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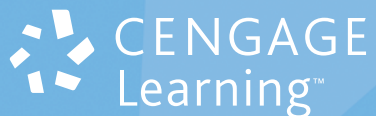
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NELSON SENIOR MATHS

FOR THE AUSTRALIAN CURRICULUM | **SPECIALIST 12**



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NELSON SENIOR MATHS

FOR THE AUSTRALIAN CURRICULUM | **SPECIALIST 12**

Nelson Senior Maths for the Australian Curriculum Specialist 12
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ABOUT THIS SERIES

There are eight books for this series currently. These cover the subjects Essential Mathematics, General Mathematics, Mathematical Methods and Specialist Mathematics.

These books have all been written specifically, from scratch, by a national author team for the Senior Australian Curriculum.

There are worked solutions available for purchase, which accompany each of the Student Books.

There are also ExamView question banks that can be purchased separately. These will contain a large number of multiple choice questions covering each course; General, Methods and Specialist. An ExamView test generator is also provided, so that teachers can construct their own tests for each topic.

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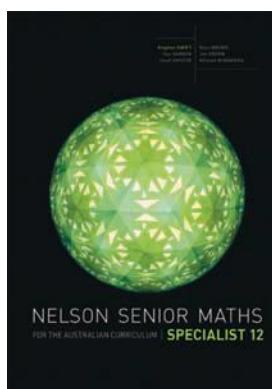
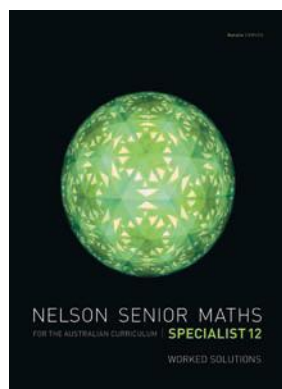
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ABOUT THIS BOOK



A clear outline of chapter contents is provided.

Links to curriculum content descriptions are included.

A Prior learning worksheet is provided, enabling teachers to assess prerequisite understanding.

○ **Example 2**

Given the points $P(-1, 3, 1)$ and $Q(3, 4, -2)$,

a find the distance between P and Q
b find the midpoint of PQ .

Solution

a Write the 3D distance formula.

Substitute for x_1, y_1 , etc.

Evaluate.

b Write the midpoint formula.

Substitute for x, y and z values.

Evaluate.

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

$$= \sqrt{(3+1)^2 + (4-3)^2 + (-2-1)^2}$$

$$= \sqrt{26}$$

$$M_{PQ} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2} \right)$$

$$= \left(\frac{-1+3}{2}, \frac{3+4}{2}, \frac{1-2}{2} \right)$$

$$= \left(1, 3\frac{1}{2}, -\frac{1}{2} \right)$$

Examples are clearly set out with reasoning and writing in separate columns.

Examples are sequenced in a logical order.

There are generally three, and no more than four, examples leading to an exercise.

Examples show solutions and steps that guide students through the use of the TI-Nspire CX and the CASIO ClassPad CAS calculators.

EXERCISE 3.01 Composition of functions

Concepts and techniques

- 1** Find $g(f(x))$ given that:
 - $g(x) = 2x - 3, f(x) = x - 1$
 - $g(x) = \log(x), f(x) = x^2$
 - $g(x) = x^2 + 2, f(x) = e^x$
 - $g(x) = \sin(x), f(x) = 2x$
- 2** Find $f(g(x))$ for each of the following and compare with $g(f(x))$ in question 1.
 - $g(x) = 2x - 3, f(x) = x - 1$
 - $g(x) = x^2 + 2, f(x) = e^x$
 - $g(x) = \log(x), f(x) = x^2$
 - $g(x) = \sin(x), f(x) = 2x$
- 3** Decompose the following functions into the form $g(f(x))$.
 - $h(x) = x + 3$
 - $h(x) = (2x + 5)^3$
 - $h(x) = x^2 + 6x + 2$
 - $h(x) = e^{x^2}$
 - $h(x) = \sin^2(x)$
 - $h(x) = \sin(2x)$

Reasoning and communication

- For the function $h(x) = 4x^2$, write two possible decompositions.
- For the following functions, find $g \circ f$ and write the domain and range of each.
 - $f(x) = 2x, g(x) = \sin(x)$
 - $f(x) = -x, g(x) = \log(x)$
 - $f(x) = \frac{1}{2x}, g(x) = 2 \sin(x)$
- Write a function that satisfies the property $g \circ f = f \circ g$.

3.02 ONE-TO-ONE FUNCTIONS

INVESTIGATION One-to-one functions

The following functions are one-to-one.

- Copy each of the graphs above into your book.
- Draw a horizontal line across each of these graphs.
- How many times does the horizontal line pass through the graph?
- Now draw a vertical line at any point on the graphs.
- How many times does the vertical line pass through the graph?

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Exercise questions are clearly divided into two categories: Concepts and techniques and Reasoning and communication.

Where appropriate, worksheets are provided for additional practice and consolidation of key concepts.

Each chapter contains at least one investigation, providing students with the opportunity to apply their understanding to a practical application.

CHAPTER SUMMARY 1

THREE-DIMENSIONAL VECTORS

- In a three-dimensional coordinate system, the position of a point is defined by an ordered triple a in $\mathbb{R}^3$, $a(x, y, z)$.
- The Cartesian (or rectangular) plane in three dimensions has three planes – the xy -plane and xz -plane. These planes partition space into eight octants.
- The distance formula in three dimensions (3D distance formula) states that for $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$
- The midpoint of the line segment from $P_1(x_1, y_1, z_1)$ to $P_2(x_2, y_2, z_2)$ is:

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2} \right)$$
- The equation of a sphere with centre $C(h, k, l)$ and radius, r , is:

$$(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2$$
 or $x^2 + y^2 + z^2 + ax + by + cz + d = 0$
- Properties of vectors:
 - If $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$ are vectors and c is a scalar:

$$a + b = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$$

$$c \cdot a = (ca_1, ca_2, ca_3)$$

$$a = (a_1, a_2, a_3)$$
- When vectors are added, the sum is called the resultant vector.
- A vector $a = (a_1, a_2, a_3)$ can be represented in the three-dimensional Cartesian system by a directed line segment $\overrightarrow{OQ}$ (or $\overrightarrow{OQ}$ with initial point (or head) $O(x_1, y_1, z_1)$ and terminal point (or tail) $Q(x_2, y_2, z_2)$, $a = \overrightarrow{OQ}$).
- The vector $\overrightarrow{OA}$ from the origin, $O(0, 0, 0)$, to $A(x_1, y_1, z_1)$ is called the position vector of A . The lower case vector symbol is also used for the position vector, so $\overrightarrow{OA} = a$.
- If $\overrightarrow{PQ} = \overrightarrow{RS}$, then $\overrightarrow{PQ}$ and $\overrightarrow{RS}$ have the same magnitude and direction. $\overrightarrow{PQ} = c\overrightarrow{RS}$, where c is a scalar, then $\overrightarrow{PQ} \parallel \overrightarrow{RS}$.
- For any two points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$:

$$\overrightarrow{P_1P_2} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$$
- The magnitude of a three-dimensional vector is called the norm of the vector. For vector a , the norm is written $|a|$. We can write $|a|$ as a .
- If $a = (a_1, a_2, a_3)$ and c is a scalar:

$$|a| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

$$|ca| = |c|a|$$
- a and vector has magnitude 1. The unit vector in the same direction as a given vector p is written as $\hat{p}$:

$$\hat{p} = \frac{1}{|p|}p$$
- The unit vectors in space are:

$$\hat{i} = (1, 0, 0)$$

$$\hat{j} = (0, 1, 0)$$

$$\hat{k} = (0, 0, 1)$$
- For any vector, $a = (a_1, a_2, a_3)$:

$$a = (a_1, a_2, a_3) = (a_1\hat{i} + a_2\hat{j} + a_3\hat{k})$$

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CHAPTER REVIEW 1

THREE-DIMENSIONAL VECTORS

Multiple choice

- 1** Which of the following points are on a line parallel to the y -axis?
 - $(-1, 3, 2)$
 - $(-1, -5, -3)$
 - $(2, 4, -3)$
 - $(-1, 4, 3)$
- 2** The distance between $M(p, q, r)$ and $N(p, q, r)$ is:
 - $|r - p|$
 - $|p - r|$
 - $|r - q|$
 - $|p - q|$

Use the diagram below to answer questions 3–4.

- 3** The point lying on the xz -plane is:
 - A
 - B
 - C
 - D
 - E
- 4** The point F is best described as lying on the:
 - xz -plane
 - xy -plane
 - yz -plane
- 5** Find the resultant of vectors p and q , where p has magnitude 25, azimuth 120° and altitude -30° , while q has magnitude 30, azimuth 60° and altitude 65° .
- 6** The dot product of $-5\hat{i} + 2\hat{j} + \hat{k}$ and $3\hat{i} - 4\hat{j} - 7\hat{k}$ is:
 - 30
 - 16
 - 12
 - 0
 - 16

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Each chapter concludes with a detailed chapter summary and a comprehensive review of concepts.

At the end of every three chapters, a comprehensive mixed revision set is provided.

ABOUT THE AUTHORS

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Stephen Swift was the lead author of the Nelson Think Maths series. He has also written as a member of an author team on successful senior mathematics textbooks. Stephen started teaching Mathematics, Science and Computing in 1973 and has taught at all levels from Year 7 to 12 in several states, in urban and country schools until retiring in 2010 from the role of Mathematics Head of Department at Wellington Point State High School in Brisbane.

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The **Digital Resources Team** includes Margaret Denham, Issam Kanaan and Roger Walter.

SYMBOLS AND ABBREVIATIONS

$\approx$	approximately equal to	$\mathbb{R}/0$ or $\mathbb{R}/\{0\}$	means the real numbers except 0
$=$	equal to	$\mathbb{C}, \mathbb{C}$	the set of complex numbers
$\equiv$	identically equal to, congruent	$\in$	is a member of
$\neq$	not equal to	$\notin$	is not a member of
$<$	less than	$\mathbf{p}, \underline{p}$	the vector $\mathbf{p}$
$>$	greater than	$ \mathbf{p} , p$	the magnitude (norm) of the vector $\mathbf{p}$
$\leq$	less than or equal to	$\overline{AB}$	displacement vector from point A to point B
$\geq$	greater than or equal to	$\overrightarrow{OA}, \mathbf{a}, \overrightarrow{OA}, \overline{OA}$	the position vector of the point A
$\pm$	plus or minus	$(r, \theta), \begin{bmatrix} r \\ \theta \end{bmatrix}$	polar form of the vector with magnitude r and direction θ
$\sqrt{\quad}$	square root	$(x, y), \begin{bmatrix} x \\ y \end{bmatrix}$	component (Cartesian) form of vector
$*$	a general operation	$-\mathbf{a}$	additive inverse of the vector $\mathbf{a}$
Σ	summation	$\hat{\mathbf{p}}$	unit vector in the direction of the vector $\mathbf{p}$
$\Rightarrow$	if ... then, implies, so $P \Rightarrow Q$ means if P then Q or P implies Q	$\mathbf{i}, \mathbf{j}, \mathbf{k}$	unit vectors in the x, y and z directions
$\Leftrightarrow$, iff	if and only if, so $P \Leftrightarrow Q$ means P if and only if Q	$\Delta \mathbf{v}$	change in the vector $\mathbf{v}$
$\overline{Q}, \neg Q$	Q' not Q	$\mathbf{a} \cdot \mathbf{b}$	the scalar product (dot product) of the vectors $\mathbf{a}$ and $\mathbf{b}$
$, \sim$	similar	$\mathbf{a} \times \mathbf{b}$	the vector product (cross product) of the vectors $\mathbf{a}$ and $\mathbf{b}$
$ $	parallel	$\frac{d\mathbf{r}}{dt}, \dot{\mathbf{r}}(t)$	the derivative of the vector $\mathbf{r}$ with respect to time t
$\perp$	perpendicular	$\frac{d^2\mathbf{r}}{dt^2}, \ddot{\mathbf{r}}(t)$	the second derivative of the vector $\mathbf{r}$ with respect to time t
RTP	required to prove	ω	angular velocity
QED	quod erat demonstrandum, indicating a proof is complete	080°	bearing of 80°
$\forall$	for all	$^\circ$	degree
$\exists$	there exists	$\angle$	angle
LHS	left-hand side	$\triangle$	triangle
RHS	right-hand side		
$\mathbb{N}, \mathbb{N}$	the set of natural numbers, $\{1, 2, 3, 4, \dots\}$		
$\mathbb{Z}, \mathbb{Z}$ or $\mathbb{I}$	the set of integers, $\{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$		
$\mathbb{Q}, \mathbb{Q}$	the set of rational numbers		
$\mathbb{Q}'$ or $\overline{\mathbb{Q}}$	the set of irrational numbers		
$\mathbb{Q}^+$	the set of positive rational numbers		
$\mathbb{R}, \mathbb{R}$	the set of real numbers		

SSS	side, side, side test of triangle congruency	$\sigma_{\bar{x}}$	the standard deviation of the sampling distribution of X
SAS	side, angle, side test of triangle congruency	ϕ	the Golden Ratio
AAS	angle, angle, side test of triangle congruency	$P(n)$	a proposition associated with the number n
RHS	right angle, hypotenuse, side test of triangle congruency	$A, \underline{A}$	the matrix A
!	factorial, so $5! = 5 \times 4 \times 3 \times 2 \times 1$	$A = (a_{ij})_{2 \times 3} = \begin{bmatrix} 3 & -2 & 5 \\ -4 & 1 & 0 \end{bmatrix}$	the 2×3 matrix A with elements a_{ij}
t_n	the n th term of a sequence	$-A$	the additive inverse of the matrix A
${}^n P_r$	the number of permutations of r things from n things	I_n	the $n \times n$ identity matrix
$\binom{n}{r}, {}^n C_r$	the number of combinations of r objects from n objects	δ_{ij}	the Kronecker delta function
$Pa_{n,r}$	element number r in row number n of Pascal's triangle	A^{-1}	the multiplicative inverse of the (square) matrix A
$A = \{2, 4, 6, 8\}$	the set A consisting of the numbers 2, 4, 6 and 8	$\det A, A , A$	the determinant of the matrix A
$A = \{x: 0 < x < 10 \text{ and } x \text{ an even number}\}$	the set A of elements x such that $0 < x < 10$ and x is an even number	$T: (x, y) \rightarrow (x', y')$	the transformation that changes the point (x, y) to (x', y')
$\subseteq$	is a subset of	D	a dilation
$\subset$	is a proper subset of	λ_1, λ_2	the dilation factors in the x and y directions for a dilation
$\cap$	intersection	R_α	a rotation around the origin through an angle α
$\cup$	union	M	a reflection across a line through the origin
$\{\}, \emptyset$	null set (empty set)	$S \circ T$	the composition of the transformations S and T
U	universal set	T^{-1}	the inverse of the transformation T
$\overline{A}, A'$	the complement of the set or event A	π (pi)	about 3.14159...
$ A $	the cardinality of the set A , for finite sets, the number of elements	$\sin(\theta)$	the sine of the angle θ
$n(A)$	for finite sets, the number of elements in the set A	$\cos(\theta)$	the cosine of the angle θ
$P(A)$	the probability of the event A	$\tan(\theta)$	the tangent of the angle θ
pdf	probability density function	$\cot(\theta)$	the cotangent of the angle θ
cdf	cumulative distribution function	$\sec(\theta)$	the secant of the angle θ
$E(X), \mu$	the expected value of the random variable X	$\operatorname{cosec}(\theta)$	the cosecant of the angle θ
$\operatorname{Var}(X)$	the variance of the random variable X	$x \rightarrow \infty$	x becomes large and positive (x approaches infinity)
$\operatorname{SD}(X), \sigma$	the standard deviation of the random variable X	$x \rightarrow -\infty$	x becomes large and negative (x approaches negative infinity)
$\bar{x}, s$	the mean and standard deviation of a sample	$\arcsin(a), \sin^{-1}(a)$	the angle $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ such that $\sin(\theta) = a$
x_α, t_α	the quantile of the proportion α		

$\arccos(a)$, $\cos^{-1}(a)$	the angle $0 \leq \theta \leq \pi$ such that $\cos(\theta) = a$	$\bar{z}$	the complex conjugate of z
$\arctan(a)$, $\tan^{-1}(a)$	the angle $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ such that $\tan(\theta) = a$	$\arg(z)$	the argument of z
i	the imaginary number $(\sqrt{-1})$	$ z $, $\text{mod}(z)$	the modulus of the complex number z
$\text{Re}(z)$	the real part of the complex number z	$\text{cis}(\theta)$	$\cos(\theta) + i \sin(\theta)$
$\text{Im}(z)$	the imaginary part of the complex number z	$ x $	the absolute value of x
		$f^{-1}(x)$	the inverse function of $f(x)$
		$gf(x)$, $(g \circ f)(x)$	the composite function $g[f(x)]$

GREEK ALPHABET

A, α	alpha	I, ι	iota	P, ρ	rho
B, β	beta	K, κ	kappa	Σ , σ	sigma
Γ , γ	gamma	Λ , λ	lambda	T, τ	tau
Δ , δ	delta	M, μ	mu	Y, υ	upsilon
E, ϵ	epsilon	N, ν	nu	Φ , ϕ	phi
Z, ζ	zeta	Ξ , ξ	xi	X, χ	chi
H, η	eta	O, $\omicron$	omicron	Ψ , ψ	psi
Θ , θ	theta	Π , π	pi	Ω , ω	omega

ICONS IN THE TEXTBOOK

Example 1

Link from question to worked example



Prior learning

CAS

CAS exercise and example



Worksheets

Worksheet



Weblink

Weblink



Spreadsheet

Interactive spreadsheet



Practice quiz

Practice quiz





TERMINOLOGY

altitude angle
azimuth angle
consistent
cross product
dependent
dot product
Gaussian elimination
homogeneous
inconsistent
independent
inner product
linear equation
non-homogeneous
norm
ordered triple
outer product
polar form
projection
resultant vector
scalar product
system of equations
unit vector
vector product
work

VECTORS IN THREE DIMENSIONS

THREE-DIMENSIONAL VECTORS

- 1.01 Three-dimensional Cartesian coordinates
- 1.02 Three-dimensional vectors
- 1.03 The scalar product
- 1.04 The vector product
- 1.05 Cartesian geometry
- 1.06 Lines in 3D
- 1.07 Equations of planes
- 1.08 Systems of linear equations

Chapter summary

Chapter review



Prior learning

THE ALGEBRA OF VECTORS IN THREE DIMENSIONS

- review the concepts of vectors from Unit 1 and extend to three dimensions including introducing the unit vectors i , j and k (ACMSM101)
- prove geometric results in the plane and construct simple proofs in three-dimensions. (ACMSM102)

VECTOR AND CARTESIAN EQUATIONS

- introduce Cartesian coordinates for three-dimensional space, including plotting points and the equations of spheres (ACMSM103)
- use vector equations of curves in two or three dimensions involving a parameter, and determine a 'corresponding' Cartesian equation in the two-dimensional case (ACMSM104)
- determine a vector equation of a straight line and straight-line segment, given the position of two points or equivalent information, in both two and three dimensions (ACMSM105)
- examine the position of two particles each described as a vector function of time, and determine if their paths cross or if the particles meet (ACMSM106)
- use the cross product to determine a vector normal to a given plane (ACMSM107)
- determine vector and Cartesian equations of a plane and of regions in a plane. (ACMSM108)

SYSTEMS OF LINEAR EQUATIONS

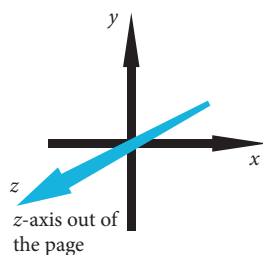
- recognise the general form of a system of linear equations in several variables, and use elementary techniques of elimination to solve a system of linear equations (ACMSM109)
- examine the three cases for solutions of systems of equations – a unique solution, no solution and infinitely many solutions – and the geometric interpretation of a solution of a system of equations with three variables. (ACMSM110)



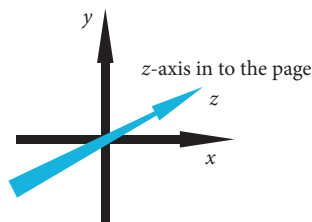
1.01 THREE-DIMENSIONAL CARTESIAN COORDINATES

You need two coordinates for points in two dimensions. For three dimensions, you need three coordinates, so you need a third axis. You can use an x -axis, y -axis and z -axis and write points as **ordered triples** like $P(x, y, z)$.

There are two possible directions in which you can put the z -axis, as shown below for an x -axis across the page and y -axis up the page. The z -axis can come out of the page or go into the page.



Right-handed system

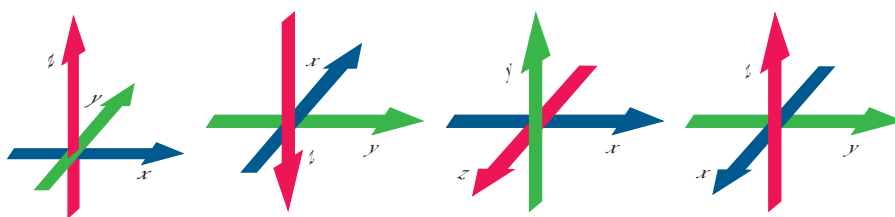


Left-handed system

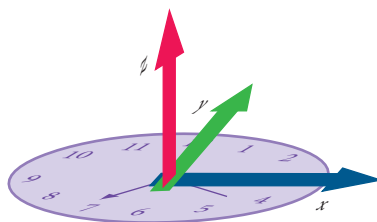
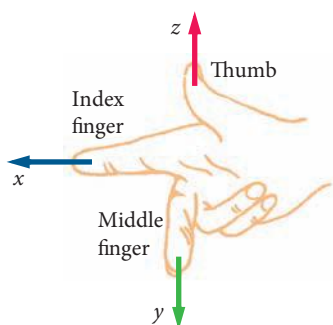


You can see why they are called the left-handed and right-handed systems. It is conventional to use a **right-handed** system. If you curl the fingers of your right hand from the x -axis to the y -axis, your thumb points in the direction of the z -axis.

As long as the relationship between the axes is the same, you can point the x -axis in any direction and still have a right-hand system.

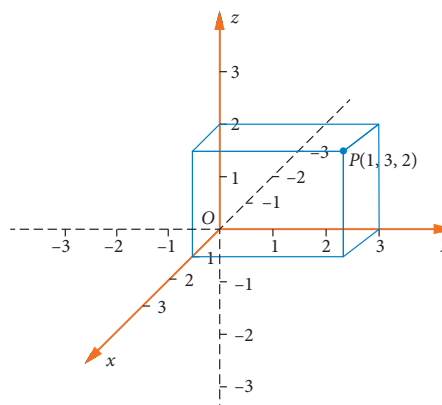


Many people like to make their index ('pointer') finger the x -axis and their middle finger ('tall-man') the y -axis so their thumb is the z -axis like the drawing below. You can also think of it in terms of a clock. If the x -axis is rotated *anticlockwise* to the y -axis, the z -axis comes out of the clock-face. Remember that angles are positive in the anticlockwise direction.



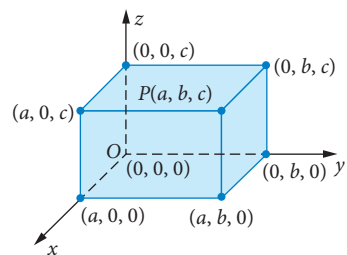
The point P in the diagram on the right has coordinates $(1, 3, 2)$, so it is:

- 1 unit in the positive direction of the x -axis
- 3 units in the positive direction of the y -axis
- 2 units in the positive direction of the z -axis



The point P is intersected by three planes – each one parallel to a coordinate plane. You can imagine P as one vertex of a rectangular box whose faces are each parallel to one of the coordinate planes.

When you plot points in 3D, you'll find a box helpful.

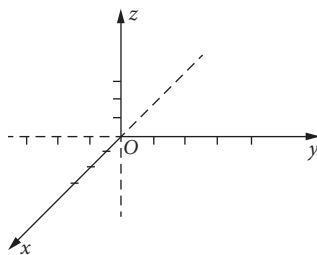


○ Example 1

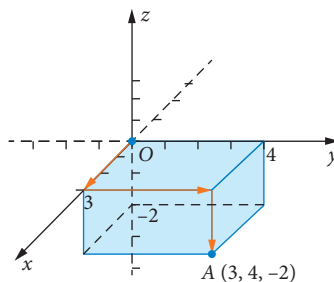
Plot the points $A(3, 4, -2)$ and $B(-1, -3, 1)$.

Solution

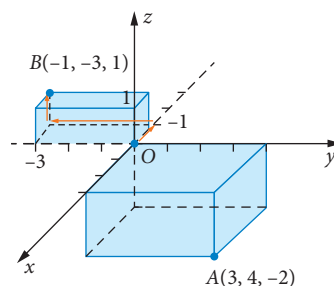
Draw a set of axes for the Cartesian space.



To locate the point $A(3, 4, -2)$, move 3 units in the positive direction of the x -axis, 4 units in the positive direction of the y -axis and 2 units in the negative direction of the z -axis.



To locate the point $B(-1, -3, 1)$, move 1 unit in the negative direction of the x -axis, 3 units in the negative direction of the y -axis and 1 unit in the positive direction of the z -axis.



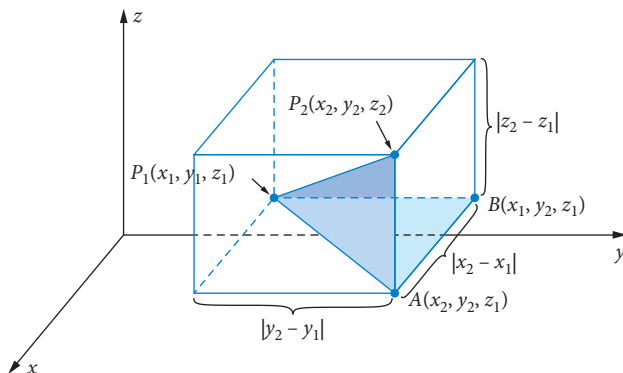


Points in 3D space

The distance formula to calculate the distance between any two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ in 2D is $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

You can develop a similar formula for the distance between any two points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ in 3D.

Construct the points $A(x_2, y_2, z_1)$ and $B(x_1, y_2, z_1)$ as shown in the diagram below.



Then $\triangle P_1BA$ is right-angled, so by Pythagoras' theorem,

$$\begin{aligned} P_1A^2 &= P_1B^2 + BA^2 \\ &= (x_2 - x_1)^2 + (y_2 - y_1)^2. \end{aligned}$$

But $\triangle P_1AP_2$ is also right-angled so

$$\begin{aligned} P_1P_2^2 &= P_1A^2 + AP_2^2 \\ &= (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 \end{aligned}$$

So for 3D, you get the formula below.

IMPORTANT

The distance between the points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ in three dimensions is given by:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

It should be noted that if P_1 and P_2 are on the xy -plane, $z_1 = z_2 = 0$, so the 3D distance formula reduces to the distance formula in two dimensions.

The midpoint formula in two dimensions for $P_1(x_1, y_1)$ to $P_2(x_2, y_2)$ is:

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$$

You can extend this to 3D as shown below.

IMPORTANT

3D midpoint formula

The midpoint of the line segment from $P(x_1, y_1, z_1)$ to $Q(x_2, y_2, z_2)$ is:

$$M_{PQ} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2} \right)$$

○ Example 2

Given the points $P(-1, 3, 1)$ and $Q(3, 4, -2)$,

- find the distance between P and Q
- find the midpoint of PQ .

Solution

- Write the 3D distance formula.

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Substitute for x_1, y_1 , etc.

$$= \sqrt{(3+1)^2 + (4-3)^2 + (-2-1)^2}$$

Evaluate.

$$= \sqrt{26}$$

- Write the midpoint formula.

$$M_{PQ} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2} \right)$$

Substitute for x, y and z values.

$$= \left(\frac{-1+3}{2}, \frac{3+4}{2}, \frac{1-2}{2} \right)$$

Evaluate.

$$= \left(1, 3\frac{1}{2}, -\frac{1}{2} \right)$$

In the two-dimensional Cartesian system, the equation of a circle is calculated using the fact that any point $P(x, y)$ on the circumference is the same distance r from the centre. If the centre is $C(h, k)$, you get $\sqrt{(x-h)^2 + (y-k)^2} = r$ so $(x-h)^2 + (y-k)^2 = r^2$.

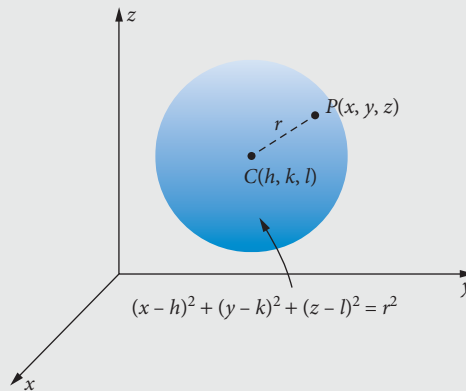
In three dimensions, points that are the same distance from a centre $C(h, k, l)$ will not be a line. They will form the surface of a sphere. For a radius r , $r = \sqrt{(x-h)^2 + (y-k)^2 + (z-l)^2}$ so $(x-h)^2 + (y-k)^2 + (z-l)^2 = r^2$

Equation of a sphere

The equation of a sphere with centre $C(h, k, l)$ and radius, r , is

$$(x-h)^2 + (y-k)^2 + (z-l)^2 = r^2$$

or $x^2 + y^2 + z^2 + ax + by + cz + d = 0.$



Example 3

- a Find the equation of a sphere with centre $(1, -2, 3)$ and radius 5.
 b Find the radius and centre of a sphere with the equation $x^2 + y^2 + z^2 - 6x + 8y + 4z + 4 = 0$

Solution

- a Write the formula.

$$(x-h)^2 + (y-k)^2 + (z-l)^2 = r^2$$

Substitute in the values.

$$(x-1)^2 + (y+2)^2 + (z-3)^2 = 5^2$$

- b Rewrite the equation.

$$x^2 + y^2 + z^2 - 6x + 8y + 4z = -4$$

Complete the squares.

$$(x^2 - 6x + 9) + (y^2 + 8y + 16) + (z^2 + 4z + 4) = -4 + 9 + 16 + 4$$

Factorise the squares.

$$(x-3)^2 + (y+4)^2 + (z+2)^2 = 5^2$$

Write the answer.

The centre is $(3, -4, -2)$ and the radius is 5.

In part a above, you would not bother expanding the brackets unless you were asked for the expanded form.

INVESTIGATION A sphere on a CAS calculator

Consider the sphere with centre $(0, 0, 0)$ and radius 5. The equation of the sphere is:

$$x^2 + y^2 + z^2 = 5^2$$

In order to graph the sphere using a calculator, you need to rearrange the equation so that it is written in terms of z . Hence, the equation becomes:

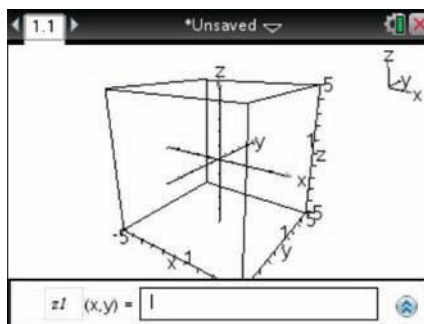
$$z = \pm\sqrt{25 - (x^2 + y^2)}$$

This needs to be graphed as two separate functions:

$$z_1 = \sqrt{25 - (x^2 + y^2)} \quad \text{and} \quad z_2 = -\sqrt{25 - (x^2 + y^2)}$$

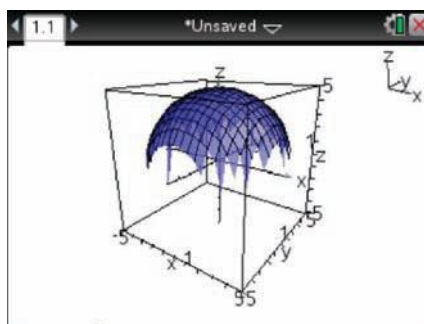
TI-Nspire CAS

Enter the graphing screen and choose 3D graphing by pressing **menu** and then selecting 2:View, 3:3d Graphing.

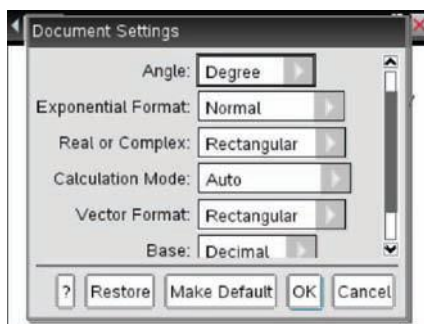


Next enter the equation $\sqrt{25 - (x^2 + y^2)}$ and press ENTER.

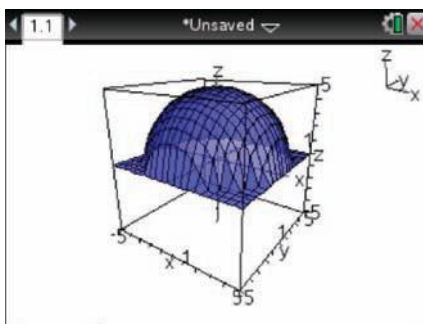
The graph does not look like a hemi-sphere. You need to adjust the calculator settings and the equation to rectify this.



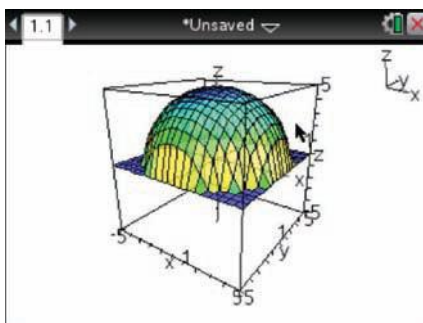
Press the 'battery' symbol to select the calculator settings. Make sure the 'Real or Complex' and 'Vector Format' settings are both 'Rectangular'.



Next you need to edit the equation. Right-click on the graph. You can do this by moving the cursor over the graph and pressing **[ctrl] [menu]**. Select 6:Edit Relation and change the relation to real ($\sqrt{25-(x^2+y^2)}$) and press OK. Now the graph looks like a hemi-sphere.



You can change the appearance of the graph to accentuate the 3D nature of the shape by right-clicking on the graph (**[ctrl] [menu]**) and selecting 8:Colour, then 4:Custom Plot Colour.

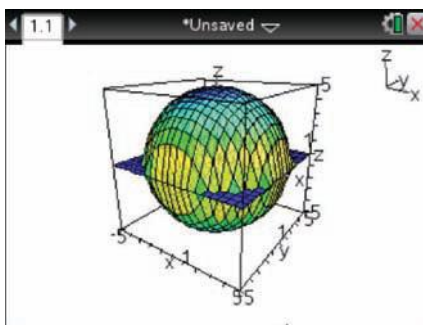


Now enter the equation for the other hemi-sphere by pressing **[ctrl] [G]** and then enter:

[(-)] z [1] [(] x [, y]]

Press ENTER.

Change the colour of this graph to match the other hemisphere.



ClassPad

3D graphs are not possible.

Experiment with the resulting graph.

- The graph can be rotated using the arrow keys.
- Pressing 'a' will auto-rotate the graph.
- The orientation of the graph can be changed by pressing 'x', 'y', 'z' or 'o'.

The magnification of the graph can be increased or decreased using the **[x]** and **[÷]** keys respectively.

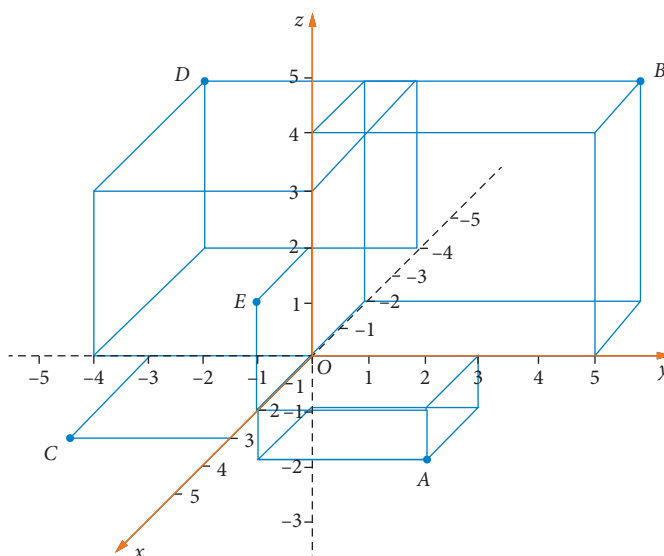
EXERCISE 1.01 Three-dimensional Cartesian coordinates



Distances, midpoints and spheres

Concepts and techniques

- Example 1** $P(2, 3, -1)$ and which of the following points are on a line parallel with the z -axis?
 A $(2, -3, -1)$ B $(4, 3, -1)$ C $(0, 3, -1)$ D $(2, 3, 4)$ E $(2, 0, -1)$
 - The distance between $A(a, b, c)$ and $P(a, d, c)$ is:
 A $|c - a|$ B $|d - b|$ C $|c - b|$ D $|d - a|$ E $|d - c|$
- Use the diagram below to answer questions 3–5.



- The point lying on the xy -plane is:
 A A B B C C D D E E
- Point E is lying on the:
 A xz -plane B x -plane C xy -plane D z -plane E yz -plane
- State the coordinates of A, B, C, D and E.
- Plot the points A and B in each case below.

a $A(4, -2, 3), B(2, 4, -5)$	b $A(3, -2, 1), B(-4, 0, 1)$	c $A(0, 0, 0), B(4, -1, -8)$
d $A(1, 1, 0), B(0, 5, -4)$	e $A(0, 1, 1), B(1, 0, 0)$	f $A(2, 4, -1), B(1, -2, 7)$

1.02 THREE-DIMENSIONAL VECTORS

A vector from the origin to a point is called a **position vector**. You should remember that it is equal to any vector with the same size and direction. It is easier to convert a vector from one form to another when it is treated as a position vector. You should recall the following about 2D vectors from last year.

IMPORTANT

2-dimensional vectors may be written in **component form**, **polar form** or in terms of the **unit vectors $\mathbf{i}$ and $\mathbf{j}$** in the x and y directions. For a **position vector $\mathbf{a}$** from the origin to the

point $A(a_1, a_2)$, $\mathbf{a} = (a_1, a_2)$, $\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$, $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$ or $\mathbf{a} = (r, \theta)$, where

$$x = r \cos(\theta), y = r \sin(\theta), r = \sqrt{a_1^2 + a_2^2}, \tan(\theta) = \frac{y}{x}, \mathbf{i} = (1, 0) \text{ and } \mathbf{j} = (0, 1).$$

a_1 and a_2 are the **components** of $\mathbf{a}$, θ is the **direction** measured anticlockwise from the x -axis and r is the **norm (magnitude)** and is also written a or $|\mathbf{a}|$.

The unit vector in the direction of $\mathbf{a}$ is written as $\hat{\mathbf{a}}$ and is given by $\hat{\mathbf{a}} = \frac{1}{|\mathbf{a}|} \mathbf{a}$.

You should see that the concepts of position vector, component form, norm and unit vectors are easy to extend to three dimensions, but the polar form requires a second angle.

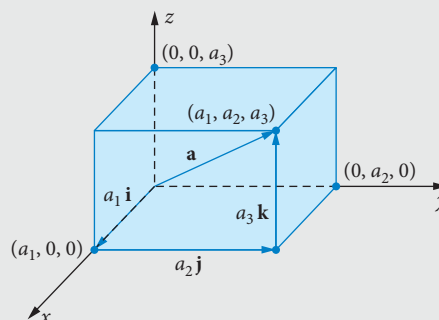
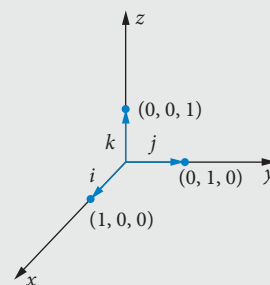
IMPORTANT

3-dimensional vectors may be written in **component form**, or in terms of the **unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$** in the x , y and z directions.

For a **position vector $\mathbf{a} = \mathbf{OA}$** from the origin to the point $A(a_1, a_2, a_3)$, $\mathbf{a} = (a_1, a_2, a_3)$, $\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ or $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, where $\mathbf{i} = (1, 0, 0)$, $\mathbf{j} = (0, 1, 0)$ and $\mathbf{k} = (0, 0, 1)$.

a_1 , a_2 and a_3 are the **components** of $\mathbf{a}$, and the **norm (magnitude)** of $\mathbf{a}$, a or $|\mathbf{a}|$, is calculated as $|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$.

The unit vector in the direction of $\mathbf{a}$ is written as $\hat{\mathbf{a}}$ and is given by $\hat{\mathbf{a}} = \frac{1}{|\mathbf{a}|} \mathbf{a}$.





The properties of addition and multiplication by a scalar for two-dimensional vectors can be extended to three dimensional vectors.

IMPORTANT

If $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ are vectors and c is a scalar:

$$\mathbf{a} + \mathbf{b} = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$$

$$c\mathbf{a} = (ca_1, ca_2, ca_3)$$

$$\mathbf{0} = (0, 0, 0)$$

$$-\mathbf{a} = -(a_1, a_2, a_3)$$

$$\mathbf{a} - \mathbf{b} = (a_1 - b_1, a_2 - b_2, a_3 - b_3)$$

Parallel vectors are scalar multiples of each other.

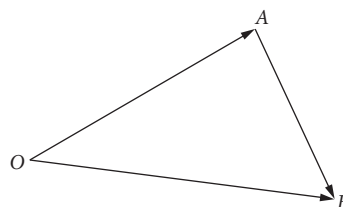
When vectors are added, their sum is called the **resultant vector**.

Consider the position vectors $\mathbf{a} = \mathbf{OA}$, $\mathbf{b} = \mathbf{OB}$ and the **displacement** vector $\mathbf{AB}$ from A to B .

From the diagram, you can see that

$$\mathbf{OA} + \mathbf{AB} = \mathbf{OB},$$

$$\text{so } \mathbf{AB} = \mathbf{OB} - \mathbf{OA} = \mathbf{b} - \mathbf{a}.$$



This is the familiar result that the displacement vector from A to B is the difference between the second position vector and the first position vector. Notice that the diagram assumes nothing about the number of dimensions.

○ Example 4

Given the position vectors $\mathbf{a} = (2, 1, -3)$ and $\mathbf{b} = (-4, 2, 7)$,

a find $2\mathbf{a} - 3\mathbf{b}$

b show that $(\mathbf{a} - \mathbf{b})$ is parallel to $(\mathbf{b} - \mathbf{a})$

c find $\hat{\mathbf{a}}$.

Solution

a Write the expression using the values of $\mathbf{a}$ and $\mathbf{b}$.

$$2\mathbf{a} - 3\mathbf{b} = 2(2, 1, -3) - 3(-4, 2, 7)$$

Expand the brackets.

$$= (4, 2, -6) + (12, -6, -21)$$

Subtract components.

$$= (16, -4, -27)$$

State the result.

$$2\mathbf{a} - 3\mathbf{b} = (16, -4, -27)$$

- b Calculate $\mathbf{a} - \mathbf{b}$.
Subtract components.
Calculate $\mathbf{b} - \mathbf{a}$.
Subtract components.
Take out -1 as a common factor.
Compare $\mathbf{a} - \mathbf{b}$ and $\mathbf{b} - \mathbf{a}$.
State the result.

- c Write the formula for the unit vector.

Use the rule for the norm of a vector.

Evaluate.

Write in the standard form.

$$\mathbf{a} - \mathbf{b} = (2, 1, -3) - (-4, 2, 7) \\ = (6, -1, -10)$$

$$\mathbf{b} - \mathbf{a} = (-4, 2, 7) - (2, 1, -3) \\ = (-6, 1, 10) \\ = -1(6, -1, -10)$$

$$\mathbf{a} - \mathbf{b} = -1(\mathbf{b} - \mathbf{a})$$

$(\mathbf{a} - \mathbf{b})$ is parallel to $(\mathbf{b} - \mathbf{a})$.

$$\hat{\mathbf{a}} = \frac{1}{|\mathbf{a}|} \mathbf{a} \\ = \frac{1}{\sqrt{2^2 + 1^2 + (-3)^2}} (2, 1, -3) \\ = \frac{1}{\sqrt{14}} (2, 1, -3) \\ = \left(\frac{\sqrt{14}}{7}, \frac{\sqrt{14}}{14}, -\frac{3\sqrt{14}}{14} \right)$$

TI-Nspire CAS

- a Enter the vectors as matrices using $\mathbf{a} :=$ (ctrl =) and commas in square brackets or Define and menu , 7 Matrix & Vector, 1: Create and 1: Matrix.
Perform the required operation.

$\mathbf{a} := [2 \ 1 \ -3]$	$[2 \ 1 \ -3]$
Define $\mathbf{b} := [-4 \ 2 \ 7]$	Done
$2 \cdot \mathbf{a} - 3 \cdot \mathbf{b}$	$[16 \ -4 \ -27]$

- b Find the differences to compare them.
Clearly $\mathbf{b} - \mathbf{a} = -1(\mathbf{a} - \mathbf{b})$

$\mathbf{a} := [2 \ 1 \ -3]$	$[2 \ 1 \ -3]$
Define $\mathbf{b} := [-4 \ 2 \ 7]$	Done
$2 \cdot \mathbf{a} - 3 \cdot \mathbf{b}$	$[16 \ -4 \ -27]$
$\mathbf{a} - \mathbf{b}$	$[6 \ -1 \ -10]$
$\mathbf{b} - \mathbf{a}$	$[-6 \ 1 \ 10]$

- c Press menu , 7: Matrix & Vector, 7: Norms and 1: Norm. Put in $\mathbf{a}$ and press enter .

$\mathbf{a} := [2 \ 1 \ -3]$	$[2 \ 1 \ -3]$
Define $\mathbf{b} := [-4 \ 2 \ 7]$	Done
$2 \cdot \mathbf{a} - 3 \cdot \mathbf{b}$	$[16 \ -4 \ -27]$
$\mathbf{a} - \mathbf{b}$	$[6 \ -1 \ -10]$
$\mathbf{b} - \mathbf{a}$	$[-6 \ 1 \ 10]$
$\text{norm}(\mathbf{a})$	$\sqrt{14}$



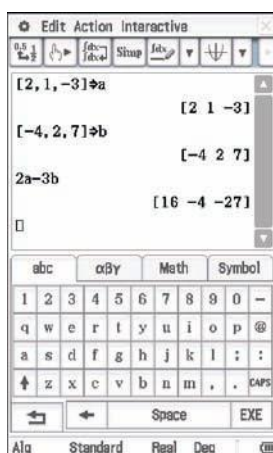
ClassPad

- a Ensure that your calculator is set to **Standard** mode.

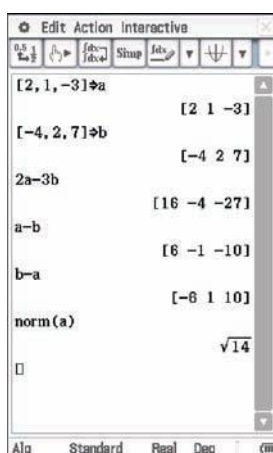
Enter the vector **a** by pressing **Keyboard**, tapping **Math3** and then the square bracket button on this keyboard.

Use the left arrow to place the cursor between the brackets and enter the values of the vector, separated by **,**. Assign this vector to variable name **a** by tapping **→** then **abc** and **a**, and press **EXE**. Enter the vector **b** in a similar manner.

Perform the required operation.



- b Find the differences to compare them.
Clearly $\mathbf{b} - \mathbf{a} = -1(\mathbf{a} - \mathbf{b})$



- c Tap **Action** then **Vector** and **norm**, type **a** then press **)** and **EXE**.

You could leave the unit vector in the form $\frac{\sqrt{14}}{14}(2, 1, -3)$. You can also write it as

$$\left(\frac{\sqrt{14}}{7}, \frac{\sqrt{14}}{14}, -\frac{3\sqrt{14}}{14} \right) \approx (0.534, 0.267, -0.802).$$



Resultant and
unit vectors
in 3D space

In three dimensions, the polar form of a vector has a magnitude r and two angles, θ and ϕ .

The **azimuth angle** θ lies between 0° and 360° and is the angle from the x -axis to the projection of the vector onto the xy -plane (between x and a).

The **altitude angle** ϕ lies between -90° and 90° and is the angle from the xy -plane to the vector (between a and r).

The azimuth and altitude angles are mainly used in astronomy and navigation.

The polar form $\mathbf{v} = (r, \theta, \phi)$ is related to the rectangular form $\mathbf{v} = (x, y, z)$ as shown in the diagram on the right. Using Pythagoras' theorem,

$$r = \sqrt{x^2 + y^2 + z^2} \text{ is just the norm.}$$

In the green triangle,

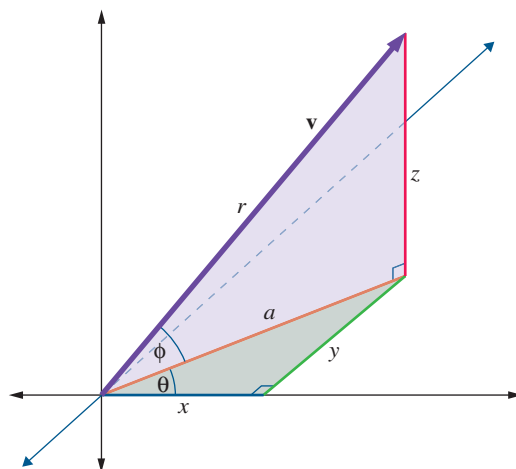
$$x = a \cos(\theta) \text{ and } y = a \sin(\theta)$$

and in the purple triangle

$$z = r \sin(\phi) \text{ and } a = r \cos(\phi),$$

$$\text{so } x = r \cos(\phi) \cos(\theta) \text{ and } y = r \cos(\phi) \sin(\theta).$$

Working backwards, $r = \sqrt{x^2 + y^2 + z^2}$ is just the norm, $\sin(\phi) = \frac{z}{r}$ and $\tan(\theta) = \frac{y}{x}$.



IMPORTANT

For the 3D vector $\mathbf{v} = (r, \theta, \phi) = (x, y, z)$:

$$x = r \cos(\phi) \cos(\theta) \quad r^2 = x^2 + y^2 + z^2$$

$$y = r \cos(\phi) \sin(\theta) \quad \tan(\theta) = \frac{y}{x}$$

$$z = r \sin(\phi) \quad \sin(\phi) = \frac{z}{r}$$

$$\text{and } r = |\mathbf{v}| = \sqrt{x^2 + y^2 + z^2}$$



Alamy/David Askham



○ Example 5

Express the following as linear combinations of **i**, **j** and **k**.

a $(2, -7, 3)$

b $\begin{bmatrix} -5 \\ 0 \\ 4 \end{bmatrix}$

c A vector of magnitude 5, azimuth 210° and altitude 66° .

Solution

a Write as multiples of **i**, **j** and **k**.

$$(2, -7, 3) = 2\mathbf{i} + (-7)\mathbf{j} + 3\mathbf{k}$$

Simplify.

$$= 2\mathbf{i} - 7\mathbf{j} + 3\mathbf{k}$$

b Write as multiples of **i**, **j** and **k**.

$$\begin{bmatrix} -5 \\ 0 \\ 4 \end{bmatrix} = -5\mathbf{i} + 0\mathbf{j} + 4\mathbf{k}$$

Simplify.

$$= -5\mathbf{i} + 4\mathbf{k}$$

c Write the given information.

$$r = 5, \theta = 210^\circ \text{ and } \phi = 66^\circ$$

Write the formula for x .

$$x = r \cos(\phi) \cos(\theta)$$

Substitute and evaluate.

$$\begin{aligned} &= 5 \cos(66^\circ) \cos(210^\circ) \\ &\approx -1.761 \end{aligned}$$

Write the formula for y .

$$y = r \cos(\phi) \sin(\theta)$$

Substitute and evaluate.

$$\begin{aligned} &= 5 \cos(66^\circ) \sin(210^\circ) \\ &\approx -1.017 \end{aligned}$$

Write the formula for z .

$$z = r \sin(\phi)$$

Substitute and evaluate.

$$\begin{aligned} &= 5 \sin(66^\circ) \\ &\approx 4.568 \end{aligned}$$

Write in terms of **i**, **j** and **k**.

$$\text{Vector} \approx -1.761\mathbf{i} - 1.017\mathbf{j} + 4.568\mathbf{k}$$

When you add three-dimensional vectors, it is difficult to calculate the resultant using the polar form. It is difficult to correctly visualise and apply trigonometry directly to the triangles of vectors because they are three-dimensional. It is far easier to convert the vectors to component form before adding.

Example 6

If **a** has magnitude 3, azimuth 47° and altitude -60° , and **b** has magnitude 6, azimuth 223° and altitude -40° , find **a + b**.

Solution

Calculate x for **a**.

$$\begin{aligned}x &= 3 \cos(-60^\circ) \cos(47^\circ) \\ &\approx 1.023\end{aligned}$$

Calculate y for **a**.

$$\begin{aligned}y &= 3 \cos(-60^\circ) \sin(47^\circ) \\ &\approx 1.097\end{aligned}$$

Calculate z for **a**.

$$\begin{aligned}z &= 3 \sin(-60^\circ) \\ &\approx -2.598\end{aligned}$$

Write **a** in terms of components.

$$\mathbf{a} \approx (1.023, 1.097, -2.598)$$

Calculate x for **b**.

$$\begin{aligned}x &= 6 \cos(-40^\circ) \cos(223^\circ) \\ &\approx -3.361\end{aligned}$$

Calculate y for **b**.

$$\begin{aligned}y &= 6 \cos(-40^\circ) \sin(223^\circ) \\ &\approx -3.135\end{aligned}$$

Calculate z for **b**.

$$\begin{aligned}z &= 6 \sin(-40^\circ) \\ &\approx -3.857\end{aligned}$$

Write **b** in terms of the components.

$$\mathbf{b} \approx (-3.361, -3.135, -3.857)$$

Calculate **a + b**.

$$\begin{aligned}\mathbf{a} + \mathbf{b} &\approx (1.023, 1.097, -2.598) + \\ &\quad (-3.361, -3.135, -3.857)\end{aligned}$$

Simplify.

$$= (-2.338, -2.038, -6.455)$$

Write the components of **a + b**.

$$x \approx -2.338, y \approx -2.038, z \approx -6.455$$

Calculate the magnitude of **a + b**.

$$\begin{aligned}r &\approx \sqrt{(-2.338)^2 + (-2.038)^2 + (-6.455)^2} \\ &\approx 7.161\end{aligned}$$

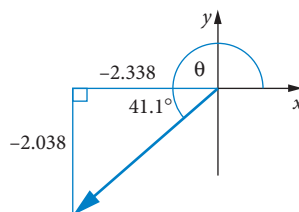
Calculate the azimuth of **a + b**.

$$\tan(\theta) \approx \frac{-2.038}{-2.338}$$

$$\theta \approx 41.1^\circ$$

Use your calculator and round off.

You can check this by making a rough sketch. From the sketch, you can see that $\theta \approx 221.1^\circ$ when the quadrant is taken into account.



State the azimuth of **a + b**.

$$\text{Azimuth of } \mathbf{a} + \mathbf{b} \approx 221.1^\circ$$

Calculate ϕ , the altitude of **a + b**.

$$\sin(\phi) \approx \frac{-6.455}{7.161}$$

$$\phi \approx -64.3^\circ$$

Use your calculator and round off.

State the result.

The resultant vector has magnitude about 7.161, azimuth 221.1° and altitude -64.3° .



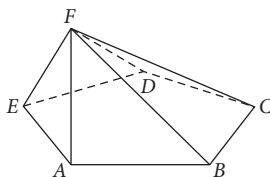
Vectors in polar form

The last part of Example 6 shows how to convert from the component form to the polar form. The calculation of θ is subject to the same problems you encountered in the two-dimensional case. In order to find the correct angle, you can draw a triangle in the same way as was done in that case. It is not necessary to include the z -component in the triangle because this is accounted for in the calculation of ϕ . In fact, for the altitude, the sign of ϕ is determined solely by the sign of z .

EXERCISE 1.02 Three-dimensional vectors

Concepts and techniques

- Example 4** If the position vectors of A and B are $\mathbf{a} = (1, 1, -1)$ and $\mathbf{b} = (3, -2, -4)$, then $|\mathbf{AB}|$ is equal to:
 A $\sqrt{3}$ B 4 C $\sqrt{22}$ D $\sqrt{26}$ E $\sqrt{29}$
- Given that $\mathbf{a} = (-2, 1, 3)$, $\mathbf{b} = (3, -1, 2)$ and $\mathbf{c} = (4, -2, -3)$, calculate:
 a $2\mathbf{a} + 3\mathbf{b}$ b $\mathbf{b} - 4\mathbf{c}$ c $2\mathbf{a} - \mathbf{b} + \mathbf{c}$
 d $2\mathbf{a} - \mathbf{b} + 2\mathbf{c}$ e $2\mathbf{c} + 3\mathbf{b} - 4\mathbf{a}$ f $5\mathbf{a} - 3\mathbf{c} + 2\mathbf{b}$
- Calculate the norm of the resultant vector in each part of question 2.
- Given that $\mathbf{a} = (4, -2, 6)$, $\mathbf{b} = (9, -12, -6)$, $\mathbf{c} = (6, -3, 9)$, $\mathbf{d} = (-6, 8, 4)$ and $\mathbf{e} = (1, -2, -1)$, determine whether the following pairs of vectors are parallel.
 a $\frac{3}{2}\mathbf{a}$ and $3\mathbf{c}$ b $(\mathbf{a} + 2\mathbf{e})$ and $(\mathbf{c} + \mathbf{b})$ c $3\mathbf{b}$ and $-2\mathbf{d}$
 d $\frac{3}{2}\mathbf{b}$ and $2\mathbf{d}$ e $(\mathbf{a} - \mathbf{c})$ and $(\mathbf{c} - \mathbf{a})$ f $(2\mathbf{a} - \mathbf{b})$ and $(3\mathbf{e} + 2\mathbf{d})$
- $ABCDE$ is a regular pentagon and F is a point not in the plane of the pentagon. The position vectors of A, B, C, D, E and F are $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}$ and $\mathbf{f}$ respectively. Express each of the following vectors in terms of $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}$ and $\mathbf{f}$.
 a $\mathbf{AB}$ b $\mathbf{AC}$ c $\mathbf{DF}$ d $\mathbf{FC}$ e $\mathbf{EB}$



- CAS** The points $A(5, 3, 6)$, $B(3, -2, 1)$, $C(6, -2, -4)$, $D(-2, -5, 4)$, and $E(-4, -3, -5)$ have position vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ and $\mathbf{e}$. Use the position vectors to find the following displacement vectors in both component and polar form.
 a $\mathbf{BE}$ b $\mathbf{CD}$ c $\mathbf{AB}$ d $\mathbf{BC}$ e $\mathbf{EA}$
- CAS** Find the norms of the following vectors.
 a $(8, -5, -7)$ b $(7, -3, 4)$ c $(-2, 5, 9)$
- Find the unit vector in the direction of each of the following, correct to 4 decimal places.
 a $(4, -7, 4)$ b $(-6, 2, -3)$ c $(-5, -3, 4)$
- If $A = (2, -3, 5)$ and $B = (-4, 2, -3)$, then $\mathbf{AB}$ is equal to:
 A $-2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ B $3\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$ C $-6\mathbf{i} + 5\mathbf{j} - 8\mathbf{k}$
 D $6\mathbf{i} - 5\mathbf{j} - 5\mathbf{k}$ E $-6\mathbf{i} + 5\mathbf{j} + 3\mathbf{k}$

- 10 **Example 5** Express the following vectors as linear combinations of **i**, **j** and **k** and state the norm of each vector.

a $(2, -3, 4)$

b $(-3, -2, 4)$

c $(3, 1, -5)$

d $(-1, 4, -4)$

e $(-3, -1, -2)$

f $\begin{bmatrix} -6 \\ 2 \\ 5 \end{bmatrix}$

g $\begin{bmatrix} 4 \\ -7 \\ 2 \end{bmatrix}$

h $\begin{bmatrix} 1 \\ 6 \\ 4 \end{bmatrix}$

i $\begin{bmatrix} -5 \\ -3 \\ 2 \end{bmatrix}$

j $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

k $(10, 294^\circ, -27^\circ)$

l $(4, 225^\circ, 32^\circ)$

m $(11, 65^\circ, 84^\circ)$

n $(16, \frac{5\pi}{3}, -\frac{\pi}{6})$

o $(8, 119^\circ, -70^\circ)$

p Magnitude 50, azimuth 110° , altitude 25°

q Magnitude 80, azimuth $\frac{5\pi}{4}$, altitude $-\frac{\pi}{3}$

- 11 Convert the following vectors to polar form.

a $(3, -5, -4)$

b $(-4, -2, 4)$

c $\begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix}$

d $\begin{bmatrix} -2 \\ -8 \\ -5 \end{bmatrix}$

- 12 Convert the following vectors to component form.

a $(10, 26^\circ, 75^\circ)$

b $(8, 120^\circ, -46^\circ)$

c $(20, \frac{11\pi}{6}, \frac{\pi}{3})$

d 28 at azimuth 210° and altitude -50°

e 32 at azimuth $\frac{3\pi}{4}$ and altitude $\frac{\pi}{6}$

- 13 **Example 6** If $\mathbf{OA} = \mathbf{a} + 3\mathbf{b}$ and $\mathbf{OB} = 5\mathbf{a} - 2\mathbf{b}$, then $\mathbf{AB}$ is equal to:

A $4\mathbf{a} - 5\mathbf{b}$

B $4\mathbf{a} - \mathbf{b}$

C $6\mathbf{a} - 5\mathbf{b}$

D $2\mathbf{a} - 3\mathbf{b}$

E $6\mathbf{a} - 5\mathbf{b}$

- 14 The position vectors $\mathbf{a} = (12, 30^\circ, -60^\circ)$ and $\mathbf{b} = (15, 150^\circ, 45^\circ)$. Find each of the following in polar form.

a $\mathbf{a} + \mathbf{b}$

b $\mathbf{a} - \mathbf{b}$

c $\mathbf{a} + 2\mathbf{b}$

d $2\mathbf{a} - \mathbf{b}$

e $3\mathbf{a} + 2\mathbf{b}$

- 15 If $\mathbf{p}$ has norm 30, azimuth $\frac{7\pi}{6}$ and altitude $\frac{\pi}{3}$, and $\mathbf{q}$ has norm 25, azimuth $\frac{7\pi}{4}$ and altitude $-\frac{\pi}{6}$, find each of the following.

a $\mathbf{p} + \mathbf{q}$

b $\mathbf{q} - \mathbf{p}$

c $3\mathbf{p} - \mathbf{q}$

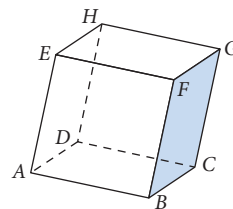
d $2\mathbf{q} - \mathbf{p}$

e $2\mathbf{p} - \mathbf{q}$



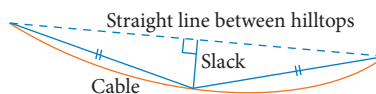
Reasoning and communication

- 16 $ABCDEFGH$ is a cube. The position vectors of A , B , C and F are $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{f}$ respectively. Find the position vectors of the other vertices and the centre of the cube in terms of $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{f}$.



- 17 An airbus is flying north-east at a height of 15 000 m and a speed of 500 km/h. It is climbing at an angle of 12° . Unfortunately, it flies into some turbulence and drops 1500 m in a time of 1 minute while moving 2000 m to the left of its path. Consider the initial position of the airbus as the origin and calculate the following.
- The vector for its flight displacement through the air in that minute.
 - The vector for its movement by the air.
 - The vector for its true displacement in that minute.

- 18 A hilltop that is 400 m above sea level is at map coordinates (50, 70) and another hilltop of elevation 320 m is at coordinates (80, 35). The map coordinates are 10 m apart. Use position and displacement vectors to find the minimum length of a power transmission cable between the hills. The cable requires slack of at least 5% of its length to prevent it snapping in high winds. Although the cable hangs in a curve, the length of the slack cable can be calculated as if it is two straight lines, as shown in the diagram.



- 19 A submarine travels in a straight line. After descending to a depth of 400 m, its position is 600 m south and 300 m east of its starting point. Find the sub's direction of travel, the angle of descent and the actual distance travelled.
- 20 A yacht tacking into the wind has a sail area of 40 m^2 . The wind pressure on the sail is 500 N/m^2 , and the sail deflects the wind up by an angle of 15° and towards the rear of the yacht by an angle of 20° . The wind is originally horizontal at an angle of 60° to the yacht.
- Find the new direction of the wind.
 - Find the total force exerted by the wind on the sail.
 - Find the components of the force acting on the wind.
 - Find the resulting forward force on the yacht (as the negative of the wind change).
 - Find the component of the force of the wind tending to sink the yacht.



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1.03 THE SCALAR PRODUCT

The **scalar product** can be found for a 3D vector in a similar way to 2D vectors.

IMPORTANT

$\mathbf{OA}$ and $\mathbf{OB}$ are position vectors such that $\mathbf{OA} = \mathbf{RS}$ and $\mathbf{OB} = \mathbf{PQ}$.

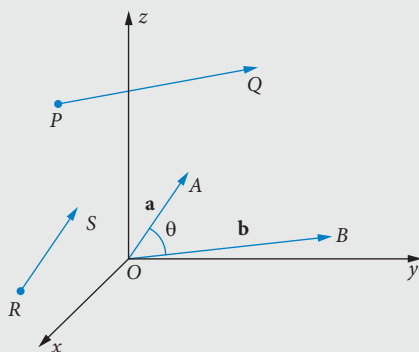
The scalar product of $\mathbf{PQ}$ and $\mathbf{RS}$ is given by

$$\mathbf{PQ} \cdot \mathbf{RS} = \mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos(\theta) = ab \cos(\theta)$$

The scalar product is also called the **inner product** or **dot product**. It is a scalar (real number), not a vector. It may be calculated from vectors in component form as

$$\mathbf{a} \cdot \mathbf{b} = (a_1, a_2, a_3) \cdot (b_1, b_2, b_3) = a_1b_1 + a_2b_2 + a_3b_3$$

where $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$.



$\mathbf{i}$ and $\mathbf{j}$ are perpendicular so

$$\mathbf{i} \cdot \mathbf{j} = |\mathbf{i}| \times |\mathbf{j}| \times \cos(90^\circ) = 1 \times 1 \times 0 = 0.$$

$\mathbf{k}$ and $\mathbf{k}$ are parallel, so

$$\mathbf{k} \cdot \mathbf{k} = |\mathbf{k}| \times |\mathbf{k}| \times \cos(0^\circ) = 1 \times 1 \times 1 = 1.$$

	$b_1\mathbf{i}$	$b_2\mathbf{j}$	$b_3\mathbf{k}$
$a_1\mathbf{i}$	$a_1\mathbf{i} \cdot b_1\mathbf{i}$	$a_1\mathbf{i} \cdot b_2\mathbf{j}$	$a_1\mathbf{i} \cdot b_3\mathbf{k}$
$a_2\mathbf{j}$	$a_2\mathbf{j} \cdot b_1\mathbf{i}$	$a_2\mathbf{j} \cdot b_2\mathbf{j}$	$a_2\mathbf{j} \cdot b_3\mathbf{k}$
$a_3\mathbf{k}$	$a_3\mathbf{k} \cdot b_1\mathbf{i}$	$a_3\mathbf{k} \cdot b_2\mathbf{j}$	$a_3\mathbf{k} \cdot b_3\mathbf{k}$

You can set out all the possible products of $a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ in a square.

In the square, it is the products on the inside that are not zero. This is why it is called the inner product.

IMPORTANT

Properties of the scalar product

1 $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos(\theta)$, where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$.

2 The scalar product is a real number, *not a vector*.

3 The scalar product is commutative: $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$

4 The scalar product is distributive over vector addition:

$$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c} \text{ and } \mathbf{a} \cdot (\mathbf{b} - \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} - \mathbf{a} \cdot \mathbf{c}$$

5 $m(\mathbf{a} \cdot \mathbf{b}) = (m\mathbf{a}) \cdot \mathbf{b} = \mathbf{a} \cdot (m\mathbf{b}) \forall m \in \mathbf{R}$ and vectors $\mathbf{a}, \mathbf{b}$.

6 $\mathbf{a} \cdot \mathbf{b} = 0 \Rightarrow \mathbf{a} = \mathbf{0}$ or $\mathbf{b} = \mathbf{0}$ or $\mathbf{a}$ and $\mathbf{b}$ are perpendicular.

In terms of components:

$$7 \mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = 1, \mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{i} = 0$$

8 If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, then $\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$.



The definition of the scalar product in terms of magnitudes and the angle between vectors can be used in reverse to find the angle.

Since $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \cos(\theta)$,

$$\cos(\theta) = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}| |\mathbf{b}|}.$$

You can use a CAS calculator to calculate scalar products. This is no problem when the coordinates are in component form. However, CAS calculators use **spherical coordinates** for the polar form. Instead of the altitude, they use the **polar angle**, which is measured anticlockwise from the z-axis. In this book, the symbol ψ will be used for the polar angle to avoid confusion. Unfortunately, the polar angle is also shown as ϕ or φ , so there can be confusion. Clearly $\phi + \psi = 90^\circ$.

○ Example 7

- Find the scalar product of vectors of magnitudes 5 and 6 at an angle of 60° to each other.
- For $\mathbf{a} = 2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$ and $\mathbf{b} = -7\mathbf{i} + \mathbf{j} + 5\mathbf{k}$, find $\mathbf{a} \cdot \mathbf{b}$.
- The vector $\mathbf{a}$ has magnitude 12, azimuth angle 102° and altitude 47° ; vector $\mathbf{b}$ has magnitude 8, azimuth angle 58° and altitude -15° . Find $\mathbf{a} \cdot \mathbf{b}$.

Solution

- Write the formula.

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \cos(\theta)$$

Substitute the values.

$$\begin{aligned} &= 5 \times 6 \times \cos(60^\circ) \\ &= 15 \end{aligned}$$

- Write the formula.

$$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$$

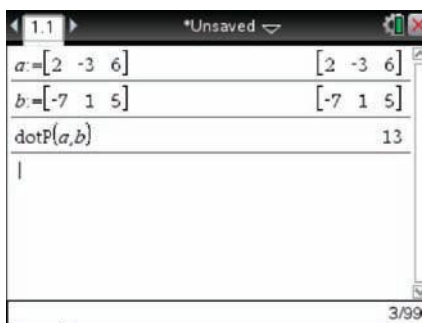
Substitute in the values.

$$\begin{aligned} &= 2 \times (-7) + (-3) \times 1 + 6 \times 5 \\ &= 13 \end{aligned}$$

TI-Nspire CAS

Set up the vectors.

Use , 7:Matrix & Vector, C:Vector and 3:Dot Product, or type dotP.

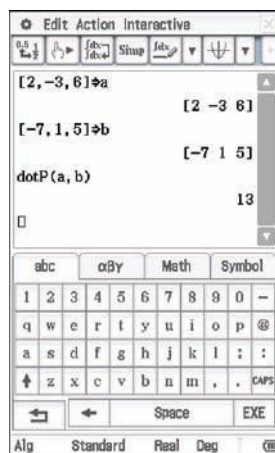


ClassPad

Set up the vectors. One way of doing this is to use the $\square$ button available under **Keyboard** **Math3**, with elements of a vector separated by a comma.

Tap $\square$, then a vector name from the keyboard under **abc**.

Tap **Action Vector dotP**, then enter a $\square$, b $\square$ and press **EXE**.



- c Use $x = r \cos(\theta) \cos(\phi)$,
 $y = r \sin(\theta) \cos(\phi)$ and
 $z = r \sin(\phi)$
 to express the vectors in component form.

Calculate the product.

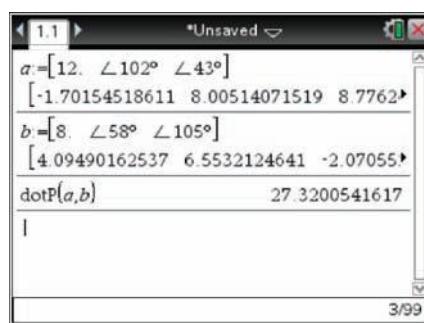
$$\mathbf{a} = (12, 102^\circ, 47^\circ) \\ \approx (-1.702, 8.005, 8.776)$$

$$\mathbf{b} = (8, 58^\circ, -15^\circ) \\ \approx (4.095, 6.553, -2.071)$$

$$\begin{aligned} \mathbf{a} \cdot \mathbf{b} &\approx (-1.702, 8.005, 8.776) \cdot (4.095, 6.553, -2.071) \\ &= -1.702 \times 4.095 + 8.005 \times 6.553 \\ &\quad + 8.776 \times (-2.071) \\ &\approx 27.320 \end{aligned}$$

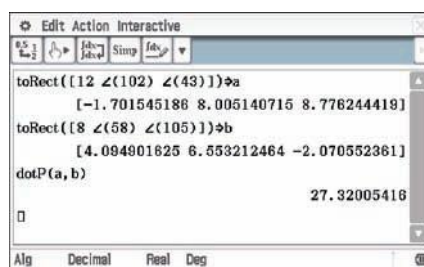
TI-Nspire CAS

Use $\phi + \psi = 90^\circ$ to find the polar angles. Set up the vectors using the angle and degree symbols from the symbol menu (**ctrl** $\square$). Force approximations by inserting decimal points.



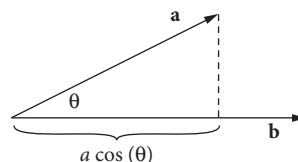
ClassPad

Use $\phi + \psi = 90^\circ$ to find the polar angles. Ensure that your calculator is set to **Decimal** mode. To express vector **a** in component form, first tap **Action Vector toRect**. Enter the vector using the angle symbol $\angle$ found under **Keyboard** **Math3** and assign the result to **a** as shown. Follow a similar process for the vector **b**. Calculate the dot product of **a** and **b** as before.





You found projections in two dimensions last year. Remember that the projection of one vector on another is the component of the vector in that direction. If $\mathbf{a}$ and $\mathbf{b}$ have the same initial point, and the angle between them is θ , then the projection of $\mathbf{a}$ in the direction of $\mathbf{b}$, namely $\mathbf{p}$, has magnitude $a \cos(\theta)$.



$$p = a \cos(\theta) \text{ or } |\mathbf{p}| = |\mathbf{a}| \cos(\theta)$$

The unit vector in the direction of $\mathbf{b}$ is $\hat{\mathbf{b}}$, so

$$\begin{aligned} \mathbf{p} &= |\mathbf{a}| \cos(\theta) \times \hat{\mathbf{b}} \\ &= |\mathbf{a}| \times 1 \times \cos(\theta) \times \hat{\mathbf{b}} \end{aligned}$$

Replace 1 by $|\hat{\mathbf{b}}|$.

$$\begin{aligned} &= |\mathbf{a}| \times |\hat{\mathbf{b}}| \times \cos(\theta) \times \hat{\mathbf{b}} \\ &= (\mathbf{a} \cdot \hat{\mathbf{b}}) \hat{\mathbf{b}} \end{aligned}$$

IMPORTANT

The magnitude p of the projection $\mathbf{p}$ of $\mathbf{a}$ on $\mathbf{b}$ can be calculated by the dot product of $\mathbf{a}$ and the unit vector in the direction of $\mathbf{b}$. Since the direction of a projection is fixed, it is often referred to by only its magnitude.

$$p = \mathbf{a} \cdot \hat{\mathbf{b}} \text{ and } \mathbf{p} = (\mathbf{a} \cdot \hat{\mathbf{b}}) \hat{\mathbf{b}}$$

This can be rewritten as

$$p = \mathbf{a} \cdot \hat{\mathbf{b}} \times \frac{|\mathbf{b}|}{|\mathbf{b}|} = \frac{(\mathbf{a} \cdot \mathbf{b})}{|\mathbf{b}|}$$

so

$$\mathbf{p} = \frac{(\mathbf{a} \cdot \mathbf{b})}{|\mathbf{b}|} \hat{\mathbf{b}}.$$

The scalar product can also be used to find the projection of a vector in the direction of another vector, to find a projection in a particular direction, or to show that vectors are perpendicular.



Example 8

- Show that the vectors $(3, -2, 3)$ and $(6, 3, -4)$ are perpendicular.
- Use the dot product to show that the vectors $\mathbf{a} = (3, 9, -6)$ and $\mathbf{b} = (-2, -6, 4)$ are parallel.
- Find the projection of $\mathbf{m} = (2, -4, 5)$ on $\mathbf{n} = (-1, 3, 4)$ using the scalar product $\mathbf{m} \cdot \mathbf{n}$.
- Find the projection of $\mathbf{c} = (-1, 3, 2)$ on $\mathbf{d} = (2, -2, 3)$ using unit vectors.
- Find the angle between $\mathbf{q} = 3\mathbf{i} + 4\mathbf{j} - \mathbf{k}$ and $\mathbf{r} = 3\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$.

Solution

- Find the scalar product.

$$(3, -2, 3) \cdot (6, 3, -4) = 18 - 6 - 12$$

Evaluate.

$$= 0$$

Use the fact that $\cos(90^\circ) = 0$ to write the answer.

Since the scalar product is zero, the vectors are perpendicular.

- Use the scalar product to calculate $\cos(\theta)$.

$$\cos(\theta) = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}| |\mathbf{b}|}$$

Substitute in the known values.

$$= \frac{-6 - 54 - 24}{\sqrt{3^2 + 9^2 + (-6)^2} \times \sqrt{(-2)^2 + (-6)^2 + 4^2}}$$

Evaluate and simplify.

$$= \frac{-84}{3\sqrt{14} \times 2\sqrt{14}} \\ = -1$$

Write the interpretation of $\cos(\theta) = -1$.

The angle is 180° ; they are parallel.

- Use the scalar product to calculate $\cos(\theta)$.

$$\cos(\theta) = \frac{\mathbf{m} \cdot \mathbf{n}}{|\mathbf{m}| |\mathbf{n}|}$$

Substitute.

$$= \frac{-2 - 12 + 20}{\sqrt{2^2 + (-4)^2 + 5^2} \times \sqrt{(-1)^2 + 3^2 + 4^2}}$$

Evaluate.

$$= \frac{6}{\sqrt{45} \times \sqrt{26}}$$

Write the formula for p .

$$p = |\mathbf{m}| \cos(\theta)$$

Substitute in the values.

$$= \sqrt{45} \times \frac{6}{\sqrt{45} \times \sqrt{26}}$$

Simplify.

$$= \frac{6}{\sqrt{26}} \\ \approx 1.18$$

Write the answer.

The projection of $(2, -4, 5)$ on $(-1, 3, 4)$ is about 1.18 in the direction of $\mathbf{n}$.



d Calculate $\hat{\mathbf{d}}$.

$$\hat{\mathbf{d}} = \frac{1}{|\mathbf{d}|} \mathbf{d}$$

Substitute for the components of $\mathbf{d}$.

$$= \frac{1}{\sqrt{2^2 + (-2)^2 + 3^2}} (2, -2, 3)$$

Simplify.

$$= \frac{1}{\sqrt{17}} (2, -2, 3)$$

Write the formula.

$$p = \mathbf{c} \cdot \hat{\mathbf{d}}$$

Substitute.

$$= (-1, 3, 2) \cdot \frac{1}{\sqrt{17}} (2, -2, 3)$$

Rearrange.

$$= \frac{1}{\sqrt{17}} (-2 - 6 + 6)$$

Evaluate.

$$= -\frac{2}{\sqrt{17}} \\ \approx -0.485$$

Write the answer.

The projection of $(-1, 3, 2)$ on $(2, -2, 3)$ is $-\frac{2}{\sqrt{17}} (2, -2, 3) \approx (-0.97, 0.97, -1.46)$

e Use the scalar product to calculate $\cos(\theta)$. $\cos(\theta) = \frac{\mathbf{q} \cdot \mathbf{r}}{|\mathbf{q}||\mathbf{r}|}$

Substitute in the known values.

$$= \frac{9 - 8 + 2}{\sqrt{3^2 + 4^2 + (-1)^2} \times \sqrt{3^2 + (-2)^2 + (-2)^2}}$$

Evaluate.

$$= \frac{3}{\sqrt{26} \times \sqrt{17}}$$

$$= \frac{3}{\sqrt{442}}$$

Use $\cos^{-1}(\theta)$ to find θ .

$$\theta \approx 81.8^\circ$$

The scalar product is important in the calculation of a number of physical quantities, such as work, energy, electrical potential and electrical power. The work (W) done when a constant force (F) moves an object through a displacement (s) is defined as the component of the force in the direction of motion multiplied by the displacement. This is simply written as $W = \mathbf{F} \cdot \mathbf{s}$.

IMPORTANT

The work done by a constant force $\mathbf{F}$ that results in a displacement $\mathbf{s}$ is $\mathbf{F} \cdot \mathbf{s}$.

○ Example 9

A constant force $\mathbf{a} = 4\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}$ Newtons moves an object from $P(2, -2, 3)$ to $R(5, 4, 0)$. The distance is measured in metres. Find the work done.

Solution

Find the displacement.

$$\mathbf{s} = \mathbf{PR} = \mathbf{r} - \mathbf{p}$$

Substitute values and simplify.

$$\begin{aligned} &= (5, 4, 0) - (2, -2, 3) \\ &= (5 - 2, 4 + 2, 0 - 3) \\ &= (3, 6, -3) \end{aligned}$$

Write the formula for work.

$$W = \mathbf{F} \cdot \mathbf{s}$$

Substitute in the known values.

$$= (4\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}) \cdot (3\mathbf{i} + 6\mathbf{j} - 3\mathbf{k})$$

Multiply the corresponding components.

$$= 12 + 18 - 18 = 12$$

State the result.

The work done is 12 Nm.

EXERCISE 1.03 The scalar product



Scalar product

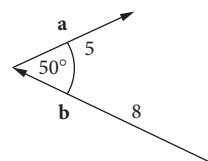
Concepts and techniques

- 1 **Example 7** The dot product of $2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ and $4\mathbf{i} - 2\mathbf{j} - \mathbf{k}$ is:

A 1 B 3 C 7 D 10 E 13

- 2 Given the two vectors shown on the right, $\mathbf{a} \cdot \mathbf{b}$ is closest to:

A 62.2 B 47.7 C 25.7
D 30.6 E 40



- 3 **Example 8** If $\mathbf{a} = 4\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}$, which of the following is perpendicular to $\mathbf{a}$?

A $-4\mathbf{i} - 3\mathbf{j} - 5\mathbf{k}$ B $5\mathbf{i} - 4\mathbf{k}$ C $5\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$
D $4\mathbf{j}$ E $4\mathbf{j} - 5\mathbf{k}$

- 4 If $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|^2$, then:

A $\mathbf{a}$ and $\mathbf{b}$ are parallel and equal in magnitude
B $\mathbf{a}$ and $\mathbf{b}$ have equal magnitudes
C $\mathbf{a}$ and $\mathbf{b}$ are perpendicular
D $\mathbf{b}$ is a multiple of $\mathbf{a}$
E None of the above is necessarily true.

- 5 If $(\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b}) = 0$, then:

A $\mathbf{a}$ and $\mathbf{b}$ are parallel
B $\mathbf{a}$ and $\mathbf{b}$ must be two-dimensional vectors
C $\mathbf{a} = \mathbf{0}$
D $\mathbf{a}$ and $\mathbf{b}$ have the same magnitude
E None of the above is true.



- 6 The angle between the vectors $\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $2\mathbf{i} - 2\mathbf{k}$ is closest to:
 A 0° B 45° C 90° D 135° E 180°
- 7 The angle between the vectors $2\mathbf{i} + 3\mathbf{j} - 5\mathbf{k}$ and $-4\mathbf{i} - 6\mathbf{j} + 10\mathbf{k}$ is closest to:
 A 0° B 45° C 90° D 180° E 270°
- 8 In each of the following, the magnitudes of two vectors and the angle between them is given.
 Find the scalar product in each case.
- | | | | |
|--------------------|----------------------|---------------------|----------------------|
| a 5, 9, 38° | b 3, 10, 45° | c 11, 7, 90° | d 15, 12, 0° |
| e 7, 2, 30° | f 6, 11, 210° | g 9, 9, 150° | h 8, 12, 300° |
- 9 Find the following scalar products.
- | | | |
|---|--|---|
| a $(9, 120^\circ) \cdot (16, 48^\circ)$ | b $(3, 185^\circ) \cdot (16, 200^\circ)$ | c $(19, 300^\circ) \cdot (6, 240^\circ)$ |
| d $(3, 160^\circ, 42^\circ) \cdot (5, 160^\circ, 12^\circ)$ | e $(9, 280^\circ, 16^\circ) \cdot (4, 280^\circ, -28^\circ)$ | f $(15, 45^\circ, -33^\circ) \cdot (8, 45^\circ, 57^\circ)$ |
- 10 Find the following scalar products.
- | | | |
|-----------------------------------|----------------------------------|----------------------------------|
| a $(3, 8) \cdot (6, -4)$ | b $(-2, 10) \cdot (5, 1)$ | c $(-6, 8) \cdot (8, -6)$ |
| d $(2, 6, 4) \cdot (5, 10, 1)$ | e $(3, -4, 5) \cdot (5, -6, -8)$ | f $(9, 2, -12) \cdot (6, -3, 4)$ |
| g $(8, -3, 5) \cdot (-2, -3, -7)$ | h $(-4, 2, -8) \cdot (-5, 1, 3)$ | i $(1, 0, 5) \cdot (-8, -3, 2)$ |
- 11 Find the following scalar products.
- | | | |
|---|--|---|
| a $\begin{bmatrix} 3 \\ -5 \end{bmatrix} \cdot \begin{bmatrix} -5 \\ 3 \end{bmatrix}$ | b $\begin{bmatrix} 2 \\ 9 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ -9 \end{bmatrix}$ | c $\begin{bmatrix} -3 \\ 8 \end{bmatrix} \cdot \begin{bmatrix} -3 \\ 8 \end{bmatrix}$ |
| d $\begin{bmatrix} 7 \\ -3 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 2 \\ -5 \end{bmatrix}$ | e $\begin{bmatrix} 2 \\ 9 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -4 \\ 6 \end{bmatrix}$ | f $\begin{bmatrix} -1 \\ 0 \\ 7 \end{bmatrix} \cdot \begin{bmatrix} -6 \\ 9 \\ 5 \end{bmatrix}$ |
- 12 Given the vectors $\mathbf{a} = 3\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$, $\mathbf{b} = 2\mathbf{i} + \mathbf{j} - 5\mathbf{k}$, $\mathbf{c} = \mathbf{i} - 5\mathbf{j} - 4\mathbf{k}$, $\mathbf{d} = \mathbf{j} - 2\mathbf{i} + 3\mathbf{k}$ and $\mathbf{e} = 4\mathbf{k} - 3\mathbf{j}$, find:
- | | | |
|---|---------------------------------|--|
| a $\mathbf{a} \cdot \mathbf{b}$ | b $\mathbf{c} \cdot \mathbf{e}$ | c $\mathbf{c} \cdot (\mathbf{d} - \mathbf{e})$ |
| d $\mathbf{c} \cdot \mathbf{d} - \mathbf{c} \cdot \mathbf{e}$ | e $\mathbf{d} \cdot \mathbf{e}$ | f $\mathbf{e} \cdot \mathbf{d}$ |
- 13 Find the angle between the following pairs of vectors.
- | | | |
|---------------------------|---------------------------------|----------------------------------|
| a $(5, -4)$ and $(-2, 1)$ | b $(3, 6, -4)$ and $(0, -2, 7)$ | c $(0, 3, -2)$ and $(6, 1, 5)$. |
|---------------------------|---------------------------------|----------------------------------|
- 14 If $\mathbf{a} = -2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$, $\mathbf{b} = 7\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$ and $\mathbf{c} = \mathbf{i} - 5\mathbf{j} + 2\mathbf{k}$, find:
- | | |
|---|---|
| a the projection of $\mathbf{b}$ on $\mathbf{c}$ | b the projection of $\mathbf{c}$ on $\mathbf{a}$ |
| c the projection of $\mathbf{a}$ on $\mathbf{b}$ | d the projection of $\mathbf{c}$ on $\mathbf{c}$ |
| e the projection of $(\mathbf{a} + \mathbf{c})$ on $\mathbf{b}$ | f the projection of $(\mathbf{a} + \mathbf{b})$ on $\mathbf{c}$ |
- 15 In each of the following cases, find the angle between $\mathbf{m}$ and $\mathbf{n}$.
- | | |
|--|---|
| a $\mathbf{m} = -4\mathbf{i} + 8\mathbf{j} - 3\mathbf{k}$, $\mathbf{n} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$ | b $\mathbf{m} = \mathbf{i} - 7\mathbf{j} + 4\mathbf{k}$, $\mathbf{n} = 5\mathbf{i} - \mathbf{k}$ |
| c $\mathbf{m} = -2\mathbf{i} - 3\mathbf{j}$, $\mathbf{n} = -6\mathbf{i} + 4\mathbf{k}$ | d $\mathbf{m} = 3\mathbf{i} - 5\mathbf{j} - \mathbf{k}$, $\mathbf{n} = 2\mathbf{i} + \mathbf{j} - 3\mathbf{k}$ |
- 16 Given the points $A(3, -2, -1)$, $B(1, 5, 4)$, $C(2, 0, -6)$ and $D(-4, 1, 5)$, calculate:
- | | |
|---|---|
| a $\mathbf{AB} \cdot \mathbf{CD}$ | b $\mathbf{BD} \cdot \mathbf{CA}$ |
| c the angle between $\mathbf{AB}$ and $\mathbf{CD}$ | d the angle between $\mathbf{BD}$ and $\mathbf{CA}$ |
| e the projection of $\mathbf{AD}$ along $\mathbf{BC}$ | f the projection of $\mathbf{AB}$ along $\mathbf{CD}$ |

- 17 **Example 9** In each of the following, calculate the work done by the force $\mathbf{u}$ in moving an object from P to Q . The force is measured in newtons and the displacement is in metres.
- a $\mathbf{u} = -\mathbf{i} + 5\mathbf{j} - 3\mathbf{k}$, $P(4, 0, -7)$ and $Q(2, 4, 0)$
- b $\mathbf{u} = 8\mathbf{i} - 4\mathbf{k}$, $P(-1, 2, 5)$ and $Q(4, 1, 0)$
- c $\mathbf{u} = 2\mathbf{i} + 3\mathbf{j}$, $P(1, -2, 4)$ and $Q(5, 0, -3)$
- 18 A force of 12 N causes an object to move 12 m at an angle of 60° to the direction of the force. What is the work done (amount of energy that is transferred)?
- 19 A constant force of magnitude 5 N in the same direction as the vector $\mathbf{a} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ moves a point from $(0, 2, 0)$ to $(0, -3, 0)$. If the displacement is measured in metres, find the work done.
- 20 A constant force of magnitude 10 N in the same direction as the positive z -axis moves a point from the origin to $(1, 3, 4)$. If the displacement is measured in metres, find the work done.

Reasoning and communication

- 21 Find the constant k , such that $\mathbf{a} = 2k\mathbf{i} + 3k\mathbf{j} - 5\mathbf{k}$ is perpendicular to $\mathbf{a} = -2\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$.
- 22 A line passes through the points $(1, 3, -4)$ and $(2, -5, -2)$.
- a Find a displacement vector parallel to the line.
- b Find the cosines of the angles between this vector and each of the axes (the *direction cosines* of the line).
- c Find the *direction angles* α , β and γ of the line—the angles between the line and each of the axes.
- 23 Use a displacement vector and the scalar product to show that, in general,
- $$\cos^2(\alpha) + \cos^2(\beta) + \cos^2(\gamma) = 1$$
- for the direction cosines of any line in three-dimensional space.
- 24 Consider the scalar product of unit vectors in two dimensions with directions α and β . Use the scalar product to find an expression for the cosine of the angle between the vectors, $\cos(\alpha - \beta)$.
- 25 Two forces of 30 N and 50 N act on an object. The 50 N force is at an angle of 35° to the 30 N force. The object moves 8 m at an angle of 15° to the 30 N force, away from the 50 N force.
- a Find the total acting force.
- b Find the energy transferred (work done) by the:
- i total force ii 30 N force iii 50 N force.
- c Compare the sizes of the answers to part **b** and interpret this information mathematically.
- 26 For any vector, prove that $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$.
- 27 Is the angle between the vectors $(9, 42^\circ, 31^\circ)$ and $(8, 12^\circ, 31^\circ)$ equal to, less than or greater than 30° ? Explain your reasoning and prove your answer.



1.04 THE VECTOR PRODUCT

You have already extended multiplication by a scalar and the scalar product from two dimensions to three dimensions. There is another kind of product of vectors that can be performed in three dimensions. It is called the **vector product** because it gives a new vector. The new vector is *perpendicular* to the original vectors.

IMPORTANT

Vector product

The vector product of two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ with an angle θ between them is defined as

$$\mathbf{v} = \mathbf{v}_1 \times \mathbf{v}_2, \text{ where } |\mathbf{v}| = |\mathbf{v}_1||\mathbf{v}_2| \sin(\theta)$$

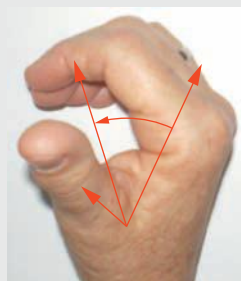
and $\mathbf{v}$ is perpendicular to both $\mathbf{v}_1$ and $\mathbf{v}_2$.

The direction of $\mathbf{v}$ is such that $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}$ form a right-handed system.

The vector product is also called the **outer product** or **cross product**.

The vector product may be calculated from vectors in component form using

$$(a_1, a_2, a_3) \times (b_1, b_2, b_3) = (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1)$$



$\mathbf{k}$ and $\mathbf{k}$ are parallel, so $\mathbf{k} \times \mathbf{k} = |\mathbf{k}| \times |\mathbf{k}| \times \sin(0^\circ) = 1 \times 1 \times 0 = 0$.

$\mathbf{i}$ and $\mathbf{j}$ are perpendicular, so $|\mathbf{i} \times \mathbf{j}| = |\mathbf{i}| \times |\mathbf{j}| \times \sin(90^\circ) = 1 \times 1 \times 1 = 1$.

$\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ make a right-handed system, so $\mathbf{i} \times \mathbf{j} = \mathbf{k}$.

If you consider all the possible orders, you get $\mathbf{j} \times \mathbf{k} = \mathbf{i}$ and $\mathbf{k} \times \mathbf{i} = \mathbf{j}$.

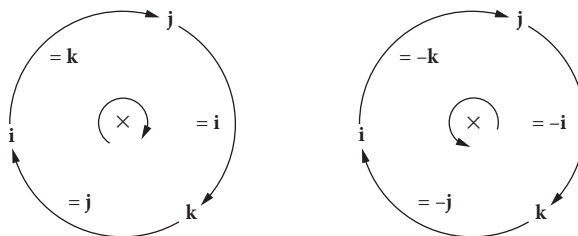
The reverse orders give negatives: $\mathbf{j} \times \mathbf{i} = -\mathbf{k}$, $\mathbf{k} \times \mathbf{j} = -\mathbf{i}$ and $\mathbf{i} \times \mathbf{k} = -\mathbf{j}$.

You can set out all the possible products of $a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ in a square.

	$b_1\mathbf{i}$	$b_2\mathbf{j}$	$b_3\mathbf{k}$
$a_1\mathbf{i}$	$a_1\mathbf{i} \times b_1\mathbf{i}$	$a_1\mathbf{i} \times b_2\mathbf{j}$	$a_1\mathbf{i} \times b_3\mathbf{k}$
$a_2\mathbf{j}$	$a_2\mathbf{j} \times b_1\mathbf{i}$	$a_2\mathbf{j} \times b_2\mathbf{j}$	$a_2\mathbf{j} \times b_3\mathbf{k}$
$a_3\mathbf{k}$	$a_3\mathbf{k} \times b_1\mathbf{i}$	$a_3\mathbf{k} \times b_2\mathbf{j}$	$a_3\mathbf{k} \times b_3\mathbf{k}$

The products that are not zero are the ones on the outside.

One way to remember which are positive and which are negative is to imagine the letters $\mathbf{i} \rightarrow \mathbf{j} \rightarrow \mathbf{k}$ as a clockwise cycle: if you multiply in order of the cycle, the result is positive; if you multiply in the opposite order, the result is negative. This is shown as a diagram below.



Some students and teachers prefer the ‘**determinant-like**’ method.

To find the vector product of $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$, write the unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ and the components in a square pattern as shown.

$$\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}$$

To find each component of $\mathbf{a} \times \mathbf{b}$, cross-multiply and subtract the components in the square *outside* the column for that component, making the $\mathbf{j}$ component negative.

$$\begin{bmatrix} \mathbf{i} & & \\ & a_2 & a_3 \\ & b_2 & b_3 \end{bmatrix} \quad \begin{bmatrix} & \mathbf{j} & \\ a_1 & & a_3 \\ b_1 & & b_3 \end{bmatrix} \quad \begin{bmatrix} & & \mathbf{k} \\ a_1 & a_2 & \\ b_1 & b_2 & \end{bmatrix}$$

$$\mathbf{a} \times \mathbf{b} = (a_2b_3 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$$

This method is a little more tedious to write out, but has the advantage of a clear pattern.

○ Example 10

- Find the vector product between two vectors of magnitudes 13 and 18 if the angle between them is 48° .
- Find the vector product of $\mathbf{u} = (1, -2, 4)$ and $\mathbf{v} = (-2, 3, -1)$.

Solution

- Find the magnitude.

$$\begin{aligned} |\mathbf{v}| &= |\mathbf{v}_1| |\mathbf{v}_2| \sin(\theta) \\ &= 13 \times 18 \times \sin(48^\circ) \\ &\approx 173.9 \end{aligned}$$

Write the answer.

The vector product is about 174 in a direction perpendicular to the plane of the original vectors.



b Write in terms of **i**, **j** and **k**.

Use the cycle to get signs.

Simplify.

You can use the determinant-like method instead. Write in the square form.

One, two, miss a few!

Write out, with the **j** component negative.

Simplify.

Ti-Nspire CAS

Enter the vectors.

Type crossP or use , 7: Matrix & Vector, C: Vector and 2: Cross Product.

ClassPad

Enter the vectors. To obtain the cross product command, tap **Action Vector** then **crossP**. Enter **u**  **v**  **EXE**.

Write the answer in the original form.

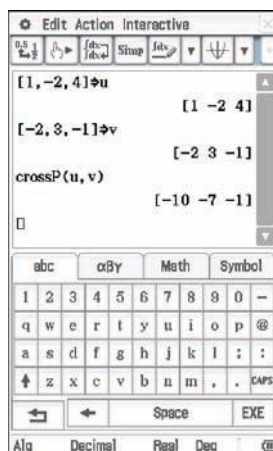
$$\mathbf{u} = \mathbf{i} - 2\mathbf{j} + 4\mathbf{k}, \mathbf{v} = -2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$$

$$\begin{aligned} \mathbf{u} \times \mathbf{v} &= [-2 \times (-1) - 4 \times 3]\mathbf{i} \\ &\quad + [4 \times (-2) - 1 \times (-1)]\mathbf{j} \\ &\quad + [1 \times 3 - (-2) \times (-2)]\mathbf{k} \\ &= -10\mathbf{i} - 7\mathbf{j} - \mathbf{k} \end{aligned}$$

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 4 \\ -2 & 3 & -1 \end{vmatrix}$$

$$\begin{vmatrix} \mathbf{i} & -2 & 4 \\ 3 & -1 & \end{vmatrix} \begin{vmatrix} \mathbf{j} & 1 & 4 \\ -2 & -1 & \end{vmatrix} \begin{vmatrix} \mathbf{k} & 1 & -2 \\ -2 & -2 & 3 \end{vmatrix}$$

$$\begin{aligned} \mathbf{u} \times \mathbf{v} &= [-2 \times (-1) - 4 \times 3]\mathbf{i} \\ &\quad - [1 \times (-1) - 4 \times (-2)]\mathbf{j} \\ &\quad + [1 \times 3 - (-2) \times (-2)]\mathbf{k} \\ &= -10\mathbf{i} - 7\mathbf{j} - \mathbf{k} \end{aligned}$$



The vector product is $(-10, -7, -1)$.

INVESTIGATION Properties of the vector product

- 1 Consider two vectors $\mathbf{v}_1 = (a_1, a_2, a_3)$ and $\mathbf{v}_2 = (b_1, b_2, b_3)$.
Use the scalar product to find the cosine of the angle between them.
Use the result that $\sin(\theta) = \sqrt{1 - \cos^2(\theta)}$ to find the sine of the angle between them.
Hence find the magnitude of the vector product.
- 2 Consider the vector product that is obtained when using the component rule method. Find the norm of the vector product.
Compare your answer with the size obtained from the scalar product method.
- 3 Now use the scalar product to show that the component form of the vector product is perpendicular to both of the original vectors.
- 4 Comment on the difficulties of proving that the component form of the vector product is the same as the form that uses the angle between the vectors.

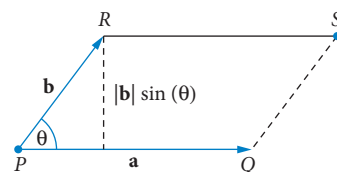
The properties of the vector product are summarised below.

IMPORTANT

Properties of the vector product

- 1 $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}| |\mathbf{b}| \sin(\theta)$, where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$.
- 2 The vector product $\mathbf{a} \times \mathbf{b}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$.
- 3 $\mathbf{a} \times \mathbf{a} = \mathbf{0}$ for any vector $\mathbf{a}$.
- 4 The vector product is left- and right-distributive over vector addition.
- 5 $\mathbf{a} \times \mathbf{b} = \mathbf{0} \Rightarrow \mathbf{a} = \mathbf{0}$ or $\mathbf{b} = \mathbf{0}$ or $\mathbf{a}$ and $\mathbf{b}$ are parallel.
- 6 $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a} \forall$ vectors $\mathbf{a}, \mathbf{b}$.
- 7 $m(\mathbf{a} \times \mathbf{b}) = (m\mathbf{a}) \times \mathbf{b} = \mathbf{a} \times (m\mathbf{b}) \forall m \in \mathbf{R}$ and vectors $\mathbf{a}, \mathbf{b}$.

You can interpret $|\mathbf{a} \times \mathbf{b}|$ geometrically. Consider two vectors $\mathbf{a} = \mathbf{PR}$ and $\mathbf{b} = \mathbf{PQ}$ with an angle of θ between them as shown here. The parallelogram formed with adjacent sides PR and QS is $PQRS$.



The height of $PQRS$ is $|b| \sin(\theta)$.

Hence, the area of $PQRS$ is $|\mathbf{a}| |b| \sin(\theta)$.

But $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}| |b| \sin(\theta)$

IMPORTANT

The magnitude of the vector product $\mathbf{a} \times \mathbf{b}$ equals the area of the parallelogram determined by $\mathbf{a}$ and $\mathbf{b}$.

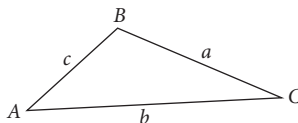


○ Example 11

Use the vector product to find the area of the triangle with vertices $A(-1, 3, 2)$, $B(3, 5, 1)$ and $C(1, 6, -2)$.

Solution

Make a general sketch of the triangle.
Label the sides and angles.



Write a formula for the area.

$$\text{Area} = \frac{1}{2}ab \sin(C)$$

Express it in terms of the vector product.

$$= \frac{1}{2}|\mathbf{BC} \times \mathbf{AC}|$$

Express the sides in terms of position vectors.

$$= \frac{1}{2}|(\mathbf{c} - \mathbf{b}) \times (\mathbf{c} - \mathbf{a})|$$

Find the sides.

$$\begin{aligned}\mathbf{c} - \mathbf{b} &= (1, 6, -2) - (3, 5, 1) \\ &= (-2, 1, -3) \\ &= -2\mathbf{i} + \mathbf{j} - 3\mathbf{k}\end{aligned}$$

$$\begin{aligned}\mathbf{c} - \mathbf{a} &= (1, 6, -2) - (-1, 3, 2) \\ &= (2, 3, -4) \\ &= 2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}\end{aligned}$$

Find the vector product.

$$\begin{aligned}(\mathbf{c} - \mathbf{b}) \times (\mathbf{c} - \mathbf{a}) &= (-2\mathbf{i} + \mathbf{j} - 3\mathbf{k}) \times (2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}) \\ &= [1 \times (-4) - (-3) \times 3]\mathbf{i} \\ &\quad + [(-3) \times 2 - (-2) \times (-4)]\mathbf{j} \\ &\quad + [-2 \times 3 - 1 \times 2]\mathbf{k} \\ &= 5\mathbf{i} - 14\mathbf{j} - 8\mathbf{k}\end{aligned}$$

Find the norm.

$$\begin{aligned}|(\mathbf{c} - \mathbf{b}) \times (\mathbf{c} - \mathbf{a})| &= \sqrt{5^2 + (-14)^2 + (-8)^2} \\ &= \sqrt{285}\end{aligned}$$

Find the area.

$$\begin{aligned}\text{Area} &= \frac{1}{2}|(\mathbf{c} - \mathbf{b}) \times (\mathbf{c} - \mathbf{a})| \\ &= \frac{1}{2} \times \sqrt{285} \\ &\approx 8.44\end{aligned}$$

Write the answer.

The area of the triangle is about 8.44 square units.

○ Example 12

Find a vector normal to the plane containing the points $A(3, -2, 1)$, $B(1, 3, 5)$ and $C(-2, 4, -1)$.

Solution

The vector product of two vectors is normal to the plane containing the vectors. You can use the product $\mathbf{AB} \times \mathbf{AC}$, say, to find a normal vector.

Find two vectors in the plane.

$$\begin{aligned}\mathbf{AB} &= \mathbf{b} - \mathbf{a} \\ &= (\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}) - (3\mathbf{i} - 2\mathbf{j} + \mathbf{k}) \\ &= -2\mathbf{i} + 5\mathbf{j} + 4\mathbf{k}\end{aligned}$$

$$\begin{aligned}\mathbf{AC} &= \mathbf{c} - \mathbf{a} \\ &= (-2\mathbf{i} + 4\mathbf{j} - \mathbf{k}) - (3\mathbf{i} - 2\mathbf{j} + \mathbf{k}) \\ &= -5\mathbf{i} + 6\mathbf{j} - 2\mathbf{k}\end{aligned}$$

Find the vector product.

$$\begin{aligned}\mathbf{AB} \times \mathbf{AC} &= (-2\mathbf{i} + 5\mathbf{j} + 4\mathbf{k}) \times (-5\mathbf{i} + 6\mathbf{j} - 2\mathbf{k}) \\ &= [5 \times (-2) - 4 \times 6]\mathbf{i} \\ &\quad - [4 \times (-5) - (-2) \times (-2)]\mathbf{j} \\ &\quad + [-2 \times 6 - 5 \times (-5)]\mathbf{k} \\ &= -34\mathbf{i} + 24\mathbf{j} + 13\mathbf{k}\end{aligned}$$

Write the answer in the original form.

The vector $(-34, 24, 13)$ is normal to the plane.

In the last two examples, you could choose different displacement vectors for the vector product. The only difference in the end result would be a possible change of sign, since the area of the triangle is unchanged.

EXERCISE 1.04 The vector product



Vector product

Concepts and techniques

- Example 10** Given $\mathbf{a} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$, $\mathbf{b} = 3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and $\mathbf{c} = -\mathbf{i} - 2\mathbf{j} - 3\mathbf{k}$, $\mathbf{a} \times \mathbf{b}$ is equal to:
 A $\mathbf{a} \cdot \mathbf{b}$ B $\mathbf{b} \times \mathbf{c}$ C $\mathbf{b} \times \mathbf{a}$ D $\mathbf{b} \cdot \mathbf{a}$ E $\mathbf{c} \times \mathbf{b}$
- Given $P(6, -3, 9)$ and $Q(-4, 2, -1)$, $\mathbf{p} \times \mathbf{q}$ is equal to:
 A $\mathbf{OP} \times \mathbf{PQ}$ B $\mathbf{PQ} \times \mathbf{p}$ C $\mathbf{PQ} \times \mathbf{q}$ D $\mathbf{0}$ E $\mathbf{PQ} \times \mathbf{OQ}$
- Given the vectors $\mathbf{m}$ and $\mathbf{n}$, $\mathbf{m} \times \mathbf{n}$ is:
 A parallel to $\mathbf{m}$ and perpendicular to $\mathbf{n}$
 B parallel to $\mathbf{m}$ and $\mathbf{n}$
 C parallel to $\mathbf{n}$ and perpendicular to $\mathbf{m}$
 D perpendicular to $\mathbf{m}$ and $\mathbf{n}$
 E None of the above is true.
- In each of the following cases, the magnitude of two vectors and the angle between them is given. Find the vector products of the pairs of vectors in each case.

a 3, 8, 120°	b 10, 2, 78°	c 12, 12, 200°	d 5, 2, 0°
e 6, 8, 47°	f 5, 14, 330°	g 7, 11, 90°	h 4, 7, 60°



5 Find the following vector products.

a $(3, 180^\circ) \times (79, 25^\circ)$

b $(6, 251^\circ) \times (32, 80^\circ)$

c $(73, 60^\circ) \times (9, -20^\circ)$

d $(6, 79^\circ, 28^\circ) \times (1, 79^\circ, 78^\circ)$

e $(3, 150^\circ, 79^\circ) \times (2, 150^\circ, -85^\circ)$

f $(7, 21^\circ, -66^\circ) \times (5, 21^\circ, 14^\circ)$

6 Find the following vector products.

a $(6, 5) \times (9, -2)$

b $(-8, 70) \times (1, 7)$

c $(-9, 5) \times (5, -9)$

d $(8, 9, 2) \times (1, 7, 7)$

e $(6, -2, 1) \times (1, -9, -5)$

f $(3, 8, -17) \times (9, -6, 2)$

g $(5, -6, 1) \times (-8, -6, -4)$

h $(-2, 8, -5) \times (-1, 7, 6)$

i $(7, 0, 1) \times (-5, -6, 8)$

7 Find the following vector products.

a $\begin{bmatrix} 3 \\ -5 \end{bmatrix} \times \begin{bmatrix} -5 \\ 3 \end{bmatrix}$

b $\begin{bmatrix} 2 \\ 9 \end{bmatrix} \times \begin{bmatrix} -2 \\ -9 \end{bmatrix}$

c $\begin{bmatrix} -3 \\ 8 \end{bmatrix} \times \begin{bmatrix} -3 \\ 8 \end{bmatrix}$

d $\begin{bmatrix} 7 \\ -3 \\ 1 \end{bmatrix} \times \begin{bmatrix} 4 \\ 2 \\ -5 \end{bmatrix}$

e $\begin{bmatrix} 2 \\ 9 \\ 5 \end{bmatrix} \times \begin{bmatrix} 3 \\ -4 \\ 6 \end{bmatrix}$

f $\begin{bmatrix} 1 \\ 0 \\ 7 \end{bmatrix} \times \begin{bmatrix} -6 \\ 9 \\ 5 \end{bmatrix}$

8 Given the vectors $\mathbf{a} = 3\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$, $\mathbf{b} = 2\mathbf{i} + \mathbf{j} - 5\mathbf{k}$, $\mathbf{c} = \mathbf{i} - 5\mathbf{j} - 4\mathbf{k}$, $\mathbf{d} = \mathbf{j} - 2\mathbf{i} + 3\mathbf{k}$ and $\mathbf{e} = 4\mathbf{k} - 3\mathbf{j}$, find:

a $\mathbf{a} \times \mathbf{b}$

b $\mathbf{c} \times \mathbf{e}$

c $\mathbf{c} \times (\mathbf{d} - \mathbf{e})$

d $\mathbf{c} \times \mathbf{d} - \mathbf{c} \times \mathbf{e}$

e $\mathbf{e} \times \mathbf{d}$

f $\mathbf{d} \times \mathbf{e}$

g $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$

h $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$

9 Given the vectors $\mathbf{a} = -\mathbf{i} + 6\mathbf{j} - 4\mathbf{k}$, $\mathbf{b} = 7\mathbf{i} - 4\mathbf{j} + 11\mathbf{k}$, $\mathbf{c} = -2\mathbf{i} + 12\mathbf{j} + 9\mathbf{k}$, $\mathbf{d} = 14\mathbf{i} - 7\mathbf{j}$, $\mathbf{e} = 12\mathbf{i} + 28\mathbf{j} + 15\mathbf{k}$, $\mathbf{f} = 12\mathbf{i} - 8\mathbf{j} - 2\mathbf{k}$, $\mathbf{g} = -3\mathbf{i} - 2\mathbf{j} - \mathbf{k}$ and $\mathbf{h} = 34\mathbf{i} - 22\mathbf{j} - 24\mathbf{k}$, find

a $\mathbf{a} \times \mathbf{b}$

b $\mathbf{c} \times \mathbf{a}$

c $\mathbf{a} \times \mathbf{c}$

d $\mathbf{c} \times \mathbf{d}$

e $\mathbf{b} \times (\mathbf{c} \times \mathbf{d})$

f $\mathbf{b} \times \mathbf{c}$

g $(\mathbf{b} \times \mathbf{c}) \times \mathbf{d}$

h $\mathbf{e} \times \mathbf{f}$

i $\mathbf{e} \times (\mathbf{g} + \mathbf{h})$

j $\mathbf{e} \times \mathbf{g} + \mathbf{e} \times \mathbf{h}$

k $\mathbf{e} \times \mathbf{g} + \mathbf{h} \times \mathbf{e}$

10 **Example 11** If a triangle has vertices at $O(0, 0, 0)$, $P(1, 2, 3)$ and $Q(-1, 2, -3)$, then the area of $\triangle OPQ$ is:

A $\sqrt{13}$

B $2\sqrt{10}$

C $2\sqrt{13}$

D 10

E $4\sqrt{10}$

11 Use the vector product to find the areas of triangles with vertices:

a $(2, 3)$, $(6, 4)$ and $(4, -2)$

b $(3, 2, 5)$, $(6, -3, 4)$ and $(-2, -3, 1)$

c $(2, 3, 4)$, $(-1, 2, 0)$ and $(3, 0, 2)$

12 Use the vector product to find the areas of parallelograms with vertices:

a $(3, -2)$, $(5, 3)$, $(7, 5)$ and $(5, 0)$

b $(4, -2, 3)$, $(-1, 3, 4)$, $(2, 4, -2)$ and $(7, -1, -3)$.

13 **Example 12** Find a vector normal to the plane containing the points:

a $(3, 2, 0)$, $(1, 6, -2)$ and $(7, 4, 5)$

b $(1, 8, 7)$, $(-2, -1, 3)$ and $(1, -2, -5)$.

Reasoning and communication

14 If $\mathbf{a} = 5\mathbf{i} + 7\mathbf{j} + 2\mathbf{k}$, $\mathbf{b} = 3\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$ and $\mathbf{c} = 2\mathbf{i} + \mathbf{j} + 5\mathbf{k}$, show that:

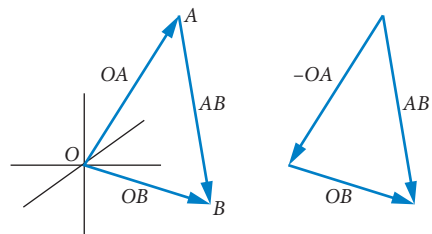
$\mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a} = \mathbf{0}$

- 15 For two vectors $\mathbf{a}$ and $\mathbf{b}$, $\mathbf{a} \times \mathbf{b} = \mathbf{b} \times \mathbf{a}$. Write a relationship between $\mathbf{a}$ and $\mathbf{b}$ and explain your reasoning.
- 16 Let $A(1, 3, 4)$, $B(2, -2, 6)$ and $C(5, 2, 3)$ be three vertices of the parallelogram $ABCD$. Find:
 - a the coordinates of D
 - b the area of $ABCD$
- 17 Find a unit vector that is normal to the plane containing the points $(1, 2, -3)$, $(-2, 4, 1)$ and $(5, -1, -3)$.
- 18 Use a counterexample to show that the vector product is not commutative.
- 19 Use a counterexample to show that the vector product is not associative.

1.05 CARTESIAN GEOMETRY

You should remember the following, which are important for this section.

You can write a vector as a linear combination of vectors in particular directions. This is called **resolution** of the vector. **Orthogonal** components are at right angles to each other.



A displacement is a change of position. The displacement $\mathbf{AB}$ is the change of position from point A to point B .

The position vector of a point is its displacement from the origin. The position vector of point A is written as $\mathbf{OA}$.

Point O is always used for the origin, and the symbol $\mathbf{a}$ may be used instead of $\mathbf{OA}$ for the position vector of point A , so $\mathbf{AB} = \mathbf{OB} - \mathbf{OA} = \mathbf{b} - \mathbf{a}$.

The scalar product can be used for geometric proofs involving perpendicularity. The following results are useful in geometric proofs.

- $k\mathbf{a}$, where $k \in \mathbf{R}^+$, is a vector in the same direction as $\mathbf{a}$ but with a magnitude of $k|\mathbf{a}|$.
- $-\mathbf{ka}$, where $k \in \mathbf{R}^+$, is a vector in the opposite direction to $\mathbf{a}$ but with a magnitude of $k|\mathbf{a}|$.
- If $\mathbf{a}$ is parallel to $\mathbf{b}$, there exists some real number k ($k \neq 0$) such that $\mathbf{b} = k\mathbf{a}$. Conversely, if $\mathbf{b} = k\mathbf{a}$, then $\mathbf{a}$ and $\mathbf{b}$ are parallel.
- If $\mathbf{AB}$ and $\mathbf{BC}$ are parallel, then A , B and C are collinear.
- If $\mathbf{a} \cdot \mathbf{b} = 0$, then $\mathbf{a} = \mathbf{0}$ or $\mathbf{b} = \mathbf{0}$ or $\mathbf{a}$ and $\mathbf{b}$ are perpendicular.

Or if $\mathbf{a}$ and $\mathbf{b}$ are perpendicular, $\mathbf{a} \cdot \mathbf{b} = 0$.

- $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$



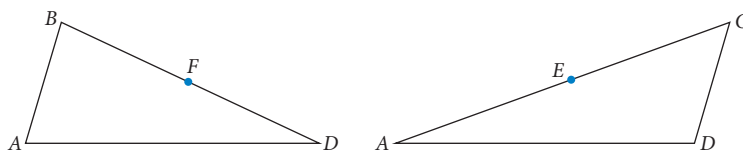
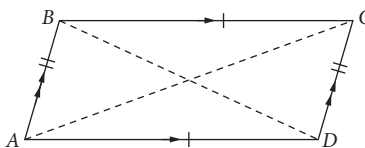
○ Example 13

Show that the diagonals of a parallelogram bisect each other.

Solution

Draw a diagram of a parallelogram $ABCD$ and diagonals BD and AC .

Show the midpoints F and E of BD and AC respectively.



Stating that the diagonals bisect each other is the same as saying that the midpoints of BD and AC coincide. Try to prove that E and F are the same point.

Find the position vector of E .

$$\mathbf{OE} = \mathbf{OA} + \frac{1}{2}\mathbf{AC}$$

Simplify.

$$= \mathbf{a} + \frac{1}{2}(\mathbf{c} - \mathbf{a})$$

$$= \mathbf{a} + \frac{1}{2}\mathbf{c} - \frac{1}{2}\mathbf{a}$$

$$= \frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{c}$$

$$= \frac{1}{2}(\mathbf{a} + \mathbf{c})$$

Find the position vector of F .

$$\mathbf{OF} = \mathbf{OB} + \frac{1}{2}\mathbf{BD}$$

Simplify as above.

$$= \frac{1}{2}(\mathbf{b} + \mathbf{d})$$

Use parallelogram properties.

$$\mathbf{AB} = \mathbf{DC}, \text{ so } \mathbf{b} - \mathbf{a} = \mathbf{c} - \mathbf{d}$$

Rearrange to suit $\mathbf{OE}$ and $\mathbf{OF}$.

$$\mathbf{b} + \mathbf{d} = \mathbf{a} + \mathbf{c}$$

Substitute in the $\mathbf{OF}$ expression.

$$\mathbf{OF} = \frac{1}{2}(\mathbf{b} + \mathbf{d})$$

$$= \frac{1}{2}(\mathbf{a} + \mathbf{c})$$

Use the $\mathbf{OE}$ expression.

$$= \mathbf{OE}$$

Write a conclusion.

Since $\mathbf{OE}$ and $\mathbf{OF}$ are the same, E and F are the same, so the diagonals of the parallelogram bisect each other.

You should remember the following important result demonstrated in Example 13.

IMPORTANT

If P is the **midpoint** of AB , its position vector is given by $\mathbf{p} = \frac{1}{2}(\mathbf{a} + \mathbf{b})$.

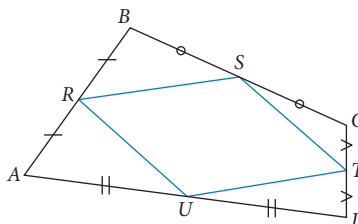
○ Example 14

Show that the midpoints of a quadrilateral form a parallelogram.

Solution

Let $ABCD$ be a quadrilateral, and let the midpoints of AB , BC , CD and DA be R , S , T and U respectively.

Draw a diagram.



Use the fact that R , S , T and U are midpoints.

$$AB = 2AR = 2RB$$

$$BC = 2BS = 2SC$$

$$CD = 2CT = 2TD$$

$$DA = 2DU = 2UA$$

The displacement from A to A is zero.

$$AB + BC + CD + DA = 0$$

Substitute.

$$2RB + 2BS + 2TD + 2DU = 0$$

$$RB + BS + TD + DU = 0$$

Rearrange.

$$RB + BS = -(TD + DU)$$

Use vector addition.

$$RS = -TU = UT$$

Write a conclusion.

The side RS is equal and parallel to the side UT . This is sufficient to show that $RSTU$ is a parallelogram by itself, but you could also show that ST is equal and parallel to RU by choosing the other halves of the sides.

Note that in the previous proof you could use the fact that the sum of the displacement vectors of any closed shape must be the zero vector. It is also useful to remember that the sum of a series of displacements 'end-to-end' can be replaced by the total displacement by using just the first and last points, as in $AB + BP + PQ + QE = AE$.

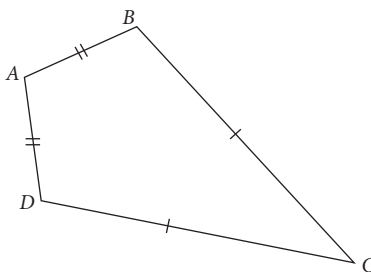


○ Example 15

Show that the diagonals of a kite are perpendicular.

Solution

Draw a kite $ABCD$ such that AC is a diagonal.
The short sides are of equal length and the long sides are of equal length.



Use equality of sides.

$$|AB| = |AD| \text{ and } |BC| = |DC|$$

Square.

$$|AB|^2 = |AD|^2 \text{ and } |BC|^2 = |DC|^2$$

Use $|\mathbf{v}|^2 = \mathbf{v} \cdot \mathbf{v}$.

$$\mathbf{AB} \cdot \mathbf{AB} = \mathbf{AD} \cdot \mathbf{AD}$$

Substitute position vectors.

$$(\mathbf{b} - \mathbf{a}) \cdot (\mathbf{b} - \mathbf{a}) = (\mathbf{d} - \mathbf{a}) \cdot (\mathbf{d} - \mathbf{a})$$

Expand the brackets.

$$\mathbf{b}^2 - 2\mathbf{a} \cdot \mathbf{b} + \mathbf{a}^2 = \mathbf{d}^2 - 2\mathbf{a} \cdot \mathbf{d} + \mathbf{a}^2$$

Rearrange.

$$2\mathbf{a} \cdot \mathbf{d} - 2\mathbf{a} \cdot \mathbf{b} = \mathbf{d}^2 - \mathbf{b}^2$$

Simplify and rearrange.

$$\mathbf{a} \cdot (\mathbf{d} - \mathbf{b}) = \frac{1}{2}(\mathbf{d}^2 - \mathbf{b}^2)$$

Do the same for the other sides.

$$\mathbf{BC} \cdot \mathbf{BC} = \mathbf{DC} \cdot \mathbf{DC}$$

$$(\mathbf{c} - \mathbf{b}) \cdot (\mathbf{c} - \mathbf{b}) = (\mathbf{c} - \mathbf{d}) \cdot (\mathbf{c} - \mathbf{d})$$

$$\mathbf{c}^2 - 2\mathbf{b} \cdot \mathbf{c} + \mathbf{b}^2 = \mathbf{c}^2 - 2\mathbf{c} \cdot \mathbf{d} + \mathbf{d}^2$$

$$\mathbf{c} \cdot (\mathbf{d} - \mathbf{b}) = \frac{1}{2}(\mathbf{d}^2 - \mathbf{b}^2)$$

Find the dot products of the diagonals.

$$\mathbf{AC} \cdot \mathbf{BD} = (\mathbf{c} - \mathbf{a}) \cdot (\mathbf{d} - \mathbf{b})$$

$$= \mathbf{c} \cdot (\mathbf{d} - \mathbf{b}) - \mathbf{a} \cdot (\mathbf{d} - \mathbf{b})$$

$$= \frac{1}{2}(\mathbf{d}^2 - \mathbf{b}^2) - \frac{1}{2}(\mathbf{d}^2 - \mathbf{b}^2)$$

$$= 0$$

Write a conclusion.

Since $\mathbf{AC} \cdot \mathbf{BD} = 0$, the vectors are perpendicular, so the diagonals of a kite intersect at right angles.

EXERCISE 1.05 Cartesian geometry



Geometric proofs using vectors

Concepts and techniques

- 1 **Examples 13–15** $ABCD$ shown on the right is a quadrilateral. Points P , Q , R and S are the midpoints of AD , DC , AB and BC respectively.

Complete the missing parts of the following proof that PQ and RS are parallel and $|PQ| = |RS|$.

Proof

Express $\mathbf{q}$ in terms of $\mathbf{c}$ and $\mathbf{d}$. $\mathbf{q} = \frac{\dots}{2} + \frac{\dots}{2}$

Express $\mathbf{p}$ in terms of $\mathbf{a}$ and $\mathbf{d}$. $\mathbf{p} = \frac{\dots}{2} + \frac{\dots}{2}$

Express $\mathbf{s}$ in terms of $\mathbf{c}$ and $\mathbf{b}$. $\mathbf{s} = \frac{\dots}{2} + \frac{\dots}{2}$

Express $\mathbf{r}$ in terms of $\mathbf{a}$ and $\mathbf{b}$. $\mathbf{r} = \frac{\dots}{2} + \frac{\dots}{2}$

Write $\mathbf{PQ}$ in terms of $\mathbf{a}$, $\mathbf{c}$ and $\mathbf{d}$. $\mathbf{PQ} = \frac{\dots}{2} + \frac{\dots}{2} - \frac{\dots}{2} - \frac{\dots}{2}$

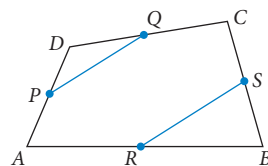
Simplify. $\mathbf{PQ} = \frac{\dots}{2} - \frac{\dots}{2}$

Write $\mathbf{RS}$ in terms of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$. $\mathbf{RS} = \frac{\dots}{2} + \frac{\dots}{2} - \frac{\dots}{2} - \frac{\dots}{2}$

Simplify. $\mathbf{RS} = \frac{\dots}{2} - \frac{\dots}{2}$

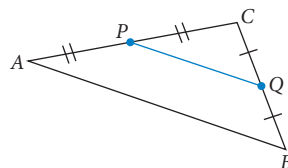
Hence. $\mathbf{PQ} = \dots$

Therefore $\mathbf{PQ} \parallel \dots$ and $|\mathbf{PQ}| = |\dots|$

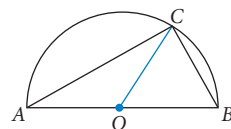


Reasoning and communication

- 2 For $\triangle ABC$, P and Q are the midpoints of AC and BC respectively. Show that $\mathbf{PQ} \parallel \mathbf{AB}$ and equal to half its length.

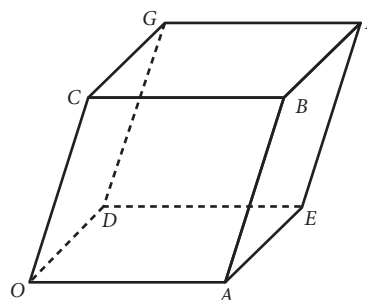
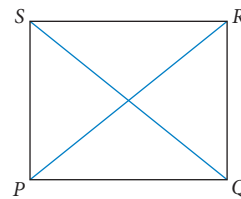


- 3 Given a semicircle with centre O and diameter AB , prove that the angle subtended at the circumference is a right angle.

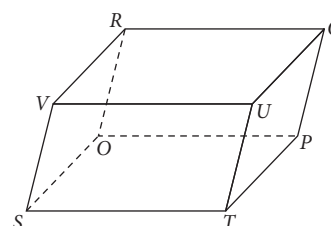




- 4 $PQRS$ is a rectangle. Use the scalar product to prove that if $\mathbf{PR} \perp \mathbf{SQ}$, then $PQRS$ is a square.
- 5 Find a vector expression in terms of $\mathbf{a}$ and $\mathbf{b}$ for the position vector of the point P on AB that divides it in the ratio $k_1 : k_2$.
- 6 In the triangle ABC , a median is drawn from A to the midpoint D of the side BC . The point M divides AD in the ratio $2 : 1$.
- Find an expression for $\mathbf{m}$ in terms of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.
 - Use the result from part **a** to show that the medians are concurrent.
 - Write an expression for the position vector of the centroid of ABC .
- 7 In a triangle ABC , the point D divides BC in the ratio $|\mathbf{AB}| : |\mathbf{AC}|$.
- Express $\mathbf{d}$ in terms of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.
 - Use the scalar product to find $\cos(\angle DAC) - \cos(\angle DAB)$.
 - Use your results to make an inference about the angle bisectors of a triangle.
- 8 If G is the centroid of a triangle ABC , prove that $\mathbf{GA} + \mathbf{GB} + \mathbf{GC} = \mathbf{0}$.
- 9 $ABCD$ is a tetrahedron. H is the point that divides the line joining A to the centroid of BCD in the ratio $3 : 1$.
- Find an expression for the position vector of H in terms of $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$.
 - Show that the four lines joining the vertices of $ABCD$ to the centroids of the opposite faces are concurrent.
- 10 Given a tetrahedron $ABCD$, find an expression for the midpoint of the line joining the midpoints of the edges AB and CD .
- 11 Use the scalar product to prove that opposite angles in a parallelogram are equal.
- 12 Use the scalar product to prove that, if the opposite sides of a quadrilateral are parallel and equal in length, the quadrilateral must be a parallelogram.
- 13 Use the vector product to prove that a quadrilateral whose diagonals bisect each other and are equal in length must be a rectangle.
- 14 Prove that one diagonal of a kite is bisected by the other.
- 15 Consider the parallelepiped $OABCDEFG$. If $\mathbf{OA} = \mathbf{a}$, $\mathbf{OD} = \mathbf{d}$ and $\mathbf{OC} = \mathbf{c}$, show that $|\mathbf{OF}|^2 = |\mathbf{a}|^2 + |\mathbf{c}|^2 + |\mathbf{d}|^2 + 2(\mathbf{d} \cdot \mathbf{a} + \mathbf{d} \cdot \mathbf{c} + \mathbf{a} \cdot \mathbf{c})$



- 16 Consider the parallelepiped $OPQRSTUV$. Show that the diagonals SQ and RT bisect each other.



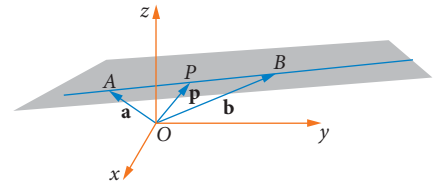
1.06 LINES IN 3D

Consider the line AB shown here in three-dimensional space.

A and B are points on the line, and have position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively. P is any other point on the line, and has position vector $\mathbf{p}$. Because A , P and B are collinear, $\mathbf{AB}$ and $\mathbf{AP}$ are parallel, so $\mathbf{AP}$ is a scalar multiple of $\mathbf{AB}$.

$$\mathbf{AP} = t\mathbf{AB} \quad \text{where } t \text{ is a real number.}$$

This gives $\mathbf{p} = \mathbf{a} + \mathbf{AP} = \mathbf{a} + t\mathbf{AB} = \mathbf{a} + t(\mathbf{b} - \mathbf{a})$



IMPORTANT

The **vector equation** of the line through points A and B with position vectors is:

$$\mathbf{p} = \mathbf{a} + t\mathbf{AB} \quad (\text{for some real number } t)$$

Alternatively, if $\mathbf{u}$ is any vector in the direction of the line AB and if $\mathbf{w}$ is any point on the line, then $\mathbf{p} = \mathbf{w} + t\mathbf{u}$. Of course, the value of t will be different for each point.

By varying the value of t , you can locate any point on the line.

- If $0 \leq t \leq 1$, point P lies on the **line segment** AB .
- If $t > 1$, point P lies on the line beyond B .
- If $t < 0$, point P lies on the line beyond A to the left, as shown in the previous diagram.

For $\mathbf{u}$ parallel to $\mathbf{AB}$, $\mathbf{p} = \mathbf{a} + t\mathbf{u}$ locates any point on the line AB . The choice of A is arbitrary, as long as it is on the line.

○ Example 16

- Find the vector equation of the line which passes through the points $A(3, 1)$ and $B(5, -2)$
- Find the vector equation of the line segment from $C(2, -3, 5)$ to $D(-1, 7, 4)$.

Solution

- Find $\mathbf{AB}$.

$$\mathbf{AB} = (5\mathbf{i} - 2\mathbf{j}) - (3\mathbf{i} + \mathbf{j})$$

Simplify.

$$= 2\mathbf{i} - 3\mathbf{j}$$

Write the vector equation.

$$\mathbf{p} = \mathbf{a} + t\mathbf{AB}$$

Substitute values.

$$\mathbf{p} = 3\mathbf{i} + \mathbf{j} + t(2\mathbf{i} - 3\mathbf{j})$$



b Find $\mathbf{CD}$.

$$\mathbf{CD} = (2\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}) - (-\mathbf{i} + 7\mathbf{j} + 4\mathbf{k})$$

Simplify.

$$= 3\mathbf{i} - 10\mathbf{j} + \mathbf{k}$$

Write the vector equation.

$$\mathbf{p} = \mathbf{c} + t\mathbf{CD}$$

Substitute values and restrict t .

$$\mathbf{p} = -\mathbf{i} + 7\mathbf{j} + 4\mathbf{k} + t(3\mathbf{i} - 10\mathbf{j} + \mathbf{k}) \text{ for } 0 \leq t \leq 1$$

Sometimes it is useful to convert the vector form of the equation of a line into the Cartesian form. You do this by eliminating the parameter t . Consider the points $A(a_1, a_2, a_3)$, $B(b_1, b_2, b_3)$ and $P(x, y, z)$ lying on the line AB .

$$\text{Then } \mathbf{p} = \mathbf{a} + t\mathbf{AB} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} + t \begin{bmatrix} b_1 - a_1 \\ b_2 - a_2 \\ b_3 - a_3 \end{bmatrix} = \begin{bmatrix} a_1 + t(b_1 - a_1) \\ a_2 + t(b_2 - a_2) \\ a_3 + t(b_3 - a_3) \end{bmatrix}$$

$$\text{So } x = a_1 + t(b_1 - a_1)$$

$$y = a_2 + t(b_2 - a_2)$$

$$z = a_3 + t(b_3 - a_3)$$

Rearranging these equations gives:

$$t = \frac{x - a_1}{b_1 - a_1}$$

$$t = \frac{y - a_2}{b_2 - a_2}$$

$$t = \frac{z - a_3}{b_3 - a_3}$$

Equating these expressions gives:

$$\frac{x - a_1}{b_1 - a_1} = \frac{y - a_2}{b_2 - a_2} = \frac{z - a_3}{b_3 - a_3}$$

IMPORTANT

The **Cartesian form** of the equation of a line in space passing through the points $A(a_1, a_2, a_3)$ and $B(b_1, b_2, b_3)$ is:

$$\frac{x - a_1}{b_1 - a_1} = \frac{y - a_2}{b_2 - a_2} = \frac{z - a_3}{b_3 - a_3}$$

You can also use the form $\mathbf{p} = \mathbf{a} + t\mathbf{u}$, where $\mathbf{u}$ is any vector in the direction of the line. In this case, you get $x = a_1 + tu_1$, etc, so the vector equation can be written as follows.

IMPORTANT

The vector equation of a line is $\mathbf{p} = \mathbf{a} + t\mathbf{u}$, where $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ is a vector in the direction of the line and $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ is a point on the line.

The corresponding Cartesian form is $\frac{x-a_1}{u_1} = \frac{y-a_2}{u_2} = \frac{z-a_3}{u_3}$.

The **parametric form** of a straight line was given in the derivation above:

$x = a_1 + t(b_1 - a_1)$, $y = a_2 + t(b_2 - a_2)$, $z = a_3 + t(b_3 - a_3)$. Since $b_1 - a_1$, $b_2 - a_2$ and $b_3 - a_3$ are constants, this can be shown as $x = a_1 + c_1t$, $y = a_2 + c_2t$, $z = a_3 + c_3t$. In the two

dimensional case of $x = a_1 + c_1t$, $y = a_2 + c_2t$, elimination of t gives $\frac{x-a_1}{c_1} = \frac{y-a_2}{c_2}$ or

$y = \frac{c_2}{c_1}x + a_2 - \frac{a_1c_2}{c_1}$ so the slope is $\frac{c_2}{c_1}$ and the y -intercept is $a_2 - \frac{a_1c_2}{c_1}$. If the parametric form of a

line is not linear (in the parameter) in at least one dimension, then the line will not be straight. For example, the simple 2-dimensional case $x = t + 1$ and $y = 4t^2 + 3$ is the parametric form of $y = 2(x - 1)^2 + 3$.

Example 17

Find the Cartesian form of the equation of the straight line which passes through the points $(8, 4, -2)$ and $(5, 2, -1)$.

Solution

Assign the points.

Let $A(a_1, a_2, a_3) = (8, 4, -2)$ and $B(b_1, b_2, b_3) = (5, 2, -1)$.

Write the Cartesian form of the equation.

$$\frac{x-a_1}{b_1-a_1} = \frac{y-a_2}{b_2-a_2} = \frac{z-a_3}{b_3-a_3}$$

Substitute for $a_1, a_2, \dots$

$$\frac{x-8}{5-8} = \frac{y-4}{2-4} = \frac{z-(-2)}{-1-(-2)}$$

Simplify.

$$\frac{x-8}{-3} = \frac{y-4}{-2} = \frac{z+2}{1}$$

The vector equation for the position of an object moving with constant velocity in a straight line is given by $\mathbf{p} = \mathbf{a} + \mathbf{v}t$, where $\mathbf{a}$ is the position when $t = 0$ and $\mathbf{v}$ is the velocity. This is the vector equation of a straight line, so you can use this to determine if the paths of two objects would cross, or even if they would collide.



○ Example 18

- a Find the vector equation of the line that passes through the point $A(1, -1, 0)$ that is parallel to the line passing through $M(-3, 2, 1)$ and $N(2, 1, 0)$.
- b Show that $B(-9, 1, 2)$ lies on the line.
- c Two objects are moving with constant velocity in straight lines. The first starts at $(0, 2, -1)$ and has velocity $(1, -1, 0)$ and the second has velocity $(0, 1, -1)$, starting from $(2, -3, 2)$. Do their paths cross and do they collide?

Solution

- a Find the vector $\mathbf{MN}$.

$$\mathbf{MN} = (2\mathbf{i} + \mathbf{j}) - (-3\mathbf{i} + 2\mathbf{j} + \mathbf{k})$$

Simplify.

$$= 5\mathbf{i} - \mathbf{j} - \mathbf{k}$$

Write the vector equation for the line.

$$\mathbf{p} = \mathbf{a} + t\mathbf{u}$$

Substitute.

$$= \mathbf{i} - \mathbf{j} + t(5\mathbf{i} - \mathbf{j} - \mathbf{k})$$

- b Write the position vector $\mathbf{b}$.

$$\mathbf{b} = -9\mathbf{i} + \mathbf{j} + 2\mathbf{k}$$

Write an equation for the value of t .

$$\text{Let } -9\mathbf{i} + \mathbf{j} + 2\mathbf{k} = (\mathbf{i} - \mathbf{j}) + t(5\mathbf{i} - \mathbf{j} - \mathbf{k})$$

Equate $\mathbf{i}$ parts.

$$-9 = 1 + 5t \text{ so } t = -2$$

Check $t = 2$ in the other equations.

Equate $\mathbf{j}$ parts.

$$1 = -1 + (-2)(-1) = 1 \checkmark$$

Equate $\mathbf{k}$ parts.

$$2 = 0 + (-2)(-1) = 2 \checkmark$$

State the result.

$B(-9, 1, 2)$ lies on the line that passes through $A(1, -1, 0)$, which is parallel to the line passing through $M(-3, 2, 1)$ and $N(2, 1, 0)$.

- c Write the position of the first object.

$$\text{Position of first, } \mathbf{p}_1 = 2\mathbf{j} - \mathbf{k} + t(\mathbf{i} - \mathbf{j})$$

Write the position of the second object.

$$\text{Position of second, } \mathbf{p}_2 = 2\mathbf{i} - 3\mathbf{j} + 2\mathbf{k} + t(\mathbf{j} - \mathbf{k})$$

Write the condition for the paths to cross.

The paths cross if $\mathbf{p}_1 = \mathbf{p}_2$ at times t and s , say.

Equate the $\mathbf{i}$ parts of $\mathbf{p}_1$ and $\mathbf{p}_2$.

$$t\mathbf{i} = 2\mathbf{i} + 0\mathbf{i}, \text{ so } t = 2$$

Substitute in the first equation.

$$\mathbf{p}_1 = 2\mathbf{j} - \mathbf{k} + 2(\mathbf{i} - \mathbf{j}) = 2\mathbf{i} - \mathbf{k}$$

Equate the $\mathbf{j}$ parts with time s in $\mathbf{p}_2$.

$$0\mathbf{j} = -3\mathbf{j} + s\mathbf{j} \text{ so } s = 3$$

Substitute in the second equation.

$$\mathbf{p}_2 = 2\mathbf{i} - 3\mathbf{j} + 2\mathbf{k} + 3(\mathbf{j} - \mathbf{k}) = 2\mathbf{i} - \mathbf{k}$$

State the result.

$2\mathbf{i} - \mathbf{k}$ is on both lines so the paths cross.

Write the times where the paths cross.

The first passes through $(2, 0, -1)$ at $t = 2$ and the second at $t = 3$.

Write the overall answer.

Their paths cross but they do not collide.

The method shown in part c above can be used to determine whether lines meet. If they are not parallel but do not meet, then they are **skew** lines.

EXERCISE 1.06 Lines in 3D



Concepts and techniques

- Example 16**
 - a Write down the vector $\mathbf{AB}$ that joins the points $A(3, 2, 7)$ and $B(-1, 2, 3)$.
 - b Find the vector equation of the line through A and B .
 - c How is the equation of the line segment AB different to the equation of the line AB ?
- 2 Write down the vector equation of the line segment between the points with position vectors $\mathbf{p} = 3\mathbf{i} + 7\mathbf{j} - 2\mathbf{k}$ and $\mathbf{q} = -3\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$. Also, find the Cartesian equation of this line.
- 3 Find the vector equation of the line passing through $(9, 1, 2)$, which is parallel to the vector $\mathbf{u} = \mathbf{i} + \mathbf{j} + \mathbf{k}$.
- Example 17** Find the Cartesian equation of the line passing through the origin and $(1, -5, 3)$.
- 5 Find the Cartesian equation of the line passing through $(1, 0, 1)$ and $(0, 1, 1)$.
- 6 Find the Cartesian form of the equation passing through the point $(1, 2, 1)$, which is parallel to $\mathbf{u} = 4\mathbf{i} + 5\mathbf{j} - \mathbf{k}$.
- 7 Find the vector equation of the line whose Cartesian equation is $\frac{x}{3} = \frac{y-1}{1} = \frac{z+1}{2}$.
- 8 A line passes through the point $(2, 3, 5)$ in the direction of $\mathbf{i} + 3\mathbf{j} - \mathbf{k}$. Find:
 - a the vector equation of the line
 - b the Cartesian equation of the line.
- 9 If $A = (1, 2, 3)$, $B = (-1, -2, -3)$ and $C = (4, -4, 4)$, find the Cartesian equation of:
 - a AB
 - b AC
- 10 Find the vector and Cartesian equations of the line passing through A , in the direction of $\mathbf{u}$ for each of the following.
 - a $A(1, 0, -1)$ and $\mathbf{u} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$
 - b $A(0, 1, -1)$ and $\mathbf{u} = \mathbf{i} + 2\mathbf{j}$
 - c $A(-6, 5, 2)$ and $\mathbf{u} = 2\mathbf{i} - 5\mathbf{k}$
 - d $A(2, 3, -3)$ and $\mathbf{u} = -\mathbf{i} + 3\mathbf{k}$
- 11 Find the vector and Cartesian equations of the line passing through A and B in each of the following cases.
 - a $A(-4, 3, 5)$ and $B(-2, 4, 1)$
 - b $A(0, 5, 0)$ and $B(5, 0, -5)$
 - c $A(9, 3, -2)$ and $B(4, 5, -1)$
 - d $A(2, -1, 3)$ and $B(1, 4, -3)$
- Example 18** The Cartesian equation of a line is given by: $\frac{x-1}{1} = \frac{y}{2} = \frac{z-1}{3}$.

Do the points $P(3, 2, 6)$ and $Q(-1, -4, -5)$ lie on the line?
- 13 Given the points $P(1, -1, 0)$, $M(-3, 2, 1)$ and $N(2, 1, 0)$:
 - a find the vector equation of the line that passes through P and is parallel to MN
 - b show that $Q(-14, 2, 3)$ lies on the line determined in part a.



Reasoning and communication

- 14 Find the Cartesian form of the lines given by the following parametric equations and describe the curve.

a $x = t - 4, y = t^3 - 2$

b $x = \cos(t), y = \sin(t)$ [Hint: use the Pythagorean identity]

c $x = 4 \cos(t), y = 3 \sin(t)$

- 15 Two lines have the vector equations:

$\mathbf{p} = \mathbf{i} + \mathbf{j} - 2\mathbf{k} + t(\mathbf{j} - \mathbf{k})$

$\mathbf{q} = 2\mathbf{j} - 2\mathbf{k} + s(\mathbf{i} - \mathbf{j})$

Find the point of intersection of the lines.

- 16 Given that the position vectors of A, B, C and D are $\mathbf{a} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}, \mathbf{b} = -\mathbf{i} - 3\mathbf{k}, \mathbf{c} = 3\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ and $\mathbf{d} = 8\mathbf{i} + \mathbf{j} + 4\mathbf{k}$ respectively, find the vector equation of the line:

a through A and B

b through D and parallel to BC

c through C and parallel to AB .

- 17 Consider the following vector equations.

Line 1: $\mathbf{p} = 7\mathbf{i} - 3\mathbf{j} + 3\mathbf{k} + t(3\mathbf{i} - 2\mathbf{j} + \mathbf{k})$

Line 2: $\mathbf{q} = 7\mathbf{i} - 2\mathbf{j} + 4\mathbf{k} + s(-2\mathbf{i} + \mathbf{j} - \mathbf{k})$

Line 3: $\mathbf{r} = \mathbf{i} + u(\mathbf{j} - \mathbf{k})$

a Show that Line 1 and Line 2 intersect and find their point of intersection.

b Show that Line 1 and Line 3 are skew lines.

c Determine if Line 2 and Line 3 intersect, are parallel or are skew lines.

- 18 A line passes through $(1, 2, 3)$ and is parallel to $\mathbf{d} = 2\mathbf{i} - \mathbf{j} - 4\mathbf{k}$.

a Find the vector and Cartesian equations of the line.

b Which of the points $P(2, 0, 1), Q(3, 1, -1)$ and $R(\frac{1}{2}, 2\frac{1}{4}, 4)$ lie on the line?

- 19 At what point does the line passing through $(3, 1, -2)$ and $(-2, 7, -4)$ intersect the xy -plane?

- 20 Two particles begin moving from positions $(5, 7, -9)$ and $(-4, -2, 9)$ with velocities $(1, 2, 3)$ and $(2, 3, 1)$ respectively. Determine if their paths cross, and if so, whether they collide.

- 21 Two aeroplanes close to an airport are moving at speeds of 306 km/h and 396 km/h respectively. The first plane is descending at a rate of 31 m/s on a SE path and the second is rising at 13 m/s on a NE path. At 9:30 p.m. the air traffic controller plots their positions as 3.8 km N and 1.1 km E of the airport and 2.2 km S and 0.55 km E of the control tower with altitudes of 6972 m and 5031 m respectively. Will they get dangerously close to each other (within 500 m)?

- 22 One aeroplane is travelling near an airport in a path given by $x = 10 \cos(6t), y = 10 \sin(6t), z = 6$ where x, y and z are in km and the airport is at $(0, 0, 0)$. Another plane is travelling on the path $x = 12 \cos(5t), y = 12 \sin(5t), z = 8$ while a third plane is following the path $x = 40(t - 0.1), y = 60(t - 0.1), z = 0.1 - t$. z is the vertical axis, x is north, y is east and t is time in hours. Describe the paths and what the aircraft are doing.

1.07 EQUATIONS OF PLANES

There are a number of ways of specifying a plane. Consider the points $P(x, y, z)$, $A(a_1, a_2, a_3)$ and Q , where AQ is perpendicular to the plane.

In component form, $\mathbf{AP} = (x - a_1, y - a_2, z - a_3)$

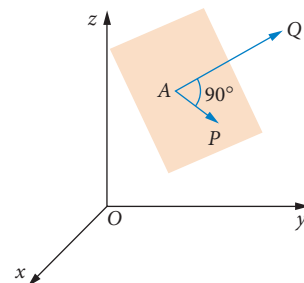
Let $\mathbf{AQ} = \mathbf{n} = (n_1, n_2, n_3)$

Because $\mathbf{n}$ and $\mathbf{AQ}$ are orthogonal, their scalar product is zero:

$$\mathbf{n} \cdot \mathbf{AP} = 0$$

So $(n_1, n_2, n_3) \cdot (x - a_1, y - a_2, z - a_3) = 0$

$$n_1(x - a_1) + n_2(y - a_2) + n_3(z - a_3) = 0$$



IMPORTANT

The **Cartesian equation of a plane** through $A(a_1, a_2, a_3)$, with normal vector $\mathbf{n} = (n_1, n_2, n_3)$ is:

$$n_1(x - a_1) + n_2(y - a_2) + n_3(z - a_3) = 0$$

Let the position vectors of P and A be $\mathbf{p}$ and $\mathbf{a}$ respectively, then, $\mathbf{AP} = \mathbf{p} - \mathbf{a}$.

$\mathbf{n}$ and $\mathbf{AP}$ are orthogonal, so:

$$\mathbf{n} \cdot \mathbf{AP} = 0$$

or $\mathbf{n} \cdot (\mathbf{p} - \mathbf{a}) = 0$

IMPORTANT

The **vector equation of the plane** through $A(a_1, a_2, a_3)$, with normal vector $\mathbf{n}$ is:

$$\mathbf{n} \cdot (\mathbf{p} - \mathbf{a}) = 0, \text{ where } P \text{ is a point in the plane.}$$

This equation may also be written as

$$\mathbf{p} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n} = k$$

where k is a constant.

A plane is often represented by a capital Greek letter such as Γ (gamma). A **region of a plane** can be specified by adding additional conditions to the equation of the plane.



○ Example 19

Consider the plane, Γ , passing through $A(2, -2, 4)$ with normal $\mathbf{n} = (1, 2, 3)$.

- Find the Cartesian equation of the plane.
- Does the point $B(-2, 3, 2)$ lie on the plane?
- Find an expression for the points inside the circle on the plane with centre A and radius 10.

Solution

- Write the equation of the plane.

$$\mathbf{n} \cdot (\mathbf{p} - \mathbf{a}) = 0$$

$$\text{so } n_1(x - a_1) + n_2(y - a_2) + n_3(z - a_3) = 0$$

Substitute in the values.

$$1(x - 2) + 2(y + 2) + 3(z - 4) = 0$$

Simplify.

$$x + 2y + 3z - 10 = 0$$

- Write the LHS of the equation.

$$\text{LHS} = x + 2y + 3z - 10$$

Substitute values for B .

$$= -2 + 2 \times 3 + 3 \times 2 - 10$$

Evaluate.

$$= 0 = \text{RHS}$$

Write the answer.

$A(-2, 3, 2)$ lies on the plane

$$x + 2y + 3z - 10 = 0$$

- Write the distance of the points inside the circle from A .

$$\sqrt{(x - 2)^2 + [y - (-2)]^2 + (z - 4)^2} < 10$$

Square both sides.

$$(x - 2)^2 + (y + 2)^2 + (z - 4)^2 < 100$$

Rewrite the equation of Γ .

But $x = 10 - 2y - 3z$ for points on Γ .

Substitute in the inequality.

$$(10 - 2y - 3z - 2)^2 + (y + 2)^2 + (z - 4)^2 < 100$$

Simplify the first bracket.

$$(8 - 2y - 3z)^2 + (y + 2)^2 + (z - 4)^2 < 100$$

Multiply out and collect terms.

$$5y^2 + 10z^2 + 12yz - 28y - 56z + 84 < 100$$

Write the answer.

The points satisfy the inequality

$$5y^2 + 10z^2 + 12yz - 28y - 56z < 16$$

In Example 19 part c, the answer could be given in terms of x and y or in terms of x and z instead of in terms of y and z . The alternate solutions are $10x^2 + 13y^2 + 4xy - 32x + 44y < 824$ or $5x^2 + 12z^2 - 4xz - 44x - 72z < 124$. Non-circular regions of an arbitrary plane could also be found, but they are beyond the scope of this course.

The equation of a plane can be written as $\mathbf{p} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n} = k$

If $\mathbf{n} = (l, m, n)$ in component form, then the equation of the plane becomes:

$$\mathbf{p} \cdot \mathbf{n} = (x, y, z) \cdot (l, m, n) = k$$

so $lx + my + nz = k$, which is an alternative Cartesian equation for the plane.

IMPORTANT

The graph of the equation $lx + my + nz = k$ is a plane with normal $\mathbf{n} = (l, m, n)$.

○ Example 20

Find the equation of the plane, Γ , through $A(3, -5, 4)$, which is parallel to the plane $3x + y - 6z + 8 = 0$.

Solution

Write the normal to the plane.

$lx + my + nz = k$ has the normal $\mathbf{n} = (l, m, n)$.

Write the normal for this plane.

$3x + y - 6z + 8 = 0$ has the normal $\mathbf{n} = (3, 1, -6)$.

Write the equation of Γ .

The equation of Γ is $3x + y - 6z = k$

Substitute $A(3, -5, 2)$ in the LHS.

LHS $= 3 \times 3 + (-5) - 6 \times 4$

Simplify.

$= -20$

Use the answer.

But A is on the plane, so $-20 = \text{RHS} = k$

State the answer.

The equation of Γ is $3x + y - 6z + 20 = 0$

You can find the equation of a plane from any three non-collinear points, as they determine a plane. For 3 non-collinear points A, B and C , the cross product of two sides of the triangle will give a normal, which can then be used with any point to determine the plane.

○ Example 21

Find the equation of the plane that contains the points $P(0, -2, 1)$, $Q(4, 1, 3)$ and $R(-1, 0, 2)$.

Solution

Use P and R to find $\mathbf{PR}$.

$\mathbf{PR} = (-1, 0, 2) - (0, -2, 1)$
 $= (-1, 2, 1)$

Use P and Q to find $\mathbf{PQ}$.

$\mathbf{PQ} = (4, 1, 3) - (0, -2, 1)$
 $= (4, 3, 2)$

Find a normal vector.

$\mathbf{n} = \mathbf{PR} \times \mathbf{PQ}$

Substitute values.

$= (-1, 2, 1) \times (4, 3, 2)$
 $= (2 \times 2, 1 \times 4, -1 \times 3) - (1 \times 3, -1 \times 2, 2 \times 4)$
 $= (4, 4, -3) - (3, -2, 8)$

Calculate the answer.

$= (1, 6, -11)$
 $= \mathbf{i} + 6\mathbf{j} - 11\mathbf{k}$

Write the Cartesian equation of the plane.

$n_1(x - a_1) + n_2(y - a_2) + n_3(z - a_3) = 0$

Use $(0, -2, 1)$ and $\mathbf{n} = (1, 6, -11)$.

$1(x - 0) + 6(y + 2) - 11(z - 1) = 0$

Simplify.

$x + 6y - 11z + 23 = 0$

State the result.

The plane has the equation $x + 6y - 11z + 23 = 0$



EXERCISE 1.07 Equations of planes

Concepts and techniques



Equations of planes

- 1 **Example 19** Which plane does not contain the point $P(10, -3, 5)$?
- A $3x + 6y - 2z - 2 = 0$ B $2x - 2y - 3z - 11 = 0$
C $x + y - z - 12 = 0$ D $4x + 5y + z - 30 = 0$
E $2x - 2y - 3z - 11 = 0$
- 2 The equation $5x + 3y - 7 = 0$ can represent
- A a Cartesian equation of a line in two dimensions
B a Cartesian equation of a plane in three-dimensional space
C a vector equation of a plane in three-dimensional space
D neither A nor B
E both A and B
- 3 Determine if the points $P(1, 3, -1)$, $Q(3, 5, 1)$, and $R(1, 6, 5)$ are on the plane defined by the equation $x + 2y - z - 8 = 0$.
- 4 Which of the following points lie on the plane $4x + 3y - 5z - 10 = 0$?
- a $A(1, 2, 0)$ b $B(-7, 6, 4)$ c $C(-2, 1, -3)$ d $D(1.2, -2.4, 6.2)$
- 5 a Find the equation of the plane that contains the point $A(3, -1, 0)$, for which the vector $\mathbf{r} = 2\mathbf{i} + 10\mathbf{j} - 7\mathbf{k}$ is perpendicular to the plane.
b Find an expression for the points inside the circle on the plane with centre $(2, 2, 4)$ and radius 6.
c Find an expression for the points on the plane on or outside the circle on the plane with centre A and radius 9.
- 6 a Find the equation of the plane that contains the point $(0, 1, 2)$ and is perpendicular to $\mathbf{n} = 2\mathbf{i} + \mathbf{j} - 3\mathbf{k}$.
b Find an expression for the points on or inside the circle on the plane with centre $(0, 1, 2)$ and radius 8.
c Find an expression for the points on the plane outside the circle on the plane with centre $(1, 2, 3)$ and radius 5.
- 7 Find the equation of the plane through the point $(5, -2, 4)$ with normal vector $\mathbf{n} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$.
- 8 Find the equation of the plane passing through $(0, 0, 0)$, with normal vector $2\mathbf{i} + 4\mathbf{j} - 5\mathbf{k}$.
- 9 A plane passes through the point $P(3, 2, 5)$ and is perpendicular to $\mathbf{n} = \mathbf{i} + \mathbf{k}$.
- a Find the vector equation of the plane.
b Find the Cartesian equation of the plane.
- 10 Find the equation of the plane that passes through $D(5, 2, -3)$ and is orthogonal to the planes $2x - y + 4z = 7$ and $3y - z = 8$.
- 11 **Example 20** Which vector is not normal to the plane $x + 2y - 3z - 4 = 0$?
- A $(-4, -8, 12)$ B $(2, -3, 4)$ C $(2, 4, -6)$
D $(3, 6, -9)$ E $(-1, -2, 3)$

- 12 Find the equation of the plane that passes through $A(5, -2, 3)$ and is parallel to the plane $3x + y - 6z + 8 = 0$.
- 13 Find the equation of the plane that passes through $M(-11, 4, -2)$ with normal vector $\mathbf{n} = 6\mathbf{i} - 5\mathbf{j} - \mathbf{k}$.
- 14 Find the equation of the plane through $D(5, -3, 2)$ that is:
 a parallel to the xy -plane b parallel to the yz -plane c parallel to the xz -plane
- 15 Find the equation of the plane that passes through $Q(2, 5, -6)$ and is parallel to the plane $3x - y + 2z = 10$.
- 16 Find the equation of the plane passing through the point $(1, 1, 1)$ and is parallel to the plane $2x + y + z = 0$.
- 17 **Example 21** Find the equation of the plane that contains the points in each of the following cases.
 a $(1, -2, 0)$, $(0, -1, 2)$ and $(3, 1, 4)$ b $(1, 0, 1)$, $(2, 4, 6)$ and $(1, 2, -1)$
 c $(1, -2, -3)$, $(4, -4, 4)$ and $(3, 2, -3)$ d $(0, 1, 2)$, $(2, 1, 3)$ and $(3, 1, 0)$
 e $(0, 0, 0)$, $(0, 2, 5)$ and $(1, 4, 0)$ f $(1, 0, 2)$, $(-1, 1, 2)$ and $(5, 0, 3)$
- 18 Find the equation of the plane that contains the points $A(2, -1, 0)$, $B(-1, 3, 4)$ and $C(3, 0, 2)$.
- 19 Given the points $A(1, 0, 0)$, $B(1, 2, 2)$ and $C(3, 0, 1)$, find the equation of the plane passing through A , B , and C .
- 20 Find the equation of the plane passing through the points $(0, -1, 1)$, $(1, 0, 1)$ and $(1, 2, 2)$.

Reasoning and communication

- 21 A plane passes through $(7, 3, -5)$ and the vector normal to the plane, $\mathbf{n} = \mathbf{i} + \mathbf{j} + \mathbf{k}$. Find the shortest distance from the plane to the origin.
- 22 What is the distance of the plane $\mathbf{p} \cdot (3\mathbf{i} + 2\mathbf{j} + \mathbf{k}) = 5$ to the origin?
- 23 Determine if the planes $-6x + 9y + 3z + 2 = 0$ and $2x - 3y - z - 5 = 0$ are parallel, perpendicular or neither.
- 24 Find the distance from the point $Q(-5, 3, 2)$ to the plane $4x - 5y + 8z - 7 = 0$.
- 25 Find the equation of the plane through the point $P(1, 2, -3)$ and the line of intersection of the planes $x + 3y - 5z - 10 = 0$ and $6x - y + z = 0$.
- 26 Determine whether the following pair of planes Γ_1 and Γ_2 are parallel or intersect.
 $\Gamma_1 : x + 2y - 4z - 2 = 0$
 $\Gamma_2 : -2x - 4y + 8z - 1 = 0$
- 27 The following equations describe a set of planes. Describe how the planes intersect.
 $\Gamma_1 : x - 5y + 2z - 10 = 0$
 $\Gamma_2 : x + 7y - 2z + 6 = 0$
 $\Gamma_3 : 8x + 5y + z - 20 = 0$



1.08 SYSTEMS OF LINEAR EQUATIONS

A **linear equation** is one that can be written with no powers, roots, reciprocals, and so on—for example, $5x - y + 4z = 8$. A **system** of equations shows the relationships between a group of variables. You already know how to solve simultaneous equations with two variables.

The main steps used to solve simultaneous equations by elimination (for two equations) are as follows.

- 1 Choose which variable to eliminate.
- 2 Make the coefficients of that variable the same size, but opposite in sign, by multiplying by suitable numbers.
- 3 Add the equations to eliminate the variable.

Once the variable is eliminated, you can divide to find the value of the remaining variable and substitute back to find the first variable. This method can be extended to multiple equations, proceeding to eliminate one variable at a time. The elimination of one variable at a time, reducing both the number of variables and the number of equations at each step, is usually called **Gaussian elimination**.

IMPORTANT

Gaussian elimination steps for multiple equations

- 1 Choose which variable to eliminate.
- 2 Put the equation with the simplest coefficient of this variable first in the list.
- 3 For each of the other equations, multiply the chosen equation by a suitable number and then add to eliminate the chosen variable.
- 4 Repeat steps 1–3 with the new equations until an equation with a single variable is obtained, if possible. Find its value.
- 5 Progressively substitute to find the values of all the variables.

Example 22

Solve the following set of equations by elimination.

$$3a - 2b - 4c = 4$$

$$2a + b + 3c = 4$$

$$2a - 3b - c = -4$$

Solution

Put the simplest equation first.

$$2a + b + 3c = 4 \quad [2]$$

$$3a - 2b - 4c = 4 \quad [1]$$

$$2a - 3b - c = -4 \quad [3]$$

In [1] and [2], make the coefficients of b the same but with opposite signs.

$$4a + 2b + 6c = 8 \quad [2] \times 2$$

$$3a - 2b - 4c = 4 \quad [1]$$

$$7a \quad + 2c = 12 \quad [4]$$

Add the equations to eliminate b .

In [1] and [3], make the coefficients of b the same but with opposite signs.

$$6a + 3b + 9c = 12 \quad [2] \times 3$$

$$2a - 3b - c = -4 \quad [3]$$

$$8a \quad + 8c = 8$$

Add the equations to eliminate b .

Simplify.

$$a + c = 1 \quad [5]$$

In [5] and [4], make the coefficients of c the same but with opposite signs.

$$-2a - 2c = -2 \quad [5] \times -2$$

$$7a + 2c = 12 \quad [4]$$

Add the equations to eliminate c .

$$5a = 10$$

Simplify.

$$a = 2 \quad [6]$$

Substitute in [5].

$$2 + c = 1$$

Solve for c .

$$c = -1$$

Now substitute in [2].

$$2 \times 2 + b + 3 \times (-1) = 4$$

Solve for b .

$$4 + b - 3 = 4$$

$$b = 3$$

Check that the solution works in all the original equations.

For $a = 2$, $b = 3$ and $c = -1$:

$$3a - 2b - 4c = 3 \times 2 - 2 \times 3 - 4 \times (-1) = 4 \quad [1] \checkmark$$

$$2a + b + 3c = 2 \times 2 + 3 + 3 \times (-1) = 4 \quad [2] \checkmark$$

$$2a - 3b - c = 2 \times 2 - 3 \times 3 - (-1) = -4 \quad [3] \checkmark$$

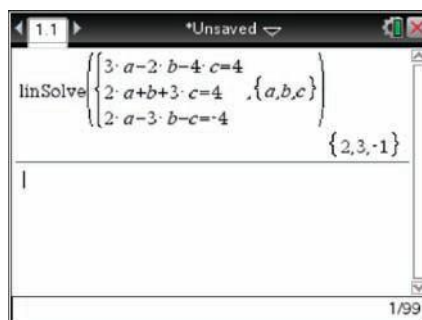
Write the answer.

$$a = 2, b = 3 \text{ and } c = -1$$

TI-Nspire CAS

Use **menu**, 3: Algebra, 7: Solve System of Equations and 2: Solve System of Linear Equations.

Make the number of variables 3 and enter a , b and c as the variables. Enter the equations and press **enter**.

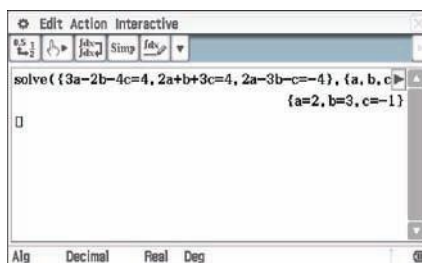




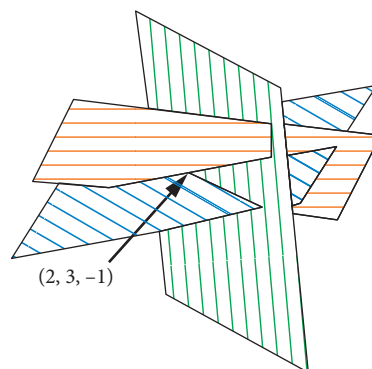
ClassPad

Tap **Action** then **Equation/Inequality**, then **solve**. To enter the list of equations, first tap the braces {} key under **Keyboard** **Math3**.

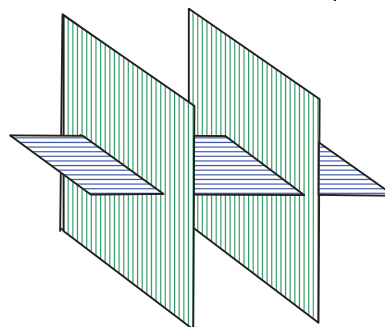
Between the braces, enter the three equations with a comma after each one. Press **,** and in a second pair of braces, list the variables a,b,c as shown and close the parentheses on the solve command. Press **(EXE)** to solve the system of equations for a, b and c.



What does Example 22 mean? $3a - 2b - 4c = 4$, $2a + b + 3c = 4$ and $2a - 3b - c = -4$ are all the equations of planes. The point $(2, 3, -1)$ is on all three planes, so they intersect at one point, as illustrated on the right. These have a *unique solution*.



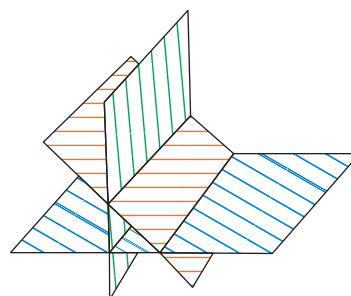
What other possibilities are there? Two planes could be parallel, as shown on the right. There would be no point of intersection so there would be no solution.



Three parallel planes have no solution either. They have the same normal vector but no common points, such as $3x + 4y - 2z = 5$, $3x + 4y - 2z = 10$ and $6x + 8y - 4z = -5$.

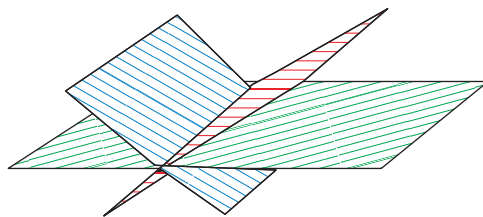
Likewise, three planes that intersect in parallel lines as shown on the right have no point of intersection.

Equations like this give an absurd result when you try to solve them. If you subtract the first equation from the second equation, you get $0 = 5$.



You could have 3 planes intersecting in a line as shown here.

These have an infinite number of solutions corresponding to points on the line. When you try to solve equations for planes like this, such as $x + 2y - 2z + 1 = 0$, $4x - y - 2z + 4 = 0$ and $3x + 3y - 4z + 3 = 0$, you can only eliminate one variable. One of the equations can be written as a combination of the other two because the normal vectors are in the same plane.



IMPORTANT

There are three cases that occur for simultaneous equations.

Inconsistent equations correspond to cases involving at least a pair of parallel planes or planes that intersect in parallel lines.

A **unique solution** corresponds to planes that intersect at a single point.

Dependent equations have an equation that can be written as a combination of the others. This corresponds to planes that intersect in a straight line.

Similar considerations apply to equations with four or more unknowns corresponding to multidimensional vector spaces, but these are beyond the scope of this course. In any case, simultaneous equations with many variables are normally solved using approximate matrix methods to obtain 'good enough' solutions.

○ Example 23

Solve the following by elimination.

$$x + 2y - 2z + 1 = 0$$

$$4x - y - 2z + 4 = 0$$

$$3x + 3y - 4z + 3 = 0$$

Solution

Put the simplest equation first.

$$4x - y - 2z + 4 = 0 \quad [2]$$

$$x + 2y - 2z + 1 = 0 \quad [1]$$

$$3x + 3y - 4z + 3 = 0 \quad [3]$$

In [1] and [2], make the coefficients of y the same but with opposite signs.

$$8x - 2y - 4z + 8 = 0 \quad [2] \times 2$$

$$x + 2y - 2z + 1 = 0 \quad [1]$$

Add the equations to eliminate y .

$$9x - 6z + 9 = 0 \quad [4]$$

Simplify.

$$3x - 2z + 3 = 0$$

In [1] and [3], make the coefficients of y the same but with opposite signs.

$$12x - 3y - 6z + 12 = 0 \quad [2] \times 3$$

$$3x + 3y - 4z + 3 = 0 \quad [3]$$

Add the equations to eliminate y .

$$15x - 10z + 15 = 0 \quad [5]$$

Simplify.

$$3x - 2z + 3 = 0 \quad [5]$$

State the situation.

Choose a value for z .

Substitute in equation [4] or [5].

Solve for x .

Substitute in equation [2].

Solve for y .

[4] and [5] are the same.

Let $z = c$, where $c \in \mathbb{R}$.

$$3x - 2c + 3 = 0$$

$$x = \frac{2c-3}{3}$$

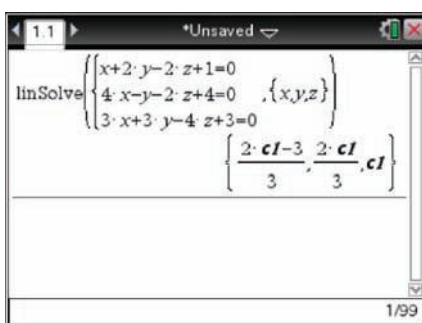
$$4 \times \frac{2c-3}{3} - y - 2c + 4 = 0$$

$$y = \frac{2c}{3}$$

TI-Nspire CAS

Enter the equations as before.

The equations do not give a unique solution.
 cI can be any real number.



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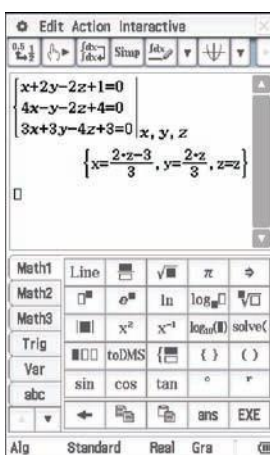
Ensure that your calculator is set to **Standard** mode. Use the solve command and enter the equations and variables as in Example 22.

Alternatively, press **Keyboard** and tap **Math1**, then **[]**. Tap **[]** again to expand the system to three equations.

Use the up arrow to enter the first line. Enter one equation on each line and use the down arrow to move down to the next line. On the third line, use the right arrow to move across to the empty box and enter the variables, with each pair separated by a comma. Press **[EXE]** when this is done to solve the system of equations for the variables specified.

The equations do not give a unique solution.
 z can be any real number.

Write the answer.



The solutions can be given as

$$x = \frac{2c-3}{3}, y = \frac{2c}{3}, z = c, \text{ where } c \in \mathbb{R}.$$

In Example 23, there are infinitely many solutions. There are different ways to express them, depending on whether x , y or z is chosen as the independent variable. For example, if y is chosen as the independent variable, the solutions are $x = c - 1$, $y = c$, $z = \frac{3c}{2}$, where $c \in \mathbf{R}$. Your teacher may want you to work out the form of the solutions if x is chosen as the independent variable.

○ Example 24

Solve the following equations.

$$\begin{aligned}p + 3q + 2m &= 4 \\7p + 7q - 6m &= 14 \\4p + 5q - 2m &= 11\end{aligned}$$

Solution

Number the equations, with the simplest one first.

$$\begin{aligned}p + 3q + 2m &= 4 & [1] \\7p + 7q - 6m &= 14 & [2] \\4p + 5q - 2m &= 11 & [3]\end{aligned}$$

In [2] and [1], make the same coefficient of p with opposite signs.

$$\begin{aligned}-7p - 21q - 14m &= -28 & [1] \times (-7) \\7p + 7q - 6m &= 14 & [2] \\ \hline -14q - 20m &= -14\end{aligned}$$

Add the equations to eliminate p .

Simplify.

$$7p + 10m = 7 \quad [4]$$

Now do equations [1] and [3].

$$-4p - 12q - 8m = -16 \quad [1] \times (-4)$$

Add the equations to eliminate p .

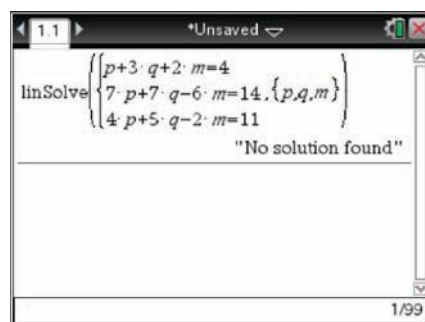
$$\begin{aligned}4p + 5q - 2m &= 11 & [3] \\ \hline -7q - 10m &= -5\end{aligned}$$

Simplify.

$$7p + 10m = 5 \quad [5]$$

TI-Nspire CAS

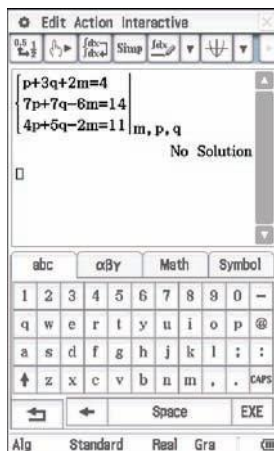
Enter the equations as before.





ClassPad

Use the solve command or  and enter the equations and variables as before.



Write the answer.

There is no solution: the equations are inconsistent.

You should use your CAS calculator for more than 3 equations or more than 3 variables. If you have 3 variables and only two equations, then unless they are inconsistent there will be an infinite number of solutions.

EXERCISE 1.08 Systems of linear equations

Concepts and techniques



Systems of equations

1 **Examples 22–24** Solve the following equations.

a $7a + 2b + 4c = 37$
 $3a + b + 2c = 16$
 $17a + 5b + 11c = 92$

b $3x + 7y + 6z = 7$
 $5x + 11y + 12z = 18$
 $x + 2y + 3z = 4$

2 **CAS** Solve the following equations.

a $6x + y + 2z = 5$
 $8x + 3y + 4z = 5$
 $2x - y - z = 6$
 $10x - 2y - z = 19$

b $2u + 5v - 3w = 8$
 $u - v + 4w = 3$
 $2u - 2v + 9w = 9$
 $u + 2v - w = 3$

c $4e - 8f - 10g + 5h = 49$
 $2e - 2f - 3g + 2h = 17$
 $2e - 3f - 4g + 2h = 20$
 $6f - 3e + 8g - 4h = -38$

d $5b + 3c - 4d + 2e = -17$
 $4b + c - 3d + 2e = -10$
 $8d - 8b - 2c - 4e = 24$
 $6b + c - 6d + 3e = -17$

e $3f - 5g + 5h - 5k = 51$
 $4g + 4k - 2f - 3h + 37 = 0$
 $6f - 11g + 10h - 12k = 113$
 $2f - 4g + 3h - 3k = 32$

Reasoning and communication

- 3 Tickets for a charity performance have different prices for children, adults and seniors. When three children were taken to the performance by their grandmother, the cost was \$22. For two children and their parents, the cost was \$32, and for four children, their father and his elderly parents, the cost was \$45.

Write equations and find the price for each type of ticket.

- 4 **CAS** To make pre-mixed concrete, dry gravel, sand and cement are mixed with water in a concrete truck. Each ingredient has a different density and different mixtures are used for different purposes. Slump, for example, is a mix of gravel, sand, cement and water in the ratio 5 : 2 : 1 : 2 (by volume). This and other mixes (by volume) for standard (5 m^3) loads, and their masses, are shown in the table below. Use the information to find the density of each material.

Type	Gravel	Sand	Cement	Water	Mass (tonne)
Slump	5	2	1	2	21.4
General mix	7.2	4	2	2.8	21.125
Foundation	10	3	3	4	20.35
Fine finish	6	5	2	3	20.5



Getty Images/Forlag

- 5 A financial adviser put together an investment portfolio of \$200 000 for a client. The portfolio was spread over three investments – shares, a unit trust and a first mortgage. The annual return for the first year was \$16 240. The shares paid 6%, the unit trust 8% and the first mortgage 10%. The dollar return from the first mortgage was twice that from the shares. Write equations and solve them to find out how much was invested in each of the investments.

CHAPTER SUMMARY

THREE-DIMENSIONAL VECTORS

- In a three-dimensional coordinate system, the position of a point is defined by an **ordered triple** as in $P(x, y, z)$.
- The Cartesian (or rectangular) plane in three dimensions has three planes – the xy -, yz - and xz -planes. These planes partition space into eight **octants**.
- The **distance formula in three dimensions (3D distance formula)** states that for $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

- The **midpoint** of the line segment from $P_1(x_1, y_1, z_1)$ to $P_2(x_2, y_2, z_2)$ is:

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2} \right)$$

- The **equation of a sphere** with centre $C(h, k, l)$ and radius, r , is:

$$(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2$$

$$\text{or } x^2 + y^2 + z^2 + ax + by + cz + d = 0$$

- Properties of vectors.

If $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ are vectors and c is a scalar:

$$\mathbf{a} + \mathbf{b} = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$$

$$c\mathbf{a} = (ca_1, ca_2, ca_3)$$

$$\mathbf{0} = (0, 0, 0)$$

$$-\mathbf{a} = -(a_1, a_2, a_3)$$

$$\mathbf{a} - \mathbf{b} = (a_1 - b_1, a_2 - b_2, a_3 - b_3)$$

- When vectors are added, their sum is called the **resultant** vector.

- A vector $\mathbf{a} = (a_1, a_2, a_3)$ can be represented in the three-dimensional Cartesian system by a directed line segment $\overline{QR}$ (or $\mathbf{QR}$) with **initial point** (or head) $Q(x, y, z)$ and **terminal point** (or tail) $R(x + a_1, y + a_2, z + a_3)$.
- The vector $\mathbf{OA}$ from the origin, $O(0, 0, 0)$, to $A(a_1, a_2, a_3)$ is called the **position vector** of A . The lower case vector symbol is also used for the position vector, so $\mathbf{OA} = \mathbf{a}$.

- If $\mathbf{PQ} = \mathbf{RS}$, then $\mathbf{PQ}$ and $\mathbf{RS}$ have the same magnitude and direction. If $\mathbf{PQ} = c\mathbf{RS}$, where c is a scalar, then $\mathbf{PQ} \parallel \mathbf{RS}$.

- For any two points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$:

$$\mathbf{P_1P_2} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$$

- The magnitude of a three-dimensional vector is called the **norm** of the vector. For vector $\mathbf{a}$, the norm is written $|\mathbf{a}|$. You can write $|\mathbf{a}|$ as a .

- If $\mathbf{a} = (a_1, a_2, a_3)$ and c is a scalar:

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

$$|c\mathbf{a}| = c|\mathbf{a}|$$

- A **unit vector** has magnitude 1. The unit vector in the same direction as a given vector $\mathbf{p}$ is written as $\hat{\mathbf{p}}$.

$$\hat{\mathbf{p}} = \frac{1}{|\mathbf{p}|}\mathbf{p}$$

- The unit vectors in space are:

$$\mathbf{i} = (1, 0, 0)$$

$$\mathbf{j} = (0, 1, 0)$$

$$\mathbf{k} = (0, 0, 1)$$

- For any vector, $\mathbf{a} = (a_1, a_2, a_3)$:

$$\mathbf{a} = (a_1, a_2, a_3) = (a_1\mathbf{i}, a_2\mathbf{j}, a_3\mathbf{k})$$

- The **polar form** of a vector has a magnitude r and two angles, θ and ϕ . The **azimuth angle** θ lies between 0° and 360° and is the angle from the x -axis to the projection of the vector onto the xy -plane (between x and a). The **altitude angle** ϕ lies between -90° and 90° and is the angle from the xy -plane to the vector (between a and r). The **spherical coordinates** have a **polar angle** ψ measured anticlockwise from the z -axis instead of the altitude angle. The polar angle lies between 0° and 180° .

■ 3D vector forms

For the 3D vector $\mathbf{v} = (r, \theta, \phi) = (x, y, z)$:

$$x = r \cos(\phi) \cos(\theta) \quad r^2 = x^2 + y^2 + z^2$$

$$y = r \cos(\phi) \sin(\theta) \quad \tan(\theta) = \frac{y}{x}$$

$$z = r \sin(\phi) \quad \sin(\phi) = \frac{z}{r} \text{ and}$$

$$r = |\mathbf{v}| = \sqrt{x^2 + y^2 + z^2} \quad \phi + \psi = 90^\circ$$

- If vectors **PQ** and **RS** have respective position vectors $\mathbf{a}$ and $\mathbf{b}$ and the angle between $\mathbf{a}$ and $\mathbf{b}$ is θ , the **scalar product** of **PQ** and **RS** is given by:

$$\mathbf{PQ} \cdot \mathbf{RS} = \mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos(\theta)$$

$$\mathbf{a} \cdot \mathbf{b} = ab \cos(\theta)$$

- In component form, if $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$

$$\mathbf{a} \cdot \mathbf{b} = (a_1, a_2, a_3) \cdot (b_1, b_2, b_3)$$

$$= a_1b_1 + a_2b_2 + a_3b_3$$

- The scalar product is also called the **inner product** or **dot product**. It is a scalar (real number), not a vector.

■ Properties of the scalar product

- 1 $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \cos(\theta)$, where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$.
- 2 The scalar product is a real number, *not* a vector.
- 3 The scalar product is commutative:

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$$

- 4 The scalar product is distributive over vector addition:

$$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$$

- 5 $m(\mathbf{a} \cdot \mathbf{b}) = (m\mathbf{a}) \cdot \mathbf{b} = \mathbf{a} \cdot (m\mathbf{b}) \quad \forall m \in \mathbf{R}$ and vectors $\mathbf{a}, \mathbf{b}$.

- 6 $\mathbf{a} \cdot \mathbf{b} = 0 \Rightarrow \mathbf{a} = \mathbf{0}$ or $\mathbf{b} = \mathbf{0}$ or $\mathbf{a}$ and $\mathbf{b}$ are perpendicular.

- The angle between two vectors can be found using the scalar product.

$$\text{Since } \mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos(\theta), \cos(\theta) = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$

- The magnitude of the **projection** of $\mathbf{a}$ on $\mathbf{b}$ can be calculated by the dot product of $\mathbf{a}$ and the unit vector in the direction of $\mathbf{b}$.

$$p = \mathbf{a} \cdot \hat{\mathbf{b}} \text{ or } p = (\mathbf{a} \cdot \hat{\mathbf{b}}) \hat{\mathbf{b}}$$

- The work done by a constant force **PQ** that results in a displacement **PR** is $\mathbf{PQ} \cdot \mathbf{PR}$.

- The **vector product** of two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ with an angle of θ between them is given by

$$\mathbf{v} = \mathbf{v}_1 \times \mathbf{v}_2, \text{ where } |\mathbf{v}| = |\mathbf{v}_1||\mathbf{v}_2| \sin(\theta).$$

$\mathbf{v}$ is perpendicular to both $\mathbf{v}_1$ and $\mathbf{v}_2$ such that the three vectors $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}$ form a right-handed system. The vector product is also called the **outer product** or **cross product**.

- The vector product may be calculated from vectors in component form by using

$$(a_1, a_2, a_3) \times (b_1, b_2, b_3)$$

$$= (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1)$$

or by using the rules: $\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0}$,

and $\mathbf{i} \times \mathbf{j} = \mathbf{k}$, $\mathbf{j} \times \mathbf{k} = \mathbf{i}$, $\mathbf{k} \times \mathbf{i} = \mathbf{j}$, $\mathbf{j} \times \mathbf{i} = -\mathbf{k}$,

$\mathbf{k} \times \mathbf{j} = -\mathbf{i}$, $\mathbf{i} \times \mathbf{k} = -\mathbf{j}$.

- Properties of the vector product

- 1 $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}| |\mathbf{b}| \sin(\theta)$, where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$.
- 2 The vector product $\mathbf{a} \times \mathbf{b}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$.
- 3 $\mathbf{a} \times \mathbf{a} = \mathbf{0}$ for any vector $\mathbf{a}$.
- 4 The vector product is left- and right-distributive over vector addition.
- 5 $\mathbf{a} \times \mathbf{b} = \mathbf{0} \Rightarrow \mathbf{a} = \mathbf{0}$ or $\mathbf{b} = \mathbf{0}$ or $\mathbf{a}$ and $\mathbf{b}$ are parallel.
- 6 $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a} \forall$ vectors $\mathbf{a}, \mathbf{b}$.
- 7 $m(\mathbf{a} \times \mathbf{b}) = (m\mathbf{a}) \times \mathbf{b} = \mathbf{a} \times (m\mathbf{b}) \forall m \in \mathbb{R}$ and vectors $\mathbf{a}, \mathbf{b}$.

- The vector product can also be calculated using the determinant-like method.

If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, then use the arrangement

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

to obtain

$$\mathbf{a} \times \mathbf{b} = (a_2b_3 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$$

- The magnitude of the vector product $\mathbf{a} \times \mathbf{b}$ equals the area of the parallelogram determined by $\mathbf{a}$ and $\mathbf{b}$.

- Point O is always used for the origin, and the symbol $\mathbf{a}$ may be used instead of $\mathbf{OA}$ for the position vector of point A . It follows that $\mathbf{AB} = \mathbf{OB} - \mathbf{OA} = \mathbf{b} - \mathbf{a}$. The sum of a series of displacements 'end-to-end' can be replaced by the total displacement using just the first and last points, as in $\mathbf{AB} + \mathbf{BP} + \mathbf{PQ} + \mathbf{QE} = \mathbf{AE}$.

- The **vector equation** of the line through points A and B with position vectors $\mathbf{a}$ and $\mathbf{b}$ is:

$$\mathbf{p} = \mathbf{a} + t\mathbf{AB} \text{ (for some real number } t\text{)}$$

Alternatively, if $\mathbf{u}$ is any vector in the direction of the line AB :

$$\mathbf{p} = \mathbf{a} + t\mathbf{u}$$

For the line segment AB , the vector equation is $\mathbf{p} = \mathbf{a} + t\mathbf{AB}$ for $0 \leq t \leq 1$.

- The **Cartesian form** of the equation of a line in space passing through the points $A(a_1, a_2, a_3)$ and $B(b_1, b_2, b_3)$ is:

$$\frac{x-a_1}{b_1-a_1} = \frac{y-a_2}{b_2-a_2} = \frac{z-a_3}{b_3-a_3}$$

- If the vector equation of a line is $\mathbf{p} = \mathbf{a} + t\mathbf{u}$, where $\mathbf{u}$ is a vector in the direction of the line, and if $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$, then the **Cartesian form** of the equation is:

$$\frac{x-a_1}{u_1} = \frac{y-a_2}{u_2} = \frac{z-a_3}{u_3}$$

- The **Cartesian equation of a plane** through $P(x_1, y_1, z_1)$, with normal vector $\mathbf{n} = (a_1, a_2, a_3)$ is:

$$a_1(x-x_1) + a_2(y-y_1) + a_3(z-z_1) = 0$$

- The **vector equation of the plane** through $P(x, y, z)$ and $A(a_1, a_2, a_3)$, with position vectors $\mathbf{a}$ and $\mathbf{p}$ respectively and with normal vector $\mathbf{n}$ is:

$$\mathbf{n} \cdot (\mathbf{p} - \mathbf{a}) = 0$$

This equation may also be written as:

$\mathbf{n} \cdot \mathbf{p} = \mathbf{n} \cdot \mathbf{a} = k$ or $\mathbf{p} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n} = k$, where k is a constant.

- The graph of the equation $lx + my + nz = k$ is a plane with normal $\mathbf{n} = (l, m, n)$.
- In the **parametric form** of a line, the coordinates are expressed as functions of the parameter. If one or more of these coordinate functions is not a linear function, the line will be curved.
- A **linear equation** is one that can be written with no powers, roots, reciprocals, and so on—for example, $5x - y + 4z = 8$. A **system** of equations shows the relationships between a group of variables.
- **Gaussian elimination** steps for multiple equations
 - 1 Choose which variable to eliminate.
 - 2 Put the equation with the simplest coefficient of this variable first in the list.
 - 3 For each of the other equations, multiply the chosen equation by a suitable number and then add to eliminate the chosen variable.
 - 4 Repeat steps 1–3 with the new equations until an equation with a single variable is obtained, if possible. Obtain its value.
 - 5 Progressively substitute to find the values of all the variables.
- There are three cases that occur for simultaneous linear equations.

A **unique solution** corresponds to planes that intersect at a single point.

Consistent equations have at least one solution.

Inconsistent equations correspond to cases involving at least a pair of parallel planes or planes that intersect in parallel lines.

Dependent equations have an equation that can be written as a combination of the others. This corresponds to planes that intersect in a straight line.
- Simultaneous equations with many variables are normally solved using approximate matrix methods to obtain ‘good enough’ solutions. CAS calculators may also be used.

CHAPTER REVIEW

THREE-DIMENSIONAL VECTORS

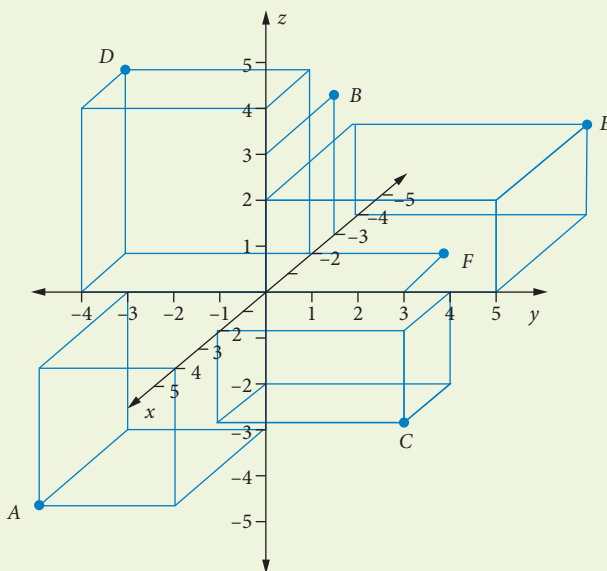
Multiple choice

- 1 **Example 1** $P(-1, 4, -3)$ and which of the following points are on a line parallel with the y -axis?
 A $(-1, 3, 2)$ B $(-1, -5, -3)$ C $(2, 4, -3)$ D $(-1, 4, 3)$ E $(1, 4, 3)$

- 2 **Example 1** The distance between $M(p, q, r)$ and $N(p, q, s)$ is:

A $|s - p|$ B $|p - r|$ C $|r - s|$ D $|r - q|$ E $|q - s|$

Use the diagram below to answer questions 3–4.



- 3 **Example 1** The point lying on the xz -plane is:

A A B B C C D D E E

- 4 **Example 1** The point F is best described as lying on the:

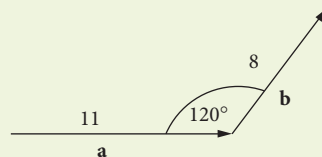
A xy -plane B y -plane C xz -plane
 D x -plane E yz -plane

- 5 **Example 7** The dot product of $-5\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $3\mathbf{i} - 4\mathbf{j} - 7\mathbf{k}$ is:

A -30 B -16 C -12 D 0 E 16

- 6 **Example 7** Given the two vectors shown on the right, $\mathbf{a} \cdot \mathbf{b}$ is closest to:

A -44 B -35.4 C 44
 D 76.2 E 88



- 7 **Example 10** Given the vectors $\mathbf{p}$ and $\mathbf{q}$, $\mathbf{p} \times \mathbf{q}$ is:
 A parallel to $\mathbf{p}$ and $\mathbf{q}$
 B parallel to $\mathbf{q}$ and perpendicular to $\mathbf{p}$
 C perpendicular to $\mathbf{p}$ and $\mathbf{q}$
 D parallel to $\mathbf{p}$ and perpendicular to $\mathbf{q}$
 E None of the above is true.
- 8 **Example 10** Given $\mathbf{a} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$, $\mathbf{b} = 3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and $\mathbf{c} = -\mathbf{i} - 2\mathbf{j} - 3\mathbf{k}$, $\mathbf{a} \times \mathbf{b}$ is equal to:
 A $\mathbf{a} \times \mathbf{c}$ B $\mathbf{c} \times \mathbf{b}$ C $\mathbf{b} \times \mathbf{a}$ D $\mathbf{c} \times \mathbf{a}$ E $-(\mathbf{c} \times \mathbf{b})$
- 9 **Example 10** Given $A(-9, 7, 2)$ and $B(-4, 3, 1)$, $\mathbf{a} \times \mathbf{b}$ is equal to:
 A $\mathbf{i} + \mathbf{j} + \mathbf{k}$ B $\mathbf{0}$ C $\mathbf{AB}$ D $\mathbf{i} - 17\mathbf{j} + \mathbf{k}$ E $\mathbf{i} - \mathbf{j} + \mathbf{k}$

Short answer

- 10 **Example 2** Given $A(1, -5, 3)$, $B(-4, 3, -2)$ and $C(5, 0, -6)$, find the straight-line distances and midpoints of the line segments:
 a AB b BC c CA
- 11 **Example 3** Find the equation of a sphere with radius 5 and centre $(3, -4, -2)$.
- 12 **Example 3** Find the radius and centre of the sphere

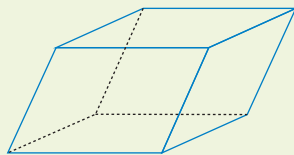
$$x^2 + y^2 + z^2 - 6x + 2y - 4z + 5 = 0$$
- 13 **Example 4** Given $\mathbf{p} = -3\mathbf{i} + \mathbf{j} + 2\mathbf{k}$, $\mathbf{q} = 4\mathbf{i} - 2\mathbf{j} + \mathbf{k}$, $A(4, 2, -3)$ and $B(5, -1, 6)$:
 a find $\mathbf{AB}$ and $|\mathbf{AB}|$
 b find $\hat{\mathbf{p}}$
- 14 **Example 4** Given $\mathbf{m} = (-6, 4, -2)$, $\mathbf{z} = (0, 3, -4)$, $\mathbf{d} = (1, 2, 5)$ and $\mathbf{e} = (3, -2, -1)$, find:
 a $\mathbf{m} + \mathbf{e}$ b $\mathbf{d} - 2\mathbf{z}$ c $\mathbf{m} - \mathbf{d}$ d $3\mathbf{e} - 4\mathbf{z}$ e $4\mathbf{d} - 3\mathbf{m}$
- 15 **Examples 4, 6** a Convert $(3, -7)$ and $(3, -2, 4)$ to polar form.
 b Convert $(8, 210^\circ)$ and $(20, 160^\circ, -36^\circ)$ to component form.
 c Find the norms of $(-5, 12)$ and $(18, 10, -15)$.
- 16 **Example 5** Write the following as linear combinations of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$.
 a $(5, -7)$ b $(25, 330^\circ)$ c $(7, 8, -2)$ d $(30, 60^\circ, 30^\circ)$
- 17 **Example 5** Express the following vectors as linear combinations of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$, and state the norm of each vector.
 a $\begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix}$ b $\begin{bmatrix} -3 \\ -5 \\ 2 \end{bmatrix}$ c $(1, -3, 0)$ d $(2, 5, -3)$
 e Magnitude 5, azimuth 240° , altitude 43°
 f Magnitude 35.7, azimuth 63° , altitude -67°
- 18 **Example 6** Find the resultant of vectors $\mathbf{p}$ and $\mathbf{q}$, where $\mathbf{p}$ has magnitude 25, azimuth 120° and altitude -30° , while $\mathbf{q}$ has magnitude 30, azimuth 60° and altitude 45° .

- 19 **Example 7** Find the scalar products of the following.
- Vectors with magnitudes 5 and 8 at an angle of 75°
 - $(3, -5)$ and $(6, 4)$
 - $(0, -3, 7)$ and $(3, 5, 2)$
- 20 **Example 7** Find the vector products of the following.
- Vectors with magnitudes 8 and 10 at an angle of 63°
 - $(14, 62^\circ)$ and $(25, 140^\circ)$
 - $(3, 5, -1)$ and $(-2, 3, 4)$
 - $2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ and $-3\mathbf{i} + 2\mathbf{k}$
- 21 **Example 8** Given $\mathbf{a} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$, $\mathbf{b} = 5\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$ and $\mathbf{c} = \mathbf{i} - \mathbf{j} + 3\mathbf{k}$, find:
- $\mathbf{a} \cdot \mathbf{c}$
 - $\mathbf{b} \cdot \mathbf{c}$
 - $\mathbf{a} + \mathbf{b}$
 - $(\mathbf{a} + \mathbf{b}) \cdot \mathbf{c}$
 - $\mathbf{c} \cdot \mathbf{b}$
- 22 **Example 8** Given $\mathbf{a} = 2\mathbf{i} + \mathbf{j} - 3\mathbf{k}$, $\mathbf{b} = \mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$, $\mathbf{c} = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$, and $\mathbf{d} = 2\mathbf{i} - 2\mathbf{j} - 3\mathbf{k}$, find:
- the projection of $\mathbf{a}$ on $\mathbf{b}$ using the scalar product $\mathbf{a} \cdot \mathbf{b}$
 - the projection of $\mathbf{c}$ on $\mathbf{d}$ using unit vectors
 - the angle between $\mathbf{a}$ and $\mathbf{c}$
- 23 **Example 9** Find the amount of work done when a constant force $\mathbf{a} = 2\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$ N moves its point of application from $P(3, 2, -2)$ to $Q(4, -4, 7)$, if displacement is measured in metres.
- 24 Find the vector product of $\mathbf{a}$ and $\mathbf{b}$ if:
- $|\mathbf{a}| = 7$, $|\mathbf{b}| = 15$ and the angle between $\mathbf{a}$ and $\mathbf{b}$ is 48°
 - $\mathbf{a} = (3, -2, 5)$ and $\mathbf{b} = (-3, 4, 7)$
- 25 **Example 10** Given $\mathbf{a} = \begin{bmatrix} 2 \\ -3 \\ -2 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} -2 \\ -2 \\ -4 \end{bmatrix}$, $\mathbf{c} = \begin{bmatrix} 4 \\ 8 \\ -5 \end{bmatrix}$, and $\mathbf{d} = \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}$, find:
- $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$
 - $\mathbf{a} \cdot \mathbf{b} \times \mathbf{c}$
 - $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$
 - $\mathbf{c} \times \mathbf{d}$
 - $\mathbf{d} \times \mathbf{c}$
 - $\mathbf{b} \times (\mathbf{b} \times \mathbf{c})$
 - $(\mathbf{b} \times \mathbf{b}) \times \mathbf{c}$
 - $\mathbf{d} \times \mathbf{d}$
- 26 **Example 11** Use the vector product to find the area of a triangle with vertices $(1, -3, 2)$, $(2, 1, 3)$ and $(-2, -1, 2)$.
- 27 **Example 12** Find a vector normal to the plane containing points $(1, -2, 3)$, $(4, 5, -2)$ and $(-1, -2, 1)$.
- 28 **Examples 16, 17** Find the vector and Cartesian equations of the line through the points $(3, -7, 4)$ and $(-5, 8, 2)$.
- 29 **Examples 16, 17** A line passes through the points $(2, -1, 5)$ and $(3, 6, -4)$.
- Find the vector and Cartesian equations of the line.
 - What is the vector equation of the line segment between the points?
 - Does the point $(0, -15, 9)$ lie on the line?
- 30 **Example 18** Find the vector equation of the line that passes through the point $P(-1, 2, 1)$ that is parallel to the line passing through $A(2, 3, -1)$ and $B(1, -2, 3)$ and show that $Q(-3, -8, 9)$ lies on the line.

- 31 **Example 18** Two lines have vector equations $\mathbf{p} = \mathbf{i} - 2\mathbf{k} + t(2\mathbf{i} + 3\mathbf{j})$ and $\mathbf{q} = 3\mathbf{i} + 6\mathbf{j} - \mathbf{k} + s(2\mathbf{i} - \mathbf{k})$.
Show that the lines intersect and find their point of intersection.
- 32 **Example 19** Consider the plane that passes through $Q(1, -1, 1)$ with normal $\mathbf{n} = \mathbf{i} + \mathbf{j} + \mathbf{k}$.
 a Find the Cartesian equation of a plane.
 b Does the point $(-1, 2, 3)$ lie on the plane?
 c Find an expression for points on the plane inside the circle of radius 5 and centre Q .
- 33 **Example 20** Find the equation of the plane that passes through $P(6, 5, -4)$ and which is parallel to the plane $5x - 2y + 3z + 10 = 0$.
- 34 **Example 21** Find the equation of the plane containing the points:
 a $(1, 0, 1)$, $(2, 4, 6)$ and $(1, 2, -1)$
 b $(1, -2, -3)$, $(4, -4, 4)$ and $(3, 2, -3)$
- 35 **Examples 22–24** Solve the following systems of equations by Gaussian elimination.
- | | | | | | |
|---|---------------------|---|--------------------|---|---------------------|
| a | $5x + 21y + 2z = 5$ | b | $4a - 6b + 3c = 0$ | c | $4p - 12q - 4r = 0$ |
| | $4x + 27y + z = 5$ | | $a - 2b + c = -6$ | | $2p - 5q - r = 3$ |
| | $2x + 12y + z = 4$ | | $2a - 2b + c = -4$ | | $8q - 3p + 2r = -3$ |

Application

- 36 Find the volume of a parallelepiped with vertices $(3, 5, 7)$, $(-2, 1, 9)$, $(4, 2, 5)$, $(9, 6, 3)$, $(1, 0, 2)$, $(-4, -4, 4)$, $(2, -3, 0)$ and $(7, 1, -2)$.



- 37 Show that for any set of points A, B, C, D, E :
 $(2\mathbf{AB} - \mathbf{AC}) \times \mathbf{DE} + \mathbf{BC} \times \mathbf{DE} + \mathbf{AB} \times (\mathbf{CD} - \mathbf{CE}) = \mathbf{0}$
- 38 Use vector methods to prove that the angle bisector of the unequal angle of an isosceles triangle is also the altitude and median.
- 39 Find counterexamples to show that $\mathbf{a} \cdot \mathbf{b}$ and $\mathbf{a} \times \mathbf{b}$ can be zero without $\mathbf{a}$ or $\mathbf{b}$ being zero vectors.

- 40 A plumber regularly buys three sizes of plastic drainpipe in 6 m lengths: 40 mm, 50 mm and 100 mm. In January, his supplier's account included the following:

9 Jan: 3×40 mm, 10×50 mm, 4×100 mm \$475.80

17 Jan: 4×40 mm, 2×50 mm, 1×100 mm \$158.20

24 Jan: 1×40 mm, 5×50 mm, 2×100 mm \$230.20

Set up equations and use Gaussian elimination to find the price of each type of pipe.



Getty Images/Bloomberg

- 41 The median of a triangle is a vector from a vertex to the midpoint of the opposite side. Show that the sum of the medians of a triangle = 0.
- 42 Show that the midpoint of the hypotenuse of a right-angled triangle is equidistant from the vertices of the triangle.
- 43 If $ABCD$ is a parallelogram and M is the midpoint of AC , prove that M is also the midpoint of BD .



Practice quiz



2

TERMINOLOGY

argument
Cartesian form
De Moivre's theorem
imaginary part
modulus
polar form
real part
rectangular

COMPLEX NUMBERS

COMPLEX NUMBERS AND DE MOIVRE'S THEOREM

- 2.01 Review of complex numbers
- 2.02 Review of complex number operations
- 2.03 Complex numbers in polar form
- 2.04 Modulus, argument and principal value
- 2.05 Operations in polar form
- 2.06 De Moivre's theorem
- 2.07 Applications of De Moivre's theorem

Chapter summary

Chapter review



Prior learning

CARTESIAN FORMS

- review real and imaginary parts $Re(z)$ and $Im(z)$ of a complex number z (ACMSM077)
- review Cartesian form (ACMSM078)
- review complex arithmetic using Cartesian forms. (ACMSM079)

COMPLEX ARITHMETIC USING POLAR FORM

- use the modulus $|z|$ of a complex number z and the argument $Arg(z)$ of a non-zero complex number z and prove basic identities involving modulus and argument (ACMSM080)
- convert between Cartesian and polar form (ACMSM081)
- define and use multiplication, division, and powers of complex numbers in polar form and the geometric interpretation of these (ACMSM082)
- prove and use De Moivre's theorem for integral powers. (ACMSM083) 

2.01 REVIEW OF COMPLEX NUMBERS

In this chapter you will review the concepts you studied in Year 11 relating to Complex Numbers and then develop further concepts. Recall the following definitions and rules.

IMPORTANT

The **imaginary number** i is the number such that $i = \sqrt{-1}$.

A **complex number** is a number that can be written in the form $a + ib$, where a and b are real numbers.

A complex number is often denoted by the letter z , so $z = a + ib$.

○ Example 1

Solve the quadratic equation $x^2 - 8x + 25 = 0$, expressing your answers in the form $a + bi$, where $a, b \in \mathbf{R}$.

Solution

Note that $x^2 - 8x + 25$ does not factorise, so it is necessary to solve by using the quadratic formula (or by completing the square).

Substitute and simplify.

$$\begin{aligned}\text{Recall that } \sqrt{-36} &= \sqrt{36 \times (-1)} \\ &= 6 \times i\end{aligned}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\begin{aligned}x &= \frac{-(-8) \pm \sqrt{(-8)^2 - 4(1)(25)}}{2(1)} \\ &= \frac{8 \pm \sqrt{-36}}{2}\end{aligned}$$

$$\begin{aligned}\therefore x &= \frac{8 \pm 6i}{2} \\ &= 4 \pm 3i\end{aligned}$$

Alternate solution

Complete the square.

Make a difference of two squares by first writing $+9 = -9i^2$.

Isolate x .

$$x^2 - 8x + 16 + 9 = 0$$

$$(x - 4)^2 + 9 = 0$$

$$(x - 4)^2 - 9i^2 = 0$$

$$(x - 4 - 3i)(x - 4 + 3i) = 0$$

$$\therefore x = 4 \pm 3i.$$

Recall the definition of the complex conjugate.

IMPORTANT

For a complex number z , where $z = a + ib$ (where a and b are real numbers), the **complex conjugate** is $\bar{z} = a - ib$.

○ Example 2

If $z = x + yi$, where $x, y \in \mathbf{R}$, prove that $z\bar{z}$ is always real.

Solution

Note that $\bar{z} = x - yi$.

Expand and simplify.

$$z\bar{z} = (x + yi)(x - yi)$$

$$z\bar{z} = x^2 - y^2i^2$$

$$= x^2 - [y^2 \times (-1)]$$

$$= x^2 + y^2$$

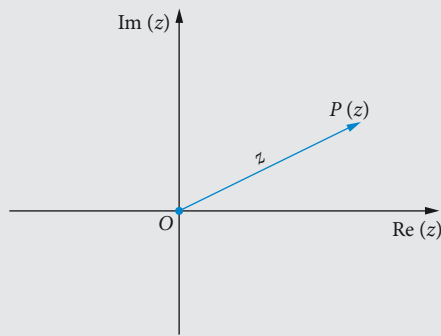
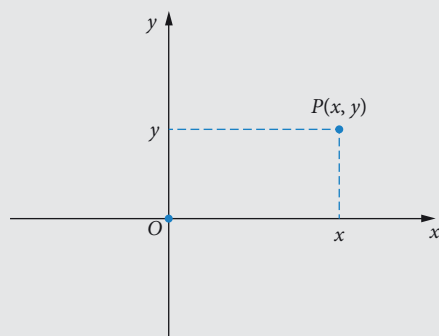
There are no terms that include an i , so all terms are real.

Since $z\bar{z} = x^2 + y^2$, $z\bar{z}$ is always real.

Recall that it is possible to represent the complex number $z = x + yi$ (where $x, y \in \mathbf{R}$) geometrically on an **Argand diagram** or **Argand plane**.

IMPORTANT

The complex number $z = x + yi$ (where $x, y \in \mathbf{R}$) can be represented geometrically on an Argand diagram as the point $P(x, y)$ or the vector $\mathbf{z}$ or $\mathbf{OP}$.



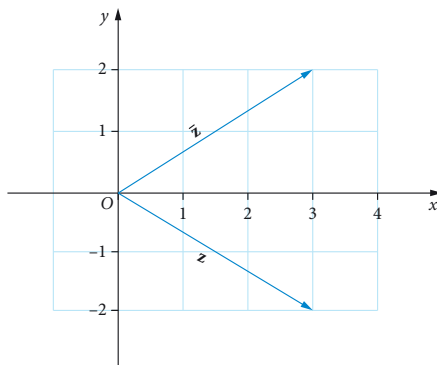
○ Example 3

For the complex number $z = 3 - 2i$,

- state the point P representing z on an Argand plane
- plot the vectors $\mathbf{z}$ and $\overline{\mathbf{z}}$ that represent z and its conjugate $\overline{z}$ on an Argand plane
- describe the relationship between the vectors $\mathbf{z}$ and $\overline{\mathbf{z}}$.

Solution

- The point P representing z is an ordered pair. $P(3, -2)$.
- The conjugate $\overline{z}$ is $3 + 2i$.



- The vectors $\mathbf{z}$ and $\overline{\mathbf{z}}$ have the same real part and opposite imaginary parts. The conjugate vectors are reflections of each other over the real axis.

EXERCISE 2.01 Review of complex numbers

Concepts and techniques

- Example 1** Use the quadratic formula to solve each equation, giving your solutions in the form $a + bi$, where $a, b \in \mathbf{R}$.
 - $x^2 + 2x + 3 = 0$
 - $x^2 - 4x + 7 = 0$
 - $z^2 - 5iz - 6 = 0$
 - $w^2 + w + 2 = 0$
- Solve each equation below by completing the square. Check that your solutions are in the form $a \pm bi$, where $a, b \in \mathbf{R}$.
 - $x^2 - 2x + 2 = 0$
 - $y^2 + 4y + 5 = 0$
 - $z^2 - 6z + 13 = 0$
 - $w^2 + 8w + 18 = 0$
- Evaluate each of the following, giving your answer in simplest form.
 - i^3
 - i^4
 - i^{50}
 - $i + i^2 + \dots + i^{11} + i^{12}$

- 4 **Example 2** Consider the complex number $w = 3 - 5i$.
- State the conjugate $\bar{w}$.
 - Find the value of $w \times \bar{w}$.
- 5 Write a quadratic equation in the form $az^2 + bz + c = 0$ with the following roots.
- $2 \pm i$
 - $\sqrt{3} \pm 2i$
 - $-1 \pm i\sqrt{5}$
- 6 Using your answers to question 2, complete the sentence.
If a quadratic equation has real coefficients and one root is $p + qi$, where $p, q \in \mathbf{R}$, then the other root will be
- 7 **Example 3** The vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ represent the complex numbers $u = 1 - 2i$, $v = 2 + i$ and $w = -3 - 3i$. Sketch $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ on an Argand diagram.

Reasoning and communication

- 8 By considering the coefficients of the equation $x^2 + 2x + 5 = 0$, decide whether or not the roots are complex conjugates. Solve the equation to see if your prediction is true.
- 9 Show that the roots of $x^2 - 3ix - 3 + i = 0$ are $1 + i$ and $2i - 1$. Explain why the roots are not complex conjugates.
- 10 Evaluate $\sum_{r=0}^{197} i^r$.
- 11 Consider the point P representing the complex number $z = -2 + 3i$.
- Sketch the vector $\mathbf{OP}$ on an Argand plane.
 - Sketch the vector $\mathbf{OQ}$ representing $w = \overline{-2 + 3i}$.
 - Describe the geometric relationship between $\mathbf{OP}$ and $\mathbf{OQ}$.
 - Explain why $\mathbf{OP} - \mathbf{OQ}$ lies on the y -axis.

2.02 REVIEW OF COMPLEX NUMBER OPERATIONS

You can perform all four operations with complex numbers: addition, subtraction, multiplication and division. This relies on grouping or equating the **real** and **imaginary parts**.

IMPORTANT

For complex numbers $a + ib$ and $c + id$ (where a, b, c and d are real numbers), $a + ib = c + id$ if and only if $a = c$ AND $b = d$.

○ Example 4

If $z = 2 + i$ and $w = -3 + 2i$, find

a $2z - w$

b w^2

Solution

a Substitute and group the real and imaginary parts.

$$\begin{aligned} 2z - w &= 2(2 + i) - (-3 + 2i) \\ &= 4 + 2i + 3 - 2i \\ &= 7 \end{aligned}$$

b Substitute and expand using the identity $(a + b)^2 = a^2 + 2ab + b^2$.

$$\begin{aligned} w^2 &= (-3 + 2i)^2 \\ &= (-3)^2 + 2(-3)(2i) + (2i)^2 \\ &= 9 - 12i + 4i^2 \\ &= 9 - 12i - 4 \\ &= 5 - 12i \end{aligned}$$

Simplify and recall that $i^2 = -1$.

Remember that you multiply by the complex conjugate to **realise the denominator**.

IMPORTANT

To **realise the denominator** of a complex number, multiply the number by 1 in the form $\frac{\bar{z}}{z}$.

○ Example 5

Express $\frac{3+i}{3-i}$ in the form $a + bi$, where a, b are real.

Solution

Realise the denominator by multiplying by the conjugate.

$$\begin{aligned} \frac{3+i}{3-i} &= \frac{3+i}{3-i} \times \frac{3+i}{3+i} \\ &= \frac{9+6i+i^2}{9-i^2} \end{aligned}$$

Simplify and split into real and imaginary parts.

$$\begin{aligned} &= \frac{9+6i-1}{9-(-1)} \\ &= \frac{8+6i}{10} \\ &= \frac{8}{10} + \frac{6}{10}i \end{aligned}$$

Write the answer in its simplest form by cancelling any common factors.

$$\frac{3+i}{3-i} = \frac{4}{5} + \frac{3}{5}i$$

Recall the notation and language used for the real and imaginary parts of a complex number.

IMPORTANT

The **real part** of $z = a + ib$ is denoted by $\operatorname{Re}(z)$, where $\operatorname{Re}(z) = a$ and the **imaginary part** by $\operatorname{Im}(z)$, where $\operatorname{Im}(z) = b$.

If $\operatorname{Re}(z) = 0$, then z is *purely imaginary*.

If $\operatorname{Im}(z) = 0$, then z is *purely real* or just *real*.

○ Example 6

If $z = 5 + 7i - i(4 - 6i)$, find

a $\operatorname{Re}(z)$

b $\operatorname{Im}(z)$

Solution

First expand and group into the real and imaginary parts.

$$\begin{aligned} z &= 5 + 7i - i(4 - 6i) \\ &= 5 + 7i - 4i + 6i^2 \\ &= 5 + 7i - 4i - 6 \\ &= -1 + 3i \end{aligned}$$

a If $z = a + bi$, where $a, b \in \mathbf{R}$, then

$$\operatorname{Re}(z) = a$$

$$\operatorname{Re}(z) = -1$$

b If $z = a + bi$, where $a, b \in \mathbf{R}$, then

$$\operatorname{Im}(z) = b$$

$$\operatorname{Im}(z) = 3$$

EXERCISE 2.02 Review of complex number operations

Concepts and techniques



Complex number operations

1 **Example 4** Simplify each of the following expressions.

a $4(2 + i) + 3(2 - i)$

b $3(5 - 4i) - i(3 - 2i)$

c $(7 + 3i)^2$

d $(\sqrt{2} + 3i)(\sqrt{2} - 3i)$

e $(9 - i)(5 + 2i)$

2 Find the values of u and v , where u and v are real, if

a $u + vi = 2(3 - 4i) + i(5 + i)$

b $u + vi = -4(1 + 2i) - 3i(2 - 4i)$

3 If $z = 2 - i$ and $w = -4 + 5i$, find:

a zw

b $\bar{z} - \bar{w}$

c $\bar{z} \times w^2$

d $(z + i)(w - 1)$

4 **Example 5** Realise the denominator for each of the following complex fractions.

a $\frac{1}{2 - i}$

b $\frac{1 + 2i}{3 - 2i}$

c $\frac{\sqrt{5} - i}{\sqrt{5} + i}$

5 Show that $\frac{1}{x - y + i(x + y)} = \frac{x - y - i(x + y)}{2(x^2 + y^2)}$.

- 6 Show that $\frac{1+i\sqrt{3}}{1-i\sqrt{3}} + \frac{1-i\sqrt{3}}{1+i\sqrt{3}}$ is always real.
- 7 **Example 6** Find $\operatorname{Re}(z)$ and $\operatorname{Im}(z)$ for each of the following.
- $z = -2\sqrt{3} - 3i\sqrt{2}$
 - $z = 2(5 + 6i) + 3(1 - 7i)$
 - $z = x - yi - 4w + vi$, where x, y, w and v are real.
 - $z = \frac{3-i\sqrt{2}}{1+i\sqrt{2}}$

Reasoning and communication

- 8 Find $\operatorname{Re}(z)$ and $\operatorname{Im}(z)$ if $z = \frac{x-1+yi}{x-1-yi}$, where x, y are real.
- 9 Given $z = u + vi$, where u, v are real, find $\frac{z}{\bar{z}}$. Hence show that

$$\operatorname{Re}\left(\frac{z}{\bar{z}}\right) = \frac{u^2 - v^2}{u^2 + v^2} \text{ and } \operatorname{Im}\left(\frac{z}{\bar{z}}\right) = \frac{2uv}{u^2 + v^2}$$

- 10 Simplify $\frac{1}{(x-yi)^2} + \frac{1}{(x+yi)^2}$.

2.03 COMPLEX NUMBERS IN POLAR FORM

You know that a complex number z can be expressed in the form $x + yi$, where $x, y \in \mathbf{R}$.

This form is known as the **Cartesian form** (or **rectangular form**) as it uses x and y coordinates that can refer to axes. It is also convenient to express a complex number z in **polar form**, which uses the size of z and the angle that z makes with the positive x -axis.

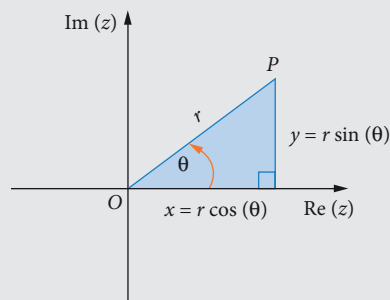
IMPORTANT

The **argument** of the complex number $z \neq 0$, $\arg(z)$, is the angle θ that OP makes with the positive real axis, where P is the point that represents z in the complex plane. The **principal value** of the argument is the one in the interval $(-\pi, \pi]$. The argument of 0 is not defined.

The **modulus** of z is the magnitude of the vector $\mathbf{z}$, given by $\operatorname{mod}(z) = |z| = \sqrt{x^2 + y^2}$.

The **polar form** of z is given by $z = r[\cos(\theta) + i \sin(\theta)]$, $\cos(\theta) + i \sin(\theta)$ is often abbreviated to **cis** (θ) .

The polar form is sometimes referred to as the **modulus-argument form**.



○ Example 7

Write down the complex numbers in polar form, given the following arguments and moduli.

a $\arg(z) = \frac{\pi}{3}, \text{mod}(z) = 6$

b $\arg(w) = \frac{3\pi}{4}, |w| = 2$

Solution

a Recall that $\arg(z) = \theta, \text{mod}(z) = r$.

Substitute these into the formula.

$$\begin{aligned} z &= r [\cos(\theta) + i \sin(\theta)] \\ &= 6 \left[\cos\left(\frac{\pi}{3}\right) + i \sin\left(\frac{\pi}{3}\right) \right] \end{aligned}$$

b Recall that $|w| = r$. Substitute in the values.

$$\begin{aligned} w &= r [\cos(\theta) + i \sin(\theta)] \\ &= 2 \left[\cos\left(\frac{3\pi}{4}\right) + i \sin\left(\frac{3\pi}{4}\right) \right] \end{aligned}$$

IMPORTANT

The Cartesian and polar forms of the complex number z are related by the following equations.

$$r = \sqrt{x^2 + y^2}, \tan(\theta) = \frac{y}{x}, x = r \cos(\theta) \text{ and } y = r \sin(\theta).$$



Modulus and argument

You can convert complex numbers between the Cartesian and polar forms using the relationships above.

○ Example 8

Convert the complex numbers to polar form.

a $z = 1 + i\sqrt{3}$

b $u = -2 - 2i$

Solution

a Find r .

$$|z| = r = \sqrt{1^2 + (\sqrt{3})^2} = 2$$

Find θ .

$$\cos(\theta) = \frac{1}{2} > 0, \sin(\theta) = \frac{\sqrt{3}}{2} > 0 \Rightarrow \theta = \frac{\pi}{3}$$

Write in polar form.

$$\begin{aligned} z &= r [\cos(\theta) + i \sin(\theta)] \\ &= 2 \left[\cos\left(\frac{\pi}{3}\right) + i \sin\left(\frac{\pi}{3}\right) \right] \end{aligned}$$

b Find r .

$$|u| = r = \sqrt{(-2)^2 + (-2)^2} = 2\sqrt{2}$$

Find the principle value of θ .

$$\cos(\theta) = \frac{-2}{2\sqrt{2}} = \frac{-1}{\sqrt{2}} < 0$$

$$\sin(\theta) = \frac{-2}{2\sqrt{2}} = \frac{-1}{\sqrt{2}} < 0 \Rightarrow \theta = -\frac{3\pi}{4}$$

Write in polar form.

$$\begin{aligned} u &= r [\cos(\theta) + i \sin(\theta)] \\ &= 2\sqrt{2} \left[\cos\left(-\frac{3\pi}{4}\right) + i \sin\left(-\frac{3\pi}{4}\right) \right] \end{aligned}$$

○ Example 9

Convert each complex number below to Cartesian form.

a $z = 4 \left[\cos\left(\frac{2\pi}{3}\right) + i \sin\left(\frac{2\pi}{3}\right) \right]$ b $v = \sqrt{2} \left[\cos\left(\frac{\pi}{4}\right) - i \sin\left(\frac{\pi}{4}\right) \right]$

Solution

a Evaluate $\cos\left(\frac{2\pi}{3}\right)$ and $\sin\left(\frac{2\pi}{3}\right)$.

$$z = 4 \left[-\frac{1}{2} + \frac{\sqrt{3}}{2}i \right]$$

Expand.

$$= -2 + 2i\sqrt{3}$$

b Evaluate $\cos\left(\frac{\pi}{4}\right)$ and $\sin\left(\frac{\pi}{4}\right)$.

$$v = \sqrt{2} \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i \right)$$

Expand.

$$= 1 - i$$

Note that a complex number in the form $z = r[\cos(\theta) - i \sin(\theta)]$ can be converted to polar form using trigonometric identities. You can write $z = r[\cos(\theta) - i \sin(\theta)] = r[\cos(-\theta) + i \sin(-\theta)]$ because $\cos(-\theta) = \cos(\theta)$ and $\sin(-\theta) = -\sin(\theta)$.

EXERCISE 2.03 Complex numbers in polar form



Complex number conversions

Concepts and techniques

1 **Example 7** Find the complex number in the form $r[\cos(\theta) + i \sin(\theta)]$ for which

a $\arg(z) = \pi, |z| = 5$

b $\arg(z) = \frac{\pi}{6}, \text{mod}(z) = 4$

c $\theta = -\frac{\pi}{2}, r = 2$

d $\arg(v) = -\frac{\pi}{7}, \text{mod}(v) = 3\sqrt{2}$

2 For each complex number below, state its argument and modulus.

a $2 \left[\cos\left(\frac{\pi}{4}\right) + i \sin\left(\frac{\pi}{4}\right) \right]$

b $2\sqrt{2} \left[\cos\left(\frac{5\pi}{6}\right) + i \sin\left(\frac{5\pi}{6}\right) \right]$

c $\cos\left(-\frac{\pi}{9}\right) + i \sin\left(-\frac{\pi}{9}\right)$

d $\frac{1}{\sqrt{3}} \left[\cos\left(-\frac{\pi}{3}\right) + i \sin\left(-\frac{\pi}{3}\right) \right]$

3 **Example 8** For each complex number z below, calculate $\text{mod}(z)$ and $\arg(z)$ in exact form.

a $z = \sqrt{3} + i$

b $z = 3 + 3i$

c $z = \frac{1}{2} - \frac{1}{2}i$

d $z = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$

e $z = 7i$

f $z = -6$

4 Convert each complex number below to polar form.

a $z = \sqrt{2} + i\sqrt{2}$

b $w = -1 + i\sqrt{3}$

c $u = \frac{\sqrt{3}}{2} - \frac{1}{2}i$

d $v = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$

e $z = -i\sqrt{5}$

f $w = 1$

5 **Example 9** Convert each complex number below to Cartesian form.

a $12 \left[\cos\left(\frac{\pi}{3}\right) + i \sin\left(\frac{\pi}{3}\right) \right]$

b $\sqrt{2} \left[\cos\left(\frac{3\pi}{4}\right) + i \sin\left(\frac{3\pi}{4}\right) \right]$

c $\cos\left(\frac{\pi}{2}\right) + i \sin\left(\frac{\pi}{2}\right)$

d $\frac{\cos\left(-\frac{7\pi}{6}\right) + i \sin\left(-\frac{7\pi}{6}\right)}{2}$

- 6 Express each complex number below, with given arg and mod values, in Cartesian form.
- a $\arg(z) = -\frac{\pi}{6}$, $|z| = 8$ b $\arg(z) = \frac{5\pi}{3}$, $\text{mod}(z) = 3$
- c $\theta = \pi$, $r = 9$ d $\arg(v) = \frac{\pi}{3}$, $\text{mod}(v) = \sqrt{27}$
- 7 Simplify each complex number below, expressing your answer in $x + yi$ form.
- a $2\cos\left(\frac{\pi}{6}\right) + 3i\sin\left(\frac{\pi}{3}\right)$ b $\sqrt{8}\left[\cos\left(-\frac{5\pi}{4}\right) + i\sin\left(\frac{\pi}{6}\right)\right]$

Reasoning and communication

- 8 Use trigonometric identities to convert each complex number to polar form.
- a $r[\cos(\theta) - i\sin(\theta)]$ b $r[-\cos(\theta) + i\sin(\theta)]$ c $-r[\cos(\theta) + i\sin(\theta)]$
- 9 a Show that the points given by $P(r\cos(\theta), r\sin(\theta))$ for $0 \leq \theta < 2\pi$ form a circle in the Cartesian plane.
- b Show that the complex number $r\sin(\theta) + ri\cos(\theta)$ can be expressed in polar form.
- c If $x + yi = 3\cos(\theta) + 4i\sin(\theta)$, show that $\frac{x^2}{9} + \frac{y^2}{16} = 1$.

2.04 MODULUS, ARGUMENT AND PRINCIPAL VALUE

$\frac{5\pi}{3}$ and $-\frac{\pi}{3}$ represent the same angle on the Cartesian or Argand plane. This means that a complex number can also be expressed in infinitely many ways. For example, $z = 2\left[\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right]$ is the same as $z = 2\left[\cos\left(-\frac{5\pi}{4}\right) + i\sin\left(-\frac{5\pi}{4}\right)\right]$. To avoid ambiguity, a complex number is usually expressed in polar form using the principal argument. In this case, $z = 2\left[\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right]$ rather than $z = 2\left[\cos\left(-\frac{5\pi}{4}\right) + i\sin\left(-\frac{5\pi}{4}\right)\right]$ is the convention since $-\pi < \frac{3\pi}{4} \leq \pi$ and $\frac{3\pi}{4}$ is the principal argument.

Multiplying and dividing complex numbers in modulus-argument form is much easier than in the Cartesian form. Adding and subtracting complex numbers is easier using Cartesian form.

Example 10

Evaluate $z_1 \times z_2$ if $z_1 = 2\left[\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right]$ and $z_2 = 5\left[\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right]$.

Solution

$$z_1 \times z_2 = 2\left[\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right] \times 5\left[\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right]$$

Use the distributive law.

$$= 10\left[\cos\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{6}\right) + i\cos\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{6}\right) + i\cos\left(\frac{\pi}{6}\right)\sin\left(\frac{\pi}{4}\right) + i^2\sin\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{6}\right)\right]$$

Rearrange into real and imaginary parts.

$$= 10\left\{\cos\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{6}\right) - \sin\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{6}\right) + i\left[\cos\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{6}\right) + \cos\left(\frac{\pi}{6}\right)\sin\left(\frac{\pi}{4}\right)\right]\right\}$$

Use the trigonometric identities.

$$= 10\left[\cos\left(\frac{\pi}{4} + \frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{4} + \frac{\pi}{6}\right)\right]$$

$$\therefore z_1 \times z_2 = 10\left[\cos\left(\frac{5\pi}{12}\right) + i\sin\left(\frac{5\pi}{12}\right)\right]$$

The example above demonstrates that when multiplying, you multiply two complex numbers in polar form by multiplying the moduli and adding the arguments.

IMPORTANT

If $z_1 = r_1[\cos(\theta_1) + i \sin(\theta_1)]$ and $z_2 = r_2[\cos(\theta_2) + i \sin(\theta_2)]$ are two complex numbers, then

- their **product** is $z_1 z_2 = r_1 r_2 [\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)]$
- $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$ and $\text{mod}(z_1 z_2) = \text{mod}(z_1) \times \text{mod}(z_2)$

○ Example 11

Find the product of $z_1 = 3\left[\cos\left(\frac{3\pi}{5}\right) + i \sin\left(\frac{3\pi}{5}\right)\right]$ and $z_2 = 2\left[\cos\left(\frac{4\pi}{5}\right) + i \sin\left(\frac{4\pi}{5}\right)\right]$.

Solution

Use the rule $z_1 z_2 = r_1 r_2 [\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)]$

$$\begin{aligned} z_1 z_2 &= 3\left[\cos\left(\frac{3\pi}{5}\right) + i \sin\left(\frac{3\pi}{5}\right)\right] \times 2\left[\cos\left(\frac{4\pi}{5}\right) + i \sin\left(\frac{4\pi}{5}\right)\right] \\ &= 3 \times 2\left[\cos\left(\frac{3\pi}{5} + \frac{4\pi}{5}\right) + i \sin\left(\frac{3\pi}{5} + \frac{4\pi}{5}\right)\right] \end{aligned}$$

Simplify. Note that the principal value of the argument must be in the interval $(-\pi, \pi]$.

$$\begin{aligned} z_1 z_2 &= 6\left[\cos\left(\frac{7\pi}{5}\right) + i \sin\left(\frac{7\pi}{5}\right)\right] \\ &= 6\left[\cos\left(-\frac{3\pi}{5}\right) + i \sin\left(-\frac{3\pi}{5}\right)\right] \end{aligned}$$

In a similar way, it is possible to develop rules for division of two complex numbers in polar form.

IMPORTANT

If $z_1 = r_1[\cos(\theta_1) + i \sin(\theta_1)]$ and $z_2 = r_2[\cos(\theta_2) + i \sin(\theta_2)]$ are two complex numbers, then

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} [\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)].$$

If $z = r[\cos(\theta) + i \sin(\theta)]$ is a complex number, then

$$z^{-1} = \frac{1}{z} = \frac{1}{r} [\cos(-\theta) + i \sin(-\theta)].$$

OR

If $z_1 = r_1[\cos(\theta_1) + i \sin(\theta_1)]$ and $z_2 = r_2[\cos(\theta_2) + i \sin(\theta_2)]$ are two complex numbers, then

$$\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2) \text{ and } \text{mod}\left(\frac{z_1}{z_2}\right) = \frac{\text{mod}(z_1)}{\text{mod}(z_2)}.$$

If $z = r[\cos(\theta) + i \sin(\theta)]$ is a complex number, then

$$\arg(z^{-1}) = -\arg(z) \text{ and } \text{mod}(z^{-1}) = \frac{1}{\text{mod}(z)}.$$

○ Example 12

If $z = \sqrt{2} \left[\cos\left(\frac{\pi}{6}\right) + i \sin\left(\frac{\pi}{6}\right) \right]$ and $w = 2\sqrt{2} \left[\cos\left(\frac{2\pi}{3}\right) + i \sin\left(\frac{2\pi}{3}\right) \right]$, find

a $\frac{z}{w}$ b z^{-1}

Solution

a Use the rule

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} \left[\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2) \right],$$

then simplify.

$$\begin{aligned} \frac{z}{w} &= \frac{\sqrt{2}}{2\sqrt{2}} \left[\cos\left(\frac{\pi}{6} - \frac{2\pi}{3}\right) + i \sin\left(\frac{\pi}{6} - \frac{2\pi}{3}\right) \right] \\ &= \frac{1}{2} \left[\cos\left(-\frac{3\pi}{6}\right) + i \sin\left(-\frac{3\pi}{6}\right) \right] \\ &= \frac{1}{2} \left[\cos\left(-\frac{\pi}{2}\right) + i \sin\left(-\frac{\pi}{2}\right) \right] \end{aligned}$$

b Use the rule $z^{-1} = \frac{1}{z} = \frac{1}{r} \left[\cos(-\theta) + i \sin(-\theta) \right]$ $z^{-1} = \frac{1}{\sqrt{2}} \left[\cos\left(-\frac{\pi}{6}\right) + i \sin\left(-\frac{\pi}{6}\right) \right]$

EXERCISE 2.04 Modulus, argument and principal value

Concepts and techniques

1 **Example 10** Expand, then use the identities $\sin(A + B) = \sin(A) \cos(B) + \cos(A) \sin(B)$ and $\cos(A + B) = \cos(A) \cos(B) - \sin(A) \sin(B)$ to simplify:

a $\left[\cos\left(\frac{\pi}{5}\right) + i \sin\left(\frac{\pi}{5}\right) \right] \left[\cos\left(\frac{\pi}{7}\right) + i \sin\left(\frac{\pi}{7}\right) \right]$

b $\left[\cos\left(\frac{2\pi}{9}\right) + i \sin\left(\frac{2\pi}{9}\right) \right] \left[\cos\left(\frac{\pi}{9}\right) + i \sin\left(\frac{\pi}{9}\right) \right]$

c $[\cos(\alpha) + i \sin(\alpha)][\cos(4\alpha) + i \sin(4\alpha)]$

d $[\cos(-3\beta) + i \sin(-3\beta)][\cos(-7\beta) + i \sin(-7\beta)]$

2 Express each of the following using a principal argument.

a $\text{cis}\left(\frac{3\pi}{2}\right)$

b $\text{cis}\left(\frac{5\pi}{4}\right)$

c $3 \text{cis}\left(\frac{9\pi}{5}\right)$

d $2 \text{cis}\left(\frac{-7\pi}{4}\right)$

3 Expand and then simplify, expressing your answer in polar form.

$$\left[\cos\left(\frac{\pi}{5}\right) - i \sin\left(\frac{\pi}{5}\right) \right] \left[\cos\left(\frac{3\pi}{7}\right) - i \sin\left(\frac{3\pi}{7}\right) \right]$$

4 **Example 11** Use the fact that for two complex numbers, $z_1 z_2 = r_1 \text{cis}(\theta_1) \times r_2 \text{cis}(\theta_2) = r_1 r_2 \text{cis}(\theta_1 + \theta_2)$ to simplify the following, leaving your answer in polar form.

a $\sqrt{2} \text{cis}\left(\frac{\pi}{3}\right) \times \sqrt{5} \text{cis}\left(\frac{\pi}{4}\right)$

b $2 \text{cis}\left(\frac{5\pi}{7}\right) \times 4 \text{cis}\left(\frac{3\pi}{7}\right)$

c $-3 \text{cis}\left(\frac{\pi}{8}\right) \times \text{cis}\left(\frac{\pi}{4}\right)$

d $\sqrt{6} \text{cis}\left(\frac{-5\pi}{9}\right) \times \sqrt{2} \text{cis}\left(\frac{-8\pi}{9}\right)$

5 Realise the denominators of the following, expressing your answer in polar form.

a $\frac{1}{\cos\left(\frac{\pi}{3}\right) - i \sin\left(\frac{\pi}{3}\right)}$

b $\frac{1}{\sin\left(\frac{3\pi}{7}\right) + i \cos\left(\frac{3\pi}{7}\right)}$

c $\frac{1}{2 \text{cis}\left(\frac{-\pi}{4}\right)}$

- 6 **Example 12** Use the rules $\frac{z_1}{z_2} = \frac{r_1 \text{cis}(\theta_1)}{r_2 \text{cis}(\theta_2)} = \frac{r_1}{r_2} \text{cis}(\theta_1 - \theta_2)$ and $z^{-1} = \frac{1}{z} = \frac{1}{r \text{cis}(\theta)} = \frac{1}{r} \text{cis}(-\theta)$ to simplify:

a $\frac{6 \text{cis}(\frac{\pi}{2})}{2 \text{cis}(\frac{\pi}{3})}$

b $\frac{12 \text{cis}(\frac{-\pi}{3})}{3 \text{cis}(\frac{2\pi}{3})}$

c $\frac{15 \text{cis}(\frac{-\pi}{5})}{5 \text{cis}(\frac{-3\pi}{5})}$

d $\frac{1}{\text{cis}(\theta)}$

e $\frac{1}{3 \text{cis}(\frac{3\pi}{2})}$

f $\frac{4}{\text{cis}(\frac{-5\pi}{4})}$

- 7 If $z = \sqrt{3} \text{cis}(\frac{\pi}{3})$ and $w = \text{cis}(\frac{3\pi}{4})$, find:

a zw

b $\frac{z}{w}$

c $\frac{1}{zw}$

d z^2

e $(zw)^{-2}$

f $\frac{z}{zw}$

- 8 Simplify each of the following, leaving your answer in polar form.

a $\left[\cos\left(\frac{\pi}{3}\right) + i \sin\left(\frac{\pi}{3}\right) \right] \left[\cos\left(\frac{5\pi}{3}\right) - i \sin\left(\frac{5\pi}{3}\right) \right]$

b $[\cos(4) - i \sin(4)][\cos(-2) - i \sin(-2)]$

c $3 \left[\cos\left(\frac{\pi}{2}\right) - i \sin\left(\frac{\pi}{2}\right) \right] \left[4 \cos\left(\frac{3\pi}{4}\right) + 4i \sin\left(\frac{3\pi}{4}\right) \right]$

Reasoning and communication

- 9 If $z = \cos(\theta) + i \sin(\theta)$, prove that $z^{-1} = \bar{z}$.
- 10 If $z = r[\cos(\theta) + i \sin(\theta)]$, prove that $z^{-1} = \frac{\bar{z}}{r^2}$.
- 11 If $z = r[\cos(\theta) + i \sin(\theta)]$ and $w = r[\cos(\alpha) + i \sin(\alpha)]$, show that $\text{Re}(zw) = r^2 \cos(\theta + \alpha)$ and $\text{Im}(zw) = r^2 \sin(\theta + \alpha)$.

2.05 OPERATIONS IN POLAR FORM

You can combine rules to solve problems.

Example 13

By first expressing $\sqrt{3} + i$ in polar form, find:

a $\arg(\sqrt{3} + i)^{-1}$

b $\left| (\sqrt{3} + i)^{-1} \right|$

Solution

Find the modulus, r , and argument, θ , of $\sqrt{3} + i$.

$$r = |\sqrt{3} + i| = \sqrt{(\sqrt{3})^2 + 1^2} = 2$$

$$\cos(\theta) = \frac{\sqrt{3}}{2}, \sin(\theta) = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}$$

$$\therefore \sqrt{3} + i = 2 \left[\cos\left(\frac{\pi}{6}\right) + i \sin\left(\frac{\pi}{6}\right) \right]$$

a Use the rule $\arg(z^{-1}) = -\arg(z)$.

$$\arg(\sqrt{3} + i)^{-1} = -\arg(\sqrt{3} + i) \\ = -\frac{\pi}{6}$$

b Use the rule $\text{mod}(z^{-1}) = \frac{1}{\text{mod}(z)}$.

$$\left|(\sqrt{3} + i)^{-1}\right| = \frac{1}{\text{mod}(\sqrt{3} + i)} = \frac{1}{2}$$

You can extend rules to multiple complex numbers.

IMPORTANT

If $z_1 = r_1[\cos(\theta_1) + i \sin(\theta_1)]$, $z_2 = r_2[\cos(\theta_2) + i \sin(\theta_2)]$, ..., $z_n = r_n[\cos(\theta_n) + i \sin(\theta_n)]$ are multiple complex numbers, then

$$z_1 z_2 \dots z_n = r_1 r_2 \dots r_n [\cos(\theta_1 + \theta_2 + \dots + \theta_n) + i \sin(\theta_1 + \theta_2 + \dots + \theta_n)]$$

OR

$$\arg(z_1 z_2 \dots z_n) = \arg(z_1) + \arg(z_2) + \dots + \arg(z_n) \text{ and } |z_1 z_2 \dots z_n| = |z_1| |z_2| \dots |z_n|.$$

If $z_1 = z_2 = \dots = z_n$, then the following rules also hold.

IMPORTANT

$$\arg(z^n) = n \times \arg(z) \text{ and } |z^n| = |z|^n$$

and

$$\arg(z^{-n}) = -n \times \arg(z) \text{ and } |z^{-n}| = |z|^{-n}.$$

○ Example 14

If $z = \sqrt{2} \text{cis}\left(\frac{3\pi}{4}\right)$, $w = \sqrt{3} \text{cis}\left(\frac{\pi}{4}\right)$ and $u = \sqrt{2} \text{cis}\left(\frac{\pi}{2}\right)$, find

a $\arg(zwu)$ b $\left|\frac{z}{w^2 u}\right|$

Solution

a Use the rule $\arg(z_1 z_2 \dots z_n) = \arg(z_1) + \arg(z_2) + \dots + \arg(z_n)$

$$\arg(zwu) = \frac{3\pi}{4} + \frac{\pi}{4} + \frac{\pi}{2} = \frac{3\pi}{2}$$

Express $\frac{3\pi}{2}$ in the domain $(-\pi, \pi]$.

$$\therefore \arg(zwu) = \frac{-\pi}{2}.$$

b Use $|z_1 z_2 \dots z_n| = |z_1| |z_2| \dots |z_n|$ and $|z^n| = |z|^n$

$$\left|\frac{z}{w^2 u}\right| = \frac{|z|}{|w|^2 |u|} = \frac{|z|}{|w|^2 |u|}$$

Now substitute and simplify.

$$\left|\frac{z}{w^2 u}\right| = \frac{\sqrt{2}}{(\sqrt{3})^2 \sqrt{2}} = \frac{1}{3}$$

A number of difficult calculations can be done by expressing complex numbers in polar form.

○ Example 15

Evaluate $\frac{1}{(1+i)(1-i\sqrt{3})}$, leaving your answer in polar form.

Solution

First convert $1+i$ and $1-i\sqrt{3}$ to polar form.

$$1+i = \sqrt{2} \left[\cos\left(\frac{\pi}{4}\right) + i \sin\left(\frac{\pi}{4}\right) \right] \text{ and}$$

$$1-i\sqrt{3} = 2 \left[\cos\left(\frac{-\pi}{3}\right) + i \sin\left(\frac{-\pi}{3}\right) \right].$$

Now use the rules $\arg(zw) = \arg(z) + \arg(w)$ and $\arg(z^{-1}) = -\arg(z)$

$$\begin{aligned} \arg \left[\frac{1}{(1+i)(1-i\sqrt{3})} \right] &= -\arg[(1+i)(1-i\sqrt{3})] \\ &= -[\arg(1+i) + \arg(1-i\sqrt{3})] \\ &= -\left(\frac{\pi}{4} - \frac{\pi}{3}\right) \\ &= \frac{\pi}{12} \end{aligned}$$

Use the rules $|zw| = |z||w|$ and $|z^{-n}| = |z|^{-n}$.

$$\begin{aligned} |(1+i)(1-i\sqrt{3})| &= |(1+i)||1-i\sqrt{3}| \\ &= \sqrt{2} \times 2 \\ &= 2\sqrt{2} \end{aligned}$$

$$\therefore \left| \frac{1}{(1+i)(1-i\sqrt{3})} \right| = \frac{1}{|(1+i)(1-i\sqrt{3})|} = \frac{1}{2\sqrt{2}}$$

Express your answer in the form $r[\cos(\theta) + i \sin(\theta)]$.

$$\therefore \frac{1}{(1+i)(1-i\sqrt{3})} = \frac{1}{2\sqrt{2}} \left[\cos\left(\frac{\pi}{12}\right) + i \sin\left(\frac{\pi}{12}\right) \right].$$

EXERCISE 2.05 Operations in polar form

Concepts and techniques

- 1 **Example 13** Convert each number to polar form and hence find $\arg(z)$ and $|z|$.

a $z = 1 - i$

b $z = \sqrt{3} + i$

c $z = -\sqrt{2} + i\sqrt{2}$

d $z = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$

- 2 Find the argument of each complex number in radians, correct to 3 significant figures.

a $z = 5 - 2i$

b $z = -2 + 3i$

c $z = -\sqrt{3} - 2i$

d $z = 1 - i\sqrt{5}$



Constructing
spirals



Polar complex number
operations

3 Evaluate each of the following.

a $\arg\left(\frac{1}{2} - \frac{i}{2}\right)$ b $|3[\cos(2) + i \sin(2)]|$ c $\arg(\operatorname{cis}(\theta) \times \operatorname{cis}(\alpha))$ d $\left| \frac{5 \operatorname{cis}\left(\frac{\pi}{9}\right)}{\sqrt{5} \operatorname{cis}\left(\frac{\pi}{3}\right)} \right|$

4 **Example 14** Use the theorems $\arg(z_1 z_2 \dots z_n) = \arg(z_1) + \arg(z_2) + \dots + \arg(z_n)$ and $|z_1 z_2 \dots z_n| = |z_1| |z_2| \dots |z_n|$ to simplify:

a $[\cos(2\alpha) + i \sin(2\alpha)][\cos(3\alpha) + i \sin(3\alpha)][\cos(6\alpha) + i \sin(6\alpha)]$
 b $2\left[\cos\left(\frac{\pi}{9}\right) + i \sin\left(\frac{\pi}{9}\right)\right] \times \sqrt{2}\left[\cos\left(\frac{\pi}{4}\right) + i \sin\left(\frac{\pi}{4}\right)\right] \times \sqrt{6}\left[\cos\left(\frac{-\pi}{18}\right) + i \sin\left(\frac{-\pi}{18}\right)\right]$
 c $\left[\cos\left(\frac{3\pi}{7}\right) + i \sin\left(\frac{3\pi}{7}\right)\right] \left[3\cos\left(\frac{\pi}{7}\right) - 3i \sin\left(\frac{\pi}{7}\right)\right] \left[\sqrt{5}\cos\left(\frac{-2\pi}{7}\right) - i\sqrt{5}\sin\left(\frac{-2\pi}{7}\right)\right]$
 d $\frac{1}{\left[4\cos\left(\frac{5\pi}{11}\right) + 4i \sin\left(\frac{5\pi}{11}\right)\right]^2 \times 3\left[\cos\left(\frac{3\pi}{5}\right) + i \sin\left(\frac{3\pi}{5}\right)\right]}$

5 Use the rules $|z^n| = |z|^n$ and $|z^{-n}| = |z|^{-n}$ to find

a $|\cos(\theta) + i \sin(\theta)|^5$ b $\left|3\left[\cos\left(\frac{\pi}{6}\right) + i \sin\left(\frac{\pi}{6}\right)\right]\right|^4$ c $\left|\frac{1}{\sqrt{7}[\cos(\theta) + i \sin(\theta)]}\right|^3$

6 Use the rules $\arg(z^n) = n \times \arg(z)$ and $\arg(z^{-n}) = -n \times \arg(z)$ to simplify

a $\arg\{[\cos(\theta) + i \sin(\theta)]^6\}$ b $\arg\left\{\left[\cos\left(\frac{3\pi}{8}\right) + i \sin\left(\frac{3\pi}{8}\right)\right]^9\right\}$ c $\arg\left\{\frac{1}{\left[\cos\left(\frac{4\pi}{9}\right) + i \sin\left(\frac{4\pi}{9}\right)\right]^4}\right\}$

7 **Example 15** First express each of the following in polar form and then simplify. Leave your answer in polar form where appropriate.

a $(\sqrt{3} - i)(1 + i)$ b $\frac{\sqrt{2} + i\sqrt{2}}{1 + i\sqrt{3}}$
 c $\frac{1}{6\left(-\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)}$ d $\frac{(2 - 2i\sqrt{3})(-\sqrt{3} + i)}{(4 + 4i)}$

Reasoning and communication

8 a Express $(1 + i)(1 + i\sqrt{3})$ in mod-arg form.

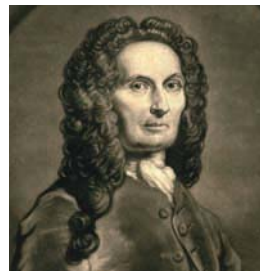
b Hence find the exact value of $\cos\left(\frac{7\pi}{2}\right)$.

9 If $r[\cos(\alpha) + i \sin(\alpha)] = \frac{3 - i}{2 + 5i}$, find the exact value of $\sin(\alpha)$.

10 If $\{r[\cos(\alpha) + i \sin(\alpha)]\}^n = x + yi$, where x and y are real, prove that $r = (x^2 + y^2)^{\frac{1}{2n}}$.

2.06 DE MOIVRE'S THEOREM

The rules in the previous section can be extended to powers of $z = r[\cos(\theta) + i \sin(\theta)]$. This work was developed by the French mathematician Abraham De Moivre (1667 – 1754) and is called De Moivre's theorem.



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IMPORTANT

De Moivre's theorem

If $z = \cos(\theta) + i \sin(\theta)$ is a complex number, then $z^n = \cos(n\theta) + i \sin(n\theta)$, $\forall n \in \mathbb{Z}$.

Proof

The proof relies on mathematical induction.

Proposition

Let $P(n)$ be the proposition that $[\cos(\theta) + i \sin(\theta)]^n = \cos(n\theta) + i \sin(n\theta)$, $\forall n \in \mathbb{N}$, $n \geq 1$.

RTP

Both $P(1)$ is true and $P(k+1)$ is true given that $P(k)$ is true.

Proof

When $n = 1$,

$$\text{LHS} = [\cos(\theta) + i \sin(\theta)]^1$$

$$= \cos(\theta) + i \sin(\theta)$$

$$\text{RHS} = [\cos(1 \times \theta) + i \sin(1 \times \theta)]$$

$$= \cos(\theta) + i \sin(\theta)$$

$\therefore \text{LHS} = \text{RHS} \Rightarrow P(1)$ is true.

Assume that $P(k)$ is true, i.e. that

$$[\cos(\theta) + i \sin(\theta)]^k = [\cos(k\theta) + i \sin(k\theta)], \text{ for some } k \in \mathbb{N}, k \geq 1.$$

$$P(k+1): [\cos(\theta) + i \sin(\theta)]^{k+1} = \cos[(k+1)\theta] + i \sin[(k+1)\theta], \text{ for some } k \in \mathbb{N}, k \geq 1.$$

Consider the LHS of $P(k+1)$:

$$\begin{aligned} [\cos(\theta) + i \sin(\theta)]^{k+1} &= [\cos(\theta) + i \sin(\theta)][\cos(\theta) + i \sin(\theta)]^k \\ &= [\cos(\theta) + i \sin(\theta)][\cos(k\theta) + i \sin(k\theta)], \text{ using } P(k) \\ &= \cos(\theta) \cos(k\theta) + \cos(\theta) i \sin(k\theta) + i \sin(\theta) \cos(k\theta) + i^2 \sin(\theta) \sin(k\theta) \\ &= \cos(\theta) \cos(k\theta) - \sin(\theta) \sin(k\theta) + i[\cos(\theta) \sin(k\theta) + \sin(\theta) \cos(k\theta)] \\ &= \cos(\theta + k\theta) + i \sin(\theta + k\theta) \\ &= \cos[(k+1)\theta] + i \sin[(k+1)\theta] \\ &= \text{RHS of } P(k+1) \end{aligned}$$

Therefore $P(k+1)$ is true.

Therefore $P(n)$ is true by mathematical induction.

QED

○ Example 16

Use De Moivre's theorem to simplify the following.

a $[\cos(\theta) + i \sin(\theta)]^6$ b $[\cos(3\alpha) + i \sin(3\alpha)]^{-8}$ c $\left[\cos\left(\frac{\pi}{4}\right) - i \sin\left(\frac{\pi}{4}\right)\right]^4$

Solution

a Apply the theorem directly using the formula $[\cos(\theta) + i \sin(\theta)]^n = \cos(n\theta) + i \sin(n\theta)$

$$[\cos(\theta) + i \sin(\theta)]^6 = \cos(6\theta) + i \sin(6\theta)$$

b Let $\theta = 3\alpha$ and apply the theorem.

$$\begin{aligned} [\cos(3\alpha) + i \sin(3\alpha)]^{-8} \\ = \cos(-24\alpha) + i \sin(-24\alpha) \end{aligned}$$

c First, write the expression in polar form.

$$\left[\cos\left(\frac{\pi}{4}\right) - i \sin\left(\frac{\pi}{4}\right)\right]^4 = \left[\cos\left(-\frac{\pi}{4}\right) + i \sin\left(-\frac{\pi}{4}\right)\right]^4$$

Now apply the theorem and simplify.

$$\begin{aligned} \left[\cos\left(-\frac{\pi}{4}\right) + i \sin\left(-\frac{\pi}{4}\right)\right]^4 &= \cos(-\pi) + i \sin(-\pi) \\ &= -1 \end{aligned}$$

De Moivre's theorem is useful for evaluating powers of complex numbers. First, convert the expressions to polar form.

○ Example 17

Convert $\frac{-1+i\sqrt{3}}{2}$ to polar form and hence evaluate $\left(\frac{-1+i\sqrt{3}}{2}\right)^8$, giving your answer in the form $a + bi$.

Solution

First, find r and θ .

$$r = \sqrt{\left(\frac{-1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = 1$$

$$\cos(\theta) = \frac{-1}{2}, \sin(\theta) = \frac{\sqrt{3}}{2} \Rightarrow \theta = \frac{2\pi}{3}$$

$$\therefore \frac{-1+i\sqrt{3}}{2} = \cos\left(\frac{2\pi}{3}\right) + i \sin\left(\frac{2\pi}{3}\right)$$

Now apply De Moivre's theorem and simplify.

$$\begin{aligned} \left[\cos\left(\frac{2\pi}{3}\right) + i \sin\left(\frac{2\pi}{3}\right)\right]^8 &= \cos\left(\frac{16\pi}{3}\right) + i \sin\left(\frac{16\pi}{3}\right) \\ &= \cos\left(\frac{4\pi}{3}\right) + i \sin\left(\frac{4\pi}{3}\right) \\ &\text{or } \cos\left(-\frac{2\pi}{3}\right) + i \sin\left(-\frac{2\pi}{3}\right) \end{aligned}$$

Convert to Cartesian form.

$$\begin{aligned} \cos\left(-\frac{2\pi}{3}\right) + i \sin\left(-\frac{2\pi}{3}\right) &= -\frac{1}{2} - \frac{\sqrt{3}}{2}i \\ \therefore \left(\frac{-1+i\sqrt{3}}{2}\right)^8 &= -\frac{1}{2} - \frac{\sqrt{3}}{2}i. \end{aligned}$$

You can now use a combination of rules to simplify expressions.

○ Example 18

Simplify
$$\frac{[\cos(2\beta) + i \sin(2\beta)]^7 \times [\cos(3\beta) + i \sin(3\beta)]^4}{[\cos(\beta) + i \sin(\beta)]^2}$$

Solution

Use De Moivre's theorem to simplify each bracket.

$$\begin{aligned} & \frac{[\cos(2\beta) + i \sin(2\beta)]^7 \times [\cos(3\beta) + i \sin(3\beta)]^4}{[\cos(\beta) + i \sin(\beta)]^2} \\ &= \frac{[\cos(14\beta) + i \sin(14\beta)] \times [\cos(12\beta) + i \sin(12\beta)]}{\cos(2\beta) + i \sin(2\beta)} \\ &= \cos(14\beta + 12\beta - 2\beta) + i \sin(14\beta + 12\beta - 2\beta) \\ &= \cos(24\beta) + i \sin(24\beta) \end{aligned}$$

Use $z_1 z_2 = r_1 r_2 [\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)]$ and

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} [\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)]$$

EXERCISE 2.06 De Moivre's theorem



Using De Moivre's theorem

Concepts and techniques

1 **Example 16** Use De Moivre's theorem to simplify each complex number.

a $[\cos(\theta) + i \sin(\theta)]^9$

b $[\cos(\theta) + i \sin(\theta)]^{-2}$

c $[\cos(\theta) - i \sin(\theta)]^4$

d $[\cos(2\theta) + i \sin(2\theta)]^3$

2 Evaluate z^4 in Cartesian form if

a $z = 2 \left[\cos\left(\frac{\pi}{4}\right) + i \sin\left(\frac{\pi}{4}\right) \right]$

b $z = 3 \left[\cos\left(\frac{\pi}{3}\right) + i \sin\left(\frac{\pi}{3}\right) \right]$

c $z = \frac{1}{\sqrt{2}} \left[\cos\left(\frac{\pi}{8}\right) + i \sin\left(\frac{\pi}{8}\right) \right]$

d $z = 5 \left[\cos\left(\frac{7\pi}{12}\right) + i \sin\left(\frac{7\pi}{12}\right) \right]$

3 Evaluate $\left[\cos\left(-\frac{3\pi}{5}\right) + i \sin\left(-\frac{3\pi}{5}\right) \right]^{-6}$.

4 **Example 17** By first converting to polar form, evaluate each of the following, giving your answer in Cartesian form.

a $(1 + i)^3$

b $(\sqrt{3} + i)^4$

c $(\sqrt{2} - i\sqrt{2})^7$

d $\left(\frac{-1 - i}{\sqrt{2}} \right)^{-6}$

5 Evaluate the following.

a $\frac{1}{(2 + 2i)^6}$

b $\frac{1}{(1 - i\sqrt{3})^8}$

6 Show that $\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i \right)^{12} \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i \right)^3 = 1$

7 **Example 18** Simplify each complex number.

a $[\cos(\theta) + i \sin(\theta)]^5 \times [\cos(\theta) + i \sin(\theta)]^{-8}$

b $[\cos(\alpha) + i \sin(\alpha)]^3 \times [\cos(\beta) + i \sin(\beta)]^4$

c $\frac{[\cos(4\phi) + i \sin(4\phi)]^5}{[\cos(7\phi) + i \sin(7\phi)]^3}$

d $\frac{[\cos(\beta) - i \sin(\beta)]^2 \times [\cos(\beta) + i \sin(\beta)]^{-3}}{[\cos(3\beta) + i \sin(3\beta)]^4}$

Reasoning and communication

8 If $z = \cos(\theta) + i \sin(\theta)$, prove that $z^2 + \frac{1}{z^2}$ is always real.

9 If $z = \cos(\theta) + i \sin(\theta)$, prove that $z^3 - \frac{1}{z^3}$ is purely imaginary.

10 If $z = \cos(\theta) + i \sin(\theta)$, what is the value of $z^n + \frac{1}{z^n}$?

2.07 APPLICATIONS OF DE MOIVRE'S THEOREM

De Moivre's theorem is useful for deriving or proving a number of trigonometric identities. These will be examined in this section.

Example 19

Use the binomial theorem to expand $[\cos(\theta) + i \sin(\theta)]^3$. Hence derive a formula for $\cos(3\theta)$ in terms of $\cos(\theta)$.

Solution

Recall the expansion for $(a + b)^3$.

$$\begin{aligned} [\cos(\theta) + i \sin(\theta)]^3 &= \cos^3(\theta) + 3 \cos^2(\theta) [i \sin(\theta)] \\ &\quad + 3 \cos(\theta) [i^2 \sin^2(\theta)] + i^3 \sin^3(\theta) \\ &= \cos^3(\theta) + 3i \cos^2(\theta) \sin(\theta) - 3 \cos(\theta) \sin^2(\theta) \\ &\quad - i \sin^3(\theta) \end{aligned}$$

Use De Moivre's theorem to find an expression with $\cos(3\theta)$.

$$[\cos(\theta) + i \sin(\theta)]^3 = \cos(3\theta) + i \sin(3\theta)$$

$$\begin{aligned} \therefore \cos(3\theta) + i \sin(3\theta) &= \cos^3(\theta) + 3i \cos^2(\theta) \sin(\theta) \\ &\quad - 3 \cos(\theta) \sin^2(\theta) - i \sin^3(\theta) \end{aligned}$$

Equate the real parts.

$$\therefore \cos(3\theta) = \cos^3(\theta) - 3 \cos(\theta) \sin^2(\theta)$$

Use a substitution to eliminate $\sin^2(\theta)$.

$$\begin{aligned} \cos(3\theta) &= \cos^3(\theta) - 3 \cos(\theta) [1 - \cos^2(\theta)] \\ &= \cos^3(\theta) - 3 \cos(\theta) + 3 \cos^3(\theta) \\ &= 4 \cos^3(\theta) - 3 \cos(\theta) \\ \therefore \cos(3\theta) &= 4 \cos^3(\theta) - 3 \cos(\theta) \end{aligned}$$

○ Example 20

By expanding $[\cos(\alpha) + i \sin(\alpha)]^2$ in two ways, derive expressions for $\cos(2\alpha)$ and $\sin(2\alpha)$ and hence find an expression for $\tan(2\alpha)$ in terms of $\tan(\alpha)$.

Solution

Use the binomial theorem.

$$[\cos(\alpha) + i \sin(\alpha)]^2 = \cos^2(\alpha) + 2i \cos(\alpha) \sin(\alpha) + i^2 \sin^2(\alpha) \\ = \cos^2(\alpha) - \sin^2(\alpha) + 2i \cos(\alpha) \sin(\alpha)$$

Use De Moivre's theorem.

$$[\cos(\alpha) + i \sin(\alpha)]^2 = \cos(2\alpha) + i \sin(2\alpha)$$

Equate the real and imaginary parts.

$$\cos(2\alpha) = \cos^2(\alpha) - \sin^2(\alpha) \\ \sin(2\alpha) = 2 \sin(\alpha) \cos(\alpha)$$

Use $\tan(A) = \frac{\sin(A)}{\cos(A)}$

$$\tan(2\alpha) = \frac{\sin(2\alpha)}{\cos(2\alpha)} \\ = \frac{2 \sin(\alpha) \cos(\alpha)}{\cos^2(\alpha) - \sin^2(\alpha)} \\ = \frac{2 \sin(\alpha) \cos(\alpha)}{\cos^2(\alpha) - \sin^2(\alpha)}$$

Convert to $\tan(\alpha)$ by dividing the RHS by $\frac{\cos^2(\alpha)}{\cos^2(\alpha)}$.

$$\tan(2\alpha) = \frac{2 \sin(\alpha) \cos(\alpha)}{\cos^2(\alpha) - \sin^2(\alpha)} \\ = \frac{2 \tan(\alpha)}{1 - \tan^2(\alpha)}$$

Expansions using De Moivre's theorem and the binomial theorem can also be used to integrate powers of trigonometric functions.

○ Example 21

Use the fact that $\cos(3\theta) = 4 \cos^3(\theta) - 3 \cos(\theta)$ to find $\int_0^{\frac{\pi}{2}} 4 \cos^3(\theta) d\theta$.

Solution

Rearrange $\cos(3\theta) = 4 \cos^3(\theta) - 3 \cos(\theta)$ to make $4 \cos^3(\theta)$ the subject.

$$\cos(3\theta) = 4 \cos^3(\theta) - 3 \cos(\theta) \\ \therefore 4 \cos^3(\theta) = \cos(3\theta) + 3 \cos(\theta)$$

Perform the integration.

$$\int_0^{\frac{\pi}{2}} 4 \cos^3(\theta) d\theta = \int_0^{\frac{\pi}{2}} \cos(3\theta) + 3 \cos(\theta) d\theta \\ = \left[\frac{\sin(3\theta)}{3} + 3 \sin(\theta) \right]_0^{\frac{\pi}{2}} \\ = \frac{\sin\left[3\left(\frac{\pi}{2}\right)\right]}{3} + 3 \sin\left(\frac{\pi}{2}\right) - \left(\frac{\sin[3(0)]}{3} + 3 \sin(0) \right) \\ = -\frac{1}{3} + 3(1) - (0) \\ = 2\frac{2}{3}$$

INVESTIGATION Euler and $e^{i\theta}$

Leonhard Euler was born in Switzerland in 1707. His father wanted him to study Theology, but Euler persuaded his father to allow him to study Mathematics. Euler was the first mathematician to use the notation $f(x)$ for a function and in 1748 he defined the formula $e^{i\theta} = \cos(\theta) + i \sin(\theta)$.

This definition was very useful in proving the various results to do with complex numbers, including De Moivre's theorem.

By expressing the complex number $z = r[\cos(\theta) + i \sin(\theta)]$ as $z = re^{i\theta}$, prove the following results.

- 1 $z_1 z_2 \dots z_n = r_1 r_2 \dots r_n [\cos(\theta_1 + \theta_2 + \dots + \theta_n) + i \sin(\theta_1 + \theta_2 + \dots + \theta_n)]$
- 2 $z^n = r^n [\cos(n\theta) + i \sin(n\theta)], \forall n \in \mathbb{Z}$
- 3 $\frac{z_1}{z_2} = \frac{r_1}{r_2} [\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)]$
- 4 $z^{-1} = \frac{1}{z} = \frac{1}{r} [\cos(-\theta) + i \sin(-\theta)]$



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EXERCISE 2.07 Applications of De Moivre's theorem

Concepts and techniques

- 1 **Example 19** a Expand $[\cos(\theta) + i \sin(\theta)]^3$ using the binomial theorem.
b Use the fact that $[\cos(\theta) + i \sin(\theta)]^3 = \cos(3\theta) + i \sin(3\theta)$ to derive an expression for
 - i $\cos(3\theta)$ in terms of $\cos(\theta)$
 - ii $\sin(3\theta)$ in terms of $\sin(\theta)$
- 2 a Expand $[\cos(\theta) + i \sin(\theta)]^2$ in two ways.
b Hence show that
 - i $\cos(2\theta) = 2 \cos^2(\theta) - 1$
 - ii $\cos(2\theta) = 1 - 2 \sin^2(\theta)$
- 3 Use expansions of $[\cos(\theta) + i \sin(\theta)]^4$ to prove that
 - a $\cos(4\theta) = 8 \cos^4(\theta) - 8 \cos^2(\theta) + 1$
 - b $\sin(4\theta) = 4 \sin(\theta) \cos(\theta) [\cos^2(\theta) - \sin^2(\theta)]$



Trigonometric identities
using De Moivre's theorem

- 4 **Example 20** a Complete the statement $\tan(3\theta) = \frac{\tan(3\theta)}{\cos(\quad)}$.
- b Using the results you derived from $[\cos(\theta) + i \sin(\theta)]^3 = \cos(3\theta) + i \sin(3\theta)$ in question 1, show that $\tan(3\theta) = \frac{3 \sin(\theta) - 4 \sin^3(\theta)}{4 \cos^3(\theta) - 3 \cos(\theta)}$.
- c Hence prove that $\tan(3\theta) = \frac{3 \tan(\theta) - \tan^3(\theta)}{1 - 3 \tan^2(\theta)}$.
- 5 For $n \geq 1$, prove that
- $$[\cos(\theta) + i \sin(\theta)] + [\cos(2\theta) + i \sin(2\theta)] + [\cos(3\theta) + i \sin(3\theta)] + \dots + [\cos(n\theta) + i \sin(n\theta)]$$
- $$= \frac{[\cos(\theta) + i \sin(\theta)][\cos(n\theta) + i \sin(n\theta) - 1]}{\cos(\theta) + i \sin(\theta) - 1}$$
- 6 If $60^\circ - 45^\circ = 15^\circ$, express $\text{cis}(60^\circ)$ and $\text{cis}(45^\circ)$ in Cartesian form and hence simplify $\frac{\text{cis}(60^\circ)}{\text{cis}(45^\circ)}$ in two ways. Use your results to find the exact value of $\cos(15^\circ)$.
- 7 **Example 21** a Use $[\cos(\theta) + i \sin(\theta)]^3 = \cos(3\theta) + i \sin(3\theta)$ to show that $\sin(3\alpha) = 3 \sin(\alpha) - 4 \sin^3(\alpha)$.
- b Hence find the exact value of $\int_0^{\frac{\pi}{4}} 4 \sin^3(\theta) d\theta$.
- 8 Using suitable expansions, find the value of $\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \cos^4(x) dx$.

Reasoning and communication

- 9 Prove by induction that $[\cos(\theta) + i \sin(\theta)]^n = \cos(n\theta) + i \sin(n\theta)$, $n \geq 1$, $n \in \mathbb{Z}$.
- 10 If $z = \cos(\theta) + i \sin(\theta)$, simplify
- a $z - \frac{1}{z}$ b $z^2 - \frac{1}{z^2}$ c $z^n - \frac{1}{z^n}$

CHAPTER SUMMARY

COMPLEX NUMBERS AND DE MOIVRE'S THEOREM

2

■ The **imaginary number** i is the number such that $i = \sqrt{-1}$.

■ A **complex number** is a number that can be written in the form $a + ib$, where a and b are real numbers.

A complex number is often denoted by the letter z , so $z = a + ib$.

■ For a complex number z , where $z = a + ib$ (where a and b are real numbers), the **complex conjugate** of z is denoted by $\bar{z}$ and $\bar{z} = a - ib$.

■ The complex number $z = x + yi$ (where $x, y \in \mathbf{R}$) can be represented geometrically on an Argand diagram as the point $P(x, y)$ or the vector z or **OP**.

■ For complex numbers $a + ib$ and $c + id$ (where a, b, c and d are real numbers), $a + ib = c + id$ if and only if $a = c$ AND $b = d$.

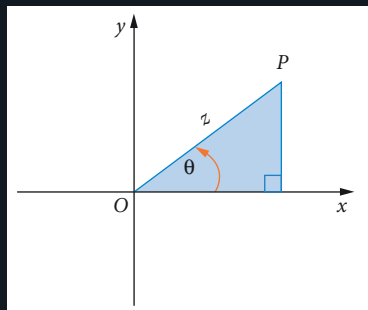
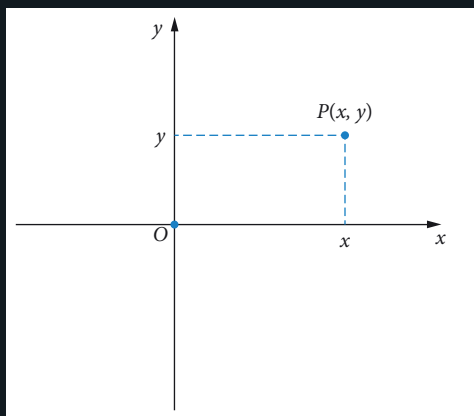
■ To **realise the denominator** of a complex number, multiply the number by 1 in the form $\frac{\bar{z}}{\bar{z}}$.

■ The **real part** of $z = a + ib$ is denoted by $\text{Re}(z)$, where $\text{Re}(z) = a$, and the **imaginary part** of $z = a + ib$ is denoted by $\text{Im}(z)$, where $\text{Im}(z) = b$.

If $\text{Re}(z) = 0$, then z is *purely imaginary*.

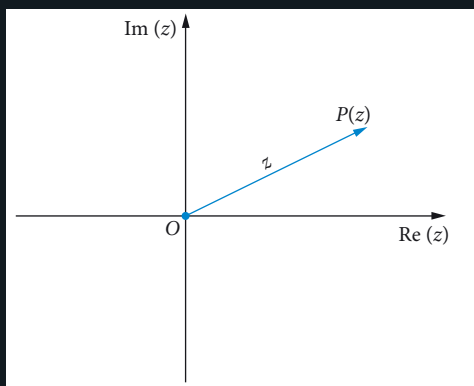
If $\text{Im}(z) = 0$, then z is *purely real* or just *real*.

■ Definition: the **Argument of z** , or **arg** (z).

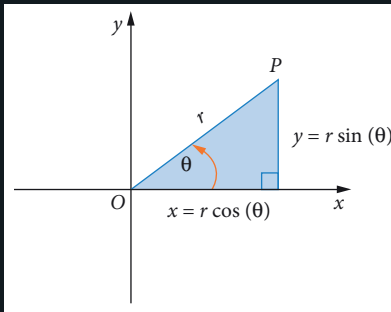


$$\arg(z) = \theta$$

If a complex number is represented by a point P in the complex plane, then the **argument of z** , denoted **arg** (z), is the angle θ that OP makes with the positive real axis Ox , with the angle measured anticlockwise from Ox . The **principal value** of the argument is the one in the interval $(-\pi, \pi]$. The argument of 0 is not defined.



■ Definition: the **Polar form of z**



Let $\arg(z) = \theta$ and $|z| = r$.

If z is a **non-zero** complex number, then $z = r[\cos(\theta) + i \sin(\theta)]$ is the **polar form** of z .

- $z = r[\cos(\theta) + i \sin(\theta)]$ is also written as $z = r \operatorname{cis}(\theta)$

- If $z_1 = r_1[\cos(\theta_1) + i \sin(\theta_1)]$ and $z_2 = r_2[\cos(\theta_2) + i \sin(\theta_2)]$ are two complex numbers, then

- their **product** is

$$z_1 z_2 = r_1 r_2 [\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)]$$
- $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$ and
 $\operatorname{mod}(z_1 z_2) = \operatorname{mod}(z_1) \times \operatorname{mod}(z_2)$

- If $z_1 = r_1[\cos(\theta_1) + i \sin(\theta_1)]$ and $z_2 = r_2[\cos(\theta_2) + i \sin(\theta_2)]$ are two complex numbers, then

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} [\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)].$$

If $z = r[\cos(\theta) + i \sin(\theta)]$ is a complex number,

$$\text{then } z^{-1} = \frac{1}{z} = \frac{1}{r} [\cos(-\theta) + i \sin(-\theta)].$$

OR

If $z_1 = r_1[\cos(\theta_1) + i \sin(\theta_1)]$ and $z_2 = r_2[\cos(\theta_2) + i \sin(\theta_2)]$ are two complex numbers, then

$$\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2) \text{ and}$$

$$\operatorname{mod}\left(\frac{z_1}{z_2}\right) = \frac{\operatorname{mod}(z_1)}{\operatorname{mod}(z_2)}.$$

- If $z = r[\cos(\theta) + i \sin(\theta)]$ is a complex number, then $\arg(z^{-1}) = -\arg(z)$ and

$$\operatorname{mod}(z^{-1}) = \frac{1}{\operatorname{mod}(z)}.$$

- If $z_1 = r_1[\cos(\theta_1) + i \sin(\theta_1)]$,
 $z_2 = r_2[\cos(\theta_2) + i \sin(\theta_2)]$, ...,
 $z_n = r_n[\cos(\theta_n) + i \sin(\theta_n)]$ are multiple complex numbers, then

$$z_1 z_2 \dots z_n = r_1 r_2 \dots r_n [\cos(\theta_1 + \theta_2 + \dots + \theta_n) + i \sin(\theta_1 + \theta_2 + \dots + \theta_n)].$$

OR

$$\arg(z_1 z_2 \dots z_n) = \arg(z_1) + \arg(z_2) + \dots + \arg(z_n) \text{ and}$$

$$|z_1 z_2 \dots z_n| = |z_1| |z_2| \dots |z_n|.$$

- $\arg(z^n) = n \times \arg(z)$ and $|z^n| = |z|^n$

and

$$\arg(z^{-n}) = -n \times \arg(z) \text{ and } |z^{-n}| = |z|^{-n}.$$

■ **De Moivre's theorem**

If $z = \cos(\theta) + i \sin(\theta)$ is a complex number, then $z^n = \cos(n\theta) + i \sin(n\theta)$, $\forall n \in \mathbb{Z}$.

- De Moivre's theorem is useful for deriving or proving a number of trigonometric identities.

- Expansions using De Moivre's theorem and the binomial theorem can also be used to integrate powers of trigonometric functions.

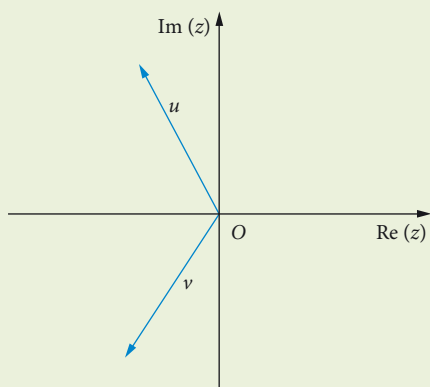
CHAPTER REVIEW

COMPLEX NUMBERS AND DE MOIVRE'S THEOREM

2

Multiple choice

- 1 **Example 1** The roots of $x^2 - 2x + 10 = 0$ are
 A $-1 \pm 3i$ B $1 \pm i\sqrt{3}$ C $1 \pm 3i$
 D $-1 \pm i\sqrt{3}$ E none of the above
- 2 **Example 1** The value of i^{74} is:
 A 1 B -1 C i D $-i$ E 74
- 3 **Example 2** The value of $\overline{3 - 2i + i(5 + 4i)}$ is:
 A $1 - 3i$ B $-1 + 3i$ C $-1 - 5i$ D $-1 - 3i$ E $7 - 3i$
- 4 **Example 3** Consider the vectors $\mathbf{u}$ and $\mathbf{v}$ representing the complex numbers u and v respectively, as shown in the diagram.



Which of the following statements is true?

- A $u = v$ B $u = -v$ C $-u = -v$ D $\bar{u} = \bar{v}$ E $u = \bar{v}$
- 5 **Example 11** The principal value of the argument of $2\text{cis}\left(\frac{31\pi}{6}\right)$ is:
 A $-\frac{5\pi}{6}$ B $-\frac{\pi}{6}$ C $\frac{7\pi}{6}$ D $\frac{\pi}{6}$ E $\frac{5\pi}{6}$

20 Find the value of each of the following in mod-arg form.

a $\frac{1}{(4-4i\sqrt{3})(-\sqrt{3}+i)}$

b $\left[(-1-i\sqrt{3})(2+2i)\right]^{-1}$

21 Simplify each of the following.

a $\frac{\left[\operatorname{cis}\left(\frac{5\pi}{7}\right)\right]^3 \left[\operatorname{cis}\left(\frac{6\pi}{7}\right)\right]^5}{\left[\operatorname{cis}\left(\frac{2\pi}{7}\right)\right]^6}$

b $\frac{[\cos(4\alpha) + i\sin(4\alpha)]^7}{[\cos(9\alpha) + i\sin(9\alpha)]^{-4} [\cos(9\alpha) - i\sin(9\alpha)]^3}$

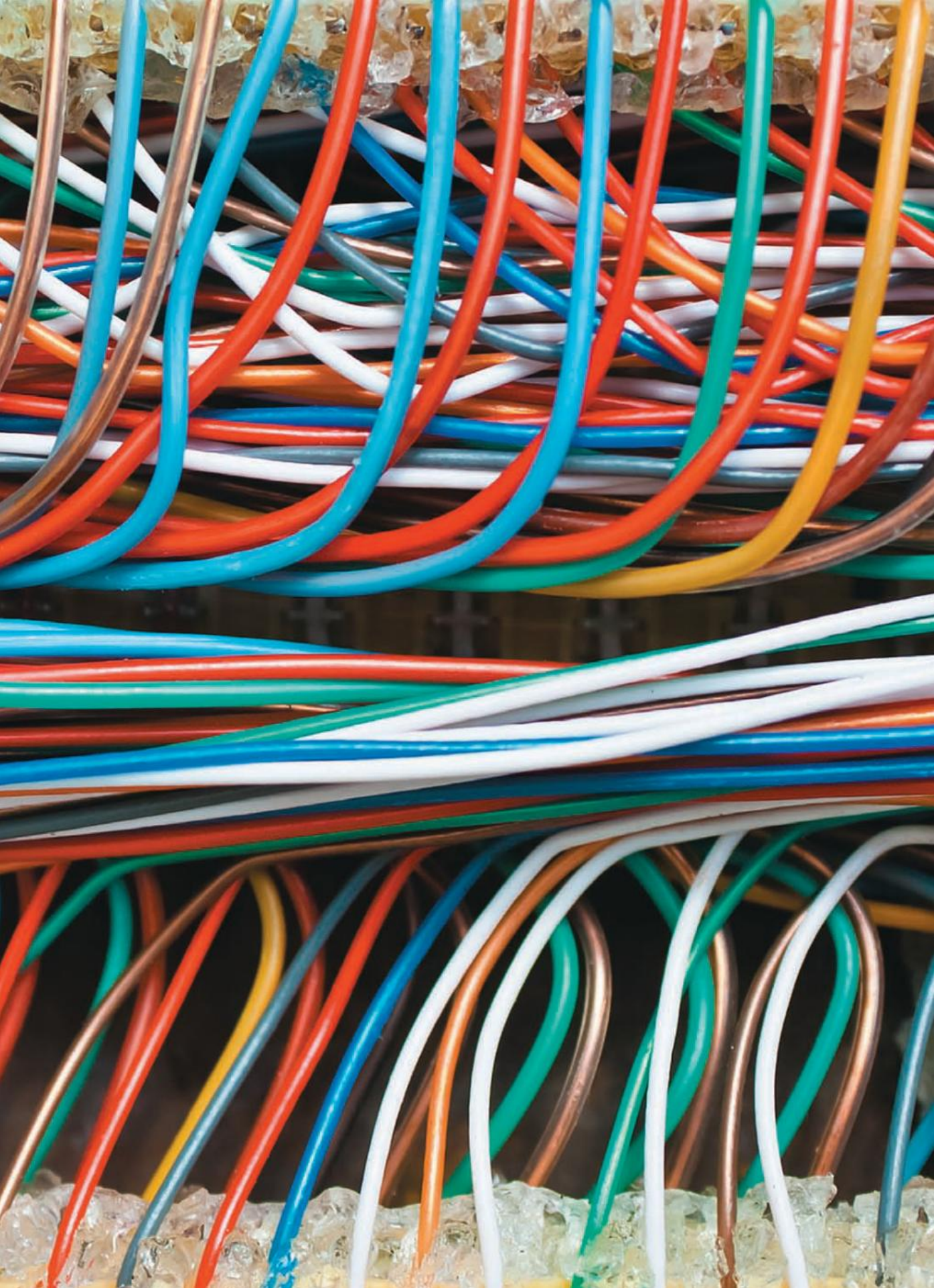
22 By expanding $[\cos(\theta) + i\sin(\theta)]^2$ in two different ways, derive expressions for $\cos(2\theta)$ and $\sin(2\theta)$. Hence show that:

a $\tan(2\alpha) = \frac{2\tan(\alpha)}{1 - \tan^2(\alpha)}$

b $\tan(15^\circ) = 2 - \sqrt{3}$



Practice quiz



3

TERMINOLOGY

absolute value
composite function
inverse function
one-to-one function
rational function
reciprocal function

FUNCTIONS AND SKETCHING GRAPHS

FUNCTIONS AND SKETCHING GRAPHS

- 3.01 Composition of functions
- 3.02 One-to-one functions
- 3.03 Inverse functions
- 3.04 Absolute value functions
- 3.05 Graphs of $y = f(x)$, $y = \frac{1}{f(x)}$, $y = |f(x)|$
- 3.06 Graphs of rational functions

Chapter summary

Chapter review



Prior learning

FUNCTIONS

- determine when the composition of two functions is defined (ACMSM092)
- find the composition of two functions (ACMSM093)
- determine if a function is one-to-one (ACMSM094)
- consider inverses of one-to-one functions (ACMSM095)
- examine the reflection property of the graph of a function and the graph of its inverse. (ACMSM096)

SKETCHING GRAPHS

- use and apply the notation $|x|$ for the absolute value for the real number x and the graph of $y = |x|$ (ACMSM098)
- examine the relationship between the graph of $y = f(x)$ and the graphs of $y = \frac{1}{f(x)}$, $y = |f(x)|$ and $y = f(|x|)$ (ACMSM099)
- sketch the graphs of simple rational functions where the numerator and denominator are polynomials of low degree. (ACMSM100) 

3.01 COMPOSITION OF FUNCTIONS

You can make more complicated functions from simple ones by combining them. You can make the output of one function the input of another. This is known as **composition** of functions. If you use $f(x) = x^2$ and $g(x) = x + 2$, then you get the function $g \circ f$ by doing $g[f(x)] = gf(x)$.

In this case, $g[f(x)] = g(x^2) = x^2 + 2$. Notice that $f[g(x)] = (x + 2)^2$, a different function.

IMPORTANT

The **composition** $g \circ f$ is the function given by $g[f(x)] = gf(x)$. This is also written as gf and is often called a 'function of a function'.

○ Example 1

Find $g[f(x)]$ given that:

a $f(x) = 2x$ and $g(x) = x^2 + 3$

b $f(x) = \sin(x)$ and $g(x) = \frac{1}{x}$

Solution

- a Substitute $f(x) = 2x$.
Substitute $2x$ into $g(x)$.

$$\begin{aligned}g[f(x)] &= g(2x) \\g[f(x)] &= (2x)^2 + 3 \\g[f(x)] &= 4x^2 + 3\end{aligned}$$

- b Substitute $f(x) = \sin(x)$.
Substitute $\sin(x)$ into $g(x)$.

$$\begin{aligned}g[f(x)] &= g[\sin(x)] \\g[f(x)] &= \frac{1}{\sin(x)} \\g[f(x)] &= \operatorname{cosec}(x)\end{aligned}$$

○ Example 2

Find $f[g(x)]$, and state the domain given that:

- a** $f(x) = 2x$ and $g(x) = x^2 + 3$ **b** $f(x) = \sin(x)$ and $g(x) = \frac{1}{x}$
c $f(x) = \sqrt{x}$, $g(x) = 4 - x^2$

Compare your results for **a** and **b** with those in Example 1.

Solution

- a** Substitute $g(x) = x^2 + 3$.

Substitute $x^2 + 3$ into $f(x)$.

$$f[g(x)] = f(x^2 + 3)$$

$$f[g(x)] = 2(x^2 + 3)$$

$$f[g(x)] = 2x^2 + 6$$

The function is valid for any value of x .

Compare the results.

The domain is $x \in \mathbf{R}$.

This is different to $g[f(x)] = 4x^2 + 3$.

- b** Substitute $g(x) = \frac{1}{x}$

Substitute $\frac{1}{x}$ into $f(x)$.

$$f[g(x)] = f\left(\frac{1}{x}\right)$$

$$f[g(x)] = \sin\left(\frac{1}{x}\right)$$

The function is not valid for 0.

Compare the results.

The domain is $x \in \mathbf{R}/0$.

This is different to $g[f(x)] = \operatorname{cosec}(x)$.

- c** Substitute $g(x) = 4 - x^2$.

Square roots of negatives are not real.

$$f[g(x)] = f(4 - x^2)$$

$$= \sqrt{4 - x^2}$$

The domain is $-2 \leq x \leq 2$.

You can write many functions as the composition of simpler functions. For example, take the function, $h(x) = e^{3x}$. This function may be written as a **composite function** $h(x) = g[f(x)]$, where $g(x) = e^x$ and $f(x) = 3x$.

○ Example 3

Decompose the following functions into the form $g[f(x)]$, and state the domain and range for each.

- a** $h(x) = 5x - 2$

- b** $h(x) = e^{2x-1}$

- c** $h(x) = x^2 + 2x + 1$

Solution

- a** Write the operations involved as functions.

Try the composition $f \circ g$.

Do it the other way round as $g \circ f$.

Try $f(x) = 5x$ and $g(x) = x - 2$

$$f[g(x)] = f(x - 2)$$

$$= 5(x - 2)$$

$$= 5x - 10$$

$$g[f(x)] = g(5x)$$

$$= 5x - 2$$

OK

Now write the domain.

The graph is a straight line, so you can get every possible value of y .

- b Write the operations involved as functions.

Try the composition $f \circ g$.

Do it the other way round as $g \circ f$.

Now write the domain.

The graph is exponential, so y values are greater than zero.

- c Write the operations involved as functions.

Try the composition $f \circ g$.

Do it the other way round as $g \circ f$.

Now write the domain.

The graph is a quadratic, so y values are greater than the minimum value of the function.

x can be anything, so the domain is $x \in \mathbf{R}$.

y can be anything, so the range is
 $y = h(x) \in \mathbf{R}$.

$$g(x) = e^x, f(x) = 2x - 1$$

$$f[g(x)] = f(e^x)$$

$$f[g(x)] = 2e^x - 1$$

$$g[f(x)] = g(2x - 1)$$

$$g(2x - 1) = e^{2x-1}$$

As required.

x can be anything, so the domain is $x \in \mathbf{R}$.

y must be positive, so the range is
 $y = h(x) > 0$

$$g(x) = x^2, f(x) = x + 1$$

$$f[g(x)] = f(x^2)$$

$$f(x + 1) = x^2 + 1$$

$$f[g(x)] = x^2 + 1$$

$$g[f(x)] = g(x + 1)$$

$$g(x + 1) = (x + 1)^2$$

$$g[f(x)] = x^2 + 2x + 1$$

As required.

x can be anything, so the domain is $x \in \mathbf{R}$.

y must be positive, so the range is $y = h(x) > 0$

Consider the composition of $f(x) = \sqrt{4 - 4x^2}$ and $g(x) = x^2 - 2x + 5$. $f \circ g$ is not a valid function because the range of g , $y \geq 4$, is outside the domain of f , $x \leq 1$. $g \circ f$ is valid only for the domain of f and the range is the same as the range of g .

IMPORTANT

Functions can only be composed if part of the range of the second function lies within the domain of the first function.

The domain of a composition of functions is either the same as the domain of the second function, or lies within it.

The range of a composition of functions is either the same as the range of the first function, or lies within it.

EXERCISE 3.01 Composition of functions

Concepts and techniques



Composition of functions

1 **Example 1** Find $g[f(x)]$ given that:

a $g(x) = 2x - 3, f(x) = x - 1$

c $g(x) = \log_e(x), f(x) = e^{2x}$

b $g(x) = x^2 + 2, f(x) = e^x$

d $g(x) = \sin(x), f(x) = 2x$

2 **Example 2** Find $f[g(x)]$ for each of the following and compare with $g[f(x)]$ in question 1.

a $g(x) = 2x - 3, f(x) = x - 1$

c $g(x) = \log_e(x), f(x) = e^{2x}$

b $g(x) = x^2 + 2, f(x) = e^x$

d $g(x) = \sin(x), f(x) = 2x$

3 **Example 3** Decompose the following functions into the form $g[f(x)]$.

a $h(x) = x + 3$

b $h(x) = (2x + 5)^3$

c $h(x) = x^2 + 6x + 2$

d $h(x) = e^{x^2}$

e $h(x) = \sin^2(x)$

f $h(x) = \sin(2x)$

Reasoning and communication

4 For the function $h(x) = 4x^2$, write two possible decompositions.

5 For the following functions, find $g \circ f$ and write the domain and range of each.

a $f(x) = 2x, g(x) = \sin(x)$

b $f(x) = -x, g(x) = \log_e(x)$

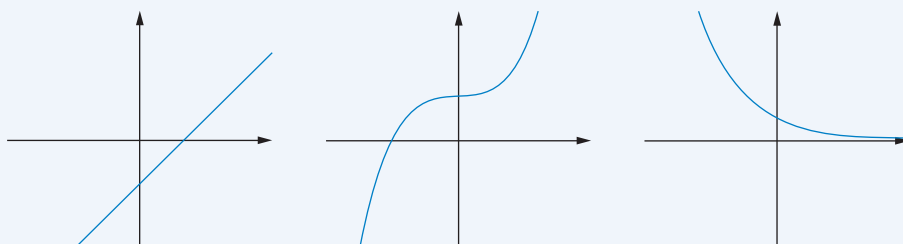
c $f(x) = \frac{1}{2x}, g(x) = 2 \sin(x)$

6 Write a function that satisfies the property $g \circ f = f \circ g$.

3.02 ONE-TO-ONE FUNCTIONS

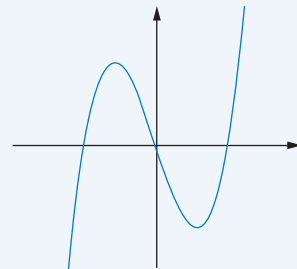
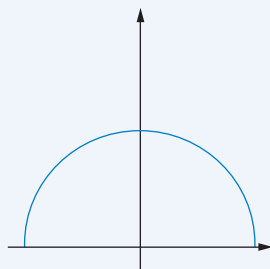
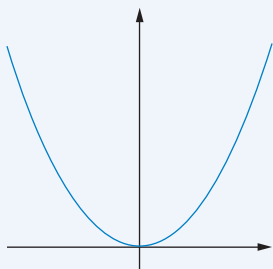
INVESTIGATION One-to-one functions

The following functions are one-to-one.



- 1 Copy each of the graphs above into your book.
- 2 Draw a horizontal line across each of these graphs.
- 3 How many times does the horizontal line pass through the graph?
- 4 Now draw a vertical line at any point on the graph.
- 5 How many times does the vertical line pass through the graph?

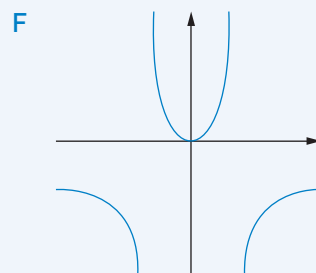
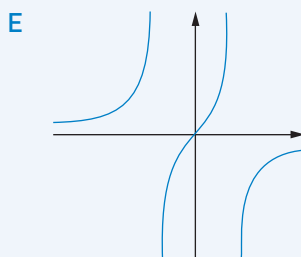
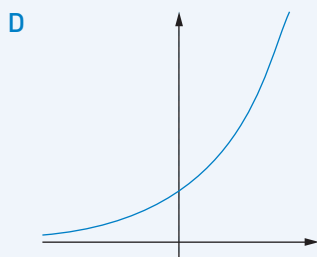
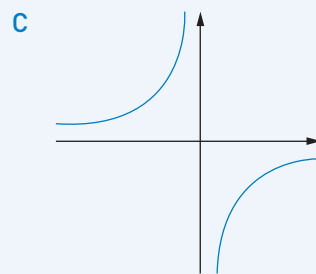
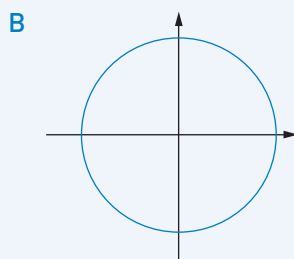
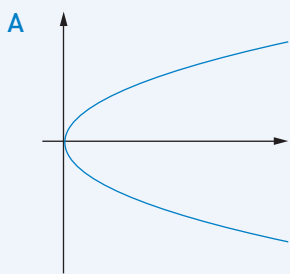
Look at the following graphs.



- 1 How many times does a horizontal line pass through each graph?
- 2 How many times does a vertical line pass through each graph?

The horizontal and vertical line tests are used to determine if a function is one-to-one or not.

Determine which of the following graphs represent a one-to-one function.



A function is **one-to-one** if it never assigns two input values to the same output value. In other words, no output value has more than one input value.

For example, to prove that $f(x) = 2x - 5$ is one-to-one, let x and y be integers and suppose that $f(x) = f(y)$. You need to show that $x = y$.

Substituting in our formula for f , $f(x) = f(y)$, so you get $2x - 5 = 2y - 5$, which reduces to $x = y$. Hence, you can say that $f(x) = 2x - 5$ is one-to-one.

To illustrate that not all functions are one-to-one, you can look at the function $f(x) = x^2$. In this case, you get $x^2 = y^2$, which simplifies to $x = \pm y$. Since there is no unique output value, you can say that the function $f(x) = x^2$ is not one-to-one.

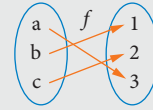
IMPORTANT

A **one-to-one** function satisfies the condition:

If $x \neq y$ then $f(x) \neq f(y)$.

It is sometimes easier to show using the contrapositive:

For all x and y , if $f(x) = f(y)$, then $x = y$.



If $f(x)$ and $g(x)$ are both one-to-one, then $f \circ g$ is one-to-one.

Since $[f \circ g](x) = [f \circ g](y) \rightarrow f[g(x)] = f[g(y)] \rightarrow g(x) = g(y) \rightarrow x = y$.

Example 4

For each of the following functions, determine whether it is one-to-one or not.

a $f(x) = 3x - 4$

b $f(x) = x^3$

c $f(x) = x^2 - x$

Solution

a Write $f(x) = f(y)$.

Simplify.

$$3x - 4 = 3y - 4$$

$$3x = 3y$$

$$x = y$$

Determine one-to-oneness.

$f(x)$ is one-to-one.

b Write $f(x) = f(y)$.

Simplify.

$$x^3 = y^3$$

$$\text{For } x > 0, x = y$$

$$\text{For } x < 0, x = y$$

Determine one-to-oneness.

$f(x)$ is one-to-one.

c Write $f(x) = f(y)$.

Simplify.

$$x^2 - x = y^2 - y$$

$$x(x - 1) = y(y - 1)$$

Determine one-to-oneness.

Since $f(0) = 0$ and $f(1) = 0$, $f(x)$ is NOT one-to-one.

It would appear from the earlier investigation that continuous functions that are either increasing or decreasing are one-to-one functions.

IMPORTANT

A function f is (strictly) **increasing** if for all x and y ($x < y$), $f(x) < f(y)$.

A function f is (strictly) **decreasing** if for all x and y ($x > y$), $f(x) > f(y)$.

IMPORTANT

If $f'(x) > 0$ for all x , then $f(x)$ is increasing.

If $f'(x) < 0$ for all x , then $f(x)$ is decreasing.

Continuous increasing functions and continuous decreasing functions are one-to-one.

○ Example 5

Show that the continuous functions below are increasing, decreasing or neither and hence determine if they are one-to-one.

a $f(x) = -x^5 - 2x^3 + 4$

b $f(x) = 2x^3 + 4x + 1$

c $f(x) = 2e^x$

d $f(x) = 3x^2 - 12$

Solution

a Find $f'(x)$.

$$f'(x) = -5x^4 - 6x^2$$

Determine if $f(x)$ is increasing or decreasing.

$$f'(x) = -(5x^4 + 6x^2) < 0$$

$f(x)$ is decreasing.

State whether the function is one-to-one or not.

Hence, $f(x)$ is one-to-one.

b Find $f'(x)$.

$$f'(x) = 6x^2 + 4$$

Determine if $f(x)$ is increasing or decreasing.

$$f'(x) > 0$$

$f(x)$ is increasing.

State whether the function is one-to-one or not.

Hence, $f(x)$ is one-to-one.

c Find $f'(x)$.

$$f'(x) = 2e^x$$

Determine if $f(x)$ is increasing or decreasing.

$$f'(x) > 0$$

$f(x)$ is increasing.

State whether the function is one-to-one or not.

Hence, $f(x)$ is one-to-one.

d Find $f'(x)$.

$$f'(x) = 6x$$

Determine if $f(x)$ is increasing or decreasing.

$$f'(x) < 0, x < 0 \text{ and } f'(x) > 0, x > 0$$

$f(x)$ is neither increasing nor decreasing.

State whether the function is one-to-one or not.

Hence, $f(x)$ is NOT one-to-one.

EXERCISE 3.02 One-to-one functions

Concepts and techniques

- 1 **Example 4** For each of the following functions, determine whether it is one-to-one or not one-to-one.
- a $f(x) = x^2 - 4$ b $f(x) = e^{2x} - 1$ c $f(x) = \frac{1}{x+3}$ d $f(x) = \frac{1}{(x-2)^2}$
- 2 Show that $f(x) = x^n + x$, $n > 0$ is not one-to-one.
- 3 **Example 5** Show that the following functions are increasing, decreasing or neither and whether they are one-to-one.
- a $f(x) = \frac{2}{x}$ b $f(x) = x^3 - 9x + 1, x > 2$ c $f(x) = \sin(x)$ d $f(x) = e^{-x}$
- 4 Show that $f(x) = \pi x - \sin(x)$ is a increasing function for all values of x .



One-to-one functions



Reasoning and communication

- 5 Show that $f(x) = 3x^3 - 2$ is one-to-one by decomposing $f(x)$.
- 6 Using decomposition, show that $h(x) = (\pi - x) - \cos(x)$ is one-to-one.
- 7 Write an increasing, one-to-one function that can be decomposed to satisfy $f \circ g = g \circ f$.

3.03 INVERSE FUNCTIONS

IMPORTANT

The **inverse** of a relation $R: x \rightarrow y$ is the relation $R^{-1}: y \rightarrow x$.

If the inverse relation of a function is also a function, it is called the **inverse function**.

A function is an **$n : 1$** (n -to-one) relation. This means that every element of the domain maps to only one element in the range. You can also think of this as meaning that every element in the domain belongs to only one ordered pair. For example, $y = x^2$ has pairs such as $(3, 9)$, $(-3, 9)$, $(0, 0)$, and so on.

By contrast, in a **$1 : n$** (one-to- n) relation, every element of the range belongs to only one ordered pair. For example, the relation $y^2 = x$ has ordered pairs such as $(25, 5)$, $(25, -5)$, and so on.

As you saw in the previous section, the range and domain elements of a $1 : 1$ function are matched, so it is both $n : 1$ and $1 : n$. This also applies in reverse.

IMPORTANT

An inverse function exists if and only if the function is one-to-one.

If the function is $1 : 1$, then its inverse relation is also $1 : 1$, so must be a function.

If the inverse is a function, it must be $n : 1$, so the function must be $1 : n$, so it is both $n : 1$ and $1 : n$, so must be $1 : 1$. **QED**

You can sometimes restrict the domain of a function to make it $1 : 1$ so that an inverse function will exist. For example, $y = x^2$ is $1 : 1$ on the positive real numbers (or on the negative real numbers).

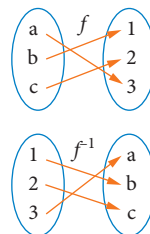
An inverse function is a second function that undoes the work of the first one.

Note: $f^{-1}(x)$ does not mean $\frac{1}{f(x)}$.

To work out an inverse function, reverse the operations that f carries out on a number.

For example, let $f(x) = 3x$, so that f takes a number x and multiplies it by 3:
 $f(x) = 3x$

To define an inverse function that takes 3 times x , and takes it back to x , is the same as saying that $f^{-1}(x)$ divides x by 3. So $f^{-1}(x) = \frac{1}{3}x$ (or $x \div 3$).



Example 6

Write the inverse relation of each of the following functions.

a $y = 2x + 1$

b $y = x^3 - 2$

c $y = e^{2x}$

Solution

a Write the given function.

$$y = 2x + 1$$

Rearrange to find x as the subject.

$$y - 1 = 2x$$

$$(y - 1) \div 2 = x$$

Write the inverse relation.

$$f^{-1}(x) = \frac{x-1}{2}$$

Write the given function.

$$y = x^3 - 2$$

Rearrange to find x as the subject.

$$y + 2 = x^3$$

$$\sqrt[3]{y+2} = x$$

Write the inverse relation.

$$f^{-1}(x) = \sqrt[3]{x+2}$$

Write the given function.

$$y = e^{2x}$$

Rearrange to find x as the subject.

$$\log_e(y) = 2x$$

$$\frac{\log_e(y)}{2} = x$$

Write the inverse relation.

$$f^{-1}(x) = \frac{\log_e(x)}{2}$$

The inverse relation of $y = f(x)$ can also be found by interchanging the x and y values of the function.

○ Example 7

Write the inverse relations of the following.

a $y = 5x - 4$ b $y = x^4 + 1$ c $y = \pm\sqrt{4-x}$

Solution

a Write the given function.

$$y = 5x - 4$$

Interchange x and y .

$$x = 5y - 4$$

Make y the subject of the equation.

$$x + 4 = 5y$$

$$y = \frac{x+4}{5}$$

Write the inverse relation.

$$f^{-1}(x) = \frac{x+4}{5}$$

b Write the given function.

$$y = x^4 + 1$$

Interchange x and y .

$$x = y^4 + 1$$

Make y the subject of the equation.

$$x - 1 = y^4$$

$$y = \sqrt[4]{x-1}$$

Write the inverse relation.

$$f^{-1}(x) = \sqrt[4]{x-1}$$

c Write the given relation.

$$y = \pm\sqrt{4-x}$$

Interchange x and y .

$$x = \pm\sqrt{4-y}$$

Make y the subject of the equation.

$$x^2 = 4 - y$$

$$y = 4 - x^2$$

Write the inverse relation.

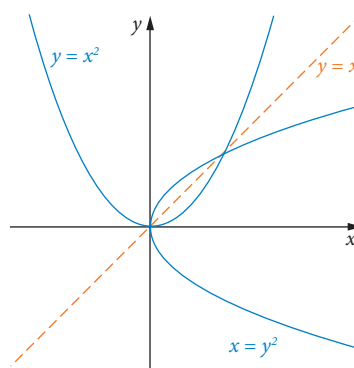
$$f^{-1}(x) = 4 - x^2$$

On the number plane, the inverse relation can be represented by a reflection of the original function in the line $y = x$.

For example, $y = x^2$ gives $x = y^2$ when reflected in the line $y = x$. Notice that, $x = y^2$ is not a function.

A function has an inverse function if it is one-to-one. That is, it is either increasing or decreasing. Generally, the original function is restricted to include the origin and is either increasing or decreasing.

In the case above, the original function would be restricted to $x \geq 0$ so that the resulting inverse relation only exists in the first quadrant and is thus an inverse function.



IMPORTANT

When an inverse function exists, the domain of the inverse function is the range of the original function, and the range of the inverse function is the domain of the original function.

○ Example 8

Draw the inverse relations of the following functions using your CAS calculator. Write a restriction, if necessary, to restrict the given function so that the resulting inverse is a function.

a $y = 5x - 3$

b $y = x^2 + 3$

c $y = e^{2x}$

Solution

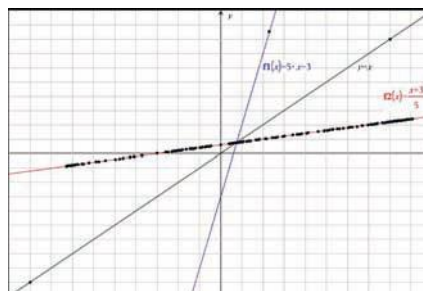
a TI-Nspire CAS

Graph/Function $y = 5x - 3$.

Draw the line in $y = x$ in **Geometry/Points and lines/Line**.

Draw the inverse by reflection in **Geometry/Transformations/Reflection**.

Turn on the trace in **Trace/Geometry Trace**.



Determine if the resulting inverse relation is a function and whether to restrict the domain of the original function.

There is no need to limit the domain of the original function as the inverse is a function.

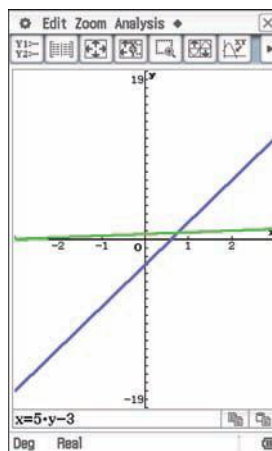
ClassPad

From **Menu** select **Graph&Table**. Tap y_1 and enter $5[x] - 3$ and **EXE**.

This will result in $y_1 = 5x - 3$. Tap $\sqrt{x}$ to draw the graph. (This is the blue line. To obtain a useful view of the function and its subsequent inverse, you may need to adjust the domain of the graph by first tapping Window and changing the maximum and minimum x and y values.)

Tap **Analysis**, then **Sketch** and **Inverse**.

The inverse graph is plotted on the screen (here, as a green line). Tap the part of the screen that shows the graph, then **Resize** to enlarge the graph of both lines so that it uses the whole screen. An expression for the inverse function is shown at the bottom of the screen.



Determine if the resulting inverse relation is a function and whether to restrict the domain of the original function.

There is no need to limit the domain of the original function as the inverse is a function.

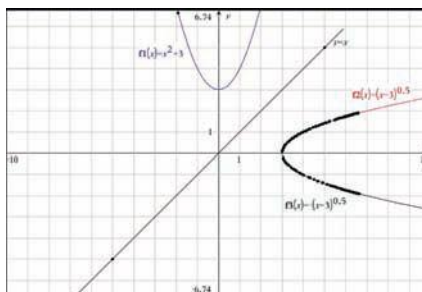
b TI-Nspire CAS

Graph/Function $y = x^2 + 3$.

Draw the line in $y = x$ in **Geometry/Points and lines/Line**.

Draw the inverse by reflection in **Geometry/Transformations/Reflection**.

Turn on the trace in **Trace/Geometry Trace**.



Determine if the resulting inverse relation is a function and whether to restrict the domain of the original function.

The resulting inverse is not a function so the original function needs to be limited.

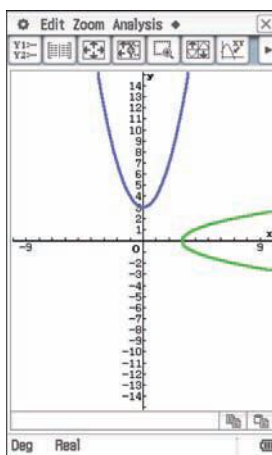
Some possible limited domains are $1 \leq x \leq 5$, $-4 \leq x \leq 0$, $-4 < x < 0$, $x \leq 0$ or $x \geq 0$.

ClassPad

Proceed in a similar manner as for part a.

At y1 enter $x^2 + 3$ and press **EXE**.

Tap $\frac{y}{x}$ to draw the graph. Tap **Analysis**, then **Sketch** and **Inverse**. The inverse graph is plotted on the screen (here, as a green line).



Determine if the resulting inverse relation is a function and whether to restrict the domain of the original function.

The resulting inverse is not a function so the original function needs to be limited.

Some possible limited domains are $1 \leq x \leq 5$, $-4 \leq x \leq 0$, $-4 < x < 0$, $x \leq 0$ or $x \geq 0$.

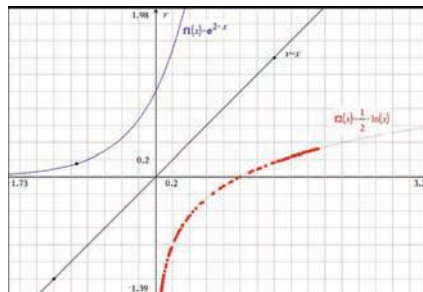
c TI-Nspire CAS

Graph/Function $y = e^{2x}$.

Draw the line in $y = x$ in **Geometry/Points and lines/Line**.

Draw the inverse by reflection in **Geometry/Transformations/Reflection**.

Turn on the trace in **Trace/Geometry Trace**.



Determine if the resulting inverse relation is a function and whether to restrict the domain of the original function.

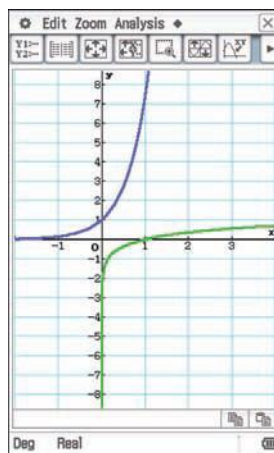
There is no need to limit the domain of the original function as the inverse is a function.

ClassPad

Proceeding in a similar manner as before, tap $y1$ and enter e^{2x} . (The exponential is obtained by first tapping **Keyboard**, then either **Math1** or **Math2**, then $e^{\square}$.) Tap $\frac{\square}{\square}$ to draw the graph.

Tap **Analysis**, then **Sketch** and **Inverse**.

The inverse graph is plotted on the screen (here, as a green line).



Determine if the resulting inverse relation is a function and whether to restrict the domain of the original function.

There is no need to limit the domain of the original function as the inverse is a function.

You can restrict the domain of any function and it will still be a function. However, you usually choose the largest possible continuous domain as 'the' domain of a function. In parts **a** and **c** of Example 8, the domains do not *need* to be restricted to make the inverse a function. In part **b**, the domain of the function does need to be restricted to make the inverse a function. The largest possible continuous domains you could choose are $x \leq 0$ or $x \geq 0$. Given a choice between a positive and a negative domain, most people choose the positive one, so it is standard practice to choose the domain $x \geq 0$ for part **b**.

EXERCISE 3.03 Inverse functions

Concepts and techniques



- 1 **Example 6** Write the inverse relation of each the following functions by making y the subject, that is, $x = f(y)$.

a $y = 2x - 1$

b $y = x^2 + 1$

c $y = 2 \log_e (x - 1)$

d $y = 1 + \sqrt{x}$

e $y = x^3 + 1$

- 2 Determine which of the inverse relations in question 1 are functions.

- 3 What is the condition for the function $y = \frac{1}{x^2}$ to have an inverse function?

- 4 **Example 7** Write the inverse relation of each the following functions by interchanging x and y and finding $y = f(x)$.

a $y = x^3$

b $y = 3x - 2$

c $y = e^x$

d $y = \frac{2}{x}$

e $y = \frac{1}{1-x}$

- 5 Restrict the domain of each of the following functions to a strictly increasing curve, find the inverse function, and state the domain and range of the inverse function.

a $y = x^2$

b $y = 3x^2 - 1$

c $y = (x - 2)^4$

d $y = \frac{3}{x^2}$

e $y = \frac{2}{x^4}$

- 6 **Example 8** Sketch the following functions and their inverse relation by reflecting each in the line $y = x$.

a $y = 2x^2$

b $y = x^2 + 2$

c $y = (x - 3)^2$

d $y = x^2 - 2x$

e $y = x^4 - 1$

- 7 By restricting the following functions to (i) strictly increasing and (ii) strictly decreasing curves, write down the inverse function.

a $f(x) = x^2 - 2x$

b $y = x^4 - 2$

c $y = 2x^4 + 1$

d $f(x) = x^2 - 6x + 1$

e $y = x^2 + 4x - 3$

Reasoning and communication

- 8 a Write down the domain and range of the function $y = \frac{1}{x-2}$.
b Find the inverse function.
c Find the domain and range of the inverse function.

- 9 By restricting the domain of $f(x) = x^2 - 4x$, find the inverse function defined over that domain.

- 10 Find where the function $y = x^2 + 6x - 1$ is strictly decreasing and state the inverse function over that domain.

3.04 ABSOLUTE VALUE FUNCTIONS

The **absolute value**, also known as **modulus**, of a number $x \in \mathbf{R}$ is written as $|x|$ and it is given by the non-negative number that defines its magnitude. For example, $|7| = 7 = |-7|$. In practical terms, the absolute value of a number is its distance from the origin.

IMPORTANT

The **absolute value** of x is given by

$$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$

Note that the effect of taking the absolute value of a number is to remove the minus sign if the number is negative and to leave the number unchanged if it is non-negative.

○ Example 9

Evaluate each of the following.

- a $|6|$ b $|-16|$ c $|4 - 6|$
d $|3 - 6| - |6 - 3|$ e $|-6| \times (-4)$

Solution

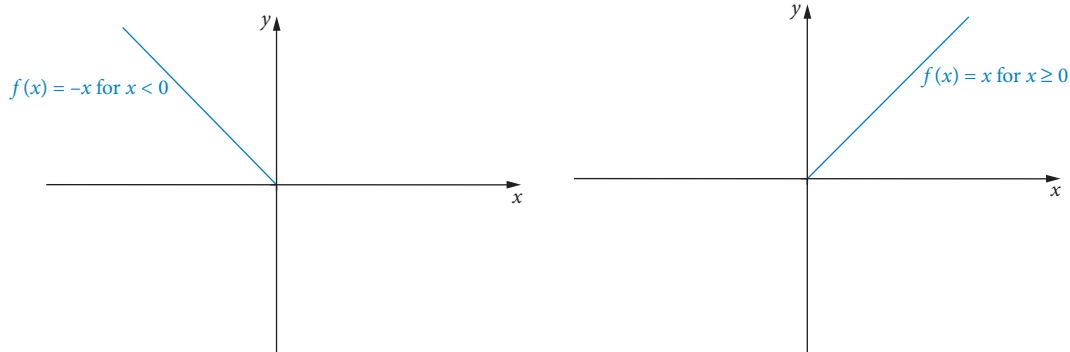
- a Write the question. $|6|$
Use the definition to determine the answer. $= 6$, since $6 > 0$.
- b Write the question. $|-16|$
Use the definition to determine the answer. $= -(-16)$
 $= 16$, since $-16 < 0$.
- c Write the question. $|4 - 6|$
Use the definition to determine the answer. $= |-2|$
 $= -(-2)$
 $= 2$, since $-2 < 0$.
- d Write the question. $|3 - 6| - |6 - 3|$
Use the definition to determine the answer. $= |-3| - |3|$
 $= -(-3) - 3$
 $= 0$, since $-3 < 0$ and $3 > 0$.
- e Write the question. $|-6| \times (-4)$
Use the definition to determine the answer. $= -(-6) \times (-4)$
 $= -24$, since $-6 < 0$.

The **absolute value function**, $y = |x|$ is a basic function. By definition, the function is represented by:

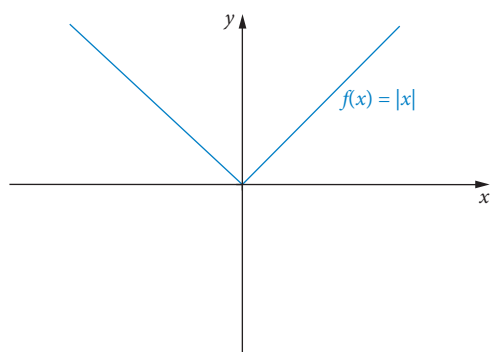
$$y = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$

This function can be thought of as two components put together to form one function. It is the combination of $y = x$ (when $x \geq 0$) and $y = -x$ (when $x < 0$).

Graphically you get:



Combining over the respective domains you get:



Of course, you can easily sketch these functions using a table of values.

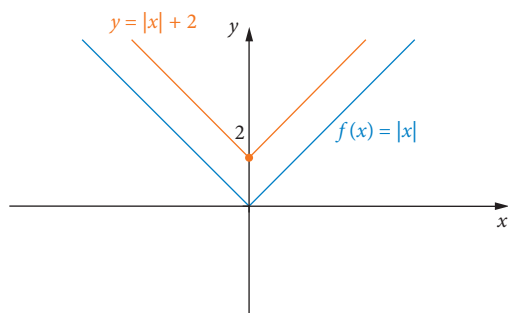
Example 10

Sketch the following absolute value functions and state their domain and range.

a $y = |x| + 2$ b $y = |x| - 2$ c $y = \frac{1}{2}|x| + 1$ d $y = |x - 3|$

Solution

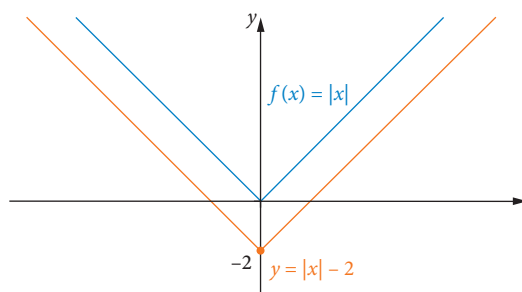
- a Sketch $f(x) = |x|$.
Add 2 to each ordinate (y-value).



State the domain and range.

Domain: $x \in \mathbf{R}$, range: $y \geq 2$

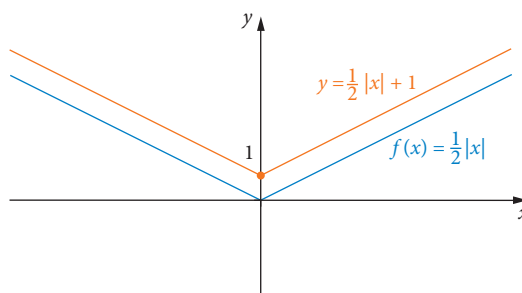
- b Sketch $f(x) = |x|$.
Subtract 2 from each ordinate (y-value).



State the domain and range.

Domain: $x \in \mathbf{R}$, range: $y \geq -2$

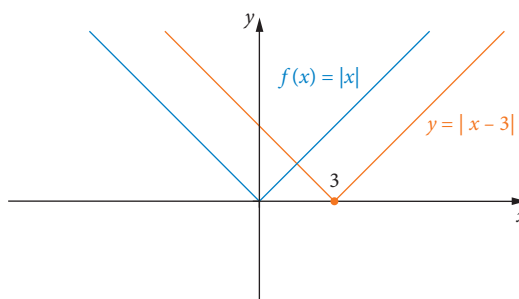
- c Sketch $f(x) = \frac{1}{2}|x|$.
Add 1 to each ordinate (y-value).



State the domain and range.

Domain: $x \in \mathbf{R}$, range: $y \geq 1$

- d Sketch $f(x) = |x|$.
Move each abscissa (x -value) 3 units to the right.



State the domain and range.

Domain: $x \in \mathbf{R}$, range: $y \geq 0$

IMPORTANT

To sketch $y = |f(x)|$, sketch $y = f(x)$ and make all negative y values positive. That is, the negative parts of the graph are reflected in the x -axis.

In sketching absolute value functions, you can solve equations like $|x - 3| = 5$ and inequations like $|2x + 3| \geq 5$ by observing where solutions lie on the number line.

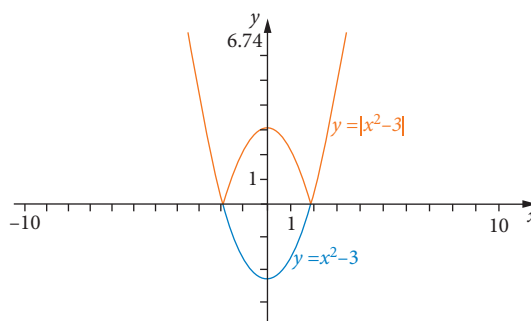
○ Example 11

Sketch the following absolute value functions and state their domain and range.

- a $y = |x^2 - 3|$ b $y = |x(x - 2)(x + 2)|$ c $y = |\sin(x)|$ d $y = \left| \frac{1}{x} \right|$

Solution

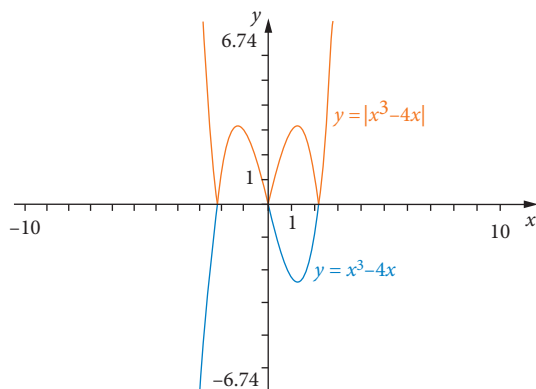
- a Sketch $y = x^2 - 3$.
Reflect the negative parts so that they are positive.



State the domain and range.

Domain: $x \in \mathbf{R}$, range: $y \geq 0$.

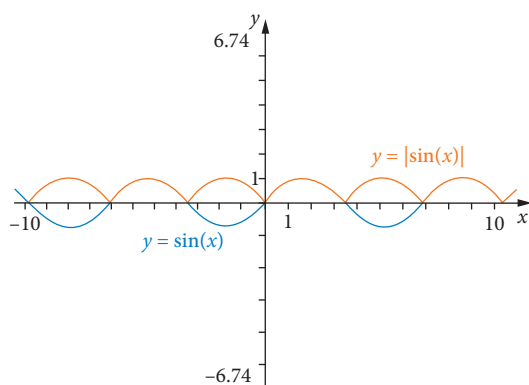
- b Sketch $y = x(x - 2)(x + 2)$.
Reflect the negative parts so that they are positive.



State the domain and range.

Domain: $x \in \mathbf{R}$, range: $y \geq 0$.

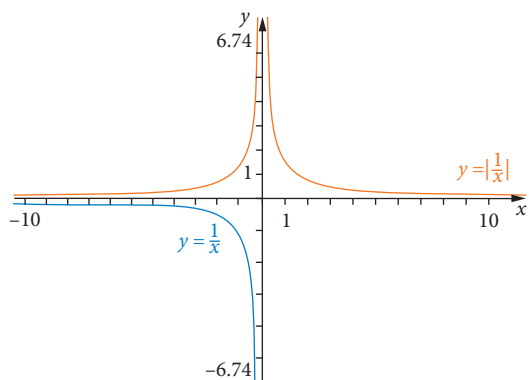
- c Sketch $y = \sin(x)$.
Reflect the negative parts so that they are positive.



State the domain and range.

Domain: $x \in \mathbf{R}$, range: $0 \leq y \leq 1$.

- d Sketch $y = \frac{1}{x}$.
Reflect the negative parts so that they are positive.



State the domain and range.

Domain: $x \in \mathbf{R}, x \neq 0$, range: $y > 0$.

EXERCISE 3.04 Absolute value functions

Concepts and techniques

1 **Example 9** Evaluate each of the following.

a $|-4|$

b $|-4 + 3|$

c $|-4| - 3|-4| - 3$

d $|-4 \times 3|$

e $-|3|^2$

f $|-4 \times 3| - 2|4 - 3|$

g $|-6 \times 2| - |-6 - 6|$

h $|-5 \times 2| \div |5 + 5|$

i $|-6 \div 3| - |-4 - 3|$

j $|-5 \times (-2)| + |-6 + 5|$

2 Solve the following equations.

a $|x - 3| = 5$

b $|2x + 3| = 5$

c $-2|x - 3| = 8$

d $3|x + 1| = 12$

e $\left|\frac{1}{x}\right| = 4$

3 Solve $|x - 3| = |2x - 1|$. Remember to check your solutions or use a graphics calculator to assist you with this one.

4 **Example 10** Sketch the following absolute value functions and state their domain and range.

a $y = |x + 4|$

b $y = |x| + 4$

c $y = 2|x + 2|$

d $y = 2|x - 2|$

e $y = |x - 2| - 2$

f $y = -|x| + 2$

g $y = -|x + 2|$

h $y = -2|x - 2| - 1$

5 Solve the equation $|x + 2| = 3$ by sketching the graphs of $y = |x + 2|$ and $y = 3$.

6 Solve the equation $|x + 2| = |3 - x|$ by sketching the appropriate graphs.

7 **Example 11** Sketch the following absolute value functions and state their domain and range.

a $y = |\cos(x)|$

b $y = |e^x|$

c $y = |x^2 - 1|$

d $y = -|x^2 - 4|$

e $y = \frac{1}{|x + 1|}$

f $y = |(x - 1)(x + 2)(x - 3)|$

Reasoning and communication

8 Solve the inequality $|(x + 2)(x - 2)| \geq 4$

9 Solve the inequality $\frac{1}{|x - 1|} \geq 3$

10 Solve the inequality $|x - 2| + |x + 2| < 5$



3.05 GRAPHS OF $y = f(x)$,

$$y = \frac{1}{f(x)}, y = f(|x|)$$

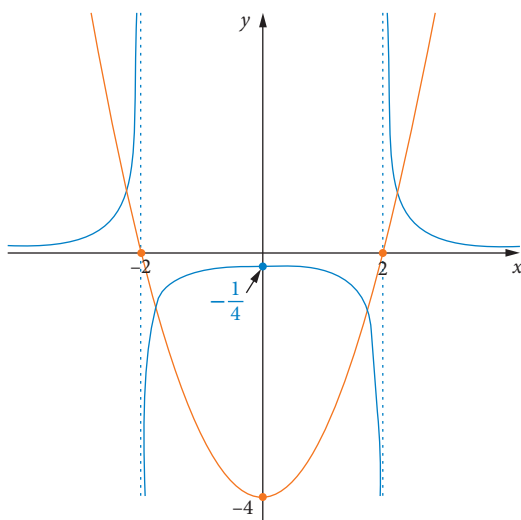
A **reciprocal function** has the form $y = \frac{1}{f(x)}$, where $f(x)$ is a polynomial and $f(x) \neq 0$.

The graphs below show the function $f(x) = x^2 - 4$ on the same plane as its reciprocal function

$$y = \frac{1}{x^2 - 4}.$$

You can see that:

- when the original function passes through the x -axis, the reciprocal function has an asymptote
- as the original function gets large, its reciprocal function gets small (approaching zero)



There are other features concerning reciprocal functions that are worth noting and a summary of these follows below.

IMPORTANT

For the function $y = f(x)$, the following features will assist in sketching $y = \frac{1}{f(x)}$.

- $f(x)$ and $\frac{1}{f(x)}$ have the same sign
- $y = f(x)$ and $y = \frac{1}{f(x)}$ intersect where $f(x) = \pm 1$
- Zeros of $y = f(x)$ correspond to vertical asymptotes of $y = \frac{1}{f(x)}$
- As $f(x)$ increases, $\frac{1}{f(x)}$ decreases and vice versa
- As $f(x) \rightarrow \pm \infty$, $\frac{1}{f(x)} \rightarrow 0$ and as $f(x) \rightarrow 0$, $\frac{1}{f(x)} \rightarrow \pm \infty$
- Maximum turning points of $y = f(x)$ correspond to minimum turning points of $y = \frac{1}{f(x)}$ and vice versa

○ Example 12

Sketch the following graphs and their reciprocal functions.

- a $f(x) = x - 2$ b $f(x) = (x - 2)(x + 1)$ c $f(x) = x(x - 3)(x + 3)$

Solution

- a Sketch $f(x) = x - 2$

Determine the y -intercept for $f(x) = x - 2$

and take the reciprocal for the y -intercept of

$$f(x) = \frac{1}{x - 2}.$$

Identify where $y = 0$ and sketch an asymptote.

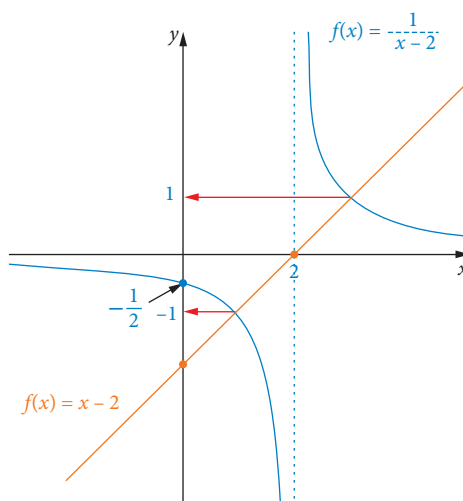
Identify where $y = 1$ and mark this shared point.

Identify where $y = -1$ and mark this shared point.

Check that where $f(x) > 0$ that $\frac{1}{f(x)} > 0$ and

that where $f(x) < 0$ that $\frac{1}{f(x)} < 0$.

Sketch the graph of $f(x) = \frac{1}{x - 2}$.



- b Sketch $f(x) = (x - 2)(x + 1)$

Determine the y -intercept and take the reciprocal for the y -intercept of

$$f(x) = \frac{1}{(x - 2)(x + 1)}.$$

Identify where $y = 0$ and add an asymptote.

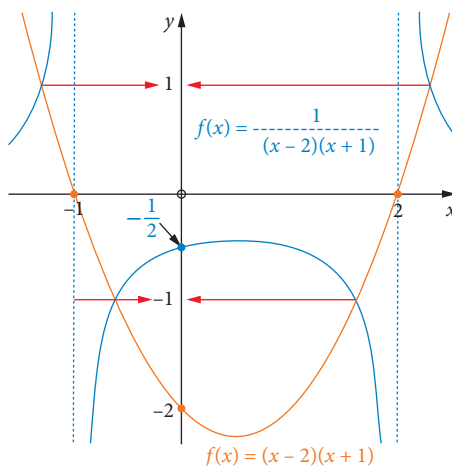
Identify where $y = 1$ and mark this shared point.

Identify where $y = -1$ and mark this shared point.

Check that where $f(x) > 0$ that $\frac{1}{f(x)} > 0$ and

that where $f(x) < 0$ that $\frac{1}{f(x)} < 0$.

Sketch the graph of $f(x) = \frac{1}{(x - 2)(x + 1)}$.



- c Sketch $f(x) = x(x - 3)(x + 3)$

Determine the y -intercept and take the reciprocal for the y -intercept of

$$f(x) = \frac{1}{x(x - 3)(x + 3)}.$$

Identify where $y = 0$ and add an asymptote.

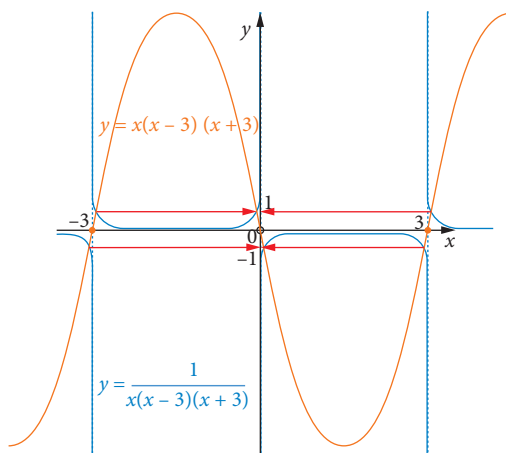
Identify where $y = 1$ and mark this shared point.

Identify where $y = -1$ and mark this shared point.

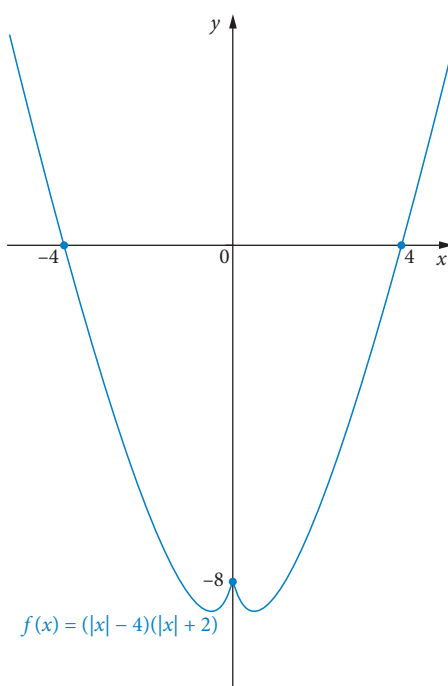
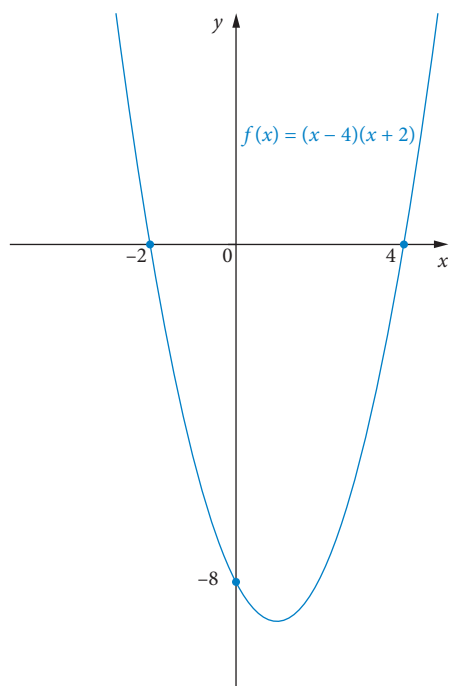
Check that where $f(x) > 0$ that $\frac{1}{f(x)} > 0$ and

that where $f(x) < 0$ that $\frac{1}{f(x)} < 0$.

Sketch the graph of $f(x) = \frac{1}{x(x - 3)(x + 3)}$.



Sketching the function $f(x) = (x - 4)(x + 2)$ below on the same plane as $f(x) = (|x| - 4)(|x| + 2)$, you can see that when $x \geq 0$ the graphs are the same and that when $x < 0$ the graph of $y = f(|x|)$ is the reflection of $y = f(x)$ in the y -axis.



Given the graph $y = f(x)$, to sketch the graph of $y = f(|x|)$, sketch the graph of $y = f(x)$ for $x \geq 0$ only and then reflect this graph in the y -axis to complete it.

○ Example 13

Sketch the graphs of $y = f(x)$ and $y = f(|x|)$ for the following equations.

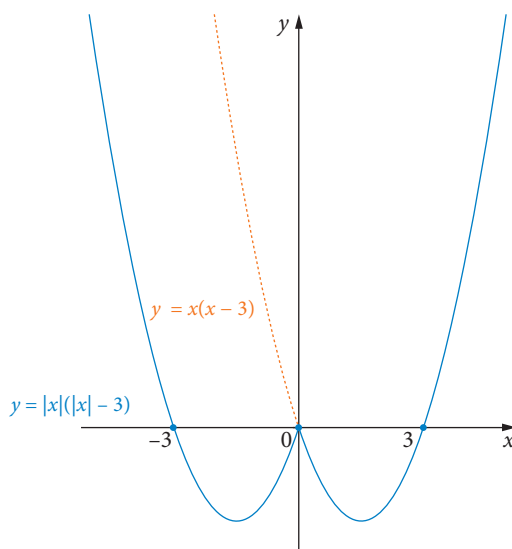
a $y = x(x - 3)$

b $y = \sin(x)$

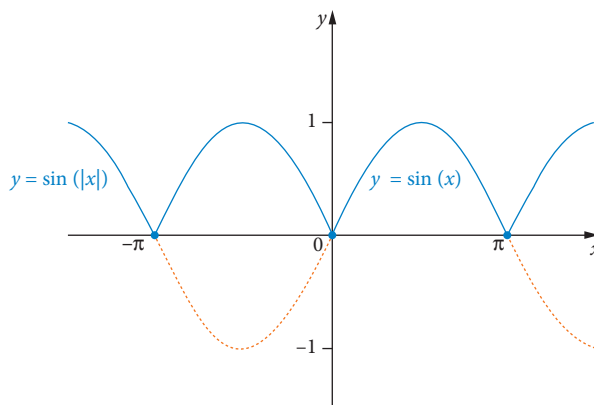
c $y = e^x$

Solution

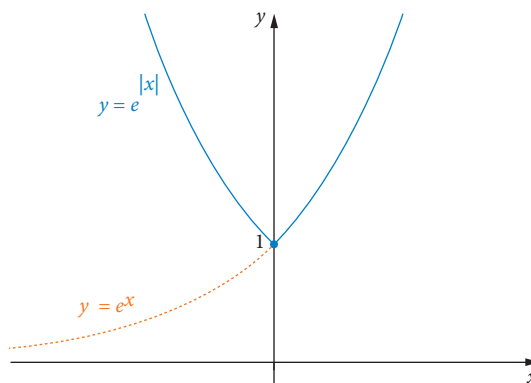
- a Sketch $y = x(x - 3)$ for $x \geq 0$ in orange.
 Reflect this graph in the y -axis to get the required graph in blue.
 Label the graph.



- b Sketch $y = \sin(x)$ for $x \geq 0$ in orange.
Reflect this graph in the y -axis to get the required graph in blue.
Label the graph.



- c Sketch $y = e^x$ for $x \geq 0$ in orange.
Reflect this graph in the y -axis to get the required graph in blue.
Label the graph.



EXERCISE 3.05 Graphs of $y = f(x)$, $y = \frac{1}{f(x)}$, $y = f(|x|)$



Transformations of
absolute values and
hyperbolas

Concepts and techniques

- 1 **Example 12** Sketch the following graphs and their reciprocal functions.

a $f(x) = x + 2$

b $f(x) = (x - 3)(x + 2)$

c $f(x) = x(x - 2)(x + 3)$

d $f(x) = e^x$

e $f(x) = \sin(x)$

- 2 Solve the inequation $\frac{1}{2x-3} \geq 1$ by first sketching $y = 2x - 3$.

- 3 Solve the inequation $\frac{1}{(x-3)(x+3)} > 0$ by first graphing $y = x^2 - 9$.

4 **Example 13** Sketch the graphs of $y = f(|x|)$ for the following graphs.

a $f(x) = x^2 + 2$

b $f(x) = \log_e(x)$

c $f(x) = 1 + x$

d $f(x) = 4x - x^2$

e $f(x) = x^2(x - 2)$

5 Solve the inequation $6|x| - x^2 = 9$ by first sketching $y = 6|x| - x^2$.

6 Solve the inequation $\log_e(|x|) < 2$ by first sketching $y = \log_e(x)$.

Reasoning and communication

7 Sketch the graph of $y = \frac{1}{1+|x|}$.

8 By first sketching $y = x^2 + 1$, sketch the graph of $y = \frac{1}{x^2 + 1}$, then $y = \frac{x}{x^2 + 1}$ and hence

$$y = \left| \frac{x}{x^2 + 1} \right|.$$

9 Using a graph of $y = x(x - 2)(x + 4)$, solve $\frac{1}{x(x - 2)(x + 4)} < 0$

3.06 GRAPHS OF RATIONAL FUNCTIONS

A reciprocal function has a denominator that is another function, but the numerator is 1. If the numerator and denominator are both polynomials, then the function is called a **rational function**. A reciprocal function is a special case of a rational function.

IMPORTANT

A **rational function** is one that can be expressed in the form $f(x) = \frac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are polynomial functions and $Q(x)$ is not the zero polynomial.

$$y = \frac{x^2}{1+x}, y = \frac{x^2-1}{x^2+1} \text{ and } y = \frac{x^3+2x^2+x+2}{x+2} \text{ are all rational functions.}$$

The first step in sketching simple rational functions is to sketch the polynomial in the denominator and then the reciprocal of this function.

Rational functions that have a constant as the numerator are the simplest ones.

○ Example 14

Sketch the following rational functions.

a $f(x) = \frac{-1}{(x-1)(x+3)}$

b $f(x) = \frac{10}{x(x+3)(x-3)}$

Solution

- a Sketch the polynomial $y = (x-1)(x+3)$. Determine any intercepts and the asymptotes of $f(x)$.

$$f(0) = \frac{1}{3}$$

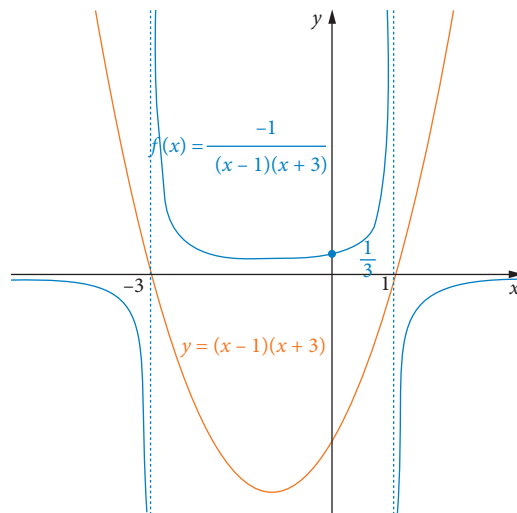
When $x = -3, 1, f(x) \rightarrow \pm \infty$ (asymptotes).

The desired function has the *opposite* signs to $y = (x-1)(x+3)$. The negative sign makes it the reflection in the x -axis of the reciprocal function.

Sketch the rational function

$$f(x) = \frac{-1}{(x-1)(x+3)} \text{ using points and features.}$$

features.



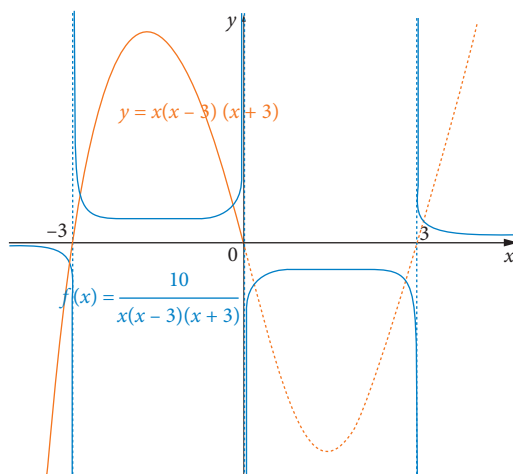
- b Sketch the polynomial $y = x(x+3)(x-3)$. Determine any intercepts and asymptotes of the reciprocal function. The desired function is stretched by the factor 10 in the vertical direction from the reciprocal function.

When $x = 0, \pm 3, f(x) \rightarrow \pm \infty$ (asymptotes).

Sketch the rational function

$$f(x) = \frac{10}{x(x-3)(x+3)} \text{ using points and features.}$$

features.



Polynomial functions are continuous, so rational functions are also continuous, except for the asymptotes. This means that rational functions can change sign only at zeros and asymptotes.

Consider $f(x) = \frac{x^2 - 3x + 5}{x^3 + x + 1}$, where the degree of the numerator is one less than the degree of the denominator. You can write the reciprocal as $\frac{1}{f(x)} = x - 3 - \frac{13x - 16}{x^2 - 3x + 5}$.

As $x \rightarrow \pm\infty$, $\frac{1}{f(x)}$ becomes very large, so $f(x) \rightarrow 0$. The same reasoning applies to any rational function whose numerator is of lower degree than the denominator.

IMPORTANT

A rational function whose numerator is of lower degree than the denominator has a horizontal asymptote of $x = 0$. You can sketch rational functions of this type using the horizontal asymptote for behaviour as $x \rightarrow \pm\infty$ and considering the sign of the function between zeroes and asymptotes.

Example 15

Sketch the following rational functions.

a $y = \frac{x}{(x+3)(x-3)}$

b $f(x) = \frac{x-1}{(x+2)(x-4)}$

c $y = \frac{(x+1)(x-2)}{(x+2)(x-1)(x-4)}$

Solution

- a The numerator is of lower degree than the denominator, so the horizontal asymptote is $x = 0$.

Determine any zeros and asymptotes.

There is a zero at $x = 0$ and asymptotes at $x = -3$ and $x = 3$. The y -intercept is the zero.

Work out the sign of the function between the zero and asymptotes.

$$\text{For } x < -3; \quad y < 0$$

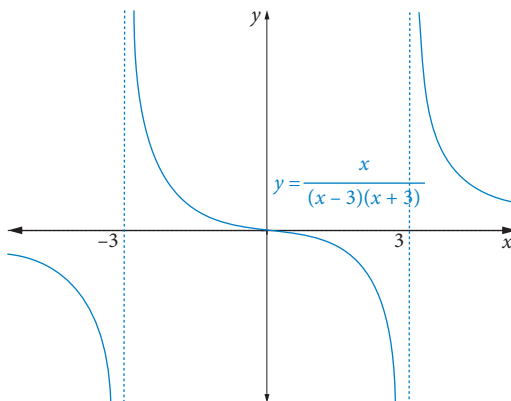
$$\text{For } -3 < x < 0; \quad y > 0$$

$$\text{For } 0 < x < 3; \quad y < 0$$

$$\text{For } x > 3; \quad y > 0$$

Sketch the rational function

$$y = \frac{x}{(x-3)(x+3)}.$$



- b The numerator is of lower degree than the denominator, so the horizontal asymptote is $x = 0$. Determine any zeros and asymptotes.

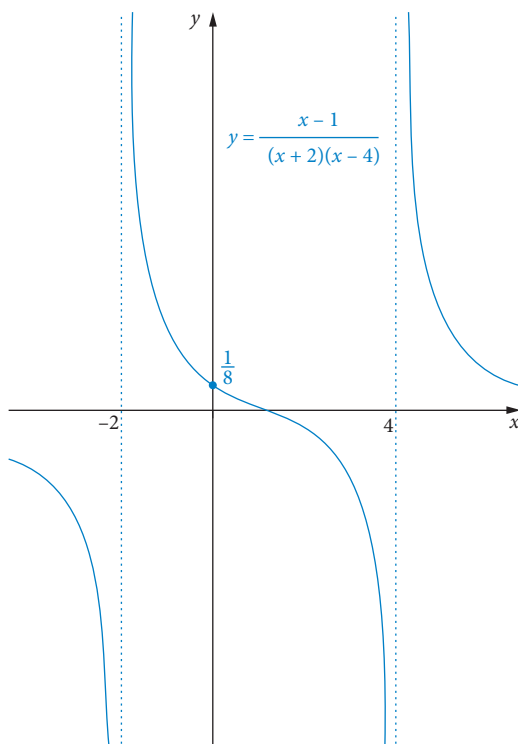
There is a zero at $x = 1$ and asymptotes at $x = -2$ and $x = 4$. The y -intercept is $\frac{1}{8}$.

Work out the sign of the function between the zero and asymptotes.

$$\begin{array}{ll} \text{For } x < -2; & y < 0 \\ \text{For } -2 < x < 1; & y > 0 \\ \text{For } 1 < x < 4; & y < 0 \\ \text{For } x > 4; & y > 0 \end{array}$$

Sketch the rational function

$$y = \frac{x-1}{(x+2)(x-4)}.$$



- c The numerator is of lower degree than the denominator, so the horizontal asymptote is $x = 0$. Determine any zeros and asymptotes.

There are zeros at $x = -1$ and $x = 2$.

The asymptotes are at $x = -2$, $x = 1$ and $x = 4$.

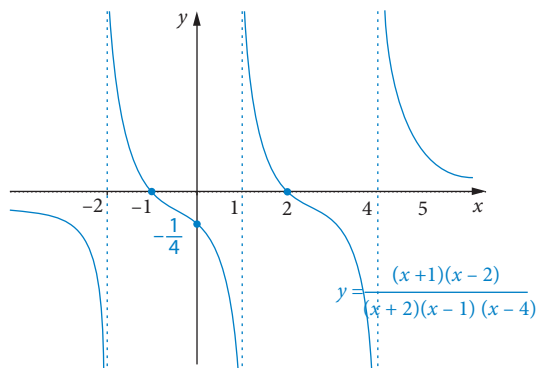
The y -intercept is $-\frac{1}{4}$.

Work out the sign of the function between the zero and asymptotes.

$$\begin{array}{ll} \text{For } x < -2; & y < 0 \\ \text{For } -2 < x < -1; & y > 0 \\ \text{For } -1 < x < 1; & y < 0 \\ \text{For } 1 < x < 2; & y > 0 \\ \text{For } 2 < x < 4; & y < 0 \\ \text{For } x > 4; & y > 0 \end{array}$$

Sketch the rational function

$$y = \frac{x^2}{(x+2)(x-1)(x-4)}.$$



The degree of the numerator of $f(x) = \frac{x^2 - 4x + 4}{x-1}$ is greater than the degree of the denominator.

You can change this to functions that you know how to sketch by dividing the numerator by the denominator.

$$\begin{array}{r}
 x-3 \\
 x-1 \overline{) x^2-4x+4} \\
 \underline{x^2-x} \\
 -3x+4 \\
 \underline{-3x+3} \\
 1
 \end{array}$$

Thus $x^2 - 4x + 4 = (x - 1)(x - 3) + 1$ and so $f(x) = \frac{x^2 - 4x + 4}{x - 1} = x - 3 + \frac{1}{x - 1}$.

This function has an asymptote at $x = 1$ and as $x \rightarrow \pm\infty$, $\frac{1}{x-1}$ becomes very small, so $f(x)$ behaves like $y = x - 3$, which is a straight line. This is called a **sloping asymptote**.

If you divide out $f(x) = \frac{3x+4}{x+2}$, you get $f(x) = 3 - \frac{2}{x+2}$, which behaves like $y = 3$ as $x \rightarrow \pm\infty$.

This is a **horizontal axis**.

IMPORTANT

You need to divide a rational function whose numerator has degree greater than or equal to the degree of the denominator. This will give the non-vertical asymptote as the dividend. You can then find the vertical asymptotes, zeros and y -intercept and sketch the function by considering the sign of the function between zeros and asymptotes.

○ Example 16

Sketch the following rational functions.

a $y = \frac{x+1}{x-1}$ b $f(x) = \frac{2x^2}{x+2}$ c $g(x) = \frac{x^2 - x + 1}{x - 1}$

Solution

a Dividing through gives

$$x + 1 = (x - 1) \times 1 + 2.$$

$$\text{This gives } y = 1 + \frac{2}{x-1}.$$

There is a horizontal asymptote at $y = 1$.

There is a vertical asymptote at $x = 1$.

The zero is at $x = -1$.

The y -intercept is at -1 .

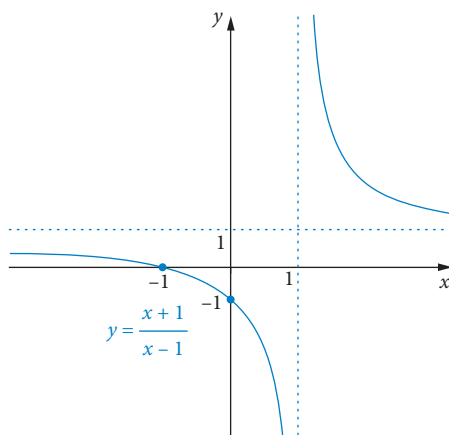
Consider the sign of y .

$$\text{For } x < -1; \quad y > 0$$

$$\text{For } -1 < x < 1, \quad y < 0$$

$$\text{For } x > 1, \quad y > 0$$

Sketch the function.



b Dividing through gives

$$2x^2 = (x+2) \times (2x-4) + 8.$$

$$\text{This gives } y = 2x - 4 + \frac{8}{x+2}.$$

This means that $y = 2x - 4$ is a sloping asymptote.

There is a vertical asymptote at $x = -2$.

There is a double zero at $x = 0$.

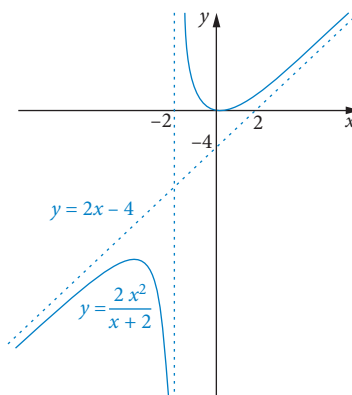
Consider the sign of y .

$$\text{For } x < -2; \quad y < 0$$

$$\text{For } -2 < x < 0, \quad y > 0$$

$$\text{For } x > 0, \quad y > 0$$

Sketch the function.



c Dividing through gives

$$x^2 - x + 1 = x(x-1) + 1$$

$$\text{Thus, } g(x) = x + \frac{1}{x-1}.$$

$y = x$ is a sloping asymptote.

There is a vertical asymptote at $x = 1$.

$x^2 - x + 1$ has no real zeros, so neither does $g(x)$.

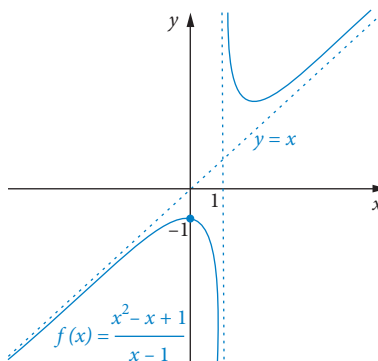
The y -intercept is at -1 .

Consider the sign of y .

$$\text{For } x < 1; \quad y < 0$$

$$\text{For } x > 1, \quad y > 0$$

Sketch the function.



IMPORTANT

To find the equations of the horizontal (or otherwise) asymptotes for a rational function, you examine the degrees of the polynomials in the numerator $P(x)$ and the denominator $Q(x)$.

- If $\deg P(x) < \deg Q(x)$, then there is a horizontal asymptote at $y = 0$.
- If $\deg P(x) = \deg Q(x)$, then there is a horizontal asymptote at $y = \frac{a}{b}$ where a is the leading coefficient of the numerator and b is the leading coefficient of the denominator.
- If $\deg P(x) > \deg Q(x)$, then the equation of the asymptote will not be horizontal, it will be a slanted line if the difference in degrees is one, and parabolic if the difference in degrees is two.

EXERCISE 3.06 Graphs of rational functions

Concepts and techniques

- 1 **Example 14** Sketch each of the following rational functions, showing any asymptotes and intercepts.

a $y = \frac{1}{x+3}$

b $y = \frac{1}{(x+3)(x-1)}$

c $y = \frac{-1}{x^2-1}$

d $y = \frac{2}{x^2+x-6}$

e $y = \frac{-2}{x^2+7x+6}$

f $y = \frac{-1}{x(x+1)(x-2)}$

- 2 **Example 15** Sketch each of the following rational functions, showing any asymptotes and intercepts.

a $y = \frac{x+1}{x^2}$

b $y = \frac{x+1}{x^2-1}$

c $y = \frac{x+2}{(x-1)(x-4)}$

d $y = \frac{x}{(x-1)(x+1)}$

e $y = \frac{x^2+1}{x(x-1)(x+1)}$

f $y = \frac{x^2+2x+1}{x(x^2-1)}$

- 3 **Example 16** Sketch each of the following rational functions, showing any asymptotes and intercepts.

a $y = \frac{x+1}{x-1}$

b $y = \frac{x-1}{x+1}$

c $y = \frac{x^2}{x-1}$

d $y = \frac{x^3-8}{(x-1)(x+1)}$

Reasoning and communication

- 4 Using the graph of the rational function $y = \frac{1}{(x-2)(x+4)}$, find the solution of the inequality $\frac{1}{(x-2)(x+4)} > 0$.

- 5 Solve the inequality $\frac{x-2}{x^2-3x-4} \leq 0$.

- 6 Solve the inequality $\frac{x^2-2}{x^2-2x+1} \leq 0$.

- 7 The value of a motorcycle (V), t years after it was bought from a store, can be estimated by the rational function $V(t) = \frac{(6+t)}{0.002t}$, for $t \geq 1$.

- What is the value of the motorcycle after 2 years?
- What is the value of the motorcycle after 10 years?
- Sketch a graph of the value of the motorcycle for the first ten years.



Stockphoto/Martin Wimmer

- 8 The annual sales (in billions of dollars) of computer games from 1990 to 2010 can be modelled by
- $$G(t) = \frac{850t^2 + 3200}{110t^2 + 1000}, \quad 0 \leq t \leq 20,$$
- where t is the number of years since 1990.

- For which year were the total sales of computer games about \$5.5 billion?
- If the sales of these computer games continued each year, what limit do these sales approach?
- Sketch a graph of this model for the sales of computer games.



Shutterstock.com/Jacek Chabaszewski

- 9 Stephen has made 60 out of his 80 free throw attempts at basketball this season.
- Write a rational expression that represents Stephen's free throw percentage if he makes his next x free throw attempts.
 - How many consecutive free throws must Stephen make in order to raise his free throw percentage to at least 90%?
 - Sketch a graph to show Stephen's rate of success with free throws this season if he makes all his free throws for the rest of the season.



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CHAPTER SUMMARY

FUNCTIONS AND SKETCHING GRAPHS

3

- The **composite** function $g \circ f$ is the function given by $g[f(x)] = gf(x)$.

This is also called a function of a function and sometimes written as gf .

- In general, $g[f(x)]$ is not equal to $f[g(x)]$.
- Functions can only be composed if part of the range of the second function lies within the domain of the first function.
- The domain of a composed function is either the same as the domain of the second function, or lies inside it.
- The range of a composed function is either the same as the range of the first function, or lies inside it.
- A **one-to-one** function satisfies the condition:
 $x \neq y \Rightarrow f(x) \neq f(y)$.
 - The contrapositive, $f(x) = f(y) \Rightarrow x = y$, may be easier to use.
- A function f is (strictly) **increasing** if for all x and y ($x < y$), $f(x) < f(y)$.
- A function f is (strictly) **decreasing** if for all x and y ($x > y$), $f(x) > f(y)$.
- If $f'(x) > 0$ for all x , then $f(x)$ is increasing.
- If $f'(x) < 0$ for all x , then $f(x)$ is decreasing.

- Continuous increasing functions and continuous decreasing functions are one-to-one.
- The **inverse** of a relation $R: x \rightarrow y$ is the relation $R^{-1}: y \rightarrow x$.
- If the inverse relation of a function f is also a function, it is called the **inverse function**, written as f^{-1} .
- An inverse function exists if and only if the function is one-to-one.
- The inverse relation of $y = f(x)$ can be found by making x the subject of the function. You can also interchange x and y first and then find y .
- On the number plane, the inverse relation can be represented by a reflection of the original function in the line $y = x$.
- When an inverse function exists, the domain of the inverse function is the range of the original function, and the range of the inverse function is the domain of the original function.
- The absolute value, also known as modulus, of a number $x \in \mathbb{R}$ is written as $|x|$ and is given by

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

- For the function $y = f(x)$, the following features will assist in sketching $y = \frac{1}{f(x)}$.
 - $f(x)$ and $\frac{1}{f(x)}$ have the same sign
 - $y = f(x)$ and $y = \frac{1}{f(x)}$ intersect, where $f(x) = \pm 1$
 - the zeros of $y = f(x)$ correspond to vertical asymptotes of $y = \frac{1}{f(x)}$
 - As $f(x)$ increases, $\frac{1}{f(x)}$ decreases and visa versa
 - As $f(x) \rightarrow \pm\infty$, $\frac{1}{f(x)} \rightarrow 0$ and as $f(x) \rightarrow 0$, $\frac{1}{f(x)} \rightarrow \pm\infty$
 - Maximum turning points of $y = f(x)$ correspond to minimum turning points of $y = \frac{1}{f(x)}$ and vice versa
- A rational function is one that can be expressed in the form $f(x) = \frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomial functions and $Q(x)$ is not the zero polynomial.

- For a rational function with numerator $P(x)$ and denominator $Q(x)$:
 - if $\deg P(x) < \deg Q(x)$, then there is a horizontal asymptote at $y = 0$
 - if $\deg P(x) = \deg Q(x)$, then there is a horizontal asymptote at $y = \frac{a}{b}$, where a is the leading coefficient of the numerator and b is the leading coefficient of the denominator
 - if $\deg P(x) > \deg Q(x)$, there may be a slanted straight line or curved asymptote. The asymptote is the dividend when the numerator is divided by the denominator.
 - you can sketch a rational function by finding the non-vertical asymptote as above, the y -intercept, the vertical asymptotes and zeros. Then use the sign between the zeros and vertical asymptotes to assist in sketching the function.

CHAPTER REVIEW

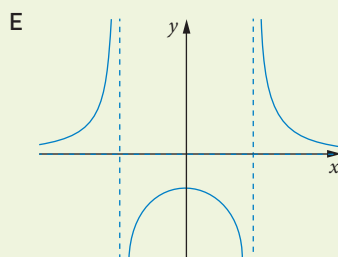
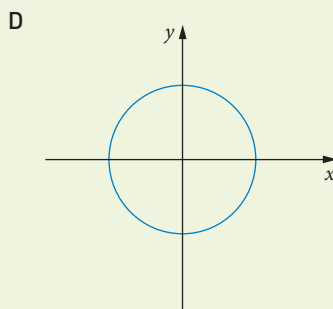
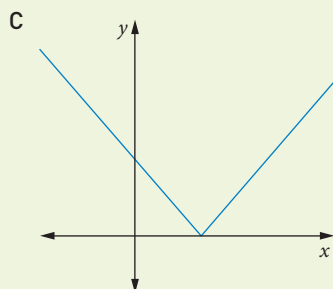
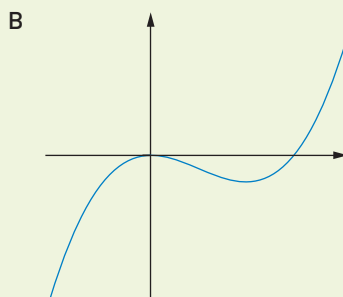
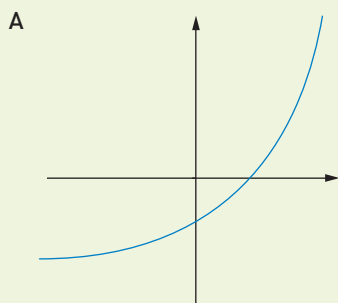
FUNCTIONS AND SKETCHING GRAPHS

3

Multiple choice

- 1 **Example 1** Which of the following gives $f[g(x)]$, where $f(x) = x^2 - 1$ and $g(x) = x + 1$?
 A $(x^2 - 1)(x + 1)$ B x^2 C $x^2 + 1$ D $x^2 + 2x$ E $x^2 + 2x + 1$

- 2 **Example 4** Which of the following is NOT a function?



- 3 **Example 2** Find $[f \circ g](x)$ given that $f(x) = 2e^x$ and $g(x) = \log_e(\frac{1}{2}x)$.
 A $\log_e(e^x)$ B x C e^x D $2 \log_e(x)$ E $\frac{1}{2}e^x$

- 4 **Example 3** Which of the following pairs of functions result when the function $h(x) = x^2 + 2x + 2$ is decomposed?
- A $f(x) = x^2, g(x) = 2x + 2$ B $f(x) = x^2 + 2x, g(x) = 2$ C $f(x) = x^2 + 1, g(x) = x + 1$
 D $f(x) = x^2 + x, g(x) = x + 2$ E $f(x) = x^2 - 1, g(x) = x - 1$
- 5 **Example 5** Determine which of the following functions is strictly decreasing.
- A $f(x) = e^x$ B $f(x) = x^2 + 2x + 1$ C $f(x) = \frac{1}{x+1}$
 D $f(x) = \sin(x)$ E $f(x) = \frac{x}{\log_e(x)}$

Short answer

- 6 **Example 6** Find the inverse relation of $y = \frac{2}{x-3}$.
- 7 **Example 7** Find the inverse relation of $y = \sqrt{5-x}$.
- 8 **Example 8** Sketch the graph of $y = 2e^{3x}$ and its inverse function.
- 9 **Example 9** Evaluate the following.
 a $|-4| - 6$ b $|6 \times (-2)| + 1$ c $|3 \times (-2)| - |6 \times (-3)|$
- 10 **Example 10** Sketch the function $y = -|x| - 3$ and state its domain and range.
- 11 **Example 11** Solve the equation $|x+1| = -\frac{1}{2}x + 1$ using graphs of $y = |x+1|$ and $y = -\frac{1}{2}x + 1$.
- 12 **Example 11** Sketch the function $y = |(x-1)(x+2)(2-x)|$. Make the function $g(x) = (x-1)(x+2)(2-x)$.
- 13 **Example 12** Sketch the reciprocal function $f(x)$ of $g(x) = (x-1)(x+2)(2-x)$.
- 14 **Example 13** Sketch the function $f(|x|)$ given that $f(x) = (x+3)(x+1)(x-2)$.
- 15 **Example 14** Sketch the function $f(x) = \frac{1}{(x-2)(x+1)}$.
- 16 **Example 15** Sketch the function $f(x) = \frac{x}{x^2-4}$.
- 17 **Example 16** Sketch the function $f(x) = \frac{x^2}{x^2-4}$.

Application

- 18 Write an equation for a rational function that has asymptotes $y = x$ and $x = -2$, and which passes through $(0, 2)$.
- 19 Solve the inequality $\frac{x^2}{x^2-4} \leq 2$.
- 20 So far this year, Ross has completed 15 out of 17 homework assignments in Mathematics. Ross intends to submit all assignments from now on. Write an expression for his percentage assignment submission rate and solve using graphing or algebra to find how many consecutive assignments he must now submit to raise his rate to 95%.



Practice quiz

MIXED REVISION

CHAPTERS 1 • 2 • 3



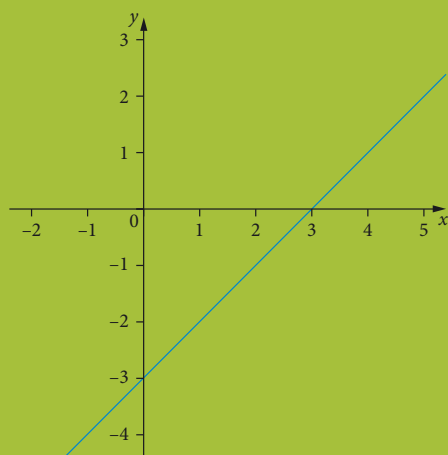
Multiple choice

- If $\mathbf{a} = \mathbf{i} + 2\mathbf{j}$ and $\mathbf{b} = 4\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$, then $|\mathbf{AB}|$ is equal to:
 A 4 B 5 C $\sqrt{35}$ D $\sqrt{41}$ E 7
- The quadrataic equation with roots $2 \pm i\sqrt{3}$ is
 A $x^2 - 4x + 7 = 0$ B $x^2 + 4x + 7 = 0$
 C $x^2 + 4x - 7 = 0$ D $x^2 - 2\sqrt{3}x + 7 = 0$
 E $x^2 + 2\sqrt{3}x + 7 = 0$
- Which of the following is the inverse function of $f(x) = \frac{x+3}{2}$?
 A $f^{-1}(x) = 2(x-3)$ B $f^{-1}(x) = 2(x+3)$ C $f^{-1}(x) = 2x+3$
 D $f^{-1}(x) = 2x-3$ E $f^{-1}(x) = \frac{x+3}{2}$
- If $z = r[\cos(\theta) + i \sin(\theta)]$, then the value of $i\bar{z}$ is
 A $r[\cos(\theta) - i \sin(\theta)]$ B $ir[\cos(\theta) + i \sin(\theta)]$
 C $r\left[\cos\left(\theta - \frac{\pi}{2}\right) + i \sin\left(\theta - \frac{\pi}{2}\right)\right]$ D $r\left[\cos\left(\theta - \frac{\pi}{2}\right) - i \sin\left(\theta - \frac{\pi}{2}\right)\right]$
 E $r\left[\cos\left(-\theta + \frac{\pi}{2}\right) + i \sin\left(-\theta + \frac{\pi}{2}\right)\right]$
- If $f(x) = x^2 - 5x - 6$, then the graph of $y = \frac{1}{f(x)}$ has:
 A x -intercepts at $x = 3$ and $x = 2$
 B x -intercepts at $x = 1$ and $x = -6$
 C asymptotes at $x = -1$ and $x = 6$
 D a local minimum at the point $\left(-\frac{3}{2}, \frac{15}{4}\right)$
 E a local maximum at the point $\left(-\frac{3}{2}, \frac{15}{4}\right)$.
- $PQRS$ is a rectangle with $|\mathbf{QR}| = 4|\mathbf{PQ}|$. If $\mathbf{PQ} = \mathbf{p}$, then $|\mathbf{PR}|$ in terms of $|\mathbf{p}|$ is:
 A $2|\mathbf{p}|$ B $3|\mathbf{p}|$ C $4|\mathbf{p}|$ D $\sqrt{17}|\mathbf{p}|$ E $5|\mathbf{p}|$
- If $\mathbf{a} = 4\mathbf{i} - \mathbf{j} + 5\mathbf{k}$ and $\mathbf{b} = 6\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$, then the projection of $\mathbf{a}$ on $\mathbf{b}$ has magnitude
 A $\frac{11}{7}$ B $\frac{11}{\sqrt{42}}$ C $\frac{22}{7}$ D $\frac{21}{\sqrt{42}}$ E $\frac{37}{7}$
- Realising the denominator in $\frac{-3-i}{3-i}$, you get
 A 1 B -1 C 0 D $\frac{-4-3i}{5}$ E $\frac{-4+3i}{5}$

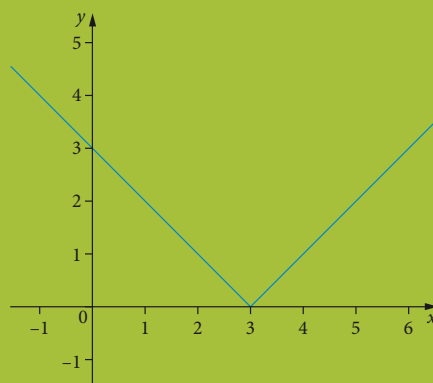
MIXED REVISION • 1 • 2 • 3

9 Sketch the function $y = |x + 3| - 1$.

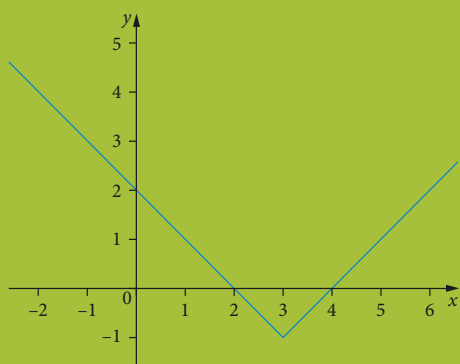
A



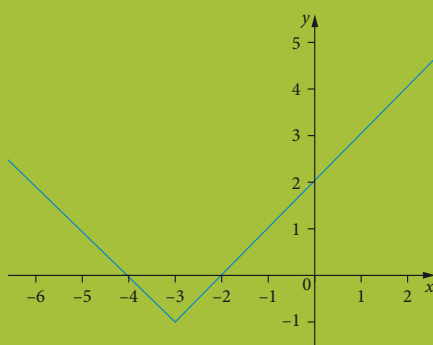
B



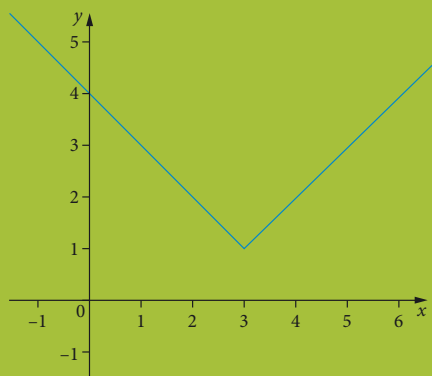
C



D

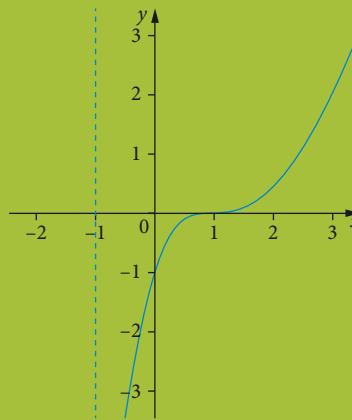


E



Short answer

- Find the value of $(1 - i\sqrt{3})^5$ in Cartesian form.
- The graph of $y = f(x)$ is given below.
Draw the graph of $y = f^{-1}(x)$.



- Given the vectors $\mathbf{a} = 3\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$, $\mathbf{b} = 5\mathbf{i} + 4\mathbf{j}$, $\mathbf{c} = 2\mathbf{i} + 4\mathbf{j} - 6\mathbf{k}$ and $\mathbf{d} = \mathbf{i} + 7\mathbf{j} + \mathbf{k}$, find:

a $\mathbf{a} \cdot \mathbf{b}$	b $\mathbf{b} \cdot \mathbf{c}$	c $\mathbf{c} \cdot \mathbf{d}$	d $\mathbf{d} \cdot \mathbf{a}$
e $\mathbf{a} \cdot \mathbf{d}$	f $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c})$	g $\mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{c}$	h $\mathbf{c} \cdot (\mathbf{a} - \mathbf{b}) + 7$
i $\mathbf{c} \cdot \mathbf{a} - \mathbf{c} \cdot \mathbf{b} + 7$	j $\mathbf{b} \cdot \mathbf{d}$		
- Use the vector product to find the area of a triangle with vertices:

a $(3, -1, 4)$, $(2, 2, 0)$ and $(3, -4, -5)$
b $(-2, 3, 5)$, $(-1, 4, -3)$ and $(0, 2, -1)$
- Solve the inequality $\frac{x}{x-2} > -1$ by sketching a rational function.
- If $z = 5\left[\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right]$ and $zw = 35\left[\cos\left(\frac{7\pi}{12}\right) + i\sin\left(\frac{7\pi}{12}\right)\right]$, find the value of w and express your answer in polar form.

Application

- Given $z = r[\cos(\theta) + i\sin(\theta)]$, prove that $z^n + (\bar{z})^n$ is real for all positive integers n .
- By sketching $f(x) = x(x-2)(x+3)$, solve the inequality $\frac{1}{x(x-2)(x+3)} > 0$.
- Prove that the line joining the midpoints of two sides of a triangle is parallel to the third side and half its length.
- Given that $z = \frac{-1+i\sqrt{3}}{-1+i}$, find z in mod-arg form. Hence find the smallest positive value of n such that z^n is purely imaginary.
- So far this season, the goal shooter in our netball team has been successful on 50 out of 75 attempts. Write a rational expression that represents the goal shooter's success rate as a percentage if she scores goals in her next x attempts. How many consecutive times must she score goals to increase her percentage to 90%? (Use a graph to assist you in solving this problem)
- Prove that the diagonals of a rhombus are perpendicular.



4

TERMINOLOGY

Argand plane
argument
complex arithmetic
complex conjugate
De Moivre's theorem
locus
polar form

COMPLEX NUMBERS

THE COMPLEX PLANE

- 4.01 Addition and subtraction in the complex plane
- 4.02 Multiplication in the complex plane
- 4.03 Division in the complex plane
- 4.04 Complex number operations as transformations
- 4.05 Graphing subsets of the complex plane: subsets involving modulus
- 4.06 Graphing subsets of the complex plane: subsets not involving modulus

Chapter summary

Chapter review



Prior learning

COMPLEX ARITHMETIC USING POLAR FORM

- define and use multiplication, division, and powers of complex numbers in polar form and the geometric interpretation of these (ACMSM082)
- prove and use De Moivre's theorem for integral powers (ACMSM083)

THE COMPLEX PLANE (THE ARGAND PLANE)

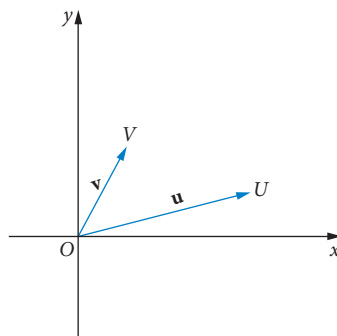
- examine and use addition of complex numbers as vector addition in the complex plane (ACMSM084)
- examine and use multiplication as a linear transformation in the complex plane (ACMSM085)
- identify subsets of the complex plane determined by relations such as $|z - 3i| \leq 4$, $\frac{\pi}{4} \leq \text{Arg}(z) \leq \frac{3\pi}{4}$, $\text{Re}(z) > \text{Im}(z)$, and $|z - 1| = 2|z - i|$ (ACMSM086) 

4.01 ADDITION AND SUBTRACTION IN THE COMPLEX PLANE

Chapter 2 was about the four operations with complex numbers and De Moivre's theorem. This chapter is about geometric interpretation of complex numbers and the Argand diagram.

○ Example 1

The vectors $\mathbf{u}$ and $\mathbf{v}$ with corresponding points U and V are shown on an Argand diagram below.



Use the parallelogram rule to draw the vectors $\mathbf{w}$ and $\mathbf{z}$, where

a $\mathbf{w} = \mathbf{u} + \mathbf{v}$

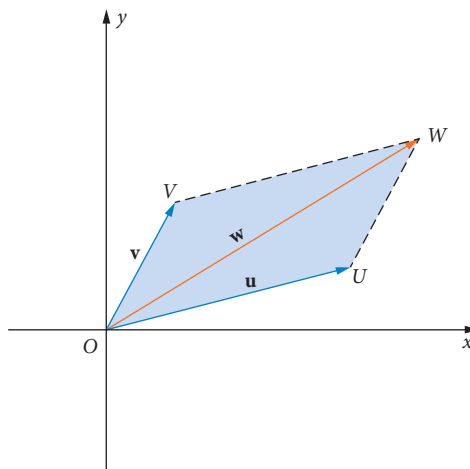
b $\mathbf{z} = \mathbf{u} - \mathbf{v}$

Plot the points W and Z that correspond to the vectors $\mathbf{w}$ and $\mathbf{z}$.

Solution

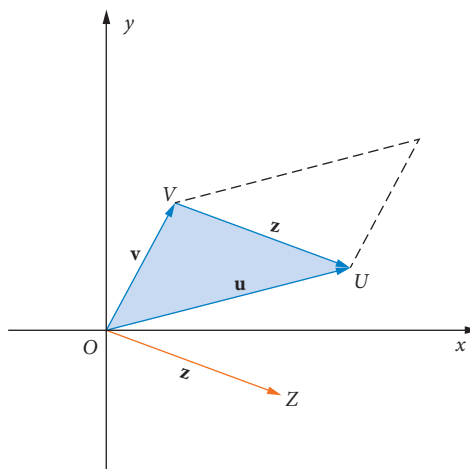
- a To find the sum of two vectors, complete the parallelogram and draw the diagonal with its tail at O .

$$\mathbf{w} = \mathbf{u} + \mathbf{v}$$



- b To find the difference of two vectors, $\mathbf{z} = \mathbf{u} - \mathbf{v}$, complete the parallelogram and draw the diagonal from V to U . Redraw $\mathbf{z}$ with its tail at the origin to find the point Z .

$$\mathbf{z} = \mathbf{u} - \mathbf{v}$$

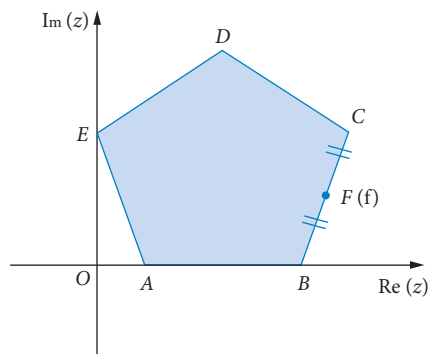


Example 2

Consider the vertices of the regular pentagon $ABCDE$ with position vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}$ as shown. F ($\mathbf{f}$) is the midpoint of BC .

Find expressions for each vector below in terms of $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ and $\mathbf{e}$.

- a $\overrightarrow{AB}$ b $\overrightarrow{AC}$ c $\overrightarrow{EB}$ d $\overrightarrow{DA}$ e $\overrightarrow{OF}$



Solution

- a The vector $\vec{AB}$ has length AB , with its arrow at B .
- b $\vec{AC}$ is the sum of vectors $\vec{AB}$ and $\vec{BC}$.
Alternatively, you can use the parallelogram rule.
- c Use the parallelogram rule.
- d Use the parallelogram rule.
- e Note that $\vec{BF} = \frac{1}{2} \vec{BC}$

$$\begin{aligned}\vec{AB} &= \vec{OB} - \vec{OA} \\ &= \mathbf{b} - \mathbf{a} \\ \vec{AC} &= \vec{AB} + \vec{BC} \\ &= (\mathbf{b} - \mathbf{a}) + (\mathbf{c} - \mathbf{b}) \\ &= \mathbf{c} - \mathbf{a} \\ \vec{EB} &= \mathbf{b} - \mathbf{e} \\ \vec{DA} &= \mathbf{a} - \mathbf{d} \\ \vec{OF} &= \vec{OB} + \frac{1}{2} \vec{BC} \\ &= \mathbf{b} + \frac{1}{2}(\mathbf{c} - \mathbf{b}) \\ &= \frac{\mathbf{b} + \mathbf{c}}{2}\end{aligned}$$

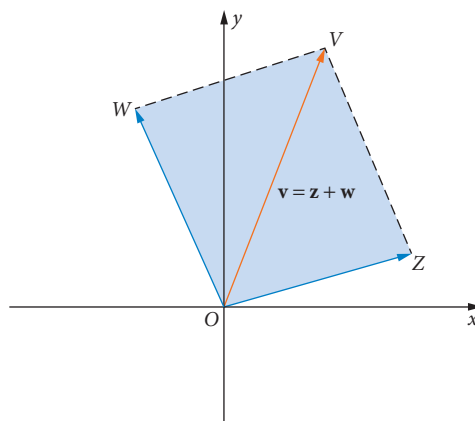
Example 3

Consider the points Z and W representing the complex numbers $z = 3 + i$ and $w = -2 + 6i$ on the Argand plane. $OZVW$ is a parallelogram.

- a Find the complex number v that represents the point V .
- b Find the length of the diagonal ZW .

Solution

- a OZ and OW are sides of $OZVW$. Use the parallelogram rule to find the complex number v that represents the point V .



Add the complex numbers.

$$\begin{aligned}v &= z + w \\ &= (3 + i) + (-2 + 6i) \\ &= 1 + 7i\end{aligned}$$

- b Again, using the parallelogram rule, the length of the diagonal ZW is represented by the modulus of the vector $\mathbf{z} - \mathbf{w}$.

$$\begin{aligned} \mathbf{z} - \mathbf{w} &= (3 + i) - (-2 + 6i) \\ &= 5 - 5i \end{aligned}$$

$$\begin{aligned} \therefore ZW &= |5 - 5i| \\ &= \sqrt{5^2 + (-5)^2} \\ &= \sqrt{50} \\ &= 5\sqrt{2} \end{aligned}$$

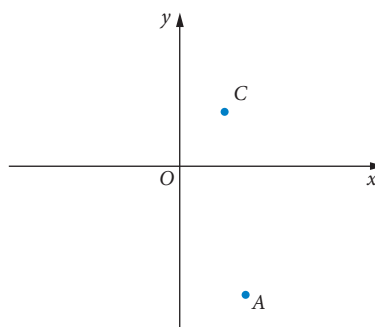
EXERCISE 4.01 Addition and subtraction in the complex plane

Concepts and techniques

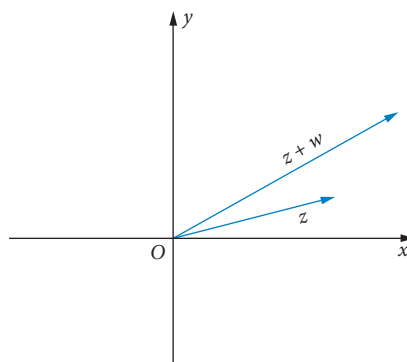


Addition and subtraction in the plane

- 1 **Example 1** The points A, B, C correspond to the complex numbers a, b, c . $ABCO$ forms a parallelogram. Copy the diagram. On your diagram,
- plot the point B
 - express the vector $\mathbf{b}$ in terms of $\mathbf{a}$ and $\mathbf{c}$
 - express the vector $\overrightarrow{AC}$ in terms of $\mathbf{a}$ and $\mathbf{c}$
 - plot the point D that corresponds to the vector $\mathbf{c} - \mathbf{a}$.



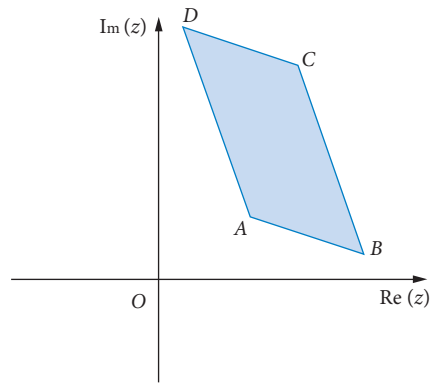
- 2 The vectors $\mathbf{z}$ and $\mathbf{z} + \mathbf{w}$ corresponding to the complex numbers z and $z + w$ are plotted on an Argand diagram.
- Draw the vector $\mathbf{w}$ and plot the location of the point W representing w .
 - With reference to the diagram, explain why $|z + w| \leq |z| + |w|$.



- 3 Use a similar technique to that in question 2 to show, for any two complex numbers z and w , that $|z - w| \geq |z| - |w|$. When does equality hold?

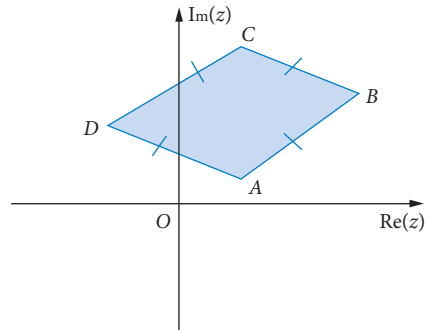
- 4 **Example 2** Consider the parallelogram $ABCD$ representing the complex numbers z_1, z_2, z_3, z_4 respectively.

- i Express each vector in terms of any two of z_1, z_2, z_3, z_4 :
- a $\vec{AB}$ b $\vec{CB}$ c $\vec{AC}$
- ii Find the complex number w in terms of z_1, z_2, z_3 or z_4 that is represented by the midpoint W of DB .

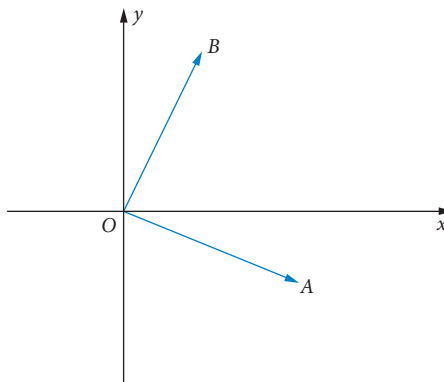


- 5 In the diagram, $ABCD$ is a rhombus. The points A, B, C and D represent the complex numbers z, w, u and v .

- i Find, in terms of z, w, u or v ,
- a $\vec{AB}$ b $\vec{CB}$ c $\vec{AC}$
- ii Find, in terms of z, w and v ,
- a u
- b the complex number m represented by the midpoint M of AC .



- 6 P, Q, R, S representing the complex numbers w_1, w_2, w_3, w_4 form a quadrilateral in the complex plane. What type of quadrilateral is $PQRS$ if $w_1 - w_2 = w_4 - w_3$?
- 7 **Example 3** Consider the complex numbers $a = \sqrt{3} - i$ and $b = 1 + i\sqrt{3}$ represented by the points A and B on an Argand diagram such that $OACB$ is a quadrilateral.



Copy the diagram and

- a plot the point C such that $OACB$ is a parallelogram
- b find the complex number c represented by C
- c show that $|a| = |b|$
- d show that $|a - b| = |c|$
- e classify the quadrilateral $OACB$.

Reasoning and communication

8 Plot the points A, B corresponding to the complex numbers $a = 4 - 6i$ and $b = 3 + 2i$.

a Find $|a|$, $|b|$ and $|a - b|$.

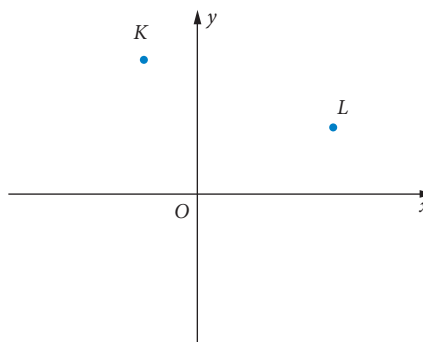
b Prove that $OA^2 + OB^2 = AB^2$.

c What type of triangle is OAB ?

9 The points L and K on the right represent the complex numbers z_1 and z_2 .

Copy the diagram and plot the points M and N representing the complex numbers $-z_1$ and $-z_2$.

What type of quadrilateral is $LKMN$? Explain.



10 The complex numbers $w = 5[\cos(60^\circ) + i \sin(60^\circ)]$ and $z = 5[\cos(30^\circ) + i \sin(30^\circ)]$ are represented by points A and D on an Argand diagram. The quadrilateral $OACD$ is a rhombus.

a Find the complex number c corresponding to the vertex C . Give your answer in Cartesian form.

b Find the area of $OACD$ in simplest form.

4.02 MULTIPLICATION IN THE COMPLEX PLANE

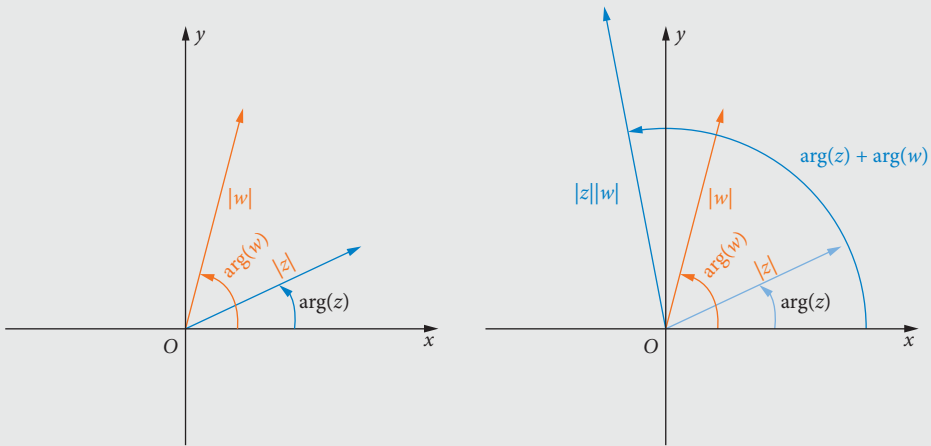
Geometric interpretation in the plane

To multiply complex numbers, you multiply the moduli and add the arguments.

IMPORTANT

If $z_1 = r_1[\cos(\theta_1) + i \sin(\theta_1)]$ and $z_2 = r_2[\cos(\theta_2) + i \sin(\theta_2)]$ are two complex numbers, then their **product** is $z_1 z_2 = r_1 r_2[\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)]$.

Geometrically, the product of two vectors $\mathbf{z}$ and $\mathbf{w}$ have a length of $|z||w|$ and an argument equal to the sum $\arg(z) + \arg(w)$. If the vector $\mathbf{z}$ is multiplied by a vector $\mathbf{w}$, its modulus increases by the factor $|w|$ and its argument is rotated by $\arg(w)$.



If $z = |z|\{\cos [\arg (z)] + i \sin [\arg (z)]\}$ and $w = |w|\{\cos [\arg (w)] + i \sin [\arg (w)]\}$ are two complex numbers, then their **product** is

$$zw = |z||w|\{\cos [\arg (z) + \arg (w)] + i \sin [\arg (z) + \arg (w)]\}$$

○ Example 4

If $z = \sqrt{2}\left[\cos\left(\frac{\pi}{6}\right) + i \sin\left(\frac{\pi}{6}\right)\right]$ and $w = i$, sketch the vector zw in the complex plane.

Solution

Convert $w = i$ to mod-arg form.

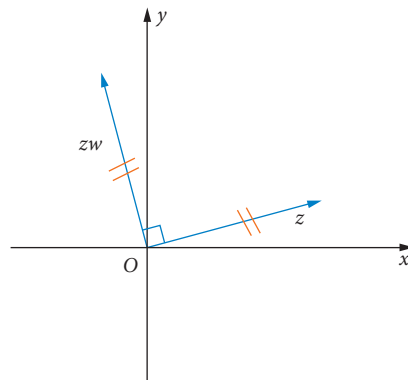
$$\text{mod } (w) = 1 \text{ and } \arg(w) = \frac{\pi}{2}$$

$$\therefore w = \cos\left(\frac{\pi}{2}\right) + i \sin\left(\frac{\pi}{2}\right).$$

Use the rule $zw = |z||w|\{\cos [\arg (z) + \arg (w)] + i \sin [\arg (z) + \arg (w)]\}$

$$\begin{aligned} zw &= \sqrt{2} \times 1 \left[\cos\left(\frac{\pi}{6} + \frac{\pi}{2}\right) + i \sin\left(\frac{\pi}{6} + \frac{\pi}{2}\right) \right] \\ &= \sqrt{2} \left[\cos\left(\frac{2\pi}{3}\right) + i \sin\left(\frac{2\pi}{3}\right) \right] \end{aligned}$$

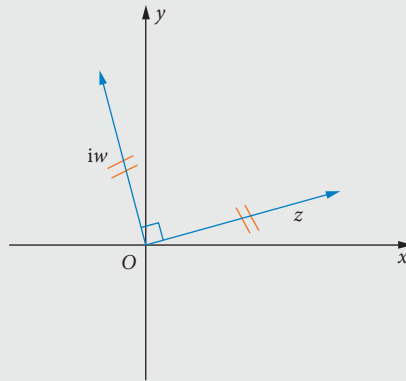
z has been rotated $\frac{\pi}{2}$ but the length remains constant.



The example above is an illustration of the following rule.

IMPORTANT

Multiplication by i is equivalent to a rotation of $\frac{\pi}{2}$ in the complex plane.



By implication, multiplication by $-i$ is equivalent to a rotation of $-\frac{\pi}{2}$ in the complex plane.

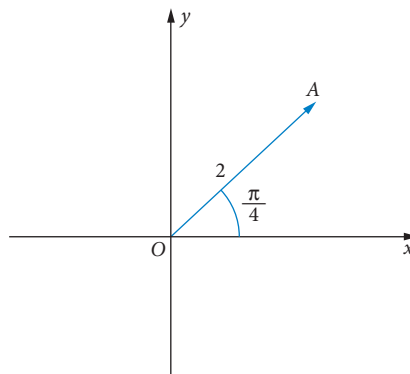
○ Example 5

If $z = 2\left[\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right]$ and $w = 3\left[\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right]$, sketch the points A, B, C, D, E , corresponding to each complex number on the complex plane.

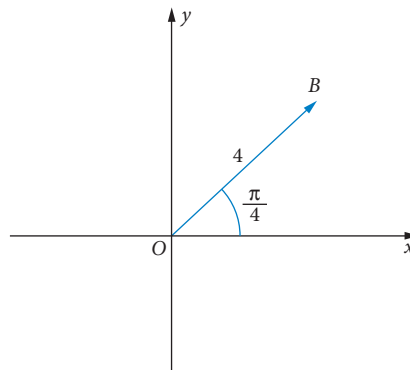
- a $A = z$ b $B = 2z$ c $C = zw$ d $D = -z$ e $E = w^2$

Solution

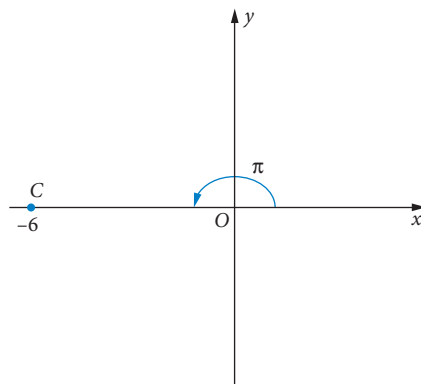
- a Sketch a complex number with modulus 2 and argument $\frac{\pi}{4}$.



- b The modulus of z is multiplied by a factor of 2.



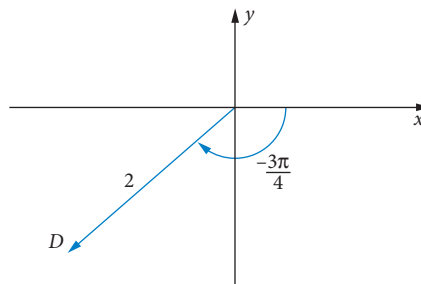
- c Using the rule $zw = |z||w|\{\cos [\arg (z) + \arg (w)] + i \sin [\arg (z) + \arg (w)]\}$,
 $\operatorname{mod}(zw) = 2 \times 3 = 6$ and
 $\arg(zw) = \frac{\pi}{4} + \frac{3\pi}{4} = \pi$



- d $-z$ is equivalent to

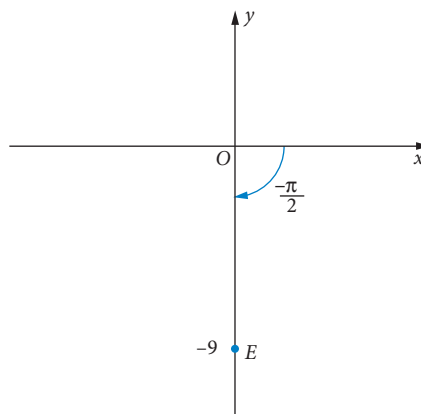
$$\begin{aligned} -1 \times z &= [\cos(\pi) + i \sin(\pi)] \times 2 \left[\cos\left(\frac{\pi}{4}\right) + i \sin\left(\frac{\pi}{4}\right) \right] \\ &= 2 \left[\cos\left(\pi + \frac{\pi}{4}\right) + i \sin\left(\pi + \frac{\pi}{4}\right) \right] \\ &= 2 \left[\cos\left(\frac{5\pi}{4}\right) + i \sin\left(\frac{5\pi}{4}\right) \right] \\ &= 2 \left[\cos\left(-\frac{3\pi}{4}\right) + i \sin\left(-\frac{3\pi}{4}\right) \right] \end{aligned}$$

Recall that the principle argument is in the interval $(-\pi, \pi]$.



- e $w^2 = w \times w$

$$\begin{aligned} &= 3 \operatorname{cis}\left(\frac{3\pi}{4}\right) \times 3 \operatorname{cis}\left(\frac{3\pi}{4}\right) \\ &= 9 \operatorname{cis}\left(\frac{3\pi}{2}\right) \\ &= 9 \operatorname{cis}\left(-\frac{\pi}{2}\right) \\ &= -9i \end{aligned}$$



In Example 5, you could use De Moivre's theorem to evaluate w^2 . In that case,

$$\begin{aligned} w^2 &= \left\{ 3 \left[\cos\left(\frac{3\pi}{4}\right) + i \sin\left(\frac{3\pi}{4}\right) \right] \right\}^2 \\ &= 3^2 \left[\cos\left(2 \times \frac{3\pi}{4}\right) + i \sin\left(2 \times \frac{3\pi}{4}\right) \right] \\ &= 9 \left[\cos\left(\frac{3\pi}{2}\right) + i \sin\left(\frac{3\pi}{2}\right) \right] \\ &= 9 \left[\cos\left(-\frac{\pi}{2}\right) + i \sin\left(-\frac{\pi}{2}\right) \right] \\ &= -9 \end{aligned}$$

○ Example 6

Given $\omega = \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)$, sketch ω , ω^2 and ω^3 on the same axes.

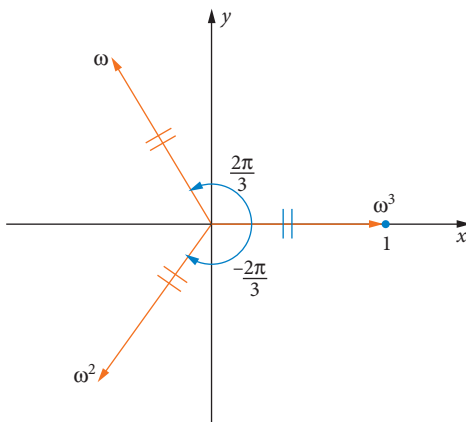
Solution

$$\arg(\omega) = \frac{2\pi}{3} \text{ and } |\omega| = 1.$$

Using De Moivre's theorem,

$$\begin{aligned}\omega^2 &= \left[\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right) \right]^2 \\ &= \cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right) \\ &= \cos\left(\frac{-2\pi}{3}\right) + i\sin\left(\frac{-2\pi}{3}\right)\end{aligned}$$

$$\begin{aligned}\omega^3 &= \left[\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right) \right]^3 \\ &= \cos\left(\frac{6\pi}{3}\right) + i\sin\left(\frac{6\pi}{3}\right) \\ &= \cos(2\pi) + i\sin(2\pi) \\ &= 1\end{aligned}$$



EXERCISE 4.02 Multiplication in the complex plane

Concepts and techniques

- 1 **Example 4** Find the resulting vector when each of the following is multiplied by the complex number i .

a $\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)$

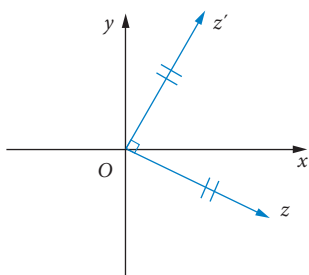
b $\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)$

c $2\left[\cos\left(\frac{-\pi}{6}\right) + i\sin\left(\frac{-\pi}{6}\right)\right]$

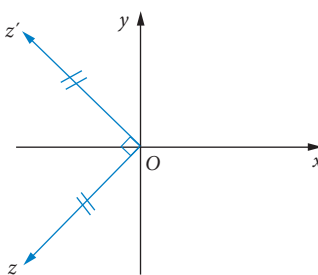
d $\sqrt{3}\left[\cos\left(\frac{-2\pi}{3}\right) + i\sin\left(\frac{-2\pi}{3}\right)\right]$

- 2 In each of the following, multiplication by which complex number takes z to z' ?

a

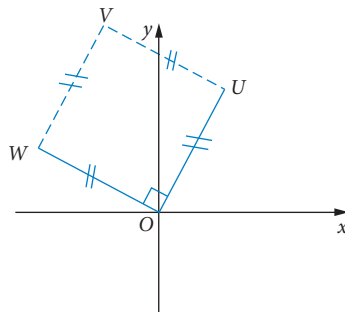


b



Multiplication in the plane

- 3 In the diagram, $UVWO$ is a square. The point U represents the complex number u .

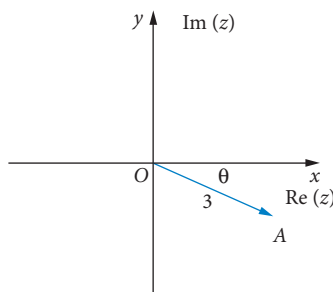


Find, in terms of u , the complex numbers represented by

a W

b V

- 4 **Example 5** The point A represents the complex number $z = 3 \operatorname{cis}(-\theta)$ on an Argand diagram.



Sketch on the same diagram:

a $-z$

b z^2

c iz

- 5 The complex numbers $z = \sqrt{2}[\cos(30^\circ) + i\sin(30^\circ)]$ and $w = \sqrt{3}[\cos(120^\circ) + i\sin(120^\circ)]$ are represented by the points Z and W on the complex plane. Sketch the point U representing the product zw .
- 6 Consider the parallelogram $OABC$, where A, B, C represent the complex numbers a, b, c . If $a = \cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)$, $OC = 2OA$ and $\angle COA = \frac{\pi}{3}$, find the complex numbers c and b .
- 7 **Example 6** Given that $z = \cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)$, sketch the following on an Argand diagram.

a z

b z^2

c z^3

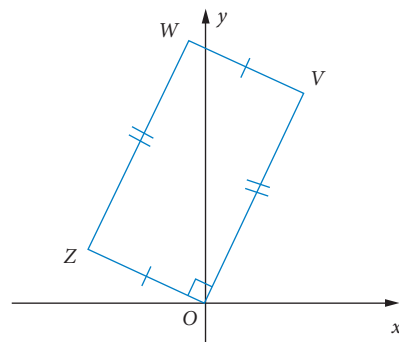
d z^4

e z^5

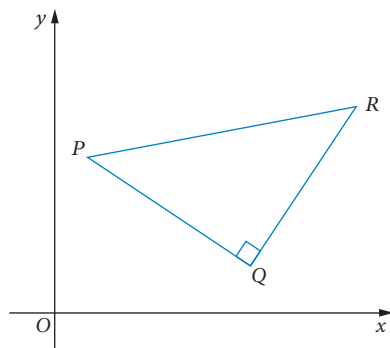
f z^6

Reasoning and communication

- 8 Consider the rectangle $OZ WV$. Given that Z represents the complex number z and $OV = 2OZ$,
- find the complex numbers represented by the points V and W in terms of z
 - find the complex number in terms of z represented by the point U , the intersection of OW and ZV .



- 9 The complex numbers w_1 , w_2 and w_3 correspond to the vertices A , B and C respectively of an arbitrary triangle.
- Draw a diagram and label the vectors $w_1 - w_2$ and $w_3 - w_2$.
 - If $w_1 - w_2 = i(w_3 - w_2)$, what type of triangle is $\triangle ABC$?
- 10 Consider the right-angled isosceles triangle PQR with corresponding complex numbers z_1, z_2, z_3 respectively. The complex number z_4 represented by the point S is defined by $z_4 = z_1 + i(z_2 - z_1)$.



- Plot the point S .
- Show that $z_4 - z_3 = z_1 - z_2$.
- What shape is $SPQR$?

4.03 DIVISION IN THE COMPLEX PLANE

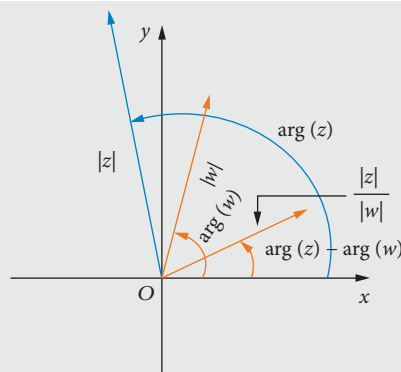
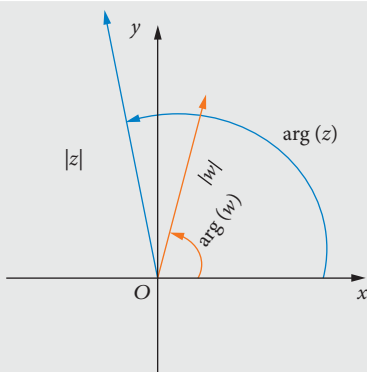
Geometric interpretation in the plane

To divide complex numbers, you divide the moduli and subtract the arguments.

IMPORTANT

If $z_1 = r_1 [\cos (\theta_1) + i \sin (\theta_1)]$ and $z_2 = r_2 [\cos (\theta_2) + i \sin (\theta_2)]$ are two complex numbers, then their **quotient** is $\frac{z_1}{z_2} = \frac{r_1}{r_2} [\cos (\theta_1 - \theta_2) + i \sin (\theta_1 - \theta_2)]$.

Geometrically, this means that the quotient of two numbers z and w have a length of $\frac{|z|}{|w|}$ and an argument equal to the difference $\arg (z) - \arg (w)$. Alternatively, if a number z is divided by a number w , then its modulus is decreased by the factor $|w|$ and its argument is rotated through the angle $\arg (w)$. If $\arg (z)$ is *positive*, the direction is *anticlockwise*. If it is negative, then the rotation will be clockwise.



If $z = |z|\{\cos [\arg (z)] + i \sin [\arg (z)]\}$ and $w = |w|\{\cos [\arg (w)] + i \sin [\arg (w)]\}$ are two complex numbers, then their **quotient** is

$$\frac{z}{w} = \frac{|z|}{|w|}\{\cos [\arg (z) - \arg (w)] + i \sin [\arg (z) - \arg (w)]\}$$

○ Example 7

If $z = 4[\cos (120^\circ) + i \sin (120^\circ)]$ and $w = 2[\cos (30^\circ) + i \sin (30^\circ)]$, sketch z and w , and show that $\frac{z}{w}$ is purely imaginary.

Solution

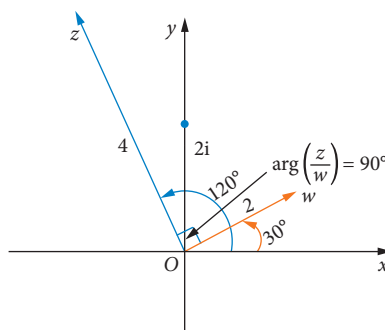
$$|z| = 4, |w| = 2$$

$$\arg (z) = 120^\circ, \arg (w) = 30^\circ$$

$$\operatorname{mod}\left(\frac{z}{w}\right) = \frac{|z|}{|w|} = \frac{4}{2} = 2$$

$$\arg \left(\frac{z}{w}\right) = 120^\circ - 30^\circ = 90^\circ$$

$\frac{z}{w} = 2[\cos (90^\circ) + i \sin (90^\circ)] = 2i$, which is purely imaginary, i.e., $2i$ lies on the y -axis.



○ Example 8

If $z = r \operatorname{cis}(\theta)$, sketch

- a $-iz$ b $\frac{z}{i}$

Solution

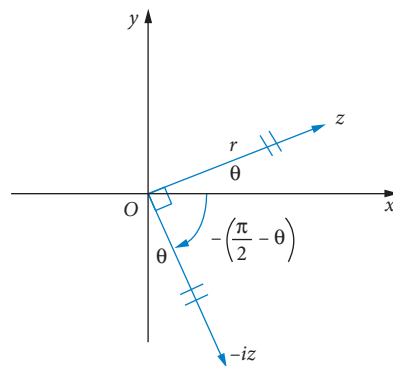
- a First express $-i$ in polar form.

Use the rule $zw = |z||w|\{\cos[\arg(z) + \arg(w)] + i \sin[\arg(z) + \arg(w)]\}$

$-iz$ is equivalent to a rotation of z through an angle of $\frac{-\pi}{2}$.

$$-i = \cos\left(\frac{-\pi}{2}\right) + i \sin\left(\frac{-\pi}{2}\right)$$

$$\begin{aligned} -iz &= \operatorname{cis}\left(\frac{-\pi}{2}\right) \times r \operatorname{cis}(\theta) \\ &= r \operatorname{cis}\left(\frac{-\pi}{2} + \theta\right) \\ &= r \operatorname{cis}\left[-\left(\frac{\pi}{2} - \theta\right)\right] \end{aligned}$$

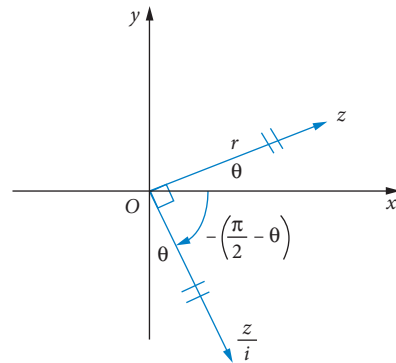


- b Express $\frac{z}{i}$ as a quotient of two complex numbers using the rule

$$\frac{z}{w} = \frac{|z|}{|w|} \{\cos[\arg(z) - \arg(w)] + i \sin[\arg(z) - \arg(w)]\}$$

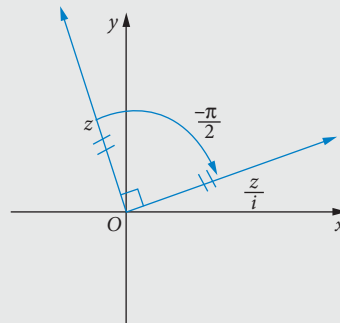
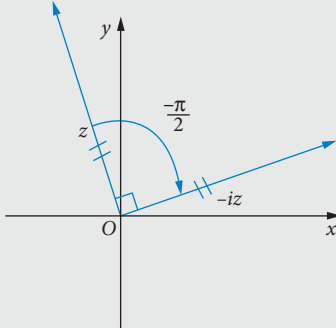
Note that both $-iz$ and $\frac{z}{i}$ rotate z through an angle of $\frac{-\pi}{2}$.

$$\begin{aligned} \frac{z}{i} &= \frac{r \operatorname{cis}(\theta)}{\operatorname{cis}\left(\frac{\pi}{2}\right)} \\ &= r \operatorname{cis}\left(\theta - \frac{\pi}{2}\right) \\ &= r \operatorname{cis}\left[-\left(\frac{\pi}{2} - \theta\right)\right] \\ &= -iz \end{aligned}$$



IMPORTANT

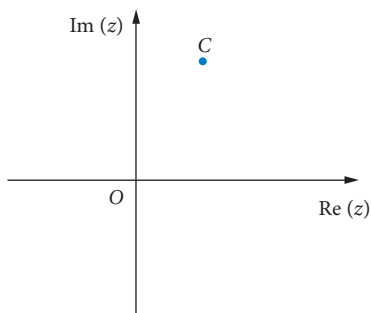
Multiplication by $-i$ or division by i are transformations equivalent to a rotation of $\frac{-\pi}{2}$ in the complex plane.



It is not difficult to see why this result is true: $\frac{z}{i} = \frac{z}{i} \times \frac{i}{i} = \frac{iz}{-1} = -iz$.

○ Example 9

If the point C represents the complex number c , plot the point R that represents $\frac{1}{c}$.

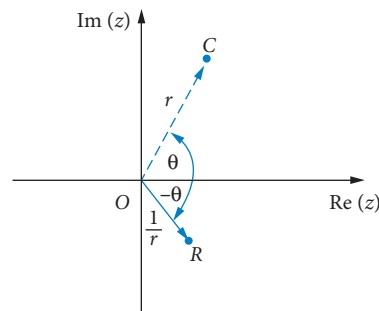


Solution

First express the complex number c as $c = r \operatorname{cis}(\theta)$.

$$\begin{aligned} \frac{1}{c} &= c^{-1} \\ &= [r \operatorname{cis}(\theta)]^{-1} \\ &= r^{-1} \operatorname{cis}(-\theta) \\ &= \frac{1}{r} \operatorname{cis}(-\theta) \end{aligned}$$

The modulus of $\frac{1}{c}$ is $\frac{1}{r}$ and the argument is $-\theta$.



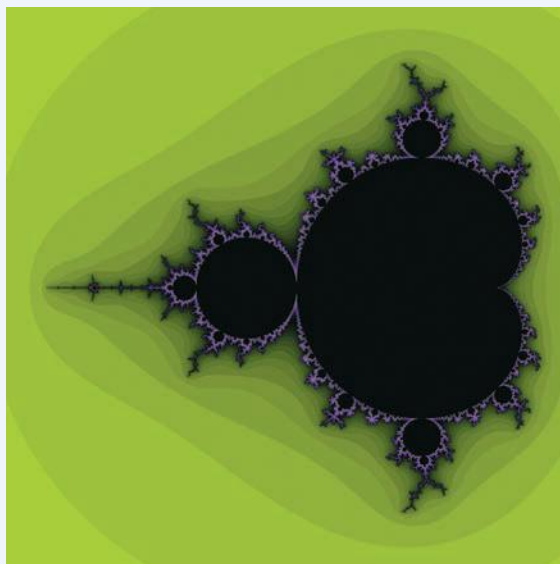
INVESTIGATION The development of chaos theory

It was not until the 1970s, with the advent of powerful computers that could do a multitude of calculations in a split second, that chaos theory was proposed. A mathematician, Benoit Mandelbrot, working for the large computer firm IBM, noticed that repeating a formula similar to Pythagoras' theorem using complex numbers generated images similar to those that occur in nature. He found that a simple equation could model ferns and trees, clouds and landforms, river systems and weather patterns. He began to wonder whether there is a pattern to observations that appear to be chaotic. Thus chaos theory was born.

Try to find out about each of the following.

- The Von Koch Curve
- The Mandelbrot set
- Fractals and their use.

There are some interesting properties of fractals. Find out what they are.



Shutterstock.com/Markus Gann



Generating the Mandelbrot set



EXERCISE 4.03 Division in the complex plane

Concepts and techniques



Division in the plane

- Example 7** For each pair of complex numbers z_1 and z_2 , find $\left| \frac{z_1}{z_2} \right|$ and $\arg\left(\frac{z_1}{z_2}\right)$. Hence sketch $\frac{z_1}{z_2}$ on an Argand diagram.
 - $z_1 = 2\left[\cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right)\right]$ and $z_2 = \sqrt{2}\left[\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right]$
 - $z_1 = 5\left[\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right]$ and $z_2 = \left[\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right]$
 - $z_1 = 9\left[\cos\left(\frac{-\pi}{4}\right) + i\sin\left(\frac{-\pi}{4}\right)\right]$ and $z_2 = 3\left[\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right]$
 - $z_1 = 4\left[\cos\left(\frac{-5\pi}{12}\right) + i\sin\left(\frac{-5\pi}{12}\right)\right]$ and $z_2 = 8\left[\cos\left(\frac{\pi}{12}\right) + i\sin\left(\frac{\pi}{12}\right)\right]$
- Find $\frac{z_1}{z_2}$ if $z_1 = 8\left[\cos\left(\frac{3\pi}{4}\right) - i\sin\left(\frac{3\pi}{4}\right)\right]$ and $z_2 = 2\sqrt{2}\left[\cos\left(\frac{\pi}{2}\right) - i\sin\left(\frac{\pi}{2}\right)\right]$. Sketch $\frac{z_1}{z_2}$.

- 3 By first expressing it in polar form, sketch $\frac{z_1 z_2}{z_3}$ on the Argand plane if

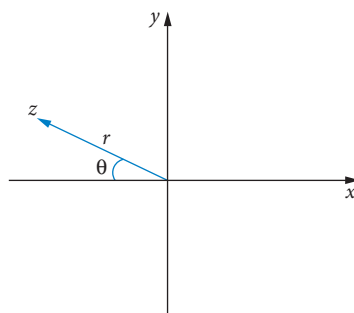
$$z_1 = 4 \left[\cos\left(\frac{-\pi}{3}\right) + i \sin\left(\frac{-\pi}{3}\right) \right], z_2 = 3 \left[\cos\left(\frac{\pi}{6}\right) - i \sin\left(\frac{\pi}{6}\right) \right] \text{ and } z_3 = 6 \left[\cos\left(\frac{2\pi}{3}\right) + i \sin\left(\frac{2\pi}{3}\right) \right].$$

- 4 **Example 8** Given the vector z representing the complex number z on the complex plane, sketch each of the following in relation to z .

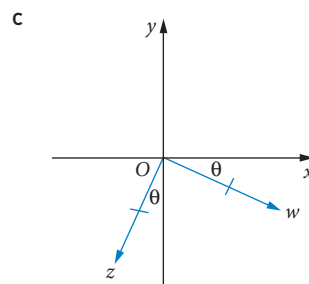
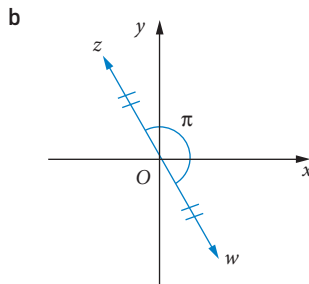
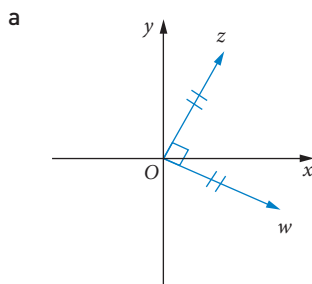
a iz b $\frac{z}{i}$ c $-iz$

d $\frac{z}{i^2}$ e $-z$

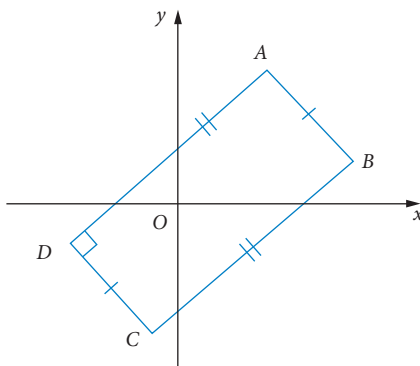
In each case, state which transformations of z are equivalent.



- 5 In each case, express the complex number w in terms of z .



- 6 Consider the complex numbers w_1, w_2, w_3, w_4 , which correspond to the vertices of a rectangle $ABCD$, where $AD = 2AB$. Explain why $\frac{w_2 - w_1}{i} = \frac{w_4 - w_1}{2}$.



- 7 **Example 9** Given each complex number z , sketch $\frac{1}{z}$.

a $z = 4 \operatorname{cis}\left(\frac{\pi}{3}\right)$

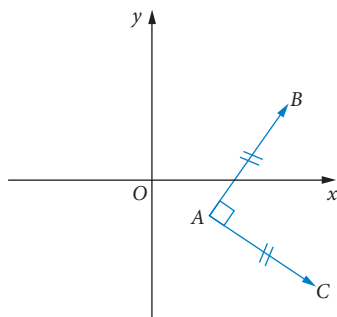
b $z = \sqrt{2} \operatorname{cis}\left(\frac{5\pi}{6}\right)$

c $z = \frac{1}{\sqrt{3}} \operatorname{cis}\left(\frac{-\pi}{4}\right)$

Reasoning and communication

- 8 Consider the points A, B, C representing the complex numbers z_1, z_2, z_3 on the complex plane.

Prove that $\frac{z_2 - z_1}{z_3 - z_1} = i$.



- 9 The arbitrary points L and M representing complex numbers w_2, w_3 together with O form a triangle in the complex plane.
- Plot the information and draw the point N representing $w_2 + w_3$.
 - Describe the quadrilateral $OLNM$.
 - Given that $|w_2 + w_3| = |w_2 - w_3|$, what can you deduce about $\frac{w_2}{w_3}$?
- 10 Prove that the complex numbers z_1, z_2 and $\frac{z_1 - iz_2}{1 - i}$ form a right-angled triangle.

4.04 COMPLEX NUMBER OPERATIONS AS TRANSFORMATIONS

Vector addition and subtraction can be used to translate plane shapes, whereas multiplication and division can be used to rotate plane shapes.

IMPORTANT

For two non-zero complex numbers z_1 and z_2 ,

$z_1 = z_2$ if and only if $|z_1| = |z_2|$ AND $\arg(z_1) = \arg(z_2)$

○ Example 10

Consider the complex numbers z_1, z_2, z_3, z_4 represented by the points A, B, C, D . What can you deduce from the following? Draw a diagram.

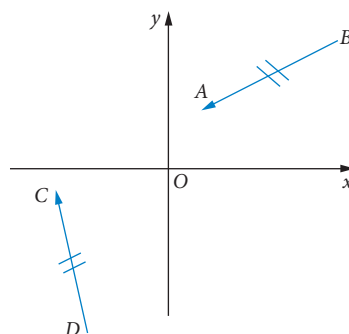
a $|z_1 - z_2| = |z_3 - z_4|$

b $\arg(z_1 - z_2) = \arg(z_3 - z_4)$

c $z_1 - z_2 = z_3 - z_4$

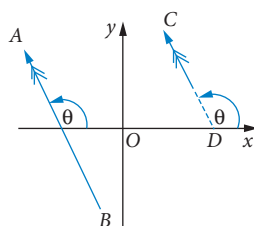
Solution

- a $|z_1 - z_2| = |z_3 - z_4|$ means that the lengths BA and DC are equal.



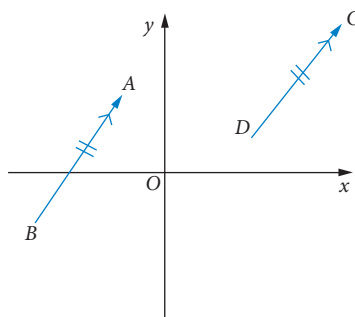
$$AB = CD.$$

- b $\arg(z_1 - z_2) = \arg(z_3 - z_4)$ means that the vectors $z_1 - z_2$ and $z_3 - z_4$ are running in the same direction.

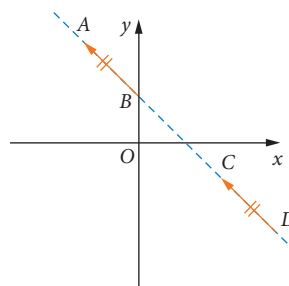


$$AB \parallel CD$$

- c $z_1 - z_2 = z_3 - z_4$ means that the vectors $z_1 - z_2$ and $z_3 - z_4$ are the same length and have the same argument.

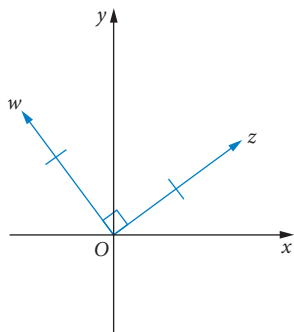


Either $ABDC$ forms a parallelogram or A, B, C, D are collinear and $AB = CD$.



○ Example 11

Consider the complex numbers with corresponding vectors sketched below.



Prove that $z^2 + w^2 = 0$.

Solution

Since the angle between them is 90° , $w = iz$.

$$w = iz.$$

Squaring both sides,

$$\begin{aligned} w^2 &= (iz)^2 \\ &= i^2 z^2 \\ &= -z^2 \end{aligned}$$

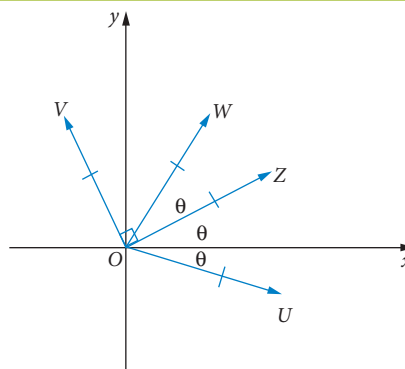
Rearrange to get the desired result.

$$\therefore z^2 + w^2 = 0.$$

○ Example 12

Consider the diagram with complex numbers z, w, u, v represented by points Z, W, U, V respectively.

Express w, u, v in terms of z .



Solution

From the diagram,

$$|z| = |w| = |u| = |v|.$$

Also, $\arg(z) = \theta$, whereas $\arg(w) = 2\theta$.

$$|z| = |w|$$

$$\arg(w) = \arg(z) + \arg(z)$$

$$\therefore w = |z| \operatorname{cis}(\theta + \theta)$$

$$= |z| \operatorname{cis}(\theta) \operatorname{cis}(\theta)$$

$$= z \operatorname{cis}(\theta)$$

$$= z \operatorname{cis}[\arg(z)]$$

U is the reflection of Z over the real axis.

$$u = \bar{z}$$

V is the result of rotating Z through 90° .

$$v = iz$$

EXERCISE 4.04 Complex number operations as transformations

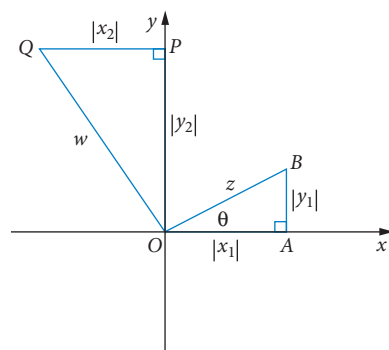
Concepts and techniques

- 1 **Example 10** Draw sketches to illustrate the following relationships for complex numbers. In each case, give a description.

a $\arg(z) = \arg(u)$ b $|z| = |w|$ c $\arg(z) = -\arg(u)$ d $\arg(z - w) = \arg(u - v)$
 e $|z - u| = |w - v|$ f $z - w = u - v$ g $\arg(z) - \arg(u) = 0$ h $\arg(z) + \arg(u) = 0$
 i $|z + u| = |z - u|$ j $z + w = u + v$ k $z - w = i(z + w)$

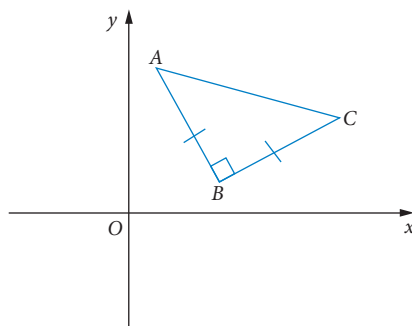
- 2 If, for a complex number z , $\arg(z) = \theta$, find $\arg(-z)$ in terms of θ .

- 3 Consider the similar triangles OAB and OPQ in the diagram on the right.
 B and Q represent the complex numbers z and w respectively. Let $\arg(z) = \theta$.
 Show that $w|y_1| = iz|x_2|$.



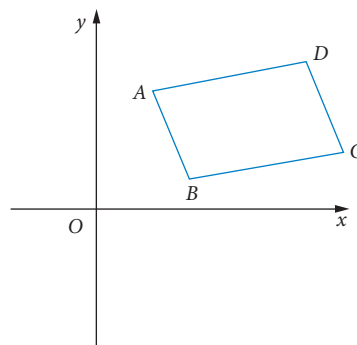
- 4 **Example 11** In the Argand diagram, ABC is a right-angled isosceles triangle.
 If A, B, C represent the complex numbers w_1, w_2, w_3 respectively, prove that

a $|w_1 - w_2| = |w_3 - w_2|$
 b $\arg(w_1 - w_2) - \arg(w_3 - w_2) = \frac{\pi}{2}$
 c $(w_1 - w_2)^2 + (w_3 - w_2)^2 = 0$
 d $|w_1 - w_2|^2 + |w_3 - w_2|^2 = |w_3 - w_1|^2$

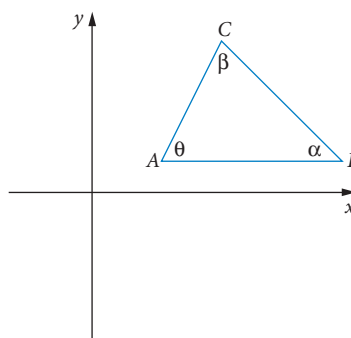


- 5 Consider the parallelogram $ABCD$ representing the complex numbers z_1, z_2, z_3, z_4 respectively.
 Plot the point:

- a E corresponding to the complex number $z_3 - z_4$
 b F corresponding to the complex number $z_1 - z_3$
 c G corresponding to the complex number $-i(z_2 - z_1)$



- $$\begin{aligned} \text{a } & \arg(z_3 - z_1) = \theta \\ \text{b } & \arg(z_3 - z_2) = \pi - \alpha \\ \text{c } & \arg\left(\frac{z_3 - z_2}{z_3 - z_1}\right) = \beta \end{aligned}$$



-

a $\arg(w) = \theta - \alpha$ **b** $\arg(w) = \theta + \alpha$
Draw a diagram in each case.

-
- The diagram shows a Cartesian coordinate system with x and y axes. Five points, labeled $z_1, z_2, z_3, z_4,$ and z_5 , are plotted in the first, second, third, and fourth quadrants respectively. These points are connected by straight lines to form a pentagon. The origin is labeled O . The angle between the positive x-axis and the line segment Oz_1 is labeled θ . Tick marks are placed on each of the line segments $Oz_1, Oz_2, Oz_3, Oz_4,$ and Oz_5 , indicating that all these segments have the same length. The point z_5 is located on the positive x-axis.

- Find the value of θ .
- Write down the complex numbers z_1, z_2, z_3, z_4, z_5 .
- Show that $z_1^2 = z_2$
- Show that $z_2^2 = z_4$
- Show that $z_3^2 = z_1$
- Show that $\bar{z}_1 = z_4$

4.05 GRAPHING SUBSETS OF THE COMPLEX PLANE: SUBSETS INVOLVING MODULUS

Subsets of the complex plane are sets of points representing complex numbers. These are described by a certain rule or condition placed on the variable complex number z . You can use a geometric approach or an algebraic approach to answering these questions. The set of points is often called the **locus** of z .

Example 13

Sketch the locus of z if

a $|z + 2i| = 4$

b $|z - 3| = 1$

c $1 \leq |z| < 9$

Solution

a Geometric approach

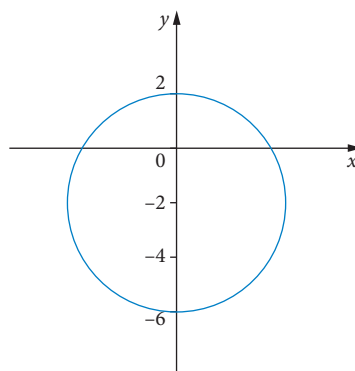
Write $|z + 2i| = |z - (0 - 2i)|$, which means the distance of z from the point $(0, -2)$.

If $|z - (0 - 2i)| = 4$, then z is the set of points 4 units from $(0, -2)$, or the circle with radius 4 and centre $(0, -2)$.

Algebraic approach

Let $z = (x, y)$, then $|z + 2i| = 4$ becomes $|x + yi + 2i| = 4$. Regrouping and using the definition of modulus,

$$\begin{aligned} |x + (y + 2)i| &= 4 \\ \sqrt{x^2 + (y + 2)^2} &= 4 \\ x^2 + (y + 2)^2 &= 16 \end{aligned}$$



The locus of z is $x^2 + (y + 2)^2 = 16$.

b Geometric approach

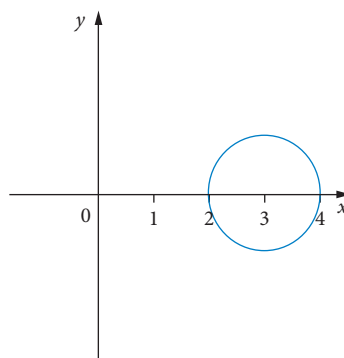
$|z - 3| = 1$ means that the distance of z from the point $(3, 0)$ is 1 unit.

The set of points on a circle of centre $(3, 0)$ and radius 1.

Algebraic approach

$|z - 3| = 1$ becomes $|(x - 3) + yi| = 1$.

So $(x - 3)^2 + y^2 = 1$ is the circle with radius 1 and centre $(3, 0)$.



The locus of z is $(x - 3)^2 + y^2 = 1$.

c *Geometric approach*

$1 \leq |z| < 9$ means the distance of z from the origin is at least 1, but less than 9 units. The set of points between the circles with centre $(0, 0)$ and radii 1 and 9, including the circle of radius 1, but not that of radius 9.

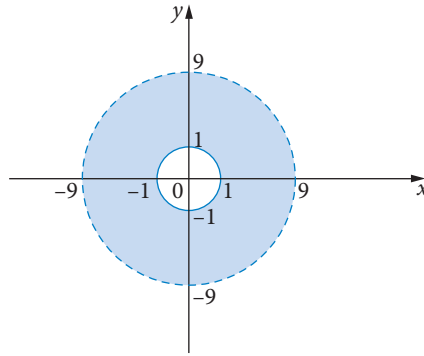
Algebraic approach

$1 \leq |z| < 9$ becomes

$$1 \leq \sqrt{x^2 + y^2} < 9$$

$$1 \leq x^2 + y^2 < 81$$

So $1 \leq x^2 + y^2 < 81$ is the region between the two circles with centre $(0, 0)$ and radii 1 and 9, including the circle of radius 1, but not that of radius 9.



The locus of z is the area between the two concentric circles. The inequality can be written as $1 \leq x^2 + y^2 < 81$.

Note the dotted circle since $|z| \neq 9$.

Some cases are more suited to the algebraic approach and other cases are better solved using the geometric approach.

○ Example 14

Sketch z if

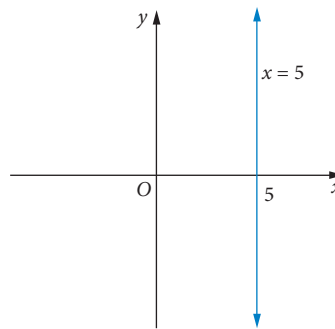
a $\operatorname{Re}(z) = 5$

b $\operatorname{Re}(z) = \operatorname{Im}(z)$

c $|z - 2 - i| = \operatorname{Re}(z)$

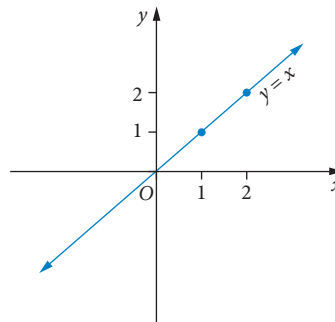
Solution

- a If $z = x + yi$, then $\operatorname{Re}(z) = x$, so if $\operatorname{Re}(z) = 5$, then the locus is the vertical line $x = 5$.



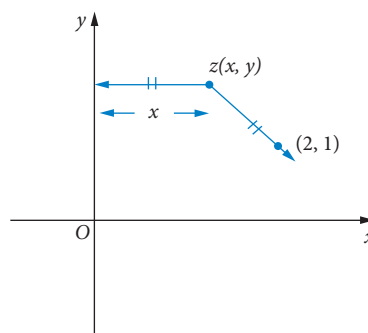
$x = 5$.

- b If $\operatorname{Re}(z) = \operatorname{Im}(z)$, replace with $x = y$. Therefore the locus is the line $y = x$.



$y = x$.

- c Geometrically, $|z - 2 - i| = \text{Re}(z)$ means that the distance of z from the point $(2, 1)$ is the same as the distance of z from the y -axis. Using the locus definition of a parabola, $(2, 1)$ is the focus and the y -axis is the directrix. This means that $(1, 1)$ is the vertex.

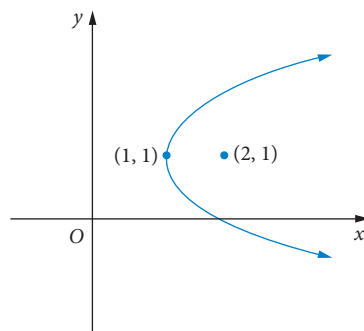


Algebraically, $\sqrt{(x-2)^2 + (y-1)^2} = x$.

Squaring both sides and simplifying,

$$\begin{aligned}(x-2)^2 + (y-1)^2 &= x^2 \\ x^2 - 4x + 4 + (y-1)^2 &= x^2 \\ (y-1)^2 &= 4x - 4\end{aligned}$$

So $(y-1)^2 = 4(x-1)$, which is a parabola with vertex $(1, 1)$ and focal length 1.



The locus of z is $(y-1)^2 = 4(x-1)$.

Example 15

Sketch z if

a $-2 < \text{Im}(z) \leq 3$

b $|z - 2| = |z|$

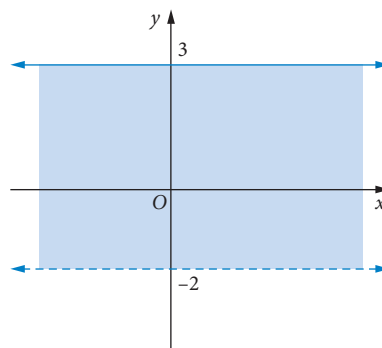
c $|z| \geq 2$ and $-\frac{\pi}{3} \leq \arg(z) < \frac{\pi}{3}$

Solution

- a $\text{Im}(z)$ represents the y -value on the complex plane.

$-2 < \text{Im}(z) \leq 3$ can be written as $-2 < y \leq 3$, which represents a region on the complex plane.

Note that the line $y = -2$ is dotted as it is not included in the inequality.

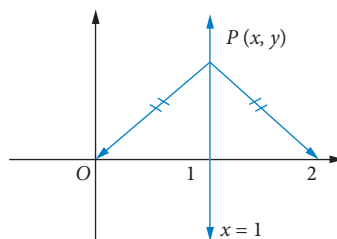


$-2 < \text{Im}(z) \leq 3$

- b $|z - 2| = |z|$ can be interpreted geometrically as the set of points equidistant from the point $(2, 0)$ and the point $(0, 0)$, or the perpendicular bisector of the interval joining those points. The equation is $x = 1$.

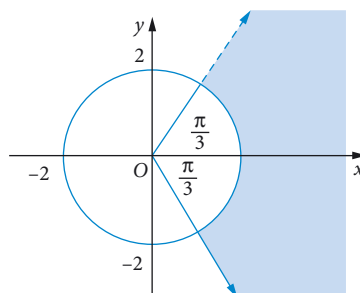
Algebraically, let $z = x + yi$ and

$$\begin{aligned}|x - 2 + yi| &= |x + yi| \\(x - 2)^2 + y^2 &= x^2 + y^2 \\4x - 4 &= 0 \\x &= 1\end{aligned}$$



z is the set of points on the vertical line $x = 1$.

- c $|z| \geq 2$ and $-\frac{\pi}{3} \leq \arg(z) < \frac{\pi}{3}$ is the intersection of points satisfying $|z| \geq 2$ and $-\frac{\pi}{3} \leq \arg(z) < \frac{\pi}{3}$. $|z| \geq 2$ is the set of points on or outside the circle $|z| = 2$. $-\frac{\pi}{3} \leq \arg(z) < \frac{\pi}{3}$ is the set of points with arguments from $-\frac{\pi}{3}$ to $\frac{\pi}{3}$, not including $\frac{\pi}{3}$.



$|z| \geq 2$ and $-\frac{\pi}{3} \leq \arg(z) < \frac{\pi}{3}$.

EXERCISE 4.05 Graphing subsets of the complex plane: subsets involving modulus

Concepts and techniques

- Example 13** Sketch the circle defined by each equation below.

a $ z = 2$	b $ z = 6$	c $ z = 2\sqrt{2}$	d $ z = \frac{1}{4}$
e $ z - 1 = 4$	f $ z + 2 = 1$	g $ z - 3i = 3$	h $ z + i = \frac{1}{2}$
- Sketch each of the following.

a $ z - (1 + i) = 1$	b $ z - (2 - 3i) = 2$	c $ z - 1 - 2i = \frac{1}{2}$	d $ z + 3 - 4i = 2$
- Sketch each region defined by the inequalities below.

a $ z \leq 3$	b $ z > 1$	c $\frac{1}{4} < z \leq 4$	d $1 \leq z - 1 < 3$
- Example 14** Sketch z for each of the following.

a $\operatorname{Re}(z) = 2$	b $\operatorname{Im}(z) = 3$	c $\operatorname{Re}(z) = -1$	d $\operatorname{Im}(z) = -2$
- Using the fact that $z = x + yi$, where x and y are real, find the Cartesian equation of each locus below and hence sketch

a $\operatorname{Re}(z) = -\operatorname{Im}(z)$	b $\operatorname{Im}(z) = 2 \operatorname{Re}(z)$	c $\operatorname{Re}(z) = 3 \operatorname{Im}(z) - 1$	d $\operatorname{Im}(z) + 4 \operatorname{Re}(z) = 12$
- Sketch the parabola defined by each equation below.

a $\operatorname{Re}(z) = z - 2 $	b $\operatorname{Im}(z) = z - 4i $	c $ z - 2 - 2i = \operatorname{Re}(z)$	d $ z - 4 + 6i = \operatorname{Im}(z)$
- Example 15** Sketch z defined by each inequality below.

a $\operatorname{Re}(z) > 4$	b $\operatorname{Im}(z) < 3$	c $-1 < \operatorname{Re}(z) \leq 1$	d $\operatorname{Im}(z) < -2$ and $\operatorname{Re}(z) \leq 4$

8 Sketch each inequality below.

a $|\operatorname{Re}(z)| \leq 1$

b $|\operatorname{Im}(z)| < 3$

c $|\operatorname{Re}(z)| \geq \frac{1}{2}$

d $|\operatorname{Re}(z)| > 2$ and $|\operatorname{Im}(z)| \leq 1$

9 Describe each of the loci below and hence sketch:

a $|z| = |z - 2i|$

b $|z - 3| = |z + 3|$

c $|z - 1 - i| = |z + 1 + i|$

d $|z - 3 + i| = |z + 5 - 3i|$

10 Sketch each of the following regions.

a $|z| \leq 2$ and $0 < \arg(z) < \frac{\pi}{2}$

b $|z - 1| \leq 1$ and $0 < \arg(z) < \frac{\pi}{4}$

c $|z - 2i| \geq 1$ and $\frac{\pi}{3} < \arg(z) < \frac{2\pi}{3}$

d $1 \leq |z| < 4$ and $-\frac{\pi}{4} < \arg(z) \leq \frac{\pi}{4}$

Reasoning and communication

11 Sketch the following.

a $\left| \frac{z+2}{z-4} \right| = 1$

b $\left| \frac{z+i}{z-i} \right| \leq 1$

c $\left| \frac{z+3}{2z} \right| > 1$

12 Show that the equation $|z - 3| = 2|z + 1|$ describes a circle. Find its centre and radius.

13 Sketch $|z - 3| + |z + 3| = 8$.

14 Find the maximum value of $\arg(z + 1)$ if $|z - 1| = 1$. Hint: draw and label a diagram.

4.06 GRAPHING SUBSETS OF THE COMPLEX PLANE: SUBSETS NOT INVOLVING MODULUS

Subsets not involving the modulus often involve the argument instead. Many of these sets are best approached geometrically. The beginning of a ray is not included as the argument for $|z| = 0$ is not defined.

Example 16

Let z be a variable complex number in the complex plane. Sketch the locus of z if

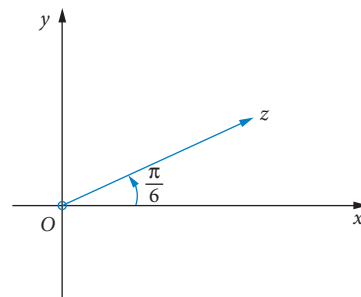
a $\arg(z) = \frac{\pi}{6}$

b $\arg(z - 1) = \frac{5\pi}{6}$

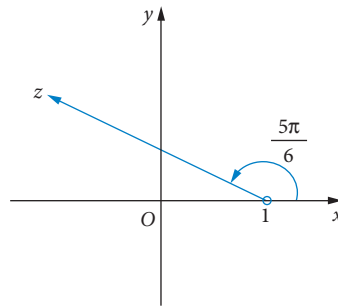
c $\arg[z - (2 + 3i)] = \frac{-\pi}{4}$

Solution

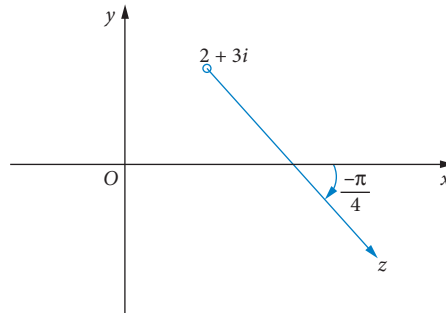
- a $\arg(z)$ is defined as the angle between the positive x -axis and the vector z , with its tail at the origin. In this case, z can be anywhere along the ray inclined at an angle of $\frac{\pi}{6}$ (going anticlockwise). Note: $z \neq 0$ as the argument of $z = 0$ is not defined.



- b $(z - 1)$ or $z - (1 + 0i)$ is the vector with its tail at $(1, 0)$. z can be anywhere along the ray inclined at $\frac{5\pi}{6}$.



- c $z - (2 + 3i)$ is the vector with its tail at $(2 + 3i)$. z can be anywhere along the ray inclined at $-\frac{\pi}{4}$.



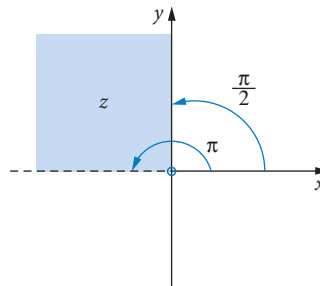
○ Example 17

Sketch the following subsets for the variable complex number z .

- a $\frac{\pi}{2} \leq \arg(z) < \pi$ b $\arg(z) = \arg(1 - i\sqrt{3})$ c $\arg(z) + \arg(\sqrt{2} + i\sqrt{2}) = 0$

Solution

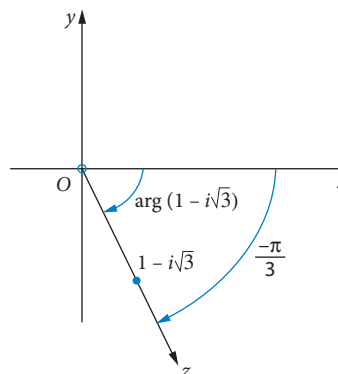
- a z can be any point such that the angle between the vector z and the positive x -axis is at least $\frac{\pi}{2}$ but less than π . Note that the x -axis is dotted since $\arg(z) \neq \pi$.



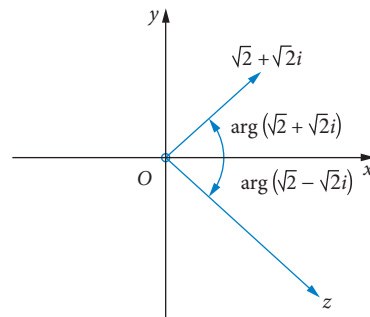
- b The vector z and the vector to $1 - i\sqrt{3}$ must be running in the same direction.

$$\arg(1 - i\sqrt{3}) = \frac{-\pi}{3}$$

The locus of z is the ray from the origin with $\arg(z) = \frac{-\pi}{3}$.



- c $\arg(z) + \arg(\sqrt{2} + i\sqrt{2}) = 0$ can be rearranged to give $\arg(z) = -\arg(\sqrt{2} + i\sqrt{2})$. The vector z lies along the ray on which the complex conjugate of $(\sqrt{2} + i\sqrt{2})$, i.e. $(\sqrt{2} - i\sqrt{2})$, lies.



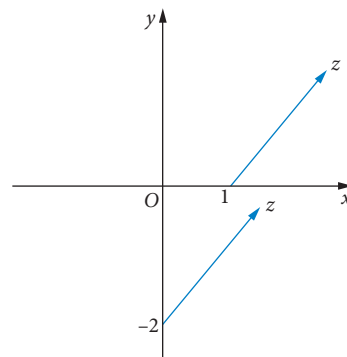
○ Example 18

Sketch the locus of z if

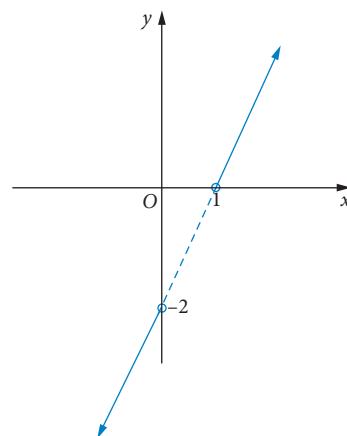
- a $\arg(z - 1) = \arg(z + 2i)$
 b $\arg(z - 2) - \arg(z - i) = \pi$
 c $\arg(z - 1) - \arg(z + 3) = \frac{\pi}{2}$

Solution

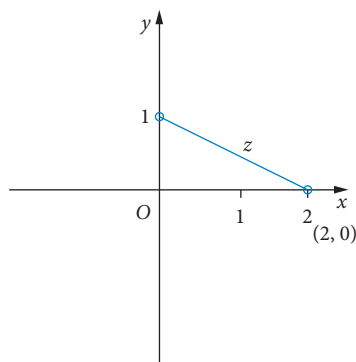
- a The vector $z - 1 = z - (1 + 0i)$ and the vector $z + 2i = z - (0 - 2i)$. z must be located so that the vectors are parallel (on the same line), going in the same direction.



z can be anywhere on the two rays on either side of the interval between $(1, 0)$ and $(0, -2)$. The axis points are not included.

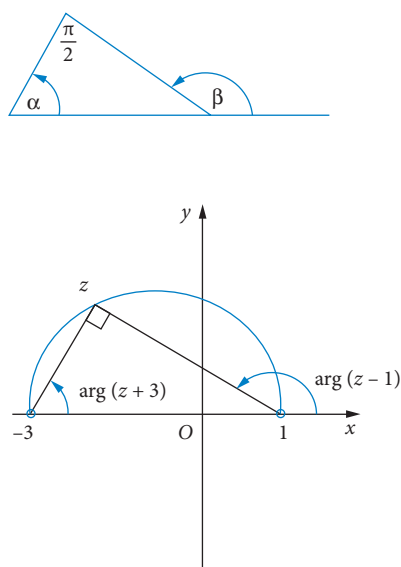


- b In this case, the vectors $(z - 2)$ and $(z - i)$ must be running in opposite directions. z is the set of points on the line interval joining $(2, 0)$ and $(0, 1)$, excluding the points on the axes.



- c In this case, it is convenient to use the exterior angle theorem and the angle in a semicircle theorem as the difference between the arguments is a constant angle, i.e., $\beta - \alpha = \frac{\pi}{2}$.

So z is the set of points that lie on the semicircle with its diameter between the points $(1, 0)$ and $(-3, 0)$.



Since the argument of 0 is not defined, the terminating points of the subsets shown in this section are not included in the subsets. This can be emphasised as shown in Example 18 with an open circle at these points, although this is sometimes omitted.



EXERCISE 4.06 Graphing subsets of the complex plane: subsets not involving modulus

Concepts and techniques

- 1 **Example 16** Sketch the following subsets of the complex plane.

a $\arg(z) = \frac{\pi}{3}$ b $\arg(z) = \frac{3\pi}{4}$ c $\arg(z) = \frac{-2\pi}{3}$ d $\arg(z) = \pi$

- 2 Sketch z if

a $\arg(z-1) = \frac{2\pi}{3}$ b $\arg(z-i) = \frac{\pi}{6}$ c $\arg(z-(1-i)) = \frac{5\pi}{6}$ d $\arg(z-2-3i) = \frac{\pi}{4}$

- 3 **Example 17** Sketch the following regions.

a $\frac{-\pi}{4} \leq \arg(z) \leq \frac{\pi}{4}$ b $\frac{\pi}{6} < \arg(z) < \frac{2\pi}{3}$ c $-\pi < \arg(z) < \frac{\pi}{2}$

- 4 Sketch each subset.

a $\arg(z) = \arg(1+i)$ b $\arg(z) = \arg(\sqrt{3}+i)$
 c $\arg(z-1) = \arg(\sqrt{2}-i\sqrt{2})$ d $\arg(z+2i) = \arg(-1+i\sqrt{3})$

- 5 Sketch z if

a $\arg(z) - \arg(-1-i) = 0$ b $\arg(z) - \arg(1+i\sqrt{3}) = 0$
 c $\arg(z) + \arg\left(\frac{1}{2} - \frac{i\sqrt{3}}{2}\right) = 0$ d $\arg(z) + \arg(-\sqrt{3}-i) = 0$

- 6 Sketch the following.

a $\arg(z-1) = \arg(-\sqrt{2}+i\sqrt{2})$ b $\arg(z-i) = \arg(1+i\sqrt{3})$

- 7 **Example 18** Find the locus of z if:

a $\arg(z) = \arg(z-3)$ b $\arg(z) = \arg(z-2i)$
 c $\arg(z+1) - \arg(z) = 0$ d $\arg(z+i) - \arg(z-1) = 0$

- 8 Sketch each of the following.

a $\arg(z-1) - \arg(z) = \pi$ b $\arg(z-2) - \arg(z+2) = \frac{\pi}{2}$
 c $\arg(z-i) - \arg(z+3) = \frac{\pi}{2}$ d $\arg(z+i) - \arg(z-1) = \pi$

Reasoning and communication

- 9 Using the theorem $\arg\left(\frac{z}{w}\right) = \arg(z) - \arg(w)$, sketch each of the following.

a $\arg\left(\frac{z}{z-2}\right) = 0$ b $\arg\left(\frac{z+1}{z-1}\right) = \frac{\pi}{2}$
 c $\arg\left(\frac{z-1}{z-i}\right) = \pi$ d $\arg\left(\frac{z-2i}{z+2i}\right) = \frac{\pi}{2}$

- 10 Describe each subset of the Argand plane defined by the equations below. Draw a diagram.

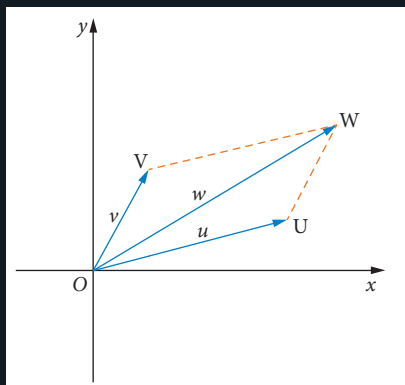
a $\arg\left(\frac{z-1}{z+i}\right) = 0$ b $\arg\left(\frac{z-2}{z+2}\right) = \frac{\pi}{4}$
 c $\arg\left(\frac{z+1}{z-1}\right) = \frac{\pi}{3}$ d $\arg\left(\frac{z-2i}{z+2}\right) = \frac{\pi}{2}$

CHAPTER SUMMARY

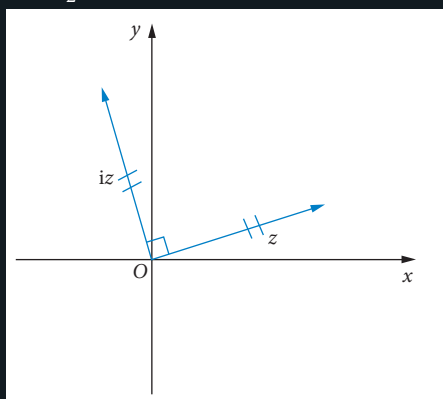
THE COMPLEX PLANE

4

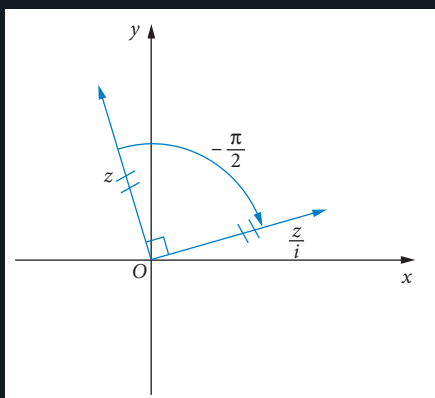
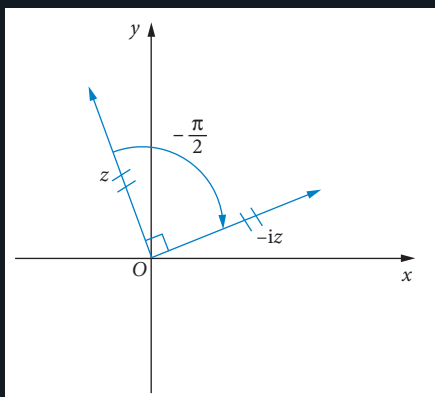
- The parallelogram rule for vectors: $\mathbf{w} = \mathbf{u} + \mathbf{v}$



- If $z_1 = r_1[\cos(\theta_1) + i \sin(\theta_1)]$ and $z_2 = r_2[\cos(\theta_2) + i \sin(\theta_2)]$ are two complex numbers, then their **product** is $z_1 z_2 = r_1 r_2 [\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)]$.
- Multiplication by i is equivalent to a rotation of $\frac{\pi}{2}$ in the complex plane.



- Multiplication by $-i$ or division by i are transformations equivalent to a rotation of $-\frac{\pi}{2}$ in the complex plane.



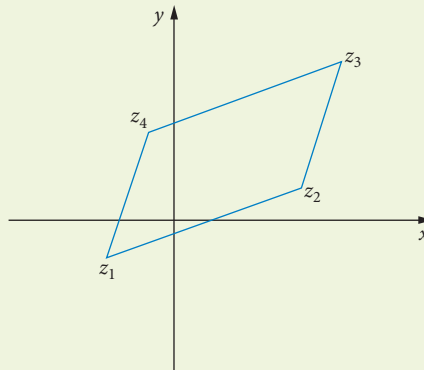
- If $z_1 = r_1[\cos(\theta_1) + i \sin(\theta_1)]$ and $z_2 = r_2[\cos(\theta_2) + i \sin(\theta_2)]$ are two complex numbers, then their **quotient** is $\frac{z_1}{z_2} = \frac{r_1}{r_2} [\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)]$.
- For two non-zero complex numbers z_1 and z_2 , $z_1 = z_2$ if and only if $|z_1| = |z_2|$ AND $\arg(z_1) = \arg(z_2)$.
- Subsets of the complex plane are sets of points representing complex numbers that are described by a certain rule or condition placed on the variable complex number z . The set of points is often called the **locus** of z . There are two approaches to finding the locus of z : *algebraic* and *geometric*.

CHAPTER REVIEW

THE COMPLEX PLANE

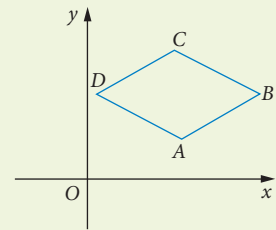
Multiple choice

- 1 **Example 1** In the diagram, $PQRS$ is a parallelogram whose vertices represent the complex numbers z_1, z_2, z_3, z_4 .

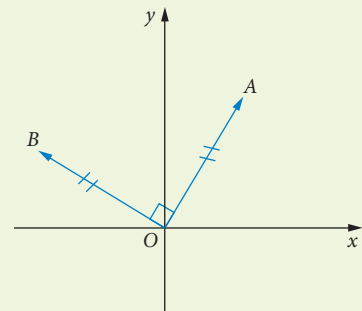


Which statement is true?

- A** $z_3 = z_4 + z_2$ **B** $z_3 - z_4 = z_2 - z_1$
C $|z_4 - z_1| = |z_3 - z_4|$ **D** $z_3 - z_1 = i(z_2 - z_4)$
E $\arg(z_3) = \arg(z_1)$
- 2 **Example 2** In the diagram, $ABCD$ is a rhombus. The points A, B, C, D represent complex numbers a, b, c, d . The vertex C represents the complex number:
- A** $d - a + b - a$ **B** $a + b - d$ **C** $d - a + b$
D $c - d + c - b$ **E** $d - a + c - b$

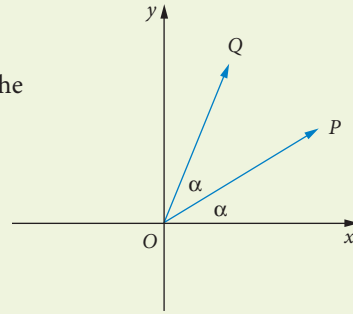


- 3 **Example 3** The points O, A, B, C form a parallelogram $OABC$, where A represents $-1 + 2i$ and B represents $1 + 5i$. The point C is represented by
- A** $7i$ **B** $-2 - 3i$ **C** $3i$ **D** $2 + 3i$ **E** $-2 + 3i$
- 4 **Example 4** Consider the two points A and B representing the complex numbers z and w respectively, where $\angle AOB = 90^\circ$. Which of the following is true?
- A** $z = iw$ **B** $z = -iw$ **C** $z = w^2$
D $z = \frac{\pi}{2}w$ **E** $z = i^2w$



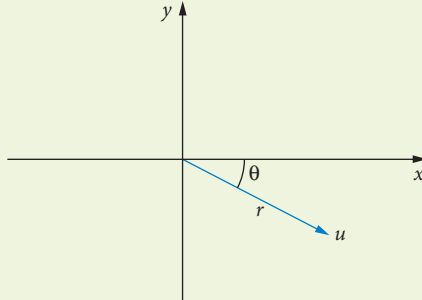
- 5 **Example 5** Consider the two points P and Q representing the complex numbers z and w respectively, where $\arg(z) = \alpha$, $\arg(w) = 2\alpha$ and $|z| = |w| = 1$. Which of the following is true?

- A $z = w^2$ B $\arg(w - z) = \alpha$
 C $\arg(w + z) = 3\alpha$ D $\arg\left(\frac{z}{w}\right) = \alpha$
 E $w = z^2$

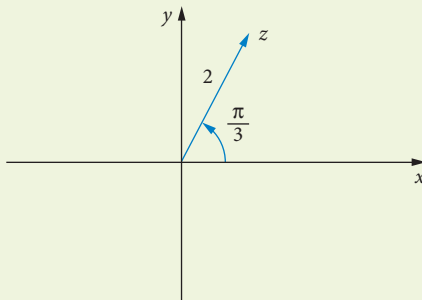


Short answer

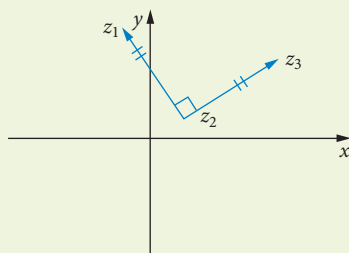
- 6 **Example 6** If $z = \cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)$, sketch z^2 on an Argand plane.
- 7 **Example 7** Consider the complex numbers $z = 2\left[\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right]$ and $w = 4\left[\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right]$.
- Explain why zw is purely real.
 - Explain why $\frac{z}{w}$ is purely imaginary.
 - Sketch zw and $\frac{z}{w}$ on an Argand diagram.
- 8 **Example 8** The complex number u is sketched below. Sketch $\frac{u}{i}$ on the same diagram.



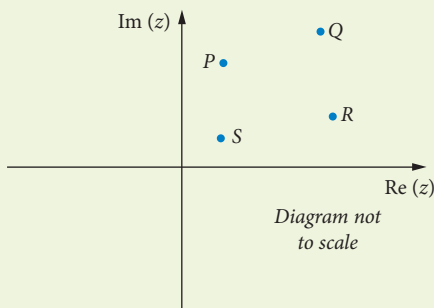
- 9 **Example 9** The complex number $z = 2\left[\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right]$ is sketched below.
- State $\text{mod}\left(\frac{1}{z}\right)$ and $\arg\left(\frac{1}{z}\right)$.
 - Sketch z and $\frac{1}{z}$ on the same diagram.



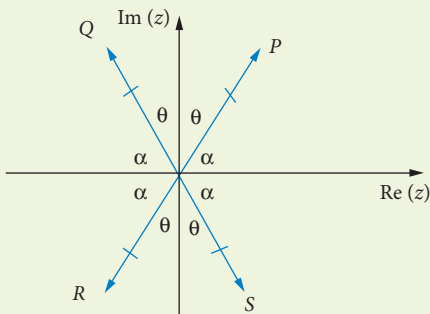
- 10 **Example 10** Consider the complex numbers z_1, z_2, z_3 . Prove that $(z_3 - z_2)^2 + (z_1 - z_2)^2 = 0$.



- 11 **Example 11** The points P, Q, R, S representing the complex numbers z_1, z_2, z_3, z_4 form a quadrilateral. State what type of quadrilateral $PQRS$ is if
- $|z_3 - z_1| = |z_2 - z_4|$
 - $z_3 - z_2 = z_4 - z_1$
 - $\arg(z_3 - z_2) = \arg(z_4 - z_1)$



- 12 **Example 12** Consider the points P, Q, R, S representing the complex numbers z_1, z_2, z_3, z_4 respectively. Express z_2, z_3, z_4 in terms of z_1 .



- 13 **Example 13** Sketch the locus of z and write the Cartesian equation or inequality for each of the following.

a $|z| = 3$

b $|z + 1| \leq 4$

c $\frac{1}{2} < |z - 1 - i| \leq 1$

- 14 **Example 16** Sketch the locus of z on an Argand plane for each case below.

a $\arg(z) = \frac{2\pi}{3}$

b $\arg(z - i) = -\frac{\pi}{4}$

c $\arg(z + 1 + i) = \frac{3\pi}{4}$

15 **Example 17** Sketch each subset of the complex plane for z defined by:

a $-\frac{\pi}{3} \leq \arg(z) < \frac{\pi}{3}$

b $\arg(z) = \arg(1 + i)$

c $\arg(z) = \arg(z - \sqrt{3} + i)$

16 **Example 18** Sketch each subset of the complex plane for z if:

a $\arg(z) = \arg(z - 2)$

b $\arg(z - 1) - \arg(z + i) = \pi$

c $\arg(z - 2i) - \arg(z - i) = \frac{\pi}{6}$

Application

17 Sketch and describe the locus of z for each of the following.

a $|z - 2| = \operatorname{Re}(z)$

b $|z + 6i| = -\operatorname{Im}(z)$

c $\operatorname{Re}(z) + \operatorname{Im}(z) = -1$

18 Sketch the locus of z for each of the following.

a $-2 < \operatorname{Re}(z) \leq 4$

b $|z + i| = |z - 1|$

c $|\operatorname{Re}(z)| \leq 2$ and $\frac{\pi}{4} < \arg(z) \leq \frac{3\pi}{4}$



Practice quiz



5

TERMINOLOGY

- angle of projection
- angular velocity
- parameter
- parametric equation
- period
- projectile
- range
- speed of projection
- time of flight
- trajectory
- vector function
- vector-valued function
- uniform circular motion

VECTORS IN THREE DIMENSIONS

VECTOR CALCULUS

- 5.01 Vector equations in the plane
- 5.02 Derivatives of vector functions in the plane
- 5.03 Integrals of vector functions in the plane
- 5.04 Straight-line motion
- 5.05 Motion with constant acceleration
- 5.06 Motion with varying acceleration

Chapter summary

Chapter review



Prior learning

VECTOR CALCULUS

- consider position of vectors as a function of time (ACMSM111)
- derive the Cartesian equation of a path given as a vector equation in two dimensions including ellipses and hyperbolas (ACMSM112)
- differentiate and integrate a vector function with respect to time (ACMSM113)
- determine equations of motion of a particle travelling in a straight line with both constant and variable acceleration (ACMSM114)
- apply vector calculus to motion in a plane including projectile and circular motion (ACMSM115) 

5.01 VECTOR EQUATIONS IN THE PLANE

You previously saw how vectors are applied to physical and geometric situations. You can also use vectors to define functions.

Functions are usually defined as $y = f(x)$ or $x = h(y)$.

The equation of a circle is not like this.

Consider a circle with centre $(0, 0)$ and radius 5. The equation of this circle is:

$$x^2 + y^2 = 25$$

You could write the equation as:

$$y = \pm\sqrt{25-x^2}$$

but this involves two functions:

$$y = \sqrt{25-x^2} \text{ — (the top half of the circle)}$$

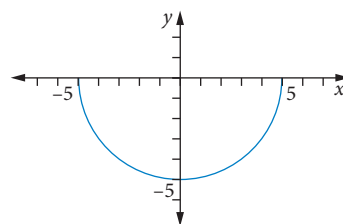
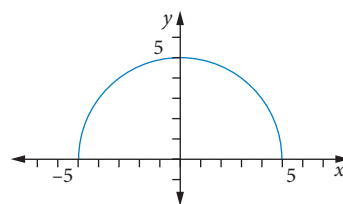
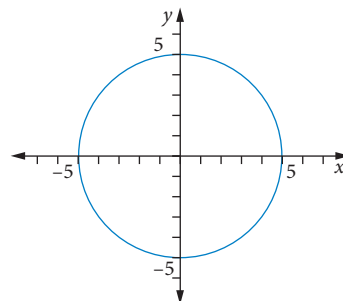
and $y = -\sqrt{25-x^2}$ — (the bottom half of the circle).

There is an alternative to defining y in terms of x , as in $y = f(x)$, or x in terms of y , as in $x = h(y)$. You can define both x and y in terms of a third variable, for example:

$$x = f(t) \text{ and } y = g(t)$$

The third variable is called a **parameter** and the equations are called **parametric equations**.

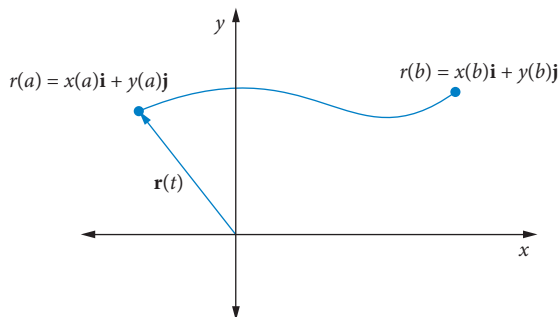
You can use parametric equations to define a **vector function** (**vector-valued function**).



Consider a position vector in two dimensions. The position vector at time t is $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$.

So, $\mathbf{r}(t)$ is a vector function with parameter t .

Consider the position vectors of the form $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$ for $a \leq t \leq b$. This is a curve whose orientation is in the direction in which the parameter (t) is increasing, as shown in the sketch below. The position $\mathbf{r}(a)$ is the initial point and the position $\mathbf{r}(b)$ is the terminal point of the curve.



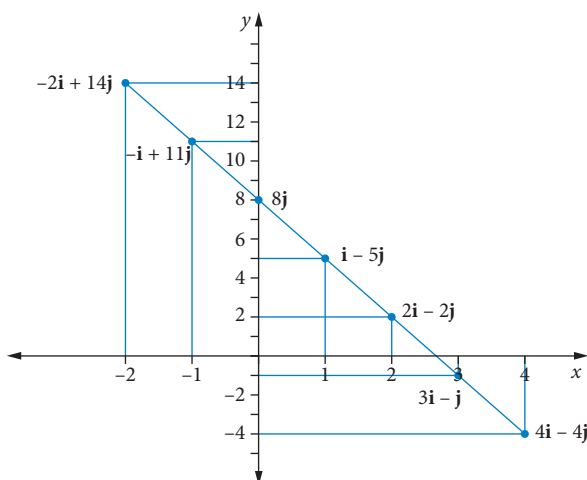
Consider the position vector $\mathbf{r}(t) = (1 + t)\mathbf{i} + (5 - 3t)\mathbf{j}$ for $-3 \leq t \leq 3$, where t represents time.

$\mathbf{r}(t)$ represents the position of a particle at various times, t , where $-3 \leq t \leq 3$.

You can graph the position of the particle on the Cartesian plane by constructing a table of values.

t	-3	-2	-1	0	1	2	3
$\mathbf{r}(t)$	$-2\mathbf{i} + 14\mathbf{j}$	$-\mathbf{i} + 11\mathbf{j}$	$8\mathbf{j}$	$\mathbf{i} + 5\mathbf{j}$	$2\mathbf{i} + 2\mathbf{j}$	$3\mathbf{i} - \mathbf{j}$	$4\mathbf{i} - 4\mathbf{j}$

The graph of the position of the particle can be drawn using the values in the table.



As $\mathbf{r}(t)$ represents the position of a particle with respect to time and $-3 \leq t \leq 3$:

- the initial point is $\mathbf{r}(-3) = -2\mathbf{i} + 14\mathbf{j}$
- the terminal point is $\mathbf{r}(3) = 4\mathbf{i} - 4\mathbf{j}$
- the direction of travel is from the initial point towards the terminal point.

It is possible to express the parametric equations as a Cartesian equation.

Let (x, y) represent any point on the graph.

$$\text{So } \mathbf{r}(t) = x\mathbf{i} + y\mathbf{j}$$

$$x\mathbf{i} + y\mathbf{j} = (1 + t)\mathbf{i} + (5 - 3t)\mathbf{j}$$

Equating the $\mathbf{i}$ components:

$$x = 1 + t$$

$$t = x - 1 \quad [1]$$

Equating the $\mathbf{j}$ components:

$$y = 5 - 3t \quad [2]$$

Substitute t from [1] into [2]:

$$y = 5 - 3(x - 1)$$

$$= 8 - 3x$$

So $y = 8 - 3x$ is the Cartesian equation for $\mathbf{r}(t) = (1 + t)\mathbf{i} + (5 - 3t)\mathbf{j}$.

○ Example 1

The position vector function for a particle is $\mathbf{r}(t) = (2t + 1)\mathbf{i} + (t^2 + t)\mathbf{j}$ for $-1 \leq t \leq 2$.

- a Find the Cartesian equation of the vector.
- b Draw a graph of the path of the particle, indicating the direction of travel.

Solution

- a Write the vector function in terms of x and y .

$$\begin{aligned}\mathbf{r}(t) &= (2t + 1)\mathbf{i} + (t^2 + t)\mathbf{j} \\ x\mathbf{i} + y\mathbf{j} &= (2t + 1)\mathbf{i} + (t^2 + t)\mathbf{j}\end{aligned}$$

Equate the $\mathbf{i}$ and $\mathbf{j}$ components.

$$x = 2t + 1 \text{ or } t = \frac{1}{2}(x - 1) \quad [1]$$

$$y = t^2 + t \quad [2]$$

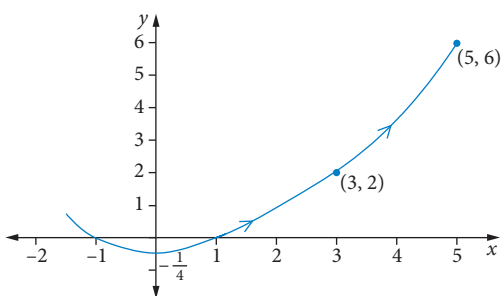
Substitute for t from [1] into [2].

$$y = \left[\frac{1}{2}(x - 1) \right]^2 + \frac{1}{2}(x - 1)$$

Expand and simplify.

$$\begin{aligned}&= \frac{1}{4}(x^2 - 2x + 1) + \frac{1}{2}(x - 1) \\ &= \frac{1}{4}x^2 - \frac{1}{2}x + \frac{1}{4} + \frac{1}{2}x - \frac{1}{2} \\ &= \frac{1}{4}x^2 - \frac{1}{4}\end{aligned}$$

- b The graph of the curve will be a parabola. Use any appropriate method to draw a diagram.



Calculate the initial point.

$$\mathbf{r}(-1) = [2 \times (-1) + 1]\mathbf{i} + [(-1)^2 + (-1)]\mathbf{j} \\ = -\mathbf{i}$$

Calculate the terminal point.

$$\mathbf{r}(2) = (2 \times 2 + 1)\mathbf{i} + (2^2 + 2)\mathbf{j} = 5\mathbf{i} + 6\mathbf{j}$$

Mark in the direction of travel using arrows.

The direction of travel is shown by the arrows.

You can use your graphics calculator to quickly check the graph of and the direction of travel for the curve in Example 1. Here, $\mathbf{r}(t) = (2t - 1)\mathbf{i} + (t^2 + t)\mathbf{j}$ and $-1 \leq t \leq 2$.

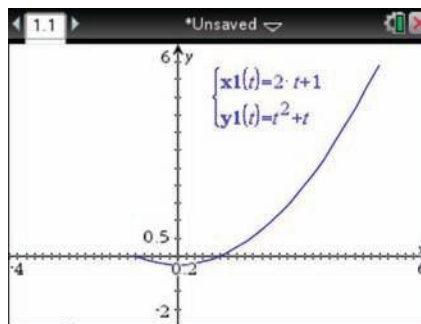
TI-Nspire CAS

Open a graphing window and select menu,

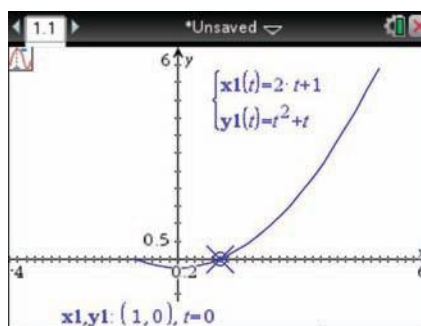
3:Graph Type, 2:Parametric.

Enter the parametric equations $2t + 1$ for $x_1(t)$ and $t^2 + t$ for $y_1(t)$ and set the bounds for t as $-1 \leq t \leq 2$.

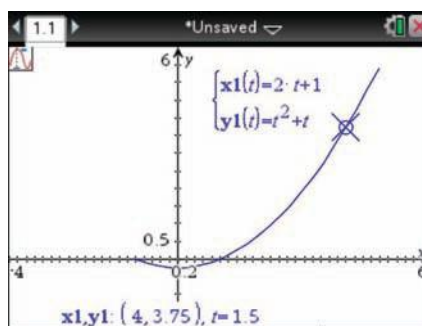
The plot is shown here.



You can check the direction of travel by using the trace function to see the position on the curve at various values of t .



Here you can see that as t increases, the position of the point on the curve moves upwards and to the right.



ClassPad

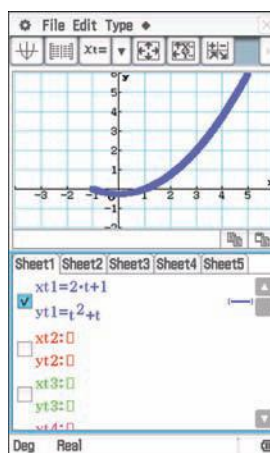
From **Menu**, select **Graph&Table**.

Tap **Type**, then **ParamType**.

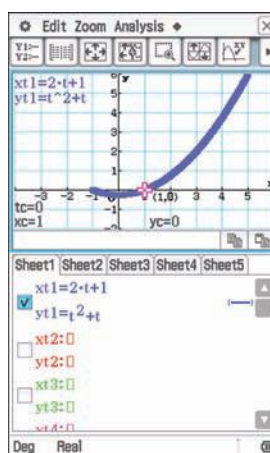
Enter the parametric equations $2t + 1$ by $xt1$ and $t^2 + t$ by $yt1$ and press **EXE**. (Remember that you will need to tap **Keyboard** and **abc** to access t .)

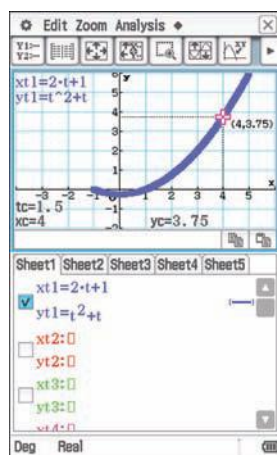
Tap **Range** to set the range of values for t . In this case, set **Start** to -1 , **End** to 2 , and **Step** to 0.01 and tap **OK**. To calculate the table of values used in plotting the graph, tap **Table**.

Tap **Y1=** to draw the graph.



You can check the direction of travel by using the trace function to see the position on the curve at various values of t . Tap **Analysis** then **Trace**, and use the arrow keys to move the crosshair along the graph. The value of t is shown on the screen as t_c .





Trigonometric identities are often used to calculate Cartesian equations of vector functions. In particular, you may find the following identities useful in this regard.

$$\sin^2(t) + \cos^2(t) = 1 \quad 1 + \tan^2(t) = \sec^2(t) \quad 1 - 2\sin^2(t) = \cos(2t)$$

$$\cosh^2(t) - \sinh^2(t) = 1 \quad 1 + \cot^2(t) = \operatorname{cosec}^2(t) \quad 2\cos^2(t) - 1 = \cos(2t)$$

$$\sin(2t) = 2\sin(t)\cos(t)$$

Consider a particle moving in a path described by $\mathbf{r}(t) = \cos(t)\mathbf{i} + \sin(t)\mathbf{j}$ for $0 \leq t \leq 2\pi$.

Let $\mathbf{r}(t) = x\mathbf{i} + y\mathbf{j}$. By equating $\mathbf{i}$ and $\mathbf{j}$ components we have $x = \cos(t)$ and $y = \sin(t)$.

Substitute for x and y in the trigonometric identity $\sin^2(t) + \cos^2(t) = 1$.

You get $x^2 + y^2 = 1$.

This is the unit circle, i.e., a circle with centre $(0, 0)$ and radius $= 1$.

The initial point is: $\mathbf{r}(t) = [\cos(0)]\mathbf{i} + [\sin(0)]\mathbf{j} = \mathbf{i}$

The terminal point is: $\mathbf{r}(2\pi) = [\cos(2\pi)]\mathbf{i} + [\sin(2\pi)]\mathbf{j} = \mathbf{i}$

The initial and terminal points are the same, so the path is a complete circle beginning at $(1, 0)$ and then moving in an anticlockwise direction before returning to $(1, 0)$.

IMPORTANT

The vector equation $\mathbf{r}(t) = \cos(t)\mathbf{i} + \sin(t)\mathbf{j}$ for $0 \leq t \leq 2\pi$ represents the path of a particle that:

- moves on the curve with equation $x^2 + y^2 = 1$
- has the position $(1, 0)$ at time $t = 0$
- moves in an anticlockwise direction
- takes 2π units of time to complete one revolution

Example 2

The position vector function for a particle is $\mathbf{r}(t) = [3 \cos(t)]\mathbf{i} + [2 \sin(t)]\mathbf{j}$ for $0 \leq t \leq 2\pi$.

- Find the corresponding Cartesian relation of the vector.
- Determine the domain and range of the Cartesian relation.
- Draw a graph of the path of the particle, indicating the direction of travel.

Solution

- Write the vector function in terms of x and y .

$$\begin{aligned}\mathbf{r}(t) &= [3 \cos(t)]\mathbf{i} + [2 \sin(t)]\mathbf{j} \\ x\mathbf{i} + y\mathbf{j} &= [3 \cos(t)]\mathbf{i} + [2 \sin(t)]\mathbf{j}\end{aligned}$$

Equate the $\mathbf{i}$ and $\mathbf{j}$ components.

$$x = 3 \cos(t) \text{ or } \cos(t) = \frac{x}{3} \quad [1]$$

$$y = 2 \sin(t) \text{ or } \sin(t) = \frac{y}{2} \quad [2]$$

Use the trig identity
 $\sin^2(t) + \cos^2(t) = 1$.

$$\sin^2(t) + \cos^2(t) = 1$$

Substitute for $\cos(t)$ and $\sin(t)$ from [1] and [2].

$$\left(\frac{y}{2}\right)^2 + \left(\frac{x}{3}\right)^2 = 1$$

Rearrange and expand.

$$\frac{x^2}{9} + \frac{y^2}{4} = 1$$

- The domain of the Cartesian relation can be determined using the vector function.

$$\begin{aligned}x\mathbf{i} + y\mathbf{j} &= [3 \cos(t)]\mathbf{i} + [2 \sin(t)]\mathbf{j} \\ \text{So, } x &= 3 \cos(t)\end{aligned}$$

The *range* of this function determines the *domain* of the Cartesian relation.

Range of $x = 3 \cos(t)$ is $[-3, 3]$, i.e., $-3 \leq x \leq 3$.

Domain of $\frac{x^2}{9} + \frac{y^2}{4} = 1$ is $[-3, 3]$.

The range of the Cartesian relation can also be determined using the vector function.

$$\begin{aligned}x\mathbf{i} + y\mathbf{j} &= [3 \cos(t)]\mathbf{i} + [2 \sin(t)]\mathbf{j} \\ \text{So, } y &= 2 \sin(t)\end{aligned}$$

The *range* of this function determines the *range* of the Cartesian relation.

Range of $y = 2 \sin(t)$ is $[-2, 2]$, i.e., $-2 \leq y \leq 2$.

Domain of $\frac{x^2}{9} + \frac{y^2}{4} = 1$ is $[-2, 2]$.

Calculate the initial point.

$$\mathbf{r}(t) = [3 \cos(0)]\mathbf{i} + [2 \sin(0)]\mathbf{j} = 3\mathbf{i}$$

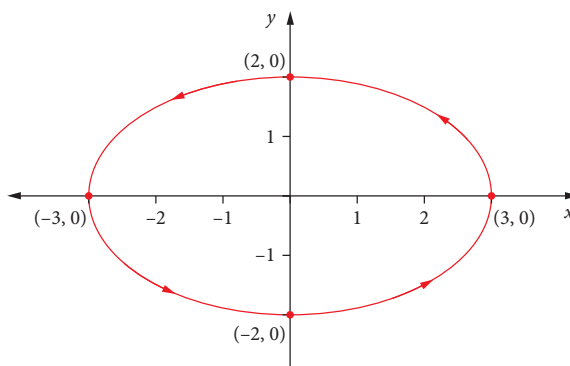
Calculate the terminal point.

$$\mathbf{r}(2\pi) = [3 \cos(2\pi)]\mathbf{i} + [2 \sin(2\pi)]\mathbf{j} = 3\mathbf{i}$$

The initial and terminal points are the same. Check another value of t .

$$\mathbf{r}\left(\frac{\pi}{2}\right) = \left[3 \cos\left(\frac{\pi}{2}\right)\right]\mathbf{i} + \left[2 \sin\left(\frac{\pi}{2}\right)\right]\mathbf{j} = 2\mathbf{j}$$

- c The graph of this curve is an ellipse. Use any appropriate method to sketch a diagram.



The path of travel is in the direction from (3, 0) to (0, 2) and so on. Mark in the direction of travel using arrows.

The direction of travel is shown with red arrows.

Example 3

The motion of two particles is given by the vector functions $\mathbf{r}(t) = (3t - 5)\mathbf{i} + 7t\mathbf{j}$ and $\mathbf{q}(t) = (t + 3)\mathbf{i} + (t^2 + 12)\mathbf{j}$.

Calculate:

- the position at which the particles collide
- the positions at which the paths of the particles intersect
- the distance between the particles when $t = 2$.

Solution

- a The particles collide when they have the same position at the same time.

$$\text{Let } \mathbf{r}(t) = \mathbf{q}(t)$$

Replace $\mathbf{r}(t)$ and $\mathbf{q}(t)$ with their respective functions.

$$(3t - 5)\mathbf{i} + 7t\mathbf{j} = (t + 3)\mathbf{i} + (t^2 + 12)\mathbf{j}$$

Equate the $\mathbf{i}$ components.

$$3t - 5 = t + 3$$

Rearrange and solve for t .

$$2t = 8$$

$$t = 4$$

Equate the $\mathbf{j}$ components.

$$7t = t^2 + 12$$

Rearrange and solve for t .

$$t^2 - 7t + 12 = 0$$

$$(t - 4)(t - 3) = 0$$

$$t = 3 \text{ or } 4$$

The particles will have the same position at the same time when $t = 4$.

Calculate the position at which the particles collide.

$$\text{Let } t = 4$$

$$\mathbf{r}(4) = (3 \times 4 - 5)\mathbf{i} + 7 \times 4\mathbf{j}$$

$$= 7\mathbf{i} + 28\mathbf{j}$$

$$\mathbf{q}(4) = (4 + 3)\mathbf{i} + (4^2 + 12)\mathbf{j}$$

$$= 7\mathbf{i} + 28\mathbf{j}$$

State the result.

- b** To find all the times at which the paths intersect, you redefine the time variables.

Equate the **i** components.

Isolate t_2 .

Equate the **j** components.

Substitute for t_2 from [1].

Simplify.

Rearrange and group like terms.

$t_1 = 4$, so $(t_1 - 4)$ is a factor. Factorise and solve for t_1 .

Substitute back into the vector functions to find the positions.

State the result.

- c** Find the positions of the particles when $t = 2$.

Find the displacement between the particles.

Use the distance formula.

State the result.

The position at which the particles collide is $7\mathbf{i} + 28\mathbf{j}$.

$$\mathbf{r}(t_1) = (3t_1 - 5)\mathbf{i} + 7t_1\mathbf{j}$$

$$\mathbf{q}(t_2) = (t_2 + 3)\mathbf{i} + (t_2^2 + 12)\mathbf{j}$$

$$(3t_1 - 5) = (t_2 + 3) \\ t_2 = 3t_1 - 8 \quad [1]$$

$$7t_1 = t_2^2 + 12 \\ = (3t_1 - 8)^2 + 12 \\ = 9t_1^2 - 48t_1 + 64 + 12$$

$$9t_1^2 - 55t_1 + 76 = 0$$

$$(t_1 - 4)(9t_1 - 19) = 0$$

$$t_1 = 4 \text{ or } \frac{19}{9}$$

When $t = 4$, position = $7\mathbf{i} + 28\mathbf{j}$.

When $t = \frac{19}{9}$, position = $\frac{4}{3}\mathbf{i} + \frac{133}{9}\mathbf{j}$

The positions at which the paths of the particles intersect are $7\mathbf{i} + 28\mathbf{j}$ and $\frac{4}{3}\mathbf{i} + \frac{133}{9}\mathbf{j}$.

$$\mathbf{r}(2) = (3 \times 2 - 5)\mathbf{i} + 7 \times 2\mathbf{j} \\ = \mathbf{i} + 14\mathbf{j}$$

$$\mathbf{q}(2) = (2 + 3)\mathbf{i} + (2^2 + 12)\mathbf{j} \\ = 5\mathbf{i} + 18\mathbf{j}$$

$$\text{Displacement} = 5\mathbf{i} + 18\mathbf{j} - (\mathbf{i} + 14\mathbf{j}) \\ = 4\mathbf{i} + 4\mathbf{j}$$

$$d = \sqrt{4^2 + 4^2} \\ = 4\sqrt{2}$$

When $t = 2$, the distance between the particles is $4\sqrt{2}$ units.

EXERCISE 5.01 Vector equations in the plane

Concepts and techniques



Parametric and
Cartesian
equations

1 **Example 1** The vector function $\mathbf{r}(t) = t^2\mathbf{i} + t^4\mathbf{j}$ has the corresponding Cartesian relation:

A $y = 2x^2$ for $x > 0$

B $y = \sqrt{x}$ for $x > 0$

C $y = \frac{x^2}{2}$ for $x > 0$

D $y = x^2$ for $x \geq 0$

E $y = \sqrt{2x}$ for $x > 0$

2 The vector function $\mathbf{r}(t) = [2 \tan(t)]\mathbf{i} + [2 \cos^2(t)]\mathbf{j}$ for $0 \leq t \leq 2\pi$ has the corresponding Cartesian relation:

A $y = x^2 + 4$

B $y = \frac{8}{x^2 + 4}$

C $y = 8x^2 + 4$

D $y = 4x^2 + 8$

E $y = \frac{x^2 + 4}{8}$

3 Which of the following best describes the curve for the vector function $\mathbf{r}(t) = \cos(t)\mathbf{i} + \sin^2(t)\mathbf{j}$, $0 \leq t \leq 2\pi$?

A A hyperbola

B A circle

C A portion of a parabola

D A portion of an ellipse

E A portion of a hyperbola

4 At how many places does the curve for the vector function $\mathbf{r}(t) = \cos(3t)\mathbf{i} + \sin(t)\mathbf{j}$ cross the x -axis?

A 0

B 1

C 2

D 3

E 4

5 Find the Cartesian equation that corresponds to each of the following vector equations.

a $\mathbf{r}(t) = t\mathbf{i} + 3t\mathbf{j}$ ($t \in \mathbf{R}$)

b $\mathbf{r}(t) = 5\mathbf{i} + 3t\mathbf{j}$ ($t \in \mathbf{R}$)

c $\mathbf{r}(t) = -2t\mathbf{i} + 6t\mathbf{j}$ ($t \in \mathbf{R}$)

d $\mathbf{r}(t) = (4 - t)\mathbf{i} + (t + 3)\mathbf{j}$ ($t \in \mathbf{R}$)

e $\mathbf{r}(t) = t^2\mathbf{i} + (5 - 2t)\mathbf{j}$ ($t \in \mathbf{R}$)

f $\mathbf{r}(t) = (t - 1)\mathbf{i} + (t^3 + 6)\mathbf{j}$ ($t \in \mathbf{R}$)

g $\mathbf{r}(t) = \frac{1+2t}{t}\mathbf{i} + (t^2 + 1)\mathbf{j}$ ($t > 0$)

h $\mathbf{r}(t) = \frac{1}{t+1}\mathbf{i} + \frac{1}{t}\mathbf{j}$ ($t > 0$)

6 State the domain and range of the Cartesian relation found in question 5.

7 A number of position vector functions are given below. For each function:

i find the corresponding Cartesian equation

ii draw a diagram of the path of the particle, indicating the direction of travel

iii state the domain and range of the Cartesian function.

a $\mathbf{r}(t) = (t - 2)\mathbf{i} + (2t + 3)\mathbf{j}$ for $0 \leq t \leq 5$

b $\mathbf{r}(t) = (1 - 2t)\mathbf{i} + (1 + t)\mathbf{j}$ for $-1 \leq t \leq 4$

c $\mathbf{r}(t) = (t^2 + 1)\mathbf{i} + (t^2 - 1)\mathbf{j}$ for $-2 \leq t \leq 2$

d $\mathbf{r}(t) = (t^3 + 1)\mathbf{i} + (t^3 - 1)\mathbf{j}$ for $-2 \leq t \leq 2$

e $\mathbf{r}(t) = (4t^2 - 5)\mathbf{i} + (2t + 3)\mathbf{j}$ for $t \in \mathbf{R}$

f $\mathbf{r}(t) = t^3\mathbf{i} + t^2\mathbf{j}$ for $t \in \mathbf{R}$

g $\mathbf{r}(t) = e^t\mathbf{i} + e^{-2t}\mathbf{j}$ for $t \in \mathbf{R}$

h $\mathbf{r}(t) = \sqrt{t}\mathbf{i} + (3t + 4)\mathbf{j}$ for $t \geq 0$

8 **Example 2** Find the Cartesian relation that corresponds to each of the following.

a $\mathbf{r}(t) = \cos(t)\mathbf{i} + \sin(t)\mathbf{j}$ for $t \in \mathbf{R}$

b $\mathbf{r}(t) = \sin(t)\mathbf{i} + \cos(t)\mathbf{j}$ for $t \in [0, \pi]$

c $\mathbf{r}(t) = \cos(t)\mathbf{i} + \sin(t)\mathbf{j}$ for $t \in [\pi, \frac{3\pi}{2}]$

d $\mathbf{r}(t) = 3 \cos(t)\mathbf{i} + 2 \sin(t)\mathbf{j}$ for $t \in \mathbf{R}$

e $\mathbf{r}(t) = 3 \cos^2(t)\mathbf{i} + 4 \sin^2(t)\mathbf{j}$ for $t \in \mathbf{R}$

f $\mathbf{r}(t) = 2 \sec(t)\mathbf{i} + 3 \tan(t)\mathbf{j}$ for $t \in [0, \frac{\pi}{2}]$

g $\mathbf{r}(t) = 3 \cos(2t)\mathbf{i} + 3 \sin(2t)\mathbf{j}$ for $t \in [\pi, \frac{3\pi}{2}]$

h $\mathbf{r}(t) = 2 \sec^2(t)\mathbf{i} + 3 \tan^2(t)\mathbf{j}$ for $t \in [-\pi, \pi]$



- 9 State the domain and range of each of the relations found in question 8.
- 10 A number of position vector functions are given below. For each function:
- find the corresponding Cartesian equation
 - draw a diagram of the path of the particle, indicating the direction of travel
 - state the domain and range of the Cartesian function.
- a $\mathbf{r}(t) = 3 \sin(t)\mathbf{i} + 2 \cos(t)\mathbf{j}$ for $0 \leq t \leq 2\pi$ b $\mathbf{r}(t) = 2 \cos(t)\mathbf{i} - 3 \sin(t)\mathbf{j}$ for $0 \leq t \leq 2\pi$
- c $\mathbf{r}(t) = \cos(t)\mathbf{i} - \sin(t)\mathbf{j}$ for $t \in [\frac{\pi}{2}, \pi]$ d $\mathbf{r}(t) = 4 \cos(2t)\mathbf{i} + 4 \sin(2t)\mathbf{j}$ for $t \in [\pi, \frac{3\pi}{2}]$
- 11 **Example 3** The position of a particle is defined by the vector function
 $\mathbf{r}(t) = (5 - 3t)\mathbf{i} + (2t + 1)\mathbf{j}$.
- Find the position of the particle when $t = 3$.
 - Find the distance between the particle and another particle located at $(5, 7)$ when $t = 3$.
- 12 The position of a particle is defined by the vector function $\mathbf{r}(t) = (3t - 2)\mathbf{i} + (1 - t)\mathbf{j}$.
- Calculate the distance between the particle and the origin when $t = 2$.
 - Find the times at which the particle is 1 unit from the origin.
- 13 The positions of two particles are defined by the vector functions $\mathbf{r}(t) = (4t + 3)\mathbf{i} + (2t^2 + 5)\mathbf{j}$ and $\mathbf{q}(t) = (8t - 7)\mathbf{i} + 7t\mathbf{j}$.
- Calculate the distance between the particles when $t = 2$.
 - When do the particles collide?
 - Where do the paths of the particles cross?

Reasoning and communication

- 14 For each of the following position vector functions, find the corresponding Cartesian equation for the function and sketch the path of the particle, indicating the direction of travel and the domain and range of the Cartesian equation.
- $\mathbf{r}(t) = \sec(t)\mathbf{i} + \tan(t)\mathbf{j}$ for $\frac{-\pi}{3} \leq t \leq \frac{\pi}{3}$
 - $\mathbf{r}(t) = \cos(2t)\mathbf{i} + \sin(t)\mathbf{j}$ for $-\pi \leq t \leq \pi$
 - $\mathbf{r}(t) = \sin^2(t)\mathbf{i} + 2 \cos(t)\mathbf{j}$ for $0 \leq t \leq 2\pi$
 - $\mathbf{r}(t) = [3 \cos(2t)]\mathbf{i} + \sin^2(2t)\mathbf{j}$ for $0 \leq t \leq \pi$
 - $\mathbf{r}(t) = [2 \sin(t) \cos(t)]\mathbf{i} + 2 \sin^2(t)\mathbf{j}$ for $0 \leq t \leq \pi$
 - $\mathbf{r}(t) = \cos(2t)\mathbf{i} + \cos(t)\mathbf{j}$ for $0 \leq t \leq 2\pi$
- 15 The positions of two particles are defined by the vector functions
 $\mathbf{r}(t) = (6t^2 - 15t)\mathbf{i} + (7t^3 - 9t^2 + 1)\mathbf{j}$ and $\mathbf{q}(t) = (4t - 10)\mathbf{i} + (4t - 2t^3 - 3)\mathbf{j}$.
 Where and when do the particles collide?

5.02 DERIVATIVES OF VECTOR FUNCTIONS IN THE PLANE

Consider the figure on the right.

The curve is defined by the vector function $\mathbf{r}(t)$, which represents the position of a particle at any time t .

P and Q lie on the path of a curve. The position vector of P is $\mathbf{r}(t)$.

Q is another point on the curve at time $(t + \delta t)$ and the position vector of Q is $\mathbf{r}(t + \delta t)$.

$\mathbf{PQ}$ is the displacement vector of the particle during the time interval δt .

Hence $\mathbf{PQ} = \mathbf{r}(t + \delta t) - \mathbf{r}(t)$.

As $\delta t \rightarrow 0$, P approaches Q .

The derivative of $\mathbf{r}$ with respect to t is denoted by $\frac{d\mathbf{r}}{dt}$ or more commonly as $\dot{\mathbf{r}}(t)$.

$$\dot{\mathbf{r}} = \lim_{\delta t \rightarrow 0} \frac{\mathbf{r}(t + \delta t) - \mathbf{r}(t)}{\delta t}$$

From the work of the previous section, you know that $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$.

So,

$$\begin{aligned} \dot{\mathbf{r}} &= \lim_{\delta t \rightarrow 0} \frac{\mathbf{r}(t + \delta t) - \mathbf{r}(t)}{\delta t} \\ &= \lim_{\delta t \rightarrow 0} \frac{[x(t + \delta t)\mathbf{i} + y(t + \delta t)\mathbf{j}] - [x(t)\mathbf{i} + y(t)\mathbf{j}]}{\delta t} \\ &= \lim_{\delta t \rightarrow 0} \frac{x(t + \delta t)\mathbf{i} - x(t)\mathbf{i}}{\delta t} + \lim_{\delta t \rightarrow 0} \frac{y(t + \delta t)\mathbf{j} - y(t)\mathbf{j}}{\delta t} \\ &= \lim_{\delta t \rightarrow 0} \frac{x(t + \delta t) - x(t)}{\delta t} \mathbf{i} + \lim_{\delta t \rightarrow 0} \frac{y(t + \delta t) - y(t)}{\delta t} \mathbf{j} \\ &= \frac{dx}{dt} \mathbf{i} + \frac{dy}{dt} \mathbf{j} \end{aligned}$$

$$\dot{\mathbf{r}}(t) = \dot{x}(t)\mathbf{i} + \dot{y}(t)\mathbf{j}$$

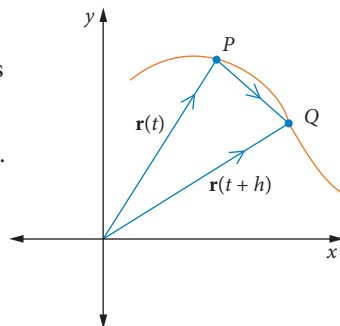
The magnitude of $\dot{\mathbf{r}}(t)$ is:

$$|\dot{\mathbf{r}}(t)| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{\dot{x}^2 + \dot{y}^2}$$

The second derivative of $\mathbf{r}(t)$ is $\frac{d^2\mathbf{r}}{dt^2}$ or, more commonly, $\ddot{\mathbf{r}}(t)$.

Hence

$$\begin{aligned} \ddot{\mathbf{r}}(t) &= \frac{d^2x}{dt^2} \mathbf{i} + \frac{d^2y}{dt^2} \mathbf{j} \\ &= \ddot{x}(t)\mathbf{i} + \ddot{y}(t)\mathbf{j} \end{aligned}$$



IMPORTANT**Derivatives of a vector function**

Given $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$:

$$\dot{\mathbf{r}}(t) = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} = \dot{x}(t)\mathbf{i} + \dot{y}(t)\mathbf{j}$$

$$|\dot{\mathbf{r}}(t)| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{\dot{x}^2 + \dot{y}^2}$$

$$\ddot{\mathbf{r}}(t) = \frac{d^2x}{dt^2}\mathbf{i} + \frac{d^2y}{dt^2}\mathbf{j} = \ddot{x}(t)\mathbf{i} + \ddot{y}(t)\mathbf{j}$$

$$|\ddot{\mathbf{r}}(t)| = \sqrt{\left(\frac{d^2x}{dt^2}\right)^2 + \left(\frac{d^2y}{dt^2}\right)^2} = \sqrt{\ddot{x}^2 + \ddot{y}^2}$$

It should be noted that differentiating three-dimensional vectors is simply an extension of differentiating vectors in the plane (two-dimensional vectors). It therefore follows that:

IMPORTANT

Given $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$,

$$\dot{\mathbf{r}}(t) = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k} = \dot{x}(t)\mathbf{i} + \dot{y}(t)\mathbf{j} + \dot{z}(t)\mathbf{k}$$

$$|\dot{\mathbf{r}}(t)| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}$$

$$\ddot{\mathbf{r}}(t) = \frac{d^2x}{dt^2}\mathbf{i} + \frac{d^2y}{dt^2}\mathbf{j} + \frac{d^2z}{dt^2}\mathbf{k} = \ddot{x}(t)\mathbf{i} + \ddot{y}(t)\mathbf{j} + \ddot{z}(t)\mathbf{k}$$

$$|\ddot{\mathbf{r}}(t)| = \sqrt{\left(\frac{d^2x}{dt^2}\right)^2 + \left(\frac{d^2y}{dt^2}\right)^2 + \left(\frac{d^2z}{dt^2}\right)^2} = \sqrt{\ddot{x}^2 + \ddot{y}^2 + \ddot{z}^2}$$

Most of the basic properties for the derivatives of functions hold for the derivatives of vector functions. In addition, there are also properties for the derivatives of scalar and vector products of vector functions.

Properties for the derivatives of vector functions

$$\begin{aligned}\frac{d}{dt}[\mathbf{u}(t) + \mathbf{v}(t)] &= \frac{d\mathbf{u}(t)}{dt} + \frac{d\mathbf{v}(t)}{dt} \\ &= \dot{\mathbf{u}} + \dot{\mathbf{v}}\end{aligned}$$

$$\begin{aligned}\frac{d[k\mathbf{u}(t)]}{dt} &= k \frac{d\mathbf{u}(t)}{dt} \\ &= k\dot{\mathbf{u}}\end{aligned}$$

$$\begin{aligned}\frac{d}{dt}[f(t)\mathbf{u}(t)] &= \frac{df(t)}{dt}\mathbf{u}(t) + f(t)\frac{d\mathbf{u}(t)}{dt} \\ &= f'(t)\mathbf{u} + f(t)\dot{\mathbf{u}}\end{aligned}$$

$$\begin{aligned}\frac{d}{dt}[\mathbf{u}(t) \cdot \mathbf{v}(t)] &= \frac{d\mathbf{u}(t)}{dt} \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \frac{d\mathbf{v}(t)}{dt} \\ &= \dot{\mathbf{u}} \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \dot{\mathbf{v}}\end{aligned}$$

$$\begin{aligned}\frac{d}{dt}[\mathbf{u}(t) \times \mathbf{v}(t)] &= \frac{d\mathbf{u}(t)}{dt} \times \mathbf{v}(t) + \mathbf{u}(t) \times \frac{d\mathbf{v}(t)}{dt} \\ &= \dot{\mathbf{u}} \times \mathbf{v}(t) + \mathbf{u}(t) \times \dot{\mathbf{v}}\end{aligned}$$

$$\begin{aligned}\frac{d}{dt}\{\mathbf{u}[f(t)]\} &= \frac{df(t)}{dt} \frac{d\mathbf{u}(t)}{dt} f(t) \\ &= f'(t)\dot{\mathbf{u}}f(t)\end{aligned}$$

○ Example 4

If $\mathbf{r}(t) = 5t^2\mathbf{i} - \cos(3t)\mathbf{j}$, find:

- a $\dot{\mathbf{r}}(t)$ b $|\dot{\mathbf{r}}(t)|$ c $\ddot{\mathbf{r}}(t)$

Solution

- a Use the rule for the first derivative.

Find the derivatives of the components.

$$\begin{aligned}\dot{\mathbf{r}}(t) &= \dot{x}(t)\mathbf{i} + \dot{y}(t)\mathbf{j} \\ &= 10t\mathbf{i} + 3\sin(3t)\mathbf{j}\end{aligned}$$

- b Use the rule to calculate the norm.

Evaluate.

$$\begin{aligned}|\dot{\mathbf{r}}(t)| &= \sqrt{(10t)^2 + [3\sin(3t)]^2} \\ &= \sqrt{100t^2 + 9\sin^2(3t)}\end{aligned}$$

- c Differentiate $\dot{\mathbf{r}}(t)$.

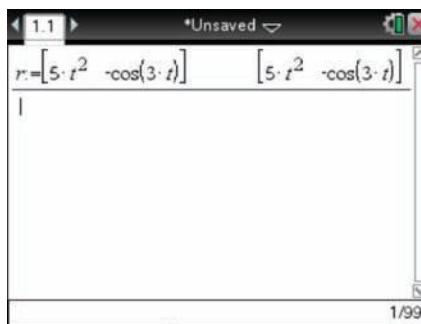
$$\ddot{\mathbf{r}}(t) = 10\mathbf{i} + 9\cos(3t)\mathbf{j}$$

You can use your graphics calculator to check calculations such as those in Example 4.

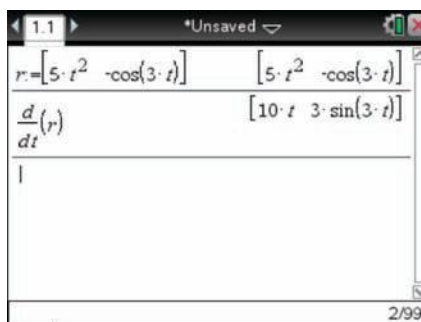


TI-Nspire CAS

First assign the vector $5t^2\mathbf{i} - \cos(3t)\mathbf{j}$ to $\mathbf{r}$.
Make sure that you assign the value to $\mathbf{r}$ using ctrl [=] and make sure that you use ctrl [] to enter the vector.



To find the derivative of $\mathbf{r}$, press menu and select 4:Calculus, 1:Derivative. Enter t in the bottom part of the derivative statement and then enter $\mathbf{r}$ as the vector to be differentiated. Press ENTER.

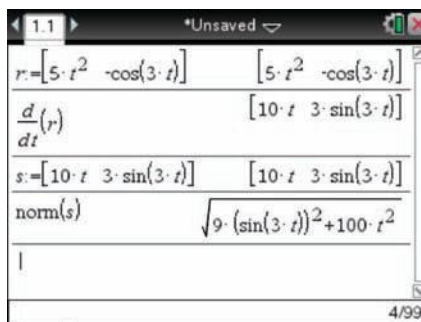


$$\dot{\mathbf{r}}(t) = 10t\mathbf{i} + 3 \sin(3t)\mathbf{j}$$

Next assign $\dot{\mathbf{r}}(t)$ as a new vector, $\mathbf{s}$.

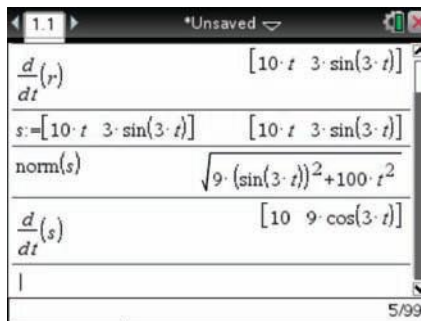
To find the norm of $\mathbf{s}$, press menu and then select 7:Matrix and Vector, 7:Norms and 1:Norm.

Enter $\mathbf{s}$ and then press ENTER.



$$|\dot{\mathbf{r}}(t)| = |\mathbf{s}| = \sqrt{100t^2 + 9 \sin^2(3t)}$$

To find $\ddot{\mathbf{r}}(t)$, calculate the derivative of $\mathbf{s}$ as described previously.



$$\ddot{\mathbf{r}}(t) = 10\mathbf{i} + 9 \cos(3t)\mathbf{j}$$

Example 5

- a If $\mathbf{r}(t) = 7t\mathbf{i} + [(5t-3)^4 + 2]\mathbf{j}$, find $\dot{\mathbf{r}}(t)$ and $\ddot{\mathbf{r}}(t)$.
- b If $\mathbf{u}(t) = 2t\mathbf{i} - t^2\mathbf{j}$ and $\mathbf{v}(t) = 3t^2\mathbf{i} + t^3\mathbf{j}$, verify that:
- $\frac{d}{dt}[\mathbf{u}(t) \cdot \mathbf{v}(t)] = \dot{\mathbf{u}} \cdot \mathbf{v} + \mathbf{u} \cdot \dot{\mathbf{v}}$
 - $\frac{d}{dt}[\mathbf{u}(t) \times \mathbf{v}(t)] = \dot{\mathbf{u}} \times \mathbf{v}(t) + \mathbf{u}(t) \times \dot{\mathbf{v}}$

Solution

- a Use the rule for the first derivative.

Find the derivatives of the components.

Simplify.

Differentiate $\dot{\mathbf{r}}(t)$.

Simplify.

- b i Calculate $\mathbf{u}(t) \cdot \mathbf{v}(t)$.

Find the derivative.

Calculate $\dot{\mathbf{u}} \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \dot{\mathbf{v}}$.

Simplify.

State the result.

- ii Calculate $\mathbf{u}(t) \times \mathbf{v}(t)$.

Differentiate.

Calculate $\dot{\mathbf{u}} \times \mathbf{v}(t)$.

$$\dot{\mathbf{r}}(t) = \dot{x}(t)\mathbf{i} + \dot{y}(t)\mathbf{j}$$

$$= 7\mathbf{i} + [5 \times 4(5t-3)^3]\mathbf{j}$$

$$\dot{\mathbf{r}}(t) = 7\mathbf{i} + 20(5t-3)^3\mathbf{j}$$

$$\ddot{\mathbf{r}}(t) = [5 \times 20 \times 3(5t-3)^2]\mathbf{j}$$

$$= 300(5t-3)^2\mathbf{j}$$

$$\begin{aligned}\mathbf{u}(t) \cdot \mathbf{v}(t) &= (2t \times 3t^2) + (-t^2 \times t^3) \\ &= 6t^3 - t^5\end{aligned}$$

$$\begin{aligned}\frac{d}{dt}[\mathbf{u}(t) \cdot \mathbf{v}(t)] &= \frac{d}{dt}(6t^3 - t^5) \\ &= 18t^2 - 5t^4\end{aligned}$$

$$\begin{aligned}\dot{\mathbf{u}} \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \dot{\mathbf{v}} &= (2\mathbf{i} - 2t\mathbf{j}) \cdot (3t^2\mathbf{i} + t^3\mathbf{j}) + \\ &\quad (2t\mathbf{i} - t^2\mathbf{j}) \cdot (6t\mathbf{i} + 3t^2\mathbf{j}) \\ &= (6t^2 - 2t^4) + (12t^2 - 3t^4) \\ &= 18t^2 - 5t^4\end{aligned}$$

$$\frac{d}{dt}[\mathbf{u}(t) \cdot \mathbf{v}(t)] = \dot{\mathbf{u}} \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \dot{\mathbf{v}}$$

$$\begin{aligned}\mathbf{u}(t) \times \mathbf{v}(t) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -t^2 & 0 \\ 3t^2 & t^3 & 0 \end{vmatrix} \\ &= (2t^4 + 3t^4)\mathbf{k} \\ &= 5t^4\mathbf{k}\end{aligned}$$

$$\frac{d}{dt}[\dot{\mathbf{u}} \times \mathbf{v}(t)] = 20t^3\mathbf{k}$$

$$\begin{aligned}\dot{\mathbf{u}} \times \mathbf{v}(t) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2t & -2t & 0 \\ 3t^2 & t^3 & 0 \end{vmatrix} \\ &= (2t^3 + 6t^3)\mathbf{k} \\ &= 8t^3\mathbf{k}\end{aligned}$$

Calculate $\mathbf{u}(t) \times \dot{\mathbf{v}}$.

$$\begin{aligned}\mathbf{u}(t) \times \dot{\mathbf{v}} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2t & -t^2 & 0 \\ 6t & 3t^2 & 0 \end{vmatrix} \\ &= (6t^3 + 6t^3)\mathbf{k} \\ &= 12t^3\mathbf{k}\end{aligned}$$

Calculate $\dot{\mathbf{u}} \times \mathbf{v}(t) + \mathbf{u}(t) \times \dot{\mathbf{v}}$.

$$\begin{aligned}\dot{\mathbf{u}} \times \mathbf{v}(t) + \mathbf{u}(t) \times \dot{\mathbf{v}} &= 8t^3\mathbf{k} + 12t^3\mathbf{k} \\ &= 20t^3\mathbf{k}\end{aligned}$$

State the result.

$$\frac{d}{dt}[\mathbf{u}(t) \times \mathbf{v}(t)] = \dot{\mathbf{u}} \times \mathbf{v}(t) + \mathbf{u}(t) \times \dot{\mathbf{v}}$$

When the curve of a vector function is plotted on the Cartesian plane, the gradient at any point $P(x, y)$ on the curve is given by $\frac{dy}{dx}$.

○ Example 6

A curve is described by the vector function $\mathbf{r}(t) = 3 \cos(t)\mathbf{i} + 2 \sin(t)\mathbf{j}$ for $0 < t < \pi$.

- Find the gradient of the curve at the point $P(x, y)$ where $x = 3 \cos(t)$ and $y = 2 \sin(t)$.
- Find the gradient of the curve when $t = \frac{\pi}{4}$.

Solution

- Use the chain rule to find an expression for $\frac{dy}{dx}$. $\frac{dy}{dx} = \frac{dy}{dt} \frac{dt}{dx}$

$$\text{Find } \frac{dy}{dt}. \quad \frac{dy}{dt} = 2 \cos(t)$$

$$\text{Find } \frac{dx}{dt}. \quad \frac{dx}{dt} = -3 \sin(t)$$

$$\text{You want } \frac{dt}{dx}. \quad \frac{dt}{dx} = \frac{-1}{3 \sin(t)}$$

$$\begin{aligned}\text{Substitute known values back into the equation for } \frac{dy}{dx}. \quad \frac{dy}{dx} &= \frac{dy}{dt} \frac{dt}{dx} \\ &= 2 \cos(t) \times \frac{-1}{3 \sin(t)}\end{aligned}$$

$$\text{Simplify.} \quad = -\frac{2}{3} \cot(t)$$

- Let $t = \frac{\pi}{4}$. When $t = \frac{\pi}{4}$

$$\begin{aligned}\frac{dy}{dx} &= -\frac{2}{3} \cot\left(\frac{\pi}{4}\right) \\ &= -\frac{2}{3}\end{aligned}$$

Evaluate.



EXERCISE 5.02 Derivatives of vector functions in the plane



Derivatives of
vectors

Concepts and techniques

- 1 **Example 4** If $\mathbf{r}(t) = (2t^2 - 7)\mathbf{i} - (5 + 3t)\mathbf{j}$, then $|\dot{\mathbf{r}}(1)|$ is:
 A 1 B 5 C 7 D 8 E 13
- 2 If $\mathbf{r}(t) = 3t^2\mathbf{i} + (3t - 4)\mathbf{j}$, then $\ddot{\mathbf{r}}(t)$ is:
 A $6t\mathbf{i} + 3\mathbf{j}$ B $6t\mathbf{i} + 0\mathbf{j}$ C $6\mathbf{i} + 0\mathbf{j}$
 D $0\mathbf{i} + 0\mathbf{j}$ E $6\mathbf{i} + 3\mathbf{j}$
- 3 If $\mathbf{r}(t) = [\sin(5t)]\mathbf{i} - [4 \cos(t)]\mathbf{j}$, then when $t = \frac{\pi}{2}$, $|\dot{\mathbf{r}}(t)|$ is:
 A -4 B 0 C 4 D $5\sqrt{2}$ E Not defined
- 4 If $\mathbf{r}(t) = (2t^3 + 3)\mathbf{i} + (t^2 - 7)\mathbf{j}$, then when $t = 1$, $\ddot{\mathbf{r}}(t)$ is:
 A $12\mathbf{i} + 2\mathbf{j}$ B $12\mathbf{i} + 0\mathbf{j}$ C $6\mathbf{i} + 2\mathbf{j}$ D $0\mathbf{i} + 0\mathbf{j}$ E $12\mathbf{i}$
- 5 Find $\dot{\mathbf{r}}(t)$ and $\ddot{\mathbf{r}}(t)$ for each of the following.
 a $\mathbf{r}(t) = t^2\mathbf{i} - 3t\mathbf{j}$ b $\mathbf{r}(t) = 10t\mathbf{i} - (7t - 3)\mathbf{j}$ c $\mathbf{r}(t) = 5t^4\mathbf{i} + 8t\mathbf{j}$
 d $\mathbf{r}(t) = e^t\mathbf{i} + e^{2t}\mathbf{j}$ e $\mathbf{r}(t) = \sin(t)\mathbf{i} - \cos(t)\mathbf{j}$ f $\mathbf{r}(t) = 5 \cos(t)\mathbf{i} - 5 \sin(t)\mathbf{j}$
 g $\mathbf{r}(t) = (2 - 3t)\mathbf{i} - 2 \cos(t)\mathbf{j}$ h $\mathbf{r}(t) = \cos(t)\mathbf{i} - 3 \sin(t)\mathbf{j}$
- 6 Use your answers to question 5 to calculate $|\dot{\mathbf{r}}(t)|$ and $|\ddot{\mathbf{r}}(t)|$ for each part.
- 7 **Example 5** Find $\dot{\mathbf{r}}(t)$ and $\ddot{\mathbf{r}}(t)$ for each of the following.
 a $\mathbf{r}(t) = (5t - 7)\mathbf{i} + \frac{3}{2}t^2\mathbf{j}$ b $\mathbf{r}(t) = 12t\mathbf{i} - (5t - 3)^2\mathbf{j}$ c $\mathbf{r}(t) = e^{8-6t}\mathbf{i} + 6e^{-7t}\mathbf{j}$
 d $\mathbf{r}(t) = -(4e^{\frac{t}{4}+5})\mathbf{i} + 2e^{6-5t}\mathbf{j}$ e $\mathbf{r}(t) = (2 - 3t)^2\mathbf{i} - \cos(3t)\mathbf{j}$ f $\mathbf{r}(t) = \sin^2(t)\mathbf{i} - \tan(t)\mathbf{j}$
- 8 Given $\mathbf{r}(t) = 9t\mathbf{i} - [(3t + 2)^5 - 3t]\mathbf{j}$, find $\dot{\mathbf{r}}(t)$ and $\ddot{\mathbf{r}}(t)$.
- 9 Given $\mathbf{r}(t) = (7 - t)^3\mathbf{i} - \sqrt{t^2 - 3}\mathbf{j}$, find $\dot{\mathbf{r}}(t)$ and $\ddot{\mathbf{r}}(t)$.
- 10 If $\mathbf{u}(t) = t^2\mathbf{i} + (3t - 4t^2)\mathbf{j}$ and $\mathbf{v}(t) = 2t^4\mathbf{i} - t^3\mathbf{j}$, verify that:
 a $\frac{d}{dt}[\mathbf{u}(t) \cdot \mathbf{v}(t)] = \dot{\mathbf{u}} \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \dot{\mathbf{v}}$ b $\frac{d}{dt}[\mathbf{u}(t) \times \mathbf{v}(t)] = \dot{\mathbf{u}} \times \mathbf{v}(t) + \mathbf{u}(t) \times \dot{\mathbf{v}}$
- 11 If $\mathbf{u}(t) = (3t^2 - 2t)\mathbf{i} + (5t - 2t^3)\mathbf{j}$ and $k = 4$, verify that $\frac{d[k\mathbf{u}(t)]}{dt} = k\dot{\mathbf{u}}$
- 12 If $\mathbf{u}(t) = (4t - 3t^2)\mathbf{i} + 7t\mathbf{j}$ and $\mathbf{v}(t) = 3t\mathbf{i} - (t + 6)\mathbf{j}$, verify that $\frac{d}{dt}[\mathbf{u}(t) + \mathbf{v}(t)] = \dot{\mathbf{u}} + \dot{\mathbf{v}}$
- 13 **Example 6** Find the gradient at the point on the curve that is determined by the value of t given for each of the following.
 a $\mathbf{r}(t) = t^2\mathbf{i} - (3t - 7)\mathbf{j}$ at $t = 1$ b $\mathbf{r}(t) = (2t^2 - 5t)\mathbf{i} + 16t\mathbf{j}$ at $t = 2$
 c $\mathbf{r}(t) = \sin(t)\mathbf{i} - \cos(t)\mathbf{j}$ at $t = \frac{\pi}{4}$ d $\mathbf{r}(t) = \cos(t)\mathbf{i} + \sin(t)\mathbf{j}$ at $t = \frac{\pi}{3}$
 e $\mathbf{r}(t) = 2e^t\mathbf{i} - e^{-2t}\mathbf{j}$ at $t = 1$ f $\mathbf{r}(t) = \sin(2\pi t)\mathbf{i} + \cos(\pi t)\mathbf{j}$ at $t = \frac{1}{3}$
- 14 A curve is described by the vector function $\mathbf{r}(t) = \tan(t)\mathbf{i} + \sec(t)\mathbf{j}$ for $-\frac{\pi}{2} < t < \frac{\pi}{2}$.
 a Find the gradient of the curve at the point $P(x, y)$, where $x = \tan(t)$ and $y = \sec(t)$.
 b Find the gradient of the curve when $t = \frac{\pi}{6}$.

Reasoning and communication

- 15 The position of a particle at time t is given by $\mathbf{r}(t) = \sin(t)\mathbf{i} + \cos(t)\mathbf{j}$ where $t > 0$. Prove that $\dot{\mathbf{r}}(t)$ and $\ddot{\mathbf{r}}(t)$ are always perpendicular.
- 16 The position of a particle at time t is given by $\mathbf{r}(t) = (t^2 - 7)\mathbf{i} + 4t\mathbf{j}$, where $t > 0$. Find when the magnitude of the angle between $\dot{\mathbf{r}}(t)$ and $\ddot{\mathbf{r}}(t)$ is 60° .

5.03 INTEGRALS OF VECTOR FUNCTIONS IN THE PLANE

If a vector depends on time, it is often necessary to integrate it with respect to time. Because $\mathbf{i}$ and $\mathbf{j}$ are constant vectors, they are treated as constants in the process of integration.

IMPORTANT

Integration of vector functions

Given $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$,

$$\begin{aligned}\int \mathbf{r}(t) dt &= \int [x(t)\mathbf{i} + y(t)\mathbf{j}] dt \\ &= \int x(t)\mathbf{i} dt + \int y(t)\mathbf{j} dt \\ &= X(t)\mathbf{i} + Y(t)\mathbf{j} + \mathbf{c}\end{aligned}$$

where $\frac{dX}{dt} = x(t)$, $\frac{dY}{dt} = y(t)$, and $\mathbf{c}$ is a constant vector.

It should be noted in the previous definition, $\frac{d\mathbf{c}}{dt} = 0$ as $\mathbf{c}$ is a constant vector.

It is also important to remember that the constant vector $\mathbf{c} = c_1\mathbf{i} + c_2\mathbf{j}$.

The integration of three-dimensional vector functions is a simple extension of the integration of two-dimensional vector functions.

IMPORTANT

Given $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$,

$$\begin{aligned}\int \mathbf{r}(t) dt &= \int [x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}] dt \\ &= \int x(t)\mathbf{i} dt + \int y(t)\mathbf{j} dt + \int z(t)\mathbf{k} dt \\ &= X(t)\mathbf{i} + Y(t)\mathbf{j} + Z(t)\mathbf{k} + \mathbf{c}\end{aligned}$$

where $\frac{dX}{dt} = x(t)$, $\frac{dY}{dt} = y(t)$, $\frac{dZ}{dt} = z(t)$ and $\mathbf{c}$ is a constant vector.



Example 7

Find $\mathbf{r}(t)$ if $\dot{\mathbf{r}}(t) = \cos(2t)\mathbf{i} - \sin(2t)\mathbf{j}$ and $\mathbf{r}(0) = 3\mathbf{i} - 2\mathbf{j}$.

Solution

Use the integral.

$$\mathbf{r}(t) = \int \dot{\mathbf{r}}(t) dt$$

Substitute for $\dot{\mathbf{r}}(t)$.

$$\begin{aligned} &= \int [\cos(2t)\mathbf{i} - \sin(2t)\mathbf{j}] dt \\ &= \int [\cos(2t) dt]\mathbf{i} + \int [-\sin(2t) dt]\mathbf{j} \end{aligned}$$

Find the integrals.

$$= \left[\frac{1}{2} \sin(2t) + c_1 \right] \mathbf{i} + \left[\frac{1}{2} \cos(2t) + c_2 \right] \mathbf{j}$$

Find $\mathbf{r}(0)$.

$$\mathbf{r}(0) = \left[\frac{1}{2} \sin(0) + c_1 \right] \mathbf{i} + \left[\frac{1}{2} \cos(0) + c_2 \right] \mathbf{j}$$

Evaluate.

$$= (0 + c_1)\mathbf{i} + \left(\frac{1}{2} + c_2 \right) \mathbf{j}$$

Use the initial condition $\mathbf{r}(0) = 3\mathbf{i} - 2\mathbf{j}$.

$$3\mathbf{i} - 2\mathbf{j} = c_1\mathbf{i} + \left(\frac{1}{2} + c_2 \right) \mathbf{j}$$

Equate $\mathbf{i}$ components.

$$c_1 = 3$$

Equate $\mathbf{j}$ components.

$$\frac{1}{2} + c_2 = -2$$

$$c_2 = -\frac{5}{2}$$

Use c_1 and c_2 to state $\mathbf{r}(t)$.

$$\mathbf{r}(t) = \left[\frac{1}{2} \sin(2t) + 3 \right] \mathbf{i} + \left[\frac{1}{2} \cos(2t) - \frac{5}{2} \right] \mathbf{j}$$

Example 8

Given $\ddot{\mathbf{r}}(t) = -8\mathbf{i}$, $\mathbf{r}(0) = \mathbf{0}$ and $\dot{\mathbf{r}}(0) = 10\mathbf{i} - 15\mathbf{j}$, calculate $\mathbf{r}(t)$.

Solution

Use the integral to find $\dot{\mathbf{r}}(t)$.

$$\dot{\mathbf{r}}(t) = \int \ddot{\mathbf{r}}(t) dt$$

Substitute for $\ddot{\mathbf{r}}(t)$.

$$= \int -8 dt \mathbf{i}$$

Find the integral.

$$\dot{\mathbf{r}}(t) = (-8t + c_1)\mathbf{i} + c_2\mathbf{j}$$

Find $\dot{\mathbf{r}}(0)$.

$$\dot{\mathbf{r}}(0) = c_1\mathbf{i} + c_2\mathbf{j}$$

Use the condition $\dot{\mathbf{r}}(0) = 10\mathbf{i} - 15\mathbf{j}$.

$$10\mathbf{i} - 15\mathbf{j} = c_1\mathbf{i} + c_2\mathbf{j}$$

Solve by equating $\mathbf{i}$ and $\mathbf{j}$ components.

$$c_1 = 10$$

$$c_2 = -15$$

State $\dot{\mathbf{r}}(t)$.

$$\dot{\mathbf{r}}(t) = (-8t + 10)\mathbf{i} - 15\mathbf{j}$$

Use the integral to find $\mathbf{r}(t)$.

$$\mathbf{r}(t) = \int \dot{\mathbf{r}}(t) dt$$

Substitute for $\dot{\mathbf{r}}(t)$.

$$= \int [(-8t + 10)\mathbf{i} - 15\mathbf{j}] dt$$

Find the integrals.

$$= \int (-8t + 10) dt \mathbf{i} + \int -15 dt \mathbf{j}$$

Use the fact that $\mathbf{r}(0) = \mathbf{0}$.

$$= (-4t^2 + 10t + c_3)\mathbf{i} - (15t + c_4)\mathbf{j}$$

$$\begin{aligned} (-4 \times 0^2 + 10 \times 0 + c_3)\mathbf{i} - (15 \times 0 + c_4)\mathbf{j} &= \mathbf{0}\mathbf{i} + \mathbf{0}\mathbf{j} \\ c_3\mathbf{i} - c_4\mathbf{j} &= \mathbf{0}\mathbf{i} + \mathbf{0}\mathbf{j} \end{aligned}$$

Solve by equating $\mathbf{i}$ and $\mathbf{j}$ components.

$$c_3 = 0$$

$$c_4 = 0$$

Write the expression for $\mathbf{r}(t)$.

$$\mathbf{r}(t) = (-4t^2 + 10t)\mathbf{i} - 15t\mathbf{j}$$

The above definition can be extended to calculate the definite integral of a vector function in space (a three-dimensional vector function).

So, if $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$

$$\begin{aligned} \int_a^b \mathbf{r}(t) dt &= \int_a^b [x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}] dt \\ &= \int_a^b x(t)\mathbf{i} dt + \int_a^b y(t)\mathbf{j} dt + \int_a^b z(t)\mathbf{k} dt \\ &= [X(t)]_a^b \mathbf{i} + [Y(t)]_a^b \mathbf{j} + [Z(t)]_a^b \mathbf{k} \end{aligned}$$

IMPORTANT

Definite integral of a vector function

Given $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$,

$$\begin{aligned} \int_a^b \mathbf{r}(t) dt &= \int_a^b [x(t)\mathbf{i} + y(t)\mathbf{j}] dt \\ &= \int_a^b x(t)\mathbf{i} dt + \int_a^b y(t)\mathbf{j} dt \\ &= [X(t)]_a^b \mathbf{i} + [Y(t)]_a^b \mathbf{j} \end{aligned}$$

where $\frac{dX}{dt} = x(t)$ and $\frac{dY}{dt} = y(t)$.

Example 9

If $\mathbf{r}(t) = \sqrt[3]{t}\mathbf{i} + e^{-t}\mathbf{j}$, evaluate $\int_0^1 \mathbf{r}(t) dt$.

Solution

Write an expression for the definite integral.

$$\int_0^1 \mathbf{r}(t) dt = \int_0^1 (\sqrt[3]{t}\mathbf{i} + e^{-t}\mathbf{j}) dt$$

Express with fractional power.

$$= \int_0^1 t^{\frac{1}{3}}\mathbf{i} dt + \int_0^1 e^{-t}\mathbf{j} dt$$

Find the integrals.

$$= \left[\frac{3}{4} t^{\frac{4}{3}} \right]_0^1 \mathbf{i} + \left[-e^{-t} \right]_0^1 \mathbf{j}$$

Evaluate.

$$= \frac{3}{4}\mathbf{i} + \left(1 - \frac{1}{e} \right) \mathbf{j}$$

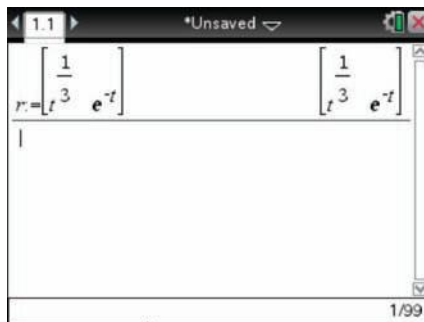
You can use your graphics calculator to check calculations such as those in Example 9.



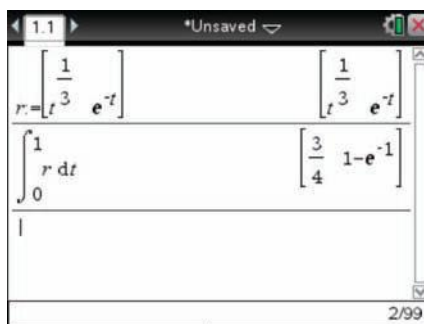
TI-Nspire CAS

First assign the vector $\sqrt[3]{t} \mathbf{i} + e^{-t} \mathbf{j}$ to $\mathbf{r}$ as described previously.

You will need to enter $\sqrt[3]{t}$ as $t^{\frac{1}{3}}$.



To find the definite integral of $\mathbf{r}$, press $\left[\text{menu} \right]$ and select 4:Calculus, 3:Integral. Enter the bounds of integration as 0 and 1 and then enter $\mathbf{r}$ as the vector to be integrated. Enter t at the end of the integration statement. Press ENTER.

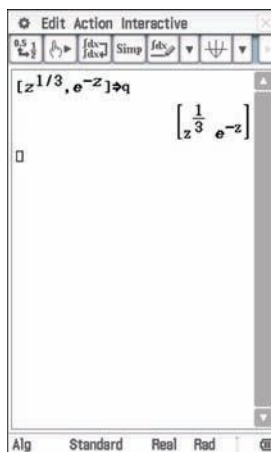


$$\int_0^1 (\sqrt[3]{t} \mathbf{i} + e^{-t} \mathbf{j}) dt = \frac{3}{4} \mathbf{i} + \left(1 - \frac{1}{e}\right) \mathbf{j}$$

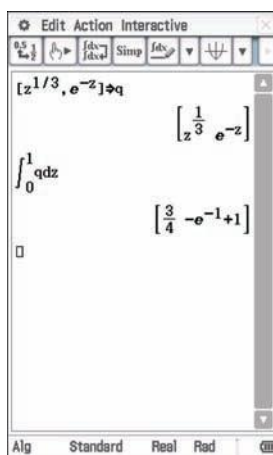
ClassPad

First assign the vector $\sqrt[3]{z} \mathbf{i} + e^{-z} \mathbf{j}$ to $\mathbf{q}$ as described previously.

You will need to enter $\sqrt[3]{z}$ as $z^{\frac{1}{3}}$.



To find the definite integral of $\mathbf{q}$, press **[Keyboard]**, then tap **[Math2]** **[$\int_a^b$]**. Use the arrow keys to move between the empty boxes to enter the details of the calculation. First enter the lower bound of the integration in the lower left box on the integration symbol as 0 and the upper bound in the upper left box as 1. In the first box in the middle of the integration symbol, enter the vector $\mathbf{q}$ and after d given on this line, enter the variable of interest, here z . Press **[EXE]** to perform the calculation.



EXERCISE 5.03 Integrals of vector functions in the plane

Concepts and techniques



- Example 7** If $\dot{\mathbf{r}}(t) = 4t\mathbf{i} - 7\mathbf{j}$ and $\mathbf{r}(0) = \mathbf{i} + 2\mathbf{j}$, then $\mathbf{r}(t)$ is:

A $2t^2\mathbf{i} - 7t\mathbf{j}$ B $(\frac{1}{2}t^2 - 1)\mathbf{i} + (2 + 7t)\mathbf{j}$ C $t\mathbf{i} + 2t\mathbf{j}$
 D $(2t^2 + 1)\mathbf{i} + (2 - 7t)\mathbf{j}$ E $4\mathbf{i}$
- If $\dot{\mathbf{r}}(t) = e^t\mathbf{i} - t\mathbf{j}$ and $\mathbf{r}(0) = 5\mathbf{j}$, then $\mathbf{r}(t)$ is:

A $e^t\mathbf{i} - \frac{1}{2}t^2\mathbf{j}$ B $(e^t - 1)\mathbf{i} - (\frac{1}{2}t^2 - 5)\mathbf{j}$ C $(1 - e^t)\mathbf{i} - \frac{1}{2}t^2\mathbf{j}$
 D $e^t\mathbf{i} + (\frac{1}{2}t^2 - 5)\mathbf{j}$ E $(2e^t + 1)\mathbf{i} - (t^2 - 5)\mathbf{j}$
- Example 8** If $\dot{\mathbf{r}}(t) = -\sin(t)\mathbf{i} + 12t\mathbf{j}$, $\dot{\mathbf{r}}(0) = \mathbf{i} - 3\mathbf{j}$ and $\mathbf{r}(0) = 2\mathbf{i}$, then $\mathbf{r}(t)$ is:

A $[2 + \sin(t)]\mathbf{i} + (2t^3 - 3t)\mathbf{j}$ B $\sin(t)\mathbf{i} - (2t^3 - 3t)\mathbf{j}$ C $[\sin(t) - 2]\mathbf{i} + (t^3 + 3t)\mathbf{j}$
 D $[1 + \sin(t)]\mathbf{i} + (\frac{1}{2}t^2 - 3)\mathbf{j}$ E $[2 - \sin(t)]\mathbf{i} + (2t^3 + 3)\mathbf{j}$
- Example 9** $\int_0^1 [\sin(t)\mathbf{i} + 6\mathbf{j}]dt$ is equal to:

A $\cos(1)\mathbf{i} + 6\mathbf{j}$ B $-\cos(1)\mathbf{i} + 6t\mathbf{j}$ C $[-\cos(1) + 1]\mathbf{i} + 6\mathbf{j}$
 D $2\mathbf{i} + 6\mathbf{j}$ E $\mathbf{i} + 6\mathbf{j}$
- $\int_0^\pi [\sin(t)\mathbf{i} + t\mathbf{j}]dt$ is equal to:

A $-2\mathbf{i} + \pi^2\mathbf{j}$ B $2\mathbf{i} + \frac{1}{2}\pi^2\mathbf{j}$ C $2\mathbf{i} + \frac{1}{2}\pi\mathbf{j}$ D $\frac{1}{2}\pi\mathbf{j}$ E $\mathbf{i} + \frac{1}{2}\pi^2\mathbf{j}$
- Find the indefinite integral for each of the following.

a $\int (2t\mathbf{i} + \mathbf{j})dt$ b $\int (4t^3\mathbf{i} - 4\sqrt{t}\mathbf{j})dt$ c $\int (6t\mathbf{i} - t^{\frac{3}{2}}\mathbf{j})dt$
 d $\int [(5 + 8t)\mathbf{i} + (2 - e^{-t})\mathbf{j}]dt$ e $\int [\cos(t)\mathbf{i} + \sin(2t)\mathbf{j}]dt$ f $\int [2e^{2t}\mathbf{i} - 3\cos(2t)\mathbf{j}]dt$

7 In each of the following, find $\mathbf{r}(t)$ for the given conditions.

a $\dot{\mathbf{r}}(t) = 4e^{2t}\mathbf{i} + 3e^t\mathbf{j}$ and $\mathbf{r}(0) = 2\mathbf{i}$

b $\dot{\mathbf{r}}(t) = 3t^2\mathbf{i} + 6\sqrt{t}\mathbf{j}$ and $\mathbf{r}(0) = \mathbf{i} + 2\mathbf{j}$

c $\dot{\mathbf{r}}(t) = \cos(t)\mathbf{i} + 3\sin(t)\mathbf{j}$ and $\mathbf{r}(0) = 2\mathbf{i} - \mathbf{j}$

d $\dot{\mathbf{r}}(t) = 2t\mathbf{i} + \sin(t)\mathbf{j}$ and $\mathbf{r}(0) = 2\mathbf{i} - \mathbf{j}$

8 In each of the following, find $\mathbf{r}(t)$ for the given conditions.

a $\ddot{\mathbf{r}}(t) = -10\mathbf{j}$, $\dot{\mathbf{r}}(0) = \mathbf{i} + \mathbf{j}$ and $\mathbf{r}(0) = \mathbf{j}$

b $\ddot{\mathbf{r}}(t) = -32\mathbf{j}$, $\dot{\mathbf{r}}(0) = 20\mathbf{i} + 5\mathbf{j}$ and $\mathbf{r}(0) = \mathbf{0}$

c $\ddot{\mathbf{r}}(t) = -4\cos(t)\mathbf{i} - 3\sin(t)\mathbf{j}$, $\dot{\mathbf{r}}(0) = 3\mathbf{j}$ and $\mathbf{r}(0) = 4\mathbf{j}$

9 Evaluate the following definite integrals.

a $\int_0^1 (8t\mathbf{i} + t\mathbf{j}) dt$

b $\int_{-1}^1 (t\mathbf{i} + t^3\mathbf{j}) dt$

c $\int_0^{\frac{\pi}{2}} [\cos(t)\mathbf{i} + \sin(t)\mathbf{j}] dt$

d $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} [2\sin(2t)\mathbf{i} - \cos(t)\mathbf{j}] dt$

e $\int_0^2 (t\mathbf{i} - e^t\mathbf{j}) dt$

f $\int_0^1 (e^{-t}\mathbf{i} + e^{2t}\mathbf{j}) dt$

10 The vector $\mathbf{r}(t)$ is defined by $\mathbf{r}(t) = t^2\mathbf{i} + e^{-t}\mathbf{j}$. Evaluate:

a $\int_0^1 \mathbf{r}(t) dt$

b $\int_2^3 \mathbf{r}(t) dt$

c $\int_1^4 \mathbf{r}(t) dt$

Reasoning and communication

11 Let $\mathbf{u}(t)$ and $\mathbf{v}(t)$ be two two-dimensional vectors. If $\times$ denotes the vector product, is the following result true?

$$\int_{t_1}^{t_2} \mathbf{u}(t) dt \times \int_{t_1}^{t_2} \mathbf{v}(t) dt = \int_{t_1}^{t_2} [\mathbf{u}(t) \times \mathbf{v}(t)] dt$$

5.04 STRAIGHT-LINE MOTION

Consider an object moving along the curve MP , as shown below. Using the reference point O , the position of the object at any time is represented by the vector $\mathbf{r}(t)$.

The time interval for the object to move from A to B is δt .

Velocity, $\mathbf{v}$, is the rate of change of displacement, and speed = $|\mathbf{v}| = v$.

The average velocity, $\bar{\mathbf{v}}$, over the path AB is given by

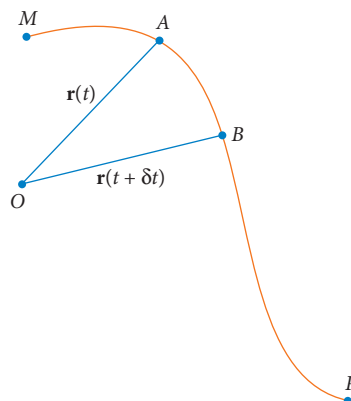
$$\bar{\mathbf{v}} = \frac{\mathbf{r}(t + \delta t) - \mathbf{r}(t)}{\delta t}$$

The instantaneous velocity is given by

$$\mathbf{v} = \lim_{\delta t \rightarrow 0} \frac{\mathbf{r}(t + \delta t) - \mathbf{r}(t)}{\delta t}$$

So $\mathbf{v} = \frac{d\mathbf{r}(t)}{dt}$

If the object's velocity is changing, it is accelerating.



It is important to remember that an accelerating body may have constant speed. The acceleration $\mathbf{a}$ of the body is defined as

$$\mathbf{a} = \lim_{\delta t \rightarrow 0} \frac{\mathbf{v}(t + \delta t) - \mathbf{v}(t)}{\delta t}$$

So $\mathbf{a} = \frac{d\mathbf{v}(t)}{dt}$

The magnitude of the acceleration is given by $|\mathbf{a}| = a$.

It is common to use a shorthand notation as follows.

IMPORTANT

Position, velocity and acceleration

$$\frac{dx(t)}{dt} = \frac{dx}{dt} = \dot{x}$$

$$\mathbf{v} = \frac{d\mathbf{r}(t)}{dt} = \frac{d\mathbf{r}}{dt} = \dot{\mathbf{r}}$$

$$\mathbf{a} = \frac{d\mathbf{v}(t)}{dt} = \frac{d\mathbf{v}}{dt} = \dot{\mathbf{v}} \quad \text{and} \quad \mathbf{a} = \ddot{\mathbf{r}}$$

Consider a particle whose position at time t is given by the vector equation $\mathbf{r}(t) = 10t\mathbf{i} + (20t - 5t^2)\mathbf{j}$.

Find the velocity by taking the derivative of $\mathbf{r}(t)$.

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = 10\mathbf{i} + (20 - 10t)\mathbf{j}$$

Now find the acceleration by finding the derivative of $\mathbf{v}$.

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = -10\mathbf{j}$$

To draw a graph of the particle's path, find the Cartesian equation of its position.

$$\mathbf{r} = 10t\mathbf{i} + (20t - 5t^2)\mathbf{j} = x\mathbf{i} + y\mathbf{j}$$

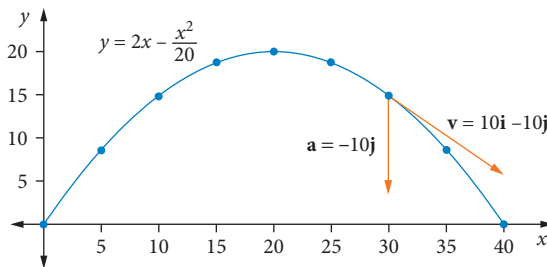
$$x = 10t \quad [1]$$

$$y = 20t - 5t^2 \quad [2]$$

Eliminate t to obtain a Cartesian equation.

$$y = 2x - \frac{x^2}{20}$$

The graph can now be drawn.



The vectors representing the acceleration and velocity are placed on the graph to help you visualise the relationship between them and the position of the particle.

You will note that the velocity is a tangent to the graph, while the acceleration is constant.

The dot product can be used to find the angles between the vectors. If the dot product of two vectors is zero, the angle between the vectors is 90° ; that is, the vectors are perpendicular.



Example 10

The position vector of a particle at time t is given by

$$\mathbf{r}(t) = \sin(2t)\mathbf{i} + \cos(2t)\mathbf{j} + 2t\mathbf{k}.$$

- Find the velocity and acceleration of the particle at time t .
- Show that the magnitudes of both the velocity and the acceleration are constant.
- Show that the acceleration is perpendicular to the velocity.

Solution

- Write the position vector.

Find the derivative.

Differentiate again.

- Write the velocity.

Find the magnitude $\sqrt{x^2 + y^2 + z^2}$.

Use $\cos^2(\theta) + \sin^2(\theta) = 1$.

Write the acceleration.

Find the magnitude.

Use $\sin^2(\theta) + \cos^2(\theta) = 1$.

Write the answer.

- Find the dot product of velocity and acceleration.

Write the answer.

$$\mathbf{r}(t) = \sin(2t)\mathbf{i} + \cos(2t)\mathbf{j} + 2t\mathbf{k}$$

$$\mathbf{v} = \dot{\mathbf{r}} = 2\cos(2t)\mathbf{i} - 2\sin(2t)\mathbf{j} + 2\mathbf{k}$$

$$\mathbf{a} = \dot{\mathbf{v}} = -4\sin(2t)\mathbf{i} - 4\cos(2t)\mathbf{j}$$

$$\mathbf{v} = 2\cos(2t)\mathbf{i} - 2\sin(2t)\mathbf{j} + 2\mathbf{k}$$

$$\begin{aligned} |\mathbf{v}| &= \sqrt{4\cos^2(2t) + 4\sin^2(2t) + 4} \\ &= \sqrt{4[\cos^2(2t) + \sin^2(2t)] + 4} \\ &= \sqrt{4 + 4} = \sqrt{8} = 2\sqrt{2} \end{aligned}$$

$$\mathbf{a} = -4\sin(2t)\mathbf{i} - 4\cos(2t)\mathbf{j}$$

$$\begin{aligned} |\mathbf{a}| &= \sqrt{16\cos^2(2t) + 16\sin^2(2t)} \\ &= \sqrt{16[\cos^2(2t) + \sin^2(2t)]} \\ &= \sqrt{16} = 4 \end{aligned}$$

The magnitudes of the velocity and the acceleration are constant.

$$\begin{aligned} \mathbf{v} \cdot \mathbf{a} &= [2\cos(2t)\mathbf{i} - 2\sin(2t)\mathbf{j} + 2\mathbf{k}] \cdot \\ &\quad [-4\sin(2t)\mathbf{i} - 4\cos(2t)\mathbf{j}] \\ &= -8\cos(2t)\sin(2t) + 8\sin(2t)\cos(2t) \\ &= 0 \end{aligned}$$

Since their dot product is zero, the velocity and acceleration are perpendicular.

Consider a particle that moves such that its position at any time t is described by

$$\mathbf{r}(t) = 5 \cos(2t)\mathbf{i} + 5 \sin(2t)\mathbf{j}$$

The velocity of the particle is found by taking the derivative.

$$\mathbf{v} = \dot{\mathbf{r}} = -10 \sin(2t)\mathbf{i} + 10 \cos(2t)\mathbf{j}$$

The dot product of $\mathbf{r}$ and $\mathbf{v}$ is:

$$\begin{aligned}\mathbf{r} \cdot \mathbf{v} &= [\cos(2t)\mathbf{i} + 5 \sin(2t)\mathbf{j}] \cdot [-10 \sin(2t)\mathbf{i} + 10 \cos(2t)\mathbf{j}] \\ &= -50 \cos(2t) \sin(2t) + 50 \sin(2t) \cos(2t) \\ &= 0\end{aligned}$$

Since the scalar product $\mathbf{r} \cdot \mathbf{v} = 0$, the position vector and the velocity are perpendicular.

Find the acceleration of the particle.

$$\mathbf{a} = \dot{\mathbf{v}} = -20 \cos(2t)\mathbf{i} - 20 \sin(2t)\mathbf{j}$$

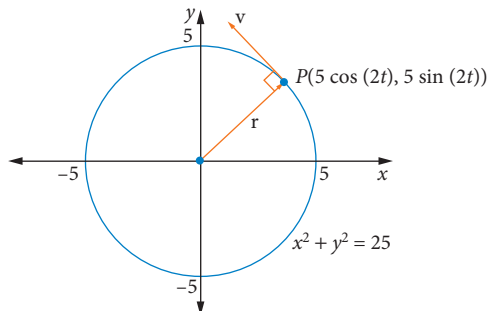
Factorise $\mathbf{a}$ to get it in terms of $\mathbf{r}$.

$$\begin{aligned}\mathbf{a} &= -4[5 \cos(2t)\mathbf{i} + 5 \sin(2t)\mathbf{j}] \\ &= -4\mathbf{r}\end{aligned}$$

Since $\mathbf{a}$ is a multiple of $\mathbf{r}$, it must be parallel to $\mathbf{r}$.

You saw in previous work in this chapter that the Cartesian equation for $\mathbf{r}$ is: $x^2 + y^2 = 25$

This is the equation of a circle centred at the origin with a radius of 5 units, as shown in the diagram below.



These results are true for all particles travelling in a circular path.

Antidifferentiating acceleration yields an expression for velocity, while antidifferentiating velocity yields an expression for displacement.

In other words, the indefinite integral is used to reverse vector derivatives.



○ Example 11

- a** The velocity of a particle at any time t is given by $\mathbf{v} = 6\mathbf{i} - 2t\mathbf{j} + 5t\mathbf{k}$.
Find its position $\mathbf{r}(t)$ at any time, given that $\mathbf{r}(0) = 7\mathbf{i} - \frac{1}{2}\mathbf{k}$.
- b** The combined effects of air and other resistance to motion and gravity on a freely falling object produce an acceleration of $-2\mathbf{i} - 3\mathbf{j} + 4t\mathbf{k}$. Four seconds after beginning to fall, the object has a velocity of $42\mathbf{i} - 12\mathbf{j} + 32\mathbf{k}$. Find an expression for its velocity at any time t .

Solution

- a** Write the expression for velocity.

$$\mathbf{v} = 6\mathbf{i} - 2t\mathbf{j} + 5t\mathbf{k}$$

Find the integral.

$$\mathbf{r}(t) = \int (6\mathbf{i} - 2t\mathbf{j} + 5t\mathbf{k}) dt$$

The constant of integration is a vector.

$$= 6t\mathbf{i} - t^2\mathbf{j} + \frac{5}{2}t^2\mathbf{k} + \mathbf{c}$$

Substitute $t = 0$.

$$\mathbf{r}(0) = \mathbf{c}$$

Use the given value of $\mathbf{r}(0)$ to find $\mathbf{c}$.

$$\mathbf{c} = 7\mathbf{i} - \frac{1}{2}\mathbf{k}$$

Substitute into $\mathbf{r}(t)$.

$$\mathbf{r}(t) = 6t\mathbf{i} - t^2\mathbf{j} + \frac{5}{2}t^2\mathbf{k} + 7\mathbf{i} - \frac{1}{2}\mathbf{k}$$

Factorise and write the answer.

$$= (6t + 7)\mathbf{i} - t^2\mathbf{j} + \left(\frac{5}{2}t^2 - \frac{1}{2}\right)\mathbf{k}$$

- b** Write the acceleration.

$$\mathbf{a}(t) = -2\mathbf{i} - 3\mathbf{j} + 4t\mathbf{k}$$

Find the integral.

$$\begin{aligned}\mathbf{v}(t) &= \int (-2\mathbf{i} - 3\mathbf{j} + 4t\mathbf{k}) dt \\ &= -2t\mathbf{i} - 3t\mathbf{j} + 2t^2\mathbf{k} + \mathbf{c}\end{aligned}$$

Substitute $t = 4$.

$$\mathbf{v}(4) = -8\mathbf{i} - 12\mathbf{j} + 32\mathbf{k} + \mathbf{c}$$

Write the given value of $\mathbf{v}(4)$.

$$\mathbf{v}(4) = 42\mathbf{i} - 12\mathbf{j} + 32\mathbf{k}$$

Equate the values of $\mathbf{v}(4)$ to find $\mathbf{c}$.

$$\mathbf{c} = 50\mathbf{i}$$

Substitute into $\mathbf{v}(t)$.

$$\mathbf{v}(t) = -2t\mathbf{i} - 3t\mathbf{j} + 2t^2\mathbf{k} + 50\mathbf{i}$$

Factorise and write the answer.

$$= (50 - 2t)\mathbf{i} - 3t\mathbf{j} + 2t^2\mathbf{k}$$

○ Example 12

A particle, P , begins from a point $10\mathbf{j}$ m and continues with constant velocity $2\mathbf{i} - \mathbf{j}$ ms⁻¹. One second later another particle, Q , starts at the point $2\mathbf{i} + 15\mathbf{j}$ m and moves with constant velocity $2\mathbf{i} - 5\mathbf{j}$ ms⁻¹.

- Show that the particles collide.
- Find the Cartesian equations of their paths.
- Find the Cartesian coordinates for the point of collision.

Solution

- Write a variable for time.

Let t s be the time after P starts.

Write the velocity of P .

$$\mathbf{v}_P = 2\mathbf{i} - \mathbf{j}$$

Integrate.

$$\mathbf{r}_P = 2t\mathbf{i} - t\mathbf{j} + \mathbf{c}_1$$

Substitute $t = 0$.

$$10\mathbf{j} = 2 \times 0\mathbf{i} - 0\mathbf{j} + \mathbf{c}_1$$

Find $\mathbf{c}_1$.

$$\mathbf{c}_1 = 10\mathbf{j}$$

Rewrite $\mathbf{r}_P$.

$$\begin{aligned}\mathbf{r}_P &= 2t\mathbf{i} - t\mathbf{j} + 10\mathbf{j} \\ &= 2t\mathbf{i} + (10 - t)\mathbf{j}\end{aligned}\quad [1]$$

Find $\mathbf{r}_Q$ in the same way.

$$\mathbf{v}_Q = 2\mathbf{i} - 5\mathbf{j}$$

Integrate.

$$\mathbf{r}_Q = 2t\mathbf{i} - 5t\mathbf{j} + \mathbf{c}_2$$

Substitute $t = 1$.

$$2\mathbf{i} + 15\mathbf{j} = 2\mathbf{i} - 5\mathbf{j} + \mathbf{c}_2$$

Find $\mathbf{c}_2$.

$$\mathbf{c}_2 = 20\mathbf{j}$$

Rewrite $\mathbf{r}_Q$.

$$\begin{aligned}\mathbf{r}_Q &= 2t\mathbf{i} - 5t\mathbf{j} + 20\mathbf{j} \\ &= 2t\mathbf{i} + (20 - 5t)\mathbf{j}\end{aligned}\quad [2]$$

If the particles collide, they will be in the same place at the same time, so $\mathbf{r}_P = \mathbf{r}_Q$ for some time t .

Write the condition in terms of the components.

$$2t\mathbf{i} + (10 - t)\mathbf{j} = 2t\mathbf{i} + (20 - 5t)\mathbf{j}$$

The components must be equal.

$$2t\mathbf{i} = 2t\mathbf{i} \text{ and } (10 - t)\mathbf{j} = (20 - 5t)\mathbf{j}$$

The $\mathbf{i}$ component is OK. Equate the coefficients of $\mathbf{j}$ to find t .

$$\begin{aligned}10 - t &= 20 - 5t \\ t &= 2.5\end{aligned}$$

Write the answer.

The particles collide 2.5 s after P starts moving.



- b To find the Cartesian equations of the paths of the particles, eliminate t from [1] and [2].

Write [1] in component terms.

$$\begin{aligned}\mathbf{r}_P &= 2t\mathbf{i} + (10 - t)\mathbf{j} \\ &= x\mathbf{i} + y\mathbf{j} \\ x &= 2t, y = 10 - t\end{aligned}$$

Eliminate t .

$$y = 10 - \frac{x}{2}$$

Write in standard form.

$$x + 2y - 20 = 0$$

Write [2] in component terms.

$$\begin{aligned}\mathbf{r}_Q &= 2t\mathbf{i} + (20 - 5t)\mathbf{j} \\ &= x\mathbf{i} + y\mathbf{j} \\ x &= 2t, y = 20 - 5t\end{aligned}$$

Eliminate t .

$$y = 20 - \frac{5x}{2}$$

Write in standard form.

$$5x + 2y - 40 = 0$$

Write the answer.

The Cartesian equations of the paths of P and Q are $x + 2y = 20$ and $5x + 2y = 40$ respectively.

- c Write the equations.

$$x + 2y = 20 \quad [3]$$

$$5x + 2y = 40 \quad [4]$$

Solve simultaneously using [4] - [3].

$$4x = 20, \text{ so } x = 5$$

Substitute into [3].

$$5 + 2y = 20, \text{ so } y = 7.5$$

Check in both equations.

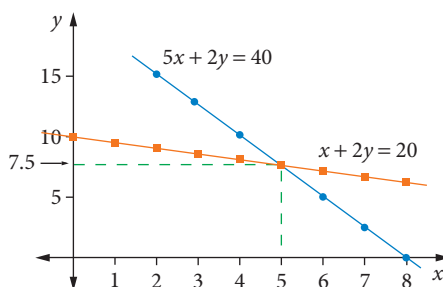
$$x + 2y = 5 + 2 \times 7.5 = 20 \quad \checkmark$$

$$5x + 2y = 5 \times 5 + 2 \times 7.5 = 40 \quad \checkmark$$

Write the answer.

The particles collide at the point $(5, 7.5)$.

In Example 12, you could find the point of intersection of the particles by drawing a graph that shows their positions. This is shown in the diagram below.



EXERCISE 5.04 Straight-line motion

Concepts and techniques

- 1 **Example 10** The position vector of a particle is given by $\mathbf{r}(t) = (t^2 - t)\mathbf{i} - (t^3 - 5)\mathbf{j}$. The magnitude of the velocity at $t = 2$ is:

A $\sqrt{13}$ B $\sqrt{120}$ C $\sqrt{135}$ D $\sqrt{153}$ E 15

- 2 **Example 11** The initial position and initial velocity of a particle are $2\mathbf{j}$ and $3\mathbf{i}$ respectively. If the acceleration of the particle is given by $\mathbf{i} + 2\mathbf{j}$, the position vector of the particle is given by:

A $(t + 3)\mathbf{i} + 2t\mathbf{j}$ B $\left(\frac{1}{2}t^2 + 3t\right)\mathbf{i} + (t^2 + 2)\mathbf{j}$ C $\frac{1}{2}t^2\mathbf{i} + t^2\mathbf{j}$
D $\left(\frac{1}{2}t^2 - 3\right)\mathbf{i} + (t^2 - 2t)\mathbf{j}$ E $3\mathbf{i} + 2\mathbf{j}$

- 3 The acceleration of a particle at any time, t , is given by $(6t - 1)\mathbf{i} + 4\mathbf{j}$. If the initial velocity of the particle is given by $2\mathbf{i} - \mathbf{j}$, the velocity vector of the particle at $t = 2$ is given by:

A $10\mathbf{i} + 8t\mathbf{j}$ B $4\mathbf{i} + 2\mathbf{j}$ C $2\mathbf{i} - \mathbf{j}$ D $14\mathbf{i} + 10\mathbf{j}$ E $12\mathbf{i} + 7\mathbf{j}$

- 4 The velocities of a number of particles are listed below. Find the magnitude of the velocity in each case.

a $5\mathbf{i} + 12\mathbf{j}$ b $3\mathbf{i} - 4\mathbf{j}$ c $2t\mathbf{i} + 5\mathbf{j}$
d $4\mathbf{i} - 7t\mathbf{j}$ e $3 \cos(\theta)\mathbf{i} + 3 \sin(\theta)\mathbf{j}$ f $\sqrt{2t}\mathbf{i} + \frac{3}{t}\mathbf{j}$

- 5 The position ($\mathbf{r}$) of a number of points at time t is given below. For each point, find:

i the velocity
ii the acceleration
iii the magnitude of the velocity.

a $\mathbf{r} = 3\mathbf{i} - 4t\mathbf{j}$ b $\mathbf{r} = \frac{a}{t}\mathbf{i} + bt\mathbf{j}$ c $\mathbf{r} = \cos(3t)\mathbf{i} + \sin(3t)\mathbf{j}$
d $\mathbf{r} = 3 \sin(2t)\mathbf{i} - 3 \cos(2t)\mathbf{j}$ e $\mathbf{r} = 8\mathbf{i} - 3t\mathbf{j} + 4t\mathbf{k}$ f $\mathbf{r} = a \sin(5t)\mathbf{i} + b \cos(5t)\mathbf{j}$
g $\mathbf{r} = 2t\mathbf{i} + t^2\mathbf{j} - 4t\mathbf{k}$ h $\mathbf{r} = 2e^t\mathbf{i} + e^{-2t}\mathbf{j}$ i $\mathbf{r} = \sin(t)\mathbf{i} - \cos(t)\mathbf{j} + 3t\mathbf{k}$
j $\mathbf{r} = 2t^2\mathbf{i} + e^{2t}\mathbf{j} + 3 \cos(2t)\mathbf{k}$

- 6 Given the following information regarding a particle at time t , determine an expression for the acceleration and position of the particle in each case.

a $\mathbf{v} = 5t\mathbf{i} - 2t\mathbf{j}$ and $\mathbf{r} = 3\mathbf{i} - 4\mathbf{j}$ at $t = 0$
b $\mathbf{v} = 4 \sin(t)\mathbf{i} - \cos(3t)\mathbf{j}$ and $\mathbf{r} = 6\mathbf{j}$ at $t = 0$
c $\mathbf{v} = \frac{2}{t}\mathbf{i} - 3\mathbf{j}$ and $\mathbf{r} = 5\mathbf{i} - 3\mathbf{j}$ at $t = 1$
d $\mathbf{v} = e^{2t}\mathbf{i} + (3t^2 + 4)\mathbf{j}$ and $\mathbf{r} = 2\mathbf{i} + \mathbf{j}$ at $t = 1$
e $\mathbf{v} = 3t^2\mathbf{i} + 4\mathbf{j}$ and $\mathbf{r} = \mathbf{i} - \mathbf{j}$ at $t = -2$

- 7 **Example 12** Find the Cartesian equations for the paths of particles with the position vectors described as follows.

a $\mathbf{r} = 2t\mathbf{i} + t\mathbf{j}$ b $\mathbf{r} = 3t\mathbf{i} - 4t\mathbf{j}$ c $\mathbf{r} = (4t + 3)\mathbf{i} + (3 - 2t)\mathbf{j}$
d $\mathbf{r} = 3t^2\mathbf{i} - 4t\mathbf{j}$ e $\mathbf{r} = 3t\mathbf{i} - (11 - 9t)\mathbf{j}$ f $\mathbf{r} = 2 \sin(t)\mathbf{i} + 5 \cos(t)\mathbf{j}$
g $\mathbf{r} = \sqrt{t}\mathbf{i} + (3t^2 - 1)\mathbf{j}$ h $\mathbf{r} = [2 + \sin(t)]\mathbf{i} + [4 - \cos(t)]\mathbf{j}$



Position, velocity and acceleration



Reasoning and communication

- 8 The displacement of a particle at time t is given by $\mathbf{r} = 3 \cos(2t)\mathbf{i} + 3 \sin(2t)\mathbf{j}$.
- Show that the velocity is always perpendicular to the displacement.
 - Describe the direction of the acceleration relative to the displacement.
 - Describe the path of the particle.
- 9 The displacement of a particle at time t is given by $\mathbf{r} = 2 \cos(t)\mathbf{i} + \sqrt{2} \sin(t)\mathbf{j} - \sqrt{2} \sin(t)\mathbf{k}$.
- Show that the speed of the particle is constant.
 - Describe the direction of the acceleration relative to the velocity.
- 10 A particle begins at $-3\mathbf{j}$ and moves with constant velocity $3\mathbf{i} + 2\mathbf{j}$. At the same time, a second particle begins from position $-4\mathbf{i} + \mathbf{j}$ with velocity $2t\mathbf{i} + \mathbf{j}$.
- Find the position of each particle at time t .
 - Show that the two particles collide.
 - Find the time and place of the collision.
- 11 The position of a particle at time t is given by $\mathbf{r} = \frac{1}{t}\mathbf{i} + 9t\mathbf{j}$.
- Describe the path taken by the particle.
 - Sketch the path of the particle from $t = 1$ to $t = 5$.
 - Calculate the velocity and acceleration of the particle at $t = 1$ and $t = 5$.
 - Show the velocity and acceleration at $t = 1$ and $t = 5$ as vectors on the sketch made in part **b**.
- 12 An object moves in a straight line so that its acceleration at t s is given by $\mathbf{a} = (3 - 4t)\mathbf{i}$. Initially, the object has a speed of 2 m s^{-1} in the $\mathbf{i}$ direction and is 5 m in the $\mathbf{i}$ direction from a fixed point, O .
- Calculate its velocity and position after t s.
 - When and where is the body instantaneously at rest?
- 13 The position of an object at time t is given by $\mathbf{r} = 4 \cos(t)\mathbf{i} + 4 \sin(t)\mathbf{j}$.
- Show that the path taken by the object is circular.
 - Find the velocity and acceleration of the object at time t .
 - Describe the relationship between the velocity and the position at any time.
 - Describe the relationship between the acceleration and the position at any time.
- 14 A particle moves so that its position is given by $\mathbf{r} = 3 \cos(t)\mathbf{i} + 5 \sin(t)\mathbf{j}$.
- Describe the path taken by the particle.
 - Determine expressions for the velocity and acceleration of the particle at time t .
 - When are the particle's velocity and acceleration perpendicular to each other?
 - Compare the direction and magnitude of the acceleration and the position of the particle.
- 15 The acceleration and initial velocity of a particle are given by $\mathbf{a} = 2t\mathbf{i} - 3\mathbf{j}$ and $\mathbf{v} = -4\mathbf{i} + 6\mathbf{j}$.
- Find an expression for the velocity of the particle.
 - When is the velocity zero?
 - What is the magnitude of its acceleration at this time?
- 16 An object is projected so that its position at any time t is given by $\mathbf{r} = 2t\mathbf{i} + (8t - t^2)\mathbf{j}$, where the components are measured in metres and the time in seconds.
- Find the point of projection of the object.
 - Calculate the speed of projection.
 - Determine the Cartesian equation for the path of the object.

- 17 The velocities of two particles P and Q , moving in the Cartesian plane, are given by $\mathbf{v}_P = 2\mathbf{i} + 2t\mathbf{j}$ and $\mathbf{v}_Q = 3\mathbf{i} + 5\mathbf{j}$.
At $t = 0$, the positions of the particles are $\mathbf{r}_P = 5\mathbf{i} - 6\mathbf{j}$ and $\mathbf{r}_Q = -\mathbf{i}$.
- Find the position of each particle at any time t .
 - Where and when do the particles collide?
 - Calculate the Cartesian equation for the path of each particle.
- 18 An object is projected in the xy plane so that its position at any time t is given by $\mathbf{r} = 4t\mathbf{i} + (6 - 5t^2)\mathbf{j}$ where the components are measured in metres and the time in seconds.
- From what position is the object projected?
 - Calculate the speed of projection.
 - At what angle to the horizontal is it projected?
 - Show that the acceleration is constant.

5.05 MOTION WITH CONSTANT ACCELERATION

When you throw a ball, shoot an arrow, or jump across a puddle, a force is applied before release. This force produces an initial velocity, but plays no further part in the motion. The path followed by the object is determined solely by gravity and air resistance. This is called **projectile motion**. For low speeds and short distances, air resistance is relatively small, so it may be neglected.



Alamy/Greg Balfour Evans

IMPORTANT

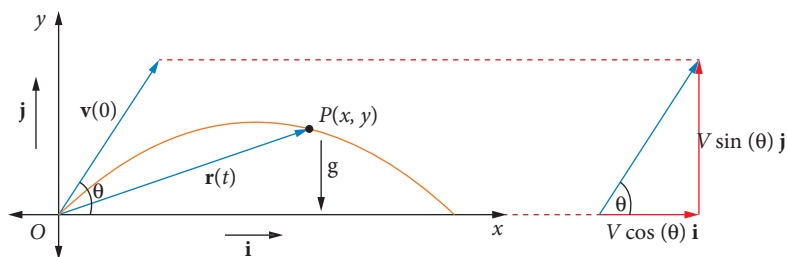
Projectile motion

A **projectile** is launched with an initial velocity comprising of its **speed of projection** and its **angle of projection**. Its motion is then determined solely by gravitational acceleration (g) and air resistance. In simple cases, air resistance may be neglected.

The path of a projectile is called its **trajectory**. Its final horizontal displacement is called its **range**.

In this section the effect of wind, air resistance, etc. on the trajectory of a projectile is ignored. Acceleration due to gravity varies, but near the surface of the Earth it is assumed to be $g \approx 9.8 \text{ m s}^{-2}$.

Consider a projectile with an initial speed V at an angle of projection θ . Choose the origin of the coordinate system to be the point of projection, the x -axis to be the horizontal, and the y -axis to be the vertical. Write the position of the projectile at time t as $P(x, y)$.



The position vector of the projectile is $\mathbf{r}(t) = x\mathbf{i} + y\mathbf{j}$.

From the diagrams, the initial vector velocity is given by $\mathbf{v}(0) = V \cos(\theta)\mathbf{i} + V \sin(\theta)\mathbf{j}$.

You can write acceleration as $\mathbf{a}(t) = a_x(t)\mathbf{i} + a_y(t)\mathbf{j}$ where $a_x(t)$ and $a_y(t)$ are the components of acceleration in the $\mathbf{i}$ and $\mathbf{j}$ directions respectively.

However, the only acceleration is downward, due to gravity, so:

$$\mathbf{a}(t) = 0\mathbf{i} - g\mathbf{j}$$

$$\text{or } \mathbf{a}(t) = -g\mathbf{j}$$

In component form:

$$a_x(t) = \ddot{x}(t) = 0 \text{ and } a_y(t) = \ddot{y}(t) = -g$$

IMPORTANT

Acceleration (projectile motion)

Vector form: $\mathbf{a}(t) = -g\mathbf{j}$

Horizontal component: $a_x(t) = \ddot{x}(t) = 0$

Vertical component: $a_y(t) = \ddot{y}(t) = -g$

To find the vector velocity $\mathbf{v}(t)$ for the projectile, find the integral of acceleration. As the projectile has an initial velocity, $\mathbf{v}(0)$, this will be the constant of integration.

$$\begin{aligned} \mathbf{v}(t) &= \int \mathbf{a}(t) dt + \mathbf{v}(0) \\ &= \int (-g\mathbf{j}) dt + [V \cos(\theta)\mathbf{i} + V \sin(\theta)\mathbf{j}] \\ &= V \cos(\theta)\mathbf{i} + [-gt\mathbf{j} + V \sin(\theta)\mathbf{j}] \end{aligned}$$

$$\mathbf{v}(t) = V \cos(\theta)\mathbf{i} + [V \sin(\theta) - gt]\mathbf{j}$$

Write this in component form:

$$\mathbf{v}(t) = v_x(t)\mathbf{i} + v_y(t)\mathbf{j}$$

where $v_x(t)$ and $v_y(t)$ are the components of velocity in the $\mathbf{i}$ and $\mathbf{j}$ directions respectively.

So $v_x(t) = \dot{x}(t) = V \cos(\theta)$ and $v_y(t) = \dot{y}(t) = V \sin(\theta) - gt$

From these equations, you can see that the horizontal component of velocity is a constant and equal to $V \cos(\theta)$. The vertical component of velocity varies and is equal to $V \sin(\theta) - gt$.

The speed of the projectile at any time, $v(t)$, is simply the magnitude of the velocity vector $\mathbf{v}(t)$.

$$v(t) = |\mathbf{v}(t)| = \sqrt{\mathbf{v}(t) \cdot \mathbf{v}(t)}$$

Once the projectile is in motion, the vertical component of velocity is reduced by the acceleration due to gravity. The projectile reaches its maximum height when the vertical component of velocity is zero. Calculate the time taken to reach the maximum height.

$$v_y(t) = V \sin(\theta) - gt = 0$$

$$gt = V \sin(\theta)$$

$$t = \frac{V \sin(\theta)}{g}$$

If the initial and terminal positions of the projectile are in the same horizontal plane, the total time it is in the air is twice the time required to reach its maximum height.

IMPORTANT

Velocity (projectile motion)

Vector form: $\mathbf{v}(t) = V \cos(\theta) \mathbf{i} + [V \sin(\theta) - gt] \mathbf{j}$

Horizontal component: $v_x(t) = \dot{x}(t) = V \cos(\theta)$

Vertical component: $v_y(t) = \dot{y}(t) = V \sin(\theta) - gt$

Speed = $v(t) = |\mathbf{v}(t)| = \sqrt{\mathbf{v}(t) \cdot \mathbf{v}(t)}$

Time of flight = $T = \frac{2V \sin(\theta)}{g}$

Time to reach maximum height = $T_{\frac{1}{2}} = \frac{V \sin(\theta)}{g}$

Find a vector equation for the position of the projectile, $\mathbf{r}(t)$, by finding the integral of the velocity vector. It is assumed (unless otherwise stated) that the initial position of the projectile is the origin, so:

$$\mathbf{r}(0) = 0\mathbf{i} + 0\mathbf{j}$$

and this is the constant of integration.

$$\mathbf{r}(t) = \int \mathbf{v}(t) + \mathbf{r}(0)$$

$$= \int [V \cos(\theta) \mathbf{i} + (V \sin(\theta) - gt) \mathbf{j}] dt + 0\mathbf{i} + 0\mathbf{j}$$

$$\mathbf{r}(t) = Vt \cos(\theta) \mathbf{i} + \left[Vt \sin(\theta) - \frac{1}{2}gt^2 \right] \mathbf{j}$$

In component form, this becomes:

$$\mathbf{r}(t) = x(t) \mathbf{i} + y(t) \mathbf{j}$$

So $x(t) = Vt \cos(\theta)$ and $y(t) = Vt \sin(\theta) - \frac{1}{2}gt^2$



IMPORTANT

Position (projectile motion)

Vector form: $\mathbf{r}(t) = Vt \cos(\theta)\mathbf{i} + \left[Vt \sin(\theta) - \frac{1}{2}gt^2 \right]\mathbf{j}$

Horizontal component: $x(t) = Vt \cos(\theta)$

Vertical component: $y(t) = Vt \sin(\theta) - \frac{1}{2}gt^2$

The last two equations stated above are the parametric equations for the position of a projectile under gravity. If desired, the equation of the trajectory can be found by eliminating time from the equations.

Write the equation for horizontal displacement. $x = [V \cos(\theta)]t$

Rearrange to obtain t . $t = \frac{x}{V \cos(\theta)}$

Substitute for t in the equation for vertical displacement and simplify.

$$\begin{aligned} y &= [V \sin(\theta)]t - \frac{1}{2}gt^2 \\ &= \frac{[V \sin(\theta)]x}{V \cos(\theta)} - \frac{1}{2}g \left[\frac{x}{V \cos(\theta)} \right]^2 \\ &= x \tan(\theta) - \frac{gx^2}{2V^2 \cos^2(\theta)} \end{aligned}$$

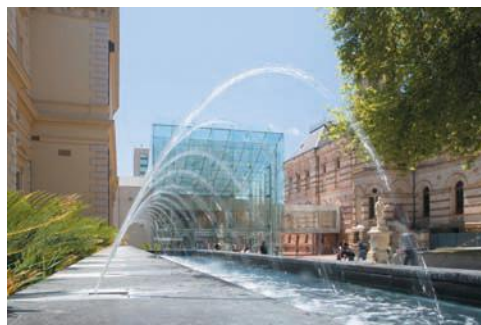
It is possible to show that:

$$\frac{1}{\cos^2(\theta)} = \tan^2(\theta) + 1$$

Hence the equation for the trajectory can also be written as

$$y = x \tan(\theta) - \frac{gx^2}{2V^2} [\tan^2(\theta) + 1]$$

so that only one trigonometric function is involved.



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IMPORTANT

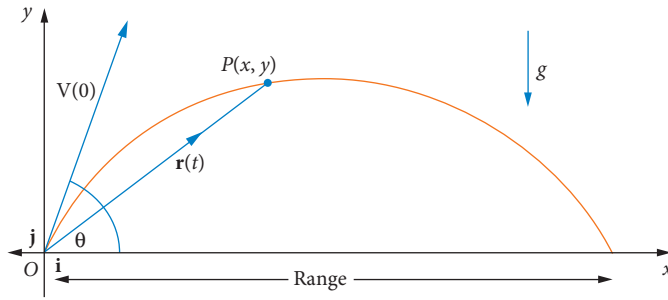
Equation of a trajectory

The general equation for a trajectory is:

$$y = x \tan(\theta) - \frac{gx^2}{2V^2 \cos^2(\theta)} \quad \text{or} \quad y = x \tan(\theta) - \frac{gx^2}{2V^2} [\tan^2(\theta) + 1]$$

A trajectory is *parabolic* in shape because the height y is a quadratic function of the horizontal displacement x .

You already know equations for the time of flight and the time taken to reach the maximum height for the special case where the projectile motion occurs above a flat plane.



At the maximum height, the vertical velocity must be zero. This occurs at $t = \frac{V \sin(\theta)}{g}$.

The maximum height is the vertical displacement at $t = \frac{V \sin(\theta)}{g}$.

Write the vertical displacement.

$$y(t) = Vt \sin(\theta) - \frac{1}{2}gt^2$$

This is a maximum at $t = \frac{V \sin(\theta)}{g}$.

$$\begin{aligned} y_{\max}(t) &= V \sin(\theta) \times \frac{V \sin(\theta)}{g} - \frac{1}{2}g \left[\frac{V \sin(\theta)}{g} \right]^2 \\ &= \frac{V^2 \sin^2(\theta)}{g} - \frac{1}{2} \frac{V^2 \sin^2(\theta)}{g} \\ &= \frac{V^2 \sin^2(\theta)}{2g} \end{aligned}$$

Write the horizontal displacement.

$$x(t) = Vt \cos(\theta)$$

The range is the horizontal displacement at $t = \frac{2V \sin(\theta)}{g}$.

Substitute for t and simplify.

$$\begin{aligned} x_{\max}(t) &= V \cos(\theta) \times \frac{2V \sin(\theta)}{g} \\ &= \frac{2V^2 \cos(\theta) \sin(\theta)}{g} \\ &= \frac{V^2 \sin(2\theta)}{g} \end{aligned}$$

Use the property $2 \sin(\theta) \cos(\theta) = \sin(2\theta)$



Special projectile motion results

For projectile motion above a flat plane, the following formulas apply.

$$\text{Maximum height: } y_{\max}(t) = \frac{V^2 \sin^2(\theta)}{2g} \text{ at time } T_{\frac{1}{2}} = \frac{V \sin(\theta)}{g}$$

$$\text{Range: } x_{\max}(t) = \frac{2V^2 \cos(\theta) \sin(\theta)}{g} = \frac{V^2 \sin(2\theta)}{g} \text{ at time } T = \frac{2V \sin(\theta)}{g}$$

$$\text{Maximum possible range (at } \theta = 45^\circ) = \frac{V^2}{g}.$$

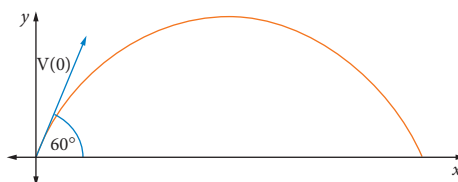
Example 13

A projectile is launched from a horizontal plane with an initial velocity of 40 m s^{-1} at a 60° angle to the horizontal. Calculate:

- the equation of the trajectory
- the time of flight
- the range
- the greatest height reached
- the position vector $\mathbf{r}(t)$ and hence the distance of the projectile from its initial position 1 second after its launch.

Solution

Draw a diagram.



- Write the horizontal displacement.
Substitute in the values and simplify.

$$\begin{aligned} x(t) &= [V \cos(\theta)]t \\ &= [40 \cos(60^\circ)]t \\ &= 20t \end{aligned} \quad [1]$$

- Write the vertical displacement.
Substitute in the values and simplify.

$$\begin{aligned} y(t) &= [V \sin(\theta)]t - \frac{1}{2}gt^2 \\ &\approx [40 \sin(60^\circ)]t - \frac{1}{2} \times 9.8t^2 \\ &= \left(40 \times \frac{\sqrt{3}}{2}\right)t - 4.9t^2 \\ &= 20\sqrt{3}t - 4.9t^2 \end{aligned} \quad [2]$$

- Rearrange [1] to obtain t .

$$t = \frac{x}{20}$$

- Substitute for t in [2] and simplify.

$$y \approx 20\sqrt{3} \times \frac{x}{20} - 4.9 \times \frac{x^2}{400}$$

- Simplify to write the equation of the trajectory.

$$= \sqrt{3}x - 0.01225x^2$$

- b Write the rule for time of flight.

Substitute in the values and simplify.

Write the answer.

- c Write the rule for range.

Substitute in the values and simplify.

Write the answer.

- d Write the rule for the maximum height.

Substitute in the values and simplify.

Write the answer.

- e Write the equation for velocity.

Substitute for values and simplify.

Express $\mathbf{r}(t)$ as the antiderivative of velocity.

Integrate. $\mathbf{r}(0) = (0\mathbf{i} + 0\mathbf{j})$, so substitute $(0\mathbf{i} + 0\mathbf{j})$ for $\mathbf{c}$ and then simplify.

Calculate $\mathbf{r}(1)$.

Calculate displacement in the first second.

The distance is equal to the magnitude of the displacement.

State the result.

$$\begin{aligned} T &= \frac{2V \sin(\theta)}{g} \\ &= \frac{2 \times 40 \sin(60^\circ)}{9.8} \\ &\approx 7.07 \end{aligned}$$

The time of flight is about 7.07 s.

$$\begin{aligned} x_{\max}(t) &= \frac{V^2 \sin(2\theta)}{g} \\ &= \frac{40^2 \sin(120^\circ)}{9.8} \\ &\approx 141.392 \end{aligned}$$

The range is about 141 m.

$$\begin{aligned} y_{\max}(t) &= \frac{V^2 \sin^2(\theta)}{2g} \\ &= \frac{40^2 \sin^2(60^\circ)}{2 \times 9.8} \\ &\approx 61.22 \end{aligned}$$

The greatest height is about 61 m.

$$\begin{aligned} \mathbf{v}(t) &= V \cos(\theta)\mathbf{i} + [V \sin(\theta) - gt]\mathbf{j} \\ &= 40 \cos(60^\circ)\mathbf{i} + [40 \sin(60^\circ) - 9.8t]\mathbf{j} \\ &= 20\mathbf{i} + (20\sqrt{3} - 9.8t)\mathbf{j} \end{aligned}$$

$$\begin{aligned} \mathbf{r}(t) &= \int [20\mathbf{i} + (20\sqrt{3} - 9.8t)\mathbf{j}] dt + \mathbf{c} \\ &= 20t\mathbf{i} + (20\sqrt{3}t - 4.9t^2)\mathbf{j} + (0\mathbf{i} + 0\mathbf{j}) \\ &= 20t\mathbf{i} + (20\sqrt{3}t - 4.9t^2)\mathbf{j} \end{aligned}$$

$$\begin{aligned} \mathbf{r}(1) &= (20 \times 1)\mathbf{i} + (20\sqrt{3} \times 1 - 4.9 \times 1^2)\mathbf{j} \\ &\approx 20\mathbf{i} + 29.74\mathbf{j} \end{aligned}$$

$$\begin{aligned} \text{Displacement} &= \mathbf{r}(1) - \mathbf{r}(0) \\ &= 20\mathbf{i} + 29.74\mathbf{j} - (0\mathbf{i} + 0\mathbf{j}) \\ &= 20\mathbf{i} + 29.74\mathbf{j} \end{aligned}$$

$$\begin{aligned} \text{Distance} &= \sqrt{20^2 + 29.74^2} \\ &\approx 35.84 \end{aligned}$$

The projectile is about 36 m from its initial position 1 second after its launch.

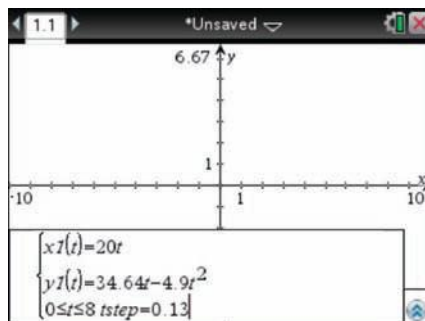


You can use a graphics calculator to examine the path of a projectile. Consider the case in Example 13.

The parametric equations for $x(t)$ and $y(t)$ can be written as $x(t) = 20t$ and $y(t) \approx 34.64t - 4.9t^2$

TI-Nspire CAS

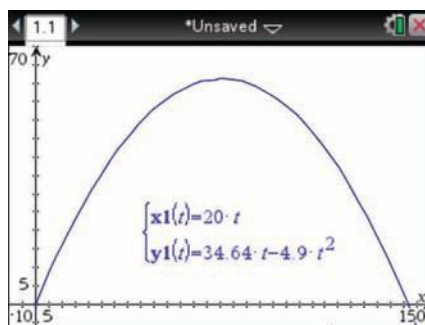
Open a graph and then press **menu** and select 3:Graph Type and 2:Parametric. Enter the equations for $x(t)$ and $y(t)$ and set the bounds for t as $0 \leq t \leq 8$.



Press ENTER and then adjust the bounds for x and y as:

$$-10 \leq x \leq 100$$

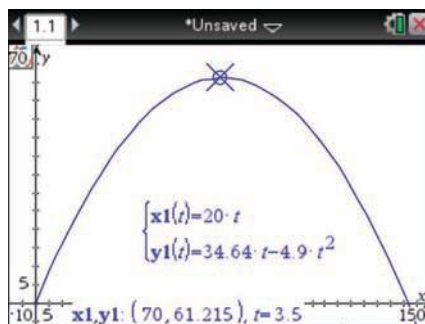
$$-5 \leq y \leq 70$$



Now you can use the Trace or Analyse features to investigate the path of the projectile.

Here, the greatest height is shown to be about 61.215 m.

You can adjust the *tstep* to a smaller number such as 0.01 to obtain more exact answers.



ClassPad

From  select  **Graph&Table**.

Tap **Type** then **ParamType**.

Enter the parametric equations for $x(t)$ and $y(t)$ by $xt1$ and $yt1$ respectively and press

EXE. (Remember that you will need to tap

Keyboard  to access t .)

Tap  to set the range of values for t . In this case, set **Start** to 0, **End** to 8, and **Step** to 0.05 and tap **OK**. To calculate the table of values used in plotting the graph, tap .

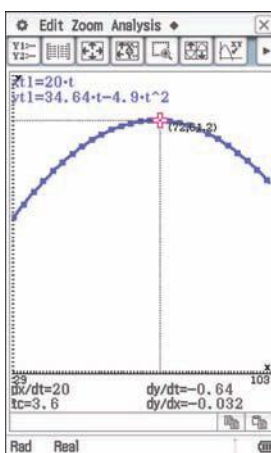
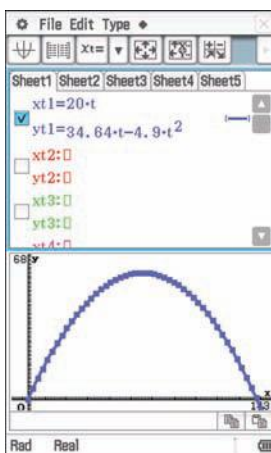
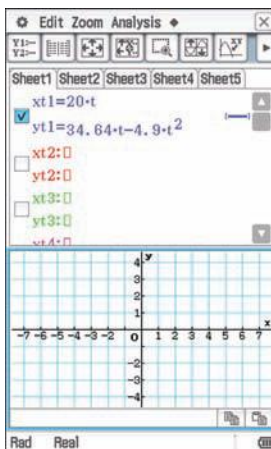
Tap  to draw the graph. To adjust the domain and range, tap  and set the bounds for x and y as:

$$-10 \leq x \leq 150$$

$$-5 \leq y \leq 70.$$

Tap the part of the screen that shows the graph. Tap  so that the graph occupies the whole screen. Now you can tap **Analyse** and **Trace** to investigate the path of the projectile. Here, the greatest height is shown to be about 61.2 m.

You can redraw the graph by tapping  and setting **Step** to a smaller value such as 0.01 to obtain more accurate answers when using the **Trace** feature.



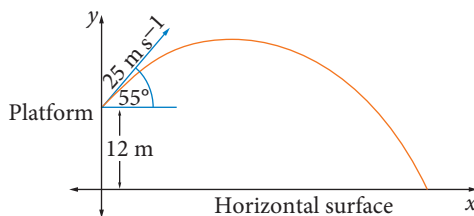
Example 14

A stone is projected from a platform 12 m above a horizontal surface with an initial velocity of 25 m s^{-1} at a 55° angle of elevation. Find:

- the position vector of the stone
- the time taken for the stone to strike the horizontal surface
- the horizontal distance travelled by the stone from its point of projection to where it meets the surface
- the greatest height above the horizontal surface reached during flight
- the distance between the point of projection and where the stone hits the horizontal surface.

Solution

Sketch a diagram.



In this case, it makes sense to place the origin on the horizontal plane, directly under the point of projection. The initial position is thus $(x_0, y_0) = (0, 12) = 0\mathbf{i} + 12\mathbf{j}$.

- a Write the equation for velocity.

$$\mathbf{v}(t) = V \cos(\theta)\mathbf{i} + [V \sin(\theta) - gt]\mathbf{j}$$

Substitute for values and simplify.

$$\begin{aligned} &= 25 \cos(55^\circ)\mathbf{i} + [25 \sin(55^\circ) - 9.8t]\mathbf{j} \\ &\approx 14.34\mathbf{i} + (20.48 - 9.8t)\mathbf{j} \end{aligned}$$

Express $\mathbf{r}(t)$ as the antiderivative of velocity.

$$\mathbf{r}(t) = \int [14.34\mathbf{i} + (20.48 - 9.8t)\mathbf{j}] dt + \mathbf{c}$$

Integrate.

$$\begin{aligned} &= 14.34t\mathbf{i} - (20.48t - 4.9t^2)\mathbf{j} + (0\mathbf{i} + 12\mathbf{j}) \\ &= 14.34t\mathbf{i} - (20.48t - 4.9t^2 + 12)\mathbf{j} \end{aligned}$$

$\mathbf{r}(0) = (0\mathbf{i} + 12\mathbf{j})$, so substitute $(0\mathbf{i} + 12\mathbf{j})$ for $\mathbf{c}$ and then simplify.

State the result.

$$\mathbf{r}(t) = 14.34t\mathbf{i} - (20.48t - 4.9t^2 + 12)\mathbf{j}$$

- b The stone strikes the horizontal surface when the vertical component of the position is zero.

$$y(t) = 20.48t - 4.9t^2 + 12$$

Let $y(t) = 0$ and solve using the quadratic formula. The negative value corresponds to the time the stone would be at $y = 0$ if the path was extended backwards from the point at which it was projected.

$$\begin{aligned} 0 &= -4.9t^2 + 20.48t + 12 \\ t &= \frac{-20.48 \pm \sqrt{(20.48)^2 - 4 \times (-4.9) \times 12}}{2 \times (-4.9)} \\ &\approx 4.70 \text{ or } -0.52 \end{aligned}$$

State the result.

The time of flight is about 4.7 s.

- c Write the horizontal position.

$$x(t) \approx 14.34t$$

Substitute $t \approx 4.7$.

$$\begin{aligned} &= 14.34 \times 4.7 \\ &\approx 67.4 \text{ m} \end{aligned}$$

Write the answer.

The stone travels about 67.4 m horizontally.

- d The greatest height is reached when $v_y(t)$ is zero. Let $v_y(t) = 0$.

Solve for t .

Write the equation for the vertical position.

Substitute $t = 2.09$.

Write the answer.

- e Draw a diagram using the known information.

$$v_y(t) \approx 20.48 - 9.8t$$

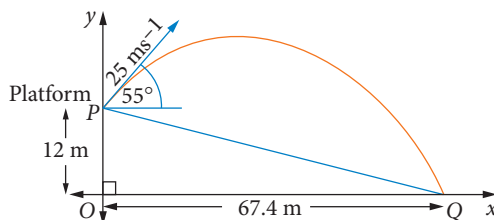
$$0 = 20.48 - 9.8t$$

$$t \approx 2.09$$

$$y(t) = 20.48t - 4.9t^2 + 12$$

$$\approx 33.4 \text{ m}$$

The greatest height reached is about 33.4 m.



Use Pythagoras' theorem to calculate the distance.

$$\text{Distance} = \sqrt{PO^2 + OQ^2}$$

$$= \sqrt{12^2 + 67.4^2}$$

$$\approx 68.46$$

State the result.

The projectile is about 68 m from its initial position when it strikes the horizontal surface.

You can use a graphics calculator to find approximate solutions for problems involving projectiles.

Consider the situation where a cannonball is fired over a horizontal plane at a projection angle of 40° at a speed of 30 m s^{-1} . There is an 8 m high obstacle 20 m from the firing point. Will the cannonball clear the obstacle?



Write the parametric equations for $x(t)$ and $y(t)$.

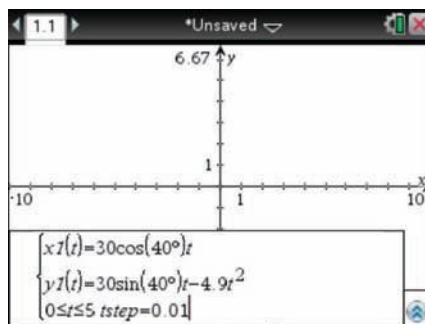
$$x(t) = [V \cos(\theta)]t = [30 \cos(40^\circ)]t$$

$$y(t) = [V \sin(\theta)]t - \frac{1}{2}gt^2 \approx [30 \sin(40^\circ)]t - \frac{1}{2} \times 9.8t^2 = [30 \sin(40^\circ)]t - 4.9t^2$$

TI-Nspire CAS

Open a graph and then press **[menu]** and select 3:Graph Type and 2:Parametric. Enter the equations for $x(t)$ and $y(t)$ and set the bounds for t as $0 \leq t \leq 5$ and set $tstep$ to 0.01.

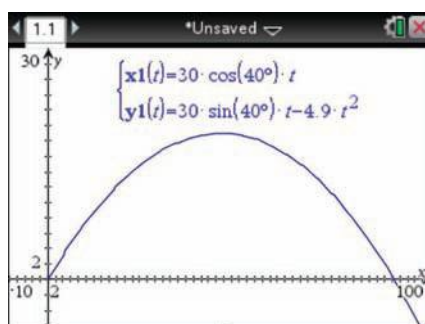
Make sure that you use the degree symbol when you type in the equations.



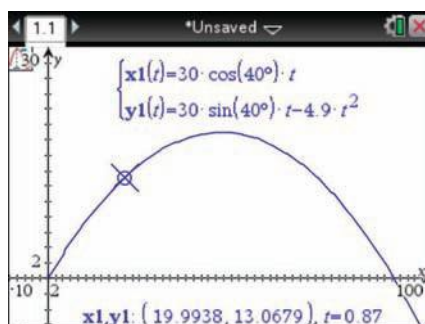
Press ENTER and then adjust the bounds for x and y as:

$$-10 \leq x \leq 100$$

$$-5 \leq y \leq 30$$



Use the Trace function to examine the point on the path of the cannonball where $x = 20$.



When $x \approx 20$, $y \approx 13$.

So the cannonball clears the 8 m high wall.

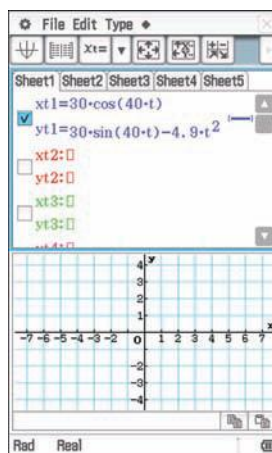
ClassPad

Ensure that your calculator is in **Deg** mode.

From **[Menu]**, select **Graph&Table**.

Tap **Type**, then **ParamType**.

Enter the equations for $x(t)$ and $y(t)$ next to **xt1** and **yt1** respectively and press **[EXE]**.



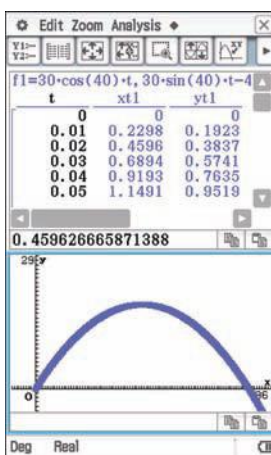
Tap to set the range of values for t . In this case, set **Start** to 0, **End** to 5, and **Step** to 0.01 and tap **OK**.

Tap and adjust the bounds for x and y as:

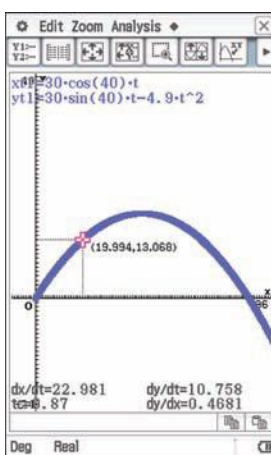
$$-10 \leq x \leq 100$$

$$-5 \leq y \leq 30$$

Tap to draw the graph.



Tap **Analysis**, then **Trace** and use the arrow keys to examine the point on the path of the cannonball where $x = 20$. You may like to tap the graph area then so that the graph uses the entire screen.

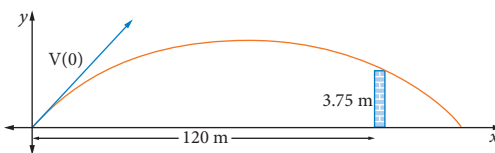


○ Example 15

A ball-bearing is fired from a slingshot from a position level with the base of a wall 120 m away. Two-and-a-half seconds later, the ball-bearing just clears the wall. If the wall is 3.75 m high, calculate the initial velocity and the projection angle of the ball-bearing.

Solution

Draw a diagram.



Write the equations for the horizontal and vertical components of position.

$$x(t) = [V \cos(\theta)]t$$

$$y(t) = [V \sin(\theta)]t - \frac{1}{2}gt^2$$

Write the known values.

$$g \approx 9.8 \text{ and at } t = 2.5, x = 120 \text{ and } y = 3.75$$

Substitute in the values.

$$120 = 2.5 V \cos(\theta) \quad [1]$$

Simplify [2].

$$3.75 = 2.5 V \sin(\theta) - 4.9 \times (2.5)^2 \quad [2]$$

Divide [3] by [1] to eliminate V .

$$34.375 = 2.5 V \sin(\theta) \quad [3]$$

Simplify.

$$\frac{2.5V \sin(\theta)}{2.5V \cos(\theta)} = \frac{34.375}{120}$$

Use $\tan^{-1}$ to find θ .

$$\tan(\theta) = 0.2864\dots$$

Substitute the exact value into [1].

$$\theta = 15.98\dots^\circ$$

Solve for V .

$$120 = 2.5 V \cos(15.98\dots^\circ)$$

Write the answer.

$$V = 49.93\dots \text{ m s}^{-1}$$

The initial velocity was about 50 m s^{-1} and the angle of projection was about 16° .

INVESTIGATION Rockets and satellites

Do some research and write a report on all the physical factors affecting the launch of a satellite or a rocket.

How is the work on projectiles that you are studying in this chapter relevant to the launching of rockets and satellites?

What things other than those introduced in this chapter need to be considered for rocket and satellite launches?

Research the history of space travel.

EXERCISE 5.05 Motion with constant acceleration



Projectile motion

Concepts and techniques

Use a graphics calculator wherever appropriate in this exercise. Use $g \approx 9.8 \text{ m s}^{-2}$ unless otherwise indicated, and assume that air resistance is negligible.

- Example 13** A projectile has an initial velocity vector given by $\mathbf{v} = 30\mathbf{i} + 20\mathbf{j}$. Its time of flight is closest to:
A 2.5 s B 3.4 s C 4.1 s D 4.3 s E 5.5 s
- A projectile has a position vector given by $\mathbf{r}(t) = 15t\mathbf{i} + (30t - 5t^2)\mathbf{j}$, where distance is measured in metres. The greatest height reached by the projectile is:
A 10 m B 15 m C 30 m D 45 m E 55 m
- A projectile is launched, with its velocity given by $\mathbf{v}(t) = 25\mathbf{i} + 40\mathbf{j}$. The range of the projectile is closest to:
A 185 m B 190 m C 205 m D 210 m E 220 m

- 4 **Example 14** A projectile is launched with a speed of $12\sqrt{3} \text{ ms}^{-1}$ at a 60° angle. If $\mathbf{i}$ and $\mathbf{j}$ are taken to be the unit vectors in the horizontal and vertical directions respectively, the projectile's initial velocity (in ms^{-1}) is:
- A $12\mathbf{i} + 18\mathbf{j}$ B $6\mathbf{i} + 18\mathbf{j}$ C $6\sqrt{3}\mathbf{i} + 18\mathbf{j}$
D $12\sqrt{3}\mathbf{i} + 18\mathbf{j}$ E $6\sqrt{3}\mathbf{i} + 12\sqrt{3}\mathbf{j}$
- 5 A projectile has an initial position given by $\mathbf{i} - \mathbf{j}$. Its velocity is given by $\mathbf{v}(t) = 4t\mathbf{i} - \mathbf{j}$. The position vector of the projectile at any time, t , $\mathbf{r}(t)$ is given by:
- A $(2t^2 - 1)\mathbf{i} + (t + 1)\mathbf{j}$ B $(2t^2 + 1)\mathbf{i} + (t - 1)\mathbf{j}$ C $(2t^2 - 1)\mathbf{i} + (t - 1)\mathbf{j}$
D $(2t^2 + 1)\mathbf{i} - (t + 1)\mathbf{j}$ E $(2t^2 + 1)\mathbf{i} + (t + 1)\mathbf{j}$
- 6 An object on a horizontal surface is projected with a velocity of 20 m s^{-1} at an angle of 30° to the horizontal. If g is taken to be 10 m s^{-2} , find:
- a the greatest height reached b the time of flight c the range.
- 7 A ball is kicked into the air with an initial velocity of 25 m s^{-1} at a 45° angle. Calculate:
- a the time of flight b the range
c the greatest height reached d the position vector for the ball
e the distance travelled by the ball in the first second after it was kicked.
- 8 A ball is hit with an initial velocity of 30 m s^{-1} at a 30° angle of elevation. Calculate:
- a the position vector for the ball
b the greatest height that the ball reaches
c the time of flight and the range
d the speed and direction of motion at a height of 4 m.
- 9 Find the greatest range possible on a horizontal plane when the projection speed is:
- a 18 m s^{-1} b 30 m s^{-1} .
- 10 A particle is projected with a speed of 19.6 m s^{-1} . Find:
- a the maximum range possible for the particle
b the angle of projection required for the range to be 12 m.



Reasoning and communication

- 11 A person standing on a flat horizontal surface can throw a stone just 60 m.
- a With what speed is the stone thrown? b How long is the stone in the air for?
- 12 An object is launched from the ground at an angle of 38° towards a vertical wall that is 15 m horizontally from the launch point. After 1.3 s, the object strikes the wall. Find the velocity with which the object was launched.
- 13 A particle is projected from the top of a platform 98 m above the sea. It is given an initial speed of 49 m s^{-1} and a projection angle of 30° .
- a How long does it take for the particle to reach the water?
b How far from the base of the platform does the particle make contact with the water?
- 14 A ball-bearing is projected from a slingshot from the top of a cliff 160 m above the sea with an initial speed of 180 m s^{-1} . If the angle of projection is 30° , find the horizontal distance from the foot of the cliff to the point where the ball-bearing meets the water.

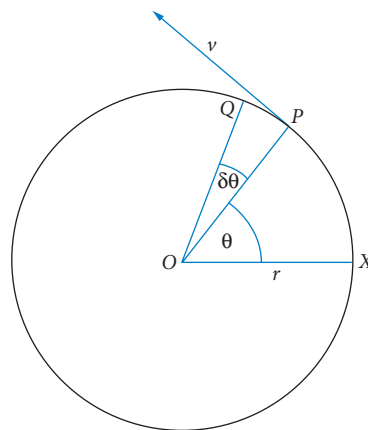
- 15 **Example 15** A ball is thrown with an initial velocity of 20 m s^{-1} at an angle of 65° . The ball is released from the thrower's hand at a height of 1.8 m above the ground. Calculate the height at which the ball will hit a vertical wall 10 m away.
- 16 A bullet is fired with a velocity that has vertical and horizontal components of $v_y(t)$ and $v_x(t)$ respectively. If the horizontal component of velocity is 600 m s^{-1} , find the angle of projection that must be used for the bullet to hit a target 2 m above the muzzle and 500 m distant from it.
- 17 A projectile must achieve a horizontal range of 100 m . Calculate the least initial speed required to obtain this outcome, and the time of flight.
- 18 A particle is projected with an initial velocity of 30 m s^{-1} at an angle of 37° . Will it clear a 10 m high wall that is at a horizontal distance of 72 m from the point of projection?
- 19 An object is projected from the top of a sea-side cliff, 30.4 m high, with a velocity of 24 m s^{-1} at an angle of 30° . Find the horizontal distance from the foot of the cliff to the point where the object hits the sea.
- 20 When a child standing on horizontal ground throws a ball, it leaves her hand from a point that is 90 cm above the ground. The ball clears a 4.5 m high wall that is 10.5 m away from where it was thrown.
- Show that the least initial speed required for this to occur is the same as the speed acquired by a body falling 7.35 m under gravity.
 - If the ball just clears the wall, how far past the wall will it first strike the ground?
- 21 Two particles, A and B , are both projected from the same point, P , with the same speed of 25 m s^{-1} . They both strike the horizontal plane through the point of projection at 60 m from P , but A reaches the point of impact before B . Calculate:
- the angles of projection of A and B
 - the time of flight of A
 - the distance between A and B at the instant when A reaches the point of impact
 - the ratio of the greatest heights reached by A and B .

5.06 MOTION WITH VARYING ACCELERATION

Uniform circular motion is motion at a constant speed around the circumference of a circle. Because the path is circular, the velocity is constantly changing and is always tangential to the path, as shown in the diagram on the right.

Consider a particle moving anticlockwise in a circle of radius r as shown in the diagram. The particle sweeps out an angle θ in time t s, beginning from the point X .

Let the particle trace out a small arc PQ in time δt so that arc $PQ = r \times \delta\theta$, where $\delta\theta$ is the corresponding change in θ .



Now, provided that δt is small, v , the tangential velocity or speed at P , is given by

$$\begin{aligned} v &= \frac{PQ}{\delta t} \\ &= \frac{r\delta\theta}{\delta t} \end{aligned}$$

So, as $\delta t \rightarrow 0$, $v = r \frac{d\theta}{dt}$

IMPORTANT

Angular velocity

The **angular velocity** ω of a particle moving in a circle is the rate of change of the angle of the radius from the particle to the centre. The tangential speed of the particle, v , is given by the product of the radius and angular velocity.

$$\omega = \frac{d\theta}{dt} \quad \text{and} \quad v = \omega r$$

Angular velocity is usually defined in radians per second (rad s^{-1}). However, for large orbits such as those in planetary motion, ω may be measured in radians per hour, per day or even per year.

In practice, angular velocity is often described as revolutions per second (rps) or revolutions per minute (rpm), and so on as appropriate.



Example 16

A large circular cog of radius 1.5 m completes 900 revolutions per minute. Calculate:

- the angular velocity in rad s^{-1}
- the speed of a point on the circumference of the cog in m s^{-1} .

Solution

- a Write the angular velocity.

$$\omega = 900 \text{ rpm}$$

Use 1 revolution = 2π radians.

$$= 900 \times 2\pi \text{ rad min}^{-1}$$

Use 1 min = 60 s.

$$\begin{aligned} &= \frac{900 \times 2\pi}{60} \\ &= 30\pi \text{ rad s}^{-1} \end{aligned}$$

Write the answer.

The angular velocity is $30\pi \text{ rad s}^{-1}$.

- b Write the equation for v .

$$v = \omega r$$

Substitute in the values.

$$\begin{aligned} &= 1.5 \times 30\pi \\ &= 45\pi \\ &\approx 141 \text{ m s}^{-1} \end{aligned}$$

Write the answer.

The speed is $45\pi \text{ m s}^{-1}$ (about 141 m s^{-1}).

As previously mentioned in this chapter, it is common to use 'dot notation' for time derivatives, so

$$\omega = \frac{d\theta}{dt} = \dot{\theta}$$

Also

$$\frac{d\omega}{dt} = \dot{\omega} = \frac{d^2\theta}{dt^2} = \ddot{\theta}$$

The angular displacement, θ , is found by calculating the integral of ω with respect to time, so

$$\theta = \int_0^t \omega dt = \omega t, \text{ if } \omega \text{ is constant.}$$

The period for circular motion is usually defined with constant speed.

IMPORTANT

Angular velocity and period

The **period** T of a body moving in uniform circular motion is the time taken to complete one complete revolution of the circle. The angular displacement and period are given by

$$\theta = \omega t \text{ and } T = \frac{2\pi}{\omega}.$$

The last equation is obtained using the fact that 1 revolution ($\theta = 2\pi$ radians) is completed in one period ($t = T$). Substitute these into $\theta = \omega t$ to obtain $2\pi = \omega T$ to get the required result.

○ Example 17

A satellite moves in a circular orbit of radius 9000 km, completing one orbit every 100 min.

- Find its angular velocity.
- Find its speed in km h^{-1} .

Solution

- Write the equation for the period.

$$T = \frac{2\pi}{\omega}$$

Transform to get ω .

$$\omega = \frac{2\pi}{T}$$

Substitute $T = \frac{100}{60}$ h.

$$\begin{aligned} &= \frac{2\pi \times 60}{100} \\ &\approx 3.77 \text{ rad h}^{-1} \end{aligned}$$

Write the answer.

The angular velocity is about 3.77 rad h^{-1} .

- Write the equation for v .

$$v = \omega r$$

Substitute in the values.

$$\begin{aligned} &\approx 3.77 \times 9000 \\ &= 33\,930 \text{ km h}^{-1} \end{aligned}$$

Write the answer.

The speed is about $33\,900 \text{ km h}^{-1}$.

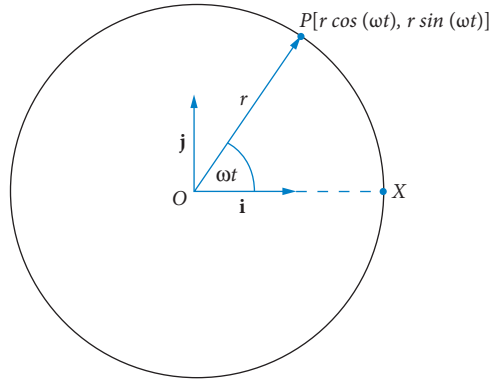
Acceleration under uniform circular motion

The magnitude of the acceleration a , the angular velocity ω and the speed v of a particle under uniform circular motion are related by the equations

$$a = \omega^2 r \quad \text{and} \quad a = \frac{v^2}{r}.$$

These may be demonstrated as follows.

Consider the situation shown in the diagram below. Assume that, at time $t = 0$, a particle is at point X on the circle and that its angular velocity is ω rad/s.



At time t s later, the particle moves to point P on the circle, thus undergoing an angular displacement of ωt . P has the coordinates $[r \cos(\omega t), r \sin(\omega t)]$, so the position vector $\mathbf{r}(t)$ is given by $\mathbf{r}(t) = r \cos(\omega t)\mathbf{i} + r \sin(\omega t)\mathbf{j}$.

Find the velocity.

$$\begin{aligned} \mathbf{v}(t) &= \dot{\mathbf{r}}(t) \\ &= -\omega r \sin(\omega t)\mathbf{i} + \omega r \cos(\omega t)\mathbf{j} \end{aligned}$$

Find the speed using $|\mathbf{v}(t)|^2 = \mathbf{v}(t) \cdot \mathbf{v}(t)$.

$$\begin{aligned} v &= \sqrt{\mathbf{v}(t) \cdot \mathbf{v}(t)} \\ &= \sqrt{(\omega r)^2 \sin^2(\omega t) + (\omega r)^2 \cos^2(\omega t)} \\ &= \sqrt{(\omega r)^2 [\sin^2(\omega t) + \cos^2(\omega t)]} \\ &= \omega r \end{aligned}$$

Use $\sin^2(\theta) + \cos^2(\theta) = 1$.

This result agrees with the result established previously.

Find the dot product of $\mathbf{v}$ and $\mathbf{r}$.

$$\begin{aligned} \mathbf{v}(t) \cdot \mathbf{r}(t) &= [-\omega r \sin(\omega t)\mathbf{i} + \omega r \cos(\omega t)\mathbf{j}] \cdot [r \cos(\omega t)\mathbf{i} + r \sin(\omega t)\mathbf{j}] \\ &= -\omega r^2 \sin(\omega t) \cos(\omega t) + \omega r^2 \sin(\omega t) \cos(\omega t) \\ &= 0 \end{aligned}$$



Since the dot product is zero, the velocity and radius are perpendicular. This analytically confirms that the velocity is tangential to the radius.

The acceleration of the particle around the circle can be found by differentiating the velocity.

$$\begin{aligned}
 \mathbf{a}(t) &= \dot{\mathbf{v}}(t) \\
 &= [-\omega r \sin(\omega t)\mathbf{i} + \omega r \cos(\omega t)\mathbf{j}] \\
 &= -\omega^2 r [\cos(\omega t)\mathbf{i} + \sin(\omega t)\mathbf{j}] \\
 &= -\omega^2 [r \cos(\omega t)\mathbf{i} + r \sin(\omega t)\mathbf{j}] \\
 &= -\omega^2 \mathbf{r}(t)
 \end{aligned}$$

Because the acceleration is negative, it must be directed towards the centre of the circle. The magnitude of $\mathbf{a}(t)$ is calculated using $|\mathbf{a}|^2 = \mathbf{a} \cdot \mathbf{a}$.

$$\begin{aligned}
 a &= \sqrt{\mathbf{a}(t) \cdot \mathbf{a}(t)} \\
 &= \sqrt{-\omega^2 \mathbf{r}(t) \cdot [-\omega^2 \mathbf{r}(t)]} \\
 &= \sqrt{\omega^4 r^2 [\cos^2(\omega t) + \sin^2(\omega t)]} \\
 &= \sqrt{\omega^4 r^2} \\
 &= \omega^2 r \\
 &= \left(\frac{v}{r}\right)^2 r \\
 &= \frac{v^2}{r}
 \end{aligned}$$

Use $\sin^2(\theta) + \cos^2(\theta) = 1$.

But $v = \omega r$, so $\omega = \frac{v}{r}$.

○ Example 18

A body attached to a string 4 m long moves on a smooth horizontal plane at a speed of 8 m s^{-1} . What is the magnitude of the body's acceleration?

Solution

Write the equation for a .

$$a = \frac{v^2}{r}$$

Substitute $v = 8 \text{ m s}^{-1}$ and $r = 4 \text{ m}$.

$$\begin{aligned}
 &= \frac{8^2}{4} \\
 &= 16 \text{ m s}^{-2}
 \end{aligned}$$

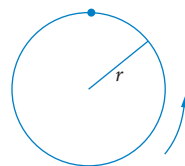
Write the answer.

The acceleration of the body is 16 m s^{-2} .

EXERCISE 5.06 Motion with varying acceleration

Concepts and techniques

The diagram on the right shows an object that moves around a circular path of constant radius at a constant speed. Use the diagram to answer questions 1–3.



- 1 **Example 16** Which of the following statements is true?
- A The object's velocity is constant.
 - B The object's acceleration is constant.
 - C The object's acceleration is zero.
 - D The object's velocity is directed toward the centre.
 - E The object's acceleration is directed toward the centre.

- 2 When the object is at the top of the circular path, what is the direction of the velocity?



- 3 When the object is at the top of the circular path, what is the direction of the acceleration?



- 4 **Example 17** An object moves around a circular path at a constant speed and makes five complete revolutions in 20 seconds. The period of rotation is:

A 4 s B 5 s C 10 s D 15 s E 20 s

- 5 **Example 18** An object travels in a circular path of radius r at a constant speed v . What happens to the object's acceleration if the radius of the circle is doubled and the speed stays unchanged?

A It doubles B It quadruples C It is cut to a half
D It is cut to a quarter E It stays unchanged

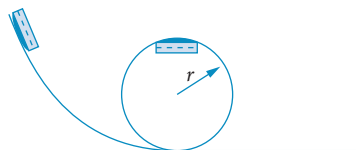
- 6 An object travels in a circular path of radius r at a constant speed v . What happens to the object's acceleration if the speed is doubled and the radius stays unchanged?

A It doubles B It quadruples C It is halved
D It is reduced to a quarter E It stays unchanged

- 7 An object moves at a constant acceleration a in a circular path of radius r . The object's velocity is:

A $a \times r$ B $\frac{a}{r}$ C $\frac{r}{a}$ D $\sqrt{a \times r}$ E $\sqrt{\frac{a}{r}}$

The diagram on the right shows a roller coaster car moving on a track with one section that is a vertical circular loop of radius r . Use the diagram to answer questions 8–9.



- 8 When the roller-coaster car is at the top of the loop, it just maintains contact with the track. What is the car's acceleration at this point?

A g upward B $0.5 g$ upward C zero
D $0.5 g$ downward E g downward

- ## Reasoning and communication

-

- 17 The diameter of a standard adult's bicycle wheel is 28 inches ($1 \text{ inch} \approx 0.0254 \text{ m}$). The bicycle is moving at a constant speed of 15 km h^{-1} . Calculate:
- the speed of a piece of chewing gum stuck on the tread of the bicycle tyre, relative to the centre of the wheel
 - the angular velocity of the wheel
 - the time taken for the wheel to complete one full revolution
 - the magnitude of the acceleration of the chewing gum.
- 18 An engine flywheel of radius 80 cm completes 400 revolutions per minute. Calculate:
- the angular velocity in rad s^{-1}
 - the speed of a point on the rim
 - the magnitude of the acceleration of a point on the rim.
- 19 A telecommunications satellite is placed in a circular orbit of radius 35 000 km. The period of its orbit is 15 h. Calculate:
- the angular velocity in rad h^{-1}
 - the speed of the satellite in km h^{-1} .



- 20 The acceleration due to gravity on Earth is approximately 9.8 m s^{-2} or 1 G. Astronauts experience much higher acceleration during space flights. To simulate high acceleration, astronauts are placed in a cockpit attached to a long arm, which is then spun in a circle. If the arm of the apparatus is 7 m long, find the angular velocity that would be necessary to create an acceleration equal in magnitude to:
- 1 G
 - 5 G
 - 10 G.
- 21 The maximum amount of acceleration that a car can achieve without slipping when travelling around a circular bend in the road is $25 \text{ km h}^{-1} \text{ s}^{-1}$. What is the minimum radius of the circle corresponding to this acceleration and a speed of 75 km h^{-1} ?

5

CHAPTER SUMMARY

VECTOR CALCULUS

- An alternative to defining y in terms of x or x in terms of y is to define both x and y in terms of a third variable, for example:

$$x = f(t) \text{ and } y = g(t)$$

The third variable is called a **parameter** and the equations formed using the parameter are called **parametric equations**.

- A **vector function** or **vector-valued function** can be defined using a parametric equation. For example, a position vector in two dimensions at time t is:

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$$

- The vector equation $\mathbf{r}(t) = \cos(t)\mathbf{i} + \sin(t)\mathbf{j}$ for $0 \leq t \leq 2\pi$ represents the path of a particle that:

- moves on the curve with equation $x^2 + y^2 = 1$
- has position $(1, 0)$ at time $t = 0$
- moves in an anticlockwise direction
- takes 2π units of time to complete one revolution

- The derivative of the vector $\mathbf{r}$ with respect to t is denoted by $\frac{d\mathbf{r}}{dt}$ or more commonly as $\dot{\mathbf{r}}(t)$.

$$\dot{\mathbf{r}} = \lim_{\delta t \rightarrow 0} \frac{\mathbf{r}(t + \delta t) - \mathbf{r}(t)}{\delta t}$$

- For $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$:

$$\dot{\mathbf{r}}(t) = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} = \dot{x}(t)\mathbf{i} + \dot{y}(t)\mathbf{j}$$

$$|\dot{\mathbf{r}}(t)| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{\dot{x}^2 + \dot{y}^2}$$

$$\ddot{\mathbf{r}}(t) = \frac{d^2x}{dt^2}\mathbf{i} + \frac{d^2y}{dt^2}\mathbf{j} = \ddot{x}(t)\mathbf{i} + \ddot{y}(t)\mathbf{j}$$

- For $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$:

$$\begin{aligned}\dot{\mathbf{r}}(t) &= \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k} \\ &= \dot{x}(t)\mathbf{i} + \dot{y}(t)\mathbf{j} + \dot{z}(t)\mathbf{k}\end{aligned}$$

$$\begin{aligned}|\dot{\mathbf{r}}(t)| &= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} \\ &= \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\end{aligned}$$

$$\begin{aligned}\ddot{\mathbf{r}}(t) &= \frac{d^2x}{dt^2}\mathbf{i} + \frac{d^2y}{dt^2}\mathbf{j} + \frac{d^2z}{dt^2}\mathbf{k} \\ &= \ddot{x}(t)\mathbf{i} + \ddot{y}(t)\mathbf{j} + \ddot{z}(t)\mathbf{k}\end{aligned}$$

$$\begin{aligned}|\ddot{\mathbf{r}}(t)| &= \sqrt{\left(\frac{d^2x}{dt^2}\right)^2 + \left(\frac{d^2y}{dt^2}\right)^2 + \left(\frac{d^2z}{dt^2}\right)^2} \\ &= \sqrt{\ddot{x}^2 + \ddot{y}^2 + \ddot{z}^2}\end{aligned}$$

- The properties of the derivatives of vector functions are:

$$\frac{d}{dt}[\mathbf{u}(t) + \mathbf{v}(t)] = \frac{d\mathbf{u}(t)}{dt} + \frac{d\mathbf{v}(t)}{dt} = \dot{\mathbf{u}} + \dot{\mathbf{v}}$$

$$\frac{d[k\mathbf{u}(t)]}{dt} = k \frac{d\mathbf{u}(t)}{dt} = k\dot{\mathbf{u}}$$

$$\begin{aligned}\frac{d}{dt}[f(t)\mathbf{u}(t)] &= \frac{df(t)}{dt}\mathbf{u}(t) + f(t)\frac{d\mathbf{u}(t)}{dt} \\ &= f'(t)\mathbf{u} + f(t)\dot{\mathbf{u}}\end{aligned}$$

$$\begin{aligned}\frac{d}{dt}[\mathbf{u}(t) \cdot \mathbf{v}(t)] &= \frac{d\mathbf{u}(t)}{dt} \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \frac{d\mathbf{v}(t)}{dt} \\ &= \dot{\mathbf{u}} \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \dot{\mathbf{v}}\end{aligned}$$

$$\begin{aligned}\frac{d}{dt}[\mathbf{u}(t) \times \mathbf{v}(t)] &= \frac{d\mathbf{u}(t)}{dt} \times \mathbf{v}(t) + \mathbf{u}(t) \times \frac{d\mathbf{v}(t)}{dt} \\ &= \dot{\mathbf{u}} \times \mathbf{v}(t) + \mathbf{u}(t) \times \dot{\mathbf{v}}\end{aligned}$$

$$\frac{d}{dt}\{\mathbf{u}[f(t)]\} = \frac{df(t)}{dt} \frac{d\mathbf{u}(t)}{dt} f(t) = f'(t)\dot{\mathbf{u}} f(t)$$

- For $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$:

$$\begin{aligned}\int \mathbf{r}(t)dt &= \int [x(t)\mathbf{i} + y(t)\mathbf{j}]dt \\ &= \int x(t)\mathbf{i}dt + \int y(t)\mathbf{j}dt \\ &= X(t)\mathbf{i} + Y(t)\mathbf{j} + \mathbf{c}\end{aligned}$$

where $\frac{dX}{dt} = x(t)$, $\frac{dY}{dt} = y(t)$, and $\mathbf{c}$ is a constant vector.

- For $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$:

$$\begin{aligned}\int \mathbf{r}(t)dt &= \int [x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}]dt \\ &= X(t)\mathbf{i} + Y(t)\mathbf{j} + Z(t)\mathbf{k} + \mathbf{c}\end{aligned}$$

where $\frac{dX}{dt} = x(t)$, $\frac{dY}{dt} = y(t)$, $\frac{dZ}{dt} = z(t)$ and $\mathbf{c}$ is a constant vector.

- For $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$:

$$\begin{aligned}\int_a^b \mathbf{r}(t)dt &= \int_a^b [x(t)\mathbf{i} + y(t)\mathbf{j}]dt \\ &= \int_a^b x(t)\mathbf{i}dt + \int_a^b y(t)\mathbf{j}dt \\ &= [X(t)]_a^b \mathbf{i} + [Y(t)]_a^b \mathbf{j}\end{aligned}$$

where $\frac{dX}{dt} = x(t)$ and $\frac{dY}{dt} = y(t)$.

- For $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$:

$$\begin{aligned}\int_a^b \mathbf{r}(t)dt &= \int_a^b [x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}]dt \\ &= [X(t)]_a^b \mathbf{i} + [Y(t)]_a^b \mathbf{j} + [Z(t)]_a^b \mathbf{k}\end{aligned}$$

- Position (x), velocity ($\mathbf{v}$) and acceleration ($\mathbf{a}$) are defined using derivatives as:

$$\frac{dx(t)}{dt} = \dot{x}$$

$$\mathbf{v} = \frac{d\mathbf{r}(t)}{dt} = \dot{\mathbf{r}}$$

$$\mathbf{a} = \frac{d\mathbf{v}(t)}{dt} = \dot{\mathbf{v}} \quad \text{and} \quad \mathbf{a} = \ddot{\mathbf{r}}$$

- Antidifferentiating acceleration yields an expression for velocity, while antidifferentiating velocity yields an expression for displacement.

- A **projectile** is launched with an initial velocity comprising its **speed of projection** and **angle of projection**. Its motion is then determined solely by gravitational acceleration (g) and air resistance. In simple cases, air resistance may be neglected. The path of a projectile is called its **trajectory**. Its maximum horizontal displacement is called its **range**.

- The equations that define projectile motion for acceleration ($\mathbf{a}$), velocity ($\mathbf{v}$) and position ($\mathbf{r}$) are:

$$\mathbf{a} = -g\mathbf{j} \text{ or } a_x = \ddot{x} = 0 \text{ and } a_y = \ddot{y} = -g$$

$$\mathbf{v} = V \cos(\theta)\mathbf{i} + [V \sin(\theta) - gt]\mathbf{j} \text{ or}$$

$$v_x = \dot{x} = V \cos(\theta) \text{ and } v_y = \dot{y} = V \sin(\theta) - gt$$

$$\mathbf{r} = Vt \cos(\theta)\mathbf{i} + \left[Vt \sin(\theta) - \frac{1}{2}gt^2 \right]\mathbf{j} \text{ or}$$

$$x = Vt \cos(\theta) \text{ and } y = Vt \sin(\theta) - \frac{1}{2}gt^2$$

- Other results for projectile motion above a flat plane include:

$$\text{Time of flight} = T = \frac{2V \sin(\theta)}{g}$$

$$\text{Maximum height: } y_{\max} = \frac{V^2 \sin^2(\theta)}{g} \text{ at time}$$

$$T_{\frac{1}{2}} = \frac{V \sin(\theta)}{g}$$

$$\text{Range: } x_{\max} = \frac{2V^2 \cos(\theta) \sin(\theta)}{g}$$

$$= \frac{V^2 \sin(2\theta)}{g}$$

$$\text{at time } T = \frac{V \sin \theta}{g}$$

$$\text{Maximum possible range (at } \theta = 45^\circ) = \frac{V^2}{g}$$

$$\text{Speed} = v(t) = |\mathbf{v}(t)| = \sqrt{\mathbf{v}(t) \cdot \mathbf{v}(t)}$$

- The general equation for a trajectory is:

$$y = x \tan (\theta) - \frac{gx^2}{2V^2 \cos^2 (\theta)} \text{ or}$$

$$y = x \tan (\theta) - \frac{gx^2}{2V^2} [\tan^2 (\theta) + 1].$$

- **Uniform circular motion** is motion at a constant speed around the circumference of a circle.
- The **angular velocity** ω of a particle moving in a circle is the rate of change of the angle of the radius between the particle and the centre. The tangential speed of the particle, v , is given by the product of the radius and angular velocity.

$$\omega = \frac{d\theta}{dt} \quad \text{and} \quad v = \omega r$$

- The **period** T of a body moving in uniform circular motion is the time taken to complete one complete revolution of the circle. The angular displacement and period are given by

$$\theta = \omega t \quad \text{and} \quad T = \frac{2\pi}{\omega}.$$

- The magnitude of the acceleration a , the angular velocity ω and the speed v of a particle under uniform circular motion are related by the equations

$$a = \omega^2 r \quad \text{and} \quad a = \frac{v^2}{r}.$$

CHAPTER REVIEW

VECTOR CALCULUS

5

Multiple choice

- 1 **Example 1** The vector function $\mathbf{r}(t) = (t^2 - 2t)\mathbf{i} + (t + 1)\mathbf{j}$ has the corresponding Cartesian relation:
 A $x = y^2 + 4y - 3$ for $x \geq -1$ B $x = y^2 - 4y + 3$ for $x \geq -1$ C $x = y^2 - 2y + 1$ for $x \geq -1$
 D $x = y^2 - 2y - 1$ for $x \geq -1$ E $x = 2y^2 - y + 1$ for $x \geq -1$
- 2 **Example 1** The vector function $\mathbf{r}(t) = [3 \sec(t)]\mathbf{i} + [4 \tan(t)]\mathbf{j}$ for $0 \leq t \leq 2\pi$ has the corresponding Cartesian relation:
 A $\frac{x^2}{16} + \frac{y^2}{9} = 1$ B $\frac{x^2}{16} - \frac{y^2}{9} = 1$ C $\frac{x^2}{9} + \frac{y^2}{16} = 1$ D $\frac{x^2}{9} - \frac{y^2}{16} = 1$ E $x^2 + y^2 = 25$
- 3 **Example 4** If $\mathbf{r}(t) = (3 - 2t)\mathbf{i} + (t^2 - 4t)\mathbf{j}$, then $|\dot{\mathbf{r}}(2)|$ is:
 A 0 B 1 C 2 D 8 E $2\sqrt{17}$
- 4 **Example 4** If $\mathbf{r}(t) = (7t - 9)\mathbf{i} + (2t^3 + 1)\mathbf{j}$, then $\ddot{\mathbf{r}}(t)$ is:
 A $7\mathbf{i} + 6t\mathbf{j}$ B $6t\mathbf{i} + 0\mathbf{j}$ C $6\mathbf{j}$ D $0\mathbf{i} + 0\mathbf{j}$ E $0\mathbf{i} + 12t\mathbf{j}$
- 5 **Example 4** If $\mathbf{r}(t) = [5 \sin(t)]\mathbf{i} - [\cos(3t)]\mathbf{j}$, then when $t = \frac{\pi}{2}$, $\dot{\mathbf{r}}(t)$ is:
 A $0\mathbf{i} - 3\mathbf{j}$ B $5\mathbf{i} - 3\mathbf{j}$ C $5\mathbf{i} + 3\mathbf{j}$ D $3\mathbf{i} + 0\mathbf{j}$ E $3\mathbf{j}$
- 6 **Example 7** If $\dot{\mathbf{r}}(t) = 3\mathbf{i} - 2t\mathbf{j}$ and $\mathbf{r}(0) = 2\mathbf{i} - \mathbf{j}$, then $\mathbf{r}(t)$ is:
 A $(3t + 2)\mathbf{i} + (t^2 - 1)\mathbf{j}$ B $3t\mathbf{i} - t^2\mathbf{j}$ C $(3t - 2)\mathbf{i} + (t^2 - 1)\mathbf{j}$
 D $(3t + 2)\mathbf{i} - (t^2 + 1)\mathbf{j}$ E $(3 + 2t)\mathbf{i} - (2t^2 + 1)\mathbf{j}$
- 7 **Example 7** If $\dot{\mathbf{r}}(t) = 2t\mathbf{i} + (e^{-t} + t)\mathbf{j}$ and $\mathbf{r}(0) = 3\mathbf{i}$, then $\mathbf{r}(t)$ is:
 A $t^2\mathbf{i} + \frac{1}{2}t^2\mathbf{j}$ B $t^2\mathbf{i} - \left(\frac{1}{2}t^2 + e^{-t} + 1\right)\mathbf{j}$ C $(t^2 + 3)\mathbf{i} + \left(\frac{1}{2}t^2 - e^{-t} + 1\right)\mathbf{j}$
 D $(t^2 - 3)\mathbf{i} - (t^2 - e^{-t} + 1)\mathbf{j}$ E $3\mathbf{i} + \left(\frac{1}{2}t^2 - e^{-t}\right)\mathbf{j}$
- 8 **Example 10** The position vector of a particle is given by $\mathbf{r}(t) = (4t - t^3)\mathbf{i} + (3 + t^2)\mathbf{j}$. The magnitude of the velocity at $t = 2$ is:
 A $3\sqrt{5}$ B $2\sqrt{15}$ C $4\sqrt{5}$ D 12 E $2\sqrt{85}$
- 9 **Example 13** A projectile has its position vector given by $\mathbf{r}(t) = 13t\mathbf{i} + (24t - 3t^2)\mathbf{j}$, where distance is measured in metres. The greatest height reached by the projectile is:
 A 21 m B 36 m C 40 m D 45 m E 48 m
- 10 **Example 13** A projectile is launched and its initial velocity is given by $\mathbf{v} = 25\mathbf{i} + 40\mathbf{j}$. The range of the projectile is closest to:
 A 175 m B 180 m C 200 m D 225 m E 240 m

- 11 **Example 17** An object moves around a circular path at a constant speed and makes ten complete revolutions in 20 seconds. The period of rotation is:
 A 0.8 s B 1.25 s C 1.5 s D 2 s E 15 s
- 12 **Example 18** An object moves at a constant acceleration of 12 m s^{-2} in a circular path of radius 3 m. The object's velocity is:
 A $\frac{1}{2} \text{ m s}^{-1}$ B $1\frac{1}{2} \text{ m s}^{-1}$ C 2 m s^{-1} D 6 m s^{-1} E 4 m s^{-1}

Short answer

- 13 **Example 1** The position vector function for a particle is $\mathbf{r}(t) = (t^2 + 2t)\mathbf{i} + (2t - 3)\mathbf{j}$ for $-4 \leq t \leq 2$.
 a Find the Cartesian equation of the vector.
 b Draw a graph of the path of the particle, indicating the direction of travel.
- 14 **Example 2** The position vector function for a particle is $\mathbf{r}(t) = [2 \cos(t)]\mathbf{i} + [3 \sin(t)]\mathbf{j}$ for $0 \leq t \leq 2\pi$.
 a Find the corresponding Cartesian relation of the vector.
 b Determine the domain and range of the Cartesian relation.
 c Draw a graph of the path of the particle, indicating the direction of travel.
- 15 **Example 3** The motion of two particles is given by the vector functions $\mathbf{r}(t) = 3t\mathbf{i} + (t^2 - 3t)\mathbf{j}$ and $\mathbf{q}(t) = (8 - t)\mathbf{i} + (3t - 8)\mathbf{j}$.
 Calculate:
 a the position at which the particles collide
 b the distance between the particles when $t = 1$.
- 16 **Example 4** If $\mathbf{r}(t) = 2e^t\mathbf{i} - \sin(5t)\mathbf{j}$, find:
 a $\dot{\mathbf{r}}(t)$ b $|\dot{\mathbf{r}}(t)|$ c $\ddot{\mathbf{r}}(t)$
- 17 **Example 5** If $\mathbf{r}(t) = [(3t + 7)^5 + 5]\mathbf{i} - 8t^2\mathbf{j}$, find $\dot{\mathbf{r}}(t)$ and $\ddot{\mathbf{r}}(t)$.
- 18 **Example 5** If $\mathbf{u}(t) = t\mathbf{i} - 3t\mathbf{j}$ and $\mathbf{v}(t) = 2t^2\mathbf{i} - t\mathbf{j}$, verify that:
 a $\frac{d}{dt}[\mathbf{u}(t) \cdot \mathbf{v}(t)] = \dot{\mathbf{u}} \cdot \mathbf{v} + \mathbf{u} \cdot \dot{\mathbf{v}}$ b $\frac{d}{dt}[\mathbf{u}(t) \times \mathbf{v}(t)] = \dot{\mathbf{u}} \times \mathbf{v}(t) + \mathbf{u}(t) \times \dot{\mathbf{v}}$
- 19 **Example 6** A curve is described by the vector function $\mathbf{r}(t) = 5 \sin(t)\mathbf{i} - 3 \cos(t)\mathbf{j}$ for $0 < t < \pi$.
 a Find the gradient of the curve at the point $P(x, y)$, where $x = 5 \sin(t)$ and $y = -3 \cos(t)$.
 b Find the gradient of the curve when $t = \frac{\pi}{3}$.
- 20 **Example 7** Find $\mathbf{r}(t)$ if $\dot{\mathbf{r}}(t) = \cos(2t)\mathbf{i} - \sin(t)\mathbf{j}$ and $\mathbf{r}(0) = 3\mathbf{i} - 2\mathbf{j}$.
- 21 **Example 8** Given $\ddot{\mathbf{r}}(t) = 12\mathbf{i} - 5\mathbf{j}$, $\mathbf{r}(0) = \mathbf{0}$ and $\dot{\mathbf{r}}(0) = 10\mathbf{i} - 15\mathbf{j}$, calculate $\mathbf{r}(t)$.

- Example 9** If $\mathbf{r}(t) = 2\sqrt{t^3}\mathbf{i} - e^{-2t}\mathbf{j}$, evaluate $\int_0^1 \mathbf{r}(t) dt$.
- Example 10** The position vectors, $\mathbf{r}$, of two particles are given below. For each particle, find:
 - i the velocity
 - ii the acceleration
 - iii the magnitude of the velocity.

a $\mathbf{r} = 3t\mathbf{i} - 2t\mathbf{j}$ b $\mathbf{r} = 5 \sin(3t)\mathbf{i} + 6 \cos(3t)\mathbf{j}$
- Example 11** Given the following information regarding particles at time t , determine expressions for the acceleration and position of the particle in each case.
 - a $\mathbf{v} = 4t\mathbf{i} - 5\mathbf{j}$ and $\mathbf{r} = 2\mathbf{i} + \mathbf{j}$ at $t = 0$
 - b $\mathbf{v} = 4 \cos(t)\mathbf{i} + 6 \sin(t)\mathbf{j}$ and $\mathbf{r} = -5\mathbf{i}$ at $t = 0$
 - c $\mathbf{v} = 5e^{0.4t}\mathbf{i} + 4\mathbf{j}$ and $\mathbf{r} = 5\mathbf{i} + \mathbf{j}$ at $t = 1$
- Example 12** Find the Cartesian equation for the path of a particle with position vector $\mathbf{r} = 5\mathbf{i} + 2t\mathbf{i} - t\mathbf{j} - \mathbf{j}$.
- Example 12** Write equations for the displacement, velocity and acceleration of a projectile where air resistance is negligible.
- Example 13** A particle is projected with an initial velocity of 39.2 m s^{-1} at a 60° angle of elevation. Assuming that $g \approx 9.8 \text{ m s}^{-2}$, find:
 - a the greatest height reached
 - b the time of flight
 - c the range
 - d the speed and direction of motion at a height of 26 m.
- Example 14** A stone is projected from the top of a tower 50 m above a horizontal surface with an initial velocity of 32 m s^{-1} at a 51.3° angle of elevation. Find:
 - a the position vector of the stone
 - b the time taken for the stone to strike the horizontal surface
 - c the horizontal distance travelled by the stone from its projection point to where it meets the horizontal surface
 - d the greatest height above the horizontal surface reached during flight
 - e the distance between the point of projection and where the stone hits the horizontal surface.
- Example 15** Give formulas for the acceleration of a particle undergoing uniform circular motion.
- Example 16** A large wheel has a radius of 2.1 m and completes 400 revolutions per minute. Calculate:
 - a the angular velocity in rad s^{-1}
 - b the speed of a point on the cog's circumference in m s^{-1} .
- Examples 17, 18** A particle moving in a circle of radius 12 m has a constant speed of 8 m s^{-1} . Find its angular velocity and acceleration.

Application

- 32 Find the corresponding Cartesian equation for the position vector function $\mathbf{r}(t) = \cot(t)\mathbf{i} + \operatorname{cosec}(t)\mathbf{j}$ for $0 < t < \pi$ and sketch the particle's path, indicating the direction of travel.
- 33 A particle's position at time t is given by $\mathbf{r}(t) = (t^2 - 4)\mathbf{i} + 2t\mathbf{j}$, where $t > 0$. Find when the magnitude of the angle between $\dot{\mathbf{r}}(t)$ and $\ddot{\mathbf{r}}(t)$ is 30° .
- 34 An object is projected so that its position at any time t is given by $\mathbf{r} = 2t\mathbf{i} + (8t - t^2)\mathbf{j}$, where the components are measured in metres and the time in seconds.
 - a Determine the object's point of projection.
 - b Calculate the speed of projection.
 - c Determine the Cartesian equation for the path of the object.
- 35 An object is thrown horizontally with a speed of $V \text{ m s}^{-1}$ from a platform h metres above a horizontal surface. The object strikes the horizontal surface d metres from the base of the tower after t seconds.
 - a Find an expression for t in terms of h and g (acceleration due to gravity).
 - b Find an expression for V_p , the speed at which the object strikes the horizontal surface, in terms of V , h and g .
 - c Determine the Cartesian equation for the path of the object.
- 36 An object moves in a straight line so that its acceleration at t s is given by $\mathbf{a} = (4 - t)\mathbf{j}$. Initially, the object has a speed of 2 m s^{-1} in the $\mathbf{i}$ direction and is 5 m in the $\mathbf{i}$ direction from a fixed point O . Calculate its velocity and position after t s.
- 37 A person standing on a flat horizontal surface can throw a stone just 45 m.
 - a With what velocity is the stone thrown?
 - b How long is the stone in the air for?



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- 38 A ball is thrown from the top of a silo 49 m above a flat wheatfield. It has an initial speed of 19 m s^{-1} and a 30° angle of projection upward.
- a How long does it take the ball to land?
 - b How far from the base of the silo does the ball land?
 - c If another ball was thrown downward at the same angle and speed, how far from the first one would it land?
- 39 A particle is projected with an initial velocity of 20 m s^{-1} at a 30° angle. Will it clear a 5 m high wall 15 m away from the point of projection?
- 40 A tyre has a diameter of 36 cm. It takes 1.2 s for each revolution. There is a stone stuck to the outside of the tyre. Find the angular velocity, speed and acceleration of the stone.
- 41 The Earth takes a year to orbit the Sun. The orbit is almost circular, with a radius of approximately $1.5 \times 10^8 \text{ km}$. Calculate the speed and acceleration of the Earth in its orbit.



Practice quiz



6

TERMINOLOGY

Argand diagram
argument
complex conjugate
complex polynomial
cubic
De Moivre's theorem
degree
discriminant
dividend
division identity
divisor
factor theorem
leading term
modulus
modulus-argument form
principal argument
quadratic
quartic
quintic
quotient
remainder
remainder theorem
roots of a complex number
roots of unity

COMPLEX NUMBERS

COMPLEX ROOTS AND POLYNOMIALS

- 6.01 Roots of unity
- 6.02 Roots of complex numbers
- 6.03 Complex polynomials
- 6.04 Remainder and factor theorems
- 6.05 Solution of polynomial equations with real coefficients
- 6.06 Solution of complex polynomial equations
- Chapter summary
- Chapter review



Prior learning

ROOTS OF COMPLEX NUMBERS

- determine and examine the n^{th} roots of unity and their location on the unit circle (ACMSM087)
- determine and examine the n^{th} roots of complex numbers and their location in the complex plane (ACMSM088)

FACTORISATION OF POLYNOMIALS

- prove and apply the factor theorem and the remainder theorem for polynomials (ACMSM089)
- consider conjugate roots for polynomials with real coefficients (ACMSM090)
- solve simple polynomial equations (ACMSM091) 

6.01 ROOTS OF UNITY

The square root of 16 means the positive square root of 16, so $\sqrt{16} = 4$. However, there are two solutions of the equation $x^2 = 16$, $x = 4$ or $x = -4$.

In the complex number system there are two solutions to $z^2 = 1$: $z = 1$ or $z = -1$.

You can use **De Moivre's theorem** to see this more clearly.

First look at co-terminal angles for a complex number z .

Write z in **modulus-argument form** as $r[\cos(\alpha) + i \sin(\alpha)]$.

You can add any multiple of 2π to the **argument** without changing the number, so you get:

$$r[\cos(\alpha) + i \sin(\alpha)] = r[\cos(\alpha + 2k\pi) + i \sin(\alpha + 2k\pi)] \text{ for any } k \in \mathbb{Z}.$$

Then, to solve $z^2 = 1$, write $z^2 = 1[\cos(0 + 2k\pi) + i \sin(0 + 2k\pi)]$.

Let the solution be $z = r[\cos(\theta) + i \sin(\theta)]$.

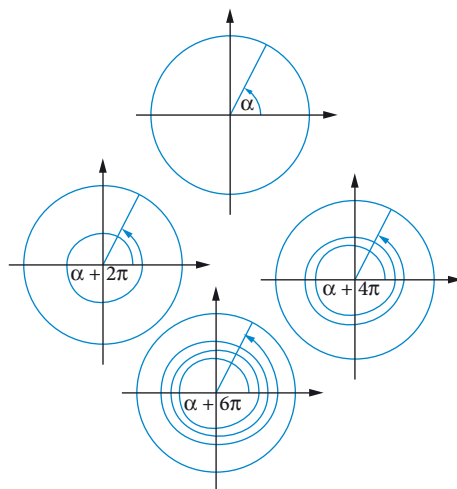
Then using De Moivre's theorem, $z^2 = r^2[\cos(2\theta) + i \sin(2\theta)]$, so the equation $z^2 = 1$ becomes

$$r^2[\cos(2\theta) + i \sin(2\theta)] = 1[(\cos(2k\pi) + i \sin(2k\pi))] \text{ for any } k \in \mathbb{Z}.$$

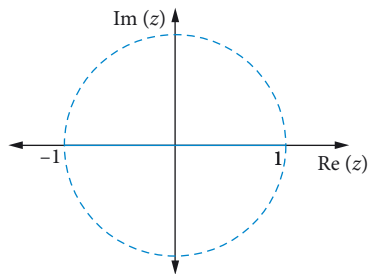
This gives $r^2 = 1$ and $2\theta = 2k\pi$, so $r = 1$ and $\theta = k\pi$ for any $k \in \mathbb{Z}$ and the possible solutions are: ..., 1 cis (-2π) , 1 cis (-1π) , 1 cis (0π) , 1 cis (1π) , 1 cis (2π) , 1 cis (3π) , ...

The answers repeat every 2π , so you should choose the answers whose argument is the **principal argument** in the domain $(-\pi, \pi)$.

For this equation: 1 cis $(0\pi) = 1$ or 1 cis $(1\pi) = -1$.



On an **Argand diagram**, these are opposite radii of the unit circle.



You can use the same procedure for higher **roots of unity** (1).

○ Example 1

- a Solve $z^5 = 1$.
b Show the roots on an Argand diagram and comment on their positions.

Solution

- a Write z in modulus-argument form.

$$\text{Let } z = r[\cos(\theta) + i \sin(\theta)]$$

Write the equation.

$$\{r[\cos(\theta) + i \sin(\theta)]\}^5 = 1 \operatorname{cis}(0)$$

Write with all possible arguments.

$$\{r[\cos(\theta) + i \sin(\theta)]\}^5 = 1 \operatorname{cis}(0 + 2k\pi) \text{ for } k \in \mathbb{Z}.$$

Use De Moivre's theorem.

$$r^5[\cos(5\theta) + i \sin(5\theta)] = 1[\cos(2k\pi) + i \sin(2k\pi)]$$

Isolate the modulus and argument.

$$r^5 = 1 \text{ and } 5\theta = 2k\pi \text{ for } k \in \mathbb{Z}$$

Solve to find them.

$$r = 1 \text{ and } \theta = \frac{2k\pi}{5} \text{ for } k \in \mathbb{Z}$$

Write the principal values.

$$r = 1 \text{ and } \theta = \frac{2k\pi}{5} \text{ for } k = -2, -1, 0, 1, 2.$$

Write out the solutions.

$$z = \cos\left(-\frac{4\pi}{5}\right) + i \sin\left(-\frac{4\pi}{5}\right) \text{ or}$$

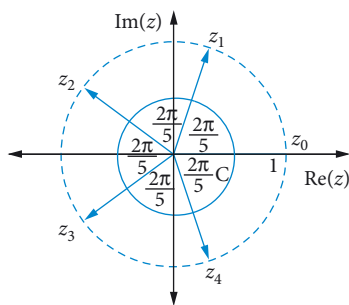
$$z = \cos\left(-\frac{2\pi}{5}\right) + i \sin\left(-\frac{2\pi}{5}\right) \text{ or}$$

$$z = \cos(0) + i \sin(0) = 1 \text{ or}$$

$$z = \cos\left(\frac{2\pi}{5}\right) + i \sin\left(\frac{2\pi}{5}\right) \text{ or}$$

$$z = \cos\left(\frac{4\pi}{5}\right) + i \sin\left(\frac{4\pi}{5}\right)$$

b Show the solutions on an Argand diagram.



Comment on the positions.

The roots are shown as z_0, z_1, z_2, z_3, z_4 .

The roots are equally spaced around the unit circle, separated by angles of $\frac{2\pi}{5}$, with one root being 1.

From the Argand diagram of the 5th roots of 1, you can see that z_1 and z_4 are **complex conjugates** because their real parts are the same but their complex parts are opposite in sign. Since both have magnitude 1, this also means that they are inverses, so $z_1 z_4 = 1$. In fact, for any n , the n th roots of unity must be 1, -1 or multiply with another root to equal 1.

○ Example 2

Show that the 5th roots of unity are either 1 or have a product with another root to equal 1.

Solution

Write the 5th roots of 1.

$$z_0 = \cos(0) + i \sin(0) = 1 \text{ or}$$

$$z_1 = \cos\left(\frac{2\pi}{5}\right) + i \sin\left(\frac{2\pi}{5}\right) \text{ or}$$

$$z_2 = \cos\left(\frac{4\pi}{5}\right) + i \sin\left(\frac{4\pi}{5}\right) \text{ or}$$

$$z_3 = \cos\left(-\frac{4\pi}{5}\right) + i \sin\left(-\frac{4\pi}{5}\right) \text{ or}$$

$$z_4 = \cos\left(-\frac{2\pi}{5}\right) + i \sin\left(-\frac{2\pi}{5}\right)$$

Write the result for z_0 .

$$z_0 = 1$$

Write out $z_1 z_4$.

$$z_1 z_4 = \left[\cos\left(\frac{2\pi}{5}\right) + i \sin\left(\frac{2\pi}{5}\right) \right] \times \left[\cos\left(-\frac{2\pi}{5}\right) + i \sin\left(-\frac{2\pi}{5}\right) \right]$$

$$= \cos(0)$$

Multiply by adding the arguments.

$$= \cos(0) + i \sin(0)$$

$$= 1$$

Write out $z_2 z_3$.

$$z_2 z_3 = \left[\cos\left(\frac{4\pi}{5}\right) + i \sin\left(\frac{4\pi}{5}\right) \right] \times \left[\cos\left(-\frac{4\pi}{5}\right) + i \sin\left(-\frac{4\pi}{5}\right) \right]$$

$$= \cos(0)$$

Multiply by adding the arguments.

$$= \cos(0) + i \sin(0)$$

$$= 1$$

Write the overall result.

$z_0 = 1$, $z_1 z_4 = 1$ and $z_2 z_3 = 1$, so the 5th roots of unity are either 1 or multiply with another root to equal 1.

You can see from the first two examples that the roots of 1 have a particular pattern.

IMPORTANT

The n th **roots of unity** are complex numbers z such that $z^n = 1$. They lie on the unit circle and have the form $\cos\left(\frac{2k\pi}{n}\right) + i \sin\left(\frac{2k\pi}{n}\right)$, where $k = 0, 1, 2, \dots, n-1$, with principal values $\cos\left(\frac{2k\pi}{n}\right) + i \sin\left(\frac{2k\pi}{n}\right)$, where $-\pi < \frac{2k\pi}{n} \leq \pi$.

You can use a CAS calculator to calculate the roots of 1, but the format of the answers can be unsimplified. It is recommended that if you use the TI-Nspire for complex numbers, you use the rectangular form of complex numbers. The ClassPad assumes this form anyway. To express an answer in the form $r[\cos(\theta) + i \sin(\theta)]$, you can find the magnitude and argument of the answer(s). The ClassPad has a conversion to this form, which is called the trigonometric form on the calculator.

Example 3

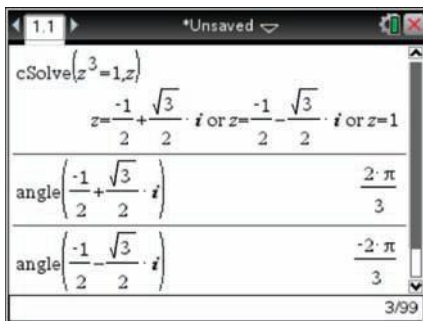
CAS Find the cube roots of 1.

Solution

TI-Nspire CAS

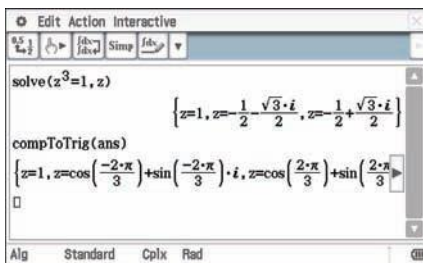
Change the document settings for Real or Complex to rectangular. Use **[menu]**, 3: Algebra, C: Complex and 1: Solve or type `csolve` ($z^3 = 1, z$).

Find the argument using **[menu]**, 2: Number, 9: Complex Number Tools, 4: Polar Angle or type `angle()` and copy and paste each answer into the brackets.



ClassPad

Ensure that the calculator is in **Cplex** (complex) mode rather than **Real**. Tap **Action**, **Advanced** and **solve** and then the equation and variable of interest, or type `solve(z^3 = 1, z)`. Change the answer to trigonometric form by tapping **Action**, **Complex**, **compToTrig**, then type `ans` **[]** or type `compToTrig(ans)`. Tap the right arrow **[]** to see the part of the solution that doesn't fit on the screen.



Write the answers, assuming that the magnitude is 1.

The cube roots are $\cos\left(-\frac{2\pi}{3}\right) + i \sin\left(-\frac{2\pi}{3}\right)$,
1 or $\cos\left(\frac{2\pi}{3}\right) + i \sin\left(\frac{2\pi}{3}\right)$.

If a question does not specify the form of an answer in complex numbers, you can choose to give the answer in either form, so choose the one that is simplest for that question.

INVESTIGATION Roots of unity

What do you get when you multiply or divide roots of unity?

- Find a cube root of unity and call it, say, z_3 .
- Find an eighth root of unity and call it, say, z_8 .
- Find the values of $z_3 \times z_8$ and $z_3 \div z_8$.
- Investigate powers of $z_3 \times z_8$ and $z_3 \div z_8$.
- What do you find?
- What suppositions can you make?
- Can you prove your suppositions?

EXERCISE 6.01 Roots of unity

Concepts and techniques

- Example 1** a Solve $z^3 = 1$.
 b Show the roots on an Argand diagram and comment on their positions.
 c Express the roots in rectangular form.
- a Solve $z^4 = 1$.
 b Show the roots on an Argand diagram and comment on their positions.
- a Solve $z^6 = 1$.
 b Show the roots on an Argand diagram and comment on their positions.
- Example 2** Show that the cube roots of unity are either 1 or multiply with another root to equal 1.
- Show that the fourth roots of unity are 1, -1 or multiply with another root to equal 1.
- Show that the 6th roots of unity are 1, -1 or when multiplied by another root are to equal 1.
- Example 3** **CAS** a Find the fourth roots of 1 in rectangular form.
 b Change the roots to polar form.
- CAS** a Find the eighth roots of 1 in rectangular form.
 b Change the roots to polar form.
- CAS** a Find the sixth roots of 1 in rectangular form.
 b Change the roots to polar form.

Reasoning and communication

- Show that if p is an n th root of 1, then p^m is also an n th root, where $n, m \in \mathbb{Z}^+$.
- Show that if p and q are n th roots of 1, then pq is also an n th root, where $n \in \mathbb{Z}^+$.

- 12 Show that if p and q are n th roots of 1, then there exist positive integers a and b , where $a \neq b$, such that $p^a = q^b$.
- 13 For any positive rational number x , show that $\cos(x\pi) + i \sin(x\pi)$ is a power of an n th root of 1, for some $n \in \mathbb{Z}^+$.
- 14 Show that the n th roots of unity are 1, -1 or multiply by another root to equal 1.

6.02 ROOTS OF COMPLEX NUMBERS

Finding the **roots of complex numbers** is similar to finding the roots of unity. For numbers with a magnitude of 1, the only difference to the roots of unity is that you add $2k\pi$ to the argument not equal to 0.

○ Example 4

Find the 6th roots of i .

Solution

Write i in modulus-argument form.

$$i = \cos\left(\frac{\pi}{2}\right) + i \sin\left(\frac{\pi}{2}\right)$$

Write z in modulus-argument form.

$$\text{Let } z = r[\cos(\theta) + i \sin(\theta)]$$

Write an equation.

$$\{r[\cos(\theta) + i \sin(\theta)]\}^6 = \cos\left(\frac{\pi}{2}\right) + i \sin\left(\frac{\pi}{2}\right)$$

Write with all possible arguments.

$$\{r[\cos(\theta) + i \sin(\theta)]\}^6 = 1 \operatorname{cis}\left(\frac{\pi}{2} + 2k\pi\right) \text{ for } k \in \mathbb{Z}$$

Use De Moivre's theorem.

$$r^6 \operatorname{cis}(6\theta) = 1 \operatorname{cis}\left(\frac{\pi}{2} + 2k\pi\right)$$

Isolate the modulus and argument.

$$r^6 = 1 \text{ and } 6\theta = \frac{\pi}{2} + 2k\pi \text{ for } k \in \mathbb{Z}$$

Solve to find them.

$$r = 1 \text{ and } \theta = \frac{\pi}{12} + \frac{2k\pi}{6} \text{ for } k \in \mathbb{Z}$$

Write the principal arguments.

$$r = 1 \text{ and } \theta = \frac{\pi}{12} + \frac{k\pi}{3} \text{ for } k = -3, -2, -1, 0, 1, 2$$

Write on a common denominator.

$$r = 1 \text{ and } \theta = \frac{(4k+1)\pi}{12} \text{ for } k = -3, -2, -1, 0, 1, 2$$

Write out the solutions.

$$z = \cos\left(\frac{-11\pi}{12}\right) + i \sin\left(\frac{-11\pi}{12}\right) \text{ or}$$

$$z = \cos\left(\frac{-7\pi}{12}\right) + i \sin\left(\frac{-7\pi}{12}\right) \text{ or}$$

$$z = \cos\left(\frac{-\pi}{4}\right) + i \sin\left(\frac{-\pi}{4}\right) \text{ or}$$

$$z = \cos\left(\frac{\pi}{12}\right) + i \sin\left(\frac{\pi}{12}\right) \text{ or}$$

$$z = \cos\left(\frac{5\pi}{12}\right) + i \sin\left(\frac{5\pi}{12}\right) \text{ or}$$

$$z = \cos\left(\frac{3\pi}{4}\right) + i \sin\left(\frac{3\pi}{4}\right)$$

The answers to Example 4 could be given as

$$z = \cos \left[\frac{(4k+1)\pi}{12} \right] + i \sin \left[\frac{(4k+1)\pi}{12} \right] \text{ for } k = -3, -2, \dots, 2.$$

You could also write it as

$$z = \cos \left[\frac{(4k+1)\pi}{12} \right] + i \sin \left[\frac{(4k+1)\pi}{12} \right] \text{ for } \{k \in \mathbb{Z}: -3 \leq k \leq 2\}$$

to avoid writing out each root separately.

Generally, when you find the roots of complex numbers, you have to take account of the magnitude as well. If the magnitude of the number is r , then the magnitude of the n th root will be $\sqrt[n]{r}$.

○ Example 5

- Find the cube roots of -125 .
- Show the roots on the complex plane.

Solution

- Write -125 in modulus-argument form.

$$-125 = 125[\cos(\pi) + i \sin(\pi)]$$

Write z in modulus-argument form.

$$\text{Let } z = r[\cos(\theta) + i \sin(\theta)]$$

Write an equation.

$$[r \operatorname{cis}(\theta)]^3 = 125 \operatorname{cis}(\pi)$$

Write with all possible arguments.

$$[r \operatorname{cis}(\theta)]^3 = 125 \operatorname{cis}(\pi + 2k\pi) \text{ for } k \in \mathbb{Z}$$

Use De Moivre's theorem.

$$r^3 \operatorname{cis}(3\theta) = 125 \operatorname{cis}(\pi + 2k\pi)$$

Isolate the modulus and argument.

$$r^3 = 125 \text{ and } 3\theta = \pi + 2k\pi \text{ for } k \in \mathbb{Z}$$

Solve to find them.

$$r = 5 \text{ and } \theta = \frac{\pi}{3} + \frac{2k\pi}{3} \text{ for } k \in \mathbb{Z}$$

Write the principal values.

$$r = 5 \text{ and } \theta = \frac{(2k+1)\pi}{3} \text{ for } k = -1, 0, 1$$

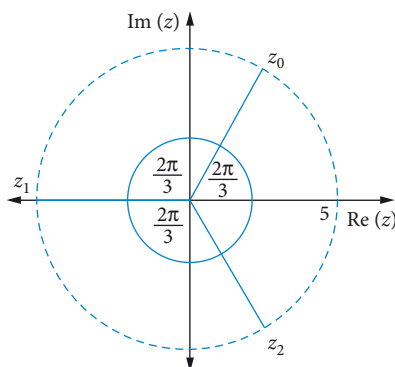
Write out the solutions.

$$z = 5 \left[\cos\left(-\frac{\pi}{3}\right) + i \sin\left(-\frac{\pi}{3}\right) \right] \text{ or}$$

$$z = 5 \left[\cos\left(\frac{\pi}{3}\right) + i \sin\left(\frac{\pi}{3}\right) \right] \text{ or}$$

$$z = 5[\cos(\pi) + i \sin(\pi)]$$

- Show the solutions on an Argand diagram. They are all on a circle of radius 5, centred at the origin.



You will only get a rational or integer magnitude for a root when the magnitude of the original number happens to be the appropriate power.

○ Example 6

- a Solve the equation $z^4 + i = 1$.
b Show the roots on the complex plane.

Solution

- a Write in the form $z^n = w$.

$$z^4 = 1 - i$$

Write $1 - i$ in modulus argument form.

$$1 - i = \sqrt{2} \left[\cos\left(-\frac{\pi}{4}\right) + i \sin\left(-\frac{\pi}{4}\right) \right]$$

Write z in modulus argument form.

$$\text{Let } z = r[\cos(\theta) + i \sin(\theta)]$$

Write an equation.

$$[r \operatorname{cis}(\theta)]^4 = \sqrt{2} \operatorname{cis}\left(-\frac{\pi}{4}\right)$$

Write with all possible arguments.

$$[r \operatorname{cis}(\theta)]^4 = \sqrt{2} \operatorname{cis}\left(-\frac{\pi}{4} + 2k\pi\right) \text{ for } k \in \mathbb{Z}$$

Use De Moivre's theorem.

$$r^4 \operatorname{cis}(4\theta) = \sqrt{2} \operatorname{cis}\left(-\frac{\pi}{4} + 2k\pi\right)$$

Isolate the modulus and argument.

$$r^4 = \sqrt{2} \text{ and } 4\theta = -\frac{\pi}{4} + 2k\pi \text{ for } k \in \mathbb{Z}$$

Solve to find them.

$$r = \sqrt[8]{2} \text{ and } \theta = -\frac{\pi}{16} + \frac{2k\pi}{4} \text{ for } k \in \mathbb{Z}$$

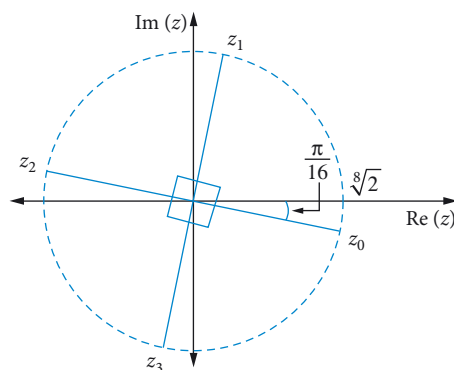
Write the principal values.

$$r = \sqrt[8]{2} \text{ and } \theta = \frac{(8k-1)\pi}{16} \text{ for } k = -1, 0, 1, 2$$

Write the solutions.

$$z = \sqrt[8]{2} \left\{ \cos\left[\frac{(8k-1)\pi}{16}\right] + i \sin\left[\frac{(8k-1)\pi}{16}\right] \right\} \\ \text{for } k = -1, 0, 1, 2$$

- b Show the solutions on an Argand diagram. They are all on a circle of radius $\sqrt[8]{2} \approx 1.0905$, centred at the origin.



You should write out the solutions in full for a small number of solutions, but for a large number you should use a variable such as k to give the solutions. Sometimes the way you express an answer can be different to someone else's expression, yet both will be correct, such as $\frac{(8k-1)\pi}{16}$ for

$$k = -1, 0, 1, 2; \frac{(8k+7)\pi}{16} \text{ for } k = -2, -1, 0, 1; k\pi - \frac{9\pi}{16}, k\pi - \frac{\pi}{16} \text{ for } k = 0, 1; \text{ or } k\pi + \frac{7\pi}{16}, k\pi + \frac{11\pi}{16} \text{ for}$$

$k = -1, 0$. You should check that each of these expressions gives the same set of answers.

IMPORTANT

The n th **roots of a complex number** z are on a circle of radius $\sqrt[n]{|z|}$, separated from each other by the angle $\frac{2\pi}{n}$ and have the form

$$\sqrt[n]{|z|} \left\{ \cos \left[\frac{\arg(z) + 2k\pi}{n} \right] + i \sin \left[\frac{\arg(z) + 2k\pi}{n} \right] \right\}, \text{ where } k = 0, 1, 2, \dots, n-1.$$

Once you have one root of a complex number, you can use the separation of the roots by the angle $\frac{2\pi}{n}$ to find the others. For example, if one 6th root of a complex number is $\sqrt{5} \operatorname{cis}(\frac{\pi}{4})$, then the other roots are separated by multiples of $\frac{\pi}{3}$ from this and each other, so they must be $\sqrt{5} \operatorname{cis}(-\frac{\pi}{12})$, $\sqrt{5} \operatorname{cis}(-\frac{5\pi}{12})$, $\sqrt{5} \operatorname{cis}(-\frac{3\pi}{4})$, $\sqrt{5} \operatorname{cis}(\frac{7\pi}{12})$ and $\sqrt{5} \operatorname{cis}(\frac{11\pi}{12})$.

EXERCISE 6.02 Roots of complex numbers



Roots of complex numbers

Concepts and techniques

- 1 **Example 4** Find the 8th roots of $-i$.
- 2 Find the cube roots of $\frac{\sqrt{3}-i}{2}$.
- 3 Solve the equation $\sqrt{2} z^4 + 1 = i$.
- 4 **Example 5** a Find the cube roots of $64i$.
b Show the roots on the complex plane and comment on their positions.
- 5 a Find the fourth roots of -81 .
b Show the roots on an Argand diagram and comment on their positions.
- 6 a Solve the equation $z^2 + 7 = 24i$, with the argument correct to 4 decimal places.
b Express in Cartesian form.
c Show the solutions on the complex plane.
- 7 **Example 6** a Find the exact sixth roots of $20 + 15i$.
b Show the roots on the complex plane and comment on their positions.
- 8 a Find the fifth roots of $-64i$.
b Show the roots on the complex plane and comment on their positions.
- 9 a Solve the equation $z^3 + 28 + 10i = 0$ and express the answers in Cartesian form correct to 3 decimal places.
b Show the solutions on the complex plane and comment on their positions.

Reasoning and communication

- 10 Show that the $(2n)$ th roots of any negative number are complex, where $n \in \mathbb{Z}^+$. That is, that none of the roots are real or purely imaginary.

6.03 COMPLEX POLYNOMIALS

You already know a lot about polynomials with real coefficients. A **complex polynomial** is exactly the same, except that the coefficients can be complex numbers. The complex numbers include purely real and purely imaginary numbers.

IMPORTANT

A polynomial is a sum of terms containing non-negative integer powers of the variable.

A **complex polynomial** in z is of the form:

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + a_{n-2} z^{n-2} + \dots + a_1 z^1 + a_0$$

where n is a positive whole number, $a_0, a_1, a_2, \dots, a_{n-1}, a_n$ may be complex numbers and $a_n \neq 0$.

The highest power of z is called the **degree** of the polynomial. The term that contains the highest power is called the **leading term**.

The degree of the polynomial above is n , and the leading term is $a_n z^n$. You can find the value of a complex polynomial at particular points by substituting in exactly the same way as for real polynomials. Real polynomials form a subset of complex polynomials.

○ Example 7

The function $m(z) = 3z^3 - iz^2 + 2iz^4 - 5z - z^4 + 3iz + 7i$

- a What is the degree of $m(z)$?
- b What is the leading term?

Solution

- a The highest power of z is z^4 .
- b Collect the coefficients in order.

Write the leading term.

$m(z)$ is of degree 4.

$$m(z) = (-1 + 2i)z^4 + 3z^3 - iz^2 + (-5 + 3i)z + 7i$$

The leading term is $(-1 + 2i)z^4$.



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○ Example 8

Find the values of $p(z) = 3z^3 - iz^2 + 2i - 5$ at the points $z = 3$, $z = -2i$ and $z = 2 + 3i$.

Solution

Substitute $z = 3$.

$$p(3) = 3 \times 3^3 - i \times 3^2 + 2i - 5$$

Simplify.

$$= 76 - 7i$$

Substitute $z = -2i$.

$$p(-2i) = 3 \times (-2i)^3 - i \times (-2i)^2 + 2i - 5$$

Simplify each term.

$$= 24i + 4i + 2i - 5$$

Simplify and write the real part first.

$$= -5 + 30i$$

Substitute $z = 2 + 3i$.

$$p(2 + 3i) = 3 \times (2 + 3i)^3 - i \times (2 + 3i)^2 + 2i - 5$$

Simplify each term.

$$= 24 + 108i - 162 - 81i - 4i + 12 + 9i + 2i - 5$$

Simplify and write the real part first.

$$= -131 + 34i$$

The arithmetic for complex polynomials is a bit more difficult than for real polynomials so you should check your calculations with a CAS calculator whenever possible.

○ Example 9

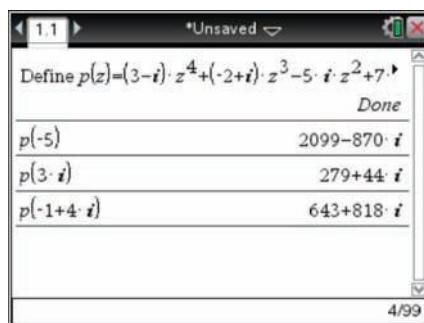
CAS Given the function $p(z) = (3 - i)z^4 + (-2 + i)z^3 - 5iz^2 + 7z + 9 + 5i$, find $p(-5)$, $p(3i)$ and $p(-1 + 4i)$.

Solution

TI-Nspire CAS

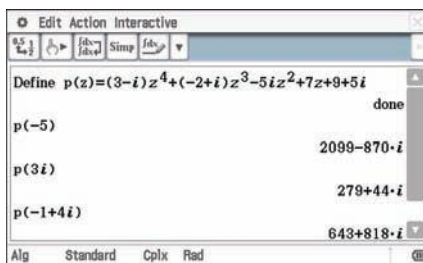
Set your calculator to rectangular (complex) mode and Define $p(z)$. Make sure that you use i from the symbols menu (π or ctrl img) for the imaginary number i .

Then enter $p(-5)$, $p(3i)$ and $p(-1 + 4i)$.



ClassPad

Make sure that your calculator is set to **Cplx** and define $p(z)$. Press **Keyboard** and use the **Math3** keyboard and **Define** $p(z)$. Make sure that you use the imaginary number i , (see **i**) on the soft keyboards **Math2** or **Math3**) not the letter i . Then enter $p(-5)$, $p(3i)$ and finally $p(-1+4i)$.



Write the answers.

$$p(-5) = 2099 - 870i, p(3i) = 279 + 44i \text{ and } p(-1+4i) = 643 + 818i$$

EXERCISE 6.03 Complex polynomials

Concepts and techniques



Complex polynomials

- Example 7** What is the degree of $p(z) = 3z^3 - iz^5 + 8iz^4 - 5z - z^3 + 7iz + 7i$?
A 3 B 4 C 5 D 7 E 8
- What is the leading term of $p(z) = 6z^2 - 2iz^3 + 2iz^5 - 5z - iz^2 + 2iz^3 + 7i$?
A $6z^2$ B $(6-i)z^2$ C $(6-2i)z^5$ D $(2i-5)z^5$ E $2iz^5$
- Example 8** Given the function $t(z) = 3z^3 - iz^2 + 2iz - 6 + 3i$, find each of the following.
a $t(2)$ b $t(2i)$ c $t(-3)$ d $t(-3i)$ e $t(2-3i)$
- Given the function $m(z) = 3iz^4 + 5z^3 + (2i-3)z + 4i$, find each of the following.
a $m(-5)$ b $m(3i)$ c $m(-5+3i)$ d $m(4-i)$ e $m(2+i)$
- Given the function $p(z) = (3-i)z^2 + (-2+5i)z + 4-2i$, find each of the following.
a $p(4)$ b $p(5i)$ c $p(4+5i)$ d $p(-5-2i)$ e $p(6+i)$
- Example 9** **CAS** Given the function $q(z) = (5+3i)z^4 + (4-3i)z^2 - 8z - 5i$, find each of the following.
a $q(4)$ b $q(5i)$ c $q(4+5i)$ d $q(-5-2i)$ e $q(6+i)$
- CAS** Given the function $f(z) = (3-2i)z^3 + (5+6i)z^2 + (-4+5i)z - 4 + 8i$, find each of the following.
a $f(-3)$ b $f(6i)$ c $f(-3+6i)$ d $f(4-7i)$ e $f(2+4i)$
- CAS** Given the function $g(z) = (-2-4i)z^5 + (5+7i)z^3 - 8iz^2 + 3z - 4 + 6i$, find each of the following.
a $g(2+3i)$ b $g(-3+4i)$ c $g(3-2i)$ d $g(4+7i)$ e $g(-2-i)$

Reasoning and communication

- In general, is it true that $p(a) + p(ib) = p(a+ib)$ for complex polynomials? Prove or disprove this proposition. Does it depend on the degree?

6.04 REMAINDER AND FACTOR THEOREMS

You can add, subtract, multiply and divide complex polynomials in the same way as real polynomials, but the coefficients are complex instead of real. When you divide one complex polynomial into another, remember to allow for both the real and imaginary part of the coefficient.

○ Example 10

Three functions are defined as follows:

$$p(z) = (2 + i)z^3 - 5z^2 + 4 + 6i,$$

$$q(z) = (6 - 2i)z^2 + (-4 - 3i)z - 6 \text{ and}$$

$$d(z) = z - 2 + 3i.$$

- Find and simplify $p(z) - q(z)$.
- Find and simplify $p(z)q(z)$.
- Find and simplify $p(z) \div d(z)$.

Solution

- Write the difference.

$$\begin{aligned} p(z) - q(z) &= [(2 + i)z^3 - 5z^2 + 4 + 6i] \\ &\quad - [(6 - 2i)z^2 + (-4 - 3i)z - 6] \\ &= [(2 + i)z^3 + [-5 - (6 - 2i)]z^2 \\ &\quad - (-4 - 3i)z + 4 + 6i - (-6)] \\ &= [(2 + i)z^3 + [-11 + 2i]z^2 + (4 + 3i)z + 10 + 6i] \end{aligned}$$

Add the coefficients of each power.

Simplify.

- Write the product.

$$\begin{aligned} p(z)q(z) &= [(2 + i)z^3 - 5z^2 + 4 + 6i] \\ &\quad \times [(6 - 2i)z^2 + (-4 - 3i)z - 6] \\ &= (2 + i)z^3 \times [(6 - 2i)z^2 + (-4 - 3i)z - 6] \\ &\quad - 5z^2 \times [(6 - 2i)z^2 + (-4 - 3i)z - 6] \\ &\quad + 4 \times [(6 - 2i)z^2 + (-4 - 3i)z - 6] \\ &\quad + 6i \times [(6 - 2i)z^2 + (-4 - 3i)z - 6] \end{aligned}$$

Use the distributive law.

Use the distributive law again.

$$\begin{aligned} &= (12 - 4i + 6i + 2)z^5 + (-8 - 6i - 4i + 3)z^4 \\ &\quad + (-12 - 6i)z^3 + (-30 + 10i)z^2 \\ &\quad + (20 + 15i)z^3 + 30z^2 + (24 - 8i)z^2 \\ &\quad + (-16 - 12i)z - 24 + (12 + 36i)z^2 \\ &\quad + (18 - 24i)z - 36i \end{aligned}$$

Collect like terms.

$$\begin{aligned} &= (14 + 2i)z^5 - 35z^4 + (8 + 9i)z^3 + (66 + 28i)z^2 \\ &\quad + (2 - 36i)z - 24 - 36i \end{aligned}$$

c Write the quotient.

$$p(z) \div d(z) = \frac{(2+i)z^3 - 5z^2 + 4 + 6i}{z - 2 + 3i}$$

Set out as long division.

$$\begin{array}{r} z - 2 + 3i \overline{) (2+i)z^3 + (-5+0i)z^2 + (0+0i)z + 4+6i} \\ \underline{(2+i)z^2} \end{array}$$

Divide the z^3 term by z .

$$\begin{array}{r} z - 2 + 3i \overline{) (2+i)z^3 + (-5+0i)z^2 + (0+0i)z + 4+6i} \\ \underline{(2+i)z^2} \end{array}$$

Multiply.

$$\begin{array}{r} z - 2 + 3i \overline{) (2+i)z^3 + (-5+0i)z^2 + (0+0i)z + 4+6i} \\ \underline{(2+i)z^3 + (-7+4i)z^2} \end{array}$$

Subtract and bring down.

$$\begin{array}{r} z - 2 + 3i \overline{) (2+i)z^3 + (-5+0i)z^2 + (0+0i)z + 4+6i} \\ \underline{(2+i)z^3 + (-7+4i)z^2} \\ (2-4i)z^2 + (0+0i)z + 4+6i \\ \underline{(2+i)z^2 + (2-4i)z} \end{array}$$

Divide the z^2 term.

$$\begin{array}{r} z - 2 + 3i \overline{) (2+i)z^3 + (-5+0i)z^2 + (0+0i)z + 4+6i} \\ \underline{(2+i)z^3 + (-7+4i)z^2} \\ (2-4i)z^2 + (0+0i)z + 4+6i \\ \underline{(2+i)z^2 + (2-4i)z} \end{array}$$

Multiply.

$$\begin{array}{r} z - 2 + 3i \overline{) (2+i)z^3 + (-5+0i)z^2 + (0+0i)z + 4+6i} \\ \underline{(2+i)z^3 + (-7+4i)z^2} \\ (2-4i)z^2 + (0+0i)z + 4+6i \\ \underline{(2-4i)z^2 + (8+14i)z} \\ (2+i)z^2 + (2-4i)z \end{array}$$

Subtract and bring down.

$$\begin{array}{r} z - 2 + 3i \overline{) (2+i)z^3 + (-5+0i)z^2 + (0+0i)z + 4+6i} \\ \underline{(2+i)z^3 + (-7+4i)z^2} \\ (2-4i)z^2 + (0+0i)z + 4+6i \\ \underline{(2-4i)z^2 + (8+14i)z} \\ -(8-14i)z + 4+6i \end{array}$$

Divide the z term.

$$\begin{array}{r} z - 2 + 3i \overline{) (2+i)z^3 + (-5+0i)z^2 + (0+0i)z + 4+6i} \\ \underline{(2+i)z^3 + (-7+4i)z^2} \\ (2-4i)z^2 + (0+0i)z + 4+6i \\ \underline{(2-4i)z^2 + (8+14i)z} \\ -(8-14i)z + 4+6i \end{array}$$



Multiply.

$$\begin{array}{r}
 (2+i)z^2 + (2-4i)z - 8 - 14i \\
 z - 2 + 3i \overline{) (2+i)z^3 + (-5+0i)z^2 + (0+0i)z + 4 + 6i} \\
 \underline{(2+i)z^3 + (-7+4i)z^2} \\
 (2-4i)z^2 + (0+0i)z + 4 + 6i \\
 \underline{(2-4i)z^2 + (8+14i)z} \\
 -(8-14i)z + 4 + 6i \\
 \underline{-(8-14i)z + 58 + 4i} \\
 -54 + 2i
 \end{array}$$

Subtract. There is no more to bring down.

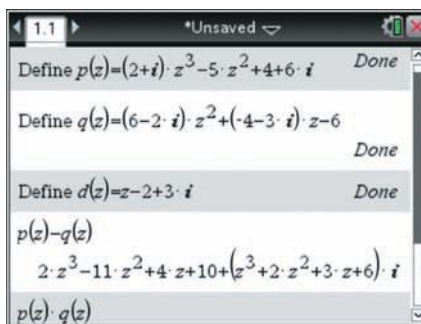
$$\begin{array}{r}
 (2+i)z^2 + (2-4i)z - 8 - 14i \\
 z - 2 + 3i \overline{) (2+i)z^3 + (-5+0i)z^2 + (0+0i)z + 4 + 6i} \\
 \underline{(2+i)z^3 + (-7+4i)z^2} \\
 (2-4i)z^2 + (0+0i)z + 4 + 6i \\
 \underline{(2-4i)z^2 + (8+14i)z} \\
 -(8+14i)z + 4 + 6i \\
 \underline{-(8+14i)z + 58 + 4i} \\
 -54 + 2i
 \end{array}$$

Write in standard form.

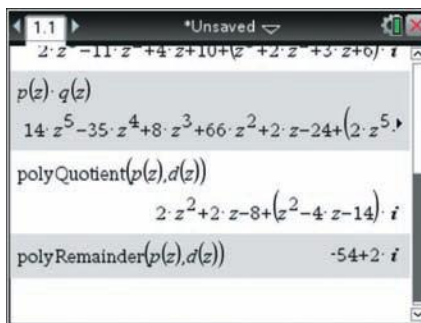
$$(p \div d)(z) = (2+i)z^2 + (2-4i)z - 8 - 14i + \frac{-54+2i}{z-2+3i}$$

TI-Nspire CAS

Set to complex (rectangular) mode and define the functions.
You can add, subtract and multiply them.

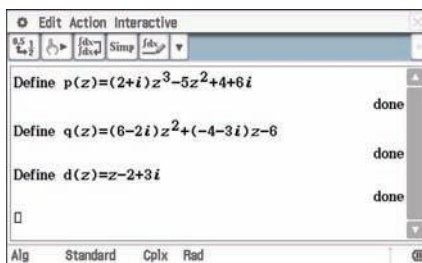


Use **menu**, 3: Algebra, 8: Polynomial Tools and 5: Quotient of Polynomial to find the quotient.
Use 4: Remainder of polynomial to get the remainder.

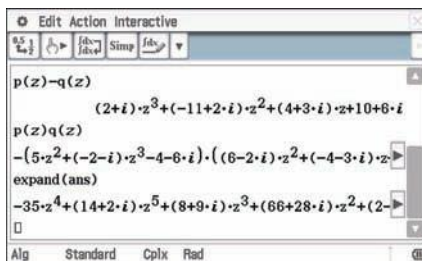


ClassPad

In **Cplx** mode, define the functions. You can add, subtract and multiply them.



Perform the subtraction and multiplication operations. Tap **Action**, **Transformation**, then **expand** to see the product in standard form.



The ClassPad does not have polynomial tools.

In Example 10, the difference of the polynomials has the higher degree of the two. The **remainder** was written in standard form as a fraction with the denominator of $z - 2 + 3i$. This is similar to writing $13 \div 5 = 2 \frac{3}{5}$ or $2504 \div 23 = 108 \frac{20}{23}$. The relationship between the **quotient**, **divisor**, **dividend** and remainder is the same.

A difference or sum of polynomials could give a result of lower degree than either polynomial if they were of the same degree and had leading terms with the same coefficient or the same but for the sign.



Polynomial operations



IMPORTANT

If a complex polynomial $P(z)$ is divided by the complex polynomial $D(z)$, then the **remainder** $R(z)$ has degree less than $D(z)$. The **quotient** $Q(z)$, **divisor**, remainder and **dividend** are related by the **division identity** $P(z) = D(z)Q(z) + R(z)$.

For a divisor of degree one, the remainder theorem for complex polynomials is the same as that for real polynomials.

IMPORTANT

Remainder theorem

If a complex polynomial $P(z)$ is divided by $D(z) = z - a$, where $a \in \mathbb{C}$, then the remainder is given by $R = P(a)$.

From the division identity, $P(z) = (z - a)Q(z) + R(z)$.

Since $R(z)$ has degree less than that of $(z - a)$, it is of degree 0, so it is a constant, say R .

Substituting $z = a$ gives $P(a) = (a - a)Q(a) + R$, which gives the required result. **QED**

○ Example 11

The function $g(z) = (6 + 2i)z^4 + (-2 + 3i)z^2 + (4 - i)z - 4 - 5i$. Find the remainders when $g(z)$ is divided by:

- a $z + 3$ b $z - 2i$ c $z - 4 + 2i$

Solution

- a Use the remainder theorem.

$$g(-3) = (6 + 2i)(-3)^4 + (-2 + 3i)(-3)^2 + (4 - i)(-3) - 4 - 5i$$

Simplify.

$$= 486 + 162i - 18 + 27i - 12 + 3i - 4 - 5i$$

Write the answer.

$$\text{Remainder} = 452 + 187i$$

- b Use the remainder theorem.

$$g(2i) = (6 + 2i)(2i)^4 + (-2 + 3i)(2i)^2 + (4 - i)(2i) - 4 - 5i$$

Simplify.

$$= 96 + 32i + 8 - 12i + 8i + 2 - 4 - 5i$$

Write the answer.

$$\text{Remainder} = 102 + 23i$$

- c Use the remainder theorem.

$$g(4 - 2i) = (6 + 2i)(4 - 2i)^4 + (-2 + 3i)(4 - 2i)^2 + (4 - i)(4 - 2i) - 4 - 5i$$

Simplify.

$$= (6 + 2i)(-112 - 384i) + (-2 + 3i)(12 - 16i) + 16 - 12i - 2 - 4 - 5i$$

Keep going.

$$= 96 - 2528i + 24 + 68i + 10 - 17i$$

Write the answer.

$$= 130 - 2477i$$

The factor theorem follows logically from the remainder theorem.

IMPORTANT

Factor theorem

The expression $z - a$, where $a \in \mathbb{C}$, is a factor of the complex polynomial $P(z)$ if and only if $P(a) = 0$.

From the remainder theorem, the remainder when $P(z)$ is divided by $z - a$ is $R = P(a)$.

If $z - a$ is a factor, then the remainder is 0, so $P(a) = 0$.

If $P(a) = 0$, then the remainder is zero, so $z - a$ is a factor. **QED**

You can use the factor theorem to show that a linear expression is a factor of a complex polynomial in the same way as for real polynomials.

○ Example 12

Show that $z + 4 - i$ is a factor of $(2 + 3i)z^3 + (11 + 11i)z^2 + (-2 - i)z - 17 - 17i$.

Solution

Write the polynomial.

$$P(z) = (2 + 3i)z^3 + (11 + 11i)z^2 + (-2 - i)z - 17 - 17i$$

Find $P(-4 + i)$

$$P(-4 + i) = (2 + 3i)(-4 + i)^3 + (11 + 11i)(-4 + i)^2 + (-2 - i)(-4 + i) - 17 - 17i$$

Simplify.

$$\begin{aligned} &= (2 + 3i)(-64 + 48i + 12 - i) \\ &\quad + (11 + 11i)(16 - 8i - 1) \\ &\quad + (8 - 2i + 4i + 1) - 17 - 17i \end{aligned}$$

Keep going.

$$= -245 - 62i + 253 + 77i - 8 - 15i$$

Still going!

$$= 0$$

Write the result.

$P(-4 + i) = 0$, so by the factor theorem, $z + 4 - i$ is a factor of $P(z)$.

EXERCISE 6.04 Remainder and factor theorems

Concepts and techniques

- 1 **Example 10** Three functions are defined as follows:

$$p(z) = (-3 - 2i)z^3 + (5 + 2i)z^2 + (1 - 2i)z + 8 + 3i,$$

$$q(z) = (4 + 5i)z^2 + (3 - 2i)z + 4 - 6i \text{ and}$$

$$g(z) = (2 + 2i)z^2 + (-2 - 4i)z + 3 + 2i.$$

Find and simplify the following.

a $p(z) + q(z)$

b $p(z) - q(z)$

c $p(z) + g(z)$

d $g(z) - q(z)$

e $q(z) \times g(z)$

f $p(z) \times q(z)$

g $p(z) \times g(z)$

- 2 The function $p(z) = (4 + i)z^3 + 3z^2 + (-1 - 3i)z - 5 + 2i$. Find and simplify the following.

a $p(z) \div (z - 3)$

b $p(z) \div (z - 2i)$

c $p(z) \div (z + 1)$

d $p(z) \div (z - 2 + i)$

e $p(z) \div (z + 3 - 4i)$

- 3 The function $p(z) = -4iz^4 + (5 + 2i)z^3 - 3z + 7$. Find and simplify the following.

a $p(z) \div (z + 2i)$

b $p(z) \div (z - 3 - i)$

c $p(z) \div (z + 2 + 2i)$

d $p(z) \div (z + 4 - 2i)$

e $p(z) \div z - 5$

- 4 **CAS** Three functions are defined as follows:

$$p(z) = (5 - 3i)z^4 + (-3 + 3i)z^3 + (2 - 4i)z^2 + (-7 - i)z - 5 + 3i$$

$$q(z) = (4 + 2i)z^5 + (7 - 5i)z^3 + (3 + 7i)z^2 + 3 - 5i \text{ and}$$

$$g(z) = (9 + i)z^3 + (-3 + 2i)z + 3 - 5i$$

Find and simplify the following.

a $p(z) + q(z)$

b $p(z) - q(z)$

c $p(z) + g(z)$

d $g(z) - q(z)$

e $q(z) \times g(z)$

f $p(z) \times q(z)$

g $p(z) \times g(z)$



Remainder and factor theorems



Your teacher will tell you when to use your CAS calculator in the rest of this exercise.

- 5 **Example 11** The function $p(z) = (3 + 5i)z^3 + (-3 - 4i)z^2 + (1 - 5i)z - 8 - 7i$. Find the remainders when $p(z)$ is divided by each of the following.

a $z + 4$ b $z - 2i$ c $z + 3 - 6i$ d $z - 1 - i$ e $z + 3 + 2i$

- 6 The function $q(z) = (1 - 3i)z^4 + 2z^2 - 5iz + 5 + 4i$. Find the remainders when $q(z)$ is divided by each of the following.

a $z - 3i$ b $z + 1 - 6i$ c $z - 3 - i$ d $z + 2 + 6i$ e $z - 4 + 5i$

- 7 The function $g(z) = 4z^5 - 2iz^4 + (-2 + 7i)z^3 + 5iz^2 - 6z - 2 + 2i$. Find the remainders when $g(z)$ is divided by each of the following.

a $z + 2i - 5$ b $z - 3i + 1$ c $z - 5 - 3i$ d $z + 6 + 4i$ e $z - 1 - i$

- 8 **Example 12** Show that $z - 2 - 3i$ is a factor of $p(z) = 3iz^3 + (14 - 6i)z^2 + (-12 - 16i)z + 1 + 8i$.

- 9 Show that $z + 2 - 4i$ is a factor of $p(z) = (3 + i)z^4 + (8 - 13i)z^3 + (-16 + 6i)z^2 + (13 + 6i)z - 14 + 8i$

- 10 Show that $z + 1 + 5i$ is a factor of

$$p(z) = (3 + 2i)z^5 + (-12 + 17i)z^4 + (-5 - 23i)z^3 + (-12 - 3i)z^2 + (23 - 12i)z - 15 + 3i.$$

- 11 Show that $z + 7$ and $z - 2i$ are both factors of

$$p(z) = z^4 + (7 - 7i)z^3 + (-7 - 45i)z^2 + (-41 + 22i)z + 56 - 42i.$$

- 12 Show that $z + 2 - 3i$ and $z - 5 + 3i$ are both factors of

$$p(z) = (1 + 3i)z^4 + (-3 - 6i)z^3 + (-69 + 9i)z^2 + (-48 - 3i)z + 5 - 105i.$$

Reasoning and communication

- 13 Write $z^2 + 4$ in factored form to show that it is a factor of

$$g(z) = (2 + 3i)z^5 + 5iz^4 + (4 + 12i)z^3 + (2 + 21i)z^2 - 16z + 8 + 4i.$$

- 14 Use the quadratic formula to write $z^2 + z + 1$ in factored form to show it is a factor of

$$p(z) = (3 + 2i)z^3 + (8 - i)z^2 + (8 - i)z + 5 - 3i.$$

- 15 Write $z^2 + 2z + 4$ in factored form to show it is a factor of

$$q(z) = (-3 + i)z^4 + (-2 + 2i)z^3 + (-4 + 2i)z^2 + (16 - 4i)z - 8i.$$

6.05 SOLUTION OF POLYNOMIAL EQUATIONS WITH REAL COEFFICIENTS

You already know the **discriminant** of the **quadratic** equation $5x^2 - 2x + 1$, $\Delta = -16$, shows that the solutions are complex. Polynomial equations with real coefficients can have solutions that are real, imaginary or complex.

○ Example 13

Solve the equation $5z^3 - 12z^2 + 5z - 2 = 0$

Solution

Write the polynomial for the equation.

$$p(z) = 5z^3 - 12z^2 + 5z - 2$$

Find a factor using the remainder theorem.

$$p(1) = -4, p(-1) = -24, p(2) = 0$$

Use the factor theorem.

Since $p(2) = 0$, $z - 2$ is a factor.

You can factorise $p(z)$ by inspection.

$$p(z) = (z - 2)(5z^2 - 2z + 1)$$

Write the factored equation.

$$(z - 2)(5z^2 - 2z + 1) = 0$$

Use the null factor law.

$$z - 2 = 0 \text{ or } 5z^2 - 2z + 1 = 0$$

Solve the first equation.

$$z - 2 = 0 \Rightarrow z = 2$$

Solve the second equation.

$$5z^2 - 2z + 1 = 0$$

Write the quadratic formula for z .

$$\Rightarrow z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Substitute values.

$$= \frac{2 \pm \sqrt{4 - 4 \times 5 \times 1}}{2 \times 5}$$

Simplify.

$$= \frac{2 \pm \sqrt{-16}}{10}$$

Use $i = \sqrt{-1}$

$$= \frac{2 \pm 4i}{10} = \frac{1 \pm 2i}{5}$$

Write the answer.

The solutions are $2, \frac{1+2i}{5}$ or $\frac{1-2i}{5}$.

Notice that the complex roots in Example 13 are complex conjugates.

The Fundamental theorem of algebra states that any polynomial equation has at least one root (perhaps in the complex number system). It follows immediately that every polynomial of degree n has exactly n roots, some of which may be equal. The proof of the theorem is beyond the scope of this course, but it is of immense importance in mathematics.

You can divide through by the coefficient of the leading term of a polynomial equation of degree n to make any polynomial equation into a monic equation with a leading coefficient of 1. In that case, the equation can be written as $(z - \alpha_1)(z - \alpha_2) \dots (z - \alpha_n) = 0$, where $\alpha_1, \alpha_2, \dots, \alpha_n$ are the roots of the equation. For the constant term of the equation to be real, the roots $\alpha_1, \alpha_2, \dots, \alpha_n$ must be real or form conjugate pairs.

IMPORTANT

Every solution of a polynomial equation with real coefficients is either real or has another root that is its complex conjugate. It follows that every real polynomial can be factored to give linear or quadratic factors with real coefficients.

It follows from the above that polynomial equations of odd degree with real coefficients must have at least one real root. The fundamental theorem and its corollaries guarantee the existence of solutions, but it doesn't help in actually finding any solutions. This also applies to factorising real polynomials. However, if you can use the factor theorem to find one real root of a **cubic** power polynomial equation, two of a **quartic** or three of a **quintic** equation, you will be able to find the other roots using the quadratic formula. When you are looking for real roots, they must be factors of the constant term.

○ Example 14

Solve $3z^4 - 3z^3 - 14z^2 = 11z + 3$

Solution

Write the polynomial function.

$$\text{Let } p(z) = 3z^4 - 3z^3 - 14z^2 - 11z - 3$$

Try factors of the constant.

$$p(1) = -28, p(-1) = 0, p(3) = 0$$

Write the factors.

$$(z + 1) \text{ and } (z - 3) \text{ are factors of } p(z).$$

Find the product.

$$(z + 1)(z - 3) = z^2 - 2z - 3$$

Write the polynomial as a product.

$$p(z) = (z^2 - 2z - 3)(az^2 + bz + c)$$

Use the z^4 and constant terms.

$$1 \times a = 3 \text{ and } -3 \times c = -3$$

Solve for a and c .

$$a = 3 \text{ and } c = 1$$

Use $p(1)$.

$$-28 = (1 - 2 - 3) \times (3 + b + 1)$$

Solve for b .

$$b = 3$$

Write the equation in factored form.

$$(z + 1)(z - 3)(3z^2 + 3z + 1) = 0$$

Solve.

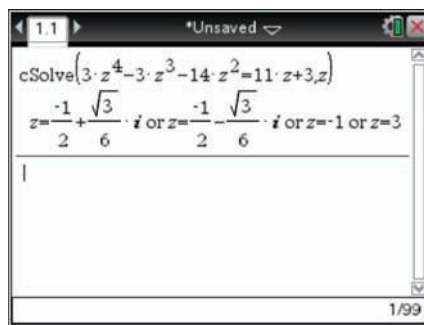
$$z = -1, z = 3, z = \frac{-3 \pm \sqrt{9 - 12}}{6}$$

Write the answers.

$$z = -1, z = 3, -\frac{1}{2} - \frac{\sqrt{3}}{6}i \text{ or } z = -\frac{1}{2} + \frac{\sqrt{3}}{6}i$$

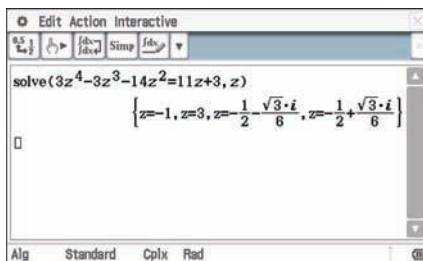
TI-Nspire CAS

Make sure that your calculator is set on complex (rectangular) mode. Use menu , 3: Algebra, C: Complex and 1: Solve or type $\text{csolve}()$ and put in the equation and variable for which to solve.



ClassPad

Make sure that your calculator is set to **Cplx** mode. Tap **Action**, **Equation/Inequality** and **solve** or type solve(and enter the equation and variable that you wish to solve for.



Notice in Example 14 that it was not necessary to test $p(-3)$ because $p(3) = 0$ so $z - 3$ was a factor, and so $z + 3$ could not be a factor as well and give the constant -3 . Also, the complex roots *are indeed* complex conjugates. You can factorise a cubic, quartic or quintic polynomial in a similar way to solving polynomial equations if you can find one, two or three real factors respectively.

Example 15

Express $z^5 - 4z^4 + 5z^3 - 6z + 4$ as a product of linear factors.

Solution

Write as a polynomial.

$$\text{Let } p(z) = z^5 - 4z^4 + 5z^3 - 6z + 4$$

Try factors of the constant term, 4.

$$p(1) = 0, p(-1) = 0, p(2) = 0, p(-2) = -120$$

Write as a product.

$$p(z) = (z - 1)(z + 1)(z - 2)(az^2 + bz + c)$$

Multiply the linear factors.

$$= (z^3 - 2z^2 - z + 2)(az^2 + bz + c)$$

Equate z^5 and z^0 terms.

$$z^5 = z^3 \times az^2 \text{ and } 4 = 2 \times c$$

Solve for a and c .

$$a = 1 \text{ and } c = 2$$

Use $p(-2)$.

$$-120 = (-8 - 8 + 2 + 2) \times [4 + b \times (-2) + 2]$$

Solve for b .

$$b = -2$$

Write the quadratic factor.

$$\text{Quadratic factor} = (z^2 - 2z + 2)$$

Use the quadratic formula.

$$= \left(z - \frac{2 + \sqrt{4 - 8}}{2} \right) \left(z - \frac{2 - \sqrt{4 - 8}}{2} \right)$$

Simplify.

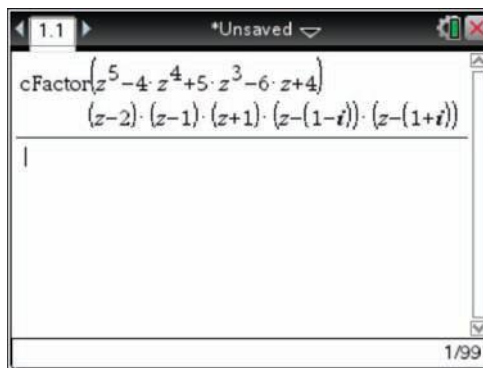
$$= [z - (1 + i)][z - (1 - i)]$$

Write the answer.

$$\begin{aligned} z^5 - 4z^4 + 5z^3 - 6z + 4 \\ = (z - 1)(z + 1)(z - 2)(z - 1 - i)(z - 1 + i) \end{aligned}$$

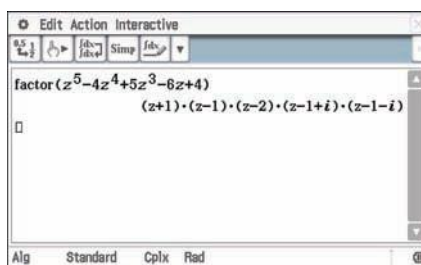
TI-Nspire CAS

In complex (rectangular) mode, use **menu**, 3: Algebra, C: Complex and 2: Factor or type `cFactor(` and put in the expression.



ClassPad

Make sure your calculator is set to **Cplx** mode. Tap **Action**, **Transformation**, **factor** and **factor** or type `factor(` and enter the expression. Press **EXE** to perform the factorisation.



In Example 15, since $p(2) = 0$, there was no need to test $p(4)$ or $p(-4)$.

EXERCISE 6.05 Solution of polynomial equations with real coefficients



Polynomials with real coefficients

Concepts and techniques

- Example 13** For $2z^3 - z^2 - 2z - 8 = (z - 2)Q(z)$, what is the quadratic factor?
 A $(z^2 + z + 4)$ B $(2z^2 + z + 4)$ C $(2z^2 - z - 4)$
 D $(2z^2 - 3z + 4)$ E $(2z^2 + 3z + 4)$
- Example 14** For $z^4 - 4z^3 - 5z^2 + 20z + 12 = (z + 2)(z - 3)Q(z)$, what is the quadratic factor?
 A $z^2 + 3z - 2$ B $z^2 - 3z - 2$ C $z^2 + 3z + 2$
 D $z^2 - 10z - 2$ E $z^2 + 5z - 2$
- Example 15** For $3z^5 - 13z^4 + 21z^3 - 19z^2 + 12z - 4 = (z - 1)(z - 1)(z - 2)Q(z)$, what is the quadratic factor?
 A $z^2 - 3z + 2$ B $3z^2 - z - 2$ C $3z^2 - z + 2$
 D $3z^2 - 3z + 2$ E $3z^2 + z + 2$

Reasoning and communication

Your teacher will tell you when to use your CAS calculator for the rest of this exercise.

- 4 Solve the equation $z^3 - 2z^2 + z - 2 = 0$.
- 5 Solve the equation $z^3 + z^2 - z + 15 = 0$.
- 6 Solve the equation $z^4 - 4z^3 + 4z^2 + 4z - 5 = 0$.
- 7 Solve the equation $z^4 - 7z^3 + 14z^2 + 2z - 20 = 0$.
- 8 Solve the equation $z^5 - 2z^4 + 40z^2 - 41z - 78 = 0$.
- 9 Solve the equation $z^5 - 8z^3 - 10z^2 - 9z + 90 = 0$.
- 10 Express $z^3 + 5z^2 + 8z + 6$ as a product of linear factors.
- 11 Express $z^4 - 7z^3 + 22z^2 - 32z + 16$ as a product of linear factors.
- 12 Express $z^5 - z^4 + 4z^3 + 14z^2 - 32z - 40$ as a product of linear factors.

6.06 SOLUTION OF COMPLEX POLYNOMIAL EQUATIONS

A complex polynomial equation can have real solutions. Just because some or all of the coefficients are complex does not mean that the equation cannot have real solutions as well as complex solutions. You should also be on the look out for possible common factors, perfect squares, difference of squares, grouping, trinomials and sums and differences when factorising polynomials and finding solutions of polynomial equations.

○ Example 16

Solve $z^3 + (2 - 6i)z^2 = (9 + 12i)z + 18$

Solution

Write as a polynomial.

Try factors of -18 .

There is a real factor!

Use long division.

$$\text{Let } p(z) = z^3 + (2 - 6i)z^2 + (-9 - 12i)z - 18 (=0)$$

$$p(1) = -24 - 18i \quad p(-1) = -6 + 6i$$

$$p(2) = -20 - 48i \quad p(-2) = 0$$

$z + 2$ is a factor.

$$\begin{array}{r} z^2 \qquad -6iz \quad -9 \\ z+2 \overline{) z^3 + (2-6i)z^2 + (-9-12i)z - 18 + 0i} \\ \underline{z^3 + (2-0i)z^2} \\ -6iz^2 + (-9-12i)z - 18 + 0i \\ \underline{-6iz^2 + (0-12i)z} \\ (-9+0i)z - 18 + 0i \\ \underline{(-9+0i)z - 18 + 0i} \\ 0 \end{array}$$

Write in factored form.

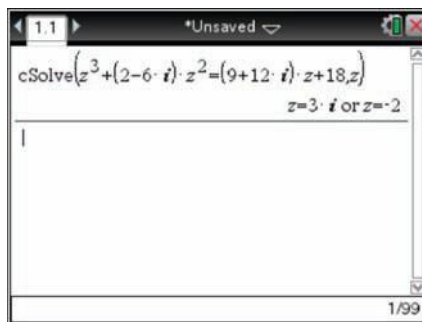
The quadratic is a perfect square.

Factorise.

TI-Nspire CAS

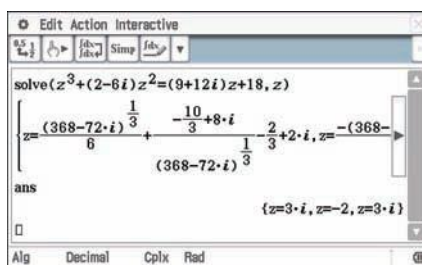
Make sure that your calculator is set in complex (rectangular) mode. Use $\boxed{\text{menu}}$, 3: Algebra, C: Complex and 1: Solve or type `csolve(` and put in the equation and variable.

$$\begin{aligned} p(z) &= (z+2)(z^2 - 6iz - 9) \\ &= (z+2)[z^2 - 2 \times 1 \times 3i + (3i)^2] \\ &= (z+2)(z-3i)^2 \end{aligned}$$



ClassPad

Make sure that your calculator is set to **Cplx** mode. Tap **Action**, **Equation/Inequality** and **solve** or type `solve(` and enter the equation and variable. If the result looks complicated, change mode from **Standard** to **Decimal** and output the answer again. In the screen shot opposite, the first expression for the solution was obtained from **Standard** mode, the second from **Decimal** mode.



The ClassPad uses the complex formula to solve cubics, which gives an answer in surd form. By changing to decimal, you can get approximate values that may be easier to compare.

If you can't find a simple real root, look for simple imaginary roots to find factors of complex polynomials.

Example 17

Solve $6iz^4 + 4iz^2 = 7z^3 + 7z + 2i$

Solution

Write as a polynomial.

Try real factors of $-2i$.

Try imaginary factors of $-2i$.

Write in factored form.

Multiply out the linear ones.

$$\text{Let } p(z) = 6iz^4 - 7z^3 + 4iz^2 - 7z - 2i$$

$$p(1) = -14 + 8i \quad p(-1) = 14 + 8i$$

$$p(2) = -70 + 110i \quad p(-2) = 70 + 110i$$

$p(i) = 0$, $p(-i) = 0$, which is enough to get a quadratic.

$$\begin{aligned} p(z) &= (z-i)(z+i)(az^2 + bz + c) \\ &= (z^2 + 1)(az^2 + bz + c) \end{aligned}$$

Use z^4 and z^0 terms.

Solve for a and c .

Use $p(1)$.

Simplify.

Solve for b .

Write $p(z)$ with all the values.

Factorise $6iz^2 - 7z - 2i$.

Write the trinomial in fraction form and cancel.

It doesn't matter which factor you multiply by $-i$.

Write $p(z)$ as fully factored.

Multiply by $-i$ to get $3z$.

Simplify.

Write the answers.

TI-Nspire CAS

Make sure your calculator is set in complex (rectangular) mode. Use **menu**, 3: Algebra, C: Complex and 1: Solve or type `csolve(` and put in the equation and variable.

$$6iz^4 = z^2 \times az^2 \text{ and } -2i = 1 \times c$$

$$a = 6i \text{ and } c = -2i$$

$$-14 + 8i = (1 + 1)(6i + b \times 1 - 2i)$$

$$2(b + 4i) = -14 + 8i$$

$$b = -7$$

$$p(z) = (z - i)(z + i)(6iz^2 - 7z - 2i)$$

$$6i \times (-2i) = 12 \text{ and } -3 \times (-4) = 12, -3 + (-4) = -7$$

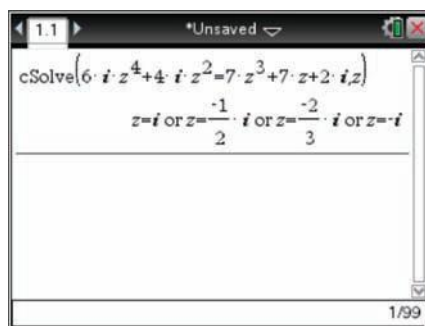
$$\begin{aligned} 6iz^2 - 7z - 2i &= \frac{(6iz - 3)(6iz - 4)}{6i} \\ &= \frac{(2iz - 1)(3iz - 2)}{i} \\ &= \frac{-i(2iz - 1)(3iz - 2)}{i \times (-i)} \\ &= (2z + i)(3iz - 2) \\ &= (2iz - 1)(-3z + 2i) \end{aligned}$$

$$p(z) = (z - i)(z + i)(2z + i)(3iz - 2) = 0$$

$$-i(z - i)(z + i)(2z + i)(3iz - 2) = 0$$

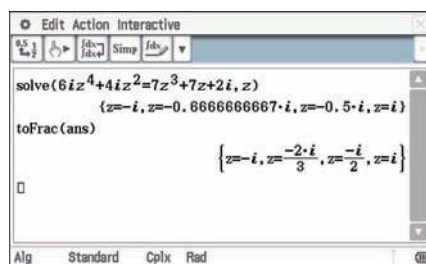
$$(z - i)(z + i)(2z + i)(3z + 2i) = 0$$

$$z = i, z = -i, z = -\frac{1}{2}i \text{ or } z = -\frac{2}{3}i$$



ClassPad

Make sure that your calculator is set to **Cplx** mode. Tap **Action**, **Equation/Inequality** and **solve** or type `solve(` and enter the equation and variable. Tap **Action**, **Transformation**, **Fraction** and **toFrac** followed by `ans` and **[EXE]** to convert the decimal expressions to fractions.



CAS calculators use approximate methods for higher power equations, so may give answers as approximations. Change to fractions for obvious cases to get exact results.

EXERCISE 6.06 Solution of complex polynomial equations



Real and imaginary factors

Concepts and techniques

Your teacher will tell you how to use your CAS calculator in this exercise.

- 1 **Example 16** The real factors of $iz^4 + 3iz^3 - 3z^2 + (-3 - 4i)z + 6$ are:
 A $(z - 1)$ and $(z + 2)$ B $(z + 1)$ and $(z - 2)$ C $(z + 1)$, $(z - 2)$ and $(z - 3)$
 D $(z + 1)$, $(z + 2)$ and $(z + 3)$ E $(z - 1)$, $(z - 2)$ and $(z - 3)$
- 2 **Example 17** The imaginary factors of $2z^4 + iz^3 - z^2 - iz - 1$ are:
 A $(z + i)$ B $(z - i)$ C $(z + i)$ and $(2z - i)$
 D $(z + i)$, $(z - i)$ and $(z + 2i)$ E $(z - i)$, $(z + i)$ and $(z - 2i)$

Reasoning and communication

- 3 Solve $z^3 - 2iz^2 + (-3 - 2i)z + 2 + 4i = 0$
- 4 Solve $z^3 + (-3 + 3i)z^2 + (-4 - 3i)z + 12 - 18i = 0$
- 5 Solve $z^3 + (-2 - 3i)z^2 + (1 + 4i)z - 6 - 3i = 0$
- 6 Solve $z^3 + (3 + 5i)z^2 + (-8 + 9i)z - 6 - 4i = 0$
- 7 Solve $z^4 + (-5 - 2i)z^3 + 4iz^2 + (20 + 10i)z - 16 - 12i = 0$
- 8 Solve $10z^4 + 13iz^3 - 37z^2 - 52iz - 12 = 0$
- 9 Solve $12z^4 - 41iz^3 + 4z^2 - 69iz - 36 = 0$
- 10 Solve $z^4 + (-1 - 4i)z^3 + (-1 + 4i)z^2 + (-11 - 42i)z - 72 + 30i = 0$

CHAPTER SUMMARY

COMPLEX ROOTS AND POLYNOMIALS



- The n th **roots of unity** are complex numbers z such that $z^n = 1$. They lie on the unit circle and have the form $\cos\left(\frac{2k\pi}{n}\right) + i \sin\left(\frac{2k\pi}{n}\right)$, where $k = 0, 1, 2, \dots, n-1$.
- The principal values of the roots of unity are given by $\cos\left(\frac{2k\pi}{n}\right) + i \sin\left(\frac{2k\pi}{n}\right)$, where $k = 0, 1, 2, \dots, n-1$, with principal values $\cos\left(\frac{2k\pi}{n}\right) + i \sin\left(\frac{2k\pi}{n}\right)$, where $\pi < \frac{2k\pi}{n} \leq \pi$.
- The polar form of a complex number is also known as the **trigonometric form**.
- The n th **roots of a complex number** z are on a circle of radius $\sqrt[n]{|z|}$, separated from each other by the angle $\frac{2\pi}{n}$ and have the form $\sqrt[n]{|z|} \left[\cos\left(\frac{\arg(z) + 2k\pi}{n}\right) + i \sin\left(\frac{\arg(z) + 2k\pi}{n}\right) \right]$, where $k = 0, 1, 2, \dots, n-1$.
- A polynomial is a sum of terms containing non-negative integer powers of the variable. A polynomial in z is of the form $P(z) = a_n z^n + a_{n-1} z^{n-1} + a_{n-2} z^{n-2} + \dots + a_1 z^1 + a_0$, where n is a positive whole number, $a_0, a_1, a_2, \dots, a_{n-1}, a_n$ are complex numbers and $a_n \neq 0$.
- The highest power of the variable in a complex polynomial is called the **degree** of the polynomial. The term that contains the highest power is called the **leading term**.
- Complex polynomials may be added, subtracted, multiplied and divided. Division can be set out as long division with allowance for real and complex parts in the coefficients.
- The **division identity** states that if a complex polynomial $P(z)$ is divided by the complex polynomial $D(z)$, then the **remainder** $R(z)$ has degree less than $D(z)$ and the **quotient** $Q(z)$; **divisor**, remainder and **dividend** are related by $P(z) = D(z)Q(z) + R(z)$.
- The **remainder theorem** states that if a complex polynomial $P(z)$ is divided by $D(z) = z - a$, where $a \in \mathbb{C}$, then the remainder is given by $R = P(a)$.
- The **factor theorem** states that $z - a$, where $a \in \mathbb{C}$, is a factor of the complex polynomial $P(z)$ if and only if $P(a) = 0$.
- Every solution of a polynomial equation with real coefficients is either real or has another root that is its conjugate complex. It follows that every real polynomial can be factored to give linear or quadratic factors with real coefficients.
- Cubic, quartic or quintic complex polynomial equations are solvable using quadratic methods for the remaining roots if one, two or three roots respectively can be found.

6

CHAPTER REVIEW

COMPLEX ROOTS AND POLYNOMIALS

Multiple choice

- Examples 1, 3** Which of the following are 6th roots of 1?
 I -1 II $\frac{1}{2}(\sqrt{2} + i\sqrt{2})$ III $\frac{1}{2}(1 - i\sqrt{3})$ IV $\frac{1}{2}(\sqrt{3} - i)$
 A I only B I and II only C I and III only
 D I and IV only E All except II
- Example 4** Which of the following are 4th roots of -1 ?
 I i II $\frac{1}{2}(\sqrt{2} + i\sqrt{2})$ III $\frac{1}{2}(-\sqrt{2} - i\sqrt{2})$ IV $-i$
 A I only B II and III only C I and III only
 D I and IV only E All of I to IV
- Example 7** The function $p(z) = -2iz^3 + (5 - 2i)z^2 - 3z^4 - 7i$. Which of the following is true?
 A $p(z)$ is of degree 4 and has leading term $-2iz^3$.
 B $p(z)$ is of degree 4 and has leading term $-7i$.
 C $p(z)$ is of degree 3 and has leading term $-2iz^3$.
 D $p(z)$ is of degree 4 and has leading term $-3z^4$.
 E $p(z)$ is of degree 2 and has leading term $(5 - 2i)z^2$.
- Examples 8, 9** The function $p(z) = z^4 - 2iz^3 + (2 + i)z^2 + 6 + i$. Which of the following is true?
 A $p(1) = 7 + i$ B $p(-1) = 9 - 4i$ C $p(-i) = 7$
 D $p(i) = 3i$ E $p(2i) = 2 + 3i$
- Example 10** Consider the functions $p(z) = (2 - i)z^3 + (5 + 3i)z^2 + (3 + 4i)z - 5 + 3i$ and $q(z) = (-2 + i)z^3 + 5iz^2 - 2z + 1$. Which of the following is true?
 A $p(z) + q(z)$ has degree 2 and $p(z)q(z)$ has degree 6
 B $p(z) - q(z)$ has degree 3 and $p(z)q(z)$ has degree 3
 C $p(z) + q(z)$ has degree 6 and $p(z)q(z)$ has degree 6
 D $p(z) + q(z)$ has degree 3 and $p(z) - q(z)$ has degree 3
 E $p(z) + q(z)$ has degree 2 and $p(z) - q(z)$ has degree 2
- Example 11** Consider the function $p(z) = z^4 - 2iz^3 + (2 + i)z^2 + 6 + i$. The remainder when $p(z)$ is divided by $z - 1 + i$ is:
 A $4 + 9i$ B $-4 - 9i$ C i
 D $8 - 7i$ E $-4 + i$
- Example 14** For $z^4 - 2z^3 - z^2 + 10z - 8 = (z - 1)(z + 2)Q(z)$, what is the quadratic factor?
 A $z^2 - 3z + 4$ B $z^2 + 3z - 4$ C $z^2 - 5z + 4$
 D $z^2 - 4z - 4$ E $z^2 + 4z + 5$
- Example 15** For $3z^5 - 7z^4 + z^3 + 3z^2 - 4z + 4 = (z + 1)(z - 1)(z - 2)Q(z)$, what is the quadratic factor?
 A $3z^2 - z - 2$ B $3z^2 - 2z + 3$ C $3z^2 - 2z + 2$
 D $3z^2 + 2z + 3$ E $3z^2 - z + 2$

- 9 **Example 16** The real factors of $6iz^4 + (-1 - 6i)z^3 + (1 - 10i)z^2 + (2 - 2i)z - 4i$ are:
 A $(z - 1)$ and $(z + 2)$ B $(z + 1)$ and $(z - 2)$ C $(z + 1)$, $(z - 2)$ and $(z - 3)$
 D $(z + 1)$, $(z + 2)$ and $(z + 3)$ E $(z - 1)$, $(z - 2)$ and $(z - 3)$
- 10 **Example 17** The imaginary factors of $6z^4 + (-1 + 6i)z^3 + (34 - i)z^2 + (-6 - 2i)z - 12$ are:
 A $(z + 2i)$ and $(z - 3i)$ B $(z - 2i)$ only C $(z + 2i)$ only
 D $(z + i)$, $(z - 2i)$ and $(z + 3i)$ E $(z - 2i)$ and $(z + 3i)$

Short answer

- 11 **Example 5** a Find the cube roots of -216 .
 b Show the roots on the complex plane.
- 12 **Example 6** a Solve the equation $z^4 + \frac{9\sqrt{2}}{2} = \frac{9\sqrt{2}}{2}i$.
 b Show the solutions on the complex plane.
- 13 **Example 8** Find the values of $g(z) = iz^3 - 2z^2 + 3z + i - 5$ at $z = 1$, $z = -2i$ and $z = 2 + i$.
- 14 **Example 9** **CAS** For $p(z) = (5 - 2i)z^4 - 3iz^3 + (2 + 3i)z^2 + 2z + 4i$, find $p(2)$, $p(-i)$ and $p(-3 + i)$.
- 15 **Example 10** For $p(z) = (1 - 2i)z^3 + 3iz^2 - 4z + 2i$, $q(z) = -3iz^2 + (4 + i)z + 3$ and $d(z) = z - 1 + i$, find and simplify the following.
 a $p(z) + q(z)$ b $p(z)q(z)$ c $p(z) \div d(z)$
- 16 **Example 11** For $p(z) = (-4 - 3i)z^3 + 5z^2 - 4iz - 8$, find the remainders when $p(z)$ is divided by
 a $z + 1$ b $z - 3i$ c $z - 2 - i$
- 17 **Example 12** Show that $z + 1 - 2i$ is a factor of $z^3 + (3 - 5i)z^2 + (-4 - 13i)z - 12 - 6i$.

Application

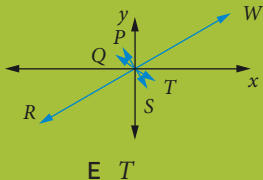
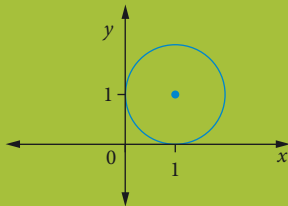
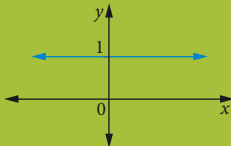
- 18 Show that no complex numbers have real roots.
- 19 Show that $z^2 + 2z + 3$ is a factor of $2z^4 + (-2 - 3i)z^3 + (-7 - 3i)z^2 + (-20 - 3i)z - 3 + 9i$.
- 20 Solve $z^3 = 2z^2 + 9$.
- 21 Solve $z^4 + (9 - 12i)z + 18 = (1 - 6i)z^3 + (11 + 6i)z^2$.
- 22 Solve $6z^4 + 30z^2 + 5iz^3 + 20iz + 24 = 0$.



2

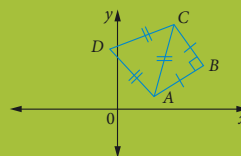
MIXED REVISION CHAPTERS 4 • 5 • 6

Multiple choice

- The position of a particle is given by $\mathbf{r}(t) = \sin(2t)\mathbf{i} - \cos(2t)\mathbf{j}$. The magnitude of the velocity of the particle is:
 A $\sqrt{\cos^2(2t) + \sin^2(2t)}$ B 2 C $\sin(2t) - \cos(2t)$
 D 4 E 1
 - The point W representing $w = \sqrt{3} + i$ is sketched on the right. Other points marked are P , Q , R , S and T . The point representing $\frac{1}{w}$ is closest to
 A P B Q C R D S E T
- 
- Which of the following are the 8th roots of 1?
 I -1 II $\frac{1}{2}(\sqrt{2} + i\sqrt{2})$ III $\frac{1}{2}(1 - i\sqrt{3})$ IV $-i$
 A I only B I and II only C I and III only
 D I and IV only E All except III
 - The position vector of a projectile at time t is $\mathbf{r}(t) = 12t\mathbf{i} + (26t - 5t^2)\mathbf{j}$ (metres). The maximum height of the projectile is closest to:
 A 12 m B 24 m C 25 m D 34 m E 48 m
 - The locus of z for $|z - 1 - i| = 1$ is sketched below. The maximum value of $|z|$ is
 A 1 B $2\sqrt{2} - 1$ C 2
 D $\sqrt{2} + 1$ E $2\sqrt{2}$
- 
- $p(z) = (1 - i)z^4 + 4z^3 + (3 + 2i)z^2 - 2iz + 10$. Which of the following is correct?
 A $p(1) = 18 + i$ B $p(-1) = 10 - 3i$ C $p(-i) = 7 + i$
 D $p(i) = 10 - 7i$ E $p(-2i) = 10 + 3i$
 - Which of the following does not describe the subset of the complex plane shown?
 A $\operatorname{Re}(z) = 1$ B $\operatorname{Im}(z) = 1$ C $|z| = |z - 2i|$
 D $z - \bar{z} = 2i$ E all of the above
 - $p(z) = z^4 - 3iz^2 + (2 - 3i)z + 8 - 2i$. What is the remainder when $p(z)$ is divided by $z - 2 - i$?
 A $12 - 27i$ B $-4 - 9i$ C $29i$ D -13 E $20 + 9i$
 - An object travels in a circular path of radius r at a constant speed v . What happens to the object's acceleration if the radius is halved?
 A It is halved. B It quadruples. C It stays unchanged.
 D It is reduced to a quarter of its previous value. E It doubles.
- 

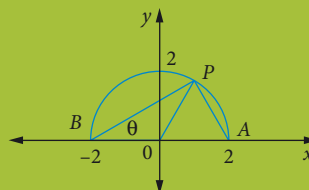
Short answer

- Sketch the region of the complex plane defined by $\frac{\pi}{4} < \arg(z - i) \leq \frac{\pi}{2}$.
- Find the cube roots of $-16 + 16i$.
 - Show the roots on the complex plane.
- Given the following information regarding a particle at time t , determine an expression for the acceleration and position of the particle in each case.
 - $\mathbf{v}(t) = 3t\mathbf{i} + 5\mathbf{j}$ and $\mathbf{r}(t) = 2\mathbf{i} - 3\mathbf{j}$ at $t = 0$
 - $\mathbf{v}(t) = 3 \cos(t)\mathbf{i} + \sin(2t)\mathbf{j}$ and $\mathbf{r}(t) = -5\mathbf{i}$ at $t = 0$
- Consider the isosceles and equilateral triangles shown in the diagram. The points A, B, C, D represent the complex numbers w_1, w_2, w_3, w_4 .
 - Find an expression for w_3 in terms of w_1 and w_2 .
 - Find an expression for w_4 in terms of w_1 and w_3 .
- A satellite in a circular orbit of radius 8000 km completes 15 revolutions per day. Calculate the velocity in km h^{-1} .
- Find and simplify $[2iz^3 + (-2 + 3i)z + 8 - i] \div (z + 2 - i)$.



Application

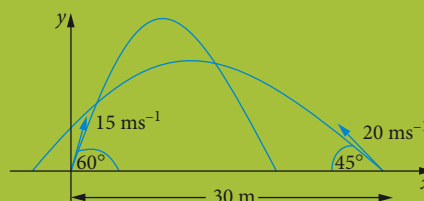
- Consider the subset of the complex plane such that $\text{Im}\left(\frac{1}{\bar{z} - i}\right) = 2$. Show that the locus of z is a circle.
- In the diagram, the point P representing the complex number w lies on the semicircle of radius 2. Prove that $2 \arg(w + 2) = \arg(w)$.



- Solve for z : $3z^4 + 5z^3 - 14z^2 - 10z - 12 = 0$.
- Solve for z : $20z^3 - 29iz^2 + 11z - 20i = 0$.
- The position of a particle at time t is given by $\mathbf{r}(t) = 4 \cos(2t)\mathbf{i} + 4 \sin(2t)\mathbf{j}$
 - Show that the velocity is always perpendicular to the displacement.
 - Describe the direction of the acceleration relative to the displacement.
 - Describe the path of the particle.

- A ball is thrown up with a velocity of 15 ms^{-1} at an angle of 60° . At the same time, another ball is thrown towards the first one, at an angle of 45° and a velocity of 20 ms^{-1} . The two balls are 30 m apart. (Use $g = 10 \text{ ms}^{-2}$).

- How far apart will they land?
- Do the paths of the balls cross over? Will they collide?





7

TERMINOLOGY

- arccosine
- arcsine
- arctangent
- double angle formulas
- integration
- integration by parts
- integration by substitution
- integration techniques
- linear factors
- logarithmic functions
- partial fractions
- Pythagorean identity
- recursive
- repeated factors
- repetition
- substitution
- trigonometric identities

INTEGRATION AND APPLICATIONS OF INTEGRATION

INTEGRATION TECHNIQUES

- 7.01 Integration using trigonometric identities
- 7.02 Integration by substitution
- 7.03 Integration of x^{-1}
- 7.04 Inverse trigonometric functions
- 7.05 Integration using partial fractions
- 7.06 Integration by parts

Chapter summary

Chapter review



Prior learning

INTEGRATION TECHNIQUES

- integrate using the trigonometric identities $\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$, $\cos^2(x) = \frac{1}{2}(1 + \cos(2x))$ and $1 + \tan^2(x) = \sec^2(x)$ (ACMSM116)
- use substitution $u = g(x)$ to integrate expressions of the form $f(g(x))g'(x)$ (ACMSM117)
- establish and use the formula $\int \frac{1}{x} dx = \ln|x| + c$, for $x \neq 0$ (ACMSM118)
- find and use the inverse trigonometric functions arcsine, arccosine and arctangent (ACMSM119)
- find and use the derivative of the inverse trigonometric functions, arcsine, arccosine and arctangent (ACMSM120)
- integrate expressions of the form $\frac{\pm 1}{\sqrt{a^2 - x^2}}$ and $\frac{a}{a^2 + x^2}$ (ACMSM121)
- use partial fractions where necessary for integration in simple cases (ACMSM122)
- integrate by parts. (ACMSM123) 

7.01 INTEGRATION USING TRIGONOMETRIC IDENTITIES

In this section, you will use your knowledge of the double angle formulas and Pythagorean trigonometric identities to integrate trigonometric expressions. You should remember the following double angle formulas from your work last year.

IMPORTANT

Double angle formulas

$$\sin(2x) = 2 \sin(x) \cos(x)$$

$$\cos(2x) = \cos^2(x) - \sin^2(x) = 2 \cos^2(x) - 1 = 1 - 2 \sin^2(x)$$

$$\tan(2x) = \frac{2 \tan(x)}{1 - \tan^2(x)}$$

You can rearrange the formula for $\cos(2x)$ to eliminate $\sin^2(x)$ in a trigonometric expression as follows.

$$\cos(2x) = 1 - 2 \sin^2(x)$$

$$\text{so } 2 \sin^2(x) = 1 - \cos(2x)$$

$$\text{so } \sin^2(x) = \frac{1}{2} [1 - \cos(2x)]$$

Similarly

$$\cos(2x) = 2 \cos^2(x) - 1$$

$$\text{so } \cos^2(x) = \frac{1}{2} [1 + \cos(2x)]$$

○ Example 1

Find $\int \cos^2(x) dx$.

Solution

Rearrange $\cos(2x) = 2 \cos^2(x) - 1$.

$$2 \cos^2(x) = 1 + \cos(2x)$$

$$\text{So } \cos^2(x) = \frac{1}{2}[1 + \cos(2x)]$$

Write the problem and replace $\cos^2(x)$.

$$\int \cos^2(x) dx = \int \frac{1}{2}[1 + \cos(2x)] dx$$

Simplify by removing the common factor.

$$= \frac{1}{2} \int [1 + \cos(2x)] dx$$

Separate the trig part.

$$= \frac{1}{2} \left[\int 1 dx + \int \cos(2x) dx \right]$$

Use the formula $\int \cos(kx) dx = \frac{1}{k} \sin(kx) + c$

$$= \frac{1}{2} \left[x + \frac{1}{2} \sin(2x) \right] + c$$

Write the answer.

$$\int \cos^2(x) dx = \frac{1}{2} x + \frac{1}{4} \sin(2x) + c$$

In Example 1, each separate step has been shown. You would normally omit some of these steps.

○ Example 2

Find $\int_0^{\frac{\pi}{12}} 3 \sin^2(6x) dx$.

Solution

Rearrange $\cos(12x) = 1 - 2 \sin^2(6x)$.

$$2 \sin^2(6x) = 1 - \cos(12x)$$

$$\text{So } \sin^2(6x) = \frac{1}{2}[1 - \cos(12x)]$$

Write the problem and replace $\sin^2(6x)$.

$$\int_0^{\frac{\pi}{12}} 3 \sin^2(6x) dx = \int_0^{\frac{\pi}{12}} \frac{3}{2} [1 - \cos(12x)] dx$$

Simplify by removing the common factor.

$$= \frac{3}{2} \int_0^{\frac{\pi}{12}} [1 - \cos(12x)] dx$$

Separate the trig part.

$$= \frac{3}{2} \left[\int_0^{\frac{\pi}{12}} 1 dx - \int_0^{\frac{\pi}{12}} \cos(12x) dx \right]$$

Use the formula $\int \cos(kx) dx = \frac{1}{k} \sin(kx) + c$.

$$= \frac{3}{2} \left[x - \frac{1}{12} \sin(12x) \right]_0^{\frac{\pi}{12}}$$

Evaluate.

$$\begin{aligned} &= \frac{3}{2} \left[\frac{\pi}{12} - \frac{1}{12} \sin(\pi) \right] \\ &= \frac{3}{2} \left(\frac{\pi}{12} - 0 \right) \\ &= \frac{\pi}{8} \end{aligned}$$

Write the answer.

$$\int_0^{\frac{\pi}{12}} 3 \sin^2(6x) dx = \frac{\pi}{8}$$

You derived the trigonometric Pythagorean identities last year.

You can reverse the derivative of $\tan(x)$ to obtain the integral of $\sec^2(x)$. Remember that $\frac{d}{dx}\tan(x) = \sec^2(x)$.

Similarly, $\frac{d}{dx}\cot(x) = -\operatorname{cosec}^2(x)$.

IMPORTANT

The Pythagorean identities

$$\sin^2(x) + \cos^2(x) = 1$$

$$\tan^2(x) + 1 = \sec^2(x)$$

$$1 + \cot^2(x) = \operatorname{cosec}^2(x)$$

IMPORTANT

$$\int \sec^2(x) dx = \tan(x) + c \text{ and so } \int \sec^2(kx) dx = \frac{1}{k} \tan(kx) + c$$

$$\int \operatorname{cosec}^2(x) dx = -\cot(x) + c \text{ and so } \int \operatorname{cosec}^2(kx) dx = -\frac{1}{k} \cot(kx) + c$$

To make things simpler when integrating, you can rewrite or substitute any trigonometric identities that you already know.

○ Example 3

- a Find $\int \tan^2(x) dx$.
b Find $\int \sin(x) \cos(x) dx$.

Solution

- a Rearrange $1 + \tan^2(x) = \sec^2(x)$.

$$1 + \tan^2(x) = \sec^2(x)$$

$$\text{So } \tan^2(x) = \sec^2(x) - 1$$

Write the problem and replace $\tan^2(x)$.

$$\int \tan^2(x) dx = \int [\sec^2(x) - 1] dx$$

Separate the trig part.

$$= \int \sec^2(x) dx - \int 1 dx$$

Use the formula $\int \sec^2(kx) dx = \frac{1}{k} \tan(kx) + c$.

$$= \tan(x) - x + c$$

Write the answer.

$$\int \tan^2(x) dx = \tan(x) - x + c$$

- b Rearrange $\sin(2x) = 2 \sin(x) \cos(x)$

$$\sin(2x) = 2 \sin(x) \cos(x)$$

$$\text{So } \sin(x) \cos(x) = \frac{1}{2} \sin(2x)$$

Write the problem and replace $\sin(x) \cos(x)$.

$$\int \sin(x) \cos(x) dx = \int \frac{1}{2} \sin(2x) dx$$

Simplify by removing the common factor.

$$= \frac{1}{2} \int \sin(2x) dx$$

Use the formula $\int \sin(kx) dx = -\frac{1}{k} \cos(kx) + c$

$$= \frac{1}{2} \left[-\frac{1}{2} \cos(2x) \right] + c$$

$$= -\frac{1}{4} \cos(2x) + c$$

Write the answer.

$$\int \sin(x) \cos(x) dx = -\frac{1}{4} \cos(2x) + c$$

EXERCISE 7.01 Integration using trigonometric identities

Concepts and techniques



Trigonometric identities

1 **Example 1** $\int \sin^2(x) dx =$

A $\frac{1}{2}x + \frac{1}{4}\sin(2x) + c$

B $\frac{1}{2}x - \frac{1}{4}\sin(4x) + c$

C $x + \frac{1}{4}\sin(4x) + c$

D $\frac{1}{2}x - \frac{1}{4}\sin(2x) + c$

E $x + \frac{1}{4}\sin(2x) + c$

2 $\int_0^{\frac{\pi}{2}} \sin^2(2x) dx$

A $\frac{\pi}{8}$

B $-\frac{\pi}{8}$

C $\frac{x}{2} - \frac{1}{8}\sin(4x) + c$

D $\frac{\pi}{8} + c$

E $\frac{\pi}{4}$

3 **Example 2** $\int \cos^2(4x) dx =$

A $\frac{1}{2}x - \frac{1}{4}\sin(2x) + c$

B $\frac{1}{2}x - \frac{1}{16}\sin(4x) + c$

C $x + \frac{1}{32}\sin(16x) + c$

D $x - \frac{1}{8}\sin(4x) + c$

E $\frac{1}{2}x + \frac{1}{16}\sin(8x) + c$

4 $\int_0^{\frac{\pi}{2}} 2\cos^2(8x) dx =$

A $-\frac{\pi}{4}$

B $\frac{\pi}{2} + c$

C $\frac{\pi}{2}$

D $x - \frac{1}{32}\sin(16x) + c$

E $x + \frac{1}{16}\sin(16x) + c$

5 **Example 3** $\int \tan^2(2x) dx =$

A $\frac{1}{2}\tan(2x) - x + c$

B $\tan(2x) - x + c$

C $\frac{1}{2}\tan(2x) + \frac{x}{2} + c$

D $\frac{1}{2}\tan(2x) - \frac{x}{2} + c$

E $\frac{1}{2}\sec(2x) + \frac{x}{2} + c$

6 $\int \sec^2(4x) dx =$

A $\frac{1}{2}\tan(2x) - x + c$

B $\frac{1}{2}\tan(2x) - x + c$

C $\frac{1}{4}\tan(4x) + c$

D $\frac{1}{4}\tan(2x) + \frac{1}{2}\sec(2x) + c$

E $\frac{1}{4}\sec(4x) + c$

7 $\int_0^{\frac{\pi}{8}} \sin(4x) \cos(4x) dx =$

A $\frac{1}{8}$

B $\frac{\pi}{8}$

C 0

D $\frac{1}{8}\sin^2(4x) + c$

E $\frac{1}{4}\sin^2(4x) + c$

Reasoning and communication

- 8 Show that $\int 2 \sin(x) \cos(x) dx = -\frac{1}{2} \cos(2x) + c$
- 9 Show that $\int \sin(3x) \cos(3x) dx = -\frac{1}{12} \cos(6x) + c$
- 10 Show that $\sin^2(x) = \frac{1}{2} [1 - \cos(2x)]$
- 11 Show that $\sin^2(3x) = \frac{1}{2} [1 - \cos(6x)]$
- 12 Show that $\int \cot^2(3x) dx = -\frac{1}{3} \cot(3x) - x + c$

7.02 INTEGRATION BY SUBSTITUTION

You already know how to use the chain rule to differentiate the function $p(x) = (2x^3 - 5x + 2)^4$ by writing $u(x) = 2x^3 - 5x + 2$ and $q(u) = u^4$.

$$\text{You get } p'(x) = q'(u) \times u'(x) = 4u^3 \times (6x^2 - 5) = 4(2x^3 - 5x + 2)^3(6x^2 - 5)$$

$$\text{You can integrate to get } \int p'(x) dx = \int 4u^3 \times (6x^2 - 5) dx = \int 4u^3 \times u'(x) dx = (2x^3 - 5x + 2)^4.$$

$$\text{But } (2x^3 - 5x + 2)^4 = u^4 = \int 4u^3 du, \text{ so } \int p'(x) dx = \int 4u^3 \times (6x^2 - 5) dx = \int 4u^3 du.$$

$$\text{By writing } p'(x) = f(x) \text{ and } 4u^3 = g(u), \text{ you get } \int f(x) dx = \int g(u) \times u'(x) dx = \int g(u) du.$$

This is the integration rule that corresponds to the chain rule.

IMPORTANT

Integration by substitution

If a function $f(x)$ can be written as $f(x) = g(u) \cdot u'(x)$, then

$$\int f(x) dx = \int g(u) u'(x) dx = \int g(u) du$$

Alternatively, if $f(x) = g[u(x)]$, then

$$\int f(x) dx = \int g(u) u'(x) du$$

To use the substitution method, you need to find a function $u(x)$ such that $u'(x)$ is part of the original and the remainder of the expression is a function of u . There is no set method for finding $u(x)$. Even so, it often helps to make $u(x)$ the 'inside' of the most complex function in the integral.

Example 4

Find $\int_0^2 (4x^2 - 3x + 1)^4 (8x - 3) dx$.

Solution

Choose a suitable substitution.

$$\text{Let } u(x) = 4x^2 - 3x + 1$$

Find $u'(x)$.

$$u'(x) = 8x - 3$$

Write the integral and substitute.

$$\int (4x^2 - 3x + 1)^4 (8x - 3) dx = \int u^4 \times u'(x) dx$$

Use the substitution rule.

$$= \int u^4 du$$

Do the integration.

$$= \frac{1}{5} u^5 + c$$

Substitute $u(x)$ back in.

$$= \frac{1}{5} (4x^2 - 3x + 1)^5 + c$$

Now do the definite integral, using your result.

$$\int_0^2 (4x^2 - 3x + 1)^4 (8x - 3) dx = \left[\frac{1}{5} (4x^2 - 3x + 1)^5 \right]_0^2$$

Calculate the answer.

$$\begin{aligned} &= \frac{1}{5} \times 11^5 - \frac{1}{5} \times 1^5 \\ &= 32\,210 \end{aligned}$$

Notice in Example 4 that you could get the same answer using $\int_1^{11} u^4 du$. This would give

$\left[\frac{1}{5} u^5 \right]_1^{11} = \frac{1}{5} \times 11^5 - \frac{1}{5} \times 1^5 = 32\,210$. For a definite integral, the substitution rule can be written as follows.

IMPORTANT

Integration by substitution

If a function $f(x)$ can be written as $f(x) = g(u) \cdot u'(x)$, then

$$\int_a^b f(x) dx = \int_a^b g(u) u'(x) dx = \int_{\alpha}^{\beta} g(u) du, \text{ where } \alpha = u(a) \text{ and } \beta = u(b).$$

Alternatively, if $f(x) = g[u(x)]$, then

$$\int_a^b f(x) dx = \int_{\alpha}^{\beta} g(u) u'(x) du, \text{ where } \alpha = u(a) \text{ and } \beta = u(b).$$

You can manipulate constants to make integration easier.

○ Example 5

Find $\int (3x-7)^5 dx$

Solution

Choose a suitable substitution.

$$\text{Let } u(x) = 3x - 7$$

Find the derivative.

$$u'(x) = 3$$

Write the integral and manipulate to get $u'(x)$.

$$\int (3x-7)^5 dx = \frac{1}{3} \int (3x-7)^5 \times 3 dx$$

Substitute in $u(x)$ and $u'(x)$.

$$= \frac{1}{3} \int u^5 \times u'(x) dx$$

Use the substitution rule.

$$= \frac{1}{3} \int u^5 du$$

Do the integration.

$$= \frac{1}{3} \times \frac{1}{6} u^6 + c$$

Simplify and substitute $u(x)$ back in.

$$= \frac{1}{18} (3x-7)^6 + c$$

You can do this wherever you can write the function to be integrated in the form $f(x) = g(u) \cdot u'(x)$.

○ Example 6

Evaluate $\int_0^{\frac{\pi}{4}} \sin(x) \cos^4(x) dx$.

Solution

Choose a suitable substitution.

$$\text{Let } u(x) = \cos(x)$$

Find the derivative.

$$u'(x) = -\sin(x)$$

Write the integral and manipulate to get $u'(x)$.

$$\int_0^{\frac{\pi}{4}} \sin(x) \cos^4(x) dx = -\int_0^{\frac{\pi}{4}} \cos^4(x) \times [-\sin(x)] dx$$

Substitute in $u(x)$ and $u'(x)$.

$$= -\int_0^{\frac{\pi}{4}} u^4 \times u'(x) dx$$

Use the substitution rule.

$$= -\int_{u(0)}^{u(\frac{\pi}{4})} u^4 du$$

Do the integration, including the change of bounds.

$$= -\left[\frac{u^5}{5} \right]_1^{\frac{1}{\sqrt{2}}}$$

Do the calculation.

$$= -\frac{1}{5} \left[\left(\frac{1}{\sqrt{2}} \right)^5 - 1^5 \right]$$

Simplify.

$$= \frac{1}{5} \left(1 - \frac{1}{4\sqrt{2}} \right)$$

Express with a rational denominator.

$$= \frac{8 - \sqrt{2}}{40}$$

You could use your CAS calculator to do the integration from the original expression. However, if the problem says to ‘use substitution or otherwise’, then you are expected to demonstrate integration techniques like those in the examples.

EXERCISE 7.02 Integration by substitution

Concepts and techniques



Integration by substitution

1 **Example 4** $\int_1^3 (x^2 - x - 1)^3 (2x - 1) dx =$

- A $\frac{x^2 - x - 1}{4}$ B $\frac{(x^2 - x - 1)^4}{4}$ C 156 D 625.2 E 312

2 $\int_0^1 6x^2 (2x^3 - 3)^4 dx =$

- A $\frac{242}{5}$ B $\frac{121}{15}$ C $\frac{363}{5}$ D $\frac{3}{10} (2x^2 - 3)^5$ E $\frac{1}{5} (2x^3 - 3)^5$

3 **Example 5** $\int (2x + 5)^7 dx =$

- A 335 886 B $\frac{1}{16} (2x + 5)^8$ C $\frac{1}{7} (2x + 5)^7$ D $\frac{1}{16} (2x + 5)^8 + c$ E $\frac{1}{8} (2x + 5)^8 + c$

4 An antiderivative for $\int \left(\frac{x}{3} - 2 \right)^4 dx$ is

- A $\frac{1}{15} (x - 6)^5$ B $\frac{1}{405} (x - 6)^5$ C $\frac{1}{5} (x - 6)$ D $\frac{2101}{405}$ E $3 \left(\frac{x}{3} - 2 \right)^4$

5 **Example 6** The integral $\int_0^{\frac{\pi}{3}} -\sin(x) \cos^3(x) dx$ can be expressed as

- A $\int_1^{\frac{1}{2}} -\sin(x) \cos^3(x) dx$ B $\int_0^{\frac{\pi}{3}} \cos^3(x) dx$ C $\int_0^{\frac{\sqrt{3}}{2}} u^3 du$
D $\int_1^{\frac{1}{2}} u^3 du$ E $\int_1^{\frac{1}{2}} \frac{u^4}{4} dx$

6 The integral $\int 3 \tan^2(2x) dx$ can be expressed as

- A $3 \int \sec^2(2x) - 1 dx$ B $\int 3 \sec^2(2x) dx$ C $\int 3 \tan^2(2x) dx$
D $\int 3 \tan^2(u) du$ E $\int 3 \tan^2(u) - 1 du$

7 Find $\int_1^3 (x^2 - x - 1)^3 (4x - 2) dx$

8 Find $\int_0^{\frac{\pi}{2}} \sin^2(x) \cos(x) dx$

Reasoning and communication

9 Show that $\int_0^{\frac{\pi}{3}} \sec^2(x) + 1 dx = \frac{\pi + 3\sqrt{3}}{3}$

10 Show that an antiderivative of the expression $\sin^2(3x)$ is $\frac{x}{2} - \frac{1}{12} \sin(6x)$.

7.03 INTEGRATION OF x^{-1}

IMPORTANT

The **natural logarithm** is given by $\log_e(x) = \ln(x)$ (for $x > 0$).

The natural logarithm is the inverse of the exponential function.

The derivative of $\log_e(x)$ is given by $\frac{d}{dx} \log_e(x) = \frac{1}{x}$ for $x > 0$.

For $y = \log_e(x)$, the definition of logarithms gives $e^y = x$.

Now from the chain rule: $\frac{d}{dx}(e^y) = \frac{de^y}{dy} \times \frac{dy}{dx} = e^y \times \frac{dy}{dx}$.

But $\frac{dx}{dy} = 1$, so taking the derivative of both sides of $x = e^y$ gives $e^y \times \frac{dy}{dx} = 1$.

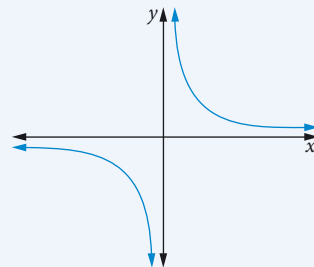
But $e^y = x$ so $x \times \frac{dy}{dx} = 1$ and thus $\frac{dy}{dx} = \frac{1}{x}$

QED

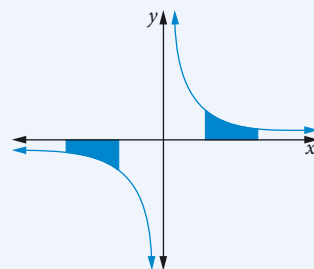
This means that $\int \frac{1}{x} dx = \log_e(x)$ for $x > 0$. What about $x < 0$?

INVESTIGATION Definite integrals of $\frac{1}{x}$

What do you notice about the shapes of the graph for $x > 0$ and for $x < 0$?



If you draw an area between the x -axis and the curve for $x > 0$ and for $x < 0$ on domains equally distant from the centre, you get the following.



What do you notice about the areas?

What can you say about $\int_{-3}^{-1} \frac{1}{x} dx$, $\int_1^3 \frac{1}{x} dx$ and $\int_3^1 \frac{1}{x} dx$?

From the investigation, it should be clear that for values of $x < 0$, you can find the integral of $\frac{1}{x}$ by using the integral for $x > 0$.

For example, $\int_{-3}^{-1} \frac{1}{x} dx = \int_3^1 \frac{1}{x} dx = [\log_e(x)]_3^1 = \log_e(1) - \log_e(3) = -1.098\dots$

The sign is negative because the area is below the axis. It is also clear that you cannot perform an integration for a region containing 0 because $\frac{1}{x}$ is not defined for $x = 0$.

IMPORTANT

The integral of $\frac{1}{x} = x^{-1}$ is given by $\int x^{-1} dx = \log_e |x| + c$, provided that $x \neq 0$.

○ Example 7

Use the method of substitution to find an expression for $\int \frac{1}{2x+3} dx$.

Solution

Choose a suitable substitution.

$$\text{Let } u(x) = 2x + 3$$

Find the derivative.

$$u'(x) = 2$$

Write the integral and manipulate to get $u'(x)$.

$$\int \frac{1}{2x+3} dx = \frac{1}{2} \int \frac{1}{2x+3} \times 2 dx$$

Substitute in $u(x)$ and $u'(x)$.

$$= \frac{1}{2} \int \frac{1}{u} \times u'(x) dx$$

Use the substitution rule.

$$= \frac{1}{2} \int \frac{1}{u} du$$

Do the integration.

$$= \frac{1}{2} \log_e |u| + c$$

Substitute $u(x)$ back in.

$$= \frac{1}{2} \log_e |2x + 3| + c$$

Write the answer.

$$\int \frac{1}{2x+3} dx = \frac{1}{2} \log_e |2x + 3| + c$$

The method of Example 7 can be used to show the following results.

IMPORTANT

$\int \frac{1}{ax+b} dx = \frac{1}{a} \log_e |ax+b| + c$ where a, b are constants and c is the constant of integration.

$$\int \frac{f'(x)}{f(x)} dx = \log_e |f(x)| + c$$

○ Example 8

Find $\int \frac{6x-2}{3x^2-2x+1} dx$.

Solution

Write in the form $\int \frac{f'(x)}{f(x)} dx$.

$$\int \frac{6x-2}{3x^2-2x+1} dx = \int \frac{f'(x)}{f(x)} dx \text{ where } f(x) = 3x^2 - 2x + 1$$

Use the rule for $\int \frac{f'(x)}{f(x)} dx$.

$$\int \frac{6x-2}{3x^2-2x+1} dx = \log_e |3x^2 - 2x + 1| + c$$

You can also manipulate expressions involving reciprocals by introducing constants to obtain the derivative of the denominator.

○ Example 9

Evaluate $\int_0^{\frac{\pi}{4}} \frac{2 \sin(x)}{\cos(x)} dx$

Solution

Choose a suitable substitution.

$$\text{Let } u(x) = \cos(x)$$

Find the derivative.

$$u'(x) = -\sin(x)$$

Write the integral and manipulate to get $u'(x)$.

$$\int_0^{\frac{\pi}{4}} \frac{2 \sin(x)}{\cos(x)} dx = -2 \int_0^{\frac{\pi}{4}} \left[\frac{-\sin(x)}{\cos(x)} \right] dx$$

Use the rule for $\int \frac{f'(x)}{f(x)} dx$.

$$= \left[-2 \log_e |\cos(x)| \right]_0^{\frac{\pi}{4}}$$

Calculate the value.

$$= -2 \log_e \left| \cos\left(\frac{\pi}{4}\right) \right| + 2 \log_e |\cos(0)|$$

Manipulate the result.

$$= \log_e \left(\frac{1}{\sqrt{2}} \right)^{-2} + 2 \log_e(1) = \log_e(2)$$

Write the answer.

$$\int_0^{\frac{\pi}{4}} \frac{2 \sin(x)}{\cos(x)} dx = \log_e(2)$$

EXERCISE 7.03 Integration of x^{-1}



Integration of $\frac{1}{x}$

Concepts and techniques

1 **Example 7** Use the method of substitution to find an expression for the following integrals.

a $\int \frac{1}{3x+2} dx$

b $\int \frac{1}{9x-7} dx$

c $\int \frac{1}{4-3x} dx$

d $\int \frac{-5}{5-2x} dx$

2 $\int_2^3 \frac{4}{3x-2} dx$ can be expressed as

- A $\int_2^3 \frac{4}{u} du$ B $\frac{4}{3} \int_2^3 \frac{1}{u} du$ C $\frac{4}{3} \int_4^7 \frac{1}{u} du$ D $4 \log_e \left(\frac{14}{3} \right)$ E $\int_4^7 \frac{4}{3x-2} dx$

3 $\int_3^4 \frac{4}{2-3x} dx$ can be expressed as

- A $-\int_2^3 \frac{4}{u} du$ B $-\frac{4}{3} \int_{-7}^{-10} \frac{1}{u} du$ C $\frac{4}{3} \int_4^7 \frac{1}{u} du$ D $4 \log_e \left(\frac{3}{14} \right)$ E $\int_{-7}^{-10} \frac{4}{3x-2} dx$

4 **Example 8** $\int \frac{2x}{x^2+3} dx =$

- A $\log_e |2x| + c$ B $\log_e |x^2+3| + c$ C $\log_e |x| + c$ D $\log_e |x^2| + c$ E $2 \log_e |x^2+3| + c$

5 $\int_1^2 \frac{x+1}{x^2+2x} dx =$

- A $\log_e |x^2+2x|$ B $\frac{1}{2} \log_e |x^2+2x| + c$ C $\frac{1}{2} \log_e \left(\frac{1}{2} \right)$ D $\frac{1}{2} \log_e \left(\frac{3}{8} \right)$ E $\frac{1}{2} \log_e \left(\frac{8}{3} \right)$

6 **Example 9** $\int_0^{\frac{\pi}{3}} \frac{\sin(x)}{\cos(x)} dx =$

- A $\log_e (2)$ B $-\log_e |\cos(x)| + c$ C $\frac{1}{2} \log_e \left(\frac{1}{2} \right)$
D $\frac{1}{2} \log_e |\sin(x)|$ E $\frac{1}{2} \log_e (2)$

7 Find expressions for the following integrals.

- a $\int \frac{2x}{3x^2+2} dx$ b $\int \frac{2 \sec^2(x)}{\tan(x)} dx$ c $\int \frac{e^{2x}}{e^{2x}+1} dx$ d $\int \frac{6x^2+4}{x^3+2x} dx$

Reasoning and communication

8 Show that $\int_0^{\frac{\pi}{4}} \tan(x) dx = \frac{1}{2} \log_e (2)$

9 Show that $\int_0^1 \frac{x}{2x^2+2} dx = \frac{1}{4} \log_e (2)$

10 Show that $\int_1^3 \frac{2e^x}{e^x-1} dx = 2 \log_e \left(\frac{e^3-1}{e-1} \right)$

7.04 INVERSE TRIGONOMETRIC FUNCTIONS

Your calculator has the functions $\sin^{-1}(x)$, $\cos^{-1}(x)$ and $\tan^{-1}(x)$ as the inverses of $\sin(x)$, $\cos(x)$ and $\tan(x)$. However, since $\sin(x)$, $\cos(x)$ and $\tan(x)$ repeat, there are actually many solutions to trigonometric equations like $\sin(x) = 0.5$. You found the general solutions of these equations last year. To make the inverses of trigonometric functions into functions, the domains of $\sin(x)$, $\cos(x)$ and $\tan(x)$ are restricted to the **principal values**.

IMPORTANT

The functions **arcsine**, **arccosine** and **arctangent** (usually abbreviated to **arcsin**, **arccos** and **arctan**) are defined as:

$$\arcsin(x) = y \text{ if and only if } \sin(y) = x \text{ and } -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$$

$$\arccos(x) = y \text{ if and only if } \cos(y) = x \text{ and } 0 \leq y \leq \pi$$

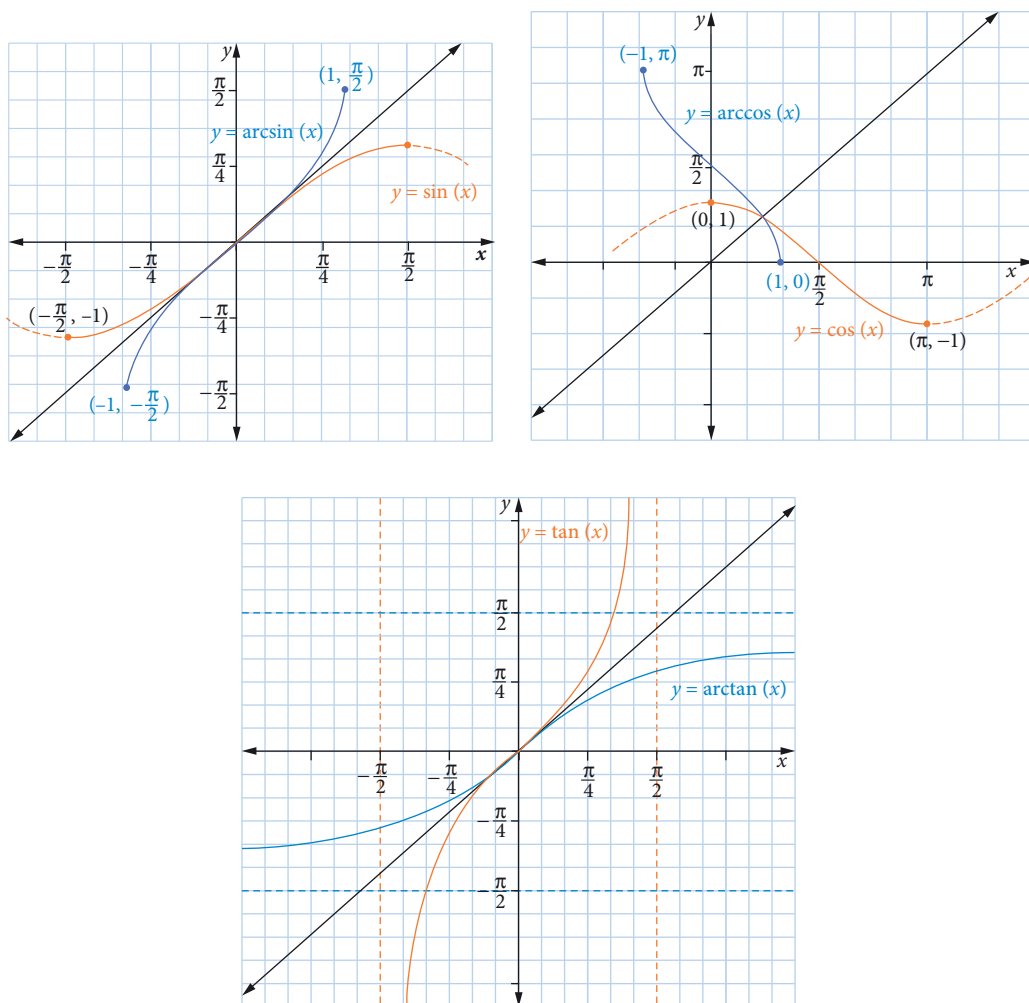
$$\arctan(x) = y \text{ where } \tan(y) = x \text{ and } -\frac{\pi}{2} < y < \frac{\pi}{2}$$

The domains of $\arcsin(x)$ and $\arccos(x)$ are both $[-1, 1]$.

The domain of $\arctan(x)$ is $(-\infty, \infty)$.

$\arcsin(x)$, $\arccos(x)$ and $\arctan(x)$ are often written as $\sin^{-1}(x)$, $\cos^{-1}(x)$ and $\tan^{-1}(x)$ respectively.

The graphs of $y = \arcsin(x)$, $y = \arccos(x)$ and $y = \arctan(x)$ are shown below. Each graph is shown with the dashed corresponding trigonometric graph. The graphs are shown as the reflections of $\sin(x)$, $\cos(x)$ and $\tan(x)$ in the line $y = x$.



The derivatives of $\arcsin(x)$, $\arccos(x)$ and $\arctan(x)$ are of considerable importance.

The rules for the derivatives can all be proven using implicit differentiation and the appropriate Pythagorean trigonometric identities.

Given $\arcsin(x) = y$, $x = \sin(y)$.

Differentiating both sides, you get $\frac{d}{dx} x = \frac{d}{dx} [\sin(y)]$.

Using the chain rule gives $1 = \cos(y) \times \frac{dy}{dx}$, so $\frac{dy}{dx} = \frac{1}{\cos(y)}$.

Using the trigonometric identity $\sin^2(x) + \cos^2(x) = 1$

gives $\frac{dy}{dx} = \frac{1}{\sqrt{1 - \sin^2(y)}}$.

Substitution of $x = \sin(y)$ then gives $\frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}}$

The other derivatives can be proven similarly.

IMPORTANT

$$\frac{d}{dx} [\arcsin(x)] = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx} [\arccos(x)] = \frac{-1}{\sqrt{1-x^2}} \text{ and}$$

$$\frac{d}{dx} [\arctan(x)] = \frac{1}{1+x^2}$$

QED

Example 10

a Find $\frac{d}{dx} \arccos(3x)$.

b Find $\frac{d}{dx} \tan^{-1}(x^2 + 1)$.

Solution

a Write the problem and use the chain rule.

Let $u = 3x$

Differentiate, using the rule for arccos.

Simplify.

b Write the problem and use the chain rule.

Let $u = x^2 + 1$

Differentiate, using the rule for arctan.

Substitute for u .

Simplify.

$$\frac{d}{dx} \arccos(3x) = \frac{d}{du} \arccos(u) \times \frac{du}{dx}$$

$$= \frac{-1}{\sqrt{1-u^2}} \times 3$$

$$= \frac{-3}{\sqrt{1-9x^2}}$$

$$\frac{d}{dx} \tan^{-1}(x^2 + 1) = \frac{d}{du} \tan^{-1}(u) \times \frac{du}{dx}$$

$$= \frac{1}{1+u^2} \times 2x$$

$$= \frac{2x}{1+(x^2+1)^2}$$

$$= \frac{2x}{x^4 + 2x^2 + 2}$$

The derivatives of the inverse trigonometric functions allow you to integrate some expressions that would otherwise be difficult to do using other means.

IMPORTANT

$$\int \frac{1}{\sqrt{1-x^2}} dx = \arcsin(x) + c$$

$$\int \frac{-1}{\sqrt{1-x^2}} dx = \arccos(x) + c$$

$$\int \frac{1}{1+x^2} dx = \arctan(x) + c$$

○ Example 11

Find $\int \frac{1}{\sqrt{1-4x^2}} dx$.

Solution

Choose a substitution.

$$\text{Let } u(x) = 2x$$

Find the derivative.

$$u'(x) = 2$$

Write the integral and rearrange to include $u'(x)$.

$$\int \frac{1}{\sqrt{1-4x^2}} dx = \frac{1}{2} \int \frac{1}{\sqrt{1-(2x)^2}} \times 2 dx$$

Substitute in $u(x)$ and $u'(x)$.

$$= \frac{1}{2} \int \frac{1}{\sqrt{1-u^2}} \times u'(x) dx$$

Use the substitution rule.

$$= \frac{1}{2} \int \frac{1}{\sqrt{1-u^2}} du$$

Do the integration.

$$= \frac{1}{2} \arcsin(u) + c$$

Substitute $u(x)$ back in.

$$= \frac{1}{2} \arcsin(2x) + c$$

Write the answer.

$$\int \frac{1}{\sqrt{1-4x^2}} dx = \frac{1}{2} \arcsin(2x) + c$$

You may also be required to integrate expressions of the type $\frac{1}{\sqrt{a^2-x^2}}$, $\frac{-1}{\sqrt{a^2-x^2}}$ and $\frac{1}{a^2+x^2}$.

Rearrange $\int \frac{1}{a^2+x^2} dx$ into a more suitable form as

$$\int \frac{1}{a^2 \left(1 + \left(\frac{x}{a} \right)^2 \right)} dx = \frac{1}{a^2} \int \frac{1}{1 + \left(\frac{x}{a} \right)^2} dx$$

Choose $u(x) = \frac{x}{a}$. Then $u'(x) = \frac{1}{a}$. Rearrange the integral to include $u'(x)$.

IMPORTANT

$$\int \frac{1}{\sqrt{a^2-x^2}} dx = \arcsin\left(\frac{x}{a}\right) + c$$

$$\int \frac{-1}{\sqrt{a^2-x^2}} dx = \arccos\left(\frac{x}{a}\right) + c$$

$$\int \frac{1}{a^2+x^2} dx = \frac{1}{a} \arctan\left(\frac{x}{a}\right) + c$$

Then $\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \int \frac{1}{1 + \left(\frac{x}{a}\right)^2} \times \frac{1}{a} dx.$

Substitute to get $\frac{1}{a} \int \frac{1}{1+u^2} \times u'(x) dx = \frac{1}{a} \int \frac{1}{1+u^2} du = \frac{1}{a} \arctan(u) + c.$

Substitute $u(x)$ back in so that $\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \arctan\left(\frac{x}{a}\right) + c.$

QED

You can prove the other integrations in a similar way.

○ Example 12

Find $\int_1^3 \frac{-1}{\sqrt{25-x^2}} dx$, correct to 3 decimal places.

Solution

Write the integral and arrange to use the rule.

$$\int_1^3 \frac{-1}{\sqrt{25-x^2}} dx = \int_1^3 \frac{-1}{\sqrt{5^2-x^2}} dx$$

Use the rule $\int \frac{-1}{\sqrt{a^2-x^2}} dx = \arccos\left(\frac{x}{a}\right) + c.$

$$= \left[\cos^{-1}\left(\frac{x}{5}\right) \right]_1^3$$

Substitute values.

$$= \cos^{-1}(0.2 \times 3) - \cos^{-1}(0.2 \times 1)$$

Simplify.

$$= \cos^{-1}(0.6) - \cos^{-1}(0.2)$$

Make sure that your calculator is set to radians.

$$= -0.442\ 143\dots$$

Round to write the answer.

$$\int_1^3 \frac{-1}{\sqrt{25-x^2}} dx \approx -0.442$$

Notice in Example 12 that $\cos^{-1}$ was preferred to $\arccos$ because it was easier to use with a calculator to obtain an approximate answer. If an exact answer had been required, it would be more usual to leave it as $\arccos(0.6) - \arccos(0.2)$. You should also check that putting the negative sign out the front and using $\int \frac{1}{\sqrt{a^2-x^2}} dx$ gives the same answer.

EXERCISE 7.04 Inverse trigonometric functions

Concepts and techniques



Inverse
trigonometric
functions

1 **Example 10** Find the following.

a $\frac{d}{dx} \sin^{-1}(3x)$

b $\frac{d}{dx} \tan^{-1}(x^3 - 2)$

c $\frac{d}{dx} \cos^{-1}(x^2)$

d $\frac{d}{dx} \tan^{-1}(2x^4 + x)$

2 $\frac{d}{dx} \cos^{-1}(4x) =$

A $\frac{1}{\sqrt{4-x^2}}$

B $\frac{1}{\sqrt{2-x^2}}$

C $\frac{1}{\sqrt{\frac{1}{2}-x^2}}$

D $\frac{-2}{\sqrt{1-4x^2}}$

E $\frac{-4}{\sqrt{1-16x^2}}$

3 $\frac{d}{dx} \tan^{-1}(4x) =$

A $\frac{1}{\frac{1}{4} + x^2}$

B $\frac{2}{4 + x^2}$

C $\frac{1}{2 + x^2}$

D $\frac{4}{1 + 16x^2}$

E $\frac{2}{1 + 4x^2}$

4 **Example 11** $\int \frac{1}{\sqrt{1-16x^2}} dx =$

A $\frac{1}{4} \sin^{-1}(4x) + c$

B $\frac{1}{4} \sin^{-1}(16x) + c$

C $\frac{1}{2} \sin^{-1}(4x) + c$

D $2 \sin^{-1}(4x) + c$

E $\frac{1}{4} \cos^{-1}(4x) + c$

5 $\int_0^1 \frac{1}{\sqrt{1-x^2}} dx =$

A $\frac{\pi}{2}$

B $-\frac{\pi}{2}$

C $\sin\left(\frac{\pi}{2}\right)$

D $\sin^{-1}\left(\frac{\pi}{2}\right)$

E $\sin^{-1}(x)$

6 $\int_0^1 \frac{1}{1+x^2} dx =$

A $\frac{\pi}{2}$

B $-\frac{\pi}{2}$

C $\tan\left(\frac{\pi}{2}\right)$

D $-\tan^{-1}(1)$

E $\frac{\pi}{4}$

7 Find the following.

a $\int \frac{1}{\sqrt{1-4x^2}} dx$

b $\int \frac{-4}{1+9x^2} dx$

c $\int_0^{\frac{1}{3}} \frac{3}{1+9x^2} dx$

d $\int_0^{\frac{1}{3}} \frac{2}{\sqrt{1-9x^2}} dx$

8 **Example 12** Find each of the following.

a $\int \frac{1}{\sqrt{36-x^2}} dx$

b $\int \frac{-1}{\sqrt{100-x^2}} dx$

c $\int \frac{1}{x^2+49} dx$

d $\int \frac{1}{\sqrt{12-x^2}} dx$

e $\int \frac{1}{x^2+6} dx$

9 Find the value of $\int_{-3}^6 \frac{1}{4+x^2} dx$.

10 Find $\int_{-1}^3 \frac{1}{\sqrt{16-x^2}} dx$ correct to 4 decimal places.

Reasoning and communication

11 Express $\frac{6}{x^2+2}$ in the form of $\frac{u'(x)}{1+u^2}$ and hence find $\int \frac{6}{x^2+2} dx$.

12 Prove that $\frac{d}{dx} [\arctan(x)] = \frac{1}{1+x^2}$.

13 Prove that $\int \frac{-1}{\sqrt{a^2-x^2}} dx = \arccos\left(\frac{x}{a}\right) + c$.

14 Prove that $\int \frac{1}{a^2+x^2} dx = \frac{1}{a} \arctan\left(\frac{x}{a}\right) + c$.

7.05 INTEGRATION USING PARTIAL FRACTIONS

Partial fractions refer to algebraic fractions that have been expressed as a sum of algebraic fractions with simpler denominators.

IMPORTANT

Fractions of the form $\frac{a}{(x-c)(x-d)}$ or $\frac{ax+b}{(x-c)(x-d)}$ can be expressed as **partial fractions** in the form $\frac{A}{(x-c)} + \frac{B}{(x-d)}$.

○ Example 13

Express $\frac{2}{x^2 - 2x - 3}$ as partial fractions.

Solution

Express $\frac{2}{x^2 - 2x - 3}$ in the form $\frac{a}{(x-b)(x-c)}$.

Write as partial fractions.

Put the RHS over a common denominator.

Equate the numerators.

Let $x = -1$ and $x = 3$ to find values of A and B .

Write the answer.

$$\begin{aligned}\frac{2}{x^2 - 2x - 3} &= \frac{2}{(x-3)(x+1)} \\ \frac{2}{(x-3)(x+1)} &= \frac{A}{(x-3)} + \frac{B}{(x+1)} \\ &= \frac{A(x+1) + B(x-3)}{(x-3)(x+1)} \\ 2 &= A(x+1) + B(x-3) \\ A = \frac{1}{2} \text{ and } B &= -\frac{1}{2} \\ \frac{2}{x^2 - 2x - 3} &= \frac{1}{2(x-3)} - \frac{1}{2(x+1)}\end{aligned}$$

You can use the method of partial fractions to find integrals where the denominator can be expressed as linear factors.

Example 14

Find $\int \frac{2x-3}{x^2-x-6} dx$.

Solution

Express $\frac{2x-3}{x^2-x-6}$ as partial fractions.

$$\begin{aligned}\frac{2x-3}{x^2-x-6} &= \frac{2x-3}{(x-3)(x+2)} \\ &= \frac{A}{x-3} + \frac{B}{x+2}\end{aligned}$$

Put the RHS over a common denominator.

$$\frac{2x-3}{x^2-x-6} = \frac{A(x+2)+B(x-3)}{(x-3)(x+2)}$$

Simplify and equate denominators.

$$2x-3 = Ax+2A+Bx-3B$$

Equate coefficients.

$$A+B=2 \text{ and } 2A-3B=-3$$

Solve for A and B .

$$A = \frac{3}{5} = 0.6 \text{ and } B = \frac{7}{5} = 1.4$$

Write the integral and express it using partial fractions.

$$\int \frac{2x-3}{x^2-x-6} dx = \int \frac{0.6}{x-3} dx + \int \frac{1.4}{x+2} dx$$

Integrate using logs.

$$\begin{aligned}&= 0.6 \log_e |x-3| \\ &\quad + 1.4 \log_e |x+2| + c\end{aligned}$$

Write the answer.

$$\begin{aligned}&\int \frac{2x-3}{x^2-x-6} dx \\ &= 0.6 \log_e |x-3| + 1.4 \log_e |x+2| + c\end{aligned}$$

You could use the properties of logarithms to write the answer in Example 14 as

$$\log_e |(x-3)^{0.6}(x+2)^{1.4}| + c \text{ or } \log_e \left| \sqrt[5]{(x-3)^3(x+2)^7} \right| + c.$$

Partial fractions of more complicated forms can be used in integration. If the denominator of an algebraic fraction can be expressed with only linear factors, with some linear factors repeated, then it can still be expressed as a sum of partial fractions.

IMPORTANT

Fractions of the form $\frac{a}{(x-c)^2(x-d)}$ or $\frac{ax+b}{(x-c)^2(x-d)}$ can be expressed as **partial fractions** in the form $\frac{A}{(x-c)^2} + \frac{B}{(x-c)} + \frac{C}{(x-d)}$.

○ Example 15

Find $\int \frac{x}{(x-2)^2(x+3)} dx$.

Solution

Express as partial fractions.

$$\frac{x}{(x-2)^2(x+3)} = \frac{A}{(x-2)^2} + \frac{B}{(x-2)} + \frac{C}{(x+3)}$$

Put the RHS over a common denominator.

$$= \frac{A(x+3) + B(x-2)(x+3) + C(x-2)^2}{(x-2)^2(x+3)}$$

Equate the numerators.

$$x = A(x+3) + B(x-2)(x+3) + C(x-2)^2$$

Use $x = 2$, -3 , and 0 to find A , B and C .

$$A = \frac{2}{5}, B = \frac{3}{25} \text{ and } C = -\frac{3}{25}$$

Write the integral and express it using partial fractions.

$$\int \frac{x}{(x-2)^2(x+3)} dx = \int \frac{\frac{2}{5}}{(x-2)^2} + \frac{\frac{3}{25}}{(x-2)} - \frac{\frac{3}{25}}{(x+3)} dx$$

Take out any common factors.

$$= \frac{2}{5} \int \frac{1}{(x-2)^2} dx + \frac{3}{25} \int \frac{1}{(x-2)} dx - \frac{3}{25} \int \frac{1}{(x+3)} dx$$

Integrate each term.

$$= \frac{2}{5} \frac{-1}{(x-2)} + \frac{3}{25} \log_e |x-2| - \frac{3}{25} \log_e |x+3|$$

Simplify terms.

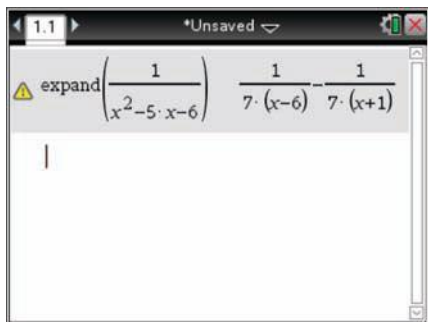
$$= \frac{3}{25} \log_e \left| \frac{x-2}{x+3} \right| - \frac{2}{5(x-2)}$$

Write the answer.

$$\int \frac{x}{(x-2)^2(x+3)} dx = -\frac{2}{5} \left(\frac{1}{x-2} \right) + \frac{3}{25} \log_e \left| \frac{x-2}{x+3} \right|$$

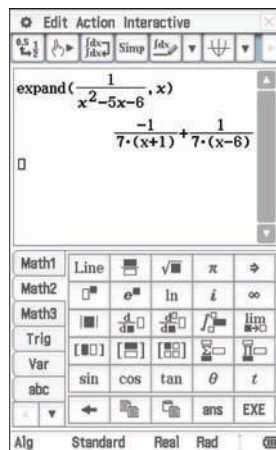
You can use a CAS calculator to change an algebraic fraction into partial fractions.

TI-Nspire CAS



ClassPad

Tap **Action**, **Transformation** and **expand** and enter the expression and the variable of interest.



EXERCISE 7.05 Integration using partial fractions



Partial fractions

Concepts and techniques

- 1 **Example 13** Express the following in partial fraction form.

a $\frac{1}{x^2 - 3x + 2}$

b $\frac{3x}{x^2 - 4x - 12}$

c $\frac{x}{2x^2 - x - 6}$

d $\frac{x^2}{(x-2)(2x^2 - x - 6)}$

- 2 **Examples 14, 15** Find the following.

a $\int \frac{1}{x^2 - 3x + 2} dx$

b $\int_1^2 \frac{3x}{x^2 - 4x - 12} dx$

c $\int \frac{1}{2x^2 - x - 6} dx$

d $\int_0^1 \frac{x}{(x-2)(2x^2 - x - 6)} dx$

3 $\int_1^2 \frac{1}{(x+1)(x-5)} dx =$

A $\frac{1}{6} \ln(2)$

B $-\frac{1}{6} \log_e(2)$

C $\frac{1}{6} \log_e\left(\frac{3}{4}\right)$

D $6 \log_e\left(\frac{1}{2}\right)$

E $-6 \log_e\left(\frac{1}{2}\right)$

4 $\int \frac{x}{(x-1)(x+4)} dx =$

A $\frac{1}{5} \log_e |(x-1)(x+4)^4| + c$

B $\frac{4}{5} \log_e |(x-1)(x+4)^4| + c$

C $\log_e |(x-1)(x+4)| + c$

D $\frac{4}{5} \log_e \left| \frac{x+4}{x-1} \right| + c$

E $\frac{1}{5} \log_e \left| \frac{(x+4)^4}{x-1} \right| + c$

5 $\frac{4x}{2x^2 - x - 3}$ can be written in partial fraction form as

A $\frac{A+4B}{2x^2 - x - 3}$

B $\frac{A}{2x^2 - x - 3} + \frac{Bx}{2x^2 - x - 3}$

C $\frac{A}{x+1} + \frac{B}{2x-3}$

D $\frac{A}{x-1} + \frac{B}{2x+3}$

E $\frac{A}{x+1} + \frac{4Bx}{2x-3}$

6 $\frac{-2}{x^3 - 2x^2 - 7x - 4}$ can be written in partial fraction form as

A $\frac{A+B+C}{x^3 - 2x^2 - 7x - 4}$

B $\frac{A}{(x+1)^2} + \frac{B}{x+1} + \frac{C}{x-4}$

C $\frac{A+B}{(x+1)^2} + \frac{C}{x-4}$

D $\frac{A+B}{(x+1)^2} \times \frac{C}{x-4}$

E $\frac{AB}{(x+1)^2} + \frac{C}{x-4}$

7 Find $\int 1 + \frac{3x-2}{x^2 - 3x + 2} dx$

8 Express $\frac{x^2}{x^2 - 1}$ as $1 + \frac{R(x)}{x^2 - 1}$ and hence find $\int \frac{x^2}{x^2 - 1} dx$.

9 Find $\int_0^{\frac{1}{2}} \frac{x^2}{x^2 - 1} dx$.

10 Find $\int_0^{\frac{1}{3}} \frac{2}{x^2 - 1} dx$.

7.06 INTEGRATION BY PARTS

Previously you used the *product rule* to calculate the derivative of products.

For $y = u(x) \cdot v(x)$ this is $\frac{dy}{dx} = v \frac{du}{dx} + u \frac{dv}{dx}$. The corresponding rule for integration is called **integration by parts**.

You use **integration by parts** to integrate expressions of the form $f(x) \cdot g'(x)$. This method allows you to integrate some products of functions where other simpler methods are not possible. In this case, it is generally not possible to remove products completely from an integral. The product is altered using **integration by parts** to a form that makes it easier to integrate.

Rearranging the product rule $\frac{dy}{dx} = v \frac{du}{dx} + u \frac{dv}{dx}$, you get $u \frac{dv}{dx} = \frac{dy}{dx} - v \frac{du}{dx}$.

Take the integral of both sides, giving

$$\int u \frac{dv}{dx} dx = \int \frac{dy}{dx} dx - \int v \frac{du}{dx} dx \text{ where } y = u(x) \cdot v(x).$$



Alamy/ACQuest Images

This gives

$$\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx.$$

Integration by parts can also be written as

$$\int f(x) \cdot g'(x) dx = f(x) \cdot g(x) - \int f'(x) \cdot g(x) dx.$$

IMPORTANT

Integration by parts

If a function can be written as $f(x) \cdot g'(x)$, then

$$\int f(x) \cdot g'(x) dx = f(x) \cdot g(x) - \int f'(x) \cdot g(x) dx$$

In practice, you select $f(x)$ when $f'(x)$ is easier to find and $g'(x)$ when $g(x)$ is no harder to find. It may take several tries to find the choice that makes the integral easier. It is not necessary to include the constant of integration because it will be part of the new integral at the end.

○ Example 16

Find $\int x \cos(x) dx$

Solution

Write $x \cos(x)$ in the form $f(x) \cdot g'(x)$.

Set $f(x) = x$ and $g'(x) = \cos(x)$

Substitute $f(x) \cdot g'(x)$ into the integral.

$$\int x \cos(x) dx = \int f(x) g'(x) dx$$

Find $f'(x)$ and $g(x)$.

$$f(x) = x \text{ so } f'(x) = 1$$

and

$$g'(x) = \cos(x) \text{ so } g(x) = \sin(x)$$

Write the problem and set it using the formula

$$\int x \cos(x) dx = x \sin(x) - \int 1 \sin(x) dx$$

$$\int f(x) \cdot g'(x) dx = f(x) \cdot g(x) - \int f'(x) \cdot g(x) dx$$

Integrate where necessary.

$$= x \sin(x) + \cos(x) + c$$

Write the answer.

$$\int x \cos(x) dx = x \sin(x) + \cos(x) + c$$

In Example 16, it would have been much harder if $f(x)$ and $g(x)$ were chosen the other way around. Integrals involving products with x , x^2 , x^3 , etc. can often be found in a similar way to Example 16. It may be necessary to apply integration by parts several times to simplify the integral. In some cases, you can use integration by parts to obtain an expression where a power is reduced. Repetition of the procedure eventually leads to an integral that can be evaluated. This is an example of a **recursive** procedure.

○ Example 17

Find $\int x^3 e^x dx$.

Solution

Step 1

Write $x^3 e^x$ in the form $f(x) \cdot g'(x)$.

Substitute $f(x) \cdot g'(x)$ into the integral.

Find $f'(x)$ and $g(x)$.

Write the problem and set it using the formula

$$\int f(x) \cdot g'(x) dx = f(x) \cdot g(x) - \int f'(x) \cdot g(x) dx$$

Repeat the procedure for the integral $\int 3x^2 e^x dx$.

Step 2

Write $3x^2 e^x$ in the form $f(x) \cdot g'(x)$.

Substitute $f(x) \cdot g'(x)$ into the integral.

Find $f'(x)$ and $g(x)$.

Write the problem and set it using the formula

$$\int f(x) \cdot g'(x) dx = f(x) \cdot g(x) - \int f'(x) \cdot g(x) dx$$

Write the answer to the original problem.

Repeat the procedure for the integral $\int 6xe^x dx$.

Step 3

Write $6xe^x$ in the form $f(x) \cdot g'(x)$.

Substitute $f(x) \cdot g'(x)$ into the integral.

Find $f'(x)$ and $g(x)$.

Write the problem and set it using the formula

$$\int f(x) \cdot g'(x) dx = f(x) \cdot g(x) - \int f'(x) \cdot g(x) dx$$

Integrate where possible.

Write the answer to the original problem.

Write the answer.

Set $f(x) = x^3$ and $g'(x) = e^x$

$$\int x^3 e^x dx = \int f(x) g'(x) dx$$

$f(x) = x^3$ so $f'(x) = 3x^2$

and

$g'(x) = e^x$ so $g(x) = e^x$

$$\int x^3 e^x dx = x^3 e^x - \int 3x^2 e^x dx$$

Set $f(x) = 3x^2$ and $g'(x) = e^x$

$$\int 3x^2 e^x dx = \int f(x) g'(x) dx$$

$f(x) = 3x^2$ so $f'(x) = 6x$

and

$g'(x) = e^x$ so $g(x) = e^x$

$$\int 3x^2 e^x dx = 3x^2 e^x - \int 6xe^x dx$$

$$\begin{aligned} \int x^3 e^x dx &= x^3 e^x - \left(3x^2 e^x - \int 6xe^x dx \right) \\ &= x^3 e^x - 3x^2 e^x + \int 6xe^x dx \end{aligned}$$

Set $f(x) = 6x$ and $g'(x) = e^x$

$$\int 6xe^x dx = \int f(x) g'(x) dx$$

$f(x) = 6x$ so $f'(x) = 6$

and

$g'(x) = e^x$ so $g(x) = e^x$

$$\int 6xe^x dx = 6xe^x - \int 6e^x dx$$

$$= 6xe^x - 6e^x + c$$

$$\int x^3 e^x dx = x^3 e^x - \left(3x^2 e^x - \int 6xe^x dx \right)$$

$$= x^3 e^x - 3x^2 e^x + 6xe^x - 6e^x + c$$

Integration by parts is very useful in finding integrals involving functions with e^x , $\log_e(x)$ and arcsine, arccosine and arctangent.

○ Example 18

Find $\int_{\frac{1}{2}}^2 \log_e(x) dx$.

Solution

Write $\log_e(x)$ in the form $f(x) \cdot g'(x)$.

Substitute $f(x) \cdot g'(x)$ into the integral.

Find $f'(x)$ and $g(x)$.

Write the problem and set it using the formula

$$\int f(x) \cdot g'(x) dx = f(x) \cdot g(x) - \int f'(x) \cdot g(x) dx$$

Separate and simplify terms.

Integrate where necessary.

Evaluate terms.

Simplify terms.

Collect terms.

Write the answer.

Set $f(x) = \log_e(x)$ and $g'(x) = 1$

$$\int_{\frac{1}{2}}^2 \log_e(x) dx = \int f(x) g'(x) dx$$

$f(x) = \log_e(x)$ so $f'(x) = \frac{1}{x}$
and

$g'(x) = 1$ so $g(x) = x$

$$\int_{\frac{1}{2}}^2 \log_e(x) dx = \left[x \log_e(x) - \int x \cdot \frac{1}{x} dx \right]_{\frac{1}{2}}^2$$

$$= [x \log_e(x)]_{\frac{1}{2}}^2 - \int_{\frac{1}{2}}^2 1 dx$$

$$= [x \log_e(x)]_{\frac{1}{2}}^2 - [x]_{\frac{1}{2}}^2$$

$$= \left[2 \log_e(2) - \frac{1}{2} \log_e\left(\frac{1}{2}\right) \right] - \left(2 - \frac{1}{2} \right)$$

$$= \left[2 \log_e(2) - \frac{1}{2} \log_e\left(\frac{1}{2}\right) \right] - \frac{3}{2}$$

$$= 2 \log_e(2) + \frac{1}{2} \log_e(2) - \frac{3}{2}$$

$$\int_{\frac{1}{2}}^2 \log_e(x) dx = \frac{5}{2} \log_e(2) - \frac{3}{2}$$

EXERCISE 7.06 Integration by parts

Concepts and techniques



1 **Example 16** $\int x \sin(x) dx =$

- A $x \sin(x) + \cos(x) + c$ B $-x \cos(x) + \sin(x) + c$ C $x \sin(x) + c$
D $x \cos(x) + \sin(x) + c$ E $\cos(x) + c$

2 $\int_0^\pi x \cos(x) dx =$

- A 2 B $-\pi$ C -2
D $x \sin(x) + \cos(x) + c$ E $x \sin(x) + c$

3 $\int_0^\pi x^2 \cos(x) dx =$

- A 2π B $-\pi$ C -2π
D $x^2 \sin(x) + 2x \cos(x) - 2 \sin(x) + c$ E $2x \cos(x) + c$

4 **Example 17** $\int x^2 e^x dx =$

- A $x^2 e^x + c$ B $x^2 e^x - 2x e^x + 2e^x + c$ C $-2x^2 e^x + 2e^x + c$
D $e - 2$ E $e^x(-2x^2 + 2) + c$

5 $\int 2x^3 e^{2x} dx =$

- A $12x^2 e^{2x} + c$ B $x^2 e^x - 2x e^x + 2e^x + c$
C $\frac{1}{2} x e^{2x} (2x^2 - 3x + 3x - 1) + c$ D $\frac{1}{2} x e^{2x} (2x^2 - 3x + 3x) + c$
E $\frac{1}{4} e^{2x} (4x^3 - 6x^2 + 6x - 3) + c$

6 Find the following.

- a $\int x e^{2x} dx$ b $\int x \cos(2x) dx$ c $\int x \log_e(2x) dx$
d $\int_0^1 x e^{2x} dx$ e $\int_0^{\frac{\pi}{2}} x \sin(2x) dx$

7 **Example 18** Find the following.

- a $\int_{\frac{1}{3}}^1 \log_e(x) dx$ b $\int \sin^{-1}(x) dx$ c $\int x \log_e(x) dx$
d $\int_1^{\frac{1}{2}} \cos^{-1}(x) dx$ e $\int \cos^{-1}(3x) dx$

Reasoning and communication

8 Use a **recursive** procedure to find $\int x^3 \sin(x) dx$.

9 Use a **recursive** procedure to find $\int_0^1 x^4 e^{2x} dx$.

7

CHAPTER SUMMARY

INTEGRATION TECHNIQUES

Double angle formulas

$$\sin(2x) = 2 \sin(x) \cos(x)$$

$$\begin{aligned}\cos(2x) &= \cos^2(x) - \sin^2(x) \\ &= 2 \cos^2(x) - 1 \\ &= 1 - 2 \sin^2(x)\end{aligned}$$

$$\tan(2x) = \frac{2 \tan(x)}{1 - \tan^2(x)}$$

$\cos(2x) = 1 - 2 \sin^2(x)$ so

$$\sin^2(x) = \frac{1}{2}[1 - \cos(2x)]$$

$\cos(2x) = 2 \cos^2(x) - 1$ so

$$\cos^2(x) = \frac{1}{2}[1 + \cos(2x)]$$

The Pythagorean identities

$$\sin^2(x) + \cos^2(x) = 1$$

$$1 + \tan^2(x) = \sec^2(x)$$

$$\cot^2(x) + 1 = \operatorname{cosec}^2(x)$$

$$\frac{d}{dx} \tan(x) = \sec^2(x)$$

$$\frac{d}{dx} \cot(x) = -\operatorname{cosec}^2(x)$$

$$\int \sec^2(x) dx = \tan(x) + c \text{ and so}$$

$$\int \sec^2(kx) dx = \frac{1}{k} \tan(kx) + c$$

$$\int \operatorname{cosec}^2(x) dx = -\cot(x) + c \text{ and so}$$

$$\int \operatorname{cosec}^2(kx) dx = -\frac{1}{k} \cot(kx) + c$$

You use the substitution $u = g(x)$ to integrate expressions of the form $f[g(x)]g'(x)$.

Integration by substitution

If a function $f(x)$ can be written as

$$f(x) = g(u) \cdot u'(x), \text{ then}$$

$$\int_a^b f(x) dx = \int_\alpha^\beta g(u) du, \text{ where } \alpha = u(a) \text{ and } \beta = u(b)$$

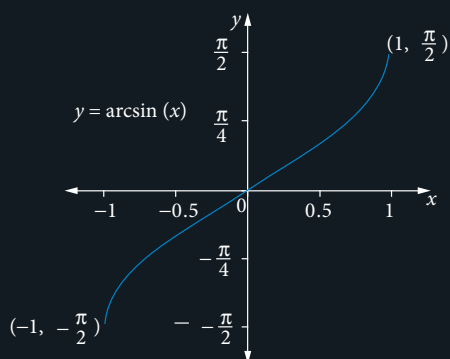
$$\frac{d}{dx} \log_e(x) = \frac{1}{x} \text{ and } \frac{d}{dx} \log_e(kx) = \frac{1}{x}$$

$$\int \frac{1}{x} dx = \log_e|x| + c$$

$$\int \frac{1}{ax+b} dx = \frac{1}{a} \log_e|ax+b| + c \text{ where } a, b \text{ are constants and } c \text{ is the constant of integration.}$$

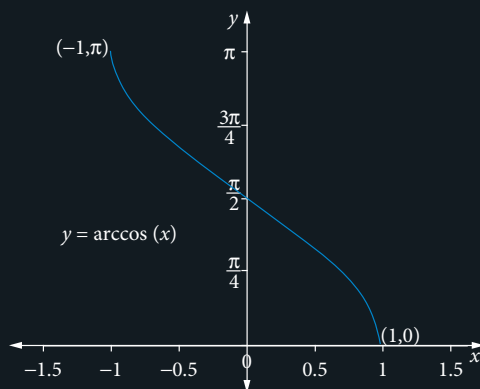
$$\int \frac{f'(x)}{f(x)} dx = \log_e|f(x)| + c, \text{ where the 'log type' is identified by the derivative of the denominator being seen on the numerator.}$$

The graphs of $y = \sin^{-1}(x)$, $y = \cos^{-1}(x)$ and $y = \tan^{-1}(x)$.



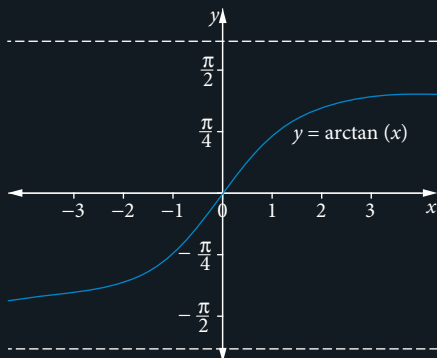
$$y = \arcsin(x) = \sin^{-1}(x)$$

$$\text{Domain } x \in [-1, 1], \text{ range } y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$



$$y = \arccos(x) = \cos^{-1}(x)$$

$$\text{Domain } x \in [-1, 1], \text{ range } y \in \left[0, \pi\right]$$



$$y = \arctan(x) = \tan^{-1}(x)$$

$$\text{Domain } \mathbf{R}, \text{ Range } y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$$\blacksquare \frac{d}{dx} \sin^{-1}\left(\frac{x}{a}\right) = \frac{1}{\sqrt{a^2 - x^2}} \text{ for } x \in (-a, a)$$

$$\blacksquare \frac{d}{dx} \cos^{-1}\left(\frac{x}{a}\right) = \frac{-1}{\sqrt{a^2 - x^2}} \text{ for } x \in (-a, a)$$

$$\blacksquare \frac{d}{dx} \tan^{-1}\left(\frac{x}{a}\right) = \frac{a}{a^2 + x^2} \text{ for } x \in \mathbf{R}$$

$$\blacksquare \int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1}(x) + c \text{ and}$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + c$$

$$\blacksquare \int \frac{-1}{\sqrt{1-x^2}} dx = \cos^{-1}(x) + c \text{ and}$$

$$\int \frac{-1}{\sqrt{a^2 - x^2}} dx = \cos^{-1}\left(\frac{x}{a}\right) + c$$

$$\blacksquare \int \frac{1}{1+x^2} dx = \tan^{-1}(x) + c \text{ and}$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + c$$

■ Fractions of the form $\frac{a}{(x-b)(x-c)}$ or

$\frac{ax}{(x-b)(x-c)}$ can be expressed as

$$\frac{A}{(x-b)} + \frac{B}{(x-c)}.$$

■ Fractions of the form $\frac{a}{(x-b)^2(x-c)}$ or

$\frac{ax}{(x-b)^2(x-c)}$ can be expressed as

$$\frac{A}{(x-b)^2} + \frac{B}{(x-b)} + \frac{C}{(x-c)}.$$

■ Remember in partial fraction problems that if the denominator is a degree less than or equal to the degree of the numerator, you must divide through first.

■ Symbolic calculators can be used to express partial fractions in one step.

■ Integration by parts

If a function can be written as $f(x) \cdot g'(x)$, then

$$\int f(x) \cdot g'(x) dx = f(x) \cdot g(x) - \int f'(x) \cdot g(x) dx$$

(Select $f(x)$ when $f'(x)$ is easier to find and $g'(x)$ when $g(x)$ is no harder to find. In cases of repeated powers, every time that you 'select $f(x)$ where $f'(x)$ is easier' reduces that power.)

CHAPTER REVIEW

INTEGRATION TECHNIQUES

Multiple choice

- 1 **Example 1** $\int 2\cos^2(x)dx =$
 - A $x + \frac{1}{4}\sin(2x) + c$
 - B $x - \frac{1}{4}\sin(4x) + c$
 - C $\frac{1}{2}x + \frac{1}{4}\sin(2x) + c$
 - D $x + \frac{1}{2}\sin(2x) + c$
 - E $\frac{1}{4}x + \frac{1}{4}\sin(2x) + c$
- 2 $\int_0^{\frac{\pi}{3}} \sin^2(3x)dx$
 - A $\frac{\pi}{4}$
 - B $-\frac{\pi}{8}$
 - C $\frac{x}{2} - \frac{1}{12}\sin(6x) + c$
 - D $\frac{x}{2} - \frac{1}{6}\sin(6x) + c$
 - E $\frac{\pi}{6}$
- 3 **Examples 2, 3** $\int \tan^2(4x)dx =$
 - A $\frac{1}{2}x - \frac{1}{4}\tan(2x) + c$
 - B $\frac{1}{2}x - \frac{1}{16}\tan(4x) + c$
 - C $x + \frac{1}{4}\sec^2(4x) + c$
 - D $-x + \frac{1}{4}\tan(4x) + c$
 - E $-x - \frac{1}{4}\tan(4x) + c$
- 4 $\int_0^{\frac{\pi}{3}} \frac{1}{4}\sin^2(8x)dx =$
 - A $\frac{\pi}{24}$
 - B $\frac{\pi}{24} + \frac{\sqrt{3}}{256}$
 - C $\frac{\pi}{16}$
 - D $\frac{1}{8}x - \frac{1}{128}\sin(16x) + c$
 - E $\frac{1}{8}x - \frac{1}{128}\cos(16x) + c$
- 5 $\int_0^{\frac{\pi}{4}} 2\sin(2x)\cos(2x)dx =$
 - A $\frac{1}{2}$
 - B $\frac{\pi}{2}$
 - C 1
 - D $-\frac{1}{4}\cos(4x) + c$
 - E $\frac{1}{4}\sin(4x) + c$
- 6 **Example 4** $\int_0^2 (x^2 - x)^3(2x - 1)dx =$
 - A $\frac{(x^2 - x)^4}{4} + c$
 - B $\frac{(x^2 - x)^4}{4}$
 - C 4
 - D 0
 - E 320
- 7 **Example 5** $\int (2x - 7)^6 dx =$
 - A $\frac{410\,678}{7}$
 - B $\frac{1093}{7}$
 - C $\frac{1}{7}(2x - 7)^7 + c$
 - D $\frac{1}{16}(2x - 7)^8 + c$
 - E $\frac{1}{14}(2x - 7)^7 + c$
- 8 **Example 8** $\int \frac{x}{x^2 + 2} dx =$
 - A $\log_e|2x| + c$
 - B $\frac{1}{2}\log_e|x^2 + 2| + c$
 - C $\log_e|x^2 + 2| + c$
 - D $\log_e|x^2| + c$
 - E $2\log_e|x^2 + 2| + c$

- 9 **Example 9** $\int_{\frac{\pi}{3}}^{\frac{\pi}{4}} \frac{2\cos(x)}{\sin(x)} dx =$
 A $\log_e\left(\frac{2}{3}\right)$ B $2\log_e|\sin(x)| + c$ C $\frac{1}{2}\log_e\left(\frac{2}{3}\right)$
 D $\frac{1}{2}\log_e|\sin(x)|$ E $\log_e\left(\frac{3}{2}\right)$
- 10 **Example 16** $\int x \cos(2x) dx =$
 A $2x \sin(2x) + \cos(2x) + c$ B $\frac{1}{2}x \sin(2x) + \frac{1}{2}\cos(2x) + c$ C $\frac{1}{2}x \sin(2x) + c$
 D $\frac{1}{2}x \sin(2x) + \frac{1}{4}\cos(2x) + c$ E $\cos(2x) + c$

Short answer

- 11 **Example 6** $\int_0^{\frac{\pi}{2}} \sin(x) \cos^2(x) dx =$
- 12 **Example 7** Use the method of substitution to find an expression for the following integrals.
 a $\int \frac{1}{2-3x} dx$ b $\int \frac{-2}{7x+2} dx$
- 13 Find an expression for the following integrals.
 a $\int \frac{x}{3x^2+3} dx$ b $\int \frac{2e^{\frac{x}{2}}}{e^{\frac{x}{2}}+1} dx$
- 14 **Example 10** Find the following.
 a $\frac{d}{dx} \cos^{-1}\left(\frac{x}{4}\right)$ b $\frac{d}{dx} \sin^{-1}\left(x^2 - \frac{1}{2}\right)$
- 15 Sketch the graph of $y = \frac{1}{3} \sin^{-1}\left(\frac{x}{4}\right)$, stating the domain.
- 16 **Example 13** Express the following in partial fraction form.
 a $\frac{1}{x^2-3x-18}$ b $\frac{x^2}{(x+1)^2(x-6)}$
- 17 **Examples 14, 15** Find the following.
 a $\int_1^2 \frac{3x}{x^2-4x-12} dx$ b $\int \frac{x^2}{3x^2-x-4} dx$
- 18 **Example 17** Find $\int x^2 e^{2x} dx$.
- 19 **Example 18** Find $\int_{\frac{1}{2}}^1 \log_e(x) dx$.

Application

- 20 Show that $\frac{1}{2} \int \sin(x) \cos(x) dx = \frac{1}{4} \sin^2(x) + c$
- 21 Show that $\int \sec^2(x) - 1 dx = \tan(x) - x + c$
- 22 Express $\sin^{-1}\left(\frac{2x}{5}\right)$ in the form of $\sin^{-1}\left(\frac{x}{a}\right)$ and hence find $\int \frac{1}{\sqrt{\frac{25}{4} - x^2}} dx$.
- 23 Use a **recursive** procedure to find $\int 2x^2 \cos(2x) dx$.





8

TERMINOLOGY

acceleration
angular velocity
differential equation
displacement
first-order
force
frequency
general solution
gradient field
implicit differentiation
logistic equation
logistic model
mass
momentum
Newton's laws of motion
position
related variables
separable variables
simple harmonic motion
slope field
velocity

RATES OF CHANGE AND DIFFERENTIAL EQUATIONS

RATES OF CHANGE AND DIFFERENTIAL EQUATIONS

- 8.01 Implicit differentiation
- 8.02 Related rates of change
- 8.03 Differential equations of the form $\frac{dy}{dx} = f(x)$
- 8.04 Differential equations of the form $\frac{dy}{dx} = g(y)$
- 8.05 First order differential equations with separable variables
- 8.06 The logistic equation
- 8.07 Force, momentum and motion
- 8.08 Motion in a straight line with constant force
- 8.09 Motion in a straight line with variable forces
- 8.10 Simple harmonic motion

Chapter summary

Chapter review



Prior learning

RATES OF CHANGE AND DIFFERENTIAL EQUATIONS

- use implicit differentiation to determine the gradient of curves whose equations are given in implicit form (ACMSM128)
- related rates as instances of the chain rule: $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$ (ACMSM129)
- solve simple first order differential equations of the form $\frac{dy}{dx} = f(x)$, differential equations of the form $\frac{dy}{dx} = g(y)$ and, in general, differential equations of the form $\frac{dy}{dx} = f(x)g(y)$ using separation of variables (ACMSM130)
- examine slope (direction or gradient) fields of a first order differential equation (ACMSM131)
- formulate differential equations including the logistic equation that will arise in, for example, chemistry, biology and economics, in situations where rates are involved. (ACMSM132)

MODELLING MOTION

- examine momentum, force, resultant force, action and reaction (ACMSM133)
- consider constant and non-constant force (ACMSM134)
- understand motion of a body under concurrent forces (ACMSM135)
- consider and solve problems involving motion in a straight line with both constant and non-constant

acceleration including simple harmonic motion and the use of expressions $\frac{dv}{dt}$, $v \frac{dv}{dx}$ and $\frac{d(\frac{1}{2}v^2)}{dx}$ for acceleration. (ACMSM136) 

8.01 IMPLICIT DIFFERENTIATION

You cannot differentiate the equation of a circle such as $x^2 + y^2 = 25$ in the usual way because it is not in the form $y = f(x)$. You can still find $\frac{dy}{dx}$ by differentiating the whole equation and using the chain rule for the parts that are functions of y .

For the equation $x^2 + y^2 = 25$, you get $2x + 2y \times \frac{dy}{dx} = 0$, which gives $\frac{dy}{dx} = -\frac{x}{y}$. This has the disadvantage of needing both the coordinates of a point on the line to calculate the derivative.

IMPORTANT

Implicit differentiation is the technique of differentiating an equation, involving the dependent and independent variables, using the chain rule for the parts that are functions of the dependent variable.

Example 1

Find $\frac{dy}{dx}$ for $x^2 + 2xy - y^2 = 4$.

Solution

Differentiate each term with respect to x .

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(2xy) - \frac{d}{dx}(y^2) = \frac{d}{dx}(4)$$

Use the product rule for the second term.

$$2x + \left[\frac{d}{dx}(2x) \times y + 2x \times \frac{d}{dx}(y) \right] - 2y \frac{dy}{dx} = 0$$

Simplify.

$$2x + 2y + 2x \frac{dy}{dx} - 2y \frac{dy}{dx} = 0$$

Now solve for $\frac{dy}{dx}$.

$$2x \frac{dy}{dx} - 2y \frac{dy}{dx} = -2x - 2y$$

Factorise the LHS.

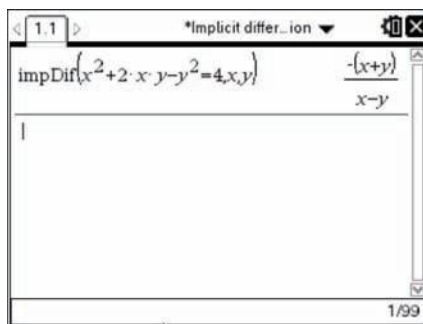
$$\frac{dy}{dx}(2x - 2y) = -2x - 2y$$

Make $\frac{dy}{dx}$ the subject and simplify.

$$\frac{dy}{dx} = \frac{-2x - 2y}{2x - 2y} = \frac{-x - y}{x - y} = \frac{x + y}{y - x}$$

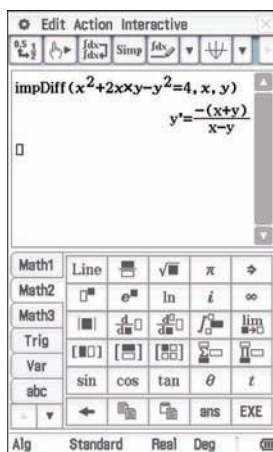
TI-Nspire CAS

Use **menu**, 4: Calculus, E: Implicit Differentiation and complete with the equation, independent and dependent variables. Make sure that you avoid the variable xy by including the $\times$ between x and y .



ClassPad

Tap **Action**, **Calculation** and **impDiff** and enter in turn the equation, independent and dependent variables, with each entry separated by a comma. Close the parentheses and press **EXE**. Make sure that you avoid the variable xy by including a $\times$ between x and y .



You don't usually write the equations of circles, ellipses and hyperbolas in the form $y = f(x)$, so you use implicit differentiation for these relations. This is particularly useful in Physics for tangents and normals in applications such as finding the reflection from curved surfaces. Remember that the normal is perpendicular to the tangent and the product of their gradients is -1 .

○ Example 2

- a Find the gradient of the tangent to the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$ at the point $(4, \frac{9}{5})$.
- b Find the equation of the normal to $\frac{x^2}{25} + \frac{y^2}{9} = 1$ at the point $(4, \frac{9}{5})$.

Solution

- a Use implicit differentiation.

$$\frac{2x}{25} + \frac{2y}{9} \frac{dy}{dx} = 0$$

Make $\frac{dy}{dx}$ the subject.

$$\frac{dy}{dx} = \frac{-9x}{25y}$$

Find the gradient of the tangent.

$$m_t = \frac{dy}{dx} = \frac{-9 \times 4}{25 \times \frac{9}{5}} = -\frac{4}{5}$$

- b Use $m_t m_n = -1$ to find the slope of the normal.

$$m_n = \frac{5}{4}$$

Write the slope-point form of a straight line.

$$y - y_1 = m(x - x_1)$$

Substitute m_n and $(4, \frac{9}{5})$.

$$y - \frac{9}{5} = \frac{5}{4}(x - 4)$$

Simplify.

$$y = \frac{5}{4}x - \frac{16}{5}$$

Write in standard form.

$$25x - 20y - 64 = 0$$

TI-Nspire CAS

- a Use implicit differentiation to find the derivative. Define $m1(x,y)$ to be the derivative.
Find the value at $(4, \frac{9}{5})$.

1.1	*Unsaved
impDif($\frac{x^2}{25} + \frac{y^2}{9} = 1, x, y$)	$\frac{-9 \cdot x}{25 \cdot y}$
Define $m1(x,y) = \frac{-9 \cdot x}{25 \cdot y}$	Done
$m1(4, \frac{9}{5})$	$-\frac{4}{5}$
$\frac{-1}{5}$	$\frac{5}{4}$
	4/99

- b Use $-1/\text{ans}$ to find the gradient of the normal. Solve the two-point form of the equation of a straight line for y and use [menu], 2: Number, 7: Fraction Tools and 1: Proper Fraction to change to the slope–intercept form.

TI-84 Plus calculator screen showing the steps to find the equation of a normal line. The screen displays the following steps:

$$\frac{-1}{\frac{5}{4}} \rightarrow m2$$

$$\text{solve}\left(y - \frac{9}{5} = m2 \cdot (x - 4), y\right) \quad y = \frac{25 \cdot x - 64}{20}$$

$$\text{propFrac}\left(y = \frac{25 \cdot x - 64}{20}\right) \quad y = \frac{5 \cdot x}{4} - \frac{16}{5}$$

ClassPad

- a Use implicit differentiation to find the derivative. Define $m1(x, y)$ to be the derivative using copy and paste. Find the value of $m1$ at $(4, \frac{9}{5})$.
- b Use $-1/\text{ans}$ to find the gradient of the normal and assign this to the name $m2$. Solve the two-point form of the equation of a straight line for y with its gradient given by $m2$.

ClassPad screen showing the steps to find the equation of a normal line using implicit differentiation. The screen displays the following steps:

$$\text{impDiff}\left(\frac{x^2}{25} + \frac{y^2}{9} = 1, x, y\right) \quad y' = \frac{-9 \cdot x}{25 \cdot y}$$

$$\text{Define } m1(x, y) = \frac{-9 \cdot x}{25 \cdot y} \quad \text{done}$$

$$m1(4, 9/5) \quad -\frac{4}{5}$$

$$-1/\text{ans} \rightarrow m2 \quad \frac{5}{4}$$

$$\text{solve}(y - 9/5 = m2 \cdot (x - 4), y) \quad \left\{ y = \frac{5 \cdot x}{4} - \frac{16}{5} \right\}$$

You can also find the equation of the tangent by using a CAS calculator to draw the relation as a conic/ellipse. You draw the tangent at the desired point to get its equation.



EXERCISE 8.01 Implicit differentiation



Implicit differentiation

Concepts and techniques

- Example 1** Find the derivative of each of the following.
 - $3x^2 - 5xy + y^3 = 3$
 - $x^2y - yx^2 = xy$
 - $\frac{2}{x} + \frac{3}{y} = 5$
 - $2x\sqrt{y} = (1-y)^2$
- Consider the function $x^2 + 2y = 5$.
 - Differentiate the function implicitly to find $\frac{dy}{dx}$.
 - Find y explicitly in terms of x and show that $\frac{dy}{dx}$ is the same as that in part a.
- Consider the relation $y^2 + 2y - x = 3$.
 - Differentiate the relation implicitly to find $\frac{dy}{dx}$.
 - Find x in terms of y .
 - Hence find $\frac{dy}{dx}$.
- Find $\frac{dy}{dx}$ in terms of x and y for each of the following by implicit differentiation.
 - $\sin(x+y) = \tan(y)$
 - $e^{2y}x - 2xy = 3$
 - $\log_e(x+y) = 5x$
 - $x^2 \cos(y) = e^y$
- Example 2**
 - Find the gradient of the tangent to the hyperbola with equation $x^2 - y^2 = 16$ at the point $(5, -3)$.
 - Hence find the equation of the normal to $x^2 - y^2 = 16$ at the point $(5, -3)$.
- Find the equation of the tangent to the curve with equation $\frac{(x+1)^2}{36} - \frac{(y-2)^2}{16} = 1$ at $\left(\frac{13}{2}, 5\right)$.
- Find the equations of the tangents to the ellipse with equation $4x^2 + 9y^2 = 40$ that have a gradient of $-\frac{2}{9}$.
- Find the equation of the normal to the curve with equation $\frac{(x-3)^2}{25} + \frac{(y+2)^2}{16} = 1$ at $\left(6, \frac{6}{5}\right)$.
- Find the equations of the tangents to the ellipse with equation $\frac{9x^2}{52} + \frac{16y^2}{52} = 1$ that are parallel to the line $y = \frac{9}{8}x - \frac{1}{8}$.
- Find the intersection of the normals to the curve with equation $\frac{x^2}{16} - \frac{y^2}{9} = 1$ at $y = 4$.

Reasoning and communication

- For the curve with equation $y^2 = 4ax$, show that the equation of the tangent at a point $P(x_1, y_1)$ on the parabola is $y_1y = 2a(x + x_1)$.
- Consider the ellipse with equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Show that the equation of the tangent to the ellipse with slope m is $y = mx \pm \sqrt{a^2m^2 + b^2}$.
- Show that the equation of the tangent, with gradient m , to the curve with equation $y^2 = 4ax$ is $y = mx + \frac{a}{m}$.
- Show that the equation of the tangent to the ellipse with equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at a point $P(x_1, y_1)$ on the ellipse is $b^2x_1x + a^2y_1y - a^2b^2 = 0$.

8.02 RELATED RATES OF CHANGE

You will sometimes need to find the rate of change of one variable with respect to time from the rate of change of a related variable with respect to time.

IMPORTANT

The relation between rates of change of **related variables** with respect to time can be found using the chain rule in the form $\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$.

○ Example 3

Given $y = 2x^2 - 3x + 1$ and $\frac{dx}{dt} = 5$, find $\frac{dy}{dt}$ when $x = -2$.

Solution

Find the rate of change of y with respect to (wrt) x .

$$\frac{dy}{dx} = 4x - 3$$

Express the rate of change of y wrt t in terms of the rate of change wrt to x .

$$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$$

Substitute the derivatives.

$$= 5(4x - 3)$$

Substitute in the value of x .

$$= 5[4 \times (-2) - 3]$$

Write the answer.

$$= -55$$

○ Example 4

A spherical metal ball is heated so that its radius is expanding at the rate of 0.04 mm per second. At what rate will its volume be increasing when the radius is 3 mm?

Solution

Write the volume of the sphere.

$$V = \frac{4}{3}\pi r^3$$

Find the rate of change with wrt to radius.

$$\frac{dV}{dr} = 4\pi r^2$$

Express the rate of change of (wrt) time in terms of the rate of change wrt to radius.

$$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$$

Substitute the values, including the radius.

$$= 4\pi \times 3^2 \times 0.04$$

Calculate the answer.

$$= 1.44\pi \text{ mm}^3/\text{s}$$



Example 5

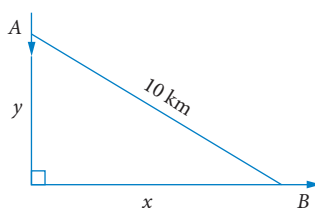
Car A is north of an intersection, while car B is moving east at 60 km/h, away from the intersection. The distance between the cars remains constant at 10 km. Find the speed of car A when car B is 8 km from the intersection.



Shutterstock.com/Tim Roberts Photography

Solution

Draw a diagram to represent the situation.



Write the relationship between x and y .

$$x^2 + y^2 = 10^2$$

Differentiate with respect to time.

$$2x \times \frac{dx}{dt} + 2y \times \frac{dy}{dt} = 0$$

Rearrange.

$$\frac{dy}{dt} = \frac{dx}{dt} \times \left(-\frac{x}{y} \right)$$

Find y when $x = 8$.

$$8^2 + y^2 = 100 \text{ so } y = 6$$

Substitute values into the rates.

$$\begin{aligned} \frac{dy}{dt} &= 60 \times \left(-\frac{8}{6} \right) \text{ km/h} \\ &= -80 \text{ km/h} \end{aligned}$$

Calculate the answer.

Comment on the minus sign.

The negative means Car A is travelling towards the intersection.

Write the answer.

Car A is travelling at 80 km/h towards the intersection.

EXERCISE 8.02 Related rates of change

Concepts and techniques



- 1 **Example 3** Find an expression for $\frac{dy}{dt}$ given
 - a $y = x^4$ and $\frac{dx}{dt} = 2$
 - b $y = e^{2x}$ and $\frac{dx}{dt} = 5$
- 2 Evaluate $\frac{dy}{dt}$ at $x = 4$, given
 - a $y = 2x^3 + 3x - 7$ and $\frac{dx}{dt} = 3$
 - b $y = (3x + 1)^3$ and $\frac{dx}{dt} = -4$
 - c $y = x \log_e(x)$ and $\frac{dx}{dt} = 3$
- 3 If $y = \sin(2x)$ and $\frac{dx}{dt} = 3$, evaluate $\frac{dy}{dt}$ when $x = \frac{\pi}{8}$.

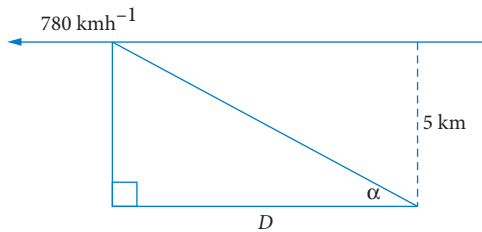
Reasoning and communication

- 4 **Example 4** A cube is expanding so that its side is increasing at the constant rate of 0.12 mm/s. Find the rate of increase in its volume when its side is 150 mm.
- 5 The radius of a cylindrical pipe 2 m long expands with heat at a constant rate of 1.2×10^{-3} mm/s. Find the rate at which the volume of the pipe will be increasing when its radius is 19 mm.
- 6 Find the rate of change of the surface area of a balloon when its radius is 6.3 cm, if its radius is expanding at a constant rate of 1.3 cm s^{-1} .
- 7 A cone's volume is given by $V = \frac{6\pi h^2}{7}$, where h is the height of liquid in the cone. If the height of the liquid is increasing at a rate of 2.3 cm/s, find the rate of increase in the volume of the liquid when its height is 12.9 cm.
- 8 A point, P , moves along a curve with equation $y = 2x^2 - 7x + 9$. What will the rate of change in the y -coordinate of P be when the x -coordinate is increasing at a rate of 8 units per second and the value of x is 3?
- 9 **Example 5** A 4 m long ladder is leaning against a wall. Its base is slipping away from the wall at a constant rate of 2 m/s. At what rate, correct to 2 decimal places, will the top of the ladder be slipping down the wall when the base is 1 m out from the wall?



Alamy/artistimages.com

- 10 A particle is moving so that its velocity is given by the formula $v = 8x^3 - 5x^2 - 3x - 1$, where x is its displacement. If the rate of change of the displacement is a constant 4.2 cm/s , find the rate of change in velocity when the displacement is -4.7 cm .
- 11 A population increases at a constant rate of 15 000 people per year. If the population has the formula $P = x^2 - 3000x + 100$, where x is the number of houses available, find the rate at which the number of houses will be increasing when there are 5000 houses.
- 12 The area of an equilateral triangle is increasing at the rate of $2 \text{ cm}^2 \text{ s}^{-1}$.
- Show that the area of the triangle is given by $A = \frac{\sqrt{3}x^2}{4}$, where x is the side of the triangle.
 - Find the rate of increase of the side of the triangle when it has side length $2\sqrt{3} \text{ cm}$.
- 13 The surface area of a spherical bubble is increasing at a constant rate of $1.9 \text{ mm}^2 \text{ s}^{-1}$. Find the rate of increase in its volume when its radius is 0.6 mm .
- 14 A spherical bath oil capsule dissolves in the bath so that its decrease in volume is proportional to its surface area. If its shape remains spherical as it dissolves, show that the radius of the capsule will decrease at a constant rate.
- 15 Water is leaking out of a right circular cone at a constant rate of $2 \text{ cm}^3 \text{ s}^{-1}$. The cone has a radius of 10 cm and height of 30 cm . Find the rate at which the radius is decreasing when the height is 3 cm .
- 16 Justin sees a plane fly directly overhead at an altitude of 5 km . The plane is moving horizontally away from Justin at a constant speed of 780 km h^{-1} with an angle of elevation of α as shown.



- Show that the horizontal distance D between the plane and Justin is given by $D = \frac{5}{\tan(\alpha)}$.
- Find the simplest expression for $\frac{dD}{d\alpha}$.
- Find the rate at which the angle of elevation is changing over time (in radians/hour) when $\alpha = \frac{\pi}{6}$.
- Is this a reliable measure of the rate in the long term?

8.03 DIFFERENTIAL EQUATIONS OF THE FORM $\frac{dy}{dx} = f(x)$

You have already solved some simple **differential equations** by finding indefinite integrals (antiderivatives). You can already solve the differential equation $\frac{dy}{dx} = 6x^2 - 2x + 9$.

IMPORTANT

A **differential equation** involves one or more derivatives of a function. A **first-order** differential equation involves only the first derivative. The general solution of a differential equation is the function itself.

The general solution of the equation $\frac{dy}{dx} = 6x^2 - 2x + 9$ is $y = 2x^3 - x^2 + 9x + c$. A specific solution can only be found if you have other information that you can use to find the value of the constant of integration, c .

Example 6

Find the general solutions of the following differential equations.

a $\frac{dy}{dx} = (1 - 2x)^3$ b $\frac{dy}{dt} = 6e^{-3t} + 4e^{2t}$ c $\frac{dp}{dy} = \frac{1}{\sqrt{9 - y^2}}$

Solution

a Find the antiderivative.

$$\int (1 - 2x)^3 dx = \frac{(1 - 2x)^4}{4 \times (-2)} + c$$

Simplify to write the answer.

$$y = -\frac{1}{8}(1 - 2x)^4 + c$$

b Find the indefinite integral.

$$\int 6e^{-3t} + 4e^{2t} dt = -2e^{-3t} + 2e^{2t} + c$$

Write in the usual order.

$$y = 2e^{2t} - 2e^{-3t} + c$$

c Use $\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + c$.

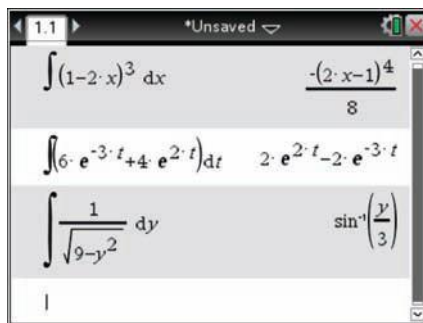
$$\int \frac{1}{\sqrt{3^2 - y^2}} dy = \sin^{-1}\left(\frac{y}{3}\right) + c$$

Write the answer.

$$p = \sin^{-1}\left(\frac{y}{3}\right) + c$$

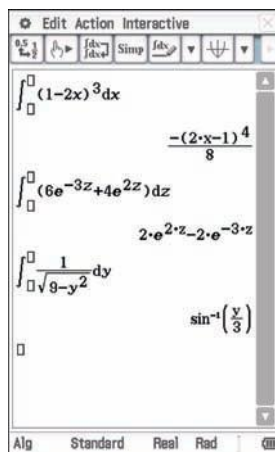
TI-Nspire CAS

Use **[menu]**, 4: Calculus, 3: Integral, but do not fill in the limits. The calculator does not put the '+ c' on the integral. Make sure that the calculator is set in radians for part c.



ClassPad

Tap **Interactive**, **Calculation**, then $\int$ and choose indefinite integral, or use $\int$ from the **Math2** soft keyboard but do not fill in the limits. The calculator does not put '+ c' on the integral. As t is a reserved variable, use z for part **b**. Make sure that the calculator is set to **Rad** mode for part **c**.



To find a particular solution to a differential equation, a point on the curve is required. To find the value of c when $y = \int (6x^2 - 2x + 9) dx = 2x^3 - x^2 + 9x + c$ and $(0, 6)$ is on the line (that is, $y(0) = 6$), substitute $x = 0$ and $y = 6$ into $y = 2x^3 - x^2 + 9x + c$, giving $c = 6$ so $y = 2x^3 - x^2 + 9x + 6$.

A general solution to a differential equation generates the equations for a whole family of curves with different constants c , and a particular solution gives the equation for one of those curves with a particular value of c .

Example 7

Find the solution to the following differential equations.

- $\frac{dy}{dx} = \frac{x-2}{x^2-4x}$ if $y(2) = 1$
- $\frac{dy}{dx} = \frac{1}{x^2+5x+6}$ if $y(1) = 0$
- $\frac{dy}{dx} = \cos^2(x)$ if $y\left(\frac{\pi}{6}\right) = 0$

Solution

a Use $\int \left(\frac{f'(x)}{f(x)} \right) dx = \log_e |f(x)| + c$

Simplify.

Use the value to find c .

Solve for c .

Write the answer.

$$\int \left(\frac{x-2}{x^2-4x} \right) dx = \frac{1}{2} \int \left(\frac{2x-4}{x^2-4x} \right) dx$$

$$y = \frac{1}{2} \log_e |x^2 - 4x| + c$$

$$1 = \frac{1}{2} \log_e |2^2 - 4 \times 2| + c$$

$$c = 1 - \log_e (2)$$

$$y = \frac{1}{2} \log_e |x^2 - 4x| + 1 - \log_e (2)$$

b Factorise $x^2 + 5x + 6$.

Change $\frac{1}{x^2 + 5x + 6}$ to partial fractions.

Solve for A and B.

Substitute $x = -3$ and $x = -2$ to find A and B.

Write the integral and use the expression with partial fractions.

Separate the terms.

Use $\int \frac{1}{x} dx = \log_e |x| + c$.

Substitute (1, 0).

Solve for c.

Write the expression.

Simplify using log laws.

Write the answer.

c Write the integral and use the formula $\cos(2x) = 2 \cos^2(x) - 1$.

Simplify.

Substitute $(\frac{\pi}{6}, 0)$ to find c.

Solve for c.

Write the answer.

$$x^2 + 5x + 6 = (x + 3)(x + 2)$$

$$\begin{aligned}\frac{1}{x^2 + 5x + 6} &= \frac{A}{x + 3} + \frac{B}{x + 2} \\ &= \frac{A(x + 2) + B(x + 3)}{(x + 2)(x + 3)}\end{aligned}$$

$$A(x + 2) + B(x + 3) = 1$$

$$A = -1 \text{ and } B = 1$$

$$\begin{aligned}\int \frac{1}{x^2 + 5x + 6} dx &= \int \left[\frac{1}{x + 2} - \frac{1}{x + 3} \right] dx \\ &= \int \frac{1}{x + 2} dx - \int \frac{1}{x + 3} dx \\ &= \log_e |x + 2| - \log_e |x + 3| + c\end{aligned}$$

$$0 = \log_e(3) - \log_e(4) + c$$

$$\begin{aligned}c &= \log_e(4) - \log_e(3) \\ &= \log_e\left(\frac{4}{3}\right)\end{aligned}$$

$$\begin{aligned}\int \frac{1}{x^2 + 5x + 6} dx &= \log_e |x + 2| - \log_e |x + 3| + \log_e\left(\frac{4}{3}\right) \\ &= \log_e \left| \frac{4(x + 2)}{3(x + 3)} \right|\end{aligned}$$

$$y = \log_e \left| \frac{4(x + 2)}{3(x + 3)} \right|$$

$$\begin{aligned}\int \cos^2(x) dx &= \frac{1}{2} \int [\cos(2x) + 1] dx \\ &= \frac{1}{4} \sin(2x) + \frac{1}{2}x + c\end{aligned}$$

$$0 = \frac{1}{4} \sin\left(2 \times \frac{\pi}{6}\right) + \frac{1}{2} \times \frac{\pi}{6} + c$$

$$c = -\frac{\sqrt{3}}{8} - \frac{\pi}{12}$$

$$y = \frac{1}{4} \sin(2x) + \frac{1}{2}x - \frac{\sqrt{3}}{8} - \frac{\pi}{12}$$



○ Example 8

A rocket is constructed so that its acceleration for a short time after take-off is given by $a = 10t + 2 \text{ ms}^{-2}$, where t is in seconds. It takes off vertically from a height of 100 m above sea level. Find its velocity and height after 4 s.

Solution

Write the integral.

$$v = \int (10t + 2) dt$$

Simplify.

$$= 5t^2 + 2t + c_1$$

Substitute $v(0) = 0$ to find c_1 .

$$c_1 = 0$$

Write the equation.

$$v = 5t^2 + 2t$$

Substitute $t = 4$ into the equation.

$$v = 88 \text{ m/s}$$

The velocity is the derivative of the height.

$$\frac{dh}{dt} = 5t^2 + 2t$$

Find the integral.

$$h = \frac{5}{3}t^3 + t^2 + c_2$$

Substitute $h = 100$ at $t = 0$ to find c_2 .

$$c_2 = 100$$

Write the equation.

$$h = \frac{5}{3}t^3 + t^2 + 100$$

Find the value at $t = 4$.

$$h = \frac{5}{3} \times 64 + 16 + 100 = 222\frac{2}{3}$$

Write the answer.

The velocity and height after 4 s are 88 m/s and $222\frac{2}{3}$ m above sea level respectively.

EXERCISE 8.03 Differential equations of the form $\frac{dy}{dx} = f(x)$



Simple differential equations

Concepts and techniques

1 **Example 6** Find the general solutions to the following differential equations.

a $\frac{dy}{dx} = 3x^2 - 4x + 7$

b $\frac{dy}{dt} = 8 \cos(3t)$

c $\frac{dy}{dw} = 24w^3 - \frac{12}{w}$

d $\frac{dy}{dx} = 6e^{-3x} - 8e^{2x}$

e $\frac{dy}{dx} = (5 + 2x)^4$

f $\frac{dy}{dx} = \frac{1}{(3-2x)^3}$

g $\frac{dx}{dy} = \frac{4}{16 + y^2}$

h $\frac{dx}{dy} = \frac{-1}{\sqrt{4-y^2}}$

2 **Example 7** Find the solutions to the following differential equations.

a $\frac{dy}{dx} = \frac{4x-10}{x^2-5x}$ if $y(6) = 0$

b $\frac{dy}{dx} = \frac{3x+1}{x^2}$ if $y(1) = 3$

c $\frac{dy}{dx} = \frac{6}{(x-2)(x+3)}$ if $y(3) = 1$

d $\frac{dy}{dx} = \frac{3}{x^2+4x+6}$ if $y(0) = 2$

e $\frac{dy}{dx} = \tan^2(x)$ if $y\left(\frac{\pi}{4}\right) = 1$

f $\frac{dy}{dx} = 8 \cos(3x) \sin(3x)$ if $y\left(\frac{\pi}{2}\right) = 2$

g $\frac{dx}{dy} = \frac{1}{\sqrt{4-9y^2}}$ if when $y = 0$, $x = 1$

h $\frac{dx}{dy} = \frac{3}{25+4y^2}$ if when $y = 0$, $x = 1$

i $\frac{dy}{dx} = 10x(5x^2-2)^4$ if $y(1) = 48$

j $\frac{dy}{dx} = -3 \cos^2(x) \sin(x)$ if $y\left(\frac{\pi}{3}\right) = 0$

Reasoning and communication

3 a Express $\frac{x^3+1}{x^2-1}$ as a partial fraction.

b Hence find the general solution to $\frac{dy}{dx} = \frac{-x^3-1}{x^2-1}$.

4 a Find the derivative of $x \log_e(x)$.

b Hence find the general solution to $\frac{dy}{dx} = \log_e(x)$.

5 **Example 8** The velocity of a stone thrown up into the air is given by the equation $v = 20 - 10t \text{ m s}^{-1}$, where t is in seconds. The stone is thrown from a height of 1.5 m. Use a differential equation to find an expression for the height of the stone, and hence find its height after 2.5 s.

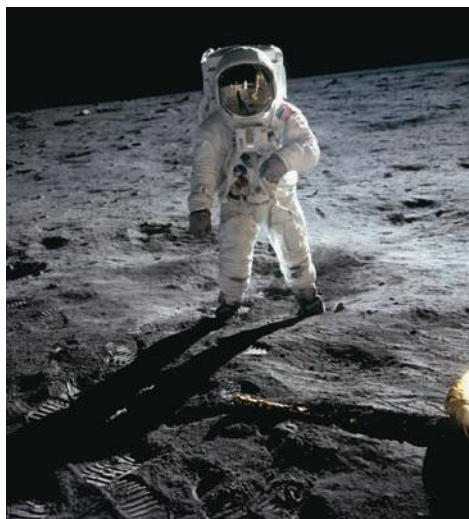
6 The marginal profit of a maker of DVD players is given by $P(n) = \frac{200n+4000}{n^2+40n+1}$, where n is the number of players made and $P(n)$ is the profit in dollars on the n th player. Use the derivative to find the total profit when n players are made, and hence the profit from 5000 players.

7 The acceleration due to gravity on the surface of the moon is such that the speed of some green cheese thrown upward by an astronaut with a strong arm is given by $v = 25 - 1.6t$, where v is in m s^{-1} and t is in seconds. Assume that the initial height is 0 m.

a Use the derivative to find the height after t s.

b Find the height after 4 s.

c Use a quadratic equation to show that the cheese never reaches a height of 200 m.



Corbis

- 8 The rate of storage of energy by a spring is given by $\frac{dQ}{dx} = 200x \text{ J m}^{-1}$, where $Q \text{ J}$ (joules) is the energy stored and $x \text{ m}$ is the compression of the spring. No energy is stored when there is no compression. Find the energy stored with a compression of $x \text{ m}$, and thus the energy stored with a compression of 12 cm .
- 9 The marginal production cost of items is the derivative of the production cost. The marginal production cost of large tents is given by $\frac{dC}{dn} = 0.002n^2 - n + 250$, where C is the cost in dollars of producing n tents. It also costs $\$20\,000$ to set up the production line before any tents are produced. Find the cost of producing n tents.
- 10 The rate of change of momentum for an object is given by the force acting on the object. An object with a momentum of 50 N s is acted upon by a force given by $F = 8 - 2t \text{ N}$ for a period of 4 s . Find the momentum at $t \text{ s}$ ($0 \leq t \leq 4$).

8.04 DIFFERENTIAL EQUATIONS OF THE FORM $\frac{dy}{dx} = g(y)$

At first glance, you may not see how to solve the differential equation $\frac{dy}{dx} = 0.05y$.

Change it to the form $\frac{1}{y} \frac{dy}{dx} = 0.05$ and integrate both sides with respect to x .

This gives $\int \frac{1}{y} \frac{dy}{dx} dx = \int 0.05 dx$.

Using the rule for integration by substitution gives $\int \frac{1}{y} dy = \int 0.05 dx$.

This becomes $\log_e |y| = 0.05x + c$, and assuming $y > 0$, this gives $y = e^{0.05x + c}$.

This is a very important type of differential equation with many applications. It is usually rewritten as $y = e^{0.05x + c} = e^{0.05x} \times e^c = y_0 e^{0.05x}$, where $y_0 = e^c$ is the value of y at $x = 0$.

You should also be able to see that this method applies to any equation of the type $\frac{dy}{dx} = g(y)$.

IMPORTANT

The differential equation $\frac{dy}{dx} = ky$ has the **general solution** $y = y_0 e^{kx}$, where y_0 is the value of y at $x = 0$.

○ Example 9

Solve the differential equation $\frac{dy}{dx} = 5y$.

Solution

Write the equation with y on the left.

$$\frac{1}{y} \frac{dy}{dx} = 5$$

Integrate both sides with respect to x .

$$\int \frac{1}{y} \frac{dy}{dx} dx = \int 5 dx$$

Simplify.

$$\log_e(y) = 5x + c$$

Change to exponential form.

$$\begin{aligned} y &= e^{5x+c} \\ &= e^c \times e^{5x} \end{aligned}$$

Let $e^c = A$ and write the answer.

$$= Ae^{5x}$$

In many real situations, the independent variable is time, so the value of A in an expression of the type $y = Ae^{kt}$ is the initial value of y .

○ Example 10

The rate of decay of radioactive material depends on the amount of material. The half-life is the time taken for half the material present at any time to decay. The carbon-14 half-life is 5730 years. In a charcoal sample taken from a midden, 0.05 g of carbon-14 was present. The archaeologist working on the site estimated that 0.27 g of carbon-14 was present at the time of the fire. Use a differential equation to find the age of the midden to the nearest year.



Corbis/Imaginedina

Solution

Choose the variables.

Let M = mass of material in g and t = time in years.

Decreasing gives a negative rate of change.

$$\frac{dM}{dt} = -kM$$

Put M on the left.

$$\frac{1}{M} \frac{dM}{dt} = -k$$

Integrate both sides with respect to time.

$$\int \frac{1}{M} \frac{dM}{dt} dt = \int -k dt$$

Simplify.

$$\log_e(M) = -kt + c$$

Change to exponential form.

$$M = e^{-kt+c} = M_0 e^{-kt}$$

Use the initial value of M .

$$M_0 = 0.27$$

Use the half-life.

$$\text{At } t = 5730, M = \frac{1}{2}M_0$$

Substitute values.

$$\frac{1}{2}M_0 = M_0 e^{-5730k}$$

Simplify.

$$e^{-5730k} = \frac{1}{2}$$

Change to logarithm form.

$$\begin{aligned} -5730k &= \log_e\left(\frac{1}{2}\right) \\ &= -\log_e(2) \end{aligned}$$

Solve for k .

$$k = \frac{\log_e(2)}{5730}$$

Write the model.

$$M = 0.27e^{-kt}, \text{ where } k = \frac{\log_e(2)}{5730}$$

Substitute the current value of M .

$$0.05 = 0.27e^{-kt}$$

Simplify.

$$e^{-kt} = \frac{0.05}{0.27}$$

Express as a logarithm.

$$-kt = \log_e\left(\frac{0.05}{0.27}\right)$$

Rearrange to find t .

$$\begin{aligned} t &= -\log_e\left(\frac{0.05}{0.27}\right) \div \frac{\log_e(2)}{5730} \\ &= 13940.857\dots \end{aligned}$$

Write the answer.

The age, correct to the nearest year, is 13 941 years.

Functions where the rate of change is a linear function of the dependent variable also arise frequently in practical applications.

○ Example 11

A piece of very hot iron was dropped into some cold water at 15°C . After 10 min, the iron had cooled to 45°C , and after a further 3 min it was at 30°C . The rate of cooling is proportional to the temperature difference between the iron and the water. Find the initial temperature of the iron in $^{\circ}\text{C}$, correct to one decimal place.

Solution

Choose the variables.

Let temperature = $T^{\circ}\text{C}$ and time = t min

Write the rate of change equation for T .

$$\frac{dT}{dt} = -k(T - 15)$$

Rearrange.

$$\frac{1}{T - 15} \frac{dT}{dt} = -k$$

Integrate both sides with respect to t .

$$\int \frac{1}{T - 15} \frac{dT}{dt} dt = \int -k dt$$

Use integration by substitution.

$$\int \frac{1}{T - 15} dT = \int -k dt$$

Simplify.

$$\log_e |T - 15| = -kt + c$$

Change to exponential form.

$$T - 15 = e^{-kt + c}$$

Express as a function of t .

$$T = 15 + Ae^{-kt}$$

Substitute in the known values.

$$45 = 15 + Ae^{-10k} \text{ and } 30 = 15 + Ae^{-13k}$$

Simplify.

$$Ae^{-10k} = 30 \text{ and } Ae^{-13k} = 15$$

Divide to isolate k .

$$\frac{Ae^{-10k}}{Ae^{-13k}} = \frac{30}{15}$$

Simplify.

$$e^{3k} = 2$$

Change to logarithmic form.

$$3k = \log_e(2), \text{ so } k = \frac{1}{3} \log_e(2)$$

Substitute into the equation for $t = 10$.

$$Ae^{-\frac{1}{3} \log_e(2) \times 10} = 30$$

Simplify.

$$A \left[e^{\log_e(2)} \right]^{-\frac{10}{3}} = 30$$

Use $e^{\log_e(2)} = 2$.

$$A \times 2^{-\frac{10}{3}} = 30$$

Find A

$$A = 30 \times 2^{\frac{10}{3}}$$

Write the model.

$$T = 15 + 30 \times 2^{\frac{10}{3}} \times e^{-\frac{1}{3} \log_e(2) \times t}$$

Simplify.

$$= 15 + 30 \times 2^{\frac{10-t}{3}}$$

Substitute $t = 0$.

$$\begin{aligned} T(0) &= 15 + 30 \times 2^{\frac{10}{3}} \\ &= 317.381 \dots \end{aligned}$$

Write the answer.

The initial temperature of the iron was 317.4°C .



INVESTIGATION Mercury pollution

A small lake of volume 1070 ML has been polluted with approximately 200 tonnes of mercury. Water flows from a river into the lake at the rate of 20 ML h^{-1} and leaves at the same rate, helping to flush the mercury out of the lake (but into the ocean further down). The river contains no mercury and, as it enters, mixes thoroughly with the water in the lake.

- Work out a differential equation for the amount of mercury remaining in the lake, expressed as a concentration in parts per million.
- Solve the equation to find the time taken for the mercury to decrease to safe levels. You will need to consult the library to find an acceptable level of mercury pollution.
- Discuss your findings as a class group, focusing on the implications of the pollution of a largely closed environment. Consider the implications of flushing pollutants into the ocean, given that ocean currents leave the continental shelf largely undisturbed.



Alamy/David Hancock

EXERCISE 8.04 Differential equations of the form $\frac{dy}{dx} = g(y)$



Differential equations and exponentials

Concepts and techniques

- 1 **Example 9** Find the general solutions of each of the following differential equations.

a $\frac{dy}{dx} = 3y$

b $\frac{dy}{dx} = 2y - 1$

c $\frac{dy}{dx} = e^{3y}$

d $\frac{dy}{dx} = y^2 + 1$

e $\frac{dy}{dx} = \sqrt{16 - y^2}$

f $\frac{dy}{dx} = y^2 + 2y$

- 2 Find the solution of each of the following differential equations.

a $\frac{dy}{dx} = y + 1$, if $y(0) = 1$

b $\frac{dy}{dx} = \frac{y^2 + 4y}{y + 4}$, if $y(2) = 4$

c $\frac{dy}{dx} = y^2 + 6y + 5$, if $y(1) = -6$

d $\frac{dy}{dx} = \sqrt{1 - y^2}$, if $y\left(\frac{5\pi}{6}\right) = -\frac{1}{2}$

Reasoning and communication

- 3 **Example 10** The amount of carbon-14 remaining in a sample of bone is 20% of the original amount assumed to be present. Carbon-14 has a half-life of 5730 years and decays at a rate proportional to the quantity present. Use a differential equation to find the quantity present after time t and hence find the age of the bone.
- 4 A lake has become polluted with phosphates and an algal bloom begins to affect it. The algae covered only 1.5 m^2 on 3 October. They grow at a rate proportional to the quantity present. On 15 October they covered 4 m^2 . Use a differential equation to find an expression for the area covered after time t . How long will it take to entirely cover the 5 km^2 lake surface, if growth continues in this fashion?



NewsPix/Bob Barker

- 5 Bacteria grow by cell division, so the growth of a bacterial colony is proportional to the number present. In a deep wound, 20 anaerobic *Clostridium* bacteria are initially present. After 2 h, the number has grown to 90. Use a differential equation to determine the number present after time t . When the number present reaches 1000, the patient begins to develop symptoms of nerve poisoning from the toxins produced by the bacteria. How long does this take?
- 6 A milk sample is taken to check the level of certain bacteria. The sample was carried in the warm boot of a car for 6 h before it was checked. When checked, the level of bacteria present was found to be 200 times the acceptable amount. After a further hour, the level had risen even further to 250 times the acceptable amount. The dairy farmer from whom the sample originated claimed that, since the sample had been stored in ideal growing conditions before testing, the results were invalid. Assuming that the growth is proportional to the number of bacteria present, use a differential equation for the level at time t to find whether the sample was outside acceptable limits at the time it was taken.
- 7 **Example 11** An oven is heated at a constant rate of 3600 J s^{-1} (watts, W) from its initial temperature of 20°C . At the same time, the oven loses heat to its surroundings at the rate of $7.5(T - 20) \text{ W}$, where T is the temperature of the oven. Each 6000 J of energy retained by the oven increases the temperature by 1°C . Use a differential equation to find the temperature of the oven after 5 minutes, and the eventual temperature.



- 8 The rate at which a hot object cools is approximately proportional to the difference between its temperature and the temperature of the surroundings. A cup of black coffee is made with boiling water. After 2 minutes, its temperature has dropped to 80°C . The room is at 25°C . Use a differential equation to find how much longer it will take to drop to 50°C .
- 9 A furnace is switched on at 9 a.m. Heat is supplied at a constant rate; but as the furnace temperature increases, heat is lost at a rate determined by the difference between its temperature and the surrounding temperature. The temperature of both was 20°C at 9 a.m. At 9:30 a.m. the furnace was at 200°C and by 10 a.m. it was at 350°C . The minimum operating temperature is 800°C . Use a differential equation to find the time when the furnace was ready for use and the highest temperature it could be expected to reach.



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- 10 A dust particle of mass m falls according to the equation

$$m \frac{dv}{dt} = mg - kv$$

where m is the mass, g is the acceleration due to gravity, v is the velocity and k is a constant. Assuming that the particle falls from rest, solve the equation and comment on the solution.

8.05 FIRST ORDER DIFFERENTIAL EQUATIONS WITH SEPARABLE VARIABLES

You can separate the dependent and independent variables in the differential equation

$$\frac{dy}{dx} = (x-1)(y+3) \text{ by writing it as } \frac{1}{y+3} \frac{dy}{dx} = x-1.$$

IMPORTANT

A differential equation with **separable variables** can be rewritten so that the dependent and independent variables are separated. These are often of the form $\frac{dy}{dx} = f(x) \cdot g(y)$.

You can integrate both sides of $\frac{1}{y+3} \frac{dy}{dx} = x-1$ to give $\int \frac{1}{y+3} \frac{dy}{dx} dx = \int (x-1) dx$, so

$\int \frac{1}{y+3} dy = \int (x-1) dx$, which gives $\log_e(y+3) = \frac{1}{2}x^2 - x + c$ and the solution $y = Ae^{\frac{1}{2}x(x-2)} - 3$.

Strictly speaking, we should write the integral of $\frac{1}{y+3}$ as $\log_e|y+3|$. If y was less than -3 , you would

have to make the integral $\log_e(-y-3)$ but this would be obvious from the context.

○ Example 12

Solve $\frac{dy}{dx} = (2x-3)(3y+4)$.

Solution

Separate the variables.

$$\frac{1}{3y+4} \frac{dy}{dx} = 2x-3$$

Integrate both sides.

$$\int \frac{1}{3y+4} \frac{dy}{dx} dx = \int (2x-3) dx$$

Use integration by substitution.

$$\int \frac{1}{3y+4} dy = \int (2x-3) dx$$

It is easiest if the coefficient of y is 1.

$$\frac{1}{3} \int \frac{1}{y+\frac{4}{3}} dy = \int (2x-3) dx$$

Simplify.

$$\frac{1}{3} \log_e \left| y + \frac{4}{3} \right| = x^2 - 3x + c$$

Multiply both sides by 3 to simplify.

$$\log_e \left| y + \frac{4}{3} \right| = 3x^2 - 9x + 3c$$

Assume $y + \frac{4}{3} > 0$ and change form.

$$y + \frac{4}{3} = Ae^{3x(x-3)}, \text{ where } A = e^{3c}$$

Express in the usual way.

$$y = Ae^{3x(x-3)} - \frac{4}{3}$$

You still need some other information to find specific solutions to differential equations with separable variables.



○ Example 13

The rate at which a new social network expands is proportional to both the size of the network and the time since its establishment. Two weeks after the establishment of a new network, there are 30 members and after another week there are 120. How long it will take to pass the 10 000 member milestone?

Solution

Choose the variables.

Let m be the number of members and t = time in weeks

Write the rate equation for m .

$$\frac{dm}{dt} = kmt$$

Separate the variables.

$$\frac{1}{m} \frac{dm}{dt} = kt$$

Integrate both sides.

$$\int \frac{1}{m} \frac{dm}{dt} dt = \int kt dt$$

Simplify.

$$\log_e(m) = \frac{1}{2}kt^2 + c$$

Write $K = \frac{1}{2}k$ and change form.

$$m = Ae^{Kt^2}$$

Substitute in the known values.

$$30 = Ae^{4K} \text{ and } 120 = Ae^{9K}$$

Divide to find K .

$$e^{5K} = 4$$

Solve for K .

$$K = 0.2 \log_e(4)$$

Substitute in $t = 2$ to find A .

$$30 = Ae^{4 \times 0.2 \log_e(4)}$$

Find A .

$$A = 30 \div 4^{0.8} = 9.896\dots$$

Write the model.

$$m = 9.896\dots e^{0.2t \times \log_e(4)}$$

Substitute $m = 10\,000$.

$$10\,000 = 9.896\dots e^{0.2 \log_e(4) \times t}$$

Solve for t .

$$t = 24.952\dots$$

Write the answer.

It will take about 25 weeks to get to 10 000 members.

In some cases, you will need to use more sophisticated techniques to solve differential equations with separable variables.

○ Example 14

Solve $\frac{dp}{dq} = 2pq(p+3)$ if $p = 10$ when $q = 3$.

Solution

Separate the variables.

$$\frac{1}{p(p+3)} \frac{dp}{dq} = 2q$$

Integrate both sides with respect to q .

$$\int \frac{1}{p(p+3)} \frac{dp}{dq} dq = \int 2q dq$$

Use integration by substitution.

$$\int \frac{1}{p(p+3)} dp = \int 2q dq$$

Express $\frac{1}{p(p+3)}$ as partial fractions.

$$\frac{1}{p(p+3)} = \frac{A}{p} + \frac{B}{p+3}$$

Put the RHS over one denominator.

$$= \frac{A(p+3) + pB}{p(p+3)}$$

Solve for A and B .

$$A = \frac{1}{3} \text{ and } B = -\frac{1}{3}$$

Write the integration.

$$\frac{1}{3} \int \left(\frac{1}{p} - \frac{1}{p+3} \right) dp = \int 2q dq$$

Simplify.

$$\frac{1}{3} \log_e \left| \frac{p}{p+3} \right| = q^2 + c$$

Assume $\frac{p}{p+3} > 0$ and rearrange.

$$\frac{p}{p+3} = Ae^{3q^2}, \text{ where } A = e^{3c}$$

Make the dependent variable p the subject.

$$p = Ape^{3q^2} + 3Ae^{3q^2}$$

Isolate p .

$$p(1 - Ae^{3q^2}) = 3Ae^{3q^2}$$

Solve for p .

$$p = \frac{3Ae^{3q^2}}{1 - Ae^{3q^2}} = \frac{3}{Be^{-3q^2} - 1}, \text{ where } B = \frac{1}{A}$$

Substitute in the known values.

$$10 = \frac{3}{Be^{-27} - 1}$$

Solve for B .

$$B = 6.916... \times 10^{11}$$

Write the answer.

$$p = \frac{3}{Be^{-3q^2} - 1}, \text{ where } B \approx 6.92 \times 10^{11}$$

You could leave the solution in the A form with $A = 1.445... \times 10^{-12}$, but the B expression is considered 'more elegant' for calculation of values.



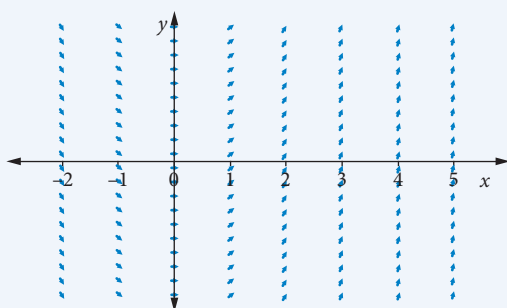
INVESTIGATION Gradient fields

Another way of visualising solutions to differential equations is to use a **gradient field**. A gradient field shows what a solution will look like by showing gradients of the function as short lines or arrows at multiple locations.

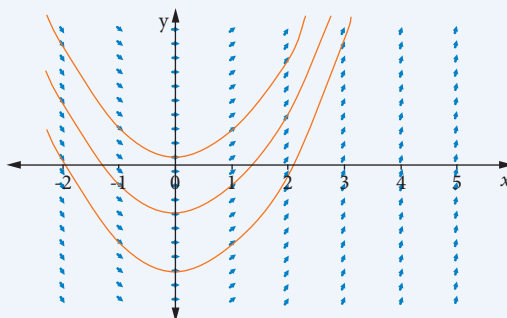
Consider the equation $\frac{dy}{dx} = 0.8x$. Find some values of the derivative.

x	-2	-1	0	1	2	3	4	5
$\frac{dy}{dx}$	-1.6	-0.8	0	0.8	1.6	2.4	3.2	4

The derivative is the slope of the tangent, so although you don't know what the value of y is, you can show the derivatives as short lines or arrows with the slope equal to the derivative.



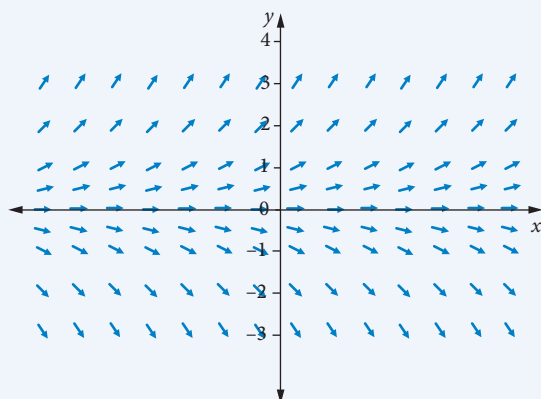
You can follow the arrows to join up possible curves.



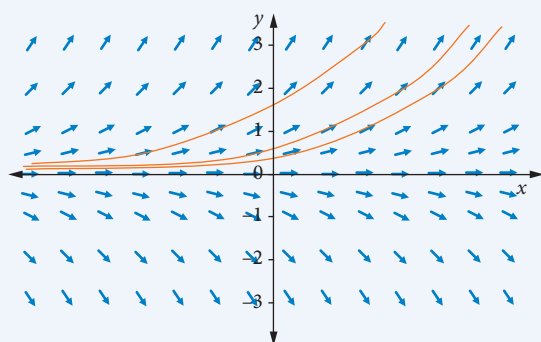
The curves that you get form a family of possible functions. It is hardly surprising that they look like parabolas.

Investigate the shapes that you get for the differential functions $\frac{dy}{dx} = 0.5 + 0.5x$ and $\frac{dy}{dx} = 0.6 - 0.6x$.

You can use the same technique, even when there is no independent variable on the RHS of the differential equation. Consider the differential equation $\frac{dy}{dx} = 0.5y$. You can draw a chart for $\frac{dy}{dx}$ against y -values and a gradient field by leaving the x -axis blank to make the arrows across the page the same slopes, as shown below.



Connecting these arrows in the same way as before gives a family of curves that look exponential. It is obvious that the x -axis is an asymptote.



Investigate the shapes you get for the differential functions $\frac{dy}{dx} = y + 0.5$ and $\frac{dy}{dx} = 1 - 0.6y$.
If you are really adventurous, try $\frac{dy}{dx} = xy$.



EXERCISE 8.05 First order differential equations with separable variables



Differential equations

Concepts and techniques

- 1 **Example 12** Solve $\frac{dy}{dx} = y(x - 3)$.
- 2 Solve $\frac{dg}{dx} = (4g - 7)(3x + 1)$.
- 3 **Example 13** Solve $\frac{dP}{dV} = P^2V$ if $P = 1$ when $V = 1$.
- 4 Given $v = 1$ at $t = 0$, solve $\frac{dv}{dt} = (4v - 1)(3t + 5)$.
- 5 Solve $\frac{dy}{dx} = x + xy$.

Reasoning and communication

- 6 **Example 14** Solve $\frac{df}{dt} = f(3f + 2)t$ if the initial value of f is 6.
- 7 Solve $\frac{dy}{dx} = \frac{(1+x^2)(1+y^2)}{xy}$
- 8 Solve $\frac{dy}{dx} + 2y = 3$ if $y = 10$ when $x = 0$
- 9 $\frac{dy}{dx} = xe^{x+y}$
- 10 $e^{2x+y} - 2e^{x-y} \frac{dy}{dx} = 0$

8.06 THE LOGISTIC EQUATION

The population of an animal introduced to a new environment will at first appear to grow exponentially. However, the population will asymptotically approach a maximum after some time because of the limitations of the resources in the environment. In this case, the rate of change of the population is proportional to two conflicting factors. Breeding will give a factor of kN , as the more there are, the greater the growth through breeding.



Alamy/Kunst and Schedulin

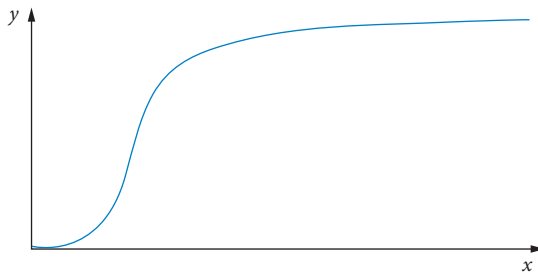
As the population grows, the growth will also be proportional to $M - N$, where M is the maximum population that can be sustained by the environment. This means that the differential equation governing the rate of growth will be

$$\frac{dN}{dt} = kN(M - N)$$

For a continuous variable y with a maximum possible value this becomes

$$\frac{dy}{dt} = ky(M - y).$$

This is called the **logistic model**. The typical graph of such a model is shown on the right.



IMPORTANT

The **logistic model** of growth is given by $\frac{dy}{dt} = ky(M - y)$, where the rate of change of a variable is proportional to both the variable and the difference between a maximum value and the variable.

INVESTIGATION Epidemics

The spread of an infection through a population can be modelled by using N for the infected population, M for the susceptible population and k as a constant related to the infectiousness of the disease. The logistic model of the rate of change of the number of people infected will be given by $\frac{dN}{dt} = kN(M - N)$.

Suppose that a population of 100 000 people are not vaccinated (not immune) against a disease and that 100 people are initially infected, with $k = 0.000\ 01$ for the time in weeks.

Plot the number infected over a period of 10 weeks.

Suppose that 20% of the population are vaccinated and k is reduced by 20% because contacts between infected and susceptible people are also reduced. Rework the model and comment on the changes.

Do this again with vaccination rates of 40%, 60%, 80% and 95% and comment on the results.



○ Example 15

In a town of 5000 people, 10 people have a new strain of cold. The spread of the cold through the town is given by the differential equation $\frac{dN}{dt} = 0.000\ 24N(5000 - N)$, where t is in weeks and N is the number of people infected. Find an expression for the number of people infected and hence find the numbers infected after 2 and 4 weeks.

Solution

Write the differential equation.

$$\frac{dN}{dt} = 0.000\ 24N(5000 - N)$$

Separate the variables.

$$\frac{1}{N(5000 - N)} \frac{dN}{dt} = 0.000\ 24$$

Integrate both sides.

$$\int \frac{1}{N(5000 - N)} \frac{dN}{dt} dt = \int 0.000\ 24 dt$$

Use integration by substitution.

$$\int \frac{1}{N(5000 - N)} dN = \int 0.000\ 24 dt$$

Write $\frac{1}{N(5000 - N)}$ as partial fractions.

$$\frac{1}{N(5000 - N)} = \frac{A}{N} + \frac{B}{5000 - N}$$

Put the fractions on the RHS over a common denominator and equate denominators.

$$A(5000 - N) + BN = 1$$

Substitute $N = 0$ and $N = 5000$ to solve.

$$A = B = \frac{1}{5000} = 0.0002$$

Substitute into the integrals.

$$\int \left[\frac{1}{5000N} + \frac{1}{5000(5000 - N)} \right] dN = \int 0.000\ 24 dt$$

Simplify.

$$\int \frac{1}{N} dN + \int \frac{1}{5000 - N} dN = 5000 \times \int 0.000\ 24 dt$$

Perform the integrations.

$$\log_e(N) - \log_e(5000 - N) = 5000 \times 0.000\ 24t + c$$

Simplify.

$$\log_e \left(\frac{N}{5000 - N} \right) = 1.2t + c$$

Change to exponential form.

$$\frac{N}{5000 - N} = Ae^{1.2t}$$

Solve for N .

$$N = \frac{5000Ae^{1.2t}}{1 + Ae^{1.2t}}$$

Multiply top and bottom by $e^{-1.2t}$.

$$\begin{aligned} &= \frac{5000Ae^{1.2t}}{1 + Ae^{1.2t}} \times \frac{e^{-1.2t}}{e^{-1.2t}} \\ &= \frac{5000A}{A + e^{-1.2t}} \end{aligned}$$

Substitute in $t = 0$ and $N = 10$.

$$10 = \frac{5000 \times A}{A + 1}$$

Solve for A .

$$A = \frac{10}{5000 - 10} = \frac{1}{499} = 0.002\,004\dots$$

Write the model.

$$N = \frac{5000A}{A + e^{-1.2t}}, \text{ where } A = \frac{1}{499}$$

Find the value at $t = 2$.

$$N(2) = 108.065\dots$$

Find the value at $t = 4$.

$$N(4) = 979.116\dots$$

Write the answer.

The number of people infected is given by

$N = \frac{5000A}{A + e^{-1.2t}}$, where $A = \frac{1}{499}$, and the numbers infected were 108 after 2 weeks and 979 after 4 weeks.

The general solution for the logistic model can be derived in exactly the same way as the solution in Example 15.

IMPORTANT

The general solution for the logistic model given by $\frac{dy}{dt} = ky(M - y)$ is

$$y = \frac{MA}{A + e^{-kMt}} \quad \text{or} \quad y = \frac{My_0}{y_0 + (M - y_0)e^{-kMt}}$$

where $A = \frac{y_0}{M - y_0}$ and y_0 is the initial value.



○ Example 16

The maximum number of possums that can be supported by the environment of a national park is estimated to be 3000. When first counted, there were found to be 1200 possums, but after 4 months the number has grown to 1300. Use the logistic model to predict the number of possums after 1 year.



Corbis/Nature Connect

Solution

Write the logistic solution.

$$N = \frac{MN_0}{N_0 + (M - N_0)e^{-kMt}}$$

Substitute in $M = 3000$, $N_0 = 1200$, $t = 4$ and $N = 1300$.

$$1300 = \frac{3000 \times 1200}{1200 + (3000 - 1200)e^{-k \times 3000 \times 4}}$$

Solve for e^{-3000k} .

$$e^{-3000k} = 0.966\,281\dots$$

Write the model.

$$N = \frac{3\,600\,000}{1200 + 1800(0.966\dots)^t}$$

Substitute $t = 12$.

$$\begin{aligned} N &= \frac{3\,600\,000}{1200 + 1800(0.966\dots)^{12}} \\ &= 1504.603\dots \end{aligned}$$

Write the answer.

After 12 months, the logistic model predicts that there will be 1504 possums in the park.

It is sometimes easier to find e^{-k} in a logistic model, as shown in Example 16, to avoid some of the very small or large numbers that can occur.

EXERCISE 8.06 The logistic equation

Reasoning and communication

- 1 **Example 15** A new viral disease was found to spread according to the equation $\frac{dN}{dt} = kN(M - N)$, where M is the susceptible population, N is the number of people infected at time t months and $k = 1.5 \times 10^{-9}$. In March 2010, it was thought that only 100 people out of a population of 18 million were infected. Use the logistic model to find the number infected in:
- a March 2011 b June 2012 c January 2017

- 2 **Example 16** A remote island with no feral animals is used to set up a breeding population of bilbies. Initially, 10 bilbies are released on the island, which has an ideal environment and is thought to be able to sustain a population of 600 bilbies. After 2 years, the population has grown to 30 bilbies. Use the logistic model to find:



Auscape/Glen Threlfo

- a when the population will reach 200
b the number of bilbies that could be taken back to the mainland in each following year while maintaining the population at 200.

- 3 A poison is dissolved in honey and set as ant baits to get rid of a nest of fire ants, estimated to have 10 000 ants. When the bait is set, 10 ants are observed to be killed immediately. The nest is observed after 1 week, when the baits are renewed, and the activity is seen to have reduced by 1%. Use the logistic model for the number of ants killed to find how long the baits must be used to ensure that the nest is exterminated. It is considered that the nest will be exterminated when the population is below 50, as this is the number needed to maintain the queen.



Shutterstock.com/Natthilajean

- 4 A phone company knows that about 40% of mobile phone owners will buy a phone that has new features whether or not they are satisfied with their current phone. Other mobile users are unlikely to change unless there is something wrong with their existing phone. Past experience suggests that initial sales will be about 1% of the market, with market penetration of 5% after 4 weeks. They estimate that they have a period of 9 weeks before their competitors bring out a phone with the new features. How much of the market can they expect to capture before this happens?
- 5 An area in the Channel Country can support about 3000 rabbits in dry times, but after a flood it can support about 30 000. The Department of Primary Industries (DPI) is interested in predicting the population in the 12 months following a flood. It is estimated that, if no control measures are taken, the population will grow to 5000 in the first month; but if normal control measures (trapping, shooting, poison) are taken, the population will grow to 3500 in the first month.
 - a Use the logistic model to determine the population each month after a flood with no control measures.
 - b Use the model to find the population each month with normal control measures.
 - c Draw a graph, showing both situations on the same axes.
 - d The DPI estimates that, if the population remains below 10 000 in the year following a flood, no serious damage will occur. Comment on your results.
- 6 It has been said that the maximum population that Australia can support is 50 million. The population reached 15 million in 1981 and 21 million in 2008. Use the logistic model to predict the population in:

a 2015	b 2020	c 2030	d 2050
- 7 The logistic model can be used to predict the death rates from a toxin. The LD-50 quantity is the amount that must be given to kill 50% of the subjects. A particular poison is tested on laboratory mice for effectiveness as a control for house mice. The LD levels were found to be 2 g for LD-50 and 3 g for LD-70. The maximum level (LD-100) is theoretically unobtainable.
 - a What quantity of toxin would be needed for:

i LD-90?	ii LD-95?	iii LD-99?
 - b What percentage of mice would have died with no poison?
- 8 A 'whisper campaign' regarding the financial affairs of an election candidate is started 4 days before the election. There are 33 420 voters in the electorate and the campaign is started by about 50 voters loyal to another candidate. After 1 day, it is estimated that 400 voters are aware of the rumour. How many people will hear the rumour before the election?
- 9 The completion of a particular chemical reaction involving a catalyst follows the logistic model. The reaction between molecules of the reactants occurs when they are adsorbed onto the catalyst sufficiently close to one another to react. As the reaction proceeds, this is less and less likely to occur. When first measured, 5% of the reaction has taken place; and after 5 minutes, another 5% has taken place. How long is it until the reaction proceeds to completion (99% reacted)?
- 10 A new service providing mini-gardens planted with vegetables was planned for inner Melbourne residents with small outdoor spaces. The service was expected to serve a maximum of 2000 residents in the suburbs where it was to be offered. A survey in Fitzroy found that 30 residents would be interested in taking the service immediately with another 10 in a fortnight's time. The rent for each garden supplied is to be \$20 per week. Use the logistic model to determine how long it would take to reach the operational break-even point of \$2000 a week in rents.

8.07 FORCE, MOMENTUM AND MOTION

Everybody has an intuitive understanding that movement does not happen without a reason. To make something move, you push or pull it in some way. You also know that ‘light’ objects respond more to the same push than ‘heavy’ objects. These ideas were formalised by Sir Isaac Newton (1642–1727). His first law of motion is essentially a definition of **force** as that ‘something’ that causes motion or changes a motion.

IMPORTANT

Newton’s first law of motion

Unless acted on by a resultant **force**, a body remains at rest or in uniform motion in a straight line.

Uniform motion means speed, so the law says that objects will either remain at rest or keep moving with constant velocity unless they are acted upon by a force. The term ‘resultant’ just means that balanced forces will not change an object’s motion. In this section, the words *body* and *particle* will be used interchangeably to describe an object. Forces are presumed to act through the centre of the object. The concept of ‘light’ and ‘heavy’ objects is encapsulated in the idea of inertial **mass**. If the same force produces twice the acceleration on one object compared to another, then you say that the mass of the object that accelerates faster is half that of the slower object.

IMPORTANT

Newton’s second law of motion

The acceleration of a body is directly proportional to the resultant force and inversely proportional to the **mass** of the object. The acceleration is in the direction of the resultant force. Units are chosen so that this law becomes $\mathbf{F} = m\mathbf{a}$, where $\mathbf{F}$ and $\mathbf{a}$ are vectors and m is a scalar.

The second law is really a definition of mass, but it does not define a unit. The international unit of mass is defined as the mass of the international prototype of the kilogram. This is made of platinum-iridium and kept in Paris under strict conditions. Copies of it are kept in National Standards Laboratories around the world. When several forces are acting at the same time, the motion is determined by the vector sum of the forces.

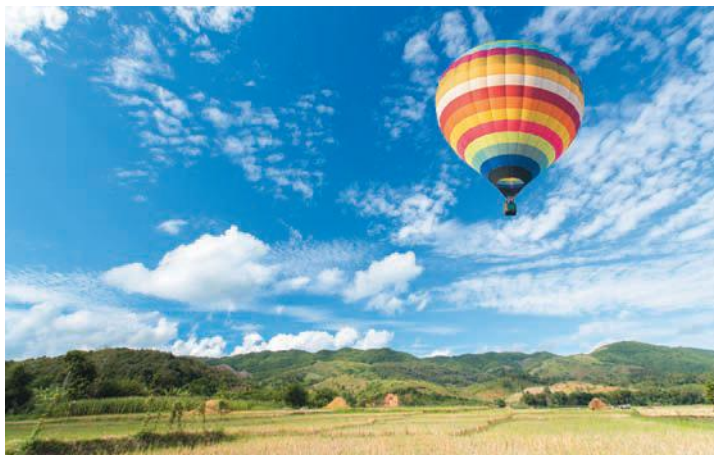
The SI unit of force is the newton (N), which is the force needed to accelerate 1 kg at 1 m/s/s.



Example 17

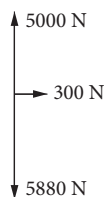
The vent in a hot-air balloon of mass 600 kg is opened to begin descent from a stationary position 120 m above level ground. Gravity is exerting a force of 5880 N downwards, the lift provided by the hot air is 5000 N upwards and a wind springs up and drives it eastwards with a force of 300 N.

- What is the acceleration of the balloon?
- How long will it take to reach the ground?
- At what speed will it hit the ground?



Solution

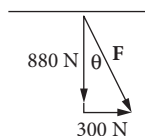
- Draw a labelled diagram of the forces.



Find the resultant downward force.

Downward force = 880 N

Draw a labelled diagram to find the total force F .



Find the magnitude of F .

$$F = \sqrt{880^2 + 300^2}$$

$$= 929.7... \text{ N}$$

Find the angle that F makes with the vertical.

$$\tan(\theta) = \frac{300}{880}$$

Solve for θ .

$$\theta = 18.82...^\circ$$

Write Newton's second law.

$$F = ma$$

Substitute and find a .

$$929.7... = 600a, \text{ so } a = 1.549... \text{ m/s}^2$$

State the answer.

The balloon will accelerate downwards at about 19° to the vertical at about 1.55 m/s^2 .

- b Find the vertical component of the acceleration.

$$a_v = a \cos(\theta)$$

Write the equation for the vertical velocity.

$$a = \frac{dv}{dt}, \text{ so } v = at + c$$

Use the stationary start.

$$\text{At } t = 0, v = 0, \text{ so } c = 0$$

Write the equation for the height.

$$\frac{dh}{dt} = a_v t, \text{ so } h = \frac{1}{2} a_v t^2 + c$$

Use $h = 120$ at $t = 0$ to find c .

$$c = 120$$

Write the equation to find t when $h = 0$.

$$0 = \frac{1}{2} a_v t^2 + 120$$

Find t (with acceleration downwards).

$$t = \sqrt{\frac{-120}{\frac{1}{2} \times (-1.549 \dots) \times \cos(18.82 \dots^\circ)}} \\ = 12.79 \dots \text{s}$$

Write the answer.

The balloon takes about 12.8 s to reach the ground.

- c Use the velocity equation.

$$v = at$$

Substitute in the values.

$$= -1.549 \dots \times 12.79 \dots \\ = -19.8 \dots \text{ m/s}$$

Write the answer.

The balloon will hit the ground at about 20 m/s.

A prudent hot-air balloon operator might close the vent some more in Example 17 as 20 m/s is about 70 km/h, which is perhaps a little fast for his passengers.

In addition to defining mass, it is useful to define **momentum**, as this also gives us an idea of the kinds of forces needed to change motion.

IMPORTANT

The **momentum**, $\mathbf{p}$, of a body is defined as the product of its mass and velocity. $\mathbf{p} = m\mathbf{a}$ is a vector. Newton's second law of motion can also be expressed as $\mathbf{F} = \frac{d\mathbf{p}}{dt}$

The equivalence is easily shown as $\frac{d\mathbf{p}}{dt} = \frac{d(m\mathbf{v})}{dt} = m \frac{d\mathbf{v}}{dt} = m\mathbf{a}$.

Newton completed his three laws of motion with the following.

IMPORTANT

Newton's third law of motion

For every action there is an equal and opposite reaction.

This means that forces occur in opposite pairs. It is an expression of the fact that you cannot push something without something to push against. It is also equivalent to saying that the total momentum of a system remains constant, or that momentum is conserved.



○ Example 18

An eight-ball with a mass of 0.12 kg struck the cushion directly on at 3 m/s and rebounded at 2 m/s. The collision with the cushion took 2 ms. What was the average force that the ball exerted on the cushion?

Solution

Find the change of momentum.

$$\begin{aligned}\Delta p &= 0.12 \times 2 - 0.12 \times (-3) \text{ kg m/s} \\ &= 0.6 \text{ kg m/s}\end{aligned}$$

Find the average rate of change of momentum.

$$\frac{\Delta p}{\Delta t} = \frac{0.6}{0.002} = 300 \text{ N}$$

Write the answer.

The average force exerted on the cushion was 300 N.

The ball in Example 18 exerted a force on the cushion, which is part of the table, so why doesn't the table move in this situation? The reason is that the Earth-ball-table system is essentially a closed system. When the player strikes the ball, a force is exerted on the ball and an equal and opposite force by the player's feet on the Earth, so the Earth moves back a miniscule amount. When the ball hits the cushion, it moves back the other way. Eventually the whole system ends up back in the same place, except for a slight change in the position of the Earth-table-player to compensate for the change of the ball's position. However, for someone outside the Earth, there is no overall change as they cancel out.

EXERCISE 8.07 Force, momentum and motion

Concepts and techniques

- Example 17** A body of mass 20 kg is acted on by two forces, $\mathbf{P} = 12\mathbf{i} - 4\mathbf{j}$ and $\mathbf{Q} = 20\mathbf{i} + 32\mathbf{j}$, where both forces are in newtons. Calculate the magnitude and direction of the acceleration.
- A block of mass 3 kg is acted on by a force $\mathbf{R} = 2\mathbf{i} + 5\mathbf{j}$ N. Calculate the magnitude and direction of the acceleration.
- A uniform force acts on a box with a mass of 5 kg and causes the box to accelerate at $(4\mathbf{i} + 8\mathbf{j}) \text{ m s}^{-2}$. Find the magnitude and direction of the force acting on the box.
- A force of 19.6 N acts on a brick of mass 2.4 kg for 5 s. Find:
 - the acceleration
 - the speed after 5 s
 - the distance travelled in this time.
- Example 18** A force of 50 newtons acts on an object of mass 8 kg, that is initially at rest, for a period of 4 seconds.
 - What is the change of momentum?
 - What is the speed of the object after the force ceases to act?
- A car of mass 1.1 tonnes (with passengers) travelling in a straight line at 100 km/h slows down to 46 km/h in a period of 4 seconds. What force was applied by the brakes?
- A force of 200 newtons acts in an easterly direction on a mass of 20 kg moving north at 10 m/s for 1.5 seconds.
 - Find the change of momentum of the mass.
 - Find the new velocity of the mass.

Reasoning and communication

- 8 A wire is stretched between two poles at a distance of 50 m apart. The tension on the wire is 400 N. A bird lands in the middle of the wire, making it sag a distance of 20 mm in the middle so that the tension increases to 1500 N. The original and new positions of the wire can be considered to be an isosceles triangle. What force does the weight of the bird exert on the wire? [Hint: the bird's weight is balanced by the extra tension acting in both directions]
- 9 A coal train weighing 23 000 tonnes is moving east at 54 km/h. Without changing speed, the train negotiates a curve over 2 minutes to moving in the ENE direction. Assuming that the force was constant, what force was required to change the train's direction?
- 10 Three identical tennis balls of mass 57 g are squashed together on flat concrete and held in place with a piece of string. The string is released by burning it through and the tennis balls move away from each other across the concrete. One of the balls goes off at 5 m/s northwards and a second one goes off at 6.5 m/s on a bearing of 120° .
- What is the change in momentum of the first ball from the original position?
 - What is the change in momentum of the second ball?
 - At what speed and in what direction does the third ball move off in?

8.08 MOTION IN A STRAIGHT LINE WITH CONSTANT FORCE

If motion is in a straight line, you can use the sign (+/−) of the acceleration a , velocity v , and position or displacement s for the direction. You do not need to write these as the vectors $\mathbf{a}$, $\mathbf{v}$ or $\mathbf{s}$.

There are a number of situations where a constant force is applied to an object, such as the force of gravity near the surface of the Earth. In this case, the acceleration is constant, so you can

write $\frac{dv}{dt} = a$, where a is the (constant) acceleration.

Then you get $v = at + c$. For an initial velocity of v (when $t = 0$) you get $v = u + at$.

You can then write $s = \int (u + at) dt = ut + \frac{1}{2}at^2 + c$.

If the initial position is s_0 , you get $s = s_0 + ut + \frac{1}{2}at^2$.

If $s_0 = 0$ or s is considered as displacement from the initial position, then you get $s = ut + \frac{1}{2}at^2$.

You can rewrite $v = u + at$ as $a = \frac{v-u}{t}$ and substitute it in $s = ut + \frac{1}{2}at^2$ to get

$$s = ut + \frac{1}{2} \times \frac{v-u}{t} \times t^2$$

$$= ut + \frac{1}{2}(v-u)t$$

$$s = \frac{1}{2}(v+u)t \text{ or } s = \left(\frac{v+u}{2}\right)t.$$

Then you can multiply $a = \frac{v-u}{t}$ and $s = \left(\frac{v+u}{2}\right)t$ to get $as = \frac{v-u}{t} \times \frac{v+u}{2} \times t$.

The ts cancel to give $as = \frac{v^2 - u^2}{2}$ or $v^2 = u^2 + 2as$.



Equations for motion in a straight line with constant force

For constant acceleration a , velocity v , initial velocity u , time t and displacement s , you can write each of the following equations for motion in a straight line.

$$\begin{aligned} v &= u + at & s &= ut + \frac{1}{2}at^2 \\ s &= \left(\frac{v+u}{2}\right)t & v^2 &= u^2 + 2as \end{aligned}$$

It is useful to be able to change from m/s to km/h and vice versa. The conversion can be worked out as $1 \text{ m/s} = 3600 \text{ m/h} = 3.6 \text{ km/h}$, so $1 \text{ m/s} = 3.6 \text{ km/h}$.

○ Example 19

A car travelling at 80 km h^{-1} is brought to rest in a distance of 50 m. The brakes are applied uniformly.

- What is the deceleration experienced by the car?
- Find the time taken for the car to stop.

Solution

- Change the speed to m/s.

$$80 \text{ km/h} = 80 \div 3.6 \text{ m/s}$$

Calculate the answer.

$$= 22.222... \text{ m/s}$$

Write the equation with speed, distance and acceleration.

$$v^2 = u^2 + 2as$$

Substitute values.

$$0^2 = (22.222...)^2 + 2a \times 50$$

Solve for a .

$$a = -4.938... \text{ m/s}^2$$

Write the answer.

The deceleration is about 4.94 m/s^2 .

- Choose the best equation.

$$s = \left(\frac{v+u}{2}\right)t$$

Substitute in the values.

$$50 = \left(\frac{0 + 22.222...}{2}\right)t$$

Solve for t .

$$t = 4.5 \text{ s}$$

Write the answer.

The car takes 4.5 s to stop.

The acceleration of gravity near the surface of the Earth is generally taken to be -9.8 m/s^2 , with the negative sign indicating the direction down.

○ Example 20

A stone is thrown vertically upward and is seen passing a point 10 m high on the face of a building 3 s after being thrown. What was the initial velocity of the stone?

Solution

Write the acceleration.

$$a = -9.8 \text{ m/s}^2$$

Choose the equation with s , t , a and u .

$$s = ut + \frac{1}{2}at^2$$

Substitute in the values.

$$10 = u \times 3 + \frac{1}{2} \times (-9.8) \times 3^2$$

Solve for u .

$$u = 18.033\dots$$

Write the answer.

The initial velocity was about 18 m/s.

There is often more than one way to solve a problem.

○ Example 21

A cyclist travelling at a constant speed of 18 km/h passes a truck just as it begins to move in the same direction. The truck accelerates uniformly at 0.4 m/s^2 for 20 s and then continues with uniform speed. How far will the truck travel before reaching the cyclist?

Solution

Change the cyclist's speed to m/s.

$$\text{Cyclist's speed} = 18 \div 3.6 = 5 \text{ m/s}$$

Find the cyclist's position after 20 s.

$$s_C = 20 \times 5 = 100 \text{ m}$$

Find the truck's position after 20 s.

$$s_T = ut + \frac{1}{2}at^2$$

Substitute in the values.

$$\begin{aligned} &= 0 + \frac{1}{2} \times 0.4 \times 20^2 \\ &= 80 \text{ m} \end{aligned}$$

Find their separation.

$$\text{Distance between them} = 100 - 80 = 20 \text{ m}$$

Find the truck's speed.

$$\begin{aligned} \text{Truck's speed} &= u + at \\ &= 0 + 0.4 \times 20 \end{aligned}$$

Calculate the value.

$$= 8 \text{ m/s}$$

You can use relative speeds.

Method 1

Find the relative speed.

$$\begin{aligned} \text{Speed of truck relative to cyclist} &= 8 - 5 \\ &= 3 \text{ m/s} \end{aligned}$$

Find the time to catch up.

$$\text{Time to catch up} = \frac{20}{3} = 6\frac{2}{3} \text{ s}$$

You can also use time.

Method 2

Write the positions t seconds later.

$$\text{Cyclist's position} = 20 + 5t; \text{ Truck} = 8t$$

Find when they coincide.

$$20 + 5t = 8t$$



Solve for t .

$$t = 6\frac{2}{3} \text{ s}$$

Find the position of the truck.

$$\begin{aligned}\text{Truck's position} &= 80 + 6\frac{2}{3} \times 8 \\ &= 133\frac{1}{3} \text{ m}\end{aligned}$$

State the answer.

The truck travels $133\frac{1}{3}$ m altogether before reaching the cyclist.

EXERCISE 8.08 Motion in a straight line with constant force

Concepts and techniques

- 1 **Example 19** A car travelling at 54 km/h comes to rest with uniform retardation in 5 s. Calculate the acceleration and the distance travelled in the time taken for the car to be brought to rest.
- 2 A motor vehicle increases its speed from 18 km/h to 72 km/h in a distance of 50 m under uniform acceleration. Calculate:
 - a the acceleration
 - b the speed of the vehicle when 25 m has been covered.
- 3 A particle with an initial velocity of 100 m/s experiences a retardation of 2 m/s^2 . How long will it take to bring the particle to rest, and how far will it travel in this time?
- 4 A monorail train begins from rest and travels with a uniform acceleration of 1.2 m/s^2 . What will be the speed of the monorail after 30 s, and how far will it have travelled?
- 5 An object moves with uniform acceleration for 3 s, in which time it travels 27 m. It then moves with uniform velocity for the next 5 s, covering 60 m. Calculate:
 - a the initial velocity
 - b the acceleration in the first part of its journey.
- 6 **Example 20** A carriage of a train moves in a straight line with uniform acceleration.
 - a How far does the carriage travel in 12 s if it moves 48 m in the first 6 s of its motion and 32 m in the last 2 s?
 - b What is the initial velocity?

Reasoning and communication

- 7 **Example 21** Cyclist X, riding at 16 km/h is overtaken and passed by cyclist Y, who is travelling at 20 km/h. At this point, X immediately increases speed with uniform acceleration. Show that X will catch Y when she reaches a speed of 24 km/h.
- 8 A roller-coaster begins from rest and moves with uniform acceleration. It travels 9.5 m in the tenth second after starting its trip. Find:
 - a the acceleration
 - b the total distance covered during the first 5 s of motion.

- 9 A cage carrying workers down a mine shaft completes the 675 m descent in 45 s. During the first quarter of the time, the cage is uniformly accelerated, while in the last quarter it is uniformly retarded, the acceleration and retardation being equal in magnitude. Calculate the uniform speed of the cage in the central portion of its descent.
- 10 A ball thrown upwards from the top of a cliff took 2 seconds longer to fall to the bottom of the cliff than it did to reach the top of its 8 second flight. How high is the cliff?
- 11 A rower has a stronger right arm so the force exerted by his right oar on the boat is an average of 60 N in a direction 20° to the left of the forward direction, while the force exerted by his left oar is 50 N at 20° to the right of the forward direction. The friction with the water is 50 N towards the rear. The rower and boat together have a mass of 280 kg.
- What is the acceleration of the boat?
 - What is its speed a minute after starting?
 - How far off course is the boat a minute after starting?
- 12 The force produced by the wind on a yacht of mass 1.4 t as it starts tacking into the wind is 60 000 N at 120° to the direction of travel.
- What force must be produced by the keel and rudder for it to move forward?
 - The drag on the yacht from the water is 20 000 N. What is the initial acceleration?

8.09 MOTION IN A STRAIGHT LINE WITH VARIABLE FORCES

Most real forces are not constant, so the accelerations they produce are not constant either. Some forces depend on the velocity of an object, some depend on distance and some depend on time. Those that depend on time are relatively easy to deal with using integration.

○ Example 22

A 5 kg body initially at rest is acted on by a force that has a magnitude that varies with time according to the expression $40t - 6t^2$, where t is in seconds. If the direction of the force remains constant, calculate the velocity of the body after 5 s.

Solution

Use Newton's second law.

$$F = ma$$

Substitute in the values.

$$40t - 6t^2 = 5a$$

Find the acceleration.

$$a = 8t - 1.2t^2$$

Integrate to find the velocity.

$$v = \int a dt = \int (8t - 1.2t^2) dt$$

Simplify.

$$v = 4t^2 - 0.4t^3 + c$$

Use $v = 0$ at $t = 0$.

$$c = 0 \text{ so } v = 4t^2 - 0.4t^3$$

Substitute in $t = 5$.

$$v = 4 \times 5^2 - 0.4 \times 5^3$$

Calculate v .

$$= 50 \text{ m/s}$$

Write the answer.

After 5 s, the velocity is 50 m/s.



In cases where the force and acceleration depend on distance or velocity, it is useful to have calculus expressions that do not involve time.

Using the chain rule, you can write

$$a = \frac{dv}{dt} = \frac{dv}{dx} \times \frac{dx}{dt}$$

But $v = \frac{dx}{dt}$, so $a = v \frac{dv}{dx}$. Using the chain rule in reverse, you get

$$v \frac{dv}{dx} = \frac{d\left(\frac{1}{2}v^2\right)}{dx} \times \frac{dx}{dv} = \frac{d\left(\frac{1}{2}v^2\right)}{dv}$$

IMPORTANT

The acceleration of an object may be expressed in any of the forms

$$a = \frac{dv}{dt} = v \frac{dv}{dx} = \frac{d\left(\frac{1}{2}v^2\right)}{dx}$$

○ Example 23

A particle moving in a straight line has an acceleration of $5 - 2x$, where x is the position at time t . The velocity at $x = 3$ is 4. Find an expression for the velocity.

Solution

Write the appropriate expression for a .

$$a = \frac{d\left(\frac{1}{2}v^2\right)}{dx}$$

Use integration.

$$\begin{aligned} \frac{1}{2}v^2 &= \int a \, dx \\ &= \int (5 - 2x) \, dx \\ &= 5x - x^2 + c \end{aligned}$$

Substitute in the values to find c .

$$\frac{1}{2} \times 4^2 = 5 \times 3 - 3^2 + c$$

Solve for c .

$$c = 2$$

Write the expression for v .

$$\frac{1}{2}v^2 = 5x - x^2 + 2$$

Find v .

$$v = \sqrt{10x - 2x^2 + 4}$$

Write the answer.

$$\text{The velocity is given by } v = \sqrt{10x - 2x^2 + 4}$$

Friction in a medium like air depends on the velocity of the object. It is a quadratic relationship, but for larger objects it can be considered as linear, while for very small objects it is proportional to v^2 . As speed increases, the total force decreases because the air friction gets closer in magnitude to the forward force. Planes can produce only a limited forward force from their engines, so eventually the drag is equal to the forward force and the plane stops accelerating. Objects falling under the influence of gravity will eventually reach a **terminal velocity**, where the force of gravity is balanced by the force due to air friction.

○ Example 24

A person free-falling has air friction of $F = kv$ acting to retard their motion. For an 85 kg person of normal body shape, the value of k is about 16. What is the terminal velocity?

Solution

Use Newton's 2nd law for gravity.

Force due to gravity = mg , where $g = 9.8 \text{ m/s}^2$

Write the total force with an open parachute.

Total force = $mg - kv$

Use Newton's 2nd law for the downward acceleration.

$$mg - kv = ma$$

Solve for a .

$$a = g - \frac{kv}{m}$$

Write as a derivative.

$$\begin{aligned} \frac{dv}{dt} &= g - \frac{kv}{m} \\ &= -\frac{k}{m} \left(v - \frac{mg}{k} \right) \end{aligned}$$

Isolate v .

$$\frac{1}{\left(v - \frac{mg}{k} \right)} \frac{dv}{dt} = -\frac{k}{m}$$

Integrate both sides with respect to time.

$$\int \frac{1}{\left(v - \frac{mg}{k} \right)} \frac{dv}{dt} dt = \int -\frac{k}{m} dt$$

Use integration by substitution.

$$\int \frac{1}{\left(v - \frac{mg}{k} \right)} dv = \int -\frac{k}{m} dt$$

Perform the integrations.

$$\log_e \left(v - \frac{mg}{k} \right) = -\frac{k}{m} t + c$$

Express as an exponential.

$$v - \frac{mg}{k} = Ae^{-\frac{kt}{m}}$$

Simplify.

$$\begin{aligned} v &= \frac{mg}{k} + Ae^{-\frac{kt}{m}} = \frac{m}{k} \left(g + Be^{-\frac{kt}{m}} \right) \\ \text{where } B &= \frac{kA}{m} \end{aligned}$$

The terminal velocity is the value as $t \rightarrow \infty$.

$$\text{As } t \rightarrow \infty, Be^{-\frac{kt}{m}} \rightarrow 0$$

Write the expression for terminal velocity.

$$\text{Terminal velocity} = \frac{mg}{k}$$

Substitute in the values.

$$\begin{aligned} &= \frac{85 \times 9.8}{16} \\ &= 52.0625 \end{aligned}$$

Write the answer.

The person's terminal velocity is about 52 m/s.

Notice in Example 24 that the constant for terminal velocity depends on the value of $\frac{m}{k}$, so for a particular body shape the terminal velocity will be about the same.



EXERCISE 8.09 Motion in a straight line with variable forces



Motion with variable forces

Concepts and techniques

- 1 **Example 22** In each of the following situations, m = mass (kg), u = initial velocity (m s^{-1}) and F = force (N). Use this information to find an expression for velocity as a function of time, $v(t)$.
 - a $m = 1$, $u = 0$ and $F = 2t - 5t^2$
 - b $m = 3$, $u = 0$ and $F = \sin(3t - 2)$
 - c $m = 8$, $u = 3$ and $F = 3e^{0.4t + 5}$
 - d $m = 2$, $u = 7$ and $F = 3v$
- 2 **Example 23** In each of the following situations, m = mass (kg), u = initial velocity (m s^{-1}) and F = force (N). Use this to find an expression for velocity as a function of position, $v(x)$, if $x = 0$ when $t = 0$.
 - a $m = 2$, $u = 4$ and $F = \cos(x)$
 - b $m = 5$, $u = 2$ and $F = 2x - 7$
 - c $m = 4$, $u = 3$ and $F = 2v$
 - d $m = 1$, $u = 2$ and $F = 4v - v^2$
 - e $m = 3$, $u = -2$ and $F = \frac{1}{x+1}$
- 3 An object of mass 2 kg moves from rest under the action of a force with magnitude after t s given by $40 - 20t$ N. When the object reaches a speed of 20 m s^{-1} , the force is removed and the object continues with constant speed for 4 s. It is then brought to rest by a constant force of 60 N. Calculate the total distance travelled.
- 4 A body of mass 5 kg is acted on by a force in a straight line. The size of the force after t s is given by $10(2t - 3t^2)$ N. If the body has an initial velocity of 3 m s^{-1} in the same direction as the force, calculate its velocity after 4 s.
- 5 **Example 24** An object of mass 4 kg is projected vertically upward against air resistance that is proportional to the square of the speed and equal to 20 N when the speed is 10 m s^{-1} . Find the greatest height reached by the object if its initial velocity is 20 m s^{-1} .
- 6 An elastic ball of mass 2 kg is dropped from a height of 50 m and experiences air resistance equal to kv^2 . The ball reaches the ground with a speed of 30 m s^{-1} .
 - a Determine the value of k in the expression for air resistance.
 - b How long after being dropped will the ball first reach the ground?
 - c If the ball rebounds vertically with a speed of 25 m s^{-1} , what is the maximum height that it will reach?

Reasoning and communication

- 7 A 5 kg body with an initial velocity of 14 m s^{-1} travels in a horizontal plane. The only force acting on the particle is the friction between it and the surface on which it moves. The magnitude of the frictional force is given by $F = 0.2v + 5$ N. Determine:
 - a an expression for velocity in terms of time
 - b the distance travelled by the body before it comes to rest.
- 8 A particle of mass m moving in a horizontal direction is subjected to air resistance given by $r = kv$, where k is a constant and v is the speed at time t and position x . Determine:
 - a an expression for velocity in terms of time if the initial velocity is u
 - b an expression for the displacement, x , at time t
 - c the limiting distance travelled by the particle.

- 9 A particle is projected vertically upward with velocity u in a medium with resistance to motion equal to kv^2 , where k is a constant and v is the velocity at time t . Find an expression for the greatest height reached by the particle.
- 10 A body of mass m is projected vertically upward with a velocity of u . If the air resistance is given by $r = kv$, where k is a constant and v is the speed of the body, find expressions for the greatest height h reached and the time t_h taken to reach that height.
- 11 A 1 kg particle falls from rest in a medium where the resistance to motion is given by $r = kv^2$, where k is a constant and v is the speed.
- a Show that the distance, x , that the particle falls down when its velocity is v , is given by $x = \frac{1}{2k} \log_e \left(\frac{g}{g - kv^2} \right)$.
- b Find the distance that the particle falls down when it reaches half of its terminal velocity.

8.10 SIMPLE HARMONIC MOTION

Simple harmonic motion is used to model vibrations. The horizontal movement of a pendulum and similar objects is also modelled by simple harmonic motion. The assumptions for simple harmonic motion are as follows.

IMPORTANT

The assumptions for simple harmonic motion are:

- 1 the motion is in a straight line
- 2 the force is given by $F = -kx$, where x is the displacement of the particle from a fixed point called the **point of equilibrium**
- 3 there is no friction

The vector equation for the force means that the force is acting to restore the particle to the point of equilibrium. You can treat the vectors as simple positive and negative numbers relative to the point of equilibrium.

Using $F = ma$, you can write $a = -\frac{k}{m}x$. This is also written as $a = -\omega^2 x$ for convenience.

Then you can write $\frac{d\frac{1}{2}v^2}{dx} = -\omega^2 x$.

Integrating both sides gives $\frac{1}{2}v^2 = -\frac{1}{2}\omega^2 x^2 + c$, so $v^2 = -\omega^2 x^2 + 2c$.

In order to simplify the formulas, $2c$ is written as $a^2\omega^2$, giving $v^2 = a^2\omega^2 - \omega^2 x^2$ with the new constant a . Note that this a is *not* the acceleration.

Simplifying gives $v = \pm\sqrt{a^2 - x^2}$.

Rearrangement gives $\frac{1}{\sqrt{a^2 - x^2}} \frac{dx}{dt} = \pm\omega$.

You can integrate both sides to get $\int \frac{1}{\sqrt{a^2 - x^2}} \frac{dx}{dt} dt = \pm\omega t + c$



Now you can use integration by substitution and $\int \frac{1}{\sqrt{a^2 - x^2}} dx = \arcsin\left(\frac{x}{a}\right)$ to get

$\arcsin\left(\frac{x}{a}\right) = \pm\omega t + c$ so $x = a \sin(\pm\omega t + c)$. Using $-\omega$ instead will give $\cos$ instead of $\sin$.

It is usual to write the phase shift as α and choose the positive value of ω , giving $x = a \sin(\omega t + \alpha)$.

a is the **amplitude** of the motion, ω is called the **angular velocity** and the time for one complete

motion is given by the **period** $T = \frac{2\pi}{\omega}$. The number of complete cycles in unit time is called the

frequency and is given by $\frac{1}{T}$. In order to avoid confusion with amplitude, you normally use the 'dot' notation for the acceleration in simple harmonic motion.

IMPORTANT

The equations for simple harmonic motion with **angular velocity** ω , **amplitude** a , **period** T , **frequency** f and **phase** α are:

Acceleration: $\ddot{x} = -\omega^2 x = -a\omega^2 \sin(\omega t + \alpha)$ or $-a\omega^2 \cos(\omega t + \alpha)$

Velocity: $v = \dot{x} = \pm\omega \sqrt{a^2 - x^2} = -a\omega \sin(\omega t + \alpha)$ or $a\omega \cos(\omega t + \alpha)$

Displacement: $x = a \cos(\omega t + \alpha)$ or $x = a \sin(\omega t + \alpha)$

Period and frequency: $T = \frac{2\pi}{\omega} = \frac{1}{f}$, $\omega = 2\pi f$

Example 25

A particle oscillates 1.2 m either side of a central position with simple harmonic motion.

The period of the motion is 8 s.

- What is the maximum speed?
- Find the speed when the particle is 0.8 m from the central position.
- The particle started its motion from the extreme position. What is the acceleration 0.3 s later?

Solution

- Write the formula for the period.

$$T = \frac{2\pi}{\omega}$$

Find ω .

$$\omega = \frac{2\pi}{T} = 0.25\pi \approx 0.785 \text{ radians/s}$$

Find the value at $\cos(0)$.

$$\text{Maximum speed} = a\omega = 1.2 \times 0.25\pi = 0.3\pi \approx 0.942 \text{ m/s}$$

Write the answer.

The maximum speed is about 0.942 m/s.

- Use $v = \pm\omega \sqrt{a^2 - x^2}$

$$\begin{aligned} \text{Speed} &= \omega \sqrt{a^2 - x^2} \\ &= 0.3\pi \times \sqrt{1.2^2 - 0.8^2} \\ &\approx 0.843 \text{ m/s} \end{aligned}$$

Substitute in the values.

Write the answer.

The speed at $x = 0.8$ is about 0.843 m/s.

- c Use $x = a \cos \dots$ for an extreme starting point. $\text{Acceleration} = -a\omega^2 \cos(\omega t + \alpha)$
 Substitute the values. $= 1.2 \times (0.25\pi)^2 \cos(0.25\pi \times 0.3)$
 Calculate the answer. $\approx 0.720 \text{ m/s}^2$
 Write the answer. The acceleration at 0.3 s is about 0.72 m/s^2 .

Make sure that you set your calculator to radians when calculating results in simple harmonic motion.

○ Example 26

The end of a spring vibrates 20 times a second. The spring compresses to 16 cm and lengthens to 20 cm while vibrating. Find the natural length of the spring and the maximum acceleration of the end.

Solution

- Find the equilibrium point. Centre of motion has length $\frac{1}{2}(16 + 20) = 18 \text{ cm}$.
 Find the angular velocity. $\omega = 2\pi f = 40\pi \approx 125.7 \text{ radians/s}$.
 Use $a\omega^2 \sin(\omega t + \alpha)$. Maximum acceleration $= a\omega^2$
 $= 3200\pi^2 \text{ cm/s}^2$
 Calculate an approximate answer. $\approx 315.8 \text{ m/s}^2$

EXERCISE 8.10 Simple harmonic motion

Concepts and techniques

- Examples 25, 26** A spring makes 12 vibrations in 3 s. Calculate the period and frequency of the vibration.
- A body in simple harmonic motion makes 90 vibrations per minute. Calculate its period and frequency.
- A mass vibrates in simple harmonic motion with an amplitude of 75 cm and a period of 3 s. Calculate:
 - the frequency
 - the maximum velocity (the velocity when displacement $x = 0$)
 - the velocity when $x = 50 \text{ cm}$
 - the maximum acceleration
 - the acceleration when $x = 50 \text{ cm}$.
- A mass vibrates up and down along a straight line, 20 cm long, in simple harmonic motion with a period of 4 s. Calculate:
 - the amplitude of the vibration
 - the velocity and acceleration of the body at the midpoint of its path
 - the velocity and acceleration of the body at the upper end of its path
 - the velocity and acceleration of the body when its displacement from the centre is 4 cm.



Reasoning and communication

- 5 A particle moves in a straight line with simple harmonic motion. Calculate the time taken to undergo a complete oscillation if:
 - a the acceleration at a distance of 1.2 m is 2.4 m s^{-2}
 - b the acceleration at a distance of 20 cm is 3.2 m s^{-2} .
- 6 A particle moving with simple harmonic motion has an amplitude of 1.5 m. If the acceleration at a distance of 60 cm from the equilibrium position is 1.2 m s^{-2} , find the velocity when the particle is:
 - a in the equilibrium position
 - b 1.2 m from its equilibrium position.
- 7 A particle moving with simple harmonic motion passes through two points P and Q that are 56 cm apart with the same velocity. It takes the particle 2 s to move from P to Q and then another 2 s to return to Q . Calculate the period and amplitude of the oscillation.
- 8 A particle undergoing simple harmonic motion completes one full oscillation every 2 s. If the amplitude of the oscillation is 90 cm, calculate the maximum velocity and the maximum acceleration.
- 9 An object in simple harmonic motion performs 150 complete oscillations every minute. If the greatest acceleration achieved is 3 m s^{-2} , calculate the greatest velocity and the distance between the extreme points of the motion.
- 10 A particle moving with simple harmonic motion performs 45 complete oscillations per minute. The velocity at a point 2.5 cm from the central point is 30 cm s^{-1} .
 - a Calculate the greatest distance reached from the central point.
 - b If P and Q are two points positioned 2.5 cm and 4 cm respectively from the centre of motion, find the time taken in moving from P to Q .
- 11 The speed of a particle is given by $v = 2\sqrt{5 + 4x - x^2} \text{ m s}^{-1}$.
 - a Show that the motion is simple harmonic.
 - b Calculate the amplitude and the period.
- 12 A component in a machine vibrates with an angular velocity of 120 radians per second and experiences a maximum acceleration of 1000 m s^{-2} . Calculate the maximum displacement and the maximum speed of the component if it is moving in simple harmonic motion.
- 13 The velocity of a particle moving in a straight line is given by $v = \sqrt{12 + 8x - 4x^2}$, where x is the displacement from a central point.
 - a Show that the motion is simple harmonic.
 - b Calculate the amplitude and the period of oscillation.
- 14 The rise and fall of the tide at a particular inlet is in simple harmonic motion, with the time difference between successive high tides being 10 h. The entrance to the inlet has a depth of 20 m at high tide and 8 m at low tide. If low tide occurs at 10:00 a.m., find the earliest time at which a vessel requiring a minimum water depth of 15 m can pass through the entrance.

CHAPTER SUMMARY

RATES OF CHANGE AND DIFFERENTIAL EQUATIONS

8

- **Implicit differentiation** is the technique of differentiating an equation, involving the dependent and independent variables, using the chain rule for the parts that are functions of the dependent variable.
- The relation between the rates of change of **related variables** with respect to time can be found using the chain rule in the form $\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$.
- A **differential equation** involves one or more derivatives of a function. A **first-order** differential equation involves only the first derivative. The **general solution** of a differential equation is the function.
- The differential equations $\frac{dy}{dx} = ky$ and $\frac{dy}{dx} = g(y)$ have the **general solutions** $y = y_0 e^{kx}$, where y_0 is the value of y at $x = 0$ and $\int \frac{1}{g(y)} dy = x + c$.
- A differential equation with **separable variables** can be rewritten so that the dependent and independent variables are separated. These are often of the form $\frac{dy}{dx} = f(x) \cdot g(y)$.
- A **gradient field** shows what a solution to a differential equation will look like by showing gradients of the functions as short lines or arrows at multiple locations.
- The **logistic model** of growth is given by $\frac{dy}{dt} = ky(M - y)$, where the rate of change of a variable is proportional to both the variable and the difference between a maximum value and the variable.
- The general solution for the logistic model given by $\frac{dy}{dt} = ky(M - y)$ is $y = \frac{MA^{\frac{1}{k}}}{A + e^{-kMt}}$ or $y = \frac{My_0}{y_0 + (M - y_0)e^{-kMt}}$ where $A = \frac{y_0}{M - y_0}$ and y_0 is the initial value.
- **Newton's first law of motion** states that 'Unless acted on by a resultant **force**, a body remains at rest or in uniform motion in a straight line.'
- **Newton's second law of motion** states that 'The acceleration of a body is directly proportional to the resultant force and inversely proportional to the **mass** of the object. The acceleration is in the direction of the resultant force.' Units are chosen so that this law becomes $\mathbf{F} = m\mathbf{a}$, where $\mathbf{F}$ and $\mathbf{a}$ are vectors and m is a scalar.
- The **momentum**, $\mathbf{p}$, of a body is defined as the product of its mass and velocity. $\mathbf{p} = m\mathbf{a}$ is a vector. Newton's second law of motion can also be expressed as $\mathbf{F} = \frac{d\mathbf{p}}{dt}$.
- **Newton's third law of motion** states that 'For every action there is an equal and opposite reaction.'
- The motion of a body depends on the resultant force acting.
- The equations of motion in a straight line with constant force are as below, where the constant acceleration is a , velocity is v , initial velocity is u , time is t and displacement is s .

$$v = u + at \quad s = ut + \frac{1}{2}at^2$$

$$s = \left(\frac{v+u}{2} \right) t \quad v^2 = u^2 + 2as$$

- The acceleration of an object may be expressed in any of the following forms.

$$a = \frac{dv}{dt} = v \frac{dv}{dx} = \frac{d\left(\frac{1}{2}v^2\right)}{dx}$$

- **Simple harmonic motion** is in a straight line with a force given by $F = -kx$, where x is the displacement of the particle from a fixed point called the **point of equilibrium**.
- The **angular velocity** of a particle undergoing simple harmonic motion is the coefficient of t in the equation $x = a \cos(\omega t + \alpha)$ or $x = a \sin(\omega t + \alpha)$.
- The **amplitude** is the maximum displacement from the equilibrium point.
- The **period** is the time taken for one complete cycle and the **frequency** is the number of cycles in unit time. The **phase** is the angular displacement at $t = 0$.
- The equations for simple harmonic motion with angular velocity ω , amplitude a , period T , frequency f and phase α are:

Acceleration:

$$\ddot{x} = -\omega^2 x = -a\omega^2 \sin(\omega t + \alpha) \text{ or } -a\omega^2 \cos(\omega t + \alpha)$$

Velocity:

$$v = \dot{x} = \pm \omega \sqrt{a^2 - x^2} = -a\omega \sin(\omega t + \alpha) \text{ or } a\omega \cos(\omega t + \alpha)$$

Displacement:

$$x = a \cos(\omega t + \alpha) \text{ or } x = a \sin(\omega t + \alpha)$$

$$\text{Period and frequency: } T = \frac{2\pi}{\omega} = \frac{1}{f}, \omega = 2\pi f$$

CHAPTER REVIEW

RATES OF CHANGE AND DIFFERENTIAL EQUATIONS

8

Multiple choice

- Example 1** What is $\frac{dy}{dx}$ for $3y^2 - 5x^3 = 4x$?
 A $\frac{5x^2}{2y}$ B $-\frac{5x^2}{2y}$ C $15x^2 + 4 - 6y$
 D $\frac{15x^2 + 4}{6y}$ E $\frac{3x^2 + 4}{2y}$
- Example 3** The function $y = 4x^2 + 5x$ and $\frac{dx}{dt} = 2$ when $x = 1$. Find $\frac{dy}{dt}$ when $x = 1$.
 A -26 B -9 C 9 D 13 E 26
- Example 6** The general solution to $\frac{dy}{dt} = 30(2t + 1)^2$ is:
 A $120(2t + 1) + c$ B $5(2t + 1)^3 + c$ C $10(t^2 + 1)^3 + c$
 D $10(t^2 + t)^3 + c$ E $5(t^2 + t)^3 + c$
- Example 8** A train leaving the station has its acceleration given by $1.8t - 0.21t^2$ for the first 10 seconds of motion. What is the speed and distance moved after 10 seconds?
 A 20 m/s and 125 m B 30 m/s and 100 m C 110 m/s and 725 m
 D 20 m/s and 380 m E 30 m/s and 200 m
- Example 9** Solve $\frac{dy}{dx} = 2 - y$
 A $y = Ae^{-x} + 2$ B $y = Ae^x - 2$ C $y = 2 - Ae^x$
 D $y = 2Ae^{-x}$ E $y = 2e^{-x}$
- Example 12** Solve $\frac{dy}{dx} = (2x - 5)(y + 3)$
 A $y = Ae^{2x-5} - 3$ B $y = Ae^{x^2-5} - 3$ C $y = 3 - Ae^{2x-5}$
 D $y = Ae^{2x^2-5} - 3$ E $y = Ae^{2x^2-5x} - 3$
- Example 17** A force of 40 newtons is applied for 4 seconds to a stationary ball of mass 10 kg. What is the acceleration of the ball 2.5 seconds after the force is applied?
 A 1 m/s^2 B 4 m/s^2 C 10 m/s^2 D 16 m/s^2 E 40 m/s^2
- Example 19** The engine of a car coasting downhill at 60 m/s applied a force to the car, increasing its speed to 84 m/s in 4 seconds. Given that the mass of the car is 900 kg, what force was applied?
 A 1500 N B 1800 N C 5400 N D $6666.\bar{6}$ N E 86 400 N

Short answer

- 9 **Example 2** An ellipse has the equation $\frac{x^2}{16} + \frac{y^2}{36} = 1$. Find the gradient and equation to the tangent at $(2\sqrt{3}, 3)$.
- 10 **Example 4** The piston in a car is at the bottom of a cylinder of diameter 8 cm. At the top of the stroke, the fuel is ignited and the expansion of the gas drives the piston downwards. When the piston is 2 cm from the top, the gas in the cylinder is expanding at a rate of 30 L/s. At what speed is the piston moving down?
- 11 **Example 7** Find the solution to $\frac{dy}{dx} = \frac{1}{x^2 - 1}$, given that when $x = 0$, $y = 3$.
- 12 **Example 9** Solve the equation $\frac{dy}{dx} = -0.5y$, given that at $x = 0$, $y = 8$.
- 13 **Example 14** Solve $\frac{dP}{dV} = P^3 V^2$.
- 14 **Example 18** A railway carriage with a mass of 3 tonnes runs into the 'bumper' at the end of the track at 15 m/s and rebounds at 5 m/s and is in contact with the bumper for 0.2 seconds. What force did the bumper exert on the carriage?
- 15 **Example 20** A diver on a high board bounces up and down several times before actually diving into the pool. The high board is 6 m above the pool and the diver takes 2.2 seconds from the time she leaves the board to the time she enters the water.
 - a How long does it take her to reach her highest point?
 - b At what speed does she enter the water?
- 16 **Example 21** Two cars in a 100 m drag race have masses of 500 kg and 600 kg and their engines can develop forward forces of 600 000 N and 690 000 N respectively. The lighter car takes off 0.02 s later than the heavier car. Which car crosses the finish line first, and how far ahead of the other car is it at that point?
- 17 **Example 22** Find the speed after 4 seconds of a 12 kg mass that moves from rest under the action of a force given by $F = 2t + 3$ newtons, where t is in seconds.
- 18 **Example 23** Find an expression for the velocity of a 3 kg particle with an initial velocity of 3 m/s and position of $x = 0$ m if the force acting on it is given by $F = 2x - 1$ newtons.
- 19 **Example 25, 26** A particle starts from rest at $x = 10$ and proceeds with simple harmonic motion about $x = 0$ so that after 2 s it reaches $x = 5$. If displacement is in mm, find:
 - a an expression for the displacement (x) at any time t
 - b the speed at $x = 0$
 - c the amplitude and period of motion
 - d the maximum speed
 - e the maximum acceleration.

Application

- 20 A particle with simple harmonic motion is moving with a velocity of 3.6 m s^{-1} as it passes through its central position. When the particle is 0.2 m from the central position, it has an acceleration of 4.8 m s^{-2} . Calculate:
 - a the amplitude of the oscillation
 - b the period of the oscillation.

- 21 An express elevator travels 30 floors in 8 s. The distance travelled is 72 m. It accelerates for the first 5 floors and decelerates at the same rate for the last 5 floors, travelling at uniform speed in between. Find that uniform speed.
- 22 The rate of change in the level of water when it is siphoned between two tanks depends on the difference in the levels and on the size of the hose connecting them. Two rainwater tanks are connected by a hose at 9 a.m., when the difference in their levels is 1.5 m. At 9:05 a.m. the difference in their levels has reduced to 1.2 m. Use a differential equation to find the time when the difference in their levels is less than 1 cm.
- 23 The maximum population of elephants that can be supported in a reserve is 6000. The population had been reduced by poaching to only 200 elephants before a very strict anti-poaching policy allowed the elephants to recover. The population increased according to the logistic model with $k = 0.000\ 016$. How many years would it take to reach 3000 elephants?



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- 24 An object of mass 4 kg is projected vertically upwards against resistive forces that are proportional to the square of the speed and equal to 10 N when the speed is 20 m/s. Assuming that $g \approx 10\text{ m/s}^2$, find the greatest height reached by the object if its initial velocity is 30 m/s.



Practice quiz

Multiple choice

- 1 $3y^3 + 4x^2 + 2y - x = 7$. Find $\frac{dy}{dx}$.
A $8x - 1$ B $\frac{8x-1}{9y+2}$ C $\frac{8-8x}{9y^2+2}$
D $\frac{1-8x}{9y^2+2}$ E $9y^2 + 8x + 2 - x$
- 2 Expressed in partial fraction form, $\frac{2}{x^2+7x-18}$ is
A $\frac{\frac{2}{11}}{x-2} - \frac{\frac{2}{11}}{x+9}$ B $\frac{\frac{2}{11}}{x+2} - \frac{\frac{2}{11}}{x-9}$ C $\frac{\frac{2}{9}}{x-2} - \frac{\frac{2}{11}}{x-9}$
D $\frac{11}{x-2} - \frac{11}{x+9}$ E $\frac{1}{11} \left(\frac{1}{x-2} - \frac{1}{x+9} \right)$
- 3 $\int \tan^2(x) dx$ can be expressed as
A $\tan(x) + x + c$ B $x \tan(x) + c$ C $\tan(x) - x + c$
D $-\log_e |\cos(x)| + c$ E $2 \tan(x) \sec^2(x)$
- 4 $\frac{dm}{dt} = 5 - 4m$ and at $t = 0$, $m = 2$. The solution for m is:
A $m = 0.25(3e^{-4t} + 5)$ B $m = 2e^{-4t}$ C $m = 2(5 - 4e^{-t})$
D $m = 2(4e^{-t} - 3)$ E $m = 2(3e^t - 1)$
- 5 The rate of change of the momentum of an object moving in a straight line is given by $\frac{dp}{dt} = 3 - 2t$ for 4 seconds. The object has mass 2 kg and initial velocity 12 m/s. The velocity after 4 seconds is:
A 2 m/s B 4 m/s C 6 m/s D 10 m/s E 14 m/s
- 6 $\int_1^2 xe^{2x} dx$ is equal to
A $\frac{3}{4}e^2(e^2 - 1)$ B $\frac{1}{4}(3e^4 - e^2)$ C $\frac{1}{4}e^{2x}(x-1)$
D $\frac{1}{2}e^{2x}(2x-1)$ E $\frac{1}{4}e^{2x}(2x-1)$

Short answer

- 1 Write an equivalent expression for $\int \frac{1}{2-7x} dx$.
- 2 A building has full length windows of height 3 m on all floors. An object falling from the roof takes 0.1 seconds to fall past a 4th floor window. All floors apart from the penthouse are 3 m apart and the 1st floor is the first one up from ground level. How high is the roof?

- 3 Some algae was washed into a lake, and grew by 25% on the tenth day afterwards to cover 5 m^2 of the lake. The rate of growth of the algae is directly proportional to the amount of water it covers.
- How much did it cover when first washed in?
 - How many more days will it take to cover 100 m^2 ?
- 4 Using the method of substitution, express in terms of a function of u an equivalent expression for $\int \sin(3x) \cos^7(3x) dx$.

Application

- Use a method of recursion to find $\int x^2 \sin(x) dx$.
 - Hence find $\int_0^{2\pi} x^2 \sin(x) dx$
- Express the rational function $\frac{x^2}{(x-1)^2(x+4)}$ as partial fractions.
 - Hence find $\int_0^{\frac{1}{2}} \frac{x^2}{(x-1)^2(x+4)} dx$.
- A small island group had a population of only 15 000 people when white settlers arrived. The native people on the island had no resistance to a disease brought by the settlers and 10 got the disease on the very first day that the settlers arrived. After severe widespread illness, they recovered and became immune, but suffered severe damage to their extremities. The rate of infection (in people per day) of native people was given by $0.000\,05N(15\,000 - N)$. How long did it take for two-thirds of the original population to catch the disease?
- The acceleration of an object moving through a viscous liquid under the influence of a magnetic field is given by $a = 5 - 0.4v$ with an initial velocity of $v = 0.5 \text{ m/s}$. Find an expression for the velocity after time t seconds and the terminal velocity.



9

TERMINOLOGY

expected value
exponential distribution
exponential random variable
Gaussian distribution
left and right rectangles
mean
median
midpoint rule
numerical integration
probability
probability density function
quantile
Simpson's rule
standard deviation
technology
trapezium rule
variance
solid of revolution

INTEGRATION AND APPLICATIONS OF INTEGRATION

APPLICATIONS OF INTEGRATION

- 9.01 Areas between curves
- 9.02 Volumes of solids of revolution
- 9.03 Numerical integration
- 9.04 The exponential probability density function
- 9.05 Applications of the exponential probability density function

Chapter summary

Chapter review



Prior learning

APPLICATIONS OF INTEGRAL CALCULUS:

- the calculation of areas between curves determined by functions (ACMSM124)
- determine volumes of solids of revolution about either axis (ACMSM125)
- use numerical integration using technology (ACMSM126)
- use and apply the probability density function, $f(t) = \lambda e^{-\lambda t}$ for $t \geq 0$, of the exponential random variable with parameter $\lambda > 0$, and use the exponential random variables and associated probabilities and quantiles to model data and solve practical problems (ACMSM127) 

9.01 AREAS BETWEEN CURVES

You can calculate the areas between the curves of known functions. For an enclosed area, it is often necessary for you to find the intersection points of the curves. The area can be calculated in two ways:

- as the difference between the areas under the two functions; or
- as the area under the **difference function**, because

$$\int_a^b f(x) dx - \int_a^b g(x) dx = \int_a^b [f(x) - g(x)] dx$$

○ Example 1

Find the area enclosed by the functions $f(x) = x^2$, $g(x) = (x - 4)^2$ and the x -axis.

Solution

Find the intersection(s).

$$x^2 = (x - 4)^2$$

Simplify.

$$= x^2 - 8x + 16$$

Solve for x .

$$x = 2$$

Substitute to find y .

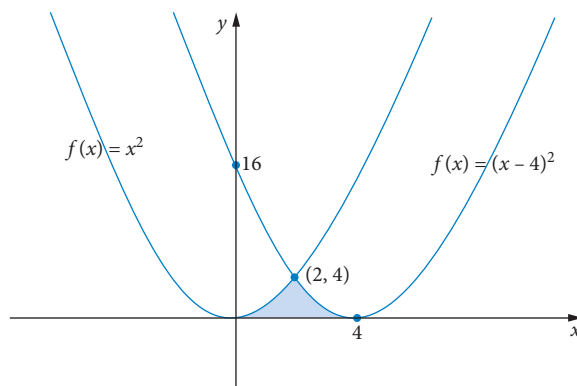
$$f(2) = 2^2 = 4$$

Write the point of intersection.

The graphs intersect at $(2, 4)$.

Sketch the graphs.

Shade the required area.



Write the integrals required.

$$\text{Area} = \int_0^2 x^2 dx + \int_2^4 (x-4)^2 dx$$

Simplify.

$$= \int_0^2 x^2 dx + \int_2^4 (x^2 - 8x + 16) dx$$

Calculate the integrals.

$$= \left[\frac{x^3}{3} \right]_0^2 + \left[\frac{x^3}{3} - 4x^2 + 16x \right]_2^4$$

$$= \left(\frac{8}{3} - 0 \right) + \left(\frac{64}{3} - \frac{56}{3} \right)$$

$$= 5\frac{1}{3}$$

Write the answer.

The area enclosed by the functions $f(x) = x^2$, $g(x) = (x-4)^2$ and the x -axis is $5\frac{1}{3}$ square units.

For areas 'enclosed by' two functions, the difference method is generally better. You should do any algebraic simplification before integrating.

○ Example 2

Find the area enclosed by $y = 3x^2 - 4x - 1$ and $y = x^2 - x - 2$.

Solution

Find the difference function.

$$\text{Let } d(x) = (3x^2 - 4x - 1) - (x^2 - x - 2)$$

Simplify.

$$= 2x^2 - 3x + 1$$

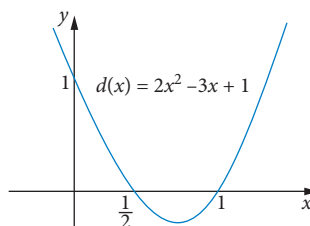
Factorise.

$$= (2x - 1)(x - 1)$$

State the zeros.

The zeros are at $x = \frac{1}{2}$ and 1.

Sketch the graph of $d(x)$.



Find the area needed.

$$\text{Enclosed area} = - \int_{\frac{1}{2}}^1 (2x^2 - 3x + 1) dx$$

Calculate the integral.

$$= - \left[\frac{2x^3}{3} - \frac{3x^2}{2} + x \right]_{\frac{1}{2}}^1$$

Evaluate.

$$= \frac{1}{24}$$

Write the answer.

The area enclosed by $y = 3x^2 - 4x - 1$ and $y = x^2 - x - 2$ is $\frac{1}{24}$ square units.

In some cases, you will need to take account of the fact that part of the area is positive and part of it is negative in order to calculate the total area enclosed.

Example 3

Find the area enclosed by $y = x^3 + x^2 - 4x + 1$ and $y = 2x + 1$.

Solution

Find the difference function.

$$\text{Let } d(x) = (x^3 + x^2 - 4x + 1) - (2x + 1)$$

Simplify.

$$= x^3 + x^2 - 6x$$

Find the zeros of $d(x)$.

$$= x(x^2 + x - 6)$$

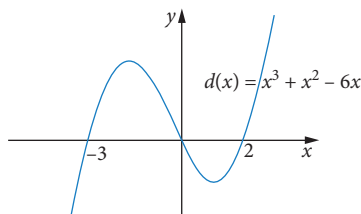
Factorise.

$$= x(x + 3)(x - 2)$$

State the zeros.

The zeros are at $x = -3, 0$ and 2 .

Sketch the graph of $d(x)$.



Take account of the signs.

$$\text{Area enclosed} = \int_{-3}^0 (x^3 + x^2 - 6x) dx - \int_0^2 (x^3 + x^2 - 6x) dx$$

Do the integrations.

$$= \left[\frac{x^4}{4} + \frac{x^3}{3} - 3x^2 \right]_{-3}^0 - \left[\frac{x^4}{4} + \frac{x^3}{3} - 3x^2 \right]_0^2$$

Do the calculations.

$$= \left[0 - \left(-\frac{63}{4} \right) \right] - \left(\frac{16}{3} - 0 \right)$$

Simplify.

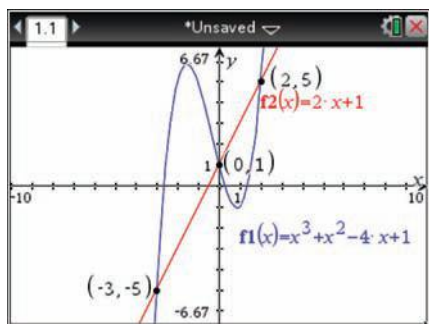
$$= \frac{63}{4} + \frac{16}{3} = 21\frac{1}{12}$$

Write the answer.

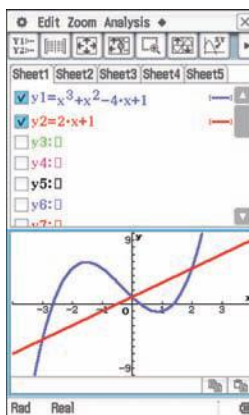
The area enclosed by $y = x^3 + x^2 - 4x + 1$ and $y = 2x + 1$ is $21\frac{1}{12}$ square units.

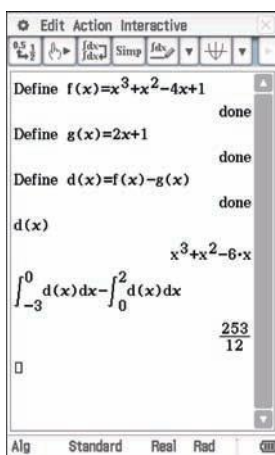
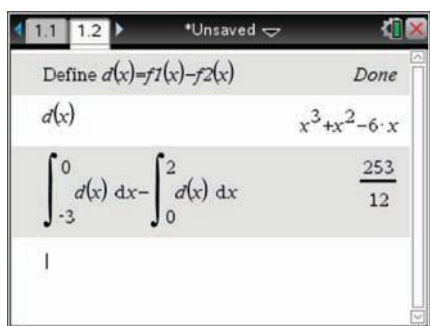
You would be wise to use your CAS calculator to check a problem such as Example 3, even if you are required to do the problem by hand.

TI-Nspire CAS



ClassPad





INVESTIGATION Lorenz curves

Not all Australians receive the same income. In fact, the poorest 10% of Australian families receive less than 2% of the total income, while the richest 10% receive 26% of the total income. *Distribution of income* is shown by a **Lorenz curve**, named after Konrad Lorenz who devised the method in 1905. The x -axis shows the proportion of the population whose incomes are the least. The y -coordinate of a point on the curve is the proportion of total income received by the proportion of the population shown on the x -axis. Since both proportions are between 0 and 1, the graph only goes from (0, 0) to (1, 1).



If everyone received the same income, the graph would be the straight line $y = x$. The area between the curve and the line $y = x$ gives a measure of the income inequality of a society. In fact, the **coefficient of inequality** is defined to compare the inequality of the distribution from one society to another.

The coefficient of inequality is given by

$$L = \frac{\text{area between Lorenz curve and the line } y = x}{\text{area under the line } y = x}$$

In 2011–2012, the net worth of Australian households was as follows:

Poorest 20%:	\$31 203
Next 20%:	\$191 207
Middle 20%:	\$437 856
Next 20%:	\$766 465
Wealthiest 20%:	\$2 215 032

Source: ABS 6523.0 Table 7

Assuming the cumulative incomes at x -values of 0.2, 0.4, 0.6, 0.8 and 1, find the total wealth by adding the amounts and then find the ratio to the total of the cumulative amounts.

Draw the Lorenz curve and use your calculator to find a line of best fit. Remember to include the points (0, 0) and (1, 1).

Calculate the coefficient of income inequality for Australia in 2011–2012.

Find data for earlier years and look at trends in Australia. Research income inequality in other countries. What do you find?

EXERCISE 9.01 Areas between curves



Areas between curves

Concepts and techniques

- Example 1** Find the area enclosed by the functions $f(x) = (2 - x)^3$, $g(x) = x^3$ and the x -axis.
- Find the area enclosed by the functions $f(x) = x^2 - 4$, $g(x) = 2x + 4$, the line $x = 1$ on the left and the x -axis.
- Example 2** Find the area enclosed by $y = 3x^2 - 2x - 3$ and $y = 2x^2 + 3x + 3$.
- Find the area enclosed by $y = 3x^2 - 3x - 1$ and $y = 2x^2 - x + 7$.
- Find the area enclosed by $y = 2x^2 + 3x + 2$ and $y = x^2 + 5x + 10$.
- Find the area enclosed by $y = 2x^2 + 12x - 16$ and $y = x + 4 - x^2$.
- Example 2** Find the area enclosed by $y = 2x^3 + 5x^2 - 9x + 5$ and $y = 3x + 5$.
- Find the area enclosed by $y = x^3 - 4x^2 - 2x + 11$ and $y = 2x - 5$.

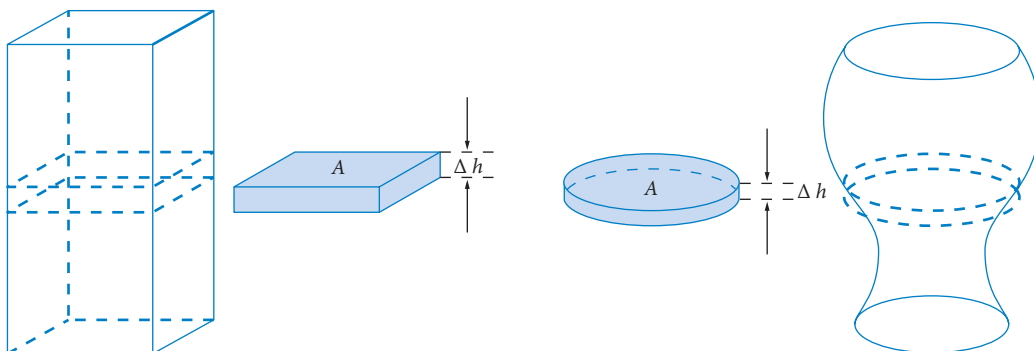
Reasoning and communication

- In Malaysia, the Lorenz curve for income is approximately

$$y = 0.06x + 0.18x^2 + 0.17x^3 + 0.35x^4 + 0.24x^5$$
 - Sketch the Lorenz curve.
 - What proportion of income is earned by the poorest 10% of the population?
 - What proportion of income is earned by the wealthiest 10% of the population?
 - What is the coefficient of income inequality in Malaysia?
- Find the area enclosed by $y = x^4 + 5x^3 + 7x^2 - 9$ and $y = 2x^2 + 5x - 3$.

9.02 VOLUMES OF SOLIDS OF REVOLUTION

You can approximate the volumes of three-dimensional shapes by slicing them into layers, finding the volume of each layer, and adding your answers together. The volume of a single slice through a three-dimensional shape is $A\Delta h$, where A is the area of the slice and Δh is the thickness of the slice, as shown below.



If successive slices of the same thickness were taken all the way up an object, the total volume would be approximately $V \approx \sum A\Delta h$ where the sum is taken over the height.

The area of each slice actually depends on its position. It would be more correct to write $A = f(h)$, since the area of the slice is a function of h . In order to obtain a more accurate value of V , make the slices thinner. As $\Delta h \rightarrow 0$, the approximation to the volume will become exact. In this case, the area of the slice becomes the cross-sectional area at the 'height' h .

Then
$$V \approx \sum A\Delta h$$

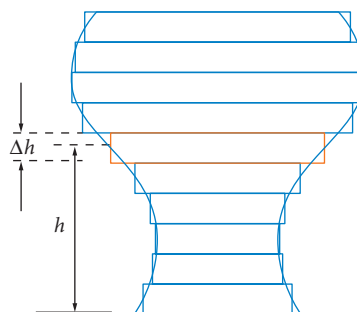
But $A = f(h)$, so
$$V \approx \sum f(h)\Delta h$$

In the limit
$$V = \lim_{\Delta h \rightarrow 0} \sum f(h)\Delta h$$

But this is the definite integral, so

$$V = \int f(h)dh = \int A(h)dh = \int A dh.$$

The function integrated is the cross-sectional area at x , which can be written as $A(h)$.



IMPORTANT

Volume of a shape

$$V = \int A(h)dh = \int A dh$$

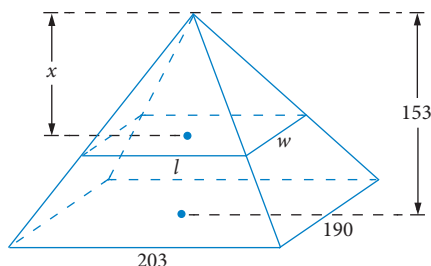
Example 4

Use integration to find the volume of a pyramid of height 153 m with a rectangular base 203 m by 190 m.

Solution

Draw a diagram.

Measure the 'height' from the top of the pyramid to the base, as shown.



Write ratios to find l and w at height x .

$$l = \frac{x}{153} \times 203 \quad \text{and} \quad w = \frac{x}{153} \times 190$$

Find an expression for the cross-sectional area.

$$A = lw$$

Write the area formula at x .

$$= \frac{x}{153} \times 203 \times \frac{x}{153} \times 190$$

Substitute and simplify.

$$= \frac{203 \times 190}{153^2} x^2$$

Write the volume formula.

$$V = \int A(x) dx$$

Integrate from $x = 0$ to $x = 153$ to find the volume.

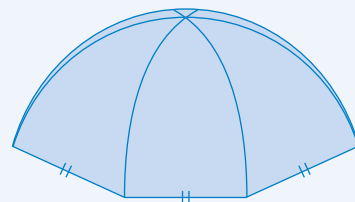
$$\begin{aligned} V &= \int_0^{153} \frac{203 \times 190}{153^2} x^2 dx \\ &= \frac{203 \times 190}{153^2} \int_0^{153} x^2 dx \\ &= \frac{203 \times 190}{153^2} \left[\frac{x^3}{3} \right]_0^{153} \\ &= \frac{203 \times 190 \times 153}{3} \\ &= 1\,967\,070 \text{ m}^3 \end{aligned}$$

Write the answer.

The volume is $1\,967\,070 \text{ m}^3$.

INVESTIGATION Pop-up tent

A lightweight 'pop-up' tent consists of six plastic struts that are inserted into pockets sewn into the joins of the fabric panels. The resulting shape has hexagonal horizontal cross-sections, while vertical cross-sections through the centre are semicircular. The overall height is 1.3 m. In order to determine whether it is safe to use a gas lamp inside, a camper needs to know the volume of air in the tent. Investigate the volume inside the tent.



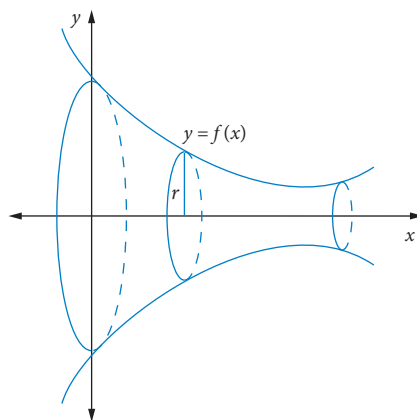
Many objects have a circular cross-section because they are manufactured by *turning* on a lathe. Objects of this shape are called **solids of revolution**, and their volumes can be found by using integration. Consider the axis of revolution as being the x -axis and write the radius as $y = f(x)$.

Then $y = f(x)$ is the radius of the cross-section, and the cross-section is a circle.

Thus $A = \pi r^2 = \pi y^2$.

The volume of a solid of revolution is given by

$$V = \int A dx = \int \pi y^2 dx = \pi \int y^2 dx.$$



IMPORTANT

Volumes of solids of revolution

The volume of a solid of revolution generated by a curve in the xy plane is given by

$$V = \pi \int y^2 dx \quad \text{when rotated about the } x\text{-axis}$$

or $V = \pi \int x^2 dy \quad \text{when rotated about the } y\text{-axis}.$

Example 5

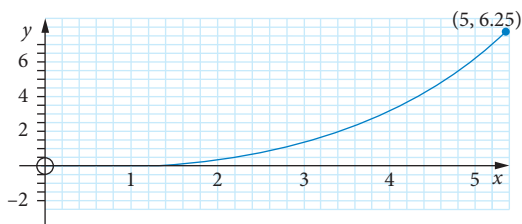
Find the volume of the bell of a French horn that could be formed by rotating the curve $y = 0.05x^3$ about the x -axis between $x = 0$ and $x = 5$. Measurements are in centimetres.



Shutterstock.com/Denis Kurylow

Solution

Draw a diagram.



Write the formula for volume rotated about the x -axis between $x = 0$ and $x = 5$.

$$V = \pi \int_0^5 y^2 dx$$

Find an expression for y^2 .

$$\begin{aligned} y^2 &= (0.05x^3)^2 \\ &= 0.0025x^6 \end{aligned}$$

Substitute into the integration formula.

$$V = \pi \int_0^5 0.0025x^6 dx$$

Integrate.

$$= 0.0025\pi \left[\frac{1}{7}x^7 \right]_0^5$$

Evaluate.

$$= 0.0025\pi \left(\frac{1}{7} \cdot 5^7 \right)$$

$$= \frac{3125\pi}{112}$$

Write the answer.

$$\text{Volume} \approx 87.656 \text{ cm}^3.$$

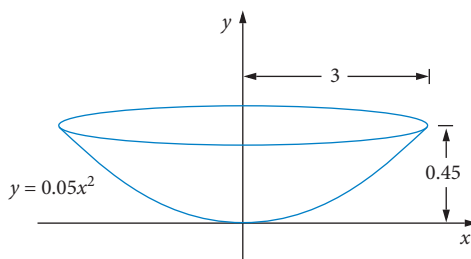
You can choose the y -axis as the axis of rotation; in this case you would swap the integration variables for y -values and find an expression for x^2 .

○ Example 6

Find the volume of a parabolic telescope mirror of radius 3 m formed by rotating the curve $y = 0.05x^2$ about the y -axis.

Solution

Draw a diagram.



The limits of integration will be the values of y that give the correct range of x .

Substitute $x = 0$ and $x = 3$.

$$x = 0 \text{ gives } y = 0.$$

$$\begin{aligned} x = 3 \text{ gives } y &= 0.05x^2 \\ &= 0.05 \times 3^2 \\ &= 0.45 \end{aligned}$$

Write the formula for volume rotated about the y -axis between $y = 0$ and $y = 0.45$.

$$V = \pi \int_0^{0.45} x^2 dy$$

Find an expression for x^2 .

$$\begin{aligned} y &= 0.05x^2 \\ \therefore x^2 &= 20y \end{aligned}$$

Substitute into the integration formula.

$$V = \pi \int_0^{0.45} 20y dy$$

Integrate.

$$= \pi \left[10y^2 \right]_0^{0.45}$$

Evaluate.

$$= \frac{81\pi}{40} \approx 6.36 \text{ m}^3$$

Write the answer.

$$\text{The volume is about } 6.36 \text{ m}^3.$$

EXERCISE 9.02 Volumes of solids of revolution



Solids of revolution

Concepts and techniques

- 1 **Example 5** Use a suitable definite integral to find the volumes, in cubic units, that are formed by rotating the following curves about the x -axis between the limits shown.
- a $y = 0.5x^2$, from $x = 0$ to $x = 3$. b $y = 5x^3$, from $x = 0$ to $x = 1$.
c $y = \log_e(x)$, from $x = 1$ to $x = 2$. d $y = \frac{1}{2}e^{\frac{1}{2}x}$, from $x = 1$ to $x = 2$.
e $y = \frac{1}{2}\cos\left(\frac{1}{2}x\right)$, from $x = \pi$ to $x = 2\pi$.
- 2 **Example 6** Use a suitable definite integral to find the volumes, in cubic units, that are formed by rotating the following curves about the y -axis between the limits shown.
- a $y = 0.5x^2$, from $y = 0$ to $y = 3$ b $y = 5x^3$, from $y = 0$ to $y = 1$
c $y = \log_e(x)$, from $y = 0$ to $y = 1$ d $y = \frac{1}{2}e^{\frac{1}{2}x}$, from $x = 1$ to $x = 2$
e $y = \frac{1}{2}\cos\left(\frac{1}{2}x\right)$, from $x = 0$ to $x = \pi$ f $y = \sqrt{x} + 1$, from $x = 1$ to $x = 2$
- 3 a A sphere of radius 5 cm can be considered to be the solid of revolution of the curve $y = \sqrt{25 - x^2}$ around the x -axis. Using integration, the volume of the hemisphere is
A $\frac{25\pi}{3}$ B 25π C 100π D $\frac{500\pi}{3}$ E 500π
b What formula could be used to get the same result?
- 4 The inside of a pottery urn that a small child could hide in can be modelled as a solid of revolution of the line $y = 0.000\,05x^3 - 0.0045x^2 + 20$ cm around the x -axis from $x = 0$ to $x = 80$ cm, with the open end at $x = 0$ and a flat base. The maximum volume, in cubic cm, of oil that could be held by the urn when it is full, is closest to
A 42.22 B 71.817 C 4222 D 22 860 E 71 817
- 5 A pointed hat is modelled by rotating the line $y = \sqrt{0.2x}$ from $x = 0$ to $x = 20$ about the y -axis. If the measurements are in cm, the volume of the hat is
A 40 B $\frac{80\pi}{3}$ C 160π D $\frac{1600}{3}$ E $\frac{1600\pi}{3}$
- 6 The interior surface of an hourglass can be considered as having the curve $y = \sqrt{1.05 - \frac{1}{1+x^2}}$ rotated about the x -axis between $x = -5$ and $x = 5$ (measurements are in cm). The volume of the hourglass is closest to
A $\tan^{-1}(x)$ B 24.36 C $\tan^{-1}\left(\frac{x}{1.05}\right)$ D 26.88 E 24.36π

Reasoning and communication

- 7 **Example 4** Find the formula for the volume of a square-based pyramid of side s and height h .
- 8 Use integration to find the volume of a pyramid of height 15 m with a rectangular base measuring 30 m by 40 m.

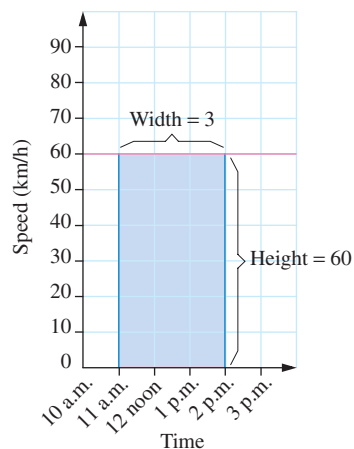
- 9 Use integration to find the volume of a cone of height 10 cm with a base diameter of 15 cm.
- 10 The interior surface of a glass *objet d'art* is formed from a molten glass sheet by allowing it to sag through a circular opening. The resulting cross-sectional shape may be modelled by the *catenary* curve $f(x) = 15.43 - 5(e^{0.05x} + e^{-0.05x})$ between $x = -20$ and $x = 20$, where all measurements are in cm. Use rotation about the x -axis between $x = -20$ and $x = 20$ to find the interior volume of the *objet d'art*.



Alamy/Radius Images

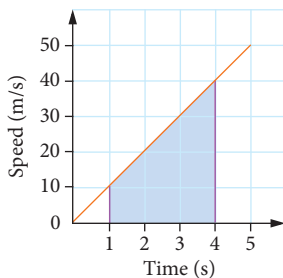
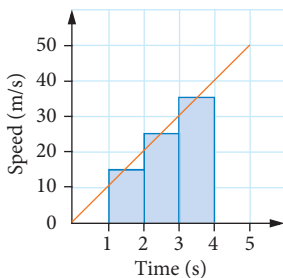
9.03 NUMERICAL INTEGRATION

The graph on the right shows the speed of a car travelling along a main road with little traffic. Between 11 a.m. and 2 p.m. it travels for 3 hours at 60 km/h, so the distance is $60 \times 3 = 180$ km. On the graph, this is shown by the area of the rectangle. Its width is 3 (hours) and its height is 60 (km/h), so its area is $60 \times 3 = 180$ (km).



Suppose that a drag-racing car smoothly accelerated from 0 to 50 m/s (180 km/h) in 5 seconds. In the 2nd second it would accelerate from 10 to 20 m/s, so its average speed would be 15 m/s and it would travel 15 m in that second. In the 3rd second it would travel 25 m, and so on.

The first graph below shows rectangles. Each rectangle's height is the average speed for that second. The area of each rectangle is the distance covered in that second. The total of the areas of the rectangles is the total distance covered from 1 to 4 seconds.



The area under a speed–time graph is the distance covered by an object during the time shown. You can find areas under curves (definite integrals) using a range of approximation techniques. Technology can be used to help you find these answers. The first graph illustrates the simplest such method, the **midpoint rule**.

The **midpoint rule** is that $\int_a^b f(x)dx \approx w \sum_{i=1}^n f(a_i)$, where the interval $[a, b]$ is divided into n equal width strips of width w and the values $a_1, a_2, \dots, a_n$ are the midpoints of the strips.

The rule can also be written as $\int_a^b f(x)dx \approx w \sum_{i=1}^n f\left(\frac{a_{i-1} + a_i}{2}\right)$, where $a_0, a_1, a_2, \dots, a_n$ are the endpoints of the strips, so $a_0 = a$ and $a_n = b$.

Example 7

Find an approximation for the definite integral of $f(x) = \log_e(5x)$ from $x = 2$ to $x = 6$ by calculating the areas of strips 0.5 units wide, using the midpoint rule.

Solution

Divide into strips.

The strips are $[2, 2.5], [2.5, 3], \dots, [5.5, 6]$.

Find the midpoints of the vertical strips.

The centres are 2.25, 2.75, ..., 5.75.

Write a rule for the midpoints.

Midpoints = $2.25 + 0.5i$ for $i = 0$ to 7.

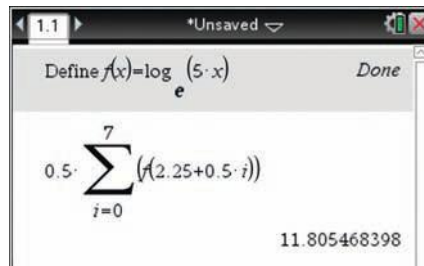
Write the required approximation.

$$\int_2^6 \log_e(5x)dx \approx 0.5 \times \sum_{i=0}^7 f(2.25 + 0.5i)$$

TI-Nspire CAS

Define the function. Make sure that you use the constant e , not the letter e .

Use the template menu $\left[\frac{\square}{\square}\right]$ to get the sum to find the approximation.

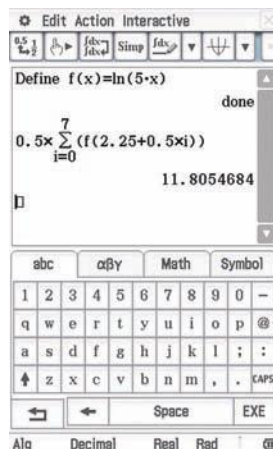


ClassPad

Define the function. Make sure that you use $\ln$ from $\left[\frac{\square}{\square}\right]$ for $\log_e$.

Enter $0.5 \times \sum_{i=0}^7 f(2.25 + 0.5i)$ using $\left[\frac{\square}{\square}\right]$

from $\left[\frac{\square}{\square}\right]$. Make sure that you use i from the $\left[\frac{\square}{\square}\right]$ menu, not i for imaginary numbers.



Round and write the answer.

$$\int_2^6 \log_e(5x)dx \approx 11.805$$

Instead of using rectangles, a better approximation can be obtained using trapeziums. This means that the tops are sloping, so they approximate the function a little better.

The area of a trapezium is given by $\frac{1}{2}h(a+b)$. For n strips of width w between a and b with endpoints a_0 ,

$a_1, a_2, \dots, a_n$ the areas of the strips will be

$$\frac{1}{2} w[f(a_0) + f(a_1)], \frac{1}{2} w[f(a_1) + f(a_2)],$$

$$\frac{1}{2} w[f(a_2) + f(a_3)], \dots, \frac{1}{2} w[f(a_{n-2}) + f(a_{n-1})],$$

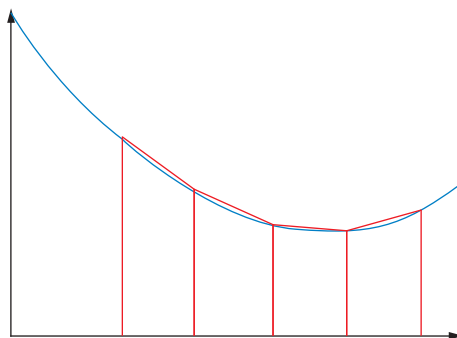
$$\frac{1}{2} w[f(a_{n-1}) + f(a_n)] \text{ so the total area of the strips will be given by}$$

$$\frac{1}{2} w[f(a_0) + f(a_1)] + \frac{1}{2} w[f(a_1) + f(a_2)] + \dots + \frac{1}{2} w[f(a_{n-2}) + f(a_{n-1})] + \frac{1}{2} w[f(a_{n-1}) + f(a_n)]$$

$$= \frac{1}{2} w[f(a_0) + f(a_1) + f(a_1) + f(a_2) + \dots + f(a_{n-2}) + f(a_{n-1}) + f(a_{n-1}) + f(a_n)]$$

$$= \frac{1}{2} w[f(a_0) + 2f(a_1) + 2f(a_2) + \dots + 2f(a_{n-1}) + f(a_n)]$$

This is called the **trapezoidal rule**.



IMPORTANT

The **trapezoidal rule** is that $\int_a^b f(x)dx \approx \frac{w}{2} \left[f(a_0) + 2 \sum_{i=1}^{n-1} f(a_i) + f(a_n) \right]$, where the interval $[a, b]$ is divided into n equal width strips of width w and end values $a_0, a_1, \dots, a_n$. Of course $a_0 = a$ and $a_n = b$.

The rule can also be written as $\int_a^b f(x)dx \approx \frac{w}{2} (E + 2M)$, where E is the sum of the values of $f(x)$ at the ends of the interval and M is the sum of all the values of $f(x)$ at the intervening points $a_1, a_2, \dots, a_{n-1}$.

Example 8

Use the trapezium rule to find an approximation for $\int_{-2}^2 e^x dx$ using 8 strips.

Solution

Find the width of each strip.

$$\text{Width of strips} = \frac{2 - (-2)}{8} = \frac{1}{2}$$

Find the points.

Endpoints are $-2, -1.5, -1, \dots, 2$.

Write the points at the ends of the interval.

$$a_0 = -2, a_8 = 2$$

Write a formula for the remaining points.

$$a_i = -2 + 0.5i \text{ for } i = 1 \text{ to } 7$$

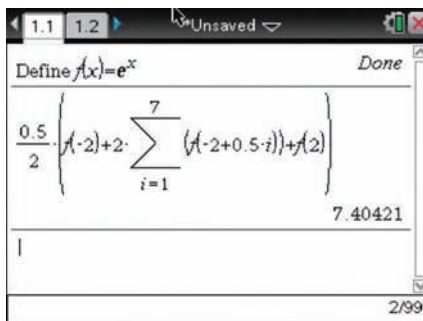
Write the required approximation.

$$\begin{aligned} & \int_{-2}^2 e^x dx \\ & \approx \frac{0.5}{2} \left[f(-2) + 2 \sum_{i=1}^7 f(-2 + 0.5i) + f(2) \right] \end{aligned}$$

TI-Nspire CAS

Define the function. Make sure you use the constant e , not the letter e .

Use the template menu  to get the sum to find the approximation.



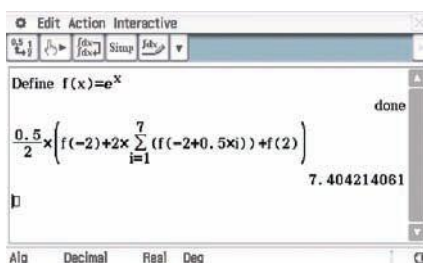
ClassPad

Define the function. Make sure that you use  from **Math1** for e^x .

$$\text{Enter } \frac{0.5}{2} \left[f(-2) + 2 \sum_{i=1}^7 f(-2 + 0.5i) + f(2) \right]$$

using  from **Math2**. Make sure that you use i from the **abc** menu, not i for imaginary numbers.

Round and write the answer.



$$\int_{-2}^2 e^x dx \approx 7.4042$$

A further improvement to the approximation of integrals using equal width strips can be made by using *double strips* with parabolas at the top instead of straight lines. In this case, for an even number of strips with endpoints $a_0, a_1, a_2, \dots, a_n$, you can show that the areas of the strips are given by $\frac{w}{3} (a_0 + 4a_1 + a_2)$, $\frac{w}{3} (a_2 + 4a_3 + a_4)$, $\frac{w}{3} (a_4 + 4a_5 + a_6)$, $\dots$, $\frac{w}{3} (a_{n-2} + 4a_{n-1} + a_n)$. Adding the areas of these strips gives **Simpson's rule**.

IMPORTANT

Simpson's rule is that $\int_a^b f(x) dx \approx \frac{w}{3} \left[f(a_0) + 2 \sum_{i=1}^{\frac{n}{2}-1} f(a_{2i}) + 4 \sum_{i=1}^{\frac{n}{2}} f(a_{2i-1}) + f(a_n) \right]$, where the

interval $[a, b]$ is divided into an even number n of equal width strips of width w and end values $a_0, a_1, \dots, a_n$. Of course $a_0 = a$ and $a_n = b$.

The rule can also be written as $\int_a^b f(x) dx \approx \frac{w}{3} (X + 2E + 4O)$, where X is the sum of the values of $f(x)$ at the ends of the interval, E is the sum of the values at the even-numbered intervening points $a_2, a_4, \dots, a_{n-2}$ and O is the sum of the values at the odd-numbered intervening points $a_1, a_3, \dots, a_{n-1}$.

Example 9

Use Simpson's rule to find $\int_1^5 \frac{5}{x} dx$ using strips of width $\frac{1}{4}$.

Solution

Find the points.

Points are 1, 1.25, 1.5, ... 4.5, 4.75, 5

Write the points at the ends of the interval.

$a_0 = 1, a_{16} = 5$

Write a formula for the remaining odd points.

$a_i = 1.25 + 0.5i$ for $i = 0$ to 7

Write a formula for the remaining even points.

$a_i = 1.5 + 0.5i$ for $i = 0$ to 6

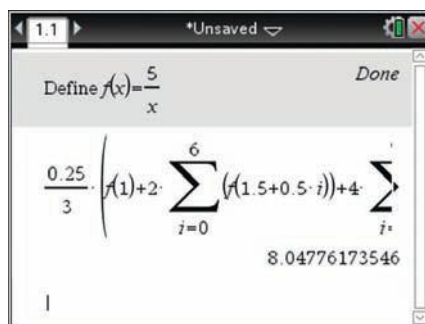
Write the required approximation.

$$\int_1^5 \frac{5}{x} dx \approx \frac{0.25}{3} \left[f(1) + 2 \sum_{i=0}^6 f(1.5 + 0.5i) + 4 \sum_{i=0}^7 f(1.25 + 0.5i) + f(5) \right]$$

TI-Nspire CAS

Define the function.

Use the template menu $\left[\frac{\square}{\square} \right]$ to get the sums to find the approximation.



ClassPad

Define the function.

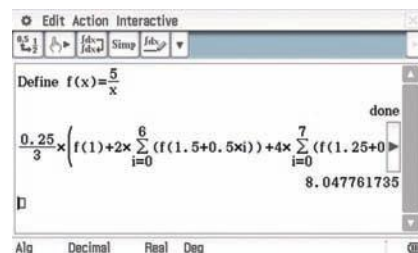
Enter

$$\frac{0.25}{3} \left[f(1) + 2 \sum_{i=0}^6 f(1.5 + 0.5i) + 4 \sum_{i=0}^7 f(1.25 + 0.5i) + f(5) \right]$$

using $\sum$ from $\left[\text{Math2} \right]$. Make sure that you use i from the $\left[\text{abc} \right]$ menu, not i for imaginary numbers.

Round and write the answer.

$$\int_1^5 \frac{5}{x} dx \approx 8.0478$$



For a cubic, quadratic or straight line, Simpson's rule actually gives the exact answer. Your CAS calculator uses methods that are more advanced than Simpson's rule to find approximations for integrals that cannot be integrated algebraically.

EXERCISE 9.03 Numerical integration

Concepts and techniques



Numerical integration



Trapezoidal rule

- 1 **Example 7** **CAS** An approximation for the definite integral of $f(x) = \frac{1}{2}x^2$, from $x = 0$ to $x = 4$ with strips 1 unit wide, using the midpoint rule is
A 7 B 10 C $10\frac{1}{2}$ D $10\frac{2}{3}$ E $12\frac{1}{8}$
- 2 **CAS** An approximation for the area under the curve $y = \frac{1}{2}x^2$ from $x = 0$ to $x = 4$ with strips $\frac{1}{2}$ a unit wide, using the midpoint rule is
A $5\frac{1}{3}$ B 6.0625 C $10\frac{1}{2}$ D 10.625 E $10\frac{2}{3}$
- 3 **Example 8** **CAS** An approximation for the area under the curve $y = \frac{1}{2}x^2$ from $x = 0$ to $x = 1$ with strips a $\frac{1}{4}$ unit wide, using the trapezoidal rule is
A $\frac{11}{64}$ B $\frac{21}{128}$ C $\frac{1}{6}$ D 10.625 E $10\frac{2}{3}$
- 4 **CAS** An approximation for the area under the curve $y = \frac{1}{2}x^2$ from $x = 0$ to $x = 4$ with strips a $\frac{1}{4}$ unit wide, using the trapezoidal rule is
A $\frac{5}{64}$ B $\frac{21}{128}$ C $\frac{1}{6}$ D $\frac{13}{64}$ E $10\frac{11}{16}$
- 5 **Example 9** **CAS** An approximation for the area under the curve $y = \frac{1}{2}x^2$ from $x = 0$ to $x = 2$ with strips a $\frac{1}{4}$ unit wide, using Simpson's rule is
A $\frac{5}{64}$ B $\frac{21}{128}$ C $\frac{4}{3}$ D $\frac{13}{64}$ E $1\frac{1}{2}$
- 6 **CAS** An approximation for the area under the curve $y = \frac{1}{2}\log_e(x)$ from $x = 1$ to $x = 4$ with strips a $\frac{1}{4}$ unit wide, using Simpson's rule is
A 1.265 B 1.582 C 1.273 D 1.908 E 1.272
- 7 **CAS** For the area under the curve $y = \log_{10}(3x)$ from $x = 1$ to $x = 6$:
a find an approximation using the midpoint rule with 10 strips
b find an approximation using the trapezoidal rule with 10 strips
c find an approximation using Simpson's rule with 10 strips
d find the area using the algebraic integral.
e Comment on your answers.
- 8 **CAS** For the area under the curve $y = 2x^3 - 4x^2 + 9x + 6$ from $x = 2$ to $x = 4$:
a find an approximation using the midpoint rule with 16 strips
b find an approximation using the trapezoidal rule with 16 strips
c find an approximation using Simpson's rule with 16 strips
d find the area using the algebraic integral.
e Comment on your answers.

Reasoning and communication

- 9 Derive the midpoint rule.
- 10 Three points are given by (a, h_1) , $(a + w, h_2)$ and $(a + 2w, h_3)$, where $h_1, h_2, h_3 > 0$.
Show that the area under the parabola passing through the three points is given by $\frac{w}{3}(h_1 + 4h_2 + h_3)$.

- 11 Given that the area of a double strip is given by $\frac{w}{3}(a_{2i} + 4a_{2i+1} + a_{2i+2})$ [for $i = 0$ to $\frac{n}{2} - 1$], derive Simpson's rule.

9.04 THE EXPONENTIAL PROBABILITY DENSITY FUNCTION

You have previously worked with the probability density function, and should remember that the area under the curve equals 1.

IMPORTANT

The **probability density function**, or pdf, of a continuous random variable is the function that when integrated over an interval gives the probability that the continuous random variable, having that pdf, lies in that interval.

You can choose any function that satisfies the conditions to be a probability density function. The choice is really governed by how you want to use the distribution you obtain. In a pdf, you expect to find the area under the curve to be equal to 1.

You can apply the particular probability density function, $f(t) = \lambda e^{-\lambda t}$ for $t \geq 0$, using the exponential random variable with parameter $\lambda > 0$, and use the exponential random variables and associated probabilities to model data and solve problems.

In this section you will consider the **exponential distribution**, with probability density function $f(t) = \lambda e^{-\lambda t}$ for $t \geq 0$.

Since the exponential function $f(t) \geq 0$ for all values of t , to show that it is a probability density function you need only show that the integral is equal to 1.

For the integral $\int_{-\infty}^{\infty} f(t) dt$, where $f(t) = \lambda e^{-\lambda t}$ for $t \geq 0$,

$$\begin{aligned}\int_{-\infty}^{\infty} f(t) dt &= \int_{-\infty}^0 f(t) dt + \int_0^{\infty} f(t) dt \\ &= \int_{-\infty}^0 0 dt + \int_0^{\infty} \lambda e^{-\lambda t} dt \\ &= 0 + \left[-e^{-\lambda t} \right]_0^{\infty} \\ &= 0 + [0 - (-1)] \\ &= 1\end{aligned}$$

QED

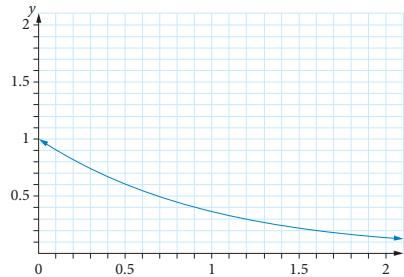
This is a distribution defined for all positive values of t . However, since it has a parameter λ , it is also a family of distributions, determined by the value of λ .

Example 10

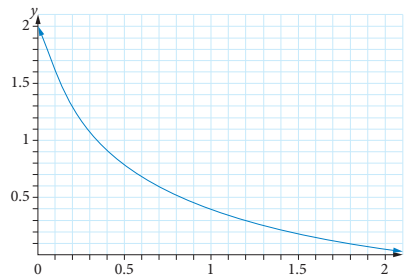
Draw graphs of the exponential distribution $f(t) = \lambda e^{-\lambda t}$ for $\lambda = 1$ and $\lambda = 2$, and comment on the shapes of the distributions.

Solution

Draw the graph of $f(t) = \lambda e^{-\lambda t}$ with $\lambda = 1$.



Draw the graph of $f(t) = \lambda e^{-\lambda t}$ with $\lambda = 2$.



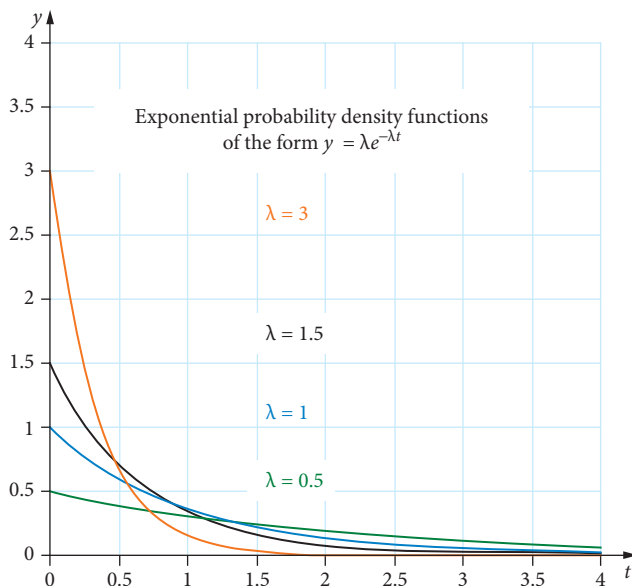
Compare the graphs.

For $\lambda = 2$, the distribution is twice as high at $t = 0$ as for $\lambda = 1$ and tapers to zero more quickly. This means that, for $\lambda = 2$, $f(t)$ has a higher probability of being close to zero.

Summarise the result.

The distribution with the higher value of λ starts higher and is skewed more to the left, so it has a higher probability of a lower value of t .

A summary of the effect of the λ value is below.



Remember that the rule for finding the probability of occurring in the interval $[a, b]$ is $P(a \leq x \leq b) = \int_a^b f(x) dx$

Example 11

A random variable X has an exponential distribution with $\lambda = 0.8$. Find $P(0 \leq x \leq 2)$.

Solution

Write the rule $P(a \leq x \leq b) = \int_a^b f(x) dx$.

$$P(0 \leq x \leq 2) = \int_0^2 f(x) dx$$

Write the rule for the exponential distribution with $\lambda = 0.8$.

$$f(x) = 0.8e^{-0.8x}$$

Substitute into the integral.

$$P(0 \leq x \leq 2) = \int_0^2 0.8e^{-0.8x} dx$$

Evaluate the integral.

$$\begin{aligned} &= \left[-\frac{0.8}{0.8} e^{-0.8x} \right]_0^2 \\ &= -e^{-1.6} + e^0 \\ &= 1 - e^{-1.6} \end{aligned}$$

Write the answer.

$$\begin{aligned} P(0 \leq x \leq 2) &= 1 - e^{-1.6} \\ &\approx 0.798 \end{aligned}$$

The exponential distribution is particularly important in calculating the probabilities of failures. You have seen in Maths Methods that the expected value of a continuous random variable X with probability density function $f(x)$ is given by $E(X) = \int_{-\infty}^{\infty} xf(x) dx$.

Expected value of an exponential distribution

The expected value of an exponential distribution with parameter λ is given by $E(X) = \frac{1}{\lambda}$.

This is easily proven using integration by parts with $f(x) = x$ and $g(x) = -e^{-\lambda x}$ to give

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx.$$

$$\begin{aligned}\int_0^{\infty} x\lambda e^{-\lambda x} dx &= \left[-xe^{-\lambda x} \right]_0^{\infty} - \int_0^{\infty} 1 \cdot (-e^{-\lambda x}) dx \\ &= \left[-xe^{-\lambda x} \right]_0^{\infty} + \int_0^{\infty} 1 \cdot (e^{-\lambda x}) dx \\ &= \left[-xe^{-\lambda x} \right]_0^{\infty} + \left[-\frac{1}{\lambda} e^{-\lambda x} \right]_0^{\infty} \\ &= (0 - 0) + \left[0 - \left(-\frac{1}{\lambda} \right) \right] \\ &= \frac{1}{\lambda}\end{aligned}$$

This means that the exponential distribution with average value β has the parameter $\lambda = \frac{1}{\beta}$. For this reason, some people prefer to use the parameter $\frac{1}{\beta}$ in place of λ .

○ Example 12

The expected life (in hours) of a light bulb can be considered to be a random variable with an exponential distribution. For incandescent bulbs, $\lambda = \frac{1}{900}$.

- Find the probability that an incandescent bulb lasts at least 1000 h.
- Find the expected time that an incandescent bulb lasts for.
- Compact fluorescent bulbs (CFLs) generally last 8 times as long as incandescent bulbs. What percentage of fluorescent bulbs would be expected to last at least 1000 h?
- LED bulbs last 5 times as long as CFLs and are cheaper to run than either CFLs or incandescent bulbs. What percentage of LED bulbs will last longer than 1000 hours?



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Solution

- a Write the required probability.

Write the rule for a probability density function.

Substitute values using $\lambda = \frac{1}{900}$,
 $f(x) = \frac{1}{900} e^{-\frac{1}{900}x}$ and $a = 1000$, $b = \infty$.

Integrate.

Write the answer.

- b Find the expected value.

Write the answer.

- c Write $E(X)$ for fluorescent bulbs.

Find λ for fluorescent bulbs.

Find the probability that $1000 \leq X \leq \infty$.

Find the integral.

Evaluate.

Write the answer.

- d Write $E(X)$ for LED bulbs.

Find λ for LED bulbs.

Find the probability that $1000 \leq X \leq \infty$.

Find the integral.

Evaluate.

Write the answer.

$P(x \geq 1000)$ is needed.

$$P(a \leq x \leq b) = \int_a^b f(x) dx$$

$$\begin{aligned} P(1000 \leq x) &= \int_{1000}^{\infty} \frac{1}{900} e^{-\frac{1}{900}x} dx \\ &= \left[-e^{-\frac{1}{900}x} \right]_{1000}^{\infty} \\ &= 0 - \left(-e^{-\frac{1000}{900}} \right) \\ &\approx 0.3292 \end{aligned}$$

The probability is about 0.329.

$$E(X) = \frac{1}{\lambda} = 900$$

The expected time that a light bulb will work for is 900 hours.

Expected life = $8 \times 900 = 7200$

$$\lambda = \frac{1}{7200}$$

$$\begin{aligned} P(1000 \leq X) &= \int_{1000}^{\infty} \frac{e^{-x/7200}}{7200} dx \\ &= \left[-e^{-x/7200} \right]_{1000}^{\infty} \\ &= 0.8703... \end{aligned}$$

About 87% can be expected to last at least 1000 hours.

Expected life = $5 \times 7200 = 36\,000$

$$\lambda = \frac{1}{36\,000}$$

$$\begin{aligned} P(1000 \leq X \leq \infty) &= \int_{1000}^{\infty} \frac{e^{-x/36\,000}}{36\,000} dx \\ &= \left[-e^{-x/36\,000} \right]_{1000}^{\infty} \\ &= 0.9726... \end{aligned}$$

About 97% can be expected to last at least 1000 hours.

EXERCISE 9.04 The exponential probability density function



Exponential distributions

Concepts and techniques

- 1 **Example 10** Draw graphs of the exponential probability density function $f(t) = \lambda e^{-\lambda t}$ for the following values of λ and compare the shapes of the functions.
a $\lambda = \frac{1}{3}$ b $\lambda = 1$ c $\lambda = 3$ d $\lambda = \frac{1}{30}$
- 2 **Example 11** A random variable T has an exponential distribution with $\lambda = 1.6$. $P(0 \leq x \leq 2)$ equals
A $1 - e^{-\frac{16}{5}}$ B $e^{-\frac{16}{5}} - 1$ C $e^{\frac{16}{5}} - 1$ D 0.959 E $1 - e^{-\frac{8}{5}}$
- 3 A random variable T has an exponential distribution with $\lambda = 2$. $P(1 \leq x \leq 2)$ equals
A 0.117 B $\frac{e^4}{e^2 - 1}$ C $\frac{e^2 - 1}{e^4}$ D $e^4 - e^{-2}$ E $1 - e^{-4}$
- 4 A random variable T has an exponential distribution with $\lambda = 8$. $P(1 \leq x \leq 3)$ equals
A 0.0003 B $\frac{e^{16} - 1}{e^{24}}$ C $e^{-8} - e^{24}$ D $e^{-8} - e^{-16}$ E $1 - e^{-16}$
- 5 A random variable T has an exponential distribution with $\lambda = \frac{1}{100}$. $P(1 \leq x \leq 10)$ equals
A 0.085 B 0.0852 C $1 - e^{-0.1}$
D $e^{-\frac{1}{100}} + e^{-\frac{1}{10}}$ E $e^{-\frac{1}{100}} - e^{-\frac{1}{10}}$
- 6 Draw graphs of the exponential probability density functions for the following values, and comment on the shapes of the functions.
a $\lambda = 0.25$ and $\lambda = 2$ b $\lambda = 0.1$ and $\lambda = 0.2$ c $\lambda = 0.7$ and $\lambda = 0.9$
- 7 A random variable X has an exponential distribution with $\lambda = 0.6$. Find each of the following.
a $P(0 \leq x \leq 3)$ b $P(3 \leq x \leq 10)$ c $P(9.5 \leq x \leq 10.5)$ d $P(x \geq 10)$
- 8 A random variable T has an exponential distribution with $\lambda = 3$. Find each of the following.
a $P(0 \leq t \leq 0.3)$ b $P(0.4 \leq t \leq 0.6)$ c $P(t \geq 0.3)$ d $P(t \geq 0.5)$
- 9 **Example 12** An exponential distribution has parameter $\lambda = 1.5$. What is the expected value of the distribution?
- 10 What is the parameter λ for an exponential distribution with an average value of 16?

Reasoning and communication

- 11 The expected life (in hours) of a printer drum can be considered to be a random variable with an exponential distribution, for which $\lambda = \frac{1}{200}$.
a Find the probability that a particular printer drum lasts at least 100 h.
b Find the expected time that a particular printer drum will last.

- 12 The average life of a standard car battery is known to be 1.5 years. Use an exponential distribution to model the life of standard car batteries to find, correct to 3 decimal places, the values of:
- λ
 - the proportion of standard batteries that fail before 1.5 years
 - the proportion of standard batteries that last for at least 4 years.
- 13 The average life of a 'calcium' car battery is known to be 2.5 years. Use an exponential distribution to model the life of 'calcium' car batteries to find (correct to 3 significant figures):
- the percentage of 'calcium' batteries that fail before 2.5 years
 - the median life of a car battery
 - the percentage of 'calcium' batteries that last for at least 4 years.

9.05 APPLICATIONS OF THE EXPONENTIAL PROBABILITY DENSITY FUNCTION

The exponential probability density function, $f(t) = \lambda e^{-\lambda t}$ for $t \geq 0$, with parameter $\lambda > 0$, has wide application in probabilities associated with lifetimes, failures and the occurrence of events at fixed average rates. It is worthwhile specifying the cumulative distribution function for an exponential probability density function.

IMPORTANT

The **cumulative distribution function** for an exponential probability function with parameter λ is given by $\text{cdf}(t) = 1 - e^{-\lambda t}$.

One of the most important properties of the exponential probability distribution is often called its 'lack of memory'. This is because the probability of occurrence within a fixed time interval is the same throughout the distribution, no matter how long it is from the start. It is the only continuous probability distribution with this property.

IMPORTANT

For events with an exponential probability density function, the probability of an event occurring in the interval $[p, p + s]$ is equal to the probability that it occurs in the interval $[0, s]$ for any values s and p .

In other words, the probabilities of occurrence in equal intervals are equal.

This can be proven using the cumulative density function parameter for the random variable with parameter λ .

Clearly, the probability of an event occurring in the interval $[p, p + s]$ is conditional on it *not having occurred* in the interval $[0, p]$.

$$\begin{aligned}
 P(p \leq x \leq p + s \text{ given } x > p) &= \frac{cdf(p+s) - cdf(p)}{1 - cdf(p)} \\
 &= \frac{(1 - e^{-\lambda(p+s)}) - (1 - e^{-\lambda p})}{1 - (1 - e^{-\lambda p})} \\
 &= \frac{e^{-\lambda p} - e^{-\lambda(p+s)}}{e^{-\lambda p}} \\
 &= \frac{e^{-\lambda p} (1 - e^{-\lambda s})}{e^{-\lambda p}} \\
 &= 1 - e^{-\lambda s}
 \end{aligned}$$

But this is $cdf(s)$.

QED

○ Example 13

At a call centre, the occurrence of calls over time follows an exponential probability distribution with a parameter of $\lambda = 2$, where time is in minutes.

- What is the probability of getting a call in the next 3 minutes?
- What is the expected waiting time to get a call?
- What is the average rate of calls per minute?

Solution

- a Write the cdf.

$$cdf(t) = 1 - e^{-\lambda t}$$

Substitute the values $t = 3$ and $\lambda = 2$.

$$cdf(3) = 1 - e^{-2 \times 3}$$

Evaluate.

$$= 0.9975\dots$$

Write the answer.

The probability is about 99.75%

- b Write the formula for the expected value.

$$E(X) = \frac{1}{\lambda}$$

Substitute $\lambda = 2$.

$$= 0.5$$

Write the answer.

The expected waiting time to the next call is half a minute.

- c Use the inverse.

The expected waiting time is 0.5 minutes, so the average rate is 2 calls per minute.

Example 13 demonstrates the relationship between the average rate of occurrences, the parameter and the expected waiting time for an exponential probability distribution.

IMPORTANT

An exponential probability distribution of occurrences over time, with parameter λ , has an average rate of occurrence of λ and an expected waiting time of $\frac{1}{\lambda}$.

Conversely, if the rate of occurrence of an event is λ , the occurrence follows an exponential probability distribution with parameter λ .

This can be proven using the same method as shown in Example 13.

Car batteries are generally guaranteed for 12 months or two years. Rechargeable batteries last for 500–800 discharge-recharge cycles. There are many other items that eventually fail, and the probability of failure over time or the number-of-use cycles is often an exponential distribution.

You are probably more interested in the probability that something will keep working than the probability that it will fail. This is called its **reliability** and is the complement of the probability of failure.

IMPORTANT

The **reliability function** $R(t) = 1 - P(\text{failure})$. The independent variable in a reliability function is usually time or the number of uses (cycles).

If the average rate of failure is λ , then $R(x) = e^{-\lambda x}$.

The reliability function $R(t)$ gives the probability that the item will keep working for time t (or for the number of times t).

Example 14

A resistor has a constant failure rate of 0.04 per hour.

- What is the probability that the resistor will work for 100 hours before it fails?
- If 100 resistors are tested, how many would be expected to fail in 25 hours?

Solution

- a Use $R(x) = e^{-\lambda x}$ where $\lambda = 0.04$.

$$R(x) = e^{-0.04x}$$

Find the reliability at 100 hours.

$$R(100) = e^{-0.04(100)} = e^{-4}$$

$$\therefore R(100) \approx 0.0183$$

Write the answer.

Probability of working for 100 hours is 0.0183.

- b Find the reliability.

$$R(t) = e^{-\lambda t}$$

Substitute in the values.

$$= e^{-0.04 \times 25}$$

$$\approx 0.3679$$

Write the probability of failure,
 $1 - R(t)$.

Probability of failing ≈ 0.6321

Find the expected number of failures out
of 100.

$$\begin{aligned}\text{Expected number} &= 100 \times 0.6321 \\ &\approx 63\end{aligned}$$

Write the answer.

About 63 resistors will fail within
25 hours.

There are often cases where you want to know a **quantile** for an exponential distribution.

Remember that a quantile t_α is the value of t such that the area under the curve below t is equal to α , so $P(-\infty \leq t \leq t_\alpha) = \alpha$. For an exponential probability function, $t \geq 0$, so the expression for a quantile becomes $P(0 \leq t \leq t_\alpha) = \alpha$.

○ Example 15

The reliability of an aircraft altimeter is 0.998 for 10 h of operation. Assuming that the life of the instrument is exponential, for what period is the probability of failure less than 5%?

Solution

Write the reliability for 10 h.

$$R(10) = 0.998$$

Write the reliability function.

$$R(t) = e^{-\lambda t}$$

Substitute in the values.

$$0.998 = e^{-\lambda \times 10}$$

Solve for λ .

$$\log_e(0.998) = -10\lambda$$

Evaluate.

$$\lambda = 0.0002...$$

Find t such that $R(t) = 0.95$.

$$e^{-0.0002...t} = 0.95$$

Solve for t .

$$-0.0002...t = \log_e(0.95)$$

Divide by -0.0002 .

$$t = 256.2...$$

Write the answer.

The probability of failure is less than 5% for about 256 h.

It is sometimes useful to know the variance and standard deviation for an exponential distribution.

You should remember from Maths Methods that for a continuous probability density function

defined on the interval $[a, b]$, $\text{Var}(X) = \int_a^b p(x)(x - \mu)^2 dx$, where μ is the expected value of X .

For an exponential probability density function with parameter λ , this gives

$$\text{Var}(X) = \int_0^\infty \lambda e^{-\lambda x} \left(x - \frac{1}{\lambda}\right)^2 dx$$

Using integration by parts *twice* gives $\text{Var}(X) = \frac{1}{\lambda^2}$, so the standard deviation is $\sigma = \frac{1}{\lambda}$.

You can calculate the median as $t_{0.5}$ using $0.5 = \text{cdf}(x) = 1 - e^{-\lambda t}$.

$e^{-\lambda t} = 0.5$, so the median is $\frac{-\log_e(0.5)}{\lambda} = \frac{\log_e(2)}{\lambda}$.

IMPORTANT

An exponential probability function with parameter λ has the following properties:

Expected value (*mean*) $\mu = \frac{1}{\lambda}$

Variance $\sigma^2 = \frac{1}{\lambda^2}$

Standard deviation $\sigma = \frac{1}{\lambda}$

Median is $\frac{\log_e(2)}{\lambda}$

EXERCISE 9.05 Applications of the exponential probability density function



Reliability

Concepts and techniques

- 1 **Example 13** A battery has a constant failure rate of 0.02 per hour.
- a The battery's reliability at 200 hours is
 A e^{-4} B $e^{-0.02}$ C $e^{0.02}$ D 0.18 E 0.018
- b If 50 batteries are tested, the number expected to be in a failed state after 20 hours is
 A $1 - e^{-0.4}$ B $1 - e^{-4}$ C $e^{0.02}$ D 16 E 17



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- 2 A machine part has a constant failure rate of 0.2 per hour.
- a The machine part's reliability at 100 hours is
 A e^{-20} B $1 - e^{-20}$ C $1 - e^{-0.2}$ D $-e^{-0.2}$ E e^{-100}
- b If 60 machine parts are tested, the number expected to be in a failed state after 10 hours is
 A $1 - e^{-2}$ B e^{-2} C 51 D 52 E $1 - e^{-20}$
- 3 The life of an item that has a mean time to fail of 100 hours is exponentially distributed. The probability of surviving through the interval 0 to 20 hours is
 A 0.8751 B $e^{-\frac{1}{5}}$ C 0.818 D $e^{-\frac{2}{5}}$ E $e^{\frac{1}{5}}$
- 4 The life of a hair dryer that has a mean time to fail of 90 hours is exponentially distributed. The probability of surviving through the interval 0 to 15 hours is
 A 0.8751 B $e^{\frac{1}{6}}$ C 0.846 D $e^{-\frac{1}{6}}$ E $e^{\frac{1}{5}}$

- 5 If you receive phone calls at an average rate of 3 per hour, how long can you expect to wait for the next call?
- 6 The average number of clicks on a Geiger counter is about 25 per minute.
 - a How long can you expect to wait for the next click?
 - b What is the standard deviation of the exponential distribution?
 - c What is the median of the exponential distribution?
 - d What is the first quartile?
 - e What is the third quartile?

Reasoning and communication

- 7 **Examples 14, 15** The average response time of students to teacher questions is about 14 seconds. Find the probability that the next question will be answered in:
 - a i less than 8 s ii more than 8 s
 - b Comment on the total of these probabilities.
- 8 For an exponential distribution with parameter λ , find the value of λ such that $P(0 \leq x \leq \lambda) = 0.8$.
- 9 The time between successive phone calls at a call centre is known to follow an exponential distribution. 90% of the intervals between calls are less than 2 minutes. What percentage of the time intervals will be less than 5 minutes?
- 10 A computer-controlled machine lathe takes 4 minutes to complete a particular job. It completes this job with a reliability of 0.99. What will be its reliability if the time allowed is:
 - a 3 min? b 5 min? c 10 min?
- 11 20% of batteries last longer than 4 years. Use an exponential model to find the guarantee period that a manufacturer should set to ensure that no more than 20% of batteries fail within the guarantee period.
- 12 Light switches have an average life of 3000 cycles and failures follow an exponential pattern.
 - a What is the parameter λ for failure?
 - b What is the probability that a light switch can be switched on and off 5000 times before it fails?
- 13 The average time taken to find a spare part for a car at a spare parts distributor is about 6 minutes (if it is in stock). Use the exponential distribution to determine:
 - a the proportion of spare parts that take less than 2 minutes to find
 - b the median time
 - c the probability that the next spare part will take longer than 10 minutes to find.



CHAPTER SUMMARY

APPLICATIONS OF INTEGRATION

- An area between two curves can be calculated in two ways:

- as the difference between the areas under the two functions; or
- as the area under the difference function.

- **Volume of a shape** $V = \int A(h)dh = \int A dh$

- **Volumes of solids of revolution**

The volume of a solid of revolution generated by a curve in the xy plane is given by

$$V = \pi \int y^2 dx \text{ when rotated about the } x\text{-axis}$$

$$\text{or } V = \pi \int x^2 dy \text{ when rotated about the } y\text{-axis.}$$

- The **midpoint rule** is that

$$\int_a^b f(x)dx \approx w \sum_{i=1}^n f(a_i), \text{ where the interval } [a, b] \text{ is divided into } n \text{ equal width strips of width } w \text{ and the values } a_1, a_2, \dots, a_n \text{ are the midpoints of the strips.}$$

- The **trapezoidal rule** is that

$$\int_a^b f(x)dx \approx \frac{w}{2} \left(f(a_0) + 2 \sum_{i=1}^{n-1} f(a_i) + f(a_n) \right),$$

where the interval $[a, b]$ is divided into n equal width strips of width w and values $a_0, a_1, \dots, a_n$. It can also be written as

$$\int_a^b f(x)dx \approx \frac{w}{2} (E + 2M), \text{ where } E \text{ is the sum of the values of } f(x) \text{ at the ends of the interval and } M \text{ is the sum of the values at the intervening points } a_1, a_2, \dots, a_{n-1}.$$

- **Simpson's rule** is that

$$\int_a^b f(x)dx \approx \frac{w}{3} \left(f(a_0) + 2 \sum_{i=1}^{\frac{n}{2}-1} f(a_{2i}) + 4 \sum_{i=1}^{\frac{n}{2}} f(a_{2i-1}) + f(a_n) \right),$$

where the interval $[a, b]$ is divided into an even number n of equal width strips of width w and end values $a_0, a_1, \dots, a_n$. It can be

$$\text{written as } \int_a^b f(x)dx \approx \frac{w}{3} (X + 2E + 4O),$$

where X is the sum of the values of $f(x)$ at the ends of the interval, E is the sum of the values at the even-numbered intervening points $a_2, a_4, \dots, a_{n-2}$ and O is the sum of the values at the odd-numbered intervening points $a_1, a_3, \dots, a_{n-1}$.

- The **probability density function, pdf**, of a continuous random variable is the function that when integrated over an interval gives the probability that the continuous random variable having that pdf, lies in that interval.

- The exponential distribution with parameter λ has the probability density function

$$f(t) = \begin{cases} \lambda e^{-\lambda t}, & \text{for } t \geq 0 \\ 0, & \text{elsewhere} \end{cases}$$

$$\text{where } P(a \leq t \leq b) = \int_a^b f(t)dt$$

- **Expected value of a continuous probability distribution**

For a continuous random variable X with probability density function $f(x)$, the expected value of the random variable is given by

$$E(X) = \int_{-\infty}^{\infty} xf(x)dx, \text{ provided the integral exists.}$$

- The exponential reliability function is

$$R(x) = e^{-\frac{x}{\theta}} \quad \text{for } x > 0$$

$$\therefore R(x) = e^{-\lambda x}$$

- A **quantile** t_α is the value of t such that the area under the curve below t is equal to α , so $P(-\infty \leq t \leq t_\alpha) = \alpha$.

- The **expected value** of a continuous probability distribution for a continuous random variable X with probability density function $f(x)$, $E(X) = \int_{-\infty}^{\infty} xf(x)dx$, provided the integral exists.

- The exponential reliability function is

$$R(x) = e^{-\lambda x} \quad \text{for } x > 0$$

- A **quantile** t_α is the value of t such that the area under the curve below t is equal to α , so $P(-\infty \leq t \leq t_\alpha) = \alpha$.

- An exponential random variable with parameter λ has the following properties:

$$\text{Expected value (mean)} \mu = \frac{1}{\lambda}$$

$$\text{Variance } \sigma^2 = \frac{1}{\lambda^2}$$

$$\text{Standard deviation } \sigma = \frac{1}{\lambda}$$

$$\text{Median is } \frac{\log_e(2)}{\lambda}$$

CHAPTER REVIEW

APPLICATIONS OF INTEGRATION

Multiple choice

- 1 **Example 2** The area enclosed by $y = 2x^2 - x - 6$ and $y = x^2 + 2x + 4$ is:
 A $93\frac{1}{3}$ B $57\frac{1}{6}$ C $36\frac{1}{6}$ D $5\frac{5}{24}$ E $7\frac{7}{24}$
- 2 **Example 4** The formula for the volume of an equilateral triangular-based pyramid of side s and height h is
 A $V = \frac{1}{2}bsh$ B $V = s^3h$ C $V = \frac{\sqrt{3}}{12}s^3h$
 D $V = \frac{\sqrt{3}}{4}s^2h$ E $V = \frac{\sqrt{3}}{12}s^2h$
- 3 **Example 7** An approximation for the definite integral of $f(x) = 2\sqrt{x-1}$ from $x = 1$ to $x = 5$ by calculating the areas of strips 1 unit wide, using the midpoint rule is closest to
 A 10.77 square units B 5.16 square units C 15.01 square units
 D 4 square units E 9.71 square units
- 4 **Example 9** An approximation for the area under the curve $y = \frac{3}{4}x^2 + 1$ from $x = 1$ to $x = 5$ with strips 1 unit wide, using the trapezium rule is
 A $\frac{35}{8}$ B $\frac{71}{2}$ C $\frac{139}{4}$ D $\frac{53}{2}$ E $\frac{89}{2}$
- 5 **Example 11** A random variable T has an exponential distribution with $\lambda = 0.6$. $P(0 \leq x \leq 3)$ equals
 A $1 - e^{-0.6}$ B $e^{-\frac{9}{5}} - 1$ C $e^{\frac{16}{5}} - 1$ D 0.835 E $1 - e^{-\frac{9}{5}}$

Short answer

- 6 **Example 1** Find the area enclosed by $f(x) = x^2 - 2x$ and $g(x) = -x^2 + x$.
- 7 **Example 2** Find the area enclosed by $f(x) = 3x^2 - 2x - 5$ and $g(x) = 2x^2 + 3x + 9$.
- 8 **Example 3** Find the area enclosed by $f(x) = x^3 - 5x^2 - x + 13$ and $g(x) = 3x - 7$.
- 9 **Example 5** Use a suitable definite integral to find the volumes, in cubic units, of solids formed by rotating the following curves about the x -axis between the limits shown.
 a $y = x^2$, from $x = 0$ to $x = 3$ b $y = 2e^{2x}$, from $x = 1$ to $x = 4$
- 10 **Example 6** Use a suitable definite integral to find the volumes, in cubic units, of solids formed by rotating the following curves about the y -axis between the limits shown.
 a $y = x^2$, from $y = 0$ to $y = 3$ b $y = \log_e(2x)$, from $y = 0$ to $y = 1$.

- [illegible]

Application

- 14 **Example 14** What is the parameter λ for an exponential distribution with an average value of 16?
- 15 **Example 13** A car radio has a constant failure rate of 0.01 per hour.
 - a What is the car radio's probability of lasting at least 200 hours?
 - b If 100 radios are tested, how many would be expected to fail within 250 hours?
- 16 **Examples 14, 15** The average response time *in seconds* of students to exam questions is an exponential distribution with $\lambda \approx 0.08$. Find the probability that the next question will be answered in:
 - a i less than 9 s ii more than 9 s
 - b Comment on the total of these probabilities.
- 17 The Lorenz curve of income distribution for a small country is approximately
$$y = 0.2x^4 + 0.4x^3 + 0.4x^2$$
 - a Sketch the Lorenz curve.
 - b What proportion of total income is earned by the poorest 20% of the population?
 - c What is the coefficient of income inequality?
- 18 Find the area enclosed by $y = x^4 - 8x^2 + x + 6$ and $y = 2x^2 + x - 3$.
- 19 For an exponential distribution with parameter λ , find the value of λ such that $P(0 \leq x \leq \lambda) = 0.9$.
- 20 A modern aircraft jet engine will operate for an average of 2000 flights before needing to be shut down in flight. This may not even be noticed by passengers because of safety margins and pilot training. What is the probability that an engine will last for 3000 flights without problems?

Practice quiz



10

TERMINOLOGY

binomial distribution
central limit theorem
confidence interval
confidence level
cumulative density function
estimator
interval estimate
margin of error
normal distribution
parameter
point estimate
population
probability density function
probability function
quantile
random sample
random variable
sample
sample mean
sampling distribution
standard deviation
statistic
triangular distribution
uniform distribution
variance

STATISTICAL INFERENCE

STATISTICAL INFERENCE

10.01 Sample means

10.02 Sampling distribution simulations

10.03 Mean and standard deviation of sample means

10.04 Confidence intervals for normal distributions

10.05 Estimation of population parameters

10.06 Survey procedures, confidence intervals and data quality

Chapter summary

Chapter review



Prior learning

SAMPLE MEANS

- examine the concept of the sample mean $\bar{X}$ as a random variable whose value varies between samples where X is a random variable with mean μ and the standard deviation σ (ACMSM137)
- simulate repeated random sampling, from a variety of distributions and a range of sample sizes, to illustrate properties of the distribution of $\bar{X}$ across samples of a fixed size n , including its mean μ , its standard deviation $\frac{\sigma}{\sqrt{n}}$ (where μ and σ are the mean and standard deviation of X), and its approximate normality if n is large (ACMSM138)
- simulate repeated random sampling, from a variety of distributions and a range of sample sizes, to illustrate the approximate standard normality of $\frac{\bar{X} - \mu}{s/\sqrt{n}}$ for large samples ($n \geq 30$), where s is the sample standard deviation. (ACMSM139)

CONFIDENCE INTERVALS FOR MEANS

- understand the concept of an interval estimate for a parameter associated with a random variable (ACMSM140)
- examine the approximate confidence interval $\left(\bar{X} - \frac{zs}{\sqrt{n}}, \bar{X} + \frac{zs}{\sqrt{n}} \right)$, as an interval estimate for μ where z is the appropriate quantile for the standard normal distribution (ACMSM141)
- use simulation to illustrate variations in confidence intervals between samples and to show that most but not all confidence intervals contain μ (ACMSM142)
- use $\bar{x}$ and s to estimate μ and σ , to obtain approximate intervals covering desired proportions of values of a normal random variable and compare with an approximate confidence interval for μ (ACMSM143)
- collect data and construct an approximate confidence interval to estimate a mean, and report on survey procedures and data quality (ACMSM144) 

10.01 SAMPLE MEANS

You have been studying the concepts of **random variables** and **random samples** in Maths Methods.

IMPORTANT

A **random variable** is a variable with a numerical value that depends on the outcome of a chance experiment. A **continuous random variable** can be any real value, or any real value from an interval. A **discrete random variable** can be a finite set, the counting numbers or an equivalent set.

A **random sample** is selected using a method that ensures that every member of the population has an equal chance of being chosen.

A random variable is usually denoted by a capital letter. The corresponding lower case letter denotes specific values of the variable. You usually calculate the **mean** of a sample as it is considered as the most important characteristic.

○ Example 1

A sample of 10 students from a Year 12 class has heights of 184, 189, 196, 182, 168, 171, 177, 165, 178 and 197 cm. What is the sample mean?

Solution

Calculate the total.

$$\text{Total} = 184 + 189 + \dots = 1807$$

Find the mean.

$$\text{Mean} = \frac{1807}{10} = 180.7 \text{ cm}$$

State the answer.

The sample mean is 180.7 cm.

The mean and standard deviation of a population use different symbols to those of a sample.

IMPORTANT

Values that are obtained from a population are called **parameters**, while those from samples are called **statistics**.

The population mean and standard deviation parameters are shown as μ and σ respectively.

Sample mean and standard deviation statistics are shown as $\bar{x}$ and s respectively.

The size of a finite population is shown as N and the number in a sample is shown as n .

INVESTIGATION Sample heights

- Put names in a hat or use other random means to get a sample of 6 people from your class.
- Calculate the mean height of the students in the sample.
- Put the names back into the hat and repeat the process.
- Keep doing this until you have at least 10 samples.
- Find the mean height and standard deviation of the whole class.
- What do you find about the sample means?
- Calculate the standard deviation of the means and compare to the standard deviation of the heights of people in the class.
- What can you conclude, if anything, about the sample means?
- How many different possible samples of 6 people can you take from your class?
- How many possible samples can you take from your class if any sample size is permitted?

If is clear that the sample mean depends on the sample. Calculators usually give two calculated standard deviations, shown as σ_n or s_n and σ_{n-1} or s_{n-1} . The theory and work in this course uses σ_n (s_n), so *make sure you use this value*.



For random samples, the sample mean $\bar{X}$ is a random variable.

The N th line of Pascal's triangle lists the number of possible combinations of 0, 1, 2, ..., N objects from N objects. The sum of the N th line is 2^N . A sample has to have at least one element, so there are $2^N - 1$ possible samples from a finite population of N . This means that for a finite population, $\bar{X}$ is a discrete random variable, but as there are usually a very large number of values in the domain, it is more commonly treated as if it were a continuous random variable. For example, a group of 30 people have over a billion values (1 073 741 823) of the sample means of their heights. You can use your CAS calculator to simulate some of these samples.

○ Example 2

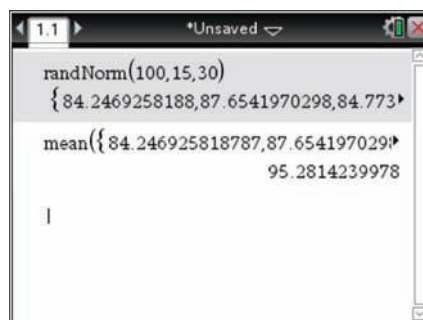
CAS IQ is standardised to a mean of 100 and a standard deviation of 15.

- Use the normal distribution to choose a random sample of 30 IQs.
- Find the mean of the sample.
- Put 20 means of random samples of 30 IQs in the first column of a spreadsheet.
- Find the mean and standard deviations of the sample means.

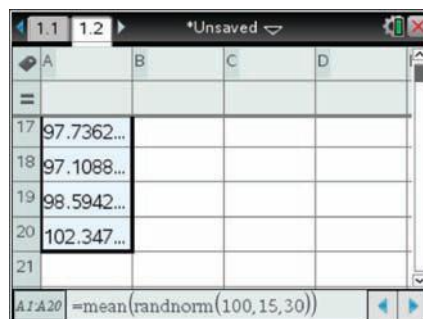
Solution

a TI-Nspire CAS

Use **menu**, 5: Probability, 4: Random and 4: Normal or type `randnorm(100, 15, 30)` to choose a random sample.



- Use **menu**, 6: Statistics, 3: List Math and 3: Mean or type `mean(ans)` to find the mean of the answer.



- c Add a spreadsheet page and type `mean(randnorm(100, 15, 30))` in cell A1. Copy this down to A20. Give the list a name, such as *s1*.

1.1 1.2 *Unsaved	
<code>randNorm(100,15,30)</code>	
{ 84.2469258188,87.6541970298,84.773	
<code>mean({ 84.2469258188,87.6541970298</code>	
	95.2814239978
<code>mean(s1)</code>	100.675187846
<code>stDevPop(s1)</code>	2.49285412155

- d Go back to the calculator page and type `mean(s1)` and `stdevpop(s1)` to find the mean and standard deviation of the sample means. Alternatively, use **menu**, 6: Statistics, 1: Stat Calculations and 1: One-Variable Statistics 'Num of ILsts' as 1, making 'X1 List' *s1*, leaving 'Frequency List' as 1 and pressing OK to get the display shown.

1.1 1.2 *Unsaved	
"Title"	"One-Variable Statist
"X"	100.675187846
"Σx"	2013.50375692
"Σx²"	202834.15539
"sx := Σn-1x"	2.55761436362
"sx := ΣnX"	2.49285412155
"n"	20.
"MinX"	96.7488285389
"Q1X"	98.7513662383
"MedianX"	100.841676404
"Q3X"	102.171794018

a ClassPad

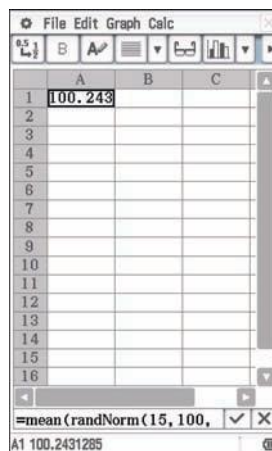
Press **Keyboard** and tap **▼** to reach the function **Catalog**. From there, scroll down to **randNorm()**, or type this command. Enter the standard deviation and mean of the required normal distribution, the number of values required from this distribution, and press **EXE**. (Remember to separate values with a comma.) Tap **▶** to see the values in the list that do not fit on the screen.

Edit Action Interactive	
<code>randNorm(15, 100, 30)</code>	
{ 89.76188104, 92.83245994	
[]	
Catalog	← M N O P Q R →
Advance	randList (Form
Number	randNorm (All
	randSeed
	rangeAppt (INPUT
	rank (
	rc
	RelGMem
	RelPic
	RelVWin
Alg	Decimal Real Rad

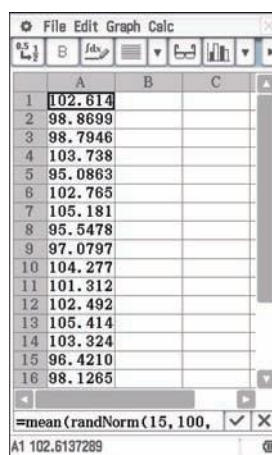
- b Either access **mean()** from **Catalog** or type `mean(` then enter **ans** **EXE**.

Edit Action Interactive	
<code>randNorm(15, 100, 30)</code>	
{ 89.76188104, 92.83245994	
<code>mean(ans)</code>	
99.82139417	
[]	
Catalog	← M N O P Q R →
Advance	matToList (Form
Number	max (All
	maxX
	maxY
	mean (INPUT
	MeanSA
	MeanSAB
	MeanSB
	MeanSErr
Alg	Decimal Real Rad

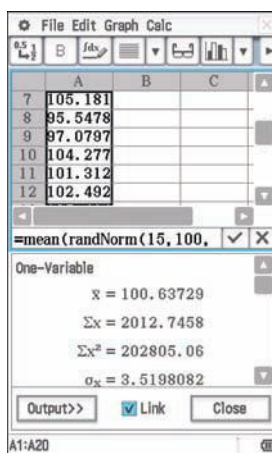
- c Tap  and select  **Spreadsheet**.
Tap the cell A1 and then tap the box at the bottom of the screen.
Enter $=\text{mean}(\text{randnorm}(15, 100, 30))$ and press **EXE**.



To fill this formula down for all cells in the column to A20, hold your stylus tip on cell A1. Drag the tip down the column to A20. You should see this range of cells surrounded by a dark outline. Tap **Edit**, **Fill**, **Fill Range** and a box will pop up. In the **Formula** field there should be the formula you entered in cell A1. In the **Range** field there should be A1:A20. Edit these fields if necessary and tap OK. (Note that this will change the random value originally in A1, however this is not important for this exercise.)



- d Select cells A1 to A20 as described before. Tap **Calc One-Variable**. The summary statistics of the 20 mean values are displayed on the screen. Use the scroll bar to examine these values. The mean of the values and their standard deviation are shown.



The results you get are very unlikely to be the same as those shown in Example 2. It is likely that everyone in the class will get different answers because you are using random samples. However, it is likely that your mean will be close to 100 and your standard deviation will be close to 2.7. The ratio of the population standard deviation and the sample means standard deviation is likely to be close to 5.5. In Example 2 it was $15 \div 2.492 \dots \approx 6$. The standard deviation of the sample means is less than the standard deviation of the population.

Example 3

- CAS** a Create a list of the following heights (or use the heights of people in your class).
 182 155 191 187 157 193 189 181 151 180 186 177
 187 186 178 167 202 156 171 195 182 173 169 166
- b Find the mean and standard deviation of the heights.
 c Use a spreadsheet to find the means of 20 samples of 9 people from the group.
 d Find the mean and standard deviation of the sample means.
 e Find the ratio of the standard deviation of the group to the standard deviation of the means.

Solution

a TI-Nspire CAS

Define the list using braces {}, with a comma separating the numbers.

The TI-Nspire CAS screen shows the definition of a list $h1$ and its statistics. The list is defined as $h1 = \{182, 155, 191, 187, 157, 193, 189, 181, 151, 180, 186, 177, 187, 186, 178, 167, 202, 156, 171, 195, 182, 173, 169, 166\}$. The mean of $h1$ is 177.541666667 and the standard deviation of the population (stDevPop($h1$)) is 13.4070229316.

Command	Result
Define $h1 = \{182, 155, 191, 187, 157, 193, 189, 181, 151, 180, 186, 177, 187, 186, 178, 167, 202, 156, 171, 195, 182, 173, 169, 166\}$	Done
$\text{mean}(h1)$	177.541666667
$\text{stDevPop}(h1)$	13.4070229316

- b Use $\text{mean}()$ and $\text{stdevpop}()$ to find the mean and standard deviation of the entire group.

- c Add a spreadsheet and type $\text{mean}(\text{randsamp}(h1, 9))$ in cell A1. Use the variable name you gave your list in place of $h1$.
 Copy this down to A20.
 Give the list a name, such as $s1$.

The TI-Nspire spreadsheet shows a list of 20 sample means calculated from random samples of size 9 from the list $h1$. The formula in cell A1 is $\text{mean}(\text{randsamp}(h1, 9))$. The results are shown in cells A1 through A20.

Cell	Value
A1	181.111...
A2	181.111111111
A3	173.
A4	178.
A5	179.888...
A6	...
A7	...
A8	...
A9	...
A10	...
A11	...
A12	...
A13	...
A14	...
A15	...
A16	...
A17	...
A18	...
A19	...
A20	...

- d Go back to the calculator page and type $\text{mean}(s1)$ and $\text{stdevpop}(s1)$ to find the mean and standard deviation of the sample means.
 e Your ratio, like the one shown, should be close to 3.

The TI-Nspire CAS screen shows the statistics of the sample means stored in list $s1$. The mean of $s1$ is 178.672222222 and the standard deviation of the population (stDevPop($s1$)) is 4.92292131981. The ratio of the standard deviation of the original data to the standard deviation of the sample means is 13.4070229316 / 4.92292131981 = 2.72338761086.

Command	Result
$\text{mean}(h1)$	177.541666667
$\text{stDevPop}(h1)$	13.4070229316
$\text{mean}(s1)$	178.672222222
$\text{stDevPop}(s1)$	4.92292131981
$\frac{13.4070229316}{4.92292131981}$	2.72338761086

a **ClassPad**

Define the list using braces {}, with commas separating the numbers and store in, say, *hl*.

- b Download the functions `stdevp` and `randsamp` from the website to your calculator. Your teacher may have already done this for you.
- Use `mean()` and `stdevp()` to find the mean and standard deviation of the entire group.

c Find a random sample of 9 values from *h1* using `randsamp(h1,9)`.

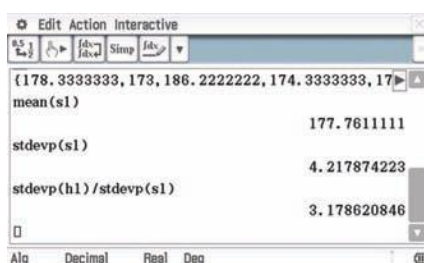
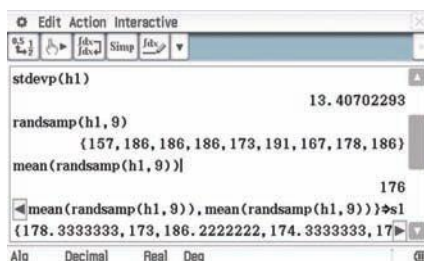
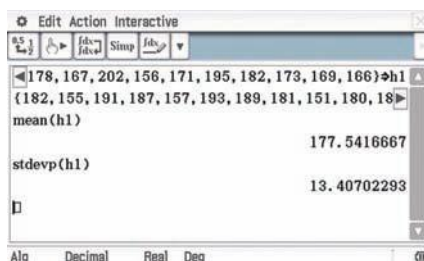
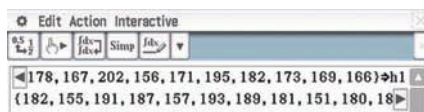
Find the mean of such a sample.

Now create a list of 20 such means using braces {}, with commas separating the numbers and store in, say, *s1*. You can copy `mean(randsamp(h1,9))` and paste this 20 times into the list to save typing.

- d Find the mean and standard deviation of the sample means.

- e Your ratio, like the one shown, should be close to 3 (between about 2.5 and 3.5).

If you wanted to, you could define a function `v(h1,n) = mean(randsamp(h1,n))` to save further typing in the creation of your list of sample means.



You can use the Excel spreadsheet ‘Sample means’ on the website to find samples, sample means and standard deviations for samples of different sizes for any list of numbers.

Use the spreadsheet to find what happens to the ratio of the population standard deviation to the sample means standard deviation as the sample size is increased.

Sample means

Instructions:

- Insert your data under the heading "Population data".
- Press the button "Insert random data".
- Then use the spinners to choose the sample size and number of samples.
- Then press get the samples.
- You also should check the samples.

Sample size 20 **Number of samples** 30

Get the samples

Insert random data samples

Population mean 15.03
Population SD 1.94
Mean of sample me 15.06
Sample means SD 0.41

Popu data

Sample No.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Sample Mean	16	15	15	15	15	15	15	16	15	16	15	15	14	15	15	15	15	15	15	15
Deen No.	Data	Data	Data	Data	Data	Data	Data	Data	Data	Data	Data	Data	Data	Data	Data	Data	Data	Data	Data	Data
1	13	17	16	13	13	17	12	17	16	18	14	12	14	13	15	17	15	18	15	15
2	16	15	15	16	18	12	15	13	16	17	13	18	16	13	18	16	13	12	15	15
3	15	16	16	16	18	12	17	12	15	17	13	18	17	13	16	17	13	16	17	13
4	18	13	15	14	13	17	16	15	14	13	15	13	15	16	12	13	15	13	15	15
5	18	15	18	13	16	16	16	14	12	17	15	13	16	13	16	13	16	13	16	13
6	18	12	18	18	13	18	17	15	12	18	15	16	16	15	15	16	15	12	12	17
7	18	15	15	12	17	13	18	16	12	18	18	14	15	15	18	16	15	15	17	13
8	16	15	15	13	16	14	18	17	18	15	17	13	12	12	14	14	16	16	17	13
9	12	16	12	15	12	16	15	13	13	12	18	16	16	13	16	12	17	14	13	15
10	18	12	15	18	15	14	16	15	12	17	12	13	13	13	16	14	13	17	13	13
11	18	12	16	14	16	15	13	14	12	15	18	12	15	16	17	16	17	16	13	12
12	13	13	12	13	14	18	17	13	15	12	13	17	13	14	16	15	15	13	16	17
13	12	17	18	18	15	15	13	17	12	15	18	12	12	13	15	15	16	15	15	15
14	18	16	18	16	17	12	16	16	18	14	14	15	17	13	16	15	14	18	18	18
15	16	14	12	14	13	13	17	12	16	12	18	17	12	17	13	13	13	18	16	16
16	18	14	12	14	15	17	13	13	12	14	15	14	15	12	16	18	15	15	13	15

EXERCISE 10.01 Sample means

Concepts and techniques



Mean, variance and
SD calculations

- 1 **Example 1** Find the sample means for each of the following samples of weights of Year 12 boys.
 - a 84, 65, 86, 84, 77, 74, 76, 61, 70, 89
 - b 58, 74, 97, 71, 81, 91, 81, 81, 85, 87
 - c 75, 52, 92, 71, 70, 82, 92, 112, 75, 90
 - d 93, 67, 70, 63, 70, 87, 73, 69, 85, 78
- 2 Find the sample means for each of the following samples of the number of fruit or vegetable serves eaten by Year 12 girls in a week.
 - a 16, 8, 23, 24, 33, 11, 27, 72, 30, 10, 23, 15
 - b 57, 26, 70, 0, 60, 9, 24, 26, 0, 25, 29, 8
 - c 66, 6, 25, 28, 29, 0, 26, 30, 11, 26, 18, 31
 - d 41, 5, 14, 38, 53, 44, 26, 53, 37, 58, 43, 39
- 3 **Example 2** **CAS** The average body mass index of Year 12 students is about 24, with a standard deviation of 3.5. Simulate 30 samples of 20 students each and calculate their means.
 - a Find the mean of the sample means.
 - b Find the standard deviation of the means.
 - c Comment on the results.
- 4 **CAS** The average weekly amount spent by Australian households on food is \$120 with a standard deviation of \$30. Simulate 40 samples of 25 households each and calculate their means.
 - a Find the mean of the sample means.
 - b Find the standard deviation of the means.
 - c Comment on the results.
- 5 **CAS** The average shoe size of Year 12 girls is $6\frac{1}{2}$, with a standard deviation of $3\frac{1}{2}$ that corresponds to a foot length of 23 cm with a standard deviation of 3 cm. Simulate 30 samples of the foot lengths of 40 students each and calculate their means.
 - a Find the mean of the sample means.
 - b Find the standard deviation of the means.
 - c Comment on the results.



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- 6 **CAS** The average weekly amount of money that Year 12 students have is \$290 with a standard deviation of \$90. Simulate 15 samples of 50 students each and calculate their means.
- Find the mean of the sample means.
 - Find the standard deviation of the means.
 - Comment on the results.

Reasoning and communication

- 7 **Example 3** **CAS** Create a list for the following hours of sleep reported by 44 Year 12 students.
 7, 7, 6, 6, 8, 7, 8, 7, 9, 8, 9, 8, 9, 7, 10, 9, 9, 8, 4, 9, 6, 8
 6, 7, 7, 9, 7, 8, 16, 7, 8, 7, 8, 7, 5, 8, 7, 9, 8, 9, 7, 11, 7, 8
- Find the mean and standard deviation of the times.
 - Use a spreadsheet to find the means of 20 samples of 9 people from the group.
 - Find the mean and standard deviation of the sample means.
 - Find the ratio of the standard deviation of the group to the standard deviation of the means.
 - Compare the ratio to $\sqrt{n}$.
- 8 **CAS** Create a list of the following times taken by 29 Year 12 students to complete a task.
 16, 13, 18, 31, 17, 18, 18, 18, 11, 11, 19, 11, 19, 18, 9,
 18, 11, 13, 8, 13, 24, 17, 10, 10, 12, 12, 23, 12, 27
- Find the mean and standard deviation of the times.
 - Use a spreadsheet to find the means of 20 samples of 16 people from the group.
 - Find the mean and standard deviation of the sample means.
 - Find the ratio of the standard deviation of the group to the standard deviation of the means.
 - Compare the ratio to $\sqrt{n}$.
- 9 **CAS** Create a list of the following test results for 50 Year 12 students.
 37, 74, 44, 58, 24, 69, 23, 60, 37, 94, 49, 86, 53, 78, 39, 64, 54,
 63, 27, 66, 37, 61, 27, 86, 51, 60, 38, 59, 50, 75, 7, 84, 36, 85,
 27, 74, 52, 67, 51, 84, 53, 78, 38, 76, 54, 69, 34, 68, 45, 59
- Find the mean and standard deviation of the scores.
 - Use a spreadsheet to find the means of 30 samples of 25 people from the group.
 - Find the mean and standard deviation of the sample means.
 - Find the ratio of the standard deviation of the group to the standard deviation of the means.
 - Compare the ratio to $\sqrt{n}$.
- 10 **CAS** Create a list of the following reaction times of some Year 12 students.
 0.34, 0.29, 0.25, 0.37, 0.45, 0.64, 0.34, 0.34, 0.33, 0.26, 0.35, 0.35, 0.44, 0.34,
 0.3, 0.49, 0.34, 0.38, 0.39, 0.4, 0.29, 0.42, 0.29, 0.44, 0.38, 0.34, 0.34, 0.36, 0.31,
 0.46, 0.56, 0.49, 0.34, 0.39, 0.4, 0.31, 0.54, 0.35, 0.42, 0.37
- Find the mean and standard deviation of the times.
 - Use a spreadsheet to find the means of 50 samples of 12 people from the group.
 - Find the mean and standard deviation of the sample means.
 - Find the ratio of the standard deviation of the group to the standard deviation of the means.
 - Compare the ratio to $\sqrt{n}$.

10.02 SAMPLING DISTRIBUTION SIMULATIONS

What does the distribution of sample means look like? Does it look like the distribution of the population? You can use your CAS calculator or spreadsheets to simulate samples to find the answer to this question. Remember the following distributions from Maths Methods.

IMPORTANT

The **binomial distribution** is the discrete probability distribution of the number of successes arising from a fixed number of Bernoulli trials.

It has the probability function $P(X = x) = \binom{n}{x} p^x q^{n-x}$, or $P(X = x) = \binom{n}{x} p^x (1-p)^{n-x}$.

A **uniform continuous probability variable** is one whose probability density function has a constant value on the domain of X . It is also called a **rectangular distribution**.

If X is defined for the domain $[a, b]$, then

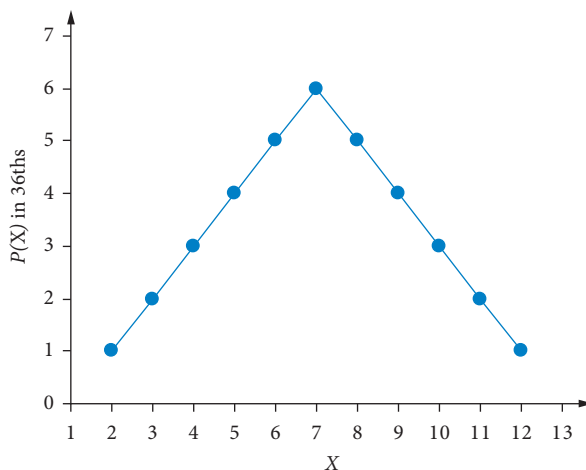
$$p(x) = \begin{cases} \frac{1}{b-a} & \text{for } a \leq x \leq b \\ 0 & \text{otherwise (i.e., } x \notin [a, b]) \end{cases}$$

A **triangular random variable** is one whose probability density function $p(x)$ or probability function has a graph in the shape of a triangle.

A **normal distribution** is a linear transformation of the standard normal distribution with a probability density function of the form $p(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$, where μ and σ are the mean and standard deviation of the random variable X . A normal distribution has a 'bell-shaped' curve.

The sum of the numbers shown on two normal dice is a familiar triangular probability function. Its probability function is shown on the right. You can obtain samples of the outcomes by using the full list of the sample space as in Example 4.

To avoid using the clumsy term 'distribution of sample means', a special name is given to this distribution.



A **sampling distribution** is the distribution of sample means of the same size from a given population.

○ Example 4

CAS a The full sample space list of the sum of the numbers on two dice is {2, 3, 3, 4, 4, 4, 5, 5, 5, 6, 6, 6, 6, 6, 7, 7, 7, 7, 8, 8, 8, 8, 9, 9, 9, 9, 10, 10, 10, 11, 11, 12}. Simulate 140 samples of 50 and find the means of each sample.

b Draw a graph of the results.

c Comment on the results.

Solution

a TI-Nspire CAS

Define the list as, say, ds .

Set the calculation mode to Approximate to simplify your answers.

The mean of a single sample can be found using $\text{mean}(\text{randSamp}(ds, 50))$.

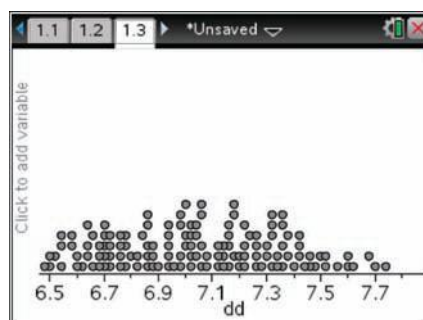
You could do this repeatedly and make notes of the results by pressing **enter** to repeat the calculations, but it is easier to do it in a Spreadsheet page in the cell A1, then copy down to A140.

Give your sampling distribution a name, say, dd .

Define $ds = \{$	
2, 3, 3, 4, 4, 4, 5, 5, 5, 6, 6, 6, 6, 6, 7, 7, 7, 7, 8, 8, 8, 8, 9, 9, 9, 9, 10, 10, 10, 11, 11, 12}	Done
$\text{mean}(\text{randSamp}(ds, 50))$	6.92
$\text{mean}(\text{randSamp}(ds, 50))$	7.16
$\text{mean}(\text{randSamp}(ds, 50))$	7.08
$\text{mean}(\text{randSamp}(ds, 50))$	6.56
$\text{mean}(\text{randSamp}(ds, 50))$	6.26

	A	B	C	D
dd				
=				
1	7.74			
2	7.78			
3	7.12			
4	6.94			
5	6.94			
A1	$\text{mean}(\text{randSamp}(ds, 50))$			

b You could plot a graph of the sampling distribution by hand, but it is easier to add a Data & Statistics page and set the horizontal variable to dd .



a ClassPad

Simulate a list of 50 rolls of one six-sided die and add these to 50 simulated rolls of another six-sided die. This approach ensures that you get 50 values in the sample space of totals of the two dice. Then take the mean of these 50 totals, and perform this 140 times.

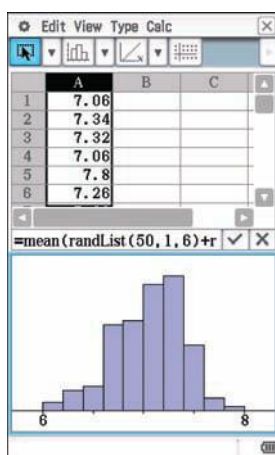
Select the  **Spreadsheet** application.

Tap cell A1, enter

`'= mean(randList(50,1,6) + randList(50,1,6))'` and press **EXE**. Fill this formula down to A140 by tapping **Edit**, **Fill** then **Fill Range**, completing the pop-up box as shown and tapping **OK**.



- b** To graph the results, tap the heading of column A (the column should now have a thick black border) followed by **Graph** and **Histogram**. To show the distribution clearly, you may need to adjust the bin size of the histogram by first tapping **Calc** then **Bin Width**. The graph shown was produced by starting from 6 with a bin width of 0.2.



- c** Compare the graph of the sampling distribution to the original triangular distribution.

The graph does not look like the original triangular distribution. The mean appears to be in about the same place, but the values are very tight around the mean.

If your class goes through Example 4, you will find that your results are not the same as those shown. Compare your results and graph to others in your class. What can you say about the results?

What about other distributions? Your CAS calculator can be used to simulate samples from other distributions in a similar way to Example 4, but some distributions are already built into the calculator.

○ Example 5

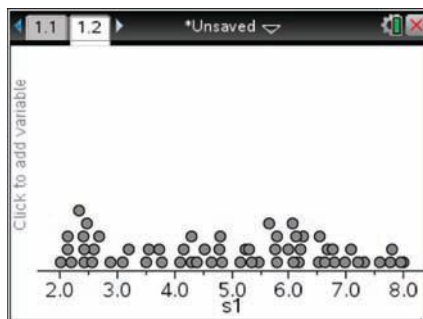
- CAS**
- Simulate a sample of 60 items from a uniform distribution on the interval $[2, 8]$ and draw a graph.
 - Simulate 100 samples of 60 items from a uniform distribution on the interval $[2, 8]$ and find the means of the samples.
 - Draw a graph of the sampling distribution (distribution of the sample means).
 - Comment on the graph.

Solution

a TI-Nspire CAS

$\text{rand}()$ gives a random number between 0 and 1, so $6 \times \text{rand}() + 2$ will give a random number from 2 to 8.

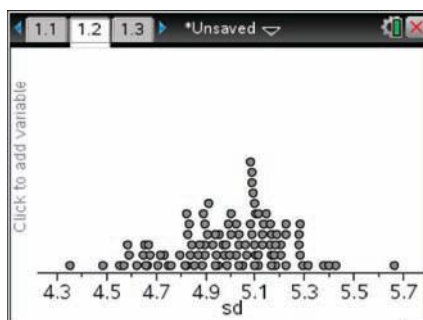
Type ' $6 \times \text{rand}() + 2$ ' into cell A1 of a Spreadsheet page and copy down to A60. Call the column s1, add a Data & Statistics page and set the horizontal variable to s1.



- Go back to the spreadsheet and type ' $=\text{mean}(6 \times \text{rand}(60) + 2)$ ' into cell B1 and copy down to B100. Call the column, say, sd.

	A s1	B sd	C	D
1	2.42604...	5.11412...		
2	4.09464...	4.88661...		
3	2.01368...	4.89563...		
4	4.29256...	5.09514...		
5	6.51230...	5.10088...		
B1	$=\text{mean}(6 \cdot \text{rand}(60) + 2)$			

- Add another Data & Statistics page and set the horizontal variable to sd. You can switch back and forth between the pages to compare the graphs.



a ClassPad

$\text{rand}()$ returns a random number between 0 and 1, so $6 \times \text{rand}() + 2$ will give a random number from 2 to 8.

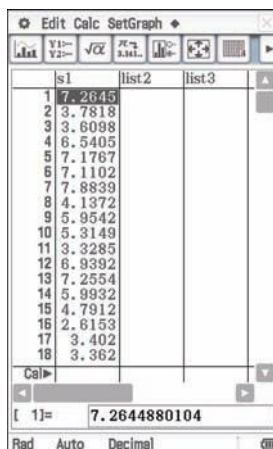
Select  **Spreadsheet**.

Type $'=6 \times \text{rand}()+2'$ into cell A1, hold the stylus on this cell and drag down to A60. Tap **Edit, Fill, Fill Range** and tap **OK** to fill this formula down to cell A60. Tap **File, Export** and a pop-up box will appear. Next to **Variable** type s1, by **Type** select LIST from the drop down menu, and ensure that in the **Range** field there is A1:A60. Tap **OK**.

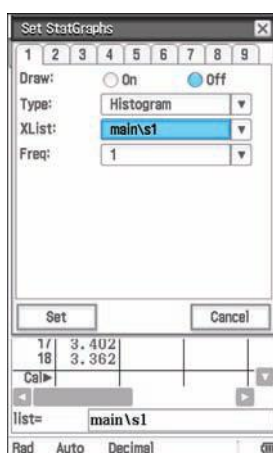
Select the  **Statistics** application and tap the column heading **list1**.

Tap **Edit, Open List**, type s1 and press **EXE**.

The simulations from the spreadsheet are now present in the first column of the sheet.



	s1	list2	list3
1	7.2645		
2	3.7818		
3	3.6098		
4	6.5405		
5	7.1767		
6	7.1102		
7	7.8839		
8	4.1372		
9	5.9342		
10	5.3149		
11	3.3285		
12	8.9392		
13	7.2554		
14	5.9932		
15	4.7912		
16	2.6153		
17	3.402		
18	3.362		



Set StatGraphs

Draw: ☐ On ☒ Off

Type: Histogram

XList: main\s1

Freq: 1

Set Cancel

1/ 3.402
18 3.362

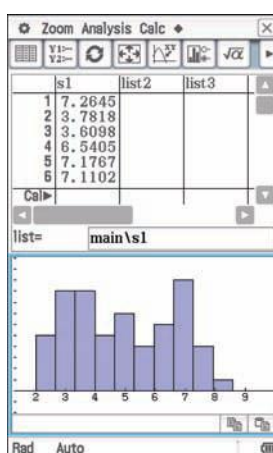
Calc

list= main\s1

Rad Auto Decimal

To graph this sample, tap **SetGraph** and ensure that the box next to StatGraph1 is checked. Tap **Setting**, complete the screen as shown and tap **Set**.

Tap  and **OK** to draw the graph.

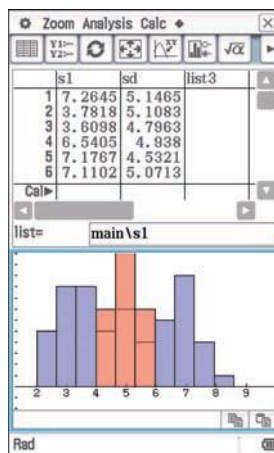


- b Go back to the spreadsheet and type
 $\text{= mean}(6 \times \text{randList}(60)+2)$ into cell B1
 and copy down to B100. Tap **File**, **Export**
 and proceed as before to name the list of
 values sd.

- c Select the  **Statistics** application and
 in a similar manner to that described in
 part a, read in the list sd into the second
 column of the sheet. Plot the series as
 described previously.

To see both graphs on the same plot, tap
SetGraph and ensure that the boxes by
StatGraph1 and **StatGraph2** are both
 checked. Tap  and **OK** to draw the graph.
 The series s1 is drawn in blue and sd is in red.

- d Comment on the appearances of the
 graphs.



The graph of a single sample is spread fairly
 between 2 and 8 with the centre at about 5, but
 the sampling distribution is concentrated about
 5 and has a distinct peak.

If your class works through the steps of Example 5, you will again find that your samples and graphs are different. What can you say about the sampling distribution?

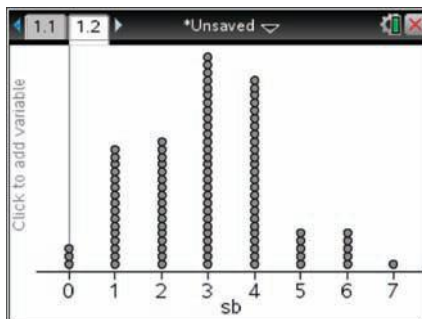
○ Example 6

- Simulate a sample of 100 items from a binomial distribution with $p = 0.3$ and $n = 10$ and draw a graph.
- Simulate 100 samples of 10 items from a binomial distribution with $p = 0.3$ and $n = 10$ and find the means.
- Draw a graph of the sampling distribution.
- Simulate 100 samples of 50 items from a binomial distribution with $p = 0.3$ and $n = 10$ and find the means.
- Draw a graph of the sampling distribution.
- Comment on the graphs.

Solution

TI-Nspire CAS

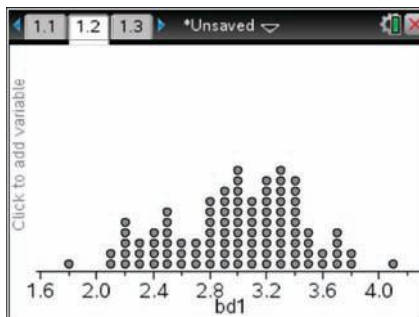
- Make sure that your calculator is set to
 Approximate calculation mode to simplify
 results.
 Type $\text{= randbin}(10,0.3,100)$ into the
 formula cell of column A of a Spreadsheet
 page. Call the column, say, sb. Add a Data
 & Statistics page and set the horizontal
 variable to sb.



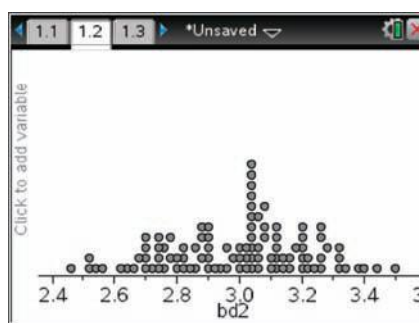
- b Go back to the spreadsheet and type
'= mean(randbin(10,0.3,10))' into cell B1
and copy down to B100.
Call the column, say, bd1.

	A sb	B bd1	C	D
1		4.	2.8	
2		4.	3.1	
3		4.	3.7	
4		3.	4.1	
5		4.	3.	

- c Add another Data & Statistics page and
set the horizontal variable to bd1.
- d Go back to the spreadsheet and type
'= mean(randbin(10,0.3,50))' into cell C1
and copy down to C100.
Call the column, say, bd2



- e Add another Data & Statistics page and
set the horizontal variable to bd2.



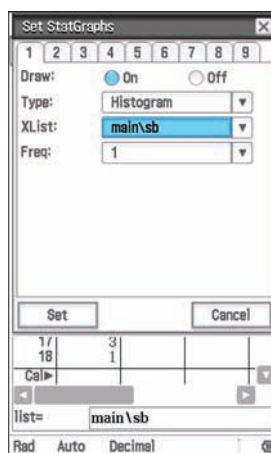
ClassPad

- a From , select the  **Spreadsheet**
application. Type '= randBin(10,0.3)' into
the formula cell of A1. Tap **Edit**, **Fill**,
Fill Range and a pop-up box will appear.
In the **Range** field type A1:A100 and tap
OK to fill the formula down to cell A100.
Tap **File**, **Export**, complete the pop-up
box as shown and tap **OK**.

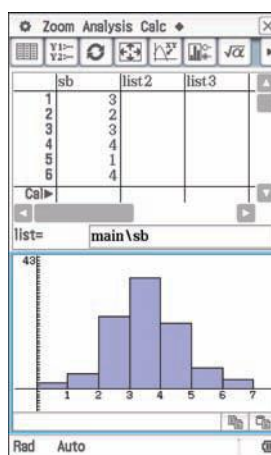
File Edit Graph Calc										
: \$ " A B C D E F										
Export										
Variable		sb								
Type		LIST								
Range		A1:A100								
OK		Cancel								

From , select the  **Statistics** application and tap **Edit**, **Open List** and type 'sb' into the field to the right of **list =** at the bottom of the screen.

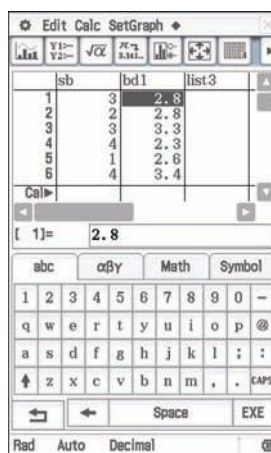
To graph these simulations from the binomial distribution specified, tap **SetGraph**, ensure that the box by **StatGraph1** is checked, and tap **Setting....** Complete the pop-up box as shown and tap **Set**.



Tap , by **HStep** enter 1 and tap **OK** to graph the values of sb.

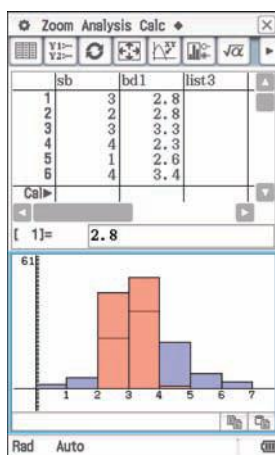


- b Go back to **Spreadsheet**. Type ' $= \text{mean}(\text{randbin}(10,0.3,10))$ ' into cell B1 and fill this formula down to B100 as described in part a. Export this list as described earlier, giving it the name bd1.

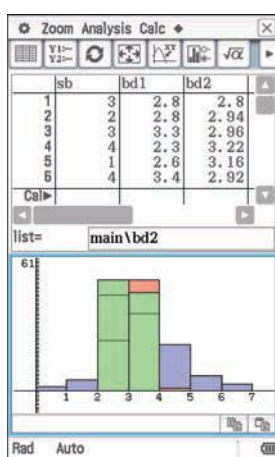


- c Return to the **Statistics** application and tap cell B1. Open the list bd1 as described earlier.

To graph these means, tap **SetGraph**, ensure the box by **StatGraph2** is checked, and tap **Setting...**. Tap tab 2, complete the pop-up box as before, changing the variable to bd1 and tap **Set**. Tap , by **HStep** enter 1 and tap **OK**. The values of sb are plotted in blue and the values of bd1 are plotted in red.



- d Go back to **Spreadsheet** and type `'=mean(randBin(10,0.3,50))'` into cell C1 and fill down to C100 as described earlier. Export this list, giving it the name bd2 and open this list in the **Statistics** application following the process described earlier.



- e To plot the graph of bd2 values, follow the steps as before, using the tab labelled 3. The results of bd2 are added to the previous plot as green columns. To view fewer than three graphs, ensure that only the results you wish to plot have their boxes checked under the **SetGraph** menu.
- f Comment on the appearances of the graphs.

The graph of a single sample is unevenly spread from 0 to 7 with a peak at 3 and is skewed to the right.

The sampling distribution for $n = 10$ is spread from about 1.8 to 4, with a peak at 3. It is more symmetrical than the single sample.

The sampling distribution for $n = 50$ is spread from about 2.5 to 3.5, with a peak at 3. It is even more symmetrical than the single sample.

Example 6 indicates that as sample size increases, the spread of a sampling distribution decreases. It is also evident that whatever the shape of the original distribution, the sampling distribution has a central peak and is concentrated towards the centre.

EXERCISE 10.02 Sampling distribution simulations

Reasoning and communication

- 1 **CAS** **Example 4** a Use the full sample space list $\{1, 2, 3, 4, 5, 6\}$ to simulate 140 samples of 40 for the number shown on a normal dice and find the means of each sample.
b Draw a graph of the results.
c Comment on the results.
- 2 **CAS** The 2, 3, 4, 5, and 6 of spades are shuffled and two of the cards are selected with replacement. The absolute difference in the numbers is calculated.
a Show the sample space as a 5×5 table.
b Use the full sample space list to simulate a sample of 25 results and draw the graph.
c Use the full sample space list to simulate 100 samples of 25 results and find the means of each sample.
d Draw a graph of the results.
e Compare the graphs.
- 3 **CAS** Four coins are tossed together and the number of heads is counted.
a Use a tree diagram to list the full sample space.
b Use the full sample space list to simulate a sample of 45 results and draw the graph.
c Use the full sample space list to simulate 200 samples of 45 results and find the means of each sample.
d Draw a graph of the results.
e Use the full sample space list to simulate 200 samples of 100 results and find the means of each sample.
f Draw a graph of the results.
g Compare the three graphs.
- 4 **CAS** **Example 5** a Simulate a sample of 30 items from a uniform distribution on the interval $[0, 12]$ and draw a graph.
b Simulate 100 samples of 30 items from a uniform distribution on the interval $[0, 12]$ and find the means of the samples.
c Draw a graph of the sampling distribution.
d Comment on the graphs.
- 5 **CAS** a Simulate a sample of 50 items from a uniform distribution on the interval $[10, 30]$ and draw a graph.
b Simulate 150 samples of 50 items from a uniform distribution on the interval $[10, 30]$ and find the means of the samples.
c Draw a graph of the sampling distribution.
d Comment on the graphs.
- 6 **CAS** a Simulate a sample of 20 items from a uniform distribution on the interval $[5, 15]$ and draw a graph.
b Simulate 100 samples of 20 items from a uniform distribution on the interval $[5, 15]$ and find the means of the samples.
c Draw a graph of the sampling distribution.

- d Simulate 100 samples of 60 items from a uniform distribution on the interval $[5, 15]$ and find the means of the samples.
 - e Draw a graph of the sampling distribution.
 - f Comment on the graphs.
- 7 **CAS** **Example 6** a Simulate a sample of 20 items from a binomial distribution with $p = 0.8$ and $n = 15$ and draw a graph.
- b Simulate 100 samples of 20 items from a binomial distribution with $p = 0.8$ and $n = 15$ and find the means.
 - c Draw a graph of the sampling distribution.
 - d Simulate 100 samples of 80 items from a binomial distribution with $p = 0.8$ and $n = 15$ and find the means.
 - e Draw a graph of the sampling distribution.
 - f Comment on the graphs.
- 8 **CAS** a Simulate a sample of 15 items from a binomial distribution with $p = 0.5$ and $n = 8$ and draw a graph.
- b Simulate 100 samples of 15 items from a binomial distribution with $p = 0.5$ and $n = 8$ and find the means.
 - c Draw a graph of the sampling distribution.
 - d Simulate 100 samples of 120 items from a binomial distribution with $p = 0.5$ and $n = 8$ and find the means.
 - e Draw a graph of the sampling distribution.
 - f Comment on the graphs.

10.03 MEAN AND STANDARD DEVIATION OF SAMPLE MEANS

In the previous sections, you will have noticed in the exercise that simulations of samples produce sample means whose mean is close to the mean of the population. However, the standard deviation of the sample means is close to $\frac{1}{\sqrt{n}}$ times the population standard deviation, where n is the sample size. This is found to be true for samples from any population.

IMPORTANT

The sampling distribution of samples of size n has mean μ but a standard deviation of $SD(\bar{X}) = \sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}}$, where μ is the mean and σ is the standard deviation of the population.

The **central limit theorem** applies to sample means and states that for sufficiently large samples, the sampling distribution of the means of a statistic approaches a normal distribution with the same mean as the population and variance given by $\frac{\sigma^2}{n}$.



○ Example 7

Samples of 20 items are taken from a population with a mean of 30 and a standard deviation of 7. What are the mean and standard deviation of the sampling distribution?

Solution

Use the rule for the mean.

$$E(\bar{X}) = \mu = 30$$

Use the rule for the standard deviation.

$$\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}} = \frac{7}{\sqrt{20}} = 1.565\dots$$

Write the answer.

The mean will be 30 and the standard deviation will be about 1.6.

○ Example 8

40 samples of packets of 50 mm BH nails were taken from a production run and the average length of the nails was worked out for each packet. The mean of the averages was 50.01 mm and the standard deviation of the averages was 0.04 mm. Each packet had 30 nails. What would you expect the mean and standard deviation of the length of nails from the production run to be?



Solution

Use $E(\bar{X}) = \mu$.

$$\mu \approx 50.01 \text{ mm}$$

Use $SD(\bar{X}) = \frac{\sigma}{\sqrt{n}}$.

$$0.04 = \frac{\sigma}{\sqrt{30}}$$

Find σ .

$$\sigma \approx 0.04 \times \sqrt{30} = 0.219\dots$$

Write the answer.

The nails would be 50 mm long with a standard deviation of about 0.2 mm.

It doesn't matter how good your sampling process is (how random), there is always some chance that a randomly selected sample will have a mean that is substantially different from the population mean.

Example 9

CAS The average time that you need to wait to speak to a consultant at a call centre is 2 minutes with a standard deviation of 25 seconds. What is the probability of 5 randomly selected calls having to wait an average of 3 minutes or more?

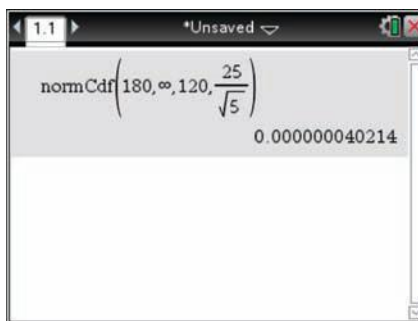
Solution

Find the standard deviation of the sample means.

$$\sigma_{\bar{x}} = \frac{25}{\sqrt{5}}$$

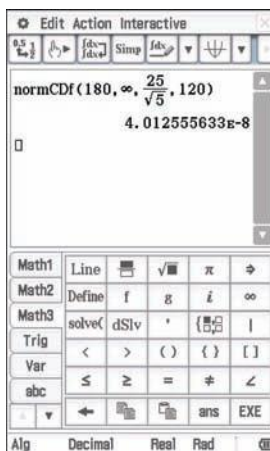
TI-Nspire CAS

Find the probability using the normal distribution.



ClassPad

Perform the calculation by using **normCDF**(and specifying the lower bound on the wait time, the upper bound (here this is positive infinity), the standard deviation of the waiting time sampling distribution and its mean, with each pair of values separated by a comma.



Write the answer.

The probability of 5 randomly selected calls having an average waiting time of 3 minutes or more is about 4×10^{-8} .

Example 9 is not the same as having to wait for 3 minutes or more on 5 calls in a row because this would not be a random sample. This could obviously happen if you rang at a busy time. You could also criticise the method in Example 9 as being outside the guideline of $n \geq 30$, but it does have the advantage of being a simple method that gives you some idea of the size of the probability, even if it is not completely accurate.

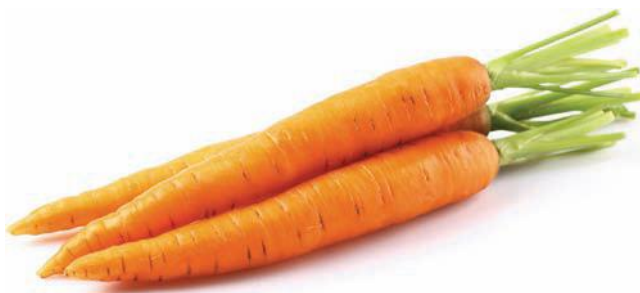
EXERCISE 10.03 Mean and standard deviation of sample means



Sampling distributions

Concepts and techniques

- 1 **Example 7** Samples of 45 items are taken from a population with a mean of 25 and a standard deviation of 4. Find the mean and standard deviation of the sampling distribution.
- 2 Samples of 50 items were taken from a population with a mean of 80 and a variance of 40. Find the mean and variance of the sample means.
- 3 The mean amount taken from ATMs is \$200, with a standard deviation of \$300. The means of samples of 60 transactions from various machines around the city were used to determine if there was systematic variation due to location. What would you expect the mean value of the sample means and the variance of the sample means to be?
- 4 **Example 8** The standard deviation of the sample means of 20 items from a population was 3. What is the standard deviation of the population?
- 5 The variance of the average lengths of samples of 30 French beans was found to be 12 cm^2 . What is the expected variance of the lengths of French beans in general?
- 6 The mean values of samples of 30 items from a population were calculated and the sampling distribution had a mean of 255 and a variance of 50. What was the mean and standard deviation of the population?
- 7 **Example 9** The average height of Year 12 boys is about 180 cm with a standard deviation of 11 cm. What is the probability that the average height of 3 Year 12 boys chosen at random will be under 165 cm?
- 8 The mean length of mature Topweight carrots is 20 cm with a variance of 5 cm^2 . What is the probability that the average length of 4 Topweight carrots selected at random will be more than 22 cm?



Shutterstock.com/DanIsvera

- 9 The average reaction time of Year 12 students is about 0.37 s with a standard deviation of 0.1 s. What is the probability that 6 students selected at random from your class have an average reaction time of less than 0.4 s?

Reasoning and communication

- 10 The average house prices in Melbourne vary from one suburb to another, with an average value of \$800 000 and a standard deviation of about \$90 000 for the suburb averages. The averages are calculated on monthly sales of about 30 houses per suburb at auction. What do you anticipate the standard deviation of house prices across Melbourne would be?

10.04 CONFIDENCE INTERVALS FOR NORMAL DISTRIBUTIONS

Sampling distributions of any kind of distribution have a similar shape to a normal distribution, and the greater the sample size, the more like a normal distribution they become. This makes the normal distribution the most important type of distribution, especially if it is not possible to find the parameters of a population. Since this is often the case, it is useful to have some estimate of how well a statistic approximates a parameter. There are two ways to quote a statistic.

IMPORTANT

A **point estimate** of a parameter is a single value obtained from a sample. The statistic used is called an **estimator** of the parameter.

An **interval estimate** of a parameter is an interval that is likely to include the value of the parameter.

You have already seen that the mean of a sampling distribution is a 'good' estimator of the population mean. What does this really mean?

IMPORTANT

An **unbiased estimator** of a parameter is one whose expected value is equal to the parameter.

An unbiased estimator is **consistent** if the difference between the estimator and parameter decreases as the sample size increases.

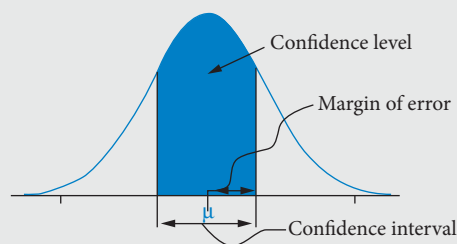
In the previous section you saw that the expected value of a sampling distribution, $\bar{X}$, is the mean of the population and that the standard deviation of $\bar{X}$, $\frac{\sigma}{\sqrt{n}}$, decreases as n increases. This means that the sample mean is an unbiased, consistent estimator of the population mean. This is not true for the sample standard deviation.

Interval estimates are based on **confidence intervals**.

IMPORTANT

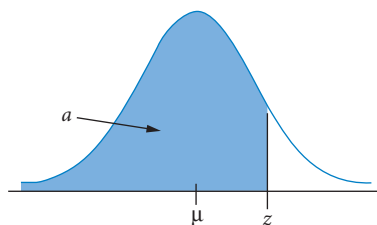
For a **confidence interval** symmetric around the mean in a statistical distribution:

- the **confidence level** is the proportion of values that are expected to lie within the interval
- the **margin of error** is the distance of the ends of the interval from the mean. It is half the confidence interval.



The confidence level is represented by the area under the curve. It is both the proportion of values and the probability of lying within the interval. You need to use the inverse cumulative normal distribution to find a confidence interval.

This is set up to calculate the value below which a given proportion of values lie. Use $a = \int_{-\infty}^z f(x)dx$, where z is the value returned by the inverse distribution function for the cumulative density function $f(x)$ and the value a . You can picture it as shown below for a normal distribution.



A confidence level of c will have an area of $\frac{c}{2}$ each side of the mean, so the 'tails' will each have an area of $0.5 - \frac{c}{2}$. The areas below each of the corresponding quantiles are $0.5 - \frac{c}{2}$ and $0.5 + \frac{c}{2}$.

○ Example 10

CAS What is the 90% confidence interval for the standard normal variable Z ?

Solution

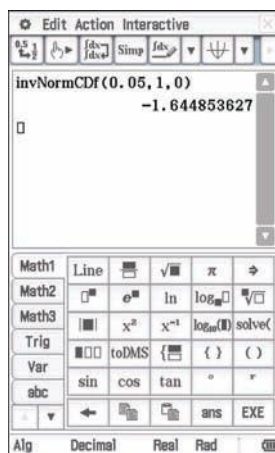
TI-Nspire CAS

Find the quantile for $0.5 - \frac{0.9}{2} = 0.05$ using **[menu]**, 5: Probability, 5: Distributions and 3: Inverse Normal. Leave the mean and standard deviation as 0 and 1.



ClassPad

Find the quantile for $0.5 - \frac{0.9}{2} = 0.05$ using the **invNormCDF**(function available by pressing **[Keyboard]** and tapping **[v]** to access the **[Catalog]**. The standard deviation is 1 and the mean is zero for the standard normal distribution.



Write the answer.

The 90% confidence interval is about -1.64 to 1.64 .

The standard normal variable has mean $\mu = 0$ and standard deviation $\sigma = 1$. For other normal variables, you need to take into account the change of origin and scale produced by the mean and standard deviation compared to the standard normal variable.

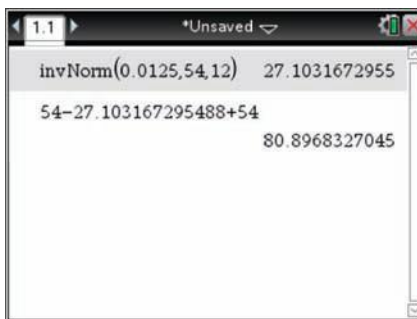
Example 11

CAS What is the 97.5% confidence interval for the normal variable X with mean 54 and standard deviation 12?

Solution

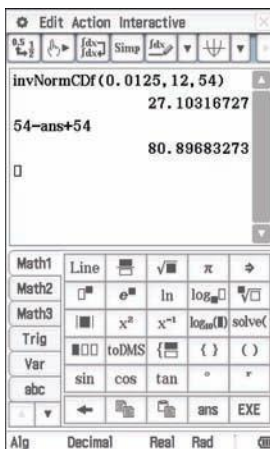
TI-Nspire CAS

Find the quantile for $0.5 - \frac{0.975}{2} = 0.0125$ using **menu**, 6: Statistics, 5: Distributions and 3: Inverse Normal. Use $\mu = 54$ and $\sigma = 12$. This is below the mean, so you need to find the same margin of error above the mean.



ClassPad

Find the quantile for $0.5 - \frac{0.975}{2} = 0.0125$ using **invNormCDF**(. Use $\sigma = 12$ and $\mu = 54$. The calculated value is below the mean, so you need to find the same margin of error above the mean.



Write the answer.

The 97.5% confidence interval is about 27.1 to 80.9.

In some cases, you will want to know what the confidence level is for a particular interval or margin of error. In this case, you need to use the normal cumulative probability function.

Example 12

CAS What is the confidence level for the interval 160–190 for a normal variable with mean 175 and standard deviation 9?

Solution

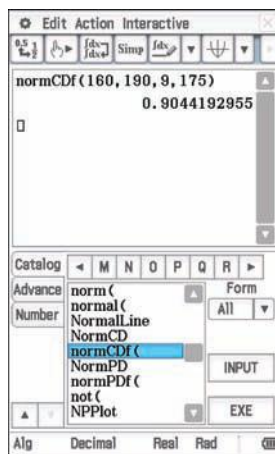
TI-Nspire CAS

Use **menu**, 5: Probability, 5: Distributions, 2: Normal Cdf or type `normalcdf()`. Put in the values for the interval, the mean and the standard deviation.



ClassPad

Use **normalCDF()** and enter the values for the interval, the standard deviation and the mean.



Write the answer.

The confidence level for the interval is about 90.4%.

In Example 12, you really should check that the interval is symmetric about the mean before proceeding. In this case, the interval was 15 on either side of the mean, so it was set up properly.

EXERCISE 10.04 Confidence intervals for normal distributions

Concepts and techniques



Normal distributions

- 1 **CAS** **Example 10** Find the 95% confidence interval for a standard normal distribution.
- 2 **CAS** Find the 99% confidence interval for a standard normal distribution.
- 3 **CAS** Find the 80% confidence interval for a standard normal distribution.
- 4 **CAS** **Example 11** Find the 90% confidence interval for the normal variable X with mean 100 and standard deviation 15.
- 5 **CAS** Find the 95% confidence interval for the normal variable X with mean 70 and standard deviation 12.
- 6 **CAS** Find the 99% confidence interval for the normal variable X with mean 55 and standard deviation 18.
- 7 **CAS** **Example 12** What is the confidence level for the interval 60–80 for a normal variable with mean 70 and standard deviation 7?
- 8 **CAS** What is the confidence level for the interval 120–160 for a normal variable with mean 140 and standard deviation 10?

Reasoning and communication

- 9 **CAS** The mean service life of high-performance tyres is about 20 000 km with a standard deviation of 8000 km. What percentage of the tyres would have a life between 10 000 and 30 000 km?



Alamy/Daniel Kessler

- 10 **CAS** A company manufacturing lithium batteries wants to give a guaranteed shelf life but doesn't want any more than 0.5% of the batteries returned under warranty. Testing has revealed that the mean shelf life is 24 months with a standard deviation of 6 months. What should their shelf-life guarantee be set at?



10.05 ESTIMATION OF POPULATION PARAMETERS

In previous sections, you have seen that the mean of a sample is an unbiased, reliable estimator of the mean of the population. In addition, as the sample size increases, the sampling distribution approaches a normal distribution, so for a sufficiently large sample, the sample statistics can be used to estimate the parameters of the population. What is meant by ‘sufficiently large’?

IMPORTANT

Samples where $n \geq 30$ are generally considered sufficiently large to use for estimating population parameters, provided that confidence levels are given.

INVESTIGATION

Confidence intervals and the mean

What does a confidence interval of 90% for the mean indicate? In principle, it means that the average value will be within the interval 90% of the time. Is this the actual case?

You can check this using a simulation with your CAS calculator.

- Use your CAS calculator to set up 30 samples of 25 from a uniform distribution on the interval 0–20.
- Find the mean of each sample.
- Use the standard deviation of the distribution. The mean of this distribution is 10 and the standard deviation is $\frac{20}{2\sqrt{3}}$. What is the (expected) standard deviation of the sampling distribution?
- Determine whether or not $\mu = 10$ is within the 90% confidence interval of the sample mean for each sample.
- Find the class average for the proportion of times that the 90% confidence interval contains the mean.
- If you have time, repeat the process for a binomial distribution, a normal distribution, 95% and 80% confidence intervals.
- What can you conclude?

If you don't know the mean of a population, it seems very unlikely that you would know the standard deviation. This means that in most cases, you are forced to rely on the sample standard deviation to estimate the confidence interval for the mean of a population. You can use the standard normal distribution to find the appropriate interval and transform the interval using

$X = \mu + z\sigma$, where $\sigma = \frac{s}{\sqrt{n}}$, is the standard deviation of the sample and n is the sample size. This

means the confidence interval for the population mean is given by $\bar{X} - \frac{zs}{\sqrt{n}}$, $\bar{X} + \frac{zs}{\sqrt{n}}$, where z is the quantile of the standard normal distribution for the level of confidence.

○ Example 13

The mean of a sample of 40 was 36, with a standard deviation of 5. Determine a 95% confidence interval for the population mean μ .

Solution

Estimate the standard deviation of the sampling distribution.

$$\sigma_{\bar{x}} \approx \frac{5}{\sqrt{40}} = 0.790\dots$$

Since $n \geq 30$, you can use the normal distribution.

The sampling distribution is approximately normal.

Find the proportion for the top of the interval.

$$a = 0.5 + \frac{0.95}{2} = 0.975$$

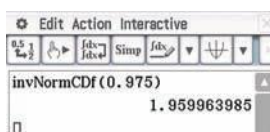
TI-Nspire CAS

Use the inverse standard normal distribution to find the quantile.



ClassPad

Use the inverse standard normal distribution to find the quantile.



Find the interval.

$$\text{The interval is } 36 \pm 1.959\dots \times \frac{5}{\sqrt{40}}.$$

Calculate and write the answer.

The 95% confidence interval is about $[34.45, 37.55]$.

The 90%, 95% and 99% confidence intervals are used so frequently that it is worth noting the quantiles for these intervals.

IMPORTANT

The 90%, 95% and 99% confidence intervals for the standard normal variable are $[-1.645, 1.645]$, $[-1.960, 1.960]$ and $[-2.576, 2.576]$ respectively.

Of course, you can use your CAS calculator to find a confidence interval without using the standard normal distribution.

Example 14

The mean of a sample of 50 was 19, with a standard deviation of 6.2. Determine a 97% confidence interval for the population mean.

Solution

Estimate the standard deviation of the sampling distribution.

$$\sigma_{\bar{x}} = \frac{6.2}{\sqrt{50}} = 0.876\dots$$

Since $n \geq 30$, you can use the normal distribution.

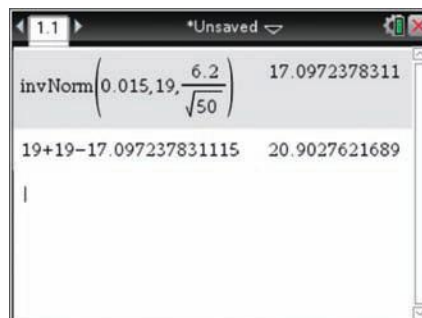
$\bar{x}$ is approximately normal.

Find the proportion needed.

$$a = 0.5 - \frac{0.97}{2} = 0.015$$

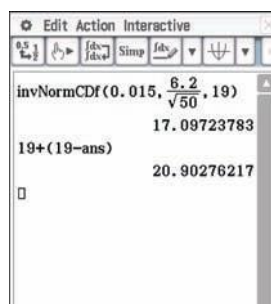
TI-Nspire CAS

Use the inverse normal distribution to find the quantile.



ClassPad

Use the inverse normal distribution to find the lower quantile and use this to find the upper quantile.



Write the answer.

The 97% confidence interval is about $[17.1, 20.9]$.

In some cases, you may have to calculate the sample mean and standard deviation. The standard deviation of a sample is not an unbiased consistent estimator of the parameter, but the difference is quite small, so for reasonably big samples it can be used. It slightly underestimates the parameter.

○ Example 15

A random sample of Year 12 girls had heights of 155, 160, 171, 165, 164, 166, 171, 165, 170, 163, 170, 178, 160, 168 and 163 cm.

- Use these results to estimate the mean and standard deviation of the heights of Australian Year 12 girls.
- State the 95% confidence interval for the mean.
- State any problems with the estimates.

Solution

- Find the total.

$$\begin{aligned}\text{Total} &= 155 + 160 + 171 + \dots \\ &= 2489\end{aligned}$$

Find the mean.

$$\begin{aligned}\text{Mean height} &= \frac{2489}{15} \\ &= 165.933\dots\end{aligned}$$

Find the sum of the squares.

$$\begin{aligned}\sum x^2 &= 155^2 + 160^2 + 171^2 + \dots \\ &= 413\,455\end{aligned}$$

Find the standard deviation of the sample.

$$\begin{aligned}s &= \sqrt{\frac{413\,455}{15} - 165.933\dots^2} \\ &= 5.458\dots\end{aligned}$$

Write the answer.

The mean height and standard deviation are estimated to be 165.9 cm and 5.5 cm respectively.

- Find the sampling distribution SD.

$$\sigma_{\bar{x}} = \frac{5.458\dots}{\sqrt{15}} = 1.409\dots$$

Find the 95% interval.

$$\text{The interval is } 165.933\dots \pm 1.96 \times 1.409\dots$$

Calculate and write the answer.

The 95% confidence interval for the mean height of Year 12 girls is [163.2 cm, 168.7cm].

- State the major problem.

$n < 30$, so the sampling distribution cannot be assumed to be normal.



EXERCISE 10.05 Estimation of population parameters



Population parameters

Concepts and techniques

- 1 **Examples 13, 14** The mean of a sample of 50 was 28, with a standard deviation of 9. Determine a 95% confidence interval for the population mean.
- 2 The mean of a sample of 30 was 135, with a standard deviation of 20. Determine a 90% confidence interval for the population mean.
- 3 The mean of a sample of 45 was 67, with a variance of 81. Determine a 99% confidence interval for the population mean.
- 4 **CAS** The mean of a sample of 100 was 420, with a standard deviation of 30. Determine an 85% confidence interval for the population mean.
- 5 **CAS** The mean of a sample of 60 was 0.7, with a standard deviation of 2. Determine a 99.5% confidence interval for the population mean.
- 6 **CAS** The mean of a sample of 80 was 52, with a variance of 64. Determine a 99.9% confidence interval for the population mean.

Reasoning and communication

- 7 **Example 15** A random sample of 40 Victorian potato farms planted the following areas of potatoes. All areas are given in hectares.
6, 19, 22, 26, 19, 15, 18, 40, 26, 25, 26, 34, 29, 21, 34, 25, 38, 21, 40, 18, 10,
3, 22, 32, 25, 2, 27, 16, 47, 15, 29, 48, 10, 8, 45, 37, 15, 5, 26, 34
 - a Use these results to estimate the mean and standard deviation of the areas planted by Australian potato farmers.
 - b State the 90% confidence interval for the mean.
 - c State any problems with the estimates.
- 8 The monthly sales of passenger vehicles in Tasmania for 2011–2013 were as follows.
979, 785, 730, 753, 749, 712, 726, 707, 809, 714, 773, 600, 667, 636, 723, 636, 688, 683,
650, 739, 537, 830, 819, 819, 746, 754, 732, 847, 760, 844, 869, 819, 588, 753, 759, 807
 - a Use these results to estimate the mean and standard deviation of passenger vehicle sales in Tasmania.
 - b State the 95% confidence interval for the mean.
 - c State any problems with the estimates.
- 9 The amounts earned in one week by a randomly selected group of Australian teachers were as follows. The amounts given are gross pay rounded to the nearest dollar before tax or other deductions.
1430, 1740, 1340, 1749, 1417, 1795, 1433, 1681, 1497, 1517, 1483, 1826, 1392,
1759, 1475, 1701, 1464, 1762, 1477, 1621, 1342, 1532, 1407, 1642, 1448
 - a Use these results to estimate the mean and standard deviation of pay for Australian teachers.
 - b State the 90% confidence interval for the mean.
 - c State any problems with the estimates.

- 10 A randomly selected group of Australian senior students said they spent the following amounts of times (in hours) on homework in a week.
- 0, 9, 21, 20, 8, 40, 8, 7, 1, 8, 20, 4, 1, 10, 3, 1, 3, 1, 4, 10, 35, 11, 5, 1, 5, 15, 12, 4, 7, 7, 0, 0, 40, 21, 0, 30, 1, 1, 3, 40
- Use these results to estimate the mean and standard deviation of senior student homework time.
 - State the 95% confidence interval for the mean.
 - State any problems with the estimates.

10.06 SURVEY PROCEDURES, CONFIDENCE INTERVALS AND DATA QUALITY

You often see statistics quoted in the media, especially polls related to politics. In some cases, the basis of the statistics may be given. In this case you can question the assertions based on the statistics quoted. In general, the 95% confidence level is taken to be the appropriate level for the calculation of confidence intervals.

○ Example 16

A poll conducted close to a Federal election was based on 1352 voter interviews by trained telephone interviewers. Home telephone numbers and the person interviewed within the household were selected at random. According to the poll, 37% favoured the ALP, 43% the Liberal/Country coalition, 9% the Greens and 11% others. On a two-party preferred basis, the support levels were calculated to be ALP 49% and Coalition 51%, with a stated margin error of 3%.

The headline was 'Coalition has a 2% lead in latest poll'.

- What does the 95% confidence level for two-party support mean?
- Based only on this poll, is the headline justified?
- Does the method of polling introduce any bias?

Solution

- The margin of error is on either side of the point estimate.
- State the possible two-party support actually meant by the figures quoted.

Write a conclusion.

- Bias is introduced if the poll favours one group of the population over another.

ALP support is between 46% and 52%.
Coalition support is between 48% and 54%.

At the 95% confidence level, the position could be anywhere between 46% to 54% or 52% to 48%.

The headline is not really justified.

There is bias towards people who have fixed lines. People with only mobiles are excluded, so it is likely that there is a bias towards older people.



Example 16 points out a dilemma of pollsters. While telephone interviews allow fast collection and processing of data at a reasonable cost, bias is inevitably introduced. The firms conducting polls in this way claim that their results are consistent with electoral results to justify their methods.

○ Example 17

The ABS interviews randomly selected people each month to compile employment data. They interview 0.32% of the population over 15 years old and change the sample by one-eighth every month. According to one such survey, males employed part-time worked 72 hours in the month. There were 1 060 400 males over 15 in part-time work at the time. Assume that the percentage interviewed is the same for full-time and part-time males and females.

- a Assuming that they are proportionally represented, how many men working part-time were interviewed?
- b If the variance of the hours worked was 369, what is the 95% confidence interval?

Solution

- a Find 0.32% of the number of men employed part-time.

Part-time males interviewed = 0.32% of 1 060 400

Calculate the number.

$$= 3393$$

Write the answer.

3393 of those interviewed were part-time.

- b Find the standard deviation.

$$SD = \sqrt{369}$$

Find the SD of the sample mean.

$$\sigma_{\bar{x}} = \frac{\sqrt{369}}{\sqrt{3393}} \approx 0.33$$

Find the 95% confidence interval.

$$\text{Confidence interval} = 72 \pm 0.33 \times 1.96$$

Calculate and write the answer.

The 95% confidence interval for the hours worked by men employed part-time was about [71.35, 72.65].



ABS

The ABS use the international definition of an employed person, which is someone who worked for payment (including contributing to a family business, working on a farm, etc.) for at least an hour in the week of the survey. While you may find this unreasonable, the methods used and the basis of the results produced by the ABS are open to public scrutiny on their website.

○ Example 18

A political poll states that the results are within a margin of error of 3.5%.

- What would the margin of error be at the 90% confidence level?
- CAS** Find the margin of error at the 99.9% confidence level.

Solution

- Assume that the stated margin is at the 95% level using the ratio of the quantiles.

$$\text{Margin of error at 90\% level} = 3.5\% \times \frac{1.645}{1.96}$$

Evaluate.

$$\approx 2.9\%$$

Write the answer.

The margin of error at the 90% confidence level would be about 2.9%.

- Find the 99.9% quantile.

$$a = 0.5 - \frac{0.999}{2} = 0.0005$$

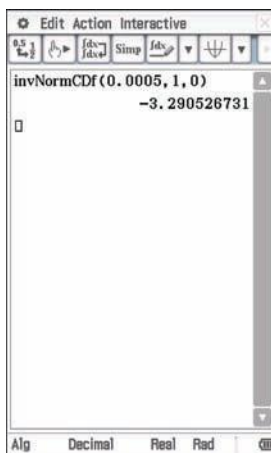
TI-Nspire CAS

Use **invNorm**. Enter the a value, mean and standard deviation.



ClassPad

Use **invNormCDF** to perform the calculation. Enter the a value, then the standard deviation and mean of the standard normal distribution.



Do the calculation.

$$99.9\% \text{ margin} = 3.5\% \times \frac{3.29}{1.96} \approx 5.9\%$$

State the answer.

The margin of error at the 99.9% confidence level would be about 5.9%.

You can see in Example 18 that results based on samples can never be certainties. The margin of error depends on how confident you want to be that the parameter is within the confidence interval. The higher the level of confidence you want, the wider the interval in which the parameter could lie. However, at any level of confidence it is still *possible* that the parameter could be outside the confidence interval.

INVESTIGATION Survey in school

Choose a topic that you are interested in and survey of a representative group of students at your school about an aspect of the topic that you can find a mean for.

You might be interested in the number of siblings, cousins, pets, rooms in the house, number of DVDs, music downloads or some other countable amount. You could be interested in height, a personal best running time, personal best jump, length of journey to school, time spent on the internet or some other measurement.

Make sure that your question(s) are not biased and that you design the survey so that it is representative and (if possible) a random sample. Make sure that the sample is large enough.

When you have done your calculations, work out suitable confidence intervals for the result.

Look at your methods and calculations and suggest how you would improve it if you did it again.

Your teacher might want you to write or present a report.



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EXERCISE 10.06 Survey procedures, confidence intervals and data quality

Reasoning and communication



- 1 **Example 16** A market research company conducted interviews to determine the best packaging pictures to use for baby food containers. Each of the proposals was presented to 40 mothers with babies under 6 months old and the mothers rated the attractiveness of the can or bottle on a scale from 1 to 10. The highest ranking picture was that of a smiling girl looking directly at the camera with an average rating of 8.5. The second and third ranked pictures had a crawling baby and a baby being nursed by a young attractive mother on 7.8 and 7.5 respectively. The standard deviations of the ratings were all about 2.4.
- Find the 90% confidence intervals for the ratings of each package.
 - State if there is any real difference between the ratings.



Alamy/Jochem Tack

- 2 **Example 17** A TV program on 'truth in sentencing' polled viewers by asking them to ring up and state the sentence they would give a driver who had killed 4 people walking on the footpath when the driver lost control of the car and ran the people down while under the influence of drugs. They also asked for the sentence if the driver had lost control of the car after sneezing on the way to the doctor. 3000 people responded to the survey with average sentences of 15 years with a 4 year standard deviation and 2 years with a 6 month standard deviation.
- Find the confidence interval at the 95% level for this survey.
 - Would you regard the result 'as powerful support for community anger about weak sentences for drug addicts' as claimed the next night when the poll results were given?

- 3 **Example 18** A researcher found that the confidence interval for the increase in scores on a standardised test of reading age after an intensive 4-week course was 10–16 months at a confidence level of 80% for the 35 participants who completed the course. The intensive course cost \$2500 for each participant. The fee was non-refundable but there were still 15 people who did not complete the course.
- a What was the average increase?
 - b What was the standard deviation of the increase for those who finished?
 - c What difference would it make if you included the 15 who dropped out and assigned them an increase of 0 months?
 - d Given the cost of the course, do you think the claim of ‘More than a year’s improvement in only 4 weeks’ is justified?
- 4 The average rainfall on King Island in August is 118 mm, with a standard deviation of 46 mm from 36 years of observations. Assume that the rainfall is randomly distributed, so the 36 years can count as a random sample.
- a What is the standard deviation of the sample mean?
 - b What is the probability that the average of the next 4 years’ rainfall in August will exceed 135 mm?
 - c The highest recorded rainfall in August was 214 mm. Does this cast any doubt on the assumption that the rainfall is randomly distributed?
- 5 The average maximum December temperature in Marble Bar in WA is 41.7°C , with a standard deviation of 2.7°C . The highest official December maximum recorded was 48.4°C on 21/12/2011. The average is based on 12 years of temperature records.
- a What is the standard deviation of the mean?
 - b What is the probability that the average over the next three years for the December maximum will exceed 44°C ?
 - c State any assumptions and limitations.

CHAPTER SUMMARY

STATISTICAL INFERENCE

10

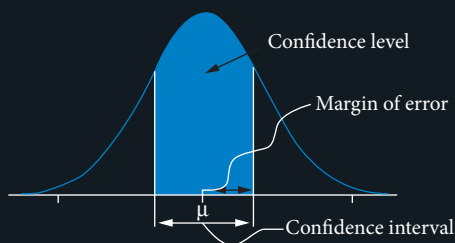
- A **random variable** is a variable with a numerical value that depends on the outcome of a chance experiment. A **continuous random variable** can be any real value, or any real value from an interval. A **discrete random variable** can be a finite set, the counting numbers or an equivalent set.
- A **random sample** is one selected using a method that ensures every member of the population has an equal chance of being chosen.
- Values obtained from a population are called **parameters**, while those from samples are called **statistics**.
- The population mean and standard deviation parameters are shown as μ and σ respectively. Sample mean and standard deviation statistics are shown as $\bar{x}$ and s respectively. The size of a finite population is denoted by N and the number in a sample is denoted by n .
- The **binomial distribution** is the discrete probability distribution of the number of successes arising from a fixed number of Bernoulli trials. It has the probability function

$$P(X = x) = \binom{n}{x} p^x q^{n-x}, \text{ or}$$

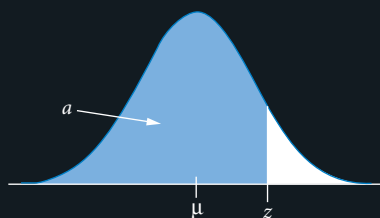
$$P(X = x) = \binom{n}{x} p^x (1 - p)^{n-x}$$
- A **uniform continuous probability variable** is one whose probability density function has a constant value on the domain of X . It is also called a **rectangular distribution**. If X is defined for the domain $[a, b]$, then
- A **triangular random variable** is one whose probability density function $p(x)$ or probability function has a graph in the shape of a triangle.
- A **normal distribution** is a linear transformation of the standard normal distribution with a probability density function of the form $p(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$, where μ and σ are the mean and standard deviation of the random variable X . A normal distribution has a 'bell-shaped' curve.
- A **sampling distribution** is the distribution of the means of samples of the same size from a given population. The sampling distribution of samples of n has mean μ but a standard deviation of $\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}}$, where μ is the mean and σ is the standard deviation of the population.
- The **central limit theorem** applies to sample means and states that for sufficiently large samples, the sampling distribution of the means of a statistic approaches a normal distribution with the same mean as the population, and its variance is given by $\frac{\sigma^2}{n}$.
- A **point estimate** of a parameter is a single value obtained from a sample. The statistic used is called an **estimator** of the parameter.
- An **interval estimate** of a parameter is an interval that is likely to include the value of the parameter.

$$p(x) = \begin{cases} \frac{1}{b-a} & \text{for } a \leq x \leq b \\ 0 & \text{otherwise (i.e., } x \notin [a, b]) \end{cases}$$

- An **unbiased estimator** of a parameter is one whose expected value is equal to the parameter.
- An unbiased estimator is **consistent** if the difference between the estimator and the parameter decreases as the sample size increases.
- For a **confidence interval** that is symmetric around the mean in a statistical distribution:



- The **confidence level** is the proportion of values that are expected to lie within the interval.
- The **margin of error** is the distance of the ends of the interval from the mean. It is half the confidence interval.
- A **quantile** from the normal distribution can be pictured as shown. In the diagram, the quantile $z = t_a$ is such that $a = \int_{-\infty}^z f(x) dx$. Quantiles are given by the inverse of the cumulative density function.



- A confidence level of c will have an area of $\frac{c}{2}$ on each side of the mean, so the 'tails' will each have an area of $0.5 - \frac{c}{2}$. The areas below each of the corresponding quantiles are $0.5 - \frac{c}{2}$ and $0.5 + \frac{c}{2}$.
- Samples where $n \geq 30$ are generally considered sufficiently large to use for estimating population parameters, provided confidence levels are given.
- The 90%, 95% and 99% confidence intervals for the standard normal variable are $[-1.645, 1.645]$, $[-1.960, 1.960]$ and $[-2.576, 2.576]$ respectively.
- The confidence interval for the mean of a population estimated from a sample is given by $\bar{X} - \frac{zs}{\sqrt{n}}$, $\bar{X} + \frac{zs}{\sqrt{n}}$, where $\bar{X}$ is the mean of the sample, z is the quantile of the standard normal distribution for the level of confidence, s is the standard deviation of the sample and n is the sample size.

CHAPTER REVIEW

STATISTICAL INFERENCE

10

Multiple choice

- 1 **Example 1** What is the mean and standard deviation of the sample below?
 3 7 9 4 6 4 5 8 3 11
 A $\mu = 6.0, \sigma \approx 2.7$ B $\mu = 2.7, \sigma \approx 6.1$ C $\mu = 6.1, \sigma \approx 2.7$
 D $\mu = 2.6, \sigma \approx 6.0$ E $\mu = 6.0, \sigma \approx 2.6$
- 2 **Example 7** A sample of 40 has a mean of 35 and a standard deviation of 8. What is the standard deviation of the sample mean?
 A 0.2 B about 0.21 C about 1.3
 D about 1.4 E 8
- 3 **Example 8** The mean and standard deviation of the weights of boxes of 60 apples were 5.7 kg and 0.2 kg. What would you expect the mean weight and standard deviation of the weights of individual apples to be?
 A $\mu = 95 \text{ g}, \sigma \approx 26 \text{ g}$ B $\mu = 95 \text{ g}, \sigma \approx 3 \text{ g}$ C $\mu = 57 \text{ g}, \sigma \approx 20 \text{ g}$
 D $\mu = 59 \text{ g}, \sigma \approx 25 \text{ g}$ E $\mu = 95 \text{ g}, \sigma \approx 20 \text{ g}$
- 4 **Example 9** **CAS** The mean time between delivery destinations for a courier driver is 11 minutes, with a standard deviation of 4 minutes. What is the probability that the next 5 stops will take more than an hour?
 A 0.106 B 0.288 C 0.401
 D 0.712 E 0.894
- 5 **Examples 10, 11** **CAS** What is the 98% confidence interval for a normal variable with mean 47 and standard deviation 5?
 A [36.2, 56.8] B [34.1, 59.9] C [36.7, 57.3]
 D [46.87, 47.13] E [35.4, 58.6]
- 6 **Example 12** **CAS** What is the confidence level for the interval 1200–1400 for a normal variable with mean 1300 and standard deviation 70?
 A 79% B 80% C 85%
 D 88% E 90%
- 7 **Examples 13, 14** What is the 95% confidence interval for the population mean if the mean of a sample of 64 is 42 and the standard deviation is 9?
 A [39.8, 44.2] B [−2.2, 2.2] C [62.6, 67.4]
 D [39.6, 44.4] E [40.1, 43.1]

Short answer

- 8 **Examples 2–6** An 8-sided dice is tossed 90 times. What is a graph of the results likely to look like?
- 9 **Examples 2–6** 60 samples of 50 are chosen from a uniform distribution over the interval $[4, 9]$ and the means are calculated. What is a graph of the means probably going to look like?
- 10 **Examples 2–6** I A sample of 36 is chosen from a binomial distribution with $p = 0.2$ and $n = 25$, and a graph of the sample is drawn.
 II 40 samples of 36 are chosen from a binomial distribution with $p = 0.2$ and $n = 25$, and a graph of the means of each sample is drawn.
 III 80 samples of 36 are chosen from a binomial distribution with $p = 0.2$ and $n = 25$, and a graph of the means of each sample is drawn.
 How will the graphs from parts I, II and III be likely to compare?
- 11 **Examples 7, 8** What is the mean and standard deviation of a sampling distribution for a binomial distribution with $p = 0.7$ and $n = 20$ for samples of 60?
- 12 **Example 9** The average total at a supermarket checkout is \$76, with a variance of 180. What is the probability that the average total for the next 4 customers to pass through is more than \$80?



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- 13 **Example 10** What is the 90% confidence interval for the standard normal variable Z ?
- 14 **Example 11** What is the 99% confidence interval for the normal variable X with mean 60 and standard deviation 8?
- 15 **Example 12** What is the confidence level for the interval $[100, 120]$ for a normal variable with a mean of 110 and a variance of 24?
- 16 **Examples 13, 14** What is the 95% confidence interval for the population mean if a sample of 45 had a mean of 40 and a standard deviation of 8?
- 17 **Example 18** A market research company stated that people would be willing to pay an extra \$2.50 for 'green' washing powder with an error margin of \$1. What would be the 90% confidence interval?

Application

- 18 40 Year 12 students at a Melbourne high school were asked how much they paid for their 'formal' outfit and the result was an average of \$135 with a standard deviation of \$35.
- State the 95% confidence interval for the amount that Year 12 students in Melbourne spend on their formal outfits.
 - Are there any problems with this estimate?
- 19 A poll of Australians that was conducted by randomly calling 600 people on mobile phones found that they were willing to pay an average of \$120 a week for fast broadband with a variance of \$200.
- What is the 99% confidence interval for the average that Australians would be willing to pay for fast broadband access?
 - Are there any problems with this result?
- 20 A survey of 247 people chosen at random from bank customers found that the confidence rating on a scale of 1–10 in the bank's internet security was 8 with an error margin of 0.2.
- What was the mean and standard deviation of the rating in the sample group?
 - Is this a cause for concern?



Practice quiz

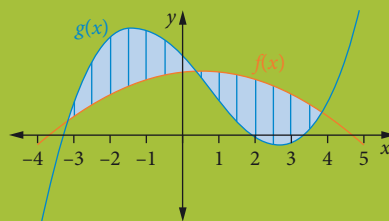
4

MIXED REVISION

CHAPTERS 9 • 10

Multiple choice

- The area under the curve $f(x) = x(x-1)(x+2)$ in square units is
 A $-\frac{15}{4}$ B 0 C $\frac{9}{4}$ D $\frac{37}{12}$ E $\frac{15}{4}$
- Using the exponential distribution $f(t) = \begin{cases} \lambda e^{-\lambda t} & \text{for } t \geq 0 \\ 0, & \text{elsewhere} \end{cases}$, the expected value is 0.1. The parameter λ is
 A λ B 0.1 λ C 10 D 0.1 E $\frac{0.1}{\lambda}$
- A sample of 49 has a mean of 57 and a standard deviation of 14. What is the standard deviation of the sample mean?
 A 1.85 B 2 C 2.64 D 3.74 E 7
- In the diagram below, the graphs of $y = f(x)$ and $y = g(x)$ are shown, where $x = a, b, c$ are the intersection points of the graphs of $y = f(x)$ and $y = g(x)$ and $a < b < c$. The area is found by the expression
 A $\int_a^b g(x) - f(x) dx + \int_b^c f(x) - g(x) dx$
 B $\int_a^b g(x) - f(x) dx + \int_a^c f(x) - g(x) dx$
 C $\int_a^c g(x) - f(x) dx$
 D $\int_a^c f(x) - g(x) dx$
 E $\int_a^b g(x) - f(x) dx - \int_b^c f(x) - g(x) dx$
- What is the 95% confidence interval for a normal variable with mean 36 and standard deviation 8?
 A [20.3, 51.7] B [28, 42] C [22.8, 49.2]
 D [15.4, 56.6] E [30.5, 41.5]
- What is the 90% confidence interval for the population mean if the mean of a sample of 50 is 25.8 and the standard deviation is 7.4?
 A [24.8, 26.8] B [13.6, 38.0] C [23.7, 27.9]
 D [24.1, 27.5] E [11.3, 40.3]



Short answer

- 1 What is the mean and standard deviation of the sampling distribution for a uniform distribution on the interval $[0, 20]$ for samples of 40?
- 2 Find the area between the curves $y = x^2 + 3x$ and $y = 4$.
- 3 What is the 95% confidence interval for the population mean if a sample of 38 had a mean of 24.7 and a standard deviation of 6.3?
- 4 Find an approximation for the area under the curve $y = \frac{1}{4}x^2 + x$ from $x = 0$ to $x = 2$ with strips of $\frac{1}{2}$ a unit wide, using the midpoint rule.

Application

- 1 a Sketch graphs of the functions $f(x) = -x^2 - 2x$ and $g(x) = x^2 + 5x + 3$.
 b Identify the points of intersection of the graphs.
 c Hence find the total areas enclosed by the graphs of functions $f(x) = -x^2 - 2x$ and $g(x) = x^2 + 5x + 3$.
- 2 The results of a survey of the times in minutes taken by a class of Year 12 students in Ballarat to complete a task are given below.

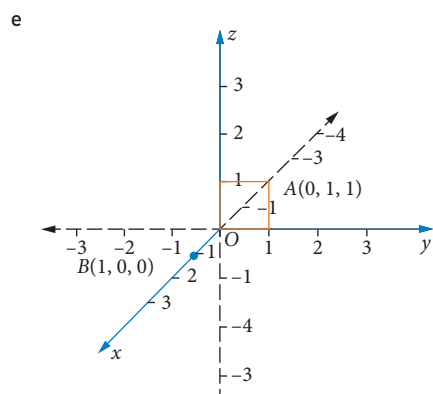
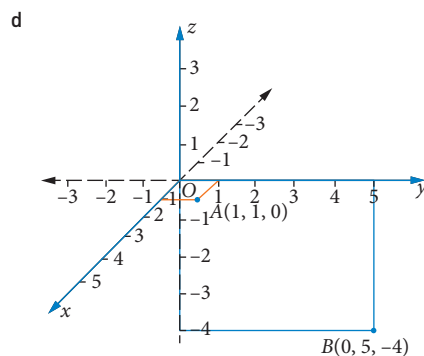
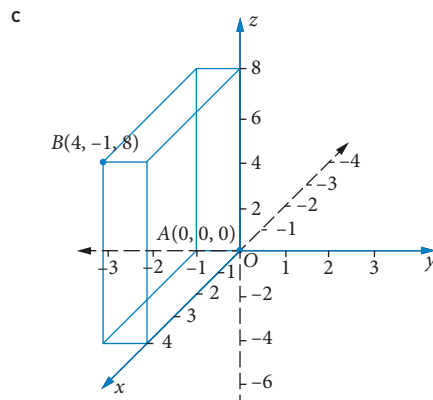
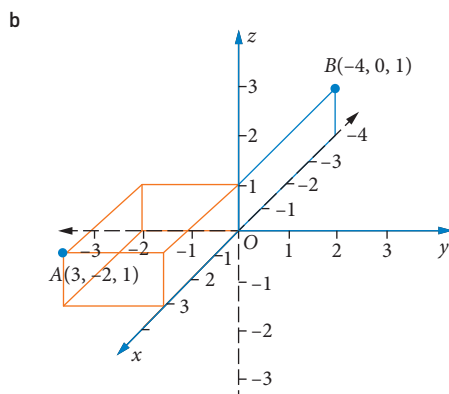
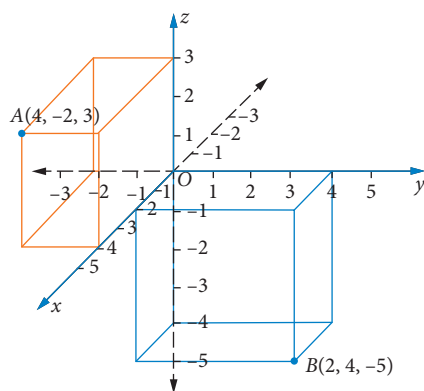
11	9	11	25	20	29	27	20	21	18	26	20
25	27	6	13	33	35	17	17	27	24	30	21

 - a Use the results to find the 99% confidence interval for the mean time for Australian Year 12 students to complete the task.
 - b List any problems with the estimate.
- 3 The reliability of a car speedometer is 0.9 for 100 hours of operation. Assuming that the life of the speedometer is exponential, for what period is the probability of failure less than 7%?
- 4 The answers of a sample of 40 people chosen at random in Collins St were used to estimate the time in minutes that people in the city centre were given for their lunch break. The 95% confidence interval for the sample was found to be $[39.0, 43.4]$. What is the probability that another 3 people asked at random would have lunch breaks totalling more than 135 minutes?

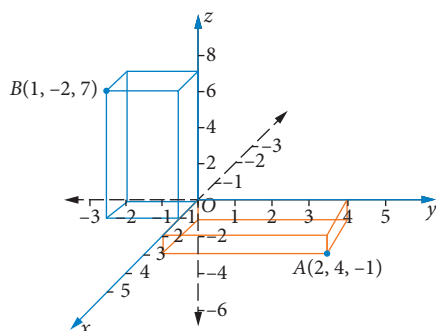
ANSWERS

1.01

- 1 D
- 2 B
- 3 C
- 4 A
- 5 $A(2, 3, -1)$, $B(-2, 5, 4)$, $C(3, -3, 0)$, $D(-4, -4, 3)$,
 $E(2, 0, 2)$
- 6 a



f



- 7 a (2, 0, 0), (2, 4, 0), (0, 4, 0), (0, 4, 3), (0, 0, 3), (2, 0, 3)
 b (2, 2, 1), (2, 4, 1), (-5, 4, 1), (-5, 2, -3),
 (-5, 4, -3), (2, 2, -3)
- 8 a $2\sqrt{26}$ b $\sqrt{53}$ c 9
 d $\sqrt{33}$ e $\sqrt{3}$ f $\sqrt{101}$
- 9 a (3, 1, -1) b $(-\frac{1}{2}, -1, 1)$ c $(2, -\frac{1}{2}, -4)$
 d $(\frac{1}{2}, 3, -2)$ e $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ f $(\frac{1}{2}, 1, 3)$
- 10 a $\sqrt{38}$ b $3\sqrt{2}$ c $3\sqrt{5}$
 d $\sqrt{105}$ e $3\sqrt{5}$ f $\sqrt{101}$
- 11 a $(2\frac{1}{2}, 0, -\frac{1}{2})$ b $(2, 1\frac{1}{2}, \frac{1}{2})$ c $(0, 2\frac{1}{2}, 3)$
 d $(\frac{1}{2}, 1, 2)$ e $(0, 4, 1\frac{1}{2})$ f $(-1, -\frac{1}{2}, 1)$
- 12 a $x^2 + y^2 + z^2 - 16 = 0$
 b $x^2 + y^2 + z^2 - 6y - 141 = 0$
 c $x^2 + y^2 + z^2 + 4x - 5 = 0$
 d $x^2 + y^2 + z^2 - 6x + 2y - 4z + 5 = 0$
 e $4x^2 + 4y^2 + 4z^2 + 40x - 8z + 103 = 0$
 f $x^2 + y^2 + z^2 - 4x + 6y - 8z - 20 = 0$
- 13 a $C = (0, 0, 0), r = 6$ b $C = (5, -3, 3), r = 3$
 c $C = (-1, \frac{1}{2}, 0), r = \sqrt{2}$ d $C = (-2, 0, 0), r = 2$
 e $C = (4, 0, -4), r = 4$ f $C = (-2, 1, -1), r = 2$
- 14 a $PR^2 = PQ^2 + QR^2$; Area of $\triangle PQR = \frac{3\sqrt{2}}{2}$ units²
 b $QR^2 = PQ^2 + PR^2$; Area of $\triangle PQR = \frac{9\sqrt{2}}{2}$ units²
- 15 $C = (2, 0, -3); r = 2\sqrt{6}; x^2 + y^2 + z^2 - 4x + 6z - 11 = 0$
- 16 $6x - 12y + 4z + 13 = 0$
- 17 a A plane parallel to the xy -plane.
 b A plane parallel to the xz -plane.
 c The xz -plane.

1.02

- 1 C
- 2 a (5, -1, 12) b (-13, 7, 14) c (-3, 1, 1)
 d (1, -1, -2) e (25, -11, -12) f (-16, 9, 28)
- 3 a $\sqrt{170}$ b $3\sqrt{46}$ c $\sqrt{11}$
 d $\sqrt{6}$ e $\sqrt{890}$ f $\sqrt{1121}$
- 4 a, c, d and e are parallel pairs, b and f are not parallel
- 5 a $b - a$ b $c - a$ c $f - d$
 d $c - f$ e $b - e$

- 6 a $(-7, -1, -6), (9.3, 188.1^\circ, -40.3^\circ)$
 b $(-8, -3, 8), (11.7, 200.6^\circ, 43.1^\circ)$
 c $(-2, -5, -5), (7.3, 248.2^\circ, -42.9^\circ)$
 d $(3, 0, -5), (5.8, 0^\circ, -59.0^\circ)$
 e $(9, 6, 11), (15.4, 33.7^\circ, 45.5^\circ)$
- 7 a 11.75 b 8.602 c 10.49
- 8 a $(0.4444, -0.7778, 0.4444)$
 b $(-0.8571, 0.2857, -0.4286)$
 c $(-0.7071, -0.4243, 0.5657)$
- 9 C
- 10 a $2i - 3j + 4k, 5.39$ b $-3i - 2j + 4k, 5.39$
 c $3i + j - 5k, 5.92$ d $-i + 4j - 4k, 5.74$
 e $-3i - j - 2k, 3.74$ f $-6i + 2j + 5k, 8.06$
 g $4i - 7j + 2k, 8.31$ h $i + 6j + 4k, 7.28$
 i $-5i - 3j + 2k, 6.16$ j $i - k, 1.41$
 k $3.624i - 8.140j - 4.540k, 10$
 l $-2.399i - 2.399j + 2.120k, 4$
 m $0.486i + 1.042j + 10.940k, 11$
 n $6.928i - 12j - 8k, 16$
 o $-1.327i + 2.393j - 7.518k, 8$
 p $-15.499i + 42.583j + 21.131k, 50$
 q $-28.284i - 28.284j - 69.282k, 80$
- 11 a $(7.071, 300.96^\circ, -34.45^\circ)$
 b $(6, 206.6^\circ, 41.81^\circ)$
 c $(8.602, 66.80^\circ, 27.72^\circ)$
 d $(9.644, 255.96^\circ, -31.23^\circ)$
- 12 a $(2.326, 1.135, 9.659)$
 b $(-2.779, 4.813, -5.755)$
 c $(8.660, -5, 17.32)$
 d $(-15.59, -8.999, -21.45)$
 e $(-19.60, 19.60, 16)$
- 13 A
- 14 a $(9.2, 115.7^\circ, 1.3^\circ)$ b $(25.6, 350.9^\circ, -55.3^\circ)$
 c $(21.8, 134.1^\circ, 29.7^\circ)$ d $(37.0, 2.0^\circ, -58.0^\circ)$
 e $(22.2, 98.1^\circ, -26.7^\circ)$
- 15 a $(26.6, 275.8^\circ, 30.5^\circ)$ b $(48.4, 344.6^\circ, -52.7^\circ)$
 c $(105.7, 187.5^\circ, 58.8^\circ)$ d $(71.0, 332.1^\circ, -45.9^\circ)$
 e $(76.6, 179.6^\circ, 57.4^\circ)$
- 16 $d = a + c - b, e = f + a - b, g = f + c - b,$
 $h = f + a - 2b + c, m = \frac{1}{2}(a + c - b + f)$
- 17 a $5.764i + 5.764k + 1.733j$ (km) or $5764i + 5764k + 1733j$ (m) with i as east, j as north and k as up.
 b $-1.414i + 1.414j - 1.5k$ (km) or $-1414i + 1414j - 1500k$ (m)
 c $4.350i + 7.178j - 0.233k$ (km) or $4349i + 7178j - 233k$ (m), or $(8.4, 58.8^\circ, 1.6^\circ)$
- 18 A length of 470.2 m allows slack of 23.4 m in the centre of the distance of 467.9 m.
- 19 Bearing 153.4° , angle of descent 30.8° and distance 781.0 m
- 20 a 40° to the yacht and 15° upwards.
 b 20 000 N
 c About 4799 N rearwards, 4903 perpendicular to the yacht and 5176 N upwards.
 d About 4799 N.

- e About 5176 N. By 'pointing the sail higher', less upward deflection of the wind is created, less force is directed downwards on the yacht and more wind energy is converted to forward (and sideways) motion of the yacht.

1.03

- 1 C
- 2 C
- 3 B
- 4 E
- 5 D
- 6 C
- 7 D
- 8 a 35.46 b 21.21 c 0 d 180
e 12.12 f -57.16 g -70.15 h 48
- 9 a 44.50 b 46.36 c 57
d 12.99 e 25.90 f 0
- 10 a -14 b 0 c -96
d 74 e -1 f 0
g -42 h -2 i 2
- 11 a -30 b -85 c 73
d 17 e 0 f 41
- 12 a -8 b -1 c -18
d -18 e 9 f 9
- 13 a 167.9° b 134.7° c 104.3°
- 14 a $-\frac{\sqrt{30}}{10}$ b $-\frac{15\sqrt{14}}{14}$ c $\frac{\sqrt{10}}{10}$
d $\sqrt{30}$ e 0 f $-\frac{3\sqrt{30}}{5}$
- 15 a 97.5° b 88.6° c 62.5° d 79.6°
- 16 a 74 b 8 c 48.2°
d 77° e -7.31 f 5.89
- 17 a 1 Nm b 60 Nm c 14 Nm
- 18 72 Nm
- 19 -14.43 Nm
- 20 40 Nm
- 21 $k = 2$
- 22 a (1, -8, 2)
b $\cos(\alpha) \approx 0.1204$, $\cos(\beta) \approx -0.9631$,
 $\cos(\gamma) \approx 0.2408$
c $\alpha \approx 83.1^\circ$, $\beta \approx 164.4^\circ$, $\gamma \approx 76.1^\circ$
- 23 Proof
- 24 $\cos(\alpha - \beta) = [\cos(\alpha), \sin(\alpha)] \cdot [\cos(\beta), \sin(\beta)]$
 $= \cos(\alpha) \cos(\beta) + \sin(\alpha) \sin(\beta)$
- 25 a 76.53 N at 13° to the 50 N force
b i 488.9 J ii 231.8 J iii 257.1 J
c $488.9 = 231.8 + 257.1$, but because of the angles, the contributions of the 30 N and 50 N forces are almost the same.
- 26 a $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}||\mathbf{a}| \cos(0^\circ)$
 $= |\mathbf{a}||\mathbf{a}| \times 1$
 $= |\mathbf{a}|^2$
- 27 It is less than 30° because, as the altitude goes from 0° to 90°, the angle goes from 30° to 0°. Changing both vectors to components and finding the angle using the dot product gives an angle of 25.64°.

INVESTIGATION: PROPERTIES OF THE VECTOR PRODUCT

- 1 $\cos(\theta) = \frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2} \sqrt{b_1^2 + b_2^2 + b_3^2}}$
 $\sin(\theta) = \frac{\sqrt{(a_1b_2 - a_2b_1)^2 + (a_1b_3 - a_3b_1)^2 + (a_2b_3 - a_3b_2)^2}}{(a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2)}$
 $|\mathbf{v}_1 \times \mathbf{v}_2| = \sqrt{(a_1b_2 - a_2b_1)^2 + (a_1b_3 - a_3b_1)^2 + (a_2b_3 - a_3b_2)^2}$
- 2 They are the same.
- 3 $[(a_2b_3 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}] \cdot [a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}] = 0$ and $[(a_2b_3 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}] \cdot [b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}] = 0$
- 4 It is easy to get lost in the detail.

1.04

- 1 B
- 2 A
- 3 D
- 4 All answers are perpendicular to both vectors.
a 20.78 b 19.56 c -49.25 d 0
e 35.10 f -35 g 77 h 24.25
- 5 a $-100.2\mathbf{k} = (100.2, 0^\circ, -90^\circ)$
b $-30.04\mathbf{k}$ c $-647.0\mathbf{k}$
d $(4.596, 11^\circ, 0^\circ)$ e $(1.654, 240^\circ, 0^\circ)$
f $(34.47, 291^\circ, 0^\circ)$
- 6 a $-57\mathbf{k}$ b $-126\mathbf{k}$ c $56\mathbf{k}$
d $(49, -54, 47)$ e $(19, 31, -52)$
f $(-86, -159, -90)$ g $(30, 12, -78)$
h $(83, 17, -6)$ i $(6, -61, -42)$
- 7 a $\begin{bmatrix} 0 \\ 0 \\ -16 \end{bmatrix}$ b $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ c $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$
d $\begin{bmatrix} 13 \\ 39 \\ 26 \end{bmatrix}$ e $\begin{bmatrix} 74 \\ 3 \\ -35 \end{bmatrix}$ f $\begin{bmatrix} -63 \\ -47 \\ 9 \end{bmatrix}$
- 8 a $18\mathbf{i} + 19\mathbf{j} + 11\mathbf{k}$ b $-32\mathbf{i} - 4\mathbf{j} - 3\mathbf{k}$
c $21\mathbf{i} + 9\mathbf{j} - 6\mathbf{k}$ d $21\mathbf{i} + 9\mathbf{j} - 6\mathbf{k}$
e $-13\mathbf{i} - 8\mathbf{j} - 6\mathbf{k}$ f $13\mathbf{i} + 8\mathbf{j} + 6\mathbf{k}$
g $38\mathbf{i} - 25\mathbf{j} - 107\mathbf{k}$ h $-21\mathbf{i} + 83\mathbf{j} - 109\mathbf{k}$
- 9 a $50\mathbf{i} - 17\mathbf{j} - 38\mathbf{k}$ b $-102\mathbf{i} - 17\mathbf{j}$
c $102\mathbf{i} + 17\mathbf{j}$ d $63\mathbf{i} + 126\mathbf{j} - 154\mathbf{k}$
e $-770\mathbf{i} + 1771\mathbf{j} + 1134\mathbf{k}$ f $-168\mathbf{i} - 85\mathbf{j} + 76\mathbf{k}$
g $532\mathbf{i} + 1064\mathbf{j} + 2366\mathbf{k}$ h $64\mathbf{i} + 204\mathbf{j} - 432\mathbf{k}$
i $-340\mathbf{i} + 765\mathbf{j} - 1156\mathbf{k}$ j $-340\mathbf{i} + 765\mathbf{j} - 1156\mathbf{k}$
k $344\mathbf{i} - 831\mathbf{j} + 1276\mathbf{k}$
- 10 B
- 11 a 11 unit² b 22.99 unit² c 8.66 unit²
- 12 a 6 unit² b 45.72 unit²
- 13 a (24, 2, -20) b (68, -36, 30)
- 14 $\mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a} = (22\mathbf{i} - 14\mathbf{j} - 6\mathbf{k}) + (11\mathbf{i} - 7\mathbf{j} - 3\mathbf{k}) + (-33\mathbf{i} + 21\mathbf{j} + 9\mathbf{k}) = \mathbf{0}$
- 15 Either $\mathbf{a} = \mathbf{0}$, $\mathbf{b} = \mathbf{0}$ or $\mathbf{b} = k\mathbf{a}$ for some real number k ($\mathbf{a}$ and $\mathbf{b}$ are parallel).
- 16 a $D(4, 7, 1)$ b $\sqrt{491}$

17 $0.599i + 0.799j + 0.0499k$

18 e.g. $(-i + 6j - 4k) \times (-2i + 12j + 9k) = 102i + 17j$
and $(-2i + 12j + 9k) \times (-i + 6j - 4k) = -102i - 17j$

19 e.g. $(7i - 4j + 11k) \times [(-2i + 12j + 9k) \times (14i - 7j)]$
 $= -770i + 1771j + 1134k$ and $[(7i - 4j + 11k) \times$
 $(-2i + 12j + 9k)] \times (14i - 7j) = 532i + 1064j +$
 $2366k$

1.05

1 $q = \frac{c+d}{2}$

$p = \frac{a+d}{2}$

$s = \frac{b+c}{2}$

$r = \frac{a+b}{2}$

$PQ = \frac{c+d}{2} - \frac{a+d}{2}$

$PQ = \frac{c-a}{2}$

$RS = \frac{b+c}{2} - \frac{a+b}{2}$

$RS = \frac{c-a}{2}$

$PQ = RS$

$PQ \parallel RS$ and $|PQ| = |RS|$

2 Proof

3 Proof

4 Proof

5 $p = \frac{1}{k_1 + k_2} (k_2 a + k_1 b)$

6 a $m = \frac{1}{3}(a + b + c)$

b Proof

c $m = \frac{1}{3}(a + b + c)$

7 a $d = \left(\frac{|c-a|}{|b-a| + |c-a|} \right) b + \left(\frac{|b-a|}{|b-a| + |c-a|} \right) c$

b 0

c The angle bisectors divide the side opposite the vertex in the ratio of the other sides.

8 Proof

9 a $h = \frac{1}{4}(a + b + c + d)$ b Proof

10 $h = \frac{1}{4}(a + b + c + d)$

11 Proof

12 Proof

13 Proof

14 Proof

15 Demonstration

16 Proof

1.06

1 $a = -4i - 4k$

b $p = 3i + 2j + 7k + t(-4i - 4k)$

c The domain is restricted to $0 \leq t \leq 1$

2 $p = 3i + 7j - 2k + t(-6i - 5j + 4k)$, for $0 \leq t \leq 1$,

$\frac{x-3}{-6} = \frac{y-7}{-5} = \frac{z+2}{4}$

3 $p = 9i + j + 2k + t(i + j + k)$

4 $\frac{x}{1} = \frac{y}{-5} = \frac{z}{3}$

5 $\frac{x-1}{-1} = \frac{y}{1}, z = 1$

6 $\frac{x-1}{4} = \frac{y-2}{5} = \frac{z-1}{-1}$

7 $p = j - k + t(3i + j + 2k)$

8 a $p = 2i + 3j + 5k + t(i + 3j - k)$

b $\frac{x-2}{1} = \frac{y-3}{3} = \frac{z-5}{-1}$

9 a $\frac{x-1}{-2} = \frac{y-2}{-4} = \frac{z-3}{-6}$

b $\frac{x-1}{3} = \frac{y-2}{-6} = \frac{z-3}{1}$

10 a $p = i - k + t(2i + 2j - k), \frac{x-1}{2} = \frac{y}{2} = \frac{z+1}{-1}$

b $p = j - k + t(i + 2j), \frac{x}{1} = \frac{y-1}{2}, z = -1$

c $p = -6i + 5j + 2k + t(2i - 5k), \frac{x+6}{2} = \frac{z-2}{-5}, y = 5$

d $p = 2i + 3j - 3k + t(-i + 3k), \frac{x-2}{-1} = \frac{z+3}{3}, y = 3$

11 a $p = -4i + 3j + 5k + t(2i + j - 4k),$
 $\frac{x+4}{2} = y-3 = \frac{z-5}{-4}$

b $p = 5j + t(5i - 5j - 5k), x = 5 - y = -z$

c $p = 9i + 3j - 2k + t(-5i + 2j + k),$
 $\frac{x-9}{-5} = \frac{y-3}{2} = \frac{z+2}{1}$

d $p = 2i - j + 3k + t(-i + 5j - 6k),$
 $\frac{x-2}{-1} = \frac{y+1}{5} = \frac{z-3}{-6}$

12 Q lies on the line.

13 a $p = i - j + t(5i - j - k)$

b Q lies on the line when $t = -3$

14 a $y = (x+4)^3 - 2$, a cubic

b $x^2 + y^2 = 1$, a circle of radius 1

c $\frac{x^2}{4^2} + \frac{y^2}{3^2} = 1$, an ellipse with a major axis of

length 8 and minor axis of length 6.

15 $(1, 1, -2)$

16 a $p = i - 2j + k + t(-2i + 2j - 4k)$

b $p = 8i + j + 4k + t(4i + j + k)$

c $p = 3i + j - 2k + t(-2i + 2j - 4k)$

- 17 a Lines intersect at (1, 1, 1) (when $t = -2$ and $s = 3$)
 b Demonstration
 c Skew lines.
- 18 a $\mathbf{p} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k} + t(2\mathbf{i} - \mathbf{j} - 4\mathbf{k})$,
 $\frac{x-1}{2} = \frac{y-2}{-1} = \frac{z-3}{-4}$
 b Q and R lie on the line.
- 19 (8, -5, 0)
- 20 They cross at (14, 25, 18) at 9 seconds, so they collide.
- 21 They would get within about 220 m about 43.4 s later unless they take avoidance action.
- 22 The first aircraft is circling the airport at a height of 6000 m and a horizontal distance of 10 km from the airport. The second is circling at a height of 8000 m and a horizontal distance of 12 km and the third is landing at the airport in 6 minutes from the direction S 33.7° W.

1.07

- 1 C
 2 E
 3 P and R are on the plane.
 4 a and c
 5 a $2x + 10y - 7z + 4 = 0$
 b $53z^2 + 104y^2 - 140yz + 154y - 144z < 0$
 c $104y^2 + 53z^2 - 140yz + 208y - 140z \geq 220$
 6 a $2x + y - 3z + 5 = 0$
 b $5x^2 + 10z^2 - 12xz + 24x - 40z \leq 24$
 c $5x^2 + 10z^2 - 12xz + 26x - 48z + 34 > 0$
 7 $x + 2y + 3z - 13 = 0$
 8 $2x + 4y - 5z = 0$
 9 a $[(x-3)\mathbf{i} + (y-2)\mathbf{j} + (z-5)\mathbf{k}] \cdot (\mathbf{i} + \mathbf{k}) = 0$
 b $x + z = 8$
 10 $11x - 2y - 6z - 69 = 0$
 11 B
 12 $3x + y - 6z + 5 = 0$
 13 $6x - 5y - z + 84 = 0$
 14 a $z = 2$ b $x = 5$ c $y = -3$
 15 $3x - y + 2z + 11 = 0$
 16 $2x + y + z - 4 = 0$
 17 a $2x - 8y + 5z - 18 = 0$
 b $9x - y - z - 8 = 0$
 c $14x - 7y - 8z - 52 = 0$
 d $y - 1 = 0$
 e $20x - 5y + 2z = 0$
 f $x + 2y - 4z + 7 = 0$
 18 $4x + 10y - 7z + 2 = 0$
 19 $x + 2y - 2z - 1 = 0$
 20 $x - y + 2z = 3$
 21 $\frac{5\sqrt{5}}{3}$
 22 $\frac{5\sqrt{14}}{14}$

- 23 The normals are parallel and therefore so are the planes.

24 $\frac{26\sqrt{105}}{105}$

25 $71x - 15y + 17z + 10 = 0$

- 26 The normals to the planes are parallel and so the planes are parallel.

- 27 The planes intersect in the line

$\mathbf{p} = (3\mathbf{i} - \mathbf{j} + \mathbf{k}) + t(-\mathbf{i} + \mathbf{j} + 3\mathbf{k})$

1.08

- 1 a $a = 5, b = -3, c = 2$
 b Inconsistent
 2 a $x = 2, y = 3, z = -5$
 b $u = -4, v = 5, w = 3$
 c $e = 2, f = -2, g = -1, h = 3$
 d $b = -1, c = -2, d = 2, e = 1$
 e $f = 2, g = -1, h = 3, k = -5$
 3 Adults \$11, children \$5, seniors \$7
 4 Gravel 6 t/m³, sand 5 t/m³, cement 0.8 t/m³, water 1 t/m³
 5 Shares = \$60 000, Unit trust = \$68 000, First mortgage = \$72 000

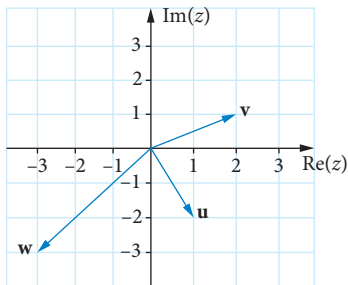
CHAPTER 1 REVIEW

- 1 B
 2 C
 3 B
 4 A
 5 B
 6 C
 7 C
 8 E
 9 A
 10 a $(-\frac{3}{2}, -1, \frac{1}{2}), \sqrt{114} \approx 10.68$
 b $(\frac{1}{2}, \frac{3}{2}, -4), \sqrt{106} \approx 10.3$
 c $(3, -\frac{5}{2}, -\frac{3}{2}), \sqrt{122} \approx 11.05$
 11 $x^2 + y^2 + z^2 - 6x + 8y + 4z + 4 = 0$
 12 Radius = 3, centre = (3, -1, 2).
 13 a $\mathbf{AB} = \mathbf{i} - 3\mathbf{j} + 9\mathbf{k}, |\mathbf{AB}| = \sqrt{91}$
 b $\hat{\mathbf{p}} = -\frac{3\sqrt{14}}{14}\mathbf{i} + \frac{\sqrt{14}}{14}\mathbf{j} + \frac{\sqrt{14}}{7}\mathbf{k}$
 14 a (-3, 2, -3) b (1, -4, 13)
 c (-7, 2, -7) d (9, -18, 13)
 e (22, -4, 26)
 15 a (7.62, 293.2°), (5.39, 326.3°, 48.0°)
 b (-6.93, -4), (-15.20, 5.53, -11.76)
 c 13, 25.48
 16 a $5\mathbf{i} - 7\mathbf{j}$ b $21.65\mathbf{i} - 12.5\mathbf{j}$
 c $7\mathbf{i} + 8\mathbf{j} - 2\mathbf{k}$ d $13\mathbf{i} + 22.5\mathbf{j} + 15\mathbf{k}$

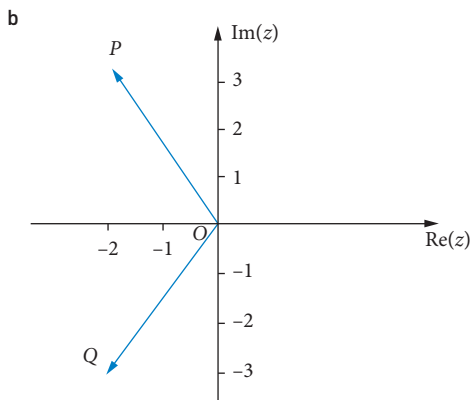
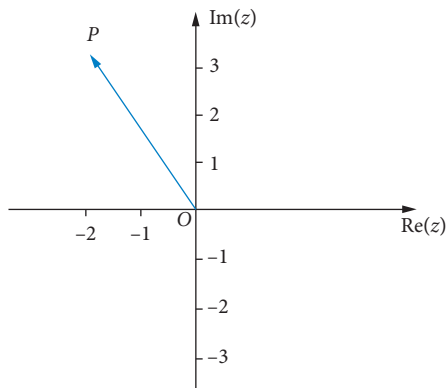
- 17 a $3\mathbf{i} - \mathbf{j} + 4\mathbf{k}$, $\sqrt{26} \approx 5.099$
 b $-3\mathbf{i} - 5\mathbf{j} + 2\mathbf{k}$, $\sqrt{38} \approx 6.164$
 c $\mathbf{i} - 3\mathbf{j}$, $\sqrt{10} \approx 3.162$
 d $2\mathbf{i} + 5\mathbf{j} - 3\mathbf{k}$, $\sqrt{38} \approx 6.164$
 e $-1.828\mathbf{i} - 3.167\mathbf{j} + 3.41\mathbf{k}$, 5
 f $6.333\mathbf{i} + 12.429\mathbf{j} - 32.862\mathbf{k}$, 35.7
- 18 $(38.13, 90.3^\circ, 13.2^\circ)$
- 19 a 10.35 b -2 c -1
- 20 a 71.28 perpendicular to the plane of the vectors
 b 342.35k c $(23, -10, 19)$ d $-6\mathbf{i} - 7\mathbf{j} - 9\mathbf{k}$
- 21 a 8 b -10 c $8\mathbf{i} + \mathbf{j} - 3\mathbf{k}$
 d -2 e -10
- 22 a -2.62 b 0.49 c 94.1°
- 23 65 Nm
- 24 a 78.0 b $-34\mathbf{i} - 36\mathbf{j} + 6\mathbf{k}$
- 25 a $\begin{bmatrix} -28 \\ -68 \\ 74 \end{bmatrix}$ b 178 c $\begin{bmatrix} 20 \\ 0 \\ 16 \end{bmatrix}$ d $\begin{bmatrix} 27 \\ -16 \\ -4 \end{bmatrix}$
 e $\begin{bmatrix} -27 \\ 16 \\ 4 \end{bmatrix}$ f $\begin{bmatrix} -88 \\ -184 \\ 136 \end{bmatrix}$ g $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ h $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$
- 26 7.23 units²
- 27 $(-14, 16, 14)$
- 28 $\mathbf{p} = 3\mathbf{i} - 7\mathbf{j} + 4\mathbf{k} + t(-8\mathbf{i} + 15\mathbf{j} - 2\mathbf{k})$; $\frac{x-3}{-8} = \frac{y+7}{15}$
 $= \frac{z-4}{-2}$
- 29 a $\mathbf{p} = 2\mathbf{i} - \mathbf{j} + 5\mathbf{k} + t(\mathbf{i} + 7\mathbf{j} - 9\mathbf{k})$; $\frac{x-2}{1} = \frac{y+1}{7} = \frac{z-5}{-9}$
 b $\mathbf{p} = 2\mathbf{i} - \mathbf{j} + 5\mathbf{k} + t(\mathbf{i} + 7\mathbf{j} - 9\mathbf{k})$ for $0 \leq t \leq 1$
 c No
- 30 $\mathbf{p} = -1\mathbf{i} + 2\mathbf{j} + \mathbf{k} + t(-\mathbf{i} - 5\mathbf{j} + 4\mathbf{k})$; $(-3, -8, 9)$ satisfies the equation for $t = 2$.
- 31 $(5, 6, -2)$
- 32 a $x + y + z - 1 = 0$ b No
 c $2y^2 + 2yz + 2z^2 + 2y - 2z < 23$
- 33 $5x - 2y + 3z - 8 = 0$.
- 34 a $9x - y - z - 8 = 0$
 b $14x - 7y - 8z - 52 = 0$
- 35 a $x = -2, y = \frac{1}{3}, z = 4$
 b Inconsistent, no solutions, the line of intersection of two planes is parallel to the third.
 c Infinite solutions, $p = 9 - 2c, q = 3 - c, r = c$, corresponding to a straight line intersection
- 36 83 units³
- 37 Proof
- 38 Proof
- 39 e.g. $(2, 4, -6) \cdot (7, 4, 5) = 0$ and $(2, 4, -6) \times (1, 2, -3) = \mathbf{0}$
- 40 Prices are: 40 mm, \$15.40; 50 mm, \$21.60; 100 mm, \$53.40
- 41 Proof
- 42 Proof
- 43 Proof

2.01

- 1 a $x = -1 \pm i\sqrt{2}$ b $x = 2 \pm i\sqrt{3}$
 c $z = 3i, 2i$ d $w = \frac{-1 \pm i\sqrt{7}}{2}$
- 2 a $x = 1 \pm i$ b $y = -2 \pm i$
 c $z = 3 \pm 2i$ d $w = -4 \pm i\sqrt{2}$
- 3 a $-i$ b 1 c -1 d 0
- 4 a $3 + 5i$ b 34
- 5 a $z^2 - 4z + 5 = 0$ b $z^2 - 2\sqrt{3}z + 7 = 0$
 c $z^2 + 2z + 6 = 0$
- 6 $p - qi$
- 7



- 8 $\Delta < 0$ and the coefficients (negative sum and product of the roots) are real, so yes. $x = -1 \pm 2i$
- 9 Use substitution or $x^2 - (\alpha + \beta)x + \alpha\beta = 0$. The coefficients are not all real.
- 10 $1 + i$
- 11 a



- c $\overrightarrow{OQ}$ is the reflection of $\overrightarrow{OP}$ in the x -axis.
 d $\overrightarrow{OP} - \overrightarrow{OQ} = -2 + 3i - (-2 - 3i) = 6i$ which is purely imaginary.

2.02

- 1 a $14 + i$ b $13 - 15i$ c $40 + 42i$
 d 11 e $47 + 13i$
 2 a $u = 5, v = -3$ b $u = -16, v = -14$
 3 a $-3 + 14i$ b $6 + 6i$
 c $22 - 89i$ d $-10 + 10i$
 4 a $\frac{2+i}{5}$ b $\frac{-1+8i}{13}$ c $\frac{2-i\sqrt{5}}{3}$
 5 Proof – realise the denominator
 6 $= -1$, which is real.
 7 a $\operatorname{Re}(z) = -2\sqrt{3}, \operatorname{Im}(z) = -3\sqrt{2}$
 b $\operatorname{Re}(z) = 13, \operatorname{Im}(z) = -9$
 c $\operatorname{Re}(z) = x - 4w, \operatorname{Im}(z) = v - y$
 d $\operatorname{Re}(z) = \frac{1}{3}, \operatorname{Im}(z) = -\frac{4\sqrt{2}}{3}$
 8 $\operatorname{Re}(z) = \frac{(x-1)^2 - y^2}{(x-1)^2 + y^2}, \operatorname{Im}(z) = \frac{2(x-1)y}{(x-1)^2 + y^2}$
 9 $\frac{u^2 + 2uvi - v^2}{u^2 + v^2}$
 10 $\frac{2(x^2 - y^2)}{(x^2 + y^2)^2}$

2.03

- 1 a $z = 5[\cos(\pi) + i \sin(\pi)]$
 b $z = 4\left[\cos\left(\frac{\pi}{6}\right) + i \sin\left(\frac{\pi}{6}\right)\right]$
 c $2\left[\cos\left(\frac{-\pi}{2}\right) + i \sin\left(\frac{-\pi}{2}\right)\right]$
 d $v = 3\sqrt{2}\left[\cos\left(\frac{-\pi}{7}\right) + i \sin\left(\frac{-\pi}{7}\right)\right]$
 2 a $\theta = \frac{\pi}{4}, r = 2$ b $\theta = \frac{5\pi}{6}, r = 2\sqrt{2}$
 c $\theta = \frac{-\pi}{9}, r = 1$ d $\theta = \frac{-\pi}{3}, r = \frac{1}{\sqrt{3}}$
 3 a $\operatorname{mod}(z) = 2, \arg(z) = \frac{\pi}{6}$
 b $\operatorname{mod}(z) = 3\sqrt{2}, \arg(z) = \frac{\pi}{4}$
 c $\operatorname{mod}(z) = \frac{\sqrt{2}}{2}, \arg(z) = \frac{-\pi}{4}$
 d $z = -\frac{1}{2} + \frac{i\sqrt{3}}{2}, \operatorname{mod}(z) = 1, \arg(z) = \frac{2\pi}{3}$
 e $z = 7i, \operatorname{mod}(z) = 7, \arg(z) = \frac{\pi}{2}$
 f $z = -6, \operatorname{mod}(z) = 6, \arg(z) = \pi$

- 4 a $z = 2 \operatorname{cis}\left(\frac{\pi}{4}\right)$ b $w = 2 \operatorname{cis}\left(\frac{2\pi}{3}\right)$
 c $u = \operatorname{cis}\left(\frac{-\pi}{6}\right)$ d $v = \operatorname{cis}\left(\frac{-3\pi}{4}\right)$
 e $z = \sqrt{5} \operatorname{cis}\left(\frac{-\pi}{2}\right)$ f $w = \operatorname{cis}(0)$
 5 a $6 + 6i\sqrt{3}$ b $-1 + i$
 c i d $\frac{-\sqrt{3}}{4} + \frac{i}{4}$
 6 a $4\sqrt{3} - 4i$ b $\frac{3}{2} - \frac{3i\sqrt{3}}{2}$
 c -9 d $\frac{3\sqrt{3}}{2} + \frac{9}{2}i$
 7 a $\sqrt{3} + \frac{3\sqrt{3}}{2}i$ b $-2 + i\sqrt{2}$
 8 a $r[\cos(-\theta) + i \sin(-\theta)]$
 b $r[\cos(\pi - \theta) + i \sin(\pi - \theta)]$
 c $r[\cos(\theta - \pi) + i \sin(\theta - \pi)]$
 9 a Let $x = r \cos(\theta), y = r \sin(\theta)$, then eliminate θ .
 b $r\left[\cos\left(\frac{\pi}{2} - \theta\right) + i \sin\left(\frac{\pi}{2} - \theta\right)\right]$
 c Eliminate θ .

2.04

- 1 a $\operatorname{cis}\left(\frac{12\pi}{35}\right)$ b $\operatorname{cis}\left(\frac{\pi}{3}\right)$
 c $\operatorname{cis}(5\alpha)$ d $\operatorname{cis}(-10\beta)$
 2 a $\operatorname{cis}\left(\frac{-\pi}{2}\right)$ b $\operatorname{cis}\left(\frac{-3\pi}{4}\right)$
 c $3 \operatorname{cis}\left(\frac{-\pi}{5}\right)$ d $2 \operatorname{cis}\left(\frac{\pi}{4}\right)$
 3 $\operatorname{cis}\left(\frac{-22\pi}{35}\right)$
 4 a $\sqrt{10} \operatorname{cis}\left(\frac{7\pi}{12}\right)$ b $8 \operatorname{cis}\left(\frac{-6\pi}{7}\right)$
 c $-3 \operatorname{cis}\left(\frac{3\pi}{8}\right) = 3 \operatorname{cis}\left(\frac{-5\pi}{8}\right)$ d $2\sqrt{3} \operatorname{cis}\left(\frac{5\pi}{9}\right)$
 5 a $\cos\left(\frac{\pi}{3}\right) + i \sin\left(\frac{\pi}{3}\right)$
 b $\cos\left(\frac{-3\pi}{7}\right) + i \sin\left(\frac{-3\pi}{7}\right)$
 c $\frac{1}{2}\left[\cos\left(\frac{\pi}{4}\right) + i \sin\left(\frac{\pi}{4}\right)\right]$
 6 a $3 \operatorname{cis}\left(\frac{\pi}{6}\right)$ b $4 \operatorname{cis}(\pi)$ c $3 \operatorname{cis}\left(\frac{2\pi}{5}\right)$
 d $\operatorname{cis}(-\theta)$ e $\frac{1}{3} \operatorname{cis}\left(\frac{\pi}{2}\right)$ f $4 \operatorname{cis}\left(\frac{-3\pi}{4}\right)$
 7 a $\sqrt{3} \operatorname{cis}\left(\frac{-11\pi}{12}\right)$ b $\sqrt{3} \operatorname{cis}\left(\frac{-5\pi}{12}\right)$
 c $\frac{1}{\sqrt{3}} \operatorname{cis}\left(\frac{11\pi}{12}\right)$ d $3 \operatorname{cis}\left(\frac{2\pi}{3}\right)$
 e $\frac{1}{3} \operatorname{cis}\left(\frac{-\pi}{6}\right)$ f $\sqrt{3} \operatorname{cis}\left(\frac{11\pi}{12}\right)$

8 a $\operatorname{cis}\left(\frac{2\pi}{3}\right)$ b $\operatorname{cis}(-2)$ c $12 \operatorname{cis}\left(\frac{\pi}{4}\right)$

9 Proof

10 Proof

11 Proof

2.05

1 a $\arg(z) = \frac{-\pi}{4}, |z| = \sqrt{2}$ b $\arg(z) = \frac{\pi}{6}, |z| = 2$

c $\arg(z) = \frac{3\pi}{4}, |z| = 2$ d $\arg(z) = \frac{-2\pi}{3}, |z| = 1$

2 a $\arg(z) = -0.381$ b $\arg(z) = 2.16$

c $\arg(z) = -2.28$ d $\arg(z) = -1.15$

3 a $\frac{-\pi}{4}$ b 3 c $\theta + \alpha$ d $\sqrt{5}$

4 a $\cos(11\alpha) + i \sin(11\alpha)$

b $4\sqrt{3} \left[\cos\left(\frac{11\pi}{36}\right) + i \sin\left(\frac{11\pi}{36}\right) \right]$

c $3\sqrt{5} \left[\cos\left(\frac{4\pi}{7}\right) + i \sin\left(\frac{4\pi}{7}\right) \right]$

d $\frac{1}{48} \left[\cos\left(\frac{27\pi}{55}\right) + i \sin\left(\frac{27\pi}{55}\right) \right]$

5 a 1 b 81 c $\frac{1}{7\sqrt{7}}$

6 a 6θ b $\frac{-5\pi}{8}$ c $\frac{2\pi}{9}$

7 a $2\sqrt{2} \operatorname{cis}\left(\frac{\pi}{12}\right)$ b $\operatorname{cis}\left(\frac{-\pi}{12}\right)$

c $\frac{1}{6} \operatorname{cis}\left(\frac{-3\pi}{4}\right)$ d $\sqrt{2} \operatorname{cis}\left(\frac{\pi}{4}\right)$

8 a $2\sqrt{2} \operatorname{cis}\left(\frac{7\pi}{12}\right)$ b $\frac{1-\sqrt{3}}{2\sqrt{2}}$

9 $\frac{-17}{\sqrt{290}}$

10 Proof

2.06

1 a $\cos(9\theta) + i \sin(9\theta)$ b $\cos(2\theta) - i \sin(2\theta)$

c $\cos(4\theta) - i \sin(4\theta)$ d $\cos(6\theta) + i \sin(6\theta)$

2 a -16 b $\frac{-81}{2} - \frac{81i\sqrt{3}}{2}$

c $\frac{i}{4}$ d $\frac{625}{2} + \frac{625i\sqrt{3}}{2}$

3 $\cos\left(\frac{2\pi}{5}\right) - i \sin\left(\frac{2\pi}{5}\right)$

4 a $-2 + 2i$ b $-8 + 8i\sqrt{3}$

c $64\sqrt{2} + 64i\sqrt{2}$ d i

5 a $\frac{i}{512}$ b $-\frac{1}{512} + i\frac{\sqrt{3}}{512}$

6 Proof

7 a $\cos(3\theta) - i \sin(3\theta)$

b $\cos(3\alpha + 4\beta) + i \sin(3\alpha + 4\beta)$

c $\cos(\phi) - i \sin(\phi)$

d $\cos(17\beta) - i \sin(17\beta)$

8 Proof

9 Proof

10 $2 \cos(n\theta)$

INVESTIGATION: EULER AND $e^{i\theta}$

Use index laws for each proof.

2.07

1 a $[\cos(\theta) + i \sin(\theta)]^3 = \cos^3(\theta) + 3i \cos^2(\theta) \sin(\theta) - 3 \cos(\theta) \sin^2(\theta) - i \sin^3(\theta)$

b i $\cos(3\theta) = 4 \cos^3(\theta) - 3 \cos(\theta)$

ii $\sin(3\theta) = 3 \sin(\theta) - 4 \sin^3(\theta)$

2 a $\cos(2\theta) + i \sin(2\theta)$

$= \cos^2(\theta) + 2i \cos(\theta) \sin(\theta) - \sin^2(\theta)$

b Proofs

3 Proofs

4 a $\tan(3\theta) = \frac{\sin(3\theta)}{\cos(3\theta)}$ b Proof c Proof

5 Proof

6 $\operatorname{cis}(60^\circ) = \frac{1}{2} + i\frac{\sqrt{3}}{2}$ and $\operatorname{cis}(45^\circ) = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$,

$\cos(15^\circ) = \frac{\sqrt{2} + \sqrt{6}}{4}$

7 a Proof b $\frac{8-5\sqrt{2}}{3}$

8 $-\frac{7\sqrt{3}}{32} + \frac{\pi}{8}$

9 Proof

10 a $2i \sin(\theta)$ b $2i \sin(2\theta)$ c $2i \sin(n\theta)$

CHAPTER 2 REVIEW

1 C

2 B

3 D

4 E

5 A

6 $32 - 10i$

7 $\frac{9+7i}{5}$

8 $\frac{1+4i\sqrt{3}}{49}$ a $\frac{1}{49}$ b $\frac{4\sqrt{3}}{49}$

9 $2\sqrt{5} \left[\cos\left(\frac{8\pi}{17}\right) + i \sin\left(\frac{8\pi}{17}\right) \right]$

10 $2 \left[\cos\left(\frac{-\pi}{6}\right) + i \sin\left(\frac{-\pi}{6}\right) \right]$

11 $2 + 2i\sqrt{3}$

12 a $\frac{-11\pi}{18}$ b 30

13 $\frac{1-i}{\sqrt{2}}$

- 14 a $\cos(A \pm B) = \cos(A)\cos(B) \pm \sin(A)\sin(B)$
 b $\sin(A \pm B) = \sin(A)\cos(B) \pm \cos(A)\sin(B)$
- 15 $\cos(3\theta) + i\sin(3\theta) = \cos^3(\theta) + 3i\cos^2(\theta)\sin(\theta) - 3\cos(\theta)\sin^2(\theta) - i\sin^3(\theta)$, 0.0483
- 16 $-2^{15} + 0i$
- 17 Proof
- 18 a $\cos(5x) - i\sin(5x)$ b $\frac{3}{4}\text{cis}\left(\frac{3\pi}{10}\right)$
 c $3[\cos(3\beta) - i\sin(3\beta)]$
- 19 $2\left[\cos\left(\frac{-3\pi}{4}\right) + i\sin\left(\frac{-3\pi}{4}\right)\right]$
 a $\frac{3\pi}{4}$ b $\frac{1}{2}$
- 20 a $\frac{1}{16}\left[\cos\left(\frac{-\pi}{2}\right) + i\sin\left(\frac{-\pi}{2}\right)\right]$
 b $\frac{1}{4\sqrt{2}}\left[\cos\left(\frac{5\pi}{12}\right) + i\sin\left(\frac{5\pi}{12}\right)\right]$
- 21 a $\text{cis}\left(\frac{5\pi}{7}\right)$ b $\text{cis}(91\alpha)$
- 22 $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$, $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$
 a Proof
 b Hint: let $\alpha = 15^\circ$ and use $\tan(2\alpha) = \frac{2\tan(\alpha)}{1 - \tan^2(\alpha)}$.

3.01

- 1 a $g[f(x)] = 2x - 5$ b $g[f(x)] = e^{2x} + 2$
 c $g[f(x)] = \log_e(e^{2x}) = 2x$ d $g[f(x)] = \sin(2x)$
- 2 a $f[g(x)] = 2x - 4$ b $f[g(x)] = e^{x^2+2}$
 c $f[g(x)] = e^{2\log_e(x)} = x^2$ d $f[g(x)] = 2\sin(x)$
- 3 a $g(x) = x + c$, $f(x) = x + 3 - c$ where c is a constant
 b $g(x) = x^3$, $f(x) = 2x + 5$
 c $g(x) = x^2 - 7$, $f(x) = x + 3$
 d $g(x) = e^x$, $f(x) = x^2$
 e $g(x) = x^2$, $f(x) = \sin(x)$
 f $g(x) = \sin(x)$, $f(x) = 2x$
- 4 $g \circ f: g(x) = 4x$, $f(x) = x^2$; $g(x) = x^2$, $f(x) = 2x$;
 $g(x) = 4x - 16$, $f(x) = x^2 + 4$
- 5 a $g \circ f = \sin(2x)$; Domain: $x \in \mathbf{R}$, Range: $-1 \leq y \leq 1$
 b $g \circ f = \log_e(-x)$; Domain: $x < 0$, Range: $y \in \mathbf{R}$
 c $g \circ f = 2\sin\left(\frac{1}{2x}\right)$; Domain: $x \in \mathbf{R}$, $x \neq 0$,
 Range: $-2 \leq y \leq 2$
- 6 There are many possibilities, e.g. $f(x) = \sqrt{x}$,
 $g(x) = x^2$; $f(x) = x + 1$, $g(x) = x - 1$

INVESTIGATION: ONE-TO-ONE FUNCTIONS

The horizontal and vertical lines pass through each of the first three graphs at most once.

The horizontal lines pass through some parts of each of the second three graphs more than once, but the vertical lines pass through at most once.

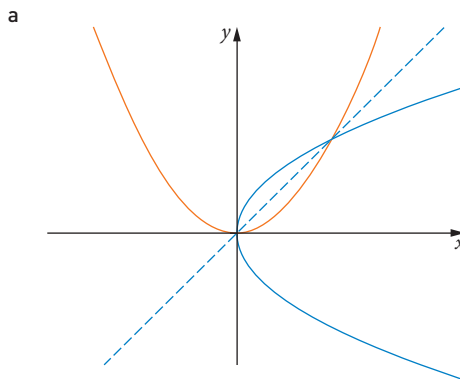
C and D are 1 : 1 functions; A and B are not functions; E and F are not 1 : 1.

3.02

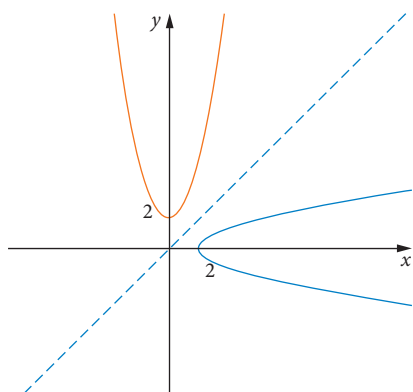
- 1 a NOT one-to-one b One-to-one
 c One-to-one d NOT one-to-one
- 2 Proof
- 3 a Strictly decreasing, and one-to-one
 b Neither increasing or decreasing and NOT one-to-one
 c Neither increasing nor decreasing, NOT one-to-one
 d Strictly decreasing, and one-to-one
- 4 Proof
- 5 For example, $g \circ f$ with $f(x) = x^3$, $g(x) = 3x - 2$
- 6 For example, $g \circ f$ with $f(x) = \pi - x$, $g(x) = x + \cos(x)$
- 7 For example, $g \circ f$ with $f(x) = e^x$, $g(x) = \log_e(x)$

3.03

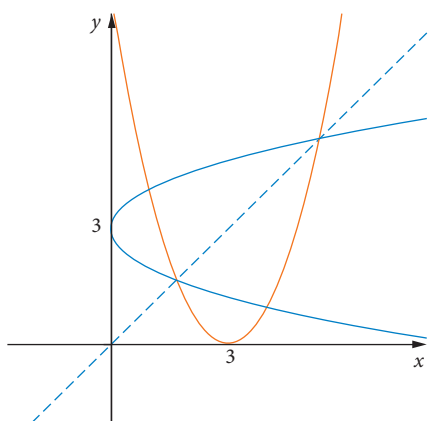
- 1 a $f^{-1}(x) = \frac{x+1}{2}$ b $f^{-1}(x) = \pm\sqrt{x-1}$
 c $f^{-1}(x) = 1 + e^{\frac{x}{2}}$ d $f^{-1}(x) = (x-1)^2$
 e $f^{-1}(x) = \sqrt[3]{x-1}$
- 2 Inverse functions exist for a, c, d, e
- 3 Either $x > 0$ or $x < 0$.
- 4 a $f^{-1}(x) = \sqrt[3]{x}$ b $f^{-1}(x) = \frac{x+2}{3}$
 c $f^{-1}(x) = \log_e(x)$ d $f^{-1}(x) = \frac{2}{x}$
 e $f^{-1}(x) = 1 - \frac{1}{x}$
- 5 a $f^{-1}(x) = \sqrt{x}$, $x \geq 0$, $y \geq 0$
 b $f^{-1}(x) = \sqrt{\frac{x+1}{3}}$, $x \geq -1$, $y \geq 0$
 c $f^{-1}(x) = \sqrt[4]{x} + 2$, $x \geq 0$, $y \geq 2$
 d $f^{-1}(x) = -\sqrt{\frac{3}{x}}$, $x > 0$, $y < 0$
 e $f^{-1}(x) = -\sqrt[4]{\frac{2}{x}}$, $x > 0$, $y < 0$
- 6 Functions in orange, inverse relations in blue.



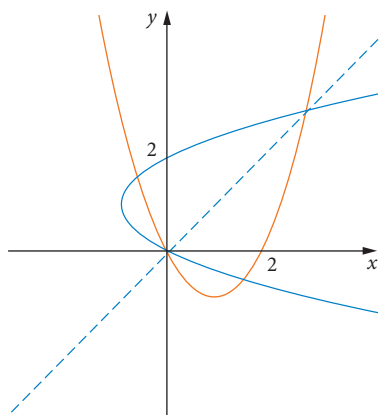
b



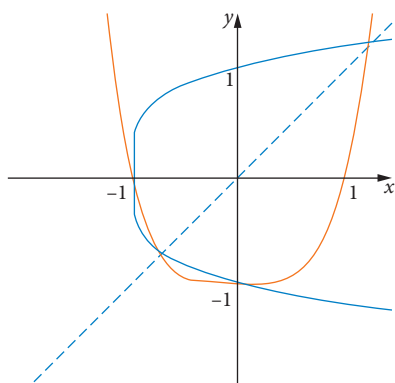
c



d



e



7 a i $f^{-1}(x) = 1 + \sqrt{x+1}$ ii $f^{-1}(x) = 1 - \sqrt{x+1}$

b i $f^{-1}(x) = +\sqrt[4]{x+2}$ ii $f^{-1}(x) = -\sqrt[4]{x+2}$

c i $f^{-1}(x) = +\sqrt[4]{\frac{x-1}{2}}$ ii $f^{-1}(x) = -\sqrt[4]{\frac{x-1}{2}}$

d i $f^{-1}(x) = 3 + \sqrt{x+8}$ ii $f^{-1}(x) = 3 - \sqrt{x+8}$

e i $f^{-1}(x) = -2 + \sqrt{x+7}$ ii $f^{-1}(x) = -2 - \sqrt{x+7}$

8 a $x \in \mathbf{R}, x \neq 2; y \in \mathbf{R}, y \neq 0$

b $f^{-1}(x) = 2 + \frac{1}{x}$

c $x \in \mathbf{R}, x \neq 0; y \in \mathbf{R}, y \neq 2$

9 $f^{-1}(x) = 2 + \sqrt{x+4}$ when $x \geq 2$

10 $x \leq -3, f^{-1}(x) = -3 - \sqrt{x+10}$

3.04

1 a 4

b 1

c -11

d 12

e -9

f 10

g 0

h 1

i -5

j 11

2 a $x = -2, 8$

b $x = -4, 1$

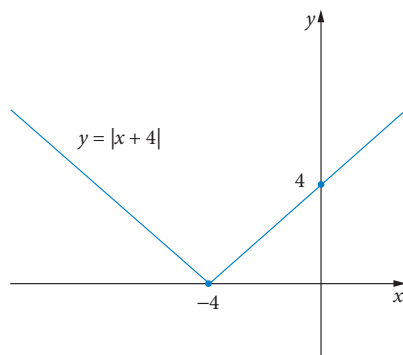
c no solution

d $x = -5, 3$

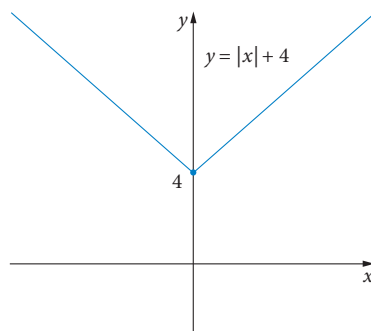
e $x = \pm \frac{1}{4}$

3 $x = -2, \frac{4}{3}$

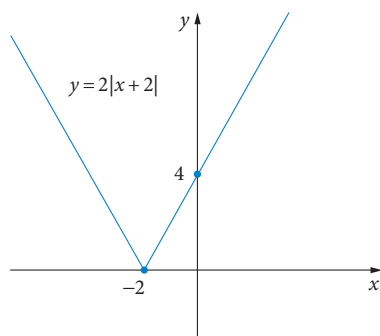
4 a $x \in \mathbf{R}, y \geq 0$



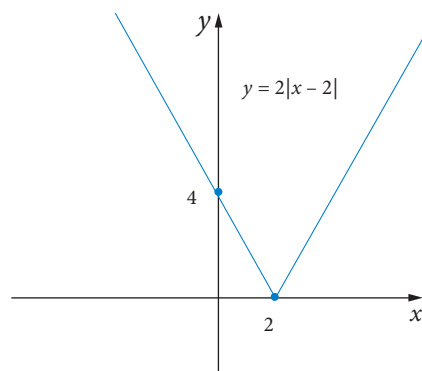
b $x \in \mathbf{R}, y \geq 4$



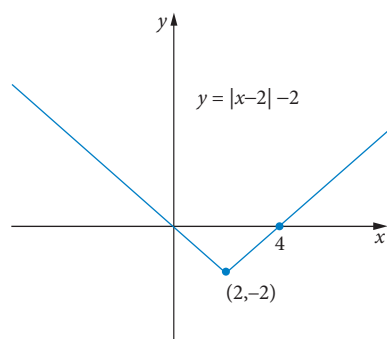
c $x \in \mathbf{R}, y \geq 0$



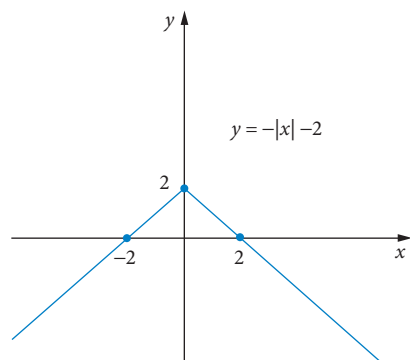
d $x \in \mathbf{R}, y \geq 0$



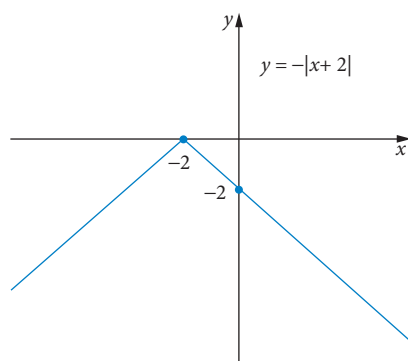
e $x \in \mathbf{R}, y \geq -2$



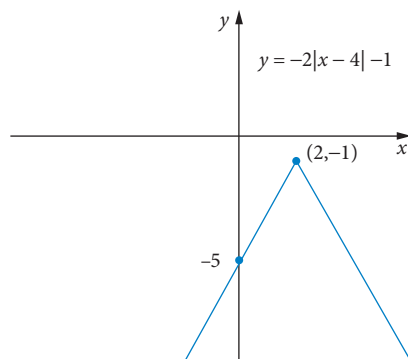
f $x \in \mathbf{R}, y \leq 2$



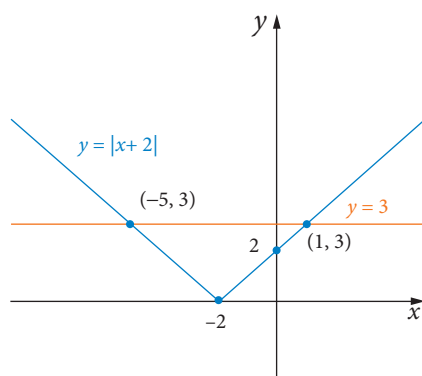
g $x \in \mathbf{R}, y \leq 0$



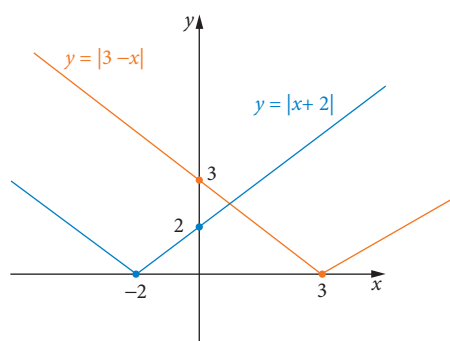
h $x \in \mathbf{R}, y \leq -1$



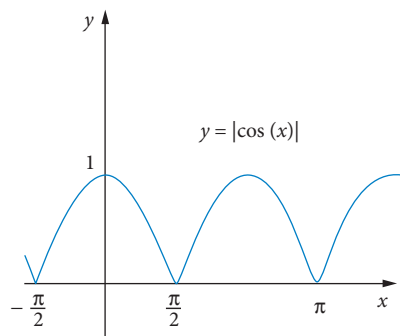
5 $x = -5, 1$



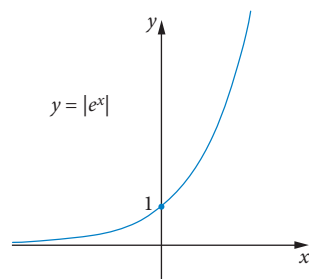
6 $x = \frac{1}{2}$



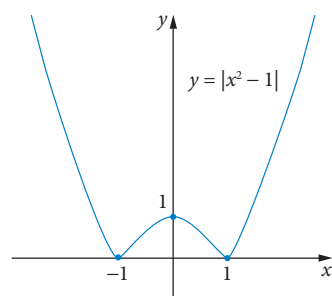
7 a $x \in \mathbf{R}, 0 \leq y \leq 1$



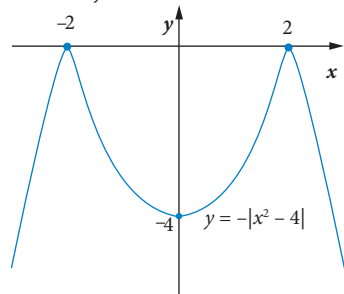
b $x \in \mathbf{R}, y > 0$



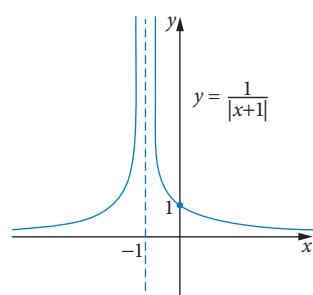
c $x \in \mathbf{R}, y \geq 0$



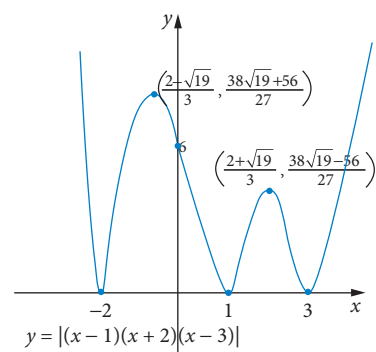
d $x \in \mathbf{R}, y \leq 0$



e $x \in \mathbf{R} (x \neq -1), y > 0$



f $x \in \mathbf{R}, y \geq 0$



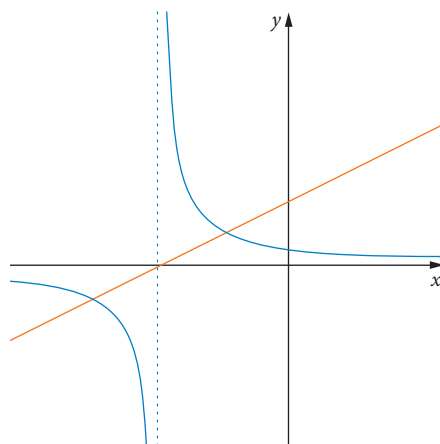
8 $x = 0, x \leq -2\sqrt{2}, x \geq 2\sqrt{2}$

9 $x \leq \frac{2}{3}, x \geq 1\frac{1}{3}$

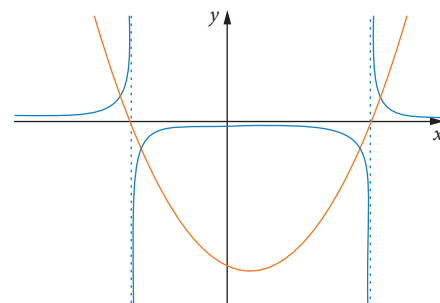
10 $-2\frac{1}{2} < x < 2\frac{1}{2}$

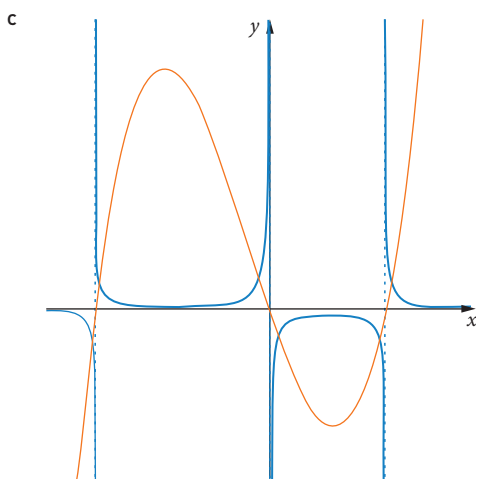
3.05

1 a

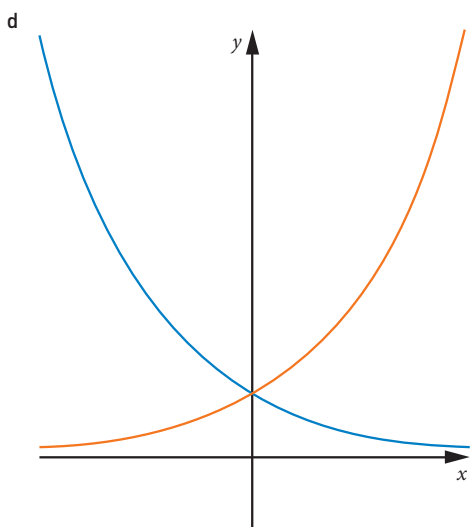
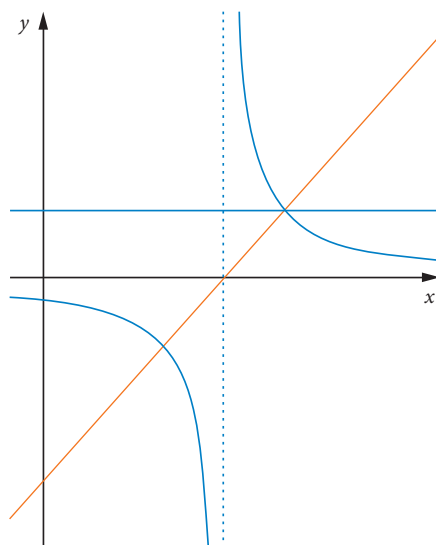


b

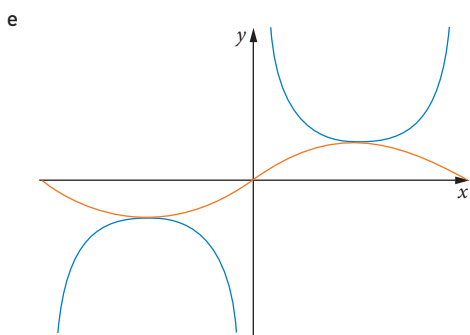
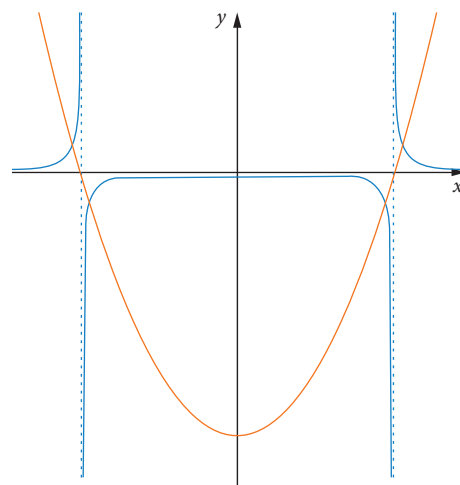




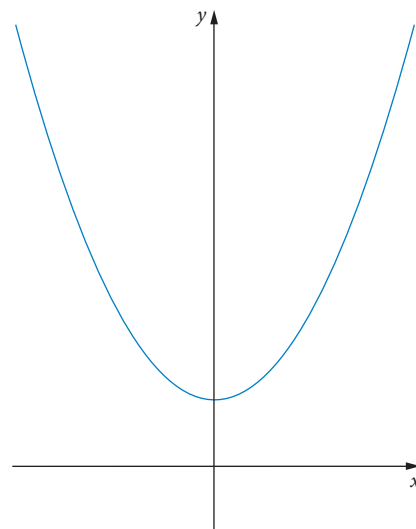
2 $1\frac{1}{2} < x \leq 2$



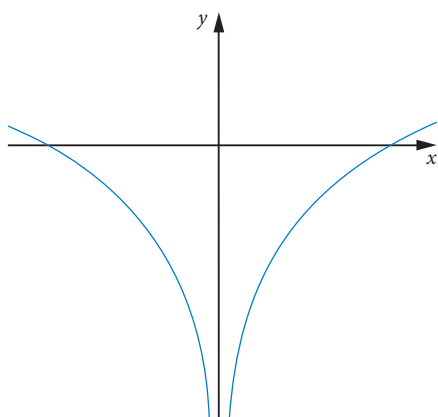
3 $x < -3, x > 3$



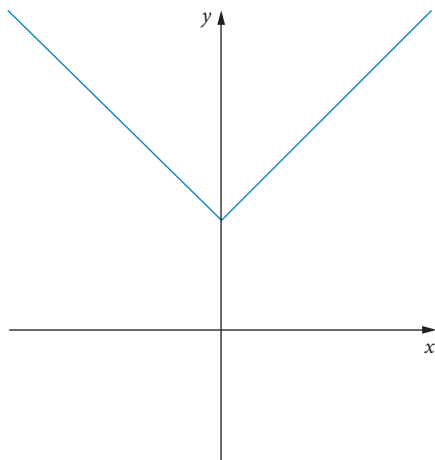
4 a



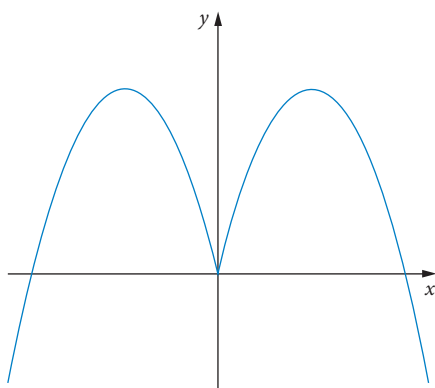
b



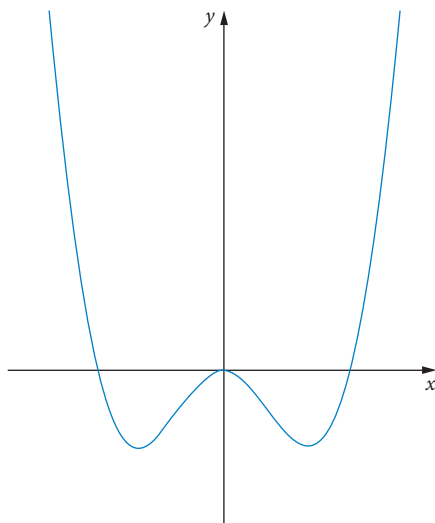
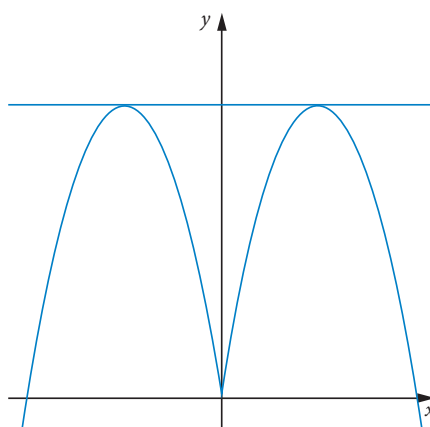
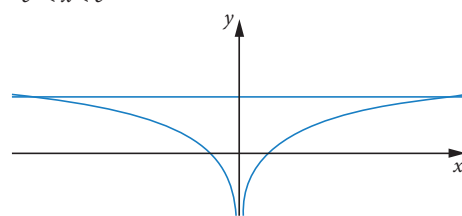
c



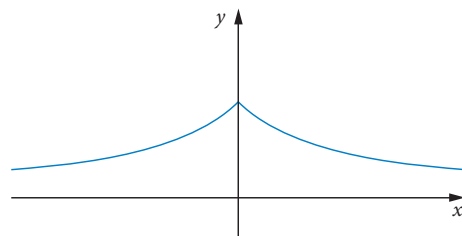
d



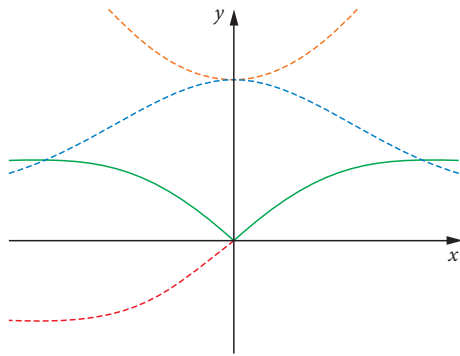
e

5 $x = \pm 3$ 6 $-e^2 < x < e^2$ 

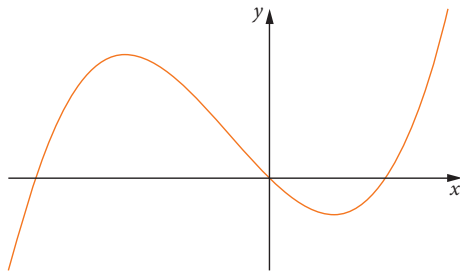
7



8

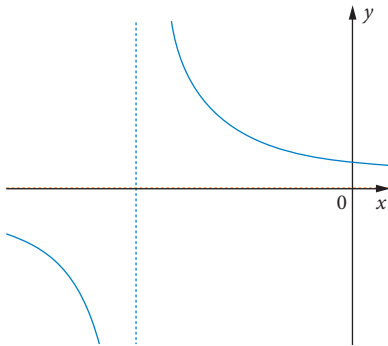


9 $x < -4, 0 < x < 2$

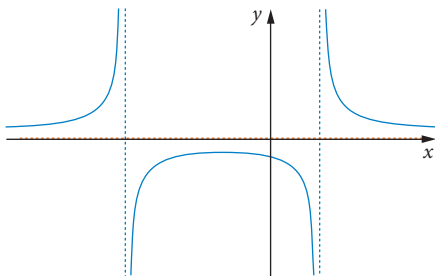


3.06

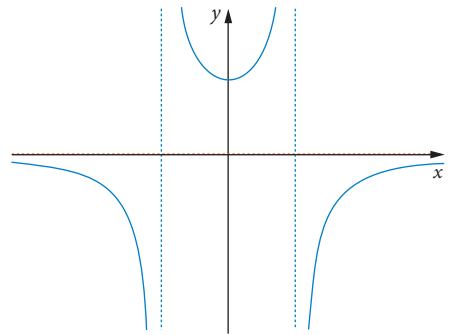
1 a



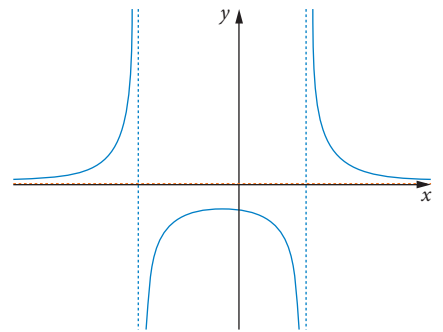
b



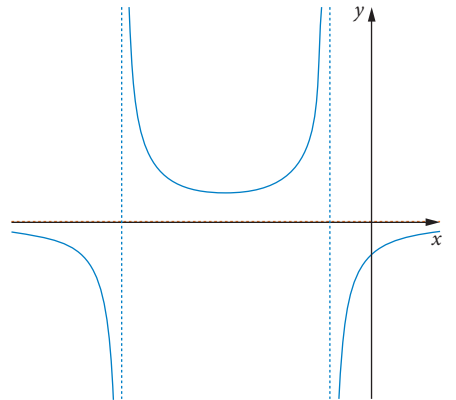
c



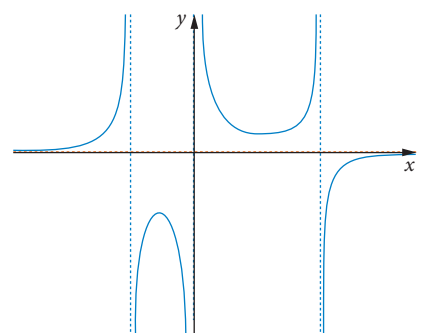
d



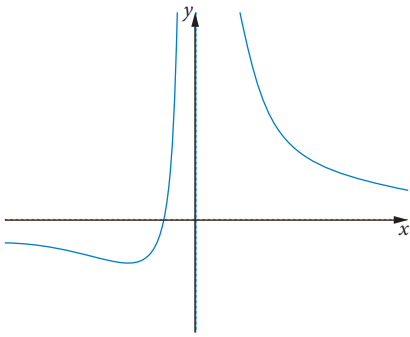
e



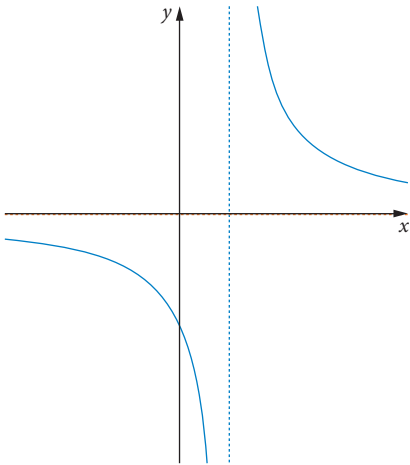
f



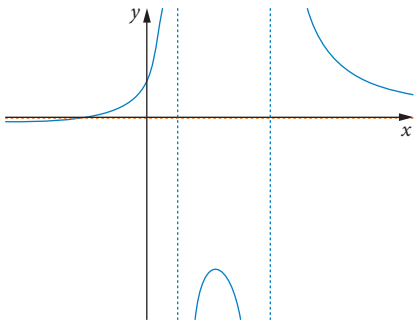
2 a



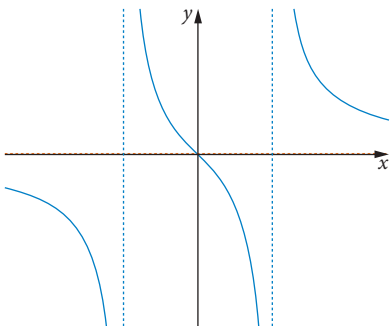
b



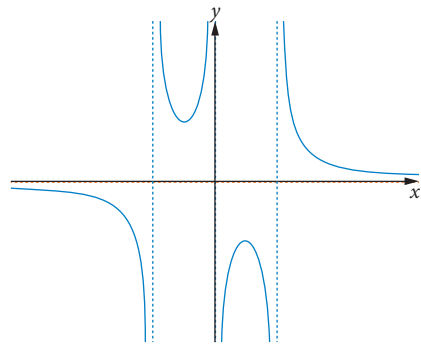
c



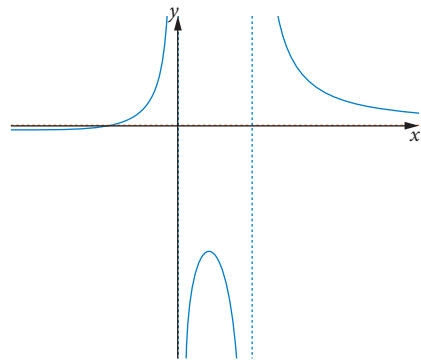
d



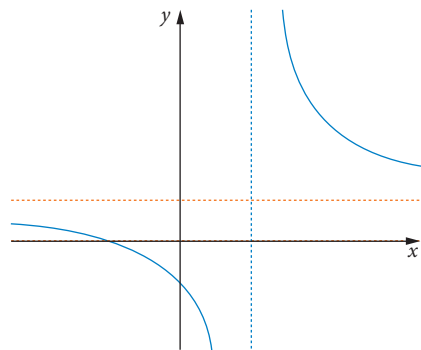
e



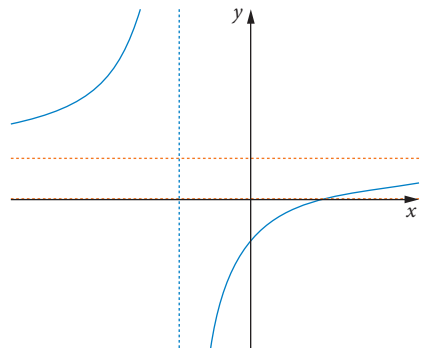
f

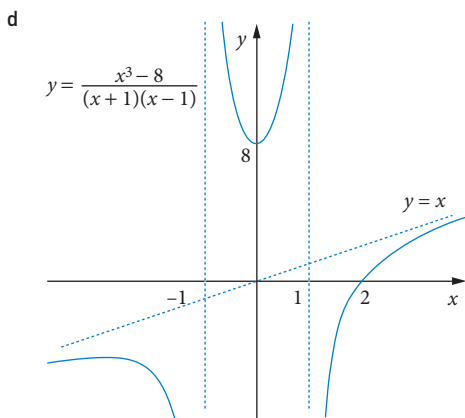
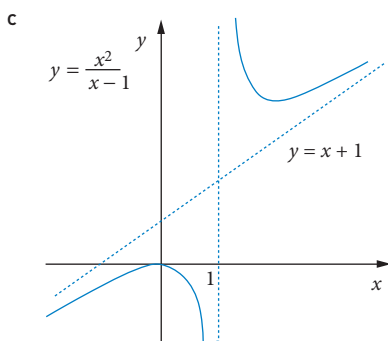


3 a

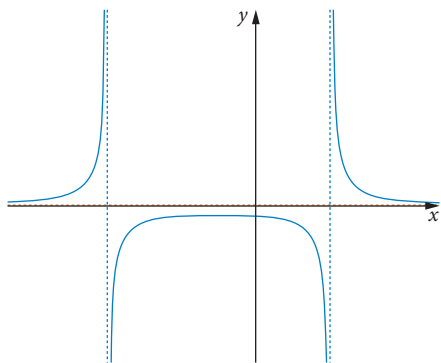


b

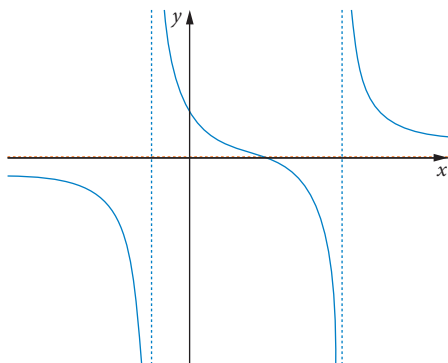




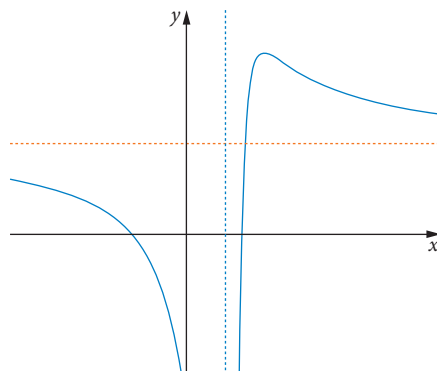
4 $x < -4, x > 2$



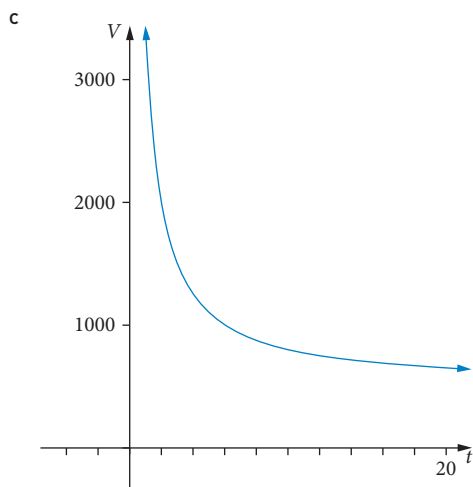
5 $x < -1, -1 \leq x < 4$



6 $-\sqrt{2} \leq x < 1, 1 < x \leq \sqrt{2}$

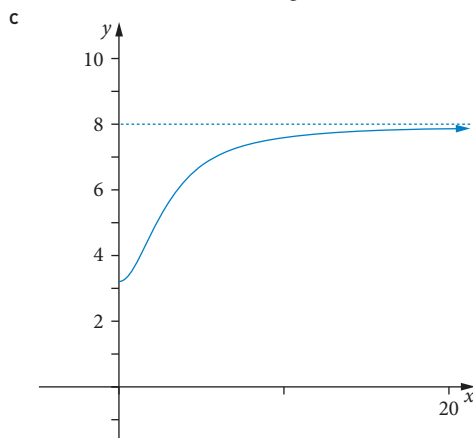


7 a $V(2) = \$2000$ b $V(10) = \$800$

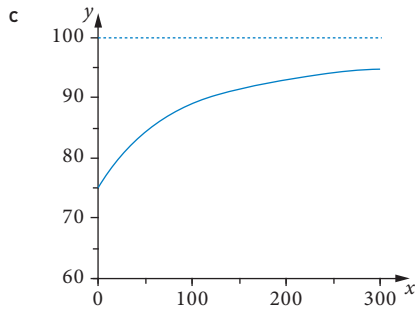


8 a $t = 3.06$ years, late January in 1993

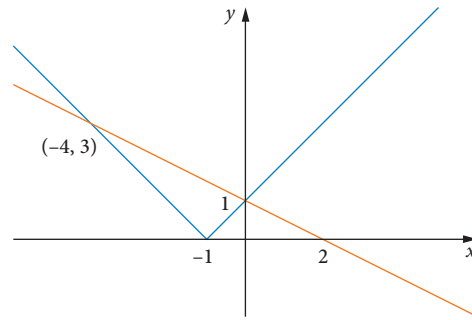
b As $t \rightarrow \infty$, sales $\rightarrow 8$ billion games



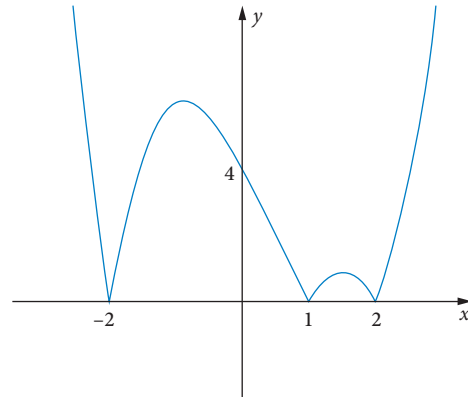
9 a $\frac{x+60}{x+80} \times 100\%$ b $\frac{x+60}{x+80} \times 100 > 90, x \geq 120$



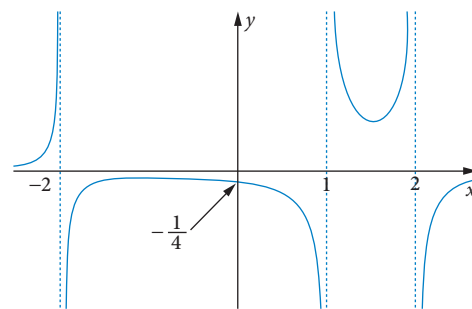
11 $x = 0, -4$



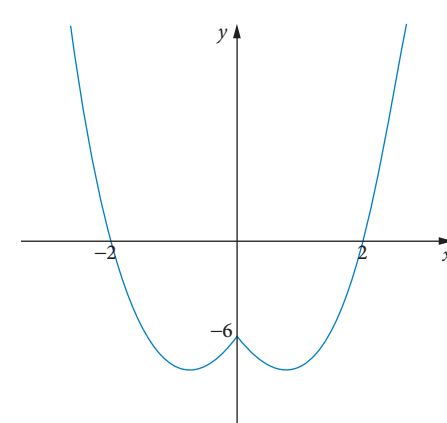
12



13



14



CHAPTER 3 REVIEW

1 D

2 D

3 B

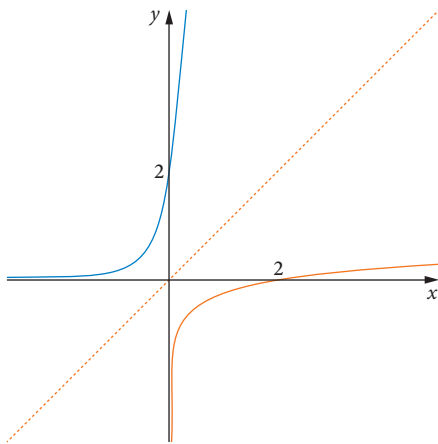
4 C

5 C

6 $x = \frac{2}{y} + 3, f^{-1}(x) = 3 + \frac{2}{x}$

7 $x = \sqrt{5-y}, y = 5-x^2$

8

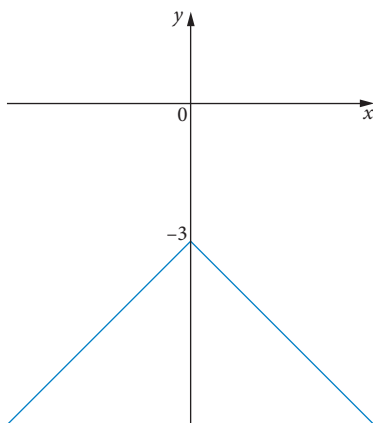


9 a -2

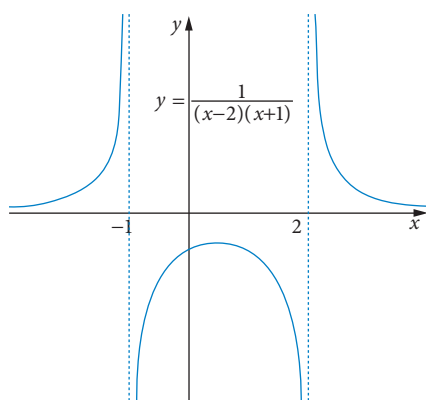
b 13

c -12

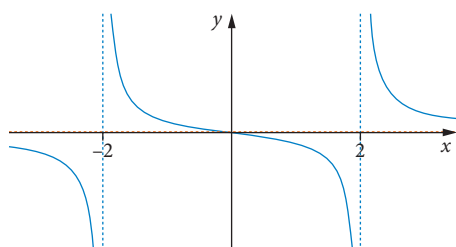
10 Domain: $x \in \mathbb{R}$, Range: $y \leq -3$



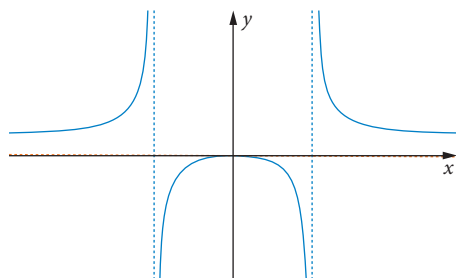
15



16

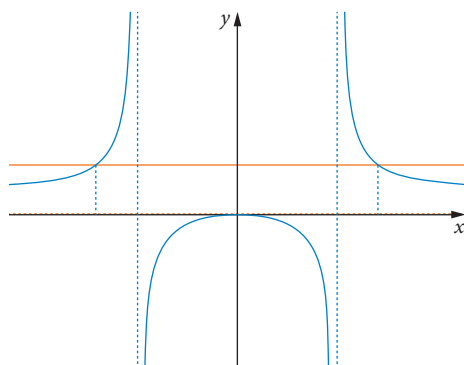


17



18 $(y-x)(x+2) = c, (y-x)(x+2) = 4$

19 $x \leq -2\sqrt{2}, -2 < x < 2, x \geq 2\sqrt{2}$



- 20 Rate = $\frac{100(x+15)}{x+17}$. Ross must submit the next 23 assignments to increase his return rate to 95%.

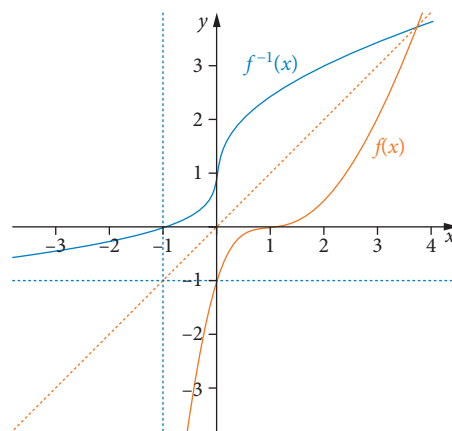
MIXED REVISION 1

Multiple choice

- 1 B
- 2 A
- 3 D
- 4 E
- 5 C
- 6 D
- 7 A
- 8 D
- 9 D

Short answer

- 1 $16 + 16i\sqrt{3}$
- 2



- 3 a 7 b 26 c 24 d -6
 e -6 f -25 g 33 h -51
 i -51 j 33

4 a 20.07 sq.u. b 8.73 sq.u.

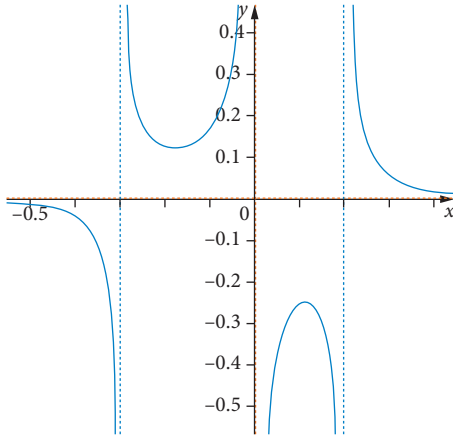
5 $\frac{x}{x-2} > -1, x < 1, x > 2$

6 $w = 7 \left[\cos\left(\frac{\pi}{4}\right) + i \sin\left(\frac{\pi}{4}\right) \right]$

Application

- 1 Proof. Use $z^n = r^n [\cos(n\theta) + i \sin(n\theta)]$ and $(\bar{z})^n = r^n [\cos(\theta) - i \sin(\theta)]^n = r^n [\cos(-\theta) + i \sin(-\theta)]^n$.

2 $\frac{1}{x(x-2)(x+3)} > 0$



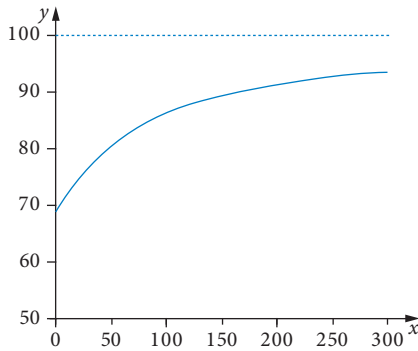
$-3 < x < 0, x > 2$

3 Proof

4 $z = \sqrt{2} \operatorname{cis}\left(-\frac{\pi}{12}\right), z^6 = \left[\sqrt{2} \operatorname{cis}\left(-\frac{\pi}{12}\right)\right]^6 = 8 \operatorname{cis}\left(-\frac{\pi}{2}\right)$

so $n = 6$.

5 $\frac{100(50+x)}{75+x} = 90$

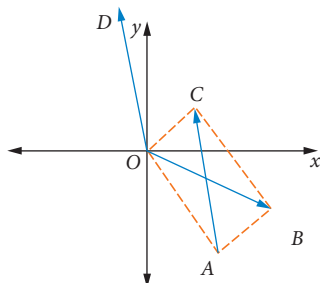


$x = 175$

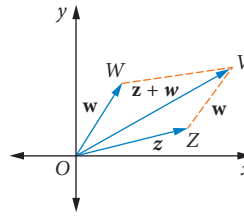
6 Proof

4.01

- 1 a see diagram
c $\vec{AC} = \vec{c} - \vec{a}$
b $b = c + a$
d see diagram



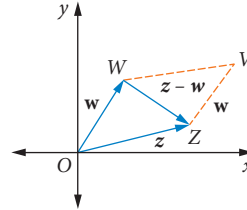
2 a



The point V represents the sum $z + w$.

- b The sum of the lengths of 2 sides of a triangle is greater than the length of the 3rd side.

3



$|z - w| + |w| \geq |z|$. Equality holds when WZ and V are collinear.

- 4 i a $\vec{AB} = \vec{z}_2 - \vec{z}_1$ b $\vec{CB} = \vec{z}_2 - \vec{z}_3$
c $\vec{AC} = \vec{z}_3 - \vec{z}_1$

ii $w = z_4 + \frac{1}{2}(z_2 - z_4)$ (many other answers)

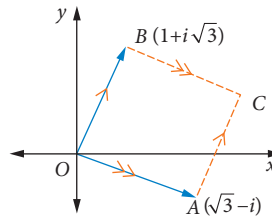
- 5 i a $\vec{AB} = \vec{w} - \vec{z}$ b $\vec{CB} = \vec{w} - \vec{u}$
c $\vec{AC} = \vec{u} - \vec{z}$

ii a $u = v + w - z$

b $m = \frac{1}{2}(u + z)$ or $m = \frac{1}{2}(v + w)$

6 Parallelogram, opposite sides parallel and equal in length

7 a



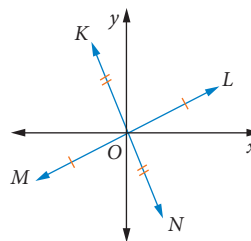
b $c = (\sqrt{3} + 1) + i(\sqrt{3} - 1)$ c Proof

d Proof e Square

8 a $|a| = 2\sqrt{13}, |b| = \sqrt{13}, |a - b| = \sqrt{65}$

b Proof c Right-angled

9



parallelogram since diagonals bisect each other.

10 a $c = \frac{5}{2}(1 + \sqrt{3}) + \frac{5}{2}(1 + \sqrt{3})i$

b $12\frac{1}{2}$ square units.

4.02

1 a $\cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right)$

b $\cos\left(-\frac{3\pi}{4}\right) + i\sin\left(-\frac{3\pi}{4}\right)$

c $2\left[\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right]$

d $\sqrt{3}\left[\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right]$

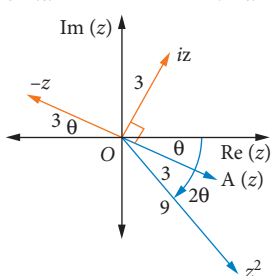
2 a i

b $-i$

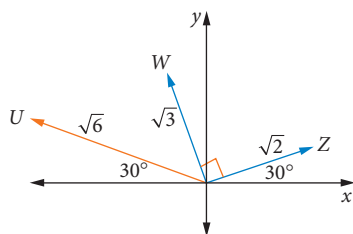
3 a iu

b $u + iu$

4



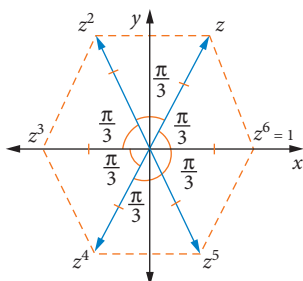
5



6 $c = 2\left[\cos\left(\frac{7\pi}{12}\right) + i\sin\left(\frac{7\pi}{12}\right)\right]$,

$b = c + a = 2\operatorname{cis}\left(\frac{7\pi}{12}\right) + \operatorname{cis}\left(\frac{\pi}{4}\right)$

7

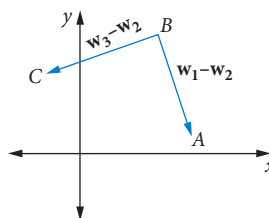


Note that connecting z_1 to z_6 forms a regular hexagon.

8 a V is $-2iz$, W is $z(1 - 2i)$

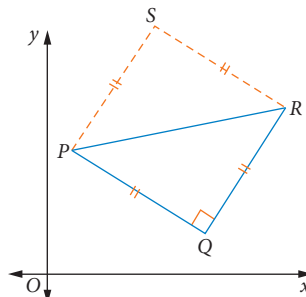
b U is $\frac{z - 2iz}{2}$

9 a



b Right-angled at B .

10 a



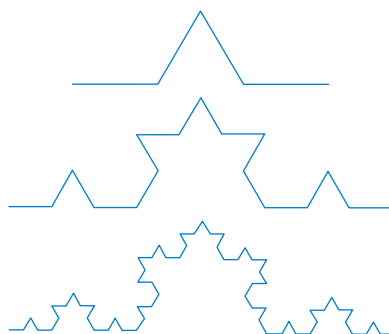
b Proof

c $PQRS$ is a square.

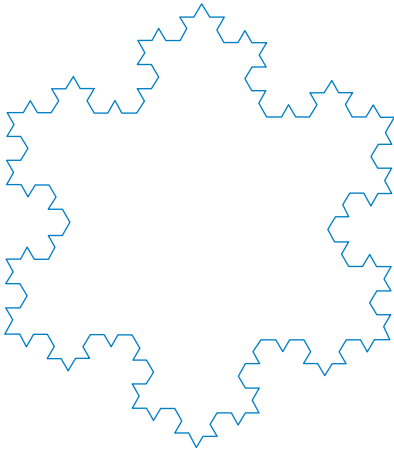
INVESTIGATION: THE DEVELOPMENT OF CHAOS THEORY

The Von Koch Curve is constructed starting with a line segment.

The line segment is divided into three equal line segments and an equilateral triangle constructed on the middle segment. The base of the triangle is then erased. The process is repeated with every line segment. Continuation produces the Koch curve.



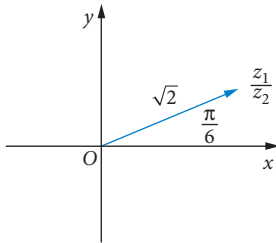
Starting with an equilateral triangle produces a Koch snowflake.



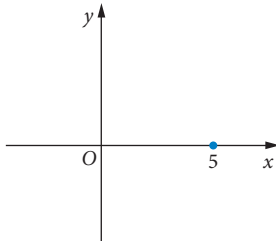
The Mandelbrot set is the set of points C in the complex plane such that the sequence $z_{n+1} = z_n^2 + C$ is bounded. Fractals are objects and sets like the Koch curve and the Mandelbrot set that are apparently chaotic but are produced by a relatively simple rule.

4.03

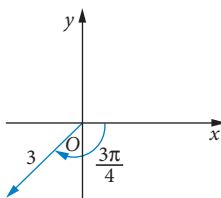
1 a $\left| \frac{z_1}{z_2} \right| = \sqrt{2}, \arg\left(\frac{z_1}{z_2}\right) = \frac{\pi}{6}$



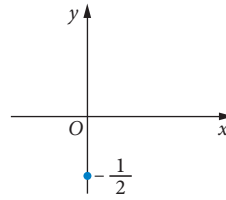
b $\left| \frac{z_1}{z_2} \right| = 5, \arg\left(\frac{z_1}{z_2}\right) = 0$



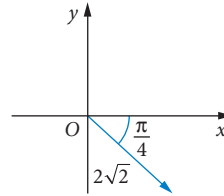
c $\left| \frac{z_1}{z_2} \right| = 3, \arg\left(\frac{z_1}{z_2}\right) = -\frac{3\pi}{4}$



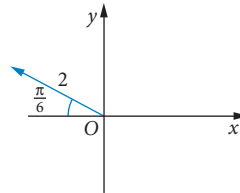
d $\left| \frac{z_1}{z_2} \right| = \frac{1}{2}, \arg\left(\frac{z_1}{z_2}\right) = -\frac{\pi}{2}$



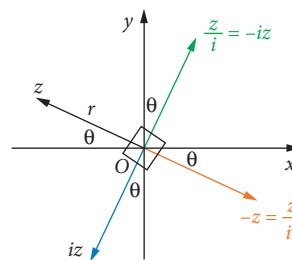
2 $\frac{z_1}{z_2} = 2\sqrt{2} \operatorname{cis}\left(-\frac{\pi}{4}\right)$



3 $\frac{z_1 z_2}{z_3} = 2 \operatorname{cis}\left(\frac{5\pi}{6}\right)$



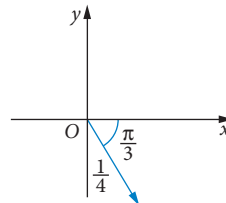
4

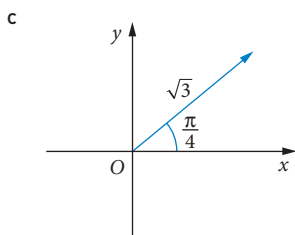
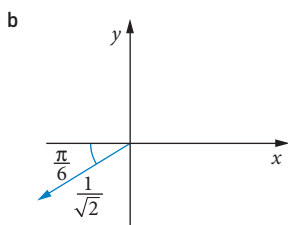


5 a $w = -iz = \frac{z}{i}$ b $w = -z$ c $w = iz$

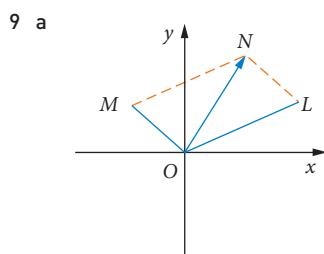
6 Hint: $|w_4 - w_1| = 2|w_2 - w_1|$,
 $\arg(w_4 - w_1) - \arg(w_2 - w_1) = \frac{\pi}{2}$

7 a





8 Proof



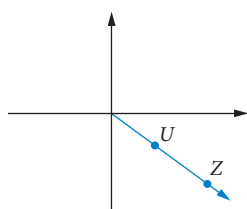
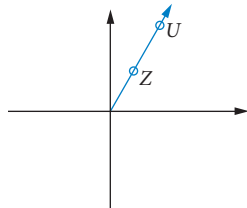
b Parallelogram

c $\frac{w_2}{w_3}$ is imaginary

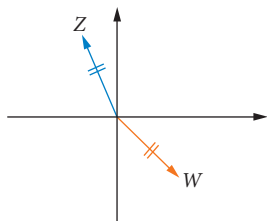
10 Proof

4.04

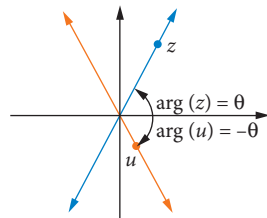
- 1 a z and u lie on the same line through the origin, on the same side of O .



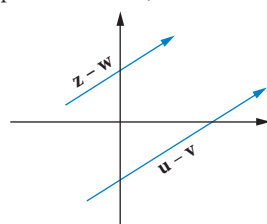
- b z and w are the same distance from O .



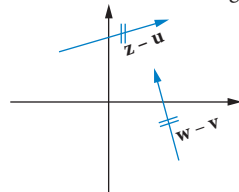
- c z and u lie on lines through O that are reflections over the x -axis



- d parallel vectors (same direction)



- e vectors with the same length



- f vectors same length and parallel

- g same as (a)

- h same as (c)

- i diagonals of quadrilateral equal, rectangle

- j $z - u = v - w$, same as (f)

- k quadrilateral has equal diagonals, perpendicular, square.

- 2 $\theta - \pi$

- 3 Proof

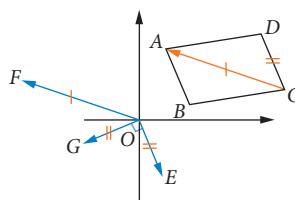
- 4 a Proof

- b Proof

- c Proof

- d Proof

5



- 6 a Proof

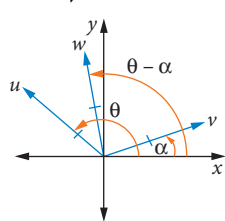
- b Proof

- c Proof

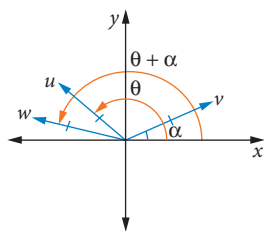
- 7 $z = u \operatorname{cis}\left(\frac{\pi}{4}\right)$, $w = -2u \operatorname{cis}\left(\frac{\pi}{4}\right)$, $v = \bar{u}$

- 8 Diagrams will vary.

- a $w = \frac{u}{v}$



b $w = uv$



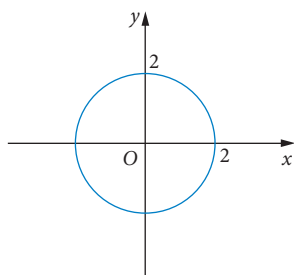
9 a $\theta = \frac{2\pi}{5}$

b $z_1 = \text{cis}\left(\frac{2\pi}{5}\right), z_2 = \text{cis}\left(\frac{4\pi}{5}\right), z_3 = \text{cis}\left(\frac{-4\pi}{5}\right),$

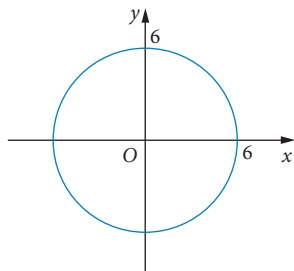
$z_4 = \text{cis}\left(\frac{-2\pi}{5}\right), z_5 = 1$

4.05

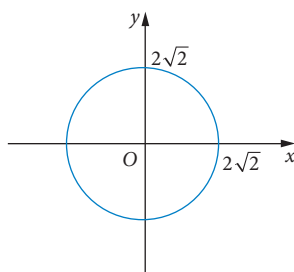
1 a



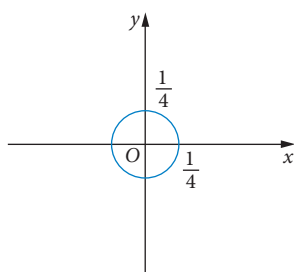
b



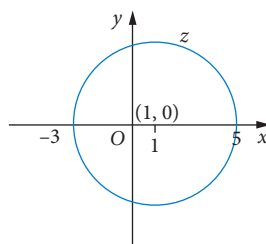
c



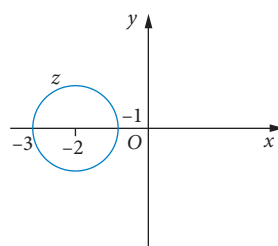
d



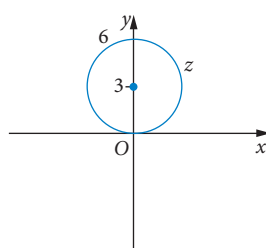
e



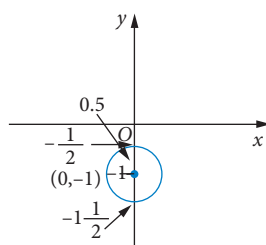
f



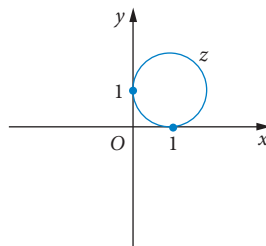
g



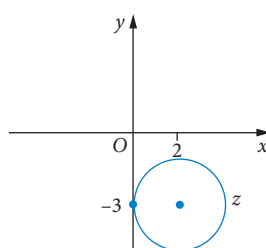
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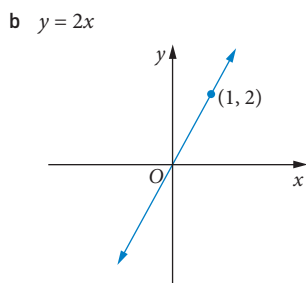
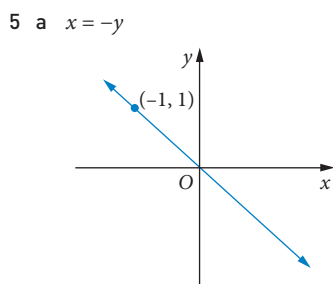
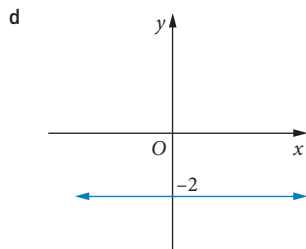
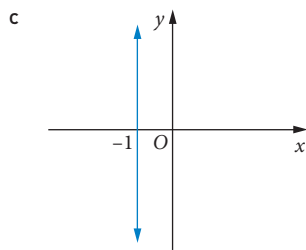
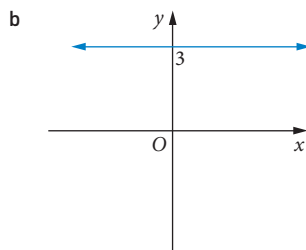
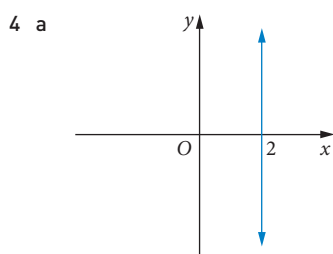
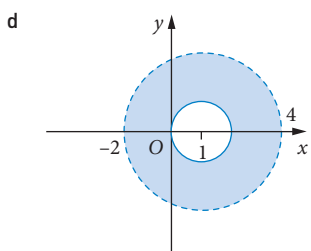
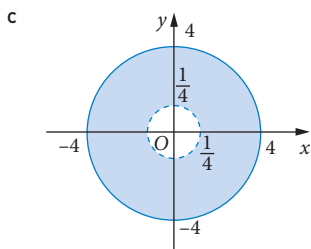
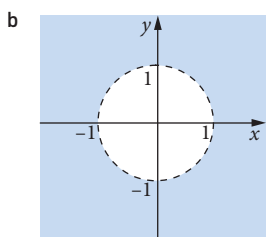
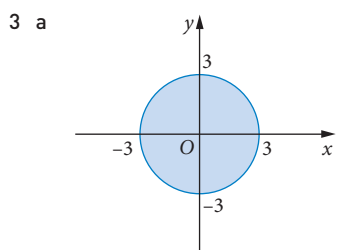
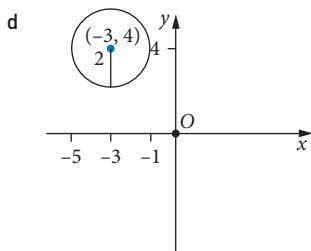
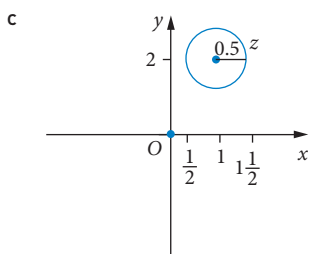


2 a

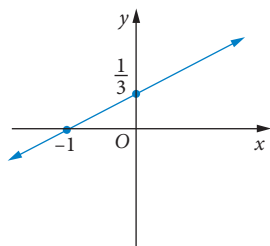


b

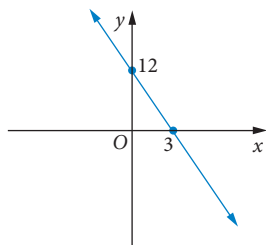




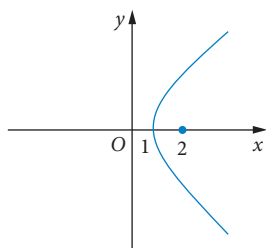
c $x = 3y - 1$



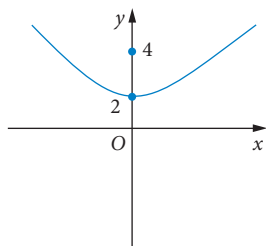
d $y + 4x = 12$



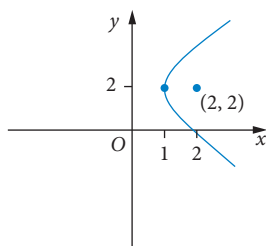
6 a



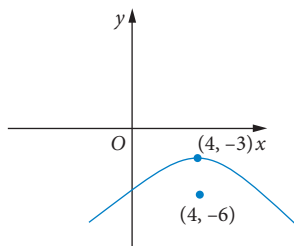
b



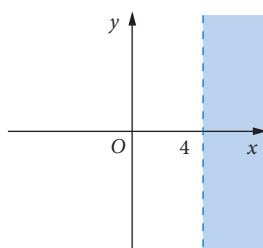
c



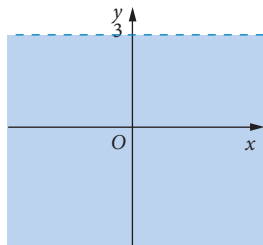
d



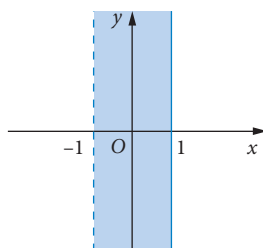
7 a



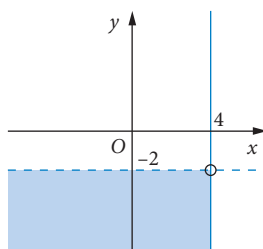
b



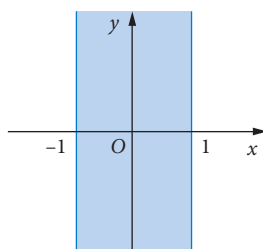
c



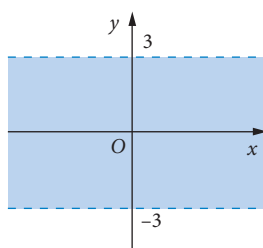
d

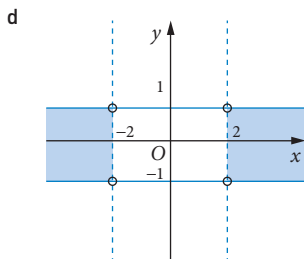
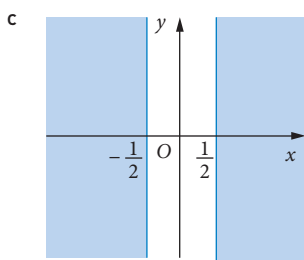


8 a

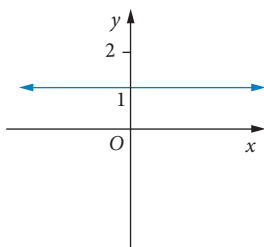


b

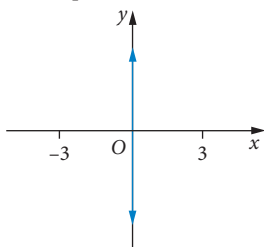




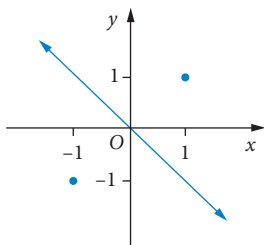
- 9 a A line equidistant from $(0, 0)$ and $(0, 2)$.



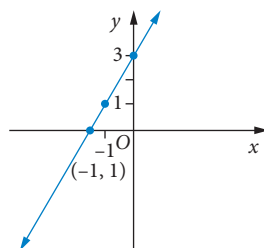
- b A line equidistant from $(3, 0)$ and $(-3, 0)$.



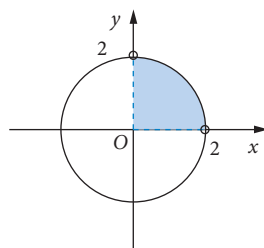
- c A line equidistant from $(1, 1)$ and $(-1, -1)$.



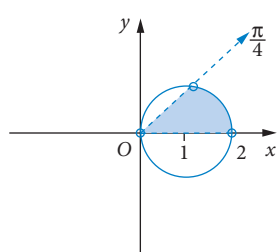
- d A line equidistant from $(3, -1)$ and $(-5, 3)$.



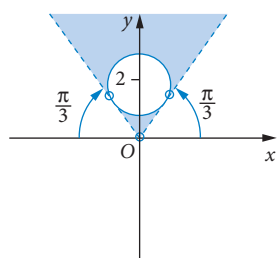
- 10 a



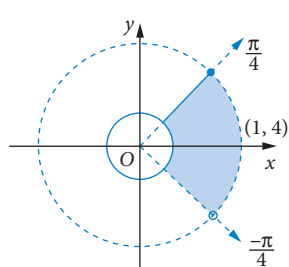
- b



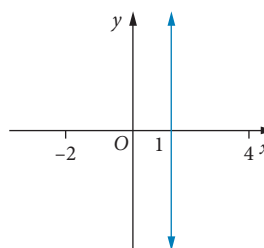
- c

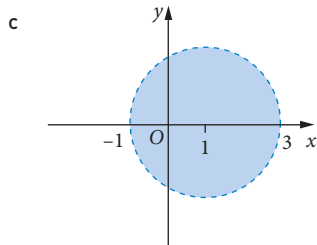
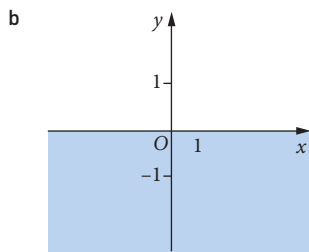


- d

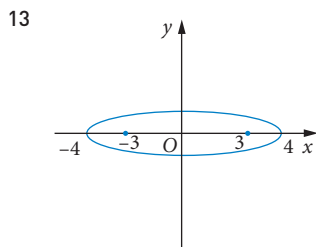


- 11 a



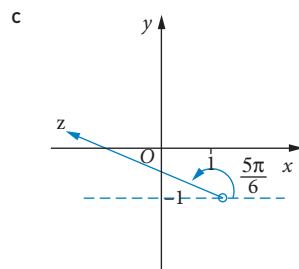
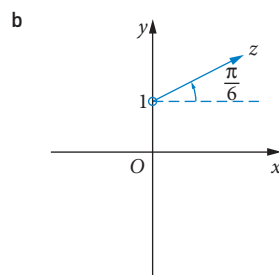
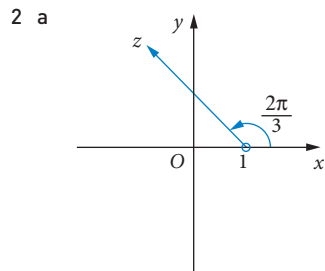
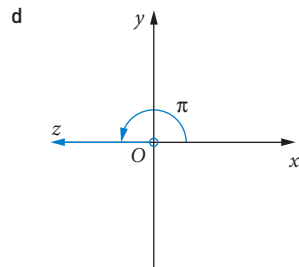
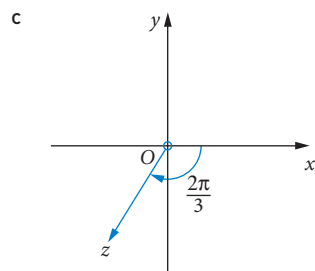
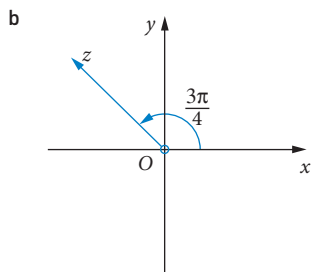
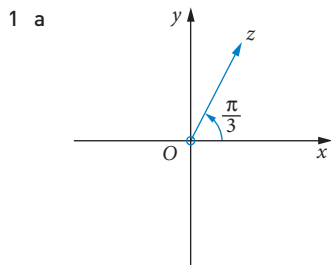


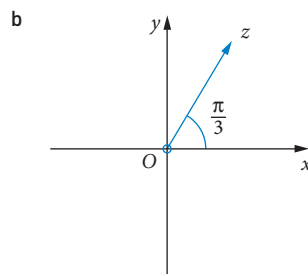
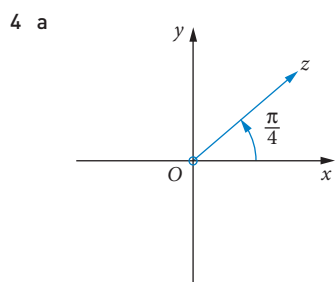
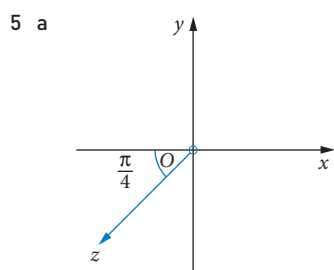
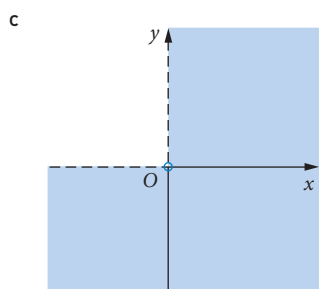
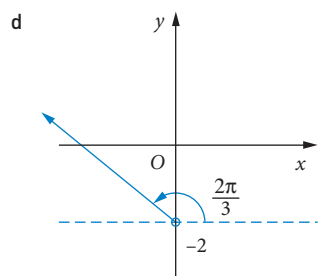
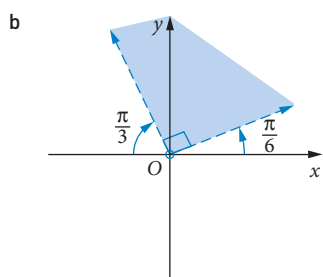
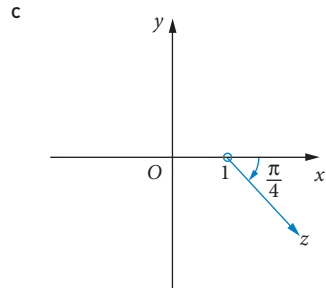
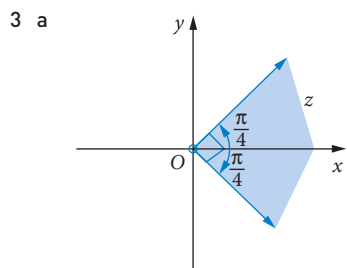
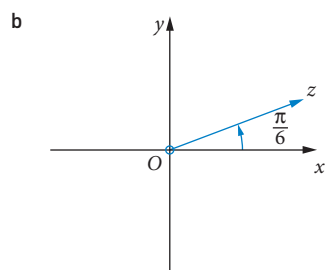
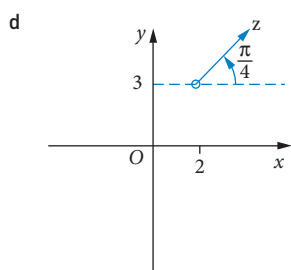
12 Centre $(-5, 0)$, radius $4\sqrt{2}$ units

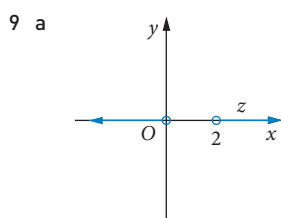
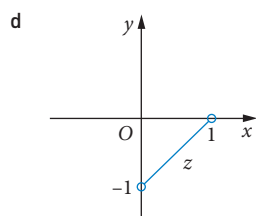
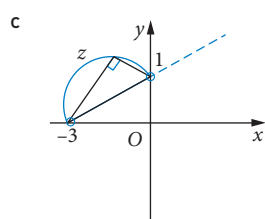
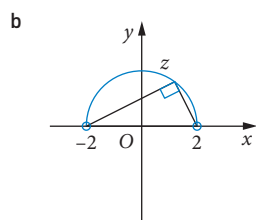
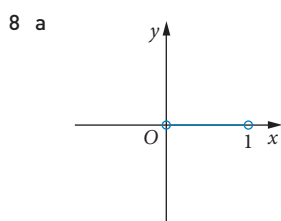
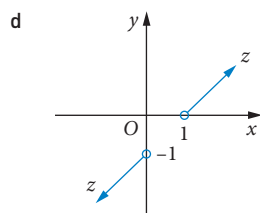
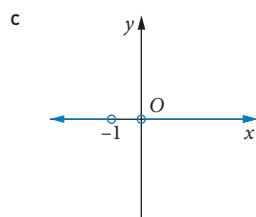
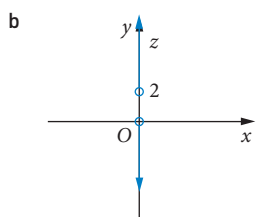
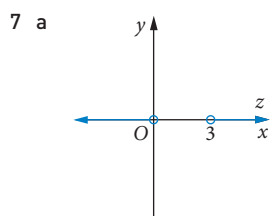
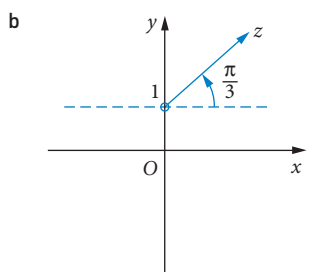
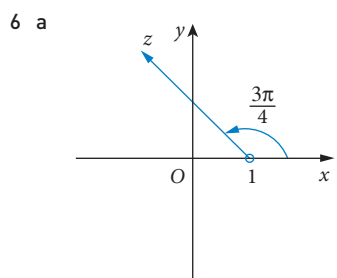
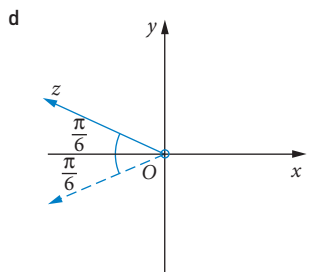
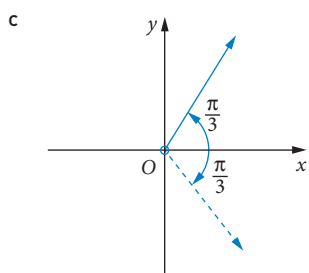


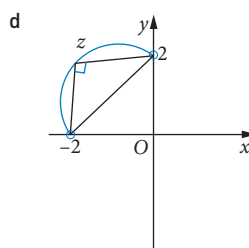
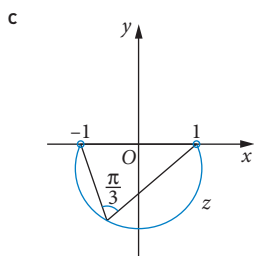
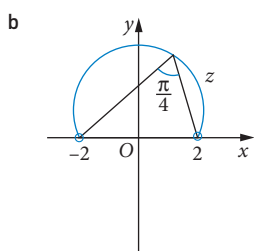
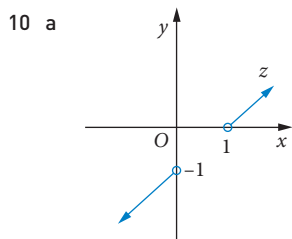
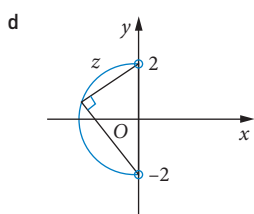
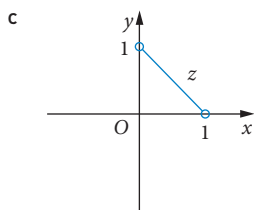
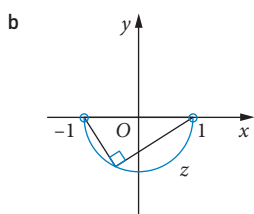
14 $\frac{\pi}{6}$

4.06



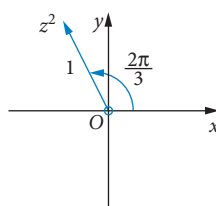




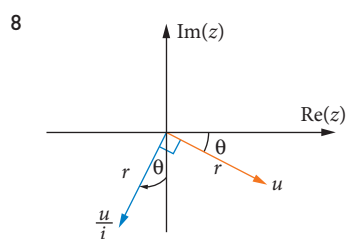
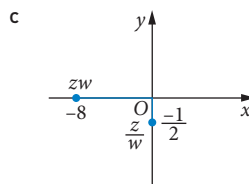


CHAPTER 4 REVIEW

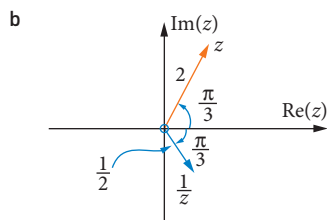
- 1 B
- 2 C
- 3 D
- 4 B
- 5 E
- 6



7 a $zw = -8$, $\text{Im}(zw) = 0$ b $\frac{z}{w} = -\frac{i}{2}$, $\text{Re}\left(\frac{z}{w}\right) = 0$



9 a $\text{mod}\left(\frac{1}{z}\right) = \frac{1}{2}$, $\arg\left(\frac{1}{z}\right) = -\frac{\pi}{3}$

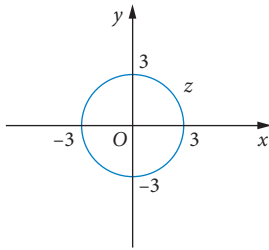


10 Proof

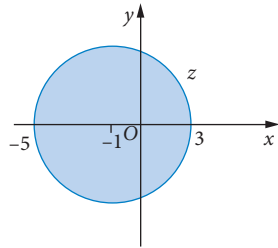
- 11 a Rectangle b Parallelogram
c Trapezium

12 $z_2 = -\bar{z}_1$, $z_3 = -z_1$, $z_4 = \bar{z}_1$

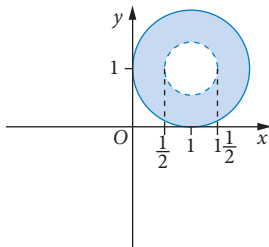
13 a $x^2 + y^2 = 9$



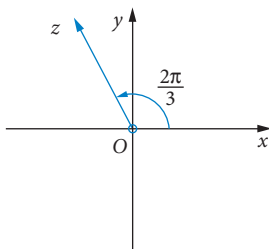
b $(x+1)^2 + y^2 \leq 16$



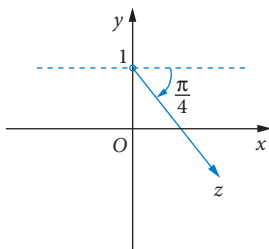
c $\frac{1}{4} < (x-1)^2 + (y-1)^2 \leq 1$



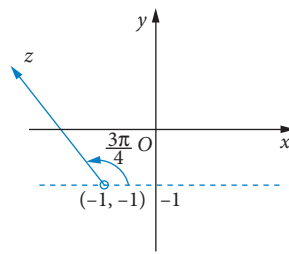
14 a



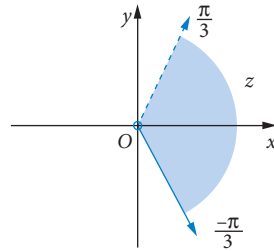
b



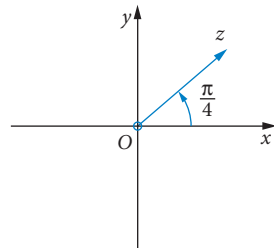
c



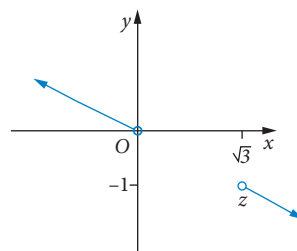
15 a



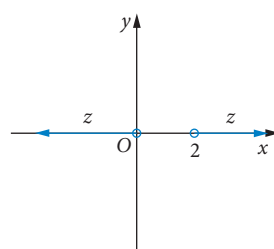
b



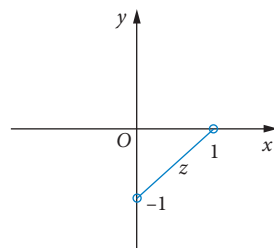
c

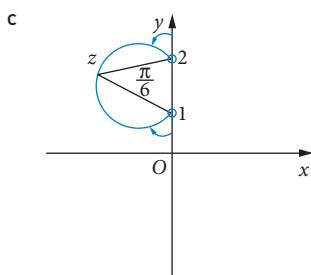


16 a

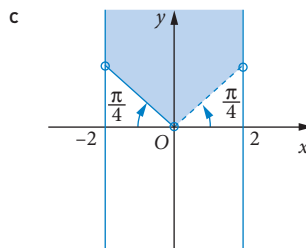
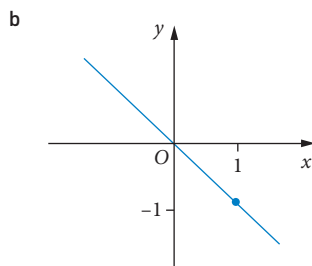
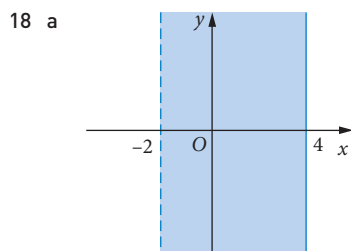
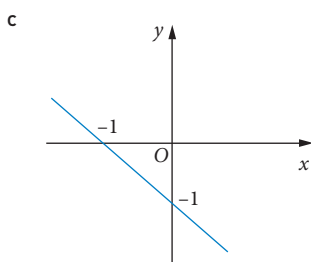
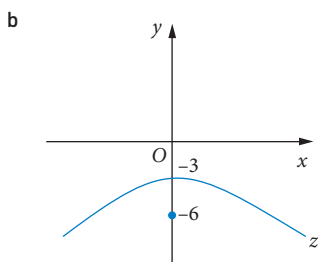
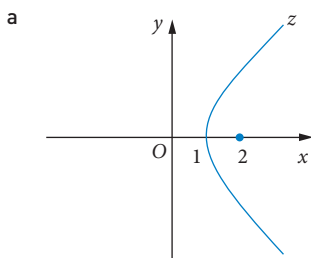


b



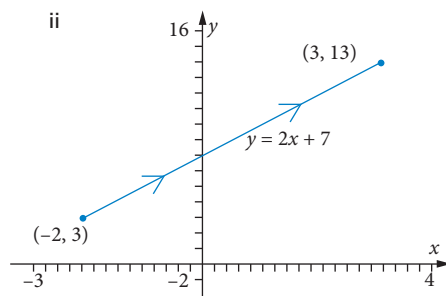


- 17 a Parabola $V(1, 0)$ focus $(2, 0)$
 b Parabola $V(0, -3)$ focus $(0, -6)$
 c Line through $(-1, 0)$ and $(0, -1)$



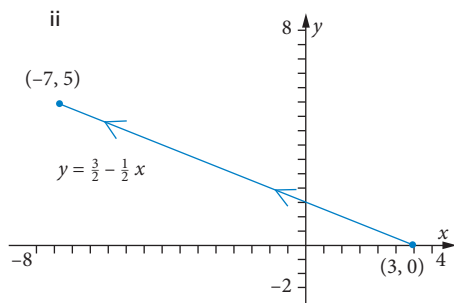
5.01

- 1 D
- 2 B
- 3 C
- 4 C
- 5 a $y = 3x$ b $x = 5$ c $y = 6$
 d $y = 7 - x$ e $x = \frac{1}{4}(5 - y^2)$ f $y = (x + 1)^3 + 6$
 g $y = \left(\frac{1}{x - 2}\right)^2 + 1$ h $y = \frac{x}{1 - x}$
- 6 a Domain: $x \in \mathbf{R}$, Range: $y \in \mathbf{R}$
 b Domain: $x = 5$, Range: $y \in \mathbf{R}$
 c Domain: $x \in \mathbf{R}$, Range: $y = 6$
 d Domain: $x \in \mathbf{R}$, Range: $y \in \mathbf{R}$
 e Domain: $x \geq 0$, Range: $y \in \mathbf{R}$
 f Domain: $x \in \mathbf{R}$, Range: $y \in \mathbf{R}$
 g Domain: $x > 2$, Range: $(1, \infty)$
 h Domain: $(0, 1)$, Range: $(0, \infty)$
- 7 a i $y = 2x + 7$



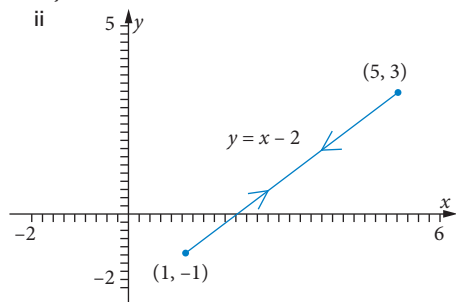
- iii Domain: $-2 \leq x \leq 3$; Range: $3 \leq y \leq 13$

b i $y = \frac{3}{2} - \frac{1}{2}x$



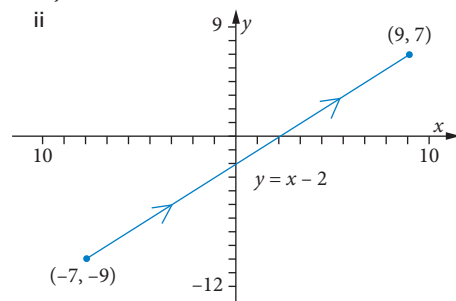
iii Domain: $-7 \leq x \leq 3$; Range: $0 \leq y \leq 5$

c i $y = x - 2$



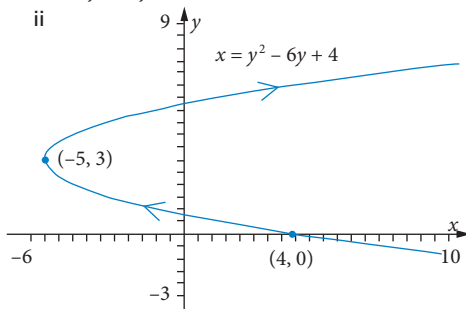
iii Domain: $1 \leq x \leq 5$; Range: $-1 \leq y \leq 3$

d i $y = x - 2$



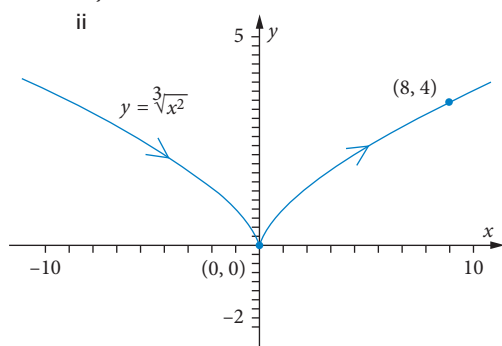
iii Domain: $-7 \leq x \leq 9$; Range: $-9 \leq y \leq 7$

e i $x = y^2 - 6y + 4$



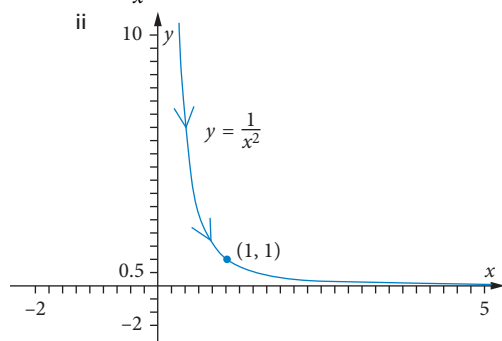
iii Domain: $x \geq -5$; Range: $y \in \mathbb{R}$

f i $y = \sqrt[3]{x^2}$



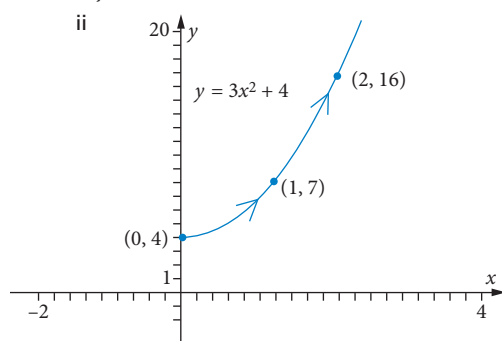
iii Domain: $x \in \mathbb{R}$; Range: $y \geq 0$

g i $y = \frac{1}{x^2}$



iii Domain: $x > 0$; Range: $y \in \mathbb{R}^+$

h i $y = 3x^2 + 4$

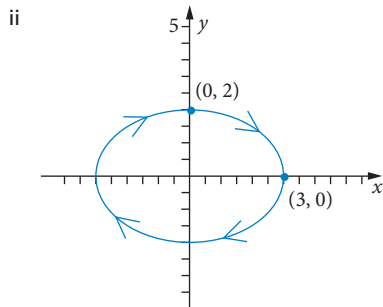


iii Domain: $x \geq 0$; Range: $y \geq 4$

- 8 a $x^2 + y^2 = 1$ b $x^2 + y^2 = 1$
 c $x^2 + y^2 = 1$ d $\frac{x^2}{9} + \frac{y^2}{4} = 1$
 e $y = 4 - \frac{4}{3}x$ f $\frac{x^2}{4} - \frac{y^2}{9} = 1$
 g $x^2 + y^2 = 9$ h $y = \frac{3}{2}x - 3$
- 9 a Domain: $[-1, 1]$, Range: $[-1, 1]$
 b Domain: $[0, 1]$, Range: $[-1, 1]$
 c Domain: $[-1, 0]$, Range: $[-1, 0]$

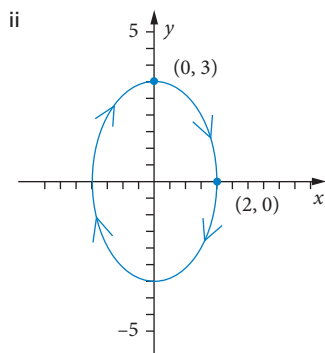
- d Domain: $[-3, 3]$, Range: $[-2, 2]$
 e Domain: $[0, 3]$, Range: $[0, 4]$
 f Domain: $[2, \infty)$, Range: $[0, \infty)$
 g Domain: $[-3, 3]$, Range: $[0, 3]$
 h Domain: $[2, \infty)$, Range: $[0, \infty)$

10 a i $\frac{x^2}{9} + \frac{y^2}{4} = 1$



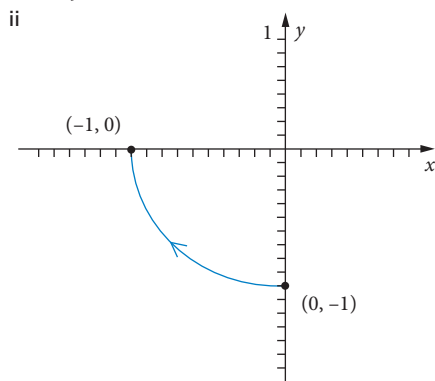
iii Domain: $[-3, 3]$; Range: $[-2, 2]$

b i $\frac{x^2}{4} + \frac{y^2}{9} = 1$



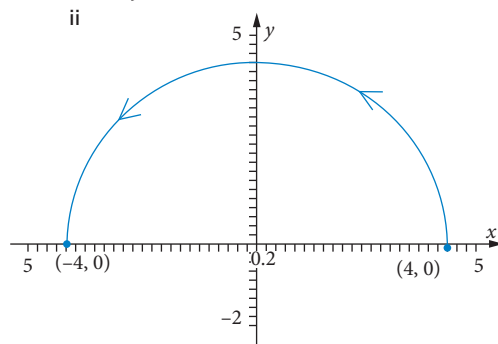
iii Domain: $[-2, 2]$; Range: $[-3, 3]$

c i $x^2 + y^2 = 1$



iii Domain: $[-1, 0]$; Range: $[-1, 0]$

d i $x^2 + y^2 = 16$



iii Domain: $[-4, 4]$; Range: $[0, 4]$

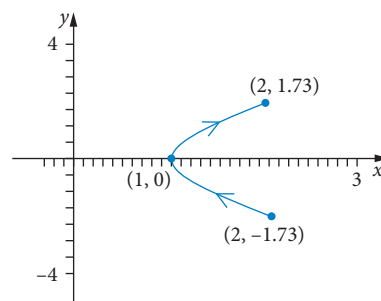
11 a $(-4, 7)$ b 9

12 a $\sqrt{17}$ b $t = \frac{2}{5}$ or 1

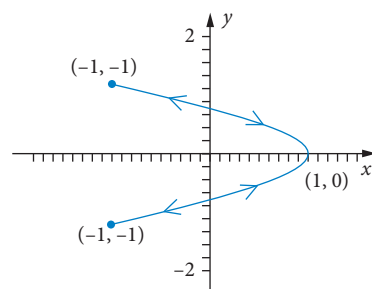
13 a $\sqrt{5}$ b $t = 2\frac{1}{2}$

c $(13, 17\frac{1}{2})$ and $(0, 6\frac{1}{8})$

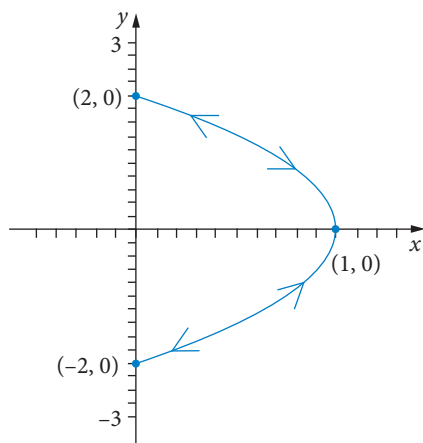
14 a $x^2 - y^2 = 1$: Domain: $[1, 2]$, Range: $[-1.73, 1.73]$



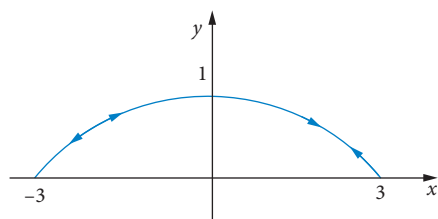
b $x = 1 - 2y^2$: Domain: $[-1, 1]$, Range: $[-1, 1]$



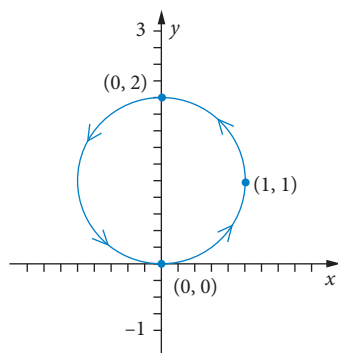
c $x = 1 - \frac{1}{4}y^2$: Domain: $[0, 1]$, Range: $[-2, 2]$



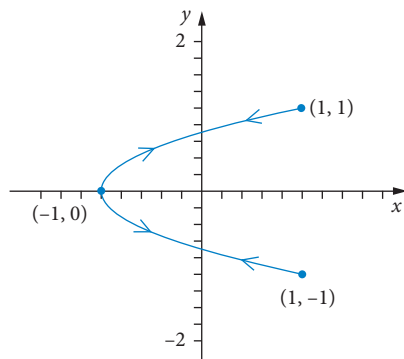
d $y = 1 - \frac{1}{9}x^2$: Domain: $[-3, 3]$, Range: $[0, 1]$



e $x^2 + (y-1)^2 = 1$: Domain: $[-1, 1]$, Range: $[0, 2]$



f $x = 2y^2 - 1$: Domain: $[-1, 1]$, Range: $[-1, 1]$



15 The particles collide at $t = \frac{2}{3}$ at $(-\frac{7}{3}, -\frac{25}{27})$.

5.02

- 1 B
- 2 C
- 3 C
- 4 A
- 5 a $\mathbf{r}(t) = 2t\mathbf{i} - 3\mathbf{j}$; $\mathbf{r}(t) = 2t\mathbf{i}$
b $\mathbf{r}(t) = 10t\mathbf{i} - 7\mathbf{j}$; $\mathbf{r}(t) = 0$
c $\mathbf{r}(t) = 20t^3\mathbf{i} + 8\mathbf{j}$; $\mathbf{r}(t) = 60t^2\mathbf{i}$
d $\mathbf{r}(t) = e^t\mathbf{i} + 2e^{2t}\mathbf{j}$; $\mathbf{r}(t) = e^t\mathbf{i} + 4e^{2t}\mathbf{j}$
e $\mathbf{r}(t) = \cos(t)\mathbf{i} + \sin(t)\mathbf{j}$; $\mathbf{r}(t) = -\sin(t)\mathbf{i} + \cos(t)\mathbf{j}$
f $\mathbf{r}(t) = -5\sin(t)\mathbf{i} - 5\cos(t)\mathbf{j}$;
 $\mathbf{r}(t) = -5\cos(t)\mathbf{i} + 5\sin(t)\mathbf{j}$
g $\mathbf{r}(t) = -3\mathbf{i} + 2\sin(t)\mathbf{j}$; $\mathbf{r}(t) = 2\cos(t)\mathbf{j}$
h $\mathbf{r}(t) = -\sin(t)\mathbf{i} - 3\cos(t)\mathbf{j}$;
 $\mathbf{r}(t) = -\cos(t)\mathbf{i} + 3\sin(t)\mathbf{j}$
- 6 a $|\mathbf{r}(t)| = \sqrt{4t^2 + 9}$; $|\mathbf{r}(t)| = 2$
b $|\mathbf{r}(t)| = \sqrt{149}$; $|\mathbf{r}(t)| = 0$
c $|\mathbf{r}(t)| = \sqrt{400t^6 + 64}$; $|\mathbf{r}(t)| = 60t^2$
d $|\mathbf{r}(t)| = e^t\sqrt{4e^{2t} + 1}$; $|\mathbf{r}(t)| = e^t\sqrt{16e^{2t} + 1}$
e $|\mathbf{r}(t)| = 1$; $|\mathbf{r}(t)| = 1$;
f $|\mathbf{r}(t)| = 5$; $|\mathbf{r}(t)| = 5$
g $|\mathbf{r}(t)| = \sqrt{9 + 4\sin^2(t)}$; $|\mathbf{r}(t)| = 2|\cos(t)|$
h $|\mathbf{r}(t)| = \sqrt{1 + 8\cos^2(t)}$; $|\mathbf{r}(t)| = \sqrt{1 + 8\sin^2(t)}$
- 7 a $\mathbf{r}(t) = 5t\mathbf{i} + 3t\mathbf{j}$; $\mathbf{r}(t) = 3\mathbf{j}$
b $\mathbf{r}(t) = 12t\mathbf{i} - 10(5t - 3)\mathbf{j}$; $\mathbf{r}(t) = -50\mathbf{j}$
c $\mathbf{r}(t) = -6e^{8-6t}\mathbf{i} - 42e^{-7t}\mathbf{j}$; $\mathbf{r}(t) = 36e^{8-6t}\mathbf{i} + 294e^{-7t}\mathbf{j}$
d $\mathbf{r}(t) = -e^{\frac{t}{4}+5}\mathbf{i} - 10e^{6-5t}\mathbf{j}$; $\mathbf{r}(t) = -\frac{1}{4}(e^{\frac{t}{4}+5})\mathbf{i} + 50e^{6-5t}\mathbf{j}$
e $\mathbf{r}(t) = -6(2-3t)\mathbf{i} + 3\sin(3t)\mathbf{j}$;
 $\mathbf{r}(t) = 18t\mathbf{i} + 9\cos(3t)\mathbf{j}$
f $\mathbf{r}(t) = \sin(2t)\mathbf{i} - \sec^2(t)\mathbf{j}$;
 $\mathbf{r}(t) = 2\cos(2t)\mathbf{i} - 2\tan(t)\sec^2(t)\mathbf{j}$
- 8 $\mathbf{r}(t) = 9t\mathbf{i} - [15(3t+2)^4 - 3]\mathbf{j}$; $\mathbf{r}(t) = -180(3t+2)^3\mathbf{j}$
- 9 $\mathbf{r}(t) = -3(t-7)^2\mathbf{i} - \frac{t}{\sqrt{t^2-3}}\mathbf{j}$;
 $\mathbf{r}(t) = -6(t-7)\mathbf{i} + \frac{3}{(t^2-3)^{\frac{3}{2}}}\mathbf{j}$

10-12 Demonstrations

- 13 a $-\frac{3}{2}$ b $\frac{16}{3}$ c 1
d $-\frac{\sqrt{3}}{3}$ e $e^{-3}(0.0498)$ f $\frac{\sqrt{3}}{2}$
- 14 a $\sin(t)$ b $\frac{1}{2}$
- 15 Proof
- 16 $t = \frac{2\sqrt{3}}{3}$

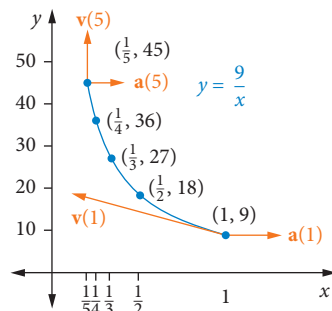
5.03

- 1 D
- 2 B
- 3 A
- 4 C
- 5 B

- 6 a $t^2\mathbf{i} + t\mathbf{j} + \mathbf{c}$
 b $t^4\mathbf{i} - \frac{8}{3}t^{\frac{3}{2}}\mathbf{j} + \mathbf{c}$
 c $3t^2\mathbf{i} - \frac{2}{5}t^{\frac{5}{2}}\mathbf{j} + \mathbf{c}$
 d $(5t + 4t^2)\mathbf{i} + (2t + e^{-t})\mathbf{j} + \mathbf{c}$
 e $\sin(t)\mathbf{i} - \frac{1}{2}\cos(2t)\mathbf{j} + \mathbf{c}$
 f $e^{2t}\mathbf{i} - \frac{3}{2}\sin(2t)\mathbf{j} + \mathbf{c}$
- 7 a $\mathbf{r}(t) = 2e^{2t}\mathbf{i} + (3e^t - 3)\mathbf{j}$
 b $\mathbf{r}(t) = (t^3 + 1)\mathbf{i} + (4t^{\frac{3}{2}} + 2)\mathbf{j}$
 c $\mathbf{r}(t) = [2 + \sin(t)]\mathbf{i} + [2 - 3\cos(t)]\mathbf{j}$
 d $\mathbf{r}(t) = (t^2 + 2)\mathbf{i} - \cos(t)\mathbf{j}$
- 8 a $\mathbf{r}(t) = t\mathbf{i} - (5t^2 - t - 1)\mathbf{j}$
 b $\mathbf{r}(t) = 20t\mathbf{i} - (16t^2 - 5t)\mathbf{j}$
 c $\mathbf{r}(t) = [-4 + 4\cos(t)]\mathbf{i} + [4 + 3\sin(t)]\mathbf{j}$
- 9 a $4\mathbf{i} + \frac{1}{2}\mathbf{j}$ b $\mathbf{0}$
 c $\mathbf{i} + \mathbf{j}$ d $-2\mathbf{j}$
 e $2\mathbf{i} + (1 - e^2)\mathbf{j}$ f $(1 - e^{-1})\mathbf{i} + \frac{1}{2}(e^2 - 1)\mathbf{j}$
- 10 a $\frac{1}{3}\mathbf{i} + (1 - e^{-1})\mathbf{j}$
 b $\frac{19}{3}\mathbf{i} + (e^{-2} - e^{-3})\mathbf{j}$
 c $21\mathbf{i} + (e^{-1} - e^{-4})\mathbf{j}$
- 11 Yes

5.04

- 1 D
 2 B
 3 E
 4 a 13 b 5
 c $\sqrt{4t^2 + 25}$ d $\sqrt{16 + 49t^2}$
 e 3 f $\sqrt{2t + \frac{9}{t^2}}$
- 5 a $\mathbf{v} = -4\mathbf{j}$, $\mathbf{a} = \mathbf{0}$, $|\mathbf{v}| = 4$
 b $\mathbf{v} = -\frac{a}{t^2}\mathbf{i} + b\mathbf{j}$, $\mathbf{a} = \frac{2a}{t^3}\mathbf{i}$, $|\mathbf{v}| = \sqrt{\frac{a^2}{t^4} + b^2}$
 c $\mathbf{v} = -3[\sin(3t)\mathbf{i} - \cos(3t)\mathbf{j}]$,
 $\mathbf{a} = -9[\cos(3t)\mathbf{i} + \sin(3t)\mathbf{j}]$, $|\mathbf{v}| = 3$
 d $\mathbf{v} = 6[\cos(2t)\mathbf{i} + \sin(2t)\mathbf{j}]$,
 $\mathbf{a} = -12[\sin(2t)\mathbf{i} - \cos(2t)\mathbf{j}]$, $|\mathbf{v}| = 6$
 e $\mathbf{v} = -3\mathbf{j} + 4\mathbf{k}$, $\mathbf{a} = \mathbf{0}$, $|\mathbf{v}| = 5$
 f $\mathbf{v} = 5[a\cos(5t)\mathbf{i} - b\sin(5t)\mathbf{j}]$,
 $\mathbf{a} = -25[a\sin(5t)\mathbf{i} + b\cos(5t)\mathbf{j}]$,
 $|\mathbf{v}| = 5\sqrt{a^2\cos^2(5t) + b^2\sin^2(5t)}$
 g $\mathbf{v} = 2\mathbf{i} + 2t\mathbf{j} - 4\mathbf{k}$, $\mathbf{a} = 2\mathbf{j}$, $|\mathbf{v}| = 2\sqrt{5 + t^2}$
 h $\mathbf{v} = 2e^t\mathbf{i} - 2e^{-2t}\mathbf{j}$, $\mathbf{a} = 2e^t\mathbf{i} + 4e^{-2t}\mathbf{j}$, $|\mathbf{v}| = 2e^t\sqrt{1 + e^{-6t}}$
 i $\mathbf{v} = \cos(t)\mathbf{i} + \sin(t)\mathbf{j} + 3\mathbf{k}$, $\mathbf{a} = -\sin(t)\mathbf{i} + \cos(t)\mathbf{j}$,
 $|\mathbf{v}| = \sqrt{10}$
 j $\mathbf{v} = 4t\mathbf{i} + 2e^{2t}\mathbf{j} - 6\sin(2t)\mathbf{k}$, $\mathbf{a} = 4[\mathbf{i} + e^{2t}\mathbf{j} - 3\cos(2t)\mathbf{j}]$,
 $|\mathbf{v}| = 2\sqrt{t^2 + e^{4t} + 9\sin^2(2t)}$
- 6 a $\mathbf{a} = 5\mathbf{i} - 2\mathbf{j}$, $\mathbf{r} = (\frac{5}{2}t^2 + 3)\mathbf{i} - (t^2 + 4)\mathbf{j}$
 b $\mathbf{a} = 4\cos(t)\mathbf{i} + 3\sin(3t)\mathbf{j}$,
 $\mathbf{r} = [4 - 4\cos(t)]\mathbf{i} - [\frac{1}{3}\sin(3t) - 6]\mathbf{j}$
 c $\mathbf{a} = -\frac{2}{t^2}\mathbf{i}$, $\mathbf{r} = [2\log_e|t| + 5]\mathbf{i} - 3t\mathbf{j}$
 d $\mathbf{a} = 2e^{2t}\mathbf{i} + 6t\mathbf{j}$, $\mathbf{r} = (\frac{1}{2}e^{2t} - \frac{1}{2}e^2 + 2)\mathbf{i} + (t^3 + 4t - 4)\mathbf{j}$
 e $\mathbf{a} = 6t\mathbf{i}$, $\mathbf{r} = (t^3 + 9)\mathbf{i} + (4t + 7)\mathbf{j}$
- 7 a $x - 2y = 0$ b $4x + 3y = 0$
 c $x + 2y - 9 = 0$ d $y^2 = \frac{16x}{3}$
- e $3x - y - 11 = 0$ f $\frac{x^2}{4} + \frac{y^2}{25} = 1$
 g $y = 3x^4 - 1$, $x \geq 0$ h $x^2 + y^2 - 4x - 8y + 19 = 0$
- 8 a Proof
 b Acceleration is in the opposite direction to displacement.
 c Circle centred at the origin, with radius 3 units.
- 9 a Proof
 b Acceleration is perpendicular to velocity.
- 10 a $3t\mathbf{i} + (2t - 3)\mathbf{j}$, $(t^2 - 4)\mathbf{i} + (t + 1)\mathbf{j}$
 b Proof
 c $t = 4$, $\mathbf{r} = 12\mathbf{i} + 5\mathbf{j}$
- 11 a Rectangular hyperbola
 b, d



c $\mathbf{v}(1) = -\mathbf{i} + 9\mathbf{j}$, $\mathbf{a}(1) = 2\mathbf{i}$, $\mathbf{v}(5) = -\frac{1}{25}\mathbf{i} + 9\mathbf{j}$, $\mathbf{a}(5) = \frac{2}{125}\mathbf{i}$

12 a $\mathbf{v} = (3t - 2t^2 + 2)\mathbf{i}$, $\mathbf{r} = \left(\frac{3t^2}{2} - \frac{2t^3}{3} + 2t + 5\right)\mathbf{i}$

b $t = 2$, $\mathbf{r} = 9\frac{2}{3}\mathbf{i}$

- 13 a Proof
 b $\mathbf{v} = -4[\sin(t)\mathbf{i} - \cos(t)\mathbf{j}]$, $\mathbf{a} = -4[\cos(t)\mathbf{i} + \sin(t)\mathbf{j}]$
 c Velocity is perpendicular to $\mathbf{r}$.
 d Acceleration is opposite to $\mathbf{r}$.
- 14 a Ellipse with axes 6 units and 10 units long with centre at the origin.
 b $\mathbf{v} = -3\sin(t)\mathbf{i} + 5\cos(t)\mathbf{j}$, $\mathbf{a} = -[3\cos(t)\mathbf{i} + 5\sin(t)\mathbf{j}]$
 c $t = \frac{n\pi}{2}$ ($n = 0, 1, 2, \dots$)
 d The acceleration and position vectors are parallel, equal in magnitude and opposite in direction.

- 15 a $\mathbf{v} = (t^2 - 4)\mathbf{i} - (3t - 6)\mathbf{j}$
 b $t = 2$
 c $|\mathbf{a}| = 5$
- 16 a $0\mathbf{i} + 0\mathbf{j}$
 b $2\sqrt{17}$
 c $y = 4x - \frac{1}{4}x^2$
- 17 a $\mathbf{r}_P = (2t + 5)\mathbf{i} + (t^2 - 6)\mathbf{j}$, $\mathbf{r}_Q = (3t - 1)\mathbf{i} + 5t\mathbf{j}$
 b $t = 6, 17\mathbf{i} + 30\mathbf{j}$
 c $P: y = \frac{1}{4}(x^2 - 10x + 1)$; $Q: 5x - 3y + 5 = 0$
- 18 a 6 m above the origin
 b 4 m s^{-1}
 c 0°
 d $\mathbf{a} = -10\mathbf{j}$

INVESTIGATION: ROCKETS AND SATELLITES

Some of the physical factors are the strength of the gravitational field, the mass of the rocket/satellite, the type and amount of fuel, the aerodynamics of the rocket and so on.

History should include the German World War 2 rockets, Sputnik and other orbital craft, the Moon landings, space stations and space exploration with unmanned probes and landers.

5.05

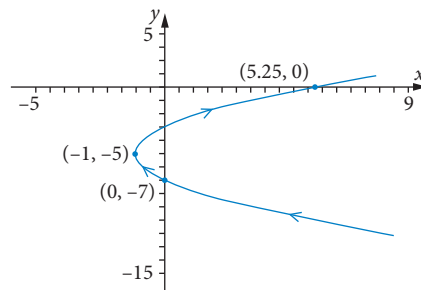
- 1 C
 2 D
 3 C
 4 C
 5 D
 6 a 5 m b 2 s c $20\sqrt{3} \text{ m}$
 7 a 3.61 s b 63.8 m c 15.9 m
 d $\mathbf{r}(t) \approx 17.7t\mathbf{i} + (17.7t - 4.9t^2)\mathbf{j}$
 e 21.8 m
 8 a $\mathbf{r}(t) \approx 15\sqrt{3}t\mathbf{i} + (15t - 4.9t^2)\mathbf{j}$
 b 11.5 m c 3.06 s, 79.5 m
 d 28.66 m s^{-1} , $\theta = 25^\circ$ (twice)
 9 a 33.1 m b 91.8 m
 10 a 39.2 m b 9° or 81°
 11 a 24.2 m s^{-1} b 3.5 s
 12 14.64 m s^{-1}
 13 a 7.6 s b 323.5 m
 14 3118 m
 15 16.4 m
 16 0.62° ($0^\circ 37'$)
 17 31.3 m s^{-1} , 4.5 s
 18 Yes, unless diameter $> 10 \text{ mm}$
 19 83.1 m
 20 a Proof b 4.02 m
 21 a A 35.1° , B 54.9°
 b 2.93 s
 c 25.2 m
 d $A : B = 0.494 : 1$

5.06

- 1 E
 2 C
 3 B
 4 A
 5 C
 6 B
 7 D
 8 E
 9 E
 10 a $1.1 \text{ rev} \left(\frac{7}{2\pi} \text{ rev} \right)$
 b 1257 rad ($400\pi \text{ rad}$)
 c 50.3 rad s^{-1} ($16\pi \text{ rad s}^{-1}$)
 11 5 rad s^{-1}
 12 a 2.24 rad s^{-1} b 5.39 m s^{-1} c 12.1 m s^{-2}
 13 a 2.1 rad s^{-1} b 20.3 rpm
 14 $23\,562 \text{ km h}^{-1}$
 15 a 31.4 rad s^{-1} b 0.628 m s^{-1}
 16 a 1.8 s b 3.491 rad s^{-1}
 c 0.53 m s^{-1} d 0.17 m s^{-1}
 e 1.86 m s^{-2}
 17 a $4.17 \text{ m s}^{-1} \approx 15 \text{ km h}^{-1}$ b 11.72 rad s^{-1}
 c 0.54 s d 48.8 m s^{-2}
 18 a 41.9 rad s^{-1} b 33.5 m s^{-1} c 1404 m s^{-2}
 19 a 0.419 rad h^{-1} b $14\,661 \text{ km h}^{-1}$
 20 a 1.18 rad s^{-1} b 2.65 rad s^{-1} c 3.74 rad s^{-1}
 21 62.5 m

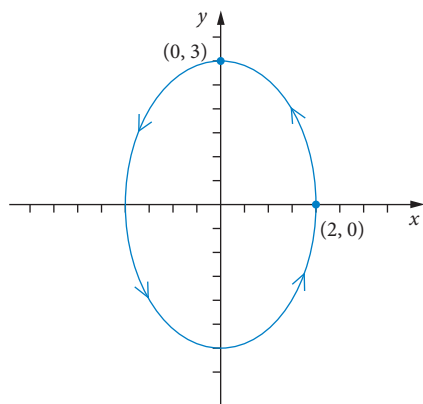
CHAPTER 5 REVIEW

- 1 B
 2 D
 3 C
 4 E
 5 A
 6 D
 7 C
 8 C
 9 E
 10 C
 11 D
 12 D
 13 a $x = \frac{1}{4}y^2 + \frac{5}{2}y + \frac{21}{4}$
 b



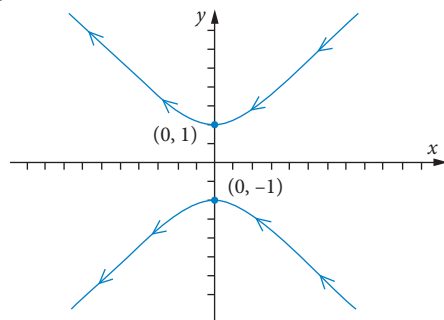
- 14 a $\frac{x^2}{4} + \frac{y^2}{9} = 1$

- b Domain = $[-2, 2]$; Range = $[-3, 3]$



- 15 a $6\mathbf{i} - 2\mathbf{j}$
b $\sqrt{51}$
- 16 a $2e^t\mathbf{i} - 5\cos(5t)\mathbf{j}$
b $\sqrt{4e^{2t} + 25\cos^2(5t)}$
c $2e^t\mathbf{i} + 25\sin(5t)\mathbf{j}$
- 17 $\mathbf{r}(t) = 15(3t+7)^4\mathbf{i} - 16t\mathbf{j}$; $\mathbf{r}(t) = 180(3t+7)^3\mathbf{i} - 16\mathbf{j}$
- 18 Demonstration
- 19 a $\frac{3}{5}\tan(t)$ b $\frac{3\sqrt{3}}{5}$
- 20 $\mathbf{r}(t) = \left[\frac{1}{2}\sin(2t) + 3\right]\mathbf{i} + [\cos(t) - 3]\mathbf{j}$
- 21 $\mathbf{r}(t) = (6t^2 + 10t)\mathbf{i} - \left[\frac{1}{2}t(5t + 30)\right]\mathbf{j}$
- 22 $\mathbf{r}(t) = \frac{4}{5}\mathbf{i} + \left[\frac{1}{2}(e^{-2} - 1)\right]\mathbf{j}$
- 23 a i $\mathbf{v} = 3\mathbf{i} - 2\mathbf{j}$ ii $\mathbf{a} = \mathbf{0}$
iii $|\mathbf{v}| = \sqrt{13}$
b i $\mathbf{v} = 15\cos(3t)\mathbf{i} - 18\sin(3t)\mathbf{j}$
ii $\mathbf{a} = -45\sin(3t)\mathbf{i} - 54\cos(3t)\mathbf{j}$
iii $|\mathbf{v}| = \sqrt{225 + 99\sin^2(3t)}$
- 24 a $\mathbf{a} = 4\mathbf{i}$, $\mathbf{r} = (2t^2 + 2)\mathbf{i} + (1 - 5t)\mathbf{j}$
b $\mathbf{a} = -4\sin(t)\mathbf{i} + 6\cos(t)\mathbf{j}$;
 $\mathbf{r} = [4\sin(t) - 5]\mathbf{i} + [6 - 6\cos(t)]\mathbf{j}$
c $\mathbf{a} = 2e^{0.4t}\mathbf{i}$; $\mathbf{r} = (12.5e^{0.4t} - 13.65)\mathbf{i} + (4t - 3)\mathbf{j}$
- 25 $x + 2y - 3 = 0$
- 26 $\mathbf{r} = [V\cos(\theta)]t\mathbf{i} + [V\sin(\theta)t - \frac{1}{2}gt^2]\mathbf{j}$,
 $\mathbf{r} = V\cos(\theta)\mathbf{i} + [V\sin(\theta) - gt]\mathbf{j}$,
 $\mathbf{r} = -g\mathbf{j}$ or $x = [V\cos(\theta)]t$, $y = [V\sin(\theta)]t - \frac{1}{2}gt^2$,
 $\dot{x} = V\cos(\theta)$, $\dot{y} = V\sin(\theta) - gt$,
 $x = 0$, $y = -g$
- 27 a 58.8 m b 6.93 s c 135.8 m
d 32.05 m s^{-1} at $\pm 52.3^\circ$
- 28 a $\mathbf{r}(t) = 20t\mathbf{i} + (50 + 25t - 4.9t^2)\mathbf{j}$
b About 6.64 s. c About 132.8 m.
d About 81.8 m. e About 141.9 m.
- 29 $a = \omega^2 r$ and $a = \frac{v^2}{r}$
- 30 a $\frac{40\pi}{3} \text{ rad s}^{-1}$ b $28\pi \text{ m s}^{-1}$ (about 88 m s^{-1})
- 31 $\omega = \frac{2}{3} \text{ rad s}^{-1}$, $a = \frac{5}{3} \text{ m s}^{-2}$

32 $y^2 - x^2 = 1$



- 33 $t = \sqrt{3} \text{ s}$
- 34 a $(0, 0)$ b 8.24 m s^{-1}
c $y = 4x - \frac{x^2}{4}$
- 35 a $t = \sqrt{\frac{2h}{g}}$ b $V_t = \sqrt{V^2 + 2hg}$
c $y = h\left(1 - \frac{x^2}{d^2}\right)$
- 36 $\mathbf{v} = 2\mathbf{i} + (4t - \frac{1}{2}t^2)\mathbf{j}$, $\mathbf{r} = (2t + 5)\mathbf{i} + \left(2t^2 - \frac{t^3}{6}\right)\mathbf{j}$
- 37 a 21 m s^{-1} b 3.03 s
- 38 a 4.28 s b 70.4 m c 31.9 m
- 39 No, it will just hit the top.
- 40 $\omega = 5.24 \text{ rad s}^{-1}$, $v = 0.942 \text{ m s}^{-1}$, $a = 4.93 \text{ m s}^{-2}$
- 41 $v \approx 29.89 \text{ km s}^{-1}$, $a \approx 0.00595 \text{ m s}^{-2}$

INVESTIGATION: ROOTS OF UNITY

The cube roots of unity are $\cos\left(\frac{2k\pi}{3}\right) + i\sin\left(\frac{2k\pi}{3}\right)$ for $k = 0, 1, 2$.

Choose, say, $z_3 = \text{cis}\left(\frac{2\pi}{3}\right)$.

Choose, say, $z_8 = \text{cis}\left(\frac{6\pi}{8}\right) = \text{cis}\left(\frac{3\pi}{4}\right)$

Then $z_3 \times z_8 = \text{cis}\left(\frac{2\pi}{3} + \frac{3\pi}{4}\right) = \text{cis}\left(\frac{17\pi}{12}\right)$

and $z_3 + z_8 = \text{cis}\left(\frac{2\pi}{3} - \frac{3\pi}{4}\right) = \text{cis}\left(-\frac{\pi}{12}\right)$

Now $(z_3 \times z_8)^{24} = \text{cis}\left(\frac{17\pi}{12} \times 24\right) = \text{cis}(17 \times 2\pi) = 1$

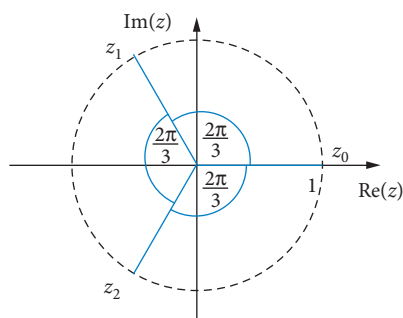
and $(z_3 + z_8)^{24} = \text{cis}\left(-\frac{\pi}{12} \times 24\right) = \text{cis}(-2\pi) = 1$

Thus $z_3 \times z_8$ and $z_3 + z_8$ are both roots of unity. Any product or sum of n th and m th roots of unity can be proven to be a root of unity by raising it to the power that is twice the lowest common multiple of n and m .

6.01

- 1 a $z = \cos\left(\frac{-2\pi}{3}\right) + i\sin\left(\frac{-2\pi}{3}\right)$ or $z = 1$ or
 $z = \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)$

b

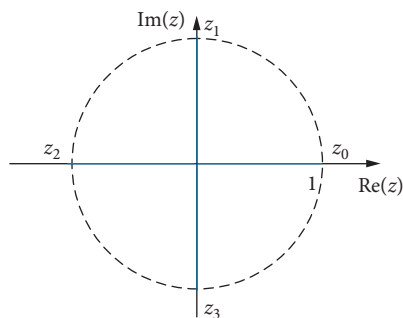


The roots are equally spaced on the unit circle, separated by angles of $\frac{2\pi}{3}$, with one root being 1.

$$c \quad z = -\frac{1}{2} - \frac{\sqrt{3}}{2}i \text{ or } z = 1 \text{ or } z = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$$

$$2 \quad a \quad z = -1 \text{ or } z = i \text{ or } z = 1 \text{ or } z = -i$$

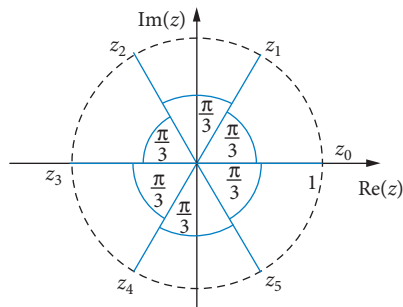
b



The roots are equally spaced on the unit circle separated by angles of $\frac{\pi}{2}$, with one root being 1 and another being -1 . They are the powers of i .

$$3 \quad a \quad z = \cos\left(\frac{-2\pi}{3}\right) + i \sin\left(\frac{-2\pi}{3}\right) \text{ or } z = \cos\left(\frac{-\pi}{3}\right) + i \sin\left(\frac{-\pi}{3}\right) \text{ or } z = 1 \text{ or } z = \cos\left(\frac{\pi}{3}\right) + i \sin\left(\frac{\pi}{3}\right) \text{ or } z = \cos\left(\frac{2\pi}{3}\right) + i \sin\left(\frac{2\pi}{3}\right) \text{ or } z = -1$$

b



The roots are equally spaced on the unit circle separated by angles of $\frac{\pi}{3}$, with one root being 1 and another being -1 .

4–6 Demonstrations

$$7 \quad a \quad z = -1 \text{ or } z = i \text{ or } z = 1 \text{ or } z = -i$$

$$b \quad z = \cos\left(\frac{-\pi}{2}\right) + i \sin\left(\frac{-\pi}{2}\right) \text{ or } z = \cos(0) + i \sin(0) \text{ or } z = \cos\left(\frac{\pi}{2}\right) + i \sin\left(\frac{\pi}{2}\right) \text{ or } z = \cos(\pi) + i \sin(\pi)$$

$$8 \quad a \quad z = i \text{ or } z = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i \text{ or } z = -i \text{ or } z = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i \text{ or } z = 1 \text{ or } z = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i \text{ or } z = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i \text{ or } z = -1$$

$$b \quad z = \cos\left(\frac{-3\pi}{4}\right) + i \sin\left(\frac{-3\pi}{4}\right) \text{ or } z = \cos\left(\frac{-\pi}{2}\right) + i \sin\left(\frac{-\pi}{2}\right) \text{ or } z = \cos\left(\frac{-\pi}{4}\right) + i \sin\left(\frac{-\pi}{4}\right) \text{ or } z = \cos(0) + i \sin(0) \text{ or } z = \cos\left(\frac{\pi}{4}\right) + i \sin\left(\frac{\pi}{4}\right) \text{ or } z = \cos\left(\frac{\pi}{2}\right) + i \sin\left(\frac{\pi}{2}\right) \text{ or } z = \cos\left(\frac{3\pi}{4}\right) + i \sin\left(\frac{3\pi}{4}\right) \text{ or } z = \cos(\pi) + i \sin(\pi)$$

$$9 \quad a \quad z = -\frac{1}{2} - \frac{\sqrt{3}}{2}i \text{ or } z = \frac{1}{2} - \frac{\sqrt{3}}{2}i \text{ or } z = 1 \text{ or } z = \frac{1}{2} + \frac{\sqrt{3}}{2}i \text{ or } z = -\frac{1}{2} + \frac{\sqrt{3}}{2}i \text{ or } z = -1$$

$$b \quad z = \cos\left(\frac{-2\pi}{3}\right) + i \sin\left(\frac{-2\pi}{3}\right) \text{ or } z = \cos\left(\frac{-\pi}{3}\right) + i \sin\left(\frac{-\pi}{3}\right) \text{ or } z = \cos(0) + i \sin(0) \text{ or } z = \cos\left(\frac{\pi}{3}\right) + i \sin\left(\frac{\pi}{3}\right) \text{ or } z = \cos\left(\frac{2\pi}{3}\right) + i \sin\left(\frac{2\pi}{3}\right) \text{ or } z = \cos(\pi) + i \sin(\pi)$$

10–14 Proofs

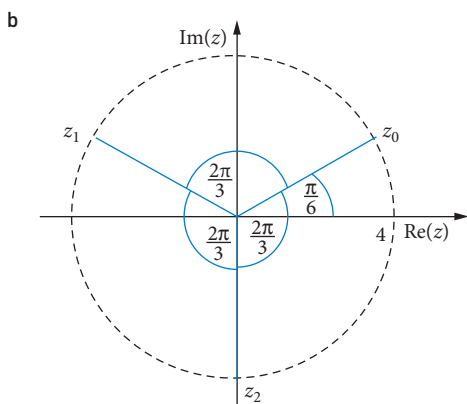
6.02

$$1 \quad \cos\left[\frac{(4k-1)\pi}{16}\right] + i \sin\left[\frac{(4k-1)\pi}{16}\right] \text{ for } k = -3, -2, \dots, 3, 4$$

$$2 \quad \cos\left(\frac{-13\pi}{18}\right) + i \sin\left(\frac{-13\pi}{18}\right), \cos\left(\frac{-\pi}{18}\right) + i \sin\left(\frac{-\pi}{18}\right) \text{ or } \cos\left(\frac{11\pi}{18}\right) + i \sin\left(\frac{11\pi}{18}\right)$$

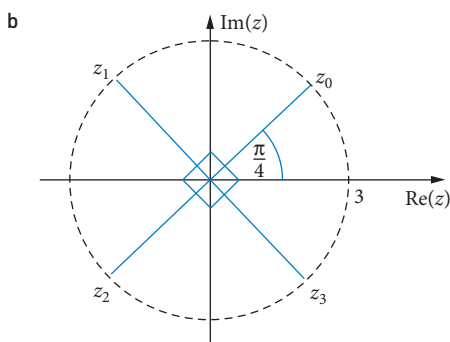
$$3 \quad \cos\left[\frac{(8k+3)\pi}{16}\right] + i \sin\left[\frac{(8k+3)\pi}{16}\right] \text{ for } k = -2, -1, 0, 1$$

$$4 \quad a \quad -4i, 4\left[\cos\left(\frac{\pi}{6}\right) + i \sin\left(\frac{\pi}{6}\right)\right] \text{ or } 4\left[\cos\left(\frac{5\pi}{6}\right) + i \sin\left(\frac{5\pi}{6}\right)\right]$$



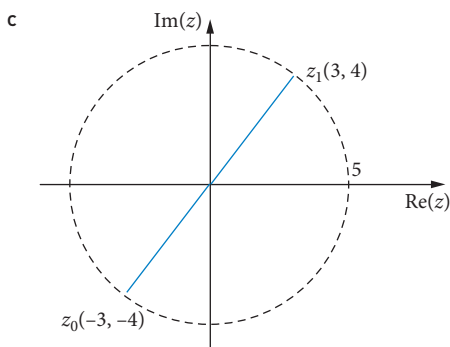
The roots are equally spaced on a circle of radius 4, centred at the origin. One of the roots is $-4i$.

5 a $3 \left\{ \cos \left[\frac{(2k+1)\pi}{4} \right] + i \sin \left[\frac{(2k+1)\pi}{4} \right] \right\}$ for $k = -2, -1, 0, 1$

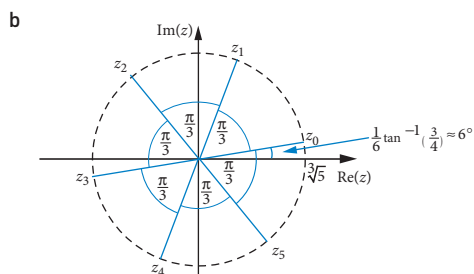


The roots are equally spaced on a circle of radius 3, centred at the origin.

6 a $5[\cos(0.9273) + i \sin(0.9273)]$ or $5[\cos(-2.2143) + i \sin(-2.2143)]$
b $-3 - 4i$ or $3 + 4i$

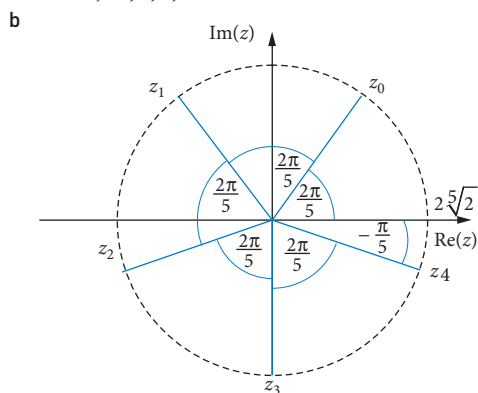


7 a $\sqrt[3]{5} \cos \left[\frac{1}{6} \left(\tan^{-1} \left(\frac{3}{4} \right) + 2k\pi \right) \right] + i \sin \left[\frac{1}{6} \left(\tan^{-1} \left(\frac{3}{4} \right) + 2k\pi \right) \right]$ for $k = -3, -2, \dots, 2$



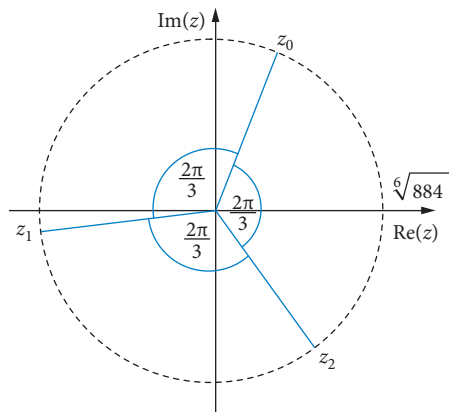
The roots are equally spaced on a circle of radius $\sqrt[3]{5} \approx 1.71$, centred at the origin.

8 a $2\sqrt[5]{2} \left\{ \cos \left[\frac{(4k-1)\pi}{10} \right] + i \sin \left[\frac{(4k-1)\pi}{10} \right] \right\}$ for $k = -2, -1, 0, 1, 2$



The roots are equally spaced on a circle of radius $2\sqrt[5]{2} \approx 2.30$, centred at the origin.

9 a $-3.078 - 0.353i$, $1.233 + 2.842i$ or $1.845 - 2.489i$
b



The roots are equally spaced on a circle of radius $\sqrt[6]{884} \approx 3.10$, centred at the origin.

10 Proof

6.03

- 1 C
2 E

- 3 a $18 + 3i$ b $-10 - 17i$
 c $-87 - 12i$ d $93i$
 e $-150 - 15i$
- 4 a $-610 + 1869i$ b $-6 + 103i$
 c $2939 - 957i$ d $970 + 263i$
 e $-70 + 39i$
- 5 a $44 + 2i$ b $-96 + 13i$
 c $-16 + 137i$ d $107 + 16i$
 e $104 + 27i$
- 6 a $1312 + 715i$ b $3025 + 1905i$
 c $-5383 - 8015i$ d $-2131 + 4351i$
 e $3013 + 7373i$
- 7 a $-28 + 101i$ b $-646 - 880i$
 c $842 - 1129i$ d $-1372 + 647i$
 e $-484 + 138i$
- 8 a $-2827 + 484i$ b $-12\ 866 + 6397i$
 c $892 + 2299i$ d $-153\ 935 - 15\ 922i$
 e $-151 - 160i$
- 9 Not true, disprove by a counterexample such as $p(z) = z^2$, $a = b = 1$. It is true for polynomials of degree 0 or 1, but not those of higher degree.

6.04

- 1 a $(-3 - 2i)z^3 + (9 + 7i)z^2 + (4 - 4i)z + 12 - 3i$
 b $(-3 - 2i)z^3 + (1 - 3i)z^2 - 2z + 4 + 9i$
 c $(-3 - 2i)z^3 + (7 + 4i)z^2 + (-1 - 6i)z + 11 + 5i$
 d $(-2 - 3i)z^2 + (-5 - 2i)z - 1 + 8i$
 e $(-2 + 18i)z^4 + (22 - 24i)z^3 + (8 + 11i)z^2 + (-19 - 4i)z + 24 - 10i$
 f $(-2 - 23i)z^5 + (-3 + 33i)z^4 + (9 + 3i)z^3 + (48 + 22i)z^2 + (22 - 21i)z + 50 - 36i$
 g $(-2 - 10i)z^5 + (4 + 30i)z^4 + (-1 - 38i)z^3 + (11 + 38i)z^2 + (3 - 42i)z + 18 + 25i$
- 2 a $(4 + i)z^2 + (15 + 3i)z + 44 + 6i + \frac{127 + 20i}{z - 3}$
 b $(4 + i)z^2 + (1 + 8i)z - 17 - i + \frac{-3 - 32i}{z - 2i}$
 c $(4 + i)z^2 + (-1 - i)z - 2i + \frac{-5 + 4i}{z + 1}$
 d $(4 + i)z^2 + (12 - 2i)z + 21 - 19i + \frac{18 - 57i}{z - 2 + i}$
 e $(4 + i)z^2 + (-13 + 13i)z - 14 - 94i + \frac{413 + 228i}{z + 3 - 4i}$
- 3 a $-4iz^3 + (-3 + 2i)z^2 + (4 + 6i)z + 9 - 8i + \frac{-9 - 18i}{z + 2i}$
 b $-4iz^3 + (9 - 10i)z^2 + (37 - 21i)z + 129 - 26i + \frac{420 + 51i}{z - 3 - i}$
 c $-4iz^3 + (-3 + 10i)z^2 + (26 - 14i)z - 83 - 24i + \frac{125 + 214i}{z + 2 + 2i}$
 d $-4iz^3 + (13 + 18i)z^2 + (-88 - 46i)z + 441 + 8i + \frac{-1773 + 850i}{z + 4 - 2i}$
 e $-4iz^3 + (5 - 18i)z^2 + (25 - 90i)z + 122 - 450i + \frac{617 - 2250i}{z - 5}$

- 4 a $(4 + 2i)z^5 + (5 - 3i)z^4 + (4 - 2i)z^3 + (5 + 3i)z^2 + (-7 - 1i)z - 2 - 2i$
 b $(-4 - 2i)z^5 + (5 - 3i)z^4 + (-10 + 8i)z^3 + (-1 - 11i)z^2 + (-7 - i)z - 8 + 8i$
 c $(5 - 3i)z^4 + (6 + 4i)z^3 + (2 - 4i)z^2 + (-10 + i)z - 2 - 2i$
 d $(-4 - 2i)z^5 + (2 + 6i)z^3 + (-3 - 7i)z^2 + (-3 + 2i)z$
 e $(34 + 22i)z^8 + (52 - 36i)z^6 + (42 + 52i)z^5 + (11 + 29i)z^4 + (5 - 107i)z^3 + (44 + 6i)z^2 + (1 + 21i)z - 16 - 30i$
 f $(26 - 2i)z^9 + (-18 + 6i)z^8 + (36 - 58i)z^7 + (4 + 44i)z^6 + (-62 - 48i)z^5 + (-20 - 4i)z^4 + (-28 + 18i)z^3 + (-50 - 48i)z^2 + (-26 + 32i)z + 34i$
 g $(48 - 22i)z^7 + (-30 + 24i)z^6 + (13 - 15i)z^5 + (-59 - 65i)z^4 + (-40 + 62i)z^3 + (9 - 33i)z^2 + (-17 + 13i)z + 34i$
- 5 a $-252 - 371i$ b $54 - 13i$
 c $1117 + 1553i$ d $-10 - 21i$
 e $269 - 143i$
- 6 a $83 - 239i$ b $3566 - 2418i$
 c $342 + 13i$ d $-4249 - 2818i$
 e $603 + 5181i$
- 7 a $-6648 - 16\ 352i$ b $-964 + 58i$
 c $-24\ 028 + 12\ 358i$ d $82\ 482 - 10\ 442i$
 e $-44 - 30i$

8–15 Proofs using the factor theorem.

6.05

- 1 E
 2 B
 3 C
 4 $z = 2$, $z = i$, or $z = -i$
 5 $z = -3$, $z = 1 - 2i$, or $z = 1 + 2i$
 6 $z = 1$, $z = -1$, $z = 2 - i$, or $z = 2 + i$
 7 $z = -1$, $z = 2$, $z = 3 - i$, or $z = 3 + i$
 8 $z = -1$, $z = 2$, $z = -3$, $z = 2 - 3i$, or $z = 2 + 3i$
 9 $z = 2$, $z = 3$, $z = -3$, $z = -1 + 2i$, or $z = -1 - 2i$
 10 $(z + 3)(z + 1 + i)(z + 1 - i)$
 11 $(z - 1)(z - 2)(z - 2 + 2i)(z - 2 - 2i)$
 12 $(z + 1)(z - 2)(z + 2)(z - 1 + 3i)(z - 1 - 3i)$

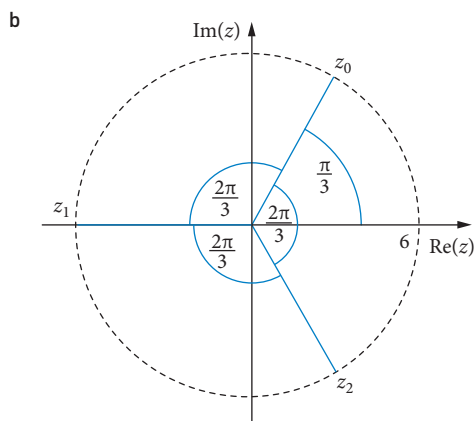
6.06

- 1 A
 2 C
 3 $z = 1$, $z = -2$ and $z = 1 + 2i$
 4 $z = -2$, $z = 3$ and $z = 2 - 3i$
 5 $z = -i$, $z = 3i$ and $z = 2 + i$
 6 $z = -i$, $z = -2i$ and $z = -3 - 2i$
 7 $z = 1$, $z = -2$ and $z = 3 + i$ (twice)
 8 $z = 2$, $z = -2$, $z = \frac{1}{5}i$ and $z = -\frac{3}{2}i$
 9 $z = i$, $z = 3i$, $z = \frac{3}{4}i$ and $z = -\frac{4}{3}i$
 10 $z = 2i$, $z = -3i$, $z = 3 + 2i$ and $z = -2 + 3i$

CHAPTER 6 REVIEW

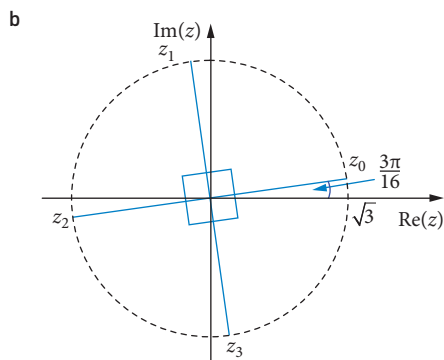
- 1 C
- 2 B
- 3 D
- 4 C
- 5 A
- 6 C
- 7 A
- 8 E
- 9 B
- 10 E

11 a $6 \left[\cos\left(\frac{\pi}{3}\right) + i \sin\left(\frac{\pi}{3}\right) \right]$, -6 , and $6 \left[\cos\left(-\frac{\pi}{3}\right) + i \sin\left(-\frac{\pi}{3}\right) \right]$



12 a $\sqrt{3} \left[\cos\left(\frac{3\pi+8k\pi}{16}\right) + i \sin\left(\frac{3\pi+8k\pi}{16}\right) \right]$

for $k = -2, -1, 0, 1$



13 $-4 + 2i, -5 - 5i, -16 - 2i$

14 $92 - 40i, 6 - 3i, 54 - 464i$

15 a $(1 - 2i)z^3 + iz + 3 + 2i$

b $(-6 - 3i)z^5 + (15 - 7i)z^4 + 18iz^3 + (-10 + 5i)z^2 + (-14 + 8i)z + 6i$

c $(1 - 2i)z^2 - z - 5 + i + \frac{-4 + 8i}{z - 1 + i}$

16 a $1 + 7i$ b $-122 + 108i$ c $36 - 38i$

17 Use $p(z + 1 - 2i) = 0$

18 Proof

19 Proof

20 $z = 3, z = -\frac{1}{2} - \frac{\sqrt{11}}{2}i$ or $z = -\frac{1}{2} + \frac{\sqrt{11}}{2}i$

21 $z = -1, z = 2$ or $z = -3i$ (twice)

22 $z = 2i, z = -2i, z = \frac{2}{3}i$ and $z = -\frac{3}{2}i$

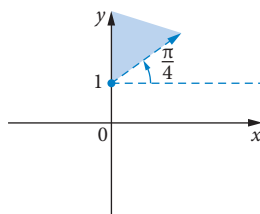
MIXED REVISION 2

Multiple choice

- 1 B
- 2 E
- 3 E
- 4 D
- 5 D
- 6 D
- 7 A
- 8 E
- 9 E

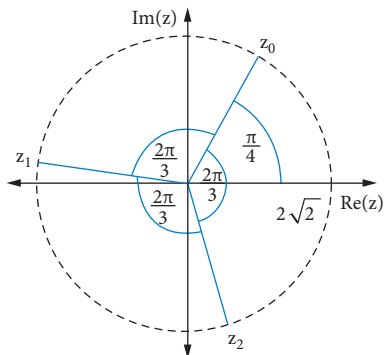
Short answer

1



2 a $2\sqrt{2} \left[\cos\left(\frac{\pi}{4}\right) + i \sin\left(\frac{\pi}{4}\right) \right]$,
 $2\sqrt{2} \left[\cos\left(\frac{11\pi}{12}\right) + i \sin\left(\frac{11\pi}{12}\right) \right]$,
 $2\sqrt{2} \left[\cos\left(-\frac{5\pi}{12}\right) + i \sin\left(-\frac{5\pi}{12}\right) \right]$

b



3 a $\mathbf{a} = 3\mathbf{i}, \mathbf{r} = \left(\frac{3}{2}t^2 + 2\right)\mathbf{i} + (5t - 3)\mathbf{j}$

b $\mathbf{a} = -3 \sin(t)\mathbf{i} + 2 \cos(2t)\mathbf{j}$,
 $\mathbf{r} = [3 \sin(t) - 5]\mathbf{i} + \frac{1}{2}[1 - \cos(2t)]\mathbf{j}$

4 a $w_3 = -i(w_1 - w_2) + w_2$

b $w_4 = (w_3 - w_1) \left(\frac{1}{2} + \frac{i\sqrt{3}}{2} \right) + w_1$

5 $31\,416 \text{ km h}^{-1}$

$$6 \quad 2iz^2 + (-2 - 4i)z + 6 + 9i + \frac{-13 - 13i}{z + 2 - i}$$

Application

- 1 Circle is $x^2 + \left(y + \frac{3}{4}\right)^2 = \frac{1}{16}$
- 2 $\arg(w + 2) = \theta$, $PO = BO$ so $\angle BPO = \theta$ (isosceles triangle) then $\angle POA = 2\theta$ (exterior angle of triangle PBO). $\therefore \arg(w) = 2\theta$.
- 3 $z = 2, z = -3, -\frac{1}{3} - \frac{\sqrt{5}}{3}i, -\frac{1}{3} + \frac{\sqrt{5}}{3}i$
- 4 $z = i, z = -\frac{4}{5}i, z = \frac{5}{4}i$
- 5 a Proof
b Acceleration is in the opposite direction of displacement.
c Circle centred at the origin with a radius of 4 units
- 6 a 29.5 m
b They do not collide, the second ball passes over the top of the first.

7.01

- 1 D
- 2 E
- 3 E
- 4 C
- 5 A
- 6 C
- 7 A

8–12 Proofs

7.02

- 1 C
- 2 A
- 3 D
- 4 B
- 5 D
- 6 A
- 7 312
- 8 $\frac{1}{3}$

9, 10 Proofs

INVESTIGATION: DEFINITE INTEGRALS OF $\frac{1}{x}$

For $x > 0$ and for $x < 0$ the graph is anti-symmetric (it is an odd function).

The physical areas between the curve and the x -axis either side of the y -axis are the same.

$$\int_{-3}^{-1} \frac{1}{x} dx = -\int_1^3 \frac{1}{x} dx = \int_3^1 \frac{1}{x} dx$$

7.03

- 1 a $\frac{1}{3} \log_e |3x + 2| + c$
b $\frac{1}{9} \log_e |9x - 7| + c$
c $-\frac{1}{3} \log_e |4 - 3x| + c$
d $\frac{5}{2} \log_e |5 - 2x| + c$
- 2 C
- 3 B
- 4 B
- 5 E
- 6 A
- 7 a $\frac{1}{3} \log_e |3x^2 + 2| + c$
b $2 \log_e |\tan(x)| + c$
c $\frac{1}{2} \log_e |e^{2x} + 1| + c$
d $2 \log_e |x^3 + 2x| + c$

8–10 Proofs

7.04

- 1 a $\frac{3}{\sqrt{1 - 9x^2}}$
c $\frac{-2x}{\sqrt{1 - x^4}}$
- b $\frac{3x^2}{1 + (x^3 - 2)^2}$
d $\frac{8x^3 + 1}{1 + (2x^4 + x)^2}$
- 2 E
- 3 D
- 4 A
- 5 A
- 6 E
- 7 a $\frac{1}{2} \sin^{-1}(2x) + c$
b $-\frac{4}{3} \tan^{-1}(3x) + c$
c $\frac{\pi}{4}$
d $\frac{\pi}{3}$

- 8 a $\arcsin\left(\frac{x}{6}\right) + c$
b $\arccos\left(\frac{x}{10}\right) + c$
c $\frac{1}{7} \arctan\left(\frac{x}{7}\right) + c$
d $\arcsin\left(\frac{x}{2\sqrt{3}}\right) + c$
e $\frac{1}{\sqrt{6}} \arctan\left(\frac{x}{\sqrt{6}}\right) + c$

$$9 \quad \frac{1}{2} [\arctan(3) + \arctan(1.5)]$$

10 1.1007

$$11 \quad \frac{6}{x^2+2} = 3\sqrt{2} \left(\frac{\frac{1}{\sqrt{2}}}{\left(\frac{x}{\sqrt{2}}\right)^2 + 1} \right) \text{ where } u(x) = \frac{x}{\sqrt{2}}.$$

$$\text{Hence } \int \frac{6}{x^2+2} dx = 3\sqrt{2} \arctan\left(\frac{x}{\sqrt{2}}\right) + c.$$

12–14 Proofs

7.05

$$1 \quad a \quad \frac{1}{x-2} - \frac{1}{x-1}$$

$$b \quad \frac{3}{4(x+2)} + \frac{9}{4(x-6)}$$

$$c \quad \frac{3}{7(2x+3)} + \frac{2}{7(x-2)}$$

$$d \quad \frac{9}{49(2x+3)} + \frac{20}{49(x-2)} + \frac{4}{7(x-2)^2}$$

$$2 \quad a \quad \log_e \left| \frac{x-2}{x-1} \right| + c \quad b \quad \frac{3}{4} \log_e \left(\frac{256}{375} \right)$$

$$c \quad \frac{1}{7} \log_e \left| \frac{x-2}{2x+3} \right| + c \quad d \quad \frac{3}{49} \log_e \left(\frac{3}{10} \right) + \frac{1}{7}$$

3 B

4 A

5 C

6 B

$$7 \quad x + \log_e \left| \frac{(x-2)^4}{x-1} \right| + c$$

$$8 \quad \frac{x^2}{(x^2-1)} = 1 + \frac{1}{(x^2-1)}, x + \frac{1}{2} \log_e \left| \frac{x-1}{x+1} \right| + c$$

$$9 \quad \frac{1}{2} - \frac{1}{2} \log_e(3)$$

$$10 \quad -\log_e(2)$$

7.06

1 B

2 C

3 C

4 B

5 E

$$6 \quad a \quad \frac{1}{4} e^{2x} (2x-1) + c$$

$$b \quad \frac{1}{4} [2x \sin(2x) + \cos(2x)] + c$$

$$c \quad \frac{1}{4} x^2 [2 \log_e(2x) - 1] + c$$

$$d \quad \frac{1}{4} (e^2 + 1) \quad e \quad \frac{\pi}{4}$$

$$7 \quad a \quad \frac{1}{3} [\log_e(3) - 2]$$

$$b \quad x \sin^{-1}(x) + \sqrt{1-x^2} + c$$

$$c \quad \frac{1}{4} x^2 [2 \log_e(x) - 1] + c$$

$$d \quad \frac{\pi}{6} - \frac{\sqrt{3}}{2}$$

$$e \quad x \cos^{-1}(3x) - \frac{1}{3} \sqrt{1-9x^2} + c$$

$$8 \quad -x^3 \cos(x) + 3x^2 \sin(x) + 6x \cos(x) - 6 \sin(x) + c$$

$$9 \quad 0.25e^2 - 0.75$$

CHAPTER 7 REVIEW

1 D

2 E

3 D

4 B

5 A

6 C

7 E

8 B

9 A

10 D

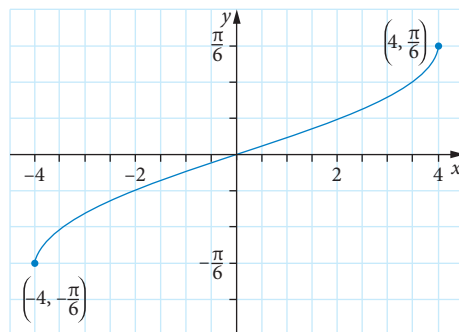
11 $\frac{1}{3}$

$$12 \quad a \quad -\frac{1}{3} \log_e |2-3x| + c \quad b \quad -\frac{2}{7} \log_e |7x+2| + c$$

$$13 \quad a \quad \frac{1}{6} \log_e |3x^2+3| + c \quad b \quad 4 \log_e \left| e^{\frac{x}{2}} + 1 \right| + c$$

$$14 \quad a \quad \frac{-1}{\sqrt{16-x^2}} \quad b \quad \frac{2x}{\sqrt{\frac{3}{4}-x^4+x^2}}$$

15



$$y = \frac{1}{3} \sin^{-1} \left(\frac{x}{4} \right)$$

Domain: $x \in [-4, 4]$

$$16 \quad a \quad \frac{1}{9(x-6)} - \frac{1}{9(x+3)}$$

$$b \quad \frac{13}{49(x+1)} + \frac{36}{49(x-6)} - \frac{1}{7(x+1)^2}$$

$$17 \quad a \quad -\frac{3}{4} \log_e \left(\frac{375}{256} \right)$$

$$b \quad \frac{1}{63} \log_e \left| \frac{(3x-4)^{16}}{(x+1)^9} \right| + \frac{1}{3} x + c$$

$$18 \quad \frac{1}{4} e^{2x} (2x^2 - 2x + 1)$$

$$19 \quad \frac{1}{2} [\log_e(2) - 1]$$

20 Proof

21 Proof

$$22 \sin^{-1}\left(\frac{2x}{5}\right) = \sin^{-1}\left(\frac{x}{\frac{5}{2}}\right)$$

$$\therefore \int \frac{1}{\sqrt{\frac{25}{4} - x^2}} dx = \sin^{-1}\left(\frac{2x}{5}\right) + c.$$

$$23 \frac{1}{2} [2x^2 \sin(2x) + 2x \cos(2x) - \sin(2x)] + c.$$

8.01

$$1 \text{ a } \frac{dy}{dx} = \frac{-6x+5y}{-5x+3y^2} \quad \text{b } \frac{dy}{dx} = \frac{-y}{x}$$

$$\text{c } \frac{dy}{dx} = -\frac{2y^2}{3x^2} \quad \text{d } \frac{dy}{dx} = \frac{-2y}{x+2\sqrt{y-2y^{\frac{3}{2}}}}$$

$$2 \text{ a } \frac{dy}{dx} = -x \quad \text{b } y = -\frac{1}{2}x^2 + \frac{5}{2}, \frac{dy}{dx} = -x$$

$$3 \text{ a } \frac{dy}{dx} = \frac{1}{2y+2}$$

$$\text{b i } x = -3 + 2y + y^2 \quad \text{ii } \frac{dy}{dx} = \frac{1}{2y+2}$$

$$4 \text{ a } \frac{dy}{dx} = \frac{\cos(x+y)\cos^2(y)}{1-\cos(x+y)\cos^2(y)}$$

$$\text{b } \frac{dy}{dx} = \frac{2y-e^{2y}}{2xe^{2y}-2x}$$

$$\text{c } \frac{dy}{dx} = 5x + 5y - 1$$

$$\text{d } \frac{dy}{dx} = \frac{2x \cos(y)}{e^y + x^2 \sin(y)}$$

$$5 \text{ a } -\frac{5}{3} \quad \text{b } 3x - 5y - 30 = 0$$

$$6 \ 10x - 9y - 20 = 0$$

$$7 \ 2x + 9y = 20, 2x + 9y = -20$$

$$8 \ 25x - 15y = 132$$

$$9 \ 9x - 8y = 26, 9x - 8y = -26$$

$$10 \left(0, \frac{100}{9}\right)$$

11–14 Proofs

8.02

$$1 \text{ a } 8x^3 \quad \text{b } 10e^{2x}$$

$$2 \text{ a } 297 \quad \text{b } -6084 \quad \text{c } 3 \log_e(4) + 3$$

$$3 \ 3\sqrt{2}$$

$$4 \ 8100 \text{ mm}^3 \text{ s}^{-1}$$

$$5 \ 91.2\pi \text{ mm}^3 \text{ s}^{-1}$$

$$6 \ \frac{1638\pi}{25} \text{ cm}^2 \text{ s}^{-1}$$

$$7 \ \frac{8901\pi}{175} \text{ cm}^3 \text{ s}^{-1}$$

$$8 \ 40 \text{ units per second}$$

$$9 \ -0.52 \text{ ms}^{-1}, \text{ i.e. moving down at the rate of } 0.52 \text{ ms}^{-1}$$

$$10 \ 2411.472 \text{ cms}^{-2}$$

$$11 \ \frac{15}{7} \text{ houses per year}$$

$$12 \text{ a } \text{Proof} \quad \text{b } \frac{2}{3} \text{ cms}^{-1}$$

$$13 \ 0.57 \text{ mm}^3 \text{ s}^{-1}$$

$$14 \ \frac{dr}{dt} = \frac{dr}{dV} \times \frac{dV}{dt} = \frac{1}{4\pi r^2} \times -k(4\pi r^2) = -k, \text{ radius will decrease at a constant rate of } k \ (k > 0)$$

$$15 \ \frac{2}{3\pi} \text{ cms}^{-1}$$

$$16 \text{ a } \text{Proof} \quad \text{b } \frac{dD}{d\alpha} = -\frac{5}{\sin^2(\alpha)}$$

$$\text{c } -39 \text{ (decreasing by 39 radians/hour)}$$

$$\text{d } \text{No, the angle will reach zero after a few seconds.}$$

8.03

$$1 \text{ a } y = x^3 - 2x^2 + 7x + c$$

$$\text{b } y = \frac{8}{3} \sin(3t) + c$$

$$\text{c } y = 6w^4 - 12 \log_e |w| + c$$

$$\text{d } y = -2e^{-3x} - 4e^{2x} + c$$

$$\text{e } y = \frac{(5+2x)^5}{10} + c$$

$$\text{f } y = \frac{1}{4(3-2x)^2} + c$$

$$\text{g } x = \tan^{-1}\left(\frac{y}{4}\right) + c$$

$$\text{h } x = -\sin^{-1}\left(\frac{y}{2}\right) + c$$

$$2 \text{ a } y = 2 \log_e |x(x-5)| - 2 \log_e(6)$$

$$\text{b } y = 3 \log_e |x| - \frac{1}{x} + 4$$

$$\text{c } y = \frac{6}{5} \log_e \left(\frac{6|x-2|}{|x+3|} \right) + 1$$

$$\text{d } y = \frac{3\sqrt{2}}{2} \tan^{-1}\left(\frac{\sqrt{2}(x+2)}{2}\right) - \frac{3\sqrt{2}}{2} \tan^{-1}(\sqrt{2}) + 2$$

$$\text{e } y = \tan(x) - x + \frac{\pi}{4}$$

$$\text{f } y = -\frac{4}{3} \cos^2(3x) + 2 \text{ or } -\frac{2}{3} \cos(6x) + \frac{4}{3}$$

$$\text{g } x = \frac{1}{3} \sin^{-1}\left(\frac{3y}{2}\right) + 1$$

$$\text{h } x = 0.3 \arctan(0.4y) + 1$$

$$\text{i } y = 0.2(5x^2 - 2)^5 - 0.6$$

$$\text{j } y = \cos^3(x) - \frac{1}{8}$$

$$3 \text{ a } x + \frac{1}{x-1}$$

$$\text{b } y = -\log_e |x-1| - \frac{x^2}{2} + c$$

$$4 \text{ a } \log_e(x) + 1$$

$$\text{b } y = x \log_e(x) - x + c$$

$$5 \ h = -5t^2 + 20t + 1.5, 20.25 \text{ m}$$

$$6 \ P(n) = 100 \log_e(n^2 + 40n + 1), \$1704$$

$$7 \text{ a } h = 25t - 0.8t^2 \quad \text{b } 87.2 \text{ m} \quad \text{c } \text{Proof}$$

$$8 \ Q = 100x^2, 1.44 \text{ J}$$

$$9 \quad C = \frac{2n^3}{3000} - \frac{n^2}{2} + 250n + 20000 \text{ dollars}$$

$$10 \quad \text{Momentum} = 8t - t^2 + 50 \text{ N s}$$

INVESTIGATION: MERCURY POLLUTION

Original amount of mercury = 200 in 1.7 million $\approx$ 117.6 ppm

$$\frac{dA}{dt} = -\frac{20}{1700}A \approx -0.01176A \text{ ppm/h}$$

$$A = 117.6e^{-0.01176t} \text{ ppm after } t \text{ hours.}$$

'Safe' level in fish varies between countries but is about 1 ppm. 'Safe' level in drinking water is much less; about 0.002 ppm.

If the lake is used as a source of drinking water, it would take about 40 days to reduce to a safe level, so it could not be used for this period.

Pollution of the continental shelf leads to higher levels of mercury in seafood.

8.04

$$1 \quad a \quad y = \pm Ae^{3x}$$

$$b \quad y = \pm Ae^{2x} + \frac{1}{2}$$

$$c \quad y = \frac{1}{3} \log_e \left[\frac{-1}{3(x+c)} \right], x+c < 0$$

$$d \quad y = \tan(x+c), -\frac{\pi}{2} - c < x < \frac{\pi}{2} - c$$

$$e \quad y = 4 \sin(x+c), -\frac{\pi}{2} - c \leq x \leq \frac{\pi}{2} - c$$

$$f \quad y = \frac{-2e^{2x}}{e^{2x} - c}$$

$$2 \quad a \quad y = 2e^x - 1$$

$$b \quad y = 4e^{x-2}$$

$$c \quad y = \frac{e^4 - 25e^{4x}}{5e^{4x} - e^4}, \frac{1}{5e^{4x} - e^4} > 0$$

$$d \quad y = \sin(x - \pi), \frac{\pi}{2} \leq x \leq \frac{3\pi}{2}$$

$$3 \quad Q \approx 100e^{-0.000121t} \% \text{ (} t \text{ in years), about 13 300 years}$$

$$4 \quad A \approx 1.5e^{0.0817t} \text{ m}^2 \text{ (} t \text{ in days), 183 or 184 days}$$

$$5 \quad N \approx 20e^{0.752t} \text{ (} t \text{ in hours), 5.2 h}$$

$$6 \quad L \approx 52.43e^{0.223t} \text{ times the acceptable level (} t \text{ in hours).}$$

At t_0 , $L \approx 52.43$ times the acceptable level, the sample was unacceptable.

$$7 \quad \frac{dT}{dt} = \frac{3750 - 7.5T}{6000}, 170^\circ\text{C, } 500^\circ\text{C}$$

$$8 \quad T = 25 + 75e^{-0.155t} ^\circ\text{C (} t \text{ in min), another 5 min 5 s}$$

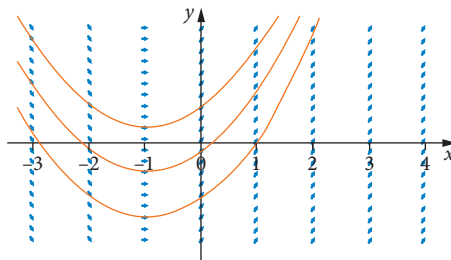
$$9 \quad T = 1100 - 1080e^{-0.3646t} ^\circ\text{C (} t \text{ in min), 12:31 p.m., } 1100 ^\circ\text{C}$$

$$10 \quad v = \frac{gm}{k} \left(1 - e^{-\frac{k}{m}t} \right). \text{ It reaches a final velocity of } \frac{gm}{k}.$$

INVESTIGATION: GRADIENT FIELDS

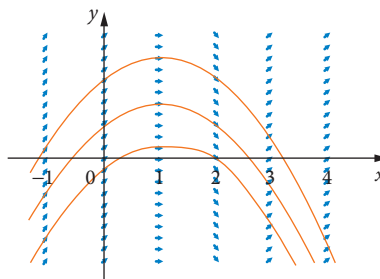
$\frac{dy}{dx} = 0.5 + 0.5x$ is shifted to the left and flatter than

$$\frac{dy}{dx} = 0.8x.$$

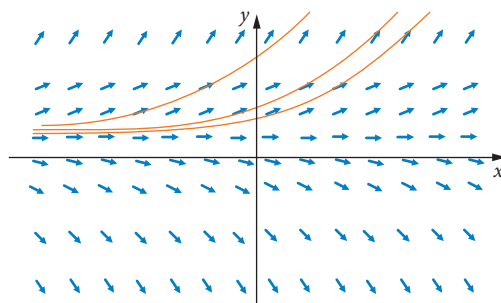


$\frac{dy}{dx} = 0.6 - 0.6x$ is shifted to the left, flatter and upside

$$\text{down compared to } \frac{dy}{dx} = 0.8x.$$

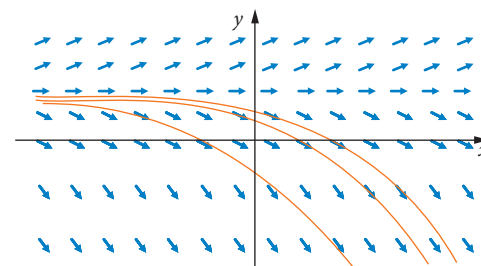


$\frac{dy}{dx} = y + 0.5$ is shifted up and steeper compared to $\frac{dy}{dx} = 0.5y$.

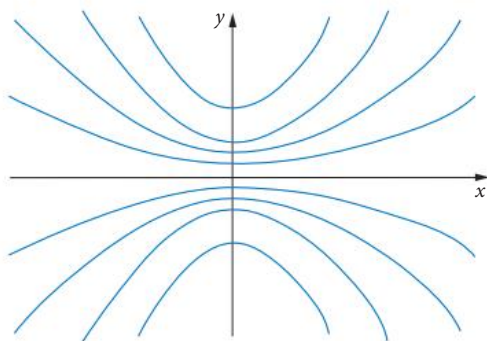


$\frac{dy}{dx} = 1 - 0.6y$ is upside down, shifted up and a little

$$\text{steeper compared to } \frac{dy}{dx} = 0.5y.$$



$\frac{dy}{dx} = xy$ is somewhat like the following.

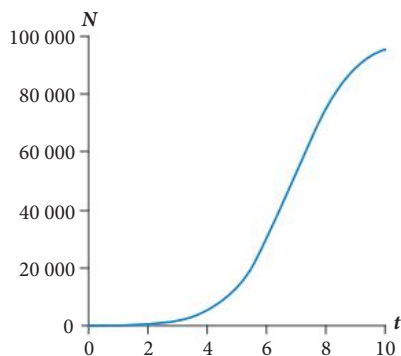


8.05

- 1 $y = Ae^{0.5x^2 - 3x}$
- 2 $g = Ae^{6x^2 + 4x} + \frac{7}{4}$
- 3 $P = \frac{-2}{V^2 - 3}$
- 4 $v = \frac{3}{4}e^{6t^2 + 20t} + \frac{1}{4}$
- 5 $y = Ae^{0.5x^2} - 1$
- 6 $f = \frac{6e^{t^2}}{10 - 9e^{t^2}}$
- 7 $y = \pm \sqrt{Ax^2 e^{x^2} - 1}$
- 8 $y = 8.5e^{-2x} + 1.5$
- 9 $y = -\log_e [e^x(1-x) + c]$
- 10 $y = -0.5 \log_e (c - e^x)$

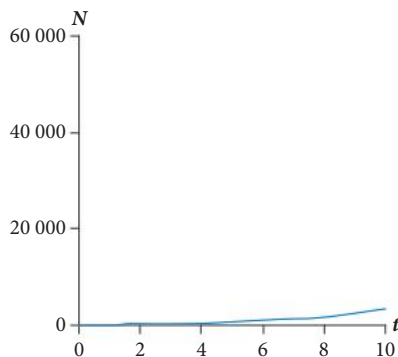
INVESTIGATION: EPIDEMICS

First model



With no vaccination, 95 000 are infected after 10 weeks. As the number vaccinated increases, the rate at which the disease spreads decreases.

Graph for 40% vaccination, so 60 000 are susceptible.

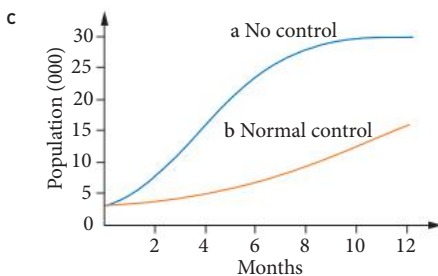


When 60% are vaccinated, only 490 are infected (out of the possible 40 000) after 10 weeks.

At 95% vaccination, only an extra 2 are infected after 10 weeks, even though 100 were infected at the start. Vaccination reduces the speed with which a disease spreads through the population, so large-scale vaccination helps everyone, not just those vaccinated.

8.06

- 1 a 138 b 207 c 915
- 2 a About 6 years (5.973).
b About 83 each subsequent year.
- 3 5.278 weeks (50 left), so they can be removed after 6 weeks.
- 4 22%
- 5 a 3000, 5000, 7941, 11 796, 16 152, 20 321, 23 722, 26 155, 27 734, 28 697, 29 262, 29 585, 29 768
b 3000, 3500, 4070, 4718, 5446, 6259, 7159, 8143, 9208, 10 346, 11 547, 12 796, 14 077



- d Some extra control measures will be needed.
- 6 a 22.7 million b 23.9 million
- c 26.3 million d 31.0 million
- 7 a i 4.6 g ii 5.5 g iii 7.4 g
b 15.5%
- 8 28 904
- 9 50.45 min
- 10 9 weeks (8.47)

8.07

- 1 2.13 m/s^2 at $\tan^{-1}\left(\frac{7}{8}\right) \approx 41.2^\circ$
- 2 1.8 m/s^2 at $\tan^{-1}\left(\frac{5}{2}\right) \approx 68.2^\circ$
- 3 $20\sqrt{5} \text{ N}$ at $\tan^{-1}(2) \approx 63.4^\circ$
- 4 a 8.17 m/s^2 b 40.83 m/s c 102.1 m
- 5 a $200 \text{ N}\cdot\text{s}$ b 25 m/s
- 6 4125 N
- 7 a $\Delta p = 300 \text{ N}\cdot\text{s}$ to the east
b About 18 m/s in the direction $\text{N } 56^\circ \text{ E}$
- 8 About 1.8 N (so the bird has a mass of about 180 g)
- 9 About $1\,122\,000 \text{ N}$ in the direction $\text{N } 11.2^\circ \text{ W}$
- 10 a $0.285 \text{ kg}\cdot\text{m/s}$ north
b $0.3705 \text{ kg}\cdot\text{m/s}$ at a bearing of 120° .
c About 5.89 m/s at a bearing of about 253° .

8.08

- 1 $-3 \text{ m/s}^2, 37.5 \text{ m}$
- 2 a 3.75 m/s^2 b 14.6 m/s
- 3 $50 \text{ s}, 2500 \text{ m}$
- 4 $36 \text{ m/s}, 540 \text{ m}$
- 5 a 6 m/s b 2 m/s^2
- 6 a 132 m b 5 m/s
- 7 Proof
- 8 a 1 m/s^2 b 12.5 m
- 9 20 m/s
- 10 78.4 m
- 11 a About 0.19 m/s^2 .
b About 11.5 m/s (which is unrealistic).
c About 22 m to the left.
- 12 a About $52\,000 \text{ N}$ perpendicular to the direction of travel towards the windward side.
b About 7.1 m/s^2 .

8.09

- 1 a $v = t^2 - \frac{5t^3}{3}$
b $v = -\frac{1}{9}[\cos(3t-2) - \cos(2)]$
c $v = \frac{15}{16}(e^{0.4t+5} - e^5) + 3$
d $v = 7e^{\frac{3}{t}}$
- 2 a $v = \sqrt{\sin(x)+16}$ b $v = \sqrt{\frac{2}{5}x^2 - \frac{14}{5}x + 4}$
c $v = \frac{1}{2}x + 3$ d $v = 4 - 2e^{-x}$
e $v = \sqrt{\frac{2}{3}\log_e(x+1)+4}$
- 3 $113\frac{1}{3} \text{ m}$
- 4 -93 m s^{-1}
- 5 11.48 m

- 6 a $k = 0.003\,456$ b 3.24 s
c 30.25 m
- 7 a $v = -25 + 39e^{-\frac{t}{25}}$ b 72.1 m
- 8 a $v = ue^{-\frac{k}{m}t}$ b $x = \frac{m}{k}u\left(1 - e^{-\frac{k}{m}t}\right)$
c Limiting distance $= \frac{mu}{k}$
- 9 $x = \frac{m}{2k}\log_e\left(1 + \frac{ku^2}{mg}\right)$
- 10 $h = \frac{mu}{k} - \frac{m^2g}{k^2}\log_e\left(1 + \frac{ku}{mg}\right), t_h = \frac{m}{k}\log_e\left(1 + \frac{ku}{mg}\right)$
- 11 a Proof b $\frac{1}{2k}\log_e\left(\frac{4}{3}\right)$

8.10

- 1 $\frac{1}{4} \text{ s}, 4 \text{ s}^{-1}$
- 2 $0.67 \text{ s}, 1.5 \text{ s}^{-1}$
- 3 a $\frac{1}{3} \text{ s}^{-1}$ b 1.57 m s^{-1} c 1.17 m s^{-1}
d 3.29 m s^{-2} e 2.19 m s^{-2}
- 4 a 10 cm
b $15.7 \text{ cm s}^{-1}, 0 \text{ cm s}^{-2}$
c $0 \text{ cm s}^{-1}, 24.7 \text{ cm s}^{-2}$
d $14.4 \text{ cm s}^{-1}, 9.87 \text{ cm s}^{-2}$
- 5 a $\pi\sqrt{2} \text{ s}$ b $\frac{\pi}{2} \text{ s}$
- 6 a $1.5\sqrt{2} \text{ m s}^{-1}$ b $0.9\sqrt{2} \text{ m s}^{-2}$
- 7 $8 \text{ s}, 28\sqrt{2} \text{ cm}$
- 8 $0.9\pi \text{ m s}^{-1}, 0.9\pi^2 \text{ m s}^{-2}$
- 9 $\frac{3}{5\pi} \text{ m s}^{-1}, \frac{6}{25\pi^2} \text{ m}$
- 10 a 6.84 cm b 0.053 s
- 11 a Proof b $3 \text{ m}, \pi \text{ s}$
- 12 $0.0694 \text{ m}, 8.33 \text{ m s}^{-1}$
- 13 a Proof b $a = 2, T = \pi$
- 14 $12:46 \text{ p.m.}$

CHAPTER 8 REVIEW

- 1 D
- 2 E
- 3 B
- 4 A
- 5 A
- 6 E
- 7 B
- 8 C
- 9 $-\frac{3\sqrt{3}}{2}, 3\sqrt{3}x + 2y - 24 = 0$
- 10 About 5.97 m/s .
- 11 $y = \frac{1}{2}\log_e\left|\frac{x-1}{x+1}\right| + 3$

12 $y = 8e^{-0.5x}$

13 $P = \pm \sqrt{\frac{3}{c-2V^3}}$

14 300 000 N

15 a About 0.82 s. b About 13.5 m/s.

16 The heavier car wins by 5.42 m.

17 $\frac{1}{3}$ m/s

18 $v = \sqrt{\frac{1}{3}(2x^2 - 2x + 27)}$

19 a $x = 10 \cos\left(\frac{\pi t}{6}\right)$ b 5.24 mm/s

c 12 s, 10 mm d 5.24 mm/s

e 2.74 mm/s^2

20 a 0.735 m b 1.28 s

21 12 m/s

22 About 10:52 a.m.

23 About 35 years

24 About 35.7 m

MIXED REVISION 3

Multiple choice

1 D

2 A

3 C

4 A

5 D

6 B

Short answer

1 $-\frac{1}{7} \log_e |7x-2| + c$

2 About 59.4 m (59.430...)

3 a About 0.5 m^2 (0.536...)

b 14 days (13.425...)

4 $-\frac{1}{3} \int u^7 du$

Application

1 a $\int x^2 \sin(x) dx = -x^2 \cos(x) + 2x \sin(x) + 2 \cos(x) + c$

b $\int_0^{2\pi} x^2 \sin(x) dx = -4\pi^2$

2 a $\frac{x^2}{(x-1)^2(x+4)} = \frac{1}{5(x-1)^2} + \frac{9}{25(x-1)} + \frac{16}{25(x+4)}$

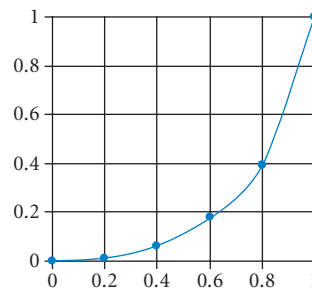
b $\frac{32}{25} \log_e(3) - \frac{57}{25} \log_e(2) + \frac{1}{5}$

3 11 days (10.674... days)

4 $12.5 - 12e^{-0.4t}$, 12.5 m/s

INVESTIGATION: LORENZ CURVES

x	0.2	0.4	0.6	0.8	1
Wealth	31203	191207	437856	766465	2215032
Cumulative	31203	222410	660266	1426731	3641763
Ratio	0.008568	0.061072	0.181304	0.391769	1



Income inequality coefficient for Australia ≈ 0.58

The coefficient has increased in recent years.

The coefficient is higher in most other countries.

9.01

1 0.5

2 $16\frac{1}{3}$

3 $57\frac{1}{6}$

4 36

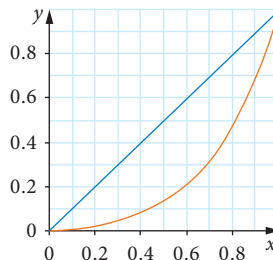
5 36

6 $127\frac{1}{54}$

7 $80\frac{1}{96}$

8 $49\frac{1}{3}$

9 a



b About 0.8%. c About 30.5%. d 0.515

10 $9\frac{23}{30}$

INVESTIGATION: POP-UP TENT

The volume is about 4 m^3 .

9.02

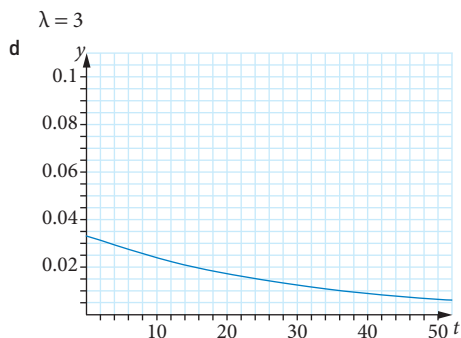
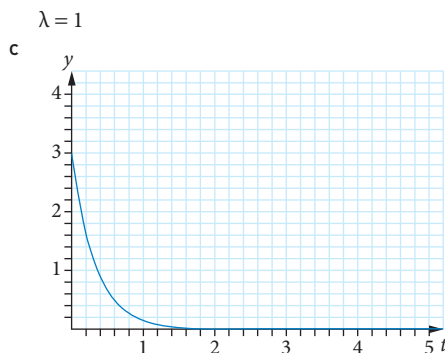
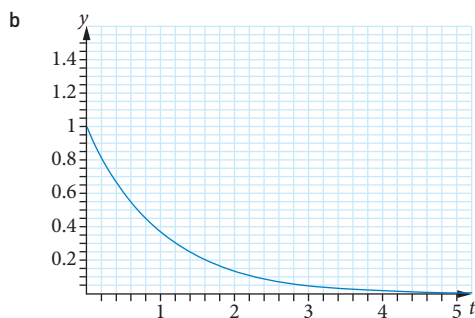
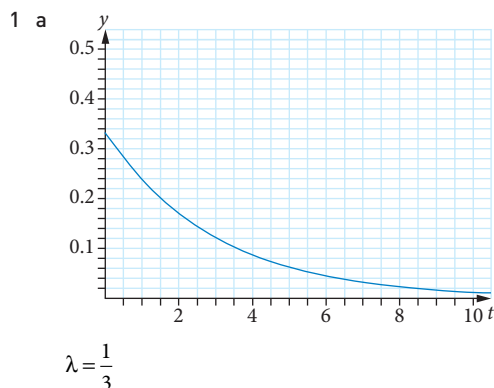
- 1 a $\frac{243\pi}{20}$ b $\frac{25\pi}{7}$ c $2\pi[1 - \log_e(2)]^2$
d $\frac{\pi e(e-1)}{4}$ e $\frac{\pi^2}{4}$
- 2 a 9π b $\frac{3\pi}{5\sqrt[3]{25}}$ c $\frac{\pi}{2}(e^2 - 1)$
d $\frac{\pi}{2}(4e - 5\sqrt{e})$ e approximately 7.17
f approximately 2.93
- 3 a D b $V = \frac{4}{3}\pi r^3$ where $r = 5$. Vol = $\frac{500}{3}\pi$
- 4 E
- 5 C
- 6 B
- 7 $V = \frac{s^2 h}{3}$
- 8 6000 m^3
- 9 $\frac{375\pi}{2}$
- 10 2021.84

9.03

- 1 C
- 2 D
- 3 A
- 4 E
- 5 C
- 6 E
- 7 a 4.8868 b 4.8756 c 4.8828
d 4.8830 e Simpson's rule is the closest.
- 8 a 111.2969 b 111.4063 c $111\frac{1}{3}$ d $111\frac{1}{3}$
e Simpson's rule is exact.

9–11 Proofs

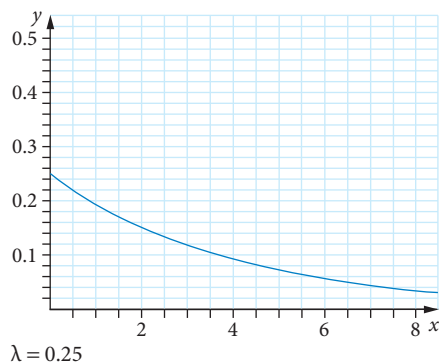
9.04

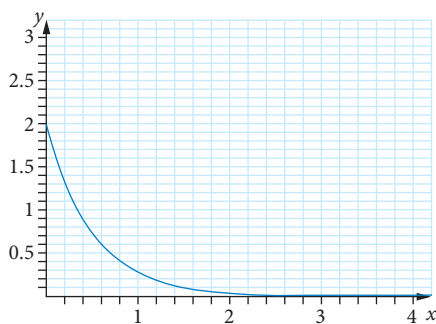


$$\lambda = \frac{1}{30}$$

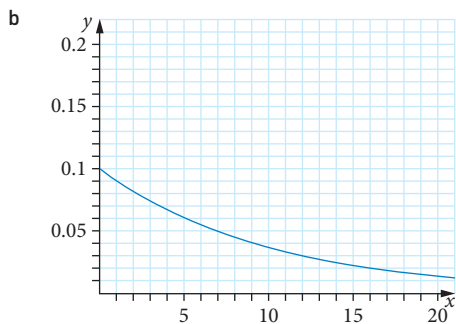
The function with the higher value of λ starts higher and is skewed more to the left.

- 2 A
- 3 C
- 4 B
- 5 E
- 6 a

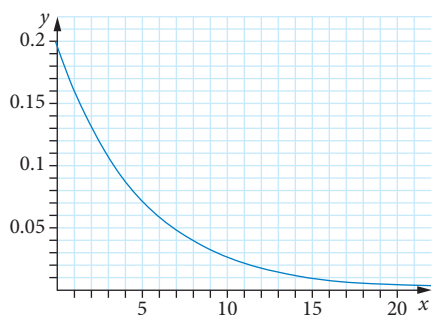




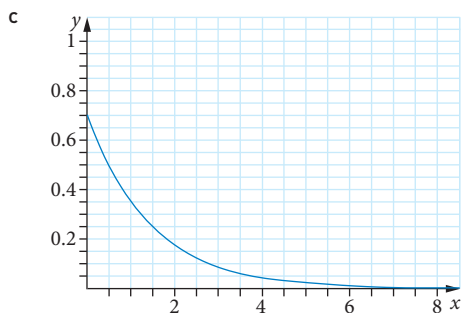
$$\lambda = 2$$



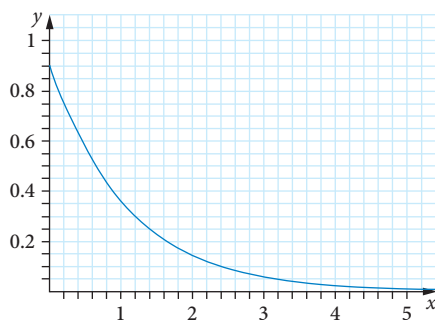
$$\lambda = 0.1$$



$$\lambda = 0.2$$



$$\lambda = 0.7$$



$$\lambda = 0.9$$

The function with the higher value of λ starts higher and is skewed more to the left.

7 a $1 - e^{-\frac{9}{5}}$ b $e^{-\frac{9}{5}} - e^{-6}$ c $e^{-\frac{57}{10}} - e^{-\frac{63}{10}}$ d e^{-6}

8 a $1 - e^{-\frac{9}{10}}$ b $e^{-\frac{6}{5}} - e^{-\frac{9}{5}}$ c $e^{-\frac{9}{10}}$ d $e^{-\frac{3}{2}}$

9 $\frac{2}{3}$

10 0.0625

11 a $P(T \geq 100) = e^{-\frac{1}{2}}$ b 200 hours

12 a $\lambda = \frac{2}{3}$ b 0.632 c 0.069

13 a About 63.2%. b About 1.73 years.
c About 20.2%.

9.05

1 a A b D

2 a A b D

3 B

4 C

5 20 minutes

6 a 2.4 s

b $\sigma = \frac{1}{25}$

c median = $\frac{\log_e(2)}{\lambda}$, so

median = $\frac{1}{25} \log_e(2) \approx 0.02773$

d $\frac{1}{25} \log_e\left(\frac{4}{3}\right) \approx 0.01 \text{ min}$

e $\frac{1}{25} \log_e(4) \approx 0.06 \text{ min}$

7 a i $P(0 \leq x \leq 8) \approx 0.4353$ ii $P(8 \leq x) \approx 0.5647$

b Total = 1 as expected.

8 $\lambda = \sqrt{\log_e(5)}$

9 $P(0 \leq x \leq 5) = 0.9968$

10 $\frac{1}{4} \log_e \frac{100}{99}$

a $R(3) = 0.9925$ b $R(5) = 0.9875$

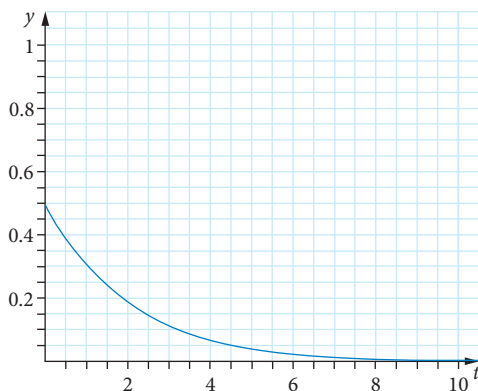
c $R(10) = 0.9752$

11 About 6.66 months

- 12 a $\frac{1}{3000}$ b 0.1889
 13 a About 0.2835 b About 4.159
 c About 0.811

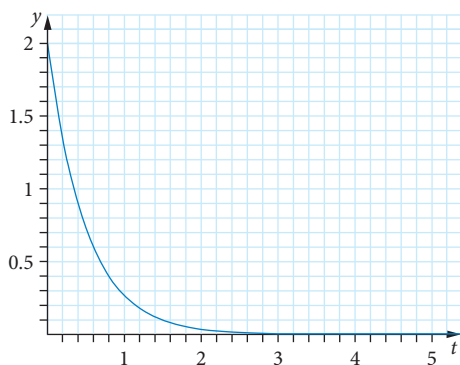
CHAPTER 9 REVIEW

- 1 B
 2 E
 3 A
 4 B
 5 E
 6 $\frac{9}{8}$ square units
 7 121.5 square units
 8 $78\frac{1}{12}$ square units
 9 a $\frac{243\pi}{5}$ b $\pi e^4(e^{12} - 1)$
 10 a $\frac{9\pi}{2}$ b $\frac{\pi}{8}(e^2 - 1)$
 11 818.05
 12 a $f(t) = \frac{1}{2}e^{-\frac{1}{2}t}$



$$\lambda = \frac{1}{2}$$

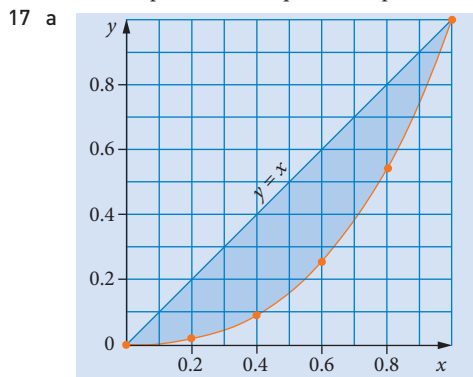
b $f(t) = 2e^{-2t}$



$$\lambda = 2$$

The lower the value of λ the flatter the slope is at the beginning.

- 13 a 15 b 30
 14 $E(X) = 16 = \frac{1}{\lambda}$ so $\lambda = \frac{1}{16}$
 15 a About 0.1353
 b About 92 (8.2 will last)
 16 $f(t) = 0.08e^{-0.08t}$ with $\lambda \approx 0.08$
 a i $P(X < 9) = 0.5132$ ii $P(X > 9) = 0.4868$
 b Total of probabilities equal 1 as expected.



- b 1.95% c About 0.453.
 18 $52\frac{4}{15}$
 19 $\sqrt{\log_e(10)}$
 20 0.2231

INVESTIGATION: SAMPLE HEIGHTS

The standard deviation of the sample means is less than the standard deviation for the whole class.
 The number of samples possible from your class is $2^n - 1$, where n is the number of people.

10.01

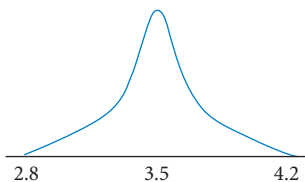
- 1 a 76.6 b 80.6 c 81.1 d 75.5
 2 a 24.3 b 27.8 c 24.7 d 37.6
 3 a Results will vary, but it will be close to 24.
 b It will vary around 0.8.
 c The sample mean is close to the mean, but the sample standard deviation is smaller.
 4 a Results will vary, but it will be close to \$120.
 b It will vary around \$6.
 c The sample mean is close to the mean, but the sample standard deviation is smaller.
 5 a Results will vary, but it will be close to 23 cm.
 b It will vary around 0.5.
 c The sample mean is close to the mean, but the sample standard deviation is smaller.
 6 a Results will vary but it will be close to \$290.
 b It will vary around \$13.
 c The sample mean is close to the mean, but the sample standard deviation is smaller.

- 7 a 7.84, 1.78
 b Results will vary, mostly between 6.6 and 9.1.
 c The mean will be close to 7.8 and the standard deviation will be close to 0.6.
 d It will be close to 3.
 e It is almost the same.
- 8 a 15.8, 5.4
 b Results will vary, mostly between 12.8 and 18.8.
 c The mean will be close to 15.8 and the standard deviation will be close to 1.4.
 d It will be close to 4.
 e It is almost the same.
- 9 a 55.68, 19.65
 b Results will vary, mostly between 48 and 64.
 c The mean will be close to 56 and the standard deviation will be close to 4.
 d It will be close to 5.
 e It is almost the same.
- 10 a 0.38, 0.08
 b Results will vary, mostly between 0.33 and 0.43.
 c The mean will be close to 0.38 and the standard deviation will be close to 0.023.
 d It will be close to 3.5.
 e It is almost the same.

10.02

Specific answers for graphs in this exercise will vary, but patterns will be similar. The graph outlines show only the *general shapes* of the histograms and dot graphs.

- 1 a, b Similar to:

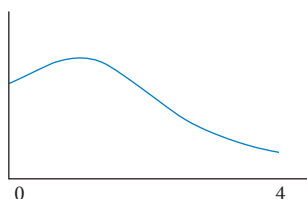


- c The graph is not a uniform distribution like the original. It is more bell-shaped and extends only from about 2.8 to 4.2.

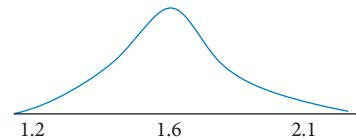
2 a

	2	3	4	5	6
2	0	1	2	3	4
3	1	0	1	2	3
4	2	1	0	1	2
5	3	2	1	0	1
6	4	3	2	1	0

- b Similar to:

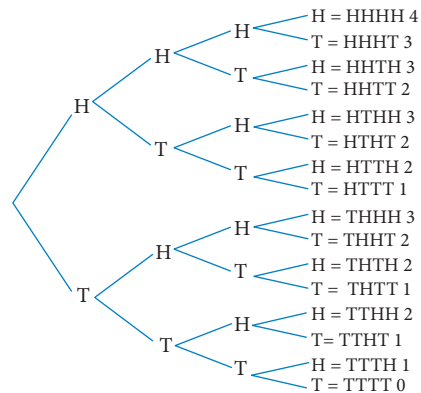


- c, d Similar to:

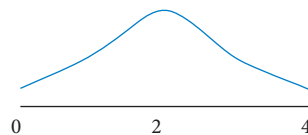


- e The graph of a single sample is unevenly spread from 0 to 4 with a peak at 0 or 1 and is skewed to the right.
 The sampling distribution is spread from about 1.1 to 2.2 with a peak at about 1.6. It is bell-shaped.

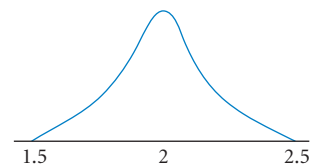
- 3 a



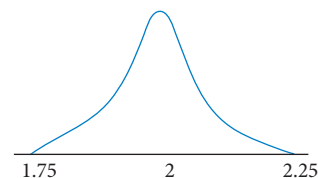
- b Similar to:



- c, d Similar to:

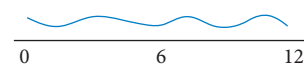


- e, f Similar to:

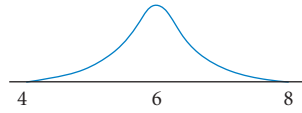


- g The single sample graph has a central peak, but the sampling distributions are more bell-shaped and much more concentrated.

- 4 a Uneven.

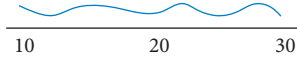


b, c Similar to:

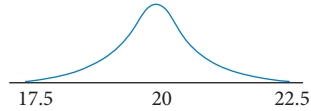


d The single sample is distributed from 0 to 12, but the sampling distribution is only from 4 to 8 with a peak about 6 in a bell-shaped curve.

5 a Uneven.

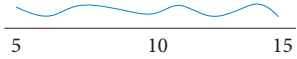


b, c Similar to:

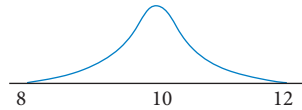


d The single sample is distributed from 10 to 30, but the sampling distribution is only from 17.5 to 22.5 with a peak about 20 in a bell-shaped curve.

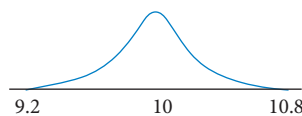
6 a Uneven.



b, c Similar to:

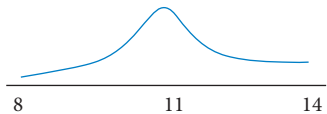


d, e Similar to:

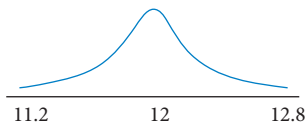


f The single sample is distributed from 5 to 15, but the sampling distributions are only from 8 to 12 and 9.2 to 10.8, with peaks at about 10 in bell-shaped curves. The larger sample size has a more concentrated distribution.

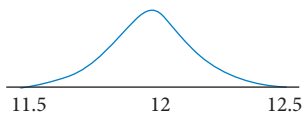
7 a Similar to:



b, c Similar to:

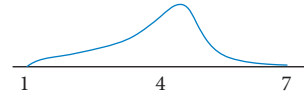


d, e Similar to:

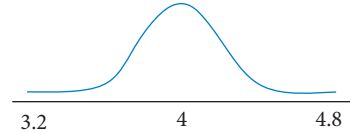


f The single sample is unevenly distributed from about 8 to 14, but the sampling distributions are only from 11.2 to 12.8 and 11.5 to 12.5, with peaks at about 12 in bell-shaped curves. The larger sample size has a more concentrated distribution.

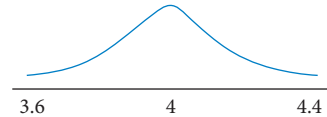
8 a Uneven.



b, c Similar to:



d, e Similar to:



f The single sample is unevenly distributed from about 1 to 7, but the sampling distributions are only from 3.2 to 4.8 and 3.6 to 4.4, with peaks at about 4 in bell-shaped curves. The larger sample size has a more concentrated distribution.

10.03

- 1 About 25, 0.6
- 2 About 80, 0.8
- 3 About \$200, 1500
- 4 About 13.4
- 5 About 360
- 6 About 255 and 38.7
- 7 0.009
- 8 0.0368
- 9 0.7688
- 10 About \$500 000

10.04

- 1 $[-1.96, 1.96]$
- 2 $[-2.57, 2.57]$
- 3 $[-1.28, 1.28]$
- 4 $[75.3, 124.7]$
- 5 $[46.5, 93.5]$
- 6 $[8.6, 101.4]$
- 7 About 84.7%.
- 8 About 95.4%.
- 9 About 78.9%.
- 10 8 months (8.545...)

INVESTIGATION: CONFIDENCE INTERVALS AND THE MEAN

The expected standard deviation of the sampling distribution is about 1.15.

The result for the class is likely to be between about 8% and 12%.

10.05

- 1 [25.5, 30.5]
- 2 [129, 141]
- 3 [63.5, 70.5]
- 4 [415.7, 424.3]
- 5 [-0.02, 1.42]
- 6 [49.1, 54.9]
- 7 a 23.95, 11.81
b About [20.9 ha, 27 ha].
c Only Victorian potato farms have been used.
- 8 a 743, 85.7
b About [715, 771].
c The data is not a random sample.
- 9 a \$1557.20, \$150.83
b About [\$1508, \$1607].
c The sample is less than 30.
- 10 a 10.4, 11.7
b About [6.8, 14.1].
c There are a lot of 'round figures' in the data that suggest some students may have been giving the recommended times at their schools instead of the actual times.

INVESTIGATION: SURVEY IN SCHOOL

Results will vary.

10.06

- 1 a [7.9, 9.1], [7.2, 8.4], [6.9, 8.1]
b Even at the 90% confidence level, the top 3 overlap so the difference could just be sampling variation.
- 2 a [14.9 years, 15.1 years], [23.8 months, 24.2 months]
b The samples are far from random and are likely to be very unrepresentative because of self-selection bias.
- 3 a 13 months
b 13.8 months
c The mean would be 9.1 months and the standard deviation would be 13 months.
d If you count only the people who finished, it is justified, but maybe the ones who dropped out did so because they weren't benefitting..

- 4 a 7.7 mm b 0.23
c No, because if it is random, it could be very high in any year. However, we do know that rainfall is not entirely random (e.g., El Nino).
- 5 a 0.78°C b 7%
c It is assumed that the data over the last 12 years is a random sample, and the sample size is too small to approximate using the normal distribution.

CHAPTER 10 REVIEW

- 1 E
- 2 C
- 3 A
- 4 B
- 5 E
- 6 C
- 7 A
- 8 The graph is likely to be uneven from 1 to 8, but with no particular peak.
- 9 The graph will probably be a bell-shaped curve with the centre at 6.5 but not extending over the whole domain (probably about 5.7 to 7.3).
- 10 The centres of all three graphs will be at about 5. I will be skewed to the right, while II and III will be more symmetrical. The spread of the results will decrease from I to II to III.
- 11 $\mu = 14$, $\sigma_{\bar{x}} \approx 0.26$
- 12 About 0.275.
- 13 [-1.645, 1.645]
- 14 [39.4, 80.6]
- 15 About 0.959.
- 16 [37.7, 42.3]
- 17 [\$1.66, \$3.34]
- 18 a [\$124.15, \$145.85]
b The sample is not random because only students from one school were used.
- 19 a [\$118.51, \$121.49]
b The poll is likely to be biased towards people who want fast access because mobile phones are used more by younger people than older people. It is not random.
- 20 a $\mu = 8$, $\sigma \approx 1.6$
b The results mean that only 7 or 8 out of the 247 surveyed rated the security less than 5 out of 10, so probably not a cause for concern.

MIXED REVISION 4

Multiple choice

- 1 D
- 2 C
- 3 B
- 4 A
- 5 A
- 6 D

Short answer

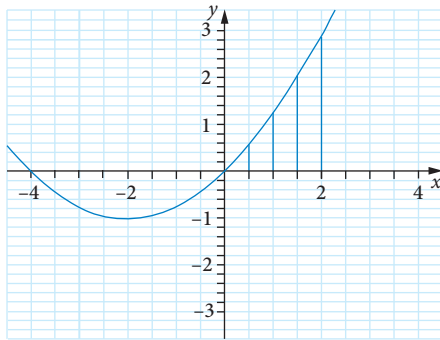
1 $\mu = 10, \sigma_{\bar{x}} = \frac{\sqrt{30}}{6} \approx 0.9129$

2
$$\begin{aligned} \text{Area} &= \int_{-4}^1 4 - (x^2 + 3x) dx \\ &= \int_{-4}^1 4 - x^2 - 3x dx \\ &= \left[4x - \frac{1}{3}x^3 - \frac{3}{2}x^2 \right]_{-4}^1 \\ &= \left(4 - \frac{1}{3} - \frac{3}{2} \right) - \left(-16 + \frac{64}{3} - 24 \right) \\ &= \frac{125}{6} \end{aligned}$$

Area = $20\frac{5}{6}$ square units

3 [22.7, 26.7]

4 $f(x) = \frac{1}{4}x^2 + x$ from $x = 0$ to $x = 2$ with strips $\frac{1}{2}$ unit wide

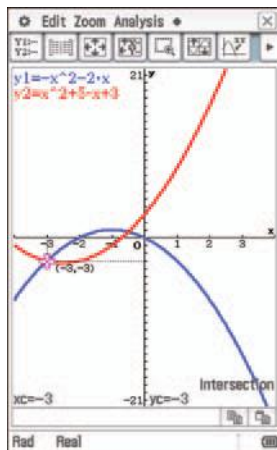


Area = $\frac{1}{2} [f(0.25) + f(0.75) + f(1.25) + f(1.75)]$

Area = $\frac{85}{32}$ square units

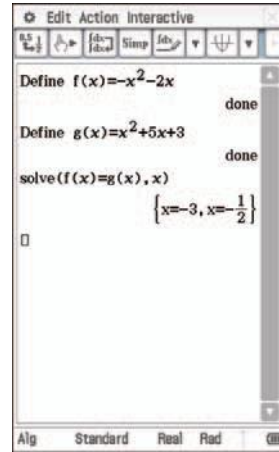
Application

1 a



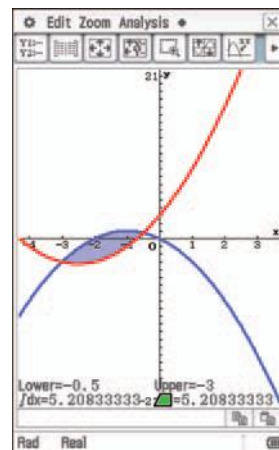
$f(x) = -x^2 - 2x$ and $g(x) = x^2 + 5x + 3$.

b Solve $-x^2 - 2x = x^2 + 5x + 3$
gives $-2x^2 - 7x - 3 = 0$
so $2x^2 + 7x + 3 = 0$ and $(2x + 1)(x + 3) = 0$
so $x = -3, x = -\frac{1}{2}$



c
$$\begin{aligned} \text{Area} &= \int_{-3}^{-\frac{1}{2}} (-x^2 - 2x) - (x^2 + 5x + 3) dx \\ &= \int_{-3}^{-\frac{1}{2}} -2x^2 - 7x - 3 dx \\ &= \left[-\frac{2}{3}x^3 - \frac{7}{2}x^2 - 3x \right]_{-3}^{-\frac{1}{2}} \\ &= \left(\frac{1}{12} - \frac{7}{8} + \frac{3}{2} \right) - \left(18 - \frac{63}{2} + 9 \right) \\ &= \frac{125}{24} \end{aligned}$$

Area = $\frac{125}{24}$ square units



2 a [17.4, 25.3]

b The sample is not random and there are less than 30 people in the sample.

3 The probability of failure is less than 7% for about 69 h.

4 0.1769

GLOSSARY

Chapter 1

altitude angle For the polar form of a vector, the altitude angle, ϕ , lies between -90° and 90° and is the angle from the xy -plane to the vector.

azimuth angle For the polar form of a vector, the azimuth angle, θ , lies between 0° and 360° and is the angle from the x -axis to the projection of the vector onto the xy -plane.

Cartesian equation of a line The Cartesian form of the equation of a line in space passing through the points $A(a_1, a_2, a_3)$ and $B(b_1, b_2, b_3)$ is:

$$\frac{x - a_1}{b_1 - a_1} = \frac{y - a_2}{b_2 - a_2} = \frac{z - a_3}{b_3 - a_3}$$

If the vector equation of a line is $\mathbf{p} = \mathbf{a} + t\mathbf{u}$, where $\mathbf{u}$ is a vector in the direction of the line and if $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$, then the Cartesian form of the equation is:

$$\frac{x - a_1}{u_1} = \frac{y - a_2}{u_2} = \frac{z - a_3}{u_3}$$

Cartesian equation of a plane The Cartesian equation of a plane through $P(x_1, y_1, z_1)$, with normal vector $\mathbf{n} = (a_1, a_2, a_3)$ is:

$$a_1(x - x_1) + a_2(y - y_1) + a_3(z - z_1) = 0$$

consistent The term used to describe a system of linear equations that have at least one solution.

cross product Another name for **vector product**.

dependent The term used to describe a system of linear equations that can be derived from a combination of others.

distance formula in three dimensions (3D

distance formula) The distance between

$P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is:

$$d(P_1, P_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

dot product Another name for **scalar product**.

Gaussian elimination The process used to solve a system of linear equations by progressively eliminating variables.

homogeneous The term used to describe a system of linear equations that has no constant terms: every equation in the system can be written in the form

$$a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 + \cdots + a_nx_n = 0.$$

inconsistent The term used to describe a system of linear equations that has no solutions.

independent The term used to describe a system of linear equations that has no equation that can be eliminated as a combination of the others.

initial point The point where a vector starts. For example, vector $\mathbf{QR}$ has initial point (or head) Q .

inner product Another name for **scalar product**.

linear equation A linear equation is one that can be written with no powers, roots, reciprocals, and so on—for example, $5x - y + 4z = 8$.

midpoint The midpoint of the line segment from $P_1(x_1, y_1, z_1)$ to $P_2(x_2, y_2, z_2)$ is:

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2} \right)$$

non-homogeneous The term used to describe a system of linear equations that has at least one equation with a constant term.

norm The magnitude of a three-dimensional vector is called the norm of the vector. For vector $\mathbf{a}$, the norm is written as $|\mathbf{a}|$ or a . If $\mathbf{a} = (a_1, a_2, a_3)$, then

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

ordered triple A set of three numbers (x, y, z) used to locate the position of a point in a three-dimensional coordinate system.

outer product Another name for **vector product**.

polar form The polar form of a vector has a magnitude r and two angles—the **azimuth angle** (θ) and the **altitude angle** (ϕ).

position vector The vector $\mathbf{OP}$ from the origin, $O(0, 0, 0)$, to $P(a_1, a_2, a_3)$ is called the position vector of P . The lower case vector symbol is also used for the position vector, so $\mathbf{OP} = \mathbf{p}$.

projection The projection of $\mathbf{a}$ on $\mathbf{b}$ is the component of $\mathbf{a}$ in the direction of $\mathbf{b}$ and is given by $\mathbf{a} \cdot \hat{\mathbf{b}}$

resultant vector The vector formed when vectors are added.

scalar product For vectors $\mathbf{a}$ and $\mathbf{b}$ with the angle θ between them, their **scalar product** is given by:

$$\begin{aligned}\mathbf{a} \cdot \mathbf{b} &= |\mathbf{a}||\mathbf{b}| \cos(\theta) \text{ or} \\ \mathbf{a} \cdot \mathbf{b} &= ab \cos(\theta)\end{aligned}$$

In component form, if $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$, then

$$\begin{aligned}\mathbf{a} \cdot \mathbf{b} &= (a_1, a_2, a_3) \cdot (b_1, b_2, b_3) \\ &= a_1b_1 + a_2b_2 + a_3b_3\end{aligned}$$

sphere The equation of a sphere with centre $C(h, k, l)$ and radius, r , is:

$$\begin{aligned}(x-h)^2 + (y-k)^2 + (z-l)^2 &= r^2 \\ \text{or } x^2 + y^2 + z^2 + ax + by + cz + d &= 0\end{aligned}$$

system of equations A system of equations shows the relationships between a group of variables.

terminal point The point where a vector ends. For example, vector $\mathbf{QR}$ has terminal point (or tail) R .

unit vector A unit vector has magnitude 1. The unit vector in the same direction as a given vector $\mathbf{p}$ is symbolised as

$$\hat{\mathbf{p}} = \frac{1}{|\mathbf{p}|} \mathbf{p}$$

unit vectors in space The unit vectors in space are $\mathbf{i} = (1, 0, 0)$, $\mathbf{j} = (0, 1, 0)$ and $\mathbf{k} = (0, 0, 1)$.

vector equation of a line The vector equation of the line through points A and B with

position vectors $\mathbf{a}$ and $\mathbf{b}$ is $\mathbf{p} = \mathbf{a} + t(\mathbf{b} - \mathbf{a})$ (for some real number t).

If $\mathbf{u}$ is any vector in the direction of the line AB :

$$\mathbf{p} = \mathbf{a} + t\mathbf{u}$$

vector equation of a plane The vector equation of the plane through $A(x, y, z)$ and $P(x_1, y_1, z_1)$, with position vectors $\mathbf{a}$ and $\mathbf{p}$ respectively and normal vector $\mathbf{n}$ is:

$$\mathbf{n} \cdot (\mathbf{p} - \mathbf{a}) = 0$$

This equation may also be written as:

$$\mathbf{n} \cdot \mathbf{p} = \mathbf{n} \cdot \mathbf{a} = k \text{ or } \mathbf{p} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n} = k,$$

where k is a constant.

vector product The vector product of two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ with an angle of θ between them is given by

$$\mathbf{v} = \mathbf{v}_1 \times \mathbf{v}_2, \text{ where } |\mathbf{v}| = |\mathbf{v}_1||\mathbf{v}_2| \sin(\theta).$$

In component form, the vector product of $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ is:

$$\begin{aligned}(a_1, a_2, a_3) \times (b_1, b_2, b_3) \\ = (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1)\end{aligned}$$

Using the determinant method, the vector product of $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ is:

$$\mathbf{a} \times \mathbf{b} = (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

work The work done by a constant force $\mathbf{F}$ that results in a displacement $\mathbf{s}$ is $\mathbf{F} \cdot \mathbf{s}$.

Chapter 2

argument For a complex number z represented by a point P in the complex plane, the **argument** of z , denoted $\arg(z)$, is the angle θ between the positive real axis O_x and OP , measured anticlockwise from O_x . The **principal value** of the argument is the one in the interval $(-\pi, \pi]$.

Cartesian (form) A complex number z expressed in the form $x + yi$, where $x, y \in \mathbf{R}$.

De Moivre's theorem The identity $[\cos(\theta) + i \sin(\theta)]^n = \cos(n\theta) + i \sin(n\theta)$ for all integral values of n .

mod-arg (form) A complex non-zero number z expressed in terms of its modulus r and argument θ , i.e.,

$$z = r [\cos(\theta) + i \sin(\theta)] \text{ or } z = r \operatorname{cis}(\theta).$$

polar (form) Another name for **mod-arg form**.

rectangular (form) Another name for **Cartesian form**.

Chapter 3

absolute value function The **absolute value function**, $y = |x|$ is a function defined by:

$$y = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$

composite function Is often called a 'function of a function' and results when a function is embedded within another function, e.g. $gf(x) = g[f(x)] = g \circ f$

inverse function The inverse relation of a function in the case where the inverse is also a function.

inverse relation For a relation $R: x \rightarrow y$, the relation $R^{-1}: y \rightarrow x$.

one-to-one function A function is said to be one-to-one if it never assigns two input values to the same output value.

rational function A rational function is one that can be expressed in the form $f(x) = \frac{P(x)}{Q(x)}$,

where $P(x)$ and $Q(x)$ are polynomial functions and $Q(x)$ is not the zero polynomial.

reciprocal function A reciprocal function is a special case of the rational function. For a reciprocal function, the numerator is always 1.

A reciprocal function has the form $y = \frac{1}{f(x)}$,

where $f(x)$ is a polynomial and $f(x) \neq 0$.

Chapter 4

Argand plane A two-dimensional plane representing complex numbers as points with the horizontal axis as $\operatorname{Re}(z)$ and the vertical axis as $\operatorname{Im}(z)$ for the real and imaginary parts.

argument For a complex number z represented by a point P in the complex plane, the **argument of z** , denoted $\arg(z)$, is the angle θ between the positive real axis O_x and OP , measured anticlockwise from O_x . The **principal value** of the argument is the one in the interval $(-\pi, \pi]$.

complex arithmetic The four operations $+$, $-$, $\times$, $\div$ involving complex numbers.

complex conjugate The complex number $\bar{z}$, where $\bar{z} = a - ib$ for the complex number $z = a + ib$ (where a and b are real numbers).

De Moivre's theorem The identity $[\cos(\theta) + i \sin(\theta)]^n = \cos(n\theta) + i \sin(n\theta)$ for all integral values of n .

locus A set of points that obeys a certain condition.

polar form Another name for **mod-arg form**.

root of unity A solution to the equation $z^n = 1$.

Chapter 5

angle of projection The angle at which a projectile is launched.

angular velocity (ω) For a particle moving in a circle, the rate of change of the angle of the radius from the particle to the centre.

parameter For Cartesian equations, a variable, usually t , used to express x and y .

parametric equation For a Cartesian equation in x and y , a parametric equation expresses x and y in terms of a third variable, t , e.g. a position vector in two dimensions at time t is:

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$$

period The period (T) of a body moving in uniform circular motion is the time taken to complete one revolution of the circle.

projectile An object subject to acceleration due to gravity whose initial (launch) velocity usually has a horizontal component.

range The maximum horizontal displacement of a projectile.

speed of projection The initial speed of a projectile.

time of flight For projectile motion above a flat plane, the time of flight is the time taken for the object to land on the horizontal plane after its launch.

trajectory The path of a projectile.

vector function or vector-valued function
A vector defined using a parametric equation. The parameter is usually time, t .

uniform circular motion Motion at a constant speed around the circumference of a circle.

Chapter 6

Argand diagram A two-dimensional plane representing complex numbers as points with the horizontal axis as $\text{Re}(z)$ and the vertical axis as $\text{Im}(z)$ for the real and imaginary parts.

argument The angle for a complex number represented in polar form as
 $z = r[\cos(\theta) + i \sin(\theta)]$, written as $\arg(z)$.

complex conjugate The complex number $\bar{z} = a - ib$ with its imaginary part of opposite sign to the given number $z = a + ib$.

complex polynomial A polynomial whose domain is the complex numbers,
 $P(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z^1 + a_0$

where n is a positive whole number and $a_0, a_1, a_2, \dots, a_n \in \mathbb{C}, a_n \neq 0$.

cubic A polynomial of degree 3.

De Moivre's theorem The identity

$$[\cos(\theta) + i \sin(\theta)]^n = \cos(n\theta) + i \sin(n\theta)$$

for all integral values of n .

degree The highest power of the variable in a polynomial.

discriminant For the quadratic equation $az^2 + bz + c = 0$, the quantity $\Delta = b^2 - 4ac$.

dividend The number or polynomial that is divided by another.

division identity The identity that $P = QD + R$ for numbers or polynomials when P is divided by D to give Q with a remainder of R .

divisor The number or polynomial that divides another.

factor theorem The statement that $z - a$ is a factor of $p(z)$ if and only if the remainder, $p(a) = 0$.

leading term The term containing the highest power of a polynomial.

modulus The size of a complex number $z = x + iy$, given by

$$\text{mod}(z) = r = |z| = \sqrt{x^2 + y^2}$$

modulus-argument form The polar form of the complex number $z = r[\cos(\theta) + i \sin(\theta)]$.

principal argument The argument of a complex number z in the domain $-\pi < \theta \leq \pi$.

quadratic A polynomial of degree 2, usually written as $az^2 + bz + c$.

quartic A polynomial of degree 4.

quintic A polynomial of degree 5.

quotient The result of dividing one number or polynomial by another.

remainder The part left over that will not divide exactly when one number or polynomial is divided by another.

remainder theorem The statement that the remainder when a polynomial $p(z)$ is divided by $z - a$ is given by $R = P(a)$.

roots of a complex number The n th roots are any of the complex numbers given by $z^n = a$ for a given complex number a .

roots of unity The n th roots are any of the complex numbers given by $z^n = 1$.

Chapter 7

double angle formulas The formulas that express $\sin(2x)$, $\cos(2x)$ and $\tan(2x)$ in terms of $\sin(x)$, $\cos(x)$ and $\tan(x)$.

integration by parts The integral of $f(x) \cdot g'(x)$ can be written as the difference between $f(x) \cdot g(x)$ and the integral of $f'(x) \cdot g(x)$:

$$\int f(x) \cdot g'(x) dx = f(x) \cdot g(x) - \int f'(x) \cdot g(x) dx$$

integrals giving inverse trigonometric

functions Standard integration formulas that give inverse trigonometric functions, such as

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + c,$$

$$\int \frac{-1}{\sqrt{a^2 - x^2}} dx = \cos^{-1}\left(\frac{x}{a}\right) + c \text{ and}$$

$$\int \frac{a}{a^2 + x^2} dx = \tan^{-1}\left(\frac{x}{a}\right) + c$$

integration by substitution The integral of a function $f(x) = g[u(x)] \cdot u'(x)$ can be calculated as the integral of $g(u)$: $\int_a^b f(x) dx = \int_{\alpha}^{\beta} g(u) du$, where $\alpha = u(a)$ and $\beta = u(b)$ or

$$\int_a^b g[u(x)] \frac{du}{dx} dx = \int_{u(a)}^{u(b)} g(u) du$$

inverse trigonometric functions The inverse functions of $\sin(x)$, $\cos(x)$ and $\tan(x)$ on restricted domains, namely $\sin^{-1}(x)$ or $\arcsin(x)$, $\cos^{-1}(x)$ or $\arccos(x)$, $\tan^{-1}(x)$ or $\arctan(x)$.

natural logarithm function The function

$$\log_e(x) = \int_1^x \frac{1}{t} dt \text{ used to integrate } \frac{1}{x}.$$

partial fractions The expression of a rational function in the form of a sum of functions whose denominators are the factors of the denominator of the original function, such as

$$\frac{x+1}{x^2-3x+2} = \frac{3}{x-2} - \frac{2}{x-1}$$

Pythagorean identities The identities $\sin^2(x) + \cos^2(x) = 1$, $1 + \tan^2(x) = \sec^2(x)$ and $\cot^2(x) + 1 = \operatorname{cosec}^2(x)$

reciprocal trigonometric functions The trigonometric functions $\operatorname{cosec}(x) = \frac{1}{\sin(x)}$,

$\sec(x) = \frac{1}{\cos(x)}$ and $\cot(x) = \frac{1}{\tan(x)}$ that are the reciprocals of the basic trigonometric functions.

recursive procedure A method of calculation or simplification of a formula involving n that reduces the value of n by 1 at each application of the procedure. In integration, the value of n is often a power.

Chapter 8

acceleration The derivative of the velocity with respect to time: the rate of change of the velocity of an object.

amplitude The maximum displacement from the point of equilibrium of a particle undergoing simple harmonic motion.

angular velocity The rate of change of angle for a particle undergoing simple harmonic motion, given by ω in the equation $x = a \cos(\omega t + \alpha)$ or $x = a \sin(\omega t + \alpha)$.

differential equation An equation involving one or more derivatives.

displacement The change in the position of an object from an (initial) fixed point.

first-order The simplest kind. For a differential equation, one that has no higher derivatives, only first derivatives.

force The cause of a change of motion in an object from rest or constant velocity.

frequency The number of cycles completed in unit time by a particle undergoing simple harmonic motion.

general solution For a differential equation, the function that is the solution of the equation.

gradient field A graph showing the gradients of the general solution of a first order differential equation on a xy -plane. Also called a slope field.

implicit differentiation Differentiation of an equation involving two variables with respect to one, using the chain rule for parts that are functions of the other variable

logistic equation The solution to the logistic model, $y = \frac{MA}{A + e^{-kMt}}$ or $y = \frac{My_0}{y_0 + (M - y_0)e^{-kMt}}$, where M is the maximum, y_0 is the initial value and $A = \frac{y_0}{M - y_0}$.

logistic model A model of limited growth given by

$$\frac{dy}{dt} = ky(M - y),$$

where M is the maximum value of y .

mass The property of an object that inversely determines the acceleration of an object produced by a standard force.

momentum The vector that is the product of the mass and velocity of an object.

Newton's laws of motion The three laws of motion relating the ideas of force, acceleration, mass and momentum formulated by Sir Isaac Newton.

period The time taken for one complete cycle by a particle undergoing simple harmonic motion.

phase The angular displacement at time $t = 0$ by a particle undergoing simple harmonic motion; the value of α in the equation $x = a \cos(\omega t + \alpha)$ or $x = a \sin(\omega t + \alpha)$.

point of equilibrium The fixed central point around which simple harmonic motion occurs.

position The place where an object is relative to some fixed point or coordinate system.

related variables Variables having a mathematical relationship expressible as an equation.

separable variables Variables that can be separated (to different sides) in a differential equation.

simple harmonic motion Motion in a straight line with a force given by $F = -kx$, where x is the displacement of the particle from a fixed point.

slope field Another name for a **gradient field**.

velocity The derivative of displacement or position with respect to time: the rate of change of the displacement or position of an object.

Chapter 9

algebraic area The signed area between a curve and the x -axis or the sum of such signed areas.

continuous random variable A continuous variable X with a numerical value that depends on the outcome of a chance experiment.

definite integral The **definite integral** of the function $f(x)$ from $x = a$ to $x = b$ is the area enclosed by the curve $f(x)$, the lines $x = a$ and $x = b$, and the x -axis.

It is shown by the symbol $\int_a^b f(x) dx$.

expected value The mean value of a continuous random variable X obtained using the formula

$$\mu = E(X) = \int_{-\infty}^{\infty} x \cdot p(x) dx.$$

exponential distribution A probability distribution on $[0, \infty)$ with probability density function $f(t) = \lambda e^{-\lambda t}$.

Gaussian distribution The **normal** (or **Gaussian**) **distribution** is a continuous probability distribution defined by the formula

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

midpoint rule The approximation for an integral given by the sum of areas of rectangular strips, calculated as

$A \approx w \sum_{i=1}^n f(a_i)$, where the interval is divided into n strips of width w and a_i are the midpoints of the strips.

parameter of an exponential distribution The value of λ in the probability density function $f(t) = \lambda e^{-\lambda t}$.

physical area The area of a shape, taken as positive. It can be obtained using integration, but signed areas must be separated to find the sum of their magnitudes.

probability density function (pdf) A probability density function $f(x)$ on the domain $[a, b]$ is a non-negative function with an integral of 1 over the domain. Integrals of the form $\int_c^d f(x)dx$ are the probabilities $P(c \leq x \leq d)$ of the associated continuous random variable X defined on the domain.

probability distribution function (pdf) The function $p(x)$ such that

$$P(c \leq X \leq d) = \int_c^d p(x)dx$$

for all intervals (c, d) on which the continuous random variable X is defined. $p(x)$ is non-negative and has an integral of 1 over the whole domain.

quantile The value t_α such that $P(X \leq t_\alpha) = \alpha$ for the continuous random variable X , where $0 < \alpha < 1$.

reliability The function $R(t) = e^{-\lambda t}$ for the probability that something with a failure rate given by an exponential probability function lasts for time t .

Simpson's rule The approximation for an integral given by the sum of areas of strips with parabolic tops. The sum is obtained by dividing the area under a curve into an even number of equal strips of width w with end values numbered from 0 onwards and calculating the value $\frac{w}{3}(X + 2E + 4O)$, where X is the sum of the values of the function at the ends (a_0 and a_n), E the sum of the values of the function at the even-numbered points and O the sum of the values of the function at the odd-numbered points.

solid of revolution A solid obtained by spinning part of a curve around the x - or y -axis.

standard deviation The square root of the variance: $\sigma = \text{SD}(X) = \sqrt{\text{Var}(X)}$

trapezium rule The approximation for an integral given by the sum of areas of trapezoidal strips calculated as $A \approx \frac{w}{2} \{f(a_0) + f(a_n) + 2[f(a_1) + f(a_2) + \dots + f(a_{n-1})]\}$, where the interval is divided into n strips with ends numbered from 0 to n .

variance The value given by

$$\text{Var}(X) = \int_{-\infty}^{\infty} p(x)(x - \mu)^2 dx$$

where $p(x)$ is the probability density function and μ is the expected value of a continuous random variable.

Chapter 10

binomial distribution The (discrete) distribution of the number of successes x in n independent trials of an experiment with only two outcomes (success and failure) with a fixed probability of success:

$$P(X = x) = \binom{n}{x} p^x q^{n-x}, \text{ where } x = 0, 1, 2, \dots, n$$

Central limit theorem The result that for 'relatively large' samples, the sampling distribution of a statistic is approximately normal, and the larger the samples, the better the approximation becomes.

confidence interval An interval estimate of a parameter such that the parameter will occur in a particular proportion of such intervals.

confidence level The proportion of confidence intervals of a sampling distribution that will include the parameter.

cumulative distribution function The integral of a probability density function $f(x)$ from the lower bound to the value x : $\int_{-\infty}^x f(x)dx$.

estimator The value of a characteristic of a population calculated from a sample.

interval estimate An interval likely to include the value of a parameter.

margin of error Half the width of a confidence interval.

normal distribution A distribution of a continuous random variable X with a probability density function of the form

$\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$, where μ and σ are the mean and standard deviation respectively.

parameter A characteristic value of the population, such as the mean.

point estimate A single value estimate of a parameter: a single value statistic.

population The whole group from which data could be collected.

probability density function The function $p(x)$ such that $P(c \leq X \leq d) = \int_c^d p(x)dx$ for all intervals (c, d) on which the continuous random variable X is defined.

quantile The value t_α such that $P(X \leq t_\alpha) = \alpha$ for the continuous random variable X , where $0 < \alpha < 1$.

random sample A sample selected such that every member of the population has an equal chance of being chosen.

random variable A variable with a numerical value that depends on the outcome of a chance experiment.

sample Part of the population under consideration.

sample mean The mean of a sample.

sampling distribution The distribution of a statistic for different samples (of the same size).

standard deviation The square root of the variance: $\sigma = \text{SD}(X) = \sqrt{\text{Var}(X)}$.

statistic An estimate of a parameter obtained using a sample.

triangular distribution A distribution whose probability (density) function has a graph in the shape of a triangle.

uniform distribution A distribution whose probability density function is constant on the interval $[a, b]$. The probability density function is defined by:

$$p(x) = \begin{cases} \frac{1}{b-a} & \text{for } a \leq x \leq b \\ 0 & \text{otherwise (i.e., } x \notin [a, b]) \end{cases}$$

variance The value given by

$$\text{Var}(X) = \int_{-\infty}^{\infty} p(x)(x-\mu)^2 dx$$

where $p(x)$ is the probability density function and μ is the expected value.

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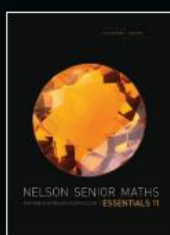
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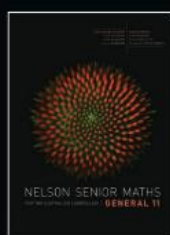
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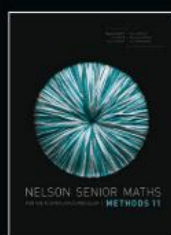
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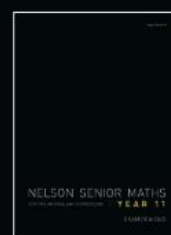
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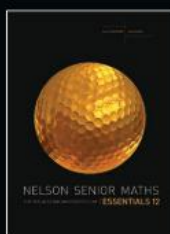
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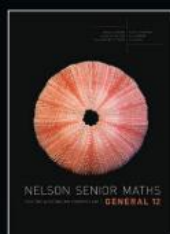
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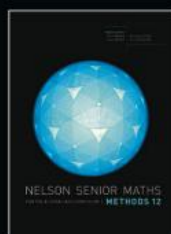
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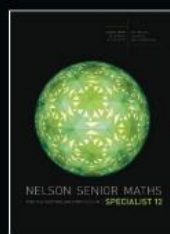
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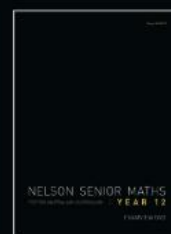
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