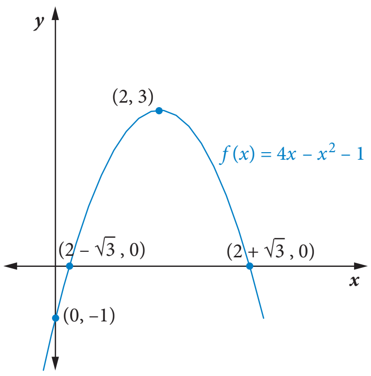

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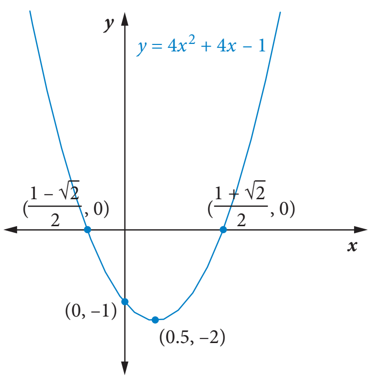

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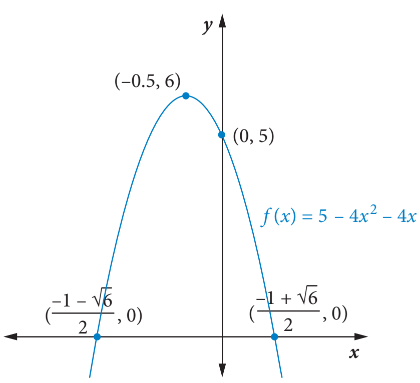

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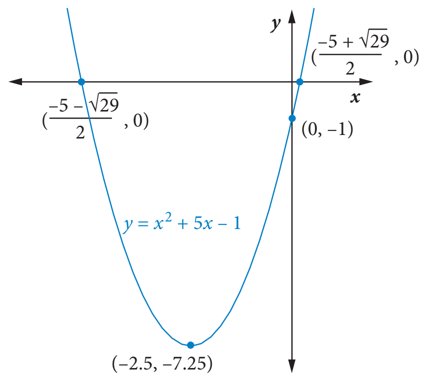

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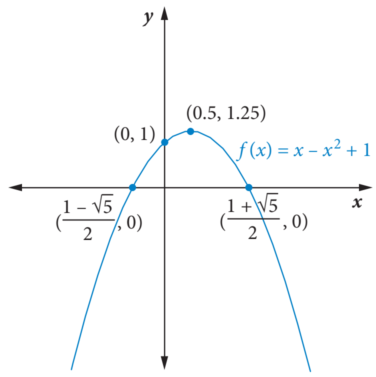

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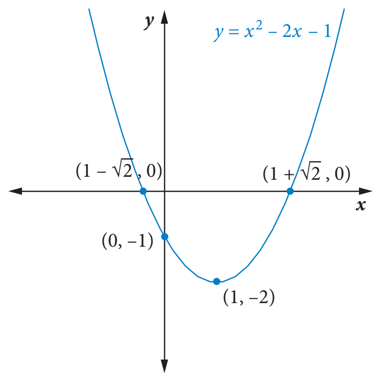

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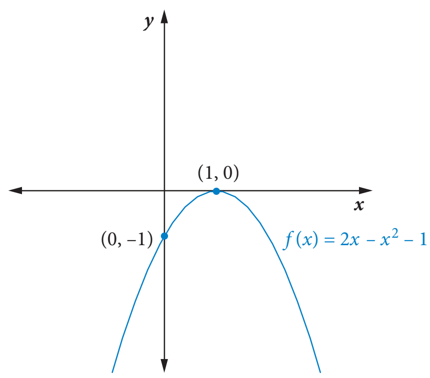

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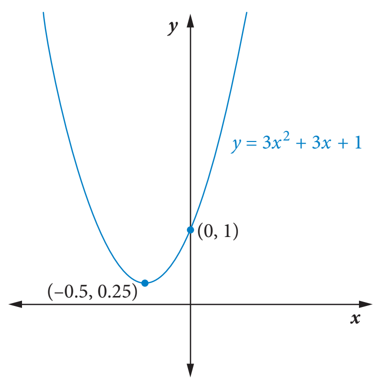

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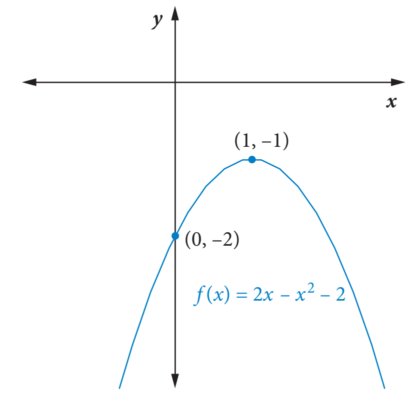
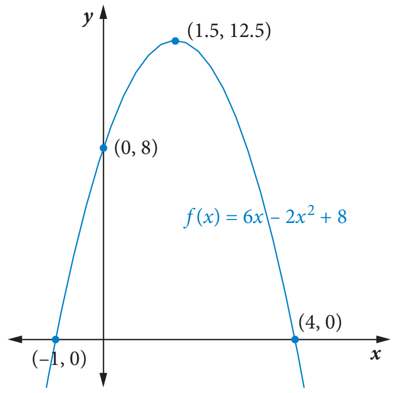


8 $k > 6$

9 $(-\infty, -3) \cup (7, \infty)$

10



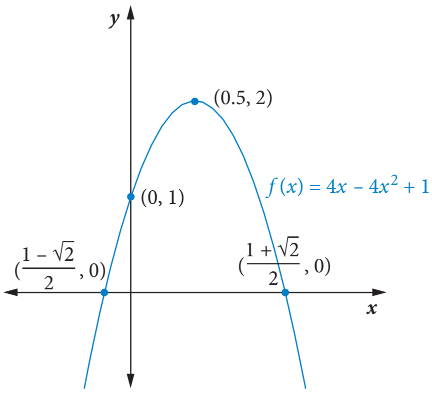
11 B **12** D

CUMULATIVE EXAMINATION 1

1 $a = -1 + \sqrt{5}$ **2** $x = -2, y = 3$

3 $(-\infty, -4) \cup (4, \infty)$

4



CUMULATIVE EXAMINATION 2

Section A

1 C **2** E **3** C

4 B 53% **5** C

Section B

1 a $x = \frac{29}{11}, y = \frac{1}{11}$

b $x = \frac{ab-5}{3a-2}, y = \frac{15-2b}{3a-2}$

c $a = \frac{2}{3}, b \neq \frac{15}{2}$

d $a = \frac{2}{3}, b = \frac{15}{2}$

2 a Range of f is R, range of g is $R\backslash\{-0\}$.

b Proof: see worked solutions.

c $a^2b^2 + 8ab + 8a + 16 = 0$

CHAPTER 3

EXERCISE 3.1

1 a, b, e

2 a 5 **b** 4 **c** –3 **d** 0

3 a $a = 0$ **b** $b = 10$ **c** $c = -6$

d $a = -1, b \neq 7$ **e** $a = 4$

4 a i $-2x^3 + 5$ **ii** $-5x^3 - 12x^2 + 5$

b Yes, because LHS = RHS.

5 a i $f(1) = 4$ **ii** $f(-3) = 0$

b i –2 **ii** 2 **iii** 14

c 9 **d** $x = \pm 5$

e $2x + 2h - 9$ **f** $x^2 + 2$ **g** $t = -1$

h $t = -4, t = 2$

6 a i 2 **ii** 8 **b i** 34 **ii** –8

7 $k = 2$ **8** $b = 29$

9 a 30 m **b** 10 m **c** 2.4 seconds

10 a 10 m by 30 m **b** 20 m by 20 m

11 D **12** B

EXERCISE 3.2

1 B **2** C

3 $a = 46, b = -28$

4 $A = 6, B = -2, C = -10$

5 $m = 3, n = -2, p = 13$

6 a Proof: see worked solutions

b $p = -2, k = 1$

7 a Proof: see worked solutions

b $\frac{1}{4}x^4 - 2x^2 + 4$

8 a $x(x-2)((x+1)^2 + 3)$

b translation of 1 unit in the negative direction of the x-axis

c i $[1, 3)$ **ii** $(-\infty, 1)$

EXERCISE 3.3

1 $a = 1, b = 6$ **2** $a = 2, b = 2$

3 a $(x+5)(x-3)$ **b** $(3x-4)(2x-3)$

c $(x-2)(x^2 - 3x - 3)\ 8$ **d** $(x+2)(2x^2 - 5x - 3)\ 15$

e $(3x+4)(2x^2 - 7 + 8)\ 3$ **f** $(5x-2)(x^2 + x + 2)\ -4$

g $(2x-1)\left(2x^2 + \frac{7}{2}x + \frac{7}{4}\right) + \frac{17}{4}$

h $(x-2)(x^2 + 2x + 4)$

4 **a** quotient = 1, remainder = $4x - 11$

b quotient = 3, remainder = $-6x - 13$

c quotient = $4x - 2$, remainder = $13x - 5$

d quotient = $x - 3$, remainder = $-6x + 14$

e quotient = $2x + \frac{4}{3}$, remainder = $-10x - 7$

f quotient = $\frac{1}{2}x + \frac{5}{6}$, remainder = $7x - 13$

g quotient = $\frac{1}{2}x - \frac{5}{4}$, remainder = $\frac{5}{2}x - \frac{55}{4}$

5 **a** quotient = $2x^2 - 5x + 7$, remainder = -3

b quotient = $2x^3 + x + 1$, remainder = 0

6 **a** $(x-2)(x+5)+9$

b $(x+1)(x^2-5x+10)-11$

c $(x+3)(2x^2-10x+33)-104$

d $(x-4)(5x^2+26x+107)+426$

7 **a** $(2x-1)(x-1.5)-0.5$

b $(3x+1)\left(2x^2 - \frac{11}{3}x - \frac{17}{9}\right) + \frac{332}{9}$

c $(2x-3)\left(2x^2 - \frac{1}{2}x + \frac{7}{4}\right) - \frac{23}{4}$

d $(2x+5)(2x^2-2x+5)-35$

8 **a** $1 + \frac{3}{x-2}$ **b** $1 + \frac{2}{x+1}$

c $1 - \frac{2}{2x-3}$ **d** $1 - \frac{12}{3x+5}$

EXERCISE 3.4

1 $4x^2 - 6x + 9$ **2** $\frac{1}{3}$

3 **a** 41 **b** -3 **c** -43

d 9424 **e** 0 **f** 37

4 **a** 47 **b** 2321 **c** 31 174 **d** -3

5 **a** $k = 8$ **b** $k = \frac{2}{7}$ **c** $k = 15\,299$

d $k = 7\frac{1}{9}$ **e** $k = \pm 2$

6 $a = -6$

7 **a** Proof: see worked solutions

b $a = \frac{29}{5}, b = -\frac{48}{5}$

8 $a = -1\frac{7}{12}, b = -1\frac{17}{48}$

9 **a** 0 **b** yes

c $x^3 - 4x^2 + x + 6 = (x-2)(x^2-2x-3)$

d $f(x) = (x-2)(x-3)(x+1)$

10 **a** $(x+5)(x+1)(x-2)$ **b** $(x+7)(x+1)(x-6)$

c $(x+2)^2(x-2)$ **d** $(x+4)(x-2)(2x+3)$

11 **a** $(x-2)(x+4)(2x-3)$ **b** $(x-5)(x-2)(2x+3)$

c $(x-5)(x-1)(x+1)$ **d** $(x-4)(x-3)(x-2)$

e $(x-2)(x+3)(2x-1)$ **f** $(x-1)(x+1)(x+2)$

12 **a** $(x-7)(x+2)(3x-1)$ **b** $(x-3)(x+2)(2x-1)$

c $(x+2)(x-3)(x+1)$ **d** $-(x-2)(x+3)(4x-3)$

e $-(x-3)(x-2)(5x-1)$ **f** $-(x-5)(x-3)(x+2)$

13 **a** $(x+1)(x-2)(x-6)$ **b** $(x+2)(x-3)(x+3)$

c $(x+2)(2x+1)(x-5)$ **d** $(x-2)(2x+1)(x+3)$

14 **a** $(x+3)(x^2-3x+9)$ **b** $(1-x)(1+x+x^2)$

c $8(t+5)(t^2-5t+25)$

d $\left(\frac{x}{2} - 3\right)\left(\frac{x^2}{4} + \frac{3x}{2} + 9\right)$

e $(x+y+3)(y^2-3y-xy+9+6x+x^2)$

15 C **16** B **17** D

18 E **19** D **20** E

EXERCISE 3.5

1 $2x - 3$ is not a factor. **2** $-2x^3 - 7x^2 + 1$

3 **a** $\pm 1, \pm\frac{1}{3}$ **b** $\pm 1, \pm 2, \pm 5, \pm 10$

c $\pm 1, \pm 2, \pm 4, \pm 8, \pm\frac{1}{5}, \pm\frac{2}{5}, \pm\frac{4}{5}, \pm\frac{8}{5}$

d $\pm 1, \pm 3, \pm 9, \pm 27$

e $\pm 1, \pm 7, \pm\frac{1}{2}, \pm\frac{7}{2}$

f $\pm 1, \pm 3, \pm 7, \pm 21, \pm\frac{1}{5}, \pm\frac{3}{5}, \pm\frac{7}{5}, \pm\frac{21}{5}$

4 **a** **i** $\pm 1, \pm 3$ **ii** $(x-1)^2(x+3)$

b **i** $\pm 1, \pm 11$ **ii** $(x-1)^2(x-11)$

c **i** $\pm 1, \pm\frac{1}{2}, \pm\frac{1}{4}$ **ii** $(x-1)^2(4x-1)$

d **i** $\pm 1, \pm 3, \pm\frac{1}{3}$ **ii** $(3x-1)(x+3)(x+1)$

e **i** $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12, \pm\frac{1}{3}, \pm\frac{2}{3}, \pm\frac{4}{3}$

ii $(x+3)(3x^2+4)$

f **i** $\pm 1, \pm 3, \pm 5, \pm 15, \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \pm\frac{15}{2}$

ii $(2x+3)(x^2+3x+5)$

g **i** $\pm 1, \pm 7, \pm\frac{1}{5}, \pm\frac{7}{5}$ **ii** $(x-7)(x-1)^2(5x-1)$

h **i** $\pm 1, \pm 5, \pm\frac{1}{3}, \pm\frac{5}{3}$ **ii** $(x-5)(x+1)^2(3x-1)$

5 7 is a factor of the constant term. 5 is a coefficient of the leading term.

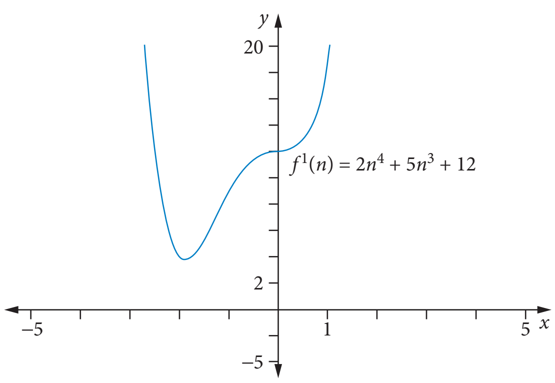
6 No, because the coefficient of x is not an integer.

7 **a** $\pm 1, \pm\frac{1}{7}, \pm\frac{2}{7}, \pm 2, \pm 7, \pm 14$

b None. Actual zeros are irrational.

8 **a** $\pm 1, \pm\frac{1}{2}, \pm 2, \pm\frac{3}{2}, \pm 3, \pm 4, \pm 6, \pm 12$

b $f(n) > 0$ for all x



EXERCISE 3.6

1 $\frac{1}{2}$

2 $(x + 3)(x - 1)^2$

3 **a** $x = 6$ **b** $x = -8$ **c** $x = 1$ **d** $x = \frac{4}{3}$

4 **a** $x = -3, 0, 2.5$ **b** $x = -4, -2, 3$

c $x = -3, -1, 3$ **d** $x = 1, 3, 4$

5 **a** $x = -4, 1, 3$ **b** $x = -3, 2, 3$

c $x = -2, 2$ **d** $x = 1, 2$

6 **a** $x = 0, -2, 3, 9$ **b** $x = -2, -1, 0$

c $x = -1, 0, 3$ **d** $x = -2, 0, 2, 3$

e $x = -2, -0.5, 0, 0.5$

7 **a** $(x + 2)(x^2 - 4x + 3)$ **b** $-2, 1, 3$

8 $a = 1 - \sqrt{2}$

9 C **10** D **11** D

12 **a** $f(0) = p(0)^3 + q(0)^2 + r(0) + s = 7$, so $s = 7$.

b $D(4, 0)$, $F(7, 0)$

EXERCISE 3.7

1 $x = 7$ **2** $x = -2, -1, 0$

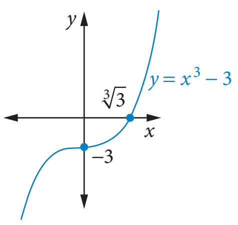
3 **a** $x \approx 0.2$ **b** $x \approx 1.6$ **c** $x \approx 1.9$

d $x \approx -1.3$ **e** $x \approx 4.7$

EXERCISE 3.8

1 **a** $x = -2.2$ **b** $x \approx 0.6$

2 **a**



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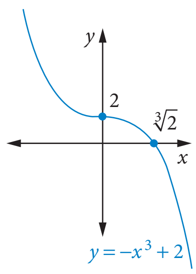

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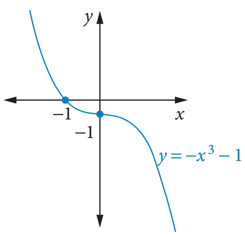

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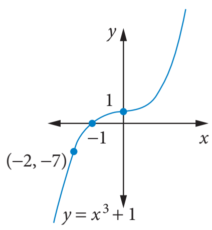

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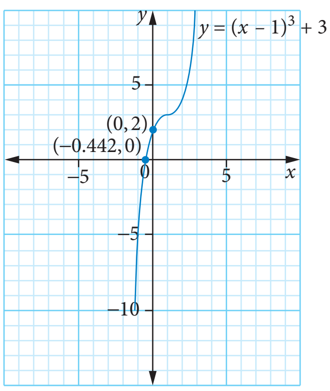

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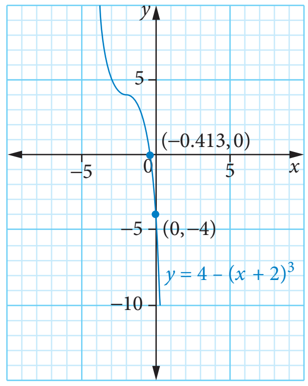

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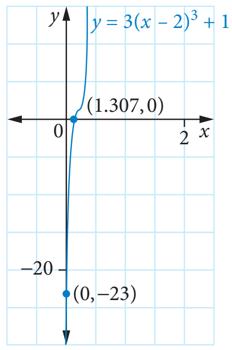

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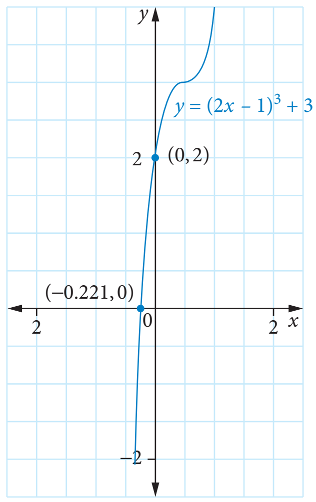

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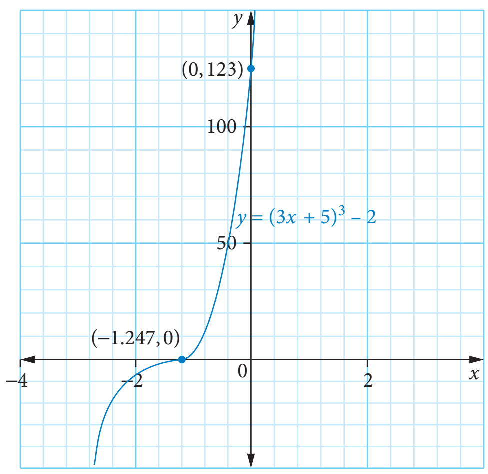

ANSWERS

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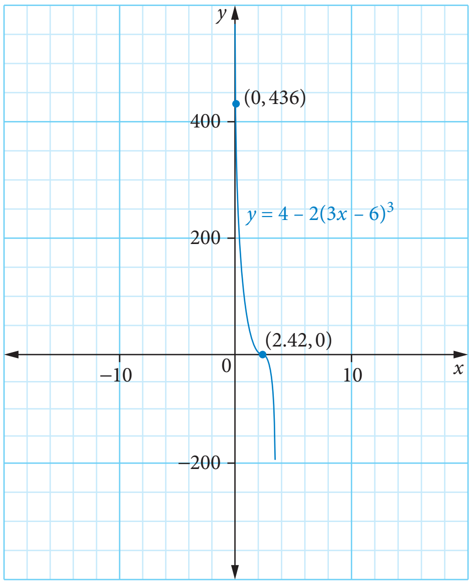

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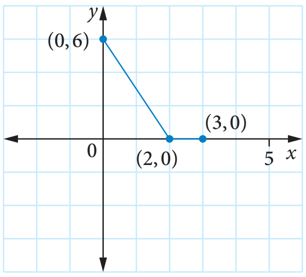

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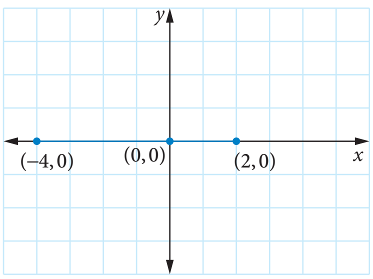

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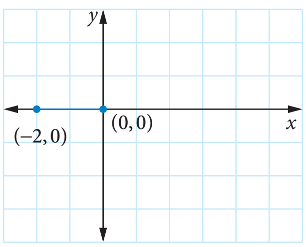

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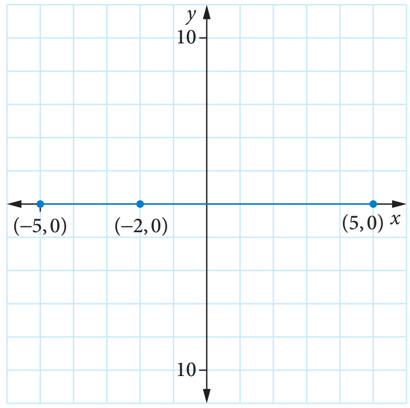

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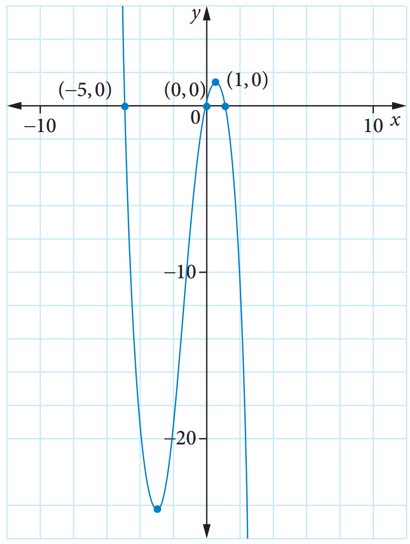

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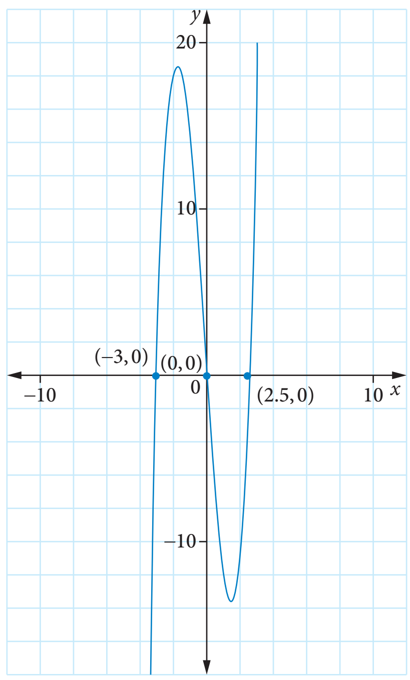

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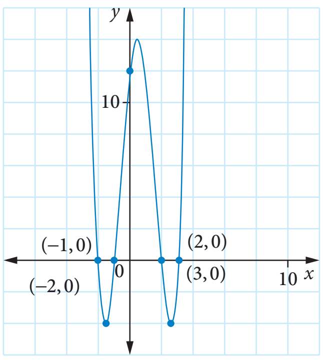

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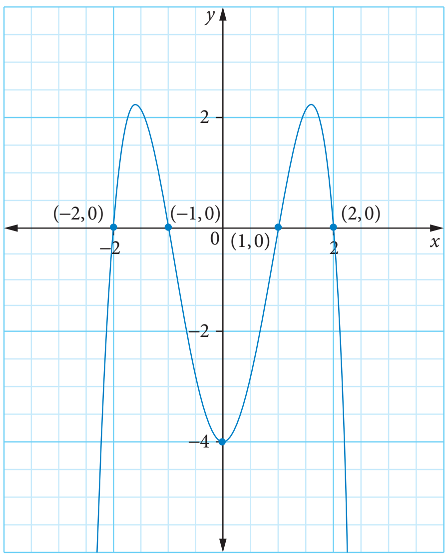

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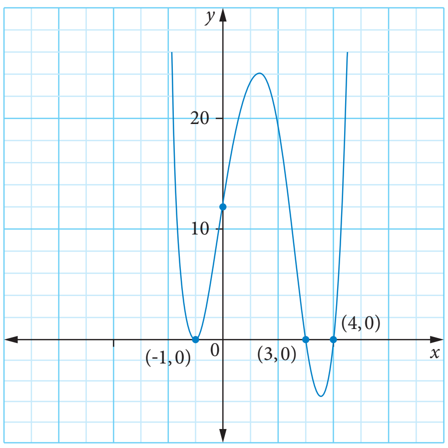

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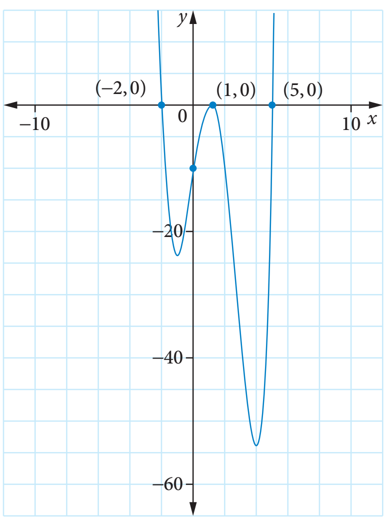

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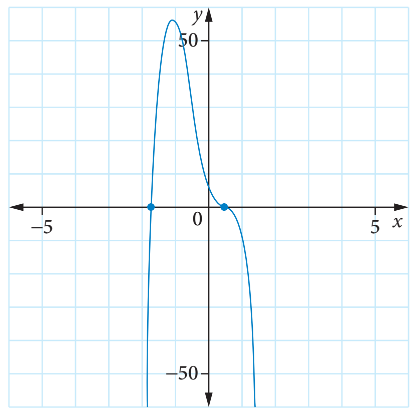

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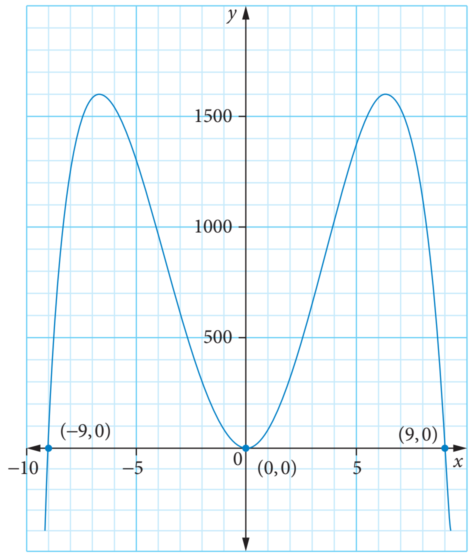


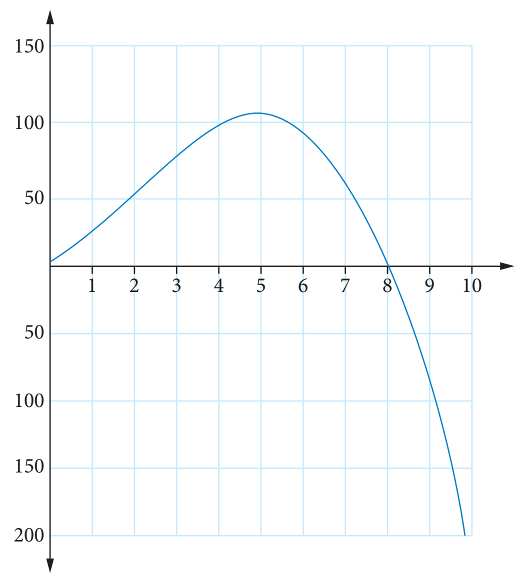
5 $y = (x + 2)(x - 1)(x - 4)$

6 $h(x) = 0.5x^2(x + 3)(x - 3)$

7 $x + 8$ and $x + 3$

8 a 0.5 cm^2

b



c after $t = 8.006$ **d** 85.32%

e 93.31%

9 A **10** B **11** C

12 B **13** C **14** C

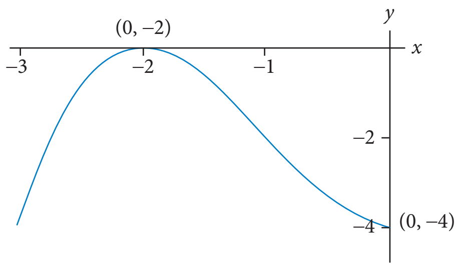
15 a $(-2, -32)$ **b** $b > 32$

16 a 8 square units **b** \$880 000 **c** $a = 1$

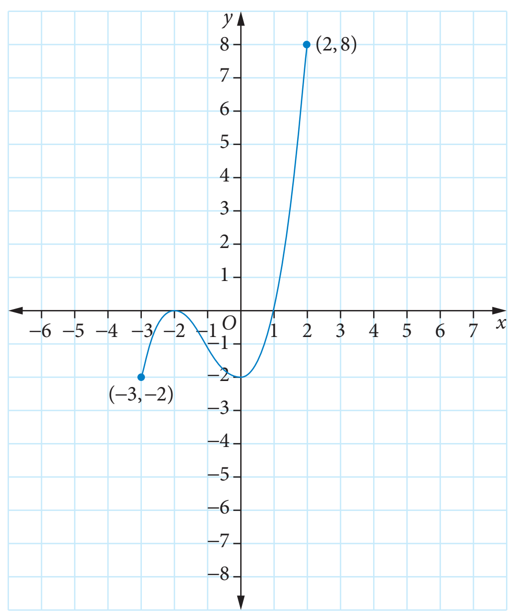
CUMULATIVE EXAMINATION 1

1 a 79% Proof: see worked solutions

b 68%



2



69%

3 a $y = \dfrac{4000\sqrt{3}}{3x^2}$ 56%

b Proof: see worked solutions 45%

CUMULATIVE EXAMINATION 2

Section A

1 C 92% **2** D 80% **3** A 74%

4 B 61% **5** A 56% **6** C 20%

Section B

1 a $c = 2, d = 64$ **b** $(-\infty, -2) \cup (2, \infty)$

c Proof: see worked solutions

d $-2 < h \le 1$ **e** $k > \dfrac{729}{64}$

2 a $a = 2, b = 4, c = 4$

b $m = -2, n = -1$

c i $y = 4x - 8$

ii $y = -11$

iii $P\left(-\dfrac{3}{4}, -11\right), Q(1, -11)$

CHAPTER 4

EXERCISE 4.1

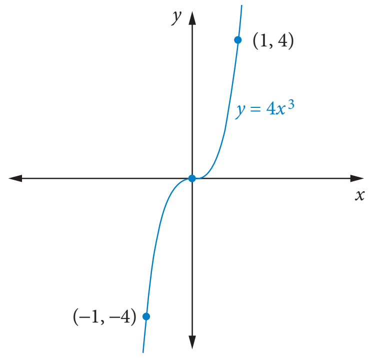
1 a Changing the parameter moves the function in the x direction, left for $b > 0$ and right for $b < 0$.

b Changing the parameter moves the function in the y direction, up for $c > 0$ and down for $c < 0$.

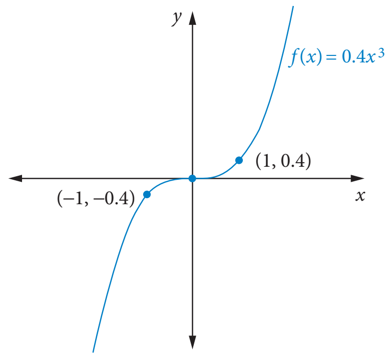
c Changing the parameter to negative reflects the function in the x-axis. The function is compressed towards the x-axis for $-1 \le a < 1$ and stretched from the x-axis for other values.

d Changing the parameter moves the function in the x direction, left for $b > 0$ and right for $b < 0$.

2 a



b



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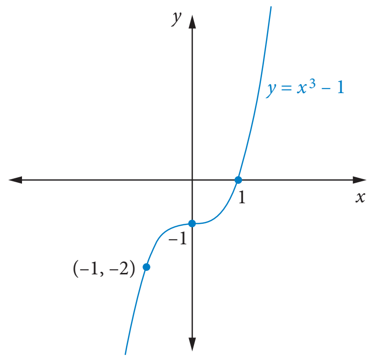

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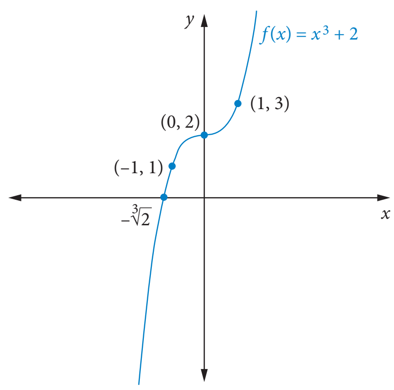

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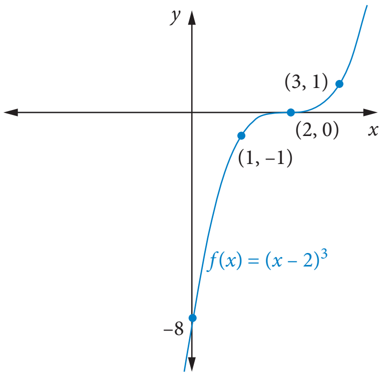

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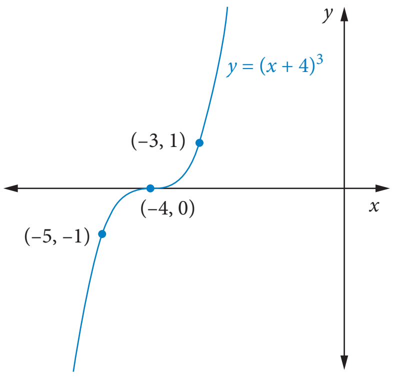
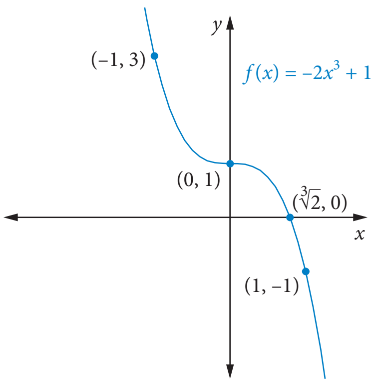


y-intercept at (0, 64)

g



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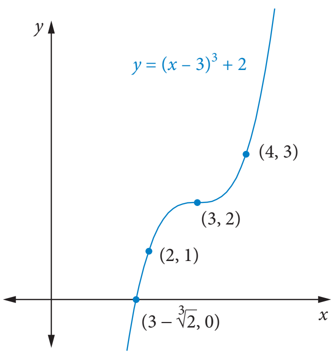
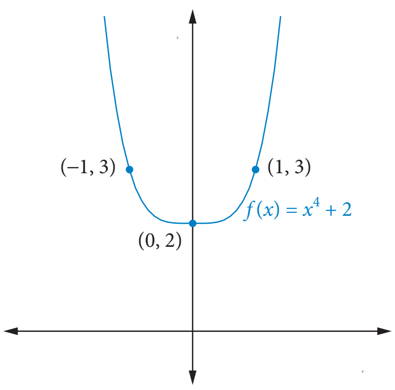


y-intercept is (0, –25)

3 a



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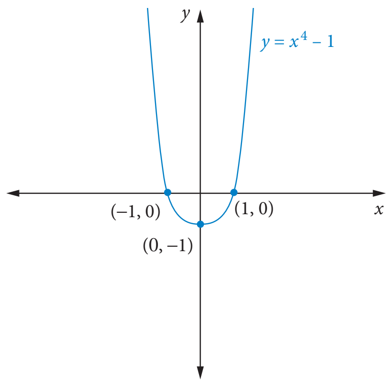

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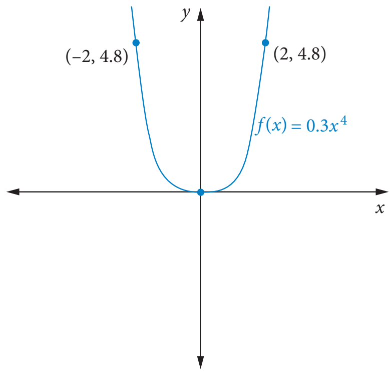

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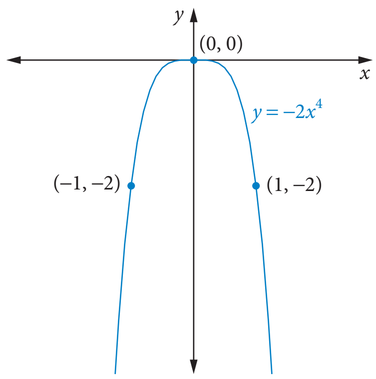

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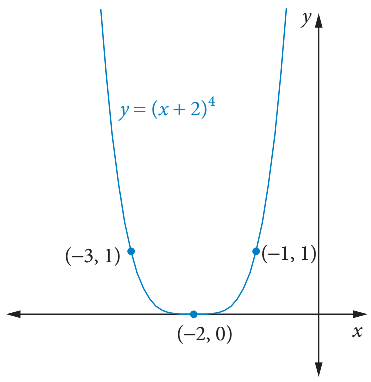
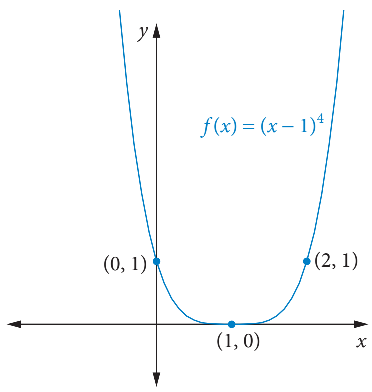


y-intercept (0, 16)

f



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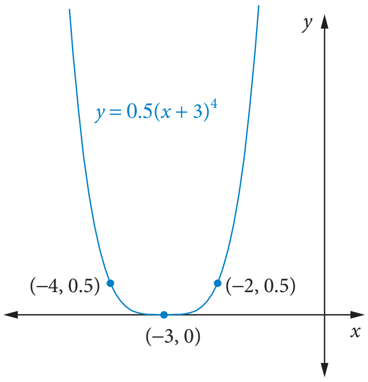
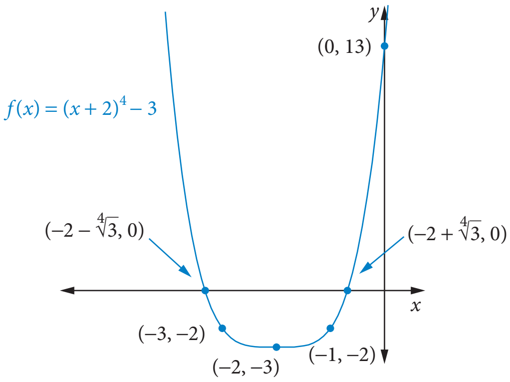
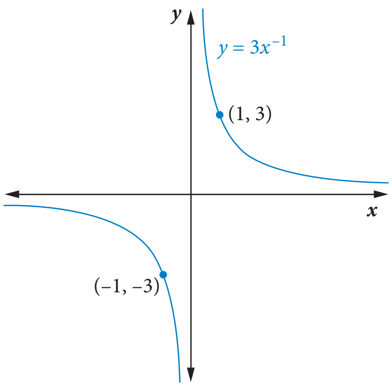


y-intercept (0, 40.5)

h



4 a



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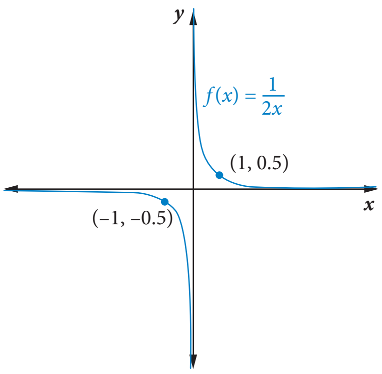

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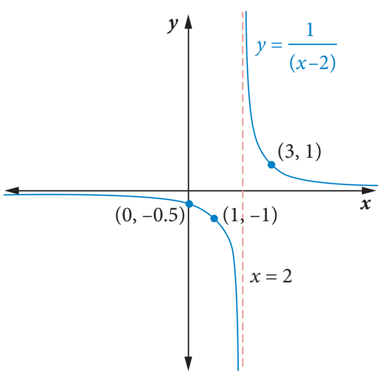

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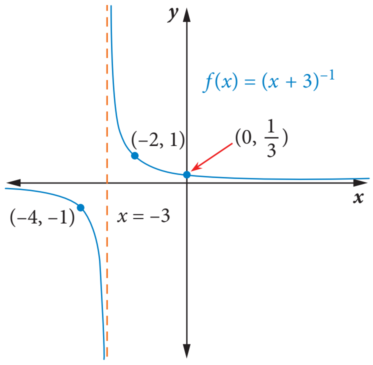

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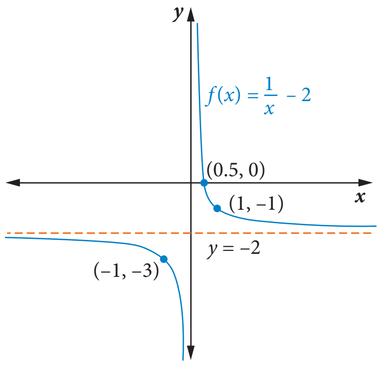

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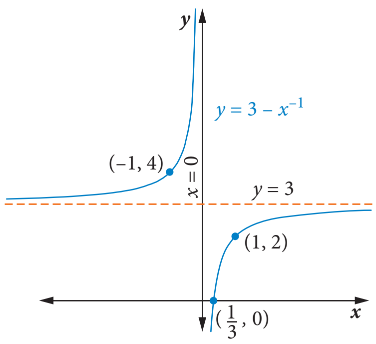

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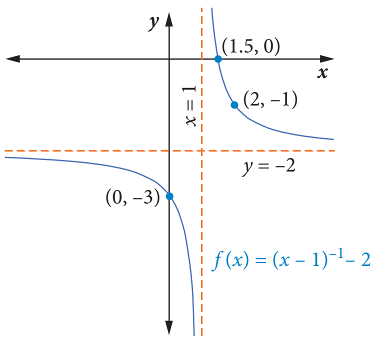

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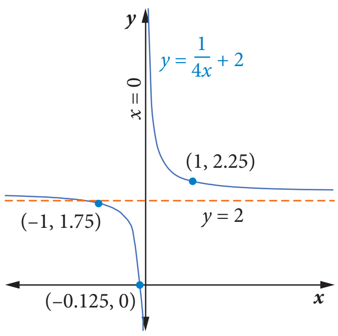

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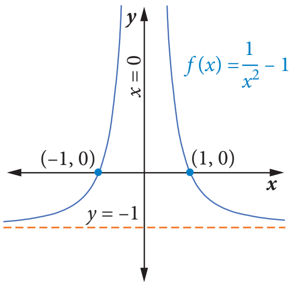

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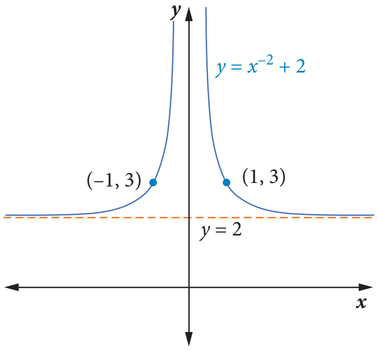

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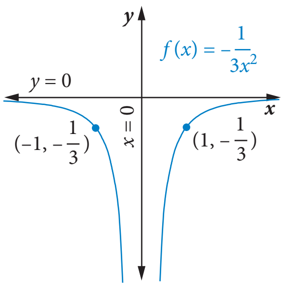

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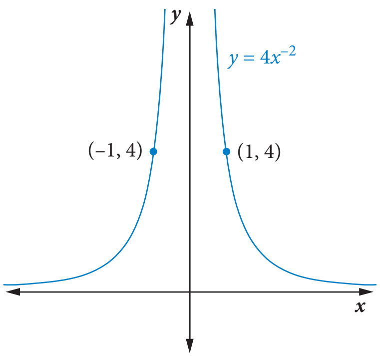

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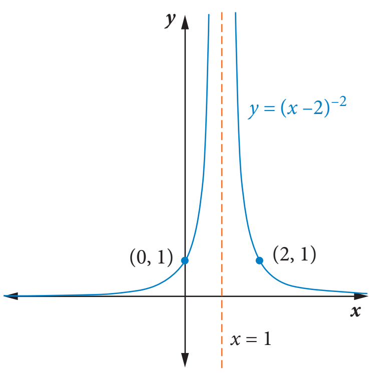

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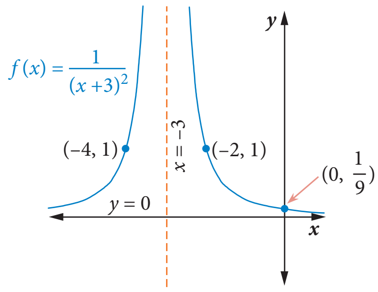

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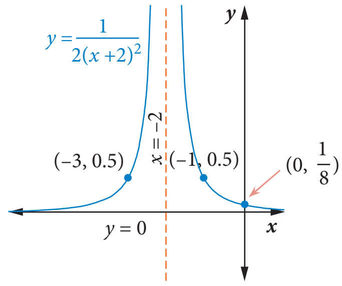

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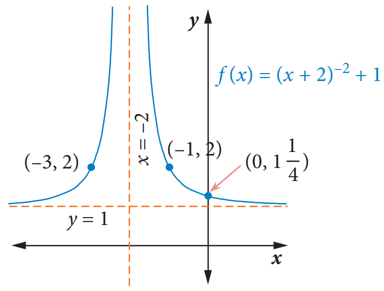

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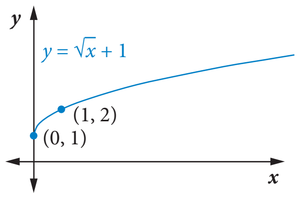

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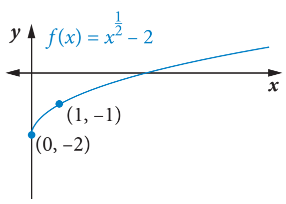

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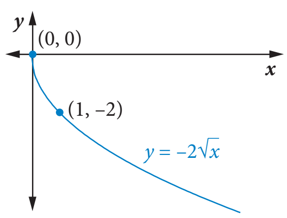

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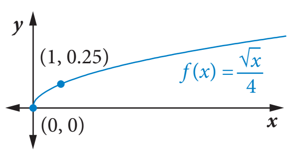

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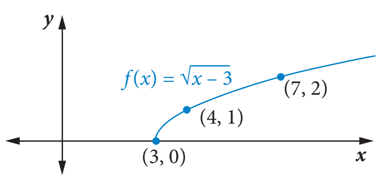

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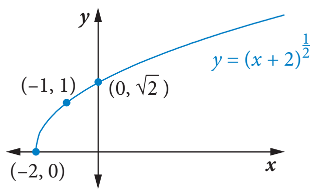

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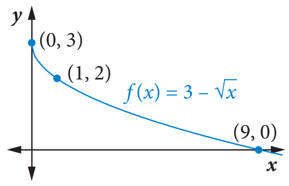

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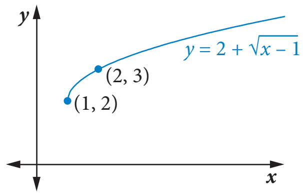

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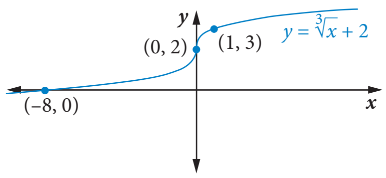

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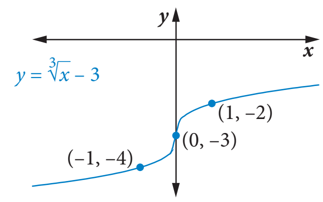
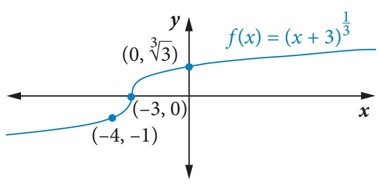


Zero at (27, 0)

c



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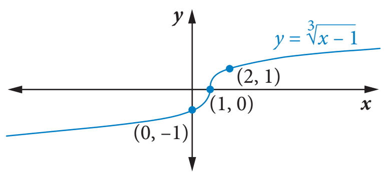

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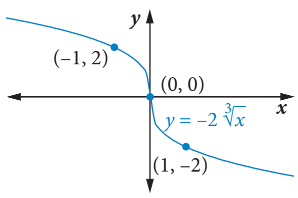

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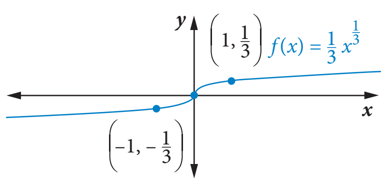

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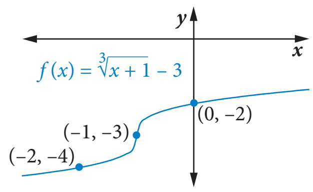
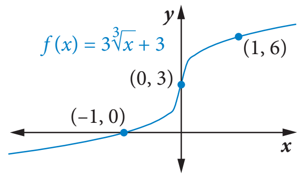
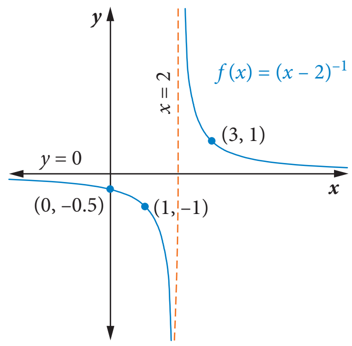


zero at (26, 0)

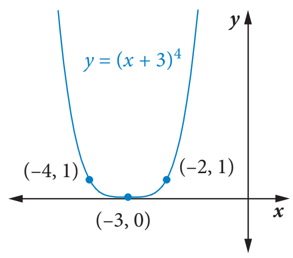
h



8 a



b



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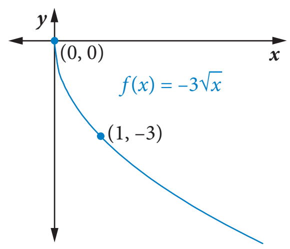

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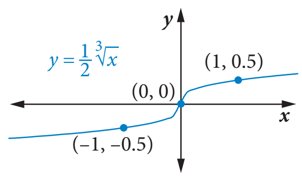

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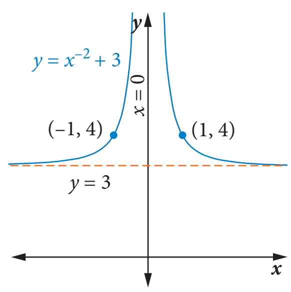

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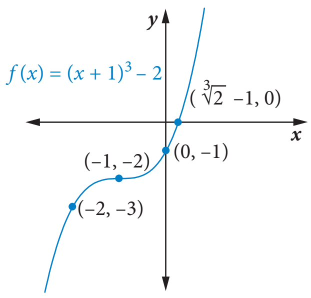

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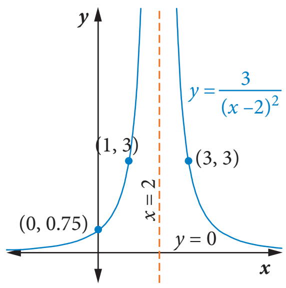

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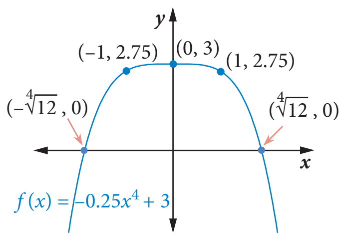


9 **a** domain: $R\backslash\{-2\}$, range: R^+

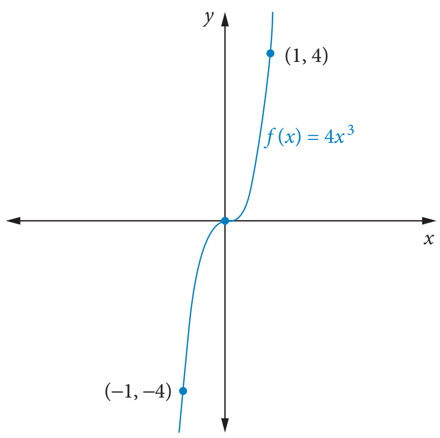
b domain: $[4, \infty)$, range: $[0, \infty)$

10 A **11** C **12** D

EXERCISE 4.2

1 C **2** A

3 **a**



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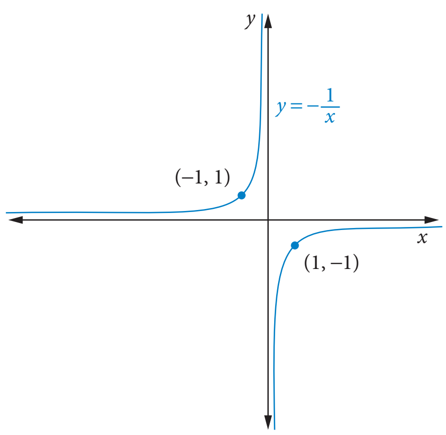

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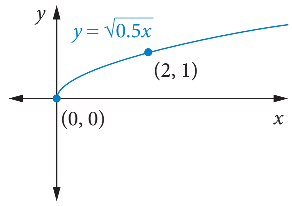

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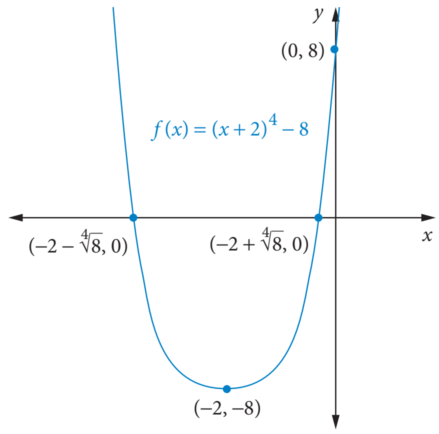

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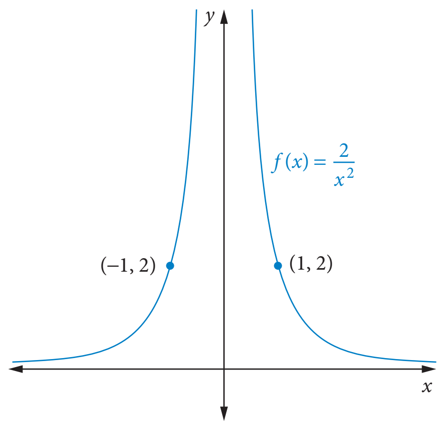

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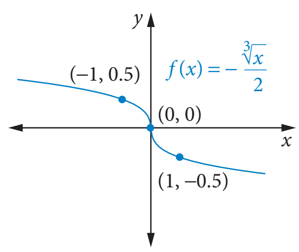

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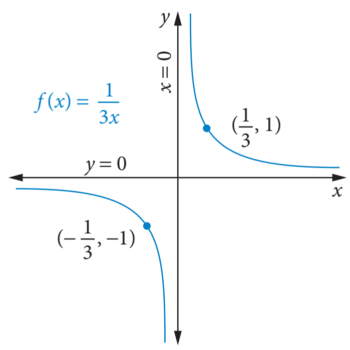

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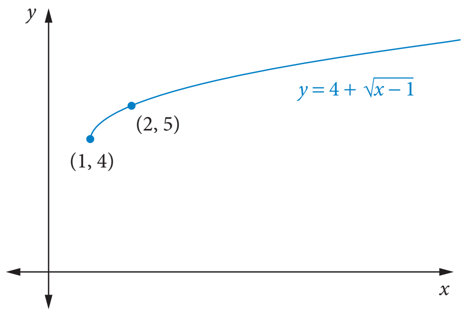

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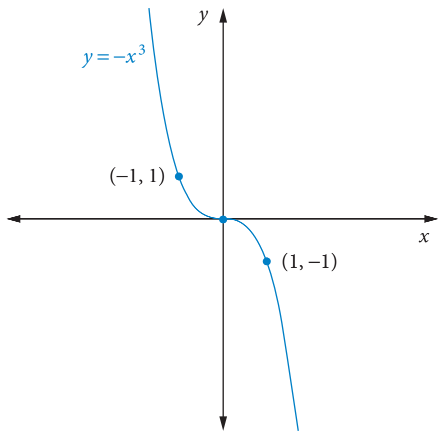

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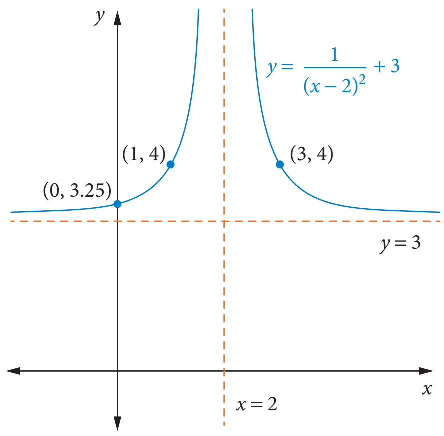

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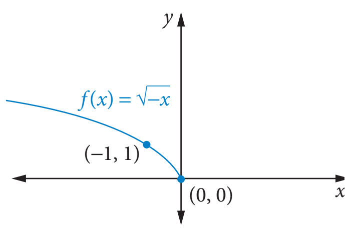

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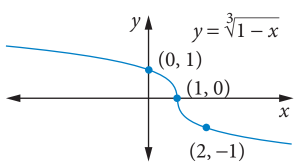


4 a $A(-8, -2), B(6, 22), C(-6, 19)$

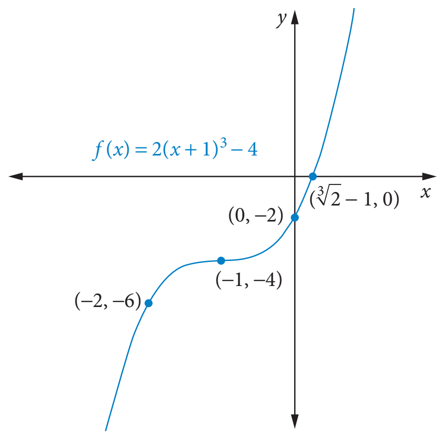
b $A(-10, 2), B(18, -2), C(-6, -1.5)$

c $A(21, 2), B(0, -6), C(18, -5)$

d $A(1, 12), B(8, -12), C(2, -9)$

e $A(1, \frac{11}{3}), B(8, -1), C(2, -\frac{4}{3})$

5 a



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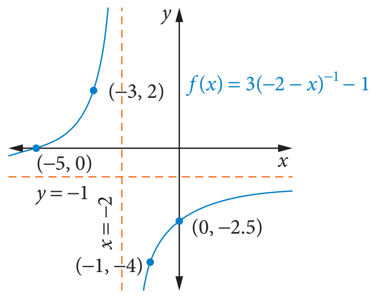

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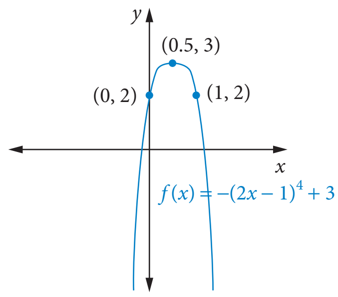

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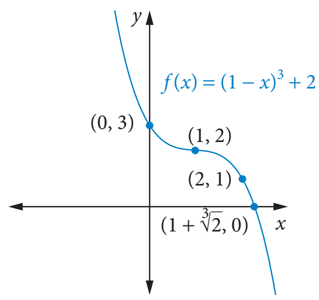

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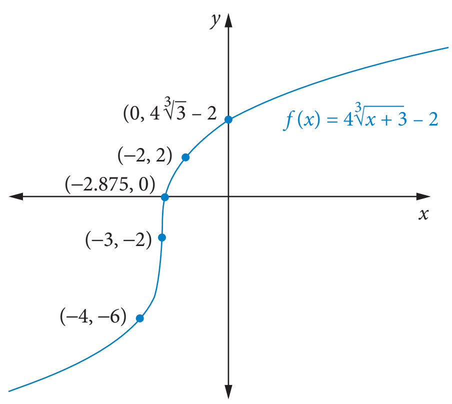

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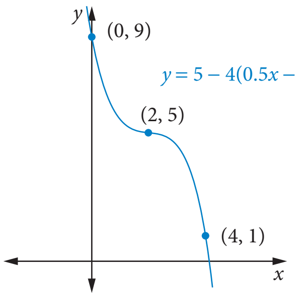

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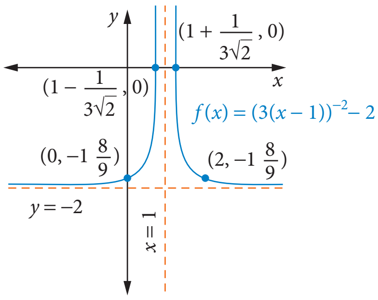

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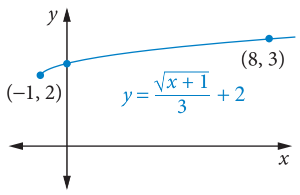
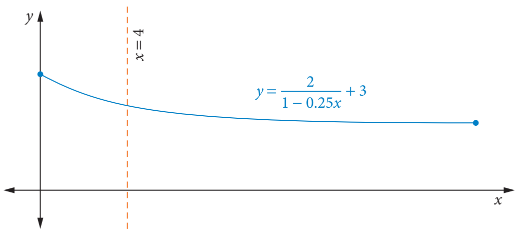


i



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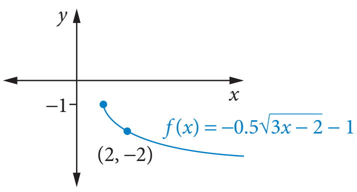

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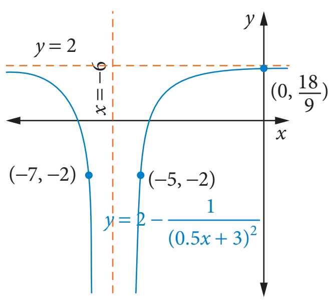

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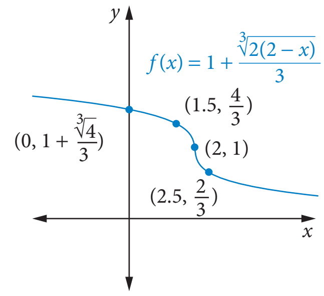


6 **a** $g(x) = -3(x-4)^2 - 1$

b $g(x) = 1 - 4(x+3)^4$

c $g(x) = 3 - \dfrac{1}{3(x-2)^2}$

d $g(x) = -2(4x+4)^2 - 1$

e $g(x) = 3\sqrt[3]{-4-x} - 5$

f $g(x) = 3(2x+1)^{-1} - 6$

g $g(x) = 3(4-x)^3 + 2.5$

h $g(x) = \sqrt{x} + 9$

7 **a** Dilation by a factor of 0.2 from the y-axis and 3 from the x-axis, translation 4 up and 2 to the right.

b Reflection in the x-axis, dilation by a factor of 3 from the y-axis, translation 2 up and 4 to the left.

c Dilation by a factor of 0.5 from the y-axis and 4 from the x-axis, reflection in the y-axis, translation 1 down and 3 to the right.

d Reflection in the x-axis, dilation by a factor of 3 from the x-axis, translation 7 up and 3 to the right.

e Dilation by a factor of 2 from the y-axis, reflection in the x-axis, translation 4 down and 2 to the right.

f Reflection in the x-axis, dilation by a factor of 6 from the x-axis, translation 3 up and 2 to the right.

g Dilation by a factor of 3 from the y-axis, reflection in the y-axis and translation 9 to the right and 5 up.

h Dilation by a factor $\frac{1}{3}$ of from the y-axis and $\frac{1}{4}$ from the x-axis and translation $\frac{13}{4}$ up and $\frac{4}{3}$ to the right.

8 **a** Reflection in the y-axis, dilation by a factor of 2 from the x-axis, translation 3 up and 2 to the right; $g(x) = 4\sqrt{2-x} + 3$.

b Reflection in an axis, dilation by a factor of 2 from the x-axis, translation 1 to the left and 1 down; $g(x) = -2(x+1)^3 - 1$.

c Reflection in an axis, translation 1 to the left and 1 down; $g(x) = -(x+1)^3 - 1$.

d Reflection in an axis, dilation by a factor of 2 from the x-axis, translation 2 to the right and 4 up; $g(x) = 4 - 2\sqrt[3]{x-2}$.

e Reflection in the x-axis, dilation by a factor of 0.5 from the x-axis, translation 3 to the left and 2 down; $g(x) = 2(x+3)^2$.

f Dilation by a factor of 0.25 from the x-axis, dilation of a factor of 0.5 from the y-axis; $g(x) = 0.5(2x+3)^4$.

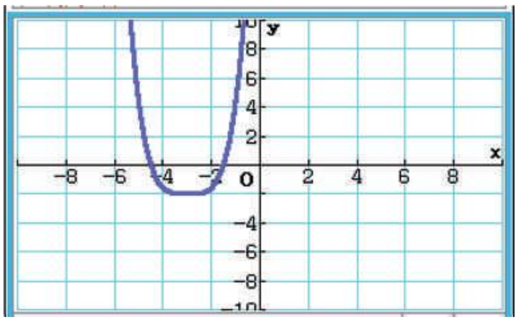
g Reflection in the x-axis, translation 2 to the right and 3 down; $g(x) = -\dfrac{2}{(x-3)^2} - 6$.

h Reflection in both axes, dilation by a factor of 0.25 from the x-axis with a translation 4 to the left (or dilation by a factor of 0.5 from the y-axis with a translation 2 to the left and a translation 5 up; $g(x) = -0.25\sqrt{x+6} + 9$.

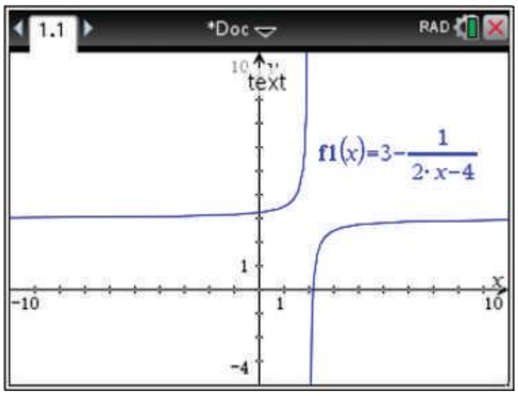
i Reflection in the y-axis, dilation from the y-axis by the factor 2 and translation 1 to the right, dilation from the x-axis by the factor 0.125 and translation 2.125 up; $g(x) = 0.5\sqrt[3]{2-x} + 2$ OR reflection in the x-axis, dilation from the x-axis by the factor 0.125 and translation 1.875 up, dilation by the factor 2 from the y-axis and translation 2.125 up; $g(x) = -0.5\sqrt[3]{x+2} + 2$.

j $g(x) =$ Reflection in the x-axis and dilation from the x-axis by the factor 0.5, translation 0.5 down, dilation by the factor 0.25 from the y-axis and translation 2 to the left; $\dfrac{1}{x+1} - 2$.

9 a



b



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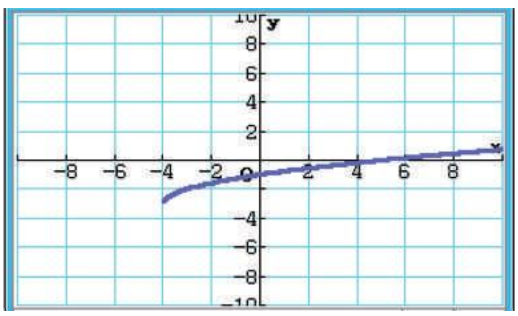

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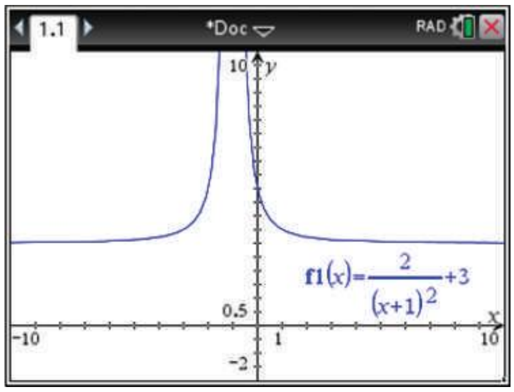

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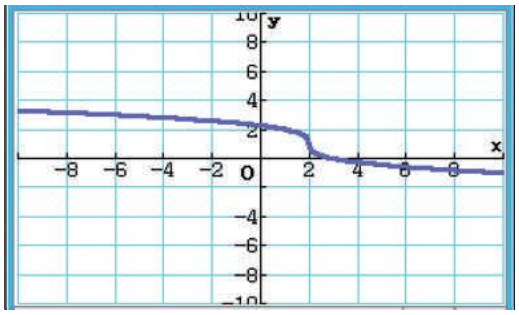

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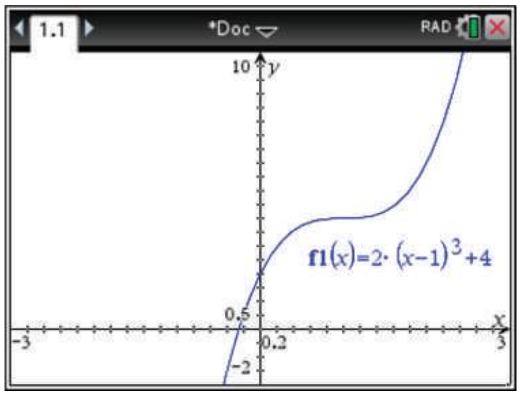

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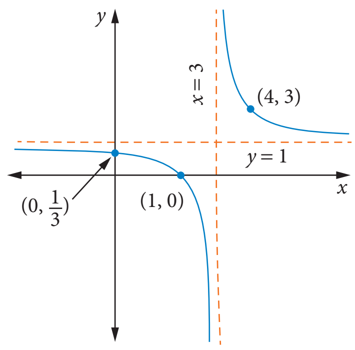

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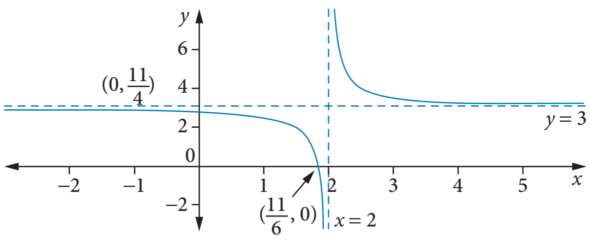


12 D **13** B **14** C **15** C

EXERCISE 4.3

1 D

2 C

3 **a** yes **b** no

c no **d** yes

e no **f** yes

g yes **h** yes

4 **a** Inverse is $g: R \to R, g(x) = \dfrac{x+8}{3}$, a function.

b Inverse is $g: R \to R, g(x) = \pm\sqrt[4]{\dfrac{x+6}{4}}$, not a function.

c Inverse is $g: R \to R, g(x) = \left(\dfrac{5-x}{2}\right)^3 - 4$, a function.

d Inverse is $g: R \to R, g(x) = \pm\sqrt{\dfrac{2}{x-3}} - 3$, not a function.

e Inverse is $g: [-6, \infty) \to R^+, g(x) = \sqrt{x+6} - 2$, a function.

f Inverse is $g: R\backslash\{4\} \to R^+, g(x) = 2 - \dfrac{1}{x-4}$, a function.

g Inverse is $g: [1, \infty) \to R, g(x) = \sqrt[3]{x+7} + 3$, a function.

h Inverse is $g: [4, \infty) \to R, g(x) = \dfrac{\sqrt{(y-4)^2 + 36}}{3}$, a function.

5 **a** $y = \sqrt[3]{x+8} - 2$, a function, domain: R, range: R.

b $y = 5 \pm \sqrt{x-3}$, not a function, domain: R, range: $[3, \infty)$.

c $y = 1 - \dfrac{2}{x-3}$, a function, domain: $R\backslash\{1\}$, range: $R\backslash\{3\}$.

d $y = \dfrac{4-x}{3}$, a function, domain: $(2, 5]$, range: $[-11, -2)$.

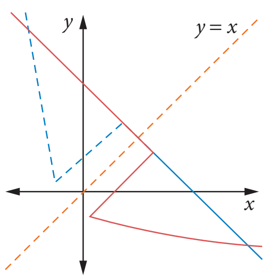
e $y = 4 + \dfrac{3}{x-1}$, a function, domain: $R^+\backslash\{4\}$, range: $R\backslash(0.25, 1]$.

f $y = \sqrt[3]{\dfrac{7-x}{5}}$, a function, domain: $(-\infty, 7]$, range: R^+.

g $y = \dfrac{(x-1)^2}{16} + 5$, a function, domain: $[5, \infty)$, range: $[1, \infty)$.

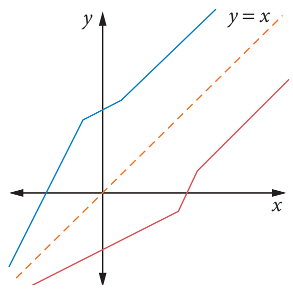
h $y = 2 \pm \dfrac{\sqrt{1-x}}{2}$, not a function, domain: $(-\infty, 1)$, range: $[-2, \infty)$.

6 **a**



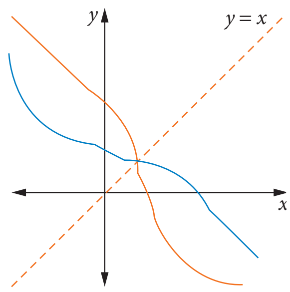
not a function

b



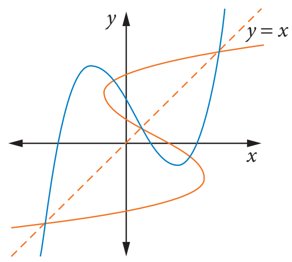
a function

c



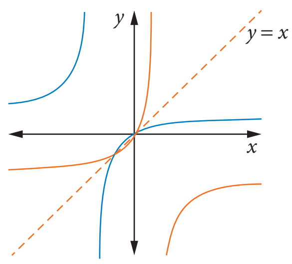
a function

d



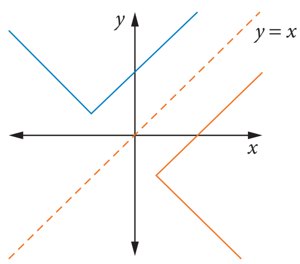
not a function

e



a function

f



not a function

7 $h^{-1}(x) = \sqrt[3]{\dfrac{x-3}{4}}, x \in R$

8 **a** $f^{-1}(x) = \dfrac{(x+1)^2 - 3}{2}$, domain: $f^{-1} = [-1, \infty)$

b $x = \sqrt{2}$ only

c **i** $c < 1$

ii $c = 1$ or $c > 2$

9 A

10 A

EXERCISE 4.4

1 D

2 C

3 **a** 3×3

b $a_{13} = -6$, $a_{23} = -7$, $a_{31} = -7$ and $a_{32} = -5$

c 2 or 2×2

d $b_{11} = 0$, $b_{12} = -1$ and $b_{21} = 5$

e 3×2

f $c_{31} = -4$, $c_{32} = 0$, c_{13} doesn't exist and $c_{21} = 6$

4 **a** $\begin{bmatrix} 2 & 2 \\ -5 & 5 \end{bmatrix}$

b $\begin{bmatrix} 3 & -1 \\ -6 & 5 \end{bmatrix}$

c $\begin{bmatrix} 0 & -10 \\ -10 & 20 \end{bmatrix}$

d $\begin{bmatrix} 1 & 23 \\ 3 & -10 \end{bmatrix}$

e not possible

f not possible

g $\begin{bmatrix} 0 & 34 \\ 12 & -26 \end{bmatrix}$

h $\begin{bmatrix} 2 & 8 \\ 27 & -13 \\ -15 & -3 \end{bmatrix}$

i $\begin{bmatrix} 1 & 9 \\ 0 & -3 \end{bmatrix}$

j $\begin{bmatrix} 1 & 9 \\ 0 & -3 \end{bmatrix}$

k $\begin{bmatrix} -3 & 1 \\ 6 & -5 \end{bmatrix}$

l $\begin{bmatrix} -8 & -20 \\ 8 & 4 \end{bmatrix}$

m $\begin{bmatrix} -1 & 2 \\ 9 & -4 \\ -3 & -3 \end{bmatrix}$

n $\begin{bmatrix} -3 & -5 \\ 0 & 7 \end{bmatrix}$

o $\begin{bmatrix} 9 & 11 \\ -15 & 8 \end{bmatrix}$

5 **a** $\begin{bmatrix} -2 & -3 \\ -1 & 1 \end{bmatrix}$

b not possible

c not possible

d $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$

e $\begin{bmatrix} 2 \\ 21 \end{bmatrix}$

f not possible

g $\begin{bmatrix} 10 & -7 \\ -15 & 3 \\ 0 & 4 \end{bmatrix}$

h $\begin{bmatrix} 46 & -16 \\ 11 & 11 \\ -73 & 1 \end{bmatrix}$

i $\begin{bmatrix} 10 & -7 \\ 15 & -11 \end{bmatrix}$

j not possible

k $\begin{bmatrix} 5 & 8 \\ 0 & 3 \\ -4 & -8 \end{bmatrix}$

l not possible

m $\begin{bmatrix} -1 \\ 24 \\ -12 \end{bmatrix}$

n not possible

o $\begin{bmatrix} -5 & 3 \\ -15 & 4 \end{bmatrix}$

6 **a** $\begin{bmatrix} -1 & -8 \\ -3 & 8 \end{bmatrix}$

b $\begin{bmatrix} -2 & -2 \\ 0 & 4 \end{bmatrix}$

c $\begin{bmatrix} -4 & -9 \\ 3 & 29 \end{bmatrix}$

d $\begin{bmatrix} 1 & -26 \\ 3 & 17 \end{bmatrix}$

e $\begin{bmatrix} 8 & 1 \\ -15 & 10 \end{bmatrix}$

f $\begin{bmatrix} 0 & 5 \\ -11 & -18 \end{bmatrix}$

g $\begin{bmatrix} -8 & -9 \\ 1 & -2 \end{bmatrix}$

h $\begin{bmatrix} -2 & -31 \\ 10 & 10 \end{bmatrix}$

i $\begin{bmatrix} 10 & 11 \\ -5 & 3 \end{bmatrix}$

j $\begin{bmatrix} -3 & -28 \\ 3 & -9 \end{bmatrix}$

7 **a** AC, BA, BC and CB

b $CB = \begin{bmatrix} -11 & 13 \\ -5 & -6 \end{bmatrix}$

8 D

9 A

10 A

11 E

EXERCISE 4.5

1 C

2 C

3 **a** $T: R^2 \to R^2, T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} x \\ y \end{bmatrix}$,

$y = (x-2)^3 + 3$

b $T: R^2 \to R^2, T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -1 \\ -2 \end{bmatrix} + \begin{bmatrix} x \\ y \end{bmatrix}$,

$y = \dfrac{1}{x+1} - 2$

c $T: R^2 \to R^2, T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 3 \\ -4 \end{bmatrix} + \begin{bmatrix} x \\ y \end{bmatrix}$,

$y = \sqrt{x-3} - 4$

d $T: R^2 \to R^2, T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -3 \\ 4 \end{bmatrix} + \begin{bmatrix} x \\ y \end{bmatrix}$,

$y = (x+5)^4 + 3$

4 **a** $T: R^2 \to R^2, T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0.5 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}, y = 48x^4$

b $T: R^2 \to R^2, T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}, y = 3\sqrt[3]{x}$

c $T: R^2 \to R^2, T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0.25 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$,

$y = 16x - 3$

d $T: R^2 \to R^2, T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 3 & 0 \\ 0 & 0.5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}, y = 4.5x^2$

5 **a** $T: R^2 \to R^2, T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$,

$y = \dfrac{1}{(x-3)^2} - 2$

b $T: R^2 \to R^2, T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}, y = -x^3$

c $T: R^2 \to R^2, T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$,

$y = 1 - (2-x)^2$

d $T: R^2 \to R^2, T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$,

$y = \sqrt[3]{1-x} - 2$

6 **a** $T: R^2 \to R^2, T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}\left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -2 \\ -1 \end{bmatrix}\right),$

$y = 2\sqrt{2-x} - 2$

b $T: R^2 \to R^2, T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0.5 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 3 \\ 2 \end{bmatrix},$

$y = 2 - (2x-6)^3 = 2 - 8(x-3)^3$

c $T: R^2 \to R^2, T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -2 & 0 \\ 0 & -0.5 \end{bmatrix}\left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 2 \\ -4 \end{bmatrix}\right), y = 4 - \dfrac{2}{x-4}$

d $T: R^2 \to R^2, T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -4 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\left(\begin{bmatrix} 2 \\ 0 \end{bmatrix} + \begin{bmatrix} x \\ y \end{bmatrix}\right),$

$y = 8\left(3 - \dfrac{x}{3}\right)^2 + 2$

7 **a** Reflection in the y-axis and translation 2 left and 3 up.

b Translation 1 to the right and 2 up and dilation by a factor of 3 from the x-axis.

c Reflection in the x-axis, dilation by a factor of 2 from the x-axis and translation 3 to the left and 2 down.

d Translation 3 units to the right, dilation by a factor of 2 from the y-axis and a factor of 0.5 from the x-axis and a translation of 1 down.

8 $T: R^2 \to R^2, T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2 \\ -3 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}, y = 12(2-x)^3 - 3$

9 C

10 E

11 E

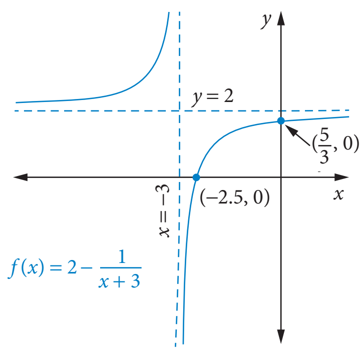
12 $a = 2, c = -4, d = -3$

CUMULATIVE EXAMINATION 1

1 –5

2 $x = -1$ 61%

3



4 $g(x) = 4(2-x)^3 - 4$

CUMULATIVE EXAMINATION 2

Section A

1 E

2 D

3 B

4 C 65%

5 A

Section B

1 **a** $b = a - 14, c = 12 - 6a$

b 12

c $2x^3 + 3x^2 - 11x - 6$

2 **a** $f(x) = 6\sqrt[3]{x+4}$

b $T: R^2 \to R^2, T: (x, y) \to \left(x - 4, 1 - \dfrac{1}{2}y\right)$ or

$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & -0.5 \end{bmatrix}\left(\begin{bmatrix} -4 \\ -2 \end{bmatrix} + \begin{bmatrix} x \\ y \end{bmatrix}\right)$

c $T: R^2 \to R^2, T: (x, y) \to \left(-x + 7, -\dfrac{1}{3}(y - 10)\right)$ or

$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -1 & 0 \\ 0 & \dfrac{-1}{3} \end{bmatrix}\left(\begin{bmatrix} 7 \\ -10 \end{bmatrix} + \begin{bmatrix} x \\ y \end{bmatrix}\right)$

CHAPTER 5

EXERCISE 5.1

1 **a** $\dfrac{11}{15}$ **b** $\dfrac{1}{2}$

2 **a** **i** $\dfrac{9}{20}$ **ii** $\dfrac{11}{20}$ **b** **i** $\dfrac{1}{2}$ **ii** $\dfrac{1}{2}$

3 **a** $\dfrac{11}{40}$ **b** $\dfrac{1}{3}$

4 **a** 0.18 **b** 0.22 **c** 0.34 **d** 0.8

5 E

6 C

7 **a**

Contestant	1	2	3	4	5	6	7	8
Experimental probability	$\frac{2}{35}$	$\frac{16}{175}$	$\frac{23}{175}$	$\frac{9}{35}$	$\frac{1}{35}$	$\frac{36}{175}$	$\frac{1}{7}$	$\frac{3}{35}$

b 1

8 **a** 0.251 **b** 0.084 **c** 0.147

d 0.139 **e** 0.050

9 **a** 0.0381 **b** 0.0105 **c** 0.0851

d 0.0738 **e** 0.7021

10 **a** $\dfrac{1}{5}$ **b** $\dfrac{1}{2}$ **c** $\dfrac{1}{15}$

d 42 **e** 90 **f** 126

11 1 red, 1 blue, 1 yellow, 4 green, 5 white

EXERCISE 5.2

1 E

2 A

3 **a** $\dfrac{1}{4}$ **b** $\dfrac{3}{10}$

4 **a** $\dfrac{5}{36}$ **b** $\dfrac{13}{18}$

5 **a** $\frac{1}{5}$ **b** $\frac{8}{25}$ **c** $\frac{12}{25}$

6 **a** $\frac{11}{18}$ **b** $\frac{5}{18}$ **c** $\frac{1}{9}$

7 **a** $S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$

b **i** $\frac{1}{8}$ **ii** $\frac{1}{2}$

8 **a** $\frac{1}{6}$ **b** $\frac{5}{12}$

9 B **10** E

11 **a** $\frac{1}{12}$ **b** $\frac{1}{18}$ **c** $\frac{5}{12}$ **d** $\frac{7}{18}$

12 **a** $\frac{1}{8}$ **b** $\frac{1}{2}$ **c** $\frac{3}{8}$

EXERCISE 5.3

1 B **2** C

3 {3, 2, 1, 3, 4, 1, 4, 1, 4, 2, 1, 6, 6, 2, 5, 2, 1, 5, 2, 4}
Results will vary.

4 $\frac{11}{20}$

5 0.35 (Answers may vary.)

6 0.3 **7** 0.8 **8** B **9** B

10 0.14

11 0.5 (Answers may vary.)

12 11.4 (Answers may vary.)

13 **a** 9.5 (Answers may vary.)

b 10 residents produce 5.5 infections, 20 people produce 7.2 infections. (Answers may vary.)

EXERCISE 5.4

1 C **2** A **3** 22 **4** 120

5 35 **6** $\frac{1}{48}$ **7** 9 **8** 30

9 90 **10** B **11** D **12** A

13 E **14** D **15** B

EXERCISE 5.5

1 E **2** A

3 ${}^6P_3 = 120$ **4** $5! = 120$

5 **a** 1320 **b** 15 **c** 6720

6 **a** 362 880 **b** 5040 **c** 3 628 800

7 **a** 120 **b** 20 **c** 132 **d** 16 **e** 1

8 144 **9** 240 **10** $\frac{3}{20}$

11 **a** 20 **b** $\frac{1}{5}$

12 **a** $6! = 720$ **b** 240 **c** 480

13 0.2

14 E **15** B **16** C **17** D **18** E

19 **a** 504 **b** 140 **c** 84

20 **a** 6 **b** 120 **c** 4320 **d** 30 240

EXERCISE 5.6

1 D **2** B

3 **a** 10 **b** 35 **c** 4 **d** 21
e 70

4 **a** 11 **b** 66 **c** 100 **d** 780
e 105

5 **a** 5985 **b** 24 **c** $\frac{10}{21}$

6 70 **7** 350 **8** $\frac{3}{7}$

9 ${}^{15}C_{12} = {}^{15}C_3 = 455$

10 28 matches

11 21 **12** B **13** E **14** E

15 E **16** A **17** 0.672 **18** 10

EXERCISE 5.7

1 D **2** D

3 $\frac{1}{35}$ **4** $\frac{781}{816}$

5 **a** $\frac{1}{35}$ **b** $\frac{12}{35}$

6 **a** $\frac{3}{7}$ **b** $\frac{3}{7}$ **c** $\frac{1}{2}$

7 **a** $\frac{9}{16}$ **b** $\frac{3}{32}$

8 $\frac{1}{210}$ **9** 0.770 **10** B

11 D **12** C **13** $\frac{103}{133}$

14 $\frac{7}{99}$ **15** $\frac{1}{66}$

CUMULATIVE EXAMINATION 1

1 **a** 120 **b** $\frac{1}{10}$

2 $\left(3+2\sqrt{2}, 4+2\sqrt{2}\right)$ and $\left(3-2\sqrt{2}, 4-2\sqrt{2}\right)$

3 $-\frac{3}{8}$

4 $\frac{20}{63}$

CUMULATIVE EXAMINATION 2

Section A

1 A **2** D **3** D **4** C **5** A

Section B

1 **a** 75 287 520 **b** 57 940 519
c 0.8119 **d** 0.1765
e 0.1498 **f** 0.9617

2 **a** $c = 2, d = 64$ **b** $(-\infty, -2) \cup (2, \infty)$

c $\text{LHS} = f(1+m) = m^3(m+3)^3$
$\text{RHS} = f(-2-m) = (-m-3)^3(-m)^3 = m^3(m+3)^3$

d $-2 < h \le 1$ **e** $k > 11.391$

CHAPTER 6

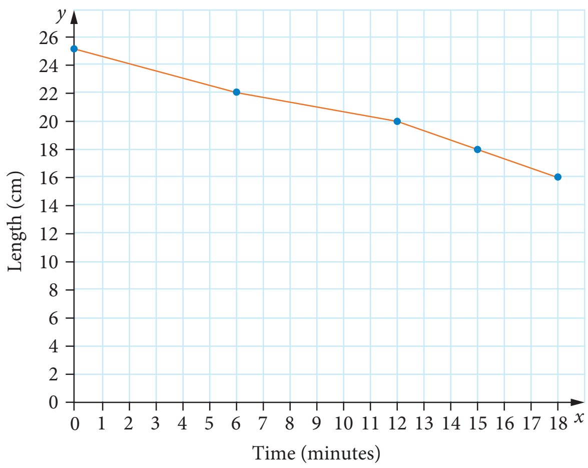
EXERCISE 6.1

1 a $\frac{10-4}{1-4} = -2$ km/s

b $\frac{1-0}{8-4} = 0.25$ kg/cm

c Not possible because the points when connected do not give a straight line.

2 a



b If the points are connected, they do not form a straight line.

c As time increases, the length of the candle decreases uniformly.

d $\frac{16-25}{18-0} = -0.5$ cm/min

e $\frac{16-18}{18-15} \approx -0.67$ cm/min

The rate using the last two points is slightly higher than the rate using the first and last points because the line is steeper.

3 −5

4 0.4 seconds/metre

5 A **6** D **7** A **8** C

9 E **10** E **11** C **12** $-ab$

13 a 3 m/s **b** 9 m/s **c** $t_1 + t_2$

14 a 250 units/day **b** 506 units/day

c Ratio is 5.065 or 1.5^4. This is a daily factor increase by 1.5 from Monday to Friday.

15 $V(t_1+1) - V(t_1) = (t_1+1)^3 - t_1^3 = 3t_1^2 + 3t_1 + 1$

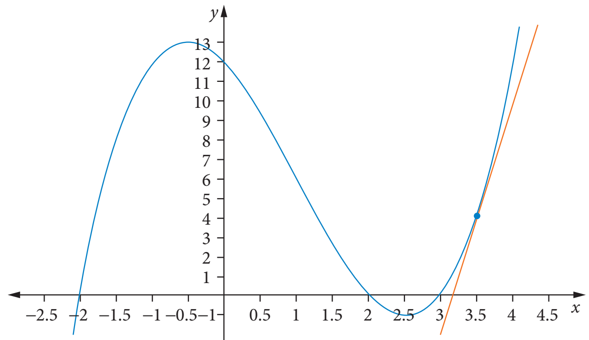
EXERCISE 6.2

1 A **2** D

3 a −2.1 lumens/m **b** −0.1 lumens/m

4 $A = 0, B = 2, C = -2$

5



Gradient at $x = 3.5$ is 11.75.

6 a $(6+h)a$ **b** 3

7 a $\frac{f(a+h)-f(a)}{h} = \frac{\frac{1}{a+h+1} - \frac{1}{a+1}}{h}$

$= -\frac{1}{(a+1)(a+h+1)}$

b As h approaches 0, $-\frac{1}{(a+1)(a+h+1)}$

approaches $-\frac{1}{(a+1)(a+1)} = -\frac{1}{(a+1)^2}$.

8 B **9** B **10** A **11** C **12** D

13 a $x = 5, y = 5^3 = 125$

$x = 5.1, y = 5.1^3 = 132.651$

rate $= \frac{132.651 - 125}{5.1 - 5} = 76.51$

$x = 5.01, y = 125.751\,501$, rate $= 75.1501$

$x = 5.001$, rate $= 75.015$

$x = 5.0001$, rate $= 75.0015$

b 75

14 $m = -\frac{1}{3}$

15 $-\frac{7}{a}$

16 a -0.6 m/s^2

b -0.86 m/s^2

c i There is zero acceleration when the rate of change of speed is zero. This occurs when the gradient is zero ($t = 15$ $t = 28$).

ii The car had constant speed.

17 $-\frac{1}{2}$

EXERCISE 6.3

1 C **2** A **3** 55.64 **4** 31.7

5 $h = 0.1, \frac{\Delta y}{\Delta x} = 25.22, h = 0.01, \frac{\Delta y}{\Delta x} = 24.001202, h = 0.001,$

$\frac{\Delta y}{\Delta x} = 24.012002, h = 0.0001, \frac{\Delta y}{\Delta x} = 24.00120002$

$f'(-2) = 24$

ANSWERS

6 7

7 $y = 4 - 6x$

8 (1.42, 0.39), (2.58, −0.39)

9 $x = 0 \quad R(x) = 0.8$

$x = 1 \quad R(x) = 4.8$

$x = 2 \quad R(x) = 8.8$

$x = 3 \quad R(x) = 12.8$

10 $\lim\limits_{h\to 0} \dfrac{f(a+h) - f(a)}{h} = \lim\limits_{h\to 0} \dfrac{(a+h)^3 - a^3}{h}$

$= \lim\limits_{h\to 0}(3a^2 + 3ah + h^2) = 3a^2$

11 E **12** B **13** A **14** A

15 D **16** B

17 a 4 m/s **b** 3.99 m/s

18 a

x	$f(x)$	$x - h$	$f(x - h)$	$\frac{\Delta y}{\Delta x}$ (rounded to nearest whole number)
0	1	−0.03	0.9109	3
2	11	10.97	10.7909	7
4	29	28.97	28.6709	11

b $R(x) = 2x + 3$

19 a $\dfrac{\Delta y}{\Delta x} = \dfrac{[a(x+hx)^2 + b(x+hx) + c] - [ax^2 + bx + c]}{hx}$

$= \dfrac{2axhx + (hx)^2 + bhx}{hx}$

$= 2ax + b + hx$

b $R(x) = 2ax + b$

20 a $\dfrac{\Delta y}{\Delta x} = 3x^2 - 3hx + h^2$

b $\dfrac{\Delta y}{\Delta x} = 3x^2 + 3hx + h^2$

c $\dfrac{\Delta y}{\Delta x} = 3x^2$

21 $\dfrac{f(x) - f(x-h)}{h} = \dfrac{\left(x^3 - x\right) - \left((x-h)^3 - (x-h)\right)}{h}$

$= 3x^2 - 3xh + h^2 - 1$

$\lim\limits_{h\to 0} 3x^2 - 3xh + h^2 - 1 = 3x^2 - 1$

$\dfrac{f(x+h) - f(x)}{h} = \dfrac{\left((x+h)^3 - (x+h)\right) - \left(x^3 - x\right)}{h}$

$= 3x^2 + 3xh + h^2 - 1$

$\lim\limits_{h\to 0} 3x^2 + 3xh + h^2 - 1 = 3x^2 - 1$

$\dfrac{f(x+h) - f(x-h)}{2h}$

$= \dfrac{\left((x+h)^3 - (x+h)\right) - \left((x-h)^3 - (x-h)\right)}{2h}$

$= 3x^2 + h^2 - 1$

$\lim\limits_{h\to 0} 3x^2 + h^2 + 1 = 3x^2 - 1$

EXERCISE 6.4

1 A **2** D

3 a 3 am Tuesday

b $\dfrac{50}{6} = 8.33$ km/h

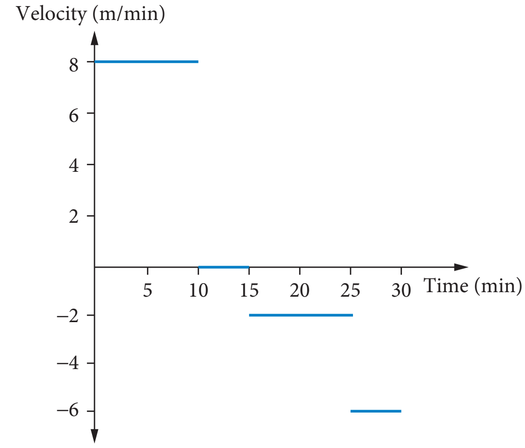
c 15 km/h

d 7 pm Monday to 1 am Tuesday

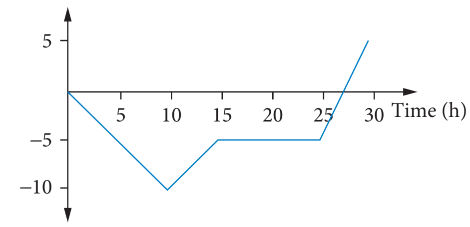
e 50 − 20 = −30 m

f $\dfrac{100}{12} = 8.33$ km/h

4 Velocity (m/min)



5 Displacement (km)



6 C **7** D **8** C **9** E **10** B

11 5.26, 4.90, 4.76, 4.85 and 4.85 m/s

12 a Between 2 seconds and 4 seconds

b 10.6 m/s **c** −4 m/s^2

d 12 m/s^2

13 a Section A **b** 10 km/h

c 15 minutes **d** about 6.7 km/h

e 9.1 km/h

14 a Using gradient function on CAS, the point of intersection $x = 1.64$ years.

Rate of A is −12.37 i.e. −123 700 units/year.
Rate of B is 6.72, i.e. 67 200 units/year.

b Minimum of A: 6.7 years, or 6 years 8 months.
This is August 2016.

c 200 corresponds to 11.51 on the x-axis. The gradient at $x = 11.51$ is 83.6, which is 836 000 units/year.

d 30 000 units/year

CUMULATIVE EXAMINATION 1

1 $\frac{3\sqrt{3}}{4}$°C/h 1.2°C/h

2 1

3 $= \frac{(r^{b-1} - r^{a-1})}{b-a} = \frac{r^{-1}(r^b - r^a)}{b-a}$

$= \frac{r^b - r^a}{r(b-a)}$

4 $a = -\frac{1}{2}$

5 $\frac{ab + ac + bc}{2(a+b+c) - 1}$

6 All real numbers except ±5

CUMULATIVE EXAMINATION 2

Section A

1 A **2** A **3** E **4** B **5** E

Section B

1 a $\frac{k^2 + k - 6}{k - 2} = 7$ **b** $k = 4$

2 a 0.61 mg/L 75%

b −0.23 mg/L/h 63%

3 a (0, 0), (0.67, −1.48) **b** −2.21

4 $a = \sqrt{2}, b = \sqrt{3}$

5 $P(-k) = 6$, so $4k^2 + 2k + 4 = 6$ or $2k^2 + k - 1 = 0$

$(2k - 1)(k + 1) = 0 \Rightarrow k = 0.5, k = -1$. The sum is −0.5.

6 $y = -2.1x + 25.1$

CHAPTER 7

EXERCISE 7.1

1 D

2 a 8 **b** 1 **c** 2 **d** 1

e 6 **f** 9 **g** 1 **h** 9

i 343 **j** 216 **k** 16 **l** 64

m $\frac{1}{4}$ **n** $\frac{16}{9}$ **o** 65 536 **p** 1

q $\frac{64}{27}$ **r** $\frac{1}{36}$ **s** 125 **t** $\frac{2}{5}$

u $\frac{27}{8}$ **v** 4 **w** 1000 **x** $\frac{1}{10\,000}$

3 a 192 **b** 196 608 **c** 24

d 768 **e** $\frac{3}{16}$

4 a 51.7706 **b** 9.7865 **c** 7663.9046

d 92.7454 **e** 0.8043

5 a 7.9222 **b** 4.1354 **c** 8.5928

d 2.5397 **e** 21.0055

6 a 2^{5x+1} **b** 5^{9n-4} **c** 7^{7m-10}

d 3^{5y-7} **e** 11^{5p+12} **f** 13^{2q+4}

g 2^{14v+1} **h** 3^{15m+16} **i** $3^{16-x} \times 5^{6-4x}$

j 2^{36n-43} **k** $2^{12-5r} \times 3^{r+5}$

l $3^{4z-8} \times 5^{2z+11} \times 7^{6z+3}$

7 a 2^4 **b** 5^{k+1} **c** 3×2^3 **d** $2^m 3^{2n+3m}$

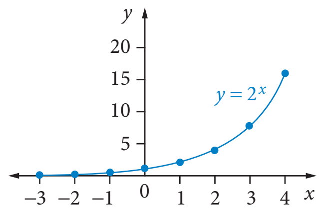
e $3^{2i+2} \times 7^3$ **f** 2^3

EXERCISE 7.2

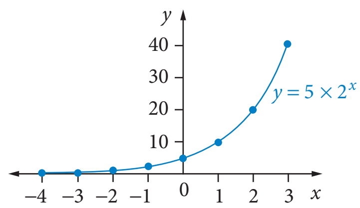
1 5^{9n-4}

2 343

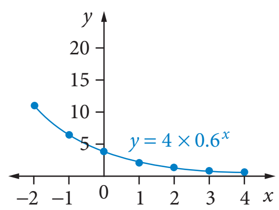
3 a The y-intercept is 1, the graph is increasing with increasing gradient, and is always positive.



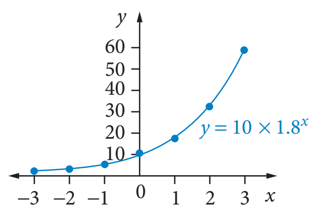
b The y-intercept is 5, the graph is increasing with increasing gradient, and is always positive.



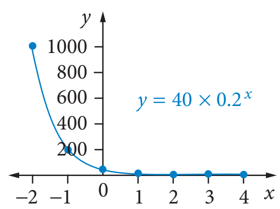
c The y-intercept is 4, the graph is decreasing with decreasing gradient, and is always positive.



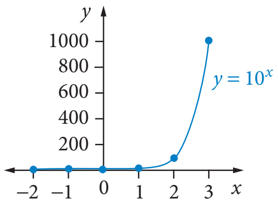
d The y-intercept is 10, the graph is increasing with increasing gradient, and is always positive.



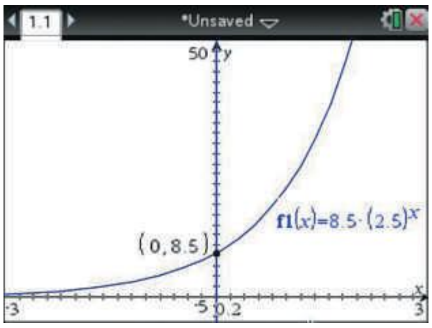
e The y-intercept is 40, the graph is decreasing with decreasing gradient, and is always positive.



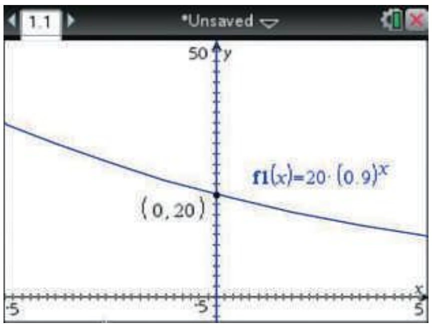
f The y-intercept is 1, the graph is increasing with increasing gradient, and is always positive.



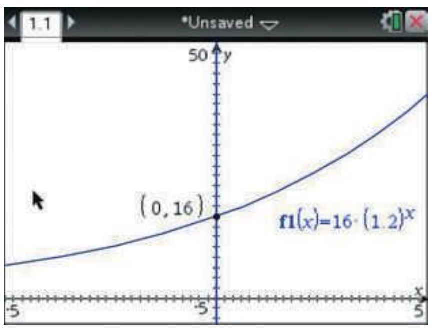
4 a The y-intercept is 8.5, the graph is increasing with increasing gradient, and is always positive.



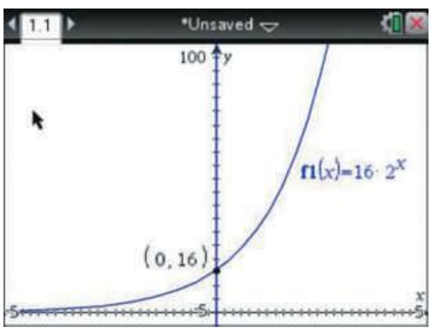
b The y-intercept is 20, the graph is decreasing with decreasing gradient, and is always positive.



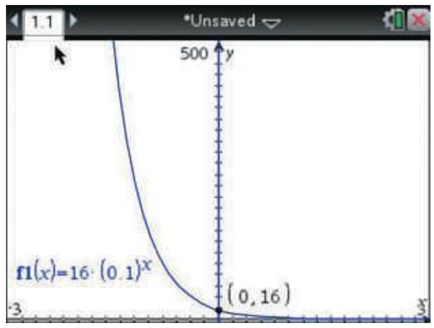
c The y-intercept is 16, the graph is increasing with increasing gradient, and is always positive.



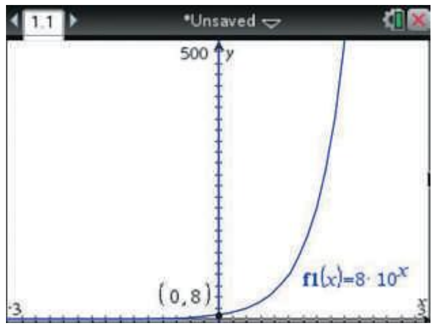
d The y-intercept is 16, the graph is increasing with increasing gradient, and is always positive.



e The y-intercept is 16, the graph is decreasing with decreasing gradient, and is always positive.

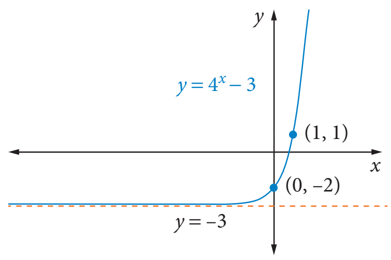


f The y-intercept is 8, the graph is increasing with increasing gradient, and is always positive.

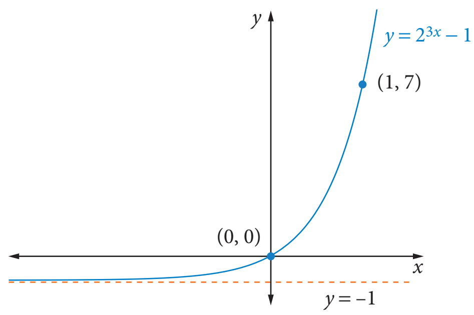


5 a D **b** C **c** A **d** B

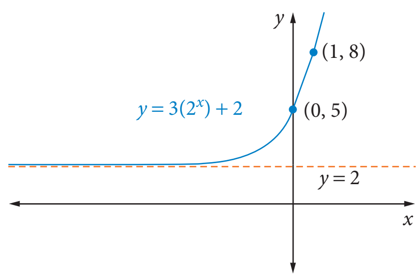
6 a y-intercept −2, asymptote $y = -3$



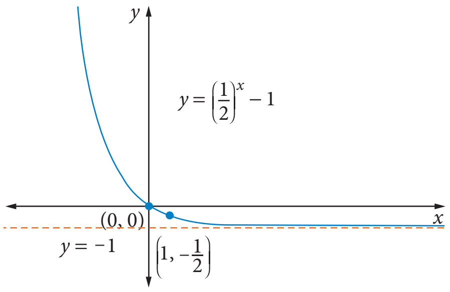
b y-intercept 0, asymptote $y = -1$



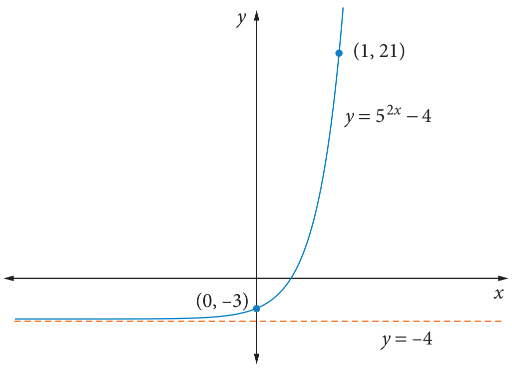
c y-intercept 5, asymptote $y = 2$



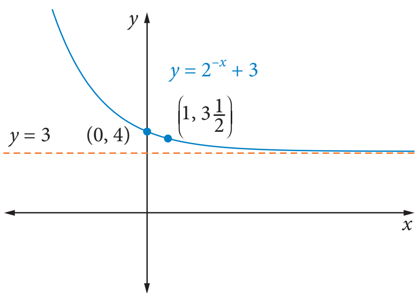
d y-intercept 0, asymptote $y = -1$



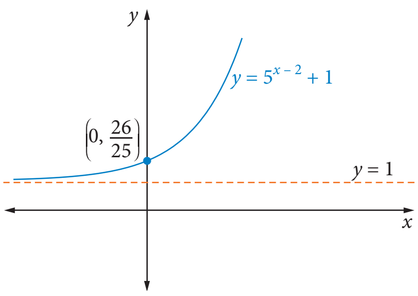
e y-intercept -3, asymptote $y = -4$



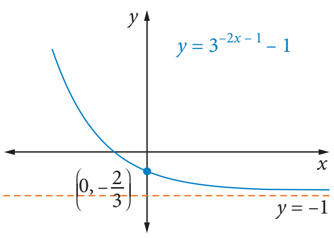
f y-intercept 4, asymptote $y = 3$



g



h



7 C **8** D **9** E

10 $A = 1, c = 1, y = 3^x + 1$

11 $A = 1, c = -2, y = 2^{-x} - 2$

12 B **13** E **14** $c = -1, d = -2$

EXERCISE 7.3

1 E **2** C

3 **a** **i** 45 **ii** 1.15
b **i** 1100 **ii** 0.82

4 **a** $A = 3 \times 1.05^d$
b **i** $3.4729\ \text{m}^2$ **ii** $4.8867\ \text{m}^2$ **iii** $7.9599\ \text{m}^2$
c 104.9 days

5 **a** $V = 3000 \times 1.08^y$
b **i** \$3779.14 **ii** \$6476.77 **iii** \$20 545.43
c 9.01 years

6 **a** $V = 24\,900 \times 0.85^y$
b **i** \$15 291.71 **ii** \$11 048.26 **iii** \$4902.17
c 19.8 years

7 **a** $T = 500 - 480 \times 0.9^m$
b **i** 216.6°C **ii** 332.6°C **iii** 465.5°C
c 14.9 min

8 **a** $p = 3 \times 2^m$ **b** 768

9 **a** $q = 24 \times 1.8^x$ **b** 117.3402

10 **a** 6191 **b** 4%

11 \$160.10

12 16 777 216

13 43 872

EXERCISE 7.4

1 $y = 2000(1.09)^x$ **2** $y = 2000(0.91)^x$

3 **a** $\log_7(49) = 2$ **b** $\log_3(27) = 3$
c $\log_2(16) = 4$ **d** $\log_5(125) = 3$
e $\log_{11}(1) = 0$ **f** $\log_2(1) = 0$
g $\log_5\left(\frac{1}{25}\right) = -2$ **h** $\log_4\left(\frac{1}{16}\right) = -2$
i $\log_{10}\left(\frac{1}{1000}\right) = -3$ **j** $\log_3\left(\frac{1}{81}\right) = -4$
k $\log_4\left(\frac{1}{64}\right) = -3$ **l** $\log_2\left(\frac{1}{8}\right) = -3$
m $\log_6\left(\sqrt[3]{36}\right) = \frac{2}{3}$ **n** $\log_7\left(\sqrt[5]{343}\right) = \frac{3}{5}$
o $\log_a(k) = r$ **p** $\log_b(3) = d$

4 **a** $5^2 = 25$ **b** $4^2 = 16$ **c** $5^3 = 125$
d $2^4 = 16$ **e** $3^1 = 3$ **f** $7^2 = 49$
g $2^7 = 128$ **h** $5^0 = 1$ **i** $8^{\frac{1}{3}} = 2$
j $4^{-\frac{1}{2}} = \frac{1}{2}$ **k** $4^{\frac{1}{4}} = \sqrt{2}$ **l** $3^{-\frac{1}{3}} = \frac{1}{\sqrt[3]{3}}$
m $2^{-\frac{3}{2}} = \frac{\sqrt{2}}{4}$ **n** $a^c = b$ **o** $c^{3m} = \sqrt{a}$

5 **a** 6 **b** 2 **c** 4 **d** 3
e 3 **f** 0 **g** 1 **h** 5
i 5 **j** 5 **k** 4 **l** −3
m −2 **n** 4 **o** −7 **p** −9
q 4 **r** −1 **s** −3 **t** 3

6 **a** $\frac{1}{2}$ **b** $\frac{3}{2}$ **c** $\frac{3}{2}$ **d** $\frac{3}{2}$

e $\frac{2}{3}$ **f** $\frac{3}{2}$ **g** $\frac{3}{2}$ **h** $\frac{4}{3}$

i $\frac{5}{6}$ **j** $\frac{5}{3}$

EXERCISE 7.5

1 0 **2** $5^2 = 25$

3 **a** 0 **b** 1 **c** 5 **d** 0

e 1 **f** 2 **g** −1 **h** −3

i undefined **j** undefined **k** 0 **l** −2

m 2 **n** −4 **o** 1 **p** a

q undefined **r** 2 **s** 6 **t** −8

4 **a** 6 **b** 3 **c** $\frac{1}{2}$ **d** $\frac{3}{2}$

e $-\frac{1}{2}$ **f** −3 **g** −4 **h** 6

5 **a** 1 **b** 1 **c** $-\frac{1}{3}$ **d** 9

e 3 **f** 0 **g** 1 **h** 2

6 **a** 1.631 **b** 0.2789 **c** 1.683

d 0.4307 **e** 0.1262 **f** 6.644

7 **a** $\log_4(x^4)$ **b** $\log_7(x^2)$ **c** $\log_6\left(\frac{1}{x}\right)$

d $\log_2\left(\frac{1}{x+2}\right)$ **e** $\log_4(x-1)$ **f** $\log_3(1) = 0$

8 **a** $\log_{10}(12) + \log_{10}(a) - 1$

b $5\log_6(a) - 5\log_6(b)$

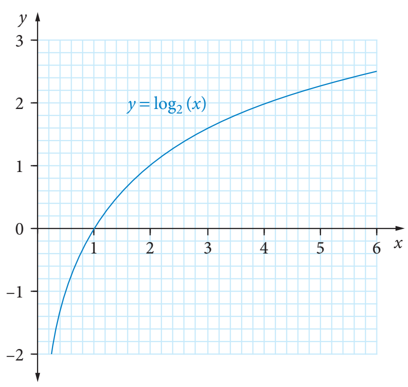
c $\frac{3}{5}\log_3(10) + \frac{3}{5}\log_3(x)$

d $\frac{2}{3}\log_4(x) + \frac{1}{3}\log_4(a) - 2\log_4(y)$

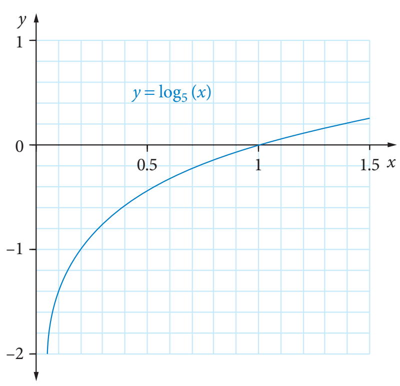
EXERCISE 7.6

1 0 **2** $2\log_4(x)$

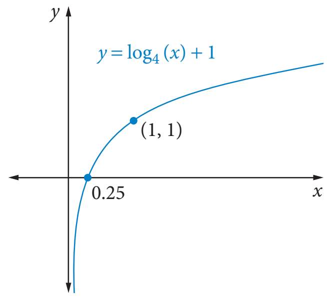
3 **a**



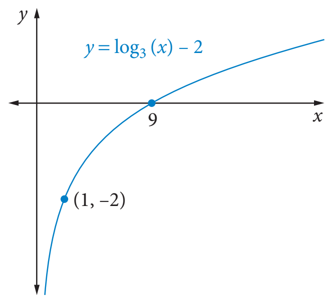
b



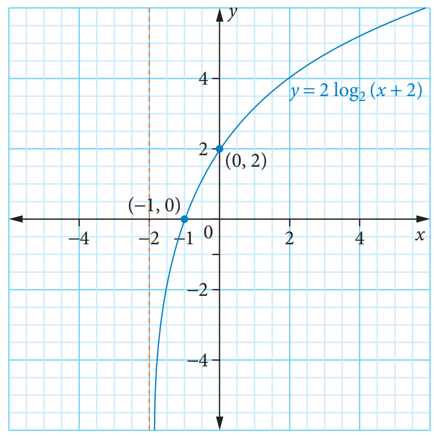
c



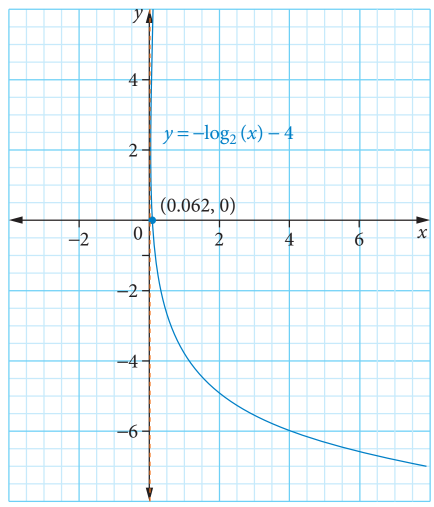
d



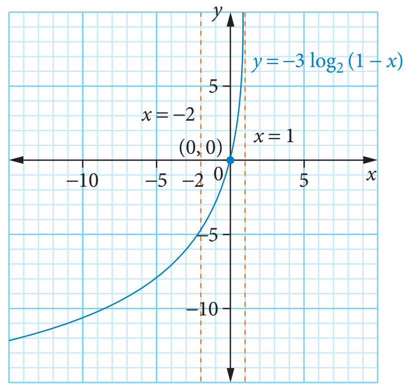
4 **a**



b

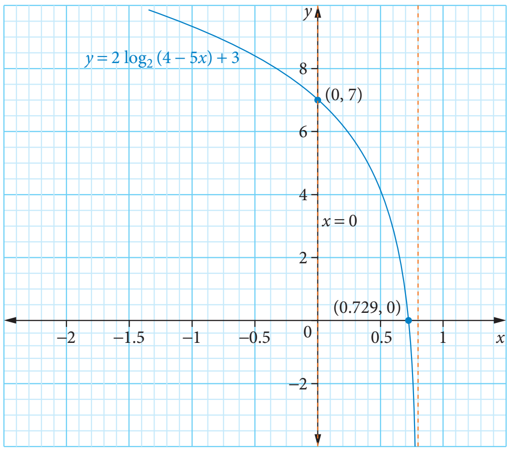


c



9780170448260

d



5 $a = -1, b = 2, y = \log_2(2 - x)$

6 $y = -\log_{10}(x - 2)$

7 125 dB

8 **a** 2799 **b** 2487

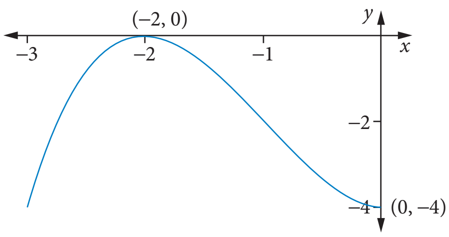
9 12.9686 **10** 8 **11** 41.25 L

12 7 **13** B **14** B

CUMULATIVE EXAMINATION 1

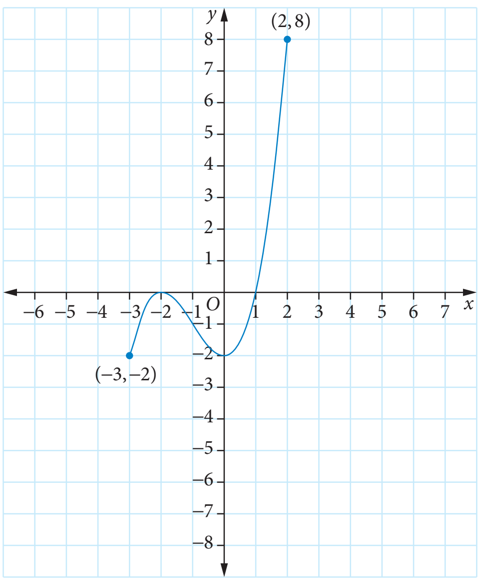
1 **a** Proof: see worked solutions 79%

b



68%

2



69%

3 $\log_{10}(6x + 10)$ 75%

4 **a** $y = \frac{1}{2}\log_4\left(\frac{x+3}{2}\right)$ 67%

b $(-1, \infty)$ 45%

CUMULATIVE EXAMINATION 2

Section A

1 D 78% **2** E 56% **3** B 61%

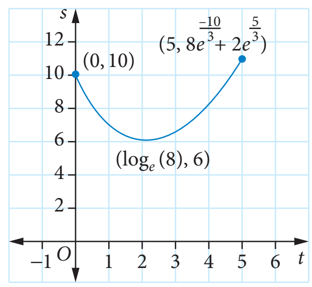
4 A 56% **5** C 44% **6** B 67%

Section B

1 **a** $S(0) = 10, S(5) = 2e^{\frac{5}{3}} + 8e^{-\frac{10}{3}}$ 68%

b $c = 8$, min $= 6$ 65%

c



75%

2 **a** $a = 2, b = 4, c = 4$

b $x = -1.29\ldots, m = -2, n = -1$

c **i** $y = 4x - 8$

ii $y = -11$

iii $P\left(-\frac{3}{4}, -11\right), Q(1, -11)$

d **i** $d = -\frac{7}{4}$

ii $m' = -\frac{15}{4}, n' = -\frac{11}{4}$

CHAPTER 8

EXERCISE 8.1

1 E **2** A **3** $\frac{1}{5}$ **4** $\frac{19}{20}$

5 0.33 **6** $p = 0.35, 0.05$

7 0.1 **8** 0.486

9 **a** $\frac{5}{24}$ **b** $\frac{1}{3}$

10 0.01

11 **a** $\frac{7}{40}$ **b** $\frac{1}{40}$ **c** $\frac{3}{5}$

12 0.12 **13** D **14** C **15** C **16** C

EXERCISE 8.2

1 D **2** A

3 Pr(2 the same colour) $= \frac{3}{5} \times \frac{3}{5} + \frac{2}{5} \times \frac{2}{5} = \frac{13}{25}$

4 0.096

5 Pr(TTTH) $= \frac{27}{256}$

6 $\frac{5}{11}$

7 0.12 (Answers may vary.)

8 0.18 (Answers may vary.)

9 0.227 (Answers may vary.)

10 **a** $\frac{1}{12}$ **b** $\frac{1}{3}$

11 0.3

12 **a i** p^3 **ii** $3p^2(1-p)$ **b** $p=0, p=\frac{3}{4}$

13 A 14 D 15 E 16 E

17 E 18 C 19 E 20 $\frac{1}{64}$

21 0.1678 22 $4p(1-p)^3$

23 **a** 0.486 **b** 0.034 **c** 0.930

EXERCISE 8.3

1 D 2 C 3 $\frac{5}{9}$ 4 $\frac{3}{7}$

5 $\frac{2}{3}$ 6 0.8 7 0.9 8 $\frac{12}{17}$

9 0.2178 10 0.735 11 0.032 12 0.272

13 0.399

14 **a** $\frac{1}{4}$ **b** $\frac{1}{12}$ **c** $\frac{5}{6}$

15 0.336

16 **a** $\frac{5}{6}$ **b i** $\frac{19}{32}$ **ii** $\frac{15}{19}$

17 **a** $\frac{1}{15}$ **b** $\frac{1}{3}$

18 **a** 0.008 **b** $\frac{29}{64}$

19 **a** $\frac{2p+1}{5}$ **b i** $\frac{1-p}{2p+1}$ **ii** $\frac{7}{16}$

20 **a** $\frac{13}{24}$ **b** $\frac{9}{13}$

21 $\frac{1}{4}$ 22 A 23 B 24 E

25 **a** 0.2951 **b** 0.2849

26 **a** 0.12 **b** 0.729 **c** 0.875

27 **a** 0.036 **b** 0.268

28 $\frac{17}{64}$

29 **a** $\frac{1}{9}$ **b** $\frac{2}{9}$

EXERCISE 8.4

1 B 2 B 3 0.69 4 0.77

5 0.94 6 0.3 7 $\frac{8}{13}$ 8 0.85

9 D 10 E 11 D 12 C

13 **a** $\frac{1}{4}$ **b** $\frac{3}{100}$ **c** $\frac{79}{100}$ **d** $\frac{97}{100}$

14 **a** 0.1775 **b** 0.54 **c** 0.8825

15 **a** 0.09 **b** 0.56 **c** 0.69 **d** 0.45

CUMULATIVE EXAMINATION 1

1 domain = $(1, \infty)$, $f^{-1}(x) = \frac{-1}{\sqrt{x-1}} - 2$

2 **a** $\frac{3}{4}$ 85% **b** $\frac{2}{3}$ 70%

3 **a** $a = 2$

b i $g(x) = (x+3)^2 - 14$ **ii** $c = 1$

CUMULATIVE EXAMINATION 2

Section A

1 D 2 A 64% 3 C

4 D 5 D

Section B

1 **a** $6000 = N_0 \times 3^{2k}$ and $18\,000 = N_0 \times 3^{4k}$

b $N_0 = 2000, k = 0.5$

c 54 000 **d** 12 days

e $M = 5000 \times 2^{\frac{t}{3}}$, 9 days

2 **a i** 0.4395 **ii** 0.643

b i 0.375 **ii** 0.34375

CHAPTER 9

EXERCISE 9.1

1 **a** 36° **b** 120° **c** 225°

d 210° **e** 540° **f** 140°

g 240° **h** 420° **i** 20°

j 50° **k** 62.5° **l** 44.0°

2 **a** 0.98 **b** 1.19 **c** 2.22

d 5.04 **e** 5.45

3 **a** $\frac{3\pi}{4}$ **b** $\frac{\pi}{6}$ **c** $\frac{5\pi}{6}$ **d** $\frac{4\pi}{3}$

e $\frac{5\pi}{3}$ **f** $\frac{7\pi}{20}$ **g** $\frac{\pi}{12}$ **h** $\frac{5\pi}{2}$

i $\frac{5\pi}{4}$ **j** $\frac{2\pi}{3}$

4 **a** $x = 6.359$ **b** $c = 10.723$ **c** $x = 13.347$

d $b = 2.309$ **e** $e = 6.467$ **f** $k = 7.071$

5 **a i** 40° **ii** 0.7^c

b i 35° **ii** 0.6^c

c i 38° **ii** 0.7^c

6 **a** 4π cm **b** π m **c** $\frac{25\pi}{3}$ cm

d $\frac{\pi}{2}$ cm **e** $\frac{7\pi}{4}$ mm **f** 0.65 m

g 3.92 cm **h** 6.91 mm

7 **a** 1.75 **b** 0.30 **c** 1.5

d 1.56 **e** 1.12 **f** 0.67

g 0.63 **h** 2

8 **a** 12.5 cm **b** 5.0 cm

9 19.94 m 10 2 11 E

12 E 13 29 cm

EXERCISE 9.2

1 D 2 D

3 **a** $\frac{3}{5}$ **b** $-\frac{4}{3}$

4 $-\frac{24}{7}$

5 **a** $-\cos(80°)$ **b** $-\tan\left(\frac{\pi}{7}\right)$ **c** $\sin(3°)$

6 **a** $\sin(80°)$ **b** $\cos\left(\frac{\pi}{4}\right)$ **c** $-\tan\left(\frac{\pi}{6}\right)$

7 $\dfrac{\sin(2\pi - x) + \cos(\pi + x)}{\sin(x) + \cos(x)}$

$= \dfrac{-\sin(x) - \cos(x)}{\sin(x) + \cos(x)}$

$= \dfrac{-(\sin(x) + \cos(x))}{\sin(x) + \cos(x)} = -1$

8 **a** $-\dfrac{12}{13}$ **b** $-\dfrac{5}{13}$

9 D **10** C **11** C **12** B **13** B

14 $\dfrac{4\pi}{3}, \dfrac{5\pi}{3}$

15 Use $\cos(2\pi + \theta) = \cos(\theta)$ and $\sin^2(\theta) = 1 - \cos^2(\theta)$.

EXERCISE 9.3

1 E **2** D

3 **a** $-\dfrac{1}{2}$ **b** $-\dfrac{\sqrt{3}}{2}$ **c** $-\dfrac{1}{\sqrt{3}}$ (or $\dfrac{\sqrt{3}}{3}$)

4 **a** $\dfrac{\pi}{6}$ **b** $\dfrac{\pi}{4}$ **c** $\dfrac{\pi}{3}$

5 **a** $\cos^2(\theta) - \sin^2(\theta)$

b $\cos(2\theta)$

c $\dfrac{1}{2}\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$ is an identity.

6 **a** $a = \pi$

b $\cos\left(\dfrac{7\pi}{12}\right) = \cos\left(\dfrac{\pi}{3} + \dfrac{\pi}{4}\right)$

$= \cos\left(\dfrac{\pi}{3}\right)\cos\left(\dfrac{\pi}{4}\right) - \sin\left(\dfrac{\pi}{3}\right)\sin\left(\dfrac{\pi}{4}\right)$

$= \dfrac{1}{2} \times \dfrac{1}{\sqrt{2}} - \dfrac{\sqrt{3}}{2} \times \dfrac{1}{\sqrt{2}}$

$= \dfrac{1 - \sqrt{3}}{2\sqrt{2}}$

7 **a** $-\tan(\theta)°$

b $-\dfrac{1}{\sqrt{3}}$

8 $-\dfrac{\sqrt{3}}{6}$

9 30°, 150°

10 B **11** D **12** C

13 **a** $\cos^2(\theta) + \sin^2(\phi) = 1$

$\phi = \pi - \theta$

$\cos^2(\theta) + \sin^2(\pi - \theta) = 1$

$\cos^2(\theta) + \sin^2(\theta) = 1$, using $\sin(\pi - \theta) = \sin(\theta)$

b $\cos^2(\theta) + \sin^2(\phi) = 1$

$\phi = \dfrac{\pi}{2} - \theta$

$\cos^2(\theta) + \sin^2\left(\dfrac{\pi}{2} - \theta\right) = 1.$

$\cos^2(\theta) + \cos^2(\theta) = 1$, using $\sin\left(\dfrac{\pi}{2} - \theta\right) = \cos(\theta)$

$2\cos^2(\theta) = 1$

$\cos(\theta) = \dfrac{1}{\sqrt{2}} \Rightarrow \theta = \dfrac{\pi}{4}$, from reference triangles.

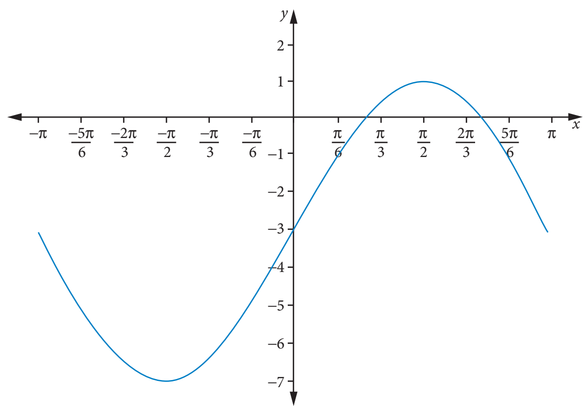
Hence $\phi = \dfrac{\pi}{4}$, since $\phi = \dfrac{\pi}{2} - \theta$.

EXERCISE 9.4

1 C **2** B

3 **a** amplitude 4, period 2π, mean value -3

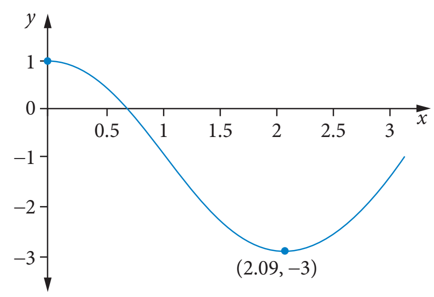
b



4 amplitude 10, period $\dfrac{\pi}{4}$, mean value 15

5 1.516%

6 **a**



b maximum at (0, 1) minimum at (2.09, −3)

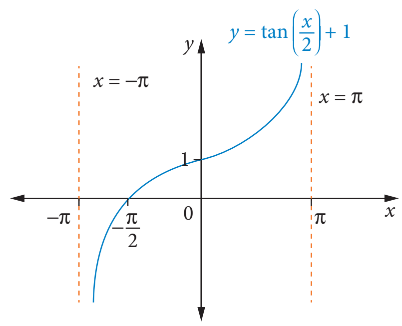
7 range = [1, 7], period = 4

8 $p = 4$ **9** $p = 2$

10 C **11** C **12** E **13** C

14 B **15** B **16** D

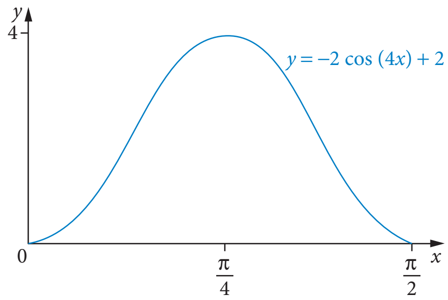
17



18 **a** amplitude 3, period $\frac{2\pi}{2} = \pi$

b dilation by a factor of $\frac{1}{2}$ from the y-axis; dilation by a factor of 3 from the x-axis; reflection in the x-axis; translation 5 units up

19



EXERCISE 9.5

1 D

2 B

3 dilation from y-axis by factor 2, translation 2 units up

4 **a** $y = a \sin(nx) + c$, $a = 2$, $n = 2$, $c = -3$

$a = 2$ dilation by factor of 2 from x-axis.

$n = 2$ dilation by a factor of $\frac{1}{2}$ from the y-axis.

$c = -3$ translation down 3 units.

c domain: R; range: $[-1, -5]$

5 **a** $a = -2$; dilation by factor of 2 from x-axis and reflection in the x-axis.

$c = 1$; translation 1 unit up.

b $x = -\frac{\pi}{2}$, $x = \frac{\pi}{2}$

d domain: $R \backslash \left\{-\frac{\pi}{2}, \frac{\pi}{2}\right\}$, range: R

6 **a** $y = -\sin(x)$ is a reflection in the x-axis. Its graph is a mirror image of $y = \sin(x)$ in the y-axis at their points of intersection. Hence, $y = \sin(-x)$, which is a reflection in the y-axis, becomes the graph of $y = \sin(x)$.

b No. The reflection in the x-axis does not produce symmetry in the y-axis at the points of intersection of $y = -\cos(x)$ with $y = \cos(x)$.

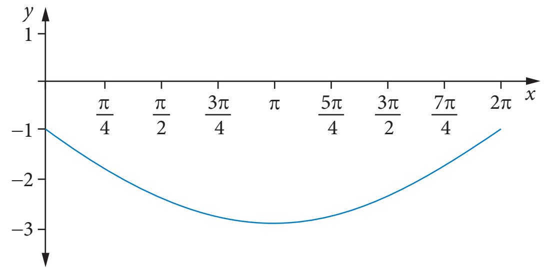
7 $y = 6 \sin(3\pi x) + 5$

8 $(0, -1)$

9 A **10** A **11** E **12** C

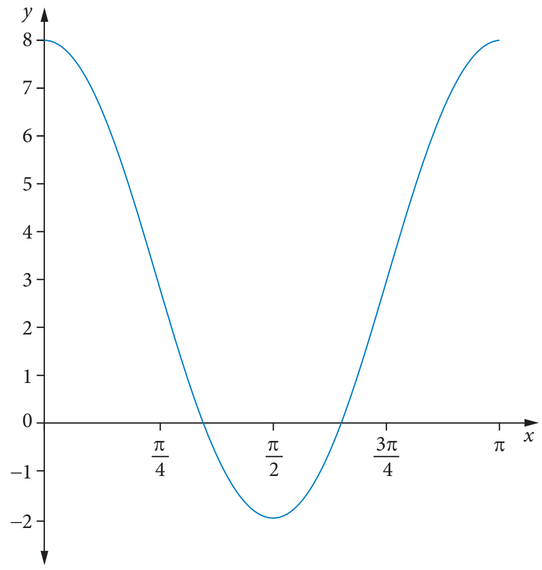
13 **a** $y = 2 \sin(-0.5x) - 1$

b



14 **a** $y = 5 \cos(2x) + 3$

b



EXERCISE 9.6

1 D

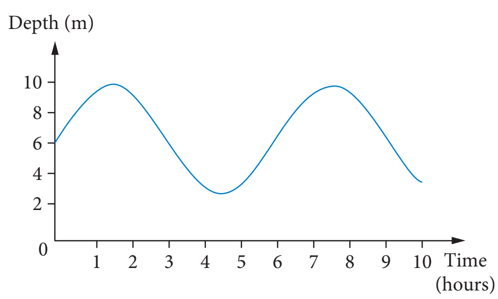
2 C

3 **a** 13

b 43 cm below the rest position

c 6.14 s

4 **a**

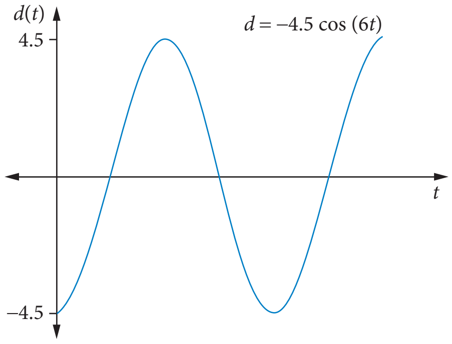


b minimum 3 m, maximum 10 m

c 3.25 pm

d 6.06 m

5 a



b The mass is suspended 4.5 cm below the unstretched position.

c 0.52 s **d** −4.3 cm

6 Maximum 35°C at 6 am, 2 pm, 6 pm

Minimum 15°C at 10 am, 6 pm

7 16 s **8** B **9** D

10 C **11** E **12** B

13 a Period is 1.795 s, number of swings is 33.

b 4.97 cm

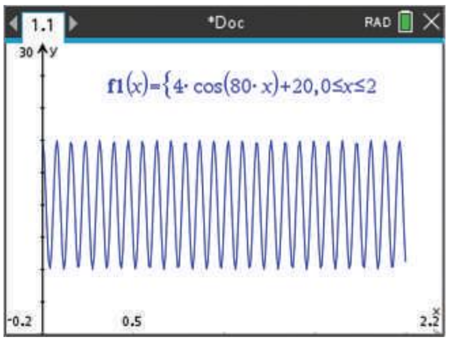
c 1.17 s

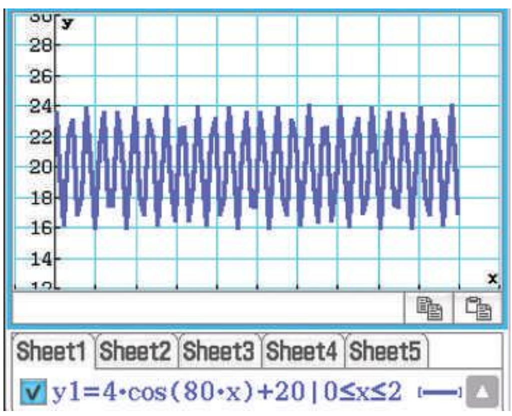
14 a 4 cm **b** 9 cycles

c 24 cm **d** $\frac{1}{3}$ milliseconds

15 a $r = 20$, $a = 80$

b $y = 4\cos(80t) + 20$ in $[0, 2]$





c 25 **d** 19.3 cm above G

CUMULATIVE EXAMINATION 1

1 0 (or undefined where $\cos(\theta) = 0$)

2 a 6 m 69%

b $14 + 4\sqrt{3}$ m

3 a $h = 0.1$, $\frac{\Delta y}{\Delta x} = -0.1$, $h = 0.01$, $\frac{\Delta y}{\Delta x} = -0.01$, $h = 0.001$, $\frac{\Delta y}{\Delta x} = -0.001$

b $h = 0.1$, $\frac{\Delta y}{\Delta x} = 0.1$, $h = 0.01$, $\frac{\Delta y}{\Delta x} = 0.01$, $h = 0.001$, $\frac{\Delta y}{\Delta x} = 0.001$

c The instantaneous rate of change (gradient) approaches 0.

4 $(x - 1)$ is a factor of $P(x)$. By division,
$P(x) = (x - 1)(x^2 - ax + b)$
$x^2 - ax + b$ has two factors when $a^2 - 4ab > 0$ or $b < \frac{a}{4}$.

CUMULATIVE EXAMINATION 2

Section A

1 C **2** B 90% **3** C 85%

4 D **5** A

Section B

1 a amplitude 400, period 6 months 90%

b minimum 800, maximum 1600 89%

c 1000 wombats 91%

2 a minimum 10 m, maximum 120 m 89%

b 30 min 84%

c 2.62 m

3 a 5.36 mm/s

b 5.60 mm/s

c The approximation for the instantaneous rate of change is close to the average rate of change because the shape of the graph is smooth and so the gradient at each point does not change significantly.

4 $\frac{12}{25}$

CHAPTER 10

EXERCISE 10.1

1 a 23 **b** 4

2 a Limit does not exist. **b** Limit does not exist.

3 a not continuous **b** continuous

c continuous

4

$$\lim_{x\to 3}\frac{x^2 - 7x + 12}{(x-3)(x+4)} = \lim_{x\to 3}\frac{(x-3)(x-4)}{(x-3)(x+4)} = \lim_{x\to 3}\frac{x-4}{x+4} = -\frac{1}{7}$$

5 $\lim_{x\to k}(f(x) + g(x)) = \lim_{x\to k}(x^2 + x + 1) = k^2 + k + 1$

$\lim_{x\to k} f(x) + \lim_{x\to k} g(x) = \lim_{x\to k} x^2 + \lim_{x\to k}(x + 1) = k^2 + k + 1$

6 B **7** E **8** C **9** C

10 $\lim_{n\to 20}\left(1 + \frac{1}{n}\right)^n = 2.653\ldots$, percentage error is

$$\frac{2.718\ldots - 2.653\ldots}{2.718\ldots} \times 100 = 2.39\%$$

11 There are no 'gaps' in [0, 3], so $f(x)$ is continuous.

12 No, $f(x)$ is not continuous at $x = \frac{1}{2}$.

13 **a** $x = 0$

b There are two different limits, with $\lim_{x\to 0^-} f(x) = -\infty$ and $\lim_{x\to 0^+} f(x) = +\infty$.

EXERCISE 10.2

1 B 2 C

3 $f'(x) = x^2$

4 $f'(x) = 6,\ f'\left(\frac{2}{3}\right) = 6$

5 $f'(x) = -2x \quad f'(-1) = 2$

6 Given $f(x) = x - a$, $f'(x) = 1$

7 B 8 A 9 A

10 $f'(x) = m$

11 $f'(x) = 2ax + b$

12 $f(x) = x^4$

$$\begin{aligned} f(x+h) &= (x+h)^4 \\ &= x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4 \end{aligned}$$

$$\begin{aligned} \frac{f(x+h)-f(x)}{h} &= \frac{x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4 - x^4}{h} \\ &= \frac{4x^3h + 6x^2h^2 + 4xh^3 + h^4}{h} \\ &= 4x^3 + 6x^2h + 4xh^2 + h^3 \end{aligned}$$

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0}(4x^3 + 6x^2h + 4xh^2 + h^3) \\ &= 4x^3 \end{aligned}$$

13 $\frac{f(x+h)-f(x)}{h}$

$$= \frac{(x^n + nx^{n-1}h^1 + k_2x^{n-2}h^2 + k_3x^{n-3}h^3 + \ldots + h^n) - x^n}{h}$$

$$= \frac{h(nx^{n-1} + k_2x^{n-2}h + k_3x^{n-3}h + \ldots + h^{n-1})}{h}$$

$$= nx^{n-1} + k_2x^{n-2}h + k_3x^{n-3}h + \ldots + h^{n-1}$$

$$\lim_{h\to 0}(nx^{n-1} + k_2x^{n-2}h + k_3x^{n-3}h + \ldots + h^{n-1})$$

$$= nx^{n-1}$$

EXERCISE 10.3

1 D 2 C

3 **a** yes; $f(x) = x\ (x \neq -2)$ is continuous at $x = 2$ and a unique tangent exists at $x = 2$.

b no; $f(x)$ is not continuous at $x = \sqrt{5}$.

c yes; $f(x) = \tan(x)$ is continuous at $x = \pi$ and a unique tangent exists at $x = \pi$.

4 **a** $3x^2 + 2x + 1$ **b** $42x^6 + 90x^8 - 60x^{11} + 48x^5$

c $-12x^5 + 72x^7 + 11x^{10}$ **d** $24x^7 - 60x^{11}$

e $72x^{11} + 143x^{12} - 120x^4$ **f** $36x^2 + 28x^3 - 66x^5$

g $32x^7 - 15 + 56x + 42x^2$ **h** $-6 + 14x + 42x^2 - 36x^3$

i $120x^7 - 35x^4 + 28x^6 + 12$ **j** $32x - 36x^3 + 21x^2 + 18$

5 **a** $-12x^{-4}$ **b** $-x^{-\frac{5}{9}}$

c $2x^{-\frac{1}{2}}\left(\text{or } \frac{2}{\sqrt{x}}\right)$

6 **a** $\frac{1}{6}$ **b** $\frac{1}{20}$

7 $\frac{dy}{dx} = 4x + x^{-3}$, $\frac{dy}{dx} = 24\frac{1}{216}$ at $x = 6$

8 $a = 3$

9 $a = 5,\ b = -1$

10 **a** $12x^3 - 12x$ **b** $12u^3 - 12u$

c $30m^5 + 40m^4 - 24$ **d** $9 + 30y^4 - 12y^2 + 14y^6$

e $6z - 27z^2 + 38$ **f** $35t^6 + 36t^3 - 16$

g $27r^2 - 13 - 12r$ **h** $4a - 24a^3$

i $8 - 12p^3 - 42p^6 - 45p^8$ **j** $8t - 6t^2 + 44t^3$

11 $a = 3,\ b = -4$

12 $f'(x) = 10x + 10$

13 A 14 A 15 E 16 C

17 D 18 A 19 A 20 E

21 $2a - 12$

22 **a** $f'(x) = 12x^2 + 5$

b $x^2 \geq 0$ for all x; hence, $12x^2 + 5 \geq 5$ for all x.

EXERCISE 10.4

1 C 2 D 3 40

4 **a** **i** -40 cm/s **ii** -80 cm/s

b -8 cm/s^2

5 0.26 m/s^2

6 $2a - 1$

7 Proof: see worked solutions

8 C 9 D 10 C 11 (5, 25)

12 -0.56 m/s^2

13 $\frac{f(a+h)-f(a-h)}{2h} = \frac{\sqrt{t+h}-\sqrt{t-h}}{2h}$

$t = h,\ \frac{\sqrt{t+h}-\sqrt{t-h}}{2h} = \frac{\sqrt{2h}}{2h} = \frac{1}{\sqrt{2h}}$

14 **a** A: 3.142, B: 4.189 **b** -0.06 cm/s^2

15 **a** – **c** Proofs: see worked solutions

d nx^{n-1}

16–18 Proofs: see worked solutions

EXERCISE 10.5

1 E 2 B

3 $y = -13x - 10$

4 **a** $y = 8x - 9$ **b** $y = -11x + 11$

c $y = -2x - 1$ **d** $y = 6x - 19$

5 **a** $(-1, -3), (3, 13)$ **b** $(0, 4), (2, 6)$

c $(-2, -22), (4, 32)$ **d** $(1, 5)$

6 $f'(-1) = -8(-1)^3 + 6(-1)^2 + 10(-1) - 4 = 0$

7 $\left(\frac{3}{10}, \frac{3}{20}\right)$

8 Proof: see worked solutions

9 E 10 E 11 $y = 10x - 40$

12 Gradient of tangent at $x = 5$ is $f'(5) = 2(5) - 8 = 2$. Gradient of line $y = 2x + 1$ is 2. Both lines have the same gradient, so they are parallel.

13 The gradient at any point is $f'(x) = 2ax + b$. The gradient is zero if it is to be parallel to the x-axis.

$2ax + b = 0$

$x = -\frac{b}{2a}$

14 (1, 1)

15 (0, −2) and (−2, 2)

CUMULATIVE EXAMINATION 1

1 a $x = -4$ b 8

2 $a = -\frac{3}{2}, b = 5$ 48% 3 $x = 3$ or $x = 4$

4 65 5 $a = 1, b = -2$

CUMULATIVE EXAMINATION 2

Section A

1 E 2 B 3 C 4 B 5 A

Section B

1 a 4 b $2x$

c $x = 2$. We find the derivative of the function at $x = 2$.

2 $m = 10, c = -21$

3 [2.9,3.2]

4 a $3y^2 - 18y + 24$ b $(3y - 6)(y - 4)$

c $y = 2, y = 4$ d $x = 1, x = 2$

5 $x = \frac{1}{2}$

CHAPTER 11

EXERCISE 11.1

1 a $\frac{5\pi}{6}, \frac{7\pi}{6}$ b $\frac{\pi}{3}, \frac{2\pi}{3}$ c $\frac{5\pi}{4}, \frac{7\pi}{4}$ d $\frac{3\pi}{4}, \frac{7\pi}{4}$

2 a $\frac{\pi}{12}, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{11\pi}{12}, \frac{17\pi}{12}, \frac{19\pi}{12}$

b $\frac{\pi}{6}, \frac{2\pi}{3}, \frac{7\pi}{6}, \frac{5\pi}{3}$ c $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$

d $x = \frac{5\pi}{18}, \frac{7\pi}{18}, \frac{17\pi}{18}, \frac{19\pi}{18}, \frac{29\pi}{18}, \frac{31\pi}{18}$

3 a 0.412, 2.730 b 1.139, 2.002

4 a $x = -\frac{7\pi}{12}, -\frac{\pi}{12}, \frac{5\pi}{12}, \frac{11\pi}{12}$ b $x = \frac{5\pi}{12}, \frac{7\pi}{12}$

c $x = -\frac{11\pi}{6}, -\frac{7\pi}{6}, -\frac{5\pi}{6}, -\frac{\pi}{6}$

d $x = \frac{3\pi}{16}, \frac{7\pi}{16}, \frac{11\pi}{16}, \frac{15\pi}{16}$

5 a $x = \frac{\pi}{4}, \frac{5\pi}{4}$ b $x = -\frac{\pi}{3}, \frac{2\pi}{3}$

c $x = \frac{\pi}{12}, \frac{7\pi}{12}$ d $x = \frac{\pi}{4}, \frac{7\pi}{12}, \frac{11\pi}{12}$

6 a $a = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$ b $\frac{4\pi}{3}$

7 $x = \frac{5\pi}{12}, \frac{7\pi}{12}$

8 $x = \frac{\pi}{4}, \frac{5\pi}{12}, \frac{11\pi}{12}, \frac{13\pi}{12}, \frac{19\pi}{12}, \frac{7\pi}{4}$

9 $x = \frac{2\pi}{3}, \frac{5\pi}{6}, \frac{5\pi}{3}, \frac{11\pi}{6}$

10 $\frac{7\pi}{3}, \frac{11\pi}{3}$ 11 $-\frac{2\pi}{3}, \frac{2\pi}{3}$

12 $x = \left\{\frac{\pi}{6}, \frac{2\pi}{3}\right\}$ 13 $x = \frac{1}{3}, \frac{5}{6}$

14 $\frac{\pi}{18}, \frac{5\pi}{18}$ 15 $x = \left\{-\frac{2\pi}{9}, \frac{2\pi}{9}\right\}$

16 $x = 2, \frac{5}{2}$

17 D 18 A 19 A 20 C
21 E 22 E 23 B 24 A
25 D 26 B 27 D 28 E

EXERCISE 11.2

1 B 2 A

3 a $\frac{5}{4}$ b $\frac{9}{4}$ c $\frac{2}{3}$ d $-\frac{2}{3}$

4 a $\frac{15}{11}$ b 4 c $-\frac{3}{19}$ d 6

5 a $\frac{5}{3}$ b −2 c $\frac{1}{9}$ d 2

6 a $x = 2, 3$ b $x = 0, 4$ c $x = 1$ d $x = 2$

7 a 0.77 b 1.77 c 13.23

8 a $x = -2, 1$ b $x = -2, 0$ c $x = -1, 2$ d $x = -3$

9 $x = \frac{3}{2}$ 10 $x = -6$

11 a $x = -\frac{3}{8}$ b $x = \frac{8}{11}$

c $x = -2, 0$ d $x = 2$

12 a $x \approx 1.209$ b $x \approx 191.689$

13 B 14 E 15 D
16 B 17 D 18 E

EXERCISE 11.3

1 C 2 C

3 a 3 b 19 c 121

4 a $\frac{13}{2}$ b $\frac{7}{9}$ c $\frac{130}{3}$

5 a 4 b 7 c $\frac{8}{17}$

6 a 1 b 4 c 1

7 a $8e^5$ b $\frac{e^3}{4} - 3$ c $\frac{e^4}{4}$

8 a $\log_e(3)$ b $\log_e(3), \log_e(4)$ c $\log_e(4)$

9 a −1.291 b −1.172, 2.271 c −1.270

10 **a** 12 **b** 20 **c** 128

11 $\frac{18}{25}$ **12** $x = \frac{e^2 - 10}{6}$ **13** 4

14 $x = -1$ **15** e^6 **16** $x = \frac{3}{2}$

17 C **18** A **19** B

20 C **21** C **22** D

23 B **24** E **25** E

EXERCISE 11.4

1 E **2** B

3 **a** $x_1 = 2$ **b** $x_2 = \frac{5}{3}$ **c** $x_3 = \frac{34}{21}$

4 **a** $x_1 \approx 2.3333$ **b** $x_2 \approx 1.8617$ **c** $x_1 \approx 1.7220$

5 First iteration $x_1 \approx -1.8571$

Second iteration $x_2 \approx -1.8475$

6 B **7** C **8** C **9** B **10** B

CUMULATIVE EXAMINATION 1

1 $\frac{8\pi}{3}, \frac{16\pi}{3}$ **2** $\frac{3}{2}$ **3** $\frac{e^3 + 14}{4}$

4 $y = 8x - 7$ **5** $\frac{2}{27}$

CUMULATIVE EXAMINATION 2

Section A

1 A **2** C **3** E **4** A **5** A

Section B

1 **a** $a = 1, b = 1, c = -12$

b $x = -3, 3, 4$

c $m = -4, n = 5$

d Proof: see worked solutions

e (2, 10)

2 **a** Let $k = e^x$,

$$3 - k - \frac{1}{k} = 0$$

$$3k - k^2 - 1 = 0$$

$$k^2 - 3k + 1 = 0$$

b $k = \frac{3 \pm \sqrt{5}}{2}, x = \log_e\left(\frac{3 \pm \sqrt{5}}{2}\right), a = \log_e\left(\frac{3 + \sqrt{5}}{2}\right)$

CHAPTER 12

EXERCISE 12.1

1 **a** $v(t) = 4t$ **b** 12

2 **a** $v(t) = 6t + 2$ **b** 6

3 D **4** B

5 **a** $\frac{2}{(x + 2)^2}$ **b** $g'(1) = -9$

6 **a** $\frac{dy}{dx} = 3(-9x^2 + 2x)(-3x^3 + x^2 - 64)^2$

b $f'(\pi) = -e^{\pi}$

7 $h(t) = 0.6, v(3) = -0.6, a(3) = -0.2$

8 **a** $x(0) = 2$ **b** $t = 1$ **c** $x(2) = -4$

d 6 m **e** $v(3) = -7$

9 **a** $v(0) = 3$ **b** $t = 0, t = 1, t = 3$ **c** $a(3) = 10$

10 **a** $v = 3t^2 + 12t - 2, a = 6t + 12$

b $x(5) = 266$ **c** $v(5) = 133$ **d** $a(5) = 42$

11 **a** 4 m/s **b** −2 m/s **c** 14 m/s

d 6 m/s. Part **a** is the average rate of change of displacement = average velocity $\frac{x(4) - x(2)}{4 - 2} = 4$ m/s.

e $a(2) = 2$ m/s^2 **f** $a(5) = 20$ m/s^2

12 **a** $h(0) = 4$ m **b** $v(0) = 40$ m/s

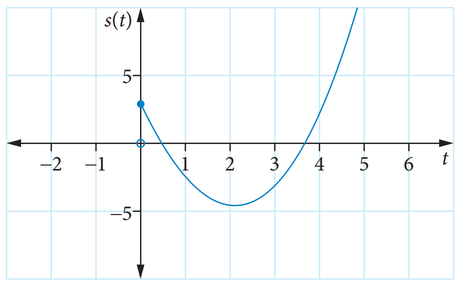
c $h(1) = 39$ m **d** at $t = 4$, $h(4) = 84$

13 **a** $v(0) = -8$ m/s **b** $a(t) = 4$

c $s(5) = 13$ m **d** $t = 2$

e $s(2) = -5$ m

f



14 **a** $v(3) = 18$ **b** $a(2) = 12$

15 **a** $v(5) = 120$ **b** $a(5) = 20$

16 $a(5) = 20$

17 B **18** D **19** E

EXERCISE 12.2

1 **a** $v(t) = 4t$ **b** $a(t) = 4$

2 C **3** $x \in (0, \infty)$ **4** $x \in (-\infty, 0)$

5 $x \in [-2, 0] \cup [2, \infty)$

6 B

7 C

8 **a** $x \in (-\infty, 1) \cup (1, \infty)$

b $x \in (-\infty, \infty)$

9 Proof: see worked solutions

10 $x \in \left(-\infty, -\frac{\sqrt{2}}{2}\right] \cup \left[0, \frac{\sqrt{2}}{2}\right]$

11 **a** strictly increasing **b** strictly decreasing

c strictly decreasing

12 **a** strictly decreasing **b** strictly decreasing

c strictly decreasing **d** strictly decreasing

e strictly increasing

13 **a** $x \in \left(-\infty, \frac{3}{2}\right)$ **b** $x \in \left(\frac{3}{2}, \infty\right)$ **c** $x = \frac{3}{2}$

14 $(-\infty, 0]$

15 $\left(-\infty, \frac{1}{4}\right]$

16 C **17** D **18** D

19 C **20** C **21** D

EXERCISE 12.3

1 **a** $x=-\frac{3}{2}$ **b** $\left[-\frac{3}{2}, \infty\right)$ **c** $\left(-\frac{3}{2}, \infty\right)$

2 $8m-16$

3 $(0,-2)$ and $\left(-\frac{2}{3}, -\frac{50}{27}\right)$

4 $(-2,-2)$ **5** E **6** C

7 **a** $(2,-5)$ **b** $(2,-2)$ and $\left(-\frac{4}{3}, \frac{446}{27}\right)$

8 $(2, 0)$ **9** $p=-12$ **10** C **11** A

12 D **13** D **14** B **15** C

16 C **17** A **18** C **19** B

20 D **21** C **22** B **23** B

24 A

25 **a** **i** $a=c+2, b=-2c-3$ **ii** $c=-3$

b **i** $f'(x)=3x^2-8x+5$ **ii** $m=1, n=\frac{5}{3}$

26 **a** **i** $f'(x)=12x^2+5$ **ii** $x^2 \geq 0$, so $12x^2+5 \geq 5$

b 0, 1, 2 **c** $c=\frac{4}{3}$

27 **a** maximum at $x=\frac{13}{8}$ **b** $x=\frac{1}{a}$ or $x=\frac{ab+1}{4a}$

c $a=0$ **d** $a=\frac{3}{b}$

e 2 **f** $p=4$

EXERCISE 12.4

1 D **2** B

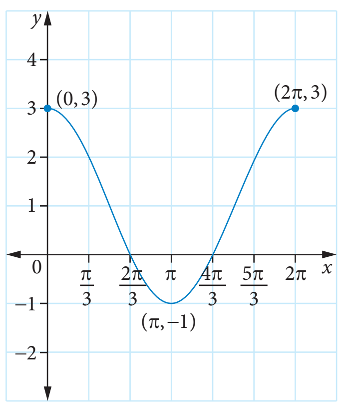
3 $(5, 0)$ **4** $x=\pm5$ **5** $x=\pm\frac{5}{4}$

6 $(-2,-9)$ **7** $(4, 0)$ **8** $\left(\frac{8}{3}, \frac{32}{27}\right)$

9 B **10** C **11** E

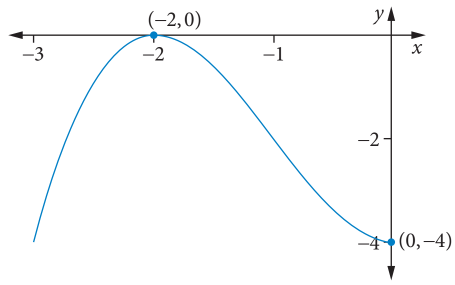
12 **a** $x=\frac{2\pi}{3}, \frac{4\pi}{3}$

b



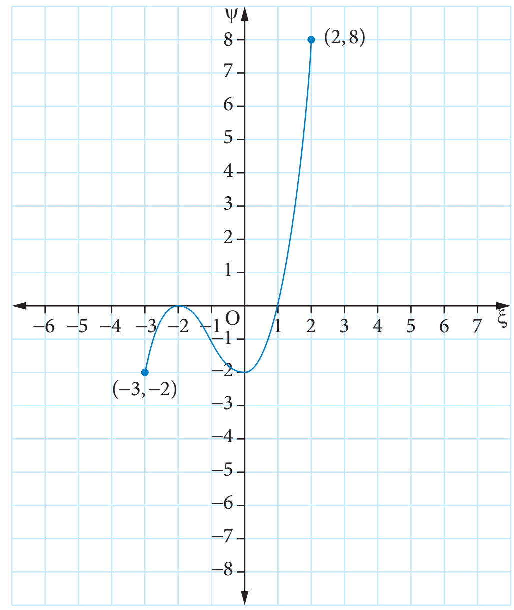
13 **a** Proof: see worked solutions

b



14 **a** $(-2, 0), (0, -2)$

b



15 **a** $(-0.3, 93), (6.2, -33)$ approximation

b $y=79$ approximation

c -26 approximation

16 **a** $(-1.5, -7.5), (0.5, 1.5)$ approximation

b $(-\infty, -1.5)$ and $(0.5, \infty)$ approximation

17 E **18** D **19** B

20 C **21** E **22** B

23 D **24** C **25** E

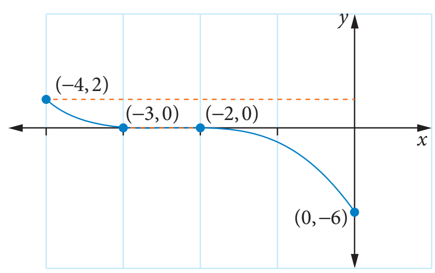
26 **a** $G\left(\frac{28}{9}, -\frac{50}{243}\right)$ **b** $k=8$ m **c** $x=2^{\frac{2}{3}}$

d 75.6 km/h **e** $m=10\times 2^{\frac{2}{3}}$

EXERCISE 12.5

1 E **2** B

3 **a**

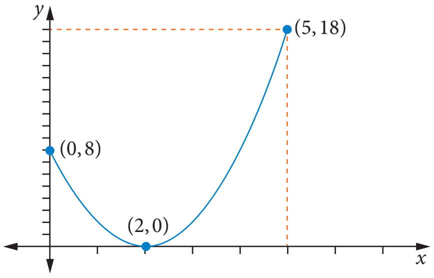


b $[-6, 2]$

4 $x = -2, x = -3$

5 -6 6 D 7 D

8



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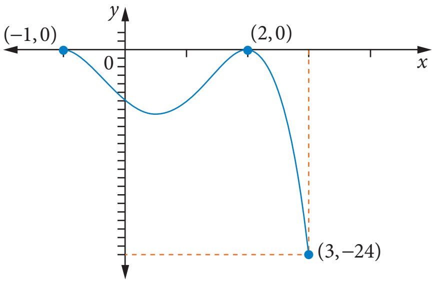


10 a $(-0.5, -6.25)$ b $y = 6$

c $y = -4$ d $(-\infty, -0.5]$

e $[-0.5, \infty)$ f -4

g $[-6.25, 14]$

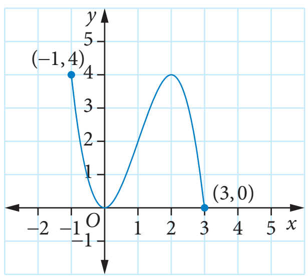
11 a $(-3, 40)$ and $(3.5, 240)$ b $y = 40$

c $y = 120$ d $(-\infty, -3)$ and $(-0.5, 3.5)$

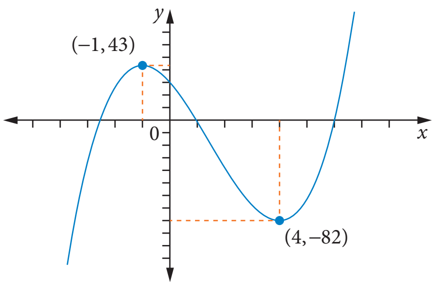
e $(-3, -0.5)$ and $(3.5, \infty)$

12 a $(0, 0)$ and $(2, 4)$

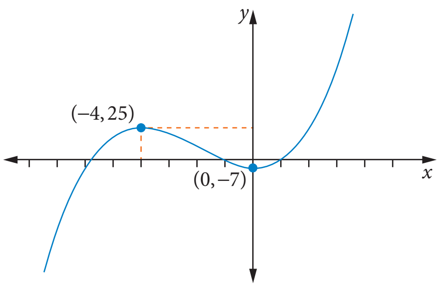
b



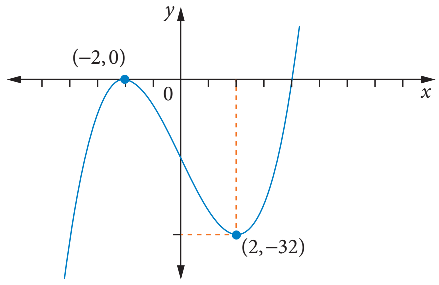
13 $(-1, 43)$ and $(4, -82)$



14 $(-4, 25)$ and $(0, -7)$



15 $(-2, 0)$ and $(2, -32)$



16 D 17 A 18 C 19 A

EXERCISE 12.6

1 1

2 a $x = -10, x = 5$ b $x = e^4 - 1$

c $x = 0, x = 2\log_e(2)$

3 max $= (-3, 0)$ and min $= \left(-\frac{1}{3}, -\frac{256}{27}\right)$

4 max = 80 m and min = 40 m

5 $P(4, 2), 2\sqrt{5}$ 6 125 000 m^2

7 225 m^2 8 14, 14

9 a $y = \dfrac{4000\sqrt{3}}{3x^2}$

b Proof: see worked solutions

c $x = 4000^{\frac{1}{3}}$

10 a $A = 18a - 6a^3$ b $A = 12, a = 1$

11 2048 cm^3

12 7.5 m

13 $t = 2$ s, $h = 16$ m

14 a $h = \dfrac{5r}{2}$ b $S = 10\pi r - 3\pi r^2$ c $r = \dfrac{5}{3}$

15 highest = 24°C, lowest = 4°C

16 C 17 B 18 A

19 A 20 D 21 B

22 a Proof: see worked solutions

b $x = \dfrac{20}{3}$ c 7407.4 cm^3

23 a Proof: see worked solutions

b $0 < x < 36$

c $\dfrac{dv}{dx} = -\dfrac{75}{14}x^2 + \dfrac{16200}{7}$

d $x = 12\sqrt{3}, h = \dfrac{120\sqrt{3}}{7}$

24 a Proof: see worked solutions

b i $q = \dfrac{-17 - 12p}{8}$

ii $f(1) = -\dfrac{p}{2} + \dfrac{73}{8}$

iii $a = \dfrac{17}{12p}$

iv Proof: see worked solutions

c D(4, 0) F(7, 0)

d 3.70 units

9780170448260

25 **a** **i** $0 = \frac{1}{200}(8a + 4b + c)\ldots(1), -0.06$

$= \frac{1}{200}(12a + 4b)\ldots(2), 0$

$= \frac{1}{200}(48a + 8b)\ldots(3)$

ii Proof: see worked solutions

b **i** $P\left(2 - 2\sqrt{3}, 0\right), M\left(2 + 2\sqrt{3}, 0\right)$

ii $2\sqrt{3}$ km **iii** 80 m

EXERCISE 12.7

1 **a** (−5, −112.5), (1, −4.5), (−0.5, 1.40625)

b (0, −25)

2 A

3 $\frac{3}{4}x^4 + \frac{2}{3}x^3$

4 $f(x) = \frac{1}{3}x^3 + \frac{3}{2}x^2 + 7x - \frac{41}{6}$

5 E

6 C

7 $f(x) = \frac{2}{3}x^3 - \frac{3}{4}x^{\frac{1}{3}} - \frac{5}{3}$

8 $f(x) = \frac{x}{2} - \frac{1}{2}\log_e(2x - 2) - 1 + \frac{1}{2}\log_e(2)$

9 $-\frac{1}{2(x-1)^2}$

10 $g(x) = x^3 + x$

11 **a** $\frac{x^3}{3}$ **b** $\frac{x^4}{4} + \frac{x^3}{3}$

c $x^4 + x^2 - 4x$ **d** $x^3 + \frac{5x^2}{2} + 7x$

12 C

13 C

EXERCISE 12.8

1 $2x^5 + x^4 - x^3 - x^2 + c$

2 C

3 $f(x) = \frac{3}{4}x^4 + \frac{2}{3}x^3 - \frac{17}{12}$

4 $x(t) = \frac{1}{3}t^3 + \frac{3}{2}t^2 + 2$

5 E 6 D 7 E

8 **a** 95 m/s **b** 175 m

9 **a** 570 m/s^2 **b** 135 m **c** $t = 0.5$ s

10 **a** 20 m/s^2 **b** $\frac{746}{9}$ m

11 163 m 12 262 m 13 28 m

14 −42.5 m 15 12 m 16 D

17 A 18 B 19 D

CUMULATIVE EXAMINATION 1

1 **a** 6 **b** 6

2 128

3 **a** $-\frac{6}{x^4} - 1$ **b** $\frac{1}{4}$

4 $\frac{x^3}{8} - 20\sqrt{x} + x + 53$

CUMULATIVE EXAMINATION 2

Section A

1 A 2 C 90% 3 A 67%

4 D 5 A 6 A 63%

Section B

1 **a** $t = \frac{10\log_e(4.5)}{7}$ 77%

b −33.5 mg/h 78%

c 191 mg

d 9.5 mg/h

2 **a** domain: $(-\infty, -1)$, range: R 55%

b **i** $y = -x - 2$ 74%

ii $y = x + 2$ 65%

c p is not a one-to-one function because it fails the horizontal line test.

It is a many-to-one function. 66%

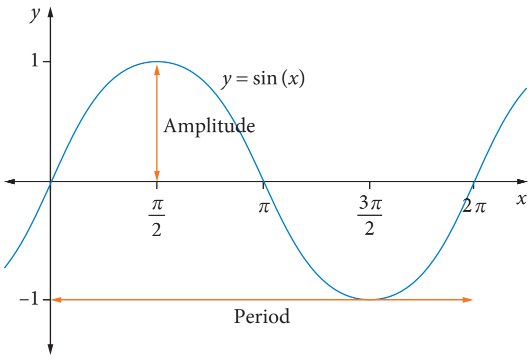
d $p'(a) = 2\left(e^a - 1\right)e^{-2a} = 2e^{-a} - 2e^{-2a}$ 67%

Glossary and index

acceleration The rate of change of velocity of a moving object, represented by the function $a = \frac{dv}{dt}$, where v is the velocity ('signed speed'). Conversely, $v = \int a(t)\,dt$. (pp. 244, 441)

addition rule for probabilities
$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$. (p. 315)

amplitude The vertical distance from the centre of a circular function to its maximum and minimum points. The amplitude of $y = \sin(x)$ and $y = \cos(x)$ is 1. (p. 347)

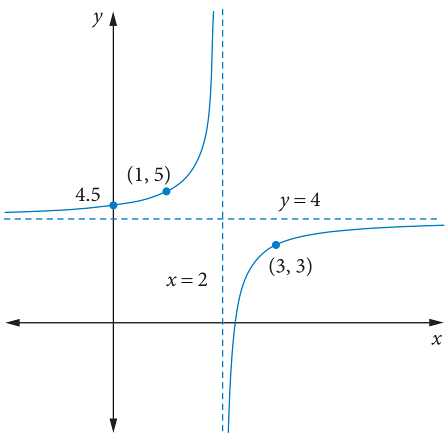


anti-derivative (or **integral** or **primitive**) The opposite of the derivative. The anti-derivative of $f(x)$ is a function $F(x)$ whose derivative is $f(x)$: $F'(x) = f(x)$. (p. 476)

anti-differentiation (or **integration**) The opposite of differentiation; that is, the process of finding the anti-derivative or integral. (p. 476)

ASTC rule An acronym (All Students Take Calculus) to remember the quadrants where sine, cosine and tangent ratios are positive. (p. 336)

asymptote A line towards which a graph gets closer but never touches. In the graph below, the lines $x = 2$ and $y = 4$ are asymptotes. (p. 46)



average rate of change The rate of change between point $A(x_1, y_1)$ and point $B(x_2, y_2)$ on the graph of a function, given by the **gradient** $m = \frac{y_2 - y_1}{x_2 - x_1}$ of the straight line connecting the two points. (p. 222)

axis of symmetry A line about which a graph is reflected equally on both sides. (p. 64) See **parabola** for diagram.

base The number being raised to a power; for example, in 2^5, the base is 2. (p. 4)

binomial expression An algebraic expression with two terms, such as $x + 4$ or $3x^2 - 6$. (p. 6)

binomial product Two or more binomial expressions multiplied together; for example, $(3x + 2)(x - 1)$. (p. 6)

bisection method An approximation method for finding the zeros of a function (or the roots of an equation) that involves examining the signs of the values of the function repeatedly to narrow in on the zero. (p. 107)

central difference approximation A formula that approximates the gradient, m, of the tangent at a point $(x, f(x))$ on the graph of $y = f(x)$ using the values of $f(x)$ a small distance h units on either side of x, so $f'(x) \neq \frac{f(x+h) - f(x-h)}{2h}$. (pp. 236, 393)

central value The mean value of a circular function, the average y value of the function. (p. 347)

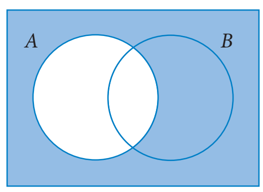
circular function A trigonometric function such as $y = \sin(x)$, $y = \cos(x)$ or $y = \tan(x)$. (p. 335)

co-domain The set of values that the range of a function or relation can fall in. For example, the function $f: [0, 7] \to R$, $g(x) = x^2 + 4$ has domain $[0, 7]$, co-domain R and range $[4, \infty)$. (p. 39)

coefficient The number multiplying a variable. For example, in $x^3 - 3x^2 + 6x$ the coefficient of x^2 is -3. (p. 78)

combination An unordered selection from a group of items; for example, choose 5 players for a basketball team from a group of 12 players. ${}^nC_r$ represents the number of possible combinations of r items chosen from n items.
${}^nC_r = \frac{n!}{(n-r)!\,r!}$ (p. 205)

complement of a set The set containing all the elements that are *not* in the original set. On a Venn diagram, A' is everything except set A. (p. 292)



completing the square A method of solving a quadratic equation by rewriting part of the equation in the form $a^2 + 2ab + b^2 = (a + b)^2$ or $a^2 - 2ab + b^2 = (a - b)^2$. (p. 14)

compound event An event consisting of two or more simple events that occur in succession, such as tossing a coin three times. Different from **simple event**. (p. 188)

concave The shape of the curve of a graph in an upward or downward direction. (p. 45)

concave down Upside-down U-shaped, describing the shape of a curve, especially a parabola. (p. 45)

concave up U-shaped, describing the shape of a curve, especially a parabola. (p. 45)

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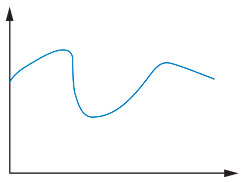
conditional probability The probability that an event occurs given that another event also occurs; for example, the probability that a soccer team will win a match given that it is a rainy day. $\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$, where $\Pr(A|B)$ means 'the probability of A occurring given B occurs'. (p. 306)

constant A value that is fixed. A number without a variable. For example, in $4x^2 + 3\sqrt{2}x + 9$ the constant term is 9. (p. 78)

continuous function A function without a break in its graph. Its graph is smooth, and can be drawn without taking the pen off the paper. (p. 378)



cubic polynomial A polynomial of degree 3; for example $P(x) = -2x^3 + x^2 - 10x - 7$. (p. 79)

degree of a polynomial The highest power of the variable in a polynomial. For example, $x^3 - 3x^2 + 6x$ has degree 3. (p. 78)

dependent variable A variable that is calculated from another variable according to a rule or equation. The variable on the vertical scale of a graph. For $y = f(x)$, y is the dependent variable. (p. 39)

derivative For the function $y = f(x)$, the derivative of y, $\frac{dy}{dx}$ or $f'(x)$, is the function of the instantaneous rate of change of $f(x)$, also called the **gradient function**. (p. 235)

difference of two cubes The expression $a^3 - b^3$ that can be factorised into $(a - b)(a^2 + ab + b^2)$. (p. 97)

difference of two squares The expression $a^2 - b^2$ that can be factorised into $(a - b)(a + b)$. (p. 8)

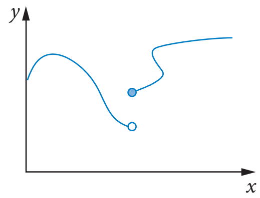
differentiation The process of finding the derivative of a function. (p. 387)

differentiation by first principles Finding the derivative of a function using the formula $\frac{dy}{dx} = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$. (p. 382)

dilation Stretching or squashing (compressing) of a graph, either from the x-axis (vertically) or y-axis (horizontally). (p. 144)

dimensions of a matrix The number of rows and columns in a matrix. For example, $\begin{bmatrix} 2 & -4 & 60 & 1 \\ 5 & 2.8 & -70 & 0 \end{bmatrix}$ has dimensions 2×4. (p. 162)

discontinuous A function with a break in its graph. Its graph cannot be drawn without taking the pen off the paper. (p. 379)



discriminant The part $\Delta = b^2 - 4ac$ of the quadratic formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ that gives information about the number of roots of the quadratic equation, $ax^2 + bx + c = 0$. (p. 17)

displacement The 'signed distance' from the origin of a moving object, represented by the function $x(t)$, where displacement is a function of time. (pp. 244, 441)

domain The set of values of the independent variable x in a relation or function. (p. 39)

***e* (Euler's number)** An irrational constant (number) like π that is used in exponential and logarithmic functions. $e = 2.71828$ (p. 424)

elements (of a matrix) The values in a matrix. For example, for the matrix $\begin{bmatrix} 2 & -4 & 60 & 1 \\ 5 & 2.8 & -70 & 0 \end{bmatrix}$, element $a_{21} = 5$ because it is in row 2, column 1. (p. 162)

elimination method An algebraic method of solving simultaneous equations by combining the equations to eliminate one of the variables x or y. (p. 23)

equating coefficients (of identical polynomials) Applying the rule that if $P(x) = a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_1x^1 + a_0$ and $Q(x) = b_nx^n + b_{n-1}x^{n-1} + b_{n-2}x^{n-2} + \ldots + b_1x^1 + b_0$ are equal for all values of x, then $a_n = b_n$, $a_{n-1} = b_{n-1}$, $a_{n-2} = b_{n-2}$, and so on up to $a_0 = b_0$. (p. 83)

exact values The trigonometric ratios of $30° \left(\frac{\pi}{6}\right)$, $45° \left(\frac{\pi}{4}\right)$, $60° \left(\frac{\pi}{3}\right)$ and their multiples, which can be expressed as exact rational or surd values such as $\frac{1}{2}$ or $\frac{\sqrt{3}}{2}$. (p. 342)

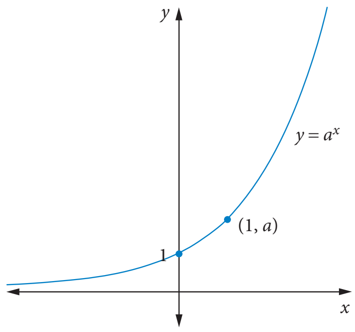
experimental (or **empirical**) **probability** The likelihood of an event occurring based on the results of an experiment or empirical data. (p. 183)

exponent Another name for power; for example, in 2^5, the exponent is 5. (p. 3)

exponential decay A decrease that is happening quickly at first, then gradually more slowly, according to the function $y = Aa^x$, where y is the quantity, A is the initial value, a is the growth factor $(0 < a < 1)$ and x is the time. Natural exponential decay follows the function $y = Ae^{kt}$ where t is the time and k is a negative constant. (p. 267)

exponential equation An equation involving a^x where the variable is in the exponent (power). (p. 4)

exponential function A function of the form $y = Aa^x$ where the variable is in the exponent (power). (p. 258)



exponential growth An increase that is happening slowly at first, then gradually more quickly, according to the function $y = Aa^x$, where y is the quantity, A is the initial value, a is the growth factor ($a > 1$) and x is the time. Natural exponential growth follows the function $y = Ae^{kt}$ where t is the time and k is a positive constant. (p. 267)

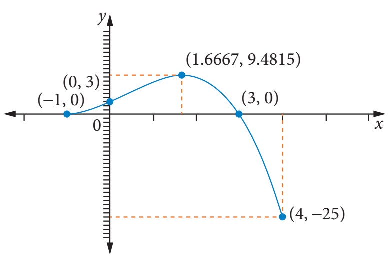
factor theorem For a polynomial $P(x)$, if $P(a) = 0$, then $x - a$ is a factor of the polynomial. (p. 94)

factorial (*n!*) symbol: $n!$ for 'n factorial.' The product of consecutive whole numbers from 1 to n. For example, $4! = 4 \times 3 \times 2 \times 1$. $0!$ is defined to be equal to 1. $n! = n \times (n-1) \times (n-2) \times \ldots \times 3 \times 2 \times 1$. $n!$ is the number of arrangements of n different objects. (p. 200)

function A relation such that each value in the domain has only one ordered pair; each x value matches with only one y value. (p. 40)

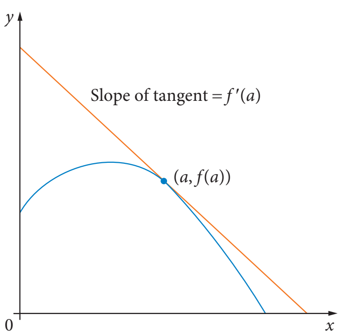
general form (of a linear equation) $ax + by + c = 0$, where a, b and c are whole numbers and a is positive. (p. 59)

global maximum/minimum point The maximum/minimum point of a function over a given interval. It is either a **local maximum/minimum point** or one of the endpoints of the interval. For the function graphed over the given interval below, the global maximum point is (1.6667, 9.4815) and the global minimum point is (4, –25). (p. 465)



gradient The slope of a line, given by $m = \dfrac{\text{rise}}{\text{run}} = \dfrac{y_2 - y_1}{x_2 - x_1}$.

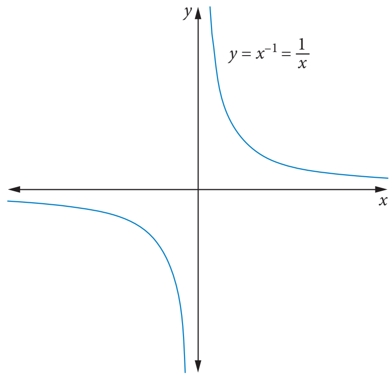
For a curve, the gradient is the slope of the tangent to the curve at a particular point. (p. 59)



gradient function *See* **derivative**.

gradient–intercept form (of a linear equation) $y = mx + c$, where m is the gradient and c is the y-intercept. (p. 59)

hyperbola A discontinuous graph with equation of the form $y = \dfrac{a}{x + b}$. (p. 138)



implied domain *See* **maximal domain**. (p. 53)

independent events Events whose outcomes (and probabilities) do not depend on each other; for example, the number rolled on a second die does not depend on the number rolled on the first die. For two independent events A and B, $\Pr(A \cap B) = \Pr(A) \times \Pr(B)$. (p. 296)

independent variable A variable that is used to calculate the value of another variable. The variable on the horizontal scale of a graph. For $y = f(x)$, x is the independent variable. (p. 39)

index (or **exponent**) Another name for power; for example, 5 is the index in 2^5. (p. 3)

instantaneous rate of change The rate of change at a point (x_1, y_1) on the graph of a function $f(x)$, given by the gradient of the tangent to the graph at that point, which is the derivative of the function at that point, $f'(x_1)$. (p. 227) *See* **gradient** for diagram.

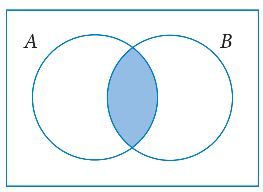
integral (or **anti-derivative** or **primitive**) The opposite of the derivative. The anti-derivative of $f(x)$ is a function $F(x)$ whose derivative is $f(x)$: $F'(x) = f(x)$. (p. 476)

integrand The function being integrated; the $f(x)$ in $\int f(x)dx$. (p. 476)

integration (or **anti-differentiation**) The opposite of differentiation, the process of finding the anti-derivative or integral. (p. 476)

intercept form (of a linear equation) $\dfrac{x}{a} + \dfrac{y}{b} = 1$, where a is the **x**-intercept and b is the y-intercept. (p. 59)

intersection of sets A set containing elements common to two or more sets. On a Venn diagram, $A \cap B$ is where sets A and B overlap. (p. 292)



interval notation A notation used to describe the domain and range of a function or solution using round and square brackets. A square bracket means the boundary value is included, while a round bracket means the boundary value is excluded. For example, $(0, 10]$ means the real numbers from 0 to 10, excluding 0 but including 10, or $0 < x \le 10$. (p. 53)

inverse function The function $y = f^{-1}(x)$ that is the inverse of $y = f(x)$, which links the values of y in $y = f(x)$ back to x. So if $f(3) = -5$, for example, then $f^{-1}(-5) = 3$. An inverse function exists only if the original function is one-to-one. (p. 47)

iteration The number of times a step is repeated in a calculation and approximation method such as the bisection method or Newton's method. (p. 428)

Karnaugh maps See **two-way table.**

leading coefficient (of a polynomial) The coefficient of the term of a polynomial that contains the highest power. For example, $x^3 - 3x^2 + 6x$ has a leading coefficient of 1 (the coefficient of x^3). (p. 78)

leading term (of a polynomial) The term of a polynomial that contains the highest power. For example, $x^3 - 3x^2 + 6x$ has a leading term of x^3. (p. 78)

limit The value of the function $f(x)$ at $x = a$ as x gets infinitely closer to a. (p. 378)

linear function A function of the form $y = mx + c$, where the power of x is 1. The graph of a linear function is a straight line. (p. 59)

linear polynomial A polynomial of degree 1. (p. 79)

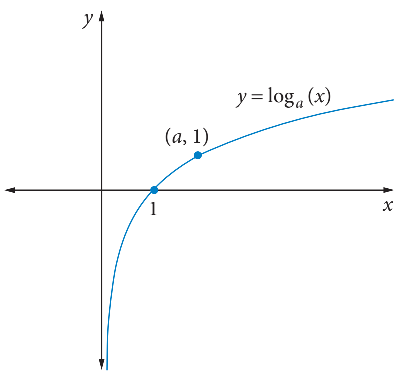
literal equation An equation that has many variables, such as $ax + b = c$. (p. 20)

local maximum/minimum point *See* **turning point**.

logarithm The exponent (power) to which a fixed number, the base, must be raised to give a number or variable. For example, $2^5 = 32$ so $\log_2(32) = 5$. (p. 270)

logarithmic equation An equation involving $y = \log_a(x)$. (p. 423)

logarithmic function A function of the form $y = \log_a(x)$. (p. 278)



long division A method of dividing one polynomial by another, using a repeated process. (p. 85)
See also **synthetic division.**

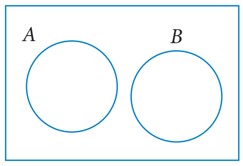
maximal domain The largest set of real numbers for which a function rule is defined. Also called **natural** or **implied domain**. (p. 53)

maximum/minimum point *See* **turning point**.

mean value *See* **central value**.

monic polynomial A polynomial whose leading coefficient is 1; for example, $x^3 - 3x^2 + 6x$. (p. 78)

mutually exclusive events Events or categories that have no items in common. Sets A and B are mutually exclusive if $A \cap B$ has no elements. (p. 293)



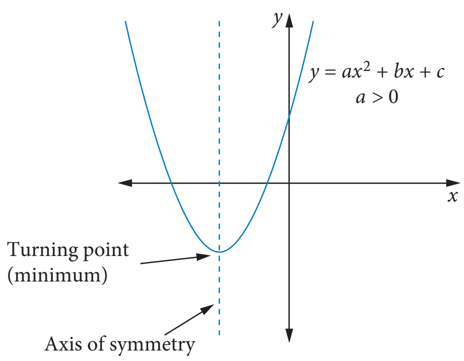
***n*!** *See* **factorial**.

Newton's method A formula for finding the zeros of a function (or the roots of an equation) that involves making a first guess, then using the formula $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$ repeatedly to improve on the approximation of the zero. (p. 428)

null factor law If $ab = 0$, then either $a = 0$, $b = 0$ or $a = b = 0$. (p. 14)

one-to-one function A function such that each value in the range has only one ordered pair (x, y); each y value matches with only one x value. (p. 41)

parabola The graph of a quadratic function $y = ax^2 + bx + c$. (p. 64)



parameter A quantity that influences the behaviour of a mathematical function. (p. 134)

Pascal's triangle A triangular pattern of numbers, where the values in each row are the sums of the two numbers in the row above it. (p. 206)

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

perfect square $(a+b)^2$ or $(a-b)^2$. $(a+b)^2$ expands to $a^2+2ab+b^2$, and $(a-b)^2$ expands to $a^2-2ab+b^2$. (p. 7)

period The horizontal distance a circular function moves before repeating itself. The period of $y=\sin(x)$ and $y=\cos(x)$ is 2π. (p. 347) See **amplitude** for diagram.

permutation An ordered arrangement of a group of items; for example, the first three placegetters in a race. ${}^nP_r$ represents the number of possible arrangements of r items chosen from n items. ${}^nP_r=\dfrac{n!}{(n-r)!}$. (p. 199)

point–gradient form (of a linear equation) $y-y_1=m(x-x_1)$, where m is the gradient and the line passes through the point (x_1, y_1). (p. 59)

point of inflection A point on a graph where the curve changes concavity; for example, the point (0, 0) on the graph of $y=x^3$. (p. 45)

polynomial An expression involving a sum of terms of the form $P(x)=a_nx^n+a_{n-1}x^{n-1}+a_{n-2}x^{n-2}+\ldots+a_1x^1+a_0$, where n is a positive whole number and $a_0, a_1, a_2 \ldots a_{n-1}, a_n$ are real numbers and $a_n \neq 0$. (p. 78)

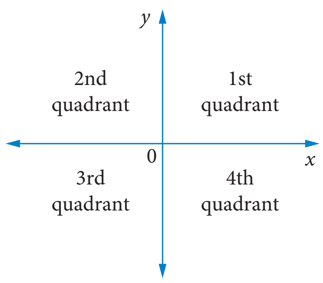
power function A function of the form x^n, where n is a real number. (p. 133)

primitive *See* **integral**.

probability *See* **theoretical probability**.

Pythagorean identity The trigonometric identity $\sin^2(\theta)+\cos^2(\theta)=1$. (p. 336)

quadrant One-quarter of the number plane, bounded by the x- and y-axes. For example, quadrant 1 is the region enclosed by the positive x-axis and the positive y-axis. (p. 335)



quadratic equation An equation of the form $ax^2+bx+c=0$, involving the variable squared. (p. 14)

quadratic expression An algebraic expression where the highest power of the variable is 2; for example, $-5x^2+2x+1$. (p. 6)

quadratic formula The formula $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$ that gives the solutions of the quadratic equation $ax^2+bx+c=0$. (p. 14)

quadratic function A function of the form $y=ax^2+bx+c$, where the power of x is 2. The graph of a quadratic function is a parabola. (p. 64)

quadratic polynomial A polynomial of degree 2. (p. 79)

quartic polynomial A polynomial of degree 4. (p. 79)

radian A measure of angle defined as arc lengths around the circumference of a circle: π radians = 180°. (p. 327)

random variable A variable, X, whose value is determined by the outcome of a random experiment. (p. 187)

range The set of values of the dependent variable y in a relation or function $y=f(x)$. (p. 39)

rational root theorem A rule for finding the rational roots (zeros) of a polynomial using factors of the constant term and the leading coefficient. (p. 100)

reference angle A first quadrant solution for a trigonometric equation like $\cos(nx)=c$ where the sign of c is ignored. (p. 413)

reflection Mirror-image or 'flipping' of a graph so that it is back-to-front or bottom-to-top. (p. 146)

relation A set of ordered pairs (x, y) that describe a relationship between the variables x and y. (p. 39)

remainder theorem If a polynomial $P(x)$ is divided by $x-a$, the remainder is $P(a)$. (p. 91)

root A value of an equation which, when substituted into the equation, satisfies the equation. It is a solution of the equation. (p. 11)

sample space The set of all possible outcomes in a probability experiment. (p. 184)

secant A straight line passing through two points on a curve. (p. 227)

simple event An event that has a single outcome, like getting a tail on the toss of one coin or rolling a number greater than 3 on the roll of one die. Different from **compound event**. (p. 187)

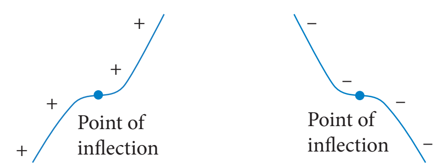
simultaneous equations A set of equations that are all satisfied by the same values. Graphically, the solution is the point of intersection between the graphs of the equations. (p. 22)

speed Rate of change of distance with respect to time. (p. 244)

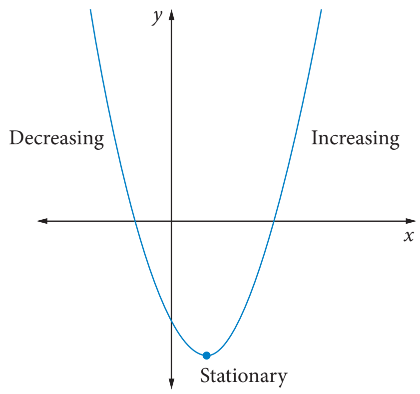
square matrix A matrix that has the same number of rows and columns. (p. 162)

stationary point A point on a graph where the gradient equals zero. The graph is flat there, neither increasing nor decreasing. $\dfrac{dy}{dx}=f'(x)=0$. A stationary point is either a **turning point** (local maximum or minimum point) or a **stationary point of inflection**. (pp. 45, 449) See **strictly decreasing** for diagram.

stationary point of inflection A stationary point on a graph where the concavity of the graph changes: from concave down to concave up, or concave up to concave down. Also, the sign of the gradient stays the same on both sides. (p. 449)



strictly decreasing A function is strictly decreasing when its y values are getting smaller and its graph is moving downwards to the right. If $a<b$, then $f(a)>f(b)$. (p. 445)



strictly increasing A function is strictly increasing when its y values are getting larger and its graph is moving upwards to the right. If $a < b$, then $f(a) < f(b)$. (p. 445)

subject (of a formula) The variable on the left-hand-side of a formula, the quantity for which the formula is used. For example, in the formula for the area of a triangle, $A = \frac{1}{2}bh$, the subject is A, the area. (p. 20)

substitution method An algebraic method of solving simultaneous equations by making one of the variables x or y the subject of one equation and then substituting it into the other equation. (p. 23)

sum of two cubes The expression $a^3 + b^3$ that can be factorised into $(a + b)(a^2 - ab + b^2)$. (p. 97)

synthetic division A method of dividing one polynomial by another, using a repeated process on a table of values, based on **long division**. (p. 88)

tangent A straight line that touches a curve at only one point. (p. 227) *See* **gradient** for diagram.

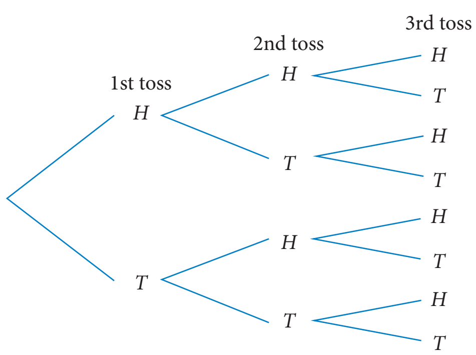
theoretical probability The probability of an event occurring based the knowledge of a situation. (p. 184)

$$\Pr(E) = \frac{\text{number of favourable outcomes for event } E}{\text{number of possible outcomes}}$$

transformation A change to a function and its graph through **dilation**, **reflection** or **translation**.
(p. 144)

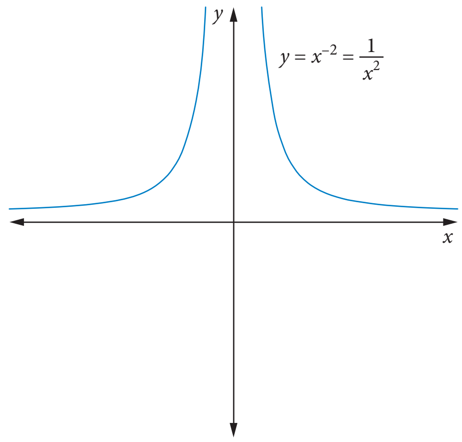
translation Shifting of a graph in the direction of the x-axis (horizontally) or y-axis (vertically). (p. 147)

tree diagram A diagram of branches for listing all of the possible outcomes of a multi-step chance experiment. (p. 299)

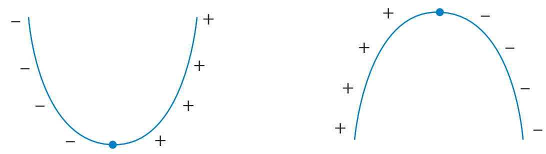


trigonometric function *See* **circular function**.

truncus A discontinuous graph of an equation of the form $y = \frac{a}{(x + b)^2}$. (p. 139)



turning point A stationary point on a graph where the sign of the gradient changes on either side. If it changes from negative (decreasing) to positive (increasing), it is a **local minimum point**. If it changes from positive (increasing) to negative (decreasing), it is a **local maximum point**.
(pp. 45, 449)

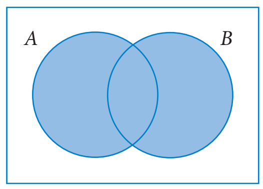


Local minimum point

Local maximum point

two-way table (or **Karnaugh map**) A table that shows amounts or probabilities belonging to two intersecting sets.
(p. 295)

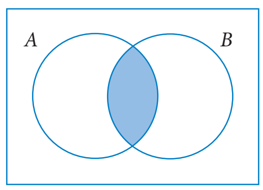
union of sets A set combining the elements of two or more sets. On a Venn diagram, $A \cup B$ are sets A and B combined.
(p. 292)



unit circle A circle of radius 1 unit on the number plane, used to define the circular (trigonometric) function of values of any size (p. 335)

velocity The 'signed speed' of a moving object, represented by the function $v = \frac{dx}{dt}$, the rate of change of the displacement, x. Conversely, $x = \int v(t)\,dt$. (pp. 244, 441)

Venn diagram A diagram of circles (usually overlapping) showing how the elements of a set are related. (p. 292)



zeros The x values of a function $f(x)$ where $f(x) = 0$. (p. 45)