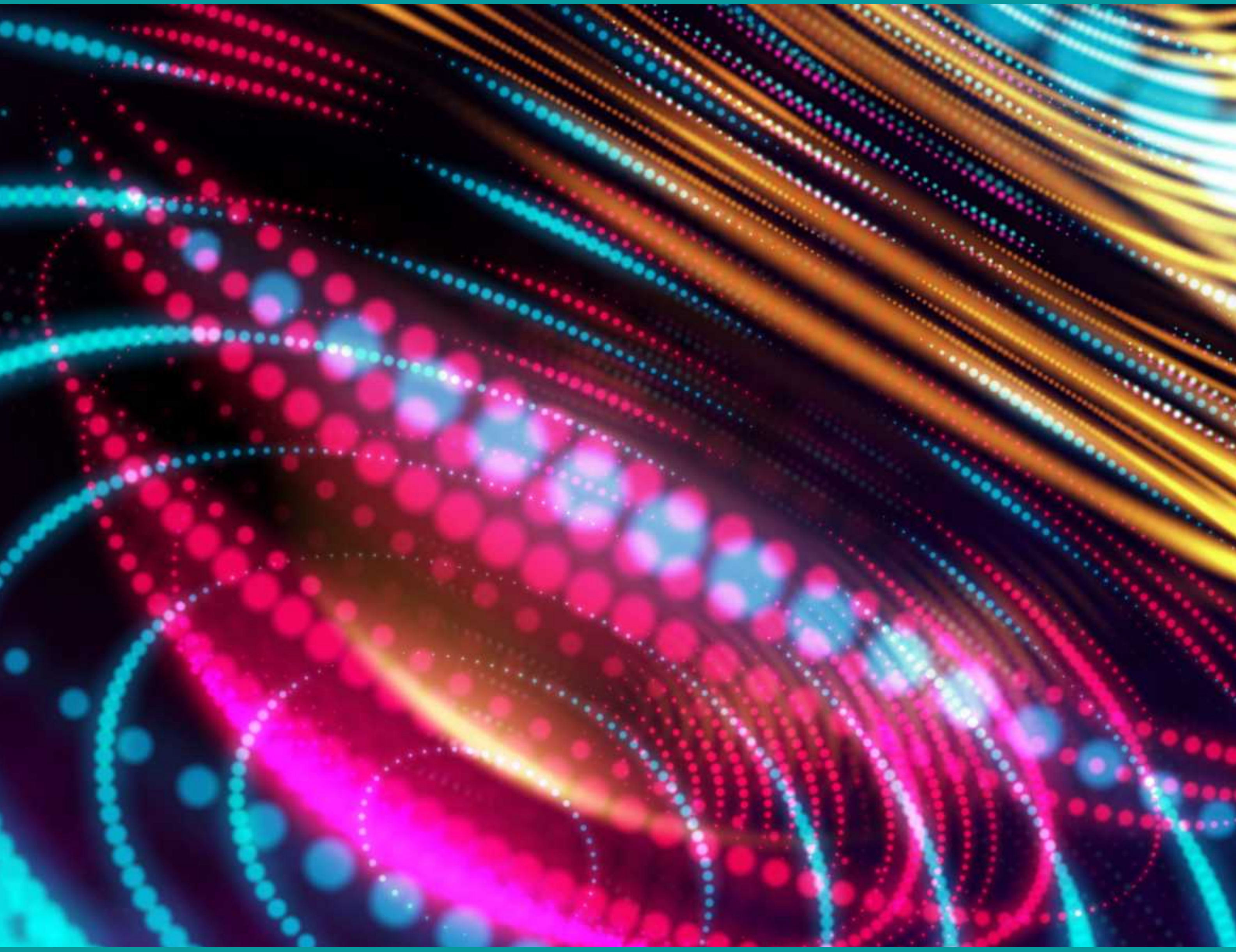
ACADEMIC ASSOCIATES

STUDY GUIDE

MATHEMATICS APPLICATIONS

YEAR 12 ATAR COURSE – UNITS 3 & 4

REVISED EDITION



Greg Hill

ACADEMIC ASSOCIATES

WACE Study Guide

MATHEMATICS APPLICATIONS

YR 12 ATAR COURSE
UNITS 3 AND 4

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About the Author

Greg Hill has been teaching mathematics for over 30 years at various government and independent schools. Greg has been an ATAR marker and author of study guides and texts, with a keen understanding of the different mathematics levels.

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PREFACE

The purpose of this book is to assist students with their preparation for tests and examinations in the new Mathematics Applications course for Units 3 and 4.

The *Syllabus Checklist* indicates to students which skills they must have acquired and the objectives they need to meet under each of the major headings of the course.

The *Worked Examples* are presented in a detailed manner, with brief notes and explanations being used to amplify the understanding required for the particular question. Some of these worked examples could be used in the written notes that students are permitted to take into an examination.

The *Problems to Solve* section in each chapter provides students with a broad range of questions without the repetitive nature of problems usually associated with a course textbook.

The *Trial Tests* are an additional component to this book, and allow students to familiarise themselves with test questions. Suggested times are given for these tests, and students are encouraged to adhere to these times to prepare properly for final examinations. Fully worked solutions are provided for students to receive immediate, accurate and useful feedback on their performance. Tests are provided for both the calculator free and calculator assumed components of the assessment.

The **Examination Style Questions** are provided enabling students to practice questions similar to those found in examinations. A wide variety of resource free and resource rich questions on each of the topics in this course are available for students. A fully worked set of solutions are given for each of these questions.

I hope this study guide will help students to better understand the concepts they will encounter and to achieve greater success in this course.

Greg Hill
April 2021

BIVARIATE DATA

1

Syllabus Checklist

By the end of this chapter, you should be able to:

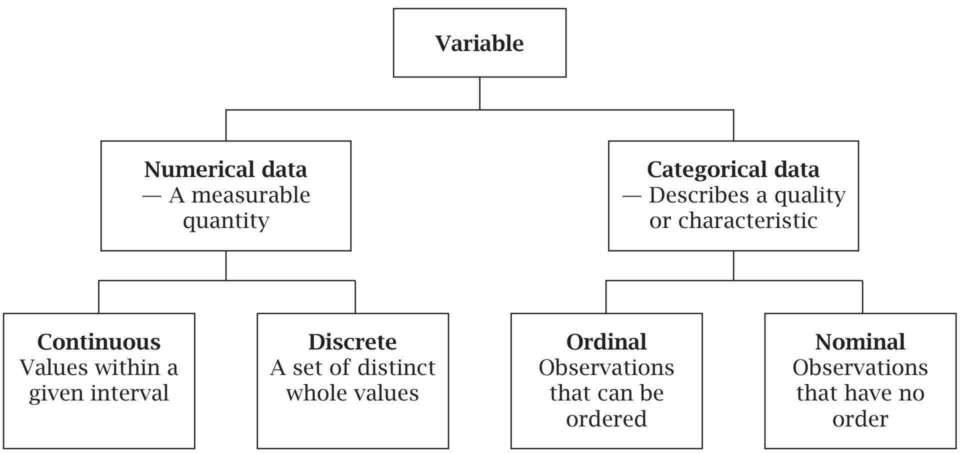
- ☐ review statistical investigation process
- ☐ identify the response and explanatory variable
- ☐ construct two-way frequency tables and determine sums and percentages of rows and columns
- ☐ identify patterns suggesting an association from percentaged two-way frequency tables
- ☐ describe an association
- ☐ construct a scatterplot
- ☐ describe an association between two numerical variables in terms of direction and strength
- ☐ calculate and interpret the correlation coefficient (r)

FORMULAE AND DEFINITIONS

REVIEW OF STATISTICS

Types of data

Collected information is known as *data*. A *variable* is any characteristic, number or quantity that can be measured or counted.



REPRESENTATION OF NUMERICAL DATA

Frequency table

A frequency table is a tabular representation of the data.

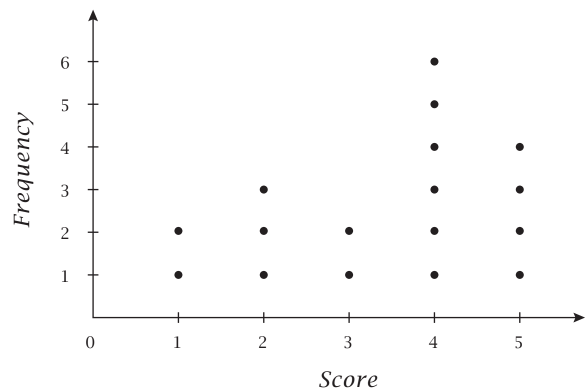
Ungrouped Data		
Score	Tally	Frequency
1	II	2
2	III	3
3	II	2
4	~~IIII~~ I	6
5	IIII	4
		17

Grouped Data	
Score	Frequency
0–4	2
5–9	3
10–14	1
15–19	8
20–24	5
	19

Dot frequency plot

A dot frequency plot gives a clear visual picture of the set of data. From this, data can be analysed to see if it contains:

- *Gaps* spaces between points
- *Outliers* points that are particularly large or small and away from the main data
- *Clusters* isolated groups of points
- *Spread* how far the points are spread or if they are tightly packed

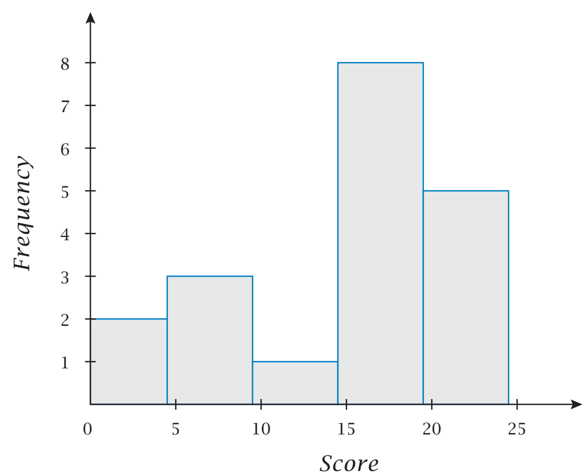


Frequency histograms

A graphical representation of a frequency table. A frequency histogram has:

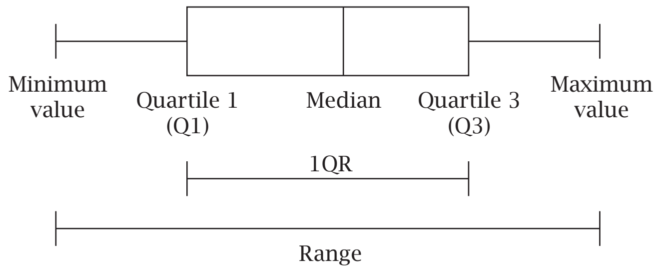
- The frequency on the vertical axis
- No gaps between the columns
- Columns with equal widths

Histograms can be drawn for ungrouped data or grouped data and are usually used to represent continuous distributions.



Boxplots

A boxplot or box and whisker plot is another graphical display of numerical data. Five summary statistics are used to construct a boxplot.



Stem and leaf plot

A visual representation of numerical data. 5|0 represents the number 50.

Example

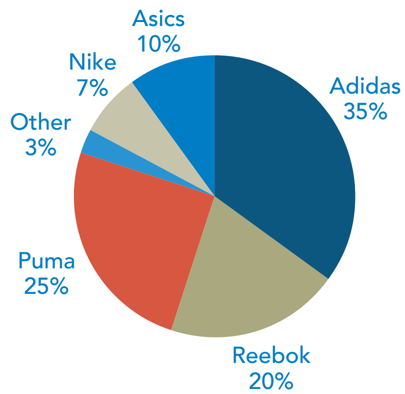
Stem	Leaf
5	0 1 5 9
6	2 4
7	3
8	1 4 7

REPRESENTATION OF CATEGORICAL DATA

Circle graphs

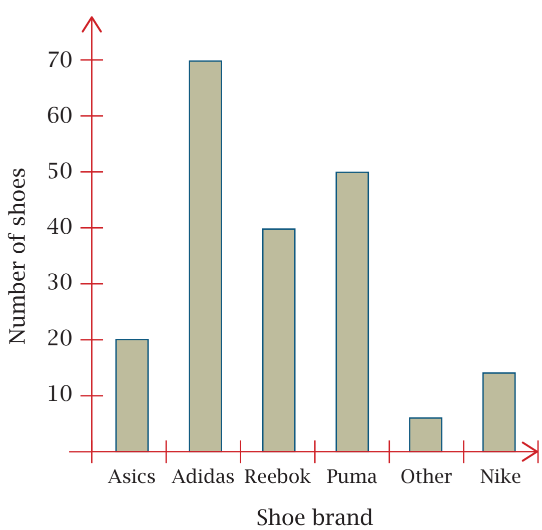
A circle graph is a graph in the form of a circle that is divided into sectors, with each sector representing a part of the data.

Brands of shoes worn by students



Column graph (or bar graph)

Shoe Brand	Number
Asics	20
Adidas	70
Reebok	40
Puma	50
Other	6
Nike	14



Statistics

Measures of Central Tendency

- Mean – the average of a set of data
- Mode – the score which occurs most often
- Median – the middle score when the scores are arranged in ascending or descending order

Measures of Spread

- Range – the difference between the highest and lowest score
- Interquartile Range = Quartile 3 (Q_3) – Quartile 1 (Q_1)
  (IQR) The middle 50% of all scores.
- Standard Deviation – a measure of the dispersion of a set of scores from the mean

Outlier

An outlier is an extreme score that lies outside the other scores. Outliers lie outside the interval

$$Q_1 - 1.5\text{ IQR} \leq \text{score} \leq Q_3 + 1.5\text{ IQR}$$

BIVARIATE DATA

Bivariate data is the study of *two* variable data. Analysis will determine if a relationship or association exists between the two variables.

Explanatory (Independent) variable versus Response (Dependent) variable

The **response** variable measures an outcome of a study.

An **explanatory** variable is the one that explains or influences change in the response variable.

CATEGORICAL DATA

Two-way Frequency Table

Two-way tables are used to investigate the relationship between two categorical variables.

Example

A survey was conducted on the relationship between gender and TV show preference (movies or sport). Results are shown below:

response variable (Sport, Movies); explanatory variable (Male, Female)

	Sport	Movies	Total
Male	41	19	60
Female	14	26	40
Total	55	45	100

Percentage Two-way Table

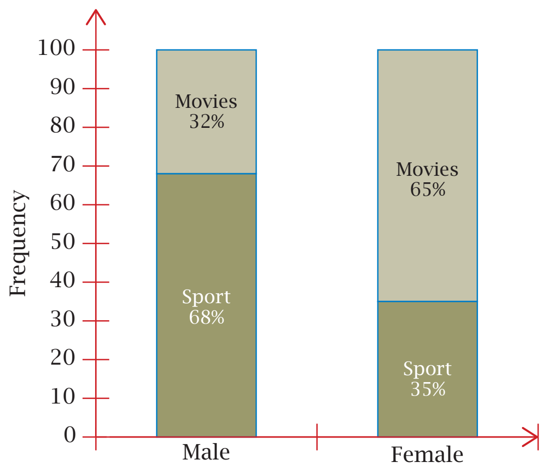
A percentage two-way table will minimise the effect of different gender numbers allowing for better comparisons to be made.

	Sport	Movies	Total
Male	68%	32%	100%
Female	35%	65%	100%

explanatory variable total 100%

Stacked Column Graph

A stacked column graph can be drawn with the *explanatory variable* being the *columns* of the graph.



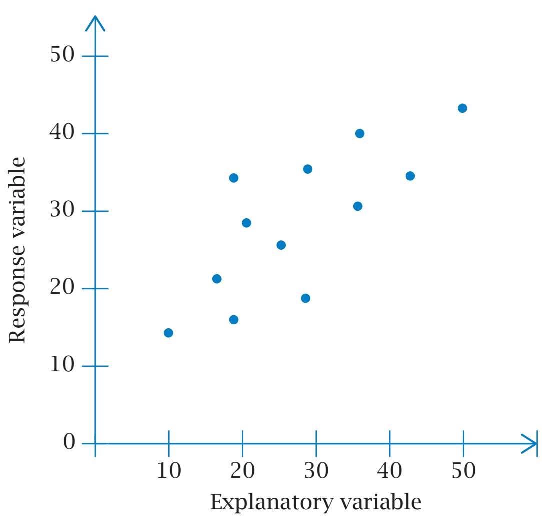
An *association* exists between the two variables if there is a significant difference in the proportions of the variables.

NUMERICAL DATA

Scattergraphs

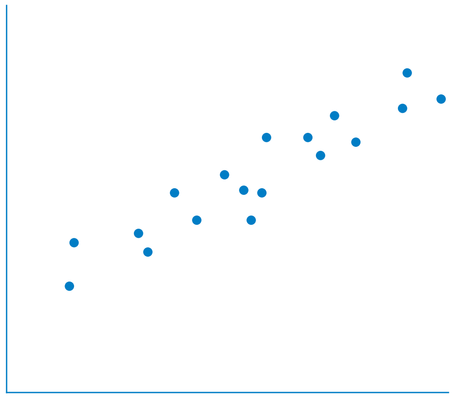
A scattergraph is a graphical display used to compare two numerical variables. The explanatory variable is graphed on the x axis and the response variable on the y axis.

A scattergraph can determine if a correlation or association exists between the two variables.

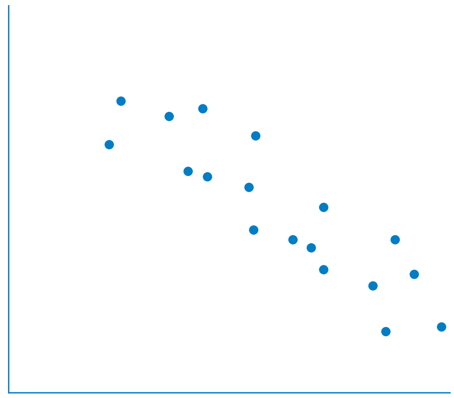


Types of Correlation

– Direction

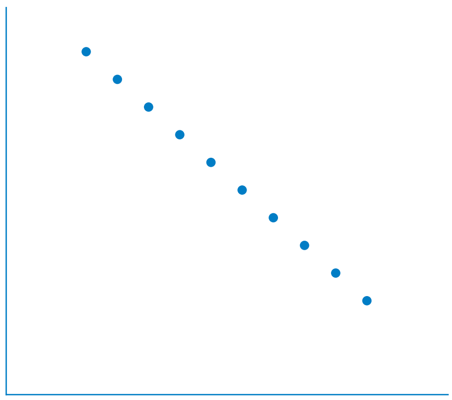


Positive correlation

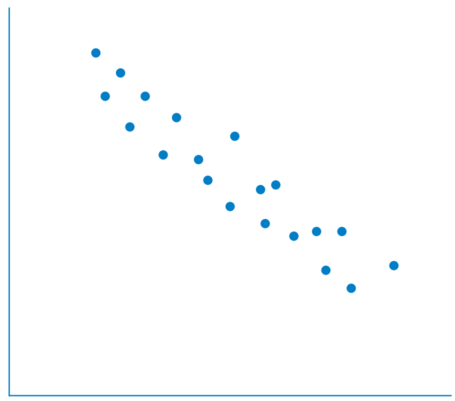


Negative correlation

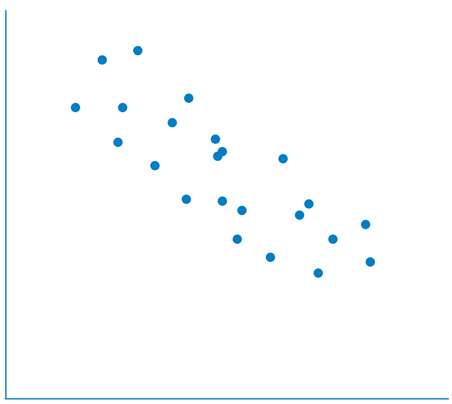
– Strength



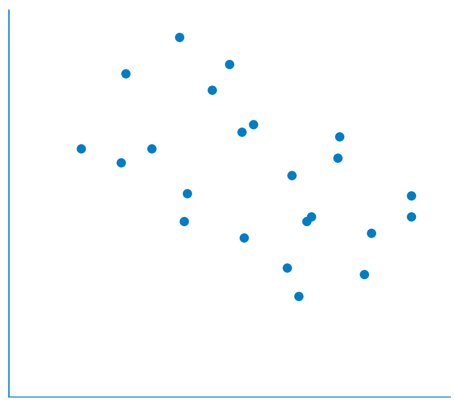
Perfect correlation



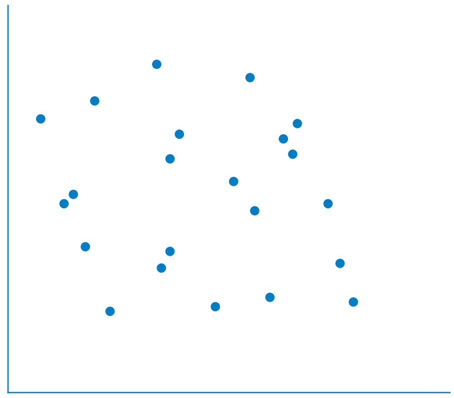
Strong correlation



Moderate correlation



Weak correlation



No relationship

Always comment on *Direction* and *Strength* for the linear model.

A *positive* correlation indicates that an increase in one variable generally results in an increase in the other.

A *negative* correlation indicates that an increase in one variable generally results in a decrease in the other.

Pearson's Correlation Coefficient

The correlation coefficient (r) is a value between –1 and 1. i.e. $-1 \le r \le 1$ and is a measure of the relationship between the two variables. A value of 1 is a perfect positive linear relationship. The correlation coefficient can be found using a *CAS calculator*.

Formula:

$$r_{xy} = \frac{S_{xy}}{S_x S_y}$$

S_{xy} ← Covariance

r_{xy}: Correlation Coefficient between x and y

S_x: Standard deviation of x

S_y: Standard deviation of y

Interpretation of Correlation Coefficients

r	Description
1	Perfect, positive linear relationship
$0.70 \le r < 1$	Strong, positive linear relationship
$0.50 \le r < 0.70$	Moderate, positive linear relationship
$0.30 \le r < 0.50$	Weak, positive linear relationship
$0 \le r < 0.30$	No linear relationship

Worked Examples

1.1 State each of the following variables as:

numerical → discrete / continuous

or

categorical → nominal / ordinal

(a) Eye colour
(b) Weight of a person
(c) Star movie rating
(d) The number of people who live in a house

(a) Categorical and nominal

(b) Numerical and continuous

(c) Categorical and ordinal

(d) Numerical and discrete

1.2 Find

(i) mean
(ii) mode
(iii) median
(iv) range
(v) interquartile range
(vi) standard deviation

(a)

Stem	Leaf
8	0 1
9	3 5 8 9
10	0 0 2 4 5
11	1 7 8
12	2 6

(b)

Score	Frequency
10–14	2
15–19	7
20–24	10
25–29	5
30–34	1

By calculator:

(a) (i) Mean = 103.1875
(ii) Mode = 100
(iii) Median = 101
(iv) Range = 46
(v) IQR = 17.5
(vi) Standard Deviation = 12.788

(b) (i) Mean = 21.2
(ii) Modal Class = 20–24
(iii) Median Class = 20–24
(iv) Range = 20
(v) IQR = 7.5
(vi) Standard Deviation = 4.833

1.3 The following represents the mathematics exam results of 20 year 12 students.

45, 72, 55, 62, 73, 84, 16, 65, 80, 58,
70, 61, 73, 51, 49, 60, 65, 75, 79, 81

(a) Calculate the five number summary.
(b) Construct a boxplot to represent the above data.
(c) Determine if any outliers exist.
(d) Describe the shape of the data.

Using calculator

(a) Minimum = 16

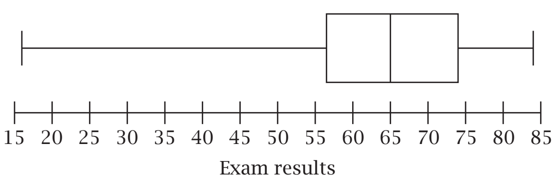
Q_1 = 56.5

Median = 65

Q_3 = 74

Maximum = 84

(b)



(c) *Outliers:*

Outliers lie outside the interval:

$Q_1 - 1.5\text{ IQR} \le x \le Q_3 + 1.5\text{ IQR}$

$Q_1 = 56.5$

$Q_3 = 74$

$\text{IQR} = 17.5$

$\therefore 56.5 - 1.5(17.5) \le x \le 74 + 1.5(17.5)$

$30.25 \le x \le 100.25$

∴ The outlier is 16 as it falls outside the interval.

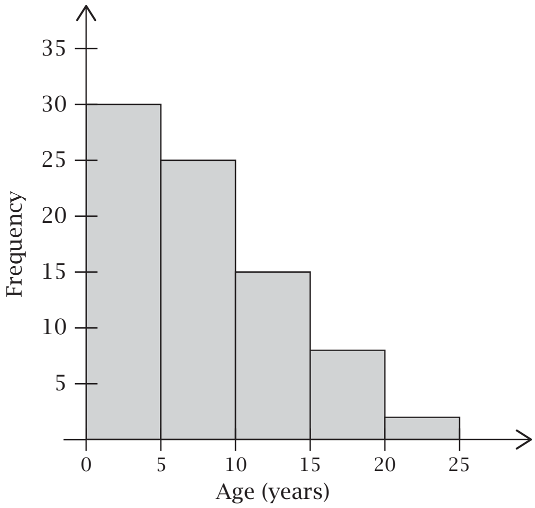
(d) The data is skewed to the left or negatively skewed.

1.4 The following frequency table shows the age of cars (in years) driven to school by students.

Age (in years)	Frequency
0–5	30
5–10	25
10–15	15
15–20	8
20–25	2

(a) Draw a histogram to represent the above data.
(b) Describe the shape of the data.
(c) What percentage of students drive to school in a car that is at least 15 years old?

(a)



(b) The data is skewed positively

(c) Percentage over 15 years old

$$\frac{10}{80} \times 100 = 12.5\%$$

1.5 The number of points scored by two basketballers on opposing teams over a 10 week period are listed below

Player 1: 6, 12, 23, 18, 27, 5, 14, 17, 21, 2

Player 2: 15, 18, 20, 25, 28, 17, 14, 22, 19, 13

(a) Display the data as a back to back stem and leaf plot.

(b) Display the data as parallel boxplots.

(c) Calculate the mean and standard deviation of the number of points scored for each player.

(d) Compare the two players using

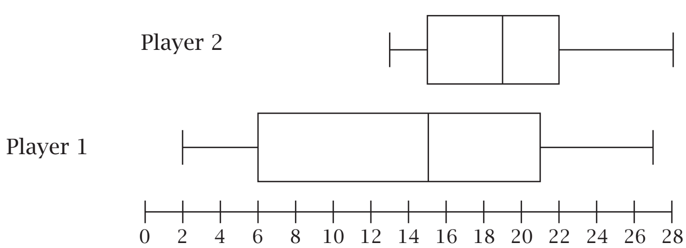
(i) the parallel boxplots.

(ii) the means and standard deviations.

(a)

Player 1		Player 2
6 5 2	0	
4 2	1	3 4
8 7	1	5 7 8 9
3 1	2	0 2
7	2	5 8

(b)



(c)

	Player 1	Player 2
Mean	14.5	19.1
Standard Deviation	7.84	4.57

(d) (i) * Higher median for Player 2 indicates higher scores.
* Smaller range for Player 2 indicating a more consistent player.

(ii) * Higher mean for Player 2 indicating higher scores.
* Lower standard deviation for Player 2 indicating a more consistent player.

1.6 State the explanatory variable and the response variable for each of the following:

(a) Amount of fertiliser and plant growth

(b) Arm length and height

(c) Hours of study and exam results.

(a) Explanatory variable: Amount of fertiliser
Response variable: Plant growth

(b) Explanatory variable: Height
Response variable: Arm length

(c) Explanatory variable: Hours of study
Response variable: Exam results

1.7 A survey was conducted to investigate whether preferred pets were determined by a school year group. The results are shown in the table below:

		Preferred Pet			
		Dog	Cat	Fish	Bird
Year Group	8	57	28	10	7
	9	59	25	7	7

(a) Determine the explanatory and response variable.

(b) Create a percentage two way table.

(c) Construct a segmented column graph appropriate to the explanatory variable.

(d) Is there an association between the variables? Give reasons.

(a) Explanatory variable: Year group
Response variable: Preferred pet

(b) As Year Group is the explanatory variable construct the percentage two-way table with *Year Group* as *columns.*

To calculate percentages

		Preferred Pet				
		Dog	Cat	Fish	Bird	Total
Year Group	8	57	28	10	7	102
	9	59	25	7	7	98

Convert to a %

$$\frac{57}{102} \times 100$$

$= 56\%$

Convert to a %

$$\frac{25}{98} \times 100$$

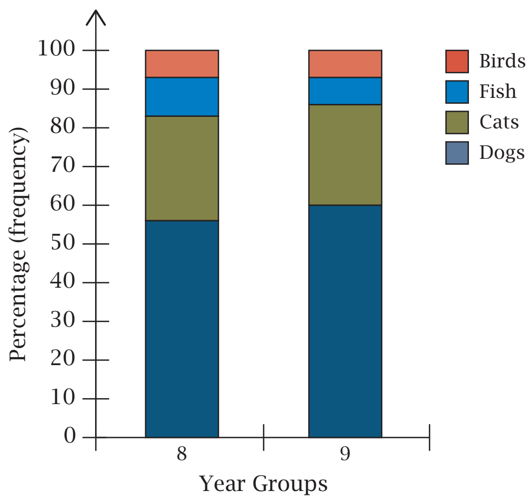
$= 26\%$

Explanatory variable = Year group.

Divide each value by the corresponding *total* of each Year group and multiply by 100.

		Year Group	
		8	9
Preferred Pet	Dog	56%	60%
	Cat	27%	26%
	Fish	10%	7%
	Bird	7%	7%
		100%	100%

(c) *Preferred Pets*



(d) As the proportions are similar in each column there is no association between Year group and preferred pet.

1.8 315 students were interviewed about their favourite subject – Mathematics or English. Of the 170 girls, 70 preferred Mathematics, while 45 boys preferred English.

(a) Construct a two-way table showing the above information.

Does gender influence preferred subject?

(b) Determine the explanatory and response variable.

(c) Construct an appropriate percentage two-way table.

(d) Construct a segmented column graph.

(e) Is there an association between the two variables? Explain.

(a)

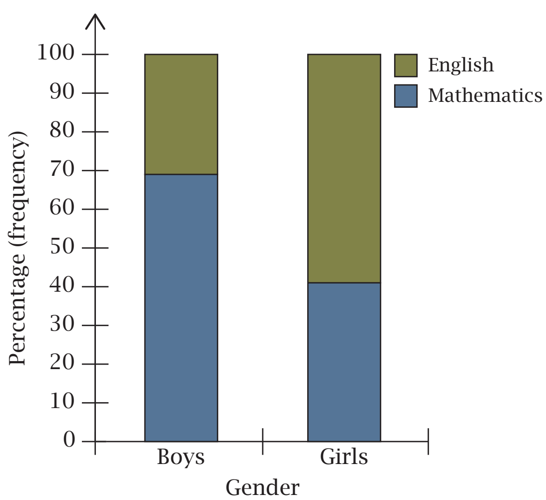
	Mathematics	English	Total
Boys	100	45	145
Girls	70	100	170
Total	170	145	315

(b) Explanatory variable = Gender
Response variable = Favourite subject

(c)

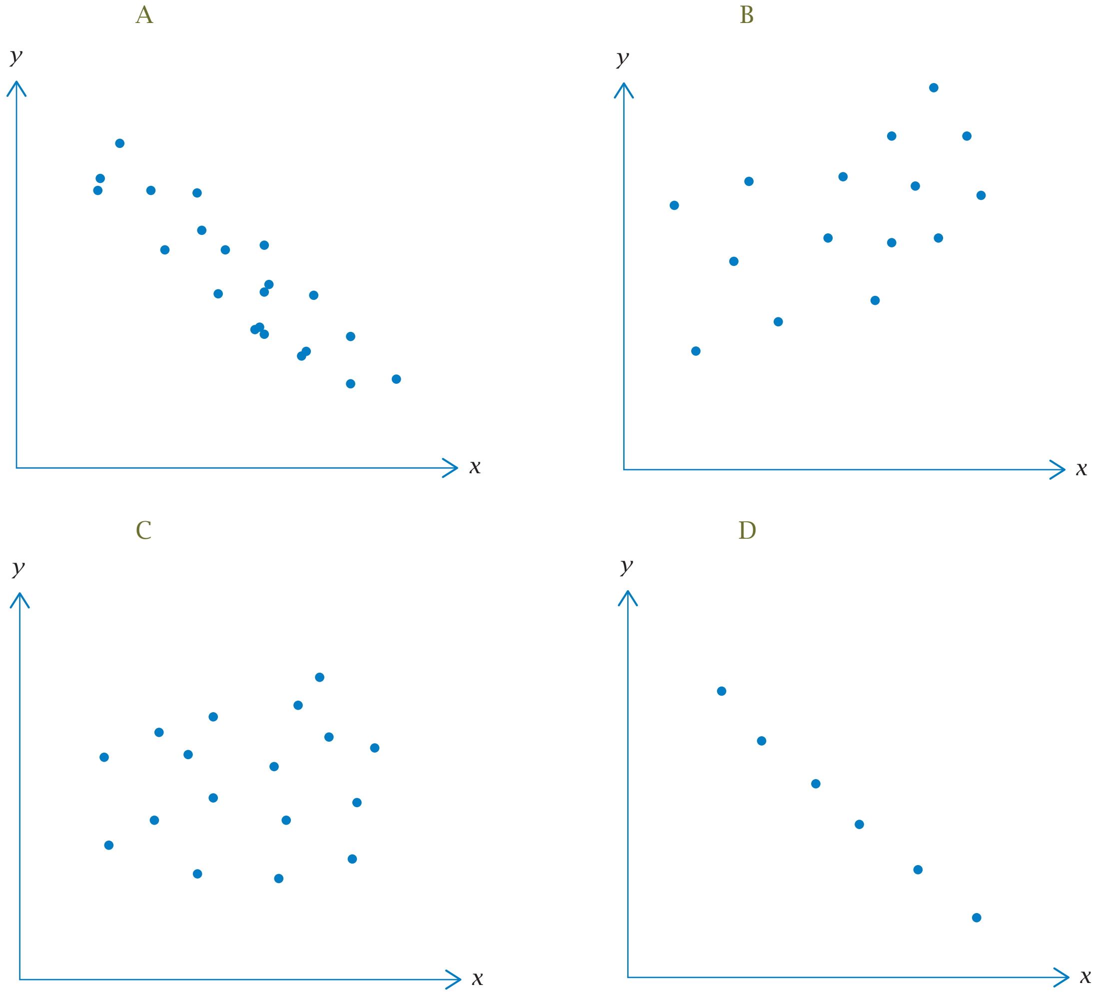
		Gender	
		Boys	Girls
Favourite subject	Mathematics	69%	41%
	English	31%	59%
		100%	100%

(d) Favourite Subject



(e) The change in the proportions from boys to girls suggests that there is an association between favourite subjects and gender. The data suggests that girls prefer English and boys prefer Mathematics.

1.9 Match each of the scattergraphs below with the corresponding correlation coefficient 'r'.



(a) $r_{xy} = 1$

(b) $r_{xy} = -1$

(c) $r_{xy} = -0.05$

(d) $r_{xy} = 0.4$

(e) $r_{xy} = -0.7$

(f) $r_{xy} = 0.8$

Graph A : $r_{xy} = -0.7$

Graph B : $r_{xy} = 0.4$

Graph C : $r_{xy} = -0.05$

Graph D : $r_{xy} = -1$

1.10 Mathematics and Science examination marks were obtained from 10 students and recorded in the table below.

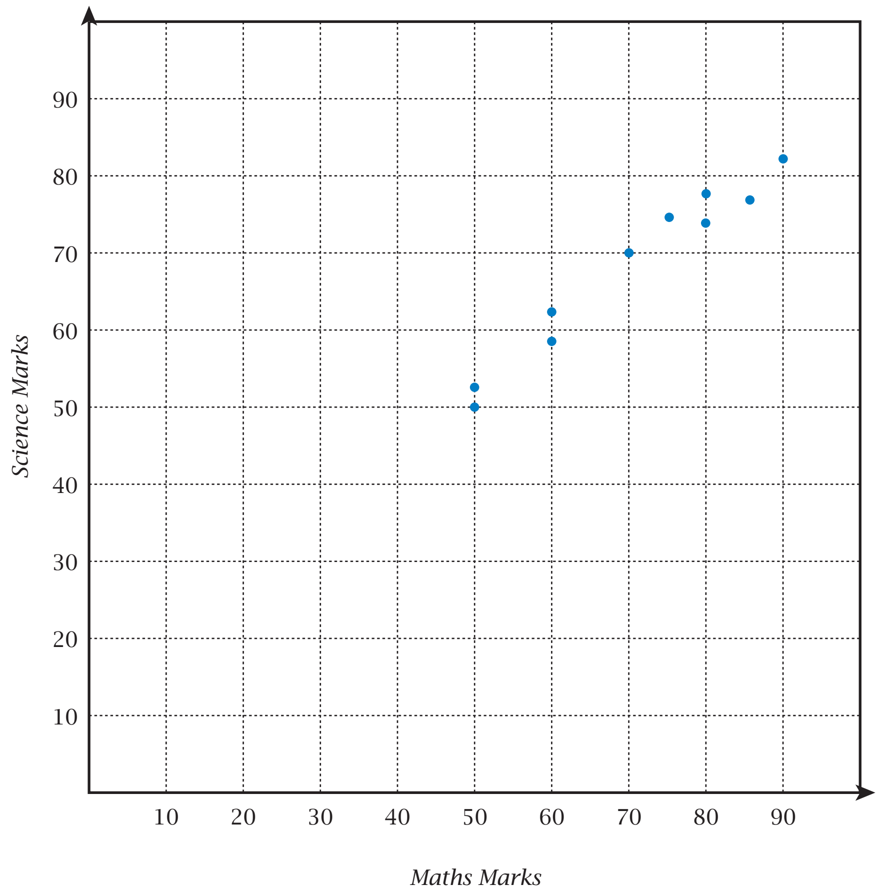
Maths marks	75	60	50	85	80	50	60	70	80	90
Science marks	74	63	54	77	78	50	58	70	74	82

(a) Draw a scatterplot to represent the above information.

(b) Calculate the correlation coefficient 'r'.

(c) Is there a relationship between maths and science marks?

(a)



(b) Correlation coefficient $r = 0.9788$

(c) As $r = 0.9788$, there is a strong, positive linear relationship between maths and science marks

PROBLEMS TO SOLVE

CHAPTER 1: BIVARIATE DATA

1. From the stem and leaf plot below calculate the:
   (a) mean
   (b) median
   (c) mode
   (d) standard deviation
   (e) range
   (f) IQR

Stem	Leaf
3	0 1
4	2 5 8
5	3 7
6	4
7	1 7 8 9
8	2 2
9	7 8 9

2. Individuals attending a rock concert had their ages (in years) recorded. The table below summarises this data.

Age (years)	18–24	25–31	32–38	39–45	46–52	53–59	60–66
Frequency	35	25	20	18	0	1	1

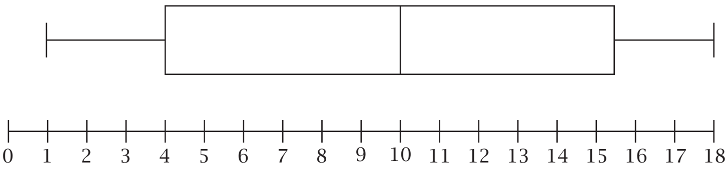
   (a) Calculate the average age of people attending the concert.
   (b) Determine the most common age.
   (c) Calculate the median age of those attending the concert.
   (d) Which indicator of the average age is better? Mean or Median. Why?
   (e) Calculate the standard deviation.

3. Research was conducted on the rental prices in the local suburb. The results for the 20 properties surveyed are listed below:

   $460, $200, $350, $400, $200, $220, $1550, $325, $750, $650,
   $500, $480, $500, $550, $600, $220, $660, $590, $700, $650.

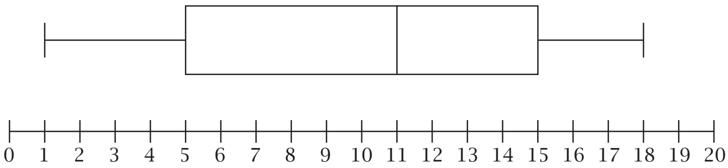
   (a) Draw a boxplot to display the data.
   (b) Describe the shape of the data.
   (c) Determine the highest rental price for the bottom 25% of rental properties.
   (d) Explain how to identify any outliers and calculate the outliers, if any.
   (e) State the effect on the mean and standard deviation if the outliers are removed.

4. The scores in ascending order 1, 3, p, 7, q, 14, r, 16, s are represented below on a median boxplot.



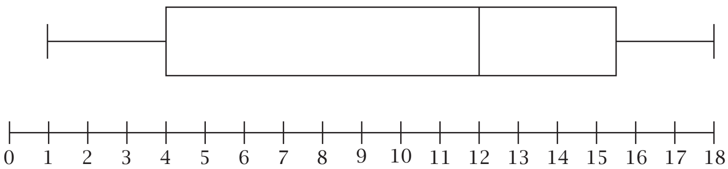
(a) Determine the values of p, q, r and s.

(b) When another score is added the boxplot becomes:



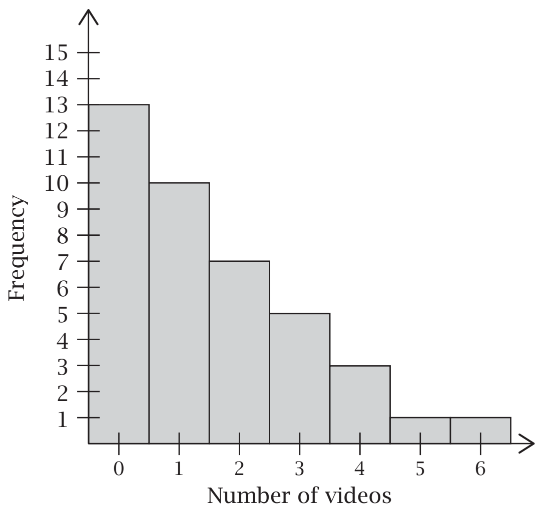
Determine the value of the added score.

(c) When a value is removed from the original set of scores, the boxplot becomes:



Determine the value of the score which has been removed.

5. The number of videos watched by a group of adults over a 1 week period is summarised in the histogram below:



(a) How many adults were surveyed.

(b) Describe the shape of the data.

(c) Calculate the average number of videos watched.

(d) What percentage of adults watched more than 3 videos in the week?

6. Determine the explanatory variable and response variable for each of the following:

(a) Temperature °C and the amount of snow.

(b) Weekly pay and the number of hours worked.

(c) Favourite movie and gender.

(d) Consumption of coffee and heart rate.

7. The results of the Applications exam for 300 students was such that:

- 75% of the boys passed the exam
- 50% of those who passed were girls
- 60% of the students passed the exam.

(a) Complete the following two-way table.

	Passed exam	Did not pass exam	Total
Boys			
Girls			
Total			

(b) Determine the total number of:

(i) boys

(ii) boys who passed the exam

(iii) girls who did not pass the exam

(iv) girls or those who failed the exam

8. 120 students were surveyed on their favourite milk flavour – 'chocolate' or 'other flavours'. The number of males who preferred chocolate milk was three times the number of females who preferred chocolate milk. Overall there were 18 more females surveyed than males and only a total of 24 students preferred chocolate milk.

(a) Draw a two-way table to illustrate the above information.

(b) Determine the number of:

(i) students who preferred 'other flavours'

(ii) females who preferred 'other flavours'

(iii) males who preferred chocolate milk.

9. A sample of 250 students were asked whether they had passed their driving test. The results are shown in the two-way table below:

	Gender		
	Male	Female	Total
Passed test	112	61	173
Failed test	28	49	77
Total	140	110	250

(a) Which is the explanatory variable and which is the response variable?

(b) Convert the two-way table into a percentage two-way table.

(c) Construct a segmented column graph.

(d) Determine if there is an association between the two variables explaining the association.

10. Does age influence the type of movie watched by an individual? A survey was conducted on movie preferences of 391 people. The results are shown below in the two-way table.

	Preferred Movie type					
Age	Animation	Horror	Drama	Thriller	Love Story	Comedy
6–15	20	0	3	2	2	2
16–25	10	12	10	12	8	3
26–35	5	15	12	15	11	11
36–45	2	3	14	20	15	12
46–55	1	2	8	25	15	13
56–65	1	3	8	16	17	12
66–75	1	2	15	12	7	14

(a) What is the explanatory variable and response variable?

(b) Convert the two-way table into a percentage two-way table.

(c) Construct a segmented column graph.

(d) Determine if there is an association between the two variables, explaining the association.

11. Does a specific year group influence whether a student has a part-time job? A survey of 296 students obtained the following information:

- 42 Year 12 students had no job
- The number of students who had no job was 4 fewer than those who had a job
- 73 Year 10 students had a job
- There were twice as many Year 10 students as Year 11 students and 12 more Year 12 students than Year 11 students

(a) Complete the following two-way table

	Job	No Job	Total
Year 10			
Year 11			
Year 12			
Total			

(b) Determine how many Year 11 students had a job.

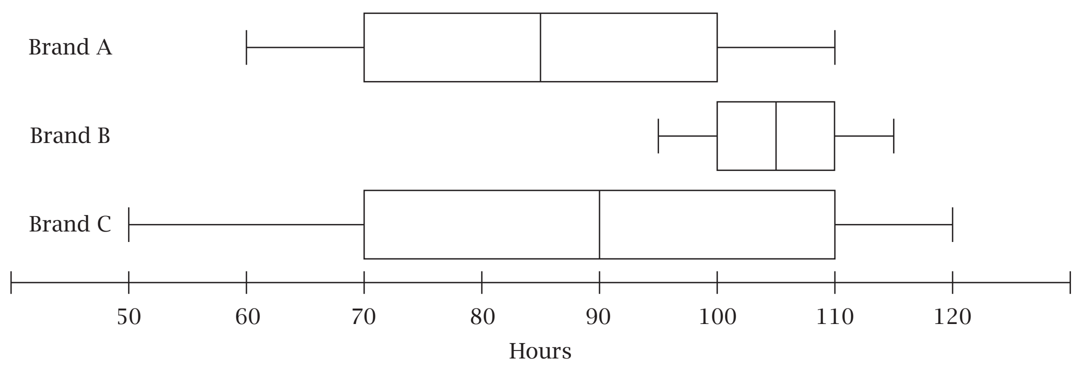
(c) What is the explanatory variable and response variable?

(d) Convert the two-way table above into a percentage two-way table.

(e) Construct a segmented column graph.

(f) Determine if there is an association between the two variables, explaining the association?

12. A study was conducted on 3 brands of light bulbs. The length of time (in hours) each bulb lasted was recorded for each brand and the results displayed as parallel boxplots.



(a) What was the median length of time for Brand B?

(b) What is the range of time for Brand A?

(c) Which light bulb is most effective? Explain.

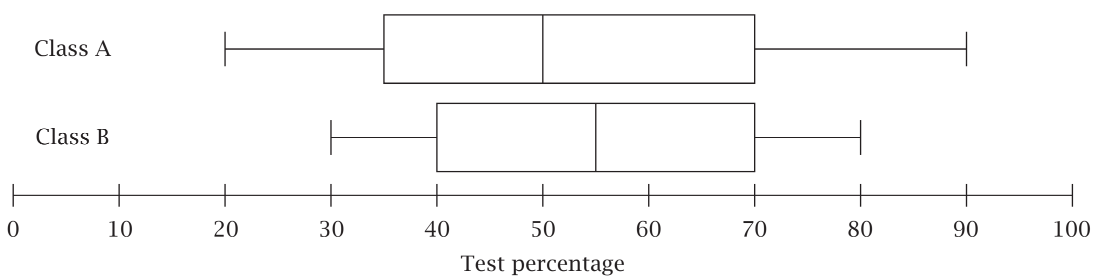
(d) What percentage of bulbs lasted:

(i) between 110 and 120 hours for Brand C?

(ii) less than 100 hours for Brand A?

(e) Is there an association between the brand of the light bulb and how long it lasts?

13. The boxplots below show the test results from Class A and Class B. Each class had the same number of students and an identical test.



Compare the distributions shown in the boxplots above

14. Draw scatterplots for each of the following sets of data. Comment on the relationship that exists between the two quantities?

(a)

Height (cm)	Weight (kg)
174	72
180	84
192	90
170	67
165	73
154	58
183	80
177	84
181	96
190	101

(b)

Maths test %	SOSE test %
56	51
80	32
72	54
64	61
96	37
66	61
84	50
75	60
45	87
90	45

15. The table below shows the predicted and actual ATAR scores for 12 students.

Predicted score	60	85	65	80	40	80	45	72	66	95	50	64
Actual score	68	86	77	79	53	84	60	56	66	97	58	69

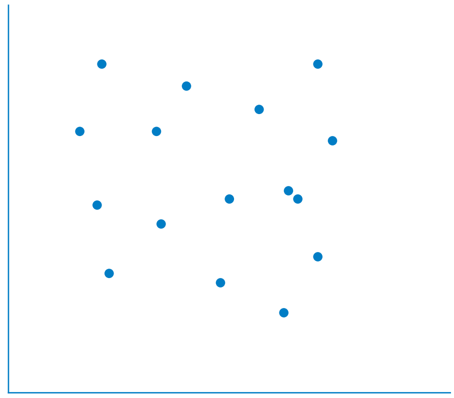
(a) Plot this information on a scatterplot.

(b) Calculate the correlation coefficient and interpret this value.

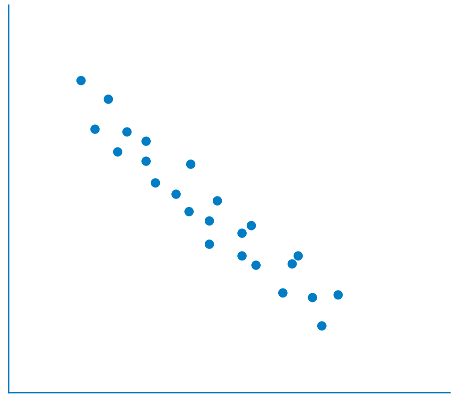
16. From the list below match the correlation coefficient (*r*) with the scatterplot.

1.02, 0, 0.4, –0.9, 1, 0.6

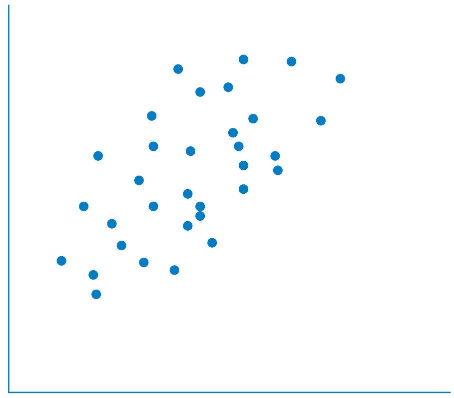
(a)



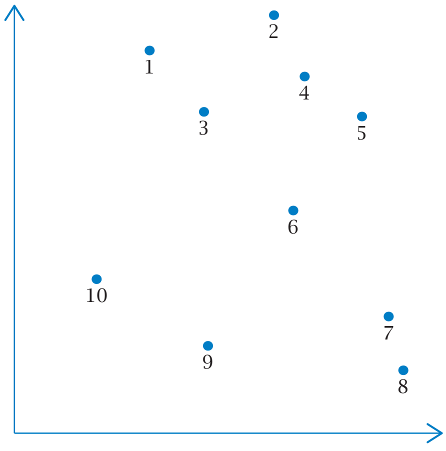
(b)



(c)



17. A set of values labelled 1–10 are plotted on a scattergraph.



Match the correlation coefficients with the appropriate sets.

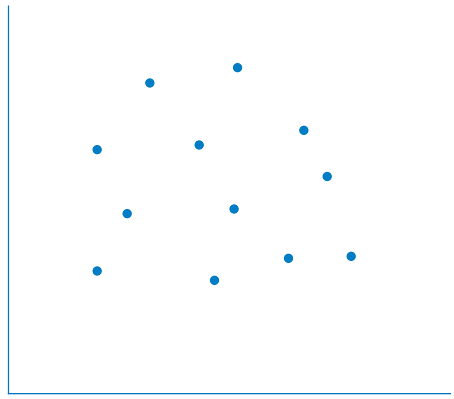
Sets
A = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}
B = {1, 3, 6, 7}
C = {1, 3, 6, 7, 8}
D = {1, 2, 3, 4, 5, 6, 7, 8}
E = {3, 4, 5, 6, 9, 10}

Correlation Coefficients
–1
–0.19
–0.95
1
0.95
–0.72
0.65

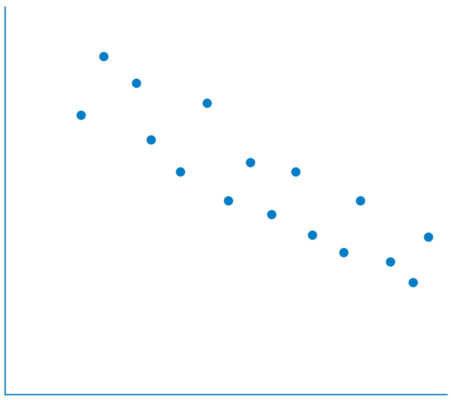
18. The three scattergraphs below represent

A: The height and weight of a person
B: The brand and price of a car
C: A person's weight and the number of pets they own

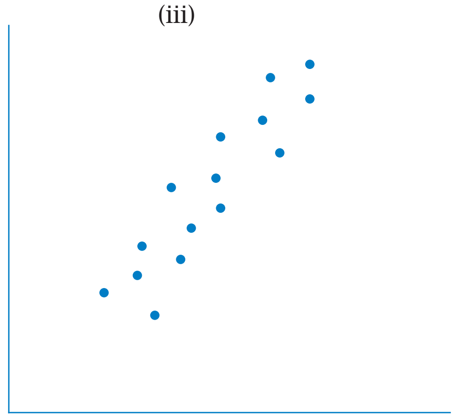
(i)



(ii)



(iii)



Match the graphs and the correlation coefficients to each of the descriptions above.

Correlation Coefficients

–1 1 2 –0.8 0.9 0.12

Graph	(i)	(ii)	(iii)
Description			
Correlation Coefficient			

19. A set of data produced the following summary statistics:

$S_{xy} = 28.925$ $S_x = 12.45$ $S_y = 3.78$

Determine

(a) r_{xy}

(b) r_{xx}

(c) If $S_{xy} = -28.925$ find r_{xy}

(d) the error if $S_y = -3.78$

20. Analysis of annual incomes (in thousands) and house prices (in thousands) produced the following results:

Person	I	II	III	IV	V	VI	VII	VIII	IX	X
Income (I)	92	250	190	100	95	80	75	140	275	170
House Price (P)	250	800	710	320	340	300	270	990	770	500

(a) Draw a scattergraph between income and house prices

(b) Calculate the correlation coefficient r_{IP}

One point on the scattergraph was different to all the others

(c) Which person was it and what is the name for this type of data value?
Remove the data point in (c) above.

(d) Recalculate the new correlation coefficient.

(e) Comment on the relationship between income and house prices.

BIVARIATE DATA – LINEAR MODELS

2

Syllabus Checklist

By the end of this chapter, you should be able to:

- ☐ use a scatterplot to identify the nature of the relationship between variables
- ☐ fit a least squares line to the data
- ☐ use a residual plot to assess the appropriateness of the linear model
- ☐ interpret the intercept and slope of the fitted line
- ☐ use the coefficient of determination to assess the strength of a linear association
- ☐ use the equation of a fitted line to make predictions
- ☐ distinguish between interpolation and extrapolation, recognising the dangers of extrapolation
- ☐ recognise observed association versus causal relationship
- ☐ identify non-causal explanations including coincidence and confounding

FORMULAE AND DEFINITIONS

The Coefficient of Determination: r^2

The coefficient of determination, r^2, is the percent of variation *explained* by the linear relationship.

As r^2 increases in value

- the line of regression becomes more appropriate for the data
- the relationship between the variables becomes stronger.

r^2 is calculated by using a CAS calculator.

$0\% \leq r^2 \leq 100\%$

Note: If $r^2 = k$
$r = \pm\sqrt{k}$

If $r^2 = 0.7$, then 70% of the variation in the dependent variable can be explained by the variation in the independent variable. or
r^2 represents the proportion of the total variation in the dependent variable that can be explained by the linear relationship between the independent and dependent variables.

Regression Line

The line of best fit or least squares regression line reduces the sum of the squares of the deviations of each point from the line.

The equation is:

$$\hat{y} = ax + b \qquad \text{where gradient } a = \frac{Sxy}{S^2_x}$$

$$y \text{ intercept } b = \bar{y} - a\bar{x}$$

$\bar{y}$ = mean of y
$\bar{x}$ = mean of x

This can be calculated using a CAS calculator.

Predictions From a Regression Line

Reliable prediction

The following conditions must be met:

- **Interpolation** – predict within the given data range
- There must be a strong linear correlation between the variables.

Less reliable prediction

- **Extrapolation** – predicting beyond the given data range.

Cause and Effect

A *confounding* or *lurking* variable is a variable that is an additional variable to the explanatory or response variable. This confounding variable can effect the relationship between the other variables.

The confounding variable might effect both the explanatory and response variables. This effect creates an observed association between the explanatory and response variables even though there is *no* 'cause and effect'. This is called a *common* response.

An association **does not** imply causation.

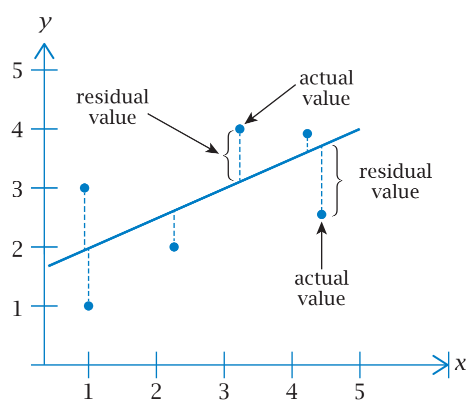
An association between two variables may also be *coincidental*. A person may find that every time they open the front door it starts raining. This is purely coincidental and not some causal relationship between door opening and rain.

Residual Plots

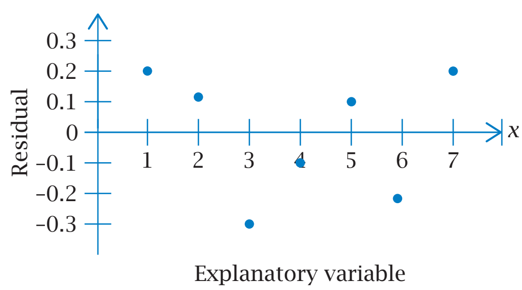
A residual is the difference between the actual y value and the predicted $\hat{y}$ value, i.e. $(y - \hat{y})$

The predicted y value $(\hat{y})$ is calculated using the least squares regression line.

Residuals



Example of a residual plot



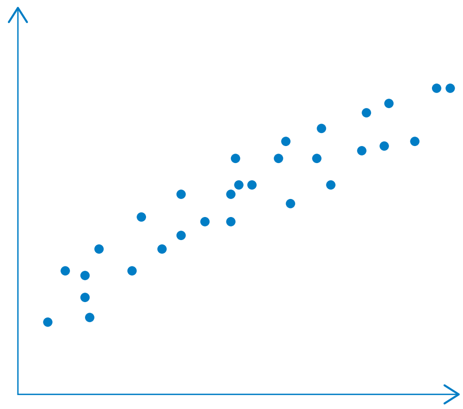
Residual Plot Conclusions

- If the residuals are RANDOM then the linear model would be appropriate.
- If the residuals form a PATTERN then the linear model would *not* be appropriate.

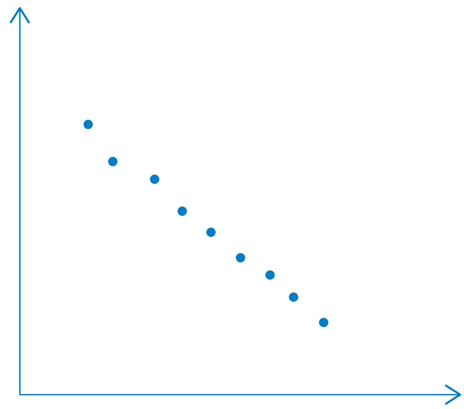
Worked Examples

2.1 Identify the nature and strength of the relationship between the two variables for each scattergraph below.

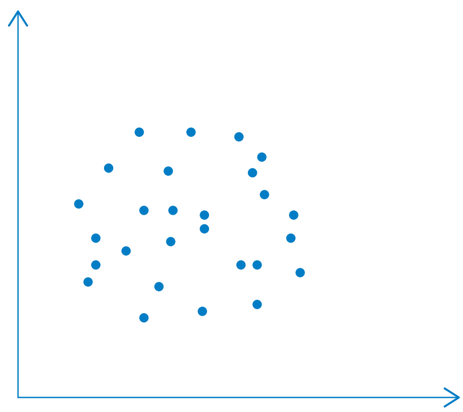
(a)



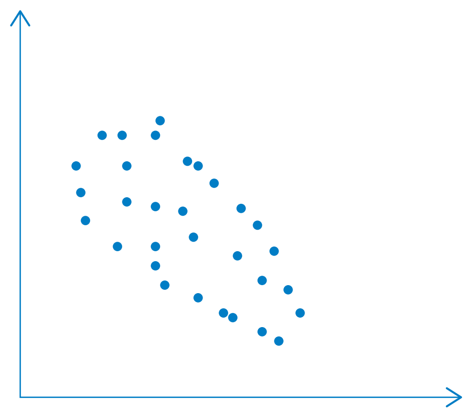
(b)



(c)



(d)



(a) strong, positive linear relationship

(b) perfect, negative linear relationship

(c) no relationship

(d) moderately negative linear relationship

2.2 Interpret the value of the gradient from each of the regression line equations below:

(a) Length of spring (cm) = 48.37 + 0.2891 × mass of object (kg)

(b) Production cost ($) = –0.895 × number of items + 327.2

(a) Gradient = 0.2891
There is an increase of 0.2891 kg for every 1 cm increase in the length of the spring.

(b) Gradient = –0.895
For every $1 increase in production costs there is a 0.895 decrease in the number of items produced.

2.3 The heights and weights of a group of 15 students are shown below:

Height (cm) h	164	173	180	161	182	175	170	188	182	176	171	185	178	161	167
Weight (kg) w	71	74	91	72	86	89	80	95	73	81	72	83	77	63	58

(a) Calculate the correlation coefficient and comment on any relationship that exists between height and weight.

(b) Find the equation of the least squares line of regression.

(c) Calculate the coefficient of determination to establish if the regression line is an appropriate model.

(d) Predict and comment on the reliability for each situation below.

(i) the weight of a student who is 184 cm tall

(ii) the weight of a student who is 200 cm tall

(a) Correlation coefficient:

$$r = 0.7426$$

A moderately strong positive linear relationship.

(b) Regression line:

$$w = 0.8931\ h - 77.9181$$

(c) Coefficient of determination

$$r^2 = 0.5514$$

i.e. 55.14% of the change in weight can be attributed to the change in the height.

A low percentage indicating it may *not* be an appropriate model.

(d) (i) $w = 0.8931\ (184) - 77.9181$

$= 86.4123$ kg

A reliable model as interpolation and a moderately strong correlation coefficient.

(ii) $w = 0.8931\ (200) - 77.9181$

$= 100.7019$ kg

An unreliable model as this is an example of extrapolation.

2.4 The marks for Chemistry and Physics tests were recorded in the table below for 10 students:

Chemistry (c)	75	80	65	87	84	64	71	79	68	69
Physics (p)	82	78	72	91	89	70	80	80	72	69

(a) Draw a scatterplot to represent this information.

(b) Describe the relationship.

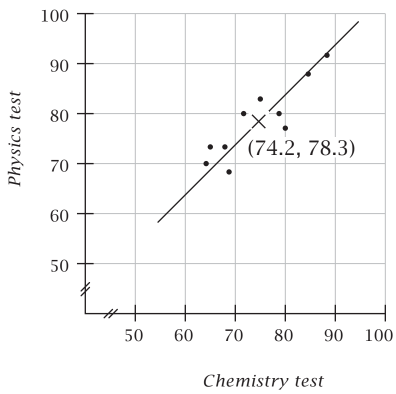
(c) Find the equation of the least squares line of regression and plot this on the scatterplot.

(d) Construct a table of residuals.

(e) Draw a residual plot.

(f) Using the residual plot, decide if the model chosen is appropriate.

(a)



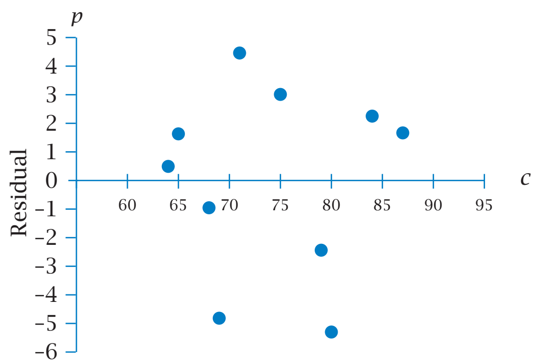
(b) Strong, positive linear relationship

(c) $p = 0.862\ c + 14.332$

(d) Residuals (from calculator)

c	p	Residual
75	82	3.0103
80	78	−5.3
65	72	1.6313
87	91	1.665
84	89	2.2513
64	70	0.4934
71	80	4.4587
79	80	−2.438
68	72	−0.954
69	69	−4.817

(e)



(f) As the residuals form a random pattern, the model is appropriate

2.5 A study found that people with two or more cars live a longer life. Can we conclude that having two or more cars will enable us to live longer? What might be the confounding variable?

There is no 'cause and effect'. The *confounding* variable here might be greater wealth. Wealth to buy multiple cars and obtain the best of everything including medical assistance.

PROBLEMS TO SOLVE

CHAPTER 2: BIVARIATE DATA – LINEAR MODELS

1. Calculate the value of r and r^2 and interpret the result for each set of data below:

(a)

x	5	9	17	18	23
y	15	25	42	52	60

(b)

x	5	6	7	8	9
y	–12	–7	–20	–21	–26

2. A negative correlation exists between the number of icecreams sold and the number of flu cases reported. Does icecream prevent the flu? Comment.

3. There exists a strong positive correlation between the number of televisions and the life expectancy for the world's nations. Does having multiple televisions increase life expectancy? Comment.

4. If the line of regression is:
Weight loss (kilogram) = 0.15 × exercise (hours) + 1.2

(a) Interpret in context the value of the gradient.

(b) Calculate the residual value for Tom who exercises for 5 hours and loses 1.47 kg.

5. The depth of tread (d) in millimetres and the number of kilometres (k) a tyre has travelled has the following statistics:

$r_{dk} = -0.58$

$d = 3.92 - 0.246\,k$

(a) Is there an increasing or decreasing trend in the depth of tread? Why?

(b) What does the least squares line of regression above indicate is the rate of change of the depth of tread?

6. A survey on

- time studying (hrs) (s)
- time watching television (hrs) (t)
- test marks (%) (m)

found the following statistics

$r_{tm} = -0.82$ $r_{sm} = 0.80$

$m = 2.5s + 30$

(a) Does studying increase or decrease test marks?

(b) Does watching television increase or decrease test marks?

(c) Interpret the values of 2.5 and 30 in the least squares line of regression.

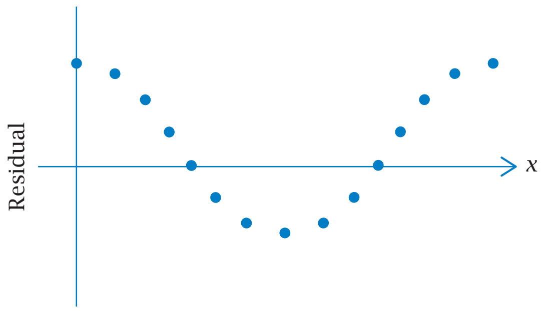
7. Fifteen data points (x, y) had the following statistics:

- $r_{xy} = 0.956$
- Line of regression $y = 3x + 7$

(a) An interpolation was calculated for an x value of 6.
Find the predicted value of y. Comment on the reliability of this prediction.

(b) For an x value of –6 an extrapolation was calculated. Find the predicted value of y. Comment on the reliability of this prediction.

(c) A residual plot for the fifteen points is shown below:



Comment on this residual plot.

8. The linear relationship between x and y has a negative correlation. The least squares regression equation is $y = ax + b$.
Which of the following must be true?

(a) both a and b must be negative

(b) both a and b must be positive

(c) a must be negative

(d) a must be positive

(e) b must be negative

9.

Student	1	2	3	4	5	7	8	9	10	11	12
Test x	35	51	80	76	51	49	88	55	62	70	68
Test y	50	45	76	87	50	61	93	54	70	67	72
Test z	47	50	59	65	37	72	87	47	54	45	65

Twelve students completed three tests. Unfortunately student 6 was absent for test y.

(a) Find

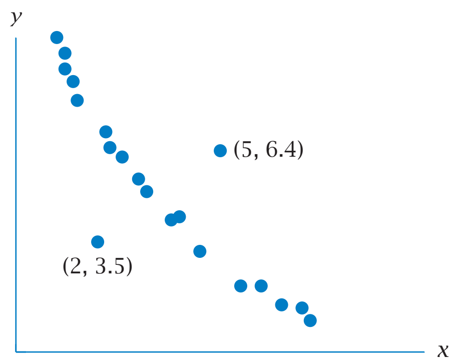
(i) r_{xy}

(ii) r_{xz}

(iii) r_{yz}

(b) Using the best indicator from part (a) estimate the test y mark for student 6 if student 6 scored 75% for test x and 72% for test z.

10. The scattergraph below displays twenty data points (x, y).



The line of regression is $y = -1.25x + 9.7$

$$r_{xy}^2 = 0.8042$$

(a) Comment on the correlation of determination and whether the regression line is appropriate.

(b) Determine r_{xy} and comment on the relationship that exists between x and y.

(c) Determine:

(i) the predicted value of y when the x value is 5

(ii) the residual value

(d) If points (5, 6.4) and (2, 3.5) are removed, determine the effect on the correlation coefficient.

(e) If $\bar{x} = 3.55$ determine the $\bar{y}$ value.

11. Data was gathered over a 24 hour period comparing the hours of daylight (d) and the number of accidents (a). This data is shown in the table below:

Hours of daylight (d)	9	9.5	10	11	11.5	12	12.5	13
Number of accidents (a)	220	195	198	184	172	157	150	132

(a) Calculate the correlation coefficient and comment on this value and the association between the two variables.

(b) Determine the least squares line of regression of a on d.

(c) If the number of hours of daylight decreased by 3 hours what is the predicted change in the number of accidents.

(d) Determine the number of accidents if the number of daylight hours is 16. Comment on the reliability of this prediction.

(e) The data suggests that the number of daylight hours causes the number of accidents. Comment.

The number of serious injuries (i) and the number of accidents (a) is linked by the equation

$$i + 0.6a = 1400$$

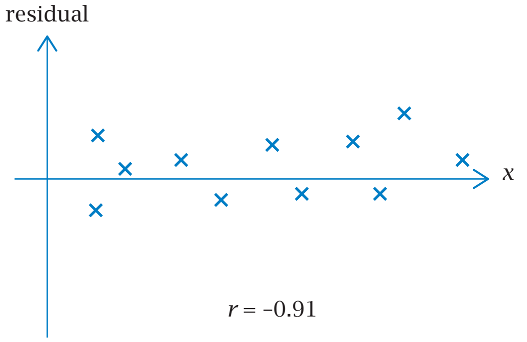
Determine

(f) r_{ia}

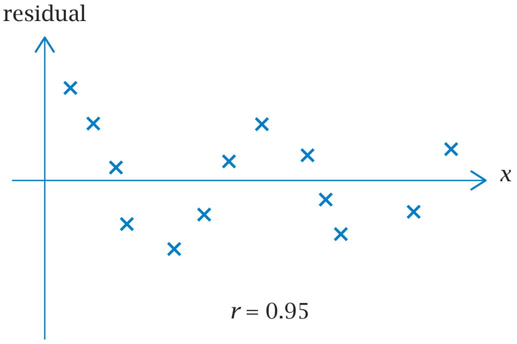
(g) r_{id}

12. The following are residual plots with an associated correlation coefficient.

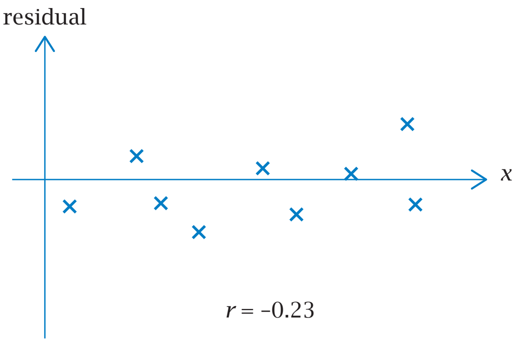
(a)



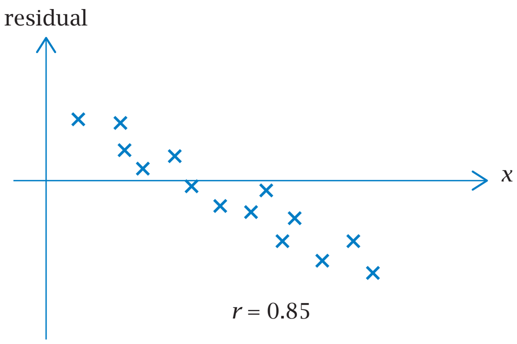
(b)



(c)



(d)



Comment on whether the linear model is appropriate for each situation above.

13. Research was conducted on the relationship between a high jumper's age, height (metres) and average height jumped (metres)

Athlete	Age (a)	Height (h)	Average height jumped (j)
A	24	1.67	1.44
B	34	1.95	2.20
C	16	1.85	2.40
D	39	1.63	1.20
E	37	1.94	1.88
F	35	1.90	2.01
G	41	1.90	1.82
H	19	1.74	1.60
I	27	1.84	1.98
J	21	1.80	1.91

Some statistics are given below:

$r_{aj} = -0.252$

$j = -0.00991a + 2.134$

(a) Calculate r_{hj} and the linear model that can be used to predict average height jumped using an athlete's height.

(b) Which is the better predictor for calculating the average height jumped? Why?

(c) Athlete C grows 20 cm. How much will the average height jumped change by?

(d) Simon a 12 year old athlete is 1.4 metres tall. Which model is suitable for predicting Simon's average height jumped? Explain.

(e) Comment on the validity of this statement. Taller athletes can jump higher.

14. Given the two variables p and q produced the following results:

p	3	4	4.5	5	5.5	6	6.5	7	7.5
q	0.1	0.2	0.25	0.32	0.33	0.35	0.47	0.49	0.53

(a) Calculate the correlation coefficient r_{pq} and comment on its value.

(b) Find the least squares regression line and use it to predict q when $p = 3.5$. Comment on the reliability of this prediction.

(c) Draw a neat sketch of the residuals for the linear regression model above. Comment on the suitability of this model.

15. The temperature and growth rate of plants in a greenhouse were measured as an experiment with the following results:

Temperature (t) °C	16	18	20	22	24	26	28
Growth rate (r)	4	8	9.5	12	15	15	14

(a) What is the response variable?

(b) Draw a scattergraph to represent the data.

(c) Calculate the correlation coefficient r_{tr}.

(d) Find the least squares line of regression and draw this on the scattergraph.

(e) If the temperature of the greenhouse increased by 1˚C what is the predicted increase in growth rate?

(f) What is the predicted growth rate of a plant in a greenhouse kept at 33˚C?

(g) Comment on the reliability of your answer to part (f).

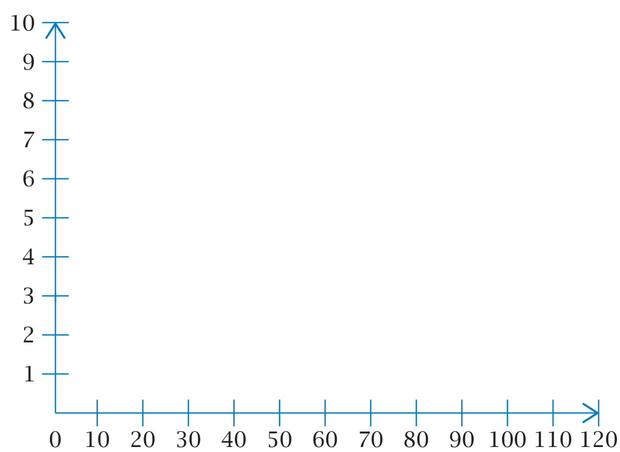
(h) Calculate the coefficient of determination.

(i) What percentage of the variation in the growth rate of plants in the greenhouse can be explained by the variation in temperature (˚C)?

16. The following table gives the average number of visits to fast food outlets per month compared to family incomes in thousand dollars.

Income (I) thousands $	40	50	60	65	70	75	80	85	90	95	100	110	120
Average visits to fast food monthly (F)	9.7	9.5	9.1	8.8	8.3	7.7	6.9	6.3	5.9	4.7	3.6	2.5	1.8

(a) Plot the values on the scattergraph below labelling the axes.



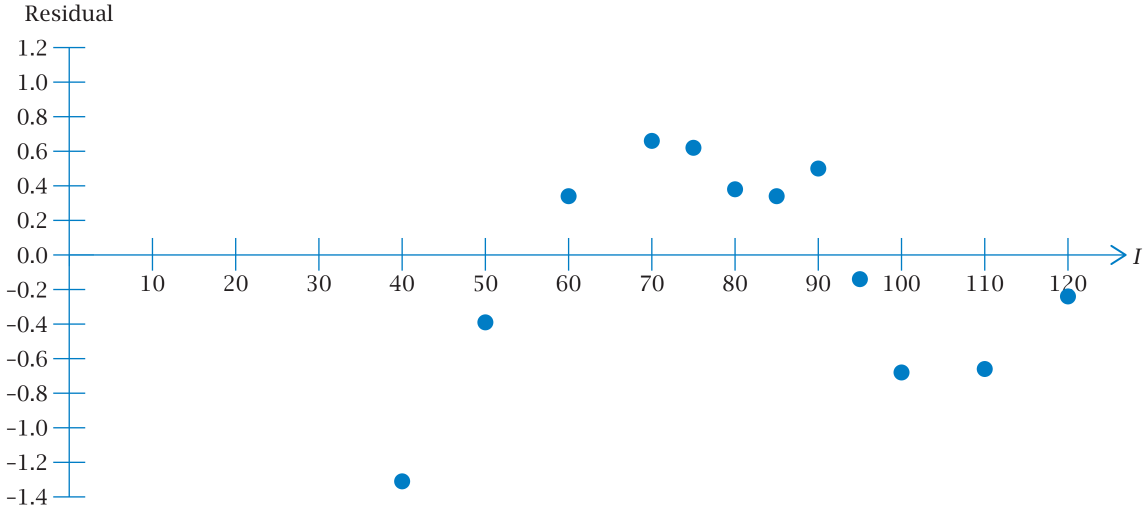
(b) What is the response variable?

(c) Calculate the correlation coefficient r_{IF}.

(d) Determine the least squares regression line and plot this on the scattergraph above.

(e) Comment on the relationship between income and visits to fast food outlets.

(f) Calculate the residual value for $I = 65$ and add this to the residual plot below.



(g) Comment on the appropriateness of the linear model.

(h) Calculate the correlation of determination.

(i) What percentage of the variation in visits to fast food outlets can be explained by the variation in income?

(j) What is the estimated average number of monthly visits for families with an income of \$55 000?

(k) If a family's income increased by \$5000 what would be the expected change in the number of monthly visits to fast food outlets?

(l) Predict the average number of monthly visits to fast food outlets with a family whose combined income is \$200 000.
Interpret this result.

SEQUENCES – ARITHMETIC, GEOMETRIC & RECURSIVE

3

Syllabus Checklist

By the end of this chapter, you should be able to:

- ☐ use recursion to generate an arithmetic or geometric sequence
- ☐ display the terms of an arithmetic or geometric sequence in both tabular and graphical form and apply in discrete situations
- ☐ deduce a rule for the *n*th term of an arithmetic or geometric sequence and use the rule to make prediction
- ☐ use arithmetic or geometric sequences in growth and decay situations
- ☐ use a general first order linear occurrence relation to generate the terms of a sequence and hence display in both tabular and graphical form
- ☐ generate a sequence defined by a first order linear recurrence relation that gives long term increasing, decreasing or steady state solutions
- ☐ use first order recurrence relations in practical problems

FORMULAE AND DEFINITIONS

Recursive Formulae

A recursive formula enables the next term of a sequence to be obtained from the previous term(s). An example is:

$T_{n+1} = T_n + 3, T_1 = 4$ $\quad$ T_1: *1st term of the sequence*

This generates the sequence

4, 7, 10, 13, …

The sequence can also be represented recursively in other forms such as:

$$T_n = T_{n-1} + 3,\ T_1 = 4$$

$$T_{n+2} = T_{n+1} + 3,\ T_1 = 4$$

$$T_{n-1} = T_{n-2} + 3,\ T_1 = 4$$

Arithmetic Sequence (Arithmetic Progression)

An arithmetic sequence is one where the *difference* between successive terms is constant. These sequences are of the form:

$a, a + d, a + 2d, a + 3d, \ldots$

where a = first term, T_1
d = common difference (($T_{n+1} - T_n$) where T_n and T_{n+1} are successive or consecutive terms)

A *graph* of an arithmetic sequence will give points that lie in a *straight* line. Slope of the line equals d.

General form: $T_n = a + (n - 1)\, d$

Recursive form: $T_{n+1} = T_n + d, T_1 = a$ $\quad$ (Recurrence relation)

Geometric Sequence (Geometric Progression)

- Each term in a geometric sequence is obtained by multiplying its predecessor by a constant.
- That is, the ratio of successive terms is constant.

$$a, ar, ar^2, ar^3, \ldots$$

where a = first term, T_1

r = common ratio $\left(\frac{T_{n+1}}{T_n}\right)$ where T_n and T_{n+1} are successive or consecutive terms

A *graph* of a geometric sequence will give points that lie in the shape of an *exponential* graph.

General form: $T_n = ar^{n-1}$

Recursive form: $T_{n+1} = r \cdot T_n, T_1 = a$

n = required term number

First Order Recurrence Relation

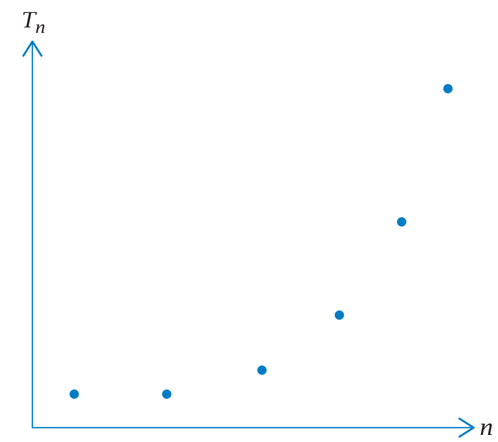
First order recurrence relations are of the form:

$$T_{n+1} = rt_n + d, T_1 = a$$

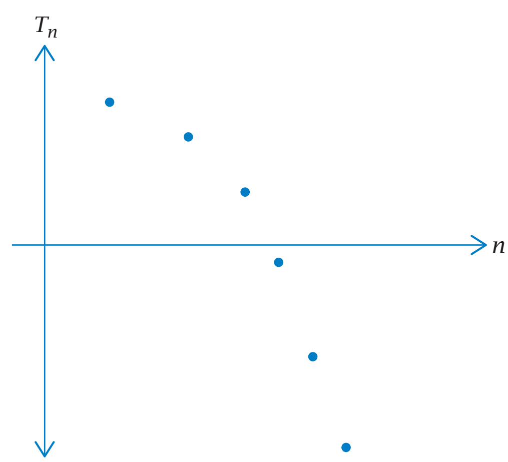
These sequences are neither arithmetic nor geometric.

The terms of these sequences may:

- increase indefinitely

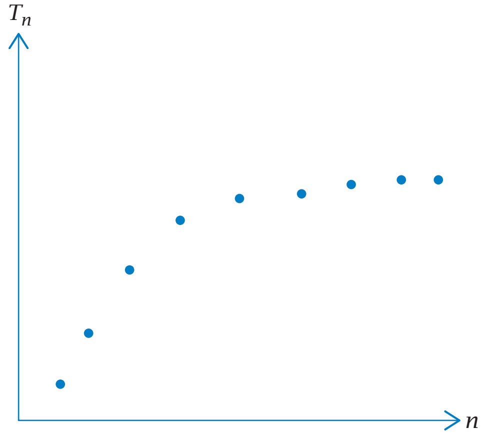


- decrease indefinitely



- approach a particular value.

This sequence has a long term *steady state* solution. That is, the sequence will approach a limiting value as $n \to \infty$. This will occur for $-1 < r < 1$ when $T_{n+1} = rT_n + d$, $T_1 = a$

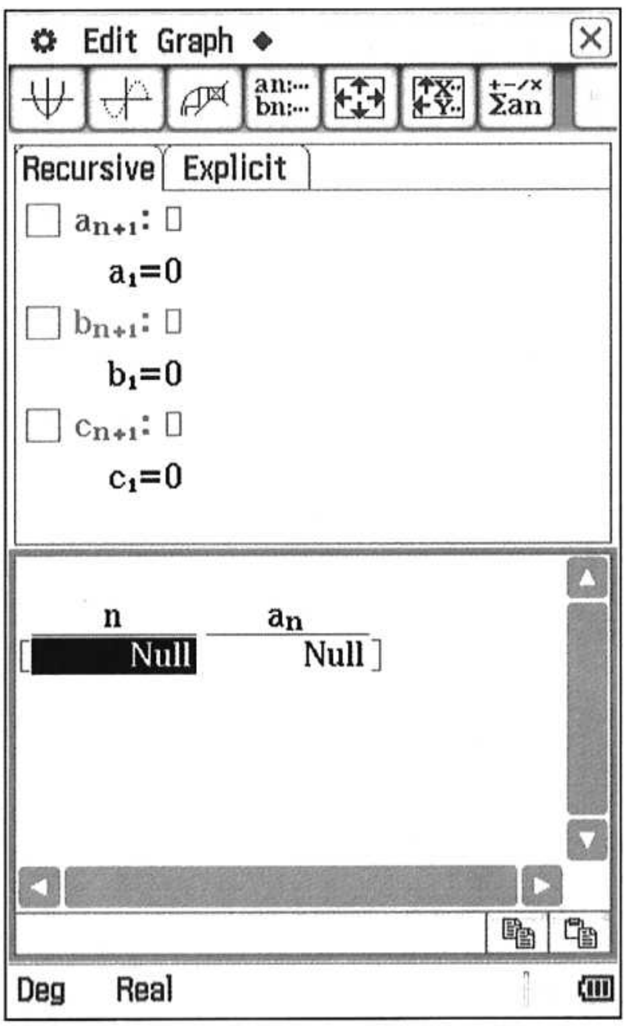


Calculating the Steady State Solution

In the long run:

$$T_{n+1} = T_n$$

The *sequence* application on a CAS calculator can be used to generate the terms of a sequence.



Worked Examples

3.1 For the following recursive equations **describe** the sequence in words.

(a) $T_{n+1} = T_n + 7$

(b) $T_n = T_{n-1} \times 2$

(c) $T_{n+3} = T_{n+2} + T_{n+1} + T_n$

(a) Each term is obtained by adding 7 to the previous term.

(b) Each term is obtained by multiplying the previous term by 2.

(c) Each term is obtained by adding each of the previous three terms.

3.2 State a recursive formula for the following sequences.

(a) 30, 27, 24, 21, …

(b) 1, 1, 2, 3, 5, 8, …

(c) 100, 50, 25, …

(a) 30, 27, 24, 21

−3 −3 −3

$T_{n+1} = T_n - 3, \qquad T_1 = 30$

(b) $T_{n+2} = T_{n+1} + T_n \quad , \quad T_1 = 1, T_2 = 1$

(c) 100, 50, 25

$\times\frac{1}{2} \quad \times\frac{1}{2}$

$T_{n+1} = \frac{1}{2}T_n, \qquad T_1 = 100$

3.3 Determine the first five terms for each of the following recursive formulas

(a) $T_{n+1} = T_n + 7,\ T_1 = 12$

(b) $T_n = 2T_{n-1} + 3,\ T_1 = 5$

(c) $T_{n+2} = T_{n+1} + T_n \quad , \quad T_1 = 3, T_2 = 7$

(a) $T_{n+1} = T_n + 7,\ T_1 = 12$

Let $n = 1$

$T_2 = T_1 + 7$

$T_2 = 12 + 7$

$T_2 = 19$

Let $n = 2$

$T_3 = T_2 + 7$

$T_3 = 19 + 7$

$T_3 = 26$

Let $n = 3$

$T_4 = T_3 + 7$

$T_4 = 26 + 7$

$T_4 = 33$

Let $n = 4$

$T_5 = T_4 + 7$

$T_5 = 33 + 7$

$T_5 = 40$

First 5 terms: 12, 19, 26, 33, 40

(b) $T_n = 2T_{n-1} + 3,\ T_1 = 5$

$T_2 = 2T_1 + 3$

$T_2 = 2(5) + 3$

$T_2 = 13$

$T_3 = 2T_2 + 3$

$T_3 = 2(13) + 3$

$T_3 = 29$

$T_4 = 2T_3 + 3$

$T_4 = 2(29) + 3$

$T_4 = 61$

$T_5 = 2T_4 + 3$

$T_5 = 2(61) + 3$

$T_5 = 125$

First 5 terms: 5, 13, 29, 61, 125

(c) $T_{n+2} = T_{n+1} + T_n \quad , \quad T_1 = 3,\ T_2 = 7$

$T_3 = T_2 + T_1$

$T_3 = 7 + 3$

$T_3 = 10$

$T_4 = T_3 + T_2$

$T_4 = 10 + 7$

$T_4 = 17$

$T_5 = T_4 + T_3$

$T_5 = 17 + 10$

$T_5 = 27$

First 5 terms: 3, 7, 10, 17, 27

3.4 Find the value of k and hence determine T_3 for each recursive formula below:

(a) $T_{n+1} = 2T_n + k \qquad T_1 = 6,\ T_2 = 16$

(b) $T_n = k\,T_{n-1} + 4 \qquad T_1 = 8,\ T_2 = 20$

(a) $T_{n+1} = 2T_n + k$

$T_2 = 2T_1 + k$

$16 = 2(6) + k$

$16 = 12 + k$

$k = 4$

$\therefore\ T_{n+1} = 2T_n + 4$

$T_3 = 2T_2 + 4$

$= 2(16) + 4$

$\therefore\ T_3 = 36$

(b) $T_n = k\,T_{n-1} + 4$

$T_2 = k\,T_1 + 4$

$20 = k(8) + 4$

$\frac{16}{8} = k$

$k = 2$

$\therefore\ T_n = 2T_{n-1} + 4$

$T_3 = 2T_2 + 4$

$= 2(20) + 4$

$\therefore\ T_3 = 44$

3.5 A sequence has a recursive formula given by:

$T_{n+1} = 0.6T_n \qquad T_1 = 150$

(a) Determine the first four terms of the sequence.

(b) What percentage increase or decrease occurs with each successive term?

(a) 150, 90, 54, 32.4

(b) Percentage decrease of 40% (multiplying each previous term by 0.6 is a decrease of 40%).

3.6 Decide whether the following sequences are arithmetic, geometric or neither.

(a) 2, 5, 8, 11, …

(b) 1, 4, 8, 12, 15, 20, …

(c) 1, 1, 2, 3, 5, 8, …

(d) 2, 6, 18, 54, …

(e) $T_{n+1} = 2T_n$, $T_1 = 10$

(f) $T_{n+1} = T_n - 5$, $T_1 = 35$

(g) $T_{n+1} = 2T_n - 3$, $T_1 = 1$

(a) 2, 5, 8, 11, … Arithmetic

+3 +3 +3

(b) 1, 4, 8, 12, 15, 20, … Neither

(c) 1, 1, 2, 3, 5, 8, … Neither

(d) 2, 6, 18, 54 Geometric

×3 ×3 ×3

(e) $T_{n+1} = 2T_n$, $T_1 = 10$

Sequence: 10, 20, 40, … Geometric

×2 ×2

(f) $T_{n+1} = T_n - 5$, $T_1 = 35$

Sequence: 35, 30, 25, 20, … Arithmetic

–5 –5 –5

(g) $T_{n+1} = 2T_n - 3$, $T_1 = 1$

Sequence: 1, –1, –5, –13, –29 Neither

3.7 Given the recursive rule

(a) $T_n = (-1)^n (T_{n-1} - 3)$
where $T_1 = 5$, find the first 4 terms

$T_n = (-1)^n (T_{n-1} - 3)$

Let $n = 2$

$T_2 = (-1)^2 (T_1 - 3)$

$T_2 = (1)(2)$

$T_2 = 2$

Let $n = 3$

$T_3 = (-1)^3 (T_2 - 3)$

$T_3 = (-1)(-1)$

$T_3 = 1$

Let $n = 4$

$T_4 = (-1)^4 (T_3 - 3)$

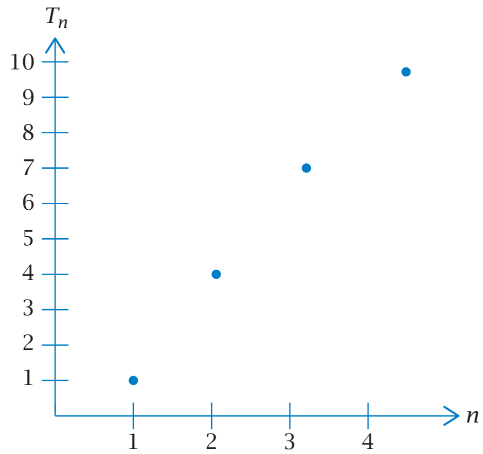
$T_4 = (1)(-2)$

$T_4 = -2$

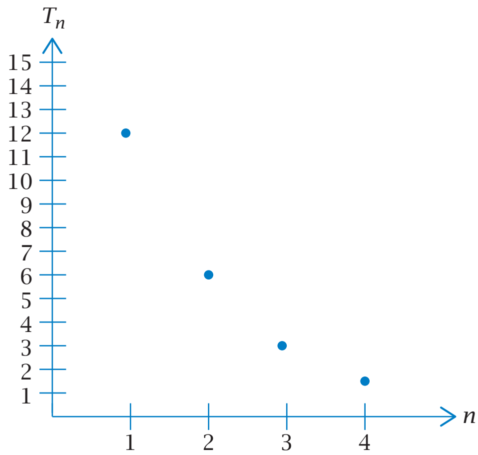
Terms: 5, 2, 1, –2

3.8 Find the recursive formula (recurrence relation) that generates the following sequences:

(a)



(b)



(a) Points lie in a straight line hence an arithmetic sequence with recurrence relation $T_{n+1} = T_n + d,\ T_1 = a$

d = difference a = first term

$d = 3$ $a = 1$

$\therefore\ T_{n+1} = T_n + 3,\ T_1 = 1$

(b) Points form a curve.

Sequence is 12, 6, 3, 1.5, …

r = common ratio a = first term

$r = \frac{1}{2}$ $a = 12$

$\therefore$ geometric sequence

$T_{n+1} = r\,.\,T_n \quad , \quad T_1 = a$

$\therefore\ T_{n+1} = \frac{1}{2}\,T_n \quad , \quad T_1 = 12$

3.9 For each of the following first order recurrence relations determine if they have a long term:

(i) increasing
(ii) decreasing or
(iii) steady state solution

(a) $T_{n+1} = 0.2\ T_n + 2,\ T_1 = 10$
(b) $T_{n+1} = 3\ T_n + 1,\ T_1 = 2$
(c) $T_{n+1} = 2.2\ T_n - 3,\ T_1 = -1$

Using a CAS calculator and either *finding* the first ten terms or *plotting* the first few terms.

(a) $T_{n+1} = 0.2\ T_n + 2,\ T_1 = 10$
long term *steady state* solution

(b) $T_{n+1} = 3\ T_n + 1,\ T_1 = 2$
long term *increasing* solution

(c) $T_{n+1} = 2.2\ T_n - 3,\ T_1 = -1$
long term *decreasing* solution

3.10 Debbie borrows $30 000 to buy a new car. The interest payable is calculated on the balance of the loan each month. Interest is 15% p.a. compounded monthly and after interest is calculated, Debbie repays $850 per month.

The recurrence relation

$T_{n+1} = p \cdot T_n + q \quad , \quad T_1 = a$

models this situation on the amount owing at the end of the *n*th amount.

(a) Determine the values *p*, *q* and *a*.

(b) How much is still owing at the start of the 6th month?

(c) At the start of which month does the balance first fall below $15 000.

(a) $T_{n+1} = p \cdot T_n + q \quad , \quad T_1 = a$

a = first term (original value)

= $30 000

Compound interest (per month)

$= \dfrac{15\%}{12}$

= 1.25%

p = 1.0125

q = repayment per month

= –$850

$\therefore T_{n+1} = 1.0125\ T_n - 850$

T_1 = $30 000

(b) At the start of the 6th month using CAS calculator
$27 564.88

(c) Using the CAS calculator
Balance first falls below $15 000 at the end of the 27th month or start of the 28th.

3.11 The sixth term of an arithmetic sequence is 12 and the eighth term is 22.

Determine:

(a) the second term

(b) the recursive formula for this sequence.

(a) $T_n = a + (n - 1)d$ $\quad T_n = a + (n - 1)d$

$T_6 = a + (6 - 1)d$ $\quad T_8 = a + (8 - 1)d$

$\boxed{12 = a + 5d}$ $\quad \boxed{22 = a + 7d}$

Solve simultaneously using the CAS calculator

$a = -13,\ d = 5$

$T_2 = -13 + (2 - 1)(5)$

$T_2 = -8$

(b) The sequence is

$-13, -8, -3, \ldots$

$\therefore T_{n+1} = T_n - 5,\ T_1 = -13$

3.12 The fifth term of a geometric progression is 81 and the second term is 24. Find the:

(a) fourth term

(b) recursive formula for this sequence

(a) $T_n = ar^{n-1}$ $\quad T_n = ar^{n-1}$

$T_5 = ar^{5-1}$ $\quad T_2 = ar^{2-1}$

$\boxed{81 = ar^4}$ $\quad \boxed{24 = ar}$

Solve simultaneously using the CAS calculator

$a = 16,$ $\quad r = \frac{3}{2}$

$T_4 = 16\left(\frac{3}{2}\right)^{4-1}$

$T_4 = 54$

(b) Sequence is

$16, 24, 36, 54, \ldots$

$\therefore T_{n+1} = \frac{3}{2}T_n, \qquad T_1 = 16$

PROBLEMS TO SOLVE

CHAPTER 3: SEQUENCES – ARITHMETIC, GEOMETRIC & RECURSIVE

1. Describe **in words** what is happening in each of the following:

   (a) 3, 7, 11, 15, …

   (b) $T_{n+1} = T_n - 5$

   (c) $T_{n+1} = 0.85T_n$

   (d) $T_n = T_{n-1} + T_{n-2}$

2. State recursive rules and T_1 for each of the following and hence find T_6.

   (a) 2, 8, 14, 20, …

   (b) 7, 5.5, 4, 2.5, …

   (c) 2, 3, 4.5, 6.75, …

   (d) 2, 5, 20, 95, …

3. Give the first 4 terms of the sequences given by the following recursive definitions.

   (a) $T_{n+1} = T_n + 5$ $\quad T_1 = 6$

   (b) $T_n = 2T_n - 6$ $\quad T_1 = 3$

   (c) $T_{n+1} = T_n + T_{n-1}$ $\quad T_1 = 2,\ T_2 = 3$

4. For the sequence whose recursive definition is given by:

   $T_{n+1} = T_n + 8$ $\quad T_1 = 2$

   List the first 6 terms.

5. For the following, find k and hence evaluate T_4.

   (a) $T_{n+1} = 3T_n + k$ $\quad T_1 = 5$, $T_2 = 12$

   (b) $T_n = kT_{n-1} - 2$ $\quad T_1 = 5$, $T_2 = -12$

6. A sequence is defined by:

   $T_n = T_{n-1} + 2n$ $\quad$ where $T_1 = 5$

   Give the first five terms of this sequence.

7. A sequence is defined by:

   $T_{n+1} - 4T_n = -3$ $\quad T_1 = 6$

   Determine the first 5 terms of the sequence.

8. The value of a share each year after an initial outlay of $100 forms a sequence with the recursive formula:

   $T_n = 1.3T_{n-1}$ $\quad T_1 = 100$

   (a) Determine the first four terms of the sequence.

   (b) What percentage increase occurs with each successive term?

9. Given the difference equation for a sequence is:

$$(T_{n+1})^2 - (T_n)(T_{n+2}) = 4 \quad \text{with } T_1 = 6 \text{ and } T_2 = 8$$

(a) Determine the first four terms.

(b) Write a simple recursive rule for the sequence.

10. Consider the sequence with nth term given by $P_n = 100 - 2n$.

(a) List the first five terms of this sequence.

(b) Determine the recursive formula for this sequence.

11. Given $T_1 = 1$ and $T_{n+1} = (T_n)^{n-1} - 2^n - 4n + 3$ determine the first five terms of the sequence.

12. A sequence is defined by $T_n = \frac{1}{2}(T_{n-1} + T_{n+1})$ where $T_1 = 2$ $T_2 = 5$

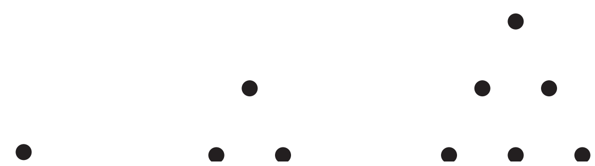
(a) Find T_3

(b) State another recursive rule for this sequence.

13. Equipment costing \$5000 depreciates at the rate of 17% per year.

Write a recursive definition to describe this situation.

14. The triangular numbers are generated by the following pattern



(a) Complete the table below.

n	1	2	3	4	5
T_n	1	3	6		

(b) For the sequence determine a recursive formula.

15. The population of Indonesia in mid 2007 was 250 million increasing at a rate of 1.8% per annum.

(a) Write a recursive formula for the sequence of the Indonesian population from 2007 onwards assuming the population growth remains the same.

(b) Estimate the population of Indonesia in 2009 if the annual growth rate remains constant.

16. Determine whether the following sequences are arithmetic, geometric or neither.

(a) 2, 7, 12, 17, …

(b) $T_{n+1} = 3T_n$, $T_1 = 5$

(c) 2, 4, –8, –16, 32, …

(d) $T_n = 2 + T_{n-1}$, $T_1 = -6$

(e) 1, 1, 2, 3, 5, …

(f) $\frac{2}{5}, \frac{3}{6}, \frac{4}{7}, \frac{5}{8}, \ldots$

(g) $p + 2, 2p + 3, 4 + 3p, \ldots$

(h) $T_n = 0.5T_{n+1}, T_1 = 2$

17. The general term of an arithmetic sequence is given by $T_n = 2n - 3$. State the recursive formula for the same sequence.

18. Calculate the number of terms in each of the following arithmetic sequences.

(a) 3, 7, 11, … , 79, 83

(b) 84, 82, 80, … , 6, 4

(c) $x, 4x, 7x, \ldots, 58x$

19. Find the 10th and *n*th terms of the following arithmetic series.

(a) 3 + 7 + 11 + 15 + …

(b) 27 + 24 + 21 + …

(c) $3x + (-2x) + (-7x) + \ldots$

20. The first term of an arithmetic sequence is 8 and the fourth term is 26. Find the common difference and the 20th term.

21. For what values of x would the sequence $-6, x^2, 7x$ form an arithmetic sequence.

22. A recurrence relation is defined by $T_{n+1} = kT_n + 2$, where k is a constant. Given $T_3 = 42$ and $T_1 = 3$ find all possible values of k.

23. The third term of an arithmetic sequence is 30 and the tenth term is 9. Find:

(a) the sixth term

(b) the first term the sequence becomes negative.

24. The third term of a geometric sequence is 4 and the sixth term 32. Find the tenth term.

25. (a) Write down the next two terms for the following sequence

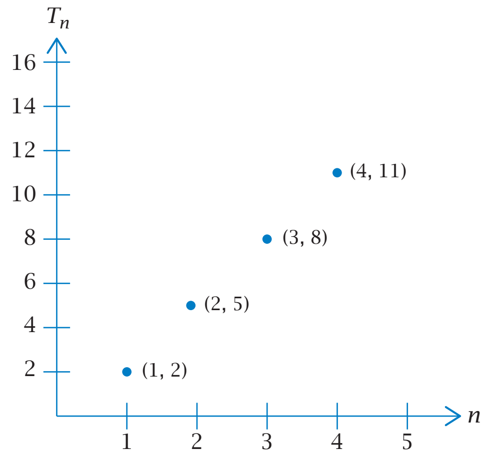
2, 7, 14, 23, 34, __, __

(b) Determine an expression for the *n*th term, T_n, for the above sequence.

26. (a) The first three terms of a geometric sequence are $x - 6$, $2x$ and x^2. Find the value(s) of x and hence the seventh term.

(b) Find the term where the sequence 2, 6, 18, 54, … first exceeds one million.

27.



(a) Which sequence is represented by the graph above.

(b) Determine the general term (T_n) for this sequence.

(c) Hence determine T_7.

28. Given:

Sequence I: $T_{n+1} = 3T_n - 2 \qquad T_1 = 2$

Sequence II: $T_{n+1} = 2T_n - 5 \qquad T_1 = 2$

Sequence III: $T_{n+1} = 2T_n - 2 \qquad T_1 = 2$

(a) Graph each of the above sequences.

(b) Are the sequences:

(i) increasing

(ii) decreasing or

(iii) steady state?

(c) In the long term, determine what happens to the terms in the sequence generated by $T_{n+1} = 0.99T_n + 15, \qquad T_1 = 4000$?

29. Two sequences are defined by

* $T_{n+1} = 3T_n - 2, \qquad T_0 = 4$
* $P_{n+1} = -3 - P_n, \qquad P_0 = 3$

(a) Determine the first four terms of each sequence.

(b) Describe each sequence.

30. John earns $52 000 in his first year of employment. He will receive an annual increase of $7500. How much will he earn in:

(a) his fifth year of employment?

(b) his twentieth year?

31. John deposits money into an account each month. He deposits \$25 in the first month and each subsequent month deposits double the amount deposited the previous month.

(a) How much did John deposit in the 5th month?

(b) How much had John deposited by the end of the 8th month?

(c) How long will it take John to save enough money to purchase a new car valued at \$25 000?

32. A block of land is valued at \$250 000. Its value will increase at 4.5% p.a. Let L_n represent the value of the land after n years.

Determine:

(a) L_1

(b) L_7

(c) L_n in terms of n

(d) n when L_n triples in its original value.

33. A body falls 16 metres in the first second of its motion, 48 metres in the second, 80 metres in the third and so on.

(a) Write down a recurrence relation that represents the total distance the body falls for every second of motion.

(b) How far does the body fall in the tenth second of motion?

34. Percy inherits \$100 000 and invests the money into an account earning 18% p.a. compounded monthly. He withdraws \$1000 each month from this account.

(a) Determine a recurrence relation for Percy's inheritance.

(b) Calculate the amount in this account after 5 years.

35. A ball is dropped from a height of 48 metres and bounces three-quarters of the distance it falls. If it continues to fall and bounce determine:

(a) a recurrence relation for the sequence generated by the bounce heights of the ball.

(b) the height of the 3rd bounce.

(c) the total distance travelled by the ball at the end of the 5th bounce.

36. A large swimming pool contains 80 000 litres of water. Each day 5% of the water is lost due to evaporation. An additional 2500 litres of water is added to the pool at the end of each day.

The recurrence relation is:

$T_{n+1} = rT_n + d, \qquad T_1 = a$

where T_n represents the amount of water in the pool at the start of the nth day.

(a) Calculate the values of r, d and a.

(b) How many litres of water will be in the swimming pool at the start of the 6th day.

(c) How long will it take for the swimming pool to fall below 60 000 litres?

UNDIRECTED GRAPHS AND NETWORKS

Syllabus Checklist

By the end of this chapter, you should be able to:

- ☐ demonstrate the meanings and use of the terms: graph, edge, vertex, loop, degree of a vertex, subgraph, simple graph, complete graph, bipartite graph, directed graph (diagraph), arc, weighted graph and network
- ☐ identify practical situations represented by a network and construct such networks
- ☐ construct an adjacency matrix and use it to solve problems
- ☐ demonstrate the meanings and use of the terms: planar graph and face
- ☐ apply Euler's formula: v + f – e = 2 to solve problems
- ☐ demonstrate the meanings and use of the terms: walk, trail, path, closed walk, closed trail, cycle, connected graph and bridge
- ☐ investigate and solve practical problems to determine the shortest path between two vertices
- ☐ demonstrate the meanings and use of the terms: Eulerian graph, Eulerian trail, semi-Eulerian graph, semi-Eulerian trail, their conditions for existence and use these concepts to solve practical problems
- ☐ demonstrate the meanings and use of the terms: Hamiltonian graph and semi-Hamiltonian graph and use these concepts to solve practical problems
- ☐ identify practical examples represented by trees and spanning trees
- ☐ identify a minimum spanning tree in a graph or by using Prim's algorithm
- ☐ use minimum spanning trees to solve minimal connector problems

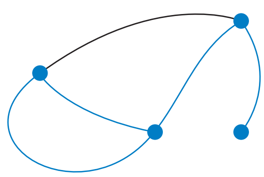
FORMULAE AND DEFINITIONS

Graphs

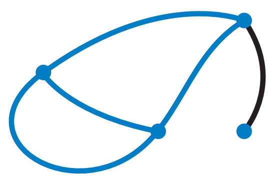
Graphs are diagrams consisting of vertices and edges. Networks are applications of graphs.

Definitions

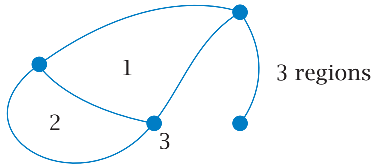
Vertices — nodes or points



Edges — lines or connections or arcs



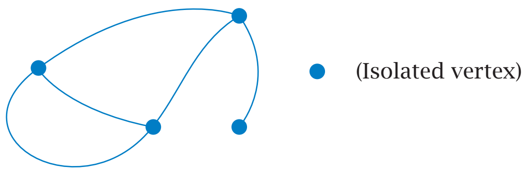
Regions — including the outside



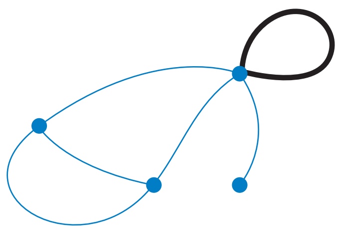
Order — the **order** of a node is the number of arcs or edges that meet at that point.

A has Order 4 B has Order 3

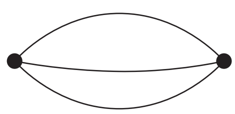
Isolated vertex — not connected to another vertex.



Loop — an edge that starts and finishes at the same vertex.



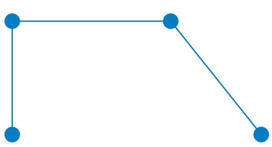
Multiple edges — when two or more edges connect the same two vertices.



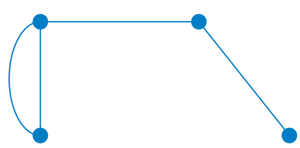
Multiple edges

Types of Graphs

Simple graph — contains no loops or multiple edges.

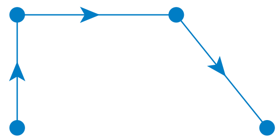


Simple graph



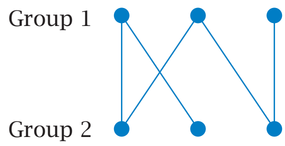
Not a simple graph (multiple edges)

Directed graph (digraph) — contains directed edges indicated by arrows.

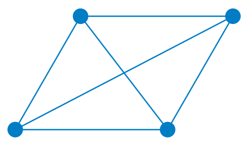


Directed graph

Bipartite graph — a graph whose vertices can be divided into two disjoint groups such that a vertex in one group is connected to at least one vertex in the second group.

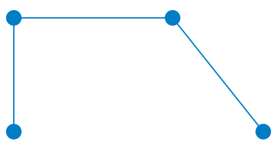


Complete graph — are simple graphs where each vertex is directly connected to every other vertex.

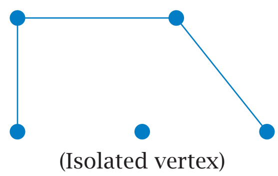


Complete graph

Connected graph — contains no isolated vertices.

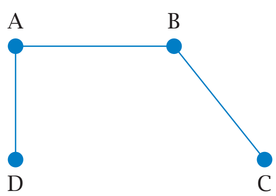


Connected graph

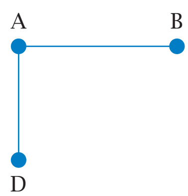


Disconnected graph

Subgraph — part of another graph with no new edges or vertices.

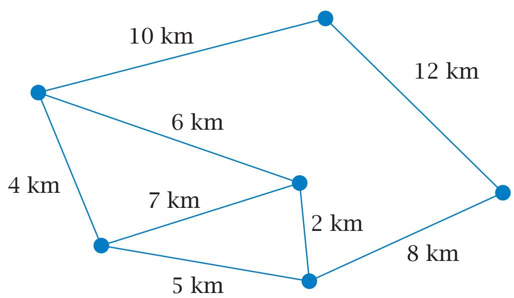


Graph



Subgraph

Weighted graph — each edge contains information such as cost, distance, time etc.



The Adjacency Matrix

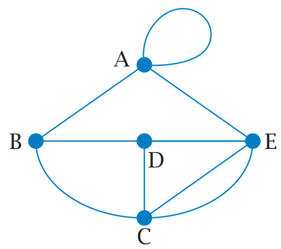
The adjacency or connection matrix is a matrix with rows or columns containing numbers which represent the number of edges connecting vertices.

* a **0** indicates that no edge connects the vertices

* a **1** indicates either 1 edge connects the vertices or a loop

* a number greater than 1 indicates multiple edges connecting the vertices.

Graph



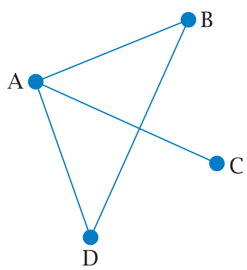
Adjacency Matrix

	A	B	C	D	E
A	1	1	0	0	1
B	1	0	1	1	0
C	0	1	0	1	2
D	0	1	1	0	1
E	1	0	2	1	0

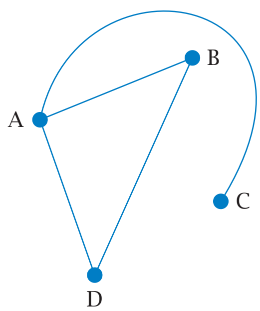
Planar Graphs

A planar graph is a graph that has *no edges* that cross.

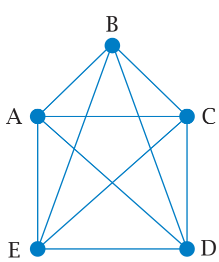
Planar



Graph can be redrawn without edges crossing



Non Planar



Traversability of a Network

A network which is traversable can be drawn without taking the pen off the paper and without tracing over the same arc more than once.

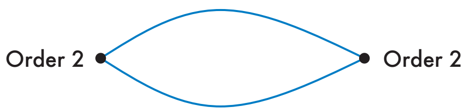
A network is traversable if:

- all the nodes are even

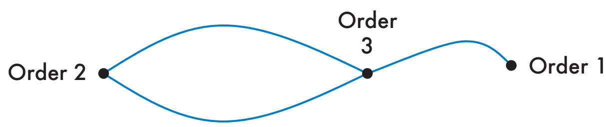
  or

- it has two odd nodes

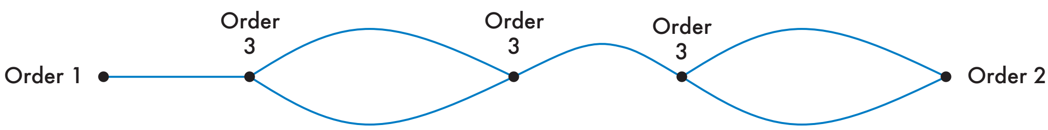
A network is not traversable if there are more than two odd nodes.



No odd vertices
Traversable



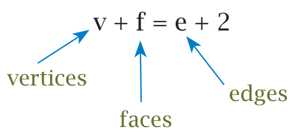
Two odd vertices
Traversable



More than two odd vertices
NOT traversable

Euler's Rule

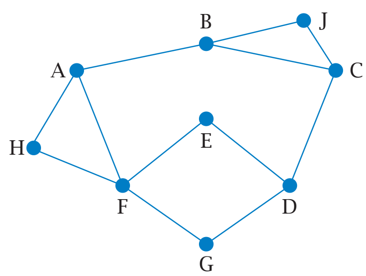
For a planar graph Euler's Rule is:



Walk

A sequence of vertices where, from each of its vertices, there is an edge to the next vertex.

The *length* of a walk is the number of *edges* used. A walk such as ABCD has length 3.



walk: ABCD

not a walk: AED

Closed Walk

A walk that starts and finishes at the same vertex.

Open Walk

A walk that starts at one vertex and finishes at a different vertex.

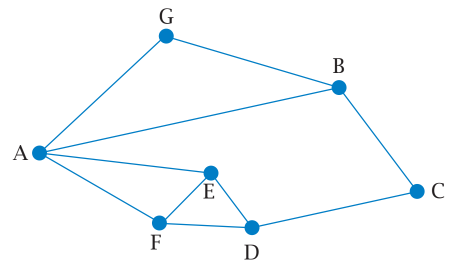
Special Walks

1. Path

A type of walk where all *vertices* and *edges* are different – no repeats.

Closed Path

A path that starts and finishes at the same vertex.



Closed path: ABCDEA

Open Path

A path that starts and finishes at different vertices.

2. Trail

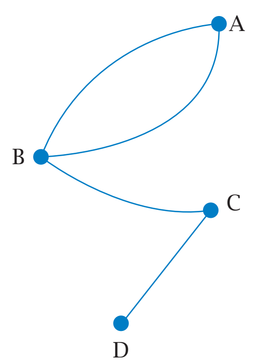
A type of walk with no repeated *edges*.

Closed Trail

A trail that starts and finishes at the same vertex.

Open Trail

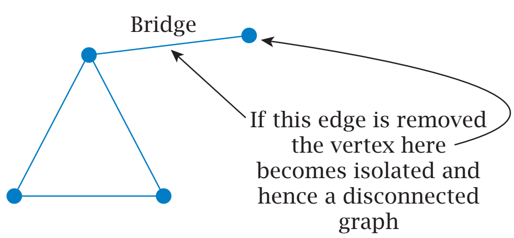
A trail that starts and finishes at different vertices.



Open trail: DCBAB

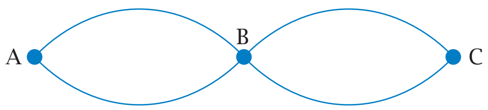
Bridge

A bridge is an edge that if removed leaves the graph disconnected.



Eulerian

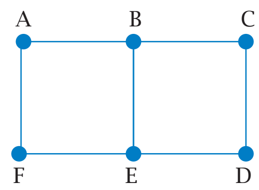
A *closed* trail in a connected graph, starting and finishing at the same vertex, travelling every edge once, and containing *no odd* vertices is said to be **Eulerian**.



- No odd vertices
- Start–finish at the same vertex

Semi-Eulerian

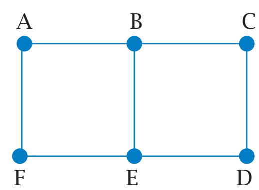
An *open* trail in a connected graph, starting and finishing at different vertices, travelling every edge once, and containing *exactly two odd vertices* is said to be **Semi-Eulerian**.



- Two odd vertices
- Start–finish at the different vertices

Hamiltonian Path

A path that starts and finishes at different vertices and includes all vertices is called a *Hamiltonian Path*.

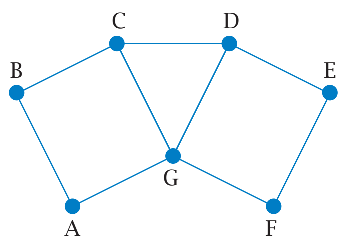


Hamiltonian Path
Start at vertex B

Path: BCDEFA

Hamiltonian Cycle

A path that starts and finishes at the same vertex and includes all vertices is called a *Hamiltonian Cycle*.

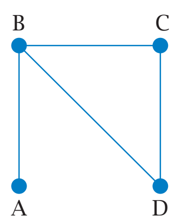


Hamiltonian Cycle
Start at vertex A

Cycle: ABCDEFGA

Semi-Hamiltonian Graph

A *Semi-Hamiltonian* graph is a graph that contains a Hamiltonian path but not a Hamiltonian cycle.



A Hamiltonian Path but *not* a Hamiltonian Cycle
Start at vertex A.
Cannot finish at A.

Path: ABCD

Shortest Path

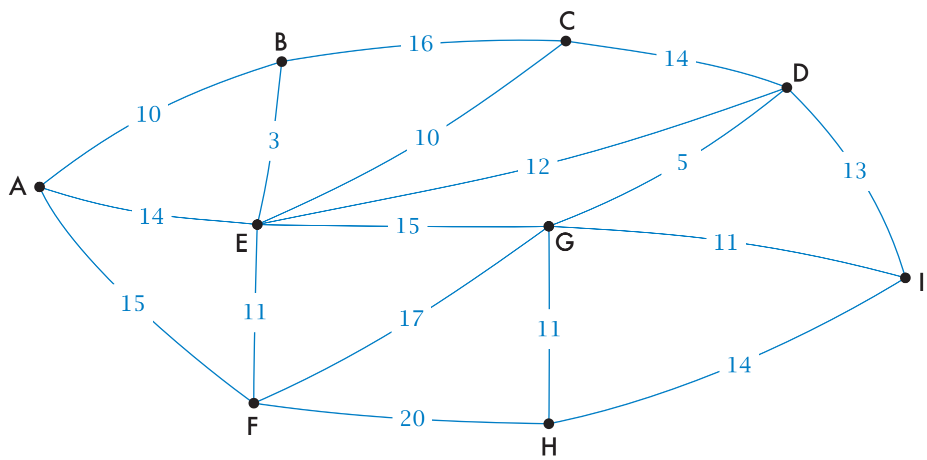
The path between two nodes with the shortest distance is called the shortest path. This may also be the shortest time, least cost or least distance.

This can be found by:

- inspection (not easy for complicated networks)
- listing all possible paths to find the one that gives the least distance travelled (tedious)
- using a logical approach detailed in the example below

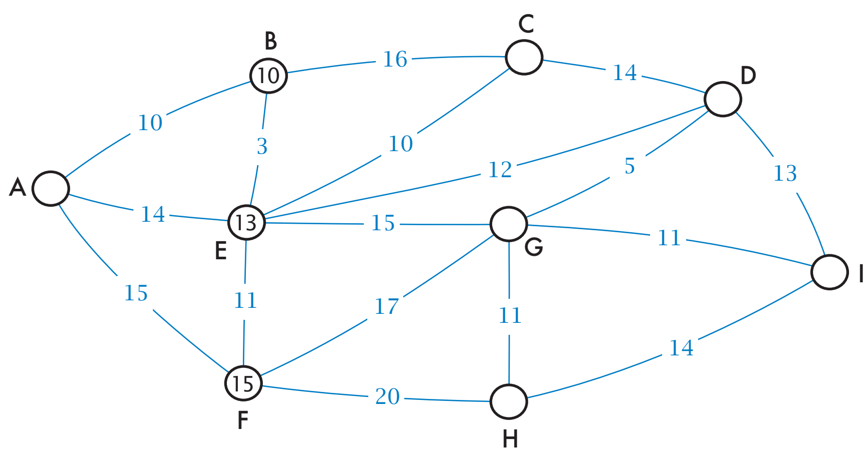
A Logical Approach

Find the shortest path from A to I.



Method:

Step 1: Place a number in each node from A to indicate the shortest distance.



- Shortest distance from A to B = 10 ✓
- Shortest distance from A to E:

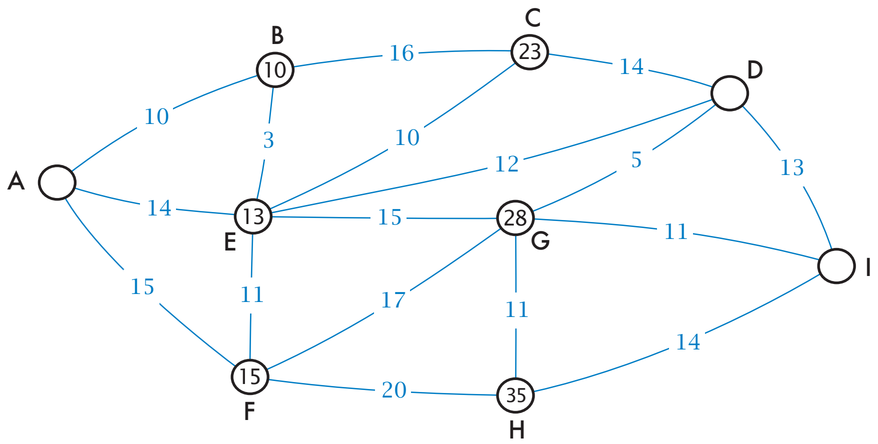
A—E = 14

A—B—E = 13 ✓

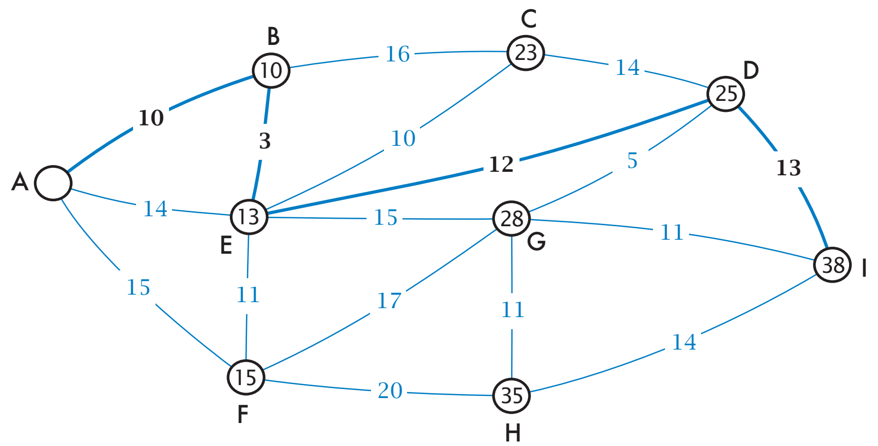
A—F—E = 26

- Shortest distance from A to F = 15 ✓

Step 2: Place a number in each node from B, E and F to indicate the shortest distance.



Step 3: From nodes C, G and H find the shortest distance to nodes D and I.

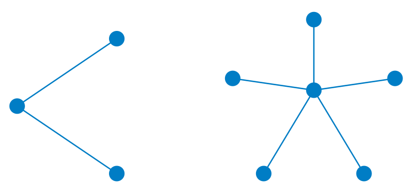


Hence the shortest path is: A—B—E—D—I and the shortest length is 38 units.

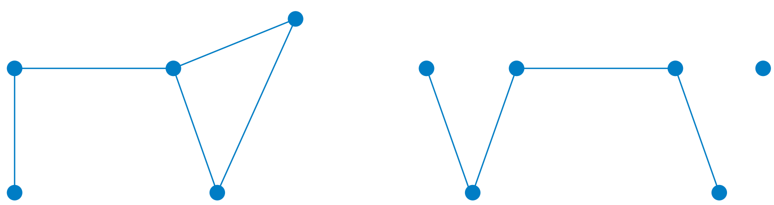
Trees

A tree is a simple connected graph that contains no cycles, loops or multiple edges.

Trees



Not trees



Contains a cycle

Contains an isolated vertex

Spanning Tree

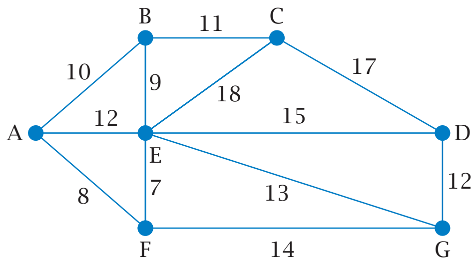
A spanning tree is a subgraph of a graph and a *tree* connecting all of the vertices.

Minimum Spanning Tree

A minimum spanning tree is a spanning tree with weights less than or equal to the weight of other spanning trees. These weights could represent distance, cost or time. A minimum spanning tree will determine the shortest length while providing a path between every pair of vertices.

Nearest Neighbour Method

When data is presented as a *Network* use this method.



Lengths in cm

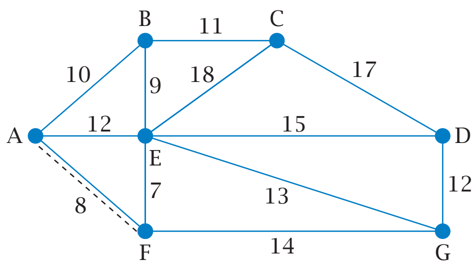
Step 1:

Choose any starting vertex
e.g. start with A.

Step 2:

Join this to the nearest vertex.

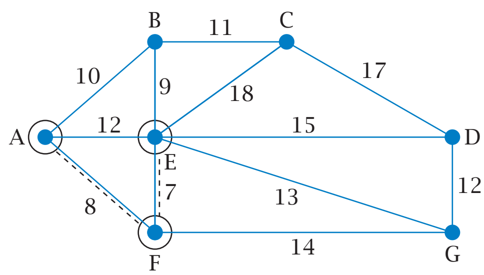
A ⟶ F



Step 3:

Find the nearest vertex to either of the other two vertices already in the solution.

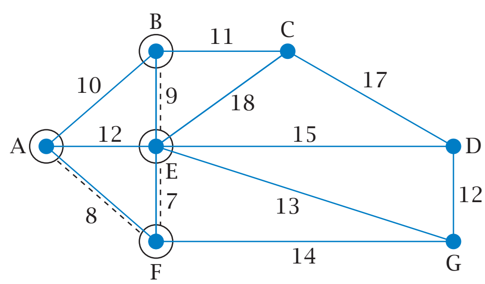
A ⟶ F
F ⟶ E



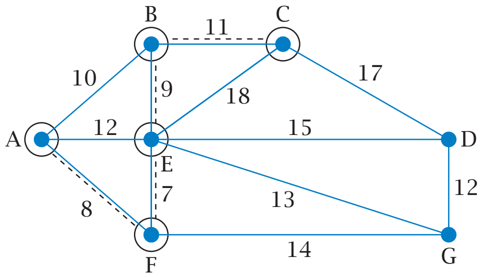
Step 4:

Continue in this manner until all vertices are connected.

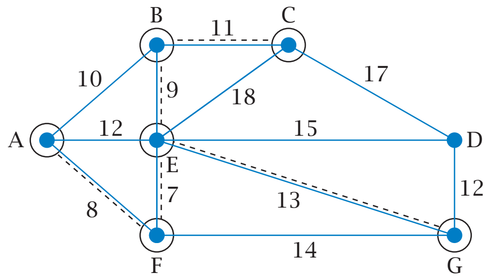
A ⟶ F
F ⟶ E
E ⟶ B



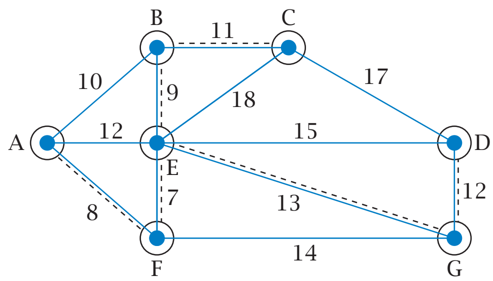
A → F
F → E
E → B
B → C



A → F
F → E
E → B
B → C
E → G



A → F
F → E
E → B
B → C
E → G
G → D



Minimum spanning tree
Length = 8 + 7 + 9 + 11 + 13 + 12
= 60 cm

Prim's Algorithm Method

When data is presented as a *Table* use this method:

	A	B	C	D	E	F	G
A	—	10	—	—	12	8	—
B	10	—	11	—	9	—	—
C	—	11	—	17	18	—	—
D	—	—	17	—	15	—	12
E	12	9	18	15	—	7	13
F	8	—	—	—	7	—	14
G	—	—	—	12	13	14	—

Units are in cm.

Step 1:

Choose any vertex to start. e.g. Start with A. Delete *row* A and circle the *smallest* value in *column* A.

	A	B	C	D	E	F	G
~~A~~	~~—~~	~~10~~	~~—~~	~~—~~	~~12~~	~~8~~	~~—~~
B	10	—	11	—	9	—	—
C	—	11	—	17	18	—	—
D	—	—	17	—	15	—	12
E	12	9	18	15	—	7	13
F	(8)	—	—	—	7	—	14
G	—	—	—	12	13	14	—

Step 2:

Delete the row circled, i.e. row F. Circle the smallest value in either of the two columns i.e. row A or F.

	A	B	C	D	E	F	G
~~A~~	~~—~~	~~10~~	~~—~~	~~—~~	~~12~~	~~8~~	~~—~~
B	10	—	11	—	9	—	—
C	—	11	—	17	18	—	—
D	—	—	17	—	15	—	12
E	12	9	18	15	—	7	13
~~F~~	~~8~~	~~—~~	~~—~~	~~—~~	~~7~~	~~—~~	~~14~~
G	—	—	—	12	13	14	—

If there is more than one smallest value, then choose either one. Delete row E.

Step 2:

Continue step 2 until all rows are deleted. Circle the smallest value in rows A, E or F.

	A	B	C	D	E	F	G
~~A~~	~~—~~	~~10~~	~~—~~	~~—~~	~~12~~	~~8~~	~~—~~
B	10	—	11	—	9	—	—
C	—	11	—	17	18	—	—
D	—	—	17	—	15	—	12
~~E~~	~~12~~	~~9~~	~~18~~	~~15~~	~~—~~	~~7~~	~~13~~
~~F~~	~~8~~	~~—~~	~~—~~	~~—~~	~~7~~	~~—~~	~~14~~
G	—	—	—	12	13	14	—

Delete row B.

Circle the smallest value in rows A, B, E or F.

	A	B	C	D	E	F	G
~~A~~	~~—~~	~~10~~	~~—~~	~~—~~	~~12~~	~~8~~	~~—~~
~~B~~	~~10~~	~~—~~	~~11~~	~~—~~	~~9~~	~~—~~	~~—~~
C	—	11	—	17	18	—	—
D	—	—	17	—	15	—	12
~~E~~	~~12~~	~~9~~	~~18~~	~~15~~	~~—~~	~~7~~	~~13~~
~~F~~	~~8~~	~~—~~	~~—~~	~~—~~	~~7~~	~~—~~	~~14~~
G	—	—	—	12	13	14	—

Delete row C.

Circle the smallest value in rows A, B, C, E or F.

	A	B	C	D	E	F	G
A	—	10	—	—	12	8	—
B	10	—	11	—	9	—	—
C	—	11	—	17	18	—	—
D	—	—	17	—	15	—	12
E	12	9	18	15	—	7	13
F	8	—	—	—	7	—	14
G	—	—	—	12	13	14	—

Delete row G.

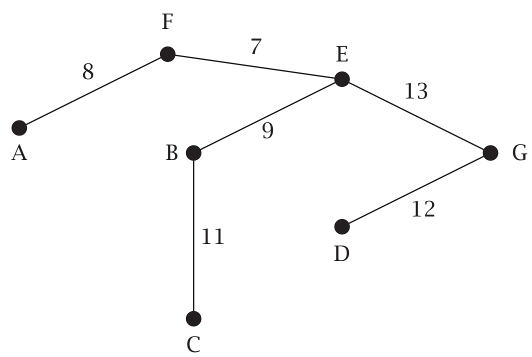
Circle the smallest value in rows A, B, C, E, F or G.

	A	B	C	D	E	F	G
A	—	10	—	—	12	8	—
B	10	—	11	—	9	—	—
C	—	11	—	17	18	—	—
D	—	—	17	—	15	—	12
E	12	9	18	15	—	7	13
F	8	—	—	—	7	—	14
G	—	—	—	12	13	14	—

Delete row D.

	A	B	C	D	E	F	G
A	—	10	—	—	12	8	—
B	10	—	11	—	9	—	—
C	—	11	—	17	18	—	—
D	—	—	17	—	15	—	12
E	12	9	18	15	—	7	13
F	8	—	—	—	7	—	14
G	—	—	—	12	13	14	—

Minimum Spanning Tree



Total length = 60 cm

Worked Examples

4.1 The table below shows the distances in metres between five towns A, B, C, D and E. Represent the information as a network diagram.

this dash means there is no arc from node A to node A

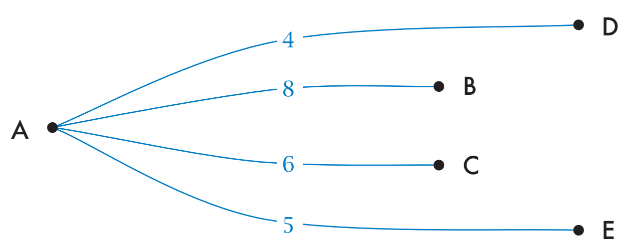
	A	B	C	D	E
A	-	8	6	4	5
B	8	-	12	10	-
C	6	12	-	7	11
D	4	10	7	-	6
E	5	-	11	6	-

this dash means there is no arc from node E to node B

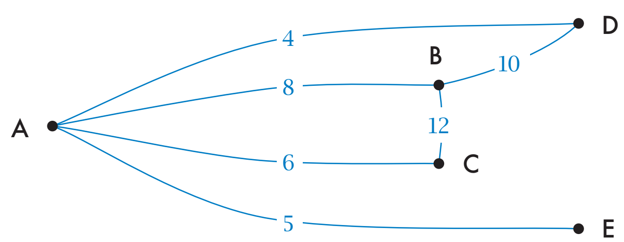
Solution:

Method

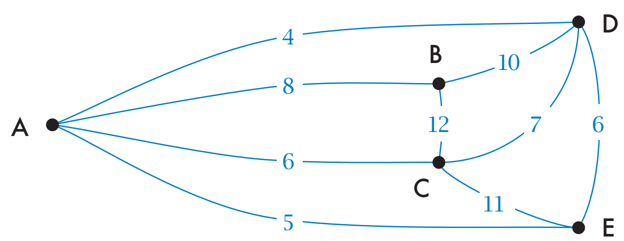
From node A draw arcs to node B, node C, node D and node E. Include values above each arc.



Continue the process joining node B to node A, node C and node D.

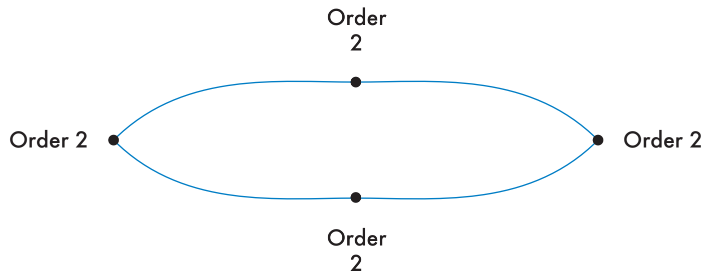


Continue the process until all nodes and arcs within the table are completed.



4.2 Determine which of the following networks are traversable. State a reason why?

(a)



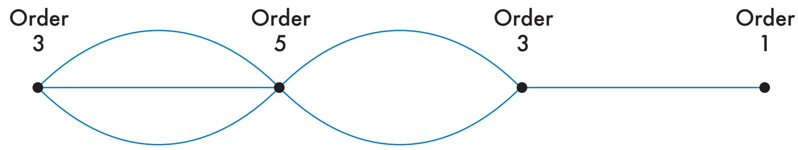
Solution:

Calculate the order of each node by counting the number of arcs entering each node.

No odd nodes.

∴ network is traversable.

(b)

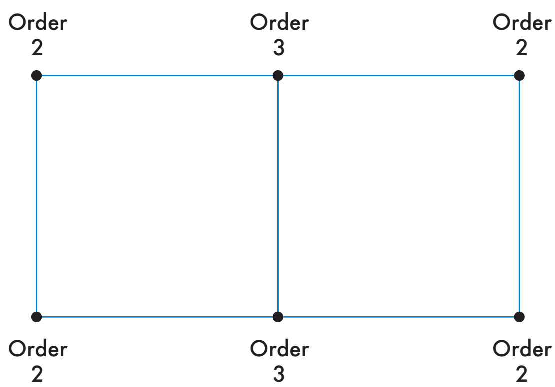


Solution:

More than two odd nodes.

∴ network is **not** traversable.

(c)



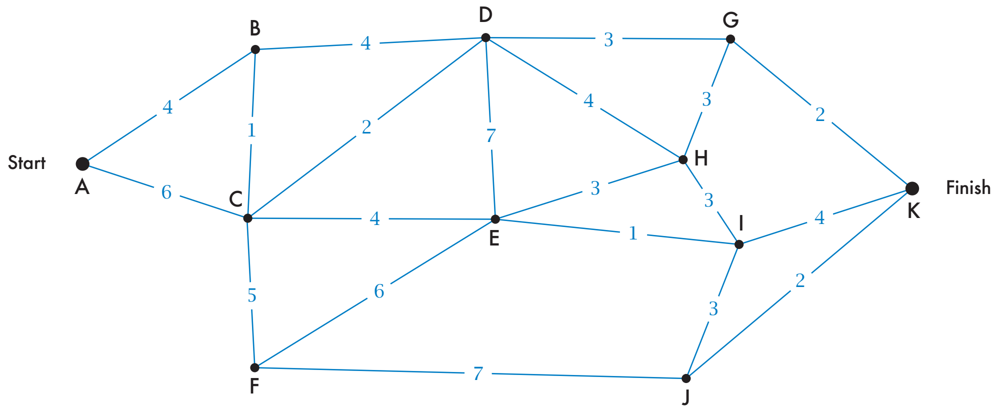
Solution:

Two odd nodes.

∴ network is traversable.

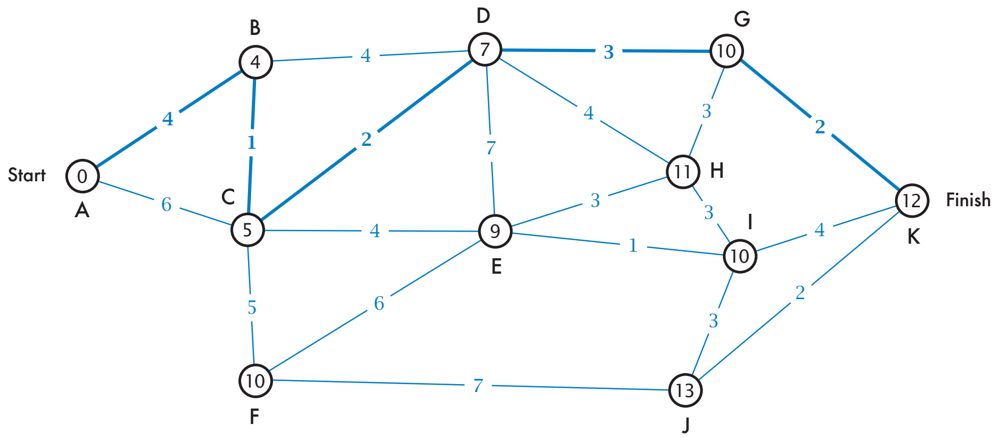
4.3 The following network shows the time (in minutes) to cycle various sections of a cross-country race from start to finish.

(a) Which path should be taken in order to complete the race in the least possible time?



Solution:

Use the shortest path logical approach to find the least amount of time from node A to node K.

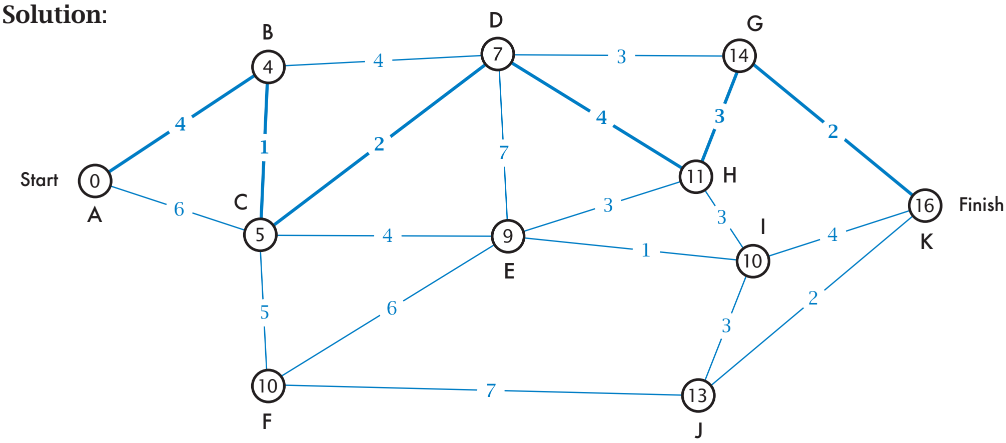


Shortest Path from: A to B = 4
A to C = 5 as A—C = 6 A—B—C = 5 *This method continues from B and C to K*

Path for least time is: A—B—C—D—G—K
Time: 12 minutes

(b) If the cyclist must pass through node H, find the path that will complete the race in the least amount of time.

Solution:



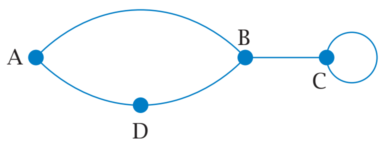
Path for least time via H is: A—B—C—D—H—G—K
Time: 16 minutes

4.4 Determine the number of:

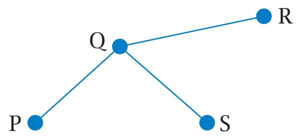
(i) vertices
(ii) edges

for each graph below:

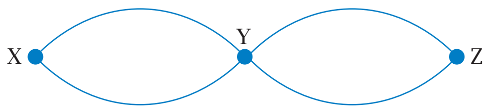
(a)



(b)



(c)



From the networks above, state the vertex:

(d) containing a loop

(e) adjacent to vertex R

(a) (i) vertices = 4
(ii) edges = 5

(b) (i) vertices = 4
(ii) edges = 3

(c) (i) vertices = 3
(ii) edges = 4

(d) Vertex C

(c) Vertex Q → connected to Vertex R by an edge

4.5 Draw a diagram which defines each of the following:

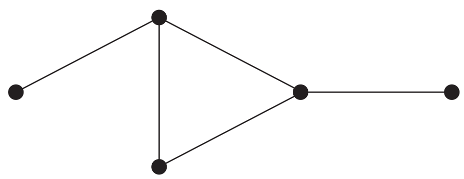
(a) a simple graph

(b) a connected graph

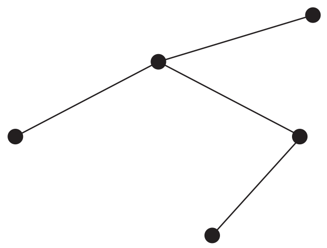
(c) a complete graph

(d) a directed graph

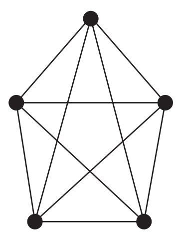
(a) Simple graphs contain no multiple edges or loops.



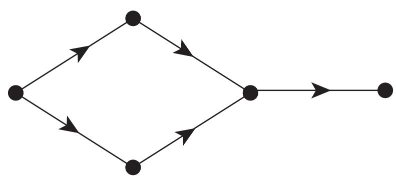
(b) Connected graphs contain no isolated vertices.



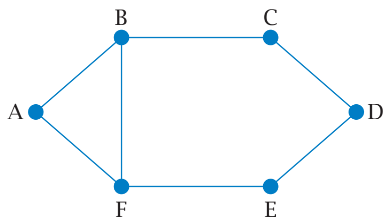
(c) Complete graphs have every vertex connected to every other vertex.



(d) Directed graphs have directions on the edges indicated by arrows.



4.6 Given the graph



determine:

(a) two paths from A to D

(b) the degree of each vertex

(c) the degree sum of the graph

(d) an open walk

(e) a closed trail

(a) Two possible paths are:

1. A — B — C — D

2. A — B — F — E — D

(others possible)

(b) Degrees of vertex

A = 2 (number of edges entering the vertex)

B = 3

C = 2

D = 2

E = 2

F = 3

(c) Degree sum $= 2 \times$ number of edges

$= 2 \times 7$

$= 14$

(d) Open walk: A — B — C — D — E

(others possible)

Start and finish at different vertices

(e) Closed Trail: B — A — F — E — D — C — B

Start and finish at same vertex. Each edge visited once.

4.7 Draw a graph representing the adjacency matrix below:

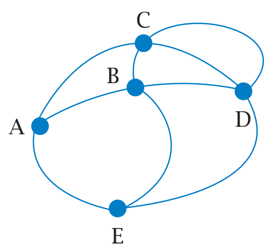
	A	B	C	D	E
A	0	1	1	0	1
B	1	0	1	1	1
C	1	1	0	2	0
D	0	1	2	0	1
E	1	1	0	1	0

A ‘0’ represents no edge joining the vertices A — A

A ‘1’ represents one edge joining the vertices B — E

A ‘2’ represents two edges joining the vertices C — D

Graph

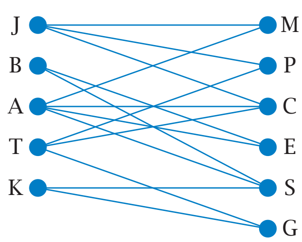


4.8 Draw a bipartite graph to represent the information below.

Five students were surveyed on their favourite subject:

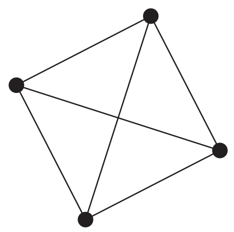
John (J)	Mathematics (M) Physics (P) Chemistry (C)
Brian (B)	English (E) Sport (S)
April (A)	Mathematics (M) English (E) Sport (S) Chemistry (C)
Tracey (T)	Physics (P) Chemistry (C) Geography (G)
Kim (K)	Sport (S) Geography (G)

Bipartite Graph

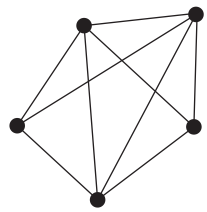


4.9 Determine whether the following graphs are planar. Check your answer by using Euler's formula.

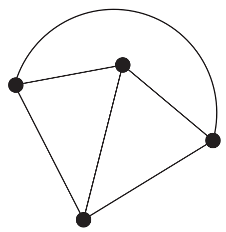
(a)



(b)



(a) Yes graph can be redrawn.

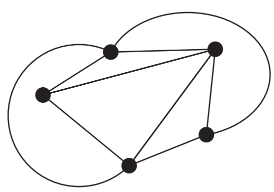


Euler's Formula: V + F = E + 2

4 + 4 = 6 + 2

Graph is planar.

(b) Yes graph can be redrawn.



Euler's Formula: V + F = E + 2

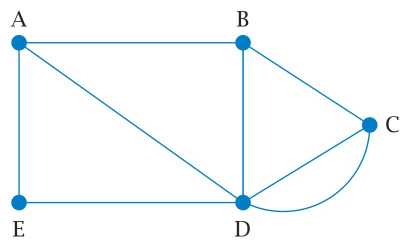
5 + 6 = 9 + 2

Graph is planar.

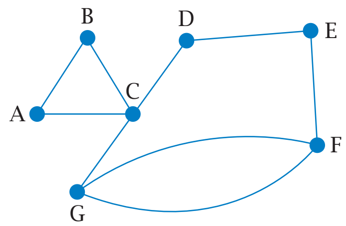
4.10 Determine if each of the graphs below are:

(i) Eulerian (ii) semi-Eulerian (iii) neither

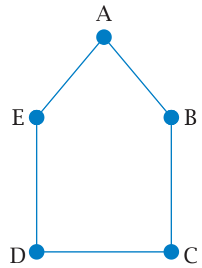
(a)



(b)



(c)



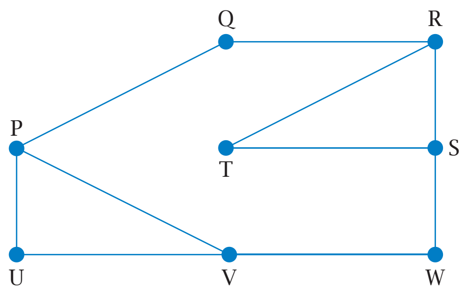
(a)	Neither —	more than 2 odd vertices
(b)	Semi-Eulerian —	exactly 2 odd vertices. Trail starts at one odd vertex (F) and finishes at the other odd vertex (G).
(c)	Eulerian —	all even vertices. Trail starts and finishes at the same vertex.

4.11 List a Hamiltonian

(a) path

(b) cycle

for the graph below



(a) A Hamiltonian path includes each vertex once and starts and finishes at different vertices.

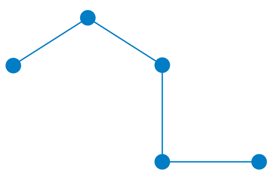
P — U — V — W — S — T — R — Q

(b) A Hamiltonian cycle includes each vertex once and starts and finishes at the same vertex.

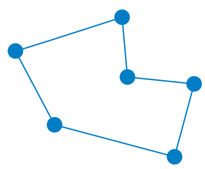
R — Q — P — U — V — W — S — T — R

4.12 Determine which of the following are trees.

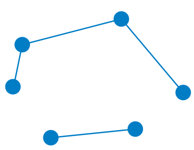
(a)



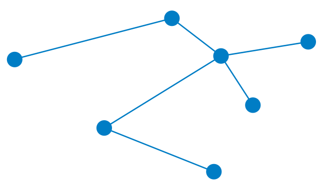
(b)



(c)



(d)



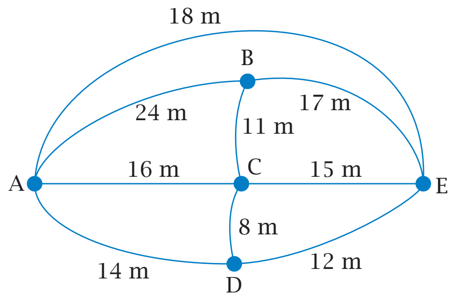
(a) Tree

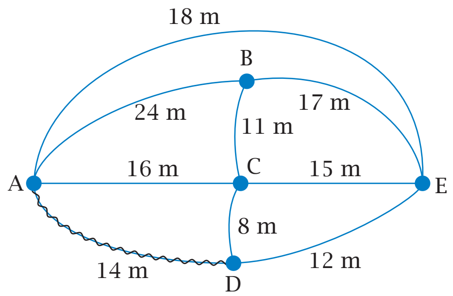
(b) Not a tree — contains a cycle

(c) Not a tree — the graph is not connected

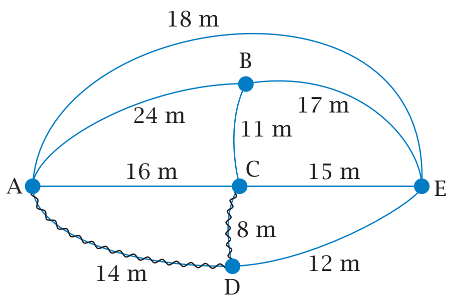
(d) Tree

4.13 For the network below, find the minimum spanning tree.





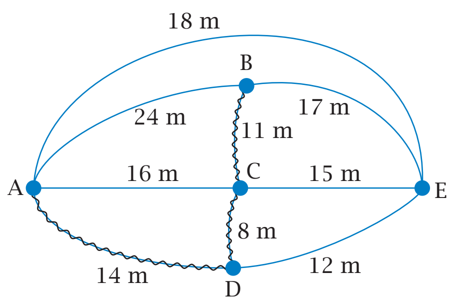
A → D Shortest distance



A → D

From A and D next shortest distance

D → C

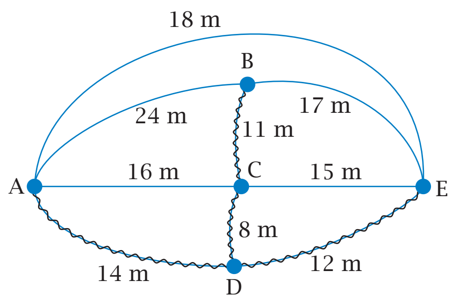


A → D

D → C

From A, C, D next shortest distance

C → B



A → D

D → C

C → B

From A, B, C and D next shortest distance

D → E

Hence minimum spanning tree is shown on the diagram above

Minimum length = 14 + 12 + 8 + 11
= 45 m

4.14 The table below shows the cost in hundreds of dollars, to install cables between seven classrooms labelled A, B, C, D, E, F and G.

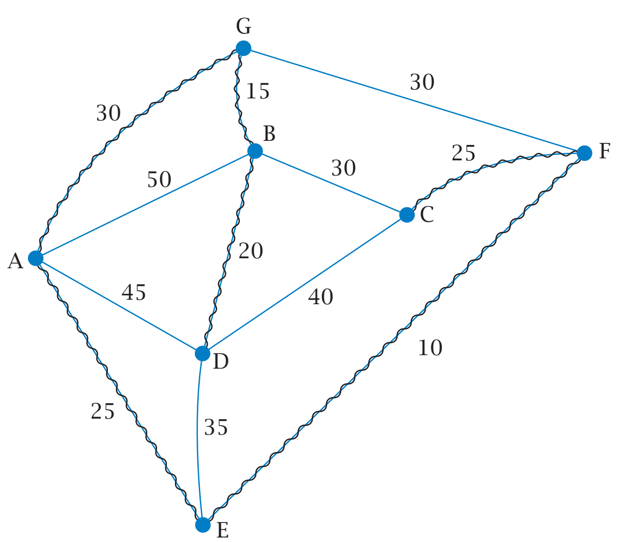
	A	B	C	D	E	F	G
A	—	50	—	45	25	—	30
B	50	—	30	20	—	—	15
C	—	30	—	40	—	25	—
D	45	20	40	—	35	—	—
E	25	—	—	35	—	10	—
F	—	—	25	—	10	—	30
G	30	15	—	—	—	30	—

(a) Draw a graph to represent the information above.

(b) Determine the minimum spanning tree using Prim's Algorithm and show this on the diagram in part (a).

(c) Hence determine the cost of installing the cables.

(a)



Note:
Other spanning trees possible.

(b)

	A	B	C	D	E	F	G
A	—	50	—	45	25	—	30
B	50	—	30	20	—	—	(15)
C	—	30	—	40	—	(25)	—
D	45	(20)	40	—	35	—	—
E	(25)	—	—	35	—	10	—
F	—	—	25	—	(10)	—	30
G	(30)	15	—	—	—	30	—

(c) Minimum cost = 25 + 30 + 20 + 10 + 25 + 15

= $12 500

PROBLEMS TO SOLVE

CHAPTER 4: UNDIRECTED GRAPHS AND NETWORKS

1. The table below shows the distances between six towns P, Q, R, S, T and U. Distances are in km. Represent the information as a network diagram.

	P	Q	R	S	T	U
P	–	42	60	–	34	–
Q	42	–	15	31	–	72
R	60	15	–	21	55	100
S	–	31	21	–	–	–
T	34	–	55	–	–	27
U	–	72	100	–	27	–

2. The table drawn below gives the cost in dollars per metre for connecting an irrigation system at a property between areas A to G.

	A	B	C	D	E	F	G
A	–	4.7	4.3	–	5.6	6.7	–
B	4.7	–	2.45	–	3	–	–
C	4.3	2.45	–	3.1	1.5	4.1	5.6
D	–	–	3.1	–	–	2.7	1.9
E	5.6	3	1.5	–	–	3.6	4.9
F	6.7	–	4.1	2.7	3.6	–	2.7
G	–	–	5.6	1.9	4.9	2.7	–

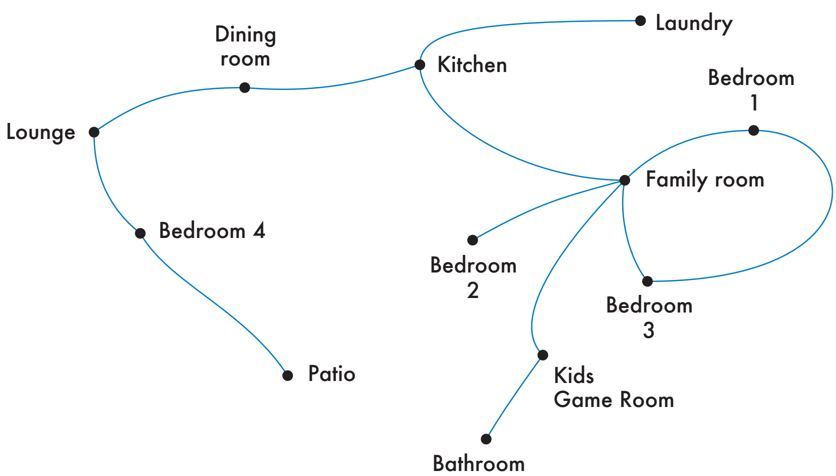
(a) What does the '–' represent in the above table?

(b) Why are the costs in $ repeated in the table?

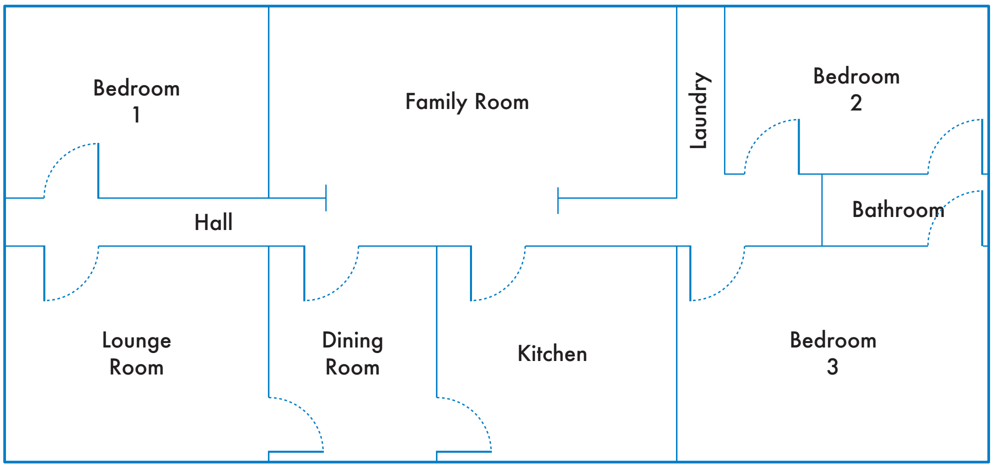
(c) Draw a network to represent the above information.

(d) What does the value 5.6 in the table represent?

3. Interpret what the following network diagram may represent.

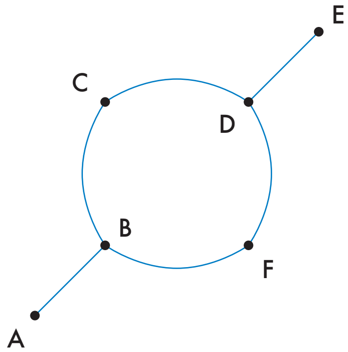


4. Draw a network diagram to represent the following floor plan of a house.

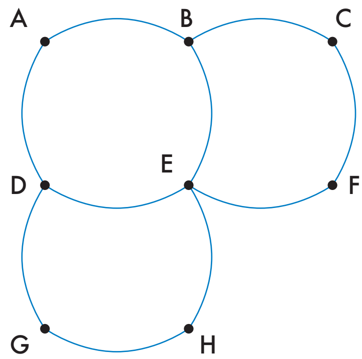


5. Determine which of the following networks can be traversed and if so state a reason why?

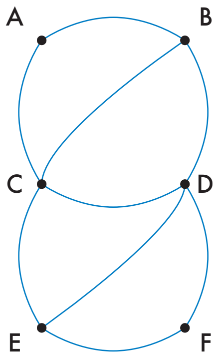
(a)



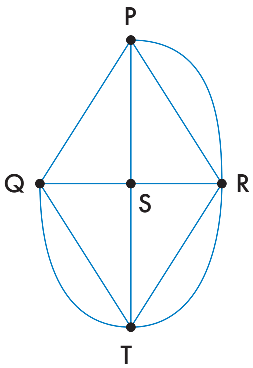
(b)



(c)

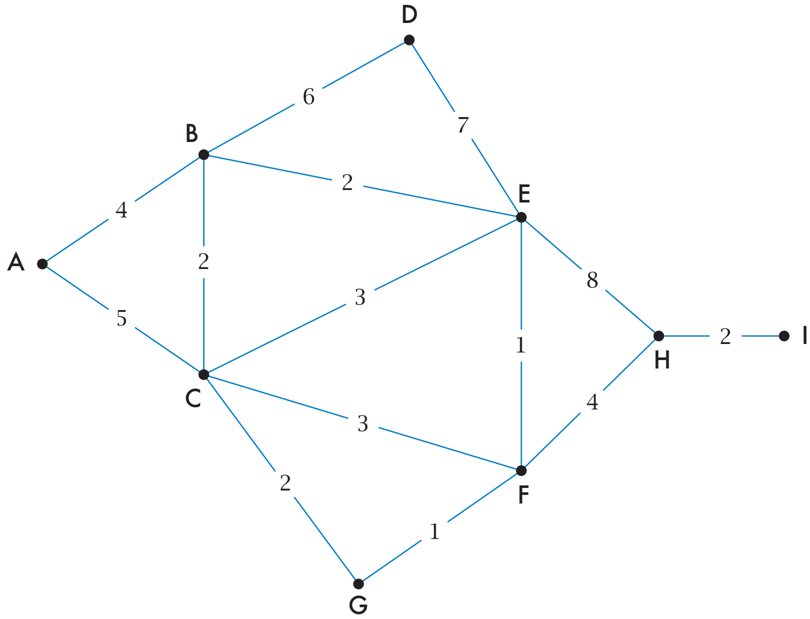


(d)

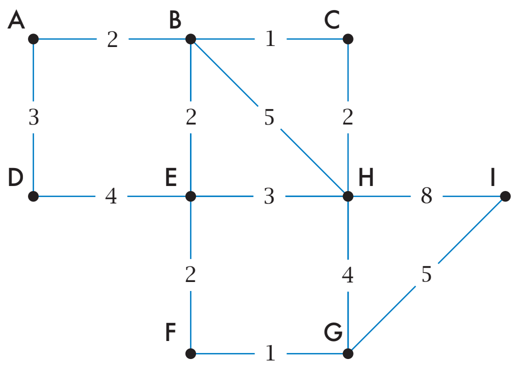


6. Determine the shortest distance from A to I, and state the path. Distances are measured in metres.

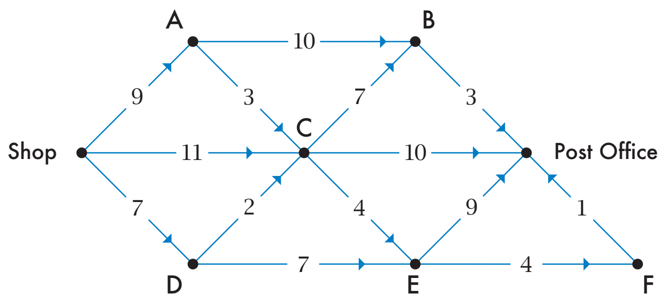
(a)



(b)



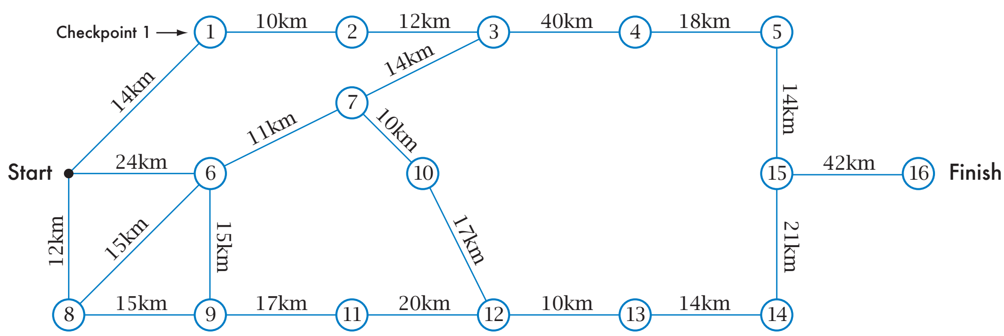
7. The network below indicates via arrows the one-way streets of a busy city CBD.



(a) A courier picks up a parcel at a shop for delivery at the post office. If the time in minutes is represented on the network, what is the shortest time taken and state the path.

(b) If there is a traffic jam at C which path should the courier take in order for the parcel to be delivered at the post office in the minimum amount of time?

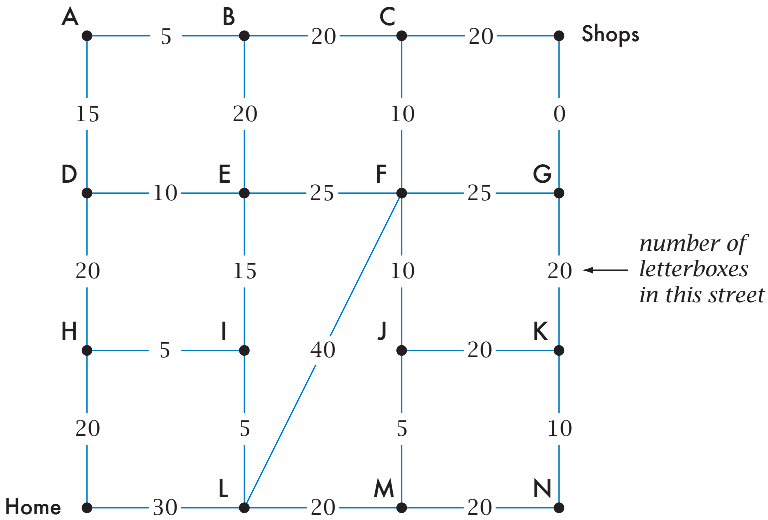
8. Drawn below is the map of the 'Escape' Rally Car Championships. Competitors must pass through various checkpoints on their way from Start to Finish.



(a) Determine the shortest distance travelled by a competitor if they pass through any number of checkpoints from Start to Finish. State the checkpoints passed and the distance travelled.

(b) If a competitor must pass through checkpoint 8 what is the shortest distance from Start to Finish?

9. Joyce and her family are employed to distribute leaflets in an area close to their home. The map below shows their suburb and the number of letterboxes in each street.



Joyce has decided to go from Home to the Shops delivering leaflets along the way. Which route should she take in order that she has to deliver to the least number of letterboxes? State the route and the number of letterboxes.

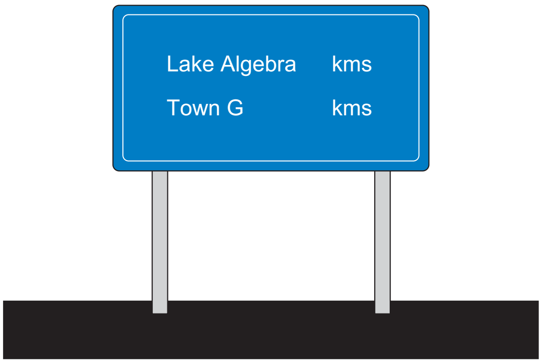
10. A system of pipes to connect seven towns is being built at a cost (in thousands of dollars) to the new dam at Lake Algebra.

The table below indicates this cost in connecting the various towns and the dam.

TOWNS

TOWNS	Lake Algebra	A	B	C	D	E	F	G
Lake Algebra	–	48	39	–	51	–	–	–
A	48	–	–	45	–	–	–	–
B	39	–	–	48	48	42	–	–
C	–	45	48	–	–	39	–	42
D	51	–	48	–	–	45	48	–
E	–	–	42	39	45	–	36	54
F	–	–	–	–	48	36	–	33
G	–	–	–	42	–	54	33	–

(a) Draw a network diagram to show all the connections and costs between the seven towns and the dam.



(b) What is the cost and cheapest connection from:

(i) Lake Algebra to Town G?

(ii) Lake Algebra to Town G via Town F?

11. Draw a graph to represent this information.
Student services is manned by five volunteers from Monday to Friday. One volunteer is required per day.

- Abbie is available Monday, Tuesday and Friday.
- Bob is available Tuesday and Thursday.
- Cathy is available Wednesday.
- Dennis is available Tuesday, Wednesday and Thursday.
- Esther is available Monday and Friday.

12. Determine the number of

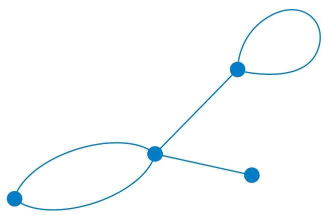
(i) vertices

(ii) edges

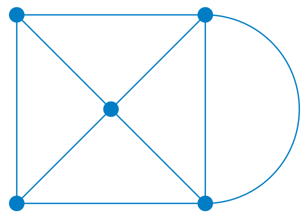
(iii) faces

for each graph below

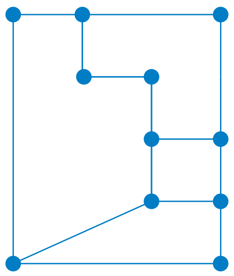
(a)



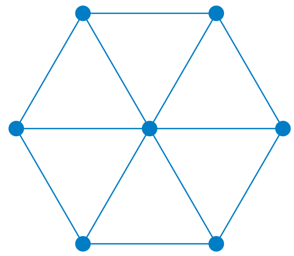
(b)



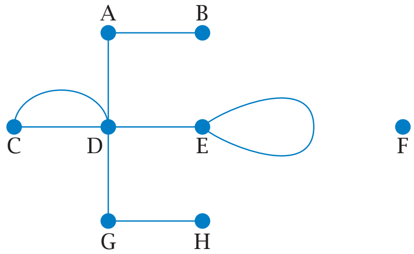
(c)



(d)



13. From the graph below state:



(a) an isolated vertex

(b) a vertex adjacent to C

(c) the vertex containing a loop

(d) two vertices connected by multiple edges.

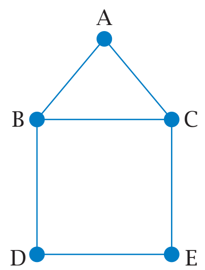
14. Which of these graphs are:

(a) simple?

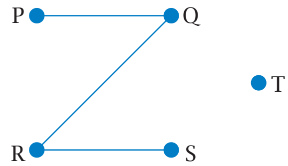
(b) complete?

(c) not connected?

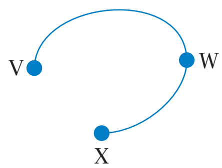
(i)



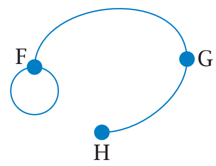
(ii)



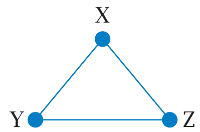
(iii)



(iv)



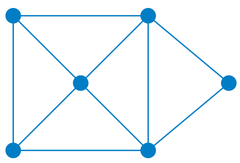
(v)



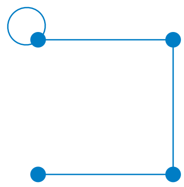
15. Draw a connected graph with two vertices of order 2, two of order 4 and two of order 3.

16. Are the graphs below simple?
Explain.

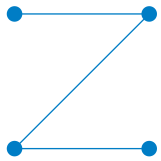
(a)



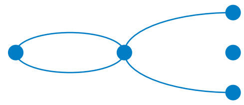
(b)



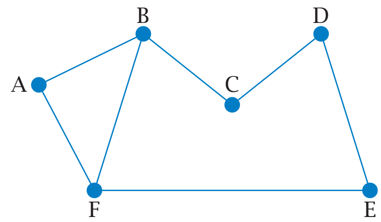
(c)



(d)



17. From the graph below state:



(a) a path from A to F

(b) the order of each vertex

Draw

(c) a subgraph from the graph above.

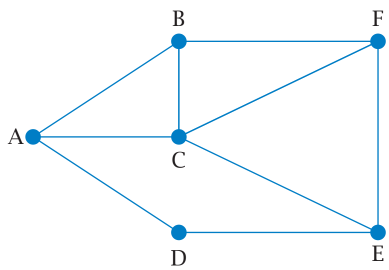
18. Using the graphs below state:

(i) the number of paths from A to B

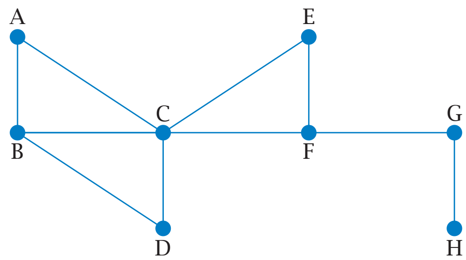
(ii) a cycle passing through D and E

(iii) the degree of each vertex.

(a)

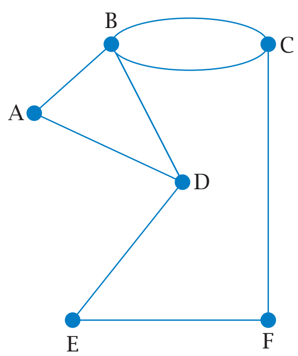


(b)

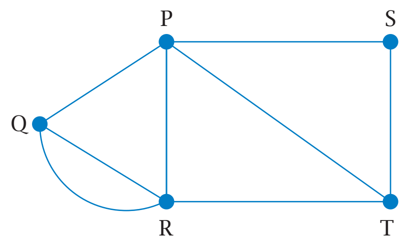


19. Determine whether the following graphs are (i) Eulerian
(ii) semi-Eulerian
(iii) neither

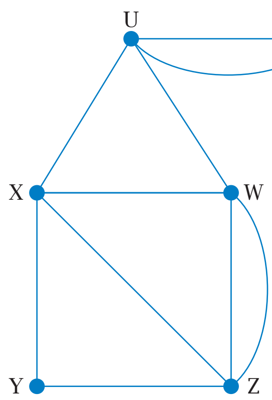
(a)



(b)

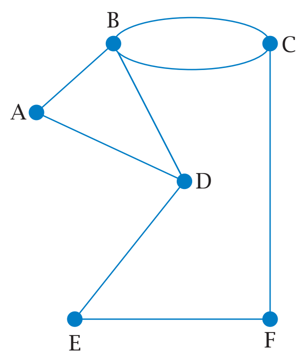


(c)

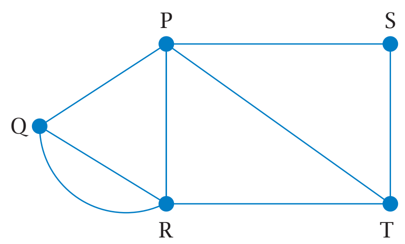


20. Determine whether the following graphs are
(i) Hamiltonian
(ii) semi-Hamiltonian
(iii) neither

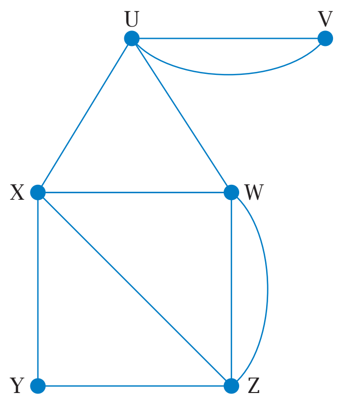
(a)



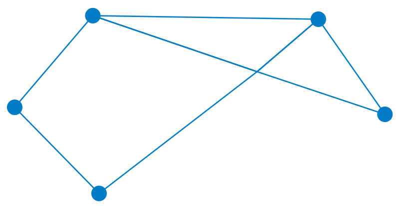
(b)



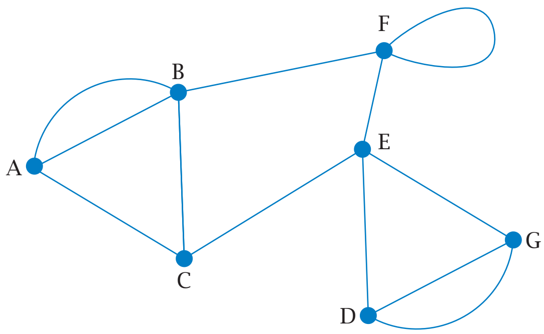
(c)



21. Determine whether the graph below is planar. Check your answer by using Euler's formula.



22. Construct an adjacency matrix for the graph below:



23. Draw the graph which corresponds to the following adjacency matrix.

(a)

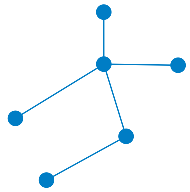
	A	B	C	D
A	0	1	2	0
B	1	0	1	1
C	2	1	1	1
D	0	1	1	0

(b)

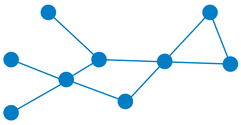
	A	B	C	D	E
A	0	1	0	1	0
B	1	0	1	1	0
C	0	1	0	2	1
D	1	1	2	0	1
E	0	0	1	1	0

24. State whether the following graphs are trees.

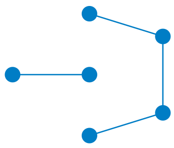
(a)



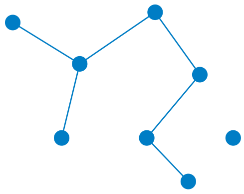
(b)



(c)



(d)



25. Given the following table:

Tree	I	II	III
Number of vertices (v)	5	8	11
Number of edges (c)	4	7	10

(a) Draw tree

(i) I

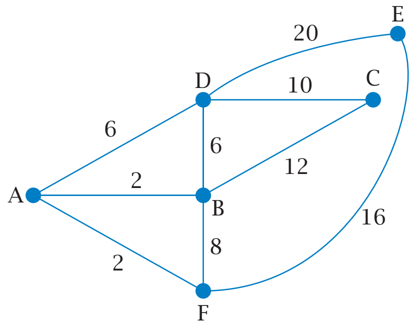
(ii) II

(iii) III

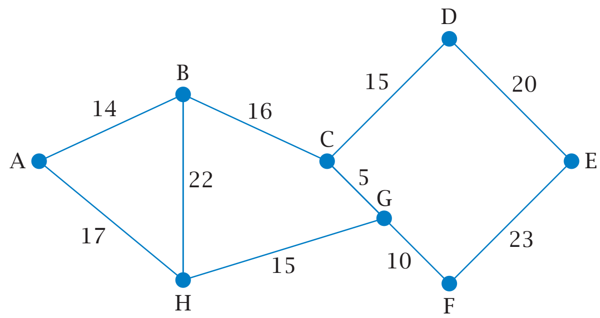
with the above conditions.

(b) Are the trees planar? Explain.

26. Determine a minimum spanning tree for the network below:



27. The road network connecting towns A, B, C, D, E, F, G and H and the travelling time (in minutes) between each is shown below.

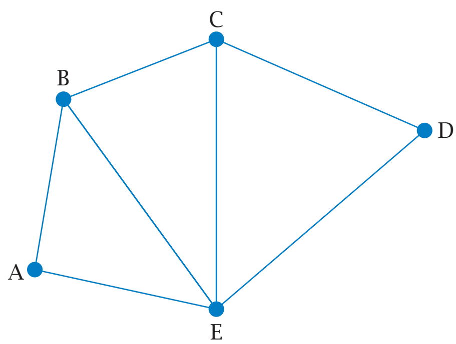


A customer representative from a major department store must visit each town once only starting at town A and finishing at a different town.

(a) Name this type of path where each vertex is visited only once.

(b) List two different possible paths and their travel times.

28.



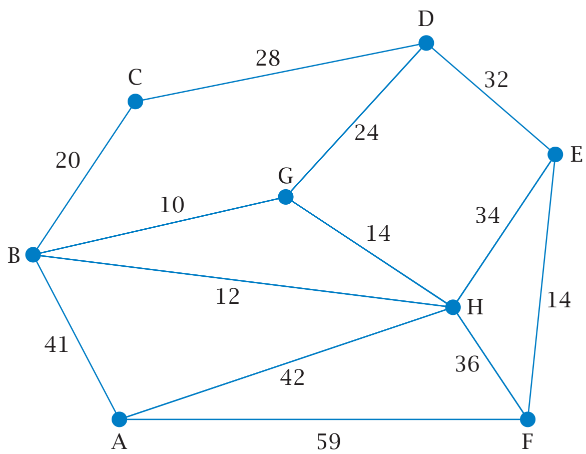
(a) Establish that Euler's formula works for the above graph.

(b) Calculate the degree of each vertex.

(c) Describe what is meant by a semi-Eulerian trail.

(d) Does the graph above have a semi-Eulerian trail? If so, state the trail.

29. Units are in metres



(a) Find a minimum spanning tree for the network above.

(b) State the length of the minimum spanning tree.

30. The following adjacency matrix represents six people A, B, C, D, E and F and six tasks 1, 2, 3, 4, 5, 6 to which they could be assigned.

	A	B	C	D	E	F
1	1	0	1	0	0	0
2	1	1	0	1	0	0
3	0	0	0	1	1	0
4	1	0	0	1	1	0
5	0	1	1	0	1	0
6	1	1	1	0	1	1

(a) Draw a bipartite graph to represent the above matrix.

(b) Is the graph a complete bipartite graph? Explain.

31. (a) A simple connected graph contains 6 vertices.
Determine the minimum and maximum number of edges.

(b) A simple connected graph contains n vertices.
Determine the mimimum and maximum number of edges.

(c) A simple graph contains 4 vertices with each vertex having the same degree d.

(i) Determine all possible values of d.

(ii) If the simple graph is also connected, determine all possible values of d.

(iii) If the simple graph is also Eulerian, determine all possible values of d.

32. A complete graph K_n where $n > 1$ has each vertex n connected to every other vertex by a single edge.

(a) Construct a complete graph for K_5.

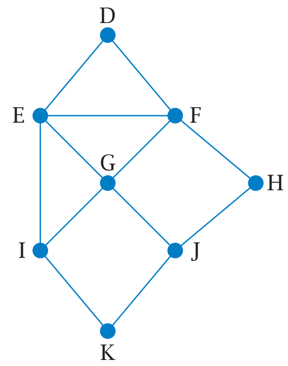
(b) Construct an adjacency matrix for K_4.

(c) Determine total number of edges for K_6.

(d) Determine if K_6 is Eulerian.

(e) State in terms of n when K_n is Eulerian

33. Eight towns are represented by the vertices on the network below. Each of these towns are connected by roads (edges).



(a) Determine the degree of each vertex.

Each town must be visited, starting at town I and travelling along each road exactly once.

(b) Which town will be visited last?

(c) How many of the towns will be visited twice?

(d) What is the name of this trail?

(e) If each town must be visited exactly once determine a path which starts at D and finishes at E.

(f) What is the name of this type of path?

34. The table below shows the cost in thousands of dollars to install underground computer cables directly between six towns A, B, C, D, E and F.

	A	B	C	D	E	F
A	—	45	—	15	—	40
B	45	—	48	—	—	—
C	—	48	—	—	39	36
D	15	—	—	—	45	40
E	—	—	39	45	—	38
F	40	—	36	40	38	—

(a) Use Prim’s algorithm to determine the minimum spanning tree.

(b) Calculate the cost of installing the cables along this spanning tree.

35. The distance in metres between eight buildings is shown in the table below. Computer cabling must be installed to connect all buildings.

	A	B	C	D	E	F	G	H
A	—	25	—	20	12	14	—	16
B	25	—	20	—	17	—	21	—
C	—	20	—	14	—	—	—	17
D	20	—	14	—	—	17	—	—
E	12	17	—	—	—	—	18	23
F	14	—	—	17	—	—	—	—
G	—	21	—	—	18	—	—	21
H	16	—	17	—	23	—	21	—

(a) Determine the minimum spanning tree for this network.

(b) What is the minimum amount of cabling needed to connect all eight buildings?

36. The table below shows the cost in thousands of dollars of constructing direct paths between seven new school buildings.

	A	B	C	D	E	F	G
A	—	45	52	—	—	51	34
B	45	—	42	63	31	32	56
C	52	42	—	41	40	—	—
D	—	63	41	—	43	67	—
E	—	31	40	43	—	30	63
F	51	32	—	67	30	—	46
G	34	56	—	—	63	46	—

(a) Use Prim's algorithm to determine the minimum spanning tree.

(b) State the cost of building the paths.

(c) Draw the minimum spanning tree.

(d) If the cost of building the path connecting F to G was reduced by 50% describe the effect on the minimum spanning tree.

TIME SERIES DATA

5

Syllabus Checklist

By the end of this chapter, you should be able to:

- ☐ construct time series plots
- ☐ describe time series plots by identifying features such as trend, seasonality, fluctuations and outliers
- ☐ smooth time series data by using a simple moving average
- ☐ calculate seasonal indices by using the average percentage method
- ☐ deseasonalise a time series by using a seasonal index
- ☐ fit a least squares line
- ☐ predict from regression lines making seasonal adjustments

FORMULAE AND DEFINITIONS

Time Series

Time Series Data involves measurements which are taken at regular time intervals. Applications are found where measurements relate to growth or change, eg the monthly sales of coffee beans.

Time Series Data can involve any of the following components:

Random Pattern

Data fluctuates unpredictably with no apparent repetitive cycle.

Trend

The overall pattern of the data (Increasing, Decreasing or Constant) found by smoothing the data.

Cyclic pattern

Data exhibits fluctuations with a repetitive nature.

Seasonal pattern

Data affected by the seasons of the year.

Moving Average

A moving average *smooths* the data. The moving average is an average of a set number of consecutive measurements. The number of data points is determined by the number of measurements in the cycle.

Residuals

A residual is the difference between the observed data and the moving average:

$$\text{Residual} = \text{actual value} - \text{moving average}$$

Seasonal Indices (Average Percentage Method)

Seasonal indices or seasonal components used to deseasonalise time series data are calculated using the Average Percentage Method.

Steps

1. Calculate the yearly average
2. Divide each month by the yearly average
3. Calculate the seasonal index for each month – the average of each month

Seasonally Adjusted Value

Seasonally adjusted value = actual value – seasonal component

Prediction

A straight line (least squares line of regression) can be fitted to the time series data to predict future values.

Predicted Value = predicted trend value × seasonal component

Extrapolation with time series data is *not* reliable due to the number of external parameters.

Deseasonalising Data

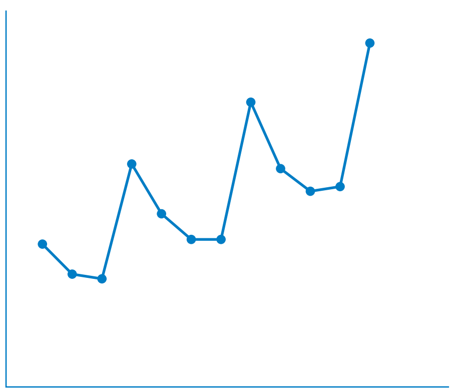
Deseasonalising smooths the data, removes the effects of seasons and allows for predictions based on the fitting of a trend line.

$$\text{Deseasonalised value} = \frac{\text{actual value}}{\text{seasonal index}}$$

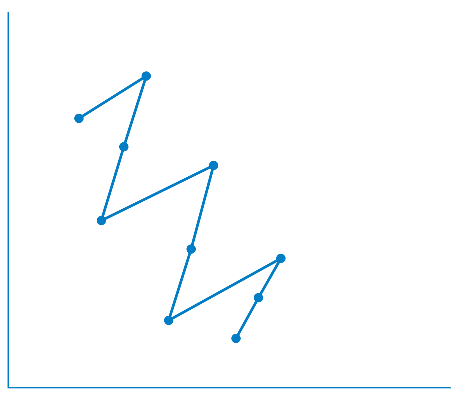
Worked Examples

5.1 Which moving average should be used to smooth the following data.

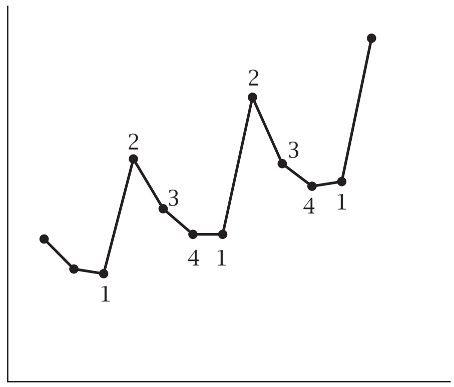
(a)



(b)

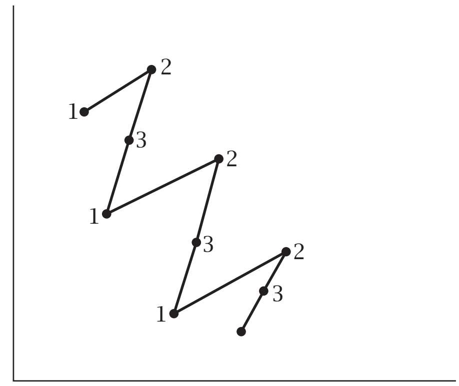


(a)



Cycle = 4
∴ 4 pt Moving Average

(b)



Cycle = 3
∴ 3 pt Moving Average

5.2 Calculate the

(a) 3 point moving average

(b) 4 point centred moving average for the following:

Time	Wages ($)
1	32
2	40
3	46
4	36
5	33
6	44
7	52
8	50
9	42
10	36

(a)

Time	Wages ($)	3 pt Moving Average
1	32	–
2	40 →	$= \frac{(32 + 40 + 46)}{3} = 39.33$
3	46	$= \frac{(40 + 46 + 36)}{3} = 40.67$
4	36	$= \frac{(46 + 36 + 33)}{3} = 38.33$
5	33	$= \frac{(36 + 33 + 44)}{3} = 37.67$
6	44 →	$= \frac{(33 + 44 + 52)}{3} = 43.00$
7	52	48.67
8	50	48.00
9	42 →	42.67
10	36	–

Odd Moving Averages

3 pt : Average data points 1, 2, 3

: Average data points 2, 3, 4

: Average data points 3, 4, 5 etc

5 pt : Average data points 1, 2, 3, 4, 5

: Average data points 2, 3, 4, 5, 6 etc

Even Moving Averages

4 pt : Use five values: sum half the first, half the last and the middle 3 values and then average by dividing by 4.

$$: \frac{\left(\frac{1}{2}\begin{bmatrix}\text{Data}\\ \text{point 1}\end{bmatrix}+2+3+4+\frac{1}{2}\begin{bmatrix}\text{Data}\\ \text{point 5}\end{bmatrix}\right)}{4}$$

$$: \frac{\left(\frac{1}{2}\begin{bmatrix}\text{Data}\\ \text{point 2}\end{bmatrix}+3+4+5+\frac{1}{2}\begin{bmatrix}\text{Data}\\ \text{point 6}\end{bmatrix}\right)}{4}$$

6 pt : Use seven values: sum half the first, half the last, and the middle 5 values and then average by dividing by 6.

(b)

Time	Wages ($)	4 pt Centred Moving Average
1	32	–
2	40	–
3	46	$= \frac{(0.5 \times 32 + 40 + 46 + 36 + 0.5 \times 33)}{4} = 38.625$
4	36	39.250
5	33	40.500
6	44	43.000
7	52	$= \frac{(0.5 \times 33 + 44 + 52 + 50 + 0.5 \times 42)}{4} = 45.875$
8	50	46.000
9	42	–
10	36	–

5.3 Given the following overseas visitor numbers (in thousands):

Year	Quarter	Visitors ('000)	Yearly average ('000)	Quarterly number of visitors as a % of mean
1998	March	42		
	June	36		
	September	38		
	December	68		
1999	March	48		
	June	43		
	September	49		
	December	72		
2000	March	58		
	June	56		
	September	58		
	December	80		

(a) Complete the entries in the table above

(b) Calculate the seasonal indices for each quarter

(c) Deseasonalise (seasonally adjust) the data

(a) Given the following overseas visitor numbers (in thousands):

Year	Quarter	Visitors ('000)	Yearly average ('000)	Quarterly number of visitors as a % of mean
1998	March	42		$\frac{42}{46} = 91.30\%$
	June	36	$\frac{42 + 36 + 38 + 68}{4}$	$\frac{36}{46} = 78.26\%$
	September	38	$= 46$	$\frac{38}{46} = 82.61\%$
	December	68		$\frac{68}{46} = 147.83\%$
1999	March	48		$\frac{48}{53} = 90.57\%$
	June	43	$\frac{48 + 43 + 49 + 72}{4}$	$\frac{43}{53} = 81.13\%$
	September	49	$= 53$	$\frac{49}{53} = 92.45\%$
	December	72		$\frac{72}{53} = 135.85\%$
2000	March	58		$\frac{58}{63} = 92.06\%$
	June	56	$\frac{58 + 56 + 58 + 80}{4}$	$\frac{56}{63} = 88.89\%$
	September	58	$= 63$	$\frac{58}{63} = 92.06\%$
	December	80		$\frac{80}{63} = 126.98\%$

(b) To calculate the seasonal indices *average* the quarterly percentages.

Year	March	June	September	December
1998	91.30%	78.26%	82.61%	147.83%
1999	90.57%	81.13%	92.45%	135.85%
2000	92.06%	88.89%	92.06%	126.98%
Seasonal Indices	91.31%	82.76%	89.04%	136.89%

(c) To remove the effect of the seasons – 'deseasonalise' – divide the *raw* data for each quarter by the appropriate seasonal index.

Year	Quarter	Visitors ('000)	Seasonally adjusted value ('000)
1998	March	42	46
	June	36	43
	September	38	43
	December	68	50
1999	March	48	53
	June	43	52
	September	49	55
	December	72	53
2000	March	58	64
	June	56	68
	September	58	65
	December	80	58

Seasonal Indices

March : 0.9131

June : 0.8276

September : 0.8904

December : 1.3689

Seasonal Adjustments

1998 March

Number of visitors = 42 000

Seasonally adjusted value $= \frac{42\,000}{0.9131}$

$= 45\,997$

$\sim 46\,000$

1999 June

Number of visitors = 43 000

Seasonally adjusted value $= \frac{43\,000}{0.8276}$

$= 51\,957$

$\sim 52\,000$

1998 September

Number of visitors $= 38\,000$

Seasonally adjusted value $= \dfrac{38\,000}{0.8904}$

$= 42\,677$

$\sim 43\,000$

2000 December

Number of visitors $= \dfrac{80\,000}{1.3689}$

$= 58\,441$

$\sim 58\,000$

5.4 The data below shows the number of guests at a hotel recorded quarterly.

Time (t)	Year	Quarter	Number of guests	4 pt Centred Moving Average (m)
1	2001	March	850	–
2		June	130	–
3		September	620	420.00
4		December	95	415.00
5	2002	March	820	410.00
6		June	120	406.88
7		September	590	412.50
8		December	100	418.75
9	2003	March	860	422.50
10		June	130	425.63
11		September	610	–
12		December	105	–
13	2004	March		

(a) Determine the least squares line of regression $M = at + b$ moving average versus time.

(b) Calculate the seasonal indices for each quarter by completing the table below.

Year	March	June	September	December	Average
2001					
2002					
2003					
Seasonal Index					

(c) Using the line of regression and the seasonal indices predict the number of guests for each quarter of 2004.

(a) Enter data into calculator i.e. columns 't' and 'm', starting from $t = 3$.

From the CAS calculator line of regression is:

$$M = 1.295t + 407.99$$

(b) *Seasonal Indices*

Year	March	June	September	December	Average
2001	2.0059	0.3068	1.4631	0.2242	423.75
2002	2.0123	0.2945	1.4479	0.2454	407.5
2003	2.0176	0.3050	1.4311	0.2463	426.25
Seasonal Index	2.0119	0.3021	1.4474	0.2386	

(c) *Predictions*

March 2004

t = 13 Seasonal Index = 2.0119

$M = 1.295t + 407.99$

$M = 1.295(13) + 407.99$

$M = 424.825$

$\therefore$ Predicted value $= 424.825 \times$ seasonal index

$= 424.825 \times 2.0119$

$= 854.71$

$\sim$ 855 guests

June 2004

t = 14 Seasonal Index = 0.3021

$M = 1.295t + 407.99$

$M = 1.295(14) + 407.99$

$M = 426.12$

$\therefore$ Predicted value $= 426.12 \times 0.3021$

$= 128.73$

$\sim$ 129 guests

September 2004

t = 15 Seasonal Index = 1.4474

$M = 1.295t + 407.99$

$M = 1.295(15) + 407.99$

$M = 427.415$

$\therefore$ Predicted value $= 427.415 \times 1.4474$

$= 618.64$

$\sim$ 619 guests

December 2004

$t = 16$ Seasonal Index = 0.2386

$M = 1.295t + 407.99$

$M = 1.295(16) + 407.99$

$M = 428.71$

$\therefore$ Predicted value $= 428.71 \times 0.2386$

$= 102.29$

$\sim$ 102 guests

5.5 The sale of heaters during the years 2007–2010 are shown below:

Quarter		t	Sales (s)	Deseasonalised sales (D)
2007	1	1	23	
	2	2	35	
	3	3	65	
	4	4	33	
2008	1	5	24	
	2	6	45	
	3	7	78	
	4	8	36	
2009	1	9	26	
	2	10	43	
	3	11	80	
	4	12	46	
2010	1	13	30	
	2	14	51	
	3	15	92	
	4	16	52	

(a) Seasonal indices are:

Q1 : 54.53%

Q2 : 91.74%

Q3 : 166.20%

Q4 : 87.53%

Calculate the values in the deseasonalised column.

(b) Determine the least squares line of regression

$D = at + b$ where

D = deseasonalised data

t = time period

(c) Use the line of regression and the seasonal indices to predict the sales for Q1 and Q4 of 2011.

(a) Deseasonalised data = $\dfrac{\text{sales}}{\text{seasonal index}}$

Quarter		t	Sales (s)	Deseasonalised sales (D)
2007	1	1	23	$\dfrac{23}{0.5453} = 42$
	2	2	35	$\dfrac{35}{0.9174} = 38$
	3	3	65	$\dfrac{65}{1.6620} = 39$
	4	4	33	$\dfrac{33}{0.8753} = 38$
2008	1	5	24	44
	2	6	45	49
	3	7	78	47
	4	8	36	41
2009	1	9	26	48
	2	10	43	47
	3	11	80	48
	4	12	46	53
2010	1	13	30	55
	2	14	51	56
	3	15	92	55
	4	16	52	59

(b) Using CAS Calculator and the values in columns t and D

$D = 1.2956t + 36.425$

(c) **Prediction**

Q1 2011

t = 17 Seasonal Index = 0.5453

$D = 1.2956t + 36.425$

$D = 1.2956(17) + 36.425$

$D = 58.4502$

∴ Predicted value = 58.4502 × seasonal index

= 58.4502 × 0.5453

= 31.87

~ 32 sales

Q4 2011

t = 20 Seasonal Index = 0.8753

$D = 1.2956t + 36.425$

$D = 1.2956(20) + 36.425$

$D = 62.337$

∴ Predicted value = 62.337 × 0.8753

= 54.56

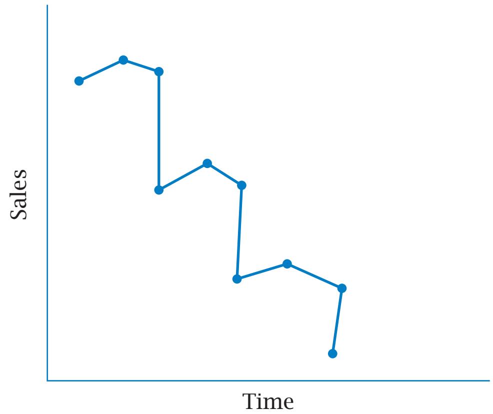
~ 55 sales

PROBLEMS TO SOLVE

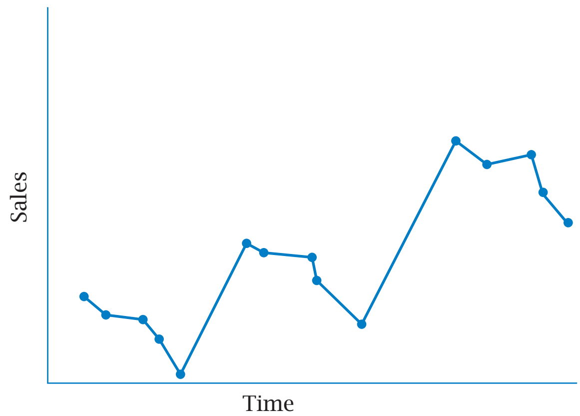
CHAPTER 5: TIME SERIES DATA

1. Describe the overall trends suggested by each of the following time series.

(a)



(b)



2. Determine the most appropriate moving average for each of the time series graphs in Question 1 above.

3. Daily petrol prices in 2009 were recorded over a 3 week period. Details are shown in the table below:

Time		Price (cents)
Week 1	Monday	86.2
	Tuesday	85.0
	Wednesday	84.7
	Thursday	84.8
	Friday	87.1
	Saturday	90.3
	Sunday	94.4
Week 2	Monday	84.3
	Tuesday	82.7
	Wednesday	84.9
	Thursday	84.9
	Friday	84.7
	Saturday	86.3
	Sunday	92.1
Week 3	Monday	84.2
	Tuesday	78.2
	Wednesday	81.4
	Thursday	82.3
	Friday	83.5
	Saturday	85.3
	Sunday	91.2

(a) Calculate the 7 point moving averages.

(b) On a graph plot the original petrol prices and the moving averages.

(c) Comment on the trend of the data.

4. A shop sales figures over a six day cycle are shown below:

Day	Sales ($)
Monday	3020
Tuesday	3130
Wednesday	3360
Thursday	3720
Friday	3090
Saturday	3870
Monday	3050
Tuesday	3240
Wednesday	3490

(a) Calculate the 6 point centred moving averages.

(b) Draw on the same set of axes the original sales and the moving averages.

5. Quarterly sales figures for wood heaters were 1250. The seasonal index for the quarter was 0.872.

Calculate the deseasonalised value.

6. Seasonal Indices for the first 3 quarters are:

Q1 : 127%
Q2 : 71%
Q3 : 114%

Determine the seasonal index for Q4.

7. A company sells and distributes computers. Its deseasonalised sales figures for the first quarter of 2014 were 1374. The seasonal index was 0.85.

Determine the actual number of computers sold.

8. Sales of icecream for 2012–2014 per quarter are shown below.

	Q1	Q2	Q3	Q4	Average
2012	140	36	61	170	101.75
2013	155	45	68	185	
2014	172	51	75	201	
Seasonal Index		0.3867			

(a) Calculate the missing entries in the table above.

(b) What does the Seasonal Index for Q1 and Q3 indicate about the sales of icecream?

9. The following data shows the airconditioner sales for the 4 seasons from 2012 to 2014:

Year	Summer	Autumn	Winter	Spring	Average
2012	1250	576	1110	620	
2013	1370	624	1234	710	
2014	1426	720	1326	830	
Seasonal Index					

(a) Calculate the average for the years 2012–2014.

(b) Calculate the Seasonal Indices for Summer, Autumn, Winter and Spring.

(c) What does the Summer Seasonal Index indicate about the sales of airconditioners?

10. Sales of records are shown in the table below:

Quarter		t	Sales	Deseasonalised Sales
2003	1	1	47	
	2	2	56	
	3	3	63	
	4	4	38	
2004	1	5	49	
	2	6	51	
	3	7	67	
	4	8	39	
2005	1	9	53	
	2	10	60	
	3	11	76	
	4	12	46	

Calculate the values in the 'deseasonalised sales' column

11. The following time series data shows the daily sales for a shop over a 4 week period:

Week	Day	Time (*t*)	Sales ('000)	Moving Average (*m*)
1	Monday	1	241	
	Tuesday	2	195	
	Wednesday	3	133	
	Thursday	4	130	
	Friday	5	235	
2	Monday	6	215	
	Tuesday	7	187	
	Wednesday	8	100	
	Thursday	9	93	
	Friday	10	210	
3	Monday	11	195	
	Tuesday	12	180	
	Wednesday	13	90	
	Thursday	14	90	
	Friday	15	191	
4	Monday	16	181	
	Tuesday	17	168	
	Wednesday	18	74	
	Thursday	19	70	
	Friday	20	175	

(a) Calculate the least squares regression line $M = at + b$.

(b) Calculate the seasonal indices for each day.

(c) Using the line of regression and seasonal indices predict the sales for each day of week 5.

12. A fast food chain collects data on quarterly profits (in thousands of dollars). The results are shown in the table below:

Quarter	Time Period (t)	Profits (p)	CMA
Mar 2012	1	C	
June 2012	2	9.65	
Sept 2012	3	12.89	10.59
Dec 2012	4	10.65	12.57
Mar 2013	5	13.04	A
June 2013	6	17.76	16.50
Sept 2013	7	20.17	18.62
Dec 2013	8	19.42	21.01
Mar 2014	9	21.23	23.31
June 2014	10	28.71	B
Sept 2014	11	27.56	
Dec 2014	12	25.83	

(a) Determine what moving average has been used in the table above.

(b) Calculate the values of A, B and C.

(c) Calculate the seasonal indices.

(d) Determine the least squares line of regression of moving averages against time.

(e) Find the predicted profit for March and September 2015.

(f) Calculate the seasonally adjusted profits for June and December 2013.

13. The following data relates to the number of visitors to a zoo recorded quarterly:

Year	Quarter	Number of visitors (v)	4 PT CMA
2010	March	870	–
	June	130	–
	September	650	586.25
	December	720	577.50
2011	March	820	A
	June	110	491.25
	September	B	426.25
	December	150	436.25
2012	March	870	447.50
	June	140	449.38
	September	660	–
	December	C	–
2013	March		

(a) Why is it appropriate to use a 4 pt centred moving average?

(b) Calculate the values of A, B and C.

The least squares line of regression is $M = -24.65t + 658.12$

(c) Explain what this equation indicates about the trend of visitors.

(d) Use the trend to find the average annual change in visitor numbers.

(e) Calculate the seasonal indices for each quarter.

(f) Use the line of regression and seasonal indices to predict visitor numbers for March 2015.

(g) Comment on the reliability of this prediction.

(h) Seasonally adjust visitor numbers for September 2011.

14. The daily sales figures for a canteen (in hundreds of dollars) are shown in the table below.

Week	Day	Sales	*D*
1	Monday	1.8	
	Tuesday	1.2	
	Wednesday	5.3	
	Thursday	12.6	
	Friday	18.3	
	Saturday	20.4	
2	Monday	3.6	
	Tuesday	1.4	
	Wednesday	8.9	
	Thursday	14.2	
	Friday	20.1	
	Saturday	23.6	
3	Monday	7.2	
	Tuesday	1.8	
	Wednesday	10.1	
	Thursday	16.3	
	Friday	22.4	
	Saturday	25.8	

(a) Seasonal Indices are:

Monday : 33.29%

Tuesday : 12.23%

Wednesday : 66.74%

Thursday : 120.83%

Friday : 170.99%

Saturday : 195.92%

Calculate the values in the deseasonalised column (*D*).

(b) Determine the least squares line of regression of deseasonalised sales against time t:

$$\text{i.e. } D = at + b$$

(c) Use the line of regression and the seasonal indices to predict the sales for each day of week 4.

15. Petrol prices (in cents/litre) were recorded in 1996 over a 3 week period. The results are recorded in the table below.

Week	Day	Price (P)	Deseasonalised price (D)
1	Monday	63.4	68.3
	Tuesday	63.5	67.2
	Wednesday	70.1	69.6
	Thursday	71.3	Q
	Friday	71.8	68.8
	Saturday	72.3	70.4
	Sunday	70.4	68.7
2	Monday	64.2	69.1
	Tuesday	65.1	68.9
	Wednesday	70.0	69.5
	Thursday	71.1	R
	Friday	73.5	70.4
	Saturday	71.9	70.0
	Sunday	70.8	69.1
3	Monday	66.4	71.5
	Tuesday	68.9	72.9
	Wednesday	70.3	69.8
	Thursday	71.2	S
	Friday	72.6	69.6
	Saturday	70.3	68.4
	Sunday	72.9	71.1

The Seasonal Indices are:

Monday : 92.88%

Tuesday : 94.54%

Wednesday : 100.74%

Thursday : P

Friday : 104.33%

Saturday : 102.72%

Sunday : 102.50%

(a) Calculate the values of P, Q, R and S

(b) Determine the least squares regression line

$$D = at + b$$

(c) Use the regression line and the seasonal indices to predict the petrol price for Sunday of Week 4.

6 FINANCE – SIMPLE AND COMPOUND INTEREST

Syllabus Checklist

By the end of this chapter, you should be able to:

- ☐ use a recurrence relation to model a compound interest loan or investment and investigate changing the interest rate and the number of compounding periods
- ☐ calculate the effective annual rate of interest and compare investment returns when interest is daily, monthly, quarterly or six monthly
- ☐ solve problems involving compound interest loans, investments and depreciating assets

FORMULAE AND DEFINITIONS

Simple Interest

Interest is constant and calculated on the initial investment.

Formula: $SI = P \times R \times T$

SI = simple interest

P = principal

R = interest rate (as a decimal)

T = time (in years)

The graph of simple interest versus time will be an increasing *linear* graph.

Compound Interest

Interest is added to the account and itself earns interest in subsequent time periods. As a result interest grows each year and the account grows exponentially.

Formula: $$A = P\left(1+\frac{r}{n}\right)^{nt}$$

A = amount in the account

P = principal

r = yearly interest rate (as a decimal)

n = number of compounding periods each year

t = time (in years)

$$\text{Compound Interest} = P\left(1+\frac{r}{n}\right)^{nt} - P$$

Recurrence Relation

$$T_{n+1} = rT_n$$

$$t_0 = a$$

$$r = \left(1 + \frac{\text{interest rate}}{100}\right)$$

$$a = \text{principal}$$

$$T_{n+1} = T_n \times \left(\frac{1 + \text{interest rate for compounding period}}{100}\right)$$

The graph of an amount earning compound interest over time will be an increasing *exponential* graph.

Effective Interest Rate

The *effective* interest rate is used to calculate the *actual* interest rate that would be paid on a loan (or investment) if the specified interest rate is affected by compounding periods.

Formula: $$e = \left(1 + \frac{r}{n}\right)^n - 1$$

e = effective interest rate

r = yearly interest rate (as a decimal)

n = number of compounding periods each year

Depreciating Assets

Depreciation refers to the decrease in the value of assets. A common method is the *Reducing Balance Depreciation.*

Worked Examples

6.1 Terry deposited $15 000 into his savings account. The savings account earns interest at a flat rate of 11.5% p.a.

Find the amount of interest in:

(a) 2 years

(b) 10 months

(a) Simple Interest $= P \times R \times T$

$= 15\,000 \times 0.115 \times 2$

$= \$3450$

(b) Simple Interest $= P \times R \times T$

$= 15\,000 \times 0.115 \times \frac{10}{12}$

$= \$1437.50$

6.2 Jody borrowed \$3600 for a period of 16 months. The amount she repaid was \$4056.

What simple interest rate was Jody charged for the loan?

$$\begin{aligned}
\text{Interest} &= \$4056 - \$3600 \\
&= \$456 \\
\text{Simple Interest} &= P \times R \times T \\
456 &= 3600 \times R \times \frac{16}{12} \\
R &= 0.095 \\
\text{Interest rate} &= 9.5\% \text{ p.a.}
\end{aligned}$$

6.3 Gwen has borrowed \$7000 at 14% p.a. simple interest. The loan is to be repaid monthly over 4 years.

(a) How much is to be repaid?

(b) Determine the monthly repayments.

(a)
$$\begin{aligned}
\text{Simple Interest} &= P \times R \times T \\
&= 7000 \times 0.14 \times 4 \\
&= \$3920 \\
\text{Total to repay} &= \$7000 + \$3920 \\
&= \$10\,920
\end{aligned}$$

(b)
$$\begin{aligned}
\text{Monthly repayments} &= \frac{\$10\,920}{48} \leftarrow \text{number of months in 4 years} \\
&= \$227.50
\end{aligned}$$

6.4 \$100 000 is invested at an interest rate of 7% p.a. compounded annually.

Determine:

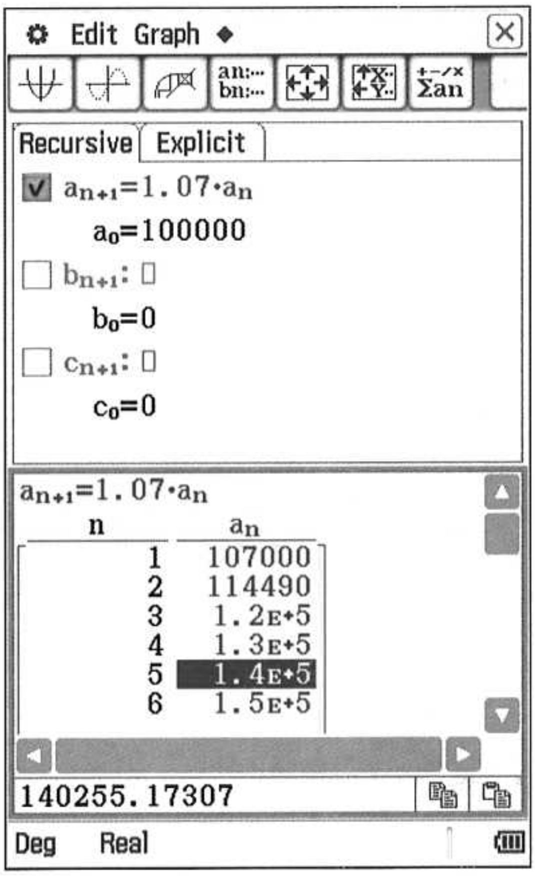
(a) The recursive formula.

(b) The value of the investment at the end of the fifth year.

(a) $T_{n+1} = 1.07\ T_n$, $T_0 = \$100\,000$

Increase of 7% (points to 1.07); Initial Value (points to T_0)

(b) Using CAS calculator



Investment is valued at $140 255.17

6.5 Determine the interest on a loan of:

(a) $15 000 at 5% p.a. compounded monthly over 3 years.

(b) $205 000 at 9.5% p.a. compounded quarterly over 20 years.

(a) Using formula:

$$A = P\left(1+\frac{r}{n}\right)^{nt}$$

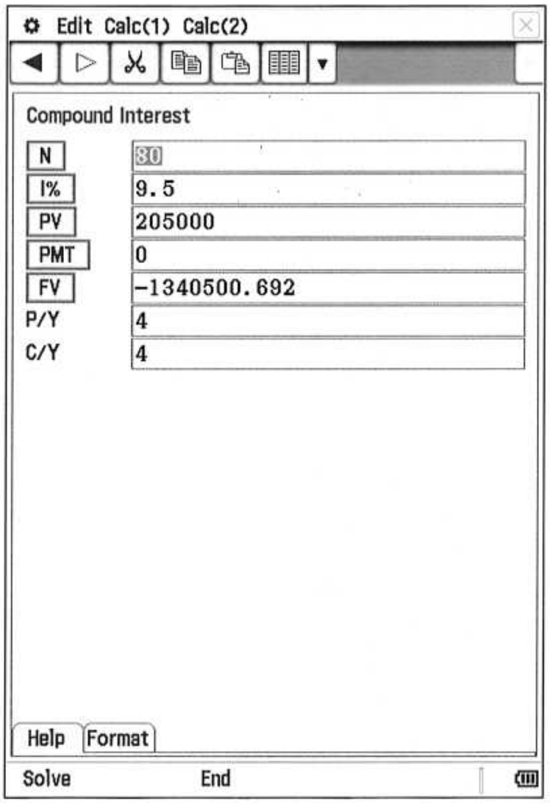
$$A = 15\,000\left(1+\frac{0.05}{12}\right)^{36}$$

$$A = \$17\,422.08$$

$$\text{Interest} = \$17\,422.08 - \$15\,000$$

$$= \$2422.08$$

(b) Using CAS calculator:



$$\text{Interest} = \$1\,340\,500.69 - \$205\,000$$

$$= \$1\,135\,500.69$$

6.6 At the start of the year, Amy deposits \$2000 into an account earning interest compounding quarterly. The effective compound interest rate is 6.80%.

Determine:

(a) The quarterly interest rate.

(b) The total interest earned in 3 years.

(a)

$$\text{Effective Interest} = \left(1 + \frac{r}{n}\right)^n - 1$$

$$0.068 = \left(1 + \frac{r}{n}\right)^4 - 1$$

$$1.068 = \left(1 + \frac{r}{n}\right)^4$$

$$\sqrt[4]{1.068} = \left(1 + \frac{r}{n}\right)$$

$$1.0166 = 1 + \frac{r}{n}$$

$$0.0166 = \frac{r}{n}$$

$$\text{Quarterly interest rate} = 1.66\%$$

(b) $$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

$$A = 2000\,(1 + 0.0166)^{(4\times3)}$$

$$A = \$2436.86$$

$$\text{Interest} = \$2436.86 - \$2000$$

$$= \$436.86$$

6.7 Veronica wishes to invest an amount of money into an account earning the best interest.

The two options are:

(a) 9.4% p.a. compounded monthly

(b) 9.5% p.a. compounded quarterly.

Calculate the *effective rate of interest* for both options.

(a) Effective interest rate $= \left(1 + \frac{r}{n}\right)^{n} - 1$

Interest rate (r)

number of compounding periods in a year (n)

$$= \left(1 + \frac{0.094}{12}\right)^{12} - 1$$

$$= 0.0982$$

$$= 9.82\% \text{ p.a.}$$

(b) Effective interest rate $= \left(1 + \frac{r}{n}\right)^{n} - 1$

$$= \left(1 + \frac{0.095}{4}\right)^{4} - 1$$

$$= 0.0984$$

$$= 9.84\% \text{ p.a.}$$

Best option is the account earning 9.5% p.a. compounded quarterly as it pays a *higher* effective rate of interest.

6.8 An asset is valued at $100 000.

Determine its value at the end of each of the first three years using a reducing balance depreciation of 15% p.a.

Initial Value:	$100 000
Value end of 1st year:	$100 000 × 0.85 = $85 000
Value end of 2nd year:	$85 000 × 0.85 = $72 250
Value end of 3rd year:	$72 250 × 0.85 = $61 412.50

Depreciation	1st year:	$15 000
	2nd year:	$12 750
	3rd year:	$10 837.50

PROBLEMS TO SOLVE

CHAPTER 6: FINANCE – SIMPLE AND COMPOUND INTEREST

1. Matt purchases a new TV by paying $350 per month for 18 months. The cash price of the TV was $5250.

   Calculate:

   (a) the total amount paid

   (b) the interest paid

   (c) the simple interest rate.

2. The price of a new washing machine is $2520. Laura pays a deposit of 20% with the remainder to be paid over 2.5 years. The monthly repayments are $72.45.

   Calculate:

   (a) the deposit

   (b) the total amount paid for the washing machine

   (c) the flat rate of interest for the monthly repayments.

3. Gerry borrows $7000. He repays monthly installments over 6 years. Simple interest is paid at a rate of 9% p.a.

   Calculate:

   (a) the total to be repaid.

   (b) the monthly repayment.

4. Alex borrowed $7000 to purchase a car. He repaid the full amount of $7262.50 which includes the simple interest over a period of 10 months.

   What was the simple interest rate?

5. Kelsey deposited $15 000 into a bank account at a rate of 8% over 5 years.

   Calculate:

   (a) the simple interest

   (b) the compound interest if compounded monthly

   (c) the difference between the simple interest and compound interest.

6. (a) Determine the length of time required for $10 400 to grow to $12 870 if the money is invested at 5% p.a. simple interest.

   (b) The interest earned on an amount of money invested for 2 years at 12% p.a. compounded annually is $1424.64.

   What is the simple interest earned on the same amount, time and interest rate.

7. Jeremy invests $12 000 into an account earning interest compounded monthly. In 4 years the account grows to $14 945.41. Determine the annual interest rate.

8. Fiona has $10 000 to invest. She has 3 options to consider.

Option 1: Simple Interest at 5% p.a.

Option 2: Compound Interest at 4% p.a. compounded monthly.

Option 3: Compound Interest at 4.5% p.a. compounded quarterly.

(a) Construct graphs that show the value of each option for *n* years, for values of *n* from 0 to 6.

(b) Describe the type of graph obtained for each of the three options.

(c) Which is the best option for Fiona if she invests the $10 000 for 6 years? Why?

9. Determine the length of time an investment will take to double in value if it is invested at 6.4% p.a. compounded monthly.

10. Matthew must save $27 000 for a holiday. How long will it take if he invests $10 000 in a term deposit earning 9.5% p.a. compounded monthly?

11. Bob wishes to invest $35 000. Which plan should he choose and why?

Plan 1: 5.95% p.a. compounded quarterly.

Plan 2: 5.9% p.a. compounded monthly.

12. A company offers a new investment scheme. For the first 4 years investors are paid 8% p.a. simple interest. From year 5 onwards investors are paid an additional 2% p.a.

(a) Joanna invests $15 000 in this new investment scheme for 3 years. How much does she accumulate?

(b) Joseph invests $*x* and, after 6 years, accumulates $3758.40 less than Joanna. What was his initial investment?

13. A bank offers a choice of two fixed term deposits.

Option A — simple interest at 7.25% p.a. for 4.5 years.

Option B — simple interest at 7.55% p.a. for 4.25 years.

David invests the same amount in both Option A and Option B. Option A earns $36.55 more than Option B.

Determine:

(a) David's original investment

(b) the interest rate if the original investment accumulates to $11 798 after 5 years.

14. Simon invests an amount of money received from his grandparents. At the end of the first year the investment is worth $7660.80 and at the end of the second year $8227.70.

If interest is compounded annually, determine:

(a) the interest rate p.a.

(b) the original investment.

15. Bruce deposits \$5000 at the beginning of the year in an account earning interest compounded quarterly. The effective annual compound interest is 8.20%.

(a) What is the quarterly interest rate?

(b) How much interest is earned in 3½ years?

16. Bree invests \$500 for 8 years. The investment plans offered by the bank are:

- Plan I Simple interest at 15% p.a.
- Plan II Compound interest at 13.75% p.a. compounded annually.
- Plan III Compound interest at 13.5% p.a. compounded quarterly.

(a) Determine with reasons the best investment plan.

(b) Calculate the effective yearly interest rate for the third plan.

(c) State the recursive formula for Plan II.

(d) Determine the value of Plan II after 4 years.

17. Leonie wishes to invest some money. The investment deals are:

- 6.75% p.a. compounded daily
- 6.8% p.a. compounded monthly
- 6.9% p.a. compounded six monthly

What is the best effective interest rate and hence the investment deal Leonie should choose?

18. Which investment provides the best return?

Investment 1: 7.12% p.a. compounded quarterly

Investment 2: 7% p.a. compounded daily.

19. A car is initially valued at \$72 000.

Determine its value at the end of the first four years using a reducing balance depreciation of 8% p.a.

20. An asset is initially valued at \$250 000.

Determine how much the asset depreciates each year in its first three years using a reducing balance depreciation of 12% p.a.

FINANCE – LOANS AND ANNUITIES

7

Syllabus Checklist

By the end of this chapter, you should be able to:

- ☐ use a recurrence relation to model a reducing balance loan and investigate the effect of interest rate and repayment amount on the loan
- ☐ solve problems involving reducing balance loans
- ☐ use a recurrence relation to model an annuity and investigate the effect of the amount invested, the interest rate and the payment amount on the duration of the annuity
- ☐ solve problems involving annuities

FORMULAE AND DEFINITIONS

Annuity

An annuity is a regular sum of money paid to someone each year for a fixed period of time. An amount of money is invested, and compounded at a fixed rate.

Perpetuity

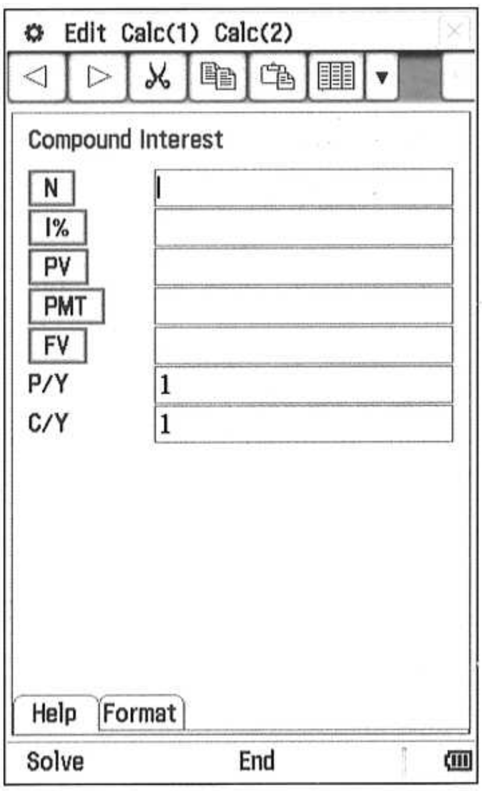
A special type of annuity where the regular sum of money paid to someone each year *lasts forever.*

Formula: Present value $= \dfrac{\text{payment}}{\text{rate}}$

Calculations

Using the financial program on a CAS calculator.

CAS calculator display



Calculator Notation

N – the number of time periods

I% – annual interest rate as a percentage

PV – present value (initial investment)

PMT – amount paid each period. Value of regular payments

FV – future value

P/Y – number of installment periods per year

C/Y – number of times interest is compounded per year.

Worked Examples

7.1 Matt borrows $35 000 for a car. He makes a monthly repayment of $750. Interest is calculated at 8.4% p.a. compounded monthly.

(a) State a recurrence relation that gives the amount still owing after *n* months.

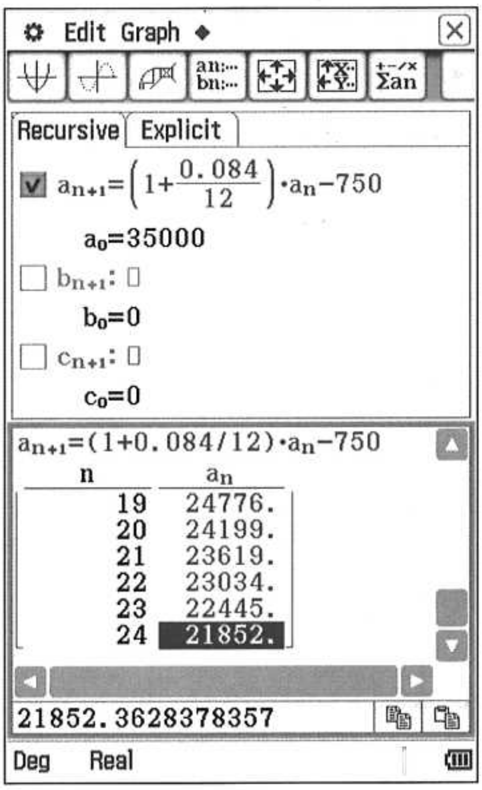
(b) How much is still owing on the loan after 2 years?

(a)
$$T_n = \left(1 + \frac{0.084}{12}\right) T_{n-1} - 750$$

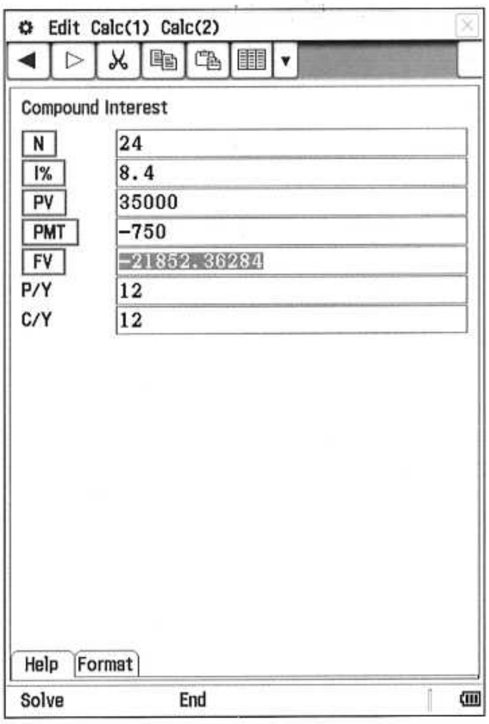
Interest p.a. (0.084); months in the year (12); payment (750)

$$T_0 = \$35\,000$$

(b) Using *SEQUENCE* on CAS calculator:



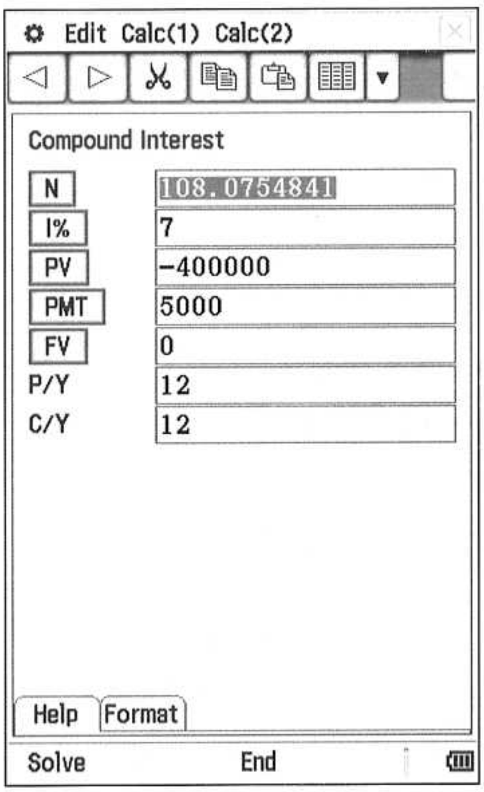
Using the *FINANCIAL* application on the CAS calculator:



The amount still owing after 2 years is $21 852.36.

7.2 Freda deposits $400 000 into an annuity earning interest at 7% p.a. compounded monthly. If monthly repayments of $5000 are made, determine how long the annuity will last.

Using the *FINANCIAL* application on the CAS calculator:



The annuity will last approximately 108 months.

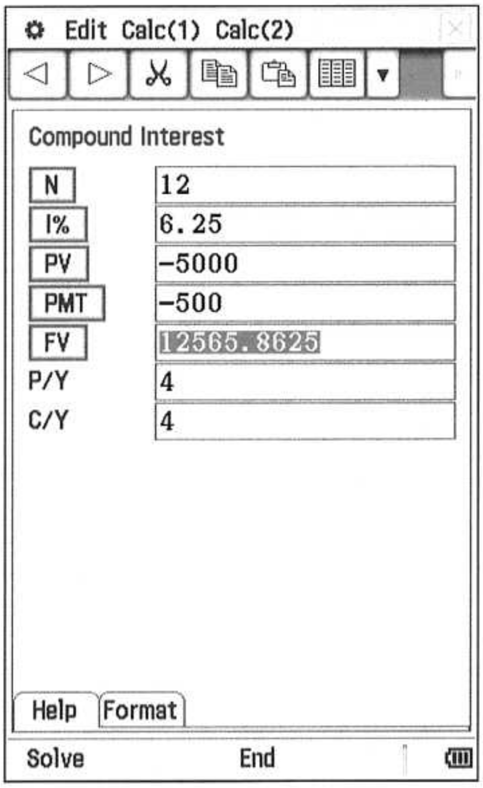
7.3 Leonie invests $5000 into an account earning 6.25% p.a. compounded quarterly. At the end of each quarter she makes further deposits of $500.

Determine:

(a) The value of the investment after 3 years.

(b) The amount of interest earned at the end of the 3rd year.

(a) Using the *FINANCIAL* application on the CAS calculator:



The value of the investment at the end of the 3rd year is $12 565.86.

(b) Total invested = 5000 + (12 × 500)

= $11 000

∴ Interest = $12 565.86 – $11 000

= $1565.86

PROBLEMS TO SOLVE

CHAPTER 7: FINANCE – LOANS AND ANNUITIES

1. Find the following:
   (a) The monthly repayment on \$90 000 borrowed for 25 years at an interest rate of 7% compounded monthly.
   (b) The amount still owing after 4 years on a loan of \$75 000 at 12% p.a. compounded monthly over 15 years if repayments of \$900.13 per month are made.
   (c) The length of time required to pay off a loan of \$20 000 at 12% p.a. compounded quarterly if \$1200 is paid quarterly.
   (d) Veronica borrows \$8000 at 12%, compounded annually. Repayments of \$2000 are made at the end of each year. How much does Veronica still owe at the end of the 3rd year?

2. Determine the regular monthly repayment on a loan of \$25 000 with an interest rate of 7% p.a. compounded monthly if the loan is to be repaid in 6 years.

3. Determine the regular monthly repayment on a loan of \$300 000 with an interest rate of 5% p.a. compounded monthly if the loan is to be repaid in 20 years.

4. Jordan invests \$800 000 into a perpetuity paying 5% p.a. compounded annually. What regular annual payment will Jordan receive?

5. How much should be invested into an annuity earning 7.5% p.a. compounded annually to provide an annual payment of \$36 000 for 20 years?

6. What interest rate is required for Ian to receive \$15 000 at the end of the first 3 months and every 3 months thereafter? \$200 000 is invested into a perpetuity earning interest compounded quarterly.

7. Adam deposits \$350 000 into an annuity earning interest at 8% p.a. compounded monthly. If monthly payments of \$3000 are made, determine how long the annuity will last?

8. How much should be invested into a perpetuity in order for Maria to receive an annual payment of \$45 000? The perpetuity returns 6.7% p.a. compounded annually.

9. An annuity returns interest at a rate of 9.5% p.a. compounded annually. A deposit of \$400 000 is invested in this annuity. If the annuity pays out \$55 000 each year:

   Calculate:
   (a) How many payments are made and the value of the final payment using a recurrence formula?
   (b) The solution to part (a) using the financial application on a CAS calculator.

10. Brian purchases a new audio visual system valued at $22 500. He borrows the money from a financial institution and pays interest at 17.25% p.a. His monthly repayments are $2500.

The spreadsheet below shows the monthly details of the loan.

Month (n)	Balance at the start of the month (T_n)	Interest
1	22 500	323.44
2	20 323.44	292.15
3	18 115.59	260.41
4	15 876.00	228.22
5	13 604.22	195.56
6	11 299.78	162.43
7	8962.21	134.43
8	6596.64	98.95
9	4195.59	62.93
10	1758.53	26.38

(a) Calculate the monthly interest rate.

(b) Determine the final repayment.

(c) Find the total interest paid for the audio-visual system.

(d) Write a recursive formula T_n for the balance at the start of the month.

(e) There was an increase in the interest rate:

(i) Which month was there a rate change?

(ii) Calculate the increase in the annual interest rate.

11. James borrows $4000 for a second hand car. He makes quarterly repayments at the beginning of each quarter except the first of $600. Interest of 3% per quarter is calculated on the balance at the end of each quarter.

(a) Calculate the amount owing at the end of the second quarter.

(b) Determine the number of repayments needed to repay the loan.

(c) What is the final repayment?

12. Morgan wishes to invest in a 15 year annuity scheme. He deposits $1200 at the beginning of each year. Interest is compounded annually at 7% p.a.

(a) Determine the amount accumulated at the end of:

(i) the first year

(ii) the second year.

(b) State a recursive formula T_n in terms of n for the amount accumulated at the end of the nth year.

(c) Calculate T_3 and T_4.

(d) Determine the total amount at the end of the 15th year.

13. Cecelia borrows $150 000 to purchase a house and land package. The interest rate is 18% p.a. with monthly repayments of $2000.

(a) Write a recursive formula for the above situation.

(b) Calculate the amount owing at the end of the first month.

(c) Is the repayment of $2000 per month adequate? Explain.

Cecelia decides to borrow only $100 000 after winning $50 000.

(d) Write another recursive formula for the amount borrowed.

(e) Is the repayment of $2000 per month now adequate? Explain.

14. Terry borrows $12 000 and repays $3500 at the end of each year. Interest is calculated at a reducible rate of 18% p.a.

(a) Complete the following table:

Year	Amount owing	Amount owing after interest	Amount owing after repayment
1	12 000	14 160	10 660
2	10 660		
3			
4			
5			

(b) If Terry wishes to pay off the loan at the end of the 5th year, determine the yearly repayment.

(c) Another financial institution offers Terry a simple interest rate of 17.5% p.a. for the amount borrowed of $12 000. Determine how much Terry still owes at the end of the 5th year. Compare this answer with the answer in part (a) above.

15. Sydney deposits $220 per month into an account paying interest at the end of each month. The table below outlines the progress of this account.

Month	Balance start of month	Deposit	Balance after deposit	Interest	Balance end of month
1	0	220	220	1.65	221.65
2	221.65	220	441.65	3.31	444.96
3	444.96	220	664.96	4.99	669.95
4	669.95	220	889.95	6.67	896.62
5	896.62	220	1116.62	8.37	1125.00
6					

(a) Calculate the annual interest rate.

(b) Complete Month 6 in the table above.

(c) Write a recursive formula for the amount in the account at the end of each month.

(d) Determine the amount in the account at the end of the 18th month.

(e) Calculate the total interest earned at the end of the 18th month.

16. Fred wishes to save money for an overseas holiday by depositing $125 a month into an interest bearing account. He deposits an initial amount of $125 into the account. Interest is calculated at 16% p.a. compounded monthly, and paid at the end of each month. The table below shows the account details.

Month	Amount at beginning of month	Interest	Deposit	Balance
n	T_n			T_{n+1}
1	125	1.67	125	251.67
2	251.67	3.36	125	380.02
3	380.02	5.07	125	510.09
4	510.09	6.80	125	641.89
5	641.89	P	125	Q

(a) Determine the values of P and Q.

(b) Write a recursive formula to determine the amount in the account at the end of each month.

(c) Determine the amount in the account at the end of the 12th month.

(d) Calculate the total interest earned at the end of the 12th month.

17. Jeremy and Debbie borrow $56 000 for an extension on their house. They are able to repay the loan at $2500 per month. Below is a summary of their repayments.

Month	Starting amount	Interest	Repayment	Final amount
1	56 000	350	2500	53 850
2	53 850	336.56	2500	51 686.56
3	51 686.56	323.04	2500	49 509.60
4	49 509.60	309.44	2500	47 319.04
5	A	B	C	D

(a) Determine the annual interest rate.

(b) What are the values A, B, C and D?

(c) Calculate:

(i) the final repayment

(ii) the total interest paid.

18. A $15 000 loan is paid off over 17 months. Interest of 9% p.a. is added at the end of the month followed by a repayment of $950.

The table below shows the amount at the end of each month and the interest payable per month:

Month	Amount at start of month	Interest	Repayment	Balance at end of month
1	15 000	112.50	950	14 162.50
2	14 162.50	106.22	950	13 318.72
3	13 318.72	99.89	950	12 468.61
4	12 468.61	93.51	950	11 612.12
5	11 612.12	87.09	950	10 749.21
6	10 749.21	80.62	950	9879.83
7	9879.83	74.10	950	9003.93
8	9003.93	67.53	950	8121.46
9	8121.46	60.91	950	7232.37
10	7232.37	54.24	950	6336.62
11	6336.62	47.52	950	5434.14
12	5434.14	40.76	950	4524.90
13	4524.90	33.94	950	3608.83
14	3608.83	27.07	950	2685.90
15	2685.90	20.14	950	1756.04
16	1756.04	13.17	950	
17				

(a) Complete the table above.

(b) Write a recursive formula to determine the amount of the loan at the end of the month.

(c) What is the final repayment?

(d) Calculate the total interest.

(e) If repayments of $1000 were made instead of $950, calculate the time taken to repay the loan.

19. Louise purchases a car for $48 000. She pays a 20% deposit and borrows the remaining amount. The interest is 6.25% p.a. and the monthly repayments are $2250. The table below shows details of the loan.

Month	Balance at start of month	Interest
1	38 400	200.00
2	36 350	189.32
3	34 289.32	178.59
4	A	B
.		
.		
.		
16	6530.20	33.87
17	4287.07	22.33

(a) Determine the values of A and B.

(b) One entry is incorrect in the table above. Identify this value.

(c) How long does it take for the loan to fall below $15 000.

(d) Calculate the final repayment.

(e) What is the total interest that Louise paid?

20. Steve borrows $130 000 for 25 years at an interest rate of 7% p.a. compounded monthly. He pays $918.82 per month in repayments. After 12 years the interest rate changes to 5% p.a. compounded monthly and remains this for the remainder of the loan.

(a) If repayments remain the same calculate the length of the loan.

(b) Calculate the total amount repaid.

(c) Determine how much Steve will save as a result of the interest rate reducing.

21. Jeremy deposits $500 000 into an annuity earning 6% p.a. compounded annually. He withdraws in the 1st year $40 000. Each subsequent year he withdraws an amount 3% more than the previous year.

(a) How much will Jeremy withdraw in the 2nd and 3rd year?

(b) How much will be left in the account after the 10th withdrawal?

(c) How many years will it take for the balance to reduce to zero?

DIRECTED NETWORKS AND DECISION MATHEMATICS

8

Syllabus Checklist

By the end of this chapter, you should be able to:

- ☐ construct a project network
- ☐ use forward and backward scanning to determine the earliest and latest starting time
- ☐ locate the critical path(s)
- ☐ use the critical path to determine the minimal time for the project to be completed
- ☐ calculate float times for non-critical activities
- ☐ solve small scale network flow problems including the maximum flow – minimum cut theorem
- ☐ use a bipartite graph/tabular or matrix form to represent an assignment/allocation problem
- ☐ determine the optimum assignment(s) by inspection or by use of the Hungarian algorithm

FORMULAE AND DEFINITIONS

Maximum Flow

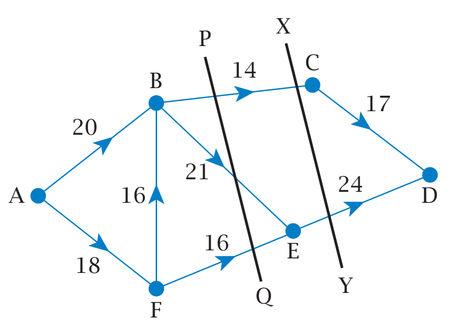
The carrying capacity or flow of a network can be found in situations such as water pipes, transportation systems and drainage systems. The flow emanates from a *source* and ends at the *sink*.

1. Values on the arcs indicate the maximum carrying capacity.
2. Direction of flow is indicated by arrows.
3. Maximum flow begins at the source and ends at the sink.

Maximum Flow – Minimum Cut

A test to determine if the flow is a maximum involves a *cut*. A cut must divide a network into two. One piece must contain the source, the other the sink.

The *value* of a cut is the sum of the capacities on the arcs in the cut.



The value of cut PQ $= 14 + 21 + 16$

$= \underline{51}$

The value of cut XY $= 14 + 24$

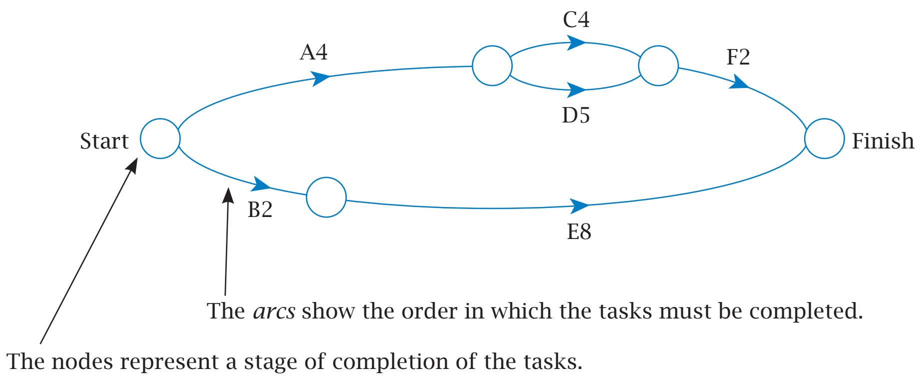
$= \underline{38}$

Maximum flow = Value of the Minimum cut

As 38 is the minimum value of the cut – the maximum flow is 38.

Project Network

A *project network* is a directed graph involving a sequence of jobs or activities. These networks indicate the *order of completion* of these events.



Critical Path

All tasks must be completed. The time taken to complete all tasks will be the *longest*. The path with the longest time is called the *critical path*.

Slack Time

This is the spare time which can allow the tasks to start late or finish early without delaying the overall finish time. There is no slack in the critical path.

Forward Scan

This is the maximum time for completing the task up to that point. The value is written in the *top* half of the node.

Backward Scan

This value written in the bottom half of the node is the latest time a task can be started. Begin at the finish node and subtract the time taken to complete a task.

Assignment Problems

The Assignment or Allocation problem is the best way of matching two groups in order to optimise

- time
- distance
- cost etc

in a bipartite graph.

The problem consists of a number of tasks and agents. One agent is assigned to exactly one task. When finding a solution to an assignment problem only relative *costs* are important. The Hungarian Algorithm will solve allocation problems.

The Hungarian Algorithm

Step 1: Subtract the smallest value in each *row* from each number in that row.

Step 2: Subtract the smallest value in each *column* from each number in that column.

Step 3: Draw the least number of horizontal and/or vertical lines through all the zeros.
If the number of lines is equal to the number of rows the process is finished → *see Step 5.*

If the number of lines is less than the number of rows then:

Step 4: Find the smallest value not covered by a line. *Subtract* this value from all *uncovered* numbers and *add* this value to all numbers that are *covered twice*.
If the number of lines is equal to the number of rows the process is finished → see Step 5.
Otherwise repeat Step 4.

Step 5: Find one zero in a row and one zero in a column. A bipartite graph showing the allocations producing the smallest value can be drawn.

Note: If the cost matrix is *not* square a dummy row and column consisting of all *zeros* can be added.

Worked Examples

8.1 The table below shows the cost in dollars of allocating workers to tasks. Use the Hungarian algorithm to find an allocation to *minimise* cost.

	Task X	Task Y	Task Z
Worker A	35	34	30
Worker B	27	30	26
Worker C	31	36	33

Cost Matrix

$$\begin{bmatrix} 35 & 34 & 30 \\ 27 & 30 & 26 \\ 31 & 36 & 33 \end{bmatrix}$$

Step 1: *Subtract* the smallest number in each **row** with every other number in that row.

$$\begin{matrix} \text{subtract } 30 \\ \text{subtract } 26 \\ \text{subtract } 31 \end{matrix} \begin{bmatrix} 5 & 4 & 0 \\ 1 & 4 & 0 \\ 0 & 5 & 2 \end{bmatrix}$$

Step 2: *Subtract* the smallest number in each **column** with every other number in that column.

$$\begin{matrix} \text{subtract } 1 & \text{subtract } 4 & \text{subtract } 2 \end{matrix}$$

$$\begin{bmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Step 3: Draw vertical and horizontal lines through the zeros – as few as possible. If the number of lines equals the number of rows or columns a solution can be found.

$$\begin{bmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Step 4: Determine the solution. **One** zero per row or column.

$$\begin{bmatrix} 4 & \boxed{0} & 0 \\ 0 & 0 & \boxed{0} \\ \boxed{0} & 1 & 0 \end{bmatrix}$$

Solution: Worker A → Task Y

Worker B → Task Z

Worker C → Task X

Total Cost: \$31 + \$34 + \$26
= \$91

8.2 The table below shows the time in minutes to complete each task. Use the Hungarian algorithm to allocate tasks so that the time is a minimum.

	Task X	Task Y	Task Z
Worker A	250	80	150
Worker B	230	90	140
Worker C	230	110	130

Time Matrix

$$\begin{bmatrix} 250 & 80 & 150 \\ 230 & 90 & 140 \\ 230 & 110 & 130 \end{bmatrix}$$

Step 1: *Subtract* the smallest number in each row with every other number in that row.

$$\begin{bmatrix} 170 & 0 & 70 \\ 140 & 0 & 50 \\ 120 & 0 & 20 \end{bmatrix}$$

Step 2: *Subtract* the smallest number in each column with every other number in that column.

$$\begin{bmatrix} 50 & 0 & 50 \\ 20 & 0 & 30 \\ 0 & 0 & 0 \end{bmatrix}$$

Step 3: Draw vertical and horizontal lines through the zeros.

$$\begin{bmatrix} 50 & 0 & 50 \\ 20 & 0 & 30 \\ 0 & 0 & 0 \end{bmatrix}$$

As there are only 2 lines yet 3 rows/columns a solution has *not* been found.

Step 4: Determine the minimum uncovered number. *Add* this to each row covered by a line.

$$\begin{bmatrix} 50 & 0 & 50 \\ 20 & 0 & 30 \\ 20 & 20 & 20 \end{bmatrix}$$

Step 5: Add this number to each column covered by a line.

$$\begin{bmatrix} 50 & 20 & 50 \\ 20 & 20 & 30 \\ 20 & 40 & 20 \end{bmatrix}$$

Step 6: Subtract the smallest number from each number in the matrix.

$$\begin{bmatrix} 30 & 0 & 30 \\ 0 & 0 & 10 \\ 0 & 20 & 0 \end{bmatrix}$$

3 lines → 3 rows/columns. Hence a solution can be found.

Step 7: Determine the solution.

$$\begin{bmatrix} 30 & \boxed{0} & 30 \\ \boxed{0} & 0 & 10 \\ 0 & 20 & \boxed{0} \end{bmatrix}$$

Solution: Worker A → Task Y

Worker B → Task X

Worker C → Task Z

Total Time: 80 + 230 + 130
= 440 minutes

8.3 The table below shows the profit in dollars of allocating workers to tasks. Use the Hungarian algorithm to find an allocation to maximise profit.

	Task X	Task Y	Task Z
Worker A	35	34	30
Worker B	27	30	26
Worker C	31	36	33

Profit Matrix

$$\begin{bmatrix} 35 & 34 & 30 \\ 27 & 30 & 26 \\ 31 & 36 & 33 \end{bmatrix}$$

Step 1: For a *maximum* – subtract every number from the largest number in the matrix.

$$\begin{bmatrix} 1 & 2 & 6 \\ 9 & 6 & 10 \\ 5 & 0 & 3 \end{bmatrix}$$

Proceed as per the minimum.

Step 2: Subtract the smallest number in each row with every other number in that row.

$$\begin{bmatrix} 0 & 1 & 5 \\ 3 & 0 & 4 \\ 5 & 0 & 3 \end{bmatrix}$$

Step 3: Subtract the smallest number in each column with every other number in that column.

$$\begin{bmatrix} 0 & 1 & 2 \\ 3 & 0 & 1 \\ 5 & 0 & 0 \end{bmatrix}$$

3 lines – 3 rows/columns. Hence a possible solution.

Step 4: Determine the solution.

$$\begin{bmatrix} \boxed{0} & 1 & 2 \\ 3 & \boxed{0} & 1 \\ 5 & 0 & \boxed{0} \end{bmatrix}$$

Step 5: Solution: Worker A → Task X

Worker B → Task Y

Worker C → Task Z

Total Profit: \$35 + \$30 + \$33
= \$98

8.4 The table below shows the time (in minutes) four workers complete three tasks. Use the Hungarian algorithm to find an allocation to minimise time.

	Task A	Task B	Task C
Worker W	13	23	16
Worker X	14	21	18
Worker Y	13	24	20
Worker Z	15	25	12

Time Matrix

As the matrix is not square – there are more workers than tasks, add in an extra column and fill it with zeros.

$$\begin{bmatrix} 13 & 23 & 16 & 0 \\ 14 & 21 & 18 & 0 \\ 13 & 24 & 20 & 0 \\ 15 & 25 & 12 & 0 \end{bmatrix}$$

Step 1: Subtract the smallest number in each row. This is not necessary since each row contains a zero.

Step 2: Subtract the smallest number in each column with every other number in that column.

$$\begin{bmatrix} \cancel{0} & \cancel{2} & \cancel{4} & \cancel{0} \\ \cancel{1} & \cancel{0} & \cancel{6} & \cancel{0} \\ \cancel{0} & \cancel{3} & \cancel{8} & \cancel{0} \\ \cancel{2} & \cancel{4} & \cancel{0} & \cancel{0} \end{bmatrix}$$

4 lines – 4 rows/columns. Hence a solution can be found.

Step 3: Determine the solution.

$$\begin{bmatrix} 0 & 2 & 4 & \boxed{0} \\ 1 & \boxed{0} & 6 & 0 \\ \boxed{0} & 3 & 8 & 0 \\ 2 & 4 & \boxed{0} & 0 \end{bmatrix}$$

Solution: Worker W → Task D (Dummy)

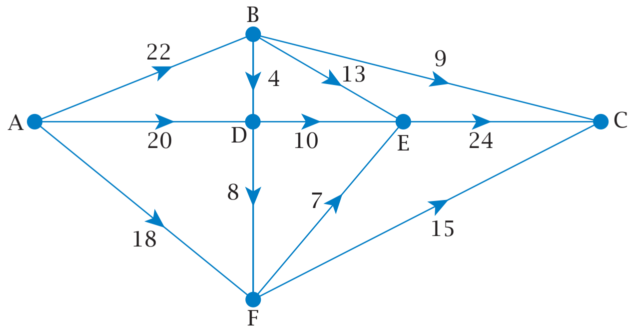
Worker X → Task B

Worker Y → Task A

Worker Z → Task C

Total time: 0 + 21 + 13 + 12
= 46 mins.

8.5 The network of pipes below shows the flow rate in litres per minute.



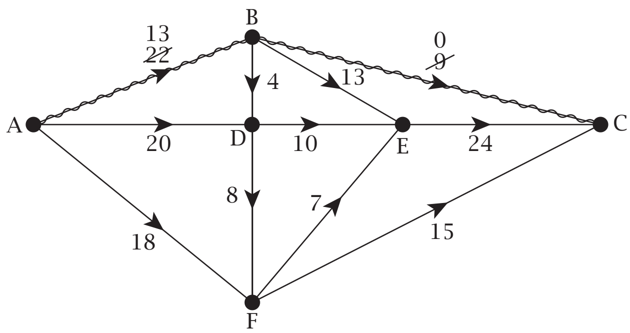
(a) Identify the source and sink.

(b) Determine the maximum flow using a systematic method.

(c) Confirm your result by finding a cut of the same value.

(a) Source : A
Sink : C

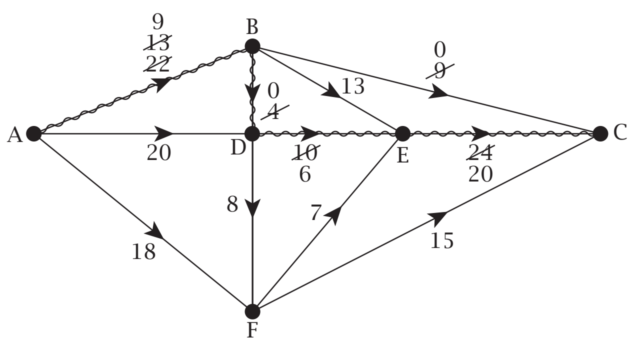
(b)



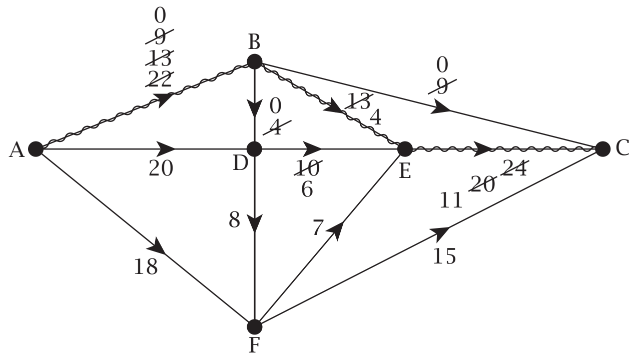
Systematic approach:

Start at the top of the network and work towards the bottom.

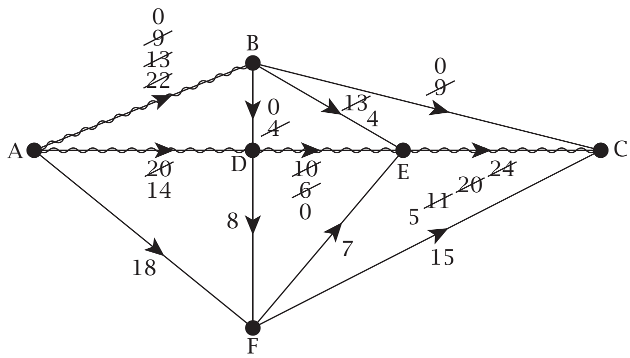
A B C: Max flow: 9
Cross off 9 leaving the remaining carrying capacity.



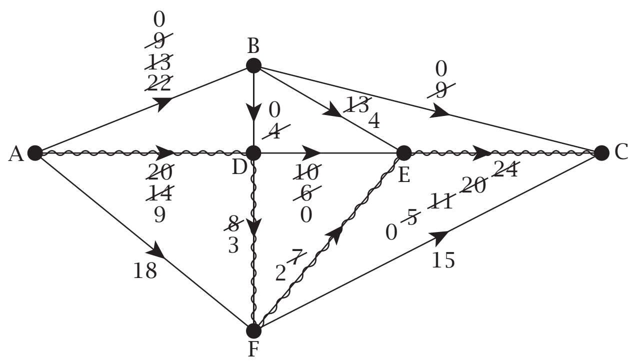
A B D E C : Max flow: 4



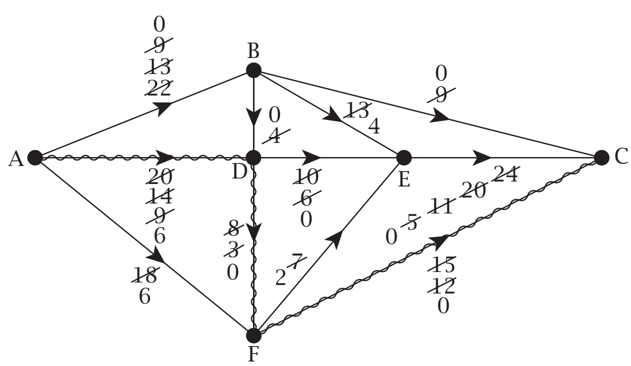
A B E C : Max flow: 9



A D E C : Max flow: 6



A D F E C : Max flow: 5



A D F C : Max flow: 3

A F C : Max flow: 12

Total flow: A B C : 9

A B D E C : 4

A B E C : 9

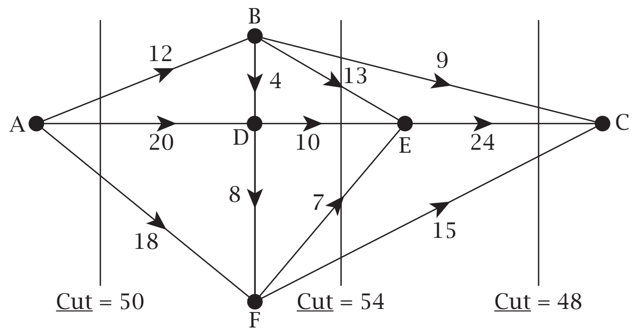
A D E C : 6

A D F E C : 5

A D F C : 3

A F C : 12

48 L/min.



(c) Maximum flow = Minimum cut

Total value of cut

= 9 + 24 + 15

= 48 L/min.

8.6 The tasks needed to complete the project and their duration in days is given in the table below:

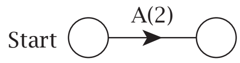
Task	Immediate Predecessors	Duration
A	–	2
B	A	6
C	B	20
D	A	2
E	C	8
F	D, E	6
G	E	4
H	F, G	10

(a) Draw the project network.

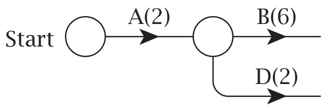
(b) State the critical path.

(c) Calculate the minimum completion time.

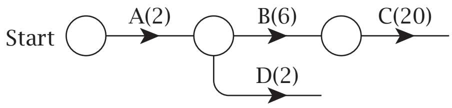
(a) Start with tasks that have no immediate predecessors.



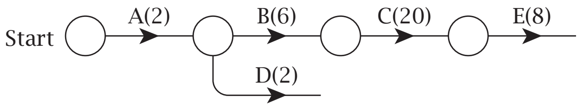
Task B and D has A as an immediate predecessor.



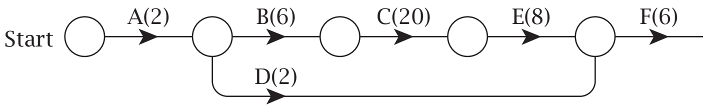
Task C has B as a predecessor.



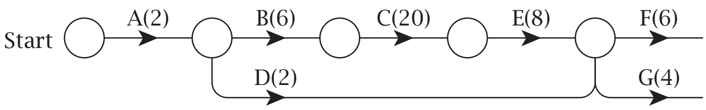
Task E follows on from C.



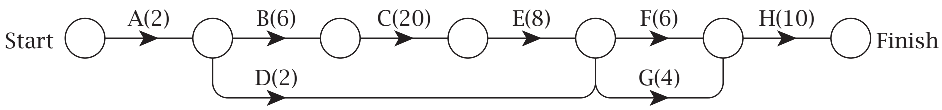
Task F follows on from D and E.



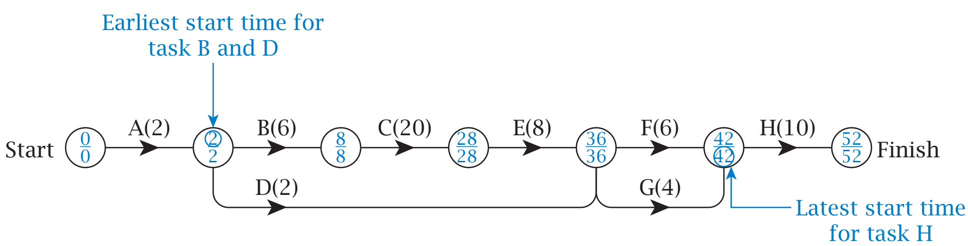
Task G follows on from E.



Task H follows on from F and G.



(b)



The difference between the earliest and latest time is called slack or float time.

The critical path is:

A — B — C — E — F — H

(c) The minimum completion time is 52 days.

PROBLEMS TO SOLVE

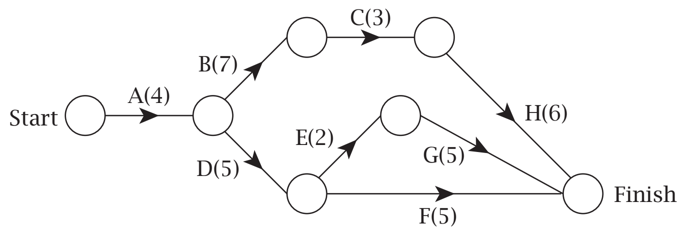
CHAPTER 8: DIRECTED NETWORKS AND DECISIONS

1. Draw a digraph to represent the following project:

   A project consists of six activities A, B, C, D, E and F.

   - A and B start immediately
   - C starts when A is finished
   - D starts when B and C are finished
   - E starts when D is finished
   - F starts when E is finished.

2. The times shown in the project network below are in days:



   (a) Determine the minimum completion time.

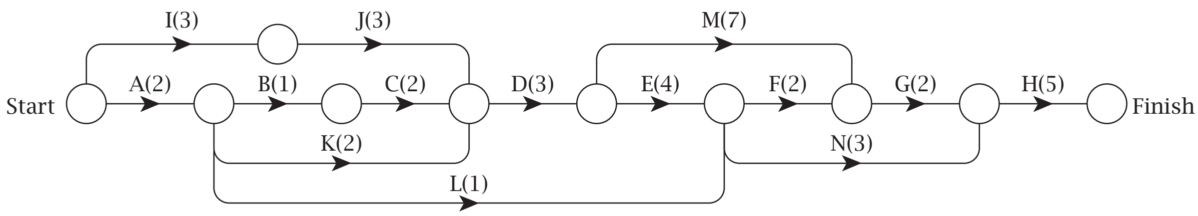
   (b) What is the critical path?

   (c) What is the earliest start time for task G?

   (d) What is the latest start time for task G?

   (e) What is the float (slack) time for task (i) C
   (ii) G

3. The times for tasks A to N in weeks are shown on the following project network.



   (a) Determine the critical path.

   (b) Calculate the minimum time required to complete the project.

   (c) What is the earliest start time for task L?

   (d) What is the latest start time for task L?

   (e) What is the slack time for task L?

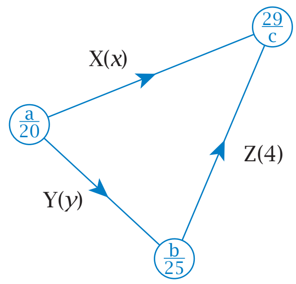
   (f) If the time for activity J is reduced by 1 week, what affect will this have on the minimum completion time?

   (g) What is the maximum time activity N can take without altering the minimum completion time?

4. The diagram below shows part of a project network where time is measured in hours.

Task X is on the critical path.

Task Y has a slack time of 3 hours.



(a) Calculate the values of a, c, x and y.

(b) Determine the minimum possible value of b.

(c) Calculate the maximum slack time for Task Z.

5. The tasks involved in a project are shown below:

Task	Immediate Predecessor	Duration (days)
A	–	10
B	–	7
C	B	8
D	B	9
E	A, C	5
F	A, C	14
G	D	6
H	E, F, G	5

(a) State:

(i) the critical path

(ii) minimum completion time

(b) If there were two people and only one person can be allocated to each task calculate the minimum time to complete all tasks.

6. The tasks needed to complete a project are shown in the table below. The duration is in weeks.

Task	Immediate Predecessor	Duration
A	–	18
B	–	30
C	I, M	8
D	A	8
E	D	6
F	D	12
G	E	4
H	G, F	6
I	B, H	4
J	I, M	10
K	G, F	6
L	G, F	8
M	L, K	4

(a) Construct a project network.

(b) State the critical path.

(c) What is the minimum time needed to complete the project?

(d) How many weeks can task B be delayed without altering the minimum completion time?

(e) If the time allocated for task C was to increase by 3 weeks what effect would this have on the minimum completion time?

7. An overnight camp resulted in the following tasks in preparation for the evening meal:

Task	Description	Predecessors	Duration (minutes)
A	Collect wood	–	15
B	Prepare fire	–	10
C	Set and light fire	A, B	6
D	Wait for embers to form	C	25
E	Collect food	–	10
F	Prepare food to be cooked	E	15
G	Prepare salad + rolls	E	10
H	Collect plates + cutlery	–	5
I	Cook food on fire	F, D	15
J	Eat food	G, I, H	20
K	Clean up	J	25
L	Put out fire	J	5

(a) Draw a project network to represent the above information.

(b) List the critical activities.

(c) Calculate the minimum completion time.

(d) How long can the preparation of food to be cooked take without affecting the completion time?

(e) If a portable gas barbeque was used, activity A and B are no longer necessary. Activity C takes only 3 minutes and activity D takes 15 minutes less. What affect does this have on the minimum completion time?

8. A list of tasks for a dinner party is shown in the table below. The time for the preparation of each task is given in minutes.

Task	Description	Immediate Predecessor	Time
A	Shop for food	–	60
B	Clean house	–	50
C	Set table	B	10
D	Prepare mains	A	20
E	Prepare dessert	A	40
F	Prepare vegetables	A	30
G	Cook mains	D	120
H	Cook vegetables	F	20
I	Plate food	G	10
J	Eat main	C, H, I	60
K	Eat dessert	E, J	10

(a) Draw a project network for the activities listed above.

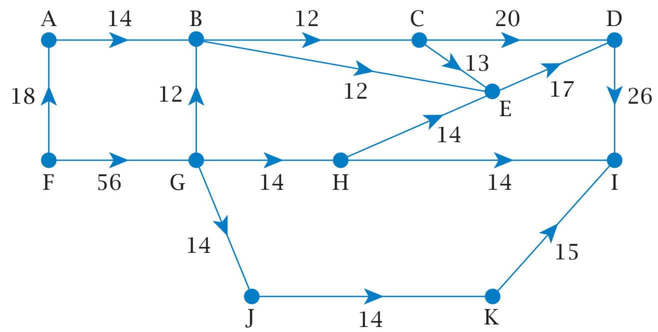
(b) List the activities on the critical path.

(c) State the minimum completion time.

(d) What is the latest time to start preparing the vegetables if mains are to be served at 7 p.m.

(e) Dessert takes 3 hours to set. How will this affect the minimum completion time?

9. The road network for a town is shown below. The numbers on the arcs represent the maximum number of vehicles that can pass along each road per minute.



(a) Identify the source and sink.

(b) Calculate the maximum flow along F G J K I.

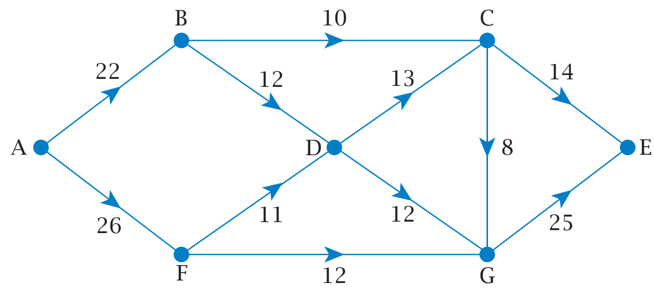
(c) Calculate the maximum flow from source to sink.

(d) Prove your flow is a maximum.

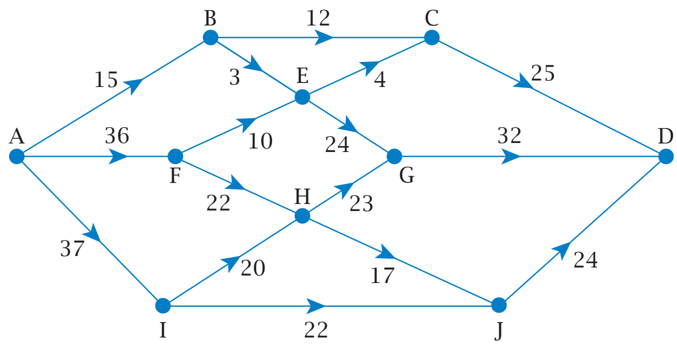
(e) Traffic flow is to be improved through the town. Which of the three roads B C, G H, or J K should be upgraded to maximise flow? Give reasons for your answer.

10. Confirm the maximum flow of each network below by finding the minimum cut.

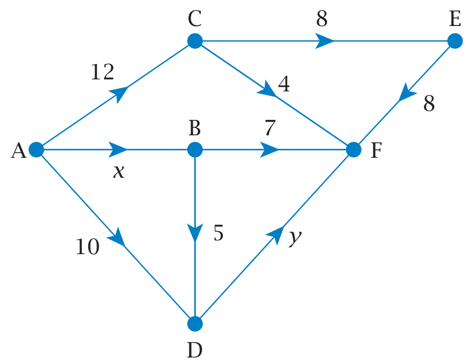
(a)



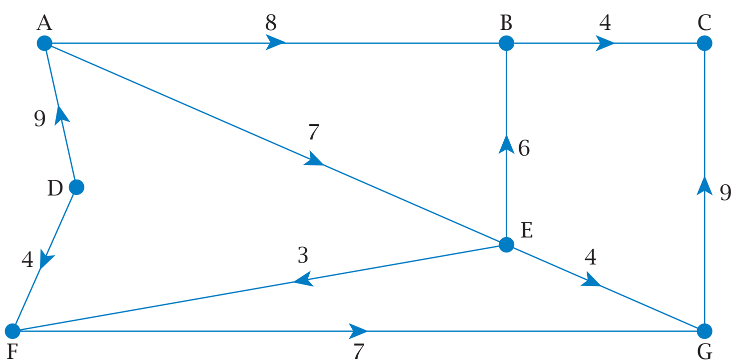
(b)



11. The diagram below shows the flow along each arc to achieve a maximum flow from source to sink. Find the values of x and y.



12. The diagram below shows a directed network where the numbers on each arc represent the flow capacity.



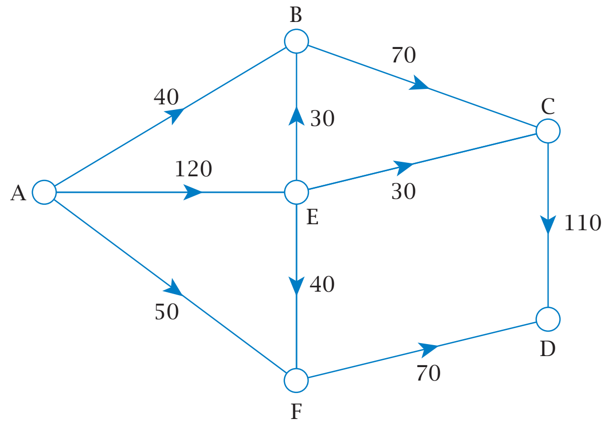
(a) Determine the source and sink.

(b) Calculate the maximum flow.

(c) Confirm the maximum flow by finding a cut of the same value.

(d) Describe the affect on the maximum flow if the capacity of A → B is reduced by 6.

13. A network of pipes are connected to a bore at A. The flow rate in litres per minute is shown in the diagram below.



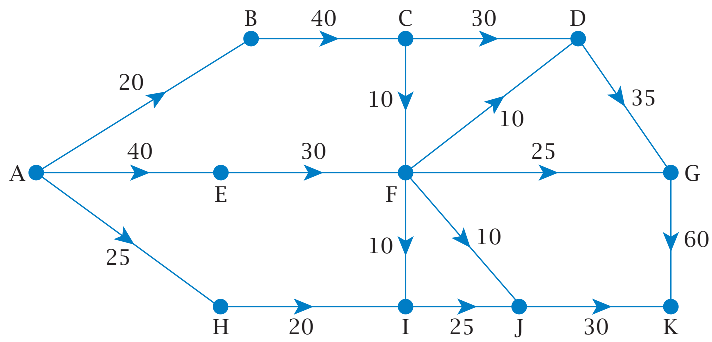
(a) Identify the source and sink.

(b) Determine the maximum flow using a systematic method.

(c) If the flow from E to C is increased to 40 litres per minute describe the effect on the maximum flow? Explain.

(d) Is the maximum flow affected if the flow from B to C is increased by 20 litres per minute? Explain.

14. Given the following network:



(a) Identify the source and sink.

(b) Find the maximum flow.

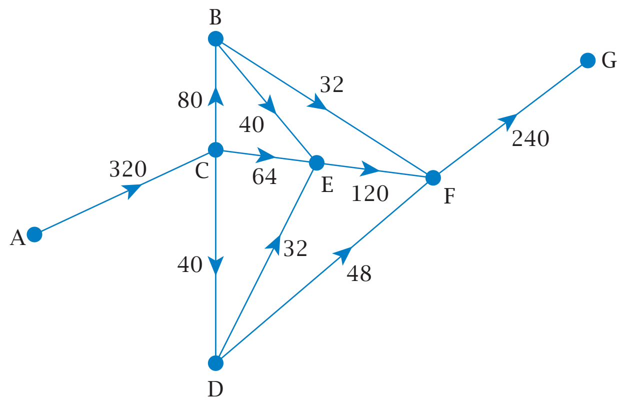
(c) Find the maximum flow if CF is reversed.

One line from the source can be increased.

(d) Which line can be increased so that the maximum flow through the network will be increased?

(e) How much should the line be increased?

15. Each arc on the diagram below represents the maximum amount of water through each pipe in kilolitres per hour.



(a) Identify the source and sink.

(b) Calculate the maximum flow.

Describe the effect on the maximum flow if:

(c) Pipe ED direction was reversed.

(d) Pipe BE was removed.

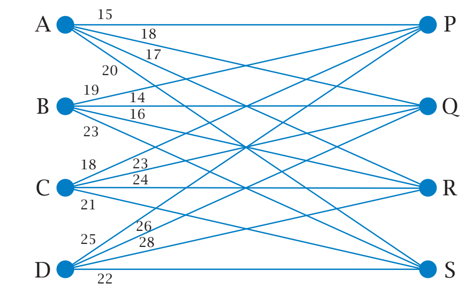
(e) Pipe EF was reduced to 75.

16. Use the Hungarian algorithm to allocate tasks to workers so as to *minimise* the cost. The tables or bipartite graph show the cost in dollars of allocating workers to tasks.

(a)

	Task P	Task Q	Task R
Worker A	37	40	38
Worker B	28	32	26
Worker C	34	38	35

(b)

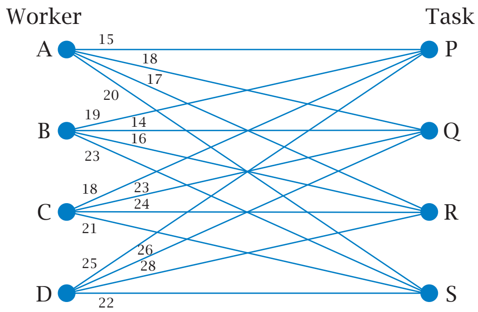


17. Use the Hungarian algorithm to allocate tasks to workers so as to *maximise* profit. The tables or bipartite graph show the profit in dollars of allocating workers to tasks.

(a)

	Task P	Task Q	Task R
Worker A	37	40	38
Worker B	28	32	26
Worker C	34	38	35

(b)



18. The table below shows the time in minutes for tasks A, B, C and D to be completed by Scott, Charlene, Bob and Taya.

	Task A	Task B	Task C	Task D
Scott	26	44	40	60
Charlene	28	48	26	20
Bob	24	46	30	80
Taya	28	42	34	40

Use the Hungarian algorithm to determine the allocation that minimises the total time.

19. Four students can be entered into a prestigious athletics competition based on 4 running events: 100 m, 200 m, 400 m, 800 m. Each student may only enter one of the four events with the winning team having the lowest total time.

Trial times in seconds are recorded in the table below for the top 4 students.

	100 m	200 m	400 m	800 m
Kai	14	25	59	131
Drew	12	26	65	159
Lachlan	18	24	62	175
Brayden	13	27	60	163

Use the Hungarian algorithm to determine which athlete should participate in which event in order to minimise the total time.

20. Four students: Adam, Ben, Connie and Dorothy are tutored by Mr Smith, Mr Brown, Mrs Green and Miss Young. Pre-testing gave the following test percentages:

	Adam	Ben	Connie	Dorothy
Mr Smith	72	79	64	68
Mr Brown	76	74	62	70
Mrs Green	64	66	70	62
Miss Young	68	70	62	60

Use the Hungarian algorithm to assign one tutor to one student to *maximise* the total improvement of all four students.

21. The table below shows the time (in minutes) for four workers P, Q, R and S to complete three tasks X, Y and Z.

	Task X	Task Y	Task Z
Worker P	10	24	15
Worker Q	11	27	18
Worker R	10	26	22
Worker S	9	28	14

Use the Hungarian algorithm to find an allocation which *minimises* the total time.

22. Four companies provide quotes for three birthday parties. The table below illustrates these quotes in dollars.

	Party X	Party Y	Party Z
Company A	224	350	442
Company B	231	364	454
Company C	246	380	456
Company D	250	388	469

One company must be assigned to one particular party. Use the Hungarian algorithm to allocate a company to a party which will minimise the total cost.

CALCULATOR FREE: TRIAL TEST 1

Calculators NOT allowed
Time Allowed: 15 minutes
Total Marks: 20

1. A car costs $30 000. It depreciates by 20% of its value each year.
After 2 years the car is worth:

(a) $24 000

(b) $6000

(c) $19 200

(d) $27 000

(e) $18 200 [1]

2. A connected planar graph has 12 vertices and 12 edges.
The number of faces for this graph is:

(a) 2

(b) 3

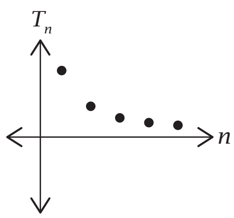
(c) 4

(d) 5

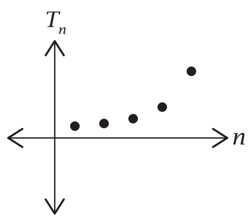
(e) 6 [1]

3. The first term of a geometric sequence is ‘a’ and the common ratio ‘r’. Which of the following represents the first 5 terms of the sequence if $a > 0$ and $0 < r < 1$.

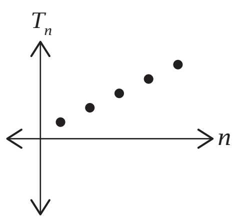
(a)



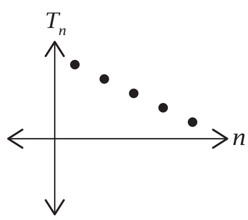
(b)



(c)



(d)



[1]

4. How many edges are needed for a graph containing 5 vertices to be:

(a) Complete?

____

____

(b) Connected?

____

______________________________ [4]

5. The relationship between two different quantities *x* and *y* produced the following correlation coefficients.

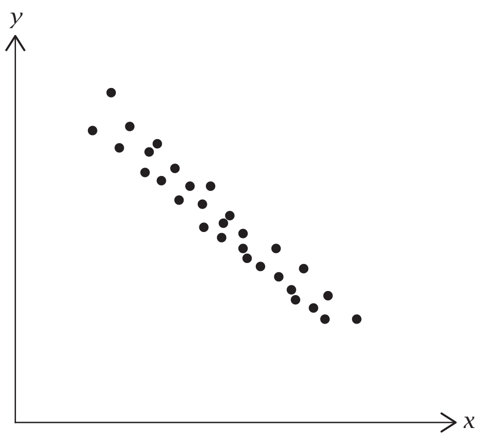
$r = 0.45$, $r = -0.4$, $r = 0.08$,

$r = 1.47$, $r^2 = 1$, $r = 0.93$,

$r = -0.8$.

For each of the scattergraphs below state the corresponding correlation coefficient from the list above.

(a)



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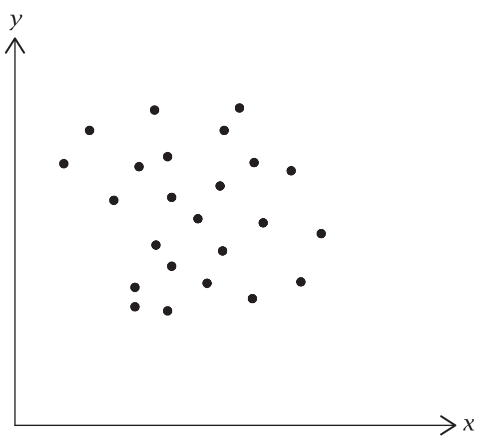
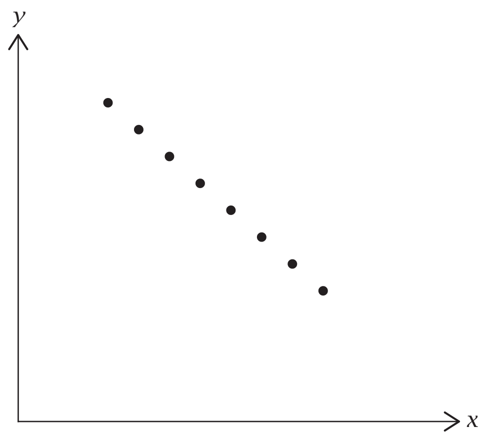
(b)

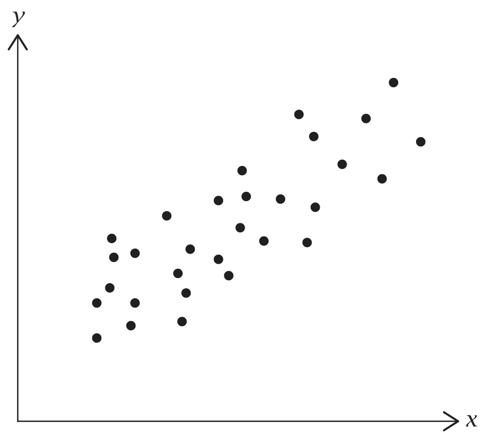


____

(c)



(d)



[4]

6. The following tasks are needed to complete a building project.

Task	Immediate Predecessors	Duration (days)
A	–	12
B	–	14
C	A	8
D	A	4
E	C, D	12
F	B	18
G	B	20
H	E, F	2
I	C, D	12
J	A	24
K	G, H, I, J	8

(a) Draw a project network for the above information.

(b) Determine the critical path.

(c) Calculate the minimum time needed to complete the project.

(d) How many days can the starting time of Task B be delayed without affecting the minimum completion time?

(e) What is the float time for Task H?

(f) If the time required for Task E increases by 3 days state the effect on:

(i) the critical path

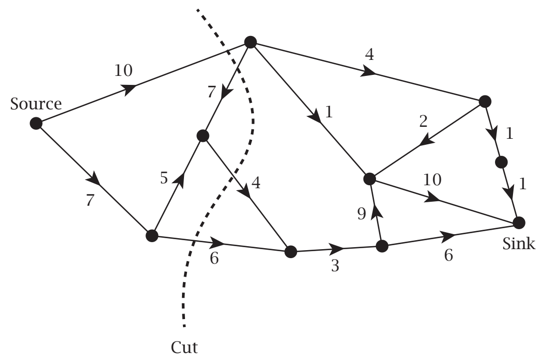
(ii) the minimum completion time

[9]

CALCULATOR FREE: TRIAL TEST 2

Calculators NOT allowed
Time Allowed: 15 minutes
Total Marks: 20

1.



The capacity of the cut is:

(a) 27

(b) 17

(c) 16

(d) 20

(e) 13 [1]

2. The maximum flow between source and sink is:

(a) 6

(b) 7

(c) 8

(d) 14

(e) 20 [1]

3. Seasonal indices for the first 3 quarters are:

Q1 : 106%
Q2 : 61%
Q3 : 158%

(a) Calculate the seasonal index for Quarter 4.

(b) If sales figures for Quarter 4 were 1200 determine the deseasonalised value.

[4]

4. Calculate the first 3 terms of the sequences given by the following recursive definitions:

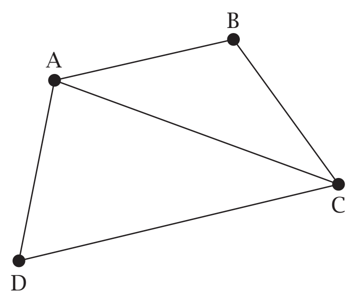
(a) $T_{n+2} = T_{n+1} - T_n$ where $T_1 = 5$
$T_2 = 3$

(b) $P_{n+1} = 4P_n + \frac{1}{P_n}$ where $P_1 = 1$

(c) $U_{n+2} = 3U_{n+1}$ where $U_1 = 3$

[5]

5. The graph below shows the roads connecting four towns A, B, C and D.



(a) Describe the mathematical term for the route which starts at D then travels to C, A and finishes at B.

(b) Describe the mathematical term for the route which starts at C then travels to B, A, C, D, A.

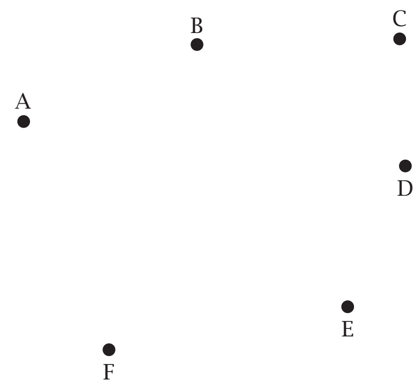
[4]

6. Six towns A, B, C, D, E and F are to be connected by a rail network. The distances between each town are in kilometres and shown below.

	A	B	C	D	E	F
A	–	27	–	–	–	24
B	27	–	19	–	–	20
C	–	19	–	34	17	20
D	–	–	34	–	24	48
E	–	–	17	24	–	18
F	24	20	20	48	18	–

(a) Use Prim's algorithm to determine the minimum length of track needed to connect all towns to a rail network.

(b) Draw the minimum spanning tree below:

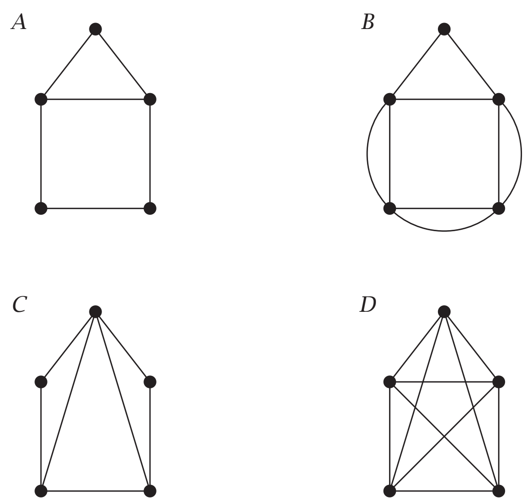


[5]

CALCULATOR FREE: TRIAL TEST 3

Calculators NOT allowed
Time Allowed: 15 minutes
Total Marks: 20

1. Given the following graphs:



(a) Which of these are graphs are:

(i) Eulerian?

(ii) Semi-Eulerian?

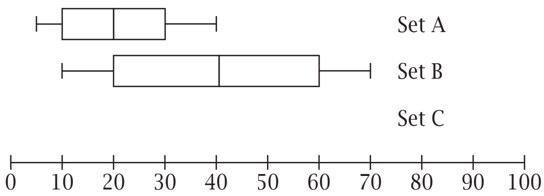
(iii) Hamiltonian?

(b) Which of these graphs are planar? Explain.

[8]

2. The boxplots for sets A and B are displayed below. A third set C has the following summary statistics:

- Median = 30
- Minimum = 15
- Maximum = 45
- LQ = 20
- UQ = 40



(a) Add Set C to the above diagram.

(b) Compare the similarities and differences between the boxplots.

[5]

3. A planar graph contains 5 vertices labelled A, B, C, D and E. The adjacency matrix below represents this planar graph.

	A	B	C	D	E
A	0	2	0	0	1
B	2	0	2	1	1
C	0	2	0	1	0
D	0	1	1	0	1
E	1	1	0	1	1

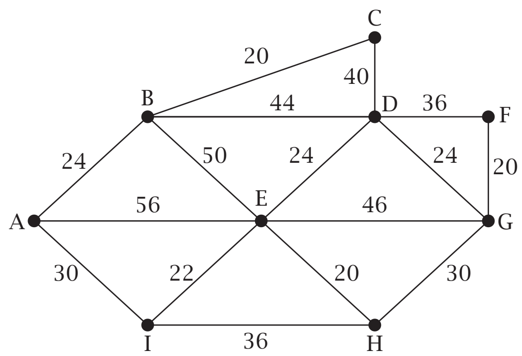
Calculate the number of

(a) faces

(b) edges

for this planar graph. [2]

4. A network connecting towns A to I is drawn below. The travel times in minutes are also indicated.



(a) Find the shortest travel time and route from Town A to Town G.

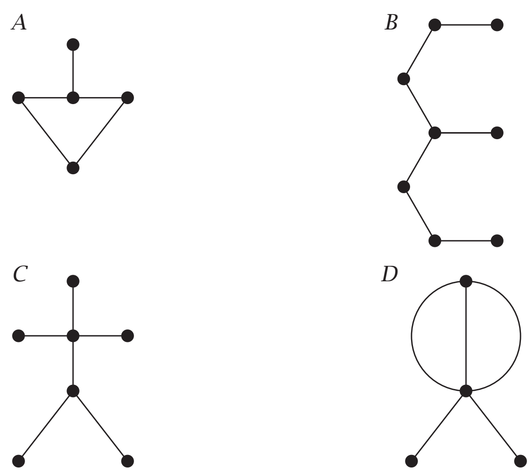
(b) The road from town A to E is being upgraded. By how much must the time in minutes decrease so that the shortest route from A to G passes through E? State the route.

[5]

CALCULATOR FREE: TRIAL TEST 4

Calculators NOT allowed
Time Allowed: 15 minutes
Total Marks: 20

1. Which of these graphs is a tree?



[1]

2. A sequence is generated using the recursive definition

$T_{n+1} = T_n + 7 \qquad T_4 = 3$

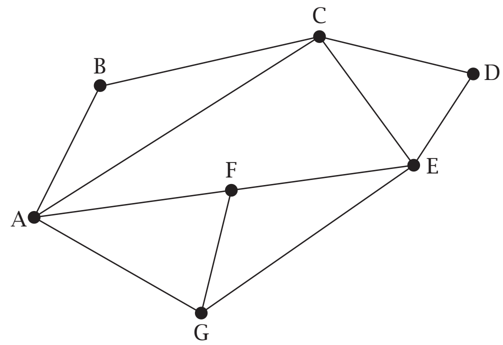
Find

(a) T_5

(b) T_1

[3]

3.



(a) Does Euler's formula work for the above network? Justify your answer.

(b) Determine the degree of each vertex.

(c) A semi-Eulerian trail starting at G must finish at which vertex. State a trail.

(d) State the name of the following walk.
G A B C D E F G

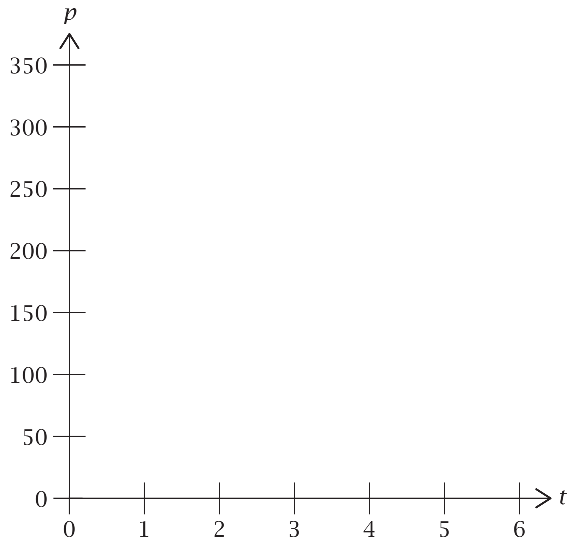
[7]

4. The study of a population (p) of a species of birds was conducted over a 6 year period. The results indicated that the population increased according to the rule $p = ka^t$, where k was the initial number, t the time in years since the study began.

The results were:

t	0	1	2	3	4	5	6
p	100	120	144	173	207	249	299

(a) Plot these points on the graph below



(b) What type of graph is produced?

(c) Use the graph to calculate the population in the 7th year.

(d) The growth rate of the species is 20% p.a. Determine the values of *k* and *a*.

(e) Describe the long term growth of the species.

(f) If another species had a poulation that followed the rule:

$p = 100 \times 0.7^t$

state the growth rate.

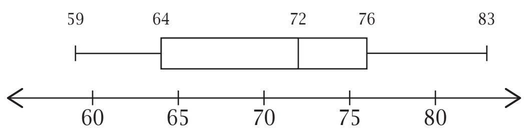
[9]

CALCULATOR ASSUMED: TRIAL TEST 5
BIVARIATE STATISTICS

Calculators allowed
Time Allowed: 30 minutes
Total Marks: 40

1. The speedometer readings of 10 cars were recorded by police cameras and displayed as a boxplot shown below:

Boxplot 1

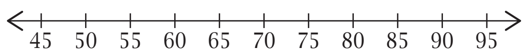


A further 10 cars speedometer readings were:

53, 48, 73, 95, 72, 64, 60, 46, 45, 63

(a) Draw a boxplot to represent this information showing the important features.

Boxplot 2



(b) Indicate the differences between the two boxplots.

(c) Determine if any outliers exist.

[6]

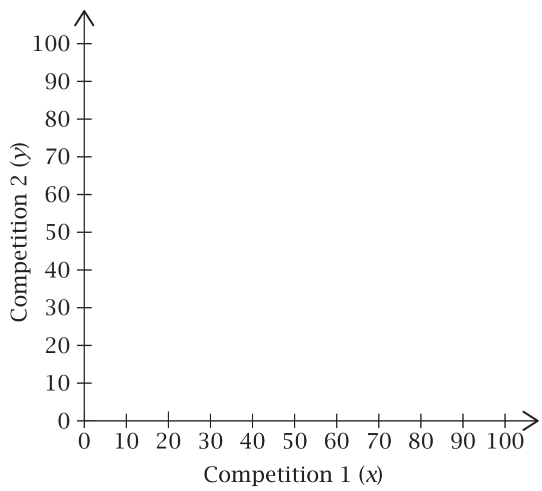
2. Fifteen students recorded their results in two competitions.

Competititon 1 (%) x	55	37	11	40	95	94	33	22	83	44	45	77	87	20
Competititon 2 (%) y	42	26	7	16	55	88	14	4	68	36	89	54	76	3

(a) Calculate the mean and standard deviation for Competition 1 and Competition 2.

(b) Would the mean and standard deviation be affected if a 16th student recorded a mark of 70% in both competitions.

(c) Draw a scatterplot of Competition 1 against Competition 2.
Identify any **outliers and remove them**.



(d) Calculate the correlation coefficient.

(e) Find the equation of line of regression.

(f) A student receives a mark of 60% for Competition 1.
Predict the student's Competition 2 mark.
Comment on the validity of this prediction.

[9]

3. Year 9 students completed an English examination. 28 boys failed the exam while 80% of the girls passed. Of the 136 students who passed, 40 were boys.

(a) Complete the two-way table.

	Boys	Girls	Total
Pass			
Fail			
Total			

(b) Determine the explanatory and response variable.

(c) Create a percentage two-way table.

(d) Construct a segmented column graph appropriate to the explanatory variable.

(e) Is there an association between the variables? Explain.

[13]

4. The statistics below show the age of a company's employees and the number of absent work days per year.

Age (a)	22	46	38	27	51	28	36	50	34	43
Absent Days (d)	10	3	6	9	4	8	37	3	7	2
Residual	−3.65	−3.30		−3.12	−0.77			−2.07	−2.97	

$\bar{a} = 37.5$ $\hat{d} = -0.306a + 20.382$

$\bar{d} = 8.9$ $r_{ad} = -0.298$

(a) Calculate the missing residuals in the table above.

(b) Determine the possible outlier from the residuals. Explain your reasoning.

Remove the outlier

(c) Calculate the new correlation coefficient.

(d) Calculate r^2 the coefficient of determination and establish if the regression line is an appropriate model.

(e) Determine the least squares line of regression for d on a.

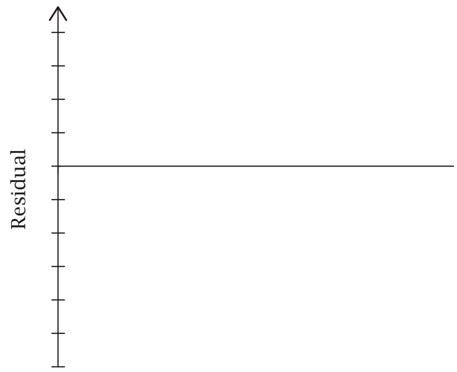
(f) Predict the number of absent work days for a 67 year old employee.

(g) Comment on the reliability of this prediction.

___

___

(h) Recalculate the residuals and plot these on the axes below.



(i) Is this linear model appropriate? Comment.

___

___ [12]

CALCULATOR ASSUMED: TRIAL TEST 6
SEQUENCES AND TIME SERIES DATA

Calculators allowed

Time Allowed: 30 minutes

Total Marks: 40

1. A sequence has a recursive formula $T_{n+1} = (-1)^n 2T_n$ with $T_1 = 2$

Determine the first four terms of this sequence.

[3]

2. A sequence is defined by the recursive formula:

$$T_n = T_{n-3} + T_{n-2} - T_{n-1}, \quad n \geq 4$$

where $T_1 = 1,\ T_2 = 2,\ T_3 = 3$

(a) Determine the next five terms of the sequence.

(b) Determine the recursive formula when n is odd.

(c) Determine the recursive formula when n is even.

[6]

3. A Plasma TV costs $12 000 and John decides to borrow the money at a rate of 2.4% per month. He agrees to repay $400 per month for the life of the loan. The amount owing after $(n+1)$ months is related to the amount owing after n months according to the rule:

$$T_{n+1} = 1.024\mathrm{T}_n - 400 \quad n \geq 0$$

where T_n is the amount owing after n months.

Find the amount owing after:

(a) the first month.

(b) the second month.

[4]

4. *(a)* An arithmetic sequence has

$T_{22} = 22$

$T_{28} = 12$

Determine T_{14}

(b) A geometric sequence has negative odd numbered terms and

$T_4 = 256$

$T_{10} = 16\,384$

Determine T_7

[5]

5. Clarisa borrows $50 000 to buy a new 4 wheel drive vehicle. The interest payable is calculated on the balance of the loan each month. Interest is calculated at 12% p.a. compounded monthly and after interest, Clarisa repays $700 per month.

The recurrence relation

$T_{n+1} = pT_n + q, T_0 = a$

where *n* is the amount owing at the end of the month models this situation.

(a) Determine the values *p*, *q* and *a*.

(b) How much is still owing at the start of the 5th month?

(c) At the start of which month does the balance first fall below $10 000?

[5]

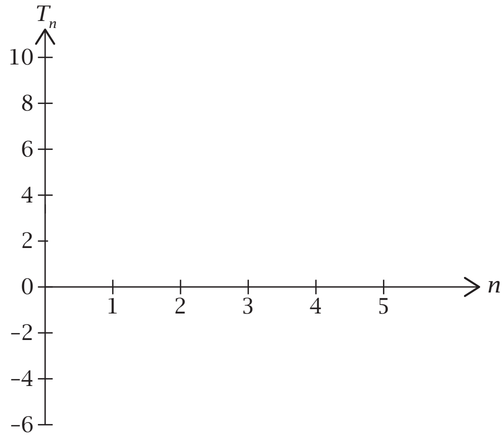
6. For each of the following first order recurrence relations determine by plotting the first 5 terms on the axes, if they have a long-term

(i) increasing

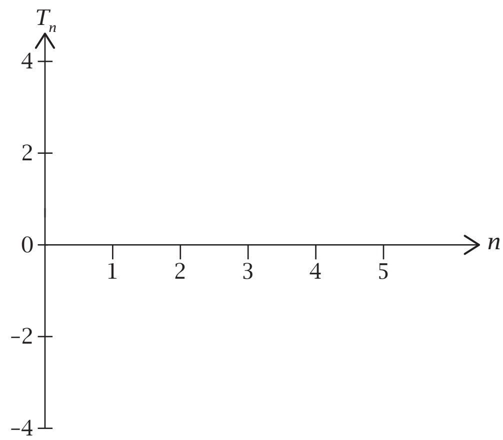
(ii) decreasing or

(iii) steady state solution

(a) $T_{n+1} = 2T_n - 1.5, T_1 = 2$



(b) $T_{n+1} = 0.5T_n - 1.5, T_1 = 2$



[4]

7. The school canteen records the icecream sales quarterly. The results are detailed below:

Year	Quarter	t	Sales	CMA (m)
2011	1	1	300	
	2	2	250	
	3	3	200	286.25
	4	4	320	B
2012	1	5	450	342.88
	2	6	390	360.50
	3	7	223	384.88
	4	8	438	407.00
2013	1	9	527	436.88
	2	10	490	470.63
	3	11	362	
	4	12	A	

(a) Determine the moving average used in the table and explain your reason.

(b) Comment on the trend of the data.

(c) Calculate the values of A and B.

(d) Calculate the missing seasonal indices in the table below.

Year	Q1	Q2	Q3	Q4	Average
2011	300	250	200	320	
2012	450	390	223	438	
2013	527	490	362		
Seasonal Indices	1.1342	0.9933			

(e) Determine the least squares line of regression $M = at + b$.

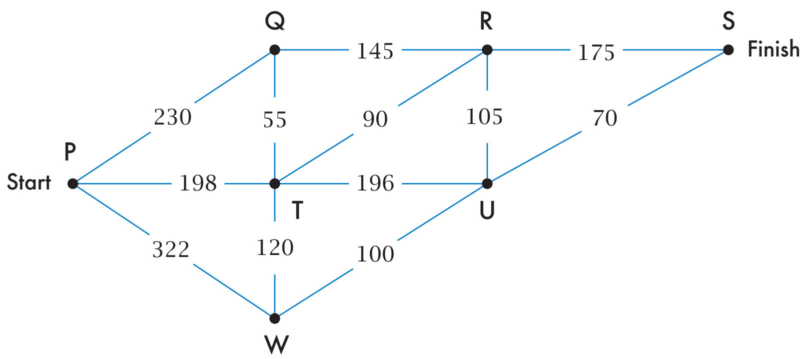
(f) Using the line of regression and the seasonal indices predict the sales for Q4, 2014.

[13]

CALCULATOR ASSUMED: TRIAL TEST 7
UNDIRECTED GRAPHS AND NETWORKS

Calculators allowed
Time Allowed: 20 minutes
Total Marks: 30

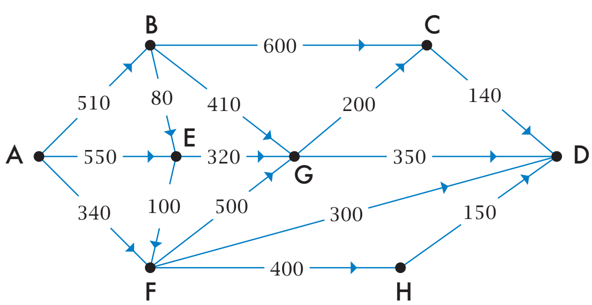
1. The network below shows the distances in kilometres.
Find the shortest path from start to finish. State the path(s) and give the length.



Can this network be traversed?
Give reason(s) as to your answer.

[6]

2. Skyhigh airlines charges passengers on the shortest distance (in km) between the departure point and the final destination. Passengers are charged 45 cents per kilometre and the network diagram below shows the distances in kilometres.

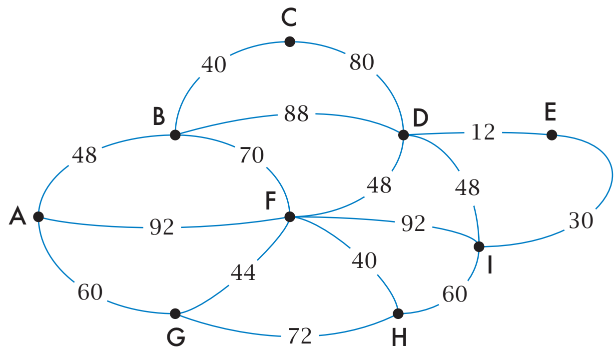


(a) Find the minimum cost to travel from town B to town D.
State the route which gives the minimum cost.

(b) If the plane must travel from town B to town D via town G determine the minimum cost and state the route.

[6]

3. The network diagram below shows 9 towns linked by roads and the travel time (in minutes).



(a) Find the shortest route from Town A to Town I.
State the route and the time taken.

(b) The local council is upgrading the road between Town A and Town F.
How much time will need to be saved so that the shortest route will be Town A—Town F—Town D—Town I ?

[7]

4. The cost, in hundreds of dollars, to deliver supplies to 5 towns: A, B, C, D and E is shown in the table below.

	A	B	C	D	E
A	–	12	–	5	12
B	12	–	12	–	10
C	–	12	–	8	7
D	5	–	8	–	6
E	12	10	7	6	–

(a) Construct a network to show the information in the table.

(b) Use Prim's Algorithm to determine the least expensive route. State the route and minimum cost.

(c) Draw a minimal spanning tree in the space below.

[5]

5. A complete graph K_n where $n > 1$ has each vertex n connected to every other vertex by a single edge.

(a) Construct a complete graph for K_4.

(b) Construct an adjacency matrix for K_5.

(c) Determine the total number of edges for K_5.

(d) Is K_5 Eulerian? Why?

(e) State in terms of n when K_n is Eulerian.

[6]

CALCULATOR ASSUMED: TRIAL TEST 8
FINANCE

Calculators allowed
Time Allowed: 30 minutes
Total Marks: 30

1. A loan of $2000 is to be repaid with 12 payments of $180 per month.
Calculate the effective annual rate of interest.

[3]

2. Joanne wishes to invest $15 000 into a savings account for 10 years. Her investment options are:

Option 1: Simple Interest at 11.5% p.a.
Option 2: 10.7% p.a. compounded quarterly.
Option 3: 9.4% p.a. compounded weekly.

Which investment option should Joanne choose? Justify your solution.

[6]

3. A financial institution approves a $120 000 loan for a customer. The loan will be repaid fully over 25 years. Interest is charged at 7.25% p.a. on the reducing monthly balance. Calculate the monthly repayments.

[3]

4. The recursive formula

$T_{n+1} = 1.25\,T_n,\ T_0 = 4000$

can be used to calculate the value of an investment after n years compounded annually.

(a) Determine the annual interest rate.

(b) Calculate the value of the investment after 10 years.

(c) What interest rate would be needed to accumulate the same value in half the time?

[6]

5. Leslie borrows $38 000 to purchase a new car. The interest rate is 12% p.a. reducible monthly. The repayment schedule is below.

Repayment	Interest ($)	Monthly repayment ($)	Amount owing ($)
1	$380	$1000	$37 380
2	$373.80	$1000	$36 753.80
3	$367.54	$1000	$36 121.34
4	A	B	C
:			
10	$321.92	$1000	$31 512.43
11	$328.26	$1000	$30 841.69
12	$321.27	$1500	$29 662.92
⋮			
34	$19.44	$1500	$385.61
35	D	E	$0

(a) The interest rate changes after repayment 10. Calculate the new interest rate.

(b) Calculate the values of A, B, C, D and E.

(c) Write a recursive formula to determine the amount of the loan at the end of each month for the first 10 repayments.

(d) Calculate the total interest.

[12]

CALCULATOR ASSUMED: TRIAL TEST 9
DIRECTED NETWORKS AND DECISION MATHEMATICS

Calculators allowed
Time Allowed: 30 minutes
Total Marks: 40

1. The table below shows cost in dollars of allocating tasks to workers.

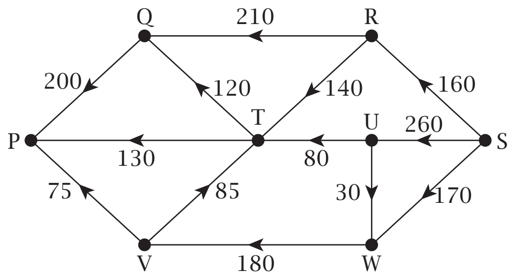
	Task X	Task Y	Task Z
Worker A	52	56	72
Worker B	47	42	48
Worker C	50	58	49

(a) Use the Hungarian algorithm to allocate tasks to workers so as to *minimise* the cost.

(b) Determine the minimum cost

[5]

2. The network below shows the train routes for a city. The numbers on each indicate the maximum number of people that can be transported along that route every 10 minutes.



(a) State the source and sink.

(b) Determine the maximum number of passengers that can be transported from source to sink.

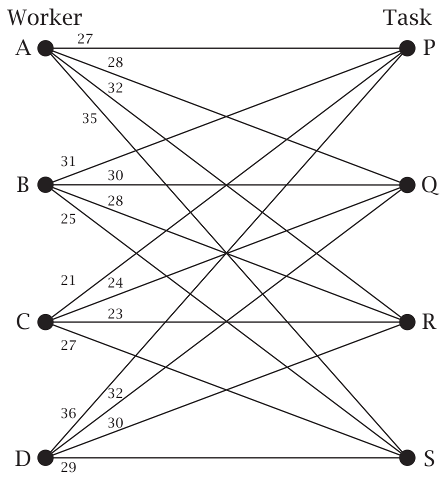
(c) Confirm the maximum number of passengers by finding the minimum cut.

(d) Is there a route that is not well used and can be removed? Explain.

(e) One route is to be upgraded by increasing the number of trains.
Which is the best route and why?
How many extra passengers can be transported along this route?

[12]

3. The bipartite graph shows the profit in dollars of allocating tasks to workers.



(a) Is this a complete bipartite graph? Why?

(b) Use the Hungarian algorithm to allocate tasks to workers to *maximise* profits.

(c) Determine the maximum profit. [8]

4. The tasks needed to complete a project are shown in the table below. The time is in hours.

Task	Time
A	4
B	$2\frac{1}{2}$
C	1
D	$\frac{1}{2}$
E	$1\frac{1}{2}$
F	$\frac{1}{2}$
G	1
H	$\frac{1}{2}$
I	8

Tasks are to be completed according to the following list.

- Tasks A and H must start together.
- Task B must follow A.
- Task B must be completed before Task C begins.
- Tasks D and F must be after Task C is completed.
- Task E follows Task D.
- Task G follows Task F.
- Task I follows Task E, G and H.

(a) Construct a project network to show the above information.

(b) Find the minimum completion time and critical path.

(c) What is the effect on the minimum completion time if Tasks E and G are increased by 30 minutes?

(d) How much extra time can be allocated to Task G without changing the minimum completion time?

[8]

5. The travel time in minutes for workers Craig, Brendan, Alice and Georgia to three depots D_1, D_2 and D_3 are given in the table below.

	D_1	D_2	D_3
Craig	42	45	48
Brendan	37	35	31
Alice	27	21	24
Georgia	51	39	45

(a) Use the Hungarian algorithm to allocate one worker to each of the depots to minimise the travel time.

(b) State the minimum travel time. [7]

EXAMINATION STYLE QUESTIONS

1. *Calculator Free*

The table below shows the cost (in hundreds of dollars) for four workers *P*, *Q*, *R* and *S* to complete three tasks *X*, *Y* and *Z*.

	Task X	Task Y	Task Z
Worker P	26	46	32
Worker Q	28	42	36
Worker R	26	48	40
Worker S	30	50	24

(a) Draw a bipartite graph to represent the above information.

(b) What is the cost if workers *P*, *Q* and *S* are allocated to tasks *Y*, *X* and *Z*. respectively?

(c) Use the Hungarian algorithm to find an allocation which minimises the total cost and hence state the minimum cost.

2. *Calculator Free*

(a) Given the following arithmetic sequence: 19, 15, 11, 7, …

(i) Calculate the 40^{th} term of the sequence.

(ii) Determine the sum of the first 40 terms.

(iii) Determine the value of n if the n^{th} term is –309.

(b) Another sequence has a recursive definition given as:

$T_{n+1} = 2T_n + 3$ *where* $T_1 = 2$.

(i) Write down the first 4 terms of this sequence.

(ii) Determine how many terms are required for the value of the sequence to be 157.

3. *Calculator Assumed*

John over 4 years invests \$8 000 in a bank at 6.25% p.a. simple interest.

(a) How much interest is earned over the 4 years?

(b) Determine how much John will have in his bank account at the end of 4 years?

(c) State a recursive formula which determines the amount John will have in his bank account at the end of each year.

4. *Calculator Free*

Saskia is planning a new project. The table below shows the activities, time (in days) and the immediate predecessors for each activity.

Activity	A	B	C	D	E	F	G
Duration (days)	10	4	6	8	2	8	6
Immediate predecessors	–	–	A, B	B	C	D, E	F

(a) Draw a project network showing all activities and durations.

(b) Determine the critical path and the minimum completion time.

(c) State the float time for Activity D.

(d) Activity D is delayed by 10 days. How does this effect the critical path and minimum completion time?

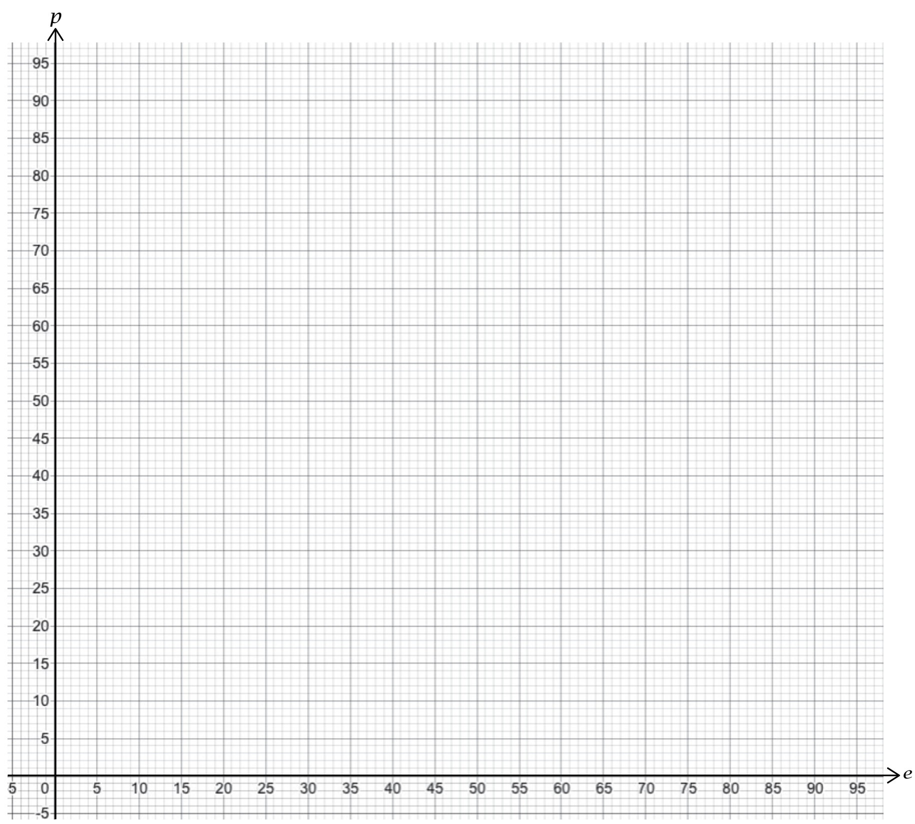
5. *Calculator Assumed*

Jonas an avid Eagle supporter measures his pulse rate during the Grand Final.

The results are shown in the table below.

Elapsed game time (mins) e	4	7	15	22	31	40	54	63	71	80
Pulse rate (bpm) p	93	91	86	84	79	73	69	63	60	56

(a) Draw a scatterplot on the axes below.



(b) Describe the strength and direction of the association between the variables.

(c) Calculate the correlation coefficient r_{ep}.

(d) Determine the equation of the least squares line for this data with elapsed game time *e* used as the explanatory variable.

(e) Calculate the coefficient of determination for the linear association and state its meaning.

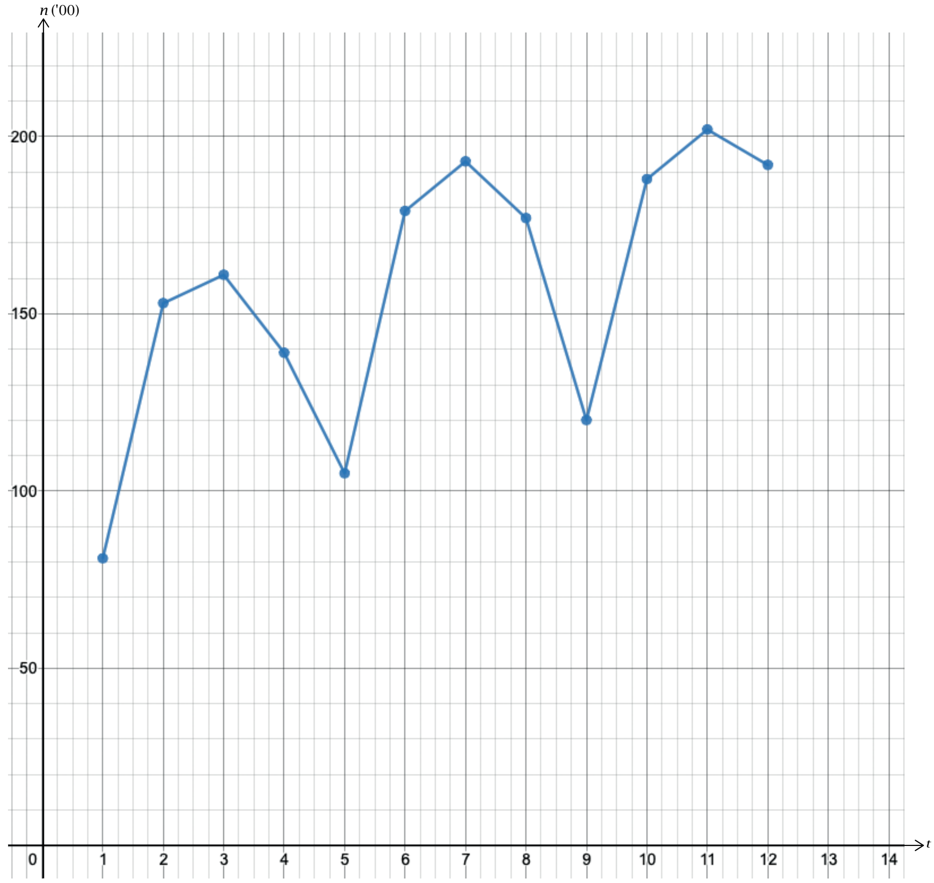
(f) Predict the pulse rate *p* when 50 mins of game time has elapsed.

(g) Comment on the validity of your prediction.

6. *Calculator Assumed*

The number of people in hundreds that visit a wildlife sanctuary is shown in the table below.

t	Year	Quarter	Number (*n*) of visitors ('00)	Quarterly mean	Visits as a percentage of the quarterly mean	Deseasonalised Values (Q) in ('00)
1	2016	1	81	133.5	60.7	125.776
2		2	153		114.6	138.587
3		3	161		120.6	136.556
4		4	139		*C*	129.543
5	2017	1	105	*B*	64.2	163.043
6		2	179		109.5	162.138
7		3	193		118.0	163.698
8		4	177		108.3	164.958
9	2018	1	120	175.5	68.4	186.335
10		2	188		107.1	*F*
11		3	*A*		115.1	171.332
12		4	192		109.4	178.938



(a) Calculate the values of A, B and C.

(b) Calculate the values of D and E in the table below.

Quarter	1	2	3	4
Seasonal Index	D	E	1.179	1.073

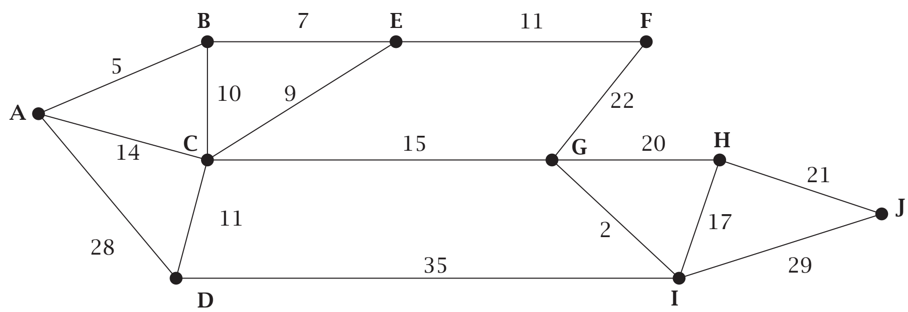
(c) Calculate the deseasonalised value of F in the table above.

(d) Determine the equation of the least squares line of Q against t.

(e) Estimate using the line from part (d) the number of visitors to the wildlife sanctuary in the first quarter of 2019.

7. *Calculator Free*

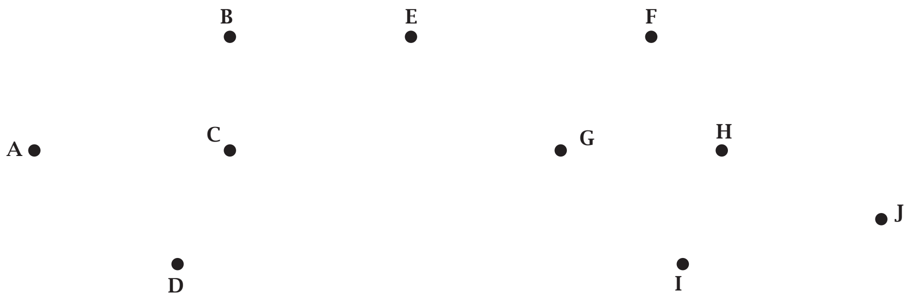
Andrew wishes to connect 10 garden lights using the least amount of wiring. The weighted graph below shows the distance between lights A, B, C, D, E, F, G, H, I and J in metres.



(a) Complete the table below showing the connections between each light.

	A	B	C	D	E	F	G	H	I	J
A	–	5	14	28	–	–	–	–	-	–
B	5	–	10	–	7		–	–	–	–
C	14	10	–	11		–		–	–	–
D	28	–		–	–	–	–	–		–
E	–				–	11	–	–	–	–
F	–					–	22	–	–	–
G	–						–	20	2	–
H	–							–	17	21
I	–								–	29
J	–									–

(b) Draw a minimum spanning tree and calculate the minimum amount of wiring needed to connect all 10 lights.



8. *Calculator Assumed*

Maureen invests \$25 000 into an account earning 8.75% p.a.

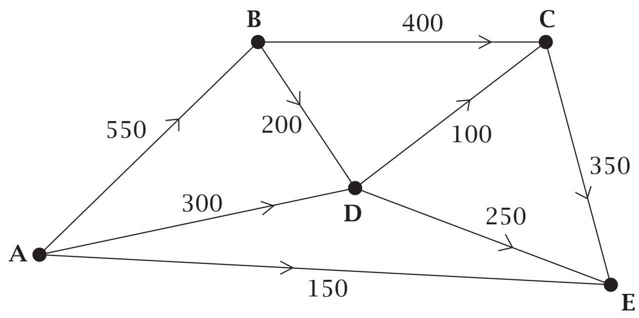
(a) Write a recursive formula which calculates the amount in the account after n months if the interest is calculated monthly.

(b) Determine the total amount in the account after three years if interest is calculated monthly.

(c) What is the effective interest rate if interest payments are calculated monthly.

9. *Calculator Free*

Traffic flow along a major network of roads, in vehicles per hour, is shown on the diagram below.



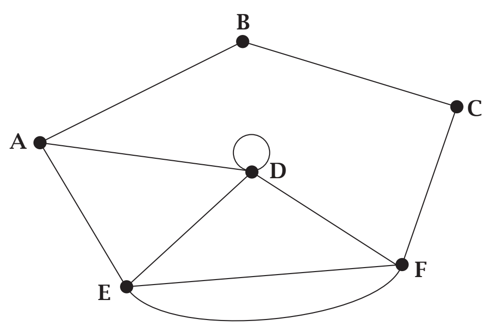
(a) Identify the source and sink.

(b) Calculate the maximum flow from *A* to *E*.

(c) The network needs to increase capacity by widening one of the roads. The direct road from *A* to *E* (150 vehicles) cannot be widened. Which one of the remaining roads should be widened in order to increase capacity and by how much will it increase the maximum flow?

10. *Calculator Free*

Six tourist attractions labelled *A*, *B*, *C*, *D*, *E* and *F* are separated by walkways according to the following network.



(a) Verify that Euler's rule works for this network.

(b) Is this graph planar? Explain.

(c) Construct an adjacency matrix for this network.

(d) Does this network contain a semi-Eulerian trail? Why?

(e) List the possible closed walks of length 2 from attraction *D*.

11. *Calculator Assumed*

A recent study determines whether a relationship exists between smoking and several diseases. The research looked at the type of smoker (none, social, regular, chain) and the following diseases (cancer, lung, cirrhosis, chronic obstructive pulmonary disease COPD). The results from 515 people are displayed in the table below.

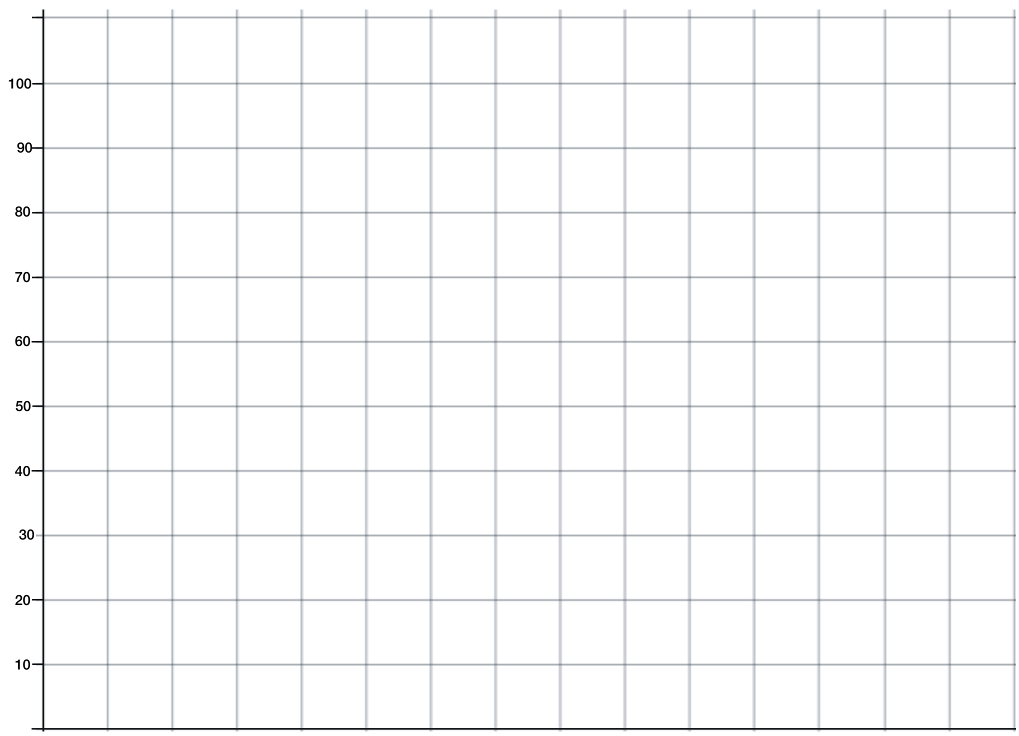
	Type of smoker			
Disease type	None	Social	Regular	Chain
Cancer	4	11	50	95
Lung	6	18	63	86
Cirrhosis	11	13	21	10
COPD	3	9	40	75
Total	24	51	174	266

(a) Which is the explanatory variable?

(b) Complete the two-way table to show the column percentages rounding your answers to the nearest whole number.

	Type of smoker			
Disease type	None	Social	Regular	Chain
Cancer				
Lung				
Cirrhosis				
COPD				
Total	100	100	100	100

(c) Use your table to construct a proportional column graph on the axes below.

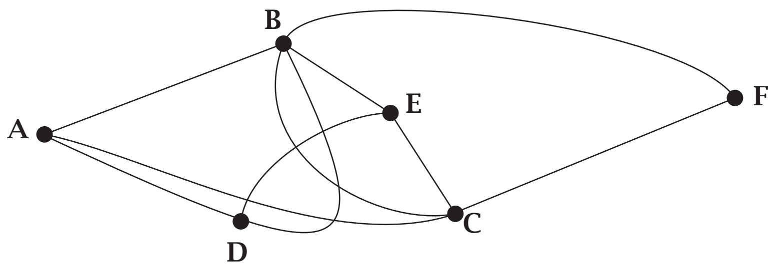


(d) Does there appear to be an association between the two variables. Explain.

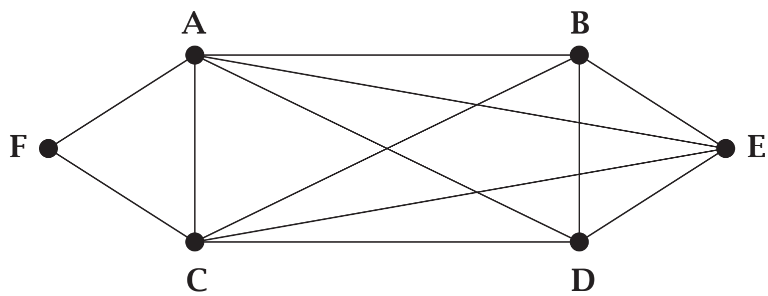
12. *Calculator Free*

For each of the following graphs, redraw them as planar graphs, if possible.

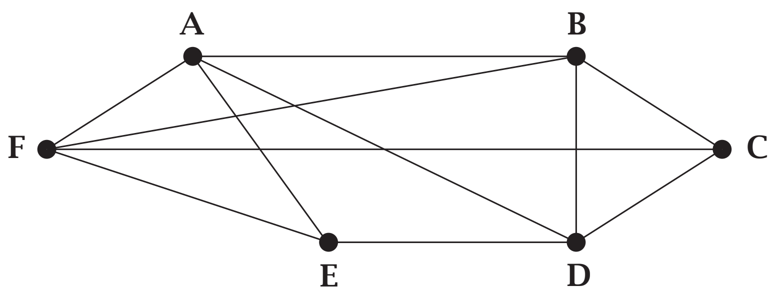
(a) *(i)*



(ii)



(iii)



(b) Is the graph in part (a) (i) traversable? Explain.

(c) Verify that Euler's rule works in part (a) (iii).

13. *Calculator Assumed*

Bob retires with a sum of \$750 000. He sets up an annuity where interest is added and \$7 250 is withdrawn at the end of each month. The table below shows the beginnings of Bob's annuity.

Month (n)	Balance at the start of the month (A_n)	Monthly interest	Amount withdrawn	Balance at the end of the month (A_{n+1})
1	\$750 000	\$3906.25	\$7250	\$746 656.25
2	\$746 656.25	\$3888.83	\$7250	*A*
3		*B*		*C*

(a) Calculate the annual interest rate.

(b) The balance of the annuity at the start of month n can be represented by the linear recurrence equation in the form $A_{n+1} = pA_n + q$, $A_1 = r$.
Determine the values of p, q and r.

(c) Determine the values of A, B and C in the table above.

(d) Calculate the balance of the annuity at the end of 5 years.

(e) How much interest did Bob's annuity earn over the 5 years?

14. *Calculator Free*

Gardens-R-Us are to complete a landscaping project. The list of activities, time (in days) and immediate predecessors for each activity are listed in the table below.

Activity	H	I	J	K	L	M	N	O	P	Q	R
Duration (days)	7	4	3	5	3	6	9	3	5	4	6
Immediate predecessors	–	H	H	H	I	I	L, J	L, J	O, K	M	N, P, Q

(a) Draw a project network showing all activities and durations.

(b) Determine the critical path and the minimum completion time.

(c) Determine the earliest starting time for Activity P.

(d) Determine the latest starting time for Activity K.

(e) State the float time for Activity Q.

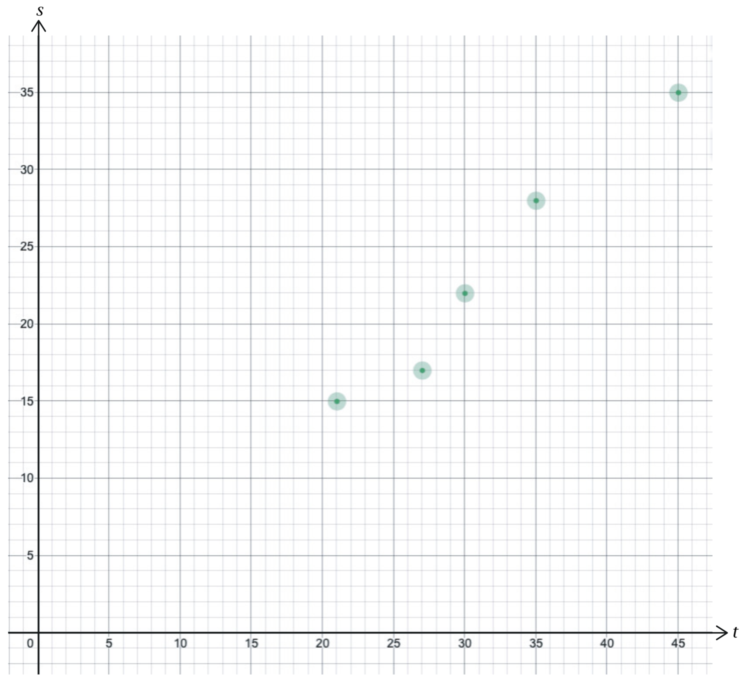
(f) Activity P is delayed by 2 days. How does this effect the critical path and minimum completion time?

15. *Calculator Assumed*

An experiment was conducted to determine the solubility of a substance based on its temperature. The results are shown in the table below.

Temperature °C t	21	27	29	30	35	38	45
Solubility (grams) s	15	17	18	22	28	31	35

(a) Complete the scatterplot below.



(b) Calculate the correlation coefficient between t and s.

(c) Determine the equation of the least square line that can be used to predict s from t and plot this on the scatterplot above.

(d) What percentage of the variation in solubility can be explained by the variation in temperature.

(e) Predict the solubility when the temperature of the substance is 10°C. Comment on the validity of this prediction.

(f) It was claimed that an increase in the temperature of this substance caused a greater solubility. Comment on this claim.

16. *Calculator Free*

The first three terms of a sequence are shown in the table below:

n	1	2	3
T_n	200	187	174

(a) Describe the type of sequence and its associated graph represented by the above data.

(b) State the next three terms.

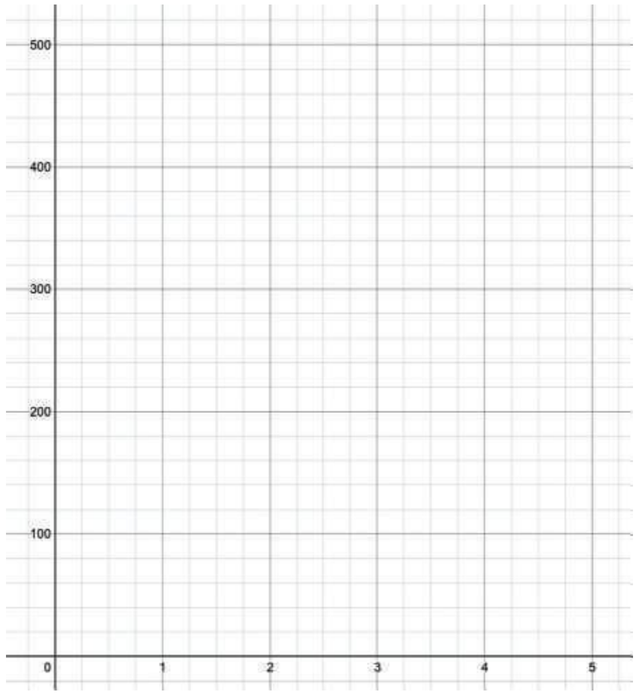
(c) Determine the n^{th} term of this sequence.

A geometric sequence is represented in the table below:

n	1	2	3	4
T_n	15		60	120

(d) Determine the missing value for $n = 2$.

(e) Plot the first 5 points on the axes below.



(f) State the recursive formula for this sequence.

Another sequence represented by the first order recurrence relation $T_{n+1} = 0.2T_n + 2$, $T_1 = 10$ has a steady state solution close to p in the long term.

(g) Determine the value of p.

17. *Calculator Assumed*

(a) Peggy invests \$500 000 into an account paying 12.25% p.a. compounded annually. How much interest will she earn over a period of 5 years?

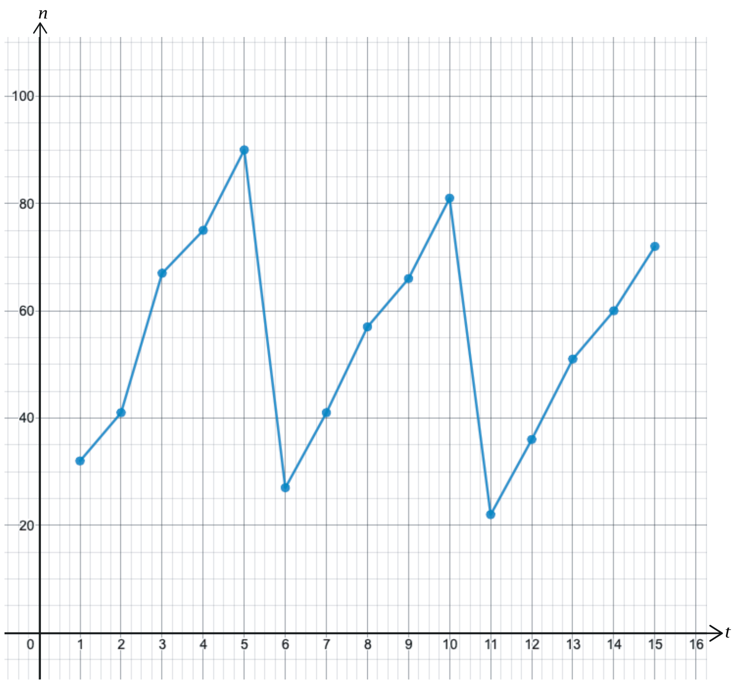
(b) Two other investment options for Peggy is to invest at a rate of 12.3% p.a. compounded monthly or 12.5% compounded daily. Justify which of these two options is better by calculating the effective interest rates.

(c) Peggy decides to set up a perpetuity for her two sons where each will receive a sum of \$25 000 to be paid at the beginning of each year. How much should Peggy invest into this perpetuity if the account pays interest at a rate of 5.75% p.a. compounded monthly?

18. *Calculator Assumed*

The number of lunches sold at the school canteen over a 3-week period is shown in the table below.

t	Day	Number of lunches (n)	Weekly mean	Percentage of the weekly mean
1	Monday	A		52.5%
2	Tuesday	41		67.2%
3	Wednesday	67	61	109.8%
4	Thursday	75		123.0%
5	Friday	90		147.5%
6	Monday	27		49.6%
7	Tuesday	41		C
8	Wednesday	57	54.4	104.8%
9	Thursday	66		121.3%
10	Friday	81		148.9%
11	Monday	22		45.6%
12	Tuesday	36		74.7%
13	Wednesday	51	B	105.8%
14	Thursday	60		124.5%
15	Friday	72		149.4%



(a) Describe the trend and seasonality of the time series.

(b) Calculate the values of A, B and C in the table above.

(c) Determine the seasonal indices for each day.

Day	Monday	Tuesday	Wednesday	Thursday	Friday
Seasonal Index					

(d) Calculate the deseasonalised number of lunches sold for Friday of week 2.

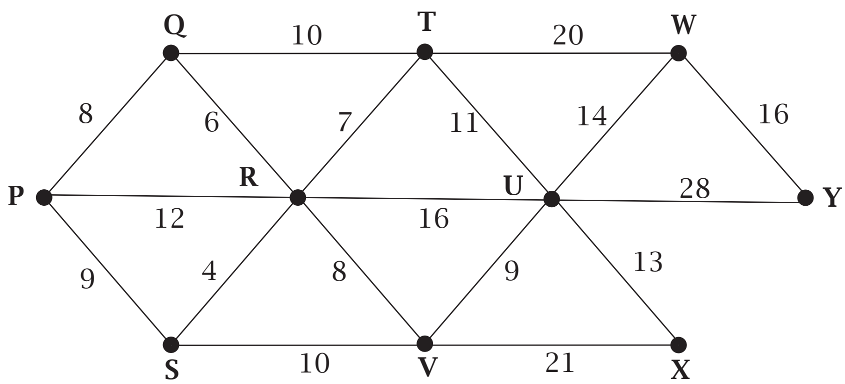
The equation of the least squares line used to estimate the number of lunches sold at the canteen is:

Deseasonalised sales = $-1.1924t + 64.0928$

(e) Estimate the actual sales of lunches sold at the canteen for Tuesday of week 4.

19. *Calculator Free*

The diagram below shows a network of roads with each length measured in kilometres and vertices *P* to *Y* representing towns.



(a) Find the minimum spanning tree and draw this on the diagram above.

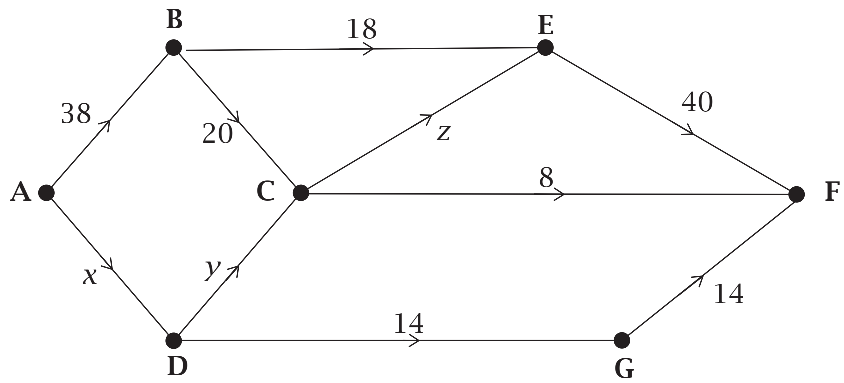
(b) Determine the length of the minimum spanning tree.

(c) Find the shortest distance from *P* to *Y*.

(d) A new road of length *x* km is added from *X* to *Y*. The shortest distance from *P* to *Y* is now reduced by using this new road. Determine the value of *x* as an inequality and hence solve.

20. *Calculator Free*

The network below shows the flow along a system of pipes with the capacity in litres per minute shown on each edge, achieving the maximum flow for the system.



(a) Find the values of *x*, *y* and *z*.

(b) Calculate the maximum flow from *A* to *F*.

21. *Calculator Assumed*

\$750 000 is deposited into a retirement fund on 1 April 2019 with a rate of 4.5% p.a. compounded monthly.

John withdraws \$75 000 on 1 April 2020 and each year thereafter.

(a) What is the balance of the retirement fund after the annual payment is made on 1 April 2024?

The retirement fund increases its interest rate to 5.25% p.a. compounded monthly for 7 years from 1 April 2024 to 1 April 2031 with annual payments to John kept at \$75 000.

(b) Determine the balance in the account after the annual payment is made to John on 1 April 2031.

(c) If John wishes to maintain the balance in the retirement fund in part (b), calculate his new annual payment.

22. *Calculator Free*

The table below shows the profit in dollars of allocating workers to tasks.

	Task A	Task B	Task C	Task D
Matt	2	13	5	20
Cheung	4	11	7	10
Leanne	3	12	10	15
Neya	4	14	3	5

Use the Hungarian algorithm to find an allocation to maximise profits and state this profit.

23. *Calculator Free*

A simple connected graph contains 9 edges and 6 vertices.

(a) Draw a planar graph with the following:
- one vertex with order 5.
- remaining vertices are odd.

Label the vertices A to F.

(b) State the number of faces for this graph.

(c) Calculate the sum of the order of the vertices.

(d) Is the graph Eulerian, semi-Eulerian or neither? Explain in detail.

(e) Determine a Hamiltonian path for this graph.

24. *Calculator Assumed*

A recent school survey asked the following questions:

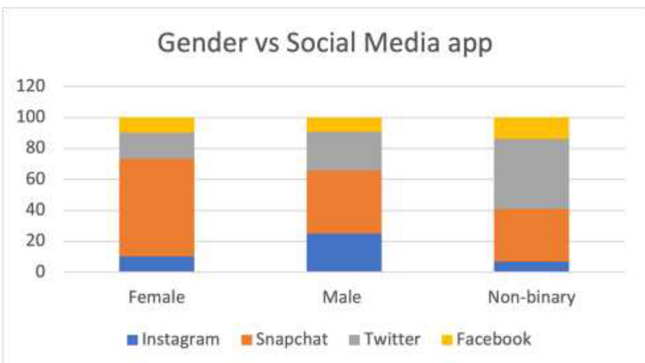
- State your gender.
- State your favourite social media app from the following list:
  - Instagram
  - Snapchat
  - Twitter
  - Facebook

Is there an association between the gender of students and their favourite social media app?

The results of 1 124 students are tabled below.

	Type of social media app			
Gender	Instagram	Snapchat	Twitter	Facebook
Female	60	370	98	58
Male	124	201	124	45
Non-binary	3	15	20	6

The data is displayed as a divided column graph below.



(a) State the explanatory variable.

(b) Construct a frequency table showing the row percentages to the nearest whole number.

	Type of social media app			
Gender	Instagram	Snapchat	Twitter	Facebook
Female				
Male				
Non-binary				

(c) Does the evidence above suggest an association between the categorical variables? Justify your response.

25. *Calculator Assumed*

Jemma borrows $18 000 to purchase a second-hand car. The table below shows the progress of the loan.

Month (n)	Amount owing at the start of the month	Monthly interest	Repayment	Amount owing at the end of the month
1	$18 000	$270	$500	$17 770
2	$17 770	$266.55	$500	$17 536.55
3	$17 536.55	$263.05	$500	$17 299.60
4	A	B	C	D

(a) Calculate the annual interest rate.

(b) Complete all entries for Month 4.

(c) How many repayments are needed to fully pay off the loan?

(d) Determine the final repayment.

(e) Calculate the <u>total</u> interest over the life of the loan.

(f) If Jemma wishes to pay off the loan in 1½ years, determine the repayments.

26. *Calculator Free*

Given the following adjacency matrix.

	A	B	C	D	E
A	0	0	0	0	0
B	0	1	2	1	1
C	0	2	0	1	0
D	0	1	2	0	0
E	0	1	0	1	0

(a) Explain what feature of the graph can be explained by the row and column of zeros.

(b) Explain why this network represented by the adjacency matrix is a directed graph.

(c) Explain why this is not a simple graph.

(d) Is the graph connected? Justify.

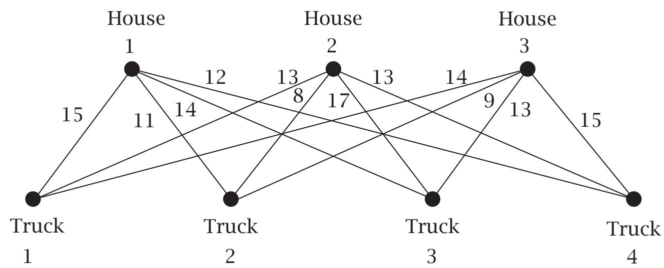
(e) Draw a graph that corresponds to this adjacency matrix.

(f) Draw a connected subgraph using only three of the vertices.

27. *Calculator Free*

A supermarket uses 4 courier trucks and needs to deliver food orders to 3 different houses.

The distance in kilometres to each house for the 4 courier trucks is shown in the graph below.



(a) State the name given to the above graph.

(b) Show the information in the above graph as a matrix.

(c) Optimise the allocation of the courier trucks to the houses using the Hungarian algorithm.

(d) What is the total minimum distance travelled by the 4 courier trucks?

28. *Calculator Assumed*

Senuka wishes to invest \$100 000 over 3 years into an account earning interest. His investment choices are:

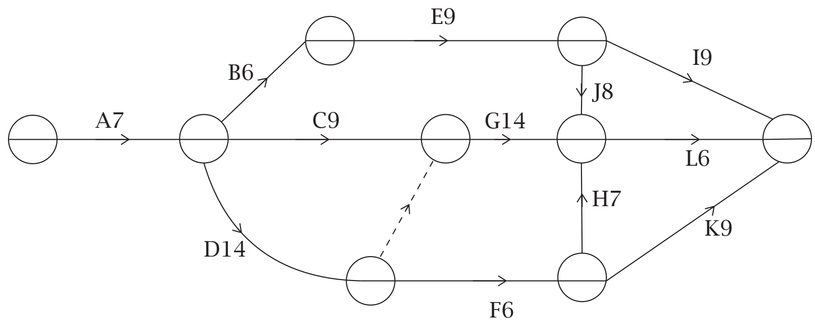
- 6.25% p.a. compounded daily
- 6.4% p.a. compounded monthly
- 6.8% p.a. compounded annually.

(a) Determine the profit achieved for each investment type.

(b) Rank in order from highest to lowest the three investments and justify your choice.

29. *Calculator Free*

A project consists of Activities A to L. The project network is drawn below with the completion time given in hours.



(a) What does the dotted line represent?

(b) Which activity(ies) immediately precede Activity K?

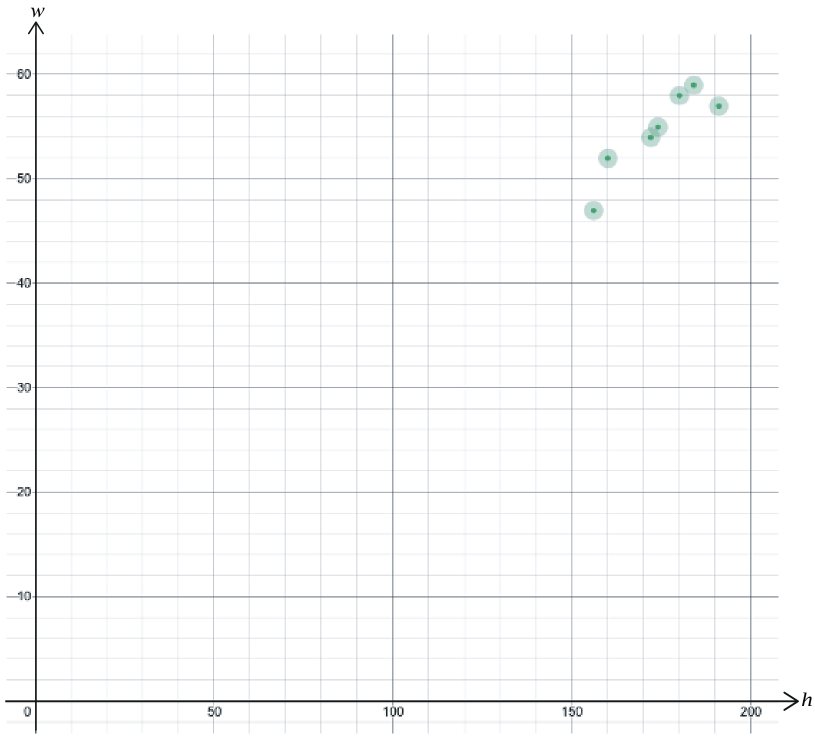
(c) Fill in the spaces with the earliest and latest starting time for each activity.

(d) State the critical path and the minimum completion time.

(e) The project needs to be shortened by 6 hours. Will altering the time for Activity G from 14 to 8 hours have the desired effect? Explain.

30. *Calculator Assumed*

The heights (*h cm*) and weights (*w kilograms*) of 7 students were recorded and their results displayed in the scatterplot below.



The equation for the least squares line for this data is: $w = 0.2946h + 3.3461$.

It was also found that 80.98% of the variation in weight (w) can be explained by the variation in height (h).

(a) Explain what the gradient of the least square line means in context of the question.

(b) Calculate the correlation coefficient r_{hw}.

(c) Predict the weight given the following heights and comment on the validity of the prediction for each student below.

(i) Marley – height: 180 cm.

(ii) Josh – height: 220 cm.

31. *Calculator Assumed*

(a) A geometric sequence has the first four terms as: 250, 200, 160, 128.

(i) Determine the n^{th} term rule for this sequence.

(ii) State the recursive formula for this sequence.

(iii) Calculate the 8^{th} term of this sequence.

(b) Another sequence has the first four terms as: –12.2, –10.1, –8, –5.9.

(i) State the type of sequence.

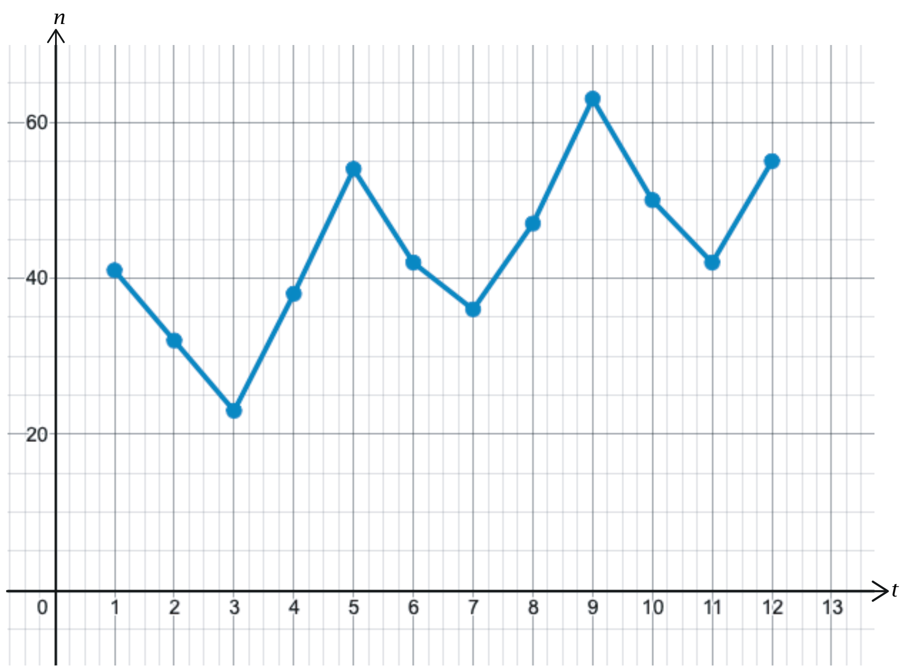
(ii) Determine the n^{th} term rule for this sequence.

(iii) State the recursive formula for this sequence.

(iv) Calculate the 26^{th} term of this sequence.

32. *Calculator Free*

The graph below shows data for a time series.



(a) Describe the trend and seasonality for this time series.

A moving average is applied to the time series.

(b) Explain the purpose of applying a moving average.

(c) What moving average should be calculated for this time series? Explain.

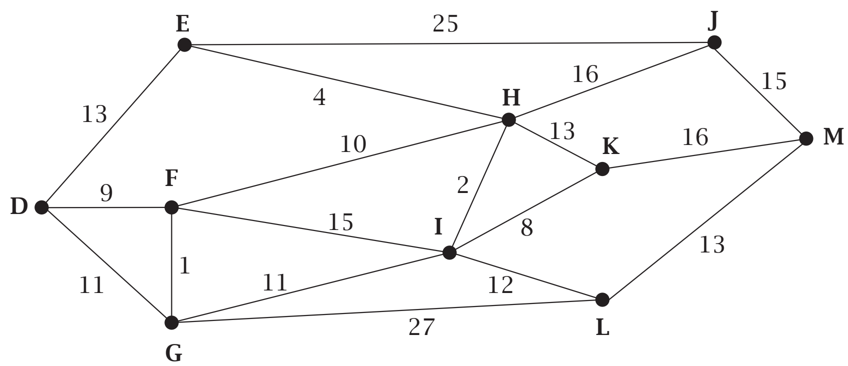
Part of the data for the graph above is shown in the table below.

t	2	3	4	5	6	7
n	32	23	38	54	42	36

(d) Calculate the appropriate moving average for $t = 4$.

33. *Calculator Free*

The diagram below represents cycle tracks between 10 tourist attractions labelled *D* to *M*. The length of each track is measured in kilometres.



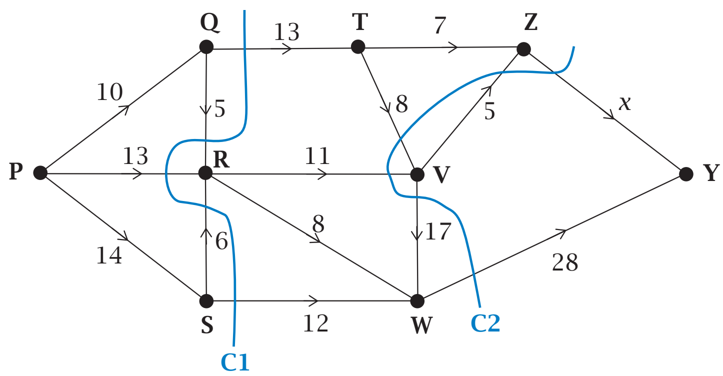
(a) State the shortest Hamiltonian path and calculate its length.

(b) State the shortest path and distance from *D* to *M*.

(c) Track I to K is closed due to repairs. How does this effect the shortest path and distance from D to M?

34. *Calculator Free*

The road network of a city is drawn below with the capacity of cars (in hundreds) per minute shown on each edge.



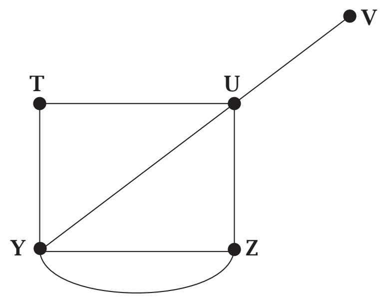
(a) Determine the value of cut $C1$.

(b) If the value of cut $C2$ is 57, determine the value of x. Show your working.

(c) By listing the different paths and their associated flow rates, determine the maximum number of cars that can pass through the network.

35. *Calculator Assumed*

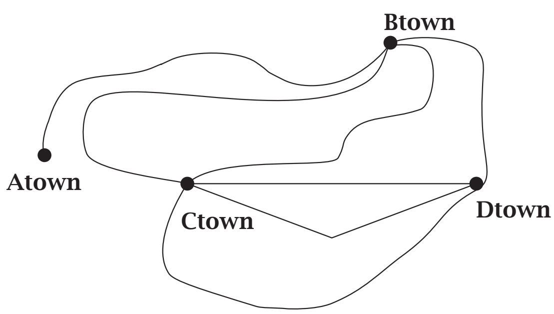
(a) Given the following network graph:



Which of the following are correct?

- The graph is simple.
- The graph is connected.
- The graph contains a cycle.
- The order of vertex *U* is 3.

(b) The diagram below shows the towns connected by four roads.



(i) Draw an adjacency matrix (*M*) for this graph.

(ii) Calculate M^2.

(iii) Use your answer in part (ii) above to calculate the number of two stage walks from Dtown to Ctown.

(iv) Explain what is found by the matrix $M^2 + M^3$.

(c) Given the following adjacency matrix.

$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \\ F \end{array}\begin{array}{c} \begin{array}{cccccc} A & B & C & D & E & F \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

(i) Draw a graph that corresponds to this adjacency matrix.

(ii) Redraw this network to show that it is bipartite, listing clearly the two distinct groups of vertices.

(iii) How many extra edges must be drawn for this to be a complete bipartite graph?

36. *Calculator Assumed*

A car is valued at \$45 900 and \$39 015 at the end of its second and third year respectively.

The depreciation of the car can be modelled by a recurrence relation: $D_{n+1} = 0.85D_n$, $D_1 = a$ where D_n represents the value of the car at the end of the n^{th} year and a represents the value of the car at the end of the first year.

(a) Show how the value of 0.85 is calculated in the recurrence relation D_{n+1}.

(b) Determine the value of a and the original price of the car.

(c) State a general rule for the n^{th} term.

(d) During which year will the value of the car fall below \$15 000.

(e) How much is the car worth at the end of the 10^{th} year?

37. *Calculator Free*

The school medley swimming team (Laura, Senuri, Chen and Adira) each swim one of four strokes (freestyle, breastroke, backstroke and butterfly). The times (in seconds) for each swimmer to swim each stroke are shown in the table below:

	Laura	Senuri	Chen	Adira
Freestyle	15	15	17	16
Breastroke	21	22	21	22
Backstroke	19	20	18	21
Butterfly	20	20	21	21

(a) Use the Hungarian algorithm to determine the allocation of swimmers to each stroke in order to minimise the team's overall time.

(b) Determine how many other possible allocations exist and state these?

(c) State the minimum team time.

38. *Calculator Free*

A building project is modelled by an activity network. The table below shows the activity, duration (in days) and the immediate predecessors for each activity.

Activity	Duration (in days)	Immediate Predecessors
A	8	----
B	9	----
C	9	----
D	7	A
E	11	C
F	5	A
G	8	C
H	2	B, D, E
I	3	F
J	9	H
K	11	B, D, E, G
L	6	H, I
M	7	J, K

(a) Draw a project network clearly showing all activities and durations.

(b) Determine the critical activities and the minimum completion time.

(c) Calculate the float times for Activities D, E and F.

(d) If Activity D is delayed by 6 days, explain the effect on the critical activities and minimum completion time.

39. *Calculator Assumed*

In preparation for retirement in 15 years, Meena deposits an amount of money monthly into a high interest savings account. The recurrence relation models the balance of the savings account after n monthly deposits according to:

$T_{n+1} = 1.024T_n + 1250 \qquad T_0 = 2500$

(a) Determine the annual interest rate.

(b) How much does Meena deposit each month?

(c) What is the balance of the savings account after 15 years?

(d) Determine the total interest earned by Meena after the final deposit at the conclusion of the 15^{th} year.

(e) Meena sets up a perpetuity at an interest rate of 9% p.a. compounded monthly based on the balance of the savings account at the end of the 15^{th} year. Calculate the amount Meena will receive annually for this perpetuity.

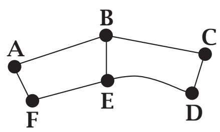
40. *Calculator Free*

(a) Draw a graph which has the following properties:

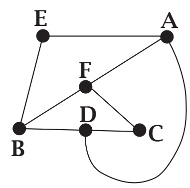
- Not Eulerian
- Has a Hamiltonian cycle.

(b) Given the following network graphs:

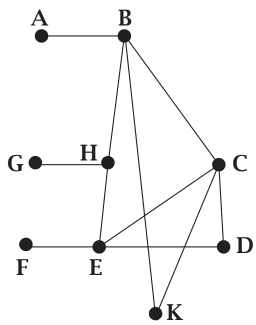
I



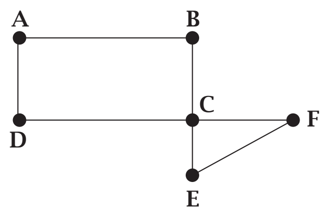
II



III



IV



(i) Which of the graphs above are a bipartite? Draw them.

(ii) Which of the graphs are Eulerian, Semi-Eulerian or neither?

(iii) State which of the graphs have a Hamiltonian path and/or Hamiltonian cycle or neither.

41. *Calculator Assumed*

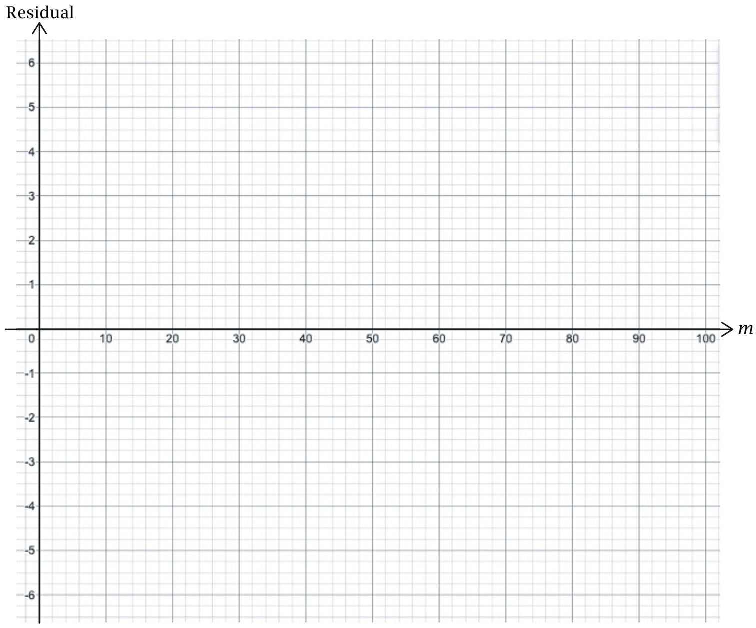
The results of a Maths (m) and Physics (p) test from a class of 10 students are shown in the table below.

Student	1	2	3	4	5	6	7	8	9	10
Maths (m)	41	78	19	89	83	56	74	51	69	45
Physics (p)	32	77	15	90	82	60	77	47	72	41
Residuals	–5.65	–1.73	1.78		–2.28	5.70	2.71	–1.75		–1.09

The equation of the least squares line for this data with Maths (m) used as the explanatory variable is $p = 1.1103m - 7.8749$. The correlation coefficient is 0.9909.

(a) Calculate the missing residuals in the table above.

(b) Construct a residual plot on the axes below.



(c) Comment on whether a linear model is appropriate.

42. *Calculator Free*

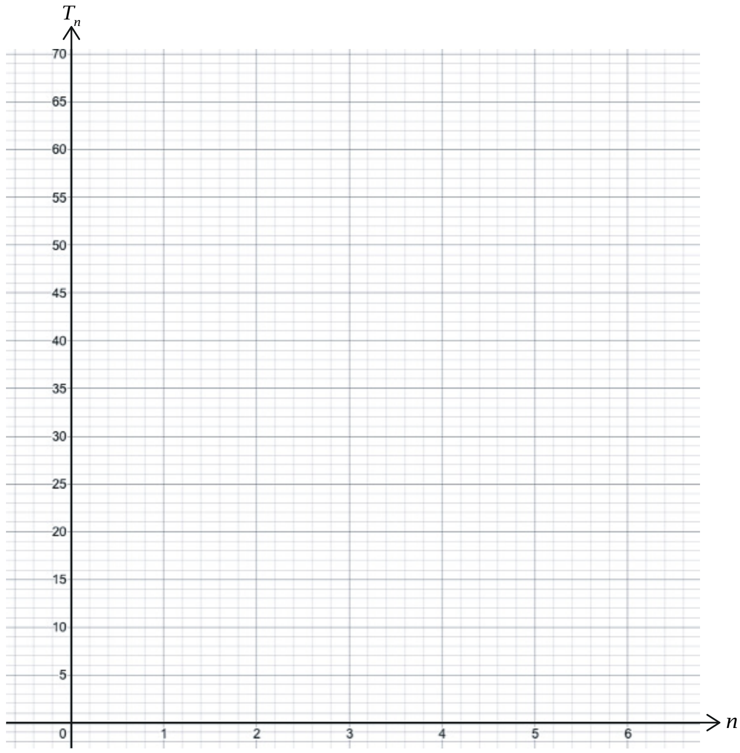
Genevieve places an amount of money into her safe each week as a way of saving money. The amount saved (in dollars) each week (n) is given by the recursive rule:

$T_{n+1} = T_n + 10, \quad T_0 = 15$

(a) Complete the table below using the recursive rule.

n	1	2	3	4	5
T_n					

(b) Plot the terms on the axes below.



(c) Determine a simplified n^{th} term rule for this sequence.

(d) If Genevieve continues to deposit money into her safe according to the recursive rule, during what week will she deposit an amount greater than $100.

43. *Calculator Assumed*

Vespa's quarterly mobile bill for the last 3 years is shown in the table below.

t	Year	Quarter	Amount ($)	4 point centred moving average (m)
1	2018	1	X	
2		2	67	
3		3	90	84.75
4		4	102	86
5	2019	1	82	87.75
6		2	73	90.375
7		3	98	93
8		4	115	95.25
9	2020	1	90	97.5
10		2	83	Y
11		3	106	
12		4	124	

(a) Why use a moving average?

(b) Calculate the values of X and Y.

(c) Calculate the least squares line for predicting m from t.

The seasonal indices for the time series are shown below.

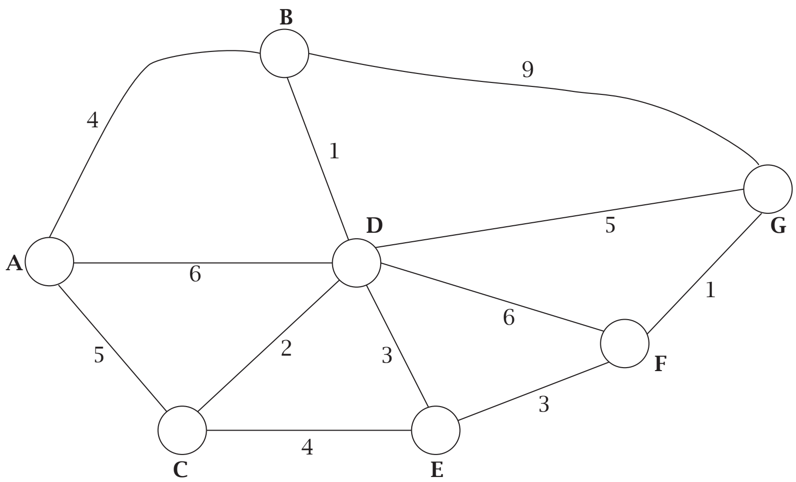
Quarter	1	2	3	4
Seasonal Index	0.903	0.804	1.062	

(d) Fill in the missing value in the table.

(e) Predict using the least squares line, Vespa's mobile bill for Quarter 4 in 2021.

44. *Calculator Free*

The attractions at the local fair are represented by the vertices and the weights on the edges represent the time (in minutes) to travel between them, are shown below.



(a) Travelling from attraction A, determine the shortest time to each other attraction and hence complete the table below.

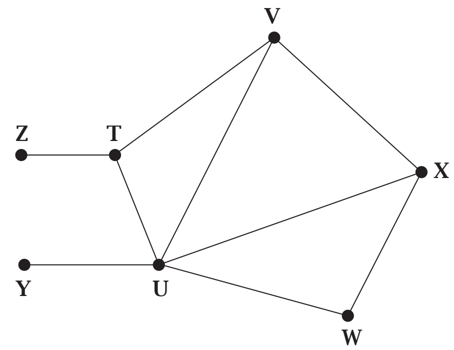
Attraction	*B*	*C*	*D*	*E*	*F*	*G*
Time (minutes)						

(b) Calculate the shortest time and route to travel from attraction *A* to *G*.

(c) Calculate the shortest time and route from attraction *A* to *G* via *F*.

45. *Calculator Free*

A courier company must deliver packages to all 7 retail shops each day. The retail shops are linked by roads according to the following network graph:



(a) The courier company must visit each retail shop once only. Is this a:

- Eulerian trail?
- Semi-Eulerian trail?
- Hamiltonian path?
- Hamiltonian cycle?

Justify your answer and state the journey.

(b) The courier company starts at retail shop *Y*, must visit each other retail shop and then return to retail shop *Y*. Add a single edge to the above diagram which will make this possible. Is this a:

- Eulerian trail?
- Semi-Eulerian trail?
- Hamiltonian path?
- Hamiltonian cycle?

Justify your answer and state the journey.

(c) The courier company must travel along each of the roads once and only once with repeated vertices (shops) permitted. Including the single edge added in part(b) above, what edge needs to be added to the diagram which will make this possible. Is this a:

- Eulerian trail?
- Semi-Eulerian trail?
- Hamiltonian path?
- Hamiltonian cycle?

Justify your answer and state the journey.

46. *Calculator Assumed*

Cameron turns 18 on the day he starts full-time employment, earning \$66 000 per annum.

He decides to deposit 5% of his income each year making equal monthly payments into a savings account.

He plans to retire at age 60.

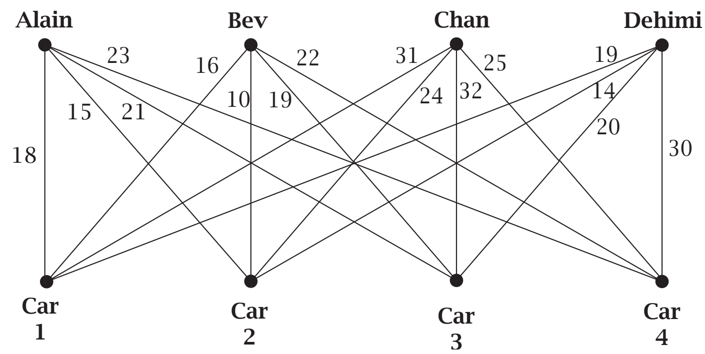
(a) How much will Cameron have in this savings account when he retires if the savings account pays 4.75% p.a. compounded monthly and his income remains constant.

(b) At age 60, Cameron invests the balance in the savings account into an annuity paying 3.25% p.a. compounded monthly. How much can he withdraw each month if he wants the money to last 20 years.

(c) Cameron wants to receive \$50 000 each year when he retires. How long will the annuity last and what will be the final payment if he receives equal monthly payments.

47. *Calculator Free*

The bipartite graph below shows car sales (in \$10 000's) per month by four salespersons (Alain, Bev, Chan, Dehemi) each selling 4 car brands (1, 2, 3 and 4).

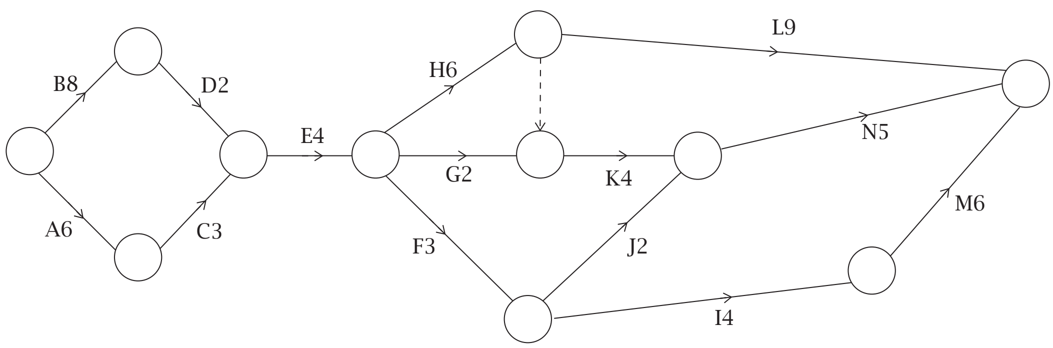


(a) Determine the allocation of each salesperson to car brand in order to maximise sales by using the Hungarian algorithm.

(b) Determine the maximum sales.

48. *Calculator Free*

A project consists of Activities A to M. The project network is drawn below with the completion time given in hours.



(a) Which activity will always lie on the critical path?

(b) State the critical activities.

(c) Determine the minimum completion time.

(d) Calculate the float time for Activity M.

Activity L was removed, and Activity T added according to the table below:

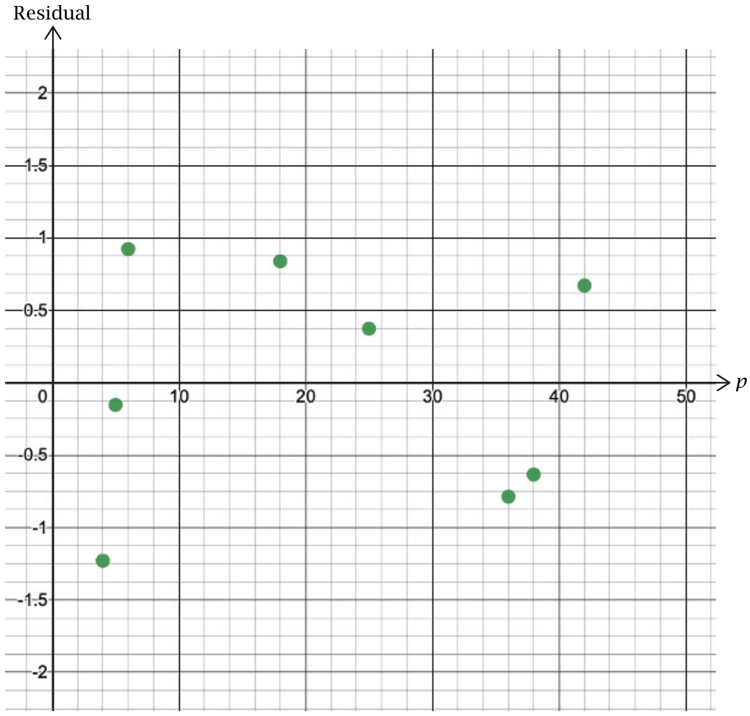
Activity	Duration (in hours)	Immediate Predecessors
N	5	J, K, T
T	1	H

(e) Redraw the project network below.

(f) Explain whether this change has altered the critical activities and minimum completion time.

49. *Calculator Assumed*

A residual plot is drawn below between the variables p and q. The equation of the least squares line to model the linear relationship between p and q is $q = -0.0829p + 15.8311$. The correlation coefficient is -0.8065.



(a) Two points are missing (32, 13) and (15, 16). Calculate the residual for each and add them to the above residual plot.

(b) Comment on the validity of this linear model.

50. *Calculator Assumed*

Ming decides to purchase a motorbike valued at \$9 800. She takes out a loan attracting interest of 10.5% p.a. compounded monthly and repays \$225 per month.

The table below illustrates her progress.

Month (n)	Amount owing at the start of the month	Monthly interest	Repayment	Amount owing at the end of the month
1	\$9800		\$225	
2			\$225	

(a) Fill in all missing values in the table above.

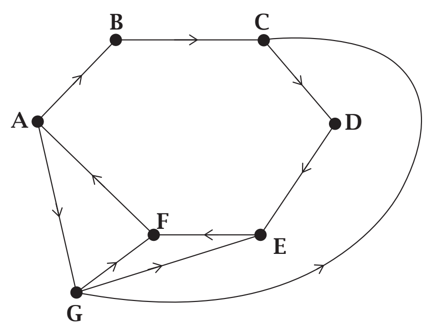
(b) State a recurrence relation for A_n the amount owing at the start of the month n.

(c) How many months are needed for Ming to pay off the loan.

(d) Calculate the total interest paid by Ming and the final repayment.

51. *Calculator Free*

A directed network is shown below.



(a) Construct an adjacency matrix M, for the above graph.

(b) Starting at vertex A find a possible solution for each of the following:

(i) A cycle of length 6.

(ii) A walk of length 4.

(iii) An open trail of length 8.

52. *Calculator Assumed*

A farmer uses a pole driver to drive star pickets into the ground to fence a new paddock. Each star picket is 2 *metres* long and must be driven in to half its length for stability of the fence. The star picket moves a distance of 30, 21.6 *and* 15.552 *cm* after each hit by the pole driver.

(a) Show that the distances the star picket moves form a geometric sequence.

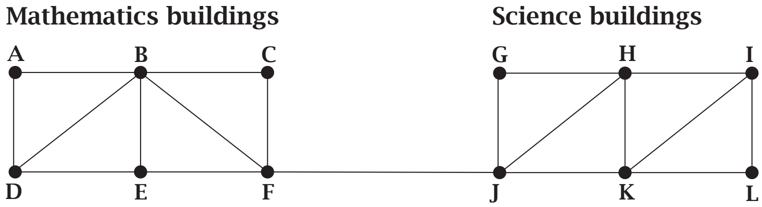
(b) State a general rule for the distance, T_n, the star picket moves for each hit, n, of the pole driver.

(c) How far does the star picket move on the 5^{th} hit of the pole driver?

(d) If the pattern continues, how many hits of the pole driver must be made by the farmer for each star picket to be stable enough for the fencing?

53. *Calculator Free*

The graph below shows a network of pathways connecting the various Science and Mathematics buildings at a university. A bridge separates the Science and Mathematics areas.



(a) If the bridge is removed, write down a cycle for the Mathematics buildings.

(b) If the bridge is removed, state, giving a reason, whether the graphs of the Mathematics and Science buildings are:

- Eulerian
- Semi-Eulerian
- Neither

(c) State whether the graph above is:

- Eulerian
- Semi-Eulerian
- Neither

(d) State the minimum number of extra edges that must be added to the above graph to make it Eulerian.

54. *Calculator Assumed*

$6 000 is to be invested into two accounts earning compound interest.
Account 1 earns 6.75% p.a. compounded monthly.

(a) Determine the final amount in Account 1 after 3 years.

(b) State a recursive rule for Account 1.

Account 2 earns interest at a rate compounded daily.

(c) What rate of interest is needed for Account 2 to earn the same interest as Account 1 after 3 years?

55. *Calculator Assumed*

The table below shows the age (*a*) in years of a company's employees and the number of days absent (*d*) from work in a year.

Age (*a*)	44	50	39	27	22	40	63	34	38	20
Days absent (*d*)	8	2	4	10	13	7	1	6	9	15

The number of days absent (*d*) can be determined by using an employee's age (*a*).

(a) Calculate r_{ad}.

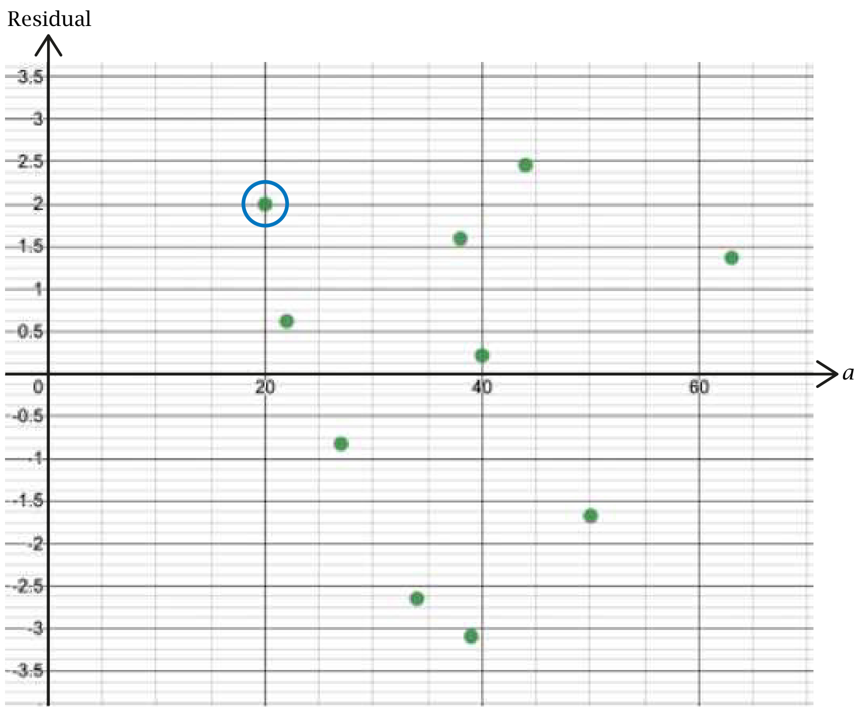
(b) Determine the least squares line between age and days absent.

(c) Interpret the value of the slope in context of the question.

(d) Predict the number of days absent from an employee who is 30 years of age.

(e) Comment on the validity of this prediction.

(f) A residual plot is drawn below. Show how the residual circled is calculated.



(g) Explain the meaning, in context, of a residual with a value of –3.09.

(h) The percentage of the variation of the days absent (*d*) can be unexplained by the variation in the age of the employee (*a*). Calculate this percentage.

56. *Calculator Assumed*

A fishpond contains 600 litres of water. Each day 40% of the water is lost due to a crack in the wall of the pond. An additional 200 litres of water is added to the pool at the end of each day.

(a) How much water is in the fishpond at the start of the 5^{th} day?

(b) Given the recurrence relation is $T_{n+1} = rT_n + d$, $T_1 = a$ where T_n is the amount of water in the pond at the start of the n^{th} day.

Calculate the values of r, d and a.

(c) Will the fishpond overflow? Explain with calculations.

57. *Calculator Assumed*

Icecream sales for a beach side café are shown below.

Year	Season	*t*	Sales	Seasonal mean	Sales as a percentage of the seasonal mean	Seasonally adjusted sales (*s*)
2018	Summer	1	256	*B*	154.4	160.7
	Autumn	2	202		*C*	*D*
	Winter	3	60		36.2	202.7
	Spring	4	145		87.5	157.1
2019	Summer	5	234	144.5	161.9	146.9
	Autumn	6	169		117.0	142.3
	Winter	7	43		29.8	145.3
	Spring	8	132		91.3	143.0
2020	Summer	9	199	123.25	161.5	124.9
	Autumn	10	145		117.6	122.1
	Winter	11	28		22.7	94.6
	Spring	12	*A*		98.2	131.1

The seasonal indices are:

Season	Summer	Autumn	Winter	Spring
Seasonal Index	1.593	*E*	0.296	0.923

(a) Calculate the values of *A*, *B*, *C*, *D* and *E*.

(b) During which season should the order for the number of icecreams supplied to the café be the smallest. Explain using information from the table/s above.

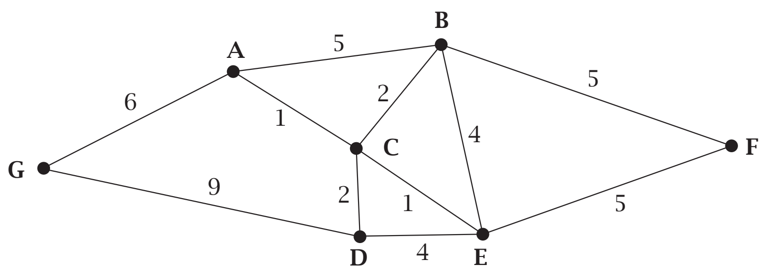
(c) Determine the least squares line using the seasonally adjusted sales.

(d) What does the slope of the least squares line indicate with respect to the long term sales figures?

(e) Use the least square line to predict the icecream sales for Autumn 2021.

58. *Calculator Free*

The school caretaker must open the 6 school gates (*A*, *B*, *C*, *D*, *E* and *G*) by 6:30 am each morning. The time taken in minutes for the caretaker to walk along the various pathways between the gates including the caretaker's house (*F*) is shown in the network diagram below.



(a) If the caretaker must check gate (*G*) and leaves the caretaker's house (*F*), determine his shortest route.

Each gate must be opened in the morning. He leaves the caretaker's house, must pass each gate exactly once, and then **returns to the house**.

(b) Describe this type of walk.

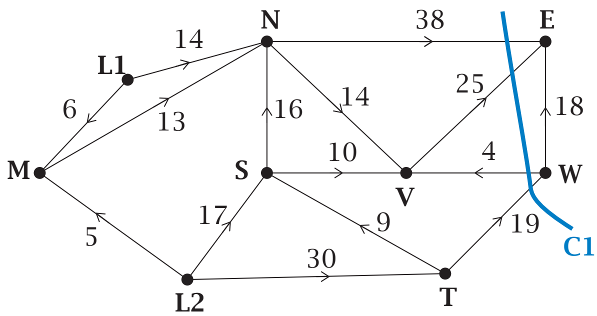
(c) Determine the caretaker's shortest path and its associated time.

(d) What time must the caretaker leave the house in order to open all the school gates on time and return to his house each morning?

(e) The paving along the pathway between *A* and *C* is being replaced and hence the pathway cannot be used. What is the shortest path and time the caretaker must leave his house in order to open all gates and return to his house by 6:30 am?

59. *Calculator Assumed*

A university has two lecture theatres (*L*1, *L*2) separated by walkways with only one exit (*E*). The weights on each edge represent the maximum number of people per minute that can travel along each walkway.



(a) Determine the value of cut *C*1.

(b) Find the maximum flow of this network from *L*1 and *L*2 to *E*.

(c) Part of the flooring at vertex *N* is being repaired so that the maximum number of people that can pass at this point is 30 per minute. The walkway joining N to V is also closed for repairs. Determine the new maximum flow through this network and describe the effect.

60. *Calculator Assumed*

An account earns 2.65% p.a. fixed for 18 years. When Dennis was born his parents deposited $10 000 into this account with a further $1000 each birthday including his 18^{th} birthday.

(a) Calculate the amount in Dennis' account on his 18^{th} birthday?

(b) Dennis invests the money received on his 18th birthday into an account paying 3.2% p.a. compounded monthly. Dennis withdraws an amount of money from this account each month. He withdraws $1000 at the end of the first month increasing the amount withdrawn by 10% per month.

State a recurrence relation for A_n the amount owing at the end of the month n.

(c) Calculate the balance after the 10^{th} withdrawal.

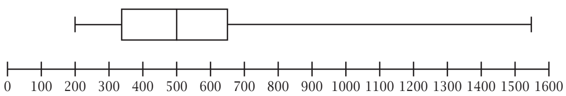
ANSWERS

CHAPTER 1: *Bivariate Data*

1. *(a)* Mean $\bar{x}$ = 66.65
   (b) Median = 71
   (c) Mode = 82
   (d) St Deviation = 21.90
   (e) Range = 69
   (f) IQR = 35.5

2. *(a)* Mean : 30.1
   (b) Modal class : 18–24
   (c) Median class : 25-31
   (d) Median is the better indicator as the mean is affected by the two outliers
   (e) St Deviation : 8.83

3. *(a)* Min : 200
   Q_1 : 337.5
   Median : 500
   Q_3 : 650
   Max : 1550



   (b) Skewed to the right
   (c) $337.50
   (d) Outliers are calculated by
   $Q_3 + 1.5\ \text{IQR} \leq \text{score} \leq Q_1 - 1.5\ \text{IQR}$
   $650 + 1.5\ (312.5) \leq \text{score} \leq 337.5 - 1.5\ (312.50)$
   $1118.75 \leq \text{score} \leq -131.25$
   Outlier is $1550
   (e) Removing the outlier will decrease both the mean and standard deviations.

4. *(a)* p = 5
   q = 10
   r = 15
   s = 18
   (b) Added score : 12
   (c) Removed score : 7

5. *(a)* 40 adults
   (b) The Histogram is skewed to the right
   (c) $\bar{x}$ = 1.55
   (d) $\frac{5}{40} \times 100 = 12.5\%$

6. *(a)* Explanatory variable – temperature °C
   Response variable – Amount of snow
   (b) Explanatory variable – number of hours worked
   Response variable – weekly pay
   (c) Explanatory variable – gender
   Response variable – favourite movie
   (d) Explanatory variable – consumption of coffee
   Response variable – heart rate.

7. *(a)*

	Passed exam	Did not pass exam	Total
Boys	90	30	120
Girls	90	90	180
Total	180	120	300

   (b) *(i)* Boys : 120
   (ii) Boys who passed : 90
   (iii) Girls who did not pass : 90
   (iv) Girls or those who failed exam : 210

8. *(a)*

	Males	Females	Total
'Chocolate' milk	18	6	24
'Other flavours' milk	33	63	96
Total	51	69	120

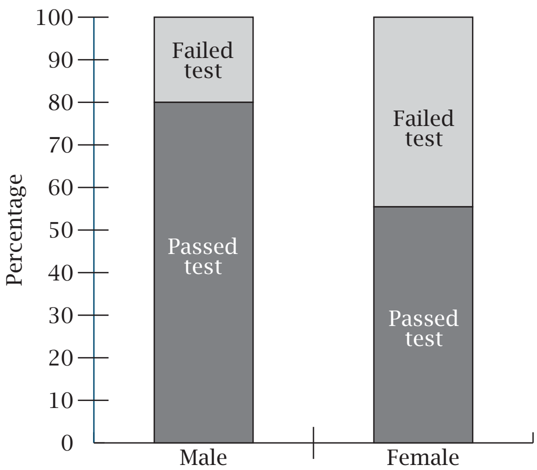
   (b) *(i)* 96
   (ii) 63
   (iii) 18

9. (a) Gender (Male/female) – explanatory variable
Passing/failing driving test – response variable

(b)

	Male	Female
Passed test	80%	55.45%
Failed test	20%	44.55%
Total	100%	100%

(c)



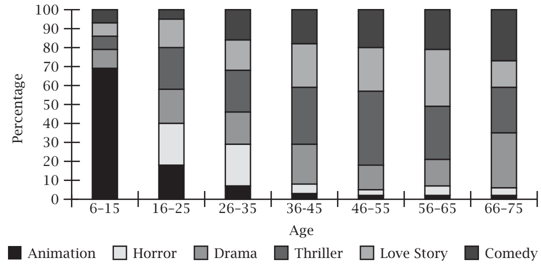
(d) The change in the proportions as we move from male to female indicates that there is an association between gender and passing the driving test. The number of males passing = 80% is significantly different to the number of females passing = 55.45%.

10. (a) Age is the explanatory variable and type of movie preference the response variable.

(b)

	Age						
Movie	6–15	16–25	26–35	36–45	46–55	56–65	66–75
Animation	69%	18%	7%	3%	2%	2%	2%
Horror	0%	22%	22%	5%	3%	5%	4%
Drama	10%	18%	17%	21%	13%	14%	29%
Thriller	7%	22%	22%	30%	39%	28%	24%
Love Story	7%	15%	16%	23%	23%	30%	14%
Comedy	7%	5%	16%	18%	20%	21%	27%
Total	100%	100%	100%	100%	100%	100%	100%

(c)



(d) The change in the proportions as we move from one age group to another indicates that there is an association between age and the preferred movie type. For example in the youngest age group, none enjoyed horror, but this increased in the 16–35 age groups and declined again in the ages after 35. The last 4 age groups have similar percentage distributions.

11. (a)

	Job	No Job	Total
Year 10	73	69	142
Year 11	36	35	71
Year 12	41	42	83
Total	150	146	296

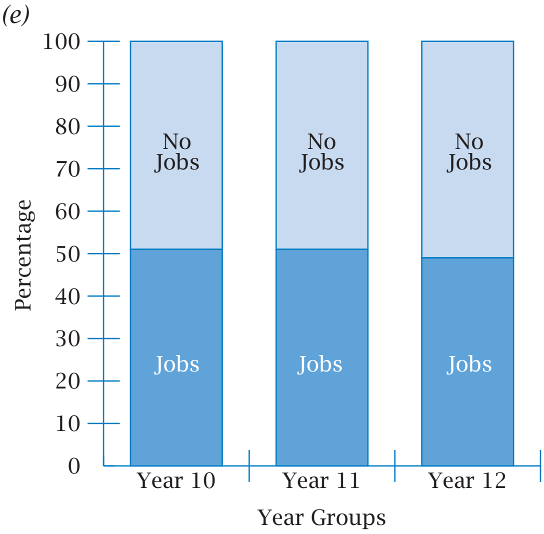
(b) 36 Year 11 students had a job.

(c) Explanatory variable – Year group i.e. Year 10, Year 11, Year 12
Response variable – Job or no job.

(d)

	Year 10	Year 11	Year 12
Job	51%	51%	49%
No Job	49%	49%	51%
Total	100%	100%	100%

(e)



(f) As the proportions are similar in each column there is no association between year groups and jobs.

12. (a) 105 hours

(b) 50 hours

(c) Brand B – higher median

– smaller IQR

(d) (i) 25%
(ii) 75%

(e) There seems to be an association between brand type and length of time the bulb lasts. Some reasons include:

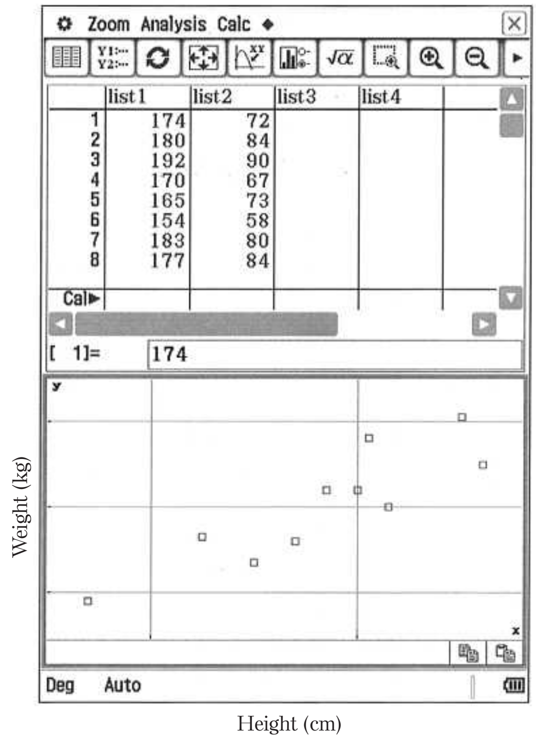
- Brand B has a higher median than Brands A and C.
- Brand B has a higher minimum value than Brands A and C.
- Brand B has a smaller range than Brands A and C.
- Brands B and C have a higher Q_3 than Brand A.

(other reasons possible)

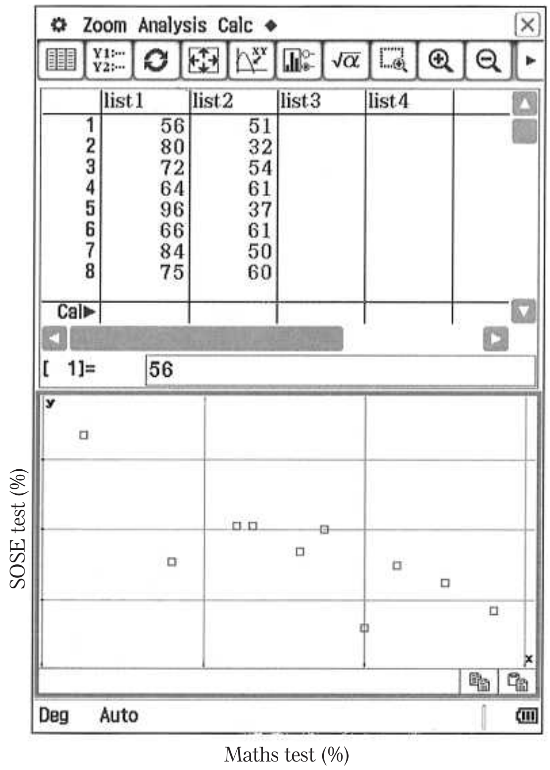
13. General comments:

- Class B has higher median.
- Class A has lowest and highest scores.
- Class A and B have both the same Q_3.
- Class B is more symmetrical than Class A.
- Class B has smaller range and IQR.
- Class B is more consistant – lower IQR, higher median.

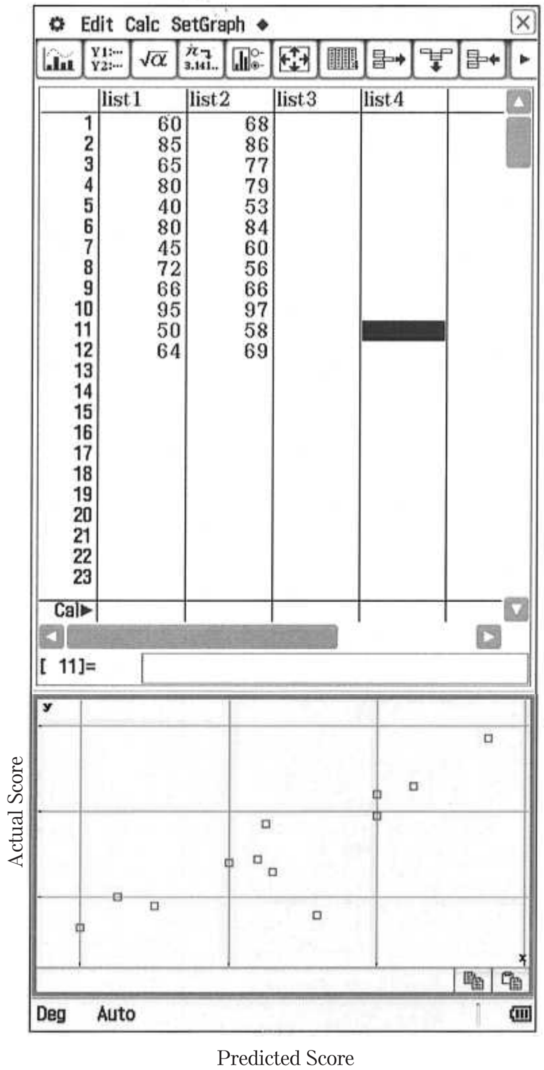
14. (a) A positive, strong linear relationship



(b) A strong, negative linear relationship



15. *(a)*



(b) A strong, positive linear relationship
$r = 0.8686$.

16. *(a)* 0

(b) –0.9

(c) 0.6

17. A : –0.19

B : –1

C : –0.95

D : –0.72

E : 0.65

18.

Graph	I	II	III
Description	C	B	A
Correlation Coefficient	0.12	–0.8	0.9

19.

(a) $r_{xy} = \dfrac{S_{xy}}{S_x . S_y}$

$= \dfrac{28.925}{(12.45)\,(3.78)}$

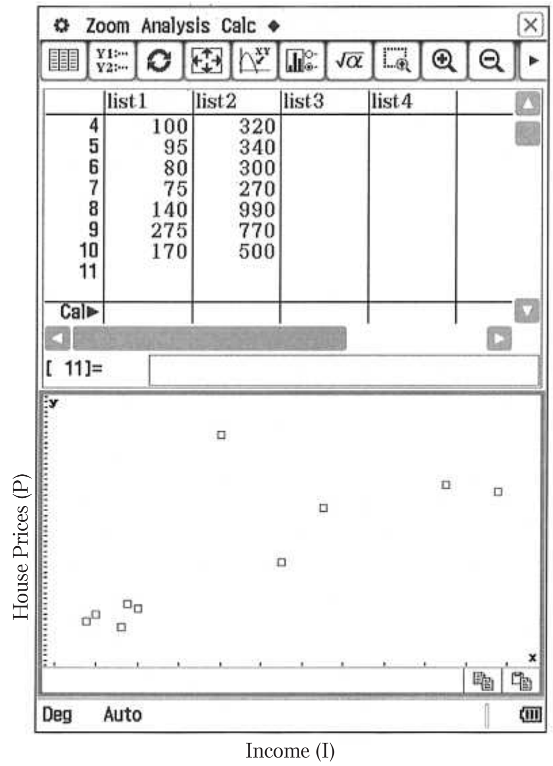
$= 0.6146$

(b) $r_{xx} = 1$

(c) $r_{xy} = -0.6146$

(d) Standard deviations cannot be negative.

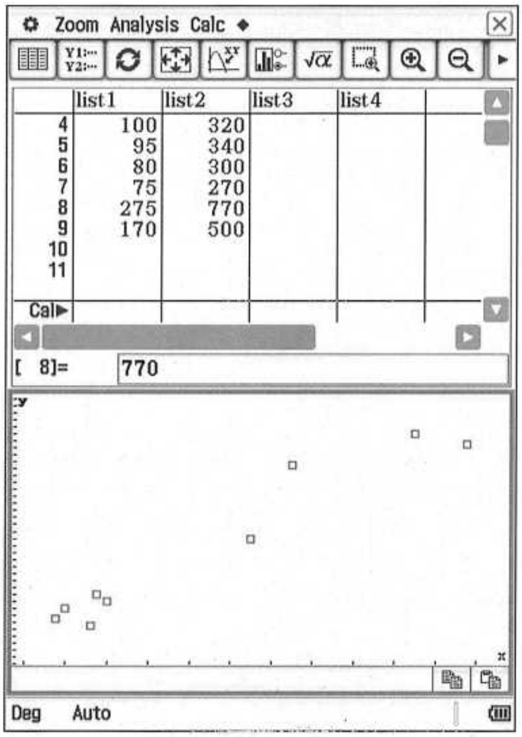
20. *(a)*



(b) $r_{IP} = 0.7504$

(c) Person VIII – an outlier

(d)



$r_{IP} = 0.9686$

(e) There is a strong, positive linear relationship between house prices and income.

CHAPTER 2: *Bivariate Data – Linear Models*

1. (a) $r_{xy} = 0.9897$

A strong, positive linear relationship

$r_{xy}^2 = 0.9795$

97.95% of the variation in y can be explained by the variation in x.

(b) $r_{xy} = -0.8742$

A strong, negative linear relationship

$r_{xy}^2 = 0.7643$

76.43% of the variation in y can be explained by the variation in x.

2. There is no cause and effect. Other factors may be involved. People eat ice cream in the warmer months when there is less chance of developing the flu. The *lurking* variable is the temperature/weather.

3. There is no cause and effect. Other factors may be involved. Rich nations have more TVs than poorer nations. Rich nations have a higher life expectancy for reasons such as health care, food, water etc. There is a *common* response between length of life and number of TV sets. The *lurking* variable is wealth.

4. (a) The value of the gradient is 0.15. This means that for every hour of exercise the weight loss is 0.15 kg.

(b) Exercise = 5 hours

Weight loss = 1.47 kg

Using line of regression

Predicted weight loss = 0.15 × 5 + 1.2

= 1.95 kg

Residual = actual – predicted

= 1.47 – 1.95

= -0.48

5. (a) A decreasing trend as the gradient of the line of regression is negative.

(b) For every 1 kilometre travelled the tyre loses 0.246 mm of tread.

Rate of change is –0.246

6. (a) Increase – positive correlation.

(b) Decrease – negative correlation.

(c) The '2.5' indicates that for every 1 hour studied the test mark increases by 2.5 %.

The '30' indicates that studying 0 hours will result in a mark of 30%.

7. (a) $\hat{y} = 3x + 7$

$\hat{y} = 3(6) + 7$

$\hat{y} = 25$

Prediction is reliable due to interpolation and a strong positive linear correlation coefficient.

(b) $\hat{y} = 3x + 7$

$\hat{y} = 3(-6) + 7$

$\hat{y} = -11$

Prediction is unreliable due to extrapolation.

(c) The residual plot shows a pattern. Hence the linear model is inappropriate.

8. (c)

9. (a) (i) $r_{xy} = 0.8960$
(ii) $r_{xz} = 0.5844$
(iii) $r_{yz} = 0.7551$

(b) Use r_{xy} as the strongest correlation

$\hat{y} = 0.8889x + 10.5528$

For test x student 6 scored 75%

$\hat{y} = 0.8889\ (75) + 10.5528$

$\hat{y} = 77.22$

Student 6 scored 77.22% for test y.

10. (a) $r_{xy}^2 = 0.8042$

This is an appropriate model due to a high r^2 value. 80.42% of the variation is explained leaving only 19.58% unexplained.

As r^2 gets closer to 1 the relationship between the variables becomes stronger and the regression line becomes more appropriate.

(b) $r_{xy} = -0.8968$

There is a strong, negative linear relationship between x and y.

(c) $\hat{y} = -1.25x + 9.7$

(i) Predicted value of y when $x = 5$

$\hat{y} = -1.25(5) + 9.7$

$= 3.45$

(ii) Residual is $y - \hat{y}$

$y - \hat{y} = 6.4 - 3.45$

Residual $= 2.95$

(d) The correlation coefficient will become stronger.

(e) $\bar{x} = 3.55$

$\bar{y} = -1.25\ (3.55) + 9.7$

$\bar{y} = 5.2625$

11. (a) $r_{da} = -0.9777$

(b) $a = -19.465d + 391.331$

(c) Predicted change

$= -19.465 \times (-3)$

$= 58.395$

Approximate increase of 58 in number of accidents.

(d) $\hat{a} = -19.465\ (16) + 391.331$

$\hat{a} = 79.891$

Approximate 80 accidents.
As this is an extrapolation prediction is unreliable.

(e) There is no cause and effect. Other factors may be involved.

(f) $r_{ia} = -1$

(g) $r_{id} = 0.9777$

12. (a) Linear model is appropriate → residuals are random.

(b) Linear model is not appropriate → residuals form a pattern.

(c) Linear model is appropriate → residuals are random.

(d) Linear model is not appropriate → residuals form a pattern.

13. (a) $r_{hj} = 0.8001$

$j = 2.5507h - 2.8034$

(b) Better predictor is height as the correlation coefficient is stronger.

(c) Rate of Change $= 2.5507$

∴ Average height change

$= 2.5507 \times (0.2)$

$= 0.51$ metre

(d) Neither model as both age (12 yrs) and height (1.4 m) are outside the data range. Prediction would be an extrapolation and hence unreliable.

(e) There is no cause and effect. Other factors may be involved.

14. (a) $r_{pq} = 0.9891$

A very strong, positive linear relationship.

(b) $q = 0.0957p - 0.1835$

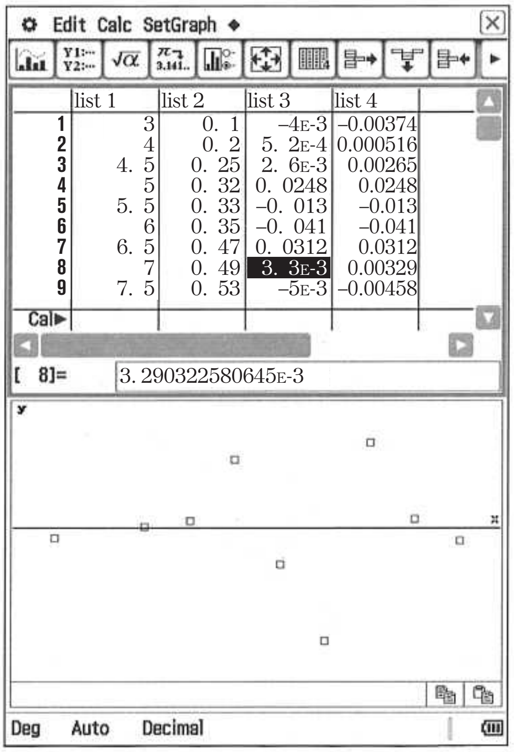
when $p = 3.5$

$\hat{q} = 0.0957\ (3.5) - 0.1835$

$\hat{q} = 0.15145$

Prediction is reliable as there is a very strong correlation coefficient and the data value is an interpolation.

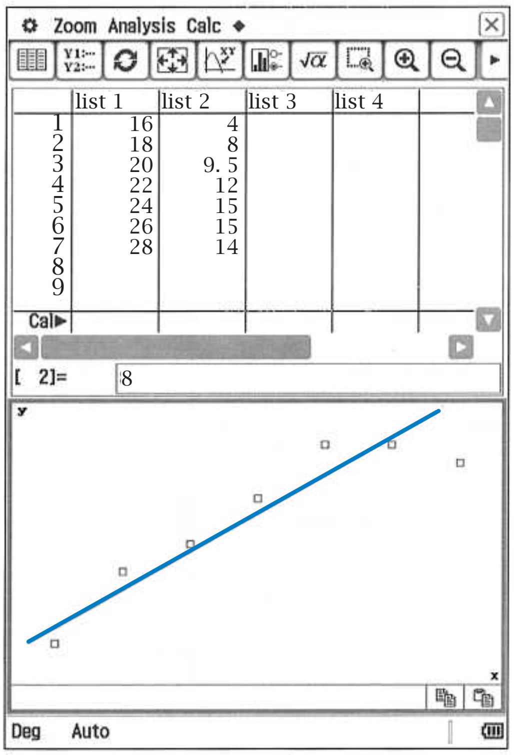
(c)



Residuals are small and random. Hence linear model is appropriate.

15. *(a)* Growth rate is the response variable

(b)



Growth rate (r)

Temperature °C (t)

(c) $r_{tr} = 0.9253$

(d) $r = 0.8839t - 8.375$

(e) Rate of change $= 0.8839$

Predicted increase $= 0.8839 \times 1$

$= 0.8839$ units

(f) $r = 0.8839\,(33) - 8.375$

$r = 20.7937$

Growth rate $= 20.7937$ units

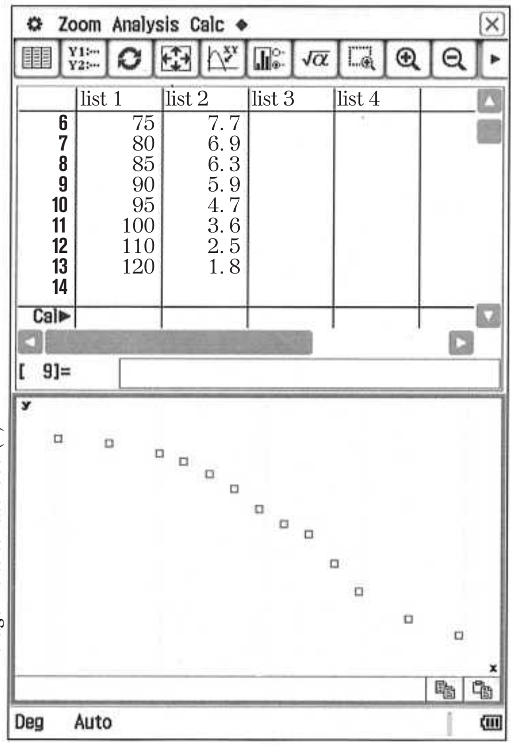
(g) This value of $t = 33$ is an extrapolation.

Hence the prediction is unreliable.

(h) $r_{tr}^{\ 2} = 0.8561$

(i) 85.61%

16. *(a)*



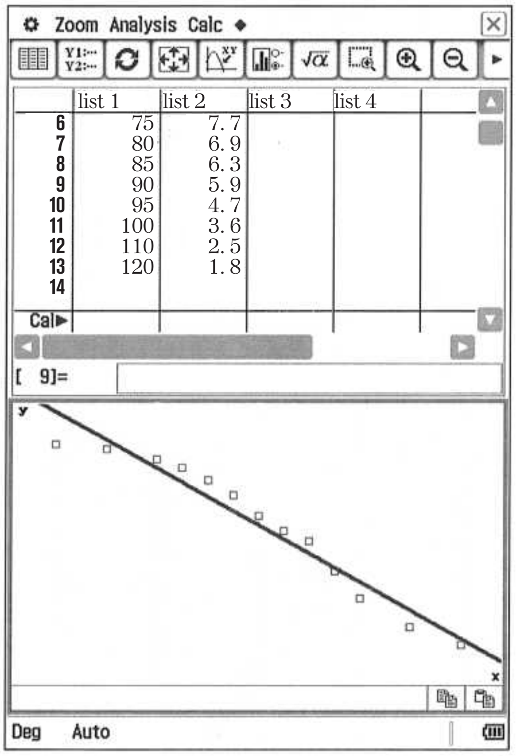
Average visits to fast food (F)

Income (I) (thousands dollars)

(b) Response variable – Average visits to fast food outlets.

(c) $r_{IF} = -0.9729$

(d)

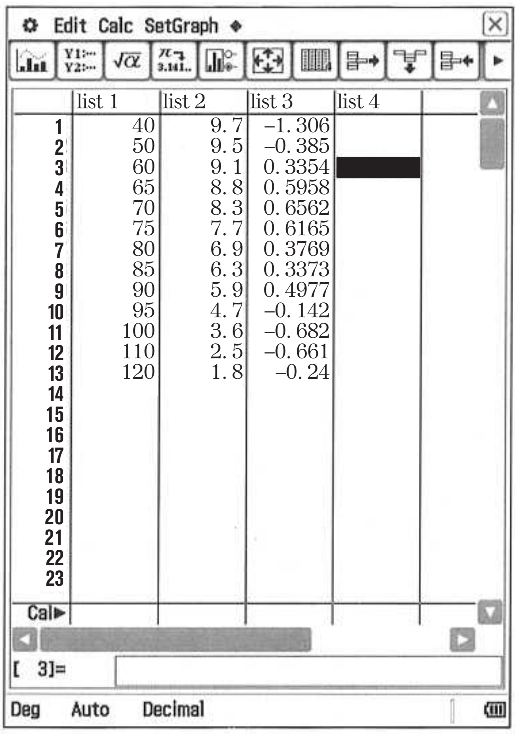


	list 1	list 2	list 3	list 4
6	75	7.7		
7	80	6.9		
8	85	6.3		
9	90	5.9		
10	95	4.7		
11	100	3.6		
12	110	2.5		
13	120	1.8		
14				

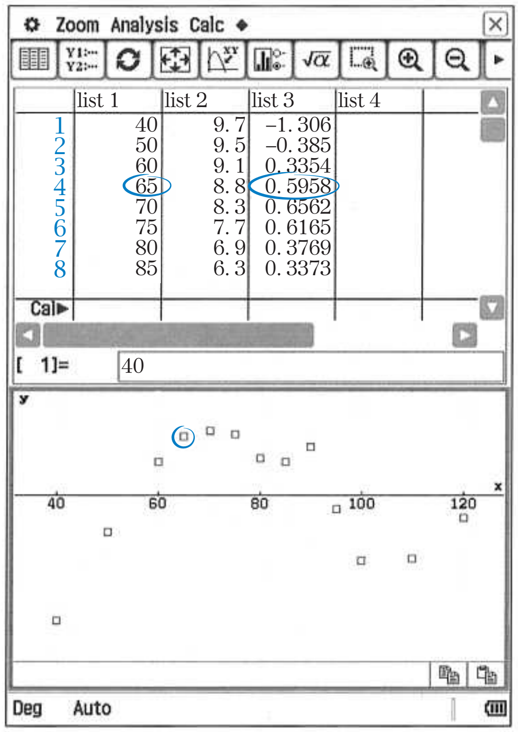
$F = -0.1121I + 15.4892$

(e) There is a strong negative linear relationship between Income and visits to fast food outlets.

(f)



	list 1	list 2	list 3	list 4
1	40	9.7	−1.306	
2	50	9.5	−0.385	
3	60	9.1	0.3354	
4	65	8.8	0.5958	
5	70	8.3	0.6562	
6	75	7.7	0.6165	
7	80	6.9	0.3769	
8	85	6.3	0.3373	
9	90	5.9	0.4977	
10	95	4.7	−0.142	
11	100	3.6	−0.682	
12	110	2.5	−0.661	
13	120	1.8	−0.24	
14				
15				
16				
17				
18				
19				
20				
21				
22				
23				



	list 1	list 2	list 3	list 4
1	40	9.7	−1.306	
2	50	9.5	−0.385	
3	60	9.1	0.3354	
4	65	8.8	0.5958	
5	70	8.3	0.6562	
6	75	7.7	0.6165	
7	80	6.9	0.3769	
8	85	6.3	0.3373	

Residual value for $I = 65$ is 0.5958.

(g) The linear model is not appropriate as the residuals form a pattern.

(h) $r_{IF}^2 = 0.9465$

(i) 94.65%

(j) $F = -0.1121I + 15.4892$

when $I = 55$

$F = 9.324$

The average number of monthly visits is 9.324.

(k) Rate of change $= -0.1121$

Expected change $= -0.1121 \times 5$

$= -0.5605$

On average a drop in monthly visits to fast food outlets by 0.5605.

(l) $F = -0.1121(200) + 15.4892$

$F = -6.93$

Unrealistic – demonstrating the dangers or unreliability of extrapolation.

CHAPTER 3: *Sequences – Arithmetic, Geometric and Recursive*

1. *(a)* Each term is obtained by adding 4 to the previous term.

(b) Each term is obtained by subtracting 5 from the previous term.

(c) Each term is obtained by multiplying the previous term by 0.85

(d) Each term is obtained by adding the two previous terms.

2. (a) $T_n = T_{n-1} + 6$, $T_1 = 2$ $T_6 = 32$

(b) $T_n = T_{n-1} - 1.5$, $T_1 = 7$ $T_6 = -0.5$

(c) $T_n = 1.5T_{n-1}$, $T_1 = 2$ $T_6 = 15.1875$

(d) $T_n = 5T_{n-1} - 5$, $T_1 = 2$ $T_6 = 2345$

3. (a) 6, 11, 16, 21

(b) 3, 0, –6, –18

(c) 2, 3, 5, 8

4. 2, 10, 18, 26, 34, 42

5. (a)
$$T_2 = 3T_1 + k$$
$$12 = 3(5) + k$$
$$k = -3$$
$$\therefore T_{n+1} = 3T_n - 3$$
$$T_4 = 96$$

(b)
$$T_2 = kT_1 - 2$$
$$-12 = k(5) - 2$$
$$-12 = 5k - 2$$
$$k = -2$$
$$\therefore T_{n+1} = -2T_1 - 2$$
$$T_4 = -46$$

6. 5, 9, 15, 23, 33

7. (a) $T_{n+1} = 4T_n - 3$ $T_1 = 6$

6, 21, 81, 321, 1281

8. (a) 100, 130, 169, 219.7

(b) 30% increase.

9. (a) 6, 8, 10, 12

(b) $T_{n+1} = T_n + 2$, $T_1 = 6$

10. (a)
$$P_n = 100 - 2n$$
$$P_1 = 100 - 2(1)$$
$$= 98$$
$$P_2 = 100 - 2(2)$$
$$= 96$$
$$P_3 = 100 - 2(3)$$
$$= 94$$
$$P_4 = 100 - 2(4)$$
$$= 92$$
$$P_5 = 100 - 2(5)$$
$$= 90$$

Sequence: 98, 96, 94, 92, 90

(b) $P_{n+1} = P_n - 2$, $P_1 = 98$

11.
$$T_1 = 1$$
$$T_2 = -2$$
$$T_3 = -11$$
$$T_4 = 104$$
$$T_5 = 1124835$$

12. (a)
$$T_n = \frac{1}{2}(T_{n-1} + T_{n+1})$$
$$T_2 = \frac{1}{2}(T_1 + T_3)$$
$$5 = \frac{1}{2}(2 + T_3)$$
$$5 = 1 + \frac{1}{2}T_3$$
$$4 = \frac{1}{2}T_3$$
$$T_3 = 8$$

(b) Sequence is : 2, 5, 8, . . .

Recursive:

$T_{n+1} = T_n + 3$, $T_1 = 2$

13. 5000, 4150, 3444.50, …

$T_{n+1} = 0.83T_n$, $T_1 = 5000$

14. (a)

n	1	2	3	4	5
T_n	1	3	6	10	15

(b) $T_n = T_{n-1} + n$, $T_1 = 1$

15. (a) $T_n = T_{n-1} \times 1.018$, $T_1 = 250$ million

2008 i.e. $n = 2$
$$T_2 = T_1 \times 1.018$$
$$= 250 \text{ million} \times 1.018$$
$$= 254.5 \text{ million}$$

(b) 2009 i.e. $n = 3$
$$T_3 = T_2 \times 1.018$$
$$= 254.5 \text{ million} \times 1.018$$
$$= 259.081 \text{ million}$$

In 2009 population is 259.081 million.

16. *(a)* Arithmetic

(b) Geometric

(c) Neither

(d) Arithmetic

(e) Neither

(f) Neither

(g) Arithmetic

(h) Geometric

17. Sequence : –1, 1, 3, 5, . . .
Recursive formula:
$T_{n+1} = T_n + 2, \quad T_1 = -1$

18. *(a)*

$$\begin{aligned} a &= 3, \quad d = 4 \\ T_n &= a + (n-1)\,d \\ 83 &= 3 + (n-1)\,(4) \\ 83 &= 3 + 4n - 4 \\ 83 &= 4n \\ n &= 21 \end{aligned}$$

$\therefore$ 21 terms

(b)

$$\begin{aligned} a &= 84 \quad d = -2 \\ T_n &= a + (n-1)\,d \\ 4 &= 84 + (n-1)\,(-2) \\ 4 &= 84 - 2n + 2 \\ 4 &= -2n + 86 \\ 2n &= 82 \\ n &= 41 \end{aligned}$$

$\therefore$ 41 terms

(c)

$$\begin{aligned} a &= x \quad d = 3x \\ T_n &= a + (n-1)\,d \\ 58x &= x + (n-1)\,(3x) \\ 58x &= x + 3xn - 3x \\ 58x &= -2x + 3xn \\ 60x &= 3xn \\ \frac{60x}{3x} &= n \\ n &= 20 \end{aligned}$$

$\therefore$ 20 terms

19. *(a)*

$$\begin{aligned} a &= 3 \quad d = 4 \\ T_{10} &= 3 + (10-1)\,(4) \\ T_{10} &= 3 + 36 \\ T_{10} &= 39 \end{aligned}$$

$$\begin{aligned} T_n &= 3 + (n-1)\,(4) \\ T_n &= 4n - 1 \end{aligned}$$

(b)

$$\begin{aligned} a &= 27 \quad d = -3 \\ T_{10} &= 27 + (10-1)\,(-3) \\ T_{10} &= 27 - 27 \\ T_{10} &= 0 \end{aligned}$$

$$\begin{aligned} T_n &= 27 + (n-1)\,(-3) \\ T_n &= 27 - 3n + 3 \\ T_n &= -3n + 30 \end{aligned}$$

(c)

$$\begin{aligned} a &= 3x \quad d = -5x \\ T_{10} &= 3x + (10-1)\,(-5x) \\ T_{10} &= 3x - 45x \\ T_{10} &= -42x \end{aligned}$$

$$\begin{aligned} T_n &= 3x + (n-1)\,(-5x) \\ T_n &= 3x - 5xn + 5x \\ T_n &= 8x - 5xn \end{aligned}$$

20.

$$\begin{aligned} T_1 &= 8 \quad T_4 = 26 \\ T_n &= a + (n-1)\,d \\ 26 &= 8 + (4-1)\,(d) \\ 26 &= 8 + 3d \\ 3d &= 18 \\ d &= 6 \end{aligned}$$

$$\begin{aligned} T_{20} &= 8 + (20-1)\,(6) \\ T_{20} &= 122 \end{aligned}$$

21.

$$\begin{aligned} x^2 - (-6) &= 7x - x^2 \\ x^2 + 6 &= 7x - x^2 \\ 2x^2 - 7x + 6 &= 0 \\ x = 2, &\quad x = 1.5 \end{aligned}$$

22.

$$T_{n+1} = kT_n + 2$$

$$T_3 = 42 \qquad T_3 = kT_2 + 2$$

$$\begin{aligned} 42 &= kT_2 + 2 \\ T_2 &= kT_1 + 2 \\ T_2 &= \boxed{3k + 2} \\ 42 &= k\,(3k + 2) + 2 \\ 42 &= 3k^2 + 2k + 2 \\ 3k^2 + 2k - 40 &= 0 \\ k = -4, &\quad k = \frac{10}{3} \end{aligned}$$

23. (a) $T_3 = 30, \qquad T_{10} = 9$

$30 = a + (3 - 1)(d) \qquad 9 = a + (10 - 1)(d)$

$30 = a + 2d \qquad 9 = a + 9d$

Solve simultaneously

$\therefore a = 36 \qquad d = -3$

$T_6 = 36 + (6 - 1)(-3)$

$T_6 = 36 + (-15)$

$T_6 = 21$

(b) $T_n = 36 + (n - 1)(-3)$

$T_n = 36 - 3n + 3$

$T_n = -3n + 39$

$-3 = -3n + 39$

$3n = 42$

$n = 14$

Fourteenth term becomes negative.

24. $T_3 = 4 \qquad T_6 = 32$

$4 = ar^2 \qquad 32 = ar^5$

Solve simultaneously

$a = 1 \qquad r = 2$

$T_{10} = ar^9$

$T_{10} = (1)(2)^9$

$T_{10} = 512$

25. (a) 47, 62

(b) $T_n = n^2 + 2n - 1$

(from classpad)

26. (a) $\dfrac{2x}{x - 6} = \dfrac{x^2}{2x}$

$x = 0, 10$ (from CAS calculator)

$\therefore x = 10$

Sequence is:

4, 20, 100, . . .

$a = 4 \qquad r = 5$

$T_7 = (4)(5)^6$

$T_7 = 62\,500$

(b) $a = 2 \qquad r = 3$

$T_n = ar^{n-1}$

$1000\,000 = (2)(3)^{n-1}$

Solve for n (from CAS calculator)

$n = 12.94$

Hence the 13th term first exceeds one million

27. (a) Arithmetic – constant difference

(b) $T_n = a + (n - 1)d$

$T_n = 2 + (n - 1)(3)$

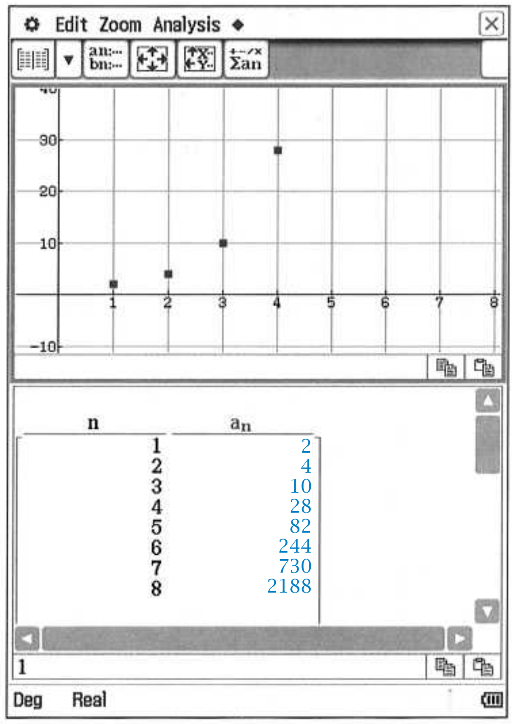
$T_n = 2 + 3n - 3$

$T_n = 3n - 1$

(c) $T_7 = 3(7) - 1$

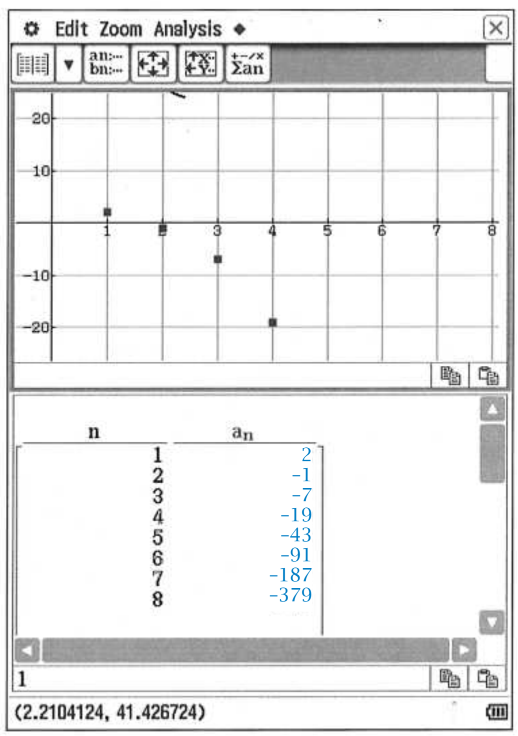
$T_7 = 20$

28. (a) Sequence I



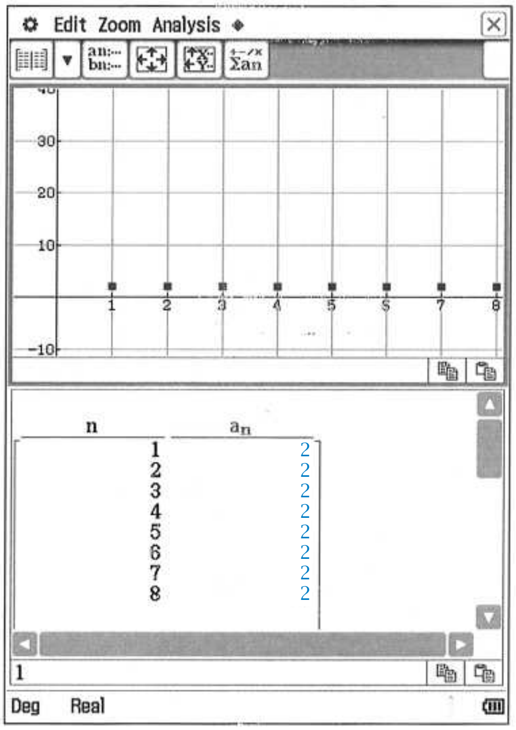
(b) Increasing sequence.

28. (a) Sequence II



(b) Decreasing sequence

28. (a) Sequence III



(b) Steady-state solution

$T_{n+1} = 0.99\ T_n + 15, \quad T_1 = 4000$

(c) In the long run, sequence is a steady-state solution as n becomes large T_n approaches 1500.

29. (a)

$T_0 = 4 \qquad P_0 = 3$

$T_1 = 10 \qquad P_1 = -6$

$T_2 = 28 \qquad P_2 = 3$

$T_3 = 82 \qquad P_3 = -6$

(b) T_{n+1} is an increasing sequence. P_{n+1} is an alternating sequence between 3 and -6.

30. $a = 52\,000 \quad , \quad d = 7500$

(a) $T_5 = 52\,000 + (5 - 1)(7500)$
$= \$82\,000$

(b) $T_{20} = 52\,000 + (20 - 1)(7500)$
$= \$194\,500$

31. $T_1 = 25, \ T_2 = 50, \ T_3 = 100, \ldots$

Geometric sequence

$a = 25 \qquad r = 2$

(a) $T_5 = 25(2)^4$
$= \$400$

(b) From CAS calculator: Total deposited by the end of the 8th month \$6375.

(c) John will take 10 months to save \$25 000.

32. (a) $L_1 = \$261\,250$

(b) $L_7 = \$340\,215.46$

(c) $L_n = 250\,000 \times 1.045^n$

(d) $750\,000 = 250\,000 \times 1.045^n$

$n = 24.96$

Will take approximately 25 years.

33. (a) 16, 48, 80

$T_{n+1} = rT_n + d \quad , \quad T_1 = a$

$T_2 = rT_1 + d$

$48 = r(16) + d$

$16r + d = 48$

$T_3 = rT_2 + d$

$80 = r(48) + d$

$48r + d = 80$

Solve simultaneously

$r = 1 \quad , \quad d = 32$

$T_{n+1} = T_n + 32, \quad T_1 = 16$

(b) $T_{10} = 304$

34. (a) $T_{n+1} = 1.015\ T_n - 1000, \quad T_0 = 100\,000$

(b) $T_{60} = \$148\,107.33$

35. (a) $T_{n+1} = \frac{3}{4} T_n \quad , \quad T_0 = 48$

(b) $T_3 = 20.25$ m

(c) Total after 5th bounce =

48 + 36 + 36 + 27 + 27 + 20.25 + 20.25 + 15.1875 + 15.1875 + 11.390625

= 256.266 m (to 3 dp).

36. (a) $T_{n+1} = 0.95\ T_n + 2500, \quad T_1 = 80\,000$

$r = 0.95$

$d = 2500$

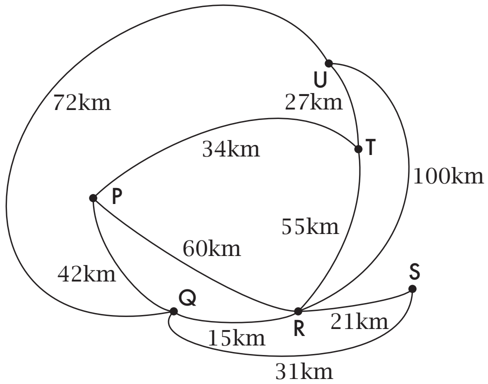
$a = 80\,000$

(b) $T_6 = 73\,213.43$ litres

(c) At the end of the 22nd day/start 23rd day the pool contains less than 60 000 L.

CHAPTER 4: *Undirected Graphs and Networks*

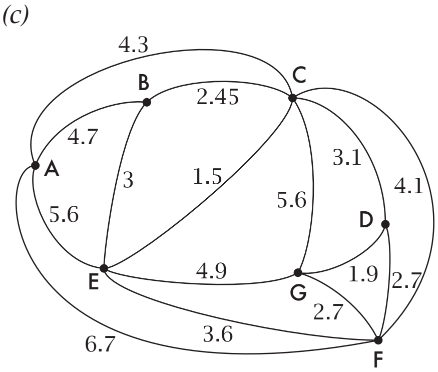
1.



2. (a) The '–' means there is no connection between these two areas.

(b) The values are repeated. As an example both A to B and B to A are the same connection.

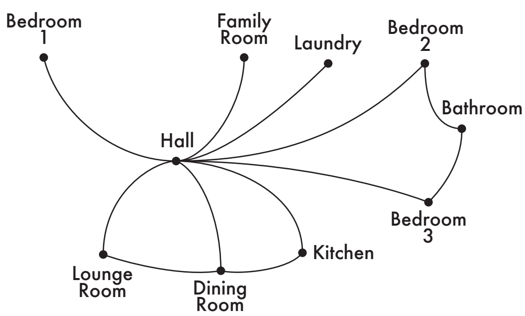
(c)



(d) The 5.6 represents the cost of $5.60 per metre for connecting an irrigation system from area C to area G.

3. A house plan showing the connections from one room to another.

4.



5. (a) No

(b) Yes, 2 odd nodes

(c) Yes, 2 odd nodes

(d) Yes, 2 odd nodes

6. (a) 13 m

A—B—E—F—H—I

(b) 12 m

A—B—E—F—G—I

7. (a) Shortest path:

Shop—D—C—E—F—Post Office

Distance: 18 minutes

(b) Shortest path:

Shop—D—E—F—Post Office

Distance: 19 minutes

8. (a) Checkpoints passed:

Start—6—7—10—12—13—14—15—16—Finish

Distance travelled: 149 km

(b) Checkpoints passed through 8

Start—8—9—11—12—13—14—15—16—Finish

Distance travelled: 151 km

9. Shortest path:

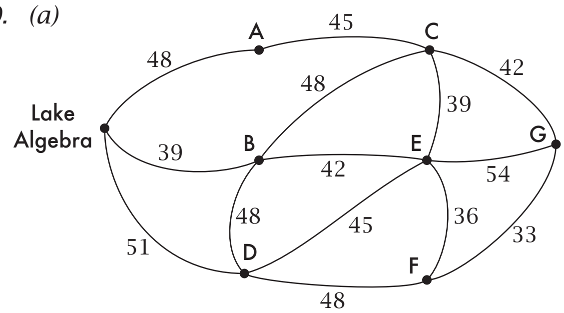
Home—L—M—J—F—G—Shops

or

Home—H—I—E—F—G—Shops

Number of letterboxes: 90

10. (a)



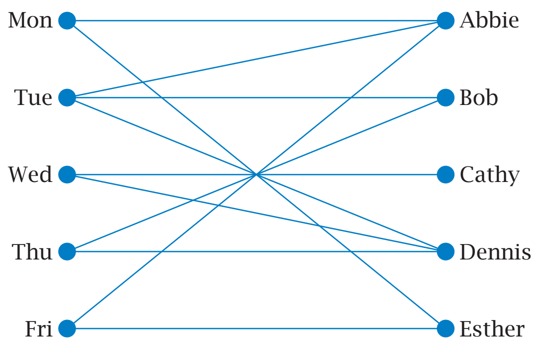
(b) (i) Lake Algebra—B—C—G

Cost: $129 000

(ii) Lake Algebra—D—F—G

Cost: $132 000

11.

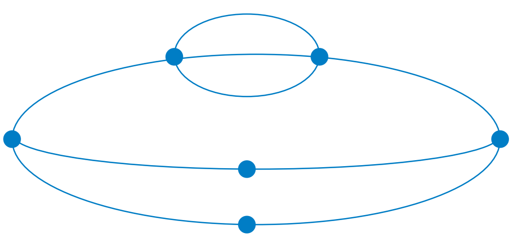


12. *(a)* *(i)* 4 *(b)* *(i)* 5
(ii) 5 *(ii)* 9
(iii) 3 *(iii)* 6
(c) *(i)* 11 *(d)* *(i)* 7
(ii) 14 *(ii)* 12
(iii) 5 *(iii)* 7

13. *(a)* F
(b) D
(c) E
(d) C and D

14. *(a)* Simple *(i)*
(ii)
(iii)
(v)
(b) Complete *(v)*
(c) Not connected *(ii)*

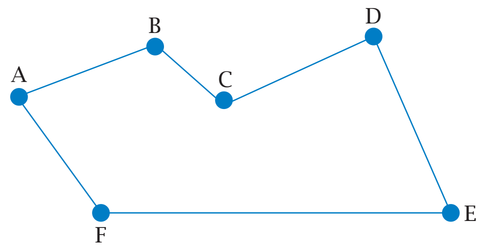
15.



Other possible graphs

16. A simple graph contains no multiple edges or loops.
(a) Simple.
(b) Not simple – contains a loop.
(c) Simple.
(d) Not simple – contains a multiple edge.

17. *(a)* A—B—C—D—E—F
(b) A—2
B—3
C—2
D—2
E—2
F—3
(c)



Example of a subgraph.
Others exist

18. *(a)* *(i)* A—B
A—C—B
A—C—F—B
A—D—E—C—B
A—D—E—C—F—B
A—D—E—F—B
A—C—E—F—B

A—D—E—F—C—B
8 possible paths

(ii) A—D—E—F—C—B—A is an example of a cycle. Others possible.

(iii) A—3
B—3
C—4
D—2
E—3
F—3

(b) *(i)* A—B
A—C—B
A—C—D—B
3 possible paths

(ii) A cycle is not possible.

(iii) A—2
B—3
C—5
D—2
E—2
F—3
G—2
H—1

19. *(a)* Semi-Eulerian
Two odd vertices – starts at one odd vertex – finishes at the other odd vertex.
(b) Semi-Eulerian
Two odd vertices – starts at one odd vertex – finishes at the other odd vertex.
(c) Eulerian
All vertices are even. Start and finish at the same vertex.

20. *(a)* Hamiltonian

– Starts and finishes at the same vertex.

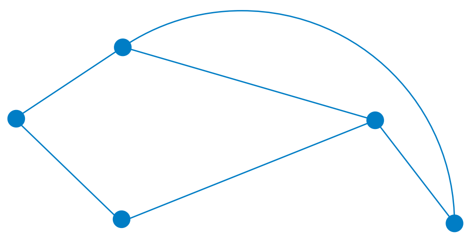
(b) Hamiltonian

– Starts and finishes at the same vertex.

(c) Semi-Hamiltonian

– Starts and finishes at different vertices.

21. Redraw diagram



Euler's formula:

$v + f = e + 2$

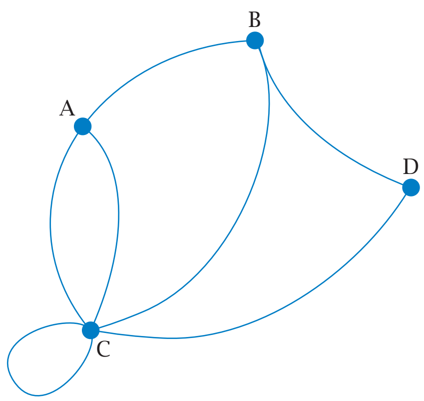
$5 + 3 = 6 + 2$

Therefore the graph is planar.

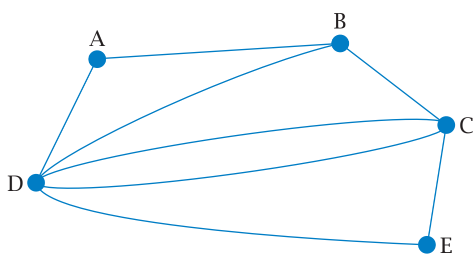
22.

	A	B	C	D	E	F	G
A	0	2	1	0	0	0	0
B	2	0	1	0	0	1	0
C	1	1	0	0	1	0	0
D	0	0	0	0	1	0	2
E	0	0	1	1	0	1	1
F	0	1	0	0	1	1	0
G	0	0	0	2	1	0	0

23. *(a)*



(b)



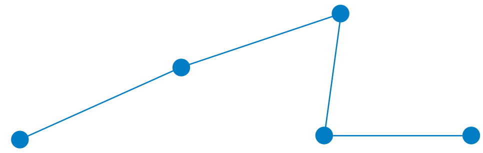
24. *(a)* Tree – connected, no loops, multiple edges or cycles.

(b) Not a tree – contains a cycle.

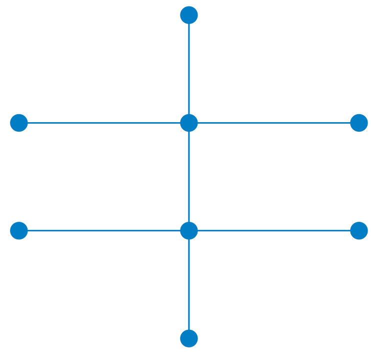
(c) Not a tree – not connected.

(d) Not a tree – not connected.

25. *(a)* *(i)*



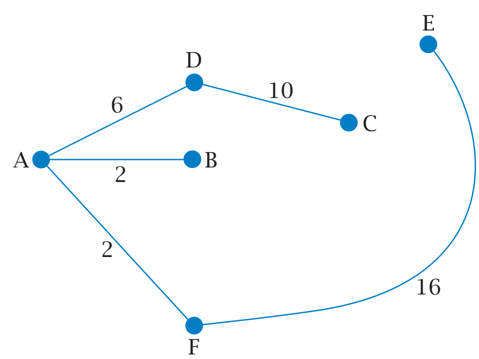
(ii)



(iii)

(b) All graphs are planar. No edges cross. Euler's formula also applies to each tree.

26.



Minimum spanning tree
Distance : 36

27. *(a)* Hamiltonian path.

(b) A—B—C—D—E—F—G—H
Travel time = 113 minutes.

A—H—B—C—G—F—E—D
Travel time = 105 minutes.

Other paths possible.

28. *(a)* Euler's formula

$v + f = e + 2$

$5 + 4 = 7 + 2$ ✔

Yes it works.

(b) A—2
B—3
C—3
D—2
E—4

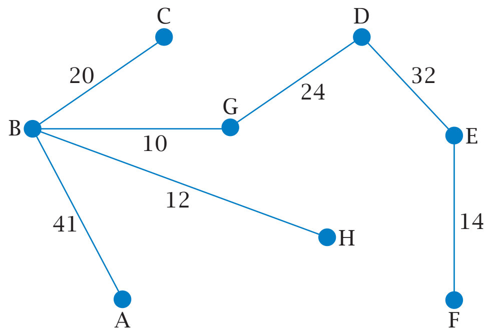
(c) Semi-Eulerian trail is an open trail with the following conditions:

- only two vertices are odd, the others are even.
- the trail must start at one of the odd vertices and finish at the other.

(d) Yes it is a Semi-Eulerian trail.

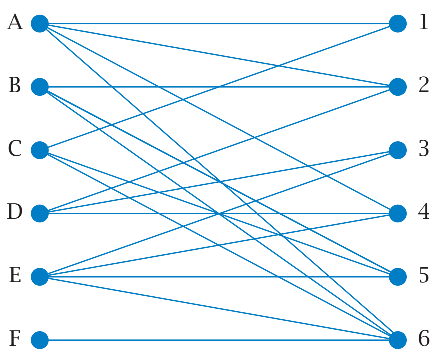
Trail : B—A—E—B—C—E—D—C

29. *(a)*



(b) 153 m.

30. *(a)*



(b) This is *not* a complete bipartite graph. Not every vertex is connected to every other vertex.

31. *(a)* Minimum number of edges = 5
Maximum number of edges = 15

(b) Minimum = $n - 1$

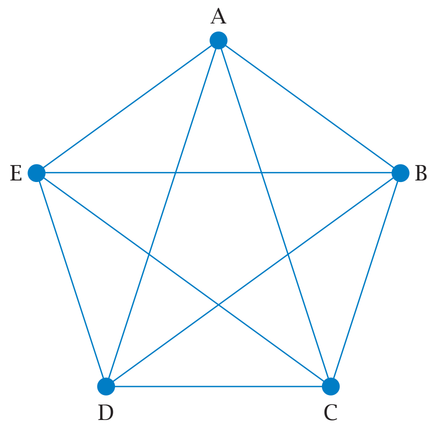
Maximum = $\dfrac{n(n-1)}{2}$

(c) *(i)* $d = 0, 1, 2, 3$

(ii) $d = 2, 3$

(iii) $d = 2$

32. *(a)*



(b)

	A	B	C	D
A	-	1	1	1
B	1	-	1	1
C	1	1	-	1
D	1	1	1	-

(c) Total number of edges = 15.

(d) K_6 is not Eulerian as each vertex is of odd degree.

(e) K_n is Eulerian when n = odd.

33. *(a)* D—2
E—4
F—4
G—4
H—2
I—3
J—3
K—2

(b) Town J

(c) 5 Towns

(d) Semi-Eulerian

(e) D—F—H—J—K—I—G—E

(f) Hamiltonian

34. *(a)*

	A	B	C	D	E	F
A	—	45	—	15	—	40
B	45	—	48	—	—	—
C	—	48	—	—	39	36
D	15	—	—	—	45	40
E	—	—	39	45	—	38
F	40	—	36	40	38	—

(b) $174 000.

35. *(a)*

	A	B	C	D	E	F	G	H
A	—	25	—	20	12	14	—	16
B	25	—	20	—	17	—	21	—
C	—	20	—	14	—	—	—	17
D	20	—	14	—	—	17	—	—
E	12	17	—	—	—	—	18	23
F	14	—	—	17	—	—	—	—
G	—	21	—	—	18	—	—	21
H	16	—	17	—	23	—	21	—

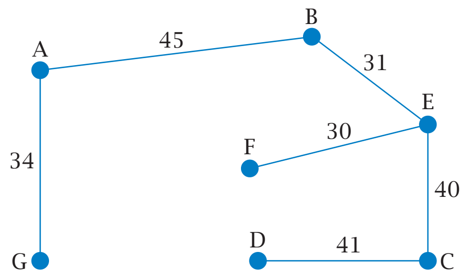
(b) 108 metres.

36. *(a)*

	A	B	C	D	E	F	G
A	—	45	52	—	—	51	34
B	45	—	42	63	31	32	56
C	52	42	—	41	40	—	—
D	—	63	41	—	43	67	—
E	—	31	40	43	—	30	63
F	51	32	—	67	30	—	46
G	34	56	—	—	63	46	—

(b) $221 000.

(c)



(d) AB no longer exists. Replaced with FG of cost $23 000.

Cost reduces to $199 000.

CHAPTER 5: *Times Series Data*

1. *(a)* Overall trend – Decreasing

(b) Overall trend – Increasing

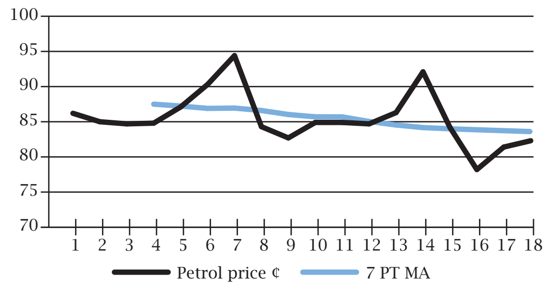
2. *(a)* 3 point moving average

(b) 5 point moving average

3. *(a)*

Price (cents)	7 PT MA
86.2	
85	
84.7	
84.8	87.50
87.1	87.23
90.3	86.90
94.4	86.93
84.3	86.94
82.7	86.60
84.9	86.03
84.9	85.70
84.7	85.69
86.3	85.04
92.1	84.54
84.2	84.17
78.2	84.00
81.4	83.86
82.3	83.73
83.5	
85.3	
91.2	

(b)

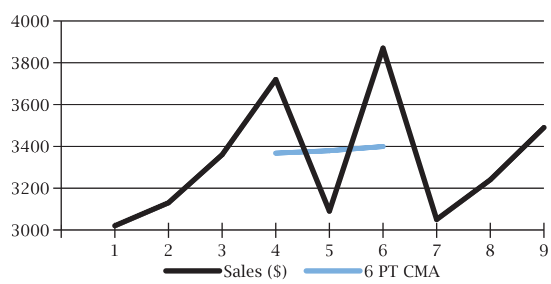


(c) Decreasing trend.

4. *(a)*

Sales $	6 PT CMA
3020	
3130	
3360	
3720	3367.500
3090	3379.167
3870	3399.167
3050	
3240	
3490	

(b)



5. Deseasonalised value $= \dfrac{\text{actual value}}{\text{seasonal index}}$

$= \dfrac{1250}{0.872}$

$= 1433.49$

6. For 4 Quarters seasonal indices must total $4 \times 100\% = 400\%$

$Q4 = 88\%$

7. Actual value = Deseasonalised value × Seasonal index

$= 1374 \times 0.85$

$= 1167.9$

Approximately 1168 computers sold.

8. (a)

	Q1	Q2	Q3	Q4	Average
2012	140	36	61	170	101.75
2013	155	45	68	185	**113.25**
2014	172	51	75	201	**124.75**
Seasonal Index	**1.3744**	**0.3867**	**0.6004**	**1.6385**	**1**

(b) The seasonal index for Q1 and Q3 are 137.44% and 60.04% respectively. This indicates that the sales of ice cream for these two quarters are above average for Q1 and below average for Q3.

9. (a) (b)

Year	Summer	Autumn	Winter	Spring	Average
2012	1250	576	1110	620	**889**
2013	1370	624	1234	710	**984.5**
2014	1426	720	1326	830	**1075.5**
Seasonal Index	**1.3745**	**0.6504**	**1.2450**	**0.7301**	**1**

(c) The seasonal index of 137.45% for summer indicates that the sales of airconditioners are 37.45% above average.

10.

Seasonal Indices					
Year	Q1	Q2	Q3	Q4	Average
2003	47	56	63	38	51
2004	49	51	67	39	51.5
2005	53	60	76	46	58.75
Seasonal Indices	0.9251	1.0366	1.2766	0.7618	1

Year	Quarter	Sales	Deseasonalised sales
2003	1	47	51
	2	56	54
	3	63	49
	4	38	50
2004	1	49	53
	2	51	49
	3	67	52
	4	39	51
2005	1	53	57
	2	60	58
	3	76	60
	4	46	60

11. (a)

Week	Time (t)	Sales ('000)	Moving Average (M)
1	1	241	
	2	195	
	3	133	186.8
	4	130	181.6
	5	235	180.0
2	6	215	173.4
	7	187	166.0
	8	100	161.0
	9	93	157.0
	10	210	155.6
3	11	195	153.6
	12	180	153.0
	13	90	149.2
	14	90	146.4
	15	191	144.0
4	16	181	140.8
	17	168	136.8
	18	74	133.6
	19	70	
	20	175	

(b) $M = -3.3624t + 192.7297$

(c) Seasonal Indices

	Monday	Tuesday	Wednesday	Thursday	Friday	Average
Week 1	241	195	133	130	235	186.8
Week 2	215	187	100	93	210	161
Week 3	195	180	90	90	191	149.2
Week 4	181	168	74	70	175	133.6
Seasonal Index	1.3218	1.1673	0.6226	0.6002	1.2881	1

(d)

Week	Time	Predicted Sales	Sales × Seasonal Index	Sales ('000)	Sales
5	21	122.1193	122.1193 × 1.3218	161.42	161421
	22	118.7569	118.7569 × 1.1673	138.63	138628
	23	115.3945	115.3945 × 0.6226	71.839	71839
	24	112.0321	112.0321 × 0.6002	67.24	67240
	25	108.6697	108.6697 × 1.2881	139.98	139978

12. *(a)* 4 point centred moving average.

(b)

$$A = \frac{(\frac{1}{2} \times 12.89 + 10.65 + 13.04 + 17.76 + \frac{1}{2} \times 20.17)}{4}$$

$= 14.50$

$$B = \frac{(\frac{1}{2} \times 19.42 + 21.23 + 28.71 + 27.56 + \frac{1}{2} \times 25.83)}{4}$$

$= 25.03$

$$10.59 = \frac{(\frac{1}{2} \times C + 9.65 + 12.89 + 10.65 + \frac{1}{2} \times 13.04)}{4}$$

$C = 5.30$

(c)

Year	March	June	September	December	Average
2012	5.3	9.65	12.89	10.65	9.6225
2013	13.04	17.76	20.17	19.42	17.5975
2014	21.23	28.71	27.56	25.83	25.8325
Seasonal Indices	0.7045	1.0412	1.1842	1.0701	1

(d) $M = 2.1t + 4.1139$

(e) **March 2015 : $t = 13$**

$M = 2.1(13) + 4.1139$

$= 31.4139$

Profit March 2015 $= 31.4139 \times 0.7045$

$= 22.1311$

Profit $=$ \$22 131.09

September 2015 : $t = 15$

$M = 2.1(15) + 4.1139$

$= 35.6139$

Profit September 2015 $=$ 35.6139×1.1842

$= 42.1740$

Profit $=$ \$42 173.98

(f) Seasonally Adjusted Profits

$$\text{June 2013} = \frac{17.76}{1.0412}$$

$= 17.057$

Profit $=$ \$17 057.24

$$\text{December 2013} = \frac{19.42}{1.0701}$$

$= 18.148$

Profit $=$ \$18 147.84

13. *(a)* Data is collected quarterly (4 times per year)

(b)

$$A = \frac{(\frac{1}{2}(650) + 720 + 820 + 110 + \frac{1}{2}(600))}{4}$$

$= 568.75$

$$426.25 = \frac{(\frac{1}{2}(820) + 110 + B + 150 + \frac{1}{2}(870))}{4}$$

$B = 600$

$$449.38 = \frac{(\frac{1}{2}(150) + 870 + 140 + 660 + \frac{1}{2}(C))}{4}$$

$C = 105$

(c) The gradient of the line of regression indicates the trend. The −24.65 indicates the number of visitors is decreasing at a rate of 24.65 per quarter.

(d) Average Annual Change

$= -24.65 \times 4$

$= -98.6$

A decline of approximately 99 visitors per year.

(e)

Year	March	June	September	December	Average
2010	870	130	650	720	592.5
2011	820	110	600	150	420
2012	870	140	660	105	443.75
Seasonal Indices	1.7938	0.2656	1.3376	0.6030	1

(f) March 2015 : $t = 21$

M = −24.65 (21) + 658.12

M = 140.47

Number of visitors = 140.47 × 1.7938

= 251.98

Approximately 252 visitors in March 2015.

(g) This is an extrapolation.
Hence prediction is unreliable.

(h) Seasonally adjusted value

$$= \frac{600}{1.3376}$$

= 448.565

Approximately 449 visitors.

14. *(a)*

Time (t)	Day	Sales	D
1	M	1.8	5.407
2	T	1.2	9.8120
3	W	5.3	7.9413
4	Th	12.6	10.4279
5	F	18.3	10.7024
6	S	20.4	10.4124
7	M	3.6	10.8141
8	T	1.4	11.4473
9	W	8.9	13.3353
10	Th	14.2	11.7520
11	F	20.1	11.7551
12	S	23.6	12.0457
13	M	7.2	21.6281
14	T	1.8	14.7179
15	W	10.1	15.1334
16	Th	16.3	13.4900
17	F	22.4	13.1002
18	S	25.8	13.1686

(b) D = $0.4389t + 7.8909$

(c) Week 4

Mon t = 19

D = 16.2303

Sales = 16.2303 × 0.3329 × 100

= $540.31

Tues t = 20

D = 16.6692

Sales = 16.6692 × 0.1223 × 100

= $203.86

Wed t = 21

D = 17.1081

Sales = 17.1081 × 0.6674 × 100

= $1141.79

Thurs t = 22

D = 17.5471

Sales = 17.5471 × 1.2083 × 100

= $2120.22

Fri t = 23

D = 17.9860

Sales = 17.9860 × 1.7099 × 100

= $3075.43

Sat t = 24

D = 18.4249

Sales = 18.4249 × 1.9592 × 100

= $3609.81

15. *(a)* P = 700 − 597.71

= 102.29%

$$Q = \frac{71.3}{1.0229}$$

= 69.7

$$R = \frac{71.1}{1.0229}$$

= 69.5

$$S = \frac{71.2}{1.0229}$$

= 69.6

(b) D = $0.0921t + 68.6110$

(c) Sunday Week 4 : $t = 28$

D = 0.0921 (28) + 68.6110

= 71.1898

Petrol price = 71.1898 × 1.0250

= 72.97 cents

CHAPTER 6: *Finance – Simple and Compound Interest*

1. *(a)* $350 × 18 = $6300

(b) Interest = $6300 − $5250

= $1050

(c) $S = PRT$

$$\$1050 = \$5250 \times R \times \frac{18}{12}$$

$$R = 13.\dot{3}\%$$

2. (a) $\text{Deposit} = 20\% \times \$2520$

$$= \$504$$

(b) $\text{Total} = \$72.45 \times 30 + \$504$

$$= \$2677.50$$

(c) $\text{Amount borrowed} = \$2016$

$$S = PRT$$

$$\$157.50 = \$2016 \times R \times 2.5$$

$$R = 3.125\%$$

3. (a) $S = PRT$

$$S = \$7000 \times 0.09 \times 6$$

$$= \$3780$$

$$\text{Total to be repaid} = \$3780 + \$7000$$

$$= \$10\,780$$

(b) $\text{Monthly repayment} = \dfrac{\$10\,780}{72}$

$$= \$149.72$$

4. $\text{Interest} = \$262.50$

$$S = PRT$$

$$\$262.50 = \$7000 \times R \times \frac{10}{12}$$

$$\frac{\$262.50}{\left(7000 \times \frac{10}{12}\right)} = R$$

$$R = 4.5\%$$

5. (a) $S = \$15\,000 \times 0.08 \times 5$

$$= \$6000$$

(b) $A = P\left(1 + \dfrac{r}{n}\right)^{nt}$

$$= \$15\,000\left(1 + \frac{0.08}{12}\right)^{(12\times5)}$$

$$= \$22\,347.69$$

$$\text{Interest} = \$22\,347.69 - \$15\,000$$

$$= \$7347.69$$

(c) $\text{Difference} = \$7347.69 - \$6000$

$$= \$1347.69$$

6. (a) $\text{Interest} = \$12\,870 - \$10\,400$

$$= \$2470$$

$$S = PRT$$

$$\$2470 = \$10\,400 \times 0.05 \times T$$

$$\frac{2470}{(10\,400 \times 0.05)} = T$$

$$T = 4.75 \text{ years}$$

(b) $A = P\left(1 + \dfrac{r}{n}\right)^{nt}$

$$CI = P\left(1 + \frac{r}{n}\right)^{nt} - P$$

$$\$1424.64 = P\left(1 + \frac{0.12}{1}\right)^{2} - P$$

$$P = \$5600$$

$$S = PRT$$

$$= 5600 \times 0.12 \times 2$$

$$= \$1344$$

Simple Interest = \$1344.

7. $CI = P\left(1 + \dfrac{r}{n}\right)^{nt} - P$

$$\$2945.41 = \$12\,000\left(1 + \frac{r}{12}\right)^{(12\times4)} - \$12\,000$$

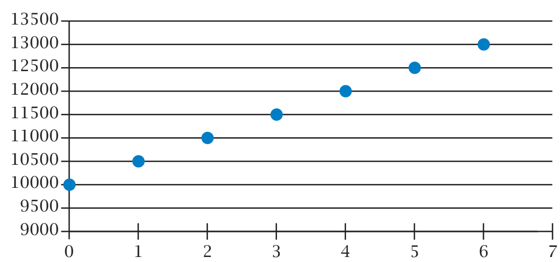
$$r = 0.055$$

Interest rate = 5.5% p.a.

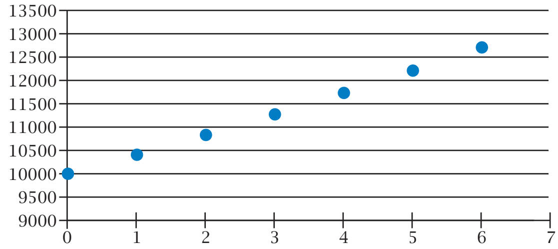
8.

	Year						
	0	1	2	3	4	5	6
Option 1	10 000	10 500	11 000	11 500	12 000	12 500	13 000
Option 2	10 000	10 407.42	10 831.43	11 272.72	11 731.99	12 209.97	12 707.42
Option 3	10 000	10 457.65	10 936.25	11 436.74	11 960.15	12 507.51	13 079.91

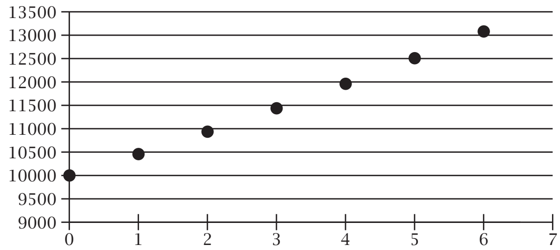
(a) Option 1



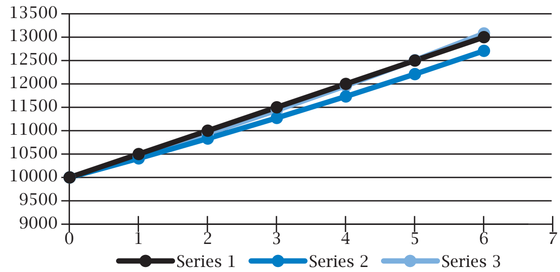
Option 2



Option 3



Option 1–3



(b) Option 1 : An increasing straight line graph

Option 2 : An increasing exponential type graph

Option 3 : An increasing exponential type graph

(c) Fiona should pick Option 3. This option earns the largest amount of interest.

9.

$$\text{CI} = \text{P}\left(1 + \frac{r}{n}\right)^{nt} - \text{P}$$

$$\text{P} = \text{P}\left(1 + \frac{0.064}{12}\right)^{0.064t} - \text{P}$$

Solve for t

$$t \approx 10.8592 \text{ years}$$

10.

$$\text{A} = \text{P}\left(1 + \frac{r}{n}\right)^{nt}$$

$$37\,000 = 10\,000\left(1 + \frac{0.095}{12}\right)^{12t}$$

$$t \approx 13.8264 \text{ years}$$

11.

After 1 year

Plan 1:

$$\text{A} = 35\,000\left(1 + \frac{0.0595}{4}\right)^{4}$$

$$\text{A} = \$37\,129.43$$

Plan 2:

$$\text{A} = 35\,000\left(1 + \frac{0.059}{12}\right)^{12}$$

$$\text{A} = \$37\,121.77$$

After 2 years

Plan 1:

$$\text{A} = 35\,000\left(1 + \frac{0.0595}{4}\right)^{8}$$

$$\text{A} = \$39\,388.41$$

Plan 2:

$$\text{A} = 35\,000\left(1 + \frac{0.059}{12}\right)^{24}$$

$$\text{A} = \$39\,372.16$$

After 3 years

Plan 1:

$$\text{A} = 35\,000\left(1 + \frac{0.0595}{4}\right)^{12}$$

$$\text{A} = \$41\,784.84$$

Plan 2:

$$\text{A} = 35\,000\left(1 + \frac{0.059}{12}\right)^{36}$$

$$\text{A} = \$41\,758.97$$

The best investment is Plan 1. Greater return compared to Plan 2.

12. *(a)*

$$\text{S} = \text{PRT}$$
$$= \$15\,000 \times 0.08 \times 3$$
$$= \$3600$$

Joanna accumulates a total of $18 600.

(b)

$$\text{S} = \text{PRT}$$

Joseph accumulates $14 841.60

$$(\text{P} \times 0.08 \times 4) + (\text{P} \times 0.10 \times 2) = (14\,841.6 - \text{P})$$

$$\text{P} = \$9764.21$$

13\. *(a)*

Option A   Option B

$(P \times 0.0725 \times 4.5) = (P \times 0.0755 \times 4.25) + \$36.55$

$P = \$6800$

Original investment \$6800

(b) $S = PRT$

$11798 - 6800 = 6800 \times R \times 5$

$R = 0.147$

Interest rate : 14.7%

14\. *(a)* $A = P(1 + r)^t$

$7660.80 = P(1 + r)^1$

$8227.70 = P(1 + r)^2$

$r = 0.074$

Interest rate = 7.4% p.a.

(b) $P = \$7132.96$

15\. *(a)* $\text{Effective Interest} = \left(1 + \frac{r}{n}\right)^n - 1$

$0.082 = \left(1 + \frac{r}{n}\right)^4 - 1$

$1.082 = \left(1 + \frac{r}{n}\right)^4$

$\sqrt[4]{1.082} = 1 + \frac{r}{n}$

$1.0199 = 1 + \frac{r}{n}$

$0.0199 = \frac{r}{n}$

∴ Quarterly interest rate is 1.99%

(b) $A = P\left(1 + \frac{r}{n}\right)^{nt}$

$A = 5000(1 + 0.0199)^{(4\times3.5)}$

$A = \$6588.34$

Interest = \$6588.34 – \$5000

= \$1588.34

16\. *(a)* *Plan I*

$S = PRT$

$= 500 \times 0.15 \times 8$

$= \$600$

Plan II

$A = 500\left(1 + \frac{0.1375}{1}\right)^8$

$= \$1401.46$

Interest = \$1401.46 – \$500

= \$901.46

Plan III

$A = 500\left(1 + \frac{0.135}{4}\right)^{(4\times8)}$

$= \$1446.33$

Interest = \$1446.33 – \$500

= \$946.33

Plan III is the best. Maximum interest earned over the 8 years by compounding quarterly.

(b) $\text{Effective Interest Rate} = \left(1 + \frac{r}{n}\right)^n - 1$

$= \left(1 + \frac{0.135}{4}\right)^4 - 1$

$= 0.14199$

Effective interest rate = 14.199%

(c) $T_{n+1} = T_n \times 1.1375$, $T_0 = \$500$

(d) $T_4 = \$837.10$

17\. Effective interest rate

(i) $\left(1 + \frac{r}{n}\right)^n - 1$

$= \left(1 + \frac{0.0675}{365}\right)^{365} - 1$

$= 0.0698$

6.98% p.a.

(ii) $\left(1 + \frac{r}{n}\right)^n - 1$

$= \left(1 + \frac{0.068}{12}\right)^{12} - 1$

$= 0.07016$

7.016% p.a.

(iii) $\left(1 + \frac{r}{n}\right)^n - 1$

$= \left(1 + \frac{0.069}{2}\right)^2 - 1$

$= 0.07019$

7.019% p.a.

Leonie should choose the 6.9% p.a. compounded six monthly as the effective interest rate is higher.

18. Effective interest rate

Investment 1

$\left(1 + \frac{r}{n}\right)^n - 1$

$= \left(1 + \frac{0.0712}{4}\right)^4 - 1$

$= 0.0731$

7.31% p.a.

Investment 2

$\left(1 + \frac{r}{n}\right)^n - 1$

$= \left(1 + \frac{0.07}{365}\right)^{365} - 1$

$= 0.0725$

7.25% p.a.

Best investment is Investment 1. The effective interest rate is higher providing a greater return on the investment.

19.

Year	Value of Car
1	\$66 240
2	\$60 940.80
3	\$56 065.54
4	\$51 580.29

20.

Year	Depreciation Value
1	\$30 000
2	\$26 400
3	\$23 232

CHAPTER 7: *Finance – Loans and Annuities*

1. (a)
N = 300
I% = 7
PV = 90 000
PMT = −636.1013
FV = 0
P/Y = 12
C/Y = 12

Monthly repayments of \$636.10.

(b)
N = 48
I% = 12
PV = 75 000
PMT = −900.13
FV = −65 808.89
P/Y = 12
C/Y = 12

Amount still owing \$65 808.89

(c)
N = 23.45
I% = 12
PV = 20 000
PMT = −1200
FV = 0
P/Y = 4
C/Y = 4

Loan will be paid off at the end of 23.45 quarters i.e. 5.8625 years.

(d)
N = 3
I% = 12
PV = 8000
PMT = −2000
FV = −4490.624
P/Y = 1
C/Y = 1

Veronica still owes \$4490.62 at the end of the 3rd year.

2.

$$\begin{aligned} N &= 72 \\ I\% &= 7 \\ PV &= 25\,000 \\ PMT &= -426.23 \\ FV &= 0 \\ P/Y &= 12 \\ C/Y &= 12 \end{aligned}$$

Regular monthly repayment = $426.23.

3.

$$\begin{aligned} N &= 240 \\ I\% &= 5 \\ PV &= 300\,000 \\ PMT &= -1979.87 \\ FV &= 0 \\ P/Y &= 12 \\ C/Y &= 12 \end{aligned}$$

Regular monthly repayment = $1979.87.

4. Regular annual payment $= 800\,000 \times \frac{5}{100}$

$= \$40\,000$

5.

$$\begin{aligned} N &= 20 \\ I\% &= 7.5 \\ PV &= -367\,001.69 \\ PMT &= 36\,000 \\ FV &= 0 \\ P/Y &= 1 \\ C/Y &= 1 \end{aligned}$$

$367 001.69 should be invested to provide an annual payment of $36 000.

6.

$$\text{Perpetuity} = \frac{\text{payment}}{\text{rate}}$$

$$\text{rate} = \frac{\text{payment}}{\text{perpetuity}}$$

$$\text{rate} = \frac{15\,000}{200\,000}$$

$$\text{rate} = 0.075$$

7.5% per quarter.
Annual interest rate is 30% p.a.

7.

$$\begin{aligned} N &= 226.36 \\ I\% &= 8 \\ PV &= -350\,000 \\ PMT &= 3000 \\ FV &= 0 \\ P/Y &= 12 \\ C/Y &= 12 \end{aligned}$$

Annuity will last for 226 months.

or

$$T_{n+1} = \frac{151}{150} T_n - 3000$$

$$T_0 = 350\,000$$

$$T_{226} = 1083.50$$

At the end of 226 months $1083.50 remain in the annuity.

8.

$$\text{Perpetuity} = \frac{\text{payment}}{\text{rate}}$$

$$\text{Perpetuity} = \frac{45\,000}{0.067}$$

$$= 671\,641.79$$

$671 641.79 should be invested into the perpetuity.

9. (a)

$$T_{n+1} = 1.095 \times T_n - 55\,000$$

$$T_0 = 400\,000$$

The annuity will last 12 years. The final payment is $51 698.21.

(b)

$$\begin{aligned} N &= 12.937 \\ I\% &= 9.5 \\ PV &= -400\,000 \\ PMT &= 55\,000 \\ FV &= 0 \\ P/Y &= 1 \\ C/Y &= 1 \end{aligned} \qquad \begin{aligned} N &= 12 \\ I\% &= 9.5 \\ PV &= -400\,000 \\ PMT &= 55\,000 \\ FV &= 47\,212.98 \\ P/Y &= 1 \\ C/Y &= 1 \end{aligned}$$

$$\begin{aligned} N &= 1 \\ I\% &= 9.5 \\ PV &= -47\,212.98 \\ PMT &= 51\,698.21 \quad \leftarrow \text{Final payment} \\ FV &= 0 \\ P/Y &= 1 \\ C/Y &= 1 \end{aligned}$$

12 payments made.

10. (a) Monthly interest rate = 1.4375%

(b) Final repayment = $1784.91

(c) Total interest

= 2500 × 9 + 1784.91 − 22 500

= $1784.91

(d) $T_n = 1.014375 \times T_{n-1} - 2500$

$T_1 = 22\,500$

(e) (i) Month 7

(ii) Annual interest rate is 18% p.a. an increase of 0.75%.

11. (a) $T_{n+1} = (T_n - 600) \times 1.03$

$T_1 = 4120$

Amount owing at end of second quarter is $3625.60.

(b) 8 repayments.

(c) Final repayment $331.68.

12. (a) (i) $1284
(ii) $2657.88

(b) $T_n = (T_{n-1} + 1200) \times 1.07$

$T_0 = 0$

(c) $T_3 = \$4127.93$

$T_4 = \$5700.89$

(d) Total = $218 060.87

13. (a) $T_{n+1} = 1.015 \times T_n - 2000$

$T_0 = 150\,000$

(b) Owing = $150 250

(c) Amount owing is increasing rather than decreasing.

(d) $T_{n+1} = 1.015T_n - 2000$

$T_0 = 100\,000$

(e) Amount owing at the end of the first month is $99 500. Repayment of $2000 is adequate as amount owing is reducing.

14. (a)

Year	Amount owing	Amount owing after interest	Amount owing after repayment
1	12 000	14 160	10 660
2	10 660	**12 578.80**	**9078.80**
3	**9078.80**	**10 712.98**	**7212.98**
4	**7212.98**	**8511.32**	**5011.32**
5	**5011.32**	**5913.36**	**2413.36**

(b) N = 5

I% = 18

PV = 12 000

PMT = −3837.33

FV = 0

P/Y = 1

C/Y = 1

Monthly repayments of $3837.33.

(c) S = PRT

S = 12 000 × 0.175 × 5

= 10 500

Total amount owing = $12 000 + $10 500

= $22 500

5 yearly repayments of
$3500 = $17 500

Amount still owing at the end of the
5th year = $5000

Terry owes less when interest is calculated at a reducible rate rather than as simple interest.

15. (a) Annual interest rate $= \dfrac{1.65}{220} \times 100 \times 12$

= 9% p.a.

(b) Month 6

$1125.00 $220 $1345 $10.09 $1355.09

(c) $T_{n+1} = (T_n + 220) \times 1.0075$

$T_0 = 0$

(d) $T_{18} = \$4254.51$

(e) Total deposits = $220 × 18

= $3960

Total interest = $4254.51 − $3960

= $294.51

16. *(a)* P = \$8.56

Q = \$775.45

(b) $T_{n+1} = \frac{76}{75} \times T_n + 125$

$T_0 = 125$

(c) $T_{12} = \$1761.57$

(d) Total deposits = 13 × \$125

= \$1625

Total interest = \$1761.57 – \$1625

= \$136.57

17. *(a)* Annual interest rate

$= \frac{350}{56\,000} \times 100 \times 12$

= 7.5% p.a.

(b) A = \$47 319.04

B = \$295.74

C = \$2500

D = \$45 114.78

$T_{n+1} = 1.00625 \times T_n - 2500$

$T_0 = 56\,000$

(c) *i.* Final repayment
\$515.55 × 1.00625
= \$518.77

ii. Total interest
= (518.77 + 24 × 2500) – 56 000
= \$4518.77

18. *(a)* Month

16	\$1756.04	\$13.17	\$950	\$819.21
17	\$819.21	\$6.14	\$825.35	\$0

(b) $T_{n+1} = 1.0075 \times T_n - 950$

$T_0 = 15\,000$

(c) Final repayment = \$825.35

(d) Total interest

= (950 × 16 + \$825.35) – \$15 000

= \$1025.35

(e) If repayments were \$1000 loan would be repaid at the end of 16 months.

19. *(a)* A = \$32 217.91

B = \$167.80

(b) Month 16 should be \$6503.20.

(c) At the end of the 12th month.

(d) Final repayment

$= \$2059.4 \times \left(1 + \frac{6.25}{1200}\right)$

= \$2070.13

(e) Total interest

= (2070.13 + 17 × 2250) – 38 400

= \$1920.13

20. *(a)* N = 144 i.e. 12 yrs

I% = 7

PV = –130 000

PMT = 918.82

FV = 93 939.45*

P/Y = 12

C/Y = 12

\$93 939.45 still owing after 12 years.

N = 133.51 * i.e. 11.13 yrs

I% = 5

PV = –93 939.45

PMT = 918.82

FV = 0

P/Y = 12

C/Y = 12

∴ 23.13 years to pay off the loan.

(b) Number of monthly repayments
144 + 133 = 277

277 × \$918.82 = \$254 513.14

Final repayment:

N = 133

I% = 5

PV = –93 939.45

PMT = 918.82

FV = 463.69*

P/Y = 12

C/Y = 12

Final repayment $= \$463.69 \times \left(1 + \frac{0.05}{12}\right)$

= \$465.62

Total repaid: \$254 513.14 + \$465.62
= \$254 978.76

(c) 25 years @ 7% p.a. compounded monthly

N = 299.99 * monthly repayments

I% = 7

PV = −130 000

PMT = 918.82

FV = 0

P/Y = 12

C/Y = 12

Total repaid: $299 \times \$918.82$
$= \$274\,727.18$

Final repayment

N = 299

I% = 7

PV = −130 000

PMT = 918.82

FV = 907.82*

P/Y = 12

C/Y = 12

$$\text{Final repayment} = \$907.82 \times \left(1 + \frac{0.07}{12}\right)$$
$$= \$913.12$$

Total repayments = \$274 727.18 + \$913.12
= \$275 640.30

Amount saved = \$275 640.30 − \$254 978.76
= \$20 661.54

21. *(a)* Second year = \$41 200
Third year = \$42 436

$$T_{n+1} = (1.06T_n - 40\,000) - (40\,000 \times 1.03^n)$$

$T_2 = \$478\,200$

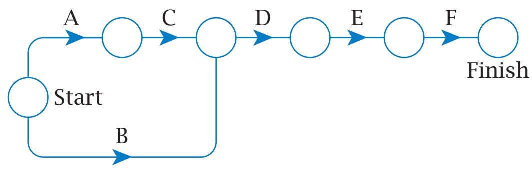
$T_3 = \$464\,456$

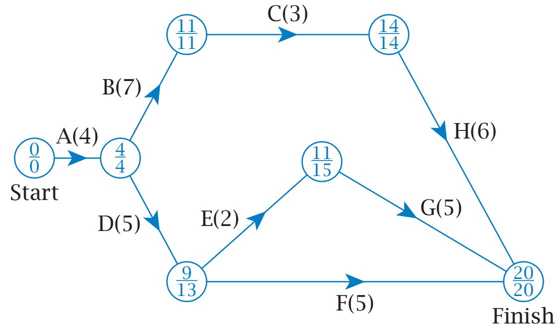
(b) $T_{10} = \$299\,515.43$

(c) Final withdrawal is the 17th year.

CHAPTER 8: *Directed Networks and Decision Mathematics*

1.





2. *(a)* 20 days

(b) A—B—C—H

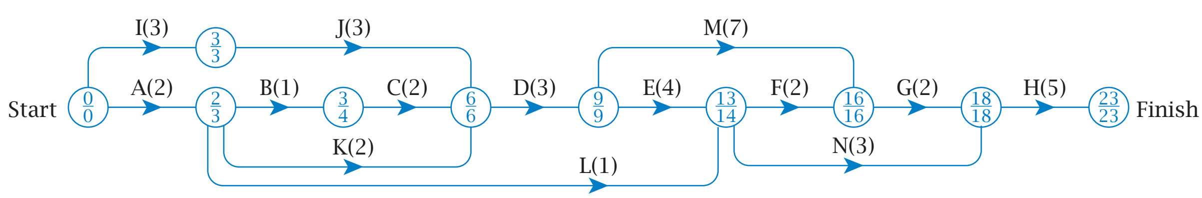
(c) After 11 days

(d) After 15 days

(e) *(i)* No slack time. C is on the critical path.

(ii) Activity G has a 4 day slack time.

3.



(a) Critical path

I—J—D—M—G—H

(b) 23 weeks.

(c) Earliest start time − after 2 weeks.

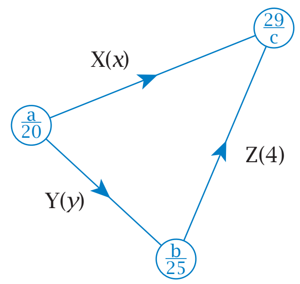
(d) Latest start time − after 13 weeks.

(e) 11 days.

(f) Reduce the minimum completion time by 1 week to 22 weeks.

(g) 5 weeks.

4.



(a) $a = 20$

$c = 29$

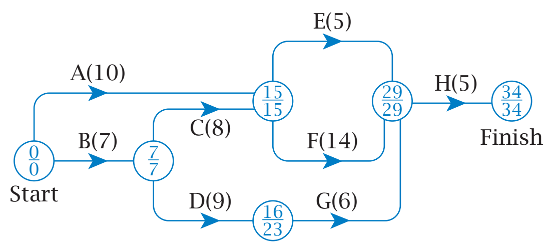
$y = 2$

$x = 9$

(b) Minimum value of $b = 22$

(c) Slack time $= 3$

5.



(a) *(i)* B—C—F—H

(ii) Minimum completion time = 34 days

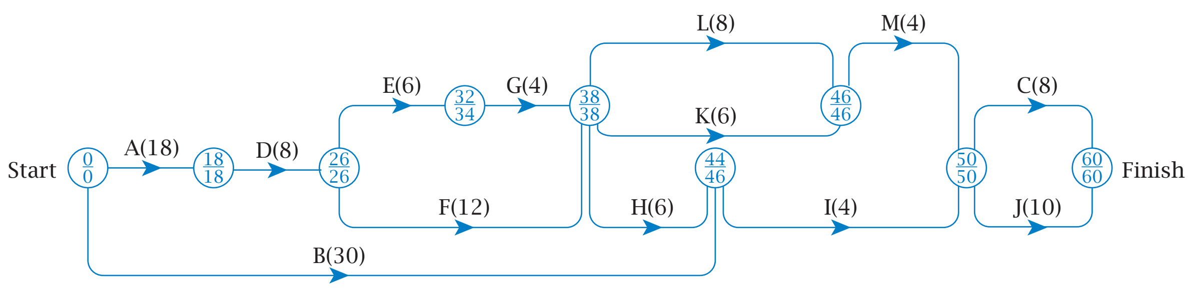
(b) 34 days

Person 1: B—C—F—H (34 days)

Person 2: A—D—G—E (30 days)

Delays the start time of activity H by 1 day.
Minimum completion time : 35 days.

6. *(a)*



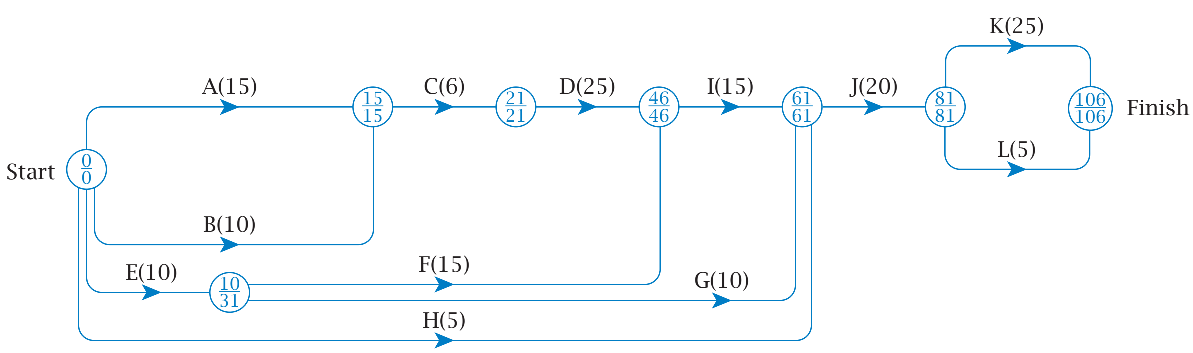
(b) A—D—F—L—M—J

(c) 60 weeks.

(d) Task B can be delayed by 16 weeks.

(e) It would increase the minimum completion time by 1 week to 61 weeks.

7. *(a)*



(b) Critical activities

A—C—D—I—J—K

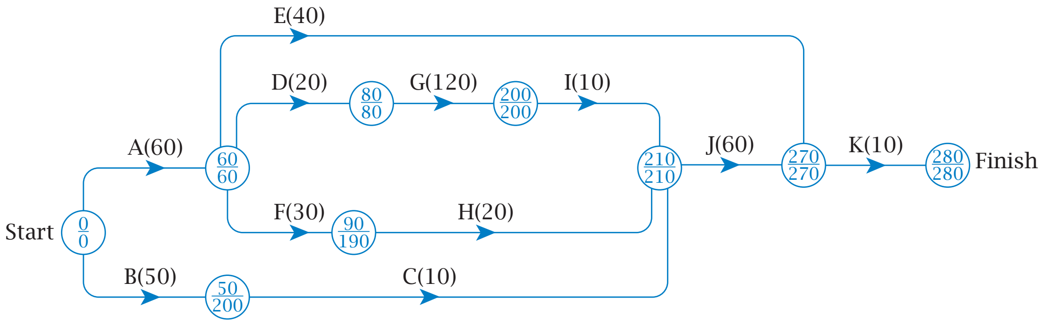
(c) Minimum completion time

= 106 minutes.

(d) The preparation of food to be cooked can take an *additional* 21 minutes i.e. a *total* 36 minutes.

(e) The minimum completion time *reduces* by 21 minutes to 85 minutes.

8. *(a)*



(b) Critical path

A—D—G—I—J—K

(c) Minimum completion time

= 280 minutes.

(d) 6.10 p.m. is the latest time to prepare the vegetables.

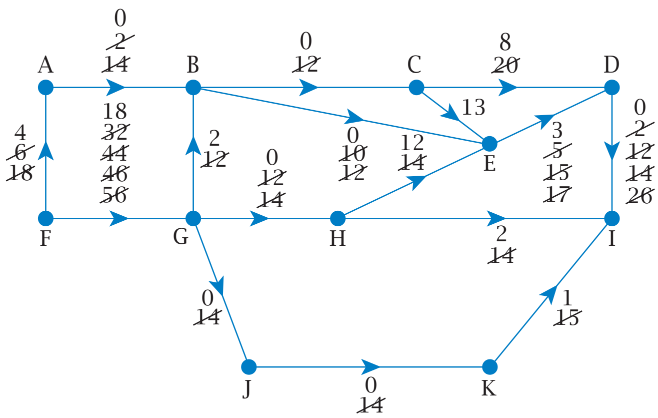
(e) This will *not* affect the minimum completion time.

9. *(a)* Source = F

Sink = I

(b) F G J K I = 14 vehicles

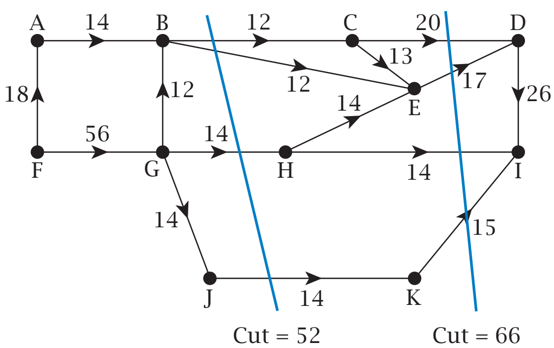
(c)



F A B C D I = 12
F A B E D I = 2
F G B E D I = 10
F G H E D I = 2
F G H I = 12
F G J K I = 14
Total: 52

Maximum flow = 52 vehicles.

(d) Maximum flow = minimum cut.



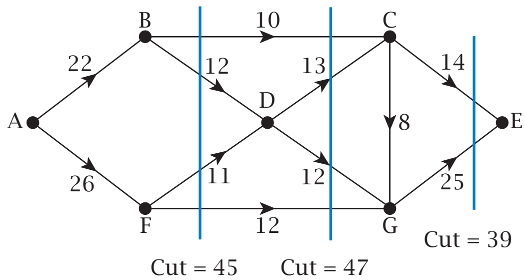
Minimum cut = 52 vehicles

∴ Maximum flow = 52 vehicles

(e)

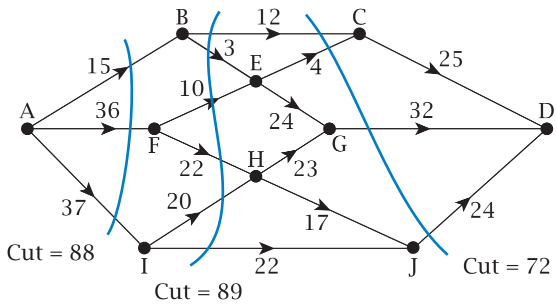
- GH is the only possible road to upgrade.
- If BC is upgraded DI is exhausted allowing no improvement in traffic flow.
- If JK is upgraded GJ is exhausted allowing no improvement in traffic flow.

10. *(a)*



Maximum flow = minimum cut
= 39

(b)



Maximum flow = 72

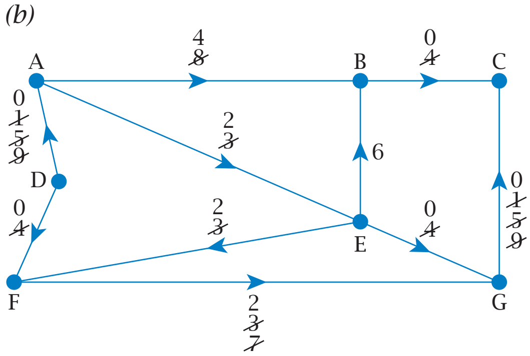
11. x = 12

y = 15

12. *(a)* Source = D

Sink = C

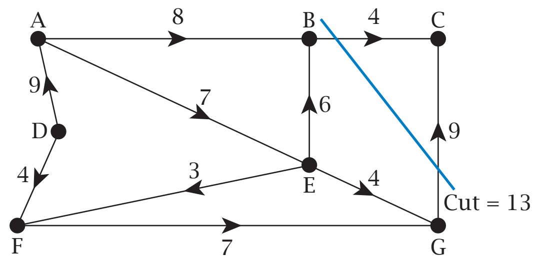
(b)



D A B C = 4
D A E G C = 4
D A E F G C = 1
D F G C = 4
Total: 13

Maximum flow = 13.

(c)



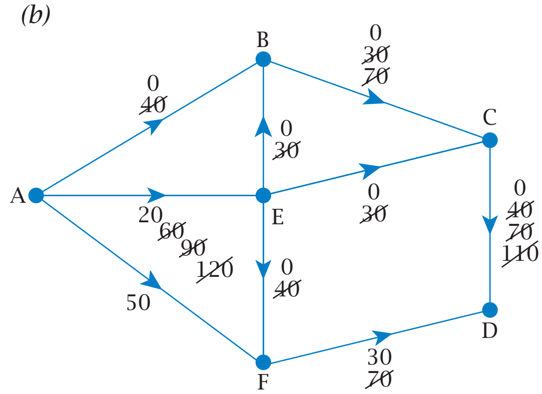
Minimum cut = maximum flow
Maximum flow = 13

(d) No effect on the maximum flow.

13. *(a)* Source = A

Sink = D

(b)



A C D = 40
A E B C D = 30
A E C D = 30
A E F D = 40
A F D = 30
Total: 170

Maximum flow = 170L/minute.

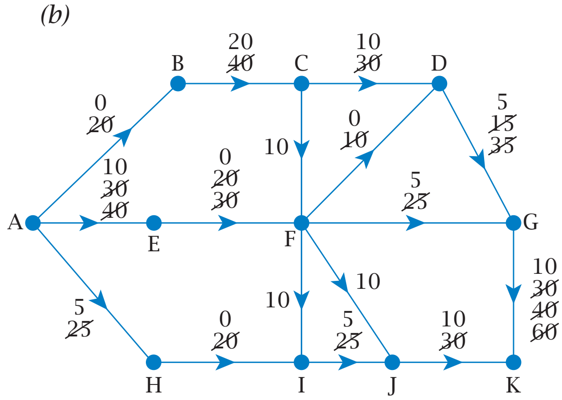
(c) Maximum flow would increase by 10L/min to 180L/min.

(d) The maximum flow is *not* affected if B—C were to increase by 20L/min. As A—B and E—B are exhausted there will be no change to the maximum flow into B—C.

14. *(a)* Source = A

Sink = K

(b)



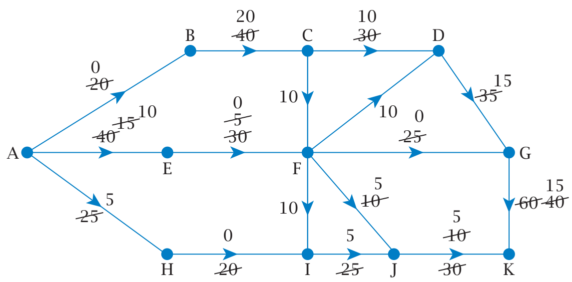
Maximum flow =

A B C D G K = 20
A E F D G K = 10
A E F G K = 20
A H I J K = 20
Total: 70

Maximum flow = 70.

OR

A B C D G K = 20
A E F G K = 25
A H I J K = 20
A E F J K = 5
Total: 70



(c) If CF is reversed there will be no effect on the maximum flow.

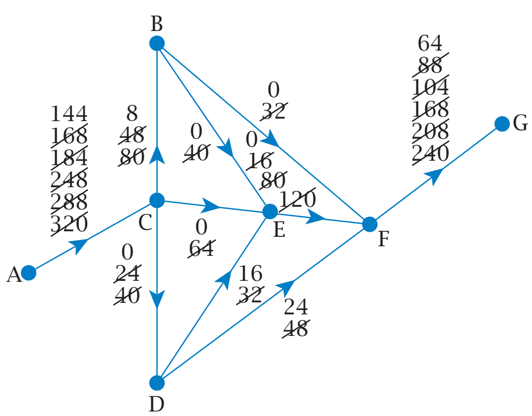
(d) Line AB.

(e) If AB is increased by 10 the maximum flow will increase to 80.

15. *(a)* Source = A

Sink = G

(b)



Maximum flow =

A C B F G = 32
A C B E F G = 40
A C E F G = 64
A C D E F G = 16
A C D F G = 24
Total: 176

Maximum flow = 176kL/hour.

(c) No effect on the maximum flow.

(d) A reduction of 40kL/hour to 136kL/hour.

(e) A reduction. Maximum flow will be 147kL/hour.

16. *(a)* Cost Matrix

$$\begin{bmatrix} 37 & 40 & 38 \\ 28 & 32 & 26 \\ 34 & 38 & 35 \end{bmatrix}$$

Step 1

$$\begin{bmatrix} 0 & 3 & 1 \\ 2 & 6 & 0 \\ 0 & 4 & 1 \end{bmatrix}$$

Step 2

$$\begin{bmatrix} 0 & 0 & 1 \\ 2 & 3 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

3 lines → 3 rows/columns.
Solution is possible.

Step 3

$$\begin{bmatrix} 0 & \boxed{0} & 1 \\ 2 & 3 & \boxed{0} \\ \boxed{0} & 1 & 1 \end{bmatrix}$$

Allocation

Worker A – Task Q

Worker B – Task R

Worker C – Task P

Minimum cost = $100.

(b) Cost Matrix

$$\begin{bmatrix} 15 & 18 & 17 & 20 \\ 19 & 14 & 16 & 23 \\ 18 & 23 & 24 & 21 \\ 25 & 26 & 28 & 22 \end{bmatrix}$$

Step 1

$$\begin{bmatrix} 0 & 3 & 2 & 5 \\ 5 & 0 & 2 & 9 \\ 0 & 5 & 6 & 3 \\ 3 & 4 & 6 & 0 \end{bmatrix}$$

Step 2

$$\begin{bmatrix} 0 & 3 & 0 & 5 \\ 5 & 0 & 0 & 9 \\ 0 & 5 & 4 & 3 \\ 3 & 4 & 4 & 0 \end{bmatrix}$$

4 lines → 4 rows/columns.
Solution is possible.

Step 3

$$\begin{bmatrix} 0 & 3 & \boxed{0} & 5 \\ 5 & \boxed{0} & 0 & 9 \\ \boxed{0} & 5 & 4 & 3 \\ 3 & 4 & 4 & \boxed{0} \end{bmatrix}$$

Allocation

Worker A – Task R

Worker B – Task Q

Worker C – Task P

Worker D – Task S

Minimum cost = \$71.

17. *(a)* Profit Matrix

$$\begin{bmatrix} 37 & 40 & 38 \\ 28 & 32 & 26 \\ 34 & 38 & 35 \end{bmatrix}$$

Step 1

$$\begin{bmatrix} 3 & 0 & 2 \\ 12 & 8 & 14 \\ 6 & 2 & 5 \end{bmatrix}$$

Step 2

$$\begin{bmatrix} 3 & 0 & 2 \\ 4 & 0 & 6 \\ 4 & 0 & 3 \end{bmatrix}$$

Step 3

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 4 \\ 1 & 0 & 1 \end{bmatrix}$$

2 lines → 3 rows/columns.
Solution is not possible.

Step 4

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 4 \\ 1 & 0 & 1 \end{bmatrix}$$

Step 5

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

3 lines → 3 rows/columns.
Solution is possible.

Step 6

$$\begin{bmatrix} \boxed{0} & 1 & 0 \\ 0 & \boxed{0} & 3 \\ 0 & 0 & \boxed{0} \end{bmatrix}$$

Allocation	or	or
Worker A – Task P	R	R
Worker B – Task Q	P	Q
Worker C – Task R	Q	P

Maximum profit = \$104.

(b) Profit Matrix

$$\begin{bmatrix} 15 & 18 & 17 & 20 \\ 19 & 14 & 16 & 23 \\ 18 & 23 & 24 & 21 \\ 25 & 26 & 28 & 22 \end{bmatrix}$$

Step 1

$$\begin{bmatrix} 13 & 10 & 11 & 8 \\ 9 & 14 & 12 & 5 \\ 10 & 5 & 4 & 7 \\ 3 & 2 & 0 & 6 \end{bmatrix}$$

Step 2

$$\begin{bmatrix} 5 & 2 & 3 & 0 \\ 4 & 9 & 7 & 0 \\ 6 & 1 & 0 & 3 \\ 3 & 2 & 0 & 6 \end{bmatrix}$$

Step 3

$$\begin{bmatrix} 2 & 1 & 3 & 0 \\ 1 & 8 & 7 & 0 \\ 3 & 0 & 0 & 3 \\ 0 & 1 & 0 & 6 \end{bmatrix}$$

3 lines → 4 rows/columns.
Solution not yet possible.

Step 4

$$\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 7 & 6 & 0 \\ 3 & 0 & 0 & 4 \\ 0 & 1 & 0 & 7 \end{bmatrix}$$

4 lines → 4 rows/columns.
Solution is now possible.

Step 5

$$\begin{bmatrix} 1 & \boxed{0} & 2 & 0 \\ 0 & 7 & 6 & \boxed{0} \\ 3 & 0 & \boxed{0} & 4 \\ \boxed{0} & 1 & 0 & 7 \end{bmatrix}$$

Allocation	or
Worker A – Task Q	S
Worker B – Task S	P
Worker C – Task R	Q
Worker D – Task P	R

Maximum profit = \$90.

18. Time Matrix

$$\begin{bmatrix} 26 & 44 & 40 & 60 \\ 28 & 48 & 26 & 20 \\ 24 & 46 & 30 & 80 \\ 28 & 42 & 34 & 40 \end{bmatrix}$$

Step 1

$$\begin{bmatrix} 0 & 18 & 14 & 34 \\ 8 & 28 & 6 & 0 \\ 0 & 22 & 6 & 56 \\ 0 & 14 & 6 & 12 \end{bmatrix}$$

Step 2

$$\begin{bmatrix} 0 & 4 & 8 & 34 \\ 8 & 14 & 0 & 0 \\ 0 & 8 & 0 & 56 \\ 0 & 0 & 0 & 12 \end{bmatrix}$$

4 lines → 4 rows/columns.
Solution is possible.

Step 3

$$\begin{bmatrix} \boxed{0} & 4 & 8 & 34 \\ 8 & 14 & 0 & \boxed{0} \\ 0 & 8 & \boxed{0} & 56 \\ 0 & \boxed{0} & 0 & 12 \end{bmatrix}$$

Allocation

Scott – Task A

Charlene – Task D

Bob – Task C

Taya – Task B

Minimum time = 118 minutes.

19. Time Matrix

$$\begin{bmatrix} 14 & 25 & 59 & 131 \\ 12 & 26 & 65 & 159 \\ 18 & 24 & 62 & 175 \\ 13 & 27 & 60 & 163 \end{bmatrix}$$

Step 1

$$\begin{bmatrix} 0 & 11 & 45 & 117 \\ 0 & 14 & 53 & 147 \\ 0 & 6 & 44 & 157 \\ 0 & 14 & 47 & 150 \end{bmatrix}$$

Step 2

$$\begin{bmatrix} 0 & 5 & 1 & 0 \\ 0 & 8 & 9 & 30 \\ 0 & 0 & 0 & 40 \\ 0 & 8 & 3 & 33 \end{bmatrix}$$

3 lines → 4 rows/columns.
Solution not yet possible.

Step 3

$$\begin{bmatrix} 0 & 4 & 0 & 0 \\ 0 & 7 & 8 & 30 \\ 1 & 0 & 0 & 41 \\ 0 & 7 & 2 & 33 \end{bmatrix}$$

Step 4

$$\begin{bmatrix} 2 & 4 & 0 & 0 \\ 0 & 5 & 6 & 28 \\ 3 & 0 & 0 & 41 \\ 0 & 5 & 0 & 31 \end{bmatrix}$$

4 lines → 4 rows/columns.
Solution possible.

Step 5

$$\begin{bmatrix} 2 & 4 & 0 & \boxed{0} \\ \boxed{0} & 5 & 6 & 28 \\ 3 & \boxed{0} & 0 & 41 \\ 0 & 5 & \boxed{0} & 31 \end{bmatrix}$$

Allocation

Kai – 800 m

Drew – 100 m

Lachlan – 200 m

Brayden – 400 m

Minimum time = 227 minutes.

20. Improvement Matrix

$$\begin{bmatrix} 72 & 79 & 64 & 68 \\ 76 & 74 & 62 & 70 \\ 64 & 66 & 70 & 62 \\ 68 & 70 & 62 & 60 \end{bmatrix}$$

Step 1

$$\begin{bmatrix} 7 & 0 & 15 & 11 \\ 3 & 5 & 17 & 9 \\ 15 & 13 & 9 & 17 \\ 11 & 9 & 17 & 19 \end{bmatrix}$$

Step 2

$$\begin{bmatrix} 7 & 0 & 15 & 11 \\ 0 & 2 & 14 & 6 \\ 6 & 4 & 0 & 8 \\ 2 & 0 & 8 & 10 \end{bmatrix}$$

Step 3

$$\begin{bmatrix} 7 & 0 & 15 & 5 \\ 0 & 2 & 14 & 0 \\ 6 & 4 & 0 & 2 \\ 2 & 0 & 8 & 4 \end{bmatrix}$$

No solution yet.

Step 4

$$\begin{bmatrix} 5 & 0 & 15 & 3 \\ 0 & 4 & 16 & 0 \\ 4 & 4 & 0 & 0 \\ 0 & 0 & 8 & 2 \end{bmatrix}$$

Solution is possible.

Step 5

$$\begin{bmatrix} 5 & \boxed{0} & 15 & 3 \\ 0 & 4 & 16 & \boxed{0} \\ 4 & 4 & \boxed{0} & 0 \\ \boxed{0} & 0 & 8 & 2 \end{bmatrix}$$

Allocation

Mr Smith – Ben

Mr Brown – Dorothy

Mrs Green – Connie

Miss Young – Adam

Maximum: 287%

21. Time Matrix

$$\begin{bmatrix} 10 & 24 & 15 & 0 \\ 11 & 27 & 18 & 0 \\ 10 & 26 & 22 & 0 \\ 9 & 28 & 14 & 0 \end{bmatrix}$$

Step 1

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 2 & 3 & 4 & 0 \\ 1 & 2 & 8 & 0 \\ 0 & 4 & 0 & 0 \end{bmatrix}$$

No solution yet.

Step 2

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 2 & 3 & 0 \\ 0 & 1 & 7 & 0 \\ 0 & 4 & 0 & 1 \end{bmatrix}$$

Solution possible.

Step 3

$$\begin{bmatrix} 1 & \boxed{0} & 1 & 1 \\ 1 & 2 & 3 & \boxed{0} \\ \boxed{0} & 1 & 7 & 0 \\ 0 & 4 & \boxed{0} & 1 \end{bmatrix}$$

Allocation

Worker P – Task Y

Worker Q – No task

Worker R – Task X

Worker S – Task Z

Minimum time = 48 minutes.

22. Cost Matrix

$$\begin{bmatrix} 224 & 350 & 442 & 0 \\ 231 & 364 & 454 & 0 \\ 246 & 380 & 456 & 0 \\ 250 & 388 & 469 & 0 \end{bmatrix}$$

Step 1

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 7 & 14 & 12 & 0 \\ 22 & 30 & 14 & 0 \\ 26 & 38 & 27 & 0 \end{bmatrix}$$

Step 2

$$\begin{bmatrix} 0 & 0 & 0 & 7 \\ 0 & 7 & 5 & 0 \\ 15 & 23 & 7 & 0 \\ 21 & 31 & 20 & 0 \end{bmatrix}$$

Step 3

$$\begin{bmatrix} 0 & 0 & 0 & 14 \\ 0 & 7 & 5 & 7 \\ 8 & 16 & 0 & 0 \\ 14 & 24 & 13 & 0 \end{bmatrix}$$

Step 4

$$\begin{bmatrix} 0 & \boxed{0} & 0 & 14 \\ \boxed{0} & 7 & 5 & 7 \\ 8 & 16 & \boxed{0} & 0 \\ 14 & 24 & 13 & \boxed{0} \end{bmatrix}$$

Allocation

Company A – Party Y

Company B – Party X

Company C – Party Z

Company D – No Party

Minimum cost = \$1037.

SOLUTIONS TO TRIAL TESTS

RESOURCE FREE: *Trial Test 1*

1. (c) ✓

2. (a) ✓

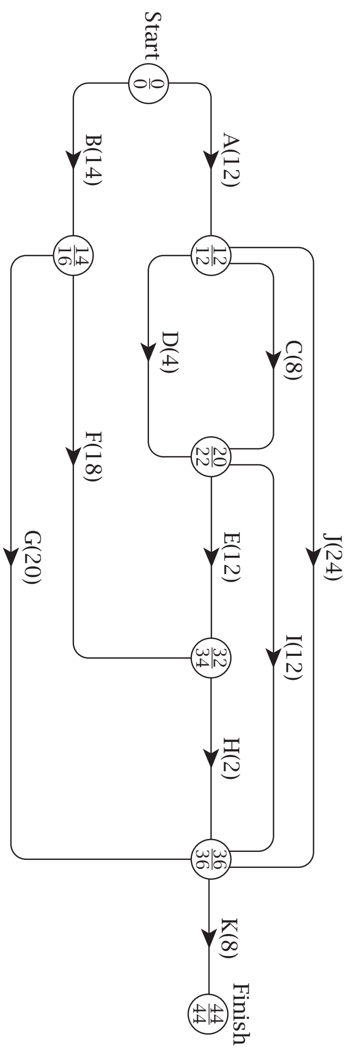
3. (a) ✓

4. (a) $\frac{n(n-1)}{2} = \frac{5(4)}{2}$ ✓
   $= 10$ edges ✓

   (b) 4 edges ✓✓

5. (a) $r = -0.8$ ✓
   (b) $r = 0.08$ ✓
   (c) $r^2 = 1$ i.e. $r = -1$ ✓
   (d) $r = 0.45$ ✓

6. (a)



✓✓✓

(b) Critical path: AJK ✓
(c) Minimum time: 44 days ✓
(d) 2 days ✓
(e) 2 days ✓
(f) (i) Critical path is now ACEHK ✓
(ii) Minimum completion time: 45 days ✓

RESOURCE FREE: *Trial Test 2*

1. (a) ✓

2. (b) ✓

3. (a) 75% ✓✓

   (b) $\frac{1200}{0.75}$ ✓
   $= 1600$ ✓

4. (a) 5, 3, −2 ✓
   (b) 1, 5, 20.2 ✓✓
   (c) 3, 9, 27 ✓✓

5. (a) Hamiltonian path ✓✓
   (b) Semi-Eulerian ✓✓

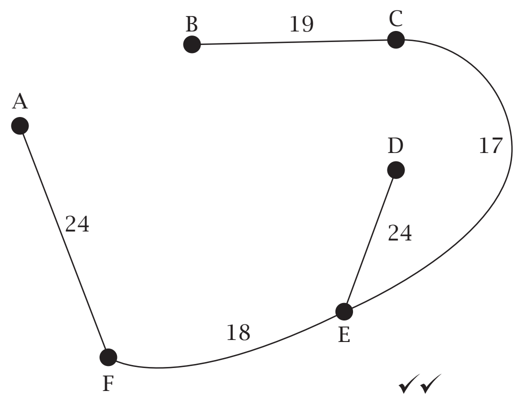
6.

	A	B	C	D	E	F
~~A~~	—	27	—	—	—	24
~~B~~	27	—	(19)	—	—	20
~~C~~	—	19	—	34	(17)	20
~~D~~	—	—	34	—	(24)	48
~~E~~	—	—	17	24	—	(18)
~~F~~	(24)	20	20	48	18	—

✓✓

Minimum length of track is 102 km ✓

(b)



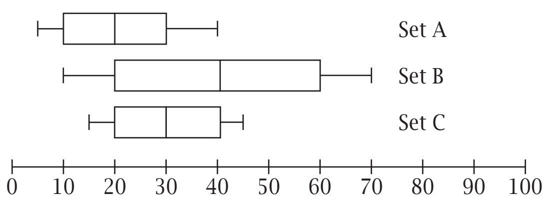
✓✓

RESOURCE FREE: *Trial Test 3*

1. *(a)* *(i)* B, D ✓✓
(ii) A, C ✓✓
(iii) A, B, C, D ✓✓

(b) Graphs A, B, and C are planar.
Graph D has edges which cross – hence non planar. ✓✓

2. *(a)*



✓✓

(b) *Similarities*

- Sets B and C have the same lower quartiles.
- Set B has the same minimum value as set A's lower quartile.

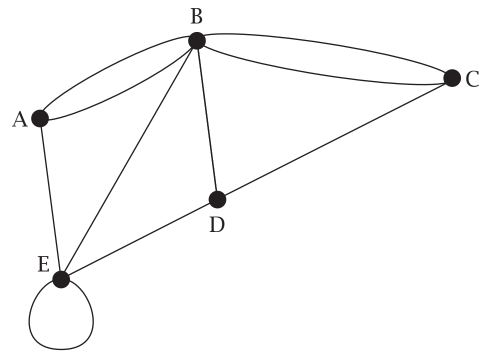
Differences

- Set B has a higher median than sets A and C.
- Set B has a larger range than sets A and C.

(Other reasons possible) ✓✓✓

(Any three)

3.



(a) faces = 7 ✓

(b) edges = 10 ✓

4. *(a)* Shortest route:

A B D G ✓✓

Time: 92 minutes ✓

(b) A E must reduce by more than 10 minutes in order to become the shortest route. ✓

Route A E G ✓

RESOURCE FREE: *Trial Test 4*

1. B and C ✓

2. *(a)* $T_5 = 10$ ✓

(b) $T_1 = -18$ ✓✓

3. *(a)* $f + v = c + 2$

$6 + 7 = 11 + 2$ ✓

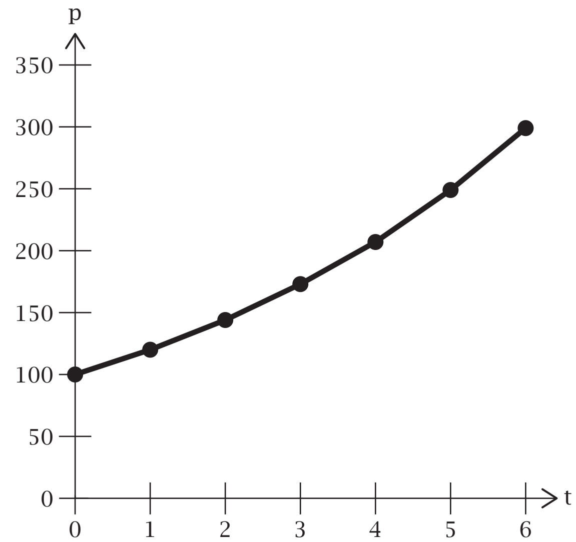
Yes it works ✓

(b) A – 4
B – 2
C – 4
D – 2
E – 4
F – 3
G – 3 ✓✓

(c) F ✓ Trail is:
G A B C A F E C D E G F ✓

(d) Hamiltonian Cycle ✓

4. (a)



✓✓

(b) Exponential graph ✓

(c) Population ≈ 360 ✓

(d) $k = 100$ ✓

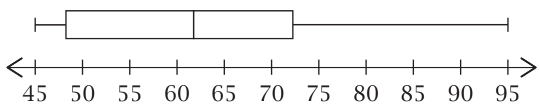
$a = 1.2$ ✓

(e) Increasing population ✓

(f) Decreasing population. Growth rate is a 30% ✓ decrease. ✓

RESOURCE RICH: *Trial Test 5*
Bivariate Statistics

1. (a)



✓✓

(b) *Differences*

- Boxplot 1 has a higher median than boxplot 2
- Boxplot 2 has a larger range and IQR than boxplot 1 ✓✓

(Other answers are possible)

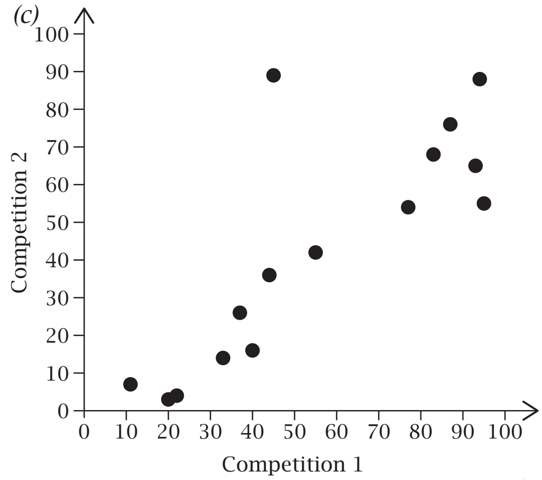
(c) $LQ - 1.5IQR \leq x \leq UQ + 1.5IQR$
$48 - 1.5(24) \leq x \leq 72 + 1.5(24)$
$12 \leq x \leq 108$ ✓
Values outside this range are outliers.
No outliers exist. ✓

2. (a) *Competition 1*
Mean : 53.071429
Standard Deviation : 27.86 ✓

Competition 2
Mean : 41.285714
Standard Deviation : 29.77 ✓

(b) Both means will increase.
Both standard deviations will decrease. ✓

(c)



✓✓

(d) $r = 0.9492$ ✓

(e) $y = 0.9114x - 11.3203$ ✓

(f) $y = 0.9114(60) - 11.3203$
$y = 43.4$ ✓

Mark for Competition 2 is ~ 43%
This is a reliable prediction → strong correlation coefficient and interpolation. ✓

3. (a)

	Boys	Girls	Total
Pass	40	96	136
Fail	28	24	52
Total	68	120	188

✓✓✓✓

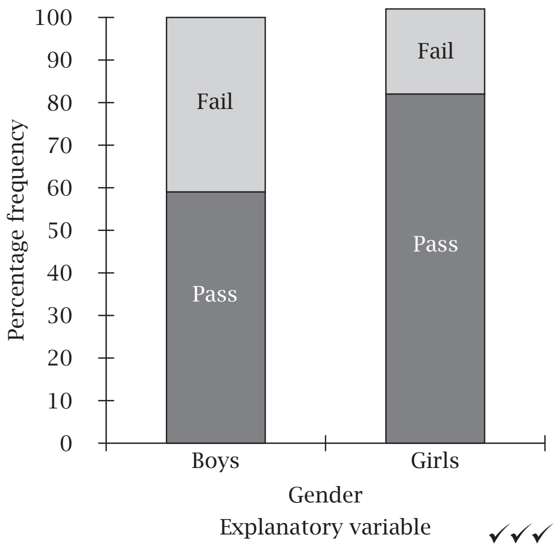
(b) Explanatory variable : Gender Boy/Girl ✓
Response variable : Pass/Fail exam ✓

(c)

		Gender	
		Boys	Girls
Exam	Pass	59%	80%
	Fail	41%	20%
		100%	100%

✓✓

(d)



✓✓✓

(e) The change in the proportions from boys to girls suggests that there is an association between gender and passing the examination. ✓

The data suggests that girls have greater success of passing the exam compared to the boys. ✓

4\. *(a)* Missing residuals in order –2.75, –3.81, 27.64, –5.22 ✓

(b) The *large* residual for the employee aged 36 years of age suggests that this is an outlier. ✓

(c) $r_{ad} = -0.926$ ✓

(d) $r^2_{ad} = 0.858$ ✓

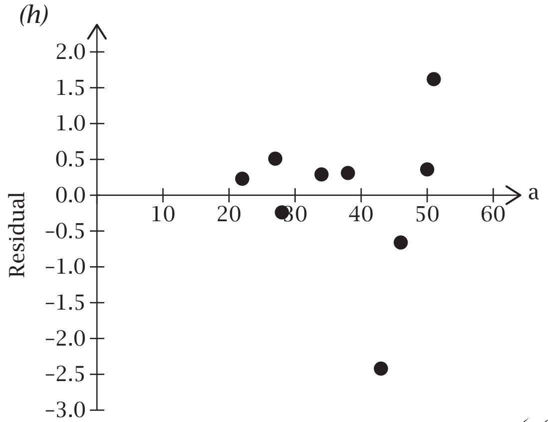
85.8% of the change in days absent can be attributed to the change in age. ✓ This high percentage indicates this is an *appropriate* model.

(e) $d = -0.255a + 15.370$ ✓

(f) $d = -0.255(67) + 15.370$
$= -1.715$ ✓

(g) Not a reliable prediction. ✓
This is an extrapolation and the value does not make sense. ✓

(h)



✓✓

(i) As the residuals are random, ✓
the linear model is appropriate.

RESOURCE RICH: *Trial Test 6*
Sequences and Time Series Data

1\. $T_{n+1} = (-1)^n 2T_n$

$T_2 = (-1)^1.2T_1$

$T_2 = (-1)^1.2(2)$

$T_2 = -4$ ✓

$T_3 = (-1)^2 2T_2$

$T_3 = (1)(2)(-4)$

$T_3 = -8$ ✓

$T_4 = (-1)^3 2T_3$

$T_4 = (-1)(2)(-8)$

$T_4 = 16$ ✓

$\therefore$ 2, –4, –8, 16

2\. *(a)* $T_4 = T_1 + T_2 - T_3$
$= 1 + 2 - 3$
$= 0$

$T_5 = T_2 + T_3 - T_4$
$= 2 + 3 - 0$
$= 5$

$T_6 = T_3 + T_4 - T_5$
$= 3 + 0 - 5$
$= -2$

$T_7 = T_4 + T_5 - T_6$
$= 0 + 5 - (-2)$
$= 7$

$T_8 = T_5 + T_6 - T_7$
$= 5 + (-2) - 7$
$= -4$

Sequence 1, 2, 3, 0, 5, –2, 7, –4 ✓✓

(b) When n is odd sequence is
1, 3, 5, 7
$T_{n+1} = T_n + 2$, ✓ $T_1 = 1$ ✓

(c) When n is even sequence is
2, 0, –2, –4
$T_{n+1} = T_n - 2$, ✓ $T_1 = 2$ ✓

3\. *(a)* $T_1 = 1.024(12\,000) - 400$ ✓

$= \$11\,888$

Amount owing after the first month is \$11 888 ✓

(b) $T_2 = 1.024(11\,888) - 400$ ✓

$= \$11\,773.31$

Amount owing after the second month is \$11 773.31 ✓

4. (a) $T_n = a + (n - 1)d$

$22 = a + (22 - 1)d$

$\boxed{22 = a + 21d}$

$T_n = a + (n - 1)d$

$12 = a + (28 - 1)d$

$\boxed{12 = a + 27d}$ ✓

Solve simultaneously using CAS calculator

$a = 57$

$d = -\frac{5}{3}$ ✓

$T_{14} = 57 + (14-1)\left(-\frac{5}{3}\right)$

$= 35\frac{1}{3}$ ✓

(b) $T_n = ar^{n-1}$

$256 = ar^3$

$16\,384 = ar^9$ ✓

Solve simultaneously

$a = -32$

$r = -2$

$T_7 = (-32)(-2)^6$

$T_7 = -2048$ ✓

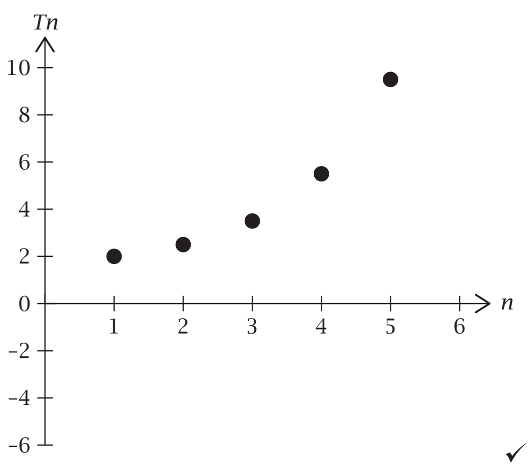
5. (a) $T_{n+1} = 1.01\ T_n - 700,\ T_0 = 50\,000$

$P = 1.01$ ✓
$q = -700$ ✓
$a = 50\,000$ ✓

(b) Still owing at the start of the 5th month is $49 187.92 ✓

(c) At the end of the 111th month. ✓

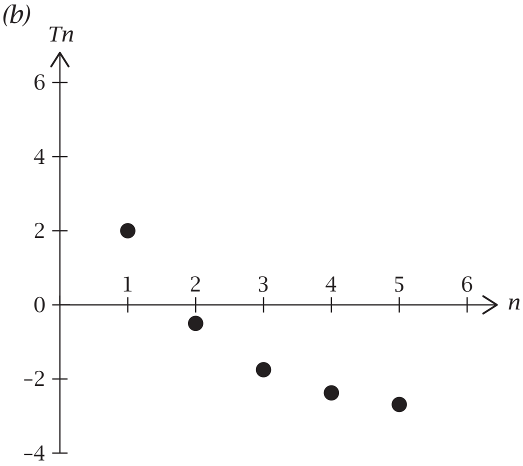
6. (a)



✓

Increasing ✓ first order recurrence relation

(b)



✓

Steady State solution first order recurrence relation ✓

7. (a) A 4 point centred moving average as the data is given quarterly – 4 times a year ✓

(b) Trend is increasing ✓

(c) A = 569 ✓
B = 322.50 ✓

(d) *Q3*
Seasonal Index = 0.6951 ✓
Q4
2013 value = 569 ✓
Seasonal Index = 1.1773 ✓
Averages
2011 = 267.5
2012 = 375.25
2013 = 487 ✓✓

(e) M = 24.754*t* + 215.542 ✓

(f) *Q4* 2014 → t = 16 ✓

M = 24.754(16) + 215.542

M = 611.61 ✓

Seasonal Index for

Q4 = 1.1773

∴ Predicted value = 611.61 × 1.1773

= 720.05 ✓

Approx 720 ice cream sales

RESOURCE RICH: *Trial Test 7 Undirected Graphs and Networks*

1. Shortest Path: P—T—R—U—S ✓✓
or P—T—R—S

Distance: 463km ✓✓

The network can not be traversed as there are 4 odd vertices. ✓✓

Networks can only be traversed if they have 0 or 2 odd vertices.

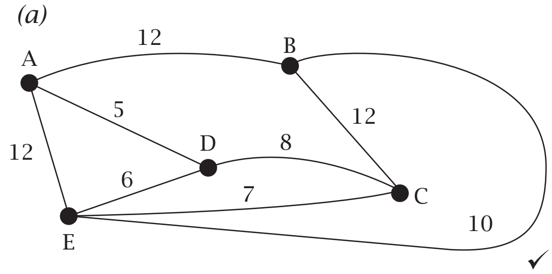
2. *(a)* Shortest path: B—E—F—D ✓
480 kilometres ✓
Cost: 480×0.45
= $216 ✓

(b) Shortest path: B—E—G—C—D ✓
740 kilometres ✓
Cost: 740×0.45
= $333 ✓

3. *(a)* Shortest route: A—B—D—E—I ✓✓
Time taken: 178 minutes ✓✓

(b) More than 10 minutes ✓✓✓

4. *(a)*



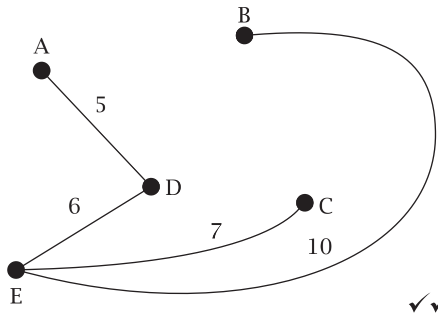
✓

(b)

	↓ A	↓ B	↓ C	↓ D	↓ E
A	—	12	—	5	12
B	12	—	12	—	(10)
C	—	12	—	8	(7)
D	(5)	—	8	—	6
E	12	10	7	(6)	—

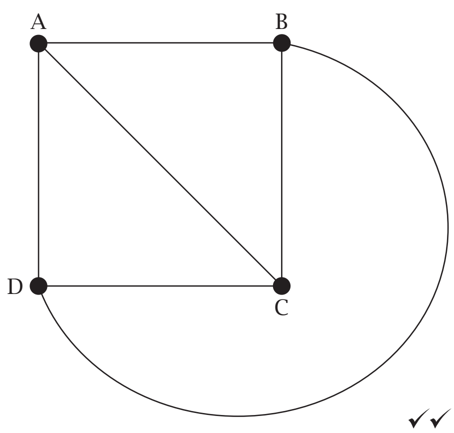
Cost = 5 + 6 + 10 + 7
= 28 ✓
Cost $2800 ✓

(c)



✓✓

5. *(a)*



✓✓

(b)

	A	B	C	D	E
A	-	1	1	1	1
B	1	-	1	1	1
C	1	1	-	1	1
D	1	1	1	-	1
E	1	1	1	1	-

✓

(c) Total number of edges $= \dfrac{n(n-1)}{2}$

$= \dfrac{5(4)}{2}$

= 10 ✓

(d) K_5 is Eulerian as each vertex is of even degree. ✓

(e) K_n is Eulerian when n is odd. ✓

RESOURCE RICH: *Trial Test 8 Finance*

1. 12 payments @ $180 = $2160
Interest paid = $2160 – $2000
= $160

Interest rate $= \dfrac{160}{2000} \times 100$

= 8% p.a. ✓

Effective annual rate of interest

$= \left(1 + \dfrac{r}{n}\right)^n - 1$

$= \left(1 + \dfrac{0.08}{12}\right)^{12} - 1$

= 0.083

= 8.3% p.a. ✓✓

2. *Option 1*

S = PRT
= 15 000 × 0.115 × 10
= $17 250

Interest = $17 250 ✓

Option 2

$A = P\left(1 + \dfrac{r}{n}\right)^{nt}$

$= 15\,000\left(1 + \dfrac{0.107}{4}\right)^{40}$

= 43 120.10 ✓

Interest = $28 120.10 ✓

Option 3

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

$$= 15\,000\left(1 + \frac{0.094}{52}\right)^{(520)}$$

$$= 38\,367.15$$ ✓

Interest = \$23 367.15 ✓

Option 2 earns the most interest and is the best investment. ✓

3. Using CAS calculator

N = 300

I % = 7.25

PV = –\$120 000

PMT = \$867.37

FV = 0

P/Y = 12

C/Y = 12 ✓✓

Monthly repayment of \$867.37 ✓

4. *(a)* Annual interest rate is 25% p.a. ✓✓

(b) T_{10} = \$37 252.90 ✓✓

(c) $4000(1 + r)^5 = 37\,252.90$ ✓

$r = 0.5625$ ✓

Interest rate = 56.25% p.a.

5. *(a)* \$31 512.43 × r = 328.26 ✓

r = 0.010417/month

r = 0.1250/annum

r = 12.5% p.a. ✓

(b) A = \$361.21 ✓

B = \$1000 ✓

C = \$35 482.55 ✓

D = \$4.02 ✓

E = \$389.63 ✓

(c) $T_{n+1} = 1.01\,T_n - 1000$ ✓

$T_0 = 38\,000$ ✓

(d) Total repayments

= 11 × 1000 + 23 × 1500 + 389.63 ✓

= \$45 889.63 ✓

Interest = \$45 889.63 – \$38 000

= \$7889.63 ✓

RESOURCE RICH: *Trial Test 9 Directed Networks and Decision Mathematics*

1. *(a)* Cost Matrix

52	56	72
47	42	48
50	58	49

✓

Step 1

0	4	20
5	0	6
1	9	0

✓

Step 2

~~0~~	~~4~~	~~20~~
~~5~~	~~0~~	~~6~~
~~1~~	~~9~~	~~0~~

3 lines → 3 rows/column
Solution is possible

Step 3

[0]	4	20
5	[0]	6
1	9	[0]

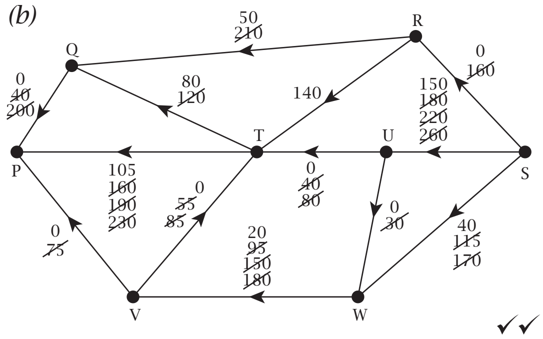
✓

Allocation: Worker A – Task X
Worker B – Task Y
Worker C – Task Z ✓

(b) *Minimum Cost* – \$143 ✓

2. *(a)* Source: S ✓, Sink : P ✓

(b)

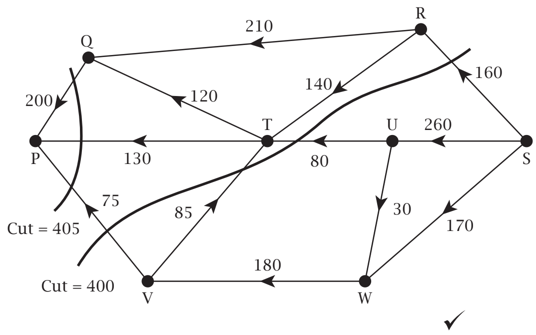


✓✓

SRQP = 160
SUTQP = 40
SUTP = 40
SUWVTP = 30
SWVTP = 55 ✓✓✓
SWVP = 75

400

(c)



Minimum cut = 400 ✓

(d) RT is not used and can be removed. ✓

(e) UT can be upgraded by 105 passengers. SU and TP are not exhausted and upgrading UT will allow more passengers through this section. 105 passengers can be transported along this route. ✓✓

3. *(a)* Yes ✓ – each worker is allocated to every task. ✓

(b) Profit matrix

$$\begin{bmatrix} 27 & 28 & 32 & 35 \\ 31 & 30 & 28 & 25 \\ 21 & 24 & 23 & 27 \\ 36 & 32 & 30 & 29 \end{bmatrix}$$ ✓

Step 1

$$\begin{bmatrix} 9 & 8 & 4 & 1 \\ 5 & 6 & 8 & 11 \\ 15 & 12 & 13 & 9 \\ 0 & 4 & 6 & 7 \end{bmatrix}$$ ✓

Step 2

$$\begin{bmatrix} 8 & 7 & 3 & 0 \\ 0 & 1 & 3 & 6 \\ 6 & 3 & 4 & 0 \\ 0 & 4 & 6 & 7 \end{bmatrix}$$ ✓

Step 3

$$\begin{bmatrix} 8 & 6 & 0 & 0 \\ 0 & 0 & 0 & 6 \\ 6 & 2 & 1 & 0 \\ 0 & 3 & 3 & 7 \end{bmatrix}$$ ✓

4 lines – 4 rows/columns
Solution is possible.

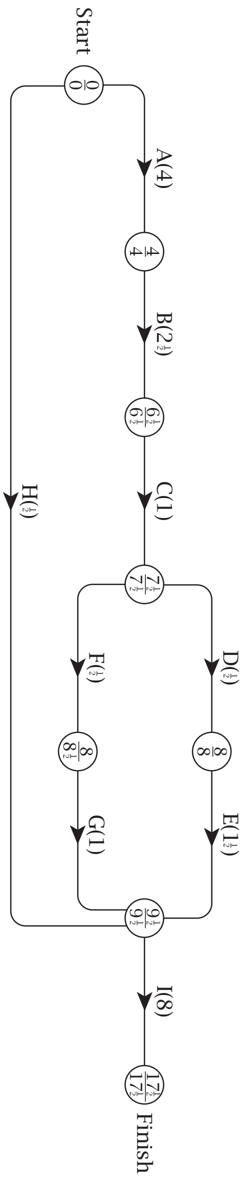
Step 4

$$\begin{bmatrix} 8 & 6 & \boxed{0} & 0 \\ 0 & \boxed{0} & 0 & 6 \\ 6 & 2 & 1 & \boxed{0} \\ \boxed{0} & 3 & 3 & 7 \end{bmatrix}$$

Allocation: Worker A – Task R
Worker B – Task Q
Worker C – Task S
Worker D – Task P ✓

(c) Maximum profit = \$125. ✓

4. *(a)*



✓✓✓✓

(b) Minimum completion time = $17\frac{1}{2}$ hours ✓
Critical path: A B C D E I ✓

(c) Critical path remains the same. Minimum completion time is now 18 hours. ✓

(d) Extra half an hour is possible. ✓

5. *(a)* Travel time matrix

$$\begin{bmatrix} 42 & 45 & 48 & 0 \\ 37 & 35 & 31 & 0 \\ 27 & 21 & 24 & 0 \\ 51 & 39 & 45 & 0 \end{bmatrix} \checkmark$$

Step 1

$$\begin{bmatrix} 15 & 24 & 24 & 0 \\ 10 & 14 & 7 & 0 \\ 0 & 0 & 0 & 0 \\ 24 & 18 & 21 & 0 \end{bmatrix} \checkmark$$

Step2

$$\begin{bmatrix} 8 & 17 & 17 & 0 \\ 3 & 7 & 0 & 0 \\ 0 & 0 & 0 & 7 \\ 17 & 11 & 14 & 0 \end{bmatrix} \checkmark$$

Step 3

$$\begin{bmatrix} 0 & 9 & 9 & 0 \\ 3 & 7 & 0 & 8 \\ 0 & 0 & 0 & 15 \\ 9 & 3 & 6 & 0 \end{bmatrix} \checkmark$$

Step 4

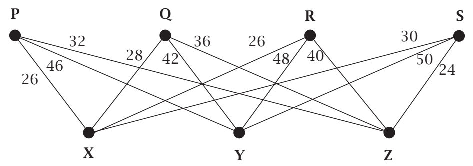
$$\begin{bmatrix} \boxed{0} & 9 & 9 & 0 \\ 3 & 7 & \boxed{0} & 8 \\ 0 & \boxed{0} & 0 & 15 \\ 9 & 3 & 6 & \boxed{0} \end{bmatrix} \checkmark$$

Allocation: Craig $\rightarrow D_1$
Brendan $\rightarrow D_3$
Alice $\rightarrow D_2$
Georgia $\rightarrow$ Dummy depot ✓

(b) Minimum travel time: 94 minutes. ✓

SOLUTIONS TO EXAMINATION STYLE QUESTIONS

1. (a)



(b) *Cost* = 46 + 28 + 24 = $9 800

(c)

$$\begin{array}{c} \\ P \\ Q \\ R \\ S \end{array}\begin{array}{c} \begin{array}{ccc} X & Y & Z \end{array} \\ \begin{bmatrix} 26 & 46 & 32 \\ 28 & 42 & 36 \\ 26 & 48 & 40 \\ 30 & 50 & 24 \end{bmatrix} \end{array}$$

$$\begin{bmatrix} 26 & 46 & 32 & 0 \\ 28 & 42 & 36 & 0 \\ 26 & 48 & 40 & 0 \\ 30 & 50 & 24 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 4 & 8 & 0 \\ 2 & 0 & 12 & 0 \\ 0 & 6 & 16 & 0 \\ 4 & 8 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 4 & 8 & \boxed{0} \\ 2 & \boxed{0} & 12 & 0 \\ \boxed{0} & 6 & 16 & 0 \\ 4 & 8 & \boxed{0} & 0 \end{bmatrix} \quad \begin{bmatrix} \boxed{0} & 4 & 8 & 0 \\ 2 & \boxed{0} & 12 & 0 \\ 0 & 6 & 16 & \boxed{0} \\ 4 & 8 & \boxed{0} & 0 \end{bmatrix}$$

Possible allocations:

Worker Q ➜ Task Y   Walker P ➜ Task X

Worker R ➜ Task X or Walker Q ➜ Task Y

Worker S ➜ Task Z   Walker S ➜ Task Z

Minimum Cost = $(42 + 26 + 24)

= $9 200

2. (a) $a = 19 \quad d = -4$

(i) $T_n = 19 + (n - 1)(-4)$
$T_{40} = 19 + (40 - 1)(-4)$
$T_{40} = -137$

(ii) $S_{40} = -2360$

(iii) $-309 = 19 + (n - 1)(-4)$
$n = 83$

(b) $T_{n+1} = 2T_n + 3$

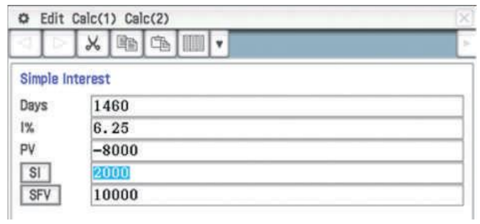
(i) $T_2 = 2(2) + 3$
$= 7$
$T_3 = 2(7) + 3$
$= 17$
$T_4 = 2(17) + 3$
$= 37$

∴ 2, 7, 17, 37

(ii) $T_5 = 2(37) + 3$
$= 77$
$T_6 = 2(77) + 3$
$= 157$

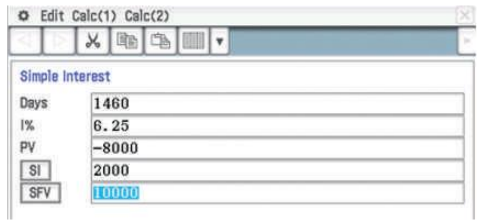
∴ 6 terms.

3. (a)



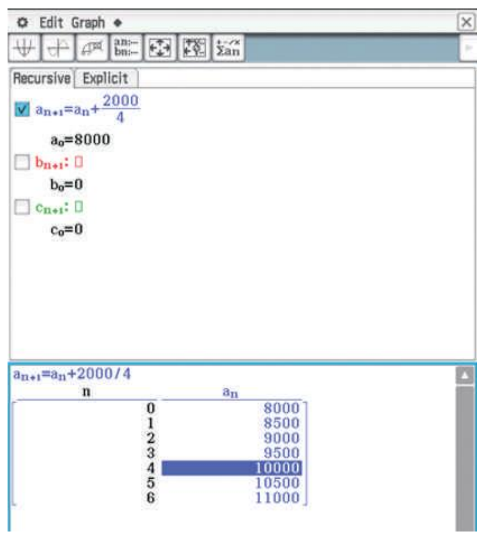
$2 000

(b)



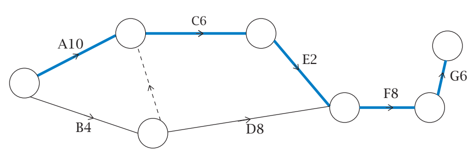
$10 000

(c)



$T_{n+1} = T_n + 500$, $\quad T_0 = 8000$

4. (a)



(b) Critical path = *ACEFG*

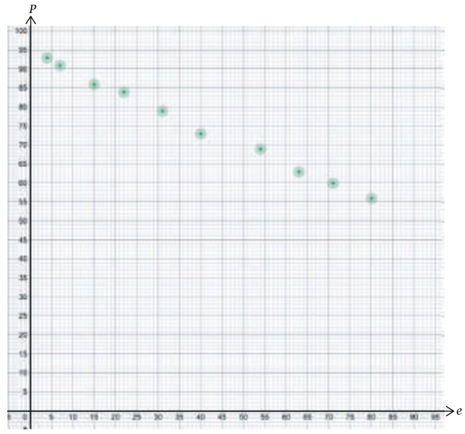
Minimum completion time = 32 days

(c) Float time for *D* = 6 days.

(d) Critical path is now *BDFG*.

Minimum completion time = 36 days.

5. (a)



(b) Very strong and negative association.

(c) $r_{ep} = -0.9979$

(d) $p = -0.4831e + 94.0943$

(e) $r^2 = 0.9957$. 99.57% of the variation in the pulse rate can be explained by the variation in the elapsed game time.

(f) $p = -0.4831(50) + 94.0943$

$p = 69.94$ bpm

(g) A valid prediction due to:

The strength of the relationship is very strong $r_{ep} = -0.9979$.

The prediction is interpolation.

6. (a) $A = 202$

$B = 163.5$

$C = 104.1$

(b) $D = 0.644$

$E = 1.104$

(c) $F = 170.290$

(d) $Q = 4.9192t + 125.625$

(e) $t = 13$

$Q = 4.9192(13) + 125.625$

$Q = 189.5746$

Estimate = 189.5746×0.644

= 122.086

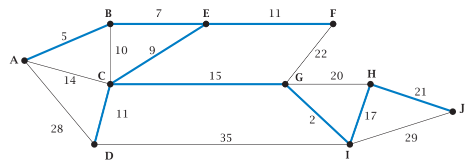
∴ Number of visitors ≈ 12 209

7. (a)

	A	B	C	D	E	F	G	H	I	J
A	-	5	14	28	-	-	-	-	-	-
B	5	-	10	-	7	-	-	-	-	-
C	14	10	-	11	9	-	15	-	-	-
D	28	-	11	-	-	-	-	-	35	-
E	-	7	9	-	-	11	-	-	-	-
F	-	-	-	-	11	-	22	-	-	-
G	-	-	15	-	-	22	-	20	2	-
H	-	-	-	-	-	-	20	-	17	21
I	-	-	-	35	-	-	2	17	-	29
J	-	-	-	-	-	-	-	21	29	-

(b)

	↓ A	↓ B	↓ C	↓ D	↓ E	↓ F	↓ G	↓ H	↓ I	↓ J
~~A~~	-	5	14	28	-	-	-	-	-	-
~~B~~	(5)	-	10	-	7	-	-	-	-	-
~~C~~	14	10	-	11	(9)	-	15	-	-	-
~~D~~	28	-	(11)	-	-	-	-	-	35	-
~~E~~	-	(7)	9	-	-	11	-	-	-	-
~~F~~	-	-	-	-	(11)	-	22	-	-	-
~~G~~	-	-	(15)	-	-	22	-	20	2	-
~~H~~	-	-	-	-	-	-	20	-	(17)	21
~~I~~	-	-	-	35	-	-	(2)	17	-	29
~~J~~	-	-	-	-	-	-	-	(21)	29	-



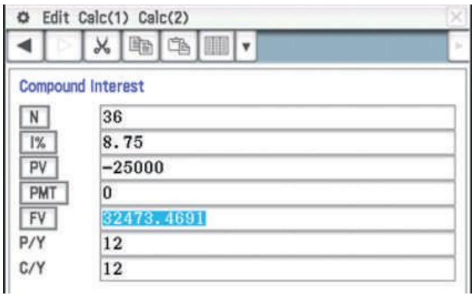
Minimum spanning tree highlighted.

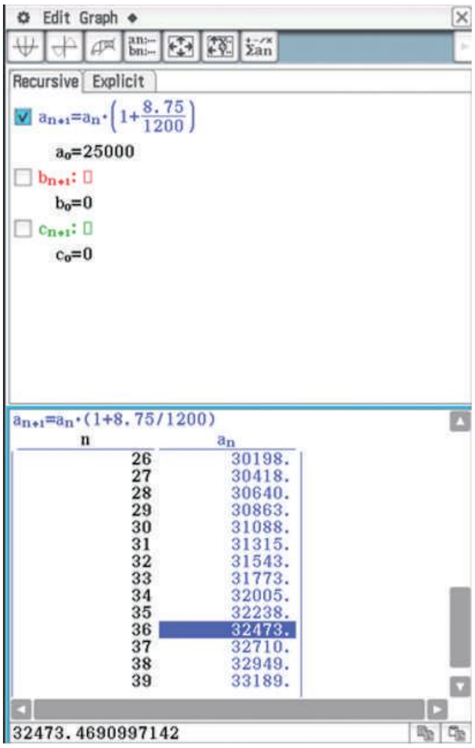
Minimum amount of wiring = 98 metres.

8. (a) $T_{n+1} = T_n \times \left(1 + \frac{8.75}{1200}\right)$

$T_0 = 25\ 000$

(b)



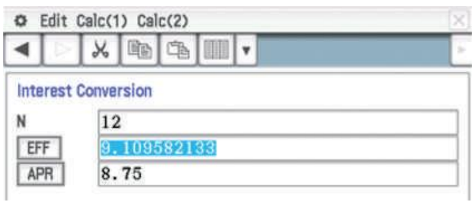


= \$32 473.47

(c) $e = \left(1 + \dfrac{0.0875}{12}\right)^{12} - 1$

= 0.091096

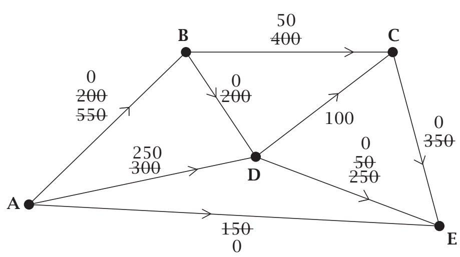
= 9.11% (2 decimal places)



9. *(a)* Source: *A*

Sink: *E*

(b) Max flow:



ABCE = 350

ABDE = 200

ADE = 50

AE = 150

Maximum flow = 750 vehicles per hour

(c) Widen road *DE* to have a capacity of 500 vehicles.

The maximum flow will now be 1000 vehicles per hour an increase of 250 vehicles per hour.

10. *(a)* $v = 6 \quad e = 10 \quad f = 6$

Euler's rule is: $v + f - e = 2$

$\therefore 6 + 6 - 10 = 2$ ✓

(b) Yes as no two lines (edges) cross.

(c)

	A	B	C	D	E	F
A	0	1	0	1	1	0
B	1	0	1	0	0	0
C	0	1	0	0	0	1
D	1	0	0	1	1	1
E	1	0	0	1	0	2
F	0	0	1	1	2	0

(d) Yes there are exactly two odd vertices.

(e) *D – A – D*

D – E – D

D – F – D

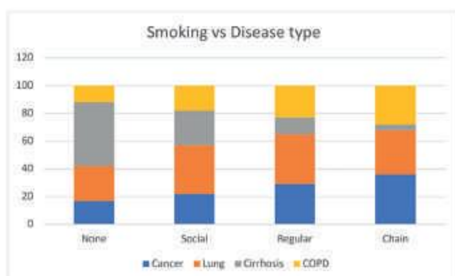
D – D – D

11. *(a)* Type of smoker

(b)

	Type of smoker			
Disease type	None	Social	Regular	Chain
Cancer	17	22	29	36
Lung	25	35	36	32
Cirrhosis	46	25	12	4
COPD	12	18	23	28
Total	100	100	100	100

(c)

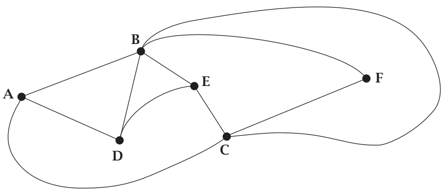


(d) As we move across the categories related to types of smokers, the proportions of each type of disease change quite noticeably suggesting an association between the two variables.

The proportions of COPD are much lower in those who don't smoke or are social smokers.

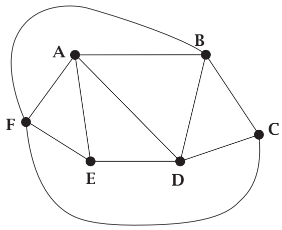
The proportions of cancer are much higher amongst regular and chain smokers.

12. (a) (i)



(ii) Not possible.

(iii)



(b) Graph (a) (i) is not traversable as it contains 4 odd vertices.

(c) $v = 7 \quad e = 11 \quad f = 6$

Euler's rule is: $v + f - e = 2$

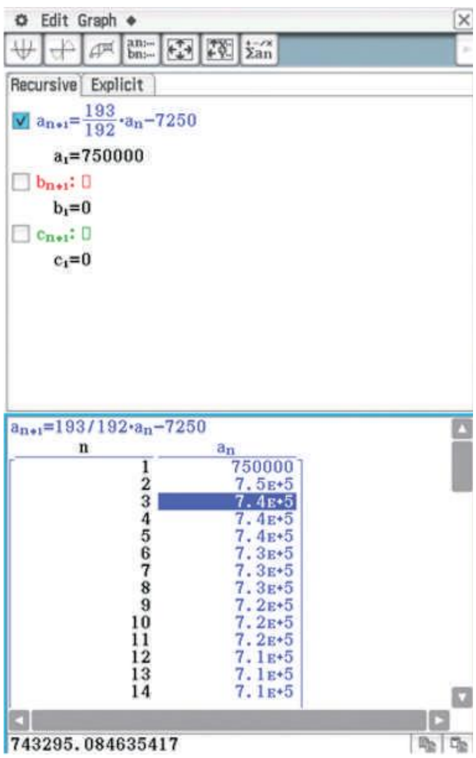
$\therefore 7 + 6 - 11 = 2$ ✓

13. (a) $\dfrac{3906.25}{750\ 000} \times 12 \times 100 = 6.25\%$

(b) $A_{n+1} = \dfrac{193}{192}A_n - 7250 \qquad A_1 = 750\ 000$

$p = \dfrac{193}{192}\ q = -7250 \qquad r = 750\ 000$

(c)

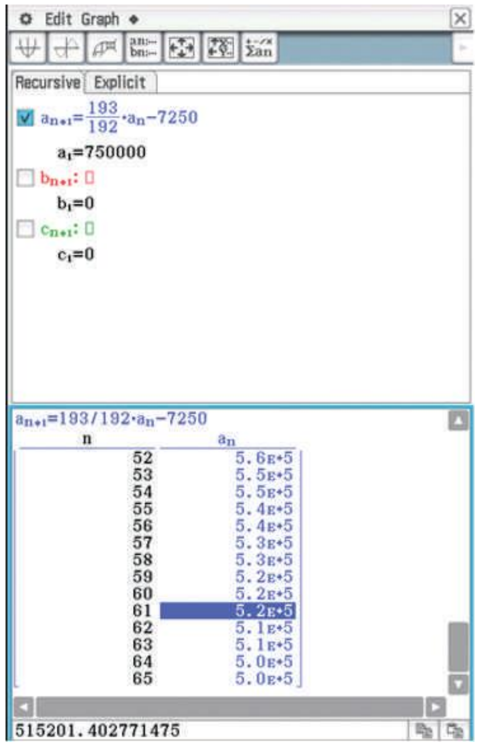


A = \$743 295.08

B = \$743 295.08 $\times \dfrac{1}{192}$ = \$3871.33

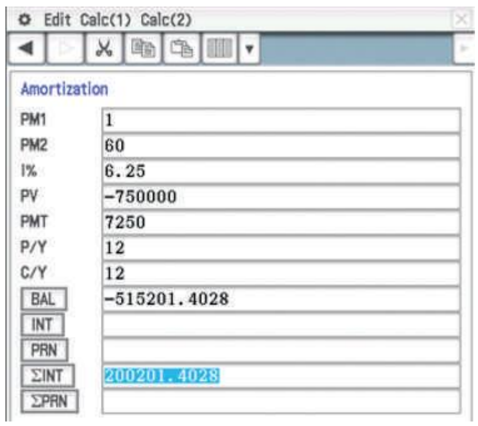
C = 743 295.08 + \$3871.33 − \$7250 = \$739 916.41

(d) Balance after 60 months:



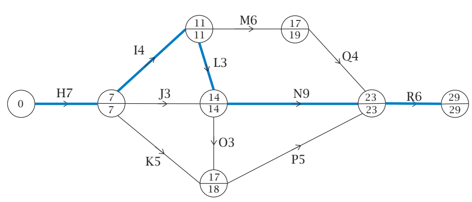
= \$515 201.40

(e) Interest earned over 60 months



= \$200 201.40

14. (a)



(b) Critical path = *HILNR*

Minimum completion time = 29 days.

(c) After 17 days.

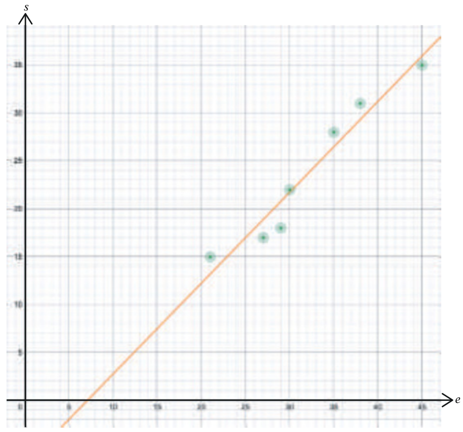
(d) After 13 days.

(e) 2 days.

(f) Critical path = *HILOPR*.

Minimum completion time = 30 days.
Completion time increases by 1 day.

15. (a)



(b) $r_{ts} = 0.9705$

(c) $s = 0.9475t - 6.7414$. See diagram above.

(d) 94.18%

(e) $s = 0.9475(10) - 6.7414$

$s = 2.734$ grams

Prediction is unreliable due to considerable extrapolation.

(f) Comment may not be valid. An association does not necessarily mean that one variable causes the other. There is no causal effect.

16. (a) Arithmetic sequence. Graph – points that lie in a straight line.

$a = 200 \quad d = -13$

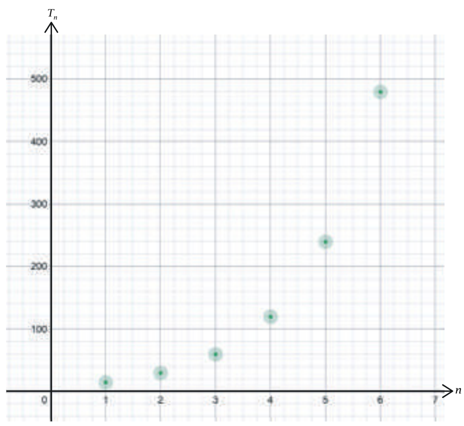
(b) 161, 148, 135

(c) $T_n = 200 + (n - 1)(-13)$

$= 213 - 13n$

(d) 30

(e)



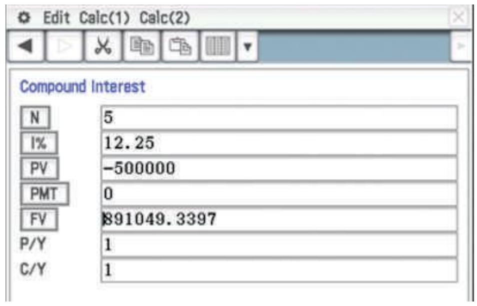
(f) $T_{n+1} = 2T_n, \quad T_1 = 15$

(g) $p = 0.2p + 2$

$0.8p = 2$

$p = 2.5$

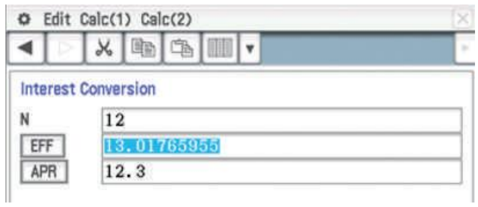
17. (a)



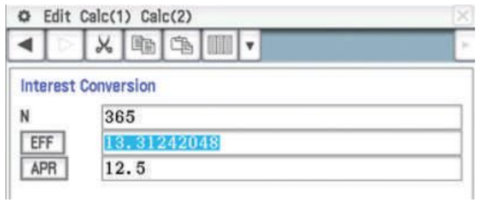
Interest earned = \$891 049.34 – \$500 000

= \$391 049.34

(b) 12.3% p.a. compounded monthly

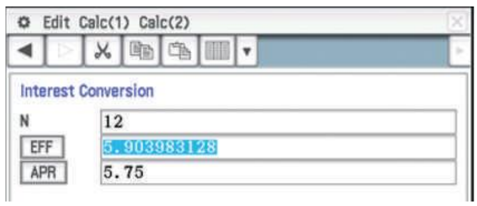


12.5% compounded daily.



Option: 12.5% compounded daily – higher effective interest rate.

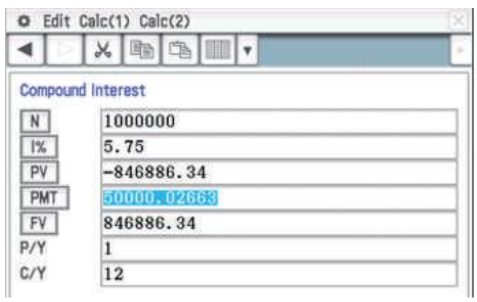
(c)



$Q = \dfrac{Pr}{100}$

$50\,000 = \dfrac{P(5.90398)}{100}$

$P = \$846\,886.34$



18. (a) Trend is decreasing.

A weekly cycle (5 day cycle) with sales that are highest (peak) on Friday and low (trough) on a Monday.

(b) $A = 32$

$B = 48.2$

$C = 75.4\%$

(c)

Day	Monday	Tuesday	Wednesday	Thursday	Friday
Seasonal Index	0.492	0.724	1.068	1.229	1.486

(d) Deseasonalised sales $= \dfrac{81}{1.486}$

$\approx$ 55 sales

(e) $t = 17$

Deseasonalised sales

$= -1.1924(17) + 64.0928$

$= 43.822$

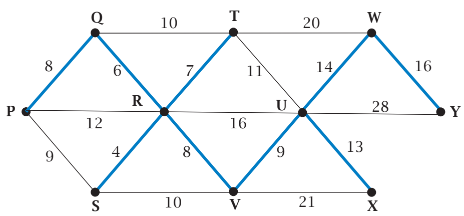
Actual number of lunches

$= 43.822 \times 0.724$

$= 31.727$

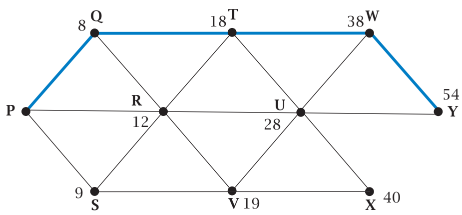
≈ 32

19. (a)



(b) Length = 85 kilometres.

(c)



Shortest distance = 54 kilometres.

(d) $40 + x < 54$

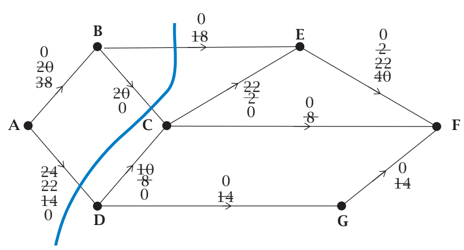
$\therefore \quad x < 14$

20. (a) $x = 24$

$y = 10$

$z = 22$

(b)



$ABEF = 18$

$ABCEF = 20$

$ADCEF = 2$

$ADCF = 8$

$ADGF = 14$

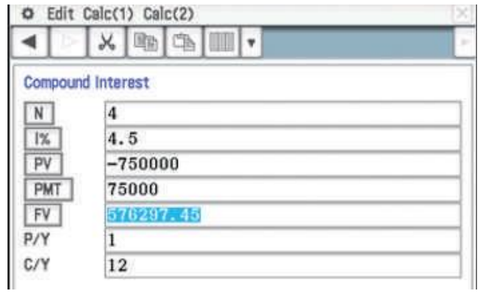
Maximum flow = 62 litres per minute

or

On diagram above.

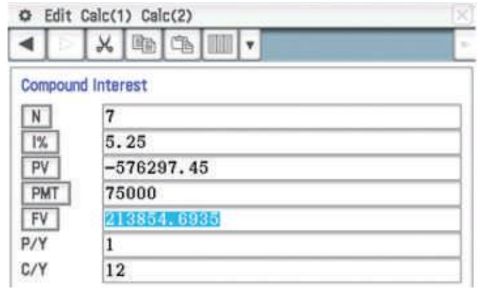
Cut = 18 + 20 + 24 = 62

21. (a)



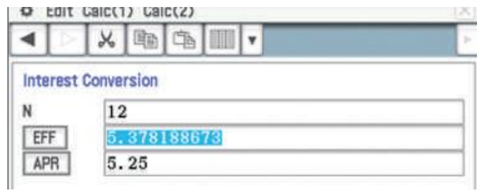
$576 297.45

(b)



$213 854.69

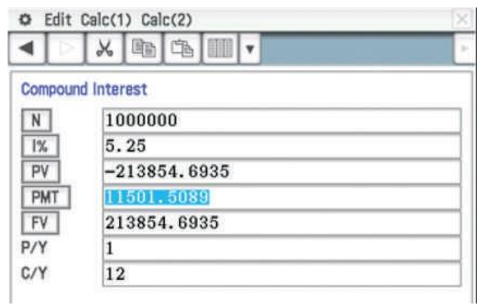
(c)



Annual payment

$= 213\,854.69 \times \dfrac{5.378188673}{100}$

= $11 501.51



22.

$$\begin{array}{c} \\ M \\ C \\ L \\ N \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 2 & 13 & 5 & 20 \\ 4 & 11 & 7 & 10 \\ 3 & 12 & 10 & 15 \\ 4 & 14 & 3 & 5 \end{bmatrix} \end{array}$$

$$\begin{bmatrix} 18 & 7 & 15 & 0 \\ 16 & 9 & 13 & 10 \\ 17 & 8 & 10 & 5 \\ 16 & 6 & 17 & 15 \end{bmatrix}$$

$$\begin{bmatrix} 18 & 7 & 15 & 0 \\ 7 & 0 & 4 & 1 \\ 12 & 3 & 5 & 0 \\ 10 & 0 & 11 & 9 \end{bmatrix}$$

$$\begin{bmatrix} 11 & 7 & 11 & 0 \\ 0 & 0 & 0 & 1 \\ 5 & 3 & 1 & 0 \\ 3 & 0 & 7 & 9 \end{bmatrix}$$

$$\begin{bmatrix} 10 & 6 & 10 & 0 \\ 0 & 0 & 0 & 2 \\ 4 & 2 & 0 & 0 \\ 3 & 0 & 7 & 10 \end{bmatrix}$$

$$\begin{bmatrix} 10 & 6 & 10 & \boxed{0} \\ \boxed{0} & 0 & 0 & 2 \\ 4 & 2 & \boxed{0} & 0 \\ 3 & \boxed{0} & 7 & 10 \end{bmatrix}$$

Allocation:

Matt → Task D

Cheung → Task A

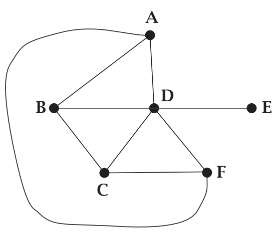
Leanne → Task C

Neya → Task B

Maximum profit = $(20 + 4 + 10 + 14)

= $48

23. *(a)*



Other solutions possible.

(b) 5 faces

(c) 3 + 3 + 3 + 3 + 1 + 5 = 18

(d) Neither as it contains six odd indices.
Eulerian requires no odd vertices.
Semi-Eulerian requires exactly two odd vertices.

(e) *EDABCF* – other solutions possible.

24. *(a)* Gender

(b)

	Type of social media app				
Gender	Instagram	Snapchat	Twitter	Facebook	Total
Female	10	63	17	10	100
Male	25	41	25	9	100
Non-binary	7	34	45	14	100

(c) As we move across the gender categories, the proportions of each social media app change quite noticeably suggesting an association between the two variables.

The proportion of Snapchat users was higher amongst females.

The proportion of Instagram users was highest in males.

The proportion of Twitter users was highest among non-binary students.

25. *(a)* $= \dfrac{270}{18\ 000} \times 12 \times 100 = 18\%$ p.a.

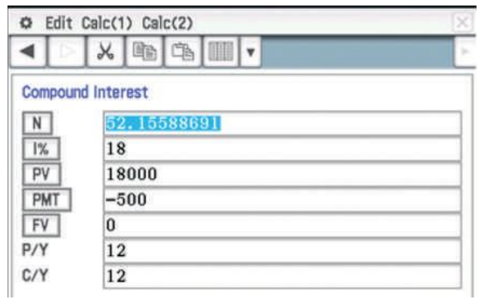
(b) A = $17 299.60

B = $17 299.60 × 0.015 = $259.49

C = $500

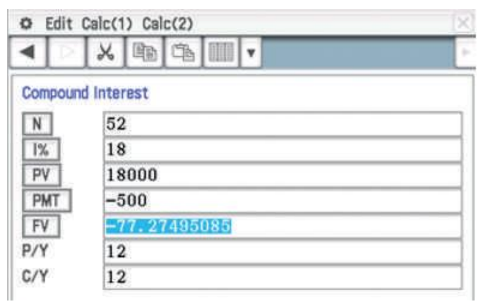
D = $17 299.60 + $259.49 – $500 = $17 059.09

(c)



53 months

(d)



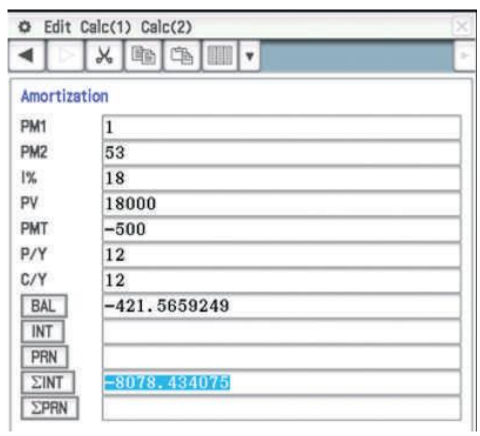
Final repayment = $77.27 × 1.015

= $78.43

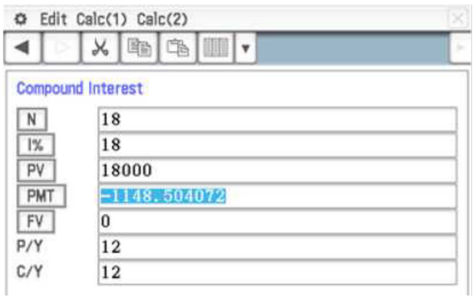
(e) Total interest = 52 × $500 + $78.43 – $18 000

= $26 078.43 – $18 000

= $8078.43



(f)



Repayments = $1148.51

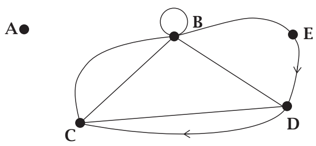
26. (a) *A* is an isolated vertex.

(b) The adjacency matrix is not symmetrical about the leading diagonal.

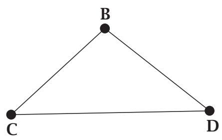
(c) There is a loop about vertex B.

(d) No – as vertex *A* is isolated.

(e)



(f)



Other solutions possible.

27. (a) Bipartite Graph.

(b)

	H1	H2	H3
T1	15	13	14
T2	11	8	9
T3	14	17	13
T4	12	13	15

(c)

15	13	14	0
11	8	9	0
14	17	13	0
12	13	15	0

4	5	5	0
0	0	0	0
3	9	4	0
1	5	6	0

3	4	4	0
0	0	0	1
2	8	3	0
0	4	5	0

1	2	2	0
0	0	0	3
0	6	1	0
0	4	5	2

1	1	1	0
1	0	0	4
0	5	0	0
0	3	4	2

1	1	1	0
1	0	0	4
0	5	0	0
0	3	4	2

Allocation:

Truck 2 → House 2

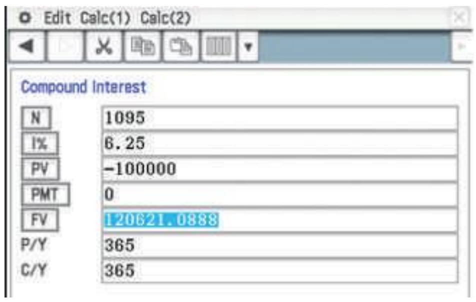
Truck 3 → House 3

Truck 4 → House 1

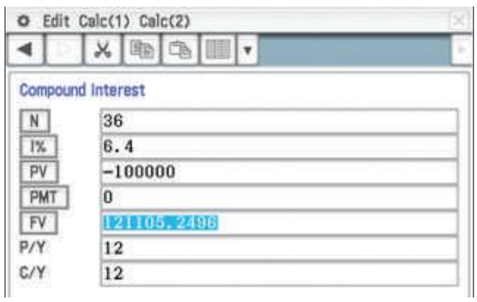
(d) Minimum distance = 12 + 8 + 13

= 33 km

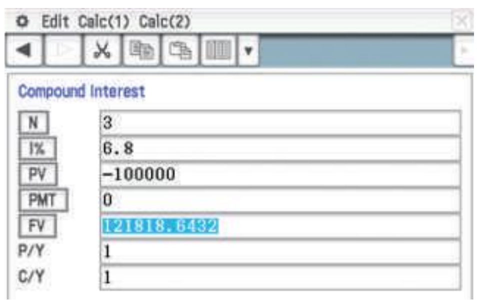
28. (a)



Profit = $20 621.09

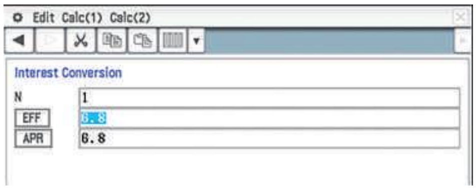


Profit = $21 105.25



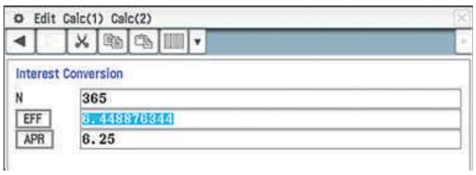
Profit = $21 818.64

(b)



6.8% Efffective 6.8%

6.4% Effective 6.591%

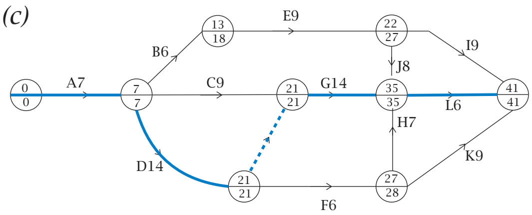


6.25% Effective 6.449%

29. (a) A dummy activity – no time or resource.

(b) Activity F

(c)



(d) Critical path = *ADGL*

Minimum completion time = 41 hours.

(e) Changing activity G will only reduce time by 1 hour. Critical path becomes *ADFHL* (40 hours).

30. (a) For every 1 cm increase in height, weight increases by 0.2946 kilograms.

(b) $r_{hw} = \sqrt{0.8098}$

$= 0.8999$

(c) (i) $w = 0.2946(180) + 3.3461$

$= 56.37$ kilograms : Reliable as interpolation and strong correlation.

(ii) $w = 0.2946(220) + 3.3461$

$= 68.16$ kilograms : Unreliable as extrapolation even though a strong correlation.

31. (a) $a = 250 \qquad r = \dfrac{200}{250} = 0.80$

(i) $T_n = 250(0.8)^{n-1}$

(ii) $T_{n+1} = 0.8T_n \;, \quad T_1 = 250$

(iii) $T_8 = 250(0.8)^7$

$= 52.4288$

(b) $a = -12.2 \qquad d = -10.1 - (-12.2) = 2.1$

(i) Arithmetic

(ii) $T_n = -12.2 + (n - 1)(2.1)$

$= -14.3 + 2.1n$

(iii) $T_{n+1} = T_n + 2.1 \;, \qquad T_1 = -12.2$

(iv) $T_{26} = -14.3 + 2.1(26)$

$= 40.3$

32. (a) Trend is increasing.

Cycles of 4 with a decrease followed by an increase.

(b) Smoothing the data to see the underlying trend.

(c) 4 point centred moving average. There are cycles of 4 in the time series graph.

(d) Moving average

$= \dfrac{0.5 \times 32 + 23 + 38 + 54 + 0.5 \times 42}{4}$

$= 38$

33. (a) *GFDEHJMLIK*

91 kilometres.

(b) *DEHIKM*

43 kilometres.

(c) Increases the shortest path by 1 km to 44 km.

Shortest path *DEHILM*.

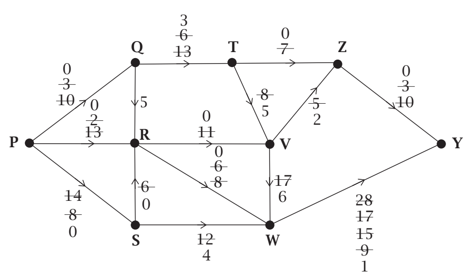
34. (a) $C1 = 13 + 5 + 13 + 6 + 12 = 49$

4900 cars

(b) $8 + 11 + 28 + x = 57$

$x = 10$

(c)



Max flow:

PQTZY = 7

PQTVZY = 3

PRVWY = 11

PRWY = 2

PSRWY = 6

PSWY = 8

Maximum flow = 37

Hence 3700 cars per minute.

35. *(a)* Graph is connected and contains a cycle.

(b) *(i)*

$$M = \begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 2 & 1 \\ 0 & 2 & 0 & 3 \\ 0 & 1 & 3 & 0 \end{bmatrix} \end{array}$$

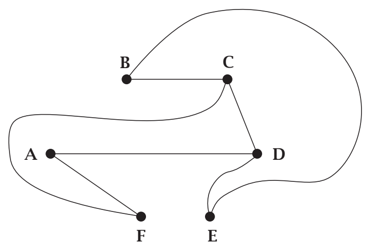
(ii)

$$M^2 = \begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 6 & 3 & 6 \\ 2 & 3 & 13 & 2 \\ 1 & 6 & 2 & 10 \end{bmatrix} \end{array}$$

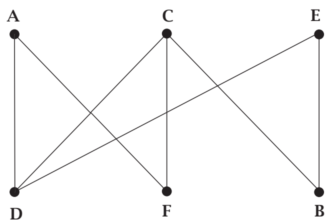
(iii) 2 two-stage walks.

(iv) The number of walks from one vertex to another of length 2 or 3, passing through 1 or 2 vertices.

(c) *(i)*



(ii)



Groups : $\{A, C, E\}$ and $\{D, F, B\}$

(iii) 2 extra edges for a total of 9.

36. *(a)* $\dfrac{39\,015}{45\,900} = 0.85$

(b) $a = \dfrac{45\,900}{0.85}$

$= \$54\,000$

$D_0 = \dfrac{54\,000}{0.85}$

$= \$63\,529.41$

(c) $D_n = 54\,000(0.85)^{n-1}$

(d) $15\,000 = 54\,000(0.85)^{n-1}$

$n = 8.88$ years

During 9^{th} year

(e) $D_{10} = 54\,000(0.85)^9$

$= \$12\,507.32$

37. *(a)*

$$\begin{array}{c} \\ F \\ Br \\ Ba \\ Bu \end{array}\begin{array}{c} \begin{array}{cccc} L & S & C & A \end{array} \\ \begin{bmatrix} 15 & 15 & 17 & 16 \\ 21 & 22 & 21 & 22 \\ 19 & 20 & 18 & 21 \\ 20 & 20 & 21 & 21 \end{bmatrix} \end{array}$$

$$\begin{bmatrix} 0 & 0 & 2 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 2 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} \boxed{0} & 0 & 2 & 0 \\ 0 & 1 & 0 & \boxed{0} \\ 1 & 2 & \boxed{0} & 2 \\ 0 & \boxed{0} & 1 & 0 \end{bmatrix}$$

Possible Allocation:

Freestyle → Laura
Breastroke → Adira
Backstroke → Chen
Butterfly → Senuri

(b) 3 other possible allocations

$$\begin{bmatrix} 0 & 0 & 2 & \boxed{0} \\ \boxed{0} & 1 & 0 & 0 \\ 1 & 2 & \boxed{0} & 2 \\ 0 & \boxed{0} & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & \boxed{0} & 2 & 0 \\ \boxed{0} & 1 & 0 & 0 \\ 1 & 2 & \boxed{0} & 2 \\ 0 & 0 & 1 & \boxed{0} \end{bmatrix} \begin{bmatrix} 0 & \boxed{0} & 2 & 0 \\ 0 & 1 & 0 & \boxed{0} \\ 1 & 2 & \boxed{0} & 2 \\ \boxed{0} & 0 & 1 & 0 \end{bmatrix}$$

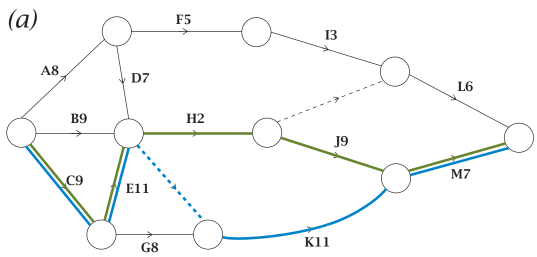
Freestyle → Adira
Breastroke → Laura
Backstroke → Chen
Butterfly → Senuri

Freestyle → Senuri
Breastroke → Laura
Backstroke → Chen
Butterfly → Adira

Freestyle → Senuri
Breastroke → Adira
Backstroke → Chen
Butterfly → Laura

(c) Minimum team time = 75 seconds

38. *(a)*



(b) Critical activities are: C, E, H, J, K, M.
Critical paths are: *CEHJM* and *CEKM*.
Minimum completion time = 38 days

(c) Float times: $D = 5$ days

$E = 0$ days

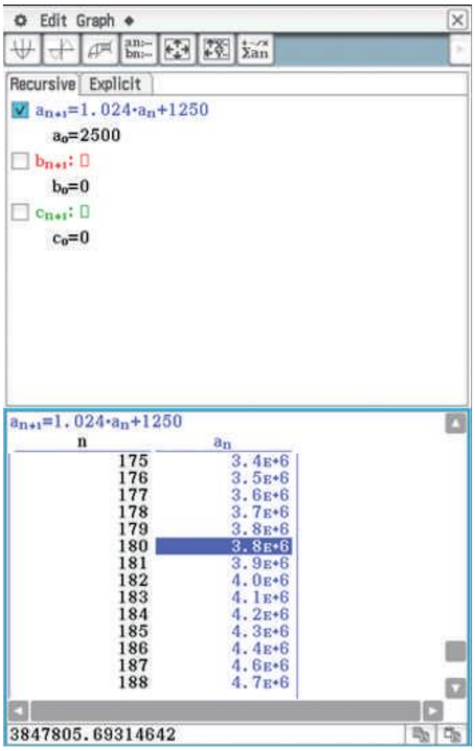
$F = 16$ days

(d) Activity D has a float time of 5 days. A delay of 6 days will change the critical activities to A, D, H, J, K, M and the minimum completion time will increase to 39 days.

39. (a) $0.024 \times 12 \times 100 = 28.8\%$ p.a.

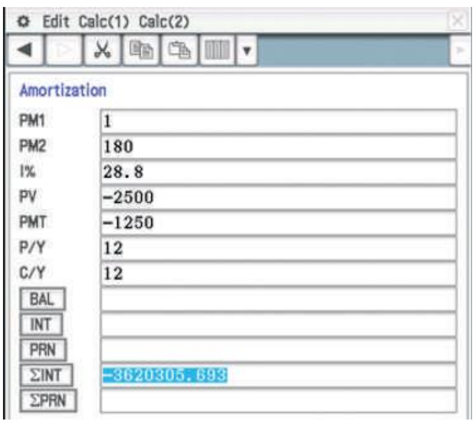
(b) $1250

(c)



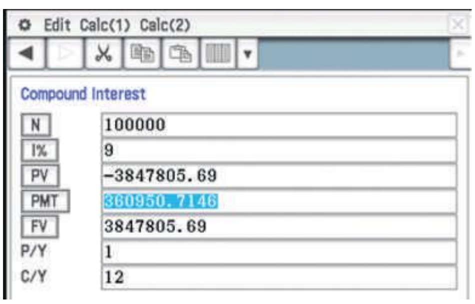
T_{180} = $3 847 805.69

(d)



Total interest = $3 620 305.69

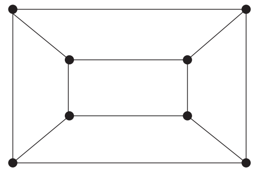
(e)



Annual perpetuity = $360 950.71

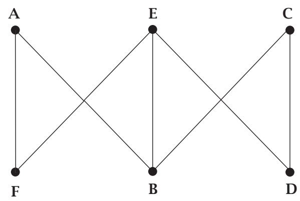
40. (a) Many examples. Graphs must have the following:

- not all vertices are even.
- visits every vertex and returns to starting vertex.

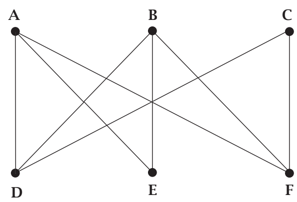


(b) (i) Bipartite

Graph I



Graph II



(ii) Eulerian – *IV*
Semi-Eulerian – *I*
Neither – *II, III*

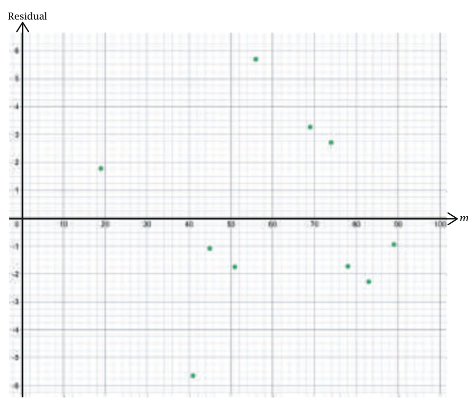
(iii) Hamiltonian Path – *I, II, IV*
Hamiltonian Cycle – *I, II*
Neither – *III*

41. (a) Missing residuals are:

Student 4: –0.94

Student 9: 3.26

(b)

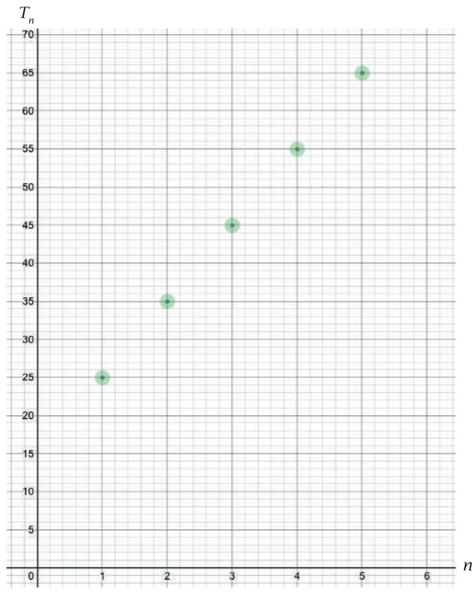


(c) The linear model is appropriate as there is no pattern evident in the residuals.

42. *(a)*

n	1	2	3	4	5
T_n	25	35	45	55	65

(b)



(c) $T_n = 25 + (n - 1)(10)$

$= 15 + 10n$

(d) $100 = 15 + 10n$

$n = 8.5$

$\therefore$ During week 9.

43. *(a)* To smooth the data in order to see the underlying trend.

(b) $\dfrac{0.5 \times X + 67 + 90 + 102 + 0.5 \times 82}{4} = 84.75$

$X = 78$

$Y = \dfrac{0.5 \times 115 + 90 + 83 + 106 + 0.5 \times 124}{4}$

$Y = 99.625$

(c) $m = 2.223t + 77.330$

(d) Quarter 4

$4 - 0.903 - 0.804 - 1.062 = 1.231$

(e) $t = 16$

$m = 2.223(16) + 77.330$

$m = 112.898$

Mobile bill $= 112.898 \times 1.231$

$= \$138.98$

44. *(a)*

Attraction	*B*	*C*	*D*	*E*	*F*	*G*
Time (minutes)	4	5	5	8	11	10

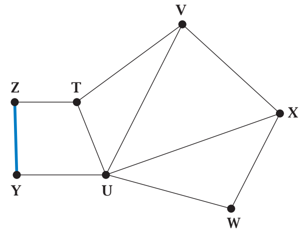
(b) *ABDG* 10 minutes

(c) *ABDEFG* or *ABDFG* 12 minutes

45. *(a)* Hamiltonian Path – starts and finishes at different vertices and visits all vertices.

Journey: *ZTVXWUY* (other possible solutions)

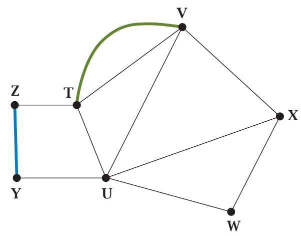
(b)



Hamiltonian Cycle – starts and finishes at the same vertex and visits all vertices.

Journey: *YZTVXWUY* (other possible solutions)

(c)

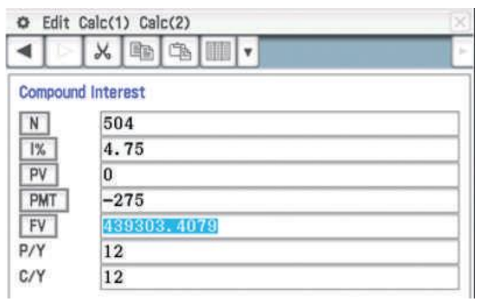


Semi-Eulerian trail – two odd vertices travelling every edge once.

Journey: *XVTVUWXUYZT* (other possible solutions)

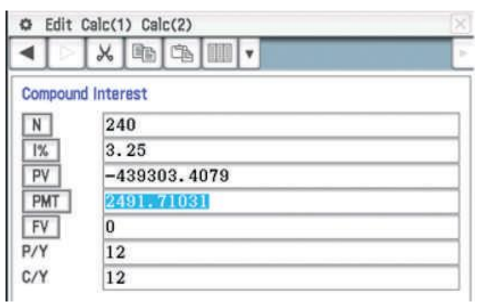
Join two of T,V,U or X.

46. *(a)*



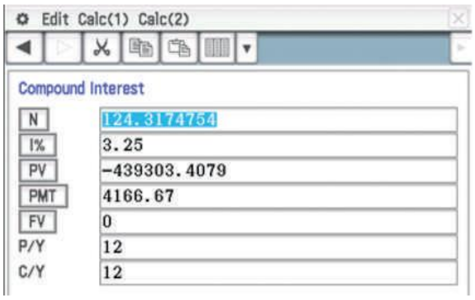
Amount = $439 303.41

(b)

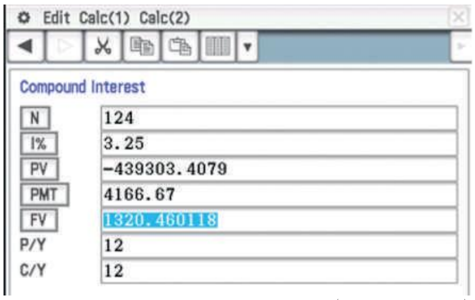


Withdrawal amount = $2 491.71

(c)



125 months



$$\text{Final payment} = \$1\,320.46 \times \left(1 + \frac{0.0325}{12}\right) = \$1\,324.04$$

47. (a)

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 18 & 16 & 31 & 19 \\ 15 & 10 & 24 & 14 \\ 21 & 19 & 32 & 20 \\ 23 & 22 & 25 & 30 \end{bmatrix} \end{array}$$

$$\begin{bmatrix} 14 & 16 & 1 & 13 \\ 17 & 22 & 8 & 18 \\ 11 & 13 & 0 & 12 \\ 9 & 10 & 7 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 13 & 15 & 0 & 12 \\ 9 & 14 & 0 & 10 \\ 11 & 13 & 0 & 12 \\ 7 & 8 & 5 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 7 & 0 & 12 \\ 2 & 6 & 0 & 10 \\ 4 & 5 & 0 & 12 \\ 0 & 0 & 5 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 5 & 0 & 10 \\ 0 & 4 & 0 & 8 \\ 2 & 3 & 0 & 10 \\ 0 & 0 & 7 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 3 & 0 & 8 \\ 0 & 4 & 2 & 8 \\ 0 & 1 & 0 & 8 \\ 0 & 0 & 9 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 2 & 0 & 7 \\ 0 & 3 & 2 & 7 \\ 0 & 0 & 0 & 7 \\ 1 & 0 & 10 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 2 & \boxed{0} & 7 \\ \boxed{0} & 3 & 2 & 7 \\ 0 & \boxed{0} & 0 & 7 \\ 1 & 0 & 10 & \boxed{0} \end{bmatrix}$$

Allocation:

Alain → Car 2
Bev → Car 3
Chan → Car 1
Dehemi → Car 4

(b) Maximum sales = $(15 + 19 + 31 + 30)
= $950 000

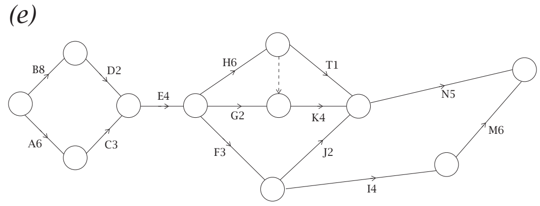
48. (a) Activity E.

(b) Critical activities are: *B*, *D*, *E*, *H*, *L*, *K*, *N*
Critical paths = *BDEHL*
= *BDEHKN*

(c) Minimum completion time = 29 hours.

(d) Float time Activity M = 2 hours.

(e)



(f) One critical path only with critical activities: *B*, *D*, *E*, *H*, *K*, *N*.
Minimum completion time has remained the same at 29 days.

49. (a) (32, 13)

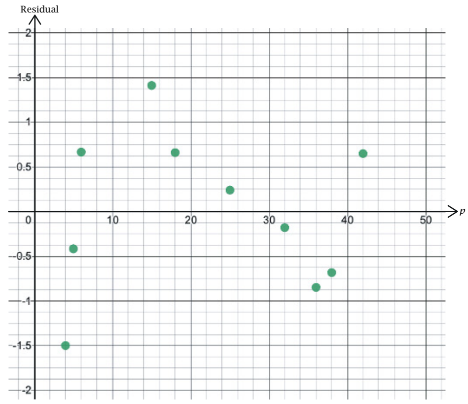
$q = -0.0829(32) + 15.8311$
$= 13.1783$

Residual = 13 – 13.1783 = −0.1783

(15, 16)

$q = -0.0829(15) + 15.8311$
$= 14.5876$

Residual = 16 – 14.5876 = 1.4124



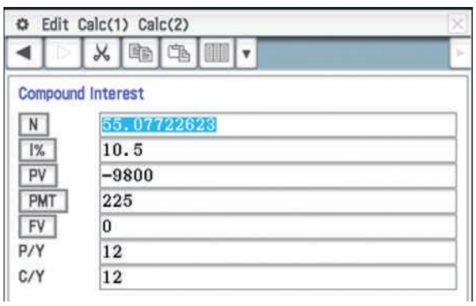
(b) A pattern is evident in the residuals hence the linear model is inappropriate.

50. (a)

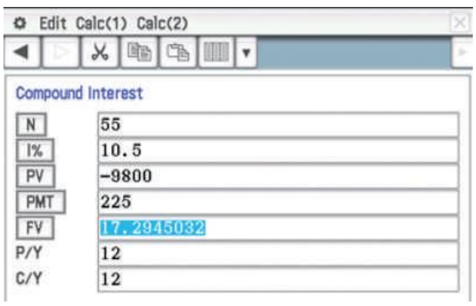
1	$9800	$85.75	$225	$9660.75
2	$9660.75	$84.53	$225	$9520.28

(b) $A_{n+1} = 1.00875A_n - 225 \qquad A_1 = 9800$

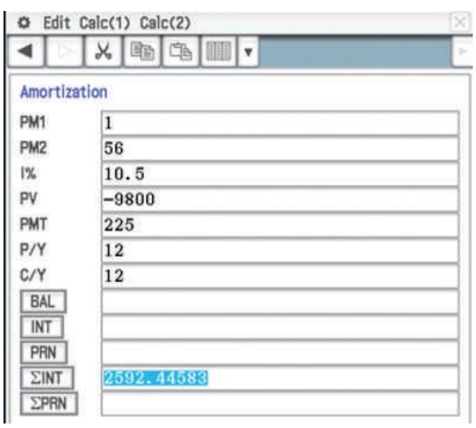
(c) 56 months



(d)



$$\text{Final repayment} = \$17.29 \times \left(1 + \frac{0.105}{12}\right) = \$17.44$$



Total interest =$2592.45

51. (a)

$$M = \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \\ F \\ G \end{array} \begin{array}{c} \begin{array}{ccccccc} A & B & C & D & E & F & G \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

(b) (i) *ABCDEFA* > (Other possible solutions)

(ii) *ABCDE* (Other possible solutions)

(iii) *AGEFABCDE* (Other possible solutions)

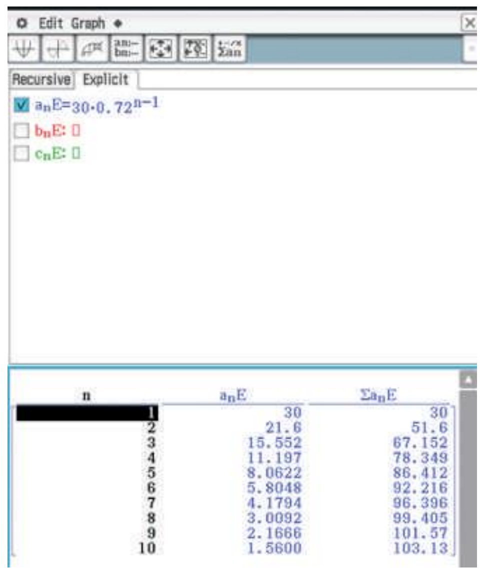
52. (a) $\frac{21.6}{30} = 0.72 \qquad \frac{15.552}{21.6} = 0.72$

Common ratio – geometric sequence.

(b) $T_n = (30)\,(0.72)^{n-1}$

(c) $T_5 = 8.0621568$ cm

(d) For stability, star picket must be driven in 1 metre (100 cm).



Require 9 hits.

53. (a) *ABEDA* (Other possible solutions)

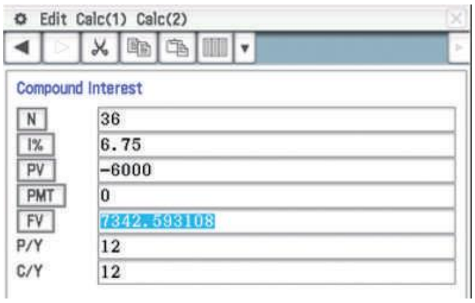
(b) Mathematics: Neither – Vertices are not all even **or** more than 2 odd vertices.

Science: Semi-Eulerian – 2 vertices are odd.

(c) Neither

(d) 2 edges – Example: *B* – *I* and *D* – *E*

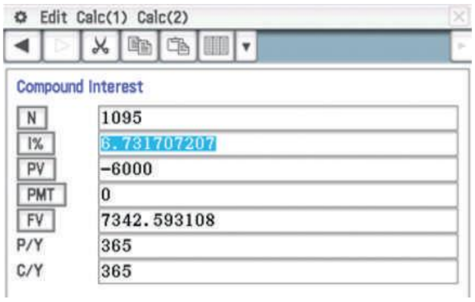
54. (a)



Final amount = \$7 342.59

(b) $T_{n+1} = 1.005625T_n$ $\quad T_0 = 6000$

(c)



Interest rate = 6.7317% p.a.

55. (a) $r_{ad} = -0.8991$.

(b) $d = -0.3109a + 19.2218$

(c) For each 1-year increase in an employee's age the number of days absent decreases by 0.3109.

(d) $d = -0.3109(30) + 19.2218$

$= 9.8948$ days

≈ 10 days

(e) Valid prediction due to strong correlation coefficient and interpolation.

(f) Point is (20, 15)

$d = -0.3109(20) + 19.2218$

$d = 13.0038$

Residual = 15 – 13.0038

= 1.9962

(g) The number of days absent is below the predicted number of days absent.

(h) Explained $r^2 = 0.8084$

Unexplained = 19.16%

56. (a) $T_5 = 512.96$ *L*

(b) $T_{n+1} = 0.6T_n + 200,$ $\quad T_1 = 600$

$r = 0.6 \quad d = 200 \quad a = 600$

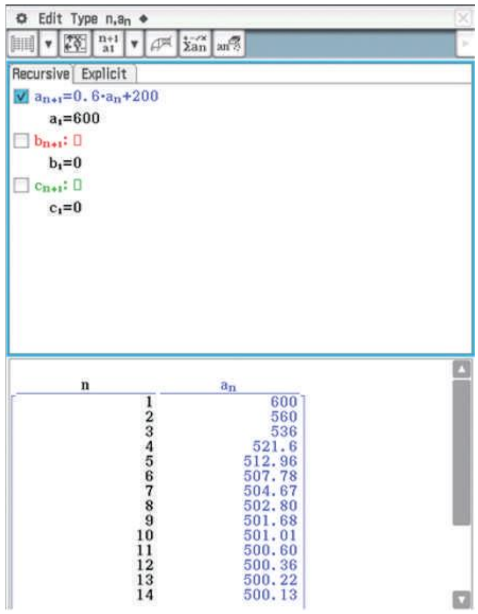
(c) Long term steady state – x

$\therefore \; x = 0.6x + 200$

$x = 500$ L

Fishpond capacity is 600 L. Long term steady state is 500 L.

Fishpond will <u>not</u> overflow.



57. (a) $A = 121$

$B = 165.75$

$C = 121.9$

$D = 170.0$

$E = 1.188$

(b) Winter – lowest seasonal index value (0.296).

(c) $s = -6.0773t + 184.561$

(d) Decreasing sales as gradient is negative (–6.0773).

(e) $t = 14$

$s = -6.0773(14) + 184.561$

$s = 99.4788$

Icecream sales = 99.4788 × 1.188

= 118.18

≈ 118 sales

58. (a) *FECAG* 13 minutes

(b) Hamiltonian Cycle

(c) *FBCAGDEF* 32 minutes.

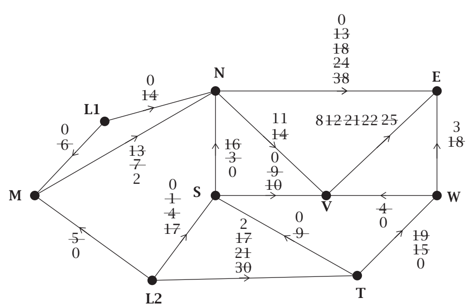
(d) 5:58 am

(e) *FECDGABF* 33 minutes

5:57 am

59. (a) $C1 = 38 + 25 + 19 = 82$

(b)



Max flow:

*L*1 *NE* = 14

*L*1 *MNE* = 6

*L*2 *MNE* = 5

*L*2 *SNE* = 13

*L*2 *SNVE* = 3

*L*2 *SVE* = 1

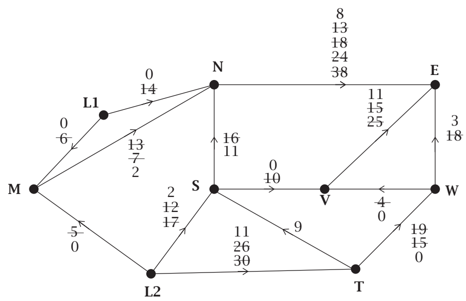
*L*2 *TSVE* = 9

*L*2 *TWVE* = 4

*L*2 *TWE* = 15

Maximum flow = 70 per minute

(c)



Max flow:

*L*1 *NE* = 14

*L*1 *MNE* = 6

*L*2 *MNE* = 5

*L*2 *SNE* = 5 Through *N* = 14 + 6 + 5 + 5 = 30

*L*2 *SVE* = 10

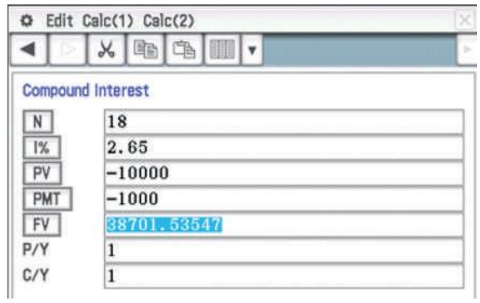
*L*2 *TWVE* = 4

*L*2 TWE = 15

Maximum flow = 59 per minute.

A decrease of 11 per minute.

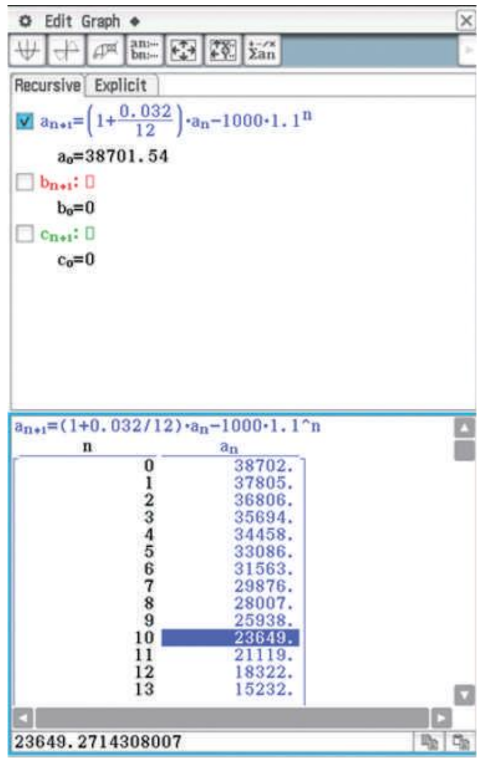
60. *(a)*



Amount = $38 701.54

(b) $A_{n+1} = \left(1 + \dfrac{0.032}{12}\right) A_n - 1000 \times 1.1^n$

(c)



Balance = $23 649.27

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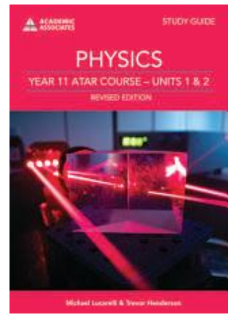

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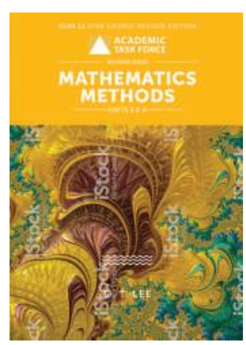

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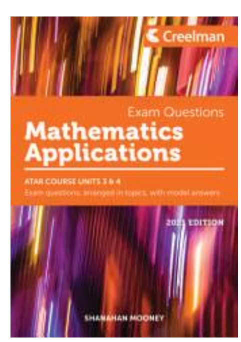

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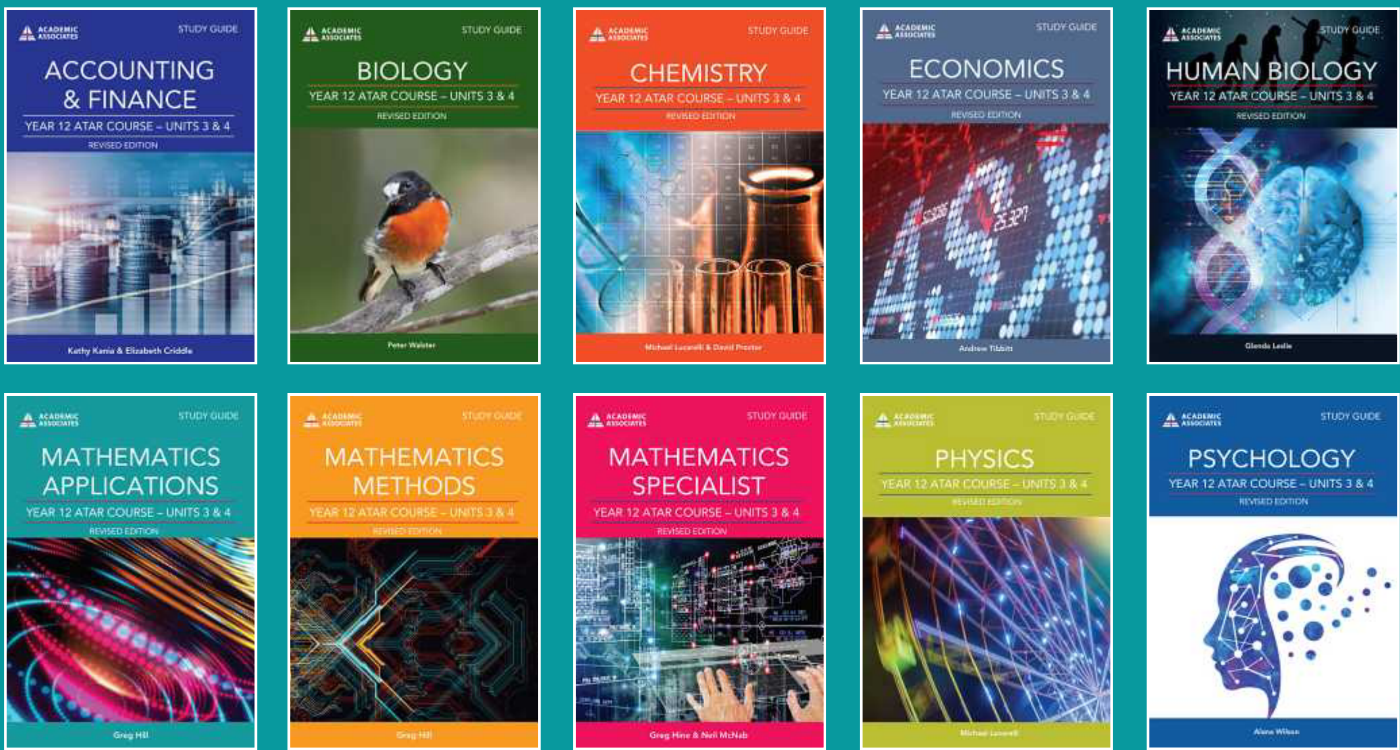

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